TWO ALGORITHMS FOR COMPUTING THE GENERAL COMPONENT OF JET SCHEME AND APPLICATIONS

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ABSTRACT. Let k be a perfect field. Let X be an integral k-variety. Let $m \in \mathbf{N}$. In this article, we study, from the theoretical and computational points of view, the component $\mathscr{G}_m(X)$ of the jet scheme $\mathscr{L}_m(X)$ defined to be the Zariski closure of the set of truncated arcs with a regular base-point. We prove that $\mathscr{G}_m(X)$ can be described from any smooth birational model of X. When X is supposed to be affine embedded in $\mathbf{A}_k^N$, this description allows us to provide an algorithm, valid in arbitrary characteristic, which computes a Gröbner basis of a presentation of $\mathscr{G}_m(X)$ in $\mathbf{A}_k^{(m+1)N}$ from the datum of a given explicit smooth affine birational model of X. Then, we show how to use the datum of a presentation of $\mathscr{G}_m(X)$ (in the suitable affine space) to deduce algebraic and geometric properties in two contexts from the theoretical and computational points of view. Firstly, we apply our main result to obtain bases for a class of homogeneous differential operators logarithmic along plane curves in arbitrary characteristics. Secondly, we introduce a motivic power series which encodes the geometry of all the $\mathscr{G}_m(X)$ and prove its rationality in the specific case of homogeneous plane curve singularities thanks to our description of $\mathscr{G}_m(X)$ *via* the normalization of X.

1. INTRODUCTION

1.1. Let k be a perfect field. Let X be a k-variety, i.e., a k-scheme of finite type. For every element $m \in \mathbf{N} \cup \{\infty\}$, one defines the *jet scheme* $\mathscr{L}_m(X)$ of level $m \in \mathbf{N}$ and the *arc scheme* $\mathscr{L}_\infty(X)$ associated with X by the functorial property

$$\mathrm{Hom}_{\mathbf{Sch}_k}(\mathrm{Spec}(A), \mathscr{L}_m(X)) \cong \mathrm{Hom}_{\mathbf{Sch}_k}(\mathrm{Spec}(A[[T]]/\langle T^{m+1}\rangle), X)$$

for every k-algebra A, and with the convention that $A[[T]]/\langle T^{m+1}\rangle = A[[T]]$ when $m = \infty$. When m runs over $\mathbf{N}$, the m-jet schemes together with the morphisms $\theta^m_{n,X}$, induced by the projections $k[[T]]/\langle T^m\rangle \to k[[T]]/\langle T^n\rangle$, with $m \geq n$, form a projective system of k-schemes whose limit exists in $\mathbf{Sch}_k$ and coincides with the arc scheme of X. We denote by $\theta_{m,X} : \mathscr{L}_\infty(X) \to \mathscr{L}_m(X)$ the canonical morphism of k-schemes. Let us stress that $\theta^1_{0,X}$ canonically is the *tangent bundle* defined from the tangent space $T_X := \mathscr{L}_1(X) \cong \mathrm{Spec}(\Omega^1_{X/k})$ of X to X.

1.2. By [19], one knows that the topology of the jet schemes is related to the singularities of X. Precisely, one has the following characterization:

Theorem ([19, Theorem 0.1]). *Let k be an algebraically closed field of characteristic zero. Let X be locally a complete intersection k-variety. The following assertions are equivalent:*

(1) The variety X only has rational singularities.
(2) The schemes $\mathscr{L}_m(X)$ are irreducible for every integer $m \geq 1$.

In the case of curves, the situation can be made more precise:

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Theorem ([15, Corollary 4.3], see also [19, Propositions 1.5, 4.12, Corollary 4.2])**.** *Let k be a perfect field. Let X be an integral k-curve. Then the following assertions are equivalent:*

(1) The tangent space T_X is irreducible.
(2) The tangent space T_X is integral.
(3) The curve X is smooth over k.

One easily observes that 1) $\Rightarrow$ 3) cannot be extended to higher dimensions in general.

1.3. Let $m \in \mathbf{N}$. If X is smooth over k, then the morphism $\theta^m_{0,X}$ is an affine bundle with fiber $\mathbf{A}_k^{m\dim(X)}$ (see [7, Ch. 3, Prop. 3.7.5]). Hence, if besides the k-variety X is assumed to be integral, then the jet scheme $\mathscr{L}_m(X)$ also is smooth and integral. In that case, the geometry and the topology of jet schemes is well understood. For any integral k-variety X, the picture is more complicated. One can observe that the Zariski closure of the open subset

$$(\theta^m_{0,X})^{-1}(\mathrm{Reg}(X)) = \mathscr{L}_m(\mathrm{Reg}(X))$$

provides an irreducible component of $\mathscr{L}_m(X)$ with dimension $(m+1)\dim(X)$; its existence is guaranteed by the perfectness of the field k (see subsection 2.1). The possible other irreducible components of $\mathscr{L}_m(X)$ dominate $\mathrm{Sing}(X)$. In this article, we are interested in understanding this *general component* of $\mathscr{L}_m(X)$ defined to be the (unique) reduced closed subscheme of $\mathscr{L}_m(X)$

$$\mathscr{G}_m(X) := \overline{(\theta^m_{0,X})^{-1}(\mathrm{Reg}(X))} = \overline{\mathscr{L}_m(\mathrm{Reg}(X))}.$$

By the former remark, we deduce that the scheme $\mathscr{L}_m(X)$ is irreducible if and only if $\mathscr{L}_m(X) = \mathscr{G}_m(X)$. In particular the general component of jet scheme plays a role in different situations (see sections 4, 8, 9) and obtaining methods to describe it from the theoretical and effective points of view seems to us a challenging problem. If $F \supset k$ is a field extension, the F-points of $\mathscr{G}_m(X)$ hence correspond to the elements of $\mathscr{L}_m(X)(F)$, i.e., *F-jets of level m*, which are specializations of m-jets with regular base points.

1.4. Let X be an integral k-variety. In section 3.1, we justify the following observation.

Theorem 1.5. *Let k be a perfect field. Let $m \geq 1$ be an integer. Let X, X' be integral k-varieties. We assume that X' is smooth over k. Let $h : X' \to X$ be a birational morphism. We denote by $h_m : \mathscr{L}_m(X') \to \mathscr{L}_m(X)$ the canonical morphism induced by h. Then, we have*

$$\mathscr{G}_m(X) = \overline{\theta_{m,X}(\mathscr{L}_\infty(X) \setminus \mathscr{L}_\infty(\mathrm{Sing}(X)))} = \overline{h_m(\mathscr{L}_m(X'))}.$$

In particular, if the arc scheme $\mathscr{L}_\infty(X)$ is irreducible, then the general component $\mathscr{G}_m(X)$ coincides with the Zariski closure of the image of $\mathscr{L}_\infty(X)$ by $\theta_{m,X}$.

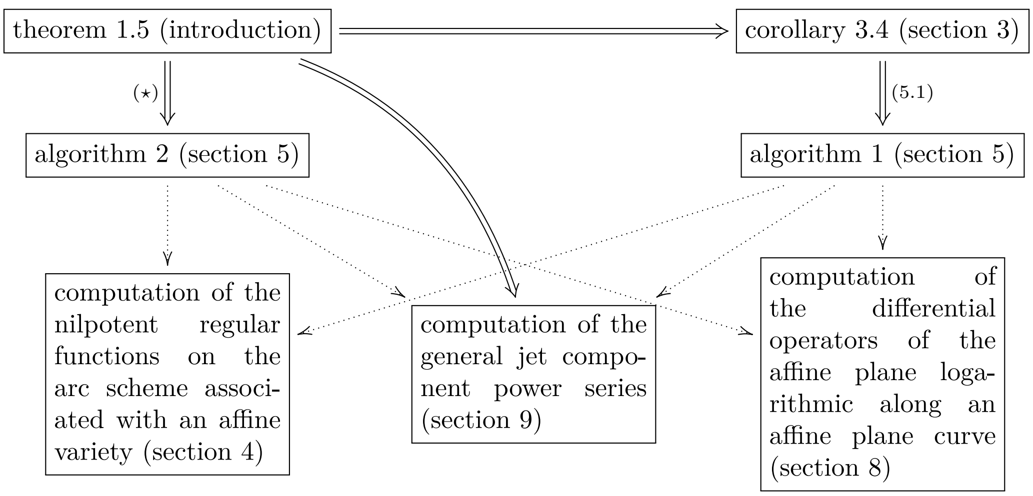
Let us stress that such a morphism h always exists for every variety X. Indeed, let us note that the inclusion $\mathrm{Reg}(X) \hookrightarrow X$ satisfies the required properties. In general, many other smooth birational models do exist. E.g., normalizations for curves, resolutions of singularities... Furthermore, if the characteristic of k is zero, then the Kolchin irreducibility theorem guarantees that $\mathscr{L}_\infty(X)$ is irreducible (but not reduced in general) when X is assumed to be irreducible; in particular, in this case, we always have the formula $\mathscr{G}_m(X) = \overline{\theta_{m,X}(\mathscr{L}_\infty(X))}$. When the characteristic of k is positive, as observed by Kolchin (see [13, IV/6/Exercise 3d)], [20, Remarque 1]), this property does not hold and, in general, there can exist irreducible components of $\mathscr{L}_\infty(X)$ contained in $\mathscr{L}_\infty(\mathrm{Sing}(X))$. However, if the dimension of X equals one, then, for any field k (of arbitrary characteristic), the arc scheme $\mathscr{L}_\infty(X)$ remains irreducible (see [7, Ch. 3, Lemma 4.3.1]).

1.6. In section 5, we provide two algorithms to compute a Gröbner basis of the ideal defining $\mathscr{G}_m(X)$ as a closed subscheme of the corresponding affine space when X is an affine integral variety defined over a perfect field of arbitrary characteristic. Algorithm 2, crucially based on theorem 1.5 and corollary 3.4, takes as input one smooth affine birational model of X. We also explain how to extend to integral varieties of arbitrary dimensions defined over fields with arbitrary characteristic the algorithm that Kpognon developed in his Ph.D. Thesis in the particular case of curves in characteristic zero (see also [14, §3.9, §4]); this is algorithm 1. It is based on formula (5.1). Somehow, algorithm 1 is a particular case of algorithm 2 where one chooses to consider any nonempty affine open subscheme of $\mathrm{Reg}(X)$ as an affine smooth birational model of X; but the algorithmic arguments are quite different. In practice, roughly speaking, algorithm 1 is based on the computation of some fixed saturated ideals whereas algorithm 2 is based on the computation of some more general kernels for which many choices exist.

1.7. In this end, we describe various situations where the understanding of the general component of jet scheme plays an important role. In section 4, we explain how the datum of a presentation of the ideal defining $\mathscr{G}_m(X)$ (for X an integral affine variety) can be interpreted as a system of generators of the nilpotent regular functions on the associated arc scheme (which only involve derivatives of order m). In section 8, we explain how the study of $\mathscr{G}_1(X)$, when $X = \mathrm{Spec}(k[x_1, x_2]/\langle f\rangle)$, can bring some information on homogeneous differential operators logarithmic along $f \in k[x_1, x_2]$ in arbitrary characteristics (see theorem 8.4). In this way our algorithms can provide system of generators of such objects. In section 9, we attach to the family $(\mathscr{G}_m(X))_{m\in\mathbf{N}}$ the *zeta function* $Z_X^{\mathrm{gen}}(T)$ *of the general components* of the jet schemes defined to be the following power series with coefficients in the *Grothendieck ring of varieties* $K_0(\mathbf{Var}_k)$:

$$Z_X^{\mathrm{gen}}(T) := \sum_{m\geq 0} [\mathscr{G}_m(X)]T^m \in K_0(\mathbf{Var}_k)[[T]],$$

by analogy with the zeta functions of Denef and Loeser. Since the latter ones are rational power series, we address the question to determine whether the zeta functions of the general components also are rational. In the case of homogeneous plane curve singularities, we prove that this power series is rational (see theorem 9.14) and state various observations. The article is organized as follows:



(⋆): up to the choice of a smooth birational model

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2. Notations, conventions

2.1. In this article, we assume that the field k is perfect with an arbitrary characteristic. A k-*variety* is a k-scheme of finite type. Because of the perfectness of the field k, every reduced k-variety X is geometrically reduced, then $\mathrm{Reg}(X)$ is not empty or, equivalently, $\mathrm{Sing}(X) \neq X$. In particular $\mathscr{L}_m(\mathrm{Reg}(X))$ is not empty for every $m \in \mathbf{N}$; hence, $\mathscr{G}_m(X)$ is well-defined as an irreducible component of $\mathscr{L}_m(X)$.

