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INTERSECTION NUMBERS
IN A HYPERBOLIC SURFACE

A DISSERTATION APPROVED FOR THE
DEPARTMENT OF MATHEMATICS

BY

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DEDICATION

to

My wife:

Johana Ortega

and daughters:

Mará Salomé Herrera-Ortega and

Mará Sofia Herrera-Ortega

For

Faithfully accompanying me during this journey.

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Abstract

For a compact surface S with constant negative curvature $-\kappa$ (for some $\kappa > 0$) and genus $\mathfrak{g} \geq 2$, we show that the tails of the distribution of $i(\alpha, \beta)/l(\alpha)l(\beta)$ (where $i(\alpha, \beta)$ is the intersection number of the closed geodesics α and β and $l(\cdot)$ denotes the geometric length) are estimated by a decreasing exponential function. As a consequence, we find the asymptotic normalized average of the intersection numbers of pairs of closed geodesics on S. In addition, we prove that the size of the sets of geodesics whose T-self-intersection number is not close to $\kappa T^2/(2\pi^2(\mathfrak{g}-1))$ is also estimated by a decreasing exponential function. And, as a corollary of the latter, we obtain a result of Lalley which states that most of the closed geodesics γ on S with $l(\gamma) \leq T$ have roughly $\kappa l(\gamma)^2/(2\pi^2(\mathfrak{g}-1))$ self-intersections, when T is large.

Chapter 1

Introduction

Before enunciating the questions that motivated this dissertation, we give some definitions and notation.

Let S be a compact hyperbolic surface of constant curvature $-\kappa$, for some $\kappa > 0$, and genus $g \geq 2$. An (**oriented**) **geodesic** (**parametrized by the arc length**) **on** S is a smooth locally distance-minimizing curve $\gamma : \mathbb{R} \to S$. A geodesic γ on S is **closed** if there exists $l > 0$ such that $\gamma(t) = \gamma(t + l)$ and $\dot{\gamma}(t) = \dot{\gamma}(t + l)$, for every $t \in \mathbb{R}$, where $\dot{\gamma}(t)$ denotes the unit vector tangent to γ at $\gamma(t)$. The minimum of such numbers is the **length of** γ and is denoted by $l(\gamma)$.

We say that two geodesics γ and γ' on S are **identical** if they both have the same trace, that is, there is $r \in \mathbb{R}$ such $\gamma(t) = \gamma'(t + r)$ and $\dot{\gamma}(t) = \dot{\gamma}'(t + r)$, for every $t \in \mathbb{R}$. Let $[\gamma]$ be the equivalence class formed by all geodesics on S that are identical to γ. By the Axiom of Choice, we choose a (representative) geodesic from each class and form a set that we denote by $\mathbb{G}(S)$. Let $C\mathbb{G}(S)$ be the subset of $\mathbb{G}(S)$ consisting of the geodesics that are closed.

In this work, we are interested on both the number of intersections of pairs of the elements of $\mathbb{G}(S)$ and in the number of self-intersections of the elements of $\mathbb{G}(S)$. Thus, let us define this notions in a precise way. First, we define the intersection number for pairs of closed geodesics.

Definition 1.1. For geodesics α and β on S with α closed, we define $i(\alpha, \beta)$, the (**geometric**) **intersection number of** α **and** β, by

$$\#\{\mathbf{s} \mid \alpha(r) = \beta(t) = \mathbf{s}; \dot{\alpha}(r), \dot{\alpha}(t) \text{ are non-parallel, for some } r, t \in \mathbb{R}\}. \quad (1.1)$$

In particular, $i(\alpha, \alpha)$ is the **self-intersection number of** α.

The issue we encounter when we try to generalize this notion of intersection to non-closed geodesics is that they may have infinitely many intersection points. In order to solve this issue, we use an extension of the intersection number of Definition (1.1) for segments of finite length of the pair of geodesics.

Definition 1.2. For $T > 0$ and two oriented non-closed geodesics γ and η on S, define $i^T(\gamma, \eta)$, the **T-intersection number of** γ and η, by

$$\#\{\mathbf{s} \mid \gamma(r) = \eta(t) = \mathbf{s}; \dot{\gamma}(r) = \dot{\eta}(t) \text{ are non-parallel, for some } r, t \in [0, T]\}. \quad (1.2)$$

In particular, $i^T(\gamma, \gamma)$ is the **T-self-intersection number of** γ.

Remark 1.3. The T-intersection number is indeed a generalization of the intersection number for pairs of closed geodesics on S, since $i^T(\alpha, \beta) = i(\alpha, \beta)$, for the closed geodesics α and β on S whenever $T \geq \max\{l(\alpha), l(\beta)\}$.

For $T > 0$, let $C\mathbb{G}_T(S) \subset C\mathbb{G}(S)$ consist of the geodesics γ with $l(\gamma) \leq T$, and $N(T) = \#C\mathbb{G}_T(S)$. This number $N(T)$ is finite for every $T > 0$.

1.1 Central Questions

The main results of this dissertation are motivated by the following questions:

- For $T > 0$, what is the average of the T-self-intersection numbers of all the geodesics on S?
- What is the average of the self-intersection numbers of the closed geodesics on S of a given length?
- What is the size of the set of geodesics whose T-self-intersection number is not close to the average?
- What is the average of the intersection numbers of pairs of closed geodesics on S as their lengths grow arbitrarily large?
- What is the size of the set formed by the pairs of closed geodesics whose intersection number is not close to the average?

1.2 Known Results

The intersection numbers have been extensively studied. Here we provide the results that we consider relevant to our work.

1.2.1 The Average of the Self-Intersection Numbers of Closed Geodesics

In [18], Lalley proves that the probability for $\gamma \in C\mathbb{G}_T(S)$ have its self-intersection number close to $\kappa l(\gamma)^2/(2\pi^2(\mathfrak{g} - 1))$ goes to one, as $T \to \infty$.

Theorem 1.4 (Lalley)**.** *There exists a constant L_S such that for every $\epsilon > 0$,*

$$\lim_{T\to\infty} \frac{\#\{\gamma \in C\mathbb{G}_T(S) : |i(\gamma,\gamma) - L_S l(\gamma)^2| < \epsilon l(\gamma)^2\}}{N(T)} = 1.$$

In fact, $L_S = \dfrac{\kappa}{2\pi^2(\mathfrak{g}-1)} = \dfrac{2}{\mathrm{Vol}(T^1(S))}$.

1.2.2 The Average Number of Angular Self-Intersections

Let us use the following notation: we write $f(T) = O\big(e^{-\delta T}\big)$ to mean that there exist some constants $C, R > 0$ such that $f(T) \leq Ce^{-\delta T}$, whenever $T > R$, in other words, f is estimated by a decreasing exponential function. In this case, we say that the function f **decays exponentially fast**.

Pollicot and Sharp defined in [22] a generalization of the self-intersection number by considering the number of self-intersections of γ whose angle of intersection θ is in the interval $[\theta_1, \theta_2]$, for $0 \leq \theta_1 < \theta_2 \leq \pi$. For a closed geodesic γ on S, they denote such number by $i_{\theta_1,\theta_2}(\gamma)$. Their main result is the following.

Theorem 1.5. *Given $0 \leq \theta_1 < \theta_2 \leq \pi$, there exists $I = I(\theta_1, \theta_2)$ and $\delta > 0$ such that, for any $\epsilon > 0$,*

$$\#\left\{\gamma \in C\mathbb{G}_T(S) \;\middle|\; \frac{i_{\theta_1,\theta_2}(\gamma)}{l(\gamma)^2} \notin (I-\epsilon, I+\epsilon)\right\} = O\big(e^{-\delta T}\big),$$

as $T \to \infty$. Moreover,

$$I(\theta_1,\theta_2) = \frac{1}{8\pi^2(\mathfrak{g}-1)} \int_{\theta_1}^{\theta_2} \sin\theta d\theta.$$

1.2.3 Distribution of the Self-Intersection Counts of Cyclic Words

In [9], Chas and Lalley defined the self-intersection count of a cyclic word α, denoted by $N(\alpha)$, as the minimum number of transversal double points among all closed curves represented by α. They prove the following.

Theorem 1.6. *There exists a constant $\Upsilon > 0$ such that*

$$\frac{N_m - m^2\kappa/(2\pi^2(\mathfrak{g}-1))}{\Upsilon m^{3/2}} \Rightarrow \; \mathit{Normal}(0,1)$$

as $m \to \infty$.

Here, Normal(0,1) denotes the standard Gaussian distribution on $\mathbb{R}$, $\Rightarrow$ is the convergence in distribution, and N_m is the random variable obtained by evaluating the self-intersection count N at a randomly chosen $\alpha \in W_m$, where W_m is the set of cyclic words of length $\leq m$.

1.2.4 The Distribution of the Self-Intersection Numbers of Closed Geodesics

In [17], Lalley proves the following.

Theorem 1.7. *Let γ_T be a closed geodesic on S of length $\leq T$ randomly chosen and let $N(\gamma_T)$ be the number of self-intersections of γ_T. Then, there exists some probability distribution Ψ on $\mathbb{R}$, as $T \to \infty$, for which*

$$\frac{N(\gamma_T) - \kappa T^2/(2\pi^2(\mathfrak{g}-1))}{T^{3/2}} \Rightarrow \Psi.$$

1.3 Our Results

First, note that Theorem 1.4 does not state anything about the size of the set of closed geodesics γ on S with $l(\gamma) \leq T$ and self-intersection number not close to $\kappa l(\gamma)^2/(2\pi^2(\mathfrak{g}-1))$.

It is precisely the questioning about the size of the set of "irregular" closed geodesics that lead us to formulate the questions in §1.1.

In order to study how to answer these questions we utilize the theory that Pollicot and Sharp use to prove their results. More specifically, we consider the identification, obtained via the exponential map, of the set of geodesics on S with the unit tangent bundle of S, i.e., $T^1(S) = \{\mathbf{v} = (x, v) \mid x \in S, v \in T_xS, \|v\| = 1\}$. Under such identification, we have that every $\mathbf{v} = (x, v) \in T^1(S)$ determines a unique oriented geodesic $\gamma_{\mathbf{v}} : \mathbb{R} \to S$ such that $\gamma_{\mathbf{v}}(0) = x$ and $\dot{\gamma}_{\mathbf{v}}(0) = v$. This space, the unit tangent bundle, is endowed with a unique regular Borel probability measure of maximum entropy, called the normalized Liouville measure on $T^1(S)$ and denoted here by $\overline{\vartheta}$, which is invariant under the geodesic flow over S. We also apply the results about the geodesic currents and the intersection form of pairs of such currents, which extend the concepts of geodesics on S and their intersection numbers.

Our first result is obtained with the help of Kifer's large deviation result of [16].

Theorem 1.8. *Let $\epsilon > 0$. There exists $\delta > 0$ such that*

$$\overline{\vartheta}\left\{\mathbf{v} \in T^1(S) : \left|\frac{i^T(\gamma_{\mathbf{v}}, \gamma_{\mathbf{v}})}{T^2} - L_S\right| \geq \epsilon\right\} = O\big(e^{-\delta T}\big),$$

as $T \to \infty$.

Given that the set of representative geodesics $\mathbb{G}(S)$ can be identified with the set $\{\mathbf{v} \in T^1(S) \mid \gamma_{\mathbf{v}} \in \mathbb{G}(S)\} \subset T^1(S)$, Theorem 1.8 and the characterization of the measure $\overline{\vartheta}$ given by Bowen in [7] imply Theorem 1.4.

In addition, the proof of the fact that the tails of the distribution of the quotients $\frac{i(\alpha,\beta)}{l(\alpha)l(\beta)}$ for the pairs of closed geodesics on S are also estimated by a decreasing exponential function is provided. For $R, T > 0$, let $C\mathbb{G}_{R,T}(S) := C\mathbb{G}_R(S) \times C\mathbb{G}_T(S)$.

Theorem 1.9. *Let $\epsilon > 0$. There exists $\delta > 0$ such that*

$$\frac{1}{N(R)N(T)} \# \left\{ (\alpha, \beta) \in C\mathbb{G}_{R,T}(S) : \left| \frac{i(\alpha,\beta)}{l(\alpha)l(\beta)} - L_S \right| \geq \epsilon \right\} = O\big(e^{-\delta R}\big),$$

as $R \to \infty$, with $T \geq R$.

For the proof of this theorem we use the fact that the intersection form function is a continuous extension of the intersection number function, a fact which Bonahon proves in [3].

Additionally, we show a bound for the intersection numbers of pairs of closed geodesics on S. Let $\varrho(S)$ be the injectivity radius of S.

Proposition 1.10. *Let α and β be closed geodesics on S. Then*

$$i(\alpha, \beta) \leq \frac{l(\alpha)l(\beta)}{\varrho(S)^2}.$$

Last, Theorem 1.9 and Proposition 1.10 allow us to find what we call the "asymptotic normalized" average of the intersection number of pairs of closed geodesics on S. We say that f and g are **asymptotically equal** and write $f(T) \sim g(T)$ if the ratio $f(T)/g(T) \to 1$, as $T \to \infty$.

