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COMPUTATION OF VIBRATION MODAL SHAPES AND NATURAL FREQUENCIES OF A GENERAL, ORTHOTROPIC-LAMINATED, THIN SHELL USING FINITE ELEMENTS

A DISSERTATION

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BY

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COMPUTATION OF VIBRATION MODAL SHAPES

AND NATURAL FREQUENCIES OF A GENERAL,

ORTHOTROPIC-LAMINATED, THIN SHELL USING FINITE ELEMENTS

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This work is dedicated to the five women of my life: First, to my wife, Charlotte, whose encouragement, patience, trust, and confidence have never wavered. Secondly, to my widowed Mother, who instilled in me whatever perseverance and tenacity I possess and who taught me early in life that anything worthwhile does not come easily. Finally, to my three beautiful daughters, Maridell, Laura Beth, and Amy Kathleen, who have had a remarkable understanding of their father's having to study during this research.

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LIST OF SYMBOLS

A	Matrix defined in Appendix A
A_{ij}, B_{ij}, D_{ij}	Stiffness constants of a laminate element
$a_1, \ldots, a_{20}$	Coefficients of the assumed polynomial modal expressions
a, b	Constants used in surface equation of a radome
e	Base of natural logarithms
AK	Matrix defined in Appendix A
C	Matrix defined in Equation (17)
e_x, e_y	Average distances across an element in the x, y directions, respectively (in.)
$\bar{E}$	Amplitude of kinetic energy of an element (in.-lb.)
E	Amplitude of kinetic energy of the shell (in.-lb.); elastic modulus of isotropic material (psi)
E_1, E_2	Major and minor elastic moduli, respectively (psi)
F	Matrix defined in Equation (25)
f	Natural frequency (hertz)
G	Shear modulus (psi)
H	Maximum height of radome shell (in.)
h	Thickness of an element (in.)
i	$\sqrt{-1}$
M	Mass matrix of the shell
$\bar{M}$	Mass matrix of an element
R	Matrix defined in Appendix A
R_m	Radius of middle surface of cylindrical or spherical shell

S	Stiffness matrix of the shell
$\overline{S}$	Stiffness matrix of an element
t	Variable time (sec.)
T	Inverse of $[AK]$
U, V, W	General displacements in x, y, z coordinate systems
u, v, w	Displacements of an element in the x, y, z directions, respectively (in.)
V	Amplitude of potential energy of the shell (in.-lb.)
$\overline{V}$	Amplitude of potential energy of an element (in.-lb.)
x, y, z	Coordinates of an element
$\overline{x}, \overline{y}, \overline{z}$	Additional coordinate set used in Figure 3.2
X	Matrix defined in Appendix A
Y	Matrix defined in Appendix A; eigenvector in the solution of the dynamic problem
$\overline{z}$	Function defining the shell geometrical surface
$\overline{z}_{,x}, \overline{z}_{,y}, \overline{z}_{,xx}, \overline{z}_{,yy}, \overline{z}_{,xy}$	First and second derivatives of $\overline{z}$
β_1, β_2	Rotations of an element in the x, y directions, respectively
γ	Vector of the coefficients $a_1, \ldots, a_{20}$
$\epsilon_1, \epsilon_2, \epsilon_6$	Strains in the x, y plane (in./in.)
$\kappa_1, \kappa_2, \kappa_6$	Curvatures in the x, y plane (in.$^{-1}$)

LIST OF SYMBOLS (CONTINUED)

ν_1, ν_2	Major and minor Poisson's ratios, respectively (dimensionless)
ξ	Vector displacements and rotations for an element
π	= 3.1415926536
ρ	Density of the shell material (lb.-sec.^{2}/in.^{4})
ω	Circular frequency (rad./sec.)
Ω	Dimensionless frequency $[= \omega R_m (E/\rho)^{-\frac{1}{2}}]$

COMPUTATION OF VIBRATION MODAL SHAPES AND NATURAL FREQUENCIES OF A GENERAL, ORTHOTROPIC-LAMINATED, THIN SHELL USING FINITE ELEMENTS

CHAPTER I

INTRODUCTION

1.1 Brief History of Shell Dynamics by Finite Element Analysis

As engineers and scientists pursue the challenges of space vehicular flight and endeavor to push aircraft flight into the high supersonic and hypersonic airspeed regimes, they require the means to predict analytically the behavior of the vehicle with an accuracy thought impossible fifteen years ago. In the structural analyses of these aircraft, two developments made what was heretofore considered a pipe dream become a reality: the advent of high-speed, large-capacity computers and the finite-element method of analysis. These two developments have complimented each other very nicely.

The development of the digital computer over the past twenty years has been astounding. As recently as 1965, many small companies and universities were very proud of their limited 8 K to 32 K (K = thousand) byte core machines, which were commonly known as second-generation hardware. Presently, third-generation computers are in widespread usage with up to 400 K bytes of core available and speeds sometimes over 100 times faster than their predecessors.

Fourth-generation machines are presently on order, with deliveries starting next year, which will have over 1,000 K bytes and are roughly ten times faster than those currently available. Economics has indeed played a vital role in this rapid growth; i.e., more could be accomplished at lower costs. However, economics has not been the sole impetus. Engineers and scientists have continually requested more available core and faster operating times to do larger, more complete and more accurate analyses.

Structural dynamic analyses have been conducted on computers since the late 1940's. The dynamic behavior of plates and shells followed along somewhat slowly, but toward the end of the 1950's, the space race with the USSR established new goals and dimensions for engineering and scientific research, and shell dynamics was no exception.

Closed-form solutions to shell problems to date are very few; practically all development has been in deriving solutions through approximate methods. These methods often require handling and assembling vast quantities of data, with the final solutions often accomplished by mathematical iteration techniques. The results of such methods could be obtained only by modeling the problem on high-speed computers.

Some of the first analysis methods to be developed were applied to shells of revolution. Kalnins [1] used such a method in which the shell boundary-value problem was reduced to an initial-value problem involving first-order differential equations which could be integrated numerically. A Stodola-type iteration technique was employed by Cohen [2], and Cooper [3] obtained results through a finite-difference solution. These results had varying degrees of success; however, they usually offered some improvements and advantages over then-existing methods.

The Galerkin method has been used successfully in several analyses. Hu [4] and Lindholm and Hu [5] used it in 1965 to analyze truncated conical shells, and Krause [6] later used a modified Galerkin method on a similar type shell. Even more recently, Wilkins, et al [7] studied the free vibration of orthotropic conical sandwich shells with this method. The Galerkin approach, in general, does have three disadvantages. First, sometimes problems are encountered in devising functions which meet all boundary conditions (both kinematic and force types) as required for convergence of solutions. Second, ill-conditioned matrices can result. Often these are associated with the modal shapes assumed and the boundary constraints applied; as a consequence, the application is restricted or excessive computer time is consumed. Third, unsymmetric matrices generally result which can limit the method used to solve the eigenvalue problem and which can consume computer space.

Probably the most popular method of analysis to date has been the Rayleigh-Ritz energy method. Finite-element methods often use assumed modal shapes for each element, and, in fact, the energy method has been the most popular method for formulating finite-element stiffness matrices; see [10] - [14]. Naumann [8] and Sewall and Naumann [9] applied this method to free-free truncated conical shells and thin cylinders with and without longitudinal stiffeners. Rayleigh-Ritz solutions are of the symmetric eigenvalue form which usually leads to faster and more accurate solutions. The user can also take advantage of the matrix symmetry by working with the upper triangle plus the diagonal; thus, computer space is conserved.

Before discussing particular applications of the finite-element method, some discussion is appropriate to point out the emphasis placed on this approach, especially during the last ten years. Possibly the strongest proponents of larger and faster computers have been the users of finite-element analyses. The development of this method of analysis has actually paralleled that of digital computers.

Four main advantages of this method have generated this rapid development. First, the method employs matrix theory which is ideally suited for digital computers. Second, the method is more flexible; i.e., it can be applied to more complex structures and offers inclusion of secondary effects such as anisotropy, material nonlinearity, thermal effects, transverse shear flexibility, finite deflections, etc. Third, users can model the structure, assemble necessary data, and make major modifications easier than before, and, in general, with fewer errors. Fourth, for the same degree of accuracy, it is often more economical in terms of computer time.

The first user of this method for the analysis of shell structure may have been Izrayelit [15] in 1956. Parikh and Norris [16] and McGrattan and North [17] appear to have been conducting similar work, apparently independently, at about the same time. Turner, et al, [18] are usually given credit for first applying the method to plane stress problems in 1956. Since then, a substantial volume of work has been published within the general area of finite-element methods for both static and dynamic structural analyses.

In the last few years, numerous books covering the fundamentals of the method have appeared. Appendix C is a bibliography of such texts.

The majority of the papers presented at the two Air Force Conferences on Matrix Methods in Structural Mechanics, [19] and [20], held in 1965 and 1968, involved finite approaches and another such conference will be held in the near future. Jones and Strome [11] recognized the need and published a survey on the applications of finite-element methods in 1966. It is estimated that since that time the literature on the subject has more than doubled. Appendix B contains a bibliography of papers on finite-element analysis of shells, prepared with the assistance of Dr. Bert.

As with the previous methods of analyses, much attention has been given to shells of revolution by the finite-element method. Jones and Strome [10], Webster [12], and Percy, et al [13] are some of the more recent publications which employed this class of geometric configuration. The analyses of Azar [21] and Bacon and Bert [14] were somewhat more complex in that they were for orthotropic sandwich shells of revolution.

Attempts to develop a finite-element method for general shell structures are more complicated. Bogner, et al [22] developed a cylindrical shell element which had 48 degrees of freedom. This is more than what is necessary and restricts the application to very large computers or a model of very few elements. Cantin and Clough [23] developed a cylindrical shell element with 24 degrees of freedom which included all six rigid-body modes being represented exactly. However, no solutions to dynamic problems were attempted. Still later, Olson and Lindberg [24] developed a similar analysis which included 28 degrees of freedom and used it to study curved fan blades. All of these methods had three major disadvantages. First, they were primarily suited for cylinders. Second, they were restricted to isotropic materials. Third, they probably contained more degrees of freedom than necessary to adequately predict the behavior of the shell.

In 1967, Connor and Brebbia [25] developed a stiffness matrix for a shallow rectangular shell element using two four-term polynomials for the in-plane displacement expressions and a twelve-term polynomial for the normal displacement. This resulted in a 20-degree-of-freedom element. About a year later, Sabir and Ashwell [26] developed the same size matrix using the same size polynomial displacement expressions by employing the analogy between doubly curved shells and plates on elastic foundations. Both of these works were restricted to isotropic materials, and both were applied only to static problems.

Perhaps the most general and versatile work to date in developing stiffness matrices for doubly curved elements was that of Adelman, et al [27]. This fine work has given insight toward the development of the present research and a similar method of approach has been used herein. The main feature of the work of Adelman, et al was that geometrically exact elements were used. In addition, the degrees of freedom were minimal, only twelve per element. However, this work was limited to shells of revolution. It also had a disadvantage in that some of the stiffness coefficients required for a general, anisotropic shell were omitted. In addition, it lacked the capability of developing the required laminate stiffness coefficients, as they had to be predetermined, independently, and read in as part of the input data. The assumed displacement functions were combinations of trigonometric expressions for the circumferential portion and third-order polynomials for the meridional part. Even with these shortcomings, it is an excellent method of analysis for axisymmetric orthotropic shells.

Up to this point, nothing has been discussed on the multilayer anisotropic aspects of this work. The reader is referred to the recent work of Bert, et al [28] and the survey paper by Bert and Egle [29] which includes an excellent discussion on the development of the dynamic analyses of multilayer shells. Also, the more recent work of Dong and Selna [30] on multilayer shells of revolution should be mentioned.

1.2 Research Objectives

The purpose of this research is to develop a general analysis, with an accompanying, documented computer program with the following capabilities and characteristics:

1. Shell Geometry - General, as long as it can be modeled by a sufficient number of shallow, quadrilateral thin-shell elements.
2. Material - Anisotropic, perfectly linearly elastic composite built up of orthotropic or isotropic layers. Each layer is of arbitrary thickness (but still thin) and material, and can be oriented at any angle. The layers can be arranged either symmetrically or unsymmetrically with respect to the middle surface.
3. Inertia Effects - All components of translational inertia are included, but all components of coupling and rotatory inertia are neglected.
4. Kinematics - Small displacements, rotations, and strains.
5. Boundary Conditions - Arbitrary within the limitations of combinations possible for the five degrees of freedom applied at each of the element's four nodes.

CHAPTER II

FORMULATION OF THE THEORY

2.1 Method of Analysis

Based on energy principles, expressions for the stiffness and mass matrices are derived for a general anisotropic, quadrilateral shallow-shell element. It is assumed that the shell can be a composite structure made of orthotropic laminates arranged in any preferred order. The total mass and stiffness matrices are then constructed by the well-known method of superimposing element matrices by requiring common nodal displacements and rotations between elements. The total strain and kinetic energies of the system are determined in terms of the total mass and stiffness matrices. The difference in energies is then minimized to form the characteristic eigenvalue-eigenvector expression for a dynamic system. The desired nodal constraints are imposed and all the eigenvalues and eigenvectors are determined for the resulting system. Zeros are relocated in the normalized eigenvectors according to the nodal constraints. Finally, additional modal deflections for a selected number of the lowest frequencies can be computed if desired.

2.2 Hypotheses

All of the following assumptions will be implicit in the analysis:

1. The shell is sufficiently thin and the thickness shear flexibility sufficiently small that the Kirchhoff-Love hypothesis is satisfied, i.e., plane cross sections remain plane, normal to the deflected shell middle surface, and suffer no thickness-direction extension.
2. All material damping, thermal, and initial-stress effects, as well as interactions with surrounding fluid, are neglected.
3. Layers in a composite shell are bonded with a perfect bond (massless, infinitesimal thickness, no relative deformation at the interface).
4. The density of the shell is uniform.
5. The shell can be divided into quadrilateral elements, each of which can be represented adequately as shallow shells (i.e., the in-surface translation terms in the curvature-change expressions are neglected).
6. All components of the local curvature tensor of the shell middle surface, i.e., $\bar{z}_{,xx}$, $\bar{z}_{,yy}$, and $\bar{z}_{,xy}$, are assumed to be constant within each element.
7. A quadrilateral element can be approximated by a rectangle whose length and width are the average value of the corresponding sides of the quadrilateral.
8. The displacement in the normal direction of the element can be approximated by a twelve-term polynomial, and the in-surface displacements can be approximated by two four-term polynomials.

2.3 Development of the Stiffness Matrix for a General Shallow-Shell Element with Double Curvature

In the following analysis, x and y denote the in-surface coordinates of a shallow-shell element and z denotes the outward normal coordinate, measured from the middle surface of the shell. Accordingly, u, v, and w will be the displacements along x, y, and z, respectively. In general, the function describing the shell element should be known and it is reasonable to assume that it can be described by the form, $\bar{z} = f(x, y)$. It then follows that $\bar{z}_{,x}$, $\bar{z}_{,y}$, $\bar{z}_{,xx}$, $\bar{z}_{,yy}$, and $\bar{z}_{,xy}$ all exist and can be determined. The subscript comma (,) variable denotes differentiation with respect to that variable.