2.2. Let k be a field and R be a k-algebra. A k-*derivation* on R is a morphism of k-modules $D : R \to R$ which satisfies the Leibniz rule, i.e., for $a, b \in R$ we have $D(ab) = aD(b) + bD(a)$. We denote by $\mathrm{Der}_k(R)$ the set of k-derivations on R, which is naturally an R-module with multiplication defined by $aD : b \mapsto a(D(b)) \in R$ for $a \in R$ and $D \in \mathrm{Der}_k(R)$. The pair (R, D) is a *differential k-algebra.* (Indeed, here, the field k is endowed with the trivial derivation.) An ideal $I \subset R$ is a *differential ideal* if and only if $D(I) \subset I$. Given $S \subset R$ we denote by $[S] := \left\langle D^i(s); s \in S, i \in \mathbf{N}\right\rangle$ the differential ideal generated by S; it is the smallest differential ideal of (R, D) containing S. The radical of $[S]$ is denoted by $\{S\}$ and it is also a differential ideal of (R, D) if k is a field of characteristic 0. As usually, if S only contains the element $a \in R$, we will simply write $[a]$ and $\{a\}$ instead of $[S]$ and $\{S\}$. An ideal I of R is called *radical* or *reduced* if $I = \sqrt{I}$. For $n \in \mathbf{N}$, $m \in \mathbf{N} \cup \{\infty\}$ let us note

$$k[x_1, \ldots, x_n]_m := k[(x_{i,j}); i \in \{1, \ldots, n\}, j \in \{0, \ldots, m\}]$$

which is a $k[x_1, \ldots, x_n]$-module via the identification of $k[x_1, \ldots, x_n]_0$ and $k[x_1, \ldots, x_n]$. The pair $(k[x_1, \ldots, x_n]_\infty, \Delta)$ is a differential k-algebra, with Δ defined by $\Delta(x_{i,j}) = x_{i,j+1}$. In the sequel, for $m \in \mathbf{N} \cup \{\infty\}$ we endow $k[x_1, \ldots, x_n]_m$ with the weighted graded structure such that $\deg(x_{i,j}) = j$ for every $1 \leq i \leq n$ and every $0 \leq j \leq m$. Hence a term of the form $\prod_{i=1}^{n} \prod_{j=0}^{m} x_{i,j}^{\alpha_{i,j}}$ has degree $\sum_{i=1}^{n} \sum_{j=0}^{m} j\alpha_{i,j}$.

2.3. Let $m \in \mathbf{N} \cup \{\infty\}$. For every polynomial $f \in k[x_1, \ldots, x_n]$, there exists a unique family $(f_s)_{0 \leq s \leq m}$ of polynomials in $k[x_1, \ldots, x_n]_m$ such that the following equality holds the ring $k[x_1, \ldots, x_n]_m[\![t]\!]$:

$$f\left(\left(\sum_{j=0}^{m} x_{i,j} t^j\right)_{0 \leq i \leq n}\right) = \sum_{s=0}^{m} f_s\left((x_{i,j})_{\substack{0 \leq i \leq n \\ 0 \leq j \leq s}}\right) t^s \pmod{t^{m+1}}. \tag{2.1}$$

Let X be an affine k-variety such that $\mathcal{O}(X) = k[x_1, \ldots, x_n]/I$. We define the ideal I_m of the ring $k[x_1, \ldots, x_n]_m$ to be

$$I_m := \langle f_s : f \in I,\ 0 \leq s \leq m\rangle.$$

Then [3, Propositions 6.1.1 and 6.5.1] assure that

$$\mathscr{L}_m(X) = \mathrm{Spec}(k[x_1, \ldots, x_n]_m / I_m)$$

(see *loc. cit.* for the relation of the family $(f_s)_{0 \leq s \leq m}$ with Hasse-Schmidt derivations). Let us stress that, by [3, Corollaries 6.2.1 and 6.5.2], when the field k is of characteristic zero, $\mathscr{L}_m(X)$

is also isomorphic to $\mathrm{Spec}(k[x_1,\ldots,x_n]_m/\langle \Delta^s(f) : f \in I, 0 \le s \le m\rangle)$. Indeed, one can easily prove the formula

$$(s!)f_s((x_{i,j}/j!)) = \Delta^s(f)$$

for every integer $s \in \mathbf{N}$. Since in this work we consider the more general setting of perfect fields, we will make use of the former presentation. Let us note that, regardless of the characteristic of the base field, for $f \in k[x_1,\ldots,x_n]$ we have $f_1 = \Delta(f)$; hence I_1 can be written as $\langle f, \Delta(f) : f \in I\rangle$.

Definition 2.4. Let k be a perfect field. Let X be an integral k-variety. Let $m \in \mathbf{N}$. We denote by $\mathscr{N}_{m,X}$ the ideal subsheaf of $\mathcal{O}_{\mathscr{L}_m(X)}$ defining $\mathscr{G}_m(X)$ as a closed subscheme of $\mathscr{L}_m(X)$.

2.5. When X is an affine variety such that $\mathcal{O}(X) = k[x_1,\ldots,x_n]/I$, in an abuse of notation we denote by $\mathscr{N}_m(X)$ the unique ideal of $k[x_1,\ldots,x_n]_m$ containing I_m whose image via the canonical quotient morphism is the ideal $\mathscr{N}_{m,X}(X)$ of $\mathcal{O}(\mathscr{L}_m(X))$. It is a prime ideal since $\mathscr{G}_m(X)$ is irreducible and reduced. The ideal $\mathscr{N}_m(X)$ has been studied for base-fields of characteristic zero, when $n = 2$ or $m = 1$ in [14, 18, 24, 12, 23, 22] (see also [4, 5]). Let us also note that any element of $\mathscr{N}_1(X)$ of degree one corresponds to a torsion Kähler differential form (e.g., see [12, Corollary 4.7], [24, Lemma 2.5], or [23, Proposition 2.3]).

3. General component and smooth birational models

3.1. Let X' be an integral smooth k-variety. Recall from [7, Ch. 3, Prop. 3.7.5] that, if U is a nonempty open subset of X', then for every $m \in \mathbf{N} \cup \{\infty\}$ the morphism $\theta^m_{0,X'}$ is an affine bundle over U with an affine space as fiber. Hence $\mathscr{L}_m(U) = (\theta^m_{0,X'})^{-1}(U)$ and it is an integral dense open subscheme of $\mathscr{L}_m(X')$.

3.2. Let U be an open subscheme of X. Let us observe that $\mathrm{Reg}(U) = \mathrm{Reg}(X) \cap U$ and $\mathrm{Reg}(U)$ is a dense open subset of $\mathrm{Reg}(X)$. From subsection 3.1, we deduce that $\mathscr{L}_m(\mathrm{Reg}(U)) := (\theta^m_{0,X})^{-1}(\mathrm{Reg}(U))$ is still a dense open subset of $\mathscr{L}_m(\mathrm{Reg}(X)) := (\theta^m_{0,X})^{-1}(\mathrm{Reg}(X))$. Thus, we conclude that $\mathscr{G}_m(X)$ coincides with the Zariski closure of $\mathscr{G}_m(U)$ in $\mathscr{L}_m(X)$, i.e.,

$$\mathscr{G}_m(X) = \overline{\mathscr{G}_m(U)}^{\mathscr{L}_m(X)}.$$

More generally, analogous arguments imply that $\mathscr{G}_m(X)$ coincides with the Zariski closure in $\mathscr{L}_m(O)$ for every dense open subscheme O of X contained in $\mathrm{Reg}(X)$.

3.3. Let us prove theorem 1.5, we begin with the first equality. Let us observe that the (irreducible) open subset $\theta^{-1}_{0,X}(\mathrm{Reg}(X))$ is an open dense subset of $\mathscr{L}_\infty(X) \setminus \mathscr{L}_\infty(\mathrm{Sing}(X))$ (e.g., see [7, Ch. 3, Corollary 4.2.3]). Since $\theta_{0,X} = \theta^m_{0,X} \circ \theta_{m,X}$, we deduce that

$$\begin{aligned}\theta_{m,X}(\overline{\theta^{-1}_{0,X}(\mathrm{Reg}(X))}) = \theta_{m,X}(\overline{\theta^{-1}_{m,X}((\theta^m_{0,X})^{-1}(\mathrm{Reg}(X)))}) &\subset \overline{\theta_{m,X}(\theta^{-1}_{m,X}((\theta^m_{0,X})^{-1}(\mathrm{Reg}(X))))} \\ &\subset \overline{(\theta^m_{0,X})^{-1}(\mathrm{Reg}(X))}. \qquad (3.1)\end{aligned}$$

Hence, by formula (3.1)

$$\theta_{m,X}(\mathscr{L}_\infty(X) \setminus \mathscr{L}_\infty(\mathrm{Sing}(X))) \subset \mathscr{G}_m(X)$$

for every integer $m \in \mathbf{N}$. By subsection 3.1 we conclude that, for every integer $m \in \mathbf{N}$,

$$\theta_{m,X}(\theta^{-1}_{0,X}(\mathrm{Reg}(X))) = \theta_{m,\mathrm{Reg}(X)}(\mathscr{L}_\infty(\mathrm{Reg}(X))) = \mathscr{L}_m(\mathrm{Reg}(X)) \subset \mathscr{G}_m(X).$$

Thus, the subset $\theta_{m,X}(\mathscr{L}_\infty(X) \setminus \mathscr{L}_\infty(\mathrm{Sing}(X)))$ contains a nonempty open subset of $\mathscr{G}_m(X)$. Since the general component is irreducible, this open subset is dense in $\mathscr{G}_m(X)$; hence, we have

$$\overline{\theta_{m,X}(\mathscr{L}_\infty(X) \setminus \mathscr{L}_\infty(\mathrm{Sing}(X)))} = \mathscr{G}_m(X).$$

Let us prove that $\mathscr{G}_m(X) = \overline{h_m(\mathscr{L}_m(X'))}$. The variety X being integral and h being birational, we can find dense open subsets $U \subset X$ and $V \subset X'$ such that h induces an isomorphism $U \cong V$. In particular, the variety X' being smooth, V also is and so is U, hence $U \subset \mathrm{Reg}(X)$. By subsection 3.2 we deduce that $\mathscr{G}_m(X) = \overline{\mathscr{L}_m(U)}$. Since h_m induces an isomorphism between $\mathscr{L}_m(V)$ and $\mathscr{L}_m(U)$, we deduce that $\mathscr{G}_m(X) = \overline{h_m(\mathscr{L}_m(V))}$.

Now, since the k-variety X' is smooth, by subsection 3.1 the m-jet scheme $\mathscr{L}_m(X')$ is irreducible and hence $\mathscr{L}_m(V)$ is a dense open subset of $\mathscr{L}_m(X')$. Then

$$\overline{h_m(\mathscr{L}_m(V))} \subset \overline{h_m(\mathscr{L}_m(X'))} = \overline{h_m\left(\overline{\mathscr{L}_m(V)}^{\mathscr{L}_m(X')}\right)}^{\mathscr{L}_m(X)} \subset \overline{h_m\left(\mathscr{L}_m(V)\right)} \ (= \mathscr{G}_m(X)),$$

which concludes the proof.

Corollary 3.4. *Let k be a perfect field and $m \geq 1$ be an integer. Let X, X' be integral k-varieties. We assume that X' is smooth over k. Let $h : X' \to X$ be a birational morphism. We denote by $h_m : \mathscr{L}_m(X') \to \mathscr{L}_m(X)$ the induced morphism. Then $\mathscr{G}_m(X)$ is the scheme theoretic image of h_m and*

$$\mathscr{N}_{m,X} = \mathrm{Ker}(\mathcal{O}_{\mathscr{L}_m(X)} \to (h_m)_*\mathcal{O}_{\mathscr{L}_m(X')}).$$

Proof. Since the k-variety X' is smooth, so is $\mathscr{L}_m(X')$ by subsection 3.1 ; it is in particular reduced. Hence by [26, Tag 056B] the first assertion follows from theorem 1.5. Now the second assertion is a consequence of [26, Tag 01R8] since the morphism h_m is quasi-compact. □

4. General component and nilpotency

In this section, we explain the link between the general component of the jet scheme and the nilpotency on the arc scheme.

4.1. Let k be a perfect field and X be an integral k-variety. We assume that the arc scheme $\mathscr{L}_\infty(X)$ is irreducible (this last condition is in particular satisfied if $\dim(X) = 1$ or $\mathbf{char}(k) = 0$). Then we deduce from theorem 1.5 that the ideal sheaf $\mathscr{N}_{m,X}$ of $\mathcal{O}_{\mathscr{L}_m(X)}$ also defines $\overline{\theta_{m,X}(\mathscr{L}_\infty(X))}$ as a closed subscheme of $\mathscr{L}_m(X)$. In particular, if the k-variety X is affine with $\mathcal{O}(X) = k[x_1, \ldots, x_n]/I$, then

$$\mathscr{N}_m(X) = \{I\} \cap k[x_1, \ldots, x_n]_m$$

(recall that $\{I\}$ is the radical of the ideal I_∞ of $k[x_1, \ldots, x_n]_\infty$). In this way, we can interpret the elements of the ideal $\mathscr{N}_m(X)$ as the polynomials in $k[x_1, \ldots, x_n]_m$ which induce nilpotent regular functions on $\mathscr{L}_\infty(X)$.