Corollary 1.11.

$$\frac{1}{N(R)N(T)} \sum_{(\alpha,\beta) \in \mathbb{G}_{R,T}(S)} \frac{i(\alpha,\beta)}{l(\alpha)l(\beta)} \sim L_S, \text{ as } R, T \to \infty.$$

1.4 Summary of Following Chapters

We divide the rest of this work into three chapters.

In Chapter 2, we review most of the definitions and results about the geodesic flow on the unit tangent bundle of a hyperbolic surface. Some of the topics included are: the Anosov property of the geodesic flow, mixing, entropy and the measure of maximum entropy of the geodesic flow.

Chapter 3 contains the theory of geodesic currents and their intersection form. We give an analytic definition as well as a geometric interpretation of such objects.

Finally, in Chapter 4, we enunciate two more tools needed for the proof of our results: a deviation result by Kifer and a bound for the intersection number of a pair of closed geodesics. Lastly, we give the proofs of our theorems with the using of the facts previously reviewed.

Chapter 2

The Geodesic Flow

In this chapter we give most of the definitions, concepts and basic facts about the geodesic flow need in the proofs of our results. Most of the material about hyperbolic surfaces (§2.1), the unit tangent bundle (§2.2) and the geodesic flow (§2.3) was taken from the paper by Hedlund [11]. The section about the geometry of the unit tangent bundle (§2.2.1) and the Anosov property of the geodesic flow (§2.3.1) is based on the paper by Parkkonen and Paulin [21]. Finally, the notions of entropy (§2.4) and mixing (§2.3.1.1) were taken from the book by Katok and Hasselblatt [14].

2.1 Hyperbolic Surfaces

Let D^2 be the unit open disk of $\mathbb{C}$ centered at the origin, $\{z = x + iy \in \mathbb{C} : |z| < 1\}$, where $x = \Re(z)$ is the real part of z and $y = \Im(z)$ the imaginary part of z, and S^1 be its boundary $\{z \in \mathbb{C} : |z| = 1\}$. We equip D^2 with the following Riemannian metric

$$\frac{4(dx^2 + dy^2)}{\kappa(1 - x^2 - y^2)^2} = \frac{4|dz|}{\kappa(1 - |z|^2)^2}, \qquad \kappa > 0. \quad (2.1)$$

That is, the length of the vector $w \in \mathbb{C}$ pointed at $z \in D^2$ induced by this metric, $\|w\|$, is $\dfrac{4|w|^2}{\kappa(1 - |z|^2)^2}$. In this way we get a simply connected Riemannian manifold $\mathbb{H}$ of constant negative curvature $-\kappa$. Its boundary S^1 is known as the **ideal boundary of** $\mathbb{H}$ and denoted by $\partial\mathbb{H}$.

The metric (2.1) assigns to a curve $\gamma : [a, b] \to \mathbb{H}$, the **hyperbolic length**

$$l(\gamma) = \int_a^b \|\dot{\gamma}(t)\| dt.$$

Angle is Euclidean angle, and the element of (hyperbolic) area is

$$\frac{4dxdy}{\kappa(1 - x^2 - y^2)^2}. \quad (2.2)$$

The geodesics defined by the metric (2.1) are arcs of circles orthogonal to $\partial\mathbb{H}$ as well as lines joining pairs of points of $\partial\mathbb{H}$ diametrically opposite, and we call them **hyperbolic lines**. Given two points P and Q of $\mathbb{H}$ there exists

a unique geodesic (or line) segment joining them, and the hyperbolic length of this segment is the **hyperbolic distance** $H(P,Q)$ between P and Q.

The **Möbius group of** $\mathbb{H}$ is the set

$$\mathrm{Mob}(\mathbb{H}) = \{f : \overline{\mathbb{H}} \to \overline{\mathbb{H}} \mid f(z) = (az+b)/(\bar{b}z+\bar{a}), a, b \in \mathbb{C}, |a|^2 - |b|^2 > 0\}.$$

The elements of $\mathrm{Mob}(\mathbb{H})$ are **linear fractional** (or **Möbius**) **transformations**. The metric (2.1) is invariant under the linear fractional transformations, that is, hyperbolic distance, angle, and area are preserved under the action of such transformations. A Möbius transformation is either **elliptic** if its unique fixed point is in $\mathbb{H}$ or **parabolic** if its unique fixed point is on $\partial\mathbb{H}$ or **hyperbolic** if its unique pair of fixed points are on $\partial\mathbb{H}$. These transformations are rigid motions of the well known hyperbolic geometry under consideration, that is, $\mathrm{Mob}^+(\mathbb{H})$ is the group of **orientation-preserving isometries of** the hyperbolic space $\mathbb{H}$.

Now, let G be a **Fuchsian** group with $\partial\mathbb{H}$ as principal circle. That is, G is a discrete subgroup of the topological group $\mathrm{Mob}(\mathbb{H})$ (with respect to the standard topology) whose elements transform $\partial\mathbb{H}$ into $\partial\mathbb{H}$ and $\mathbb{H}$ into $\mathbb{H}$. The action of G is properly discontinuous on $\mathbb{H}$, that is, for all compact subsets K of $\mathbb{H}$ there are only a finite number of $g \in G$ such that $K \cap g(K) \neq \emptyset$. Two sets of points in $\mathbb{H}$ are (G-)**congruent** if there is a transformation of G taking one of these sets into the other. Either set is said to be a **copy** of the other.

To such a group G there exist a **normal fundamental region** R. This is a simply connected region bounded by arcs of hyperbolic lines which are congruent in pairs, such that no interior points of R is congruent to some point within or on the boundary of R. If suitable conventions are made as to the inclusion of boundary points of R, no two copies of R have a common point and the totality of theses copies fills $\mathbb{H}$, that is, we have a **tesallation of** $\mathbb{H}$ by R and its copies.

If points which are congruent under G are consider identical, there is defined a (**two-dimensional orientable**) **hyperbolic manifold** or **surface** $\mathbb{H}/G$ of constant negative curvature $-\kappa$.

If $S = \mathbb{H}/G$, then G is the group of deck transformations which can be identified with $\pi_1(S)$, the fundamental group of the surface S. In the present work, we only consider groups whose surfaces are compact. In this case, there exists a positive integer $\mathfrak{g} \geq 2$, called the **genus of** S, such that the fundamental region of G is a polygon with Euler characteristic $\chi = 2 - 2\mathfrak{g}$.

2.2 The Unit Tangent Bundle

An **element** (or **vector**) $\mathbf{v}$ in $\mathbb{H}$ can be parametrized by (z, v), where z is a point of $\mathbb{H}$ and v is a unit vector tangent to $\mathbb{H}$ at z whose angle (or direction) is measured positively in the counterclockwise sense from a direction parallel to the positive real axis. The point z is the point **bearing** the element $\mathbf{v}$. The **distance between** $\mathbf{v} = (z, v)$ **and** $\mathbf{v}_1 = (z_1, v_1)$, denoted by $\text{dist}(\mathbf{v}, \mathbf{v}_1)$, is given by

$$\text{dist}(\mathbf{v}, \mathbf{v}_1) = H(z, z_1) + \measuredangle(v, v_1).$$

where $\measuredangle(v, v_1)$rad is the measure of the angle between v and v_1 in radians.

A **neighborhood** of the vector $\mathbf{v}_1 = (z_1, v_1)$ is the set of $\mathbf{v} = (z, v)$ such that $\text{dist}(\mathbf{v}, \mathbf{v}_1) < \epsilon$, for some $\epsilon > 0$. Let $T^1(\mathbb{H})$ denote the space of all elements in $\mathbb{H}$ with neighborhoods thus defined. The manifold $T^1(\mathbb{H})$ is called the **unit tangent bundle of** $\mathbb{H}$. Every $\mathbf{v} \in T^1(\mathbb{H})$ determines a unique (oriented) geodesic on $\mathbb{H}$ which we denote by $\gamma_{\mathbf{v}}$.

The fundamental group of S, that is $\pi_1(S)$ (identified with the group of deck transformations,) acts on $T^1(\mathbb{H})$ in the following way

$$g \cdot (z, v) = (g(z), \measuredangle(v, g'(z)), \tag{2.3}$$

for $g \in \pi_1(S)$ and $(z, v) \in T^1(\mathbb{H})$.

Let $T^1(S)$ be the space obtained by identifying $\pi_1(S)$-congruent vectors of $T^1(\mathbb{H})$. The space $T^1(S)$ is called the **unit tangent bundle of** S. Neighborhoods are defined in $T^1(S)$ as the correspondents of the neighborhoods in $T^1(\mathbb{H})$.

2.2.1 Geometry of the Unit Tangent Bundle

We denote by $\mathfrak{p} : T(S) \to S$ the tangent bundle of S, and again by $\mathfrak{p} : T^1(S) \to S$ its unit tangent bundle. Recall that the Levi-Civita connection ∇ of S gives a decomposition $T(T(S)) = V \oplus H$ of the vector bundle $T(T(S)) \to T(S)$ into the direct sum of two smooth vector subbundles $V \to T(S)$ and $H \to T(S)$, called **vertical** and **horizontal**, such that if $\mathfrak{p}_V : T(T(S)) \to V$ is the linear projection of $T(T(S))$ onto V parallelly to H, if $H_{\mathbf{v}}$ and $V_{\mathbf{v}}$ are the fibers of H and V above $\mathbf{v} \in T(S)$, then

- we have $V_{\mathbf{v}} = Ker T_{\mathbf{v}}\mathfrak{p} = T_{\mathbf{v}}(T_{\mathfrak{p}(\mathbf{v})}S) = T_{\mathfrak{p}(\mathbf{v})}S$;
- the restriction $T\mathfrak{p}_{|H_{\mathbf{v}}} : H_{\mathbf{v}} \to T_{\mathfrak{p}(\mathbf{v})}S$ of the tangent map of $\mathfrak{p}$ to $H_{\mathbf{v}}$ is a linear isomorphism;

- for every smooth vector field $X : S \to T(S)$ on S , we have $\nabla_{\mathbf{v}} X = \mathfrak{p}_V \circ TX(\mathbf{v})$.

The manifold $T(S)$ has a unique Riemannian metric, called Sasaki′s metric, such that for every $\mathbf{v} \in T(S)$, the map $T\mathfrak{p}_{|H_{\mathbf{v}}} : H_{\mathbf{v}} \to T_{\mathfrak{p}(\mathbf{v})}S$ is isometric, the restriction to $V_{\mathbf{v}}$ of Sasaki′s scalar product is the Riemannian scalar product on $T_{\mathfrak{p}(\mathbf{v})}S$, and the decomposition $T_{\mathbf{v}}T(S) = V_{\mathbf{v}} \oplus H_{\mathbf{v}}$ is orthogonal. We endow the smooth submanifold $T^1(S)$ of $T(S)$ with the induced Riemannian metric, also called Sasaki′s metric. The fiber $T^1_x(S)$ of every $x \in S$ is then isometric to the standard unit sphere S^1 of the standard Euclidean space $\mathbb{R}^2$.

The Riemannian measure $\vartheta = d\operatorname{Vol}_{T^1(S)}$ of $T^1(S)$, called **Liouville′s measure**, disintegrates under the fibration $\mathfrak{p} : T^1(S) \to S$ over the Riemannian measure $d\operatorname{Vol}_S$ of S, as

$$d\operatorname{Vol}_{T^1(S)} = \int_{x \in S} d\operatorname{Vol}_{T^1_x(S)} \, d\operatorname{Vol}_S(x),$$

where $d\operatorname{Vol}_{T^1_x(S)}$ is the spherical measure on the fiber $T^1_x(S)$ of $\mathfrak{p}$ above $x \in S$. In particular, by the Gauss–Bonnet theorem,

$$\vartheta(T^1(S)) = d\operatorname{Vol}(S^1) d\operatorname{Vol}(S) = \frac{4\pi^2(\mathfrak{g} - 1)}{\kappa}.$$

2.3 The Geodesic Flow

The geodesics on S are **represented** by sets of hyperbolic lines of $\mathbb{H}$, $\pi_1(S)$–congruent hyperbolic lines representing the same geodesic. The geodesics on S define a collection of $\mathbb{R}$-diffeomorphisms in $T^1(S)$ which can be described simply as follows. Let $\mathbf{v} \in T^1(S)$, and let $\tilde{\mathbf{v}}$ be one of the congruent vectors in $\mathbb{H}$ determining $\mathbf{v}$. The element $\tilde{\mathbf{v}}$ determines a directed hyperbolic line $\gamma_{\tilde{\mathbf{v}}}$. Let t be the sensed hyperbolic length on $\gamma_{\tilde{\mathbf{v}}}$ measured from the point Q bearing $\tilde{\mathbf{v}}$. Let $\tilde{\mathbf{v}}_t$ be the element of $\gamma_{\tilde{\mathbf{v}}}$ at the point with coordinate t, and let $\mathbf{v}_t$ be the vector of $T^1(S)$ determined by $\tilde{\mathbf{v}}_t$. The transformation $\varphi^t : T^1(S) \to T^1(S)$ defined by $\varphi^t \mathbf{v} = \mathbf{v}_t$ (as illustrated by Figure 2.1) is a 1–1 continuous transformation. Furthermore, this transformation is a **ϑ-preserving** transformation of $T^1(S)$ into itself, that is, $\vartheta(\varphi^t(A)) = \vartheta(A)$, for all $t \in \mathbb{R}$ and every Borel set A of $T^1(S)$.