From Novozhilov [31], the six strain-displacement relations for a shallow shell obeying the Kirchhoff-Love hypothesis are as follows:

Strain in the x direction:

$$\epsilon_1 = U_{,x} - W\bar{z}_{,xx} \tag{1a}$$

Strain in the y direction:

$$\epsilon_2 = V_{,y} - W\bar{z}_{,yy} \tag{1b}$$

In-surface shear strain:

$$\epsilon_6 = U_{,y} + V_{,x} - 2W\bar{z}_{,xy} \tag{1c}$$

Change in curvature in x direction:

$$\kappa_1 = -W_{,xx} \tag{1d}$$

Change in curvature in y direction:

$$\kappa_2 = -W_{,yy} \tag{1e}$$

Twist of the middle surface:

$$\kappa_6 = -W_{,xy} \tag{1f}$$

For a shell vibrating in a natural mode with circular frequency ω, the three displacements can be expressed as

$$U(x,y,t) = u(x,y)\, e^{i\omega t} \tag{2a}$$

$$V(x,y,t) = v(x,y)\, e^{i\omega t} \tag{2b}$$

$$W(x,y,t) = w(x,y)\, e^{i\omega t} \tag{2c}$$

Substituting these expressions into Equations (1a) through (1f), one obtains

$$\epsilon_1 = (u_{,x} - w\bar{z}_{,xx})\, e^{i\omega t} \tag{3a}$$

$$\epsilon_2 = (v_{,y} - w\bar{z}_{,yy})\, e^{i\omega t} \tag{3b}$$

$$\epsilon_6 = (u_{,y} + v_{,x} - 2w\bar{z}_{,xy})\, e^{i\omega t} \tag{3c}$$

$$\kappa_1 = -w_{,xx} e^{i\omega t} \tag{3d}$$

$$\kappa_2 = -w_{,yy} e^{i\omega t} \tag{3e}$$

$$\kappa_6 = -w_{,xy} e^{i\omega t} \tag{3f}$$

According to Ambartsumyan [32], the expression for the potential energy of deformation of an anisotropic laminar shell is:

$$
\begin{aligned}
V = \tfrac{1}{2} \iint & (A_{11}\epsilon_1{}^2 + 2\, A_{12}\epsilon_1\epsilon_2 + A_{22}\epsilon_2{}^2 + A_{66}\epsilon_6{}^2 + 2\, A_{16}\epsilon_1\epsilon_6 \\
& + 2\, A_{26}\epsilon_2\epsilon_6)\, dx\, dy \; + \tfrac{1}{2} \iint (D_{11}\kappa_1{}^2 + 2\, D_{12}\, \kappa_1\kappa_2 \\
& + D_{22}\kappa_2{}^2 + D_{66}\kappa_6{}^2 + 2\, D_{16}\kappa_1\kappa_6 + 2\, D_{66}\kappa_2\kappa_6)\, dx\, dy \qquad (4) \\
& + \iint [B_{11}\epsilon_1\kappa_1 + B_{12}(\epsilon_1\kappa_2 + \epsilon_2\kappa_1) + B_{22}\epsilon_2\kappa_2 + B_{66}\epsilon_6\kappa_6 \\
& + B_{16}(\epsilon_1\kappa_6 + \epsilon_6\kappa_1) + B_{26}(\epsilon_2\kappa_6 + \epsilon_6\kappa_6)]\, dy\, dx
\end{aligned}
$$

The A, B, and D coefficients correspond to Ambartsumyan's C, K, and D coefficients, respectively. The A, B, and D coefficients are defined by Equations (3-25), (3-26), and (3-32) in the work of Ashton, et al [33].

When the strain-displacement relations of Equations (3) are substituted into Equation (4), the strain energy can then be expressed in terms of the displacements. The magnitude of the strain energy is as follows:

$$
\begin{aligned}
V = \iint \Big(& \tfrac{1}{2} A_{11}u^2{}_{,x} - A_{11}\bar{z}_{,xx}wu_{,x} + \tfrac{1}{2} A_{11}\bar{z}^2{}_{,xx}w^2 + A_{12}u_{,x}v_{,y} \\
& - A_{12}\bar{z}_{,xx}wv_{,y} + A_{12}\bar{z}_{,xx}\bar{z}_{,yy}w^2 + \tfrac{1}{2} A_{22}v^2{}_{,y} - A_{22}\bar{z}_{,yy}wv_{,y} \\
& + \tfrac{1}{2} A_{22}\bar{z}^2{}_{,yy}w^2 + \tfrac{1}{2} A_{66}u^2{}_{,y} + A_{66}u_{,y}v_{,x} - 2\, A_{66}\bar{z}_{,xy}wu_{,y} \\
& + \tfrac{1}{2} A_{66}v^2{}_{,x} - 2\, A_{66}\bar{z}_{,xy}wv_{,x} + 2\, A_{66}\bar{z}^2{}_{,xy}w^2 + A_{16}u_{,x}u_{,y} + A_{16}u_{,x}v_{,x} \\
& - 2\, A_{16}\bar{z}_{,xy}wu_{,x} - A_{16}\bar{z}_{,xx}wu_{,y} - A_{16}\bar{z}_{,xx}wv_{,x} + 2\, A_{16}\bar{z}_{,xx}\bar{z}_{,xy}w^2 \\
& + A_{26}u_{,y}v_{,y} + A_{26}v_{,x}v_{,y} - 2\, A_{26}\bar{z}_{,xy}wv_{,y} - A_{26}\bar{z}_{,yy}wu_{,y} \\
& - A_{26}\bar{z}_{,yy}wv_{,x} + 2\, A_{26}\bar{z}_{,yy}\bar{z}_{,xy}w^2 + \tfrac{1}{2} D_{11}w^2{}_{,xx} + D_{12}w_{,xx}w_{,yy}
\end{aligned}
$$

$$+ \tfrac{1}{2} D_{22} w^2_{,yy} + \tfrac{1}{2} D_{66} w^2_{,xy} + D_{16} w_{,xx} w_{,xy} + D_{26} w_{,yy} w_{,xy}$$

$$- B_{11} w_{,xx} u_{,x} + B_{11} \bar{z}_{,xx} ww_{,xx} - B_{12} w_{,yy} u_{,x} + B_{12} \bar{z}_{,xx} ww_{,yy}$$

$$- B_{12} w_{,xx} v_{,y} + B_{12} \bar{z}_{,yy} ww_{,xx} - B_{22} w_{,yy} v_{,y} + B_{22} \bar{z}_{,yy} ww_{,yy}$$

$$- B_{66} w_{,xy} u_{,y} - B_{66} w_{,xy} v_{,x} + 2\, B_{66} \bar{z}_{,xy} ww_{,xy} - B_{16} w_{,xy} u_{,x}$$

$$+ B_{16} \bar{z}_{,xx} ww_{,xy} - B_{16} w_{,xx} u_{,y} - B_{16} w_{,xx} v_{,x} + 2\, B_{16} \bar{z}_{,xy} ww_{,xx}$$

$$- B_{26} w_{,xy} v_{,y} + B_{26} \bar{z}_{,yy} ww_{,xy} - B_{26} w_{,yy} u_{,y} - B_{26} w_{,yy} v_{,x}$$

$$+ 2\, B_{26} \bar{z}_{,xy} ww_{,yy} \Big)\, dy\, dx \qquad (5)$$

Equation (5) can then be expressed in a reduced matrix form as

$$\bar{V} = \iint \lfloor w,\ w_{,xx},\ w_{,yy},\ w_{,xy},\ u_{,x},\ u_{,y},\ v_{,x},\ v_{,y} \rfloor \,[\,R\,] \begin{Bmatrix} w \\ w_{,xx} \\ w_{,yy} \\ w_{,xy} \\ u_{,x} \\ u_{,y} \\ v_{,x} \\ v_{,y} \end{Bmatrix} dy\, dx \qquad (6)$$

where $[\,R\,]$ is an 8 X 8 symmetric matrix, the terms of which are defined in Appendix A. Studying Equation (6), one sees that an eight-term vector has been established of the terms w, $w_{,xx}$, $w_{,yy}$, $w_{,xy}$, $u_{,x}$, $u_{,y}$, $v_{,x}$, and $v_{,y}$.

The displacement amplitudes of the element were defined as u, v, and w. The rotations of the element are

$$\beta_1 = -w_{,x} - uz_{,xx} - vz_{,xy} \tag{7a}$$

$$\beta_2 = -w_{,y} - uz_{,xy} - vz_{,yy} \tag{7b}$$

The in-surface translation is neglected in Equations (7) and they become

$$\beta_1 = -w_{,x} \tag{8a}$$

$$\beta_2 = -w_{,y} \tag{8b}$$

It is assumed that the displacements u, v, and w over the element can be reasonably approximated by the following polynomial forms:

$$w = a_1 + a_2x + a_3y + a_4x^2 + a_5xy + a_6y^2 + a_7x^3 + a_8x^2y + a_9xy^2 + a_{10}y^3 + a_{11}x^3y + a_{12}xy^3 \tag{9a}$$

$$u = a_{13} + a_{14}x + a_{15}y + a_{16}xy \tag{9b}$$

$$v = a_{17} + a_{18}x + a_{19}y + a_{20}xy \tag{9c}$$

where x and y are the element coordinates. The required derivatives are:

$$w_{,x} = a_2 + 2a_4x + a_5y + 3a_7x^2 + 2a_8xy + a_9y^2 + 3a_{11}x^2y + a_{12}y^3 \tag{9d}$$

$$w_{,y} = a_3 + a_5x + 2a_6y + a_8x^2 + 2a_9xy + 3a_{10}y^2 + a_{11}x^3 + 3a_{12}xy^2 \tag{9e}$$

$$w_{,xx} = 2a_4 + 6a_7x + 2a_8y + 6a_{11}xy \tag{9f}$$

$$w_{,yy} = 2a_6 + 2a_9x + 6a_{10}y + 6a_{12}xy \tag{9g}$$

$$w_{,xy} = a_5 + 2a_8x + 2a_9y + 3a_{11}x^2 + 3a_{12}y^2 \tag{9h}$$

$$u_{,x} = a_{14} + a_{16}y \tag{9i}$$

$$u_{,y} = a_{15} + a_{16}x \tag{9j}$$

$$v_{,x} = a_{18} + a_{20}y \tag{9k}$$

$$v_{,y} = a_{19} + a_{20}x \tag{9l}$$

The vector occurring in Equation (6) can now be expressed as

$$\begin{Bmatrix} w \\ w_{,xx} \\ w_{,yy} \\ w_{,xy} \\ u_{,x} \\ u_{,y} \\ v_{,x} \\ v_{,y} \end{Bmatrix} = [X]\{\gamma\} \tag{10}$$

where

$$\{\gamma\} = \begin{Bmatrix} a_1 \\ \cdot \\ \cdot \\ \cdot \\ a_{20} \end{Bmatrix} \tag{11}$$

The terms of Matrix $[X]$ are defined in Appendix A.

Similarly, from Equations (9), a displacement-rotation vector will now be formed of the following terms:

$$\begin{Bmatrix} w \\ \beta_1 \\ \beta_2 \\ u \\ v \end{Bmatrix} = \begin{Bmatrix} w \\ -w_{,x} \\ -w_{,y} \\ u \\ v \end{Bmatrix} = [A] \begin{Bmatrix} a_1 \\ \cdot \\ \cdot \\ \cdot \\ a_{20} \end{Bmatrix} = [A]\{\gamma\} \tag{12}$$

The terms of Matrix $[A]$ are also defined in Appendix A.

The average distance across the element in the x-direction will be defined as e_x, and the average distance across the element in the y-direction will be e_y. Now, inserting x, y

$$x, y = \frac{-e_x}{2}, \frac{-e_y}{2}\ ;\ \frac{-e_x}{2}, \frac{e_y}{2}\ ;\ \frac{e_x}{2}, \frac{-e_y}{2}\ ;\ \text{and}\ \frac{e_x}{2}, \frac{e_y}{2}$$

into the appropriate locations in Equation (12) to simulate the coordinates of the four nodes, a, b, c, and, d, of the element, the following relation can be formed:

$$\left\{\begin{array}{c} w_a \\ \beta_{1a} \\ \beta_{2a} \\ u_a \\ v_a \\ --- \\ w_b \\ \beta_{1b} \\ \cdot \\ --- \\ \cdot \\ --- \\ \cdot \\ v_d \end{array}\right\} = \{\xi\} = [AK]\{\gamma\} \tag{13}$$

The terms of Matrix $[AK]$ are defined in Appendix A.

It then follows that

$$\{\gamma\} = [AK]^{-1}\{\xi\} = [T]\{\xi\} \tag{14}$$

Returning to Equation (6), the amplitude of strain energy of the element may be written as

$$\bar{V} = \int_{\frac{-e_x}{2}}^{\frac{e_x}{2}} \int_{\frac{-e_y}{2}}^{\frac{e_y}{2}} \lfloor w, \ldots, v_{,y} \rfloor \, [R] \begin{Bmatrix} w \\ \cdot \\ \cdot \\ \cdot \\ v_{,y} \end{Bmatrix} dy\, dx \tag{15}$$

From Equation (10), it follows that

$$\bar{V} = \int_{\frac{-e_x}{2}}^{\frac{e_x}{2}} \int_{\frac{-e_y}{2}}^{\frac{e_y}{2}} \{\gamma\}^T [X]^T [R] [X] \{\gamma\} \, dy\, dx$$

and since $\{\gamma\}$ is a vector of constants for any given element, it may be removed from inside the integral.

$$\bar{V} = \{\gamma\}^T \left[\int_{\frac{-e_x}{2}}^{\frac{e_x}{2}} \int_{\frac{-e_y}{2}}^{\frac{e_y}{2}} [X]^T [R] [X] \, dy\, dx \right] \{\gamma\}$$

or,

$$\bar{V} = \{\gamma\}^T [C] \{\gamma\} \tag{16}$$

where

$$[C] = \int_{\frac{-e_x}{2}}^{\frac{e_x}{2}} \int_{\frac{-e_y}{2}}^{\frac{e_y}{2}} [X]^T [R] [X] \, dy\, dx \tag{17}$$

Finally, substituting Equation (14) into Equation (16), one obtains

$$\bar{V} = \{\xi\}^T [T]^T [C] [T] \{\xi\} \tag{18}$$

and from this relation the element stiffness matrix can be identified as

$$[\,\bar{S}\,] = [\,T\,]^T\,[\,C\,]\,[\,T\,] \tag{19}$$

The strain energy of the element then becomes

$$\bar{V} = \{\,\xi\,\}^T\,[\,\bar{S}\,]\,\{\,\xi\,\} \tag{20}$$

2.4 Development of the Consistent Mass Matrix for the Element

If rotatory and coupling inertia are neglected and if the shell is vibrating in a natural mode, the magnitude of the kinetic energy for the element is

$$E = \omega^2 \int_{-\frac{e_x}{2}}^{\frac{e_x}{2}} \int_{-\frac{e_y}{2}}^{\frac{e_y}{2}} \rho\, h\, (u^2 + v^2 + w^2)\, dy\, dx \tag{21}$$

$$= \omega^2 \rho\, h \int_{-\frac{e_x}{2}}^{\frac{e_x}{2}} \int_{-\frac{e_y}{2}}^{\frac{e_x}{2}} (u^2 + v^2 + w^2)\, dy\, dx$$

where ρ is the density of the shell which is assumed uniform and h is the given element total thickness. Since h is independent of x and y for any given element, ρh can be moved outside the integral.