4.2. In the spirit of Mustaţă's result (see 1.2), the second author asks about the possible relation between the reducedness of arc scheme and the existence of rational singularities in the case of locally complete intersection integral varieties (see [14, Question 9.6]). This question can be precised as follows:

Question 4.3 (Sebag)**.** *Let k be a field of characteristic zero. Let X be a lci integral k-variety. Are the following assertions equivalent?*

1. *The arc scheme $\mathscr{L}_\infty(X)$ is reduced.*
2. *For every integer $m \in \mathbf{N}$, the jet scheme $\mathscr{L}_m(X)$ is irreducible.*
3. *For every integer $m \in \mathbf{N}$, the jet scheme $\mathscr{L}_m(X)$ is reduced.*
4. *For every integer $m \in \mathbf{N}$, the jet scheme $\mathscr{L}_m(X)$ is integral.*
5. *The k-variety X has rational singularities.*

By [19], one knows that the following properties hold true: (2) $\Leftrightarrow$ (5), (2) $\Rightarrow$ (3), (4) $\Leftrightarrow$ (5), (2) $\Rightarrow$ (1), (5) $\Rightarrow$ (1). Besides, [19, Question 4.10] raises the question about the validity of (3) $\Rightarrow$ (2) and [19, Proposition 4.12] proves it for $m = 1$. Obviously, one also has (4) $\Rightarrow$ (2)-(3), (3) $\Rightarrow$ (1) and, formally, if (2) $\Leftrightarrow$ (3) holds true, then (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) also holds true. To conclude, the remaining open problem is to know whether (1) $\Rightarrow$ (5) holds true (see [14]). This question admits a positive answer in dimension one (see [19, Corollary 4.2] and [23, Corollary 1.8]). In [23, Corollary 1.7], the second author has proved that, if $\mathscr{L}_\infty(X)$ is reduced, the k-variety X necessarily is normal. Thanks to subsection 1.2, the algebraic counterpart of question 4.3 can be formulated as follows:

Question 4.4 (Sebag)**.** *Let k be a field of characteristic zero. Let X be a lci integral affine k-variety. Are the following assertions equivalent?*

(1) The k-algebra $\mathcal{O}(\mathscr{L}_\infty(X))$ is reduced.
(2) For every integer $m \in \mathbf{N}$, we have $\mathscr{N}_m(X) = \sqrt{I_m}$.
(3) For every integer $m \in \mathbf{N}$, we have $I_m = \sqrt{I_m}$.
(4) For every integer $m \in \mathbf{N}$, we have $\mathscr{N}_m(X) = \sqrt{I_m} = I_m$.

Indeed, property (2) (resp. (3), (4)) means that $\mathscr{L}_m(X)$ is irreducible (resp. reduced, integral) for every integer $m \in \mathbf{N}$. Let us stress that one has $\sqrt{I_m} \subset \{I\} \cap k[x_1, \ldots, x_n]_m$; the other inclusion is the difficult point.

Example 4.5. Let k be a field of characteristic zero. Let us consider the polynomial $f = x_1^3 + x_2^3 + x_3^3$ in the ring $k[x_1, x_2, x_3]$, and X the associated surface in $\mathbf{A}^3_k$. It is well-known that this k-variety is a normal variety whose singularities are not rational. Let us also note that its tangent space is reduced, as every normal hypersurface of an affine space, and that the first (in order) nontrivial nilpotent function on $\mathscr{L}_\infty(X)$ appears as a function on $\mathscr{L}_2(X)$. The algorithms we discuss in section 5 allow us to exhibit such regular functions on $\mathscr{L}_2(X)$. E.g., one can consider the function induced by the polynomial $g := x_{10}^2x_{20}x_{21}x_{30}x_{32} - x_{10}x_{11}x_{20}^2x_{30}x_{32} + x_{10}^2x_{20}x_{21}x_{31}^2 - x_{10}x_{11}x_{20}^2x_{31}^2 - x_{10}^2x_{20}x_{22}x_{30}x_{31} - x_{10}^2x_{21}^2x_{30}x_{31} + x_{10}x_{12}x_{20}^2x_{30}x_{31} + x_{11}^2x_{20}^2x_{30}x_{31} + x_{10}x_{11}x_{20}x_{22}x_{30}^2 + x_{10}x_{11}x_{21}^2x_{30}^2 - x_{10}x_{12}x_{20}x_{21}x_{30}^2 - x_{11}^2x_{20}x_{21}x_{30}^2$ which appears as an element of a Groebner basis of $\mathscr{N}_2(X)$. We observe that its weighted degree is 3 in the polynomial ring $k[x_1, x_2, x_3]_2$. A computation realized with SageMath shows that $g^2 \in \langle f_0, f_1, f_2\rangle$. Because of its weighted degree, in order to see that the function induced by g in $\mathscr{L}_\infty(X)$ does not vanish it suffices to show that it does not belong to $\langle f_0, f_1, f_2, f_3\rangle$ in $k[x_1, x_2, x_3]_3$, and again this fact can be checked using SageMath. In particular, we deduce that the arc scheme associated with X is not reduced.

5. Algorithms for computing general components

Let k be a perfect field. In this section, we provide the pseudo-codes of two algorithms for computing the ideal $\mathscr{N}_m(X)$ when X is affine. Let us stress that algorithm 2 is new and require the datum of a smooth affine birational model of X. Algorithm 1 has been originally introduced in the Ph.D. Thesis of Kpognon for the study of arc scheme associated with integral affine plane curves in characteristic zero; we show here how to extend it to arbitrary integral varieties defined over perfect fields of arbitrary characteristics. Let us note that the latter algorithm can also be interpreted as a particular case of algorithm 2.

5.1. The following observation can also be deduced from [2, 3.2, 4.3]. We would like to thank Bourqui and Haiech for pointing it out to us (see also [18, Lemma 3.7]). Let $m \in \mathbf{N}$. Let k be a perfect field, X an integral affine k-variety such that $\mathcal{O}(X) = k[x_1, \ldots, x_n]/I$. Let $H \in k[x_1, \ldots, x_n]$ be a polynomial such that the associated closed subset $V(H)$ contains $\mathrm{Sing}(X)$ and $H \notin I$. By subsection 2.1, such a polynomial H exists. In this way, we observe that the standard open subset $D(H) := X \setminus V(H) \subset \mathrm{Reg}(X)$ and is a nonempty irreducible open subset of X. Similarly to the arguments used in subsection 8.1, we deduce

$$\mathscr{N}_m(X) = (I_m : H^\infty). \tag{5.1}$$

Such a formula can be interpreted as a kind of a general analog of the Rosenfeld lemma applied for algebraic equations. The Rosenfeld lemma works for polynomial differential equations in the context of Ritt-Kolchin's differential algebra theory. (See [13, Section 9].)

5.2. It is well-known that the saturation $(I_m : H^\infty)$ equals the elimination ideal of $I_m + \langle 1 - HT\rangle \subset k[x_1, \dots, x_n]_m[T]$ with respect to the variable T, which can be performed computationally using Gröbner bases (see [8, Ch. 3, §1, Theorem 2]). Hence $\mathscr{N}_m(X)$ can be computed using algorithm 1. Let us stress that the hypothesis of the base field being perfect is necessary for the choice of the element H belonging to the jacobian ideal of I, since this ideal can be computed algorithmically and in this case it defines $\mathrm{Sing}(X)$.

Algorithm 1: Computation of $\mathscr{N}_m(X)$ for X an affine integral variety

Input : A - the polynomial ring $k[x_1, \dots, x_n]$; I - an ideal of A; m - a nonnegative integer.
Output: the ideal $\mathscr{N}_m(X)$ of $k[x_1, \dots, x_n]_m$.

compute the jacobian ideal Jac of I;
choose $H \in A$ such that $H \in Jac$ and $H \notin I$;
$R := k[x_1, \dots, x_n]_m$;
$\overline{R} := R[T]$;
choose $Igen$ a list of elements of A which form a system of generators of I;
for $f \in Igen$ **do**
  for $s = 0$ **to** m **do**
    compute $f_s \in \overline{R}$ as in (2.1);
  end
end
$I_m := \langle f_s : f \in Igen, 0 \leq s \leq m\rangle$ ideal of $\overline{R}$;
$\overline{H} := H$ as an element of $\overline{R}$ via the natural injections $A \subset R \subset \overline{R}$;
$J_m := I_m + \langle 1 - \overline{H}T\rangle$ ideal of $\overline{R}$;
compute $K \subset R$ the elimination ideal of J_m with respect to the variable T;
return K

5.3. Let us now present our second algorithm, which is not based anymore on the computation of some saturated ideal. It is based on corollary 3.4, whose formulation in the case where X is an affine integral variety is the following.

Corollary 5.4. *Let k be a perfect field and $m \geq 1$ be an integer. Let X be an integral affine k-variety with $\mathcal{O}(X) = k[x_1, \dots, x_n]/I$. Let X' a smooth affine variety and $h : X' \to X$ a birational morphism. We denote by $h_m : \mathscr{L}_m(X') \to \mathscr{L}_m(X)$ the canonical morphism induced by h, by $\pi_m : k[x_1, \dots, x_n]_m \to k[x_1, \dots, x_n]_m/I_m$ the canonical morphism and by $h_m^\sharp : \mathcal{O}(\mathscr{L}_m(X)) \to \mathcal{O}(\mathscr{L}_m(X'))$ the morphism of k-algebras induced by h_m. Then we have*

$$\mathscr{N}_m(X) = \mathrm{Ker}(h_m^\sharp \circ \pi_m).$$

Let us stress that, using Gröbner bases, it is possible to compute algorithmically the kernel of a polynomial map, via elimination theory (see [1, Section 2.4]). Assume that $\mathcal{O}(X) = k[x_1, \dots, x_n]/I$ and $\mathcal{O}(X') = k[y_1, \dots, y_\ell]/J$, we also define $\pi := k[x_1, \dots, x_n] \to k[x_1, \dots, x_n]/I$ to be the canonical morphism. Then corollary 5.4 provides algorithm 2 for computing the ideal $\mathscr{N}_m(X)$ of $k[x_1, \dots, x_n]_m$.

Algorithm 2: Computation of $\mathscr{N}_m(X)$ for X an affine integral variety, given $h : X' \to X$ a birational morphism with X' smooth and affine.

Input : A - the polynomial ring $k[x_1, \dots, x_n]$; B - the polynomial ring $k[y_1, \dots, y_\ell]$; $Jgen$ - a list of elements of B which form a generating system of the ideal J; $[g_1, \dots, g_n]$ - a list of n elements of B such that $h^\sharp \circ \pi(x_i) = g_i \pmod J$ for $i \in \{1, \dots, n\}$; m - a nonnegative integer.

Output: the ideal $\mathscr{N}_m(X)$ of $k[x_1, \dots, x_n]_m$.

$R := k[x_1, \dots, x_n]_m$;
$\overline{R} := k[x_1, \dots, x_n, y_1, \dots, y_\ell]_m$;
for $j \in Jgen$ **do**
  for $s = 0$ **to** m **do**
    compute $j_s \in \overline{R}$ (via the inclusion $R \subset \overline{R}$) as in (2.1);
  end
end
$J_m := \langle j_s \,:\, j \in \, Jgen, 0 \le s \le m\rangle$ ideal of $\overline{R}$;
for $i = 1$ **to** n **do**
  $f_i := x_i - g_i \in \overline{R}$ (after injecting A and B into $\overline{R}$);
  for $s = 0$ **to** m **do**
    compute $f_{i,s} \in \overline{R}$ as in (2.1);
  end
end
$H' := \langle f_{i,s} \,:\, 1 \le i \le n,\ 0 \le s \le m\rangle$ ideal of $\overline{R}$;
$H := H' + J_m$ ideal of $\overline{R}$;
compute $K \subset R$ the elimination ideal of H with respect to the variables $y_{i,s}$ for $1 \le i \le \ell$, $0 \le s \le m$;
return K

5.5. A typical situation where we can apply algorithm 2 is when a resolution of the singularities of X is given by the normalization (e.g., for curves). Note that there exist many algorithms for computing normalizations (e.g., see [11, 25]). For example, the one developed by Greuel, Laplagne and Seelisch in [10] provides the normalization of an algebra of finite type over a field. Its implementation in SINGULAR takes as inputs the polynomial ring $k[x_1, \dots, x_n]$ and the ideal I.

5.6. Effective implementations of algorithms 1 and 2 require, in order to perform the elimination of variables which is a key ingredient in both algorithms, the computation of Gröbner bases (see [8, Ch. 3, §1, Theorem 2]). This step is computationally the hardest one and will set the order of complexity of the algorithms, which is similar in both cases. In section 6 we will test these two algorithms on several examples. Comments on possible implementations are made in section 7.

6. Examples

In this section we will compute the ideal $\mathscr{N}_m(X)$ in several examples using implementations (see [17]) of the algorithms 1 and 2 in SageMath [27]. Our implementations provide the reduced Gröbner basis (up to multiplication by constants) of $\mathscr{N}_m(X)$ for classical monomial orders (lexicographic, graded reverse lexicographic...). All the results for the examples in this section are obtained using the lexicographic order with $x_{n,m} > x_{n,m-1} > \cdots > x_{n,0} > x_{n-1,m} > \cdots > x_{0,0}$; the outputs of both algorithms are the same up to multiplication by constants. In [16] we provide further computations for these examples, using different monomial orders.

Example 6.1. We consider the integral affine plane curve

$$\mathscr{C} := \mathrm{Spec}(\mathbf{Q}[x,y]/\langle x^3 - y^2\rangle).$$

Its normalization is given by the morphism $\nu^\sharp : \mathbf{Q}[x,y]/\langle x^3-y^2\rangle \to \mathbf{Q}[T]$, $\nu^\sharp(x) = T^2$, $\nu^\sharp(y) = T^3$.