Hence, the collection $\{\varphi^t\}_{t \in \mathbb{R}}$ satisfies the following conditions

1. $\varphi^0 \mathbf{v} = \mathbf{v}$, for $\mathbf{v} \in T^1(S)$.
2. $(\varphi^r \circ \varphi^t)\mathbf{v} = \varphi^{r+t}\mathbf{v}$, for all $\mathbf{v} \in T^1(S)$ and $r, t \in \mathbb{R}$.

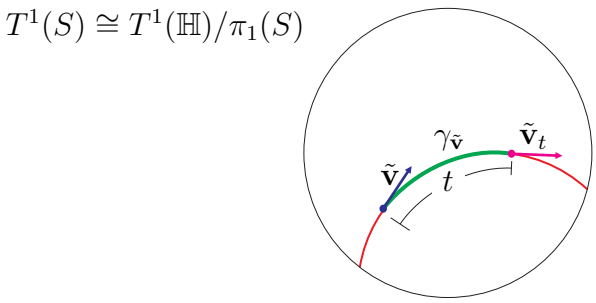

Figure 2.1: The geodesic flow.

Definition 2.1. Let

$$\begin{aligned} \varphi : \mathbb{R} \times T^1(S) &\to T^1(S) \\ (t, \mathbf{v}) &\mapsto \varphi(t, \mathbf{v}) = \varphi^t \mathbf{v}. \end{aligned}$$

The pair $(T^1(S), \varphi)$, or simply the map φ, is a **continuous dynamical system** or a **flow**, and it is called the **geodesic flow over** S.

Figure 2.2 illustrates the image of $(t, \mathbf{v})$ for some $\mathbf{v} \in T^1(S)$ and some $t > 0$ under the mapping φ, for a hyperbolic surface of genus 2.

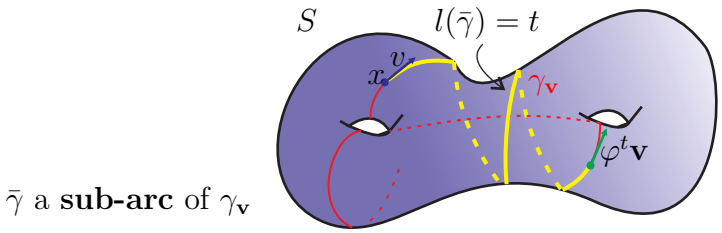

Figure 2.2: The vector $\varphi^t \mathbf{v}$

The **orbit of** $\mathbf{v} \in T^1(S)$ under the geodesic flow φ is the set $\{\varphi^t \mathbf{v} \mid t \in \mathbb{R}\}$. The orbits of the geodesic flow form a partition of $T^1(S)$, that is, $T^1(S)$ is the union of all the orbits, and each $\mathbf{v} \in T^1(S)$ belongs to one and only one orbit.

Remark 2.2. The orbits of the geodesic flow over S are in a one-to-one correspondence with the set of representative geodesics on S, that is, the elements of $\mathbb{G}(S)$. More specifically, the representative geodesic $\gamma_{\mathbf{v}}$ corresponds to the orbit of v under φ. For this reason, we refer to $\gamma_{\mathbf{v}}$ either as the orbit or the geodesic determined by $\mathbf{v}$, and the context will make it clear.

The vector $\mathbf{v} \in T^1(S)$ and its orbit are **periodic** if there exists $l > 0$ such that $\varphi^l \mathbf{v} = \mathbf{v}$. The number l is a **period**. The minimal period of $\mathbf{v}$ is precisely $l(\gamma_{\mathbf{v}})$.

2.3.1 The Anosov Property of the Geodesic Flow

Let $N = T^1(\mathbb{H})$ and consider the **bearing-point projection** $\mathfrak{b} : N \to \mathbb{H}$ defined by $\mathfrak{b}(z, v) = z$ and the **antipodal** (**flip**) $\imath : N \to N$ defined by $\imath(z, v) = (z, -v)$. Now, let $\mathbf{v} \in N$ and let $h = \gamma_{\mathbf{v}}$ be the directed hyperbolic line determined by $\mathbf{v}$. The line h has its two extreme points in $\partial\mathbb{H}$. We denote these points by $\mathbf{v}_-$ and $\mathbf{v}_+$. Consider $\mathbb{G}(\mathbb{H})$, the open subset of $\partial\mathbb{H} \times \partial\mathbb{H}$ which consists of pairs of distinct points of $\partial\mathbb{H}$. **Hopf's parametrization** of $T^1(\mathbb{H})$ is the homeomorphism from $T^1(\mathbb{H})$ to $\mathbb{G}(\mathbb{H}) \times \mathbb{R}$ sending $\mathbf{v} \in T^1(\mathbb{H})$ to the triple $(\mathbf{v}_-, \mathbf{v}_+, s) \in \mathbb{G}(\mathbb{H}) \times \mathbb{R}$, where s is the signed (algebraic) distance of $\mathfrak{b}(\mathbf{v})$ from the closest point h_0 of h to 0 (i.e., h_0 is the Euclidean mid-point of the geodesic h). In this dissertation, we identify an element $\mathbf{v}$ with its image by Hopf's parametrization. The geodesic flow acts by $\varphi^t(\mathbf{v}_-, \mathbf{v}_+, s) = (\mathbf{v}_-, \mathbf{v}_+, s + t)$ and, for every isometry g of $\mathbb{H}$, the image of $g \cdot \mathbf{v}$ is $(g(\mathbf{v}_-), g(\mathbf{v}_+), s + s_{g,\mathbf{v}})$, where $s_{g,\mathbf{v}}$ is the signed distance from $(g \cdot h)_0$ to $g(h_0)$. Furthermore, in these coordinates, the antipodal map $\imath$ is $(\mathbf{v}_-, \mathbf{v}_+, t) \mapsto (\mathbf{v}_-, \mathbf{v}_+, -t)$.

The **strong stable manifold** of $\mathbf{v} \in T^1(\mathbb{H})$ is

$$W^{ss}(\mathbf{v}) = \{\mathbf{w} \in N : H(\mathfrak{b}(\varphi^t\mathbf{v}), \mathfrak{b}(\varphi^t\mathbf{w})) \to 0 \text{ as } t \to +\infty\}$$

and the **strong unstable manifold** of $\mathbf{v} \in T^1(\mathbb{H})$ is

$$W^{\mathrm{su}}(\mathbf{v}) = \{\mathbf{w} \in N : H(\mathfrak{b}(\varphi^t\mathbf{v}), \mathfrak{b}(\varphi^t\mathbf{w})) \to 0 \text{ as } t \to -\infty\}$$

The projections in $\mathbb{H}$ of the strong unstable and strong stable manifolds of $\mathbf{v} \in N$, denoted by $H_-(\mathbf{v}) = \mathfrak{b}(W^{\mathrm{su}}(\mathbf{v}))$ and $H_+(\mathbf{v}) = \mathfrak{b}(W^{ss}(\mathbf{v}))$ are called, respectively, the **unstable and stable horospheres of $\mathbf{v}$**, and are the horospheres containing $\mathfrak{b}(\mathbf{v})$ centered at $\mathbf{v}_-$ and $\mathbf{v}_+$, respectively, as shown in the Figure 2.3.

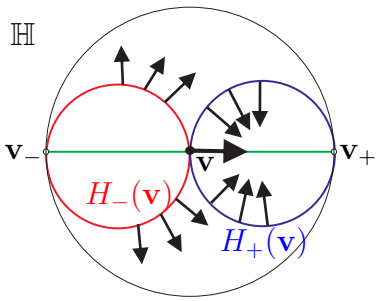

Figure 2.3: The Stable and Unstable Horospheres

The submanifold $W^s(\mathbf{v}) = \bigcup_{t\in\mathbb{R}} \varphi^t W^{ss}(\mathbf{v})$ is the **stable manifold** of $\mathbf{v}$ and consists of vectors $\mathbf{w} \in N$ with $\mathbf{w}_+ = \mathbf{v}_+$. Similarly, $W^{\mathrm{u}}(\mathbf{v}) = \bigcup_{t\in\mathbb{R}} \varphi^t W^{\mathrm{su}}(\mathbf{v})$ is

the **stable manifold of v** and consists of vectors $\mathbf{w} \in N$ with $\mathbf{w}_- = \mathbf{v}_-$.

The subspaces $W^{ss}(\mathbf{v})$ and $W^{\mathrm{su}}(\mathbf{v})$, as well as $W^{\mathrm{s}}(\mathbf{v})$ and $W^{\mathrm{u}}(\mathbf{v})$, are smooth submanifolds of N. The restrictions of φ to both $\mathbb{R} \times W^{ss}(\mathbf{v})$ and $\mathbb{R} \times W^{\mathrm{su}}(\mathbf{v})$ are smooth diffeormorphisms. In addition, $\imath W^{\mathrm{su}}(\mathbf{v}) = W^{ss}(\mathbf{v})$.

The strong stable manifolds, stable manifolds, strong unstable manifolds and unstable manifolds are the (smooth) leaves of the continuous foliations on N, invariant under the geodesic flow and the isometry group of $\mathbb{H}$.

Now, consider the vector field $Z : N \to T(N)$ defined by $\mathbf{v} \mapsto Z(\mathbf{v}) = \frac{d}{dt}\varphi^t\mathbf{v}$. The vector field Z is called the **geodesic vector field**. The geodesic flow φ on the Riemannian manifold N is a **contact Anosov flow**. That is, the vector bundle $TN \to N$ is the direct sum of three topological vector subbundles $TN = E^{su} \oplus E^0 \oplus E^{ss}$ such that are invariant under φ, where $E^0 \cap T_{\mathbf{v}}N = \mathbb{R}Z(\mathbf{v}), E^{su} \cap T_{\mathbf{v}}N = T_{\mathbf{v}}W^{su}(\mathbf{v}), E^{ss} \cap T_{\mathbf{v}}N = T_{\mathbf{v}}W^{ss}(\mathbf{v})$, and there exist two constants $C, \lambda > 0$ such that for every $t > 0$, we have

(a) $\|(D\varphi^t)\mathbf{w}\| \leq Ce^{-\lambda t}\|\mathbf{w}\|$, for every $\mathbf{w} \in E^{ss}(\mathbf{v})$,

(b) $\|(D\varphi^t)\mathbf{w}\| \geq Ce^{\lambda t}\|\mathbf{w}\|$, for every $\mathbf{w} \in E^{su}(\mathbf{v})$.

Since the measure ϑ is **invariant under** the geodesic flow φ or φ is a **ϑ-preserving transformation**, that is, $\vartheta(\varphi^t(A)) = \vartheta(A)$ for every $t \in \mathbb{R}$ and every Borel set A of $T^1(S)$, the strong stable leaves are contracted by the geodesic flow, and the strong unstable leaves are dilated.

The following figure illustrates the ideas previously described.

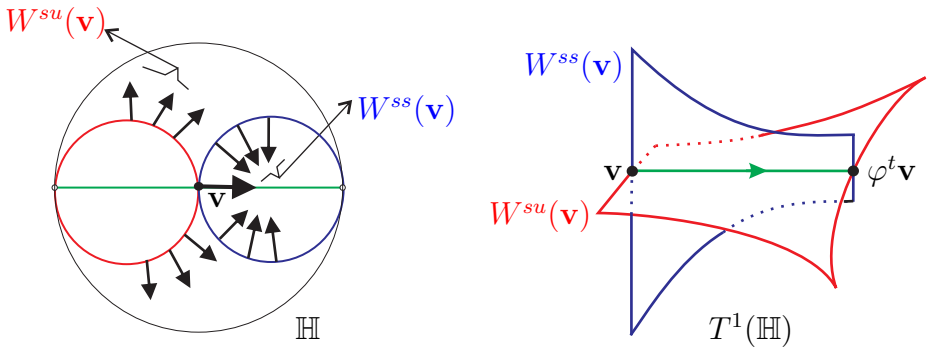

Figure 2.4: The Anosov property of the geodesic flow

Remark 2.3. All of these properties of the geodesic flow on the unit tangent bundle of $\mathbb{H}$ are induced on the unit tangent bundle of S because the action of $\pi_1(S)$ on $\mathbb{H}$ is properly discontinuous.

2.3.1.1 Mixing

One of the most important consequences of the Anosov Property is the strong mixing. Recall that the geodesic flow φ is a ϑ–preserving transformation.

The geodesic flow is **strong mixing** with respect to the measure ϑ, i.e.,

$$\lim_{T\to\infty} \vartheta(A \cap \varphi^{-T} B) = \vartheta(A) \cdot \vartheta(B),$$

or equivalently,

$$|\vartheta(A \cap \varphi^{-T} B) - \vartheta(A) \cdot \vartheta(B)| \to 0,$$

as $T \to \infty$, for all Borel sets A and B of S.