Based on the assumed displacements of Equations (9), the following relation exists:

$$\begin{Bmatrix} w \\ u \\ v \end{Bmatrix} = [\,Y\,]\,\{\,\gamma\,\} \tag{22}$$

The terms of matrix $[Y]$ are defined in Appendix A. Equation (21) can be written

$$\bar{E} = \rho h \omega^2 \int_{-\frac{e_x}{2}}^{\frac{e_x}{2}} \int_{-\frac{e_y}{2}}^{\frac{e_y}{2}} \lfloor w, u, v \rfloor \begin{Bmatrix} w \\ u \\ v \end{Bmatrix} dy\,dx \tag{23}$$

Substituting Equation (22) into Equation (23), one obtains:

$$\bar{E} = \rho h \omega^2 \{\gamma\}^T [F] \{\gamma\} \tag{24}$$

where

$$[F] = \int_{-\frac{e_x}{2}}^{\frac{e_x}{2}} \int_{-\frac{e_y}{2}}^{\frac{e_y}{2}} [Y]^T [Y]\, dy\,dx \tag{25}$$

Now using Equation (14), Equation (24) becomes

$$[E] = \rho h \omega^2 \{\xi\}^T [T]^T [F][T] \{\xi\} \tag{26}$$

The element mass matrix may be identified as

$$[\bar{M}] = (\rho h) [T]^T [F] [T] \tag{27}$$

and the final form for the magnitude of the kinetic energy becomes

$$\bar{E} = \omega^2 \{\xi\}^T [\bar{M}] \{\xi\} \tag{28}$$

2.5 Development of Modal Equations

In general, there will exist more than one element in any given analysis. Consequently, any given node may be common to only one up to four different elements. When a common node, n, exists between two elements, say the kth and the k + 1st, the following conditions of compatibility are assumed for each such juncture:

$$\left\{\begin{matrix} w_n \\ \beta_{1n} \\ \beta_{2n} \\ u_n \\ v_n \end{matrix}\right\}_{\substack{\text{kth} \\ \text{Element}}} = \left\{\begin{matrix} w_n \\ \beta_{1n} \\ \beta_{2n} \\ u_n \\ v_n \end{matrix}\right\}_{\substack{\text{k+1st} \\ \text{Element}}} \tag{29}$$

The total strain energy V and the kinetic energy E of a shell divided into K elements may be expressed as

$$V = \sum_{k=1}^{K} \bar{V}_k \tag{30}$$

$$E = \sum_{k=1}^{K} \bar{E}_k \tag{31}$$

where $\bar{V}_k$ and $\bar{E}_k$ are the elements expressed by Equations (20) and (28), respectively. If use is made of Equation (29) and the summations are carried out in Equations (30) and (31), the energy expressions may be written as follows:

$$V = \{Y\}^T [S] \{Y\} \tag{32}$$

$$E = \omega^2 \{Y\}^T [M] \{Y\} \tag{33}$$

where

$[S]$ = Stiffness matrix of the shell of order 5N

$[M]$ = Mass of matrix of the shell of order 5N

$\{Y\}$ = A vector of all the unknown displacements and rotations

N = Number of nodes in the system

The well-known procedure of superimposing element matrices is used to construct the matrices, $[S]$ and $[M]$. The superposition consists of adding those terms of the element stiffness and mass matrices that share common degrees of freedom due to sharing of a node and placing the coupling terms between nodes of a given element in their appropriate location. This has to be done to assure the compatibility imposed by Equation (29).

The modal equations for the unconstrained shell are derived by minimizing the quantity $(E - V)$ with respect to each variable in $\{Y\}$; i.e.,

$$\frac{\partial (E-V)}{\partial w_n} = 0 \qquad \frac{\partial (E-V)}{\partial \beta_{1n}} = 0 \qquad \frac{\partial (E-V)}{\partial \beta_{2n}} = 0 \qquad (34) - (36)$$

$$\frac{\partial (E-V)}{\partial u_n} = 0 \qquad \frac{\partial (E-V)}{\partial v_n} = 0 \qquad (37),(38)$$

for all $n = 1, 2, \ldots, N$.

Equations (34) through (38) can be expressed as

$$[S]\{Y\} - \omega^2 [M]\{Y\} = 0 \tag{39}$$

This is the characteristic form of the dynamic eigenvalue-eigenvector problem.

2.6 Application of Nodal Constraints

Nodal constraints are implied by deleting from the stiffness and mass matrices of Equation (39) those rows and columns which must vanish to satisfy the constraints. There are five degrees of freedom at each node. Including the completely free and completely rigid cases, there exist 32 possible different combinations of constraints which can be imposed at each node. Any of the 32 possibilities may be chosen for any node. The order of $[S]$ and $[M]$ is reduced accordingly.

After computing all the remaining eigenvalues and eigenvectors by a standard routine, zeros are reinserted in the eigenvectors in the appropriate locations according to the nodal constraints that had been imposed.

Finally, for any selected number of the lowest eigenvalues, up to the total number if desired, additional modal data can be determined. This is accomplished by determining the constants, a_1 through a_{20}, using the relation given in Equation (14), i.e.,

$$\{\gamma\} = [T]\{\xi\} \tag{14}$$

where $\{\xi\}$ is that portion of the vector $\{Y\}$ applicable to the element being studied. Once $\{\gamma\}$ is known, the modal polynomial expressions are used as given by Equations (9a) through (9c). The additional modal displacements are determined at 25 additional locations on each element.

CHAPTER III

EVALUATION OF THE THEORY

3.1 General Discussion

One of the main goals of this research was to develop a program for analyzing laminated shells with double curvature using a finite-element method. Two lesser goals were that the program would be relatively easy to use and that it could be applied to a wide range of practical problems. These three goals have been reached with varying degrees of success.

The main goal was satisfactorily attained. If the problem can be modeled within the implicit assumptions stated previously, reasonably accurate results should be obtained. The program has one major shortcoming which is characteristic of many structural and structural dynamic analyses, especially those based on finite-element theory. The program needs more computer storage space in order to include a larger number of elements to better model the shell structure.

The present work was accomplished on an IBM 360, Model 50 computer which had 230 K bytes of available storage. This program, with provisions for 16 nodes, required 225 K bytes. The largest problem analyzed to date was a cylinder modeled as 12 elements which required all 16 nodes. In the majority of problems, only 8 or 9 elements could be used, depending on the geometry of the particular problem. A few computers do exist today which have considerably more than 230 K bytes, and in about two years similar machines will be fairly commonplace. Therefore, this handicap should be overcome in the near future.

The program is definitely easy to use; the preparation of the input is very simple and straightforward. Detailed explanation for preparing the input data is provided in the opening comments of the program. A possible improvement would be to include provisions for reading in the surface curvature terms; i.e., $w_{,xx}$, $w_{,yy}$, and $w_{,xy}$, in lieu of calling them in by subroutine. The present procedure requires compiling the subroutine for each particular problem investigated. This tends to become a nuisance to the user.

It appears that the program can indeed be applied to a wide range of practical problems. The major restriction stems from the problem of available computer storage as previously mentioned. The program is capable of considering 32 different possible combinations of nodal constraints. It can handle materials ranging from those that behave in the simple isotropic fashion up to the sophisticated anisotropic laminate. In addition, the program can determine with a high degree of accuracy all the eigenvalues and eigenvectors (up to 80) for any problem that can presently be considered. The maximum time required for a single investigation was 40 minutes for an 80-degree-of-freedom problem.

3.2 Example Check Problems

Most of the examples in the literature investigated to date have been shells of revolution, especially circular cylinders. Usually, the method of solution which was developed was restricted to a particular type of problem being investigated. Often, full advantage was taken of geometry in order to obtain a high degree of accuracy in the results. With the computer storage restriction in the current analysis, it was first believed doubtful if a reasonable comparison could be made with any example from a previous work. However, three sample problems were investigated.

The theory was first evaluated for a shallow, curved panel investigated by Sewall [34]. The panel analyzed was 11 in. x 9 in. x 0.028 in. made of aluminum, with a radius of curvature of 96 in. This panel was selected as it was shallow and because experimental results were presented in addition to the analytical results for simply-supported and clamped-edge conditions. It was also believed that this would be a reasonable example for which convergence could be demonstrated with an increasing number of elements.

The results are presented in Tables 1 and 2 for the clamped and the simply-supported edge constraints, respectively. A minimum of four elements is required to investigate the clamped configuration as one unrestrained node is required in order to obtain a non-trivial solution. Consequently, this was the minimum number of elements investigated for both the clamped and the simply-supported conditions.

TABLE 1

Comparison of Frequencies for a Curved Panel

Analytical Edge Constraints: Clamped

Reference 34		Present Analysis		
		Number of Elements Used		
Experimental	Analysis	8	6	4
233.	337.0	240.2	240.7	242.3
250.	317.1	242.7	252.7	244.4
299.	358.0	255.8	258.2	265.6
351.	400.5	271.1	274.1	
405.	483.7	297.2	324.1	
497.	511.1	303.9	502.6	
507.	553.8	426.2		
532.	445.7	633.2		
640.	589.6	973.1		
656	670.4			
---	672.0			
673.	729.4			
760.	800.9			
816.	789.5			
835.	836.1			
---	917.8			
---	882.75			
999.	1055.			
1052.	1050.			
1068.	1083.			
1087.	1127.			
1222.	1243.			
1302.	1296.			
1389.	1407.			
	1618.			

TABLE 2.

Comparison of Frequencies for a Curved Panel

Analytical Edge Constraints: Simply Supported

Reference 34		Present Analysis		
		Number of Elements Used		
Experimental	Analysis	8	6	4
233.	164.1	127.9	128.3	127.9
250.	145.7	148.3	147.7	142.8
299.	273.7	174.8	176.8	240.7
351.	261.6	189.8	194.1	245.8
405.	325.7	208.5	258.7	249.8
497.	392.5	238.1	263.0	258.8
507.	399.8	265.1	269.2	266.5
532.	372.1	267.5	288.3	272.1
640.	498.9	299.2	295.0	276.8
656.	522.3	308.2	302.2	307.2
---	551.2	337.4	306.7	334.3
673.	560.1	341.0	327.6	358.4
760.	629.3	352.4	349.3	396.4
816.	274.6	367.1	362.8	405.7
835.	686.8	383.7	413.6	425.6
---	745.5	406.2	442.0	466.1
---	747.3	446.1	471.2	519.0
999.	862.3	494.1	531.3	590.0
1052.	891.6	498.6	535.1	592.8
1068.	907.3	508.0	567.6	
1087.	930.3	548.6	567.9	
1222.	1044.	550.4	662.9	
1302.	1113.	550.9	694.0	
1389.	1204.	596.1	734.9	
	1408.	693.5	816.3	

Before discussing the correlation, it should be mentioned that edge constraints described by Sewall [34] for the test values more closely resembled clamped conditions than simply-supported conditions. In studying Sewall's results [34], it is difficult to determine which constraint gave the better results. However, the clamped constraints appear to give the better results in the present analysis. The validity in the present analysis is supported by the close comparison of results in the other two example problems investigated.

Little variation was noted in the lower two frequencies when four, six, or eight elements were used. The third frequency varied considerably more, especially with the simply-supported constraint..

The theory was next applied to a 60° spherical cap as investigated by Cohen [2], who presented his results in a generalized form. This example was chosen to investigate a shell with double curvature. In order to compare, a shell with the following properties was analyzed:

$$E_1 = E_2 = 26. \times 10^6 \text{ psi}, \quad G = 10. \times 10^6 \text{ psi}, \quad \nu_1 = \nu_2 = 0.3,$$

$$\rho = 259. \times 10^{-6} \frac{\text{lb-sec}^2}{\text{in}^4}, \quad R_m = 8.08 \text{ in.}, \quad h = 0.404 \text{ in.}$$

The following is a comparison of the dimensionless frequencies, $\Omega = \omega R_m (E/\rho)^{-\frac{1}{2}}$, for the first three axisymmetric modes for a fixed-hinge edge constraint and the percent difference.

	REFERENCE 2	PRESENT ANALYSIS	PERCENT DIFFERENCE
Ω_1	.951	1.005	4.8
Ω_2	1.325	1.332	0.5
Ω_3	1.646	1.809	9.9

The modal shapes, both w and u, computed in this analysis agreed reasonably well with Cohen's [2] for the lower two modes. However, comparison of the third modal shape was somewhat difficult. The third modal shape, as reported by Cohen [2], had two interior inflection points but no interior nodes. It is doubtful if the present analysis contained sufficient nodes to predict such behavior. In general, the correlation between the two analyses is considered good.

The third example investigated was a composite cylindrical shell originally investigated by Bert, et al [28]. The cylinder was made of two-layer, cross-ply boron/epoxy material and had the following properties;

$$E_1 = 31. \times 10^6 \text{ psi}, \quad E_2 = 2.7 \times 10^6 \text{ psi}, \quad \nu_1 = 0.28$$

$$G = 0.75 \times 10^6 \text{ psi}, \quad \rho = 192. \times 10^{-6} \frac{\text{lb-sec}^2}{\text{in}^4}$$

$$R_m = 2.481 \text{ in.}, \quad L = 31.5 \text{ in.}, \quad h = 0.02 \text{ in.}$$

The shell was freely supported at both ends and a closed-form solution of the Love's first-approximation shell theory equations was obtained.

A comparison of frequencies for the two analyses is given on the following page.

FREQUENCIES (HZ)

REFERENCE 28	PRESENT ANALYSIS
240	245
255	254
532	537
440	539
450	585
500	590
675	676

The first three frequencies agree very well. The fourth through the sixth frequencies in Reference [28] were associated with wave numbers greater than two. The present analysis would require more than the limited number of sixteen nodes and twelve elements to accurately calculate these modes. The 675 hz frequency from Reference [28] was for the circumferential and meridional wave numbers both equal to two, which is the highest frequency mode the present investigation should accurately determine. This comparison is again considered good. However, it should be mentioned that the present results did have two modes associated with the in-surface circumferential degree-of-freedom with frequencies of 291 and 581 hz. Also, there were four distinct modes in the present analysis with frequencies of 668, 669, 670, and 671 hz, but none of these had a modal shape which would correspond to a circumferential wave number of 2. The 676 hz frequency did have such a modal shape; thus, it was selected for comparison.

One major objective of this work was to develop a method for analyzing general shell problems; this has been accomplished. It was therefore believed that three example problems should be investigated which would fully exercise the capability of the program and which would suggest areas of possible practical application. In addition, it was believed that by investigating different materials and edge constraints for each example, a reasonable basis of comparison would be obtained for future research.

3.3 Multilayer, Anisotropic Cylindrical Shell of Arbitrary Cross Section, Exemplified by a Fuselage Section

The theory was first evaluated for a cylinder which was not a shell of revolution. The x-direction was along the axis of the cylinder, and, as shown in Figure 1, the cross section at any axial position x was described by the function:

$$\left(\frac{y}{36}\right)^4 + \left(\frac{z}{24}\right)^3 = 1 \quad \text{where } y \text{ and } z \text{ are in inches.}$$

This section is typical of the constant sections for aircraft fuselages. The length of the cylinder was selected as 12 feet.