For the first values of $m \in \mathbf{N}$ our algorithms provide the following Gröbner bases of the ideal $\mathscr{N}_m(\mathscr{C})$ of $\mathbf{Q}[x,y]_m$ for the monomial lexicographic order $y_m > \cdots > y_0 > x_m > \cdots > x_0$.

- $\mathscr{N}_1(\mathscr{C}) = \langle y_0^2 - x_0^3, 2x_0y_1 - 3x_1y_0, 2y_0y_1 - 3x_0^2x_1, 4y_1^2 - 9x_0x_1^2\rangle$. See also [18, Theorem 7.6].
- $\mathscr{N}_2(\mathscr{C}) = \langle y_0^2-x_0^3, 2x_0y_1-3x_1y_0, 2y_0y_1-3x_0^2x_1, 4y_1^2-9x_0x_1^2, 4x_0y_2-x_1y_1-6x_2y_0, 8y_0y_2-12x_0^2x_2-3x_0x_1^2, 16y_1y_2-36x_0x_1x_2-9x_1^3\rangle$.
- $\mathscr{N}_3(\mathscr{C}) = \langle y_0^2-x_0^3, 2x_0y_1-3x_1y_0, 2y_0y_1-3x_0^2x_1, 4y_1^2-9x_0x_1^2, 4x_0y_2-x_1y_1-6x_2y_0, 8y_0y_2-12x_0^2x_2-3x_0x_1^2, 16y_1y_2-36x_0x_1x_2-9x_1^3, 6x_0y_3+x_1y_2-4x_2y_1-9x_3y_0, 27x_1^3y_3+32y_2^3-108x_1^2x_2y_2-27x_1^2x_3y_1+54x_1x_2^2y_1-108x_2^3y_0, 16y_0y_3-24x_0^2x_3-12x_0x_1x_2+x_1^3, 24y_1y_3+16y_2^2-54x_0x_1x_3-36x_0x_2^2-45x_1^2x_2\rangle$.
- The obtained Gröbner basis of $\mathscr{N}_4(\mathscr{C})$ for the given monomial order contains 18 polynomials.
- The obtained Gröbner basis of $\mathscr{N}_5(\mathscr{C})$ for the given monomial order contains 33 polynomials.

Example 6.2. Let us set $k = \mathbf{F}_3$, which is a perfect field of characteristic 3. We consider the affine plane curve

$$\mathscr{C} := \mathrm{Spec}(\mathbf{F}_3[x,y]/\langle x^3 - y^2\rangle)$$

and its normalization given as in example 6.1 after replacing the base field $\mathbf{Q}$ by $\mathbf{F}_3$.

For the first values of $m \in \mathbf{N}$ our algorithms provide the following Gröbner bases of the ideal $\mathscr{N}_m(\mathscr{C})$ of $\mathbf{F}_3[x,y]_m$ for the monomial lexicographic order $y_m > \cdots > y_0 > x_m > \cdots > x_0$.

- $\mathscr{N}_1(\mathscr{C}) = \langle y_0^2 - x_0^3, y_1\rangle$.
- $\mathscr{N}_2(\mathscr{C}) = \langle y_0^2 - x_0^3, y_1, y_2\rangle$.
- $\mathscr{N}_3(\mathscr{C}) = \langle y_0^2 - x_0^3, y_1, y_2, x_0^3y_3 + x_1^3y_0, y_0y_3 + x_1^3\rangle$.
- $\mathscr{N}_4(\mathscr{C}) = \langle y_0^2 - x_0^3, y_1, y_2, x_0^3y_3 + x_1^3y_0, y_0y_3 + x_1^3, y_4\rangle$.
- $\mathscr{N}_5(\mathscr{C}) = \langle y_0^2 - x_0^3, y_1, y_2, x_0^3y_3 + x_1^3y_0, y_0y_3 + x_1^3, y_4, y_5\rangle$.
- $\mathscr{N}_6(\mathscr{C}) = \langle y_0^2 - x_0^3, y_1, y_2, x_0^3y_3 + x_1^3y_0, y_0y_3 + x_1^3, y_4, y_5, x_0^3y_6 + x_1^3y_3 + x_2^3y_0, x_1^3y_6 + y_3^3 - x_2^3y_3, y_0y_6 - y_3^2 + x_2^3\rangle$.
- $\mathscr{N}_7(\mathscr{C}) = \langle y_0^2 - x_0^3, y_1, y_2, x_0^3y_3 + x_1^3y_0, y_0y_3 + x_1^3, y_4, y_5, x_0^3y_6 + x_1^3y_3 + x_2^3y_0, x_1^3y_6 + y_3^3 - x_2^3y_3, y_0y_6 - y_3^2 + x_2^3, y_7\rangle$.
- $\mathscr{N}_8(\mathscr{C}) = \langle y_0^2 - x_0^3, y_1, y_2, x_0^3y_3 + x_1^3y_0, y_0y_3 + x_1^3, y_4, y_5, x_0^3y_6 + x_1^3y_3 + x_2^3y_0, x_1^3y_6 + y_3^3 - x_2^3y_3, y_0y_6 - y_3^2 + x_2^3, y_7, y_8\rangle$.
- $\mathscr{N}_9(\mathscr{C}) = \langle y_0^2 - x_0^3, y_1, y_2, x_0^3y_3 + x_1^3y_0, y_0y_3 + x_1^3, y_4, y_5, x_0^3y_6 + x_1^3y_3 + x_2^3y_0, x_1^3y_6 + y_3^3 - x_2^3y_3, y_0y_6 - y_3^2 + x_2^3, y_7, y_8, x_0^3y_9 + y_3^3 - x_2^3y_3 + x_3^3y_0, x_1^3y_9 - y_3^2y_6 - x_3^3y_3, y_0y_9 + y_3y_6 + x_3^3, y_3^2y_9 - x_2^3y_9 + y_3y_6^2 + x_3^3y_6\rangle$.

Example 6.3. We consider the integral affine plane curve

$$\mathscr{C} := \mathrm{Spec}(\mathbf{Q}[x,y]/\langle y^2 - x^2 - x^3\rangle).$$

Its normalization is given by the morphism $\nu^\sharp : \mathbf{Q}[x,y]/\langle y^2-x^2-x^3\rangle \to \mathbf{Q}[T]$, $\nu^\sharp(x) = T^2 - 1$, $\nu^\sharp(y) = T^3 - T$.

For the first values of $m \in \mathbf{N}$ our algorithms provide the following Gröbner bases of the ideal $\mathscr{N}_m(\mathscr{C})$ of $\mathbf{Q}[x,y]_m$ for the monomial lexicographic order $y_m > \cdots > y_0 > x_m > \cdots > x_0$.

- $\mathscr{N}_1(\mathscr{C}) = \langle y_0^2 - x_0^3 - x_0^2, 2x_0^2y_1 + 2x_0y_1 - 3x_0x_1y_0 - 2x_1y_0, 2y_0y_1 - 3x_0^2x_1 - 2x_0x_1, 4x_0y_1^2 + 4y_1^2 - 9x_0^2x_1^2 - 12x_0x_1^2 - 4x_1^2\rangle$.
- $\mathscr{N}_2(\mathscr{C}) = \langle y_0^2 - x_0^3 - x_0^2, 2x_0^2y_1 + 2x_0y_1 - 3x_0x_1y_0 - 2x_1y_0, 2y_0y_1 - 3x_0^2x_1 - 2x_0x_1, 4x_0y_1^2 + 4y_1^2 - 9x_0^2x_1^2 - 12x_0x_1^2 - 4x_1^2, 4x_0^2y_2 + 4x_0y_2 + x0x_1y_1 - 6x_0x_2y_0 - 4x_2y_0 - 3x_1^2y_0, 8x_1y_2 + 8y_1^3 - 8x_2y_1 - 36x_0x_1^2y_1 - 12x_1^2y_1 + 27x_1^3y_0, 2y_0y_2 + y1^2 - 3x_0^2x_2 - 2x_0x_2 - 3x_0x_1^2 - x1^2, 8x_0y_1y_2 + 8y_1y_2 + 2x_1y_1^2 - 18x_0^2x_1x_2 - 24x_0x_1x_2 - 8x_1x_2 - 9x_0x_1^3 - 6x_1^3, 64x_0y_2^2 + 64y_2^2 - 16y_1^4 + 32x_2y_1^2 - 72x_1^2y_1^2 - 144x_0^2x_2^2 - 192x_0x_2^2 - 64x_2^2 - 144x_0x_1^2x_2 - 96x_1^2x_2 + 81x_0^2x_1^4 + 216x_0x_1^4 + 72x_1^4\rangle$.
- The obtained Gröbner basis of $\mathscr{N}_3(\mathscr{C})$ for the given monomial order contains 17 polynomials.

Example 6.4. We consider the Whitney umbrella

$$\mathscr{S} := \mathrm{Spec}(\mathbf{Q}[x,y,z]/\langle x^2 - y^2z\rangle),$$

which is an integral affine surface. Its normalization is given by the morphism $\nu^\sharp : \mathbf{Q}[x,y,y]/\langle x^2 - y^2z\rangle \to \mathbf{Q}[u,v]$, $\nu^\sharp(x) = uv$, $\nu^\sharp(y) = u$, $\nu^\sharp(z) = v^2$. Since the normalization is smooth, it is also a resolution of singularities of $\mathscr{S}$ and we can apply algorithm 2 in this case.

For the first values of $m \in \mathbf{N}$ our algorithms provide the following Gröbner bases of the ideal $\mathscr{N}_m(\mathscr{S})$ of $\mathbf{Q}[x,y,z]_m$ for the monomial lexicographic order $z_m > \cdots > z_0 > y_m > \cdots > y_0 > x_m > \cdots > x_0$.

- $\mathscr{N}_1(\mathscr{S}) = \langle y_0^2z_0 - x_0^2, x_0^2z_1 + 2y_0y_1z_0^2 - 2x_0x_1z_0, x_0y_0z_1 + 2x_0y_1z_0 - 2x_1y_0z_0, y_0^2z_1 + 2y_0y_1z_0 - 2x_0x_1, y_0y_1z_0z_1 + x_0x_1z_1 + 2y_1^2z_0^2 - 2x_1^2z_0\rangle$.
- $\mathscr{N}_2(\mathscr{S}) = \langle y_0^2z_0 - x_0^2, x_0^2z_1 + 2y_0y_1z_0^2 - 2x_0x_1z_0, x_0y_0z_1 + 2x_0y_1z_0 - 2x_1y_0z_0, y_0^2z_1 + 2y_0y_1z_0 - 2x_0x_1, y_0y_1z_0z_1 + x_0x_1z_1 + 2y_1^2z_0^2 - 2x_1^2z_0, x_0^2z_2 - 2x_0x_1z_1 + 2y_0y_2z_0^2 - 3y_1^2z_0^2 - 2x_0x_2z_0 + 3x_1^2z_0, 4x_0x_1z_2 + 3y_0y_1z_1^2 + 4y_0y_2z_0z_1 + 6y_1^2z_0z_1 - 2x_1^2z_1 + 8y_1y_2z_0^2 - 8x_1x_2z_0, 2x_0y_0z_2 + 3x_0y_1z_1 - x1y_0z_1 + 4x_0y_2z_0 - 4x_2y_0z_0, y_0^2z_2 + 2y_0y_1z_1 + 2y_0y_2z_0 + y_1^2z_0 - 2x_0x_2 - x_1^2, 4x_1y_0z_0z_2 - x_1y_0z_1^2 + 4x_1y_1z_0z_1 - 4x_2y_0z_0z_1 + 8x_1y_2z_0^2 - 8x_2y_1z_0^2, 4x_0y_1z_0z_2 - 3x_0y_1z_1^2 - 4x_0y_2z_0z_1 + 4x_1y_1z_0z_1 + 8x_1y_2z_0^2 - 8x_2y_1z_0^2, 4y_0y_1z_0z_2 - y_0y_1z_1^2 + 4y_1^2z_0z_1 + 4x_0x_2z_1 + 8y_1y_2z_0^2 - 8x_1x_2z_0, 4y_1^2z_0^2z_2 - 4x_1^2z_0z_2 - y_0y_1z_1^3 - 2y_0y_2z_0z_1^2 - 5y_1^2z_0z_1^2 - 2x_0x_2z_1^2 + x_1^2z_1^2 - 8y_1y_2z_0^2z_1 + 8x_1x_2z_0z_1, 4y_0y_2z_0^2z_2 + 4x_0x_2z_0z_2 - y_0y_2z_0z_1^2 - 3x_0x_2z_1^2 + 4y_1y_2z_0^2z_1 + 4x_1x_2z_0z_1 + 8y_2^2z_0^3 - 8x_2^2z_0^2\rangle$.
- The obtained Gröbner basis of $\mathscr{N}_3(\mathscr{S})$ for the given monomial order contains 40 polynomials.

Example 6.5. We consider now the surface

$$\mathscr{S} := \mathrm{Spec}(\mathbf{F}_2[x,y,z]/\langle x^2 - y^2z\rangle).$$

Its normalization is given by the morphism in example 6.4 after replacing $\mathbf{Q}$ by $\mathbf{F}_2$.

For the first values of $m \in \mathbf{N}$ our algorithms provide the following Gröbner bases of the ideal $\mathscr{N}_m(\mathscr{S})$ of $\mathbf{F}_2[x,y,z]_m$ for the monomial lexicographic order $z_m > \cdots > z_0 > y_m > \cdots > y_0 > x_m > \cdots > x_0$.