2.4 Entropy

The entropy of a dynamical system is a nonnegative number which measures the complexity of the system. Roughly, it measures the exponential growth rate of the number of distinguishable orbits as time advances.

2.4.1 Volume Entropy

In [19], Manning proves the following proposition.

Proposition 2.4. *Let M be a compact Riemannian manifold without boundary. Let $B(x,r)$ be the ball with centre x and radius r in the universal cover $\widetilde{M}$ (with the induced metric) and $V(x,r)$ be the volume of this ball. Then*

$$\lim_{r\to\infty} \frac{\log V(x,r)}{r} = \lambda,$$

for some constant $\lambda > 0$ independent of x.

In particular, if M has constant sectional curvature $-\kappa$, then $\lambda = \sqrt{\kappa}$.

The **volume entropy** (or **asymptotic volume growth**) $\mathrm{ve}(M)$ is the number λ defined in Proposition 2.4. Therefore, for the surface S, we have $\mathrm{ve}(S) = \sqrt{\kappa}$.

2.4.2 Topological Entropy

Let X be a nonempty compact Hausdorff space and $f : X \to X$ a continuous function and for $n \in \mathbb{Z}$, let $f^n = \underbrace{f \circ f \circ \cdots \circ f}_{\text{n-times}}$. Consider the **discrete** dynamical system (X, f).

In [1], Adler, Konheim and McAndrew define the topological entropy of (X, f) in the following way.

For an open cover $\mathcal{U}$ of X (i.e., a family of open sets whose union is X), let $N(\mathcal{U})$ denote the smallest cardinality of a subcover of $\mathcal{U}$ (i.e., a subfamily of $\mathcal{U}$ whose union still equals X). By compactness, $N(\mathcal{U})$ is always finite. If $\mathcal{U}$ and V are open covers of X then

$$\mathcal{U} \vee V = \{U \cap V : U \in \mathcal{U}, V \in \mathcal{V}\}$$

is called their **common refinement**. Let $U_n = \mathcal{U} \vee f^{-1}\mathcal{U} \vee \cdots \vee f^{n+1}\mathcal{U}$, where $f^k\mathcal{U} = \{f^k U : U \in \mathcal{U}\}$. Using a subadditivity argument, one shows that the limit

$$h(\mathcal{U}, f) = \lim_{n\to\infty} \frac{\log_2 N(\mathcal{U}^n)}{n}$$

exists for any open cover $\mathcal{U}$ (and equals $\inf_{n\in\mathbb{N}} \frac{\log_2 N(\mathcal{U}^n)}{n}$). The **topological entropy of** (X, T) is defined as the supremum

$$h_A(f) = \sup h(\mathcal{U}, f),$$

where the supremum ranges over all open covers $\mathcal{U}$ of X.

In [6], Bowen defined the topological entropy for (X, f), with X a metric space with distance d, in the following way.

A set $E \subseteq X$ is said to be (n, ϵ)**-separated**, if for every $x, y \in E$ with $x \neq y$ there is $i \in \{0, 1, \ldots, n-1\}$ such that $d(f^i x, f^i y) \geq \epsilon$. Let $s(n, \epsilon)$ be the maximal cardinality of an (n, ϵ)-separated set in X. Again, by compactness, this number is always finite. One defines

$$\bar{h}(\epsilon, f) = \lim_{n\to\infty} \frac{\log_2 s(n, \epsilon)}{n}.$$

The **topological entropy** is obtained as

$$h_B(f) = \sup_{\epsilon>0} \bar{h}(\epsilon, f) = \lim_{\epsilon\to 0} \bar{h}(\epsilon, f).$$

It should be noted that $s(n, \epsilon)$ can be substituted in Bowen's definition with a possibly smaller number $r(n, \epsilon)$, the minimal cardinality of an (n, ϵ)-spanning set. A set $E \subseteq X$ is (n, ϵ)-spanning if for every $x \in X$ there is $y \in E$ such that $d(f^i x, f^i y) < \epsilon$ for all $i \in \{0, 1, \ldots, n-1\}$. With such substitution one obtains the same value of $h_B(f)$.

Equality between the two notions

It is not hard to see that if U is an open cover with all elements of diameter at most ϵ and Lebesgue number 2δ then

$$s(n,\epsilon) \leq N(\mathcal{U}^n) \leq s(n,\delta),$$

which not only implies that $h_B(f) = h_A(f)$ but also that the same number $h_B(f)$ is obtained if $\bar{h}$ is replaced by $\underline{h}$ defined using lim inf in place of lim sup. From now on we use $h_{top}(f)$ to denote either $h_A(f)$ or $h_B(f)$.

Interpretation

The interpretation of the number $s(n,\epsilon)$ is the following: suppose one observes the system with a device of resolution ϵ, i.e., two points are distinguished only if the distance between them is at least ϵ. Then, after n steps of the evolution of the system, the observer will be able to distinguish at most $s(n,\epsilon)$ different orbits. Thus, the value $\bar{h}(\epsilon, f)$ is the exponential growth rate of the number of ϵ-distinguishable orbits of period n achieved as n grows to infinity. The value $h_{top}(f)$ maximizes the above over all $\epsilon > 0$. This is the precise meaning of saying that topological entropy measures the exponential complexity of the system.

In [12], Ito showed a conjecture formulated by Adler, Koneheim and McAndrew in [1].

Theorem 2.5. *Let $\psi : \mathbb{R} \times X \to X$, $(t,x) \to \psi^t(x)$, be a continuous flow on a compact metric space X. Then*

$$h_{top}(\psi^t) = |t| \cdot h_{top}(\psi^1).$$

Due to the conclusion of Theorem 2.5, for a continuous dynamical system (X,ψ), with X a compact metric space, the topological entropy of ψ is defined as the topological entropy of ψ^1.

In particular, since $T^1(S)$ is a compact metric space, $h_{top}(\varphi)$, the topological entropy of the geodesic flow over S is defined by $h_{top}(\varphi) = h_{top}(\varphi^1)$.

In [19, Theorem 2], Manning proves the following.

Theorem 2.6. *If the Riemannian manifold M has all sectional curvatures ≤ 0, then the topological entropy of the geodesic flow on the unit tangent bundle equals the the volume growth rate of M. In particular,*

$$h_{top}(\varphi) = \mathrm{ve}(S) = \sqrt{\kappa}.$$

2.4.3 Metric-Theoretic Entropy

Let $\mathcal{P}(T^1(S))$ be the space of regular Borel probability measures of $T^1(S)$ endowed with the weak*topology, that is, for $\{\mu_k\}_{k\in\mathbb{N}} \subset \mathcal{P}(T^1(S))$,

$$\mu_k \to \mu \text{ if and only if } \int_{T^1(S)} f d\mu_k \to \int_{T^1(S)} f d\mu,$$

as $k \to \infty$, for every continuous function $f : T^1(S) \to \mathbb{R}$.

Let $\mathcal{P}_\varphi(T^1(S)) = \{\mu \in \mathcal{P}(T^1(S)) \mid \mu \text{ is } \varphi\text{-invariant}\}$. Consider $\mu \in \mathcal{P}_\varphi(T^1(S))$ and J a countable set of indices. A collection of μ-measurable sets $\xi = \{A_j \in \mathcal{A} \mid j \in J\}$ is a **partition of** $T^1(S)$ if $\mu(A_i \cap A_j) = 0$, for $i \neq j$ and $\mu(T^1(S) \setminus \cup_{j\in J} A_j) = 0$.

The **entropy of** ξ is

$$H_\mu(\xi) := -\sum_{j\in J} \mu(A_j) \log_2 \max\{\mu(A_j), 1\}.$$

Given two partitions ξ and υ define the **joint** partitiion $\xi \vee \upsilon$, similar to the common refinement of two open covers, by

$$\{A \cap B \mid A \in \xi, B \in \upsilon; \mu(A \cap B) > 0\}.$$

And for $n \in \mathbb{N}$, we define ξ^φ_{-n} by

$$\xi^\varphi_{-n} = \xi \vee \varphi^{-1}(\xi) \vee \cdots \vee \varphi^{-n+1}(\xi),$$

where $\varphi^{-k}(\xi) = \{\varphi^{-k}A \mid A \in \xi\}$, for $k \in \mathbb{N}$.

If $H_\mu(\xi^\varphi_{-n}) < \infty$, for every $n \in \mathbb{N}$, then $h_\mu(\varphi, \xi) = (1/n) \lim_{n\to\infty} H(\xi^\varphi_{-n}) < \infty$. In such a case $h_\mu(\varphi, \xi)$ is called the **metric entropy** of φ **relative** to ξ.

Definition 2.7. The **entropy of** φ **with respect to** μ is defined by

$$h_\mu(\varphi) = \sup\{h_\mu(\varphi, \xi) \mid \xi \text{ is a measurable partition with } H(\xi) < \infty\}.$$

2.4.4 The Measure of Maximum Entropy

In [10], Goodman proves the following.

Theorem 2.8 (The Variational Principle)**.** *Let X be a compact Hausdorff space and $f : X \to X$ a continuous function. Then*

$$h_{top}(f) = \sup_\mu h_\mu(f),$$

where the supremum is taken over all regular f-invariant Borel probability measures on X.

In [8], Bowen shows that for the geodesic flow, the supremum of the measure-theoretic entropies was actually a maximum and that the probability measure where this maximum is achieved is the normalized Liouville measure $\overline{\vartheta}$.

Theorem 2.9 (Bowen)**.** *The measure for which the supremum in the Variational Principle becomes a maximum for the geodesic flow over S is $\overline{\vartheta}$, that is, $h_{\overline{\vartheta}}(\varphi) = h_{top}(\varphi) = \sqrt{\kappa}$, where*

$$\overline{\vartheta} = \frac{\vartheta}{\vartheta(T^1(S))} = \frac{\kappa}{4\pi^2(\mathfrak{g}-1)}\vartheta.$$

Due to the conclusion of Theorem 2.9, $\overline{\vartheta}$ is known as the **measure of maximum entropy** or the **Bowen measure** on $T^1(S)$.

Further, Bowen also proves in [8], that the measure of maximum entropy $\overline{\vartheta}$ is the weak limit of some measures associated to the periodic orbits of the geodesic flow φ.

The idea of Bowen's characterization is the following.

Definition 2.10. For a periodic orbit γ of φ, define the measure ζ_γ by

$$\zeta_\gamma(E) = \frac{1}{l(\gamma)} \int_0^{l(\gamma)} \chi_E(\varphi^t \mathbf{v}) dt, \tag{2.4}$$

for $\mathbf{v} \in \gamma$ and E a Borel set of $T^1(S)$. This measure ζ_γ is the **δ-measure** of γ.

Now, consider the sum

$$\Sigma_T = \frac{1}{N(T)} \sum_{\gamma \in C\mathbb{G}_T(S)} \zeta_\gamma.$$

Since the orbits of the geodesic flow φ form a partition of $T^1(S)$ and ζ_γ is a φ–invariant measure for a closed geodesic γ on S, Σ_T is of regular Borel measure of $T^1(S)$, which is also φ–invariant, for every $T > 0$. In addition, $h_{\Sigma_T}(\varphi) < h_{\Sigma_{T'}}(\varphi)$, whenever $T < T'$, given that $C\mathbb{G}_T(S) \subset C\mathbb{G}_{T'}(S)$. Hence, we get the following characterization of $\overline{\vartheta}$.

Theorem 2.11 (Bowen)**.** *The periodic orbits of the geodesic flow φ are equidistributed with respect to the measure of maximium entropy $\overline{\vartheta}$ as the period tends to $+\infty$. More precisely, for any Borel set E of $T^1(S)$ with $\overline{\vartheta}(\partial E) = 0$,*

$$\overline{\vartheta}(E) = \lim_{T \to \infty} \Sigma_T(E).$$

As a consequence of Theorem 2.9, we obtain the useful corollary.

Corollary 2.12. *Let A be a Borel set of $\mathcal{P}(T^1(S))$, the space of regular Borel probability measures on $T^1(S)$. Then*

$$\frac{1}{N(T)}\#\{\gamma \in C\mathbb{G}_T \mid \zeta_\gamma \in A\} \leq \overline{\vartheta}\{\mathbf{v} \in T^1(S) \mid \gamma_{\mathbf{v}} \in C\mathbb{G}; \zeta_{\gamma_{\mathbf{v}}} \in A\}.$$

Proof. Since

$$\zeta_\gamma\{\mathbf{v} \in T^1(S) \mid \gamma_{\mathbf{v}} \in C\mathbb{G}_T; \zeta_{\gamma_{\mathbf{v}}} \in A\} = \begin{cases} 1, & \zeta_\gamma \in A \\ 0, & \zeta_\gamma \notin A \end{cases}$$

we have, by Theorem 2.9, that

$$\begin{aligned} \frac{1}{N(T)}\#\{\gamma \in C\mathbb{G}_T \mid \zeta_\gamma \in A\} &= \frac{1}{N(T)}\#\{\mathbf{v} \in T^1(S) \mid \gamma_{\mathbf{v}} \in C\mathbb{G}_T; \zeta_{\gamma_{\mathbf{v}}} \in A\} \\ &= \frac{1}{N(T)} \sum_{\gamma \in C\mathbb{G}_T} \zeta_\gamma\{\mathbf{v} \in T^1(S) \mid \gamma_{\mathbf{v}} \in C\mathbb{G}_T; \zeta_{\gamma_{\mathbf{v}}} \in A\} \\ &\leq \overline{\vartheta}\{\mathbf{v} \in T^1(S) \mid \gamma_{\mathbf{v}} \in C\mathbb{G}_T; \zeta_{\gamma_{\mathbf{v}}} \in A\} \quad \square. \end{aligned}$$

The measure of maximum entropy $\overline{\vartheta}$ is also known as the **Margulis measure** because Margulis also constructed it in a different form while proving the following theorem [20, §6, Theorem 5]).