The first material was a non-woven unidirectional tape form FRP (fiberglass reinforced plastic) called XP-250 which has the following properties:

$E_1 = 5.35 \times 10^6$ psi   Filament volume fraction = 0.526

$E_2 = 1.52 \times 10^6$ psi   Void volume fraction = 0.02

$G = 0.8 \times 10^6$ psi

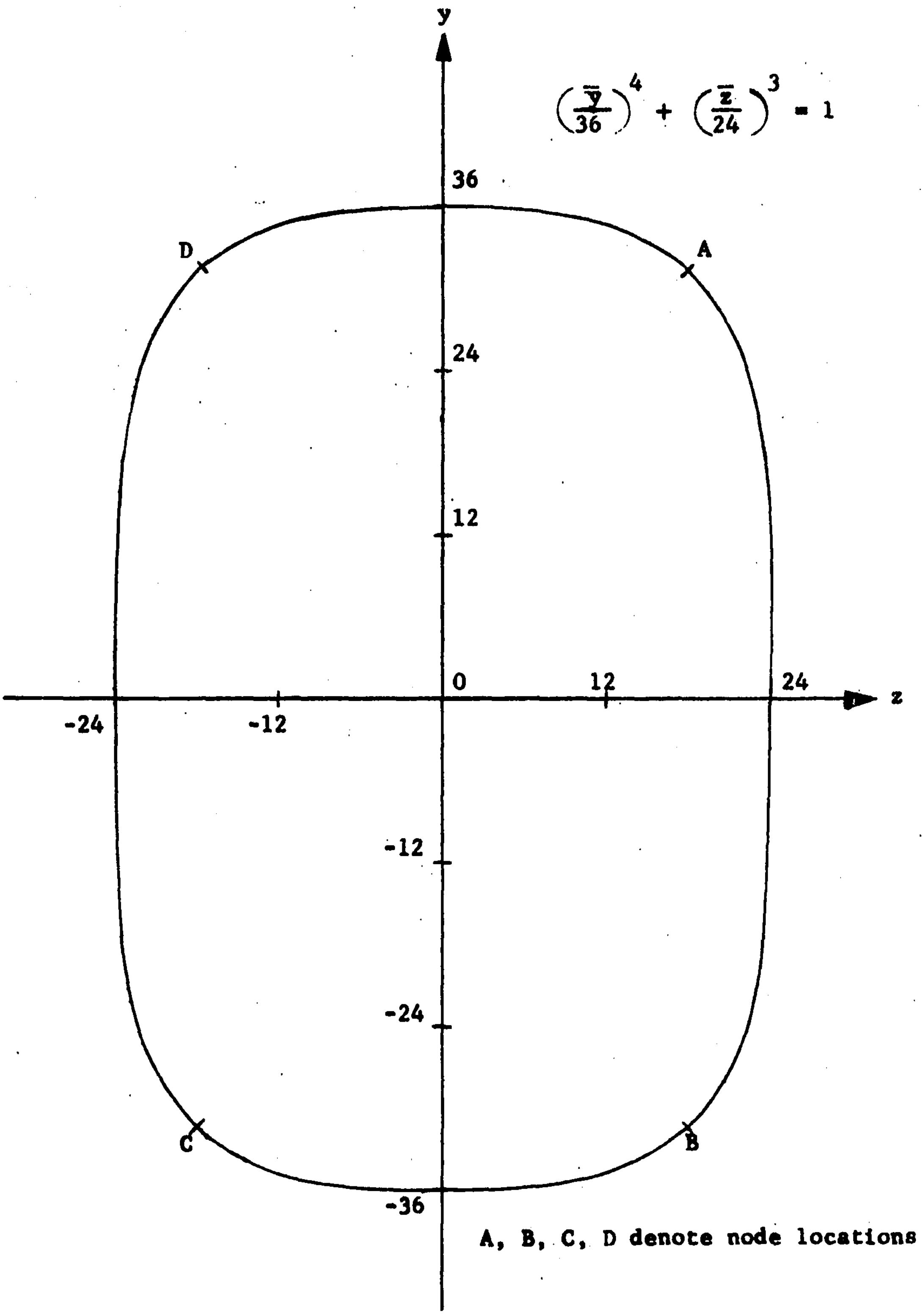


Figure 1 - Fuselage Section

$\nu_1 = 0.275$

$\nu_2 = 0.078$

$\rho = 181.3 \times 10^{-6}$ lb.-sec.^{2}/in.^{4}

For the investigation, a quasi-isotropic laminate was constructed of four layers oriented at 0, +45°, -45°, and 90°. Each layer was 0.09 inch thick.

Aluminum was selected as the other material to be investigated in order to present results for an isotropic material. The following properties were used:

$E_1 = E_2 = 10.5 \times 10^6$ psi

$G = 4 \times 10^6$ psi

$\nu_1 = \nu_2 = 0.33$

$\rho = 259.0 \times 10^{-6}$ lb.-sec.^{2}/in.^{4}

To be comparable to the XP-250 laminate, a total thickness of 0.036 inch was used.

For analysis, the cylinder was divided into three 48-inch sections. Each section was divided into four elements by placing nodes as shown in Figure 1 at Points A, B, C, and D. Three configurations of nodal constraints were investigated: clamped, freely supported (w = v = 0), and free at both ends.

The resulting ten lowest frequencies for all six configurations are shown in Table 3. All the resulting frequencies are presented in Appendix D. As expected, the lower frequencies are all quite small. This is mainly due to the small thickness which was selected primarily for the remaining two examples and was used here for consistency. Realistically, a minimum thickness of 0.10 inch would be tried for a full-monocoque structure of this size in a practical application.

One peculiarity was noted in this problem. In each of the six configurations, a large jump in frequencies occurred. The higher set of frequencies are those associated with the in-surface degrees of freedom, i.e., u and v. This large increase may be accentuated due to the fact that each cylindrical section could be divided into only four elements. A minimum of eight should be used, which would double the number of elements and nodes (and, in effect, make the resulting matrices four times as large).

Normalized modal displacements for the six lower frequency nodes associated with points in Figure 1 are presented in Figures 2. and 3 for the XP-250 cylinder in the clamped configuration. These figures show the w displacement of the shell for the two interior cross-sections of the cylinder. For the frequency shown, the left modal shape is for nodes 5, 6, 7, and 8, and the figure on the right is the corresponding modal shape for nodes 9, 10, 11, and 12.

TABLE 3.

Ten Lowest Frequencies for Fuselage Section (hz)

XP-250 Material			Aluminum Material		
Free	Freely-Supported	Clamped	Free	Freely-Supported	Clamped
3.192	3.666	3.741	5.104	5.879	5.986
3.193	4.640	4.646	5.242	7.486	7.494
3.424	6.251	6.287	5.470	10.12	10.19
4.444	6.310	6.584	7.151	10.35	10.66
4.978	6.360	7.773	8.069	10.46	12.89
6.065	6.560	7.897	9.619	10.62	13.08
6.142	7.746	9.899	9.945	12.83	15.97
6.245	7.859	11.40	10.11	13.00	18.61
6.542	9.282	18.09	10.59	15.13	29.06
6.785	9.404	18.19	11.01	15.27	29.08

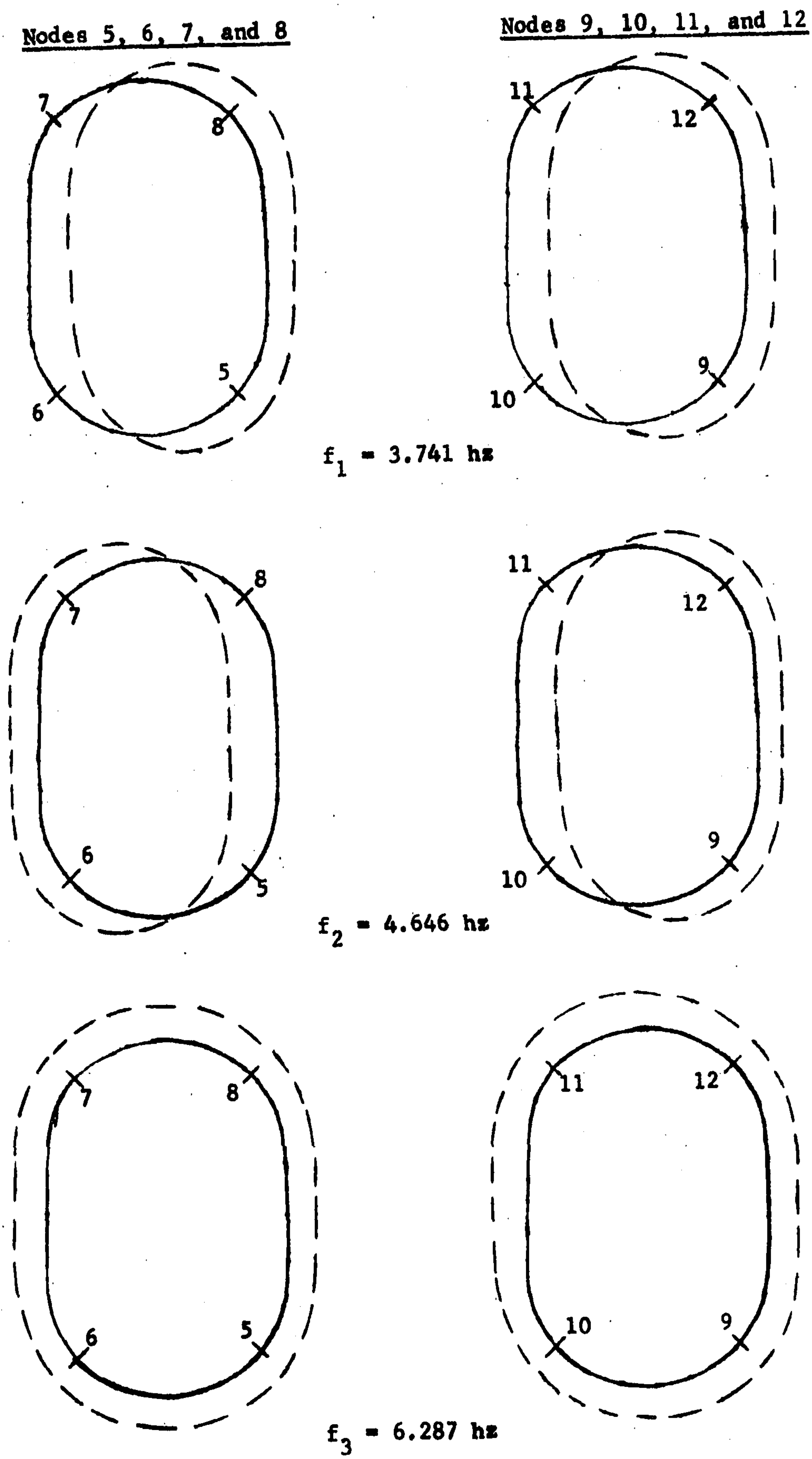


Figure 2. Modal Shapes for the Fuselage Section

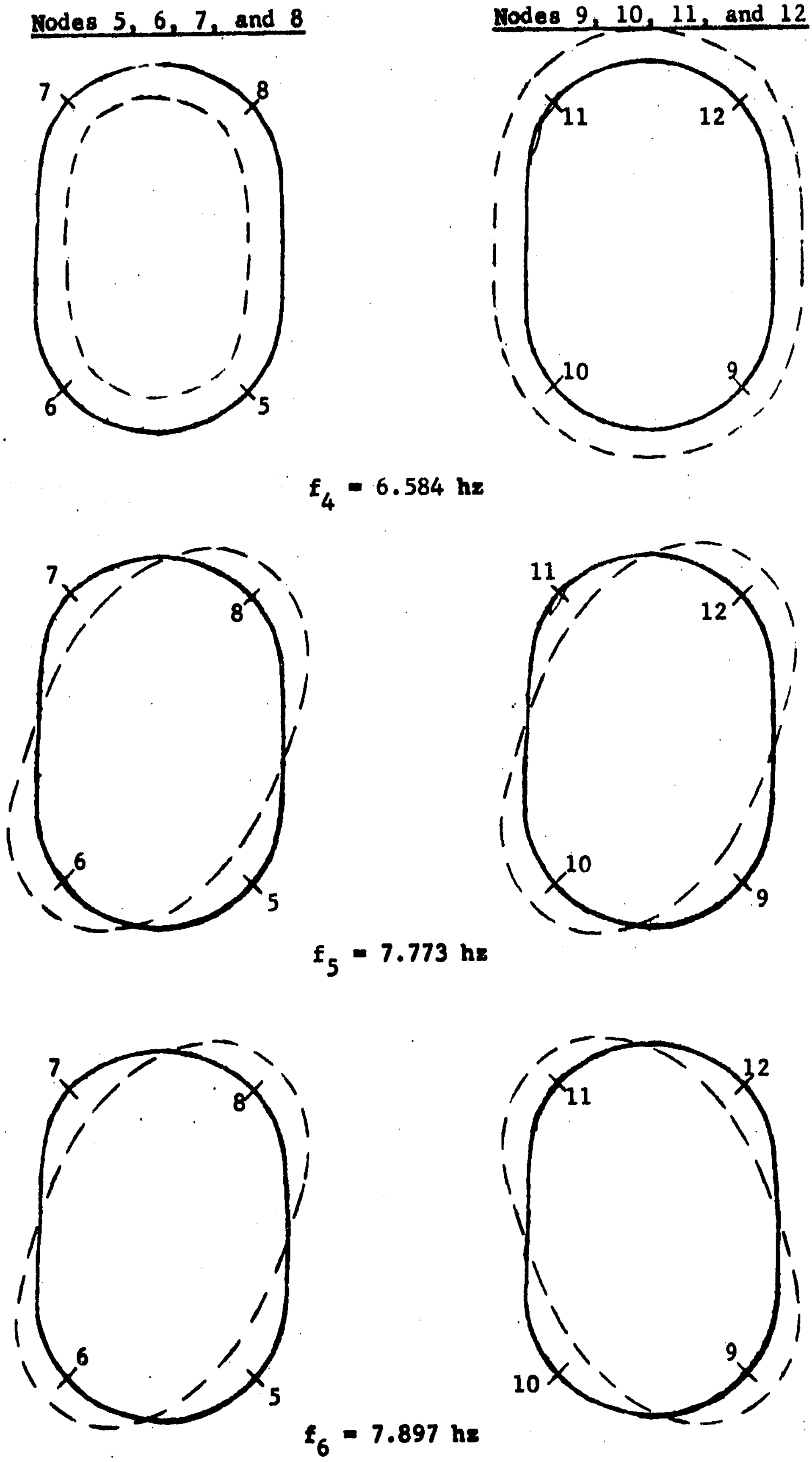


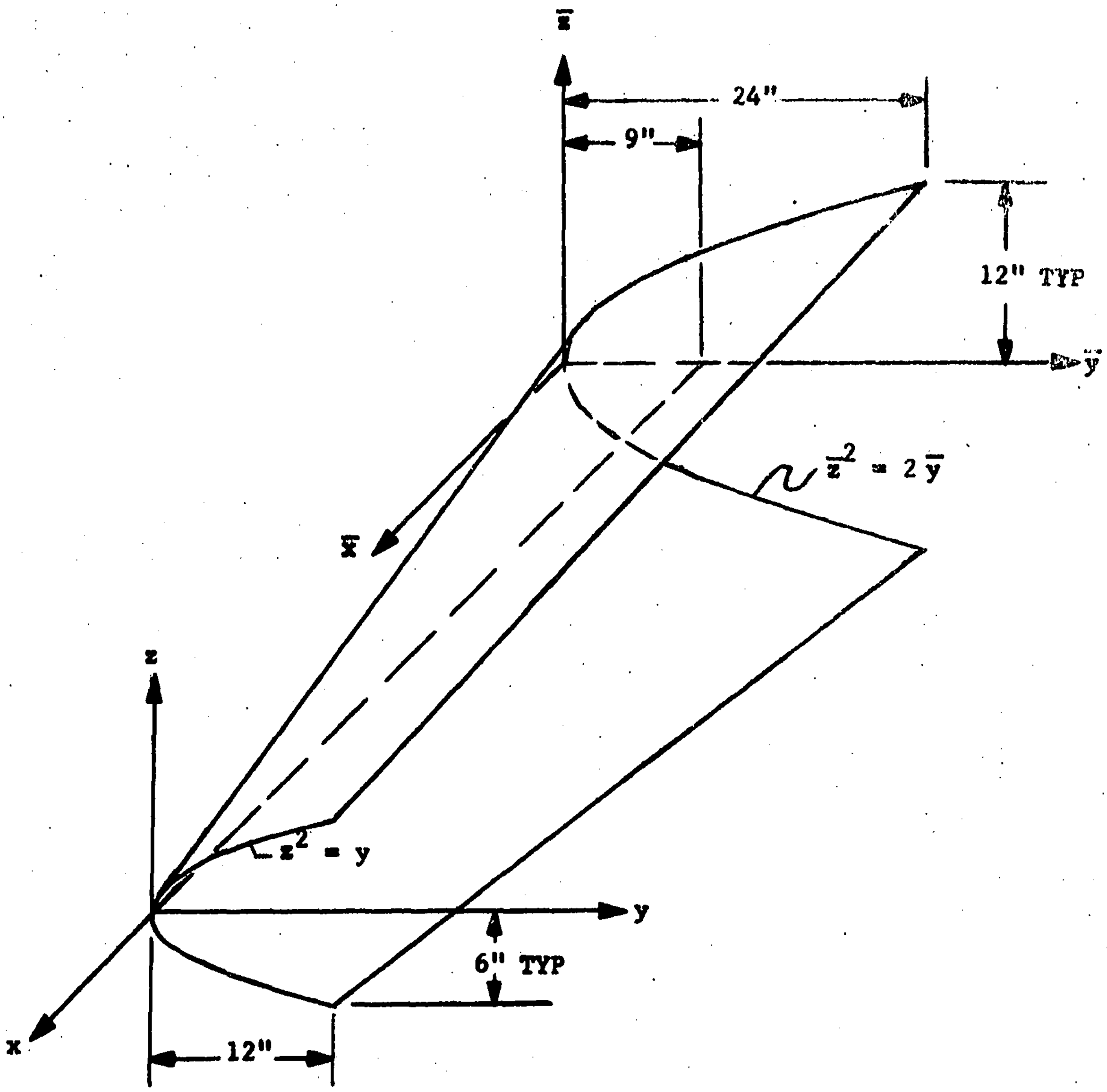
Figure 3. Modal Shapes for the Fuselage Section