- $\mathscr{N}_1(\mathscr{S}) = \langle y_0^2z_0 + x_0^2, z_1\rangle$.
- $\mathscr{N}_2(\mathscr{S}) = \langle y_0^2z_0 + x_0^2, z_1, x_0^2z_2 + y_1^2z_0^2 + x_1^2z_0, y_0^2z_2 + y_1^2z_0 + x_1^2\rangle$.
- $\mathscr{N}_3(\mathscr{S}) = \langle y_0^2z_0 + x_0^2, z_1, x_0^2z_2 + y_1^2z_0^2 + x_1^2z_0, y_0^2z_2 + y_1^2z_0 + x_1^2, z_3\rangle$.

- $\mathscr{N}_4(\mathscr{S}) = \langle y_0^2z_0 + x_0^2, z_1, x_0^2z_2 + y_1^2z_0^2 + x_1^2z_0, y_0^2z_2 + y_1^2z_0 + x_1^2, z_3, x_0^2z_4 + y_1^2z_0z_2 + y_2^2z_0^2 + x_2^2z_0, y_0^2z_4 + y_1^2z_2 + y_2^2z_0 + x_2^2, y_1^2z_0z_4 + x_1^2z_4 + y_1^2z_2^2 + y_2^2z_0z_2 + x_2^2z_2\rangle$.
- $\mathscr{N}_5(\mathscr{S}) = \langle y_0^2z_0 + x_0^2, z_1, x_0^2z_2 + y_1^2z_0^2 + x_1^2z_0, y_0^2z_2 + y_1^2z_0 + x_1^2, z_3, x_0^2z_4 + y_1^2z_0z_2 + y_2^2z_0^2 + x_2^2z_0, y_0^2z_4 + y_1^2z_2 + y_2^2z_0 + x_2^2, y_1^2z_0z_4 + x_1^2z_4 + y_1^2z_2^2 + y_2^2z_0z_2 + x_2^2z_2, z_5\rangle$.
- $\mathscr{N}_6(\mathscr{S}) = \langle y_0^2z_0 + x_0^2, z_1, x_0^2z_2 + y_1^2z_0^2 + x_1^2z_0, y_0^2z_2 + y_1^2z_0 + x_1^2, z_3, x_0^2z_4 + y_1^2z_0z_2 + y_2^2z_0^2 + x_2^2z_0, y_0^2z_4 + y_1^2z_2 + y_2^2z_0 + x_2^2, y_1^2z_0z_4 + x_1^2z_4 + y_1^2z_2^2 + y_2^2z_0z_2 + x_2^2z_2, z_5, x_0^2z_6 + x_1^2z_4 + y_1^2z_2^2 + x_2^2z_2 + y_3^2z_0^2 + x_3^2z_0, y_0^2z_6 + y_1^2z_4 + y_2^2z_2 + y_3^2z_0 + x_3^2, y_1^2z_0z_6 + x_1^2z_6 + y_1^2z_2z_4 + y_2^2z_2^2 + y_3^2z_0z_2 + x_3^2z_2, x_1^2z_2z_6 + y_2^2z_0^2z_6 + x_2^2z_0z_6 + x_1^2z_4^2 + x_2^2z_2z_4 + y_3^2z_0^2z_4 + x_3^2z_0z_4 + y_2^2z_2^3 + y_3^2z_0z_2^2 + x_3^2z_2^2, y_1^2z_2z_6 + y_2^2z_0z_6 + x_2^2z_6 + y_1^2z_4^2 + y_2^2z_2z_4 + y_3^2z_0z_4 + x_3^2z_4\rangle$.

Let us stress that $\mathscr{L}_\infty(\mathscr{S})$ is not irreducible (see e.g., [7, Ch. 3, Remark 4.3.5]), hence the ideal $\mathscr{N}_m(\mathscr{S})$ defines the irreducible component $\mathscr{G}_m(\mathscr{S})$ but it is not the ideal of nilpotent functions of $\mathscr{L}_\infty(\mathscr{S})$ which live in $\mathscr{L}_m(\mathscr{S})$, according to subsection 4.1.

Example 6.6. We consider the surface

$$\mathscr{S} := \mathrm{Spec}(\mathbf{F}_3[x,y,z]/\langle x^2 - y^2z\rangle).$$

Its normalization is given by the morphism in example 6.4 after replacing $\mathbf{Q}$ by $\mathbf{F}_3$.

For the first values of $m \in \mathbf{N}$ our algorithms provide the following Gröbner bases of the ideal $\mathscr{N}_m(\mathscr{S})$ of $\mathbf{F}_3[x,y,z]_m$ for the monomial lexicographic order $z_m > \cdots > z_0 > y_m > \cdots > y_0 > x_m > \cdots > x_0$.

- $\mathscr{N}_1(\mathscr{S}) = \langle y_0^2z_0 - x_0^2, x_0^2z_1 - y_0y_1z_0^2 + x_0x_1z_0, x_0y_0z_1 - x_0y_1z_0 + x_1y_0z_0, y_0^2z_1 - y_0y_1z_0 + x_0x_1, y_0y_1z_0z_1 + x_0x_1z_1 - y_1^2z_0^2 + x_1^2z_0\rangle$.
- $\mathscr{N}_2(\mathscr{S}) = \langle y_0^2z_0 - x_0^2, x_0^2z_1 - y_0y_1z_0^2 + x_0x_1z_0, x_0y_0z_1 - x_0y_1z_0 + x_1y_0z_0, y_0^2z_1 - y_0y_1z_0 + x_0x_1, y_0y_1z_0z_1 + x_0x_1z_1 - y_1^2z_0^2 + x_1^2z_0, x_0^2z_2 + x_0x_1z_1 - y_0y_2z_0^2 + x_0x_2z_0, x_0x_1z_2 + y_0y_2z_0z_1 + x_1^2z_1 - y_1y_2z_0^2 + x_1x_2z_0, x_0y_0z_2 + x_1y_0z_1 - x_0y_2z_0 + x_2y_0z_0, y_0^2z_2 - y_0y_1z_1 - y_0y_2z_0 + y_1^2z_0 + x_0x_2 - x_1^2, x_0y_1z_2 - x_0y_2z_1 + x_1y_1z_1 - x_1y_2z_0 + x_2y_1z_0, x_1y_0z_0z_2 - x_1y_0z_1^2 + x_1y_1z_0z_1 - x_2y_0z_0z_1 - x_1y_2z_0^2 + x_2y_1z_0^2, y_0y_1z_0z_2 - y_0y_1z_1^2 + y_1^2z_0z_1 + x_0x_2z_1 - y_1y_2z_0^2 + x_1x_2z_0, y_0y_2z_0z_2 + x_0x_2z_2 - y_0y_2z_1^2 + y_1y_2z_0z_1 + x_1x_2z_1 - y_2^2z_0^2 + x_2^2z_0, y_1^2z_0^2z_2 - x_1^2z_0z_2 - y_0y_1z_1^3 + y_0y_2z_0z_1^2 + y_1^2z_0z_1^2 + x_0x_2z_1^2 + x_1^2z_1^2 + y_1y_2z_0^2z_1 - x_1x_2z_0z_1\rangle$.
- The reduced Gröbner basis of $\mathscr{N}_3(\mathscr{S})$ for the given monomial order contains 42 polynomials.

Example 6.7. We consider the surface

$$\mathscr{S} := \mathrm{Spec}(\mathbf{Q}[x,y,z]/\langle x^3 + y^3 + z^3\rangle),$$

which is an integral normal affine surface with isolated non-rational singularities (see example 4.5). The inclusion of the standard open subset $D(x)$ in $\mathscr{S}$ induces a smooth birational model given by the morphism $h^\sharp : \mathbf{Q}[x,y,z]/\langle x^3 + y^3 + z^3\rangle \to \mathbf{Q}[u,v,w,T]/\langle 1 + v^3 + w^3, Tu - 1\rangle$, $h^\sharp(x) = u$, $h^\sharp(y) = uv$, $h^\sharp(z) = uw$. Then we can apply algorithm 2 in this case (and of course, also algorithm 1).

For the first values of $m \in \mathbf{N}$ our algorithms provide the following Gröbner bases of the ideal $\mathscr{N}_m(\mathscr{S})$ of $\mathbf{Q}[x,y,z]_m$ for the monomial lexicographic order $z_m > \cdots > z_0 > y_m > \cdots > y_0 > x_m > \cdots > x_0$.

- $\mathscr{N}_1(\mathscr{S}) = \langle z_0^3 + y_0^3 + x_0^3, y_0^3z_1 + x_0^3z_1 - y_0^2y_1z_0 - x_0^2x_1z_0, z_0^2z_1 + y_0^2y_1 + x_0^2x_1\rangle$.
- The reduced Gröbner basis of $\mathscr{N}_2(\mathscr{S})$ for the given monomial order contains 21 polynomials.

We can check that $\mathscr{N}_1(\mathscr{S})$ equals the ideal $\langle z_0^3 + y_0^3 + x_0^3, 3z_0^2z_1 + 3y_0^2y_1 + 3x_0^2x_1\rangle$ of $\mathbf{Q}[x,y,z]_1$ defining $\mathscr{L}_1(\mathscr{S})$. If $f_0 := z_0^3 + y_0^3 + x_0^3$ and $f_1 := 3z_0^2z_1 + 3y_0^2y_1 + 3x_0^2x_1$ then

$$y_0^3z_1 + x_0^3z_1 - y_0^2y_1z_0 - x_0^2x_1z_0 = z_1f_0 - \frac{z_0}{3}f_1.$$

This shows that $\mathscr{L}_1(\mathscr{S})$ is reduced (see example 4.5).

Using SageMath, we can also check that one of the elements of the obtained Gröbner basis for $\mathscr{N}_2(\mathscr{S})$, the polynomial $g := x_0^2y_0y_1z_0z_2 - x_0x_1y_0^2z_0z_2 + x_0^2y_0y_1z_1^2 - x_0x_1y_0^2z_1^2 - x_0^2y_0y_2z_0z_1 - x_0^2y_1^2z_0z_1 + x_0x_2y_0^2z_0z_1 + x_1^2y_0^2z_0z_1 + x_0x_1y_0y_2z_0^2 + x_0x_1y_1^2z_0^2 - x_0x_2y_0y_1z_0^2 - x_1^2y_0y_1z_0^2$, does not belong to the ideal $\langle z_0^3+y_0^3+x_0^3, 3z_0^2z_1+3y_0^2y_1+3x_0^2x_1, 3z_0^2z_2+3z_0z_1^2+3y_0^2y_2+3y_0y_1^2+3x_0^2x_2+3x_0x_1^2\rangle$ of $\mathbf{Q}[x,y,z]_2$ defining $\mathscr{L}_2(\mathscr{S})$. However, we can also check that g^2 does belong to this ideal (see [16]).

7. Comments on the implementations of our algorithms

In this section we address some issues related with the implementation of our algorithms (made in SageMath [27], see [17]) and the examples in section 6. In table 7.1 we resume the runtimes of our algorithms for the examples in section 6 using two different monomial orders (a lexicographic order and a graded reverse lexicographic order); the tests are provided in [16]. The computations were performed on a computer running an Intel i7-8750H 2.20GHz with 16GB of RAM. Our implementations provide the reduced (up to multiplication by a constant) Gröbner basis of $\mathscr{N}_m(X)$ for the considered monomial order, hence the outputs of both algorithms are the same (up to multiplication by a constant).

7.1. **Comments on the choice of the monomial order.** Both algorithms perform some computations involving Gröbner bases in polynomial rings of the form $k[x_1,\dots,x_n]_m$, hence the choice of a monomial order is mandatory. We have tested our algorithms implemented in SageMath [27] over several examples using two different monomial orders: on the one hand the lexicographic order with $x_{n,m} > x_{n,m-1} > \cdots > x_{n,0} > x_{n-1,m} > \cdots > x_{0,0}$; on the other hand the graded reverse lexicographic order for the ordered set of variables $x_{0,0},\dots,x_{0,m},x_{1,0},\dots,x_{n,m}$.

We observe in table 7.1 that the runtimes may vary a lot depending on the choice of the monomial order: algorithm 1 is much faster using the graded reverse lexicographic than the lexicographic order. For algorithm 2 it is not clear that one of the monomial orders is more advantageous than the other, it depends on the considered example. Let us stress that, whereas it is fairly clear that in general the fastest method to perform computations is algorithm 1 with the graded reverse lexicographic order, the situation is not clear if we want to obtain a Gröebner basis with respect to the lexicographic order. In this case algorithm 2 seems to be more convenient in characteristic 0 when a "simple" smooth birational model is known.

7.2. **Comments on the choice of the smooth birational model.** A smooth birational model is required as input for algorithm 2. Although the computation of such a smooth birational model itself is not performed in our algorithm (see subsection 5.5 for a discussion about effectively computing a smooth birational model in the case of curves) and hence it is not taken into account in our study of its efficiency, this choice has an impact on the execution time of algorithm 2. The following example illustrates this fact.