Theorem 2.13 (Margulis)**.** *The number of periodic orbits of the geodesic flow over S whose minimal period is less than or equal to T, that is the number of closed geodesics on S, satisfies*

$$N(T) \sim \frac{e^{\sqrt{\kappa}T}}{\sqrt{\kappa}T}.$$

This theorem is known as the "Prime Orbit" theorem because of its similarity with the "Prime Number" theorem and states the asymptotic growth of the number of primitive closed geodesics corresponding to indivisible conjugacy classes of curves in S. The key property of Margulis's construction of $\overline{\vartheta}$, needed in his proof, is a uniform (by factor a factor of $e^{\pm\sqrt{\kappa}t}$) exapansion/contraction of the conditional measures of $\overline{\vartheta}$ along the leaves of the unstable/stable foliations of the geodesic flow φ on $T^1(S)$.

Chapter 3

Geodesic Currents

In this chapter we provide both the analytical and geometric definitions of the geodesic currents on a surface as well as the intersection form of pairs of geodesic currents. All of these concepts were taken from papers by Bonahon, the analytical definition can be found in [3], whereas the geometric one can be found in [4]. We also enunciate a theorem of V. Kaimanovich in [13], which states the compatibility of these two notions of currents and their intersection form.

3.1 Analytical Definition

The difference between a simple curve and a nonsimple curve comes obviously from ... the self-intersections. To remove these, it is rather natural to lift the closed geodesics to the unit tangent bundle $T^1(S)$ of S. Let us disregard the orientation of the curves of the curves, that is, let us look at the tangent line bundle $\mathbb{L}(S)$, the quotient of $T^1(S)$ by the involution, which is the antipodal map on each fiber.

Since S has a hyperbolic metric, the trajectories of the geodesic flow on its unit tangent bundle are the lifts of geodesics on S by considering their tangent vectors at each point. Passing to the quotient, the geodesic flow over S induces a 1-dimensional foliation $\mathcal{F}$ on $\mathbb{L}(S)$, called the **geodesic foliation**. There is therefore a natural correspondence between closed (nonoriented) geodesics on S and compact leaves of $\mathcal{F}$.

Definition 3.1. A **geodesic current on** S is a positive transverse invariant measure for the geodesic foliation $\mathcal{F}$.

In other words, a geodesic current μ defines a positive measure supported on $V \cap \mathbb{L}(S)$ for each submanifold V of dimension 2 in $\mathbb{L}(S)$ transverse to $\mathcal{F}$, and that μ is invariant under holonomy in the following sense: if $x_1 \in V_1$ and $x_2 \in V_2$ are two points on such transverse submanifolds located on the same leaf of $\mathcal{F}$, and if $\psi : U_1 \to U_2$ is a holonomy diffeomorphism between neighborhoods of x_1 and x_2 in V_1 and V_2 (defined by following the leaves of $\mathcal{F}$), then ψ respects the measure induced by μ on U_1 and U_2. Geodesic currents are hence particular cases of geometric currents introduced by D. Ruelle and D. Sullivan in [23].

We give now a fundamental example of a geodesic current. To a given closed geodesic γ on S corresponds a compact leaf $\widetilde{\gamma}$ of $\mathcal{F}$. We associate to it the geodesic current ω_γ, which induces on each transverse manifold V the Dirac measure at the point $V \cap \widetilde{\gamma}$; invariance under holonomy is then immediate. This measure ω_γ is associated to the measure $l(\gamma)\zeta_\gamma$, with ζ_γ defined by (2.10). We equip the set $\mathcal{C}(S)$ of currents on S with the unique weak*topology, in which two currents μ and ν are close if there exists a finite family of continuous functions $f_i : V_i \to \mathbb{R}$ with compact support defined on transverse submanifolds V_i such that each $\mu(f_i)$ is close to $\nu(f_i)$ ([5, chapter 2, §1 n° 9].) We can even equip $\mathcal{C}(S)$ with a uniform space structure taking the entourages basis

$$\{(\mu, \nu) \in \mathcal{C}(S) \times \mathcal{C}(S); \forall i = 1, 2, \ldots, n, |\mu(f_i) - \nu(f_i)| < \epsilon\}$$

for all $\epsilon > 0$ and all finite families $f_i : V_i \to \mathbb{R}$ as before. We get then the following classical result in functional analysis.

Proposition 3.2. *The uniform space $\mathcal{C}(S)$ is complete.*

To understand well currents on S and their topology it is first necessary to understand well $\mathbb{L}(S)$ equipped with $\mathcal{F}$. A **flow box** B for $\mathcal{F}$ is defined by an elongated H-shape configuration on S, where the horizontal bar is a geodesic arc and where the two vertical bars are arcs transverse to the previous one and sufficiently small so that each geodesic arc joining the vertical bars which is homotopic to a path in the H, meets the vertical bars transversely. The box $B \subset \mathbb{L}(S)$ consists of the lifts of all geodesic arcs in S joining the two vertical bars that are homotopic to a path in H. Barycentric coordinates on each geodesic arc give a diffeomorphism $B \simeq Q \times [0, 1]$ for which the leaves of $B \cap \mathcal{F}$ correspond to $\{\star\} \times [0, 1]$. We point out that Q can be lifted to B as a square transverse to the foliation and that this lift is unique up to holonomy; given a geodesic current $\mu \in \mathcal{C}(S)$, we can therefore speak of the measure $\mu(B) \in \mathbb{R}^+$, defined as the measure with respect to μ of this transverse square. Likewise, if $\partial_{\mathcal{F}} B$ is the part of B corresponding to $\partial Q \times [0, 1]$ (formed by the geodesic arcs meeting one of the extremities of the H), we define $\mu(\partial_{\mathcal{F}} B)$ as the measure with respect to μ of the boundary of the transverse square, which is the lift of Q.

To illustrate this, let us investigate what this means if the geodesic current is defined by a closed geodesic γ on S, that is, ω_γ. If B is a flow box, $\omega_\gamma(B)$ is clearly the number of subarcs of γ whose lifts are leaves of $B \cap \mathcal{F}$. In other words, $\omega_\gamma(B)$ is the number of subarcs of γ that join the two vertical bars of the H.

Proposition 3.3. *A neighborhood basis for a geodesic current $\mu \in \mathcal{C}(S)$ consists of the open sets $\mathcal{U}(\mu, B_1, \ldots, B_n; \epsilon) = \{\nu \in \mathcal{C}(S) : \forall i |\mu(B_i) - \nu(B_i)| < \epsilon\}$, where $\epsilon > 0$ and the B_i are taken among all the flow boxes B such that $\mu(\partial_{\mathcal{F}} B) = 0$.*

3.1.1 The Intersection form $\mathfrak{i}$

The geometric intersection number of two closed geodesics $i(\alpha, \beta)$ is equal to the number of triples $(x, \lambda_1, \lambda_2)$, where $x \in \alpha \cap \beta$ and λ_1, λ_2 are two distinct lines in $T_x(S)$ tangent to α and β, respectively. The advantage of this definition is that it is expressed only in terms of geodesic currents, and we will exploit this observation to define the intersection form function $\mathfrak{i}$ on $\mathcal{C}(S) \times \mathcal{C}(S)$.

Starting from the line bundle $\mathbb{L}(S) \to S$, we can consider the Whitney sum $\mathbb{L}(S) \oplus \mathbb{L}(S) \to S$. In other words, $\mathbb{L}(S) \oplus \mathbb{L}(S)$ is the 4-dimensional manifold of triples $(x, \lambda_1, \lambda_2)$, where $x \in S$ and λ_1 and λ_2 are two lines in the tangent space $T_x(S)$. Forgetting the first or the second line defines two projections p_1 and p_2 from $\mathbb{L}(S) \oplus \mathbb{L}(S)$ to $\mathbb{L}(S)$. We consider the two foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ of codimension 2 in $\mathbb{L}(S) \oplus \mathbb{L}(S)$, whose leaves are the preimages of the leaves of $\mathcal{F}$ by, respectively, p_1 and p_2. These foliations are transverse outside the diagonal $\triangle = \{(x, v, v) : x \in S, v \in T_x S\}$ of $\mathbb{L}(S) \oplus \mathbb{L}(S)$.

Let μ and ν be two geodesic currents. Through p_1, μ induces a transverse invariant measure $\widehat{\mu}_1$ on $\mathcal{F}_1$, which, by transversality of $\mathcal{F}_1$ and $\mathcal{F}_2$, gives outside $\triangle$ a measure on each leaf of $\mathcal{F}_2$. Similarly, ν induces outside $\triangle$ a measure $\widehat{\nu}_2$ on each leaf of $\mathcal{F}_1$. Consider then the product measure $\widehat{\mu}_1 \times \widehat{\nu}_2$ on $\mathbb{L}(S) \oplus \mathbb{L}(S) \setminus \triangle$. The total mass of this measure is finite.

Definition 3.4. The **intersection form** of the geodesic currents μ and ν is

$$\mathfrak{i}(\mu, \nu) = (\widehat{\mu}_1 \times \widehat{\nu}_2)(\mathbb{L}(S) \oplus \mathbb{L}(S) \setminus \triangle).$$

Note that $\mathfrak{i}(\omega_\alpha, \omega_\beta) = i(\alpha, \beta)$, for $\alpha, \beta \in C\mathbb{G}(S)$.

It is always possible to add two geodesic currents, and to multiply a geodesic current by a non-negative real number. Then, the space $\mathcal{C}(S)$ appears as the completion of the space of real multiples of homotopy classes of closed curves by the following fact.

Proposition 3.5 (Bonahon)**.** *The real multiples of the currents associated to the closed geodesics on S are dense in $\mathcal{C}(S)$.*

3.2 Geometric Definition

The notion of geodesic current was designed to get a better understanding of the set of homotopy classes of unoriented closed curves on S.

Since S is a hyperbolic manifold any closed curve is homotopic to a unique multiple of a closed geodesic. Here we have to make the terminology more precise, in order to avoid any ambiguity: By convention, a closed geodesic never wraps several times around another geodesic; in other words, a closed oriented geodesic always represents an indivisible element of $\pi_1(S)$. We also accept the empty closed geodesic to deal with null-homotopic curves.

This defines a one-to-one correspondence between homotopy classes of unoriented closed curves on S and unoriented closed geodesics on S equipped with a certain positive integral multiplicity.

Consider the universal covering $\widetilde{S}$ of S. The preimage of a closed geodesic of S gives a collection of geodesics on $\widetilde{S}$ which is invariant by the (isometric) action of $\pi_1(S)$, and is discrete in the space $\mathbb{G}(\widetilde{S})$ of (unoriented) geodesics on $\widetilde{S}$, endowed with the compact-open topology. Thus, a homotopy class of closed curves on S uniquely defines a $\pi_1(S)$-invariant discrete subset of $\mathbb{G}(\widetilde{S})$, equipped with a certain integral multiplicity.

Now, to take this multiplicity into account, it is natural to identify this discrete subset of $\mathbb{G}(\widetilde{S})$ equipped with a multiplicity to the Dirac measure it defines on $\mathbb{G}(\widetilde{S})$. By construction, this measure is invariant under the action of $\pi_1(S)$.

Definition 3.6. A **geodesic current** is a (positive) measure on $\mathbb{G}(\widetilde{S})$ that is invariant under the action of $\pi_1(S)$.

The set $\widehat{\mathcal{C}}(S)$ of geodesic currents is endowed with the (metrizable) weak* uniform structure defined by the family of semidistances d_f, where f ranges over all continuous functions $f : \mathbb{G}(\widetilde{S}) \to \mathbb{R}$ with compact support and where $d_f(\mu, \nu) = |\mu(f) - \nu(f)|$ for all $\mu, \nu \in \widehat{\mathcal{C}}(S)$.

We have thus embedded the set of homotopy classes of closed curves on S in the space $\widehat{\mathcal{C}}(S)$ of geodesic currents.