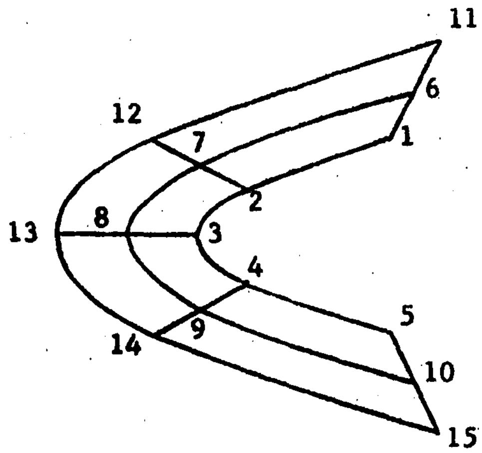
3.4 Multilayer, Anisotropic Conical Shell of Arbitrary Cross Section, Exemplified by a Wing Leading Edge

The theory was next evaluated for a typical aircraft wing leading edge. The shell has a parabolic cross section tapered as shown in Figure 4 over a length of 8 feet. The shell was divided into eight elements requiring fifteen nodes as shown. The same materials and similar edge constraints used in the first example were investigated for this example. However, for the clamped case, all the edge nodes were constrained, including nodes 6 and 10, as this was considered a practical application. In the freely-supported condition, nodes 2, 3, 4, 12, 13, and 14 were constrained in the y- and z-directions ($v = w = 0$); nodes 6 and 10 were constrained in the x- and z-directions ($u = w = 0$); and the corner nodes were constrained in all three directions, x, y, and z ($u = v = w = 0$).

The resulting ten lowest frequencies for the six configurations are presented in Table 4. All the frequencies are presented in Appendix D. All the values appear reasonable and no peculiarities were noted. Of course, the clamped condition would seem to be the most practical, with the others presented for comparison.

Modal displacements are somewhat difficult to plot due to the geometric configuration. Normalized displacements for the lower six frequencies for nodes 3, 8, and 13 in Figure 4 are presented in Figure 5 for the XP-250 leading edge in the free configuration.





End view showing node locations

Figure 4 - Wing Leading Edge Section

TABLE 4.

Ten Lowest Frequencies for Wing Leading Edge Section (hz)

XP-250 Material			Aluminum Material		
Free	Freely-Supported	Clamped	Free	Freely-Supported	Clamped
8.568	90.53	157.6	13.85	151.8	272.7
30.55	96.34	185.0	47.37	158.8	313.7
57.83	174.6	417.6	97.06	290.4	683.1
80.95	175.2	685.6	135.1	295.1	1107.7
91.87	182.1	791.1	150.3	296.1	1275.3
97.30	182.4	811.5	166.4	296.4	1315.4
104.3	186.9	891.1	170.3	302.3	1404.7
163.3	225.2	1152.5	269.3	385.9	1973.4
178.2	246.8	1203.6	291.1	407.2	1982.1
178.6	263.3	1267.4	291.3	446.9	2049.8

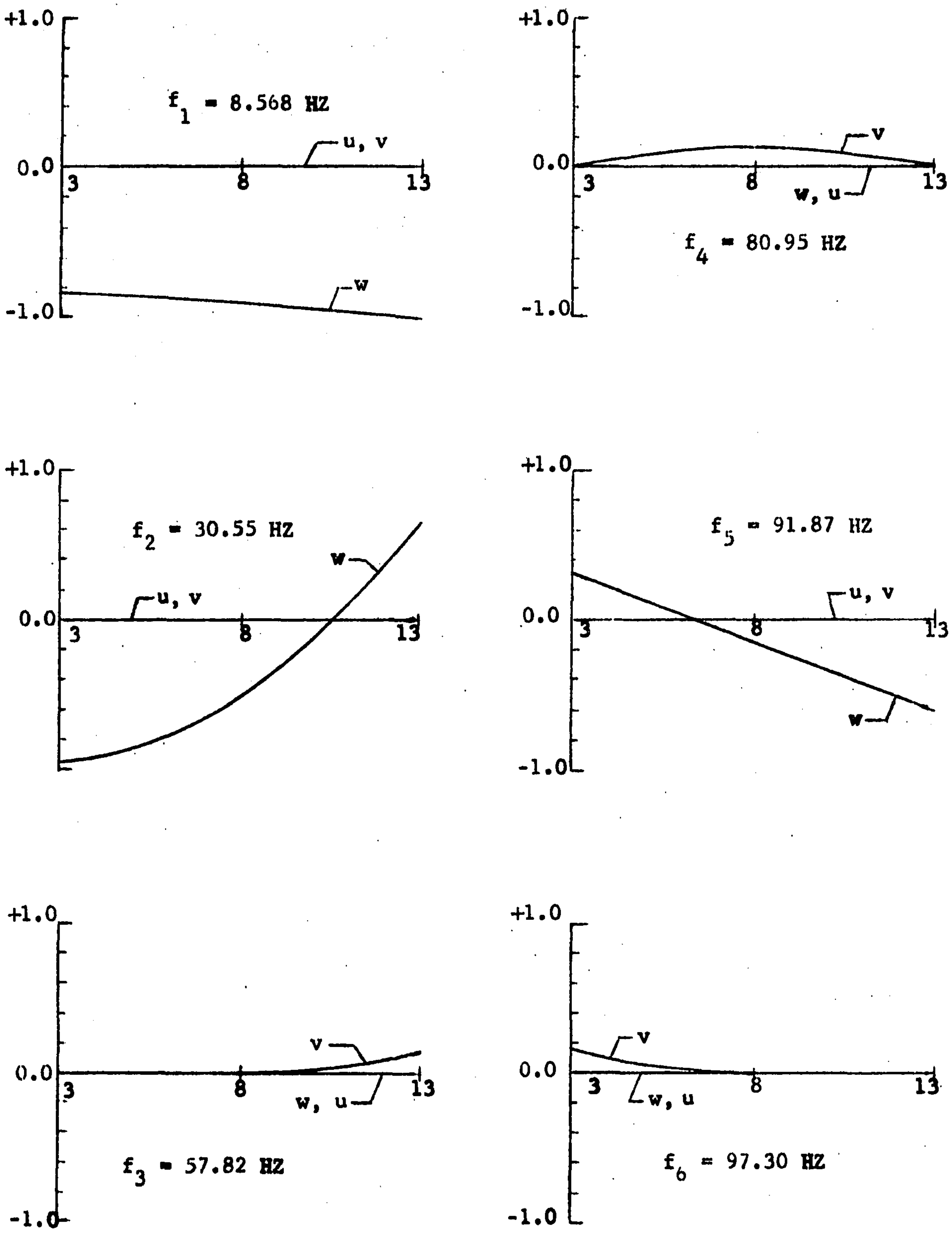


Figure 5. Modal Shapes for the Wing Leading Edge Section

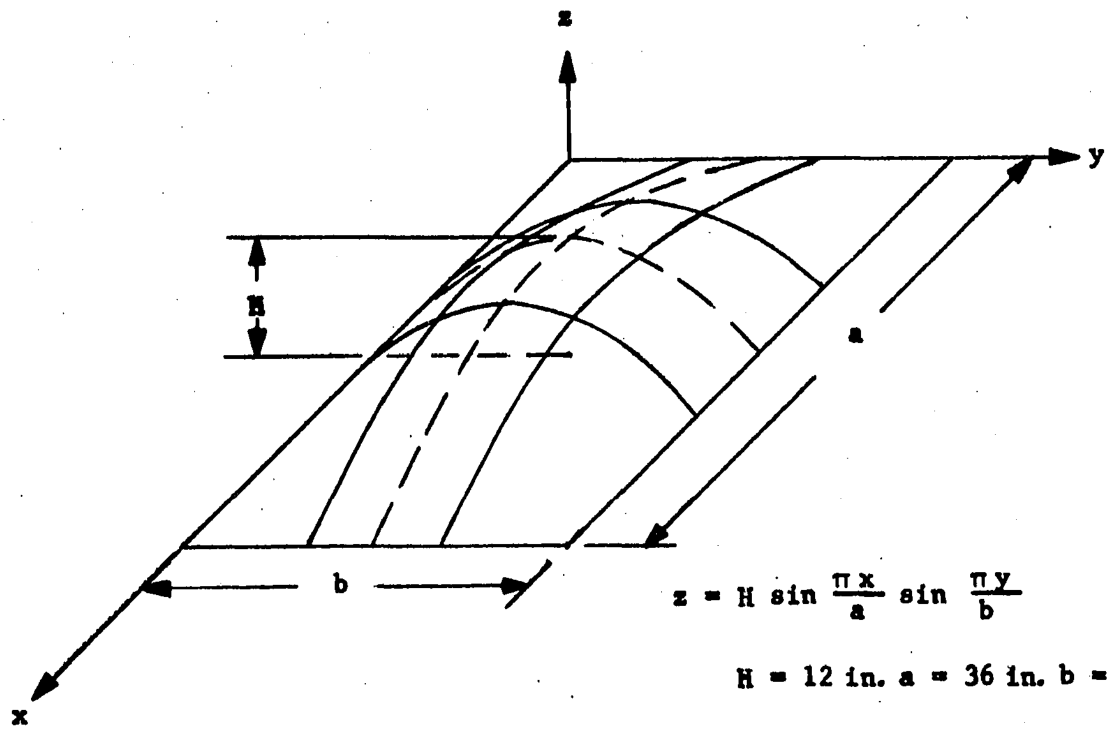
3.5 Multilayer, Anisotropic, Doubly-Curved Shell, Exemplified by a Radome

The third and last application of the theory was for a doubly-curved radome. The shell surface was described by $z = H \sin \frac{\pi x}{a} \sin \frac{\pi y}{b}$. For this example, H = 12 inches, a = 36 inches, b = 24 inches, with x, y, z in inches. As shown in Figure 6, the shell was divided into nine elements requiring all sixteen available nodes. The same materials and similar edge constraints as used in the first two examples were investigated. For the clamped case, all edge nodes were constrained. For the freely-supported case, nodes 2, 3, 14, and 15 were constrained in the x- and z-directions ($u = w = 0$); nodes 5, 8, 9, and 12 were constrained in the y- and z-directions ($v = w = 0$); and the corner nodes, 1, 4, 13, and 16, were constrained in all three directions ($u = v = w = 0$).

It is believed that this example best demonstrates the full capability of the program for the following reasons:

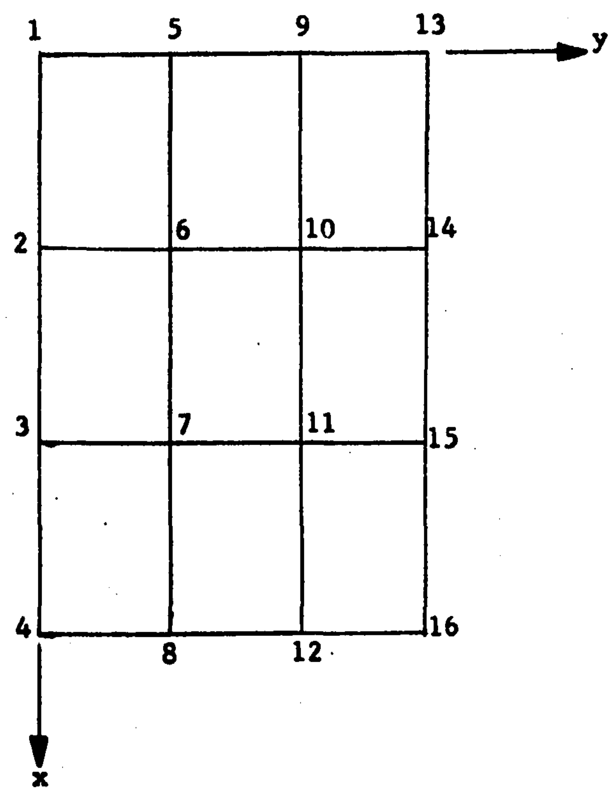
1. The structure could be modeled adequately with the 16 nodes (and 9 elements) available.
2. The shell contained double curvature.
3. Both types of material were investigated.
4. Three varied but reasonable nodal constraint configurations were investigated.

Table 5 contains the resulting ten lowest frequencies for all six cases. Again, all the frequencies are presented in Appendix D. Normalized modal displacements for the lower six frequencies for nodes 5, 6, 7, and 8 in Figure 6 are presented in Figure 7 for the XP-250 radome in the freely-supported configuration. Also, Figure 8 presents the corresponding displacements for nodes 2, 6, 10, and 14.



$$z = H \sin \frac{\pi x}{a} \sin \frac{\pi y}{b}$$

H = 12 in. a = 36 in. b = 24 in.



Plan view showing node locations

Figure 6 - Radome Geometric Configuration

TABLE 5.