Example 7.3. Let us keep the same integral affine plane curve

$$\mathscr{C} := \operatorname{Spec}(\mathbf{Q}[x,y]/\langle x^3-y^2\rangle)$$

as before but now we consider the following presentation of its normalization, which is given by the morphism $\nu^\sharp : \mathbf{Q}[x,y]/\langle x^3-y^2\rangle \to \mathbf{Q}[u,v,T]/\langle Tu-v, u^2-Tv, T^2-u, u^3-v^2\rangle$, $\nu^\sharp(x) = u$, $\nu^\sharp(y) = v$. This is the presentation given by the algorithm in [11] using the implementation in [10]. The resulting ideals (and hence their reduced Gröbner bases) are obviously the same as in example 6.1 but the computation times using algorithm 2 are much longer due to the fact that we have more variables and equations. Indeed, we obtain an "integer overflow" message in the computation of $\mathscr{N}_5(\mathscr{C})$.

Table 7.1. Wall times of algorithms 1 and 2 over several examples.

Algorithm		Algorithm 1		Algorithm 2	
Example	m	Lex	Grevlex	Lex	Grevlex
6.1	1	3.94 ms	3.85 ms	5.42 ms	7.35 ms
	2	6.64 ms	6.14 ms	10 ms	16.1 ms
	3	27.2 ms	9.8 ms	24.2 ms	29.2 ms
	4	15.1 s	21.3 ms	355 ms	385 ms
	5	$*$	2.14 s	12 min 10 s	12 min 10 s
7.3	1	3.94 ms	3.85 ms	10.6 ms	12.2 ms
	2	6.64 ms	6.14 ms	18.9 ms	23.3 ms
	3	27.2 ms	9.8 ms	60.3 ms	42.5 ms
	4	15.1 s	21.3 ms	2.77 s	405 ms
	5	$*$	2.14 s	$**$	12 min 9 s
6.2	1	3.49 ms	3.06 ms	5.12 ms	5.15 ms
	2	4.65 ms	4.7 ms	6.82 ms	17.7 ms
	3	5.09 ms	4.73 ms	12.5 ms	18 ms
	4	7.63 ms	5.78 ms	32.5 ms	24.7 ms
	5	8.46 ms	12 ms	108 ms	94.7 ms
	6	16.7 ms	7.67 ms	5.49 s	1.72 s
	7	53.7 ms	9.4 ms	1 min 54 s	1 min 10 s
	8	115 ms	15.2 ms	$*$	23 min 19 s
	9	754 ms	14.3 ms	$*$	$*$
	10	5.95 s	16.6 ms	$*$	$*$
	11	43.3 s	17.1 ms	$*$	$*$
	12	11 min 14 s	18.5 ms	$*$	$*$
	26	$*$	26 s	$*$	$*$
6.3	1	4.82 ms	4.37 ms	5.97 ms	8.35 ms
	2	12.4 ms	6.93 ms	13.6 ms	18.5 ms
	3	1.85 s	312 ms	58.6 ms	114 ms
	4	$*$	1.64 s	2.7 s	6.59 s
	5	$*$	$*$	4 h 13 min 5 s	$*$
6.4	1	4.14 ms	3.63 ms	7.72 ms	16.7 ms
	2	14.1 ms	8.49 ms	19.8 ms	38 ms
	3	6 min 2 s	225 ms	1.03 s	1.74 s
	4	$*$	2 min 45 s	1 h 34 min 35 s	2 h 19 s
6.5	1	3.43 ms	3.81 ms	6.27 ms	8.4 ms
	2	4.33 ms	4.59 ms	11.6 ms	17.4 ms
	3	5.31 ms	4.88 ms	25.7 ms	36.4 ms
	4	8.02 ms	6.51 ms	9.86 s	13.1 s
	5	17.5 ms	7.5 ms	$*$	$*$
	6	1.16 s	9.77 ms	$*$	$*$
	7	13.7 s	12.4 ms	$*$	$*$
	19	$*$	26.2 s	$*$	$*$
6.6	1	3.78 ms	3.8 ms	8.66 ms	10.3 ms
	2	9.38 ms	6.81 ms	16.7 ms	28.1 ms
	3	28.8 s	63.6 ms	215 ms	328 ms
	4	$*$	9 min 10 s	2 min 38 s	10 min 22 s
6.7	1	4.99 ms	3.65 ms	48.1 ms	30.3 ms
	2	7.88 s	46.8 ms	$*$	$*$
	3	$*$	2 min 52 s	$*$	$*$

$*$: computation did not finish after 6h.

$**$: integer overflow.

8. General component and differential operators

Thanks to section 4, we will explain the implications of the former considerations (when $m = 1$) in terms of logarithmic differential operators along an affine plane curve.

8.1. Let us assume that X is an affine hypersurface defined by an irreducible polynomial $f \in k[x_1, \ldots, x_n]$. The field k being perfect, we have $\mathrm{Sing}(X) = V(f, \partial_{x_1}(f), \ldots, \partial_{x_n}(f))$. By subsection 2.1, there exists an integer $i \in \{1, \ldots, n\}$ such that the partial derivative $\partial(f) := \partial_i(f)$ does not belong to $I := \langle f \rangle$, and then neither to the ideal I_m for every integer m. Moreover, we have that $V(\partial(f))$ contains $\mathrm{Sing}(X)$ or, equivalently, we observe that the standard open subset $D(\partial(f)) := X \setminus V(\partial(f))$ is contained in $\mathrm{Reg}(X)$. Since $(I_m : \partial(f)^\infty)$ induces the kernel of the localization morphism $\mathcal{O}(\mathscr{L}_m(X)) \to \mathcal{O}(\mathscr{L}_m(X))_{\partial(f)} = \mathcal{O}(\mathscr{L}_m(D(\partial(f))))$, we deduce from corollary 3.4 and subsection 3.2 that

$$\mathscr{N}_m(X) = (I_m : \partial(f)^\infty). \tag{8.1}$$

Let us stress that, in characteristic zero, such a statement can also be obtained as a more or less direct consequence of the Rosenfeld lemma, see [13, Section 9].

8.2. Let us now explain a link with differential operators. A k-derivation $D \in \mathrm{Der}_k(k[x_1, \ldots, x_n])$ is called *logarithmic along* f if $D(f) \in \langle f \rangle$. Every (logarithmic) k-derivation D is a linear combination (over $k[x_1, \ldots, x_n]$) of the partial derivatives of $k[x_1, \ldots, x_n]$; hence, there exist polynomials $a_1, \ldots, a_n$ such that $D = \sum_{i=1}^n a_i \partial_{x_i}$. A *differential operator of order at most* $d \geq 1$ can be identified, in a unique way in our context, as a linear combination (over $k[x_1, \ldots, x_n]$) of the form

$$D = \sum_{i_1 + \cdots + i_n \leq d} a_{i_1, \ldots, i_n} \partial_{x_1}^{i_1} \cdots \partial_{x_n}^{i_n}.$$

The set $\mathscr{D}$ of differential operators of $k[x_1, \ldots, x_n]$ over k forms a free left $k[x_1, \ldots, x_n]$-module in an obvious way. E.g., the composition of d derivations over k gives rise to a differential operator of order d. A differential operator D is said to be *homogeneous* if each of its monomials are differential operators of order d. (Let us stress that such a definition includes no condition about the homogeneity on the coefficients of D.)

8.3. Let us assume that $n = 2$ and that the partial derivative $\partial_{x_1}(f)$ is not zero, we set $\partial(f) := \partial_{x_1}(f)$ (otherwise the arguments are symmetrical). Let $d \geq 1$ be an integer. Let V_{d-1} be the left $k[x_1, x_2]$-module generated by the *homogeneous* differential operators $D = \sum_{i=0}^d a_i \partial_{x_1}^i \partial_{x_2}^{d-i}$ of $k[x_1, x_2]$ such that $D(f^d) \in \langle f \rangle$ (e.g., see [6, §1.2] for the connexion between our definition and the classical V-filtration relative to X on $\mathscr{D}$ in this context). Let us introduce the morphism of left $k[x_1, x_2]$-modules

$$(\cdot)^\sharp \colon k[x_1, x_2]_1 \to \mathscr{D}$$

defined by $\sum_{i=0}^d a_i x_{1,1}^i x_{2,1}^{d-i} \mapsto \sum_{i=0}^d (-1)^i a_i \partial_{x_2}^i \partial_{x_1}^{d-i}$. Let $P \in k[x_1, x_2]_1$ be a polynomial with (total) degree d with respect to the variables $x_{1,1}, x_{2,1}$, which is equivalent to say that its weighted degree equals d. By lemma 8.9, we observe that we have

$$(P)^\sharp(f^d) \equiv d! P(x_{1,0}, x_{2,0}, -\partial_{x_2}(f), \partial_{x_1}(f)) \pmod{\langle f \rangle}. \tag{8.2}$$

Let $\mathscr{N}_1{}^d(X)$ be the set of homogeneous elements of $\mathscr{N}_1(X)$ of degree d. In particular it holds for every $P \in \mathscr{N}_1{}^d(X)$, the set of homogeneous elements of $\mathscr{N}_1(X)$ of degree d. The preceding morphism induces by restriction a morphism of left $k[x_1, x_2]$-modules

$$(\cdot)_d^\sharp \colon \mathscr{N}_1{}^d(X) \to V_{d-1}. \tag{8.3}$$

Let us justify this assertion. In positive characteristic, if $d \geq \mathbf{char}(k)$, we have $(P)_d^\sharp(f^d) = 0$ for every $P \in \mathscr{N}_1{}^d(X)$ by (8.2), and the map is well-defined. Assume now that $d!$ is prime to the characteristic exponent of k. Then, by (8.1), for every homogeneous polynomial P, we have $P \in \mathscr{N}_1{}^d(X)$ if and only if there exists an integer $N \geq d$, such that

$$\partial(f)^N P \in \langle f, \Delta(f)\rangle. \tag{8.4}$$

But, thanks to the relation $\Delta(f) = \partial_{x_1}(f)x_{1,1} + \partial_{x_2}(f)x_{2,1}$, we deduce the equality $\partial(f)^N P = \partial(f)^{N-d}x_{2,1}^d P(x_1, x_2, -\partial_{x_2}(f), \partial_{x_1}(f)) \pmod{\langle f, \Delta(f)\rangle}$. From formula (8.4) we conclude that $P \in \mathscr{N}_1{}^d(X)$ if and only if

$$\partial(f)^{N-d}x_{2,1}^d P(x_{1,0}, x_{2,0}, -\partial_{x_2}(f), \partial_{x_1}(f)) \in \langle f, \Delta(f)\rangle. \tag{8.5}$$

Let $P \in \mathscr{N}_1{}^d(X)$. In particular, $\partial(f)^{N-d}x_{2,1}^d P(x_{1,0}, x_{2,0}, -\partial_{x_2}(f), \partial_{x_1}(f)) \in \{f\}$ which is a prime ideal by [7, Ch. 3, Lemma 4.3.1]. Since $f \neq x_2$ (recall that $\partial_{x_1}(f) \neq 0$ by assumption), we have $\partial(f)^{N-d}P(x_{1,0}, x_{2,0}, -\partial_{x_2}(f), \partial_{x_1}(f)) \in \{f\}$ and in fact, this element being of weight 0, one easily sees that it belongs to $\{f\} \cap k[x_1, x_2]_0 = \langle f\rangle$, which is a prime ideal. Thus $P(x_{1,0}, x_{2,0}, -\partial_{x_2}(f), \partial_{x_1}(f)) \in \langle f\rangle$ and (8.2) shows that $(P)_d^\sharp(f^d) \in \langle f\rangle$, hence the map is still well-defined.

Theorem 8.4. *Let k be a perfect field. Let $d \geq 1$ be an integer such that $d!$ is prime to the characteristic exponent of k. The morphism $(\cdot)_d^\sharp$ is an isomorphism.*

When $\mathbf{char}(k) = 0$, this morphism has been first considered in [24] and related to Bernstein-Sato operators in [12, §5]. If $\mathbf{char}(k) = 0$, our assumption on the integer d is always fulfilled; in positive characteristic, it simply means that $d < \mathbf{char}(k)$.

Proof. Let $D = \sum_{i=0}^d b_i \partial_{x_1}^i \partial_{x_2}^{d-i} \in V_{d-1}$ be a homogeneous differential operator with degree d. By lemma 8.9, we deduce that $D(f^d) = \sum_{i=0}^d b_i \partial_{x_1}^i(f)\partial_{x_2}^{d-i}(f) \pmod{\langle f\rangle}$, hence the the latter expression belongs to $\langle f\rangle$. Now formula (8.5) assures that $P = \sum_{i=0}^d (-1)^{d-i} b_i x_{2,1}^i x_{1,1}^{d-i}$ belongs to $\mathscr{N}_1(X)$; since $(P)_d^\sharp = D$ the map is surjective. Now let $P = \sum_{i=0}^d a_i x_{1,1}^i x_{2,1}^{d-i}$ such that $(P)_d^\sharp = \sum_{i=0}^d (-1)^i a_i \partial_{x_2}^i \partial_{x_1}^{d-i} = 0$. We apply $(P)_d^\sharp$ to $x_1^i x_2^{d-i}$ to conclude that $a_{d-i} = 0$, hence the constructed map is injective. □

Remark 8.5. By theorem 8.4, we stress that the algorithms explained in section 5 allow us to provide generator systems for V_{d-1} in arbitrary characteristics.

Remark 8.6. As in [12], one can adopt a more intrinsic point of view by replacing the notion of homogeneous differential operators with that of principal symbols of differential operators. This obvious exercise of translation is left to the reader.