Before going any further, observe that the topology of the space $\mathbb{G}(\widetilde{S})$ of geodesics on $\widetilde{S}$ is particularly simple. Indeed, by the Cartan-Hadamard theorem, $\widetilde{S}$ is isometric to the hyperbolic space $\mathbb{H}$. Recall that the model we considered for $\mathbb{H}$ is the interior of the unit disc D^2 of $\mathbb{C}$, equipped with the Riemannian metric which at $z \in D^2$ is $2/(1 - |z|^2)$ times the Euclidean metric. The geodesics on $\mathbb{H}$ are the intersection with D^2 of the circles of $\bar{D}^2$ meeting $\partial\mathbb{H} = \partial\bar{D}^2$ orthogonally. Therefore, $\mathbb{G}(\widetilde{S})$ is homeomorphic to $\mathbb{G}(\mathbb{H}^2) = (\partial\mathbb{H} \times \partial\mathbb{H} \setminus \triangle)/\mathbb{Z}_2$, where $\triangle$ is the diagonal and where $\mathbb{Z}_2$ acts on $\partial\mathbb{H} \times \partial\mathbb{H}$ by exchanging the two factors. It follows that $\mathbb{G}(\widetilde{S})$ is an open Möebius strip.

Exactly like $\partial\mathbb{H}$ sits at the infinity of $\mathbb{H}$, the surface $\widetilde{S}$ has a well-defined **circle at infinity** $\widetilde{S}_\infty$ which does not depend on the identification $\widetilde{S} \cong \mathbb{H}^2$. This $\widetilde{S}_\infty$ is the quotient of the space of all geodesic rays of $\widetilde{S}_\infty$ by the equivalence relation which identifies asymptotic rays. An isometric identification $\widetilde{S}_\infty \cong \mathbb{H}^2$ of course provides an identification $\widetilde{S}_\infty \cong \partial\mathbb{H}$. In particular, $\mathbb{G}(\widetilde{S})$ is naturally homeomorphic to $(\widetilde{S}_\infty \times \widetilde{S}_\infty \setminus \triangle)/\mathbb{Z}_2$.

The action of $\pi_1(S)$ on $\mathbb{G}(\widetilde{S})$ is not so easy to visualize. However:

Fact 3.7. The topological space $\mathbb{G}(\widetilde{S})$ equipped with the action of $\pi_1(S)$ can be abstractly described in terms of the group $\pi_1(S)$ only. In particular, it is independent of the hyperbolic metric we initially put on S.

The main corollary of Fact 3.7 is that the space $\widehat{\mathcal{C}}(S)$ of geodesic currents depends only on the group $\pi_1(S)$, and is in particular independent of the hyperbolic metric initially chosen on S.

A special geodesic current associated with the hyperbolic metric on S is the following.

Definition 3.8. The **Liouville** geodesic current ν_L of S is defined by

$$\nu_L([a,b] \times [c,d]) = \left| \log \left| \frac{(a-c)(b-c)}{(a-d)(b-c)} \right| \right|.$$

As ν_L is defined using the cross-ratio, we have that ν_L is invariant under the full isometry group $\mathrm{Iso}(\mathbb{H})$. This property uniquely defines ν_L up to a multiplicative constant (see [4, §2].)

Other type of current is the one associated to a closed geodesic on S. Let $\pi : \widetilde{S} \to S$ be the covering map obtained by taking the quotient of $\widetilde{S}$ by $\pi_1(S)$. Hence, for a closed geodesic γ on S, the set $\widetilde{\gamma} = \pi^{-1}(\gamma)$ is a $\pi_1(S)$-invariant collection of geodesics in $\widetilde{S}$. We then obtain a measure μ_γ on $G(\widetilde{S})$ by taking the Dirac measure on the discrete set $\widetilde{\gamma}$. In this way we can naturally identify closed geodesics on S with $\pi_1(S)$-invariant measures on $\mathbb{G}(\widetilde{S})$.

Let $I \times J$ be the product of two disjoint open sets of $\widetilde{S}_\infty$ whose projections under π are also disjoint (as illustrated in Figure 3.1) Then $I \times J$ is an open set of $\mathbb{G}(S)$. Then,

$$\mu_\gamma(I \times J) = \#\{\gamma \mid \gamma(-\infty) \in I, \gamma(+\infty) \in J\}.$$

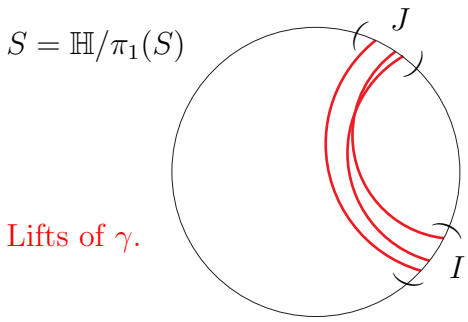

Figure 3.1: Current associated to a closed geodesic on S.

3.2.1 The Intersection Form $\mathfrak{j}$

Consider the space $D\mathbb{G}(\widetilde{S}) \subseteq \mathbb{G}(\widetilde{S}) \times \mathbb{G}(\widetilde{S})$ that consists of all couples of geodesics on S which have a transverse non-trivial intersection. Considering the intersection point of these two geodesics, $D\mathbb{G}(\widetilde{S})$ can also be interpreted as the set of triples $(x, \lambda_1, \lambda_2)$ where $x \in S$ and λ_1 and λ_2 are two distinct directions in the tangent space of S at x. As $\pi_1(S)$ acts properly discontinuously on S, its action on $D\mathbb{G}(\widetilde{S})$, defined by the Expression (2.3), is also properly discontinuous and we can consider the quotient $D\mathbb{G}(S) = D\mathbb{G}(\widetilde{S})/\pi_1(S)$. Observe that $D\mathbb{G}(S)$ is also the space of triples $(x, \lambda_1, \lambda_2)$ where $x \in S$ and where λ_1 and λ_2 are two distinct directions in the tangent space of S at x. In particular, $D\mathbb{G}(S)$ is an open 4-manifold.

The two geodesic currents $\mu, \nu \in \widehat{\mathcal{C}}(S)$ define a product measure $\mu \times \nu$ on $D\mathbb{G}(\widetilde{S}) \subseteq \mathbb{G}(\widetilde{S}) \times \mathbb{G}(\widetilde{S})$, which itself induces a measure on the quotient $D\mathbb{G}(S)$ and whose total mass is finite.

Definition 3.9. The **intersection form of μ and ν** is

$$\mathfrak{j}(\mu, \nu) = (\mu \times \nu)(D\mathbb{G}(S)).$$

3.3 Compatibility of the Intersection Forms $\mathfrak{i}$ and $\mathfrak{j}$

The following theorem states that the geometric and analytic definition of the geodesic currents as well as the intersection forms from these two approaches are compatible. This fact was proven by V. Kaimanovich in [13, Theorem 2.2]. Let $\mathcal{M}(T^1(S))$ be the space of the Radon measures on $T^1(S)$ endowed with the weak*topology and let $\mathcal{M}_\varphi(T^1(S)) \subseteq \mathcal{M}(T^1(S))$ be the subspace of $\mathcal{M}(T^1(S))$ consisting of the measures that are φ–invariant.

Given any $\widetilde{\mu} \in \mathcal{M}(T^1(S))$ we can consider the associated transverse measure μ for the foliation $\mathcal{F}$, that is, $\mu \in \widehat{\mathcal{C}}(S)$. Each $\mu \in \widehat{\mathcal{C}}(S)$ is normalized by the requirement that (locally) $\widetilde{\mu} = \mu \times dt$, where dt is the one–dimensional Lebesgue

measure along leaves in $\mathcal{F}$. Due to this fact, we identify the φ–invariant measure with its corresponding current, and also, we identify the two spaces, $\widehat{\mathcal{C}}(S)$ and $\mathcal{M}_\varphi(T^1(S))(T^1(S))$.

Theorem 3.10 (Kaimanovich)**.** *There exists a natural convex isomorphism between the cone* $(\mathcal{M}_\varphi(T^1(S)), \mathfrak{i})$ *of the invariant measures of the geodesic flow and the cone* $(\hat{\mathbb{C}}(S), \mathfrak{j})$ *of the geodesic currents on* $\widetilde{S}_\infty \times \widetilde{S}_\infty \setminus \triangle$.

In particular, Theorem 3.10 implies, given that the Riemannian measure $\vartheta \in \mathcal{M}_\varphi(T^1(S))$, that there exists a geodesic current $\nu \in \widehat{\mathcal{C}}(S)$ that corresponds to it. This geodesic current turns out to be the Liouville measure ν_L defined in (3.8). Thus, if we denote by $\overline{\nu_L}$ the geodesic current on $\widehat{\mathcal{C}}(S)$ corresponding to the normalized Liouville measure $\overline{\vartheta}$, we have

$$\overline{\nu_L} = \frac{\nu_L}{\mathfrak{j}(\nu_L, \nu_L)} = \frac{\kappa}{2\pi^2(\mathfrak{g}-1)}\nu_L.$$

3.4 Continuity of the Intersection Form

By identifying the current measures associated to closed geodesics with the corresponding geodesics, Bonahon shows the following properties of the Liouville current ϑ (or ν_L) and of the intersection form, which we use later on this work (see [4, Propositions 14 and 15] and [3, Proposition 4.5].)

Theorem 3.11 (Bonahon)**.** *The intersection form function* $\mathfrak{i}$ *(or* $\mathfrak{j}$*) is a continuous extension of the intersection number function on the set of pairs of closed geodesics.*

In particular,

$$\mathfrak{i}(\zeta_\alpha, \zeta_\beta) = \frac{i(\alpha,\beta)}{l(\alpha)l(\beta)} = \mathfrak{j}\left(\frac{\mu_\alpha}{l(\alpha)}, \frac{\mu_\beta}{l(\beta)}\right),$$

for α *and* β *closed geodesics on* S.

In addition,

$$\mathfrak{i}(\vartheta,\vartheta) = \frac{\mathrm{Vol}(T^1(S))}{2} = \frac{2\pi^2(\mathfrak{g}-1)}{\kappa} = \mathfrak{j}(\nu_L, \nu_L)$$

and

$$\mathfrak{i}(\overline{\vartheta}, \overline{\vartheta}) = \frac{\kappa}{2\pi^2(\mathfrak{g}-1)} = \frac{\vartheta(T^1(S)}{2} = \mathfrak{j}(\overline{\nu_L}, \overline{\nu_L}).$$

The intersection form $\mathfrak{i}$ can be extended to the product of pairs of positive measures (whose associated transverse measure is not necessarily invariant for the geodesic foliation $\mathcal{F}$,) and whose support is contained in $\mathbb{L}(S)$. Let us denote this extension also by $\mathfrak{i}$.

A type of measure that we utilize is a generalization of the Definition 2.10 for a periodic orbit.

Definition 3.12. For $\mathbf{v} \in T^1(S)$ and $T > 0$, define the measure $\zeta_{\mathbf{v}}^T$ by

$$\zeta_{\mathbf{v}}^T(A) = \frac{1}{T}\int_0^T \chi_A(\varphi^t \mathbf{v})dt, \tag{3.1}$$

for any Borel set A of $T^1(S)$.

In particular, if $\gamma_{\mathbf{v}}$ is a periodic orbit, we have $\zeta_{\mathbf{v}}^T = \zeta_{\gamma_{\mathbf{v}}}$, for $T \geq l(\gamma_{\mathbf{v}})$.

Fact 3.13. For $\mathbf{v} \in T^1(S)$ and $T > 0$,

$$\mathfrak{i}(\zeta_{\mathbf{v}}^T, \zeta_{\mathbf{v}}^T) = \frac{i^T(\gamma_{\mathbf{v}}, \gamma_{\mathbf{v}})}{T^2}.$$

Remark 3.14. Hereafter, $\mathfrak{i}$ will either of the intersection forms $\mathfrak{i}$ and $\mathfrak{j}$, if no ambiguity arises. In particular, $\mathfrak{i}(\overline{\vartheta}, \overline{\vartheta})$ will also denote $\mathfrak{j}(\overline{\nu_L}, \overline{\nu_L})$.

Chapter 4

Results

In this chapter we present the proofs of the results obtained while trying to answer the questions stated in Section 1.1. But first, we show that the intersection number of a pair of closed geodesics is bounded by a multiple the product of the lengths of the two geodesics. Later, we introduce one deviation result by Kifer in [15] which is an essential tool for demonstrating our two main theorems: Theorems 1.8 and 1.9.

Let L_S be the constant given by Lalley in [18], that is,

$$L_S = \frac{2}{\vartheta(T^1(S))} = \frac{\kappa}{2\pi^2(\mathfrak{g}-1)} = \mathfrak{j}(\overline{\nu_L}, \overline{\nu_L}) = \mathfrak{i}(\overline{\vartheta}, \overline{\vartheta}).$$

4.1 A Bound for the Intersection Numbers

As mentioned earlier, the intersection numbers of pairs of closed geodesics is bounded. Here, we show a bound that is a multiple of the product of the lengths of such geodesics. It is important noting that this bound can also be deduced by the techniques used by A. Basmajian in [2].

The **injectivity radius at** a point $x \in S$ is the largest radius for which the exponential map at x is a diffeomorphism. The **injectivity radius of** S, which we denote by $\varrho(S)$, is the infimum of the injectivity radii of all points of S. The injectivity radius is equivalently defined as the half of the least length of an essential loop, i.e., a loop that cannot be contracted to a point on S. Thus, if γ is a loop in S, then $l(\gamma) \geq 2\varrho(S)$.