Ten Lowest Frequencies for Doubly-Curved Radome Section (hz)

XP-250 Material			Aluminum Material		
Free	Freely-Supported	Clamped	Free	Freely-Supported	Clamped
292.6	380.1	1204.9	464.9	602.5	1963.3
328.4	452.9	1239.3	518.0	713.2	1980.0
347.6	459.8	1239.7	545.5	730.4	2016.7
421.1	1063.5	1561.9	661.2	1660.	2528.0
554.3	1077.2	1662.2	874.9	1717.	2694.1
630.9	1104.7	1679.5	1002.	1779.	2739.8
814.1	1134.0	1747.4	1282.	1802.	2796.7
922.0	1252.2	1773.1	1465.	1958.	2854.2
1106.3	1414.8	1947.8	1738.	2239.	3059.1
1183.0	1415.8	2158.5	1882.	2253.	3423.5

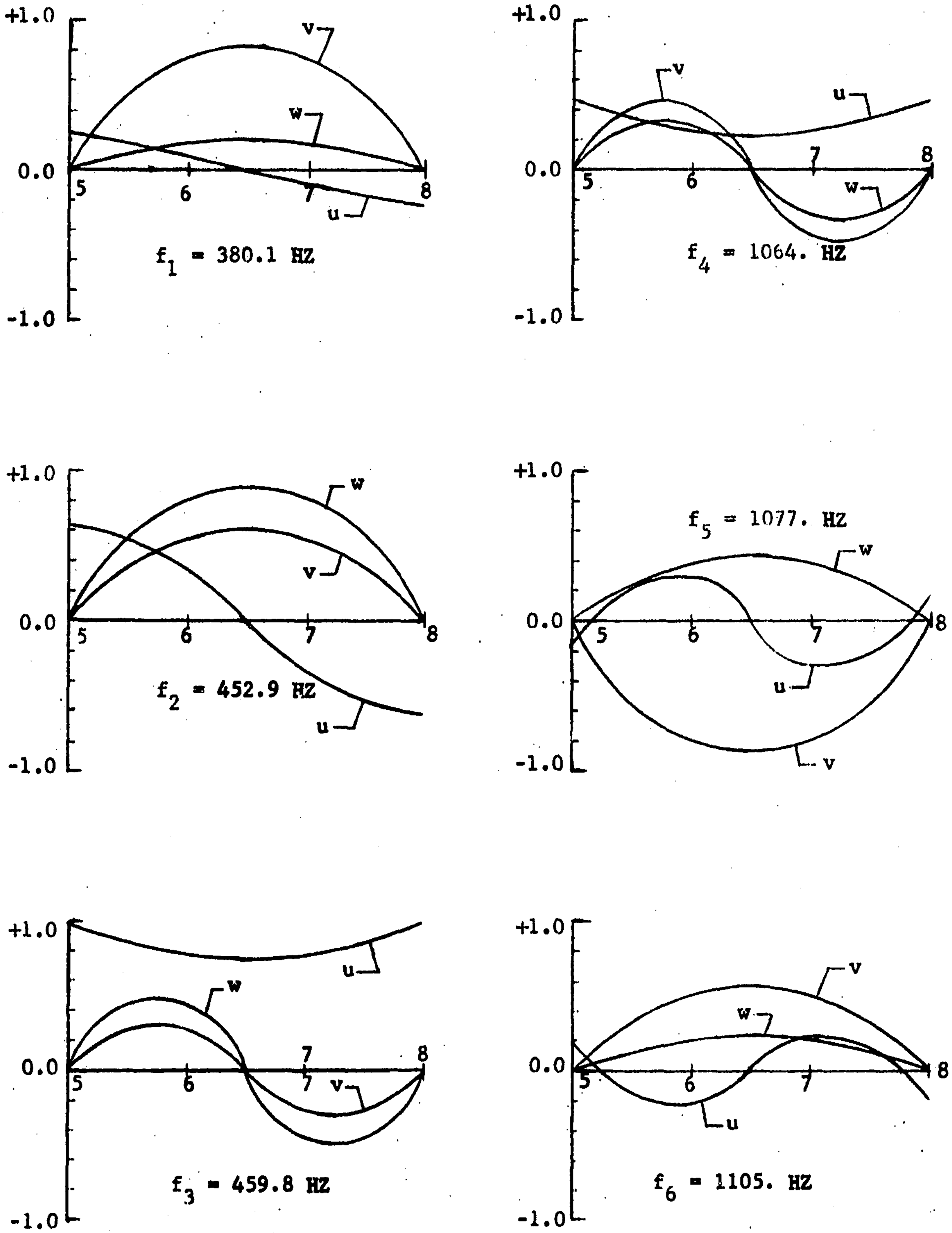


Figure 7. Modal Shapes for the Doubly-Curved Radome Section

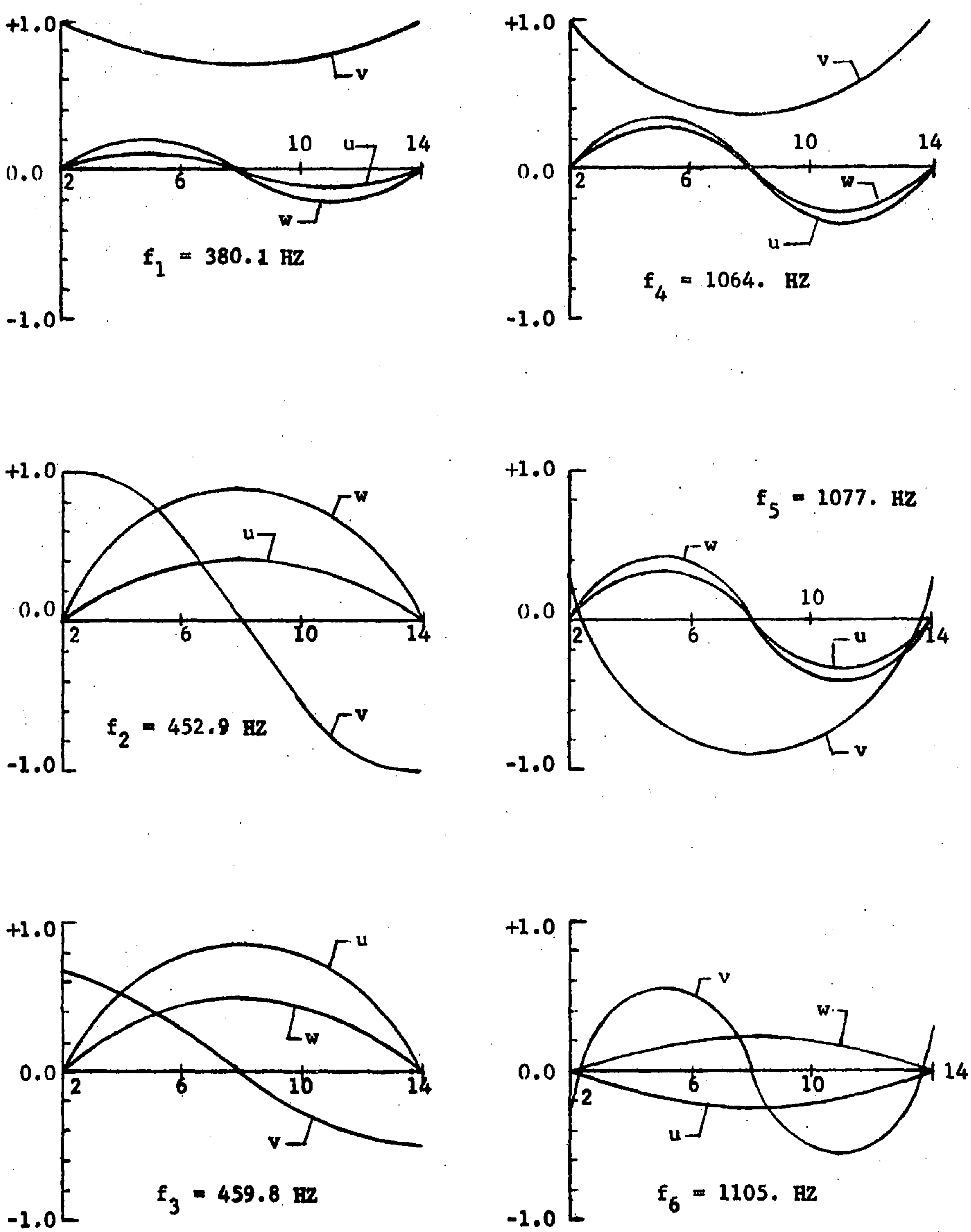


Figure 8. Modal Shapes for the Doubly-Curved Radome

CHAPTER IV

CLOSURE

The evaluation of the present theory for the shallow, curved panel shows poor agreement with the analytical results [34]. However, reasonable agreement was obtained with the experimental results [34]. The validity in the present theory is further supported by the generally good agreement obtained for the two more complicated examples, the 60° spherical cap [2] and the composite cylinder [28]. Comments on the results of the three additional problems investigated are contained in Sections 3.3, 3.4, and 3.5.

When one writes a program of this size, it is often quite difficult to know when it is finished, and this is certainly the case with this program. It is true that many goals have been attained. The program is simple to use; it is fast and accurate; it can investigate a broad range of materials and boundary conditions; and it will investigate general shells with double curvature. These are all substantial advantages and are real credits to the program.

On the minus side, the biggest disadvantage is the size of the program. It is now quite large but needs to be even larger in order to adequately investigate some shell problems. Several less important disadvantages do exist.

First, $\bar{z}_{,xx}$, $\bar{z}_{,yy}$, and $\bar{z}_{,xy}$ are obtained from a subroutine which has to be recompiled for each shell geometry. Therefore, from a hindsight point of view, these values should have been read in directly, as part of the input data. Second, the eigenvalues are determined from the largest to the smallest. Although this has been no problem to date, accuracy problems could be created with the lower values, the ones which are often of prime interest. This should definitely be investigated if the addition of more nodes is considered. Third, the program is limited to full monocoque shells. Seldom in practice is this the case; usually some stiffeners are present. Fourth, the effect of cutouts in the shell have not been considered. If an analysis of full cutouts were attempted, singularities would probably be created in both the stiffness and the mass matrices. To avoid the singularities, possibly a very, very thin "dummy" section could be assumed for the cutout. This would contribute relatively no stiffness or mass effects and may offer a satisfactory method of analysis.

Finally, when completing a program such as this, one often asks what improvements might be made in the future. And indeed, there are several. First, provisions for handling varying properties on a per-layer basis could be programmed. Presently, the same material is assumed to be used throughout the shell. Second, a better quadrature routine for the integration over each element might be used. Third, $\bar{z}_{,xx}$, $\bar{z}_{,yy}$, and $\bar{z}_{,xy}$ for each element could be read in as part of the input data. Fourth, another shell theory, in lieu of shallow shell, might be considered. Fifth, more degrees of freedom at each node could be considered. This would probably be warranted if a more complete shell theory were used. Sixth, provisions for handling stiffeners and cutouts could be made. This would be no simple task and might constitute an entirely new program.

The above-mentioned disadvantages and possible improvements should not be thought of as distracting from the present work. They are offered here as suggestions in hopes of opening the doors to future research.

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APPENDIX A

TABLES OF MATRIX ELEMENTS

A.1 Elements of Matrix [R]

$$R_{11} = [A_{11} (\bar{z}_{,xx})^2/2] + [A_{12} \bar{z}_{,xx} \bar{z}_{,yy}] + [A_{22} (\bar{z}_{,yy})^2/2]$$
$$+ [2 A_{66} (\bar{z}_{,xy})^2] + [2 A_{16} \bar{z}_{,xx} \bar{z}_{,xy}] + [2 A_{26} \bar{z}_{,yy} \bar{z}_{,xy}]$$

$$R_{12} = [B_{11} \bar{z}_{,xx} + B_{12} \bar{z}_{,yy} + 2 B_{16} \bar{z}_{,xy}] / 2$$

$$R_{13} = [B_{12} \bar{z}_{,xx} + B_{22} \bar{z}_{,yy} + 2 B_{26} \bar{z}_{,xy}] / 2$$

$$R_{14} = [B_{16} \bar{z}_{,xx} + B_{26} \bar{z}_{,yy} + 2 B_{66} \bar{z}_{,xy}] / 2$$

$$R_{15} = [-A_{11} \bar{z}_{,xx} - A_{12} \bar{z}_{,yy} - 2 A_{16} \bar{z}_{,xy}] / 2$$

$$R_{16} = [-A_{16} \bar{z}_{,xx} - A_{26} \bar{z}_{,yy} - 2 A_{66} \bar{z}_{,xy}] / 2$$

$$R_{17} = R_{16}$$

$$R_{18} = [-A_{12} \bar{z}_{,xx} - A_{22} \bar{z}_{,xy} - 2 A_{26} \bar{z}_{,xy}] /2$$

$R_{22} = D_{11}/2$ $\quad$ $R_{23} = D_{12}/2$ $\quad$ $R_{24} = D_{16}/2$

$R_{33} = D_{22}/2$ $\quad$ $R_{34} = D_{26}/2$ $\quad$ $R_{44} = D_{66}/2$

A.1 Elements of Matrix [R] (Continued)

$R_{25} = -B_{11}/2$ $\quad$ $R_{26} = R_{27} = R_{45} = -B_{16}/2$

$R_{28} = R_{35} = -B_{12}/2$ $\quad$ $R_{36} = R_{37} = R_{48} = -B_{26}/2$

$R_{38} = -B_{22}/2$ $\quad$ $R_{46} = R_{47} = -B_{66}/2$

$R_{55} = A_{11}/2$ $\quad$ $R_{56} = R_{57} = A_{16}/2$

$R_{58} = A_{12}/2$ $\quad$ $R_{66} = R_{67} = R_{77} = A_{66}/2$

$R_{88} = A_{22}/2$ $\quad$ $R_{68} = R_{78} = A_{26}/2$

[R] is symmetric, i.e., $R_{ij} = R_{ji}$.

A.2 Elements of Matrix [X]

$$
[X] = \left[\begin{array}{cccccccccccc|cccc|cccc}
1 & x & y & x^2 & xy & y^2 & x^3 & x^2y & xy^2 & y^3 & x^3y & xy^3 & & & & & & & & \\
0 & 0 & 0 & 2 & 0 & 0 & 6x & 2y & 0 & 0 & 6xy & 0 & & 0 & & & & 0 & & \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 2x & 6y & 0 & 6xy & & & & & & & & \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 2x & 2y & 0 & 3x^2 & 3y^2 & & & & & & & & \\
\hline
 & & & & & & 0 & & & & & & 0 & 1 & 0 & y & & 0 & & \\
 & & & & & & & & & & & & 0 & 0 & 1 & x & & & & \\
\hline
 & & & & & & 0 & & & & & & & 0 & & & 0 & 1 & 0 & y \\
 & & & & & & & & & & & & & & & & 0 & 0 & 1 & x
\end{array}\right]
$$

A.3 Elements of Matrix [A]

$$[A] = \left[\begin{array}{cccccccccccc|cccc|cccc}
1 & x & y & x^2 & xy & y^2 & x^3 & x^2y & xy^2 & y^3 & x^3y & xy^3 & \multicolumn{4}{c|}{} & \multicolumn{4}{c}{} \\
0 & -1 & 0 & -2x & -y & 0 & -3x^2 & -2xy & -y^2 & 0 & -3x^2y & -y^3 & \multicolumn{4}{c|}{0} & \multicolumn{4}{c}{0} \\
0 & 0 & -1 & 0 & -x & -2y & 0 & -x^2 & -2xy & -3y^2 & -x^3 & -3xy^2 & \multicolumn{4}{c|}{} & \multicolumn{4}{c}{} \\
\hline
 & & & & & & & & & & & & 1 & x & y & xy & \multicolumn{4}{c}{0} \\
\hline
 & & & & & & & & & & & & \multicolumn{4}{c|}{0} & 1 & x & y & xy
\end{array}\right]$$

$$
\left|\begin{array}{cccccccccccc}
1 & \frac{-e_x}{2} & \frac{-e_y}{2} & \frac{e_x^2}{4} & \frac{e_xe_y}{4} & \frac{e_y^2}{4} & \frac{-e_x^3}{8} & \frac{-e_x^2e_y}{8} & \frac{-e_xe_y^2}{8} & \frac{-e_y^3}{8} & \frac{e_x^3e_y}{16} & \frac{e_xe_y^3}{16} \\
0 & -1 & 0 & e_x & \frac{e_x}{2} & 0 & \frac{-3e_x^2}{4} & \frac{-e_xe_y}{2} & \frac{-e_y^2}{4} & 0 & \frac{3e_x^2e_y}{8} & \frac{e_x^3}{8} \\
0 & 0 & -1 & 0 & \frac{e_x}{2} & e_y & 0 & \frac{-e_x^2}{4} & \frac{-e_xe_y}{2} & \frac{-3e_y^2}{4} & \frac{e_x^3}{8} & \frac{3e_xe_y^2}{8} \\
\hline
\multicolumn{12}{c}{0} \\
\hline
\multicolumn{12}{c}{0} \\
\hline
1 & \frac{-e_x}{2} & \frac{e_y}{2} & \frac{e_x^2}{4} & \frac{-e_xe_y}{4} & \frac{e_y^2}{4} & \frac{-e_x^3}{8} & \frac{e_x^2e_y}{8} & \frac{-e_xe_y^2}{8} & \frac{e_y^3}{8} & \frac{-e_x^3e_y}{16} & \frac{-e_xe_y^3}{16} \\
0 & -1 & 0 & e_x & \frac{-e_y}{2} & 0 & \frac{-3e_x^2}{4} & \frac{e_xe_y}{2} & \frac{-e_y^2}{4} & 0 & \frac{-3e_x^2e_y}{8} & \frac{-e_y^3}{8} \\
0 & 0 & -1 & 0 & \frac{e_x}{2} & -e_y & 0 & \frac{-e_x^2}{4} & \frac{e_xe_y}{2} & \frac{-3e_y^2}{4} & \frac{e_x^3}{8} & \frac{3e_xe_y^2}{8} \\
\hline
\multicolumn{12}{c}{0} \\
\hline
\multicolumn{12}{c}{0} \\
\hline
\end{array}\right|
$$