Example 8.7. Let $\mathscr{C}$ be the plane curve defined by the irreducible polynomial $f := x^3 - y^2$ of $\mathbf{Q}[x, y]$. In this case, the tangent space $\mathscr{L}_1(\mathscr{C})$ is not irreducible and $\mathscr{N}_1(\mathscr{C}) \neq \langle f_0, f_1\rangle$. From a direct computation (see also [18, Theorem 7.6] for a general argument) we obtain the following Gröbner basis of the ideal $\mathscr{N}_1(\mathscr{C})$:

$$\mathfrak{B} := \left\{x_0^3 - y_0^2, 2x_0y_1 - 3x_1y_0, 2y_0y_1 - 3x_0^2x_1, 4y_1^2 - 9x_0x_1^2\right\}.$$

Obviously $(x_0^3 - y_0^2)_0^\sharp = x^3 - y^2$ so $(x_0^3 - y_0^2)_0^\sharp\left((x^3 - y^2)^0\right) \in \langle x^3 - y^2\rangle$. As for the other elements of $\mathfrak{B}$, it is easy to check that $(2x_0y_1 - 3x_1y_0)_1^\sharp\left(x^3 - y^2\right) = 6(x^3 - y^2)$, $(2y_0y_1 - 3x_0^2x_1)_1^\sharp\left(x^3 - y^2\right) = 0$ and $(4y_1^2 - 9x_0x_1^2)_2^\sharp\left((x^3 - y^2)^2\right) = 156x(x^3 - y^2)$. Conversely, let us consider the differential operator $D := 9x\partial_y^3 + 9x^4\partial_y^2\partial_x + (6x^2y - 4)\partial_y\partial_x^2 - y\partial_y$, which satisfies $D((x^3 - y^2)^3) = -6(54x^6 + 71x^3y^2 - 228xy + y^4)(x^3 - y^2)$. We check that $\sigma(D) = ((4y_1^2 - 9x_0x_1^2)(x_1 - x_0^3y_1) + (2x_0y_1 - 3x_1y_0)(2x_0^2y_1^2))^\sharp$, where $\sigma(D)$ denotes the principal symbol of D.

Example 8.8. Let $\mathscr{C}$ be the plane curve defined by the irreducible polynomial $f := y^2 - x^2(x+1)$ of $\mathbf{F}_3[x,y]$. In this case, the tangent space $\mathscr{L}_1(\mathscr{C})$ is not irreducible and $\mathscr{N}_1(\mathscr{C}) \neq \langle f_0, f_1 \rangle$. From a direct computation we obtain the following Gröbner basis of the ideal $\mathscr{N}_1(\mathscr{C})$:

$$\mathfrak{B} := \left\{ y_0^2 - x_0^2(x_0+1), y_0y_1 - x_0x_1, x_0^2y_1 + x_0y_1 - x_1y_0, x_0y_1^2 + y_1^2 - x_1^2 \right\}.$$

Obviously $(y_0^2 - x_0^2(x_0+1))_0^\sharp \left((y^2 - x^2(x+1))^0\right) = y^2 - x^2(x+1)$ and we easily check that $(y_0y_1 - x_0x_1)_1^\sharp \left(y^2 - x^2(x+1)\right) = 0$, $(x_0^2y_1 + x_0y_1 - x_1y_0)_1^\sharp \left(y^2 - x^2(x+1)\right) = 2(y^2 - x^2(x+1))$ and $(x_0y_1^2 + y_1^2 - x_1^2)_2^\sharp \left((y^2 - x^2(x+1))^2\right) = 2(x+1)(y^2 - x^2(x+1))$.

Lemma 8.9. *Let k be a field. Let $\alpha \in \mathbf{N}^2$. Let $\ell \in \mathbf{N}$ be an integer such that $\ell \geq |\alpha| := \alpha_1 + \alpha_2$. We denote by ∂_{ij}^α the differential operator on $k[x,y]$ of order $|\alpha|$ defined to be $\partial_i^{\alpha_1} \circ \partial_j^{\alpha_2}$ for every pair $(i,j) \in \{1,2\}$. Then, for every polynomials $g, P \in k[x,y]$, there exists a polynomial $q \in k[x,y]$ such that the following formula holds*

$$\partial_{ij}^\alpha(Pg^\ell) = \frac{\ell!}{(\ell - |\alpha|)!} Pg^{\ell-|\alpha|} \partial_i(g)^{\alpha_1} \partial_j(g)^{\alpha_2} + qg^{\ell-|\alpha|+1}.$$

If $\mathbf{char}(k) > 0$ the coefficient $\dfrac{\ell!}{(\ell-|\alpha|)!}$ in the previous formula should be considered as the product $\prod_{i=0}^{|\alpha|-1}(\ell - i)$.

Let us stress that, by iterating derivations, the monomial $(\partial_i(g))^{\alpha_1}(\partial_j(g))^{\alpha_2}$ appears, in the formula, only one time (possibly with zero coefficient).

Proof. We prove this assertion by induction on $|\alpha|$. If $|\alpha| \in \{0,1\}$, the formula holds with $q = 0$ if $|\alpha| = 0$ and $q = \partial_i(P)$ if $\alpha = (1,0)$ and $q = \partial_j(P)$ if $\alpha = (0,1)$. Let $d \geq 1$. Let us assume that the formula holds true for every $\beta \in \mathbf{N}^2$ with $|\beta| \leq d$. Let α' with $|\alpha'| = d+1$. Let $\ell \geq d+1$.

∘ Let us assume that $\alpha' = (\alpha'_1, \alpha'_2 + 1)$. Let us note that $\alpha'_1 + \alpha'_2 = d$. Thus we have

$$\begin{aligned} \partial_{ij}^{\alpha'}(Pg^\ell) &= \partial_i^{\alpha'_1}\left(\partial_j^{\alpha'_2}(g^\ell \partial_j(P) + \ell P g^{\ell-1}\partial_j(g))\right) \\ &= \partial_i^{\alpha'_1} \circ \partial_j^{\alpha'_2}(\partial_j(P)g^\ell) + \ell \partial_i^{\alpha'_1} \circ \partial_j^{\alpha'_2}(Pg^{\ell-1}\partial_j(g)) \end{aligned}$$

Then, we conclude the proof of this case by applying the induction hypothesis to the differential operator $\partial_i^{\alpha'_1} \circ \partial_j^{\alpha'_2}$ at each term of the former sum.

∘ Let us assume that $\alpha' = (\alpha'_1 + 1, \alpha'_2)$. Let us note that $\alpha'_1 + \alpha'_2 = d$. Thus, by the induction hypothesis, we have

$$\begin{aligned} \partial_{ij}^{\alpha'}(Pg^\ell) &= \partial_i\left(\partial_i^{\alpha'_1} \circ \partial_j^{\alpha'_2}(g^\ell P)\right) \\ &= \partial_i\left(\frac{\ell!}{(\ell-d)!} Pg^{\ell-d}(\partial_i(g))^{\alpha'_1}(\partial_j(g))^{\alpha'_2} + qg^{\ell-d+1}\right). \end{aligned}$$

By differentiating the last expression, we also obtain a formula of the required type. □

9. General components and zeta functions

9.1. If $\mathbf{Z}[\mathbf{Var}_k]$ is the free abelian group generated by the isomorphism classes of k-varieties, let us consider its quotient Γ by the submodule generated by the elements of the form $\{X\} - \{Z\} - \{U\}$ where we denote by $\{\dagger\}$ the isomorphism class of $\dagger$ in $\mathbf{Z}[\mathbf{Var}_k]$ and Z is a closed subscheme of X with $U = X \setminus Z$. The formula

$$[X] \cdot [X'] := [X \times_k X'],$$

for every pair of k-varieties (X, X') ($[\dagger]$ is the image of $\{\dagger\}$ in Γ), extended by bilinearity, endows the group Γ with a ring structure whose neutral element is $\mathbf{1} := [\mathrm{Spec}(k)]$. The class of the affine line $\mathbf{A}^1_k$ is denoted by $\mathbf{L}$. This ring is the so-called *Grothendieck ring of varieties* and it is denoted by $K_0(\mathbf{Var}_k)$.

Proposition 9.2. *Let k be a field. Let Z be a constructible subset of a k-variety X. If we have $[X] = [Z]$ in the ring $K_0(\mathbf{Var}_k)$, then $X = Z$.*

Proof. See [7, Appendix, Remark 1.1.6]. □

9.3. Let k be a field. Let X be a k-variety. One breakthrough achieved by Denef and Loeser with motivic integration was to prove the rationality of specific power series with coefficients in $\mathscr{M}_k := K_0(\mathbf{Var}_k)[\mathbf{L}^{-1}]$ which can be interpreted as geometrizations of arithmetic analogs, which encode the asymptotic of points counting in various contexts. One has the *Denef-Loeser Poincaré power series* defined as follows:

$$P(X;T) = \sum_{m\geq 0} [\theta_{m,X}(\mathscr{L}_\infty(X))]T^m$$

and, also, the analog of the Igusa-Meuser power series:

$$Q(X;T) = \sum_{m\geq 0} [\mathscr{L}_m(X)]T^m.$$

Both power series are rational when $\mathbf{char}(k) = 0$, i.e., can be identified with a quotient of two polynomials (of $\mathscr{M}_k[T]$) in a subring of $\mathscr{M}_k[[T]]$. (See [9, Theorem 5.4], [21, Corollary 4.3.6].)

9.4. Let X be an integral k-variety. In the spirit of the constructions of subsection 9.3, we introduce the zeta functions of the general components of the jet schemes as follows. For every integer $m \in \mathbf{N}$, the general component $\mathscr{G}_m(X)$ is an integral subvariety of $\mathscr{L}_m(X)$. In this way, it naturally admits a class in the ring $K_0(\mathbf{Var}_k)$. By analogy with the constructions of Denef and Loeser, we introduce the following zeta power series

$$Z^{\mathrm{gen}}(X;T) := \sum_{m\geq 0} [\mathscr{G}_m(X)]T^m \tag{9.1}$$

which belongs to the ring $K_0(\mathbf{Var}_k)[[T]]$. We set

$$Z^{\partial}(X;T) := Z^{\mathrm{gen}}(X;T) - P(X;T).$$

When the arc scheme $\mathscr{L}_\infty(X)$ is irreducible, one has

$$Z^{\partial}(X;T) = \sum_{m\geq 0} [\overline{\theta_{m,X}(\mathscr{L}_\infty(X))} \setminus \theta_{m,X}(\mathscr{L}_\infty(X))]T^m.$$

In particular, when the characteristic of k is zero, we observe that $Z^{\mathrm{gen}}(X;T)$ is rational if and only if $Z^{\partial}(X;T)$ is.

Definition 9.5. Let k be a field. Let X be an integral k-variety. The power series $Z^{\mathrm{gen}}(X;T)$ is called the *general jet components power series of X*.

9.6. We easily observe that, if the k-variety X is assumed to be smooth, the morphism $\theta_{m,X}$ is surjective for every integer m. In particular, we have $Z^{\mathrm{gen}}(X;T) = Q(X;T) = P(X;T) = [X]/(\mathbf{1}-\mathbf{L}^{\dim(X)}T)$. In addition, if we assume that the characteristic of k is zero and the k-variety X is locally a complete intersection, then we deduce from Mustaţă's theorem (see introduction or [19, Theorem 0.1]) and proposition 9.2 that $Z^{\mathrm{gen}}(X;T) = Q(X;T)$ if and only if the k-variety X has at most rational singularities. By theorem 1.5, we also deduce that, if $\mathscr{L}_\infty(X)$ is irreducible (this condition is realized in particular if $\dim(X) = 1$ or $\mathbf{char}(k) = 0$), then

$$Z^{\mathrm{gen}}(X;T) = \sum_{m\geq 0} [\overline{\theta_{m,X}(\mathscr{L}_\infty(X))}]T^m.$$

9.7. We obtain the following statement from theorem 1.5:

Proposition 9.8. *Let k be a perfect field. Let X, X' be integral k-varieties. We assume that X' is smooth over k. Let $h : X' \to X$ be a birational morphism. We assume that the scheme $\mathscr{L}_\infty(X)$ is irreducible (e.g., $\dim(X) = 1$ or $\mathbf{char}(k) = 0$) and that, for every integer $m \in \mathbf{N}$, the image of $h_m : \mathscr{L}_m(X') \to \mathscr{L}_m(X)$ is closed. Then, we have $Z^{\mathrm{gen}}(X;T) = P(X;T)$. In particular, if $\mathbf{char}(k) = 0$, the zeta function of the general components of the jet schemes then is rational in the ring $\mathscr{M}_k[[T]]$.*

Proof. Let us recall that, by the functoriality of the arc and jet schemes, the following diagram of morphisms of k-schemes commutes:

$$\begin{tikzcd}
\mathscr{L}_\infty(X') \arrow[r, "h_\infty"] \arrow[d, "\theta_{m,X'}"'] & \mathscr{L}_\infty(X) \arrow[d, "\theta_{m,X}"] \\
\mathscr{L}_m(X') \arrow[r, "h_m"'] & \mathscr{L}_m(X)
\end{tikzcd}$$

The variety X' being smooth over k, the morphism $\theta_{m,X'}$ is surjective, thus $h_m(\mathscr{L}_m(X')) \subset \theta_{m,X}(\mathscr{L}_\infty(X))$. Since $\mathscr{L}_\infty(X)$ is irreducible, by theorem 1.5 we have

$$\theta_{m,X}(\mathscr{L}_\infty(X)) \subset \overline{h_m(\mathscr{L}_m(X'))} = \mathscr{G}_m(X).$$

Now $h_m(\mathscr{L}_m(X')) = \overline{h_m(\mathscr{L}_m(X'))}$ by assumption, so $\mathscr{G}_m(X) = \theta_{m,X}(\mathscr{L}_\infty(X))$ and the power series $Z^{\mathrm{gen}}(X;T)$ and $P(X;T)$ coincide. □

The assumption that the subset $h_m(\mathscr{L}_m(X'))$ is closed in $\mathscr{L}_m(X)$ for every integer $m \in \mathbf{N}$ is a nonempty (see subsection 9.13) but quite restrictive condition as the following example shows.