Proposition 1.10. *Let α and β be closed geodesics on S. Then*

$$i(\alpha, \beta) \leq \frac{l(\alpha)l(\beta)}{\varrho(S)^2}.$$

Proof. Let $\bar{\alpha}$ be a sub-arc of α of length less than $\varrho(S)$ which contains n intersection points of α and β, and n is the greatest amount of intersection points of α and β contained in a segment of length less than $\varrho(S)$. Hence,

$$i(\alpha, \beta) \leq \left\lceil \frac{l(\alpha)}{\varrho(S)} \right\rceil n. \tag{4.1}$$

Let $x_1, \ldots, x_n$ be the points of intersection of α and β contain in $\bar{\alpha}$ listed in the order in which they appear in β, that is, $\beta^{-1}(x_i) \leq \beta^{-1}(x_{i+1})$, for $1 \leq i \leq n-1$. Let β_k be the sub-arc of β from x_k to x_{k+1}, for $1 \leq k \leq n-1$, and, β_n be the sub-arc of β joining x_1 and x_n. Similarly, let $\bar{\alpha}_k$ be the sub-arc of $\bar{\alpha}$ from x_k to x_{k+1}, for $1 \leq k \leq n-1$, and $\bar{\alpha}_n$ be the sub-arc of $\bar{\alpha}$ joining x_1 and x_n. Consider γ_k the concatenation of $\bar{\alpha}_k$ and β_k, for $1 \leq k \leq n$. Thus, γ_k is an essential loop of S, for $1 \leq k \leq n$ (as illustrated in Figure 4.1.)

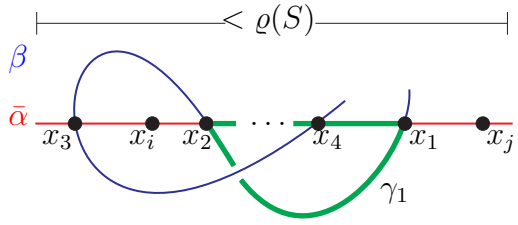

Figure 4.1: The loop γ_1.

Hence, $2\varrho(S) \leq l(\gamma_k) = l(\bar{\alpha}_k) + l(\beta_k) \leq l(\bar{\alpha}) + l(\beta_k) < \varrho(S) + l(\beta_k)$, which implies $\varrho(S) < l(\beta_k)$, for $1 \leq k \leq n$.

Consequently, $n\varrho(S) = n\sum_{k=1}^{n} \varrho(S) < \sum_{i=1}^{n} l(\beta_k) < l(\beta)$. Therefore, we have $n < \dfrac{l(\beta)}{\varrho(S)}$. Thus, by (4.1), we conclude

$$i(\alpha, \beta) \leq \left\lceil \frac{l(\alpha)}{\varrho(S)} \right\rceil i(\bar{\alpha}, \beta) \leq \frac{l(\alpha)}{\varrho(S)} \frac{l(\beta)}{\varrho(S)} = \frac{l(\alpha) l(\beta)}{\varrho(S)^2}. \quad \square$$

4.2 A Deviation Result

The following theorem is a large deviation result due to Kifer in [15]. We use it, along with the continuity of the intersection form $\mathbf{i}$, which is continuous by Theorem 3.11, to prove our main results.

Given that the unit tangent bundle is a compact metric space and the geodesic flow is a hyperbolic dynamical system, Kifer's result [16, Theorem 2.1], can be translated into our setting in the following way.

Theorem 4.1 (Kifer)**.** *For any closed subset K of $\mathcal{P}(T^1(S))$, the space of regular Borel probability measures on $T^1(S)$,*

$$\limsup_{T \to \infty} \frac{1}{T} \log \overline{\vartheta}\{\mathbf{v} \in T^1(S) \mid \zeta_{\mathbf{v}}^T \in K\} \leq -\inf_{\mu \in K} I(\mu),$$

where

$$I(\mu) = \begin{cases} h_{top}(\varphi) - h_\mu(\varphi), & \mu \in \mathcal{P}_\varphi(T^1(S)) \\ \infty, & otherwise \end{cases}.$$

4.3 Decay of the Sets of "Irregular" Geodesics

By Theorems 4.1 and Theorem 3.11 allow us to show that the T-self-intersection number of any geodesic (closed or not closed) is almost surely equal to $L_S T^2$, when T is large enough.

Theorem 1.8. *Let $\epsilon > 0$. There exists $\delta > 0$ such that*

$$\overline{\vartheta}\left\{\mathbf{v} \in T^1(S) : \left|\frac{i^T(\gamma_\mathbf{v}, \gamma_\mathbf{v})}{T^2} - L_S\right| \geq \epsilon\right\} = O\big(e^{-\delta T}\big),$$

as $T \to \infty$.

Proof. Let $\epsilon > 0$. Consider the following set

$$K := \{\mu \in \mathcal{P}(T^1(S)) : |\mathfrak{i}(\mu, \mu) - \mathfrak{i}(\overline{\vartheta}, \overline{\vartheta})| \geq \epsilon\}. \tag{4.2}$$

By Theorem 3.11, $\mathfrak{i}(\overline{\vartheta}, \overline{\vartheta}) = L_S$, and by Fact 3.13, for $\mathbf{v} \in T^1(S)$ and $T > 0$, $T^2 i^T(\gamma_\mathbf{v}, \gamma_\mathbf{v}) = i(\gamma_\mathbf{v}, \gamma_\mathbf{v})$. Consequently,

$$\left\{\mathbf{v} \in T^1(S) : \left|\frac{i^T(\gamma_\mathbf{v}, \gamma_\mathbf{v})}{T^2} - L_S\right| \geq \epsilon\right\} = \{\mathbf{v} \in T^1(S) : \zeta_\mathbf{v}^T \in K\}. \tag{4.3}$$

Therefore, by (4.3), it is enough to prove that there exists $\delta > 0$ such that $\overline{\vartheta}\{\mathbf{v} \in T^1(S) : \zeta_\mathbf{v}^T \in K\} = O\big(e^{-\delta T}\big)$, as $T \to \infty$.

The intersection form function $\mathfrak{i}$ is continuous by Theorem 3.11, then, K is a closed subset of $\mathcal{P}(T^1(S))$. Therefore, by Theorem 4.1,

$$\limsup_{T\to\infty} \frac{1}{T} \log \overline{\vartheta}\{\mathbf{v} \in T^1(S) : \zeta_\mathbf{v}^T \in K\} \leq -\inf_{\mu\in K} I(\mu).$$

If $K \cap \mathcal{P}_\varphi(T^1(S)) = \emptyset$, then $\inf_{\mu\in K} I(\mu) = \infty$. Thus,

$$\limsup_{T\to\infty} \frac{1}{T} \log \overline{\vartheta}\{\mathbf{v} \in T^1(S) : \zeta_\mathbf{v}^T \in K\} \leq -\infty.$$

Hence, $\overline{\vartheta}\Big\{\mathbf{v} \in T^1(S) : \zeta_\mathbf{v}^T \in K\Big\} \leq e^{-\infty} = 0 = O\big(e^{-\delta T}\big)$, as $T \to \infty$, for any $\delta > 0$.

If $K \cap \mathcal{P}_\varphi(T^1(S)) \neq \emptyset$, given that $\overline{\vartheta}$ is the unique Borel probability measure of $T^1(S)$ with maximum entropy $h_{\overline{\vartheta}}(\varphi) = h_{top}(\varphi) = \sqrt{\kappa}$, then

$$0 < \delta = \inf_{\mu \in K} I(\mu) = \inf_{\mu \in K \cap \mathcal{P}_\varphi(T^1(S))} I(\mu) = \inf_{\mu \in K \cap \mathcal{P}_\varphi(T^1(S))} (\sqrt{\kappa} - h_\mu(\varphi)) < \infty$$

is such that

$$\limsup_{T \to \infty} \frac{1}{T} \log \overline{\vartheta}\{\mathbf{v} \in T^1(S) \mid \zeta_{\mathbf{v}}^T \in K\} \leq -\delta,$$

that is,

$$\overline{\vartheta}\{\mathbf{v} \in T^1(S) \mid \zeta_{\mathbf{v}}^T \in K\} = O\big(e^{-\delta T}\big), \text{ as } T \to \infty. \quad \square$$

Since the set of representative geodesics is identified with a subset of $T^1(S)$, we can use the characterization of the measure of maximum entropy $\overline{\vartheta}$ given in Theorem 2.9 by Bowen and the conclusion of Theorem 1.8 to prove Lalley's result, Theorem 1.4, which states that most of the closed geodesics γ on S with $l(\gamma) \leq T$ have roughly $L_S l(\gamma)^2$ self-intersections.

Corollary 1.4 (Lalley's Theorem)**.** *For every $\epsilon > 0$,*

$$\lim_{T \to \infty} \frac{\#\{\gamma \in C\mathbb{G}_T(S) : |i(\gamma,\gamma) - L_S l(\gamma)^2| < \epsilon l(\gamma)^2\}}{N(T)} = 1.$$

Proof using Theorem 1.8. Let $T, \epsilon > 0$. Consider the set K defined in (4.2) from the proof of Theorem 1.8 and let

$$\mathcal{O}(T,\epsilon) := \left\{\gamma \in C\mathbb{G}_T(S) : \left|\frac{i(\gamma,\gamma)}{l(\gamma)^2} - L_S\right| \geq \epsilon\right\}.$$

Hence, $\mathcal{O}(T,\epsilon) = \{\gamma \in C\mathbb{G}_T(S) \mid \zeta_\gamma \in K\}$.

Note that

$$\mathcal{O}(T,\epsilon) = C\mathbb{G}_T(S) \setminus \left\{\gamma \in C\mathbb{G}_T(S) : \left|i(\gamma,\gamma) - L_S l(\gamma)^2\right| < \epsilon l(\gamma)^2\right\},$$

then, it is enough to prove that

$$\lim_{T \to \infty} \frac{\#\mathcal{O}(T,\epsilon)}{N(T)} = 0.$$

By Theorem 1.8, there exist $\delta > 0$ such that

$$\overline{\vartheta}\{\mathbf{v} \in T^1(S) \mid \zeta_{\mathbf{v}}^T \in K\} = O\big(e^{-\delta T}\big).$$

Thus, by Corollary 2.12, we conclude

$$\begin{aligned}\frac{\#\mathcal{O}(T,\epsilon)}{N(T)} &\leq \overline{\vartheta}\{\mathbf{v}\in T^1(S)\mid \gamma_{\mathbf{v}}\in C\mathbb{G}(S);\zeta_{\gamma_{\mathbf{v}}}\in K\}\\ &\leq \overline{\vartheta}\{\mathbf{v}\in T^1(S)\mid \zeta^T_{\mathbf{v}}\in K\}=O\big(e^{-\delta T}\big).\end{aligned}$$

Consequently,

$$\lim_{T\to\infty}\frac{\#\mathcal{O}(T,\epsilon)}{N(T)}\leq \lim_{T\to\infty}\overline{\vartheta}\{\mathbf{v}\in T^1(S)\mid \gamma_{\mathbf{v}}\in C\mathbb{G}_T(S);\gamma_{\mathbf{v}}\in E\}=0. \quad \square$$

4.4 Decay of the Tails of the Distribution of $\frac{i(\alpha,\beta)}{l(\alpha)l(\beta)}$

Theorem 3.11 and Theorem 4.1 enable us to prove that the tails of the distribution $\frac{i(\alpha,\beta)}{l(\alpha)(\beta)}$ are estimated by a decreasing exponential function, that is, decay exponentially fast.

Theorem 1.9. *Let $\epsilon>0$. There exists $\delta>0$ such that*

$$\frac{1}{N(R)N(T)}\#\left\{(\alpha,\beta)\in C\mathbb{G}_{R,T}(S):\left|\frac{i(\alpha,\beta)}{l(\alpha)l(\beta)}-L_S\right|\geq\epsilon\right\}=O\big(e^{-\delta R}\big),$$

as $R\to\infty$, with $T\geq R$.

Proof. Consider the map $\mathfrak{i}:\mathcal{P}(T^1(S))\times\mathcal{P}(T^1(S))\to\mathfrak{i}(\mathcal{P}(T^1(S))\times\mathcal{P}(T^1(S)))$, which is continuous since it is the restriction of the intersection form function (continuous by Theorem 3.11) to the closed $\mathcal{P}(T^1(S))\times\mathcal{P}(T^1(S))$.

Therefore, for $\epsilon>0$, the set $\mathcal{Z}=i^{-1}(L_S-\epsilon,L_S+\epsilon)$, the preimage of the ball of radius ϵ centered at

$$\mathfrak{i}(\overline{\vartheta},\overline{\vartheta})=\frac{\kappa}{2\pi^2(g-1)}=L_S,$$

is an open subset of $\mathcal{P}(T^1(S))\times\mathcal{P}(T^1(S))$.