Upper left partition (12 x 10)

A.4 Elements of Matrix [AK] (Continued)

$$
\left[\begin{array}{cccc|cccc}
 & 0 & & & & 0 & & \\
\hline
1 & \frac{-e_x}{2} & \frac{-e_y}{2} & \frac{e_x e_y}{4} & & 0 & & \\
\hline
 & 0 & & & 1 & \frac{-e_x}{2} & \frac{-e_y}{2} & \frac{e_x e_y}{4} \\
\hline
 & 0 & & & & 0 & & \\
\hline
1 & \frac{-e_x}{2} & \frac{e_y}{2} & \frac{-e_x e_y}{4} & & 0 & & \\
\hline
 & 0 & & & 1 & \frac{-e_x}{2} & \frac{e_y}{2} & \frac{-e_x e_y}{4}
\end{array}\right]
$$

Upper right partition (8 x 10)

A.4 Elements of Matrix [AK] (Continued)

$$
\left[\begin{array}{cccccccccccc}
1 & \frac{e_x}{2} & \frac{-e_y}{2} & \frac{e_x^2}{4} & \frac{-e_xe_y}{4} & \frac{e_y^2}{4} & \frac{e_x^3}{8} & \frac{-e_x^2e_y}{8} & \frac{e_xe_y^2}{8} & \frac{-e_y^3}{8} & \frac{-e_x^3e_y}{16} & \frac{-e_xe_y^3}{16} \\
0 & -1 & 0 & -e_x & \frac{e_y}{2} & 0 & \frac{-3e_x^2}{4} & \frac{e_xe_y}{2} & \frac{-e_y^2}{4} & 0 & \frac{3e_x^2e_y}{8} & \frac{e_y^3}{8} \\
0 & 0 & -1 & 0 & \frac{-e_x}{2} & e_y & 0 & \frac{-e_x^2}{4} & \frac{e_xe_y}{2} & \frac{-3e_y^2}{4} & \frac{-e_x^3}{8} & \frac{-3e_xe_y^2}{8} \\
\hline
\multicolumn{12}{c}{0} \\
\hline
\multicolumn{12}{c}{0} \\
\hline
1 & \frac{e_x}{2} & \frac{e_y}{2} & \frac{e_x^2}{4} & \frac{e_xe_y}{4} & \frac{e_y^2}{4} & \frac{e_x^3}{8} & \frac{e_x^2e_y}{8} & \frac{e_xe_y^2}{8} & \frac{e_y^3}{8} & \frac{e_x^3e_y}{16} & \frac{e_xe_y^3}{16} \\
0 & -1 & 0 & -e_x & \frac{-e_x}{2} & 0 & \frac{-3e_x^2}{4} & \frac{-e_xe_y}{2} & \frac{-e_y^2}{4} & 0 & \frac{-3e_x^2e_y}{8} & \frac{-e_y^3}{8} \\
0 & 0 & -1 & 0 & \frac{-e_x}{2} & -e_y & 0 & \frac{-e_x^2}{4} & \frac{-e_xe_y}{2} & \frac{-3e_y^2}{4} & \frac{-e_x^3}{8} & \frac{-3e_xe_y^2}{8} \\
\hline
\multicolumn{12}{c}{0} \\
\hline
\multicolumn{12}{c}{0}
\end{array}\right]
$$

Lower left partition (12 x 10)

A.4 Elements of Matrix [AK] (Continued)

$$
\left[\begin{array}{cccc|cccc}
1 & \frac{e_x}{2} & \frac{-e_y}{2} & \frac{-e_x e_y}{4} & & & 0 & \\
\hline
 & & 0 & & 1 & \frac{e_x}{2} & \frac{-e_y}{2} & \frac{-e_x e_y}{4} \\
\hline
 & & & & & & & \\
 & & 0 & & & & 0 & \\
 & & & & & & & \\
\hline
1 & \frac{e_x}{2} & \frac{e_y}{2} & \frac{e_x e_y}{4} & & & 0 & \\
\hline
 & & 0 & & 1 & \frac{e_x}{2} & \frac{e_y}{2} & \frac{e_x e_y}{4}
\end{array}\right]
$$

Lower right partition (8 x 10)

A.5 Elements of Matrix [Y]

$$[Y] = \begin{bmatrix} 1 & x & y & x^2 & xy & y^2 & x^3 & x^2y & xy^2 & y^3 & x^3y & xy^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & x & y & xy & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & x & y & xy \end{bmatrix}$$

APPENDIX B

BIBLIOGRAPHY OF PAPERS ON ANALYSIS OF THIN SHELLS BY FINITE-ELEMENT METHODS

Since the mid 1950's, many technical papers have been written on general finite-element methods. This appendix is presented as an effort to provide a means of finding the majority of such papers. As stated on Page 5, most of the presentations at the two Air Force Conferences on Matrix Methods in Structural Mechanics held in 1965 and 1968 involved finite-element approaches. The proceedings of these two conferences are listed as 1. and 2. below.

The survey paper of Jones and Strome presented at the 1965 conference mentioned above is listed as 3. These three references form an excellent basis for the early papers on general finite-element methods. The remaining papers listed in the Appendix are additional sources for information on the subject; the majority of these have been published since the survey paper of Jones and Strome. For a list of current books on the subject, see Appendix C.

1. Proceedings of the (First) Conference on Matrix Methods in Structural Mechanics, AFFDL-TR-66-80, Air Force Flight Dynamics Laboratory, Wright-Patterson Air Force Base, Ohio, 1965, AD 646300.

2. Proceedings of the Second Conference on Matrix Methods in Structural Mechanics, AFFDL-TR-68-150, Air Force Flight Dynamics Laboratory, Wright-Patterson Air Force Base, Ohio, 1968, AD-703685.

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APPENDIX C

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APPENDIX D

LISTING OF RESULTING FREQUENCIES

D.1 Fuselage Section Frequencies

XP-250 Material			Aluminum Material		
Free	Freely-Supported	Clamped	Free	Freely-Supported	Clamped
3.192	3.666	3.741	5.104	5.879	5.986
3.193	4.640	4.646	5.242	7.486	7.494
3.424	6.251	6.287	5.470	10.12	10.19
4.444	6.310	6.584	7.151	10.35	10.66
4.978	6.360	7.773	8.069	10.46	12.89
6.065	6.560	7.897	9.619	10.62	13.08
6.142	7.746	9.899	9.945	12.83	15.97
6.245	7.859	11.40	10.11	13.00	18.61
6.542	9.282	18.01	10.59	15.13	29.06
6.785	9.404	18.19	11.01	15.27	29.08
7.545	9.410	18.25	12.49	15.35	29.53
7.546	10.55	26.28	12.51	17.44	42.19
7.830	10.66	26.51	12.96	17.63	42.78
7.957	11.32	26.63	13.19	18.47	42.99
11.09	17.03	26.71	18.08	27.38	43.22
11.59	17.51	26.78	18.94	28.54	43.36
14.01	17.71	28.64	22.87	28.89	45.60
14.62	18.17	28.74	23.76	29.38	45.97
18.08	18.41	34.31	29.26	29.79	55.09
18.26	25.95	34.77	29.59	41.17	56.21
18.62	26.61	36.28	30.11	42.99	59.78
18.76	26.66	36.55	30.31	43.08	60.30
25.12	26.85	39.37	39.80	43.44	65.16
25.95	26.88	39.51	41.16	43.53	65.47
26.10	28.60	206.4	41.66	45.57	319.2
26.49	28.71	338.0	42.71	45.84	547.9
26.59	29.23	425.5	42.97	47.87	671.5
26.76	29.42	448.6	43.33	48.16	704.1
26.77	34.58	461.5	43.38	55.73	713.7
26.94	34.93	496.7	43.63	56.54	796.5
27.04	35.87	559.7	43.67	58.91	899.2
27.08	35.91	666.5	43.83	58.94	1061.
27.13	36.33	699.9	43.85	59.85	1095.
28.75	36.55	712.6	45.99	60.19	1136.
31.08	37.88	755.7	49.96	61.40	1225.
32.52	37.89	814.0	52.04	61.50	1314.
34.53	39.34	827.8	55.69	65.06	1334.
34.84	39.46	973.0	56.45	65.30	1549.
35.01	40.95	1025.	56.72	66.25	1658.
35.11	41.07	1105.	56.84	66.59	1777.
36.24	206.4		59.75	319.2	
36.41	282.2		59.98	436.3	
36.64	323.8		60.33	500.7	
36.66	338.0		60.38	587.9	
39.33	388.0		65.05	601.6	

D.1 Fuselage Section Frequencies (Continued)

XP-250 Material			Aluminum Material		
Free	Freely-Supported	Clamped	Free	Freely-Supported	Clamped
39.35	390.1		65.07	608.7	
39.51	461.5		65.39	713.7	
39.56	569.7		65.46	922.9	
206.4	602.0		319.2	942.0	
255.3	605.7		397.6	947.7	
290.0	612.9		451.9	982.1	
338.0	641.3		547.9	1008.	
378.5	699.9		595.5	1095.	
384.6	755.7		597.9	1225.	
461.5	857.0		713.7	1388.	
474.3	860.2		774.9	1391.	
499.8	973.0		783.3	1549.	
536.2	1025.		872.2	1658.	
555.0	1069.		873.6	1732.	
612.9	1104.		947.7	1777.	
652.7	1105.		1009	1786.	
691.1	1117.		1089.	1803.	
699.9	1232.		1095.	1975.	
729.0			1152.		
748.7			1197.		
755.7			1225.		
765.9			1237.		
884.2			1412.		
896.5			1426.		
973.0			1549.		
1004.			1627.		
1025.			1658.		
1069.			1732.		
1110.			1777.		
1120.			1798.		
1197.			1915.		
1232.			1975.		

D.2 Wing Leading Edge Section Frequencies

XP-250 Material			Aluminum Material		
Free	Freely-Supported	Clamped	Free	Freely-Supported	Clamped
8.568	90.53	157.6	13.85	15.18	272.7
30.55	96.34	185.0	47.37	15.88	313.7
57.83	174.6	417.6	97.06	290.4	683.1
80.95	175.2	685.6	135.1	295.1	1108.
91.87	182.1	791.1	150.3	296.1	1275.
97.30	182.4	811.5	166.4	296.4	1315.
104.3	186.9	891.1	170.3	302.3	1405.
163.3	225.2	1153.	269.3	385.9	1973.
178.2	246.8	1204.	291.1	407.2	1982.
178.6	263.3	1267.	291.3	446.9	2050.
179.8	279.1	1473.	293.2	473.3	2398.
180.0	285.9	1753.	295.0	498.7	2724.
197.1	311.9	3002.	352.6	503.6	4866.
219.1	323.6	3411.	359.8	529.3	5281.
220.0	323.7	5657.1	359.9	529.4	9172.
238.9	325.7		409.4	535.1	
246.9	337.9		437.0	563.3	
263.0	338.2		438.5	564.5	
248.8	406.7		459.6	689.0	
285.3	442.9		476.7	772.1	
300.1	457.7		517.7	791.4	
330.8	673.6		539.7	1043.	
332.0	816.9		543.5	1325.	
340.0	881.2		553.7	1432.	
342.3	919.3		560.5	1435.	
348.6	925.4		600.6	1535.	
364.2	962.4		614.6	1565.	
393.3	1159.		636.3	1851.	
400.7	1162.		725.2	1937.	
440.6	1188.		726.0	1980.	
513.8	1212.		811.0	2004.	
525.1	1251.		884.6	2023.	
678.7	1320.		1105.	2223.	
737.0	1365.		1195.	2242.	
777.2	1439.		1202.	2274.	
816.1	1466.		1322.	2360.	
886.3	1613.		1440.	2696.	
976.4	1662.		1587.	2710.	
1014.	1749.		1601.	2842.	
1045.	1887.		1600.	2941.	
1056.	2071.		1714.	3233.	
1059.	2584.		1784.	4188.	
1089.	2885.		1792.	4466.	
1178.	3495.		1918.	5425.	
1197.	3883.		1981.	6147.	
1224.	4062.		2002.	6435.	

D.2 Wing Leading Edge Section Frequencies (Continued)

XP-250 Material			Aluminum Material		
Free	Freely-Supported	Clamped	Free	Freely-Supported	Clamped
1273.	6152.		2053.	9974.	
1275.			2055.		
1385.			2263.		
1483.			2372.		
1487.			2569.		
1548.			2607.		
1661.			2657.		
1683.			2725.		
1787.			2834.		
1921.			3063.		
1965.			3070.		
2020.			3221.		
2196.			3544.		
2478.			3869.		
2581.			4122.		
2834.			4455.		
3059.			4866.		
3136.			4942.		
3312.			5355.		
3797.			5897.		
3936.			6359.		
4399.			6829.		
4634.			7508.		
5203.			8437.		
6164.			9991.		
7282.			11809.		

D.3 Doubly-Curved Radome Frequencies

XP-250 Material			Aluminum Material		
Free	Freely-Supported	Clamped	Free	Freely-Supported	Clamped
292.6	380.1	1205.	464.9	602.5	1936.
328.4	452.9	1239.	518.0	713.2	1980.
347.6	459.8	1240.	545.5	730.4	2017.
421.1	1064.	1562.	661.2	1660.	2528.
554.3	1077.	1662.	874.9	1717.	2694.
630.9	1105.	1679.	1002.	1779.	2740.
814.1	1134.	1747.	1282.	1802.	2797.
992.0	1252.	1743.	1465.	1958.	2854.
1106.	1415.	1948.	1738.	2239.	3059.
1183.	1416.	2158.	1882.	2253.	3424.
1193.	1535.	2361.	1896.	2429.	3822.
1288.	1576.	29[illegible]	2037.	2576.	4016.
1317.	1582.	[illegible]	[illegible]75.	2585.	4080.
1330.	1596.	[illegible]	[illegible]	2615.	4343.
1415.	1636.	[illegible]	[illegible]	2672.	4393.
1435.	1681.	[illegible]	[illegible]	2739.	4955.
1510.	1692.	[illegible]	[illegible]	2752.	5658.
1558.	1711.	[illegible]	[illegible]	2756.	6724.
1586.	1715.	[illegible]	[illegible]	2767.	8250.
1593.	1746.	[illegible]	[illegible]	2812.	8326.
1617.	1760.	[illegible]	[illegible]	2814.	
1667.	1761.	[illegible]	[illegible]	2815.	
1678.	1769.	[illegible]	[illegible]	2817.	
1679.	1772.	[illegible]	[illegible]	2831.	
1688.	1791.	[illegible]	[illegible]	2835.	
1693.	1792.	[illegible]	[illegible]749.	2842.	
1700.	1801.		2762.	2843.	
1731.	1802.		2766.	2844.	
1736.	1812.		2772.	2845.	
1737.	1813.		2783.	2849.	
1752.	1815.		2814.	2918.	
1768.	1816.		2821.	2923.	
1777.	1817.		2824.	2950.	
1778.	1886.		2831.	2981.	
1794.	1946.		2836.	3140.	
1795.	2026.		2837.	3141.	
1797.	2162.		2839.	3423.	
1804.	2295.		2840.	3649.	
1806.	2574.		2841.	4268.	
1809.	2691.		2844.	4368.	
1810.	2897.		2845.	4484.	
1813.	2991.		2846.	4856.	
1817.	3179.		2847.	5042.	
1818.	3301.		2863.	5796.	
1821.	3720.		2940.	6043.	
1822.	3941.		2958.	6223.	