Example 9.9. Let k be a field of characteristic zero. Let $f \in k[x,y]$ be the polynomial $f = x^3 - y^2$. We set $X = \mathrm{Spec}(k[x,y]/\langle f\rangle)$. Let K be a field extension of k. E.g., the element $(T,0) \in (K[[T]]/\langle T^2\rangle)^2$ defines an element in $\overline{\theta_{1,X}(\mathscr{L}_\infty(X))} \setminus \theta_{1,X}(\mathscr{L}_\infty(X))$. Indeed, we have the isomorphism

$$\mathscr{G}_1(X) \cong \mathrm{Spec}\left(\frac{k[x,y]_1}{\langle f, 3x_0^2x_1 - 2y_0y_1, 3x_0y_1 - 2y_0x_1, 9x_0x_1^2 - 4y_1^2\rangle}\right).$$

9.10. If the field k is assumed to be of characteristic zero, the proof of the rationality of $P(X;T)$ relies on the use of model-theoretic arguments. It seems to us to be a challenging question to understand whether one can follow the same spirit to prove the rationality of $Z^{\mathrm{gen}}(X;T)$. To the best of our knowledge and understanding, such a question is open and seems to be difficult.

Question 9.11. *Is the power series $Z^{\mathrm{gen}}(X;T)$ always rational?*

We have observed in various examples that the answer could be positive outside the class of varieties satisfying the assumptions of proposition 9.8. The following example in positive characteristic illustrates this assertion.

Example 9.12. Let $k = \mathbf{F}_2$. Let us consider $f = x^3 - y^2$, and denote $X = \mathrm{Spec}(k[x,y]/\langle f\rangle)$. The power series $Z^{\mathrm{gen}}(X;T)$ equals $\sum_{m\geq 0} \mathbf{L}^{m+1}T^m = \mathbf{L}/(\mathbf{1} - \mathbf{L}\mathbf{T})$ and is hence rational. Let us briefly explain how to compute this power series. The normalization of X is given by the morphism $\nu^\sharp : \mathbf{F}_2[x,y]/\langle x^3 - y^2\rangle \to \mathbf{F}_2[T]$, $\nu^\sharp(x) = T^2$, $\nu^\sharp(y) = T^3$. Then, for $m \in \mathbf{N}$, the induced morphism $\nu_m^\sharp : \mathcal{O}(\mathscr{L}_m(X)) \to \mathbf{F}_2[T]_m$ is given by the factorization through $\mathcal{O}(\mathscr{L}_m(X))$ of the morphism $\tilde{\nu}_m^\sharp : \mathbf{F}_2[x,y]_m \to \mathbf{F}_2[T]_m$ such that $\tilde{\nu}_m^\sharp(x_i) = 0$ if i is odd, $\tilde{\nu}_m^\sharp(x_i) = T_{i/2}$ if i is even and $\tilde{\nu}_m^\sharp(y_i) = \sum_{j=0}^{\lfloor i/2\rfloor} T_j^2 T_{i-2j}$. Now, by corollary 5.4 we have $\mathscr{N}_m(X) = \mathrm{Ker}(\tilde{\nu}_m^\sharp)$.

We consider the family $(g_i)_{i\in\mathbf{N}}$ of elements of $k[x,y]_\infty$ defined as $g_i = x_i$ if i is odd and $g_i = g_{2j} = y_j^2 + \sum_{r=0}^{\lfloor j/2\rfloor} x_{2r}^2 x_{2j-4r}$ if i is even. Using some technical arguments, one can check that $\operatorname{Ker}(\tilde{\nu}_m^\sharp) = \langle g_i : 1 \le i \le m\rangle$, hence

$$\mathscr{N}_m(X) = \langle g_i : 1 \le i \le m\rangle.$$

Now one can prove by an induction on m that, for every integer $m \in \mathbf{N}$, $[\mathscr{G}_m(X)] = \mathbf{L}^{m+1}$. One checks that it is true for $m \le 4$ and that for $m \ge 5$ odd, one has

$$k[x,y]_m/\mathscr{N}_m(X) \cong (k[x,y]_{m-1}/\mathscr{N}_{m-1}(X)) \otimes_k k[y_m]$$

which shows the result in this case. For $m \ge 5$ even, one has

$$k[x,y]_m[x_0^{-1}]/\mathscr{N}_m(X) \cong k[x,y]_0[x_0^{-1}]/\mathscr{N}_0(X) \otimes_k k[y_i; 1 \le i \le m]$$

and hence

$$[\mathscr{G}_m(X) \cap D(x_0)] = (\mathbf{L} - \mathbf{1})\mathbf{L}^m.$$

On the other hand, if we consider the morphism of k-algebras $\rho : k[x,y]_\infty \to k[x,y]_\infty$ given by $\rho(x_i) = x_{i+2}$, $\rho(y_i) = y_{i+3}$ for $i \in \mathbf{N}$, then $g_s = \rho(g_{s-6}) \pmod{x_0}$ for $s \ge 6$ even. This is useful to prove that

$$(k[x,y]_m/\mathscr{N}_m(X) + \langle x_0\rangle)_{\mathrm{red}} \cong (k[x,y]_{m-6}/\mathscr{N}_{m-6}(X))_{\mathrm{red}} \otimes_k k[x_{m-4}, x_{m-2}, y_{m-5}, y_{m-4}, y_{m-3}]$$

and we deduce that

$$[\mathscr{G}_m(X) \cap V(x_0)] = \mathbf{L}^m,$$

which proves our formula since $[\mathscr{G}_m(X)] = [\mathscr{G}_m(X) \cap D(x_0)] + [\mathscr{G}_m(X) \cap V(x_0)]$.

9.13. In this subsection, we provide a concrete class of *singular* k-varieties endowed with a resolution of singularities which satisfy the assumption of proposition 9.8.

Theorem 9.14. *Let k be a perfect field. Let $f \in k[x,y]$ be an irreducible polynomial which is assumed to be homogeneous. Let X be the associated affine plane k-curve. Let $\nu\colon X' \to X$ be the normalization of X. For every integer $m \in \mathbf{N}$, the image of the induced morphism $\nu_m\colon \mathscr{L}_m(X') \to \mathscr{L}_m(X)$ is closed.*

Let us stress that the class of curves involved in theorem 9.14 corresponds, after a base change to an algebraically closed extension of k, to a (possibly nonreduced) union of lines through the origin.

Proof. In the case where $f = x$ or $f = y$ our statement is obviously satisfied by subsection 9.6. From now on we assume that f does not coincide with x or y. For every integer $m \in \mathbf{N}$ and every k-algebra K, by the very definition, one has

$$\mathscr{L}_m(X)(K) \cong \operatorname{Hom}_{\mathbf{Sch}_k}(\operatorname{Spec}(K[[T]]/\langle T^{m+1}\rangle), X) \cong \operatorname{Hom}_{\mathbf{Alg}_k}(k[x,y]/\langle f\rangle, K[[T]]/\langle T^{m+1}\rangle).$$

So, we identify the datum of an element of $\mathscr{L}_m(X)(K)$ with that of a pair $(\varphi_m(T), \psi_m(T)) \in (K[[T]]/\langle T^{m+1}\rangle)^2$ such that $f((\varphi_m(T), \psi_m(T))) \equiv 0 \pmod{T^{m+1}}$. Let $m \ge 1$. Let $g \in k[z]$ be the polynomial $f(z,1)$. Let us set $F = k[z]/\langle g\rangle$. Since the polynomial f is assumed to be irreducible, the k-algebra F is a field, which is a nontrivial separable field extension of k. We have $\mathcal{O}(X') = F[x]$. The normalization ν is the dual (in the opposite category) of the morphism of k-algebras

$$\nu^\sharp\colon k[x,y]/\langle f\rangle \to F[x]$$

defined by $x \mapsto x, y \mapsto zx$. Let K be a k-algebra. We deduce that

$$\begin{array}{rcl} \nu_m(\mathscr{L}_m(X'))(K) & = & \{(\varphi_m(T), \psi_m(T)) \in \mathscr{L}_m(X)(K); \exists a \in F(K)\ \psi_m(T) = a\varphi_m(T)\} \\ & = & \bigcup_{a\in F(K)}\{(\varphi_m(T), a\varphi_m(T)); \varphi_m(T) \in K[[T]]/\langle T^{m+1}\rangle\} \end{array} \tag{9.2}$$

This defines a closed subset of $\mathscr{L}_m(X)(K)$. Since the construction obviously is functorial on K, that concludes the proof. $\square$

9.15. Formula (9.2) allows us to deduce an explicit expression of the power series $Z^{\mathrm{gen}}(X;T)$ in this specific case. In particular, we provide here an alternative proof of its rationality which works in every characteristic. Let k be a perfect field. Let $f \in k[x,y]$ be a homogeneous irreducible polynomial, with $\deg(f) = d \geq 1$. We set $X = \mathrm{Spec}(k[x,y]/\langle f \rangle)$. Then, for every integer $m \in \mathbf{N}$, we have

$$[\mathscr{G}_m(X)] = [F]\mathbf{L}^{m+1} - [F] + \mathbf{1}.$$

Indeed, if $s\colon \mathrm{Spec}(F) \to \mathbf{A}_F^{m+1}$ is the zero section of $\mathbf{A}_F^{m+1} \to \mathrm{Spec}(F)$ and $\mathfrak{o}$ the zero jet in $\mathscr{L}_m(X)$, we observe by formula (9.2) that $\mathscr{G}_m(X)$ is piecewise isomorphic to the k-variety

$$\mathbf{A}_F^{m+1} \setminus s(\mathrm{Spec}(F)) \sqcup \mathfrak{o}.$$

As a direct consequence of the very definition of our power series, we conclude that

$$\begin{aligned} Z^{\mathrm{gen}}(X;T) &= \frac{[F]\mathbf{L}}{\mathbf{1}-\mathbf{L}T} + \frac{\mathbf{1}-[F]}{\mathbf{1}-T} \\ &= \frac{\mathbf{1}-[F]+[F]\mathbf{L}-\mathbf{L}T}{(\mathbf{1}-\mathbf{L}T)(\mathbf{1}-T)} \end{aligned} \tag{9.3}$$

As a consequence, if the field extension $p\colon k \to k'$ defines a splitting field of f, we have

$$[\mathscr{G}_m(X)]_{k'} := p^*[\mathscr{G}_m(X)] := [\mathscr{G}_m(X) \otimes_k k'] = d\mathbf{L}^{m+1} - d + \mathbf{1} \in K_0(\mathrm{Var}_{k'}).$$

and, if Z' is the image of $Z^{\mathrm{gen}}(X;T)$ in $K_0(\mathbf{Var}_{k'})[[T]]$ by the base-change ring morphism (see subsection 9.1),

$$Z' = \frac{\mathbf{1}-d+d\mathbf{L}-\mathbf{L}T}{(\mathbf{1}-\mathbf{L}T)(\mathbf{1}-T)}.$$

Example 9.16. Let $k = \mathbf{Q}$. Let $f = x^2 + xy + y^2 \in k[x,y]$. Let $p(z) = z^2 + z + 1 \in \mathbf{Q}[t]$ and $X = \mathrm{Spec}(k[x,y]/\langle f \rangle)$. Let us set $F := k[z]/\langle p(z) \rangle$. The normalization morphism

$$\nu\colon k[x,y]/\langle f \rangle \to F[x] \cong \mathbf{A}_F^1$$

is given by $x \mapsto x, y \mapsto zx$. Theorem 9.14 provides the formula

$$Z^{\mathrm{gen}}(X;T) = \frac{\mathbf{1}-[F]+[F]\mathbf{L}-\mathbf{L}T}{(\mathbf{1}-\mathbf{L}T)(\mathbf{1}-T)}.$$

Let j be a nontrivial 3rd root of unity, and K be a field extension of $\mathbf{Q}(j)$. Thus, we also deduce from theorem 9.14 that

$$Z' = \frac{-\mathbf{1}+2\mathbf{L}-\mathbf{L}T}{(\mathbf{1}-\mathbf{L}T)(\mathbf{1}-T)}.$$

Since there exists an obvious K-isomorphism between $X_K = \mathrm{Spec}(K[x,y]/\langle f \rangle)$ and the nodal plane curve singularity $Y = \mathrm{Spec}(K[x,y]/\langle xy \rangle)$, we observe that this formula coincides with that of the Poincaré power series $P(Y;T)$ as expected.

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