Let $R,T>0$ with $R\leq T$. Consider the set

$$\mathcal{W}_{R,T}=\{(\alpha,\beta)\in C\mathbb{G}_{R,T}(S)\mid(\zeta_\alpha,\zeta_\beta)\in\mathcal{Z}\}. \tag{4.4}$$

By Theorem 3.11, $\mathfrak{i}(\zeta_\alpha,\zeta_\beta)=\dfrac{i(\alpha,\beta)}{l(\alpha)l(\beta)}$. Hence,

$$\left\{(\alpha,\beta)\in C\mathbb{G}_{R,T}(S):\left|\frac{i(\alpha,\beta)}{l(\alpha)l(\beta)}-L_S\right|\geq\epsilon\right\}$$

is the complement of $\mathcal{W}_{R,T}$ in $C\mathbb{G}_R(S) \times C\mathbb{G}_T(S)$.

Thus, it is enough to prove that there exists $\delta > 0$ such that

$$\frac{\#C\mathbb{G}_{R,T}(S) \setminus \mathcal{W}_{R,T}}{N(R)N(T)} = O\big(e^{-\delta R}\big).$$

Since $(\overline{\vartheta}, \overline{\vartheta}) \in \mathcal{Z}$ and $\mathcal{Z}$ is an open set of the product topology of $\mathcal{P}(T^1(S)) \times \mathcal{P}(T^1(S))$, there exist $\mathcal{U}, \mathcal{V} \subseteq \mathcal{P}(T^1(S))$ open neighborhoods of $\overline{\vartheta}$ such that $\mathcal{U} \times \mathcal{V} \subseteq \mathcal{Z}$.

Let $\mathcal{U}_R = \{\alpha \in C\mathbb{G}_R(S) : \zeta_\alpha \in \mathcal{U}\}$ and $\mathcal{V}_T = \{\beta \in C\mathbb{G}_T(S) : \zeta_\beta \in \mathcal{V}\}$. Observe that $\mathcal{U}_R \times \mathcal{V}_T \subseteq \mathcal{W}_{R,T}$.

Given that $\mathcal{P}(T^1(S)) \setminus \mathcal{U}$ and $\mathcal{P}(T^1(S)) \setminus \mathcal{V}$ are both closed subsets of $\mathcal{P}(T^1(S))$ and neither of them contains $\overline{\vartheta}$, we have, by Theorem 4.1, that there exist $\delta_1 > 0$ depending on $\mathcal{U}$ and $\delta_2 > 0$ depending on $\mathcal{V}$ such that

$$\overline{\vartheta}\{\mathbf{v} \in T^1(S) : \zeta_{\mathbf{v}}^R \in \mathcal{P}(T^1(S)) \setminus \mathcal{U}\} = O\big(e^{-\delta_1 R}\big),$$

as $R \to \infty$ and,

$$\overline{\vartheta}\{\mathbf{v} \in T^1(S) : \zeta_{\mathbf{v}}^T \in \mathcal{P}(T^1(S)) \setminus \mathcal{V}\} = O\big(e^{-\delta_2 T}\big),$$

as $T \to \infty$, respectively.

Hence, by Corollary 2.12, we get that

$$\frac{\#C\mathbb{G}_R(S) \setminus \mathcal{U}_R}{N(R)} \leq \overline{\vartheta}\{\mathbf{v} \in T^1(S) : \gamma_{\mathbf{v}} \in C\mathbb{G}_R(S); \zeta_{\gamma_{\mathbf{v}}} \in \mathcal{P}(T^1(S)) \setminus \mathcal{U}\} = O\big(e^{-\delta_1 R}\big),$$

as $R \to \infty$ and, and

$$\frac{\#C\mathbb{G}_T(S) \setminus \mathcal{V}_T}{N(T)} \leq \overline{\vartheta}\{\mathbf{v} \in T^1(S) : \gamma_{\mathbf{v}} \in C\mathbb{G}_T(S); \zeta_{\gamma_{\mathbf{v}}} \in \mathcal{P}(T^1(S)) \setminus \mathcal{U}\} = O\big(e^{-\delta_2 T}\big),$$

as $T \to \infty$, respectively.

Thus, as $R, T \to \infty$, we obtain

$$\frac{\#C\mathbb{G}_{R,T}(S) \setminus \mathcal{W}_{R,T}}{N(R)N(T)} \leq \frac{\#C\mathbb{G}_{R,T}(S) \setminus \mathcal{U}_R \times \mathcal{V}_T}{N(R)N(T)}$$

$$\leq \frac{\#C\mathbb{G}_R(S) \setminus \mathcal{U}_R \cdot \#C\mathbb{G}_T(S) \setminus \mathcal{V}_T}{N(R)N(T)}$$
$$+ \frac{\#C\mathbb{G}_R(S) \setminus \mathcal{U}_R \cdot \#\mathcal{V}_T}{N(R)N(T)} + \frac{\#C\mathbb{G}_T(S) \setminus \mathcal{V}_T \cdot \#\mathcal{U}_R}{N(R)N(T)}$$
$$= O\big(e^{-\delta_1 R}\big)O\big(e^{-\delta_2 T}\big) + O\big(e^{-\delta_1 R}\big) + O\big(e^{-\delta_2 T}\big) = O\big(e^{-\delta_1 R}\big). \qquad \square$$

4.5 The Normalized Average of the Intersection Numbers of Pairs of Closed Geodesics

An immediate consequence of Theorem 1.9 is the fact that the normalized average of the intersection numbers of pairs of closed geodesics on S is asymptotically equal to L_S. For the proof of this fact, we use the bound for the intersection number of a pair of closed geodesics found in Theorem 1.10.

Corollary 1.11.

$$\frac{1}{N(R)N(T)} \sum_{\substack{\alpha \in C\mathbb{G}_R(S) \\ \beta \in C\mathbb{G}_T(S)}} \frac{i(\alpha, \beta)}{l(\alpha)l(\beta)} \sim L_S,$$

as $R, T \to \infty$.

Proof. Let $\epsilon > 0$. For $R, T > 0$ with $R \leq T$, consider the set $\mathcal{W}_{R,T}$ defined in (4.4) from the proof of Theorem 1.9. In addition, let $\delta, J, C > 0$ be constants satisfying the conclusion of such theorem, that is, for $J \leq R \leq T$, we have

$$\frac{\#C\mathbb{G}_{R,T}(S) \setminus \mathcal{W}_{R,T}}{N(R)N(T)} \leq \frac{C}{e^{\delta R}}.$$

Moreover, let J be such that $Ce^{-\delta R} < \epsilon$, whenever $R > J$.

By Proposition 4.1, we have $\sup_{C\mathbb{G}_R(S)\times C\mathbb{G}_T(S)} \frac{i(\alpha,\beta)}{l(\alpha)l(\beta)} \leq \frac{1}{\varrho(S)}$. Therefore, for $J < R \leq T$, we have

$$\begin{aligned}
&\left|\frac{1}{L_S N(R)N(T)}\left(\sum_{(\alpha,\beta)\in\mathbb{G}_{R,T}(S)} \frac{i(\alpha,\beta)}{l(\alpha)l(\beta)}\right) - 1\right| \\
&\qquad = \left|\frac{1}{L_S N(R)N(T)} \sum_{(\alpha,\beta)\in\mathbb{G}_{R,T}(S)} \left(\frac{i(\alpha,\beta)}{l(\alpha)l(\beta)} - L_S\right)\right| \\
&\qquad \leq \frac{1}{L_S N(R)N(T)}\left(\sum_{(\alpha,\beta)\in\mathcal{W}_{R,T}} \left|\frac{i(\alpha,\beta)}{l(\alpha)l(\beta)} - L_S\right|\right. \\
&\qquad\qquad \left. + \sum_{(\alpha,\beta)\in C\mathbb{G}_{R,T}(S)\setminus\mathcal{W}_{R,T}} \left|\frac{i(\alpha,\beta)}{l(\alpha)l(\beta)} - L_S\right|\right) \\
&\qquad \leq \frac{1}{L_S N(R)N(T)}\left(\#\mathcal{W}_{R,T}\cdot\epsilon\right. \\
&\qquad\qquad \left. + \#C\mathbb{G}_{R,T}(S)\setminus\mathcal{W}_{R,T}\cdot \sup_{(\alpha,\beta)\in\mathbb{G}_{R,T}(S)}\left|\frac{i(\alpha,\beta)}{l(\alpha)l(\beta)} - L_S\right|\right) \\
&\qquad < \frac{1}{L_S}\left(\epsilon + \frac{C}{e^{\delta R}}\left[\frac{1}{\varrho(S)^2} + L_S\right]\right) \\
&\qquad < \frac{1}{L_S}\left(1 + \frac{1}{\varrho(S)^2} + L_S\right)\epsilon.
\end{aligned}$$

Given that ϵ was chosen arbitrarily, we conclude that

$$\lim_{R,T\to\infty} \frac{1}{L_S N(R)N(T)} \sum_{(\alpha,\beta)\in C\mathbb{G}_{R,T}(S)} \frac{i(\alpha,\beta)}{l(\alpha)l(\beta)} = 1,$$

or equivalently,

$$\frac{1}{N(R)N(T)} \sum_{(\alpha,\beta)\in C\mathbb{G}_{R,T}(S)} \frac{i(\alpha,\beta)}{l(\alpha)l(\beta)} \sim L_S,$$

as $R, T \to \infty$. □

Bibliography

[1] R. L. Adler, A. G. Konheim, and M. H. McAndrew. Topological entropy. *Trans. Amer. Math. Soc.*, 114:309–319, 1965.

[2] Ara Basmajian. Universal length bounds for non-simple closed geodesics on hyperbolic surfaces. *J. Topology*, 6(2):513–524, 2013.

[3] Francis Bonahon. Bouts des variétés hyperboliques de dimension 3. *Ann. of Math. (2)*, 124(1):71–158, 1986.

[4] Francis Bonahon. The geometry of Teichmüller space via geodesic currents. *Invent. Math.*, 92(1):139–162, 1988.

[5] N. Bourbaki. *Eleméments de Mathématiques, Integration*, volume 6 of *Hermann*. 1965.

[6] Robert Edward Bowen. *TOPOLOGICAL ENTROPY AND AXIOM A*. ProQuest LLC, Ann Arbor, MI, 1970. Thesis (Ph.D.)–University of California, Berkeley.

[7] Rufus Bowen. Periodic orbits for hyperbolic flows. *Amer. J. Math.*, 94:1–30, 1972.

[8] Rufus Bowen. Entropy for maps of the interval. *Topology*, 16(4):465–467, 1977.

[9] Moira Chas and Steven P. Lalley. Self-intersections in combinatorial topology: statistical structure. *Invent. Math.*, 188(2):429–463, 2012.

[10] T. N. T. Goodman. Relating topological entropy and measure entropy. *Bull. London Math. Soc.*, 3:176–180, 1971.

[11] Gustav A. Hedlund. The dynamics of geodesic flows. *Bull. Amer. Math. Soc.*, 45(4):241–260, 1939.

[12] Shunji Ito. An estimate from above for the entropy and the topological entropy of a C^1-diffeomorphism. *Proc. Japan Acad.*, 46:226–230, 1970.

[13] Vadim Kaimanovich. Ergodicity of harmonic invariant measures for the geodesic flow on hyperbolic spaces. Technical report, J. Reine Angew. Math, 1994.

[14] Anatole Katok and Boris Hasselblatt. *Introduction to the modern theory of dynamical systems*, volume 54 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1995. With a supplementary chapter by Katok and Leonardo Mendoza.

[15] Yuri Kifer. Large deviations in dynamical systems and stochastic processes. *Trans. Amer. Math. Soc.*, 321(2):505–524, 1990.

[16] Yuri Kifer. Large deviations, averaging and periodic orbits of dynamical systems. *Comm. Math. Phys.*, 162(1):33–46, 1994.

[17] Steven Lalley. Statistical regularities of self-intersection counts for geodesics on negatively curved surfaces. *http://arxiv.org/pdf/1111.2060v2.pdf*, 2013.

[18] Steven P. Lalley. Self-intersections of closed geodesics on a negatively curved surface: statistical regularities. In *Convergence in ergodic theory and probability (Columbus, OH, 1993)*, volume 5 of *Ohio State Univ. Math. Res. Inst. Publ.*, pages 263–272. de Gruyter, Berlin, 1996.

[19] Anthony Manning. Topological entropy for geodesic flows. *Ann. of Math. (2)*, 110(3):567–573, 1979.

[20] Grigoriy A. Margulis. *On some aspects of the theory of Anosov systems*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2004. With a survey by Richard Sharp: Periodic orbits of hyperbolic flows, Translated from the Russian by Valentina Vladimirovna Szulikowska.

[21] Jouni Parkkonen and Frederic Paulin. Counting common perpendicular arcs in negative curvature. *http://arxiv.org/pdf/1305.1332.pdf*, 2013.

[22] Mark Pollicot and Richard Sharp. Angular self-intersections for closed geodesics on surfaces. *Proc. Amer. Math. Soc.*, 134(2):419–426, 2006.

[23] David Ruelle and Dennis Sullivan. Currents, flows and diffeomorphisms. *Topology*, 14(4):319–327, 1975.