D.3 Doubly-Curved Radome Frequencies (Continued)

XP-250 Material			Aluminum Mate	
Free	Freely-Supported	Clamped	Free	Freely-Support
1831.	4599.		2973.	7480.
1997.	4833.		3227.	7841.
2092.	4167.		3265.	8292.
2321.	5192.		3613.	8399.
2329.	6674.		3731.	10837.
2338.	6803.		3757.	11007.
2524.			3936.	
2596.			4300.	
2786.			4597.	
2966.			4786.	
2999.			4848.	
3225.			5147.	
3290.			5311.	
3548.			5626.	
3936.			6161.	
4043.			6283.	
4044.			6341.	
4399.			6961.	
4422.			7093.	
4519.			7210.	
4744.			7622.	
4767.			7670.	
4862.			7861.	
5136.			8178.	
5316.			8618.	
5636.			8950.	
5876.			9344.	
6573.			10624.	
6585.			10712.	
6868.			11154.	
7052.			11376.	

D.3 Doubly-Curved Radome Frequencies

XP-250 Material			Aluminum Material		
Free	Freely-Supported	Clamped	Free	Freely-Supported	Clamped
292.6	380.1	1205.	464.9	602.5	1936.
328.4	452.9	1239.	518.0	713.2	1980.
347.6	459.8	1240.	545.5	730.4	2017.
421.1	1064.	1562.	661.2	1660.	2528.
554.3	1077.	1662.	874.9	1717.	2694.
630.9	1105.	1679.	1002.	1779.	2740.
814.1	1134.	1747.	1282.	1802.	2797.
992.0	1252.	1743.	1465.	1958.	2854.
1106.	1415.	1948.	1738.	2239.	3059.
1183.	1416.	2158.	1882.	2253.	3424.
1193.	1535.	2361.	1896.	2429.	3822.
1288.	1576.	2924.	2037.	2576.	4016.
1317.	1582.	2480.	2075.	2585.	4080.
1330.	1596.	2635.	2132.	2615.	4343.
1415.	1636.	2768.	2192.	2672.	4393.
1435.	1681.	3152.	2280.	2739.	4955.
1510.	1692.	3467.	2345.	2752.	5658.
1558.	1711.	4203.	2539.	2756.	6724.
1586.	1715.	5058.	2613.	2767.	8250.
1593.	1746.	5143.	2618.	2812.	8326.
1617.	1760.		2628.	2814.	
1667.	1761.		2696.	2815.	
1678.	1769.		2699.	2817.	
1679.	1772.		2734.	2831.	
1688.	1791.		2740.	2835.	
1693.	1792.		2749.	2842.	
1700.	1801.		2762.	2843.	
1731.	1802.		2766.	2844.	
1736.	1812.		2772.	2845.	
1737.	1813.		2783.	2849.	
1752.	1815.		2814.	2918.	
1768.	1816.		2821.	2923.	
1777.	1817.		2824.	2950.	
1778.	1886.		2831.	2981.	
1794.	1946.		2836.	3140.	
1795.	2026.		2837.	3141.	
1797.	2162.		2839.	3423.	
1804.	2295.		2840.	3649.	
1806.	2574.		2841.	4268.	
1809.	2691.		2844.	4368.	
1810.	2897.		2845.	4484.	
1813.	2991.		2846.	4856.	
1817.	3179.		2847.	5042.	
1818.	3301.		2863.	5796.	
1821.	3720.		2940.	6043.	
1822.	3941.		2958.	6223.	

D.3 Doubly-Curved Radome Frequencies (Continued)

XP-250 Material			Aluminum Material		
Free	Freely-Supported	Clamped	Free	Freely-Supported	Clamped
1831.	4599.		2973.	7480.	
1997.	4833.		3227.	7841.	
2092.	4167.		3265.	8292.	
2321.	5192.		3613.	8399.	
2329.	6674.		3731.	10837.	
2338.	6803.		3757.	11007.	
2524.			3936.		
2596.			4300.		
2786.			4597.		
2966.			4786.		
2999.			4848.		
3225.			5147.		
3290.			5311.		
3548.			5626.		
3936.			6161.		
4043.			6283.		
4044.			6341.		
4399.			6961.		
4422.			7093.		
4519.			7210.		
4744.			7622.		
4767.			7670.		
4862.			7861.		
5136.			8178.		
5316.			8618.		
5636.			8950.		
5876.			9344.		
6573.			10624.		
6585.			10712.		
6868.			11154.		
7052.			11376.		

APPENDIX E

COMPUTER PROGRAM DOCUMENTATION

The program is written in Fortran IV and was run using an IBM System 360, Model 50 (with 230 K bytes of core storage). No scratch tapes are required, as all calculations are accomplished in core with the aid of an overlay structure.

Great care has been taken to completely document the program within the source listing by the use of comment statements. The MAIN part of the program contains a lengthy set of comments. Each subroutine has a similar, but smaller, comment section.

Subroutines INVRSR, NROOT, AND EIGEN were taken from the IBM Scientific Subroutine Package and slightly modified as required for usage in this program. Subroutines PI6J, PI5J, and FOMATI were borrowed from the program library of Beech Aircraft Corporation and modified slightly for use herein. All other subroutines and MAIN were written specifically for this investigation.

APPENDIX F

COMPUTER PROGRAM LISTING

```
C
C
C
C                 COMPUTATION OF VIBRATION MODES AND FREQUENCIES
C                 OF A GENERAL, ORTHOTROPIC-LAMINATED, THIN SHELL
C                            BY THE FINITE ELEMENT METHOD
C               USING A GENERAL QUADRILATERAL, SHALLOW SHELL ELEMENT
C
C                                BY G. L. THOMPSON
C                            DEPARTMENT...ENGINEERING
C             SCHOOL OF AEROSPACE, MECHANICAL AND NUCLEAR ENGINEERING
C                             UNIVERSITY OF OKLAHOMA
C
C
C     ...........................................................................
C
C     PROGRAM DY375
C
C     PURPOSE
C        COMPUTATION OF THE VIBRATION MODES AND FREQUENCIES OF A
C        GENERAL, ORTHOTROPIC-LAMINATED, THIN SHELL BY THE FINITE
C        ELEMENT METHOD USING A GENERAL QUADRILATERAL, SHALLOW
C        SHELL ELEMENT.
C
C     DESCRIPTION OF PARAMETERS
C              H(I,J) - THE DISTANCE FROM THE MID-PLANE OF THE I-TH
C                       ELEMENT TO THE 'OUTER EDGE' OF THE J-TH
C                       LAMINATE.
C          THETA(I,J) - THE ANGLE BETWEEN THE X-AXIS OF THE I-TH
C                       ELEMENT AND THE MAJOR MATERIAL AXIS OF IT'S
C                       J-TH LAMINATE.
C           S(20,20) - THE 20 X 20 ELEMENT STIFFNESS MATRIX, USED AND
C                       REDEFINED FOR EACH ELEMENT.
C          AK(20,20) - THE 20 X 20 INVERSE OF THE MATRIX WHICH
C                       TRANSFORMS DISPLACEMENTS AND ROTATIONS AT THE
C                       NODES OF AN ELEMENT TO COEFFICIENTS OF THE
C                       POLYNOMIAL DISPLACEMENT FUNCTIONS.
C           F(20,20) - THE 20 X 20 ELEMENT MASS MATRIX, USED AND
C                       REDEFINED FOR EACH ELEMENT.
C               XE(4) - THE FOUR X-COORDINATES FOR THE WORKING ELEMENT
C               YE(4) - THE FOUR Y-COORDINATES FOR THE WORKING ELEMENT
C               HE(J) - THE H DISTANCE FOR THE J-TH LAMINATE OF THE
C                       WORKING ELEMENT.
C              THE(J) - THE THETA ANGLE FOR THE J-TH LAMINATE OF THE
C                       WORKING ELEMENT.
C               EK(I) - THE AVERAGE SIZE OF THE I-TH ELEMENT IN THE
C                       X DIRECTION.
C               EJ(I) - THE AVERAGE SIZE OF THE I-TH ELEMENT IN THE
C                       Y DIRECTION.
```

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C                X(L) - THE X COORDINATE OF THE L-TH NODE.
C                Y(L) - THE Y COORDINATE OF THE L-TH NODE.
C             ALAM(I) - THE NUMBER OF LAMINATES IN THE I-TH ELEMENT.
C              HD(18) - THE DUMMY VARIABLE USED TO STORE THE HEADING
C                       CARD.
C             DATE(3) - THE THREE VALUES USED TO STORE THE CURRENT
C                       DATE AS AUTOMATICALLY READ FROM MACHINE TAPE.
C             F1 - F8 - THESE ARE THE 8 FLAG VALUES DESCRIBED IN THE
C                       INPUT INSTRUCTIONS FOR OPTIONAL PRINT-OUT.
C                  ZX - THE SECOND DERIVATIVE OF THE Z-FUNCTION OF THE
C                       SURFACE W.R.T. X, USED TO DEFINE THE ELEMENTAL
C                       CURVATURE.
C                  ZY - SIMILAR TO ZX EXCEPT W.R.T. Y.
C                 ZXY - SIMILAR TO ZX EXCEPT W.R.T. X AND Y.
C                  E1 - MAJOR MODULUS OF ELASTICITY.
C                  E2 - MINOR MODULUS OF ELASTICITY.
C                  G1 - SHEAR MODULUS OF ELASTICITY.
C                  P1 - MAJOR POISSON'S RATIO.
C                  P2 - MINOR POISSON'S RATIO.
C                 EPK - THE EK OF THE WORKING ELEMENT.
C                 EPJ - THE EJ OF THE WORKING ELEMENT.
C                RHOH - DENSITY OF MATERIAL (LB/CU IN)
C               RHOHT - AREA DENSITY OF ELEMENT (LB/SQ IN)
C            IJK(I,M) - THE NUMBER OF THE M-TH NODE OF THE I-TH
C                       ELEMENT.
C             IJKL(M) - THE NUMBER OF THE M-TH NODE OF THE WORKING
C                       ELEMENT.
C              KEC(L) - THE NODAL COSTRAINT CODE NUMBER OF THE L-TH
C                       NODE.
C                  KE - THE NUMBER OF ELEMENTS IN THE PROBLEM.
C                  KN - THE NUMBER OF NODES IN THE PROBLEM.
C               NOLAM - THE NUMBER OF H DISTANCES IN THE WORKING
C                       ELEMENT.
C          AKG(KS,KS) - THE KS X KS GENERAL STIFFNESS MATRIX.
C          AMG(KS,KS) - THE KS X KS GENERAL MASS MASS MATRIX.
C         EIGVEC(KSM) - THE RESULTING EIGENVECTORS IN VECTOR ARRAY,
C                       KSM =KS*KS.
C        EIGVAL(KS,2) - EIGVAL(KS,1) ARE THE RESULTING EIGENVALUES.
C                       EIGVAL(KS,2) ARE THE CONVERTED FREQUENCIES.
C             Q1(KSM) - A SINGLE SUBSCRIPTED DUMMY WORKING VARIABLE.
C             Q2(KSM) - SAME AS Q1(KSM)
C                  KB - THE NUMBER OF LOWER EIGENVALUES (OTHER THAN
C                       ZERO) FOR WHICH THE MODAL DEFLECTIONS, W, U,
C                       AND V, WILL BE DETERMINED AT 25 LOCATIONS
C                       (5X5 GRID) ON EACH ELEMENT.
C               E(KB) - A VECTOR OF THE LOWEST KB FREQUENCIES.
C           EI(KB,KB) - THE CORRESPONDING EIGENVECTORS FOR E(KB).
C
C        REMARKS
```

```
C         THE 'WORKING ELEMENT' EXPRESSION USED IN THE DISCUSSION I
C         THE ELEMENT FOR WHICH THE CALCULATIONS ARE CURRENTLY BEIN
C         ACCOMPLISHED.
C         DOUBLE PRECISION ARITHMATIC IS USED THROUGHOUT FOR
C         INCREASED ACCURACY.
C         TO REDUCE THE SIZE OF THE PROGRAM, AN OVERLAY 'TREE'
C         ROUTINE HAS BEEN USED AND IS DESCRIBED BRIEFLY IN THE NEXT
C         SERIES OF COMMENTS.
C
C
C      SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
C         DIRECTLY CALLED FROM MAINLINE PROGRAM
C             INITL   INTIL   INPUT   SURFU   PRIN2   STIF1   AMASS
C             AJKLM   PI6J    EDCCN   NROOT   VNORM   EDCONI  PI5J
C             MODAL
C         OTHERS CALLED BY THE ABOVE SUBROUTINES.
C             INVRSR  STIFF   AMATR   AMAT    AMATX   AMATM   FOMATI
C             AMATF   EXIT    EIGEN
C
C         DETAILED DESCRIPTIONS ARE CONTAINED IN THE SUBROUTINES.
C         THE FOLLOWING SUBROUTINES CONTAIN THE COMMON STATEMENTS
C             MAIN    STIF1   AMASS   INITL   INPUT
C
C
C         AN OVERLAY TREE WAS FORMED WITH THE SUBROUTINES TO REDUCE
C         THE PROGRAM SIZE.  BESIDES THE STANDARD MAIN 'TRUNK', THE
C         PROGRAM HAS 11 ALPHA BRANCHES WITH ONE OF THEM HAVING 3 BETA
C         BRANCHES.  THE TREE MUST BE ARRANGED IN THE FOLLOWING ORDER.
C            ENTRY MAIN
C               OVERLAY ALPHA
C                  INSERT INPUT
C               OVERLAY ALPHA
C                  INSERT SURFU
C               OVERLAY ALPLA
C                  INSERT PRIN2
C               OVERLAY ALPHA
C                  INSERT STIF1
C                         OVERLAY BETA
C                            INSERT STIFF
C                         OVERLAY BETA
C                            INSERT AMATR
C                         OVERLAY BETA
C                            INSERT AMATX
C               OVERLAY ALPHA
C                  INSERT AMASS,AMATF
C               OVERLAY ALPHA
C                  INSERT AJKLM
C               OVERLAY ALPHA
C                  INSERT EDCON
```

```
C                 OVERLAY ALPHA
C                    INSERT NRCOT,EIGEN
C                 OVERLAY ALPHA
C                    INSERT VNCRM
C                 OVERLAY ALPHA
C                    INSERT EDCONI
C                 CVERLAY ALPHA
C                    INSERT MODAL
C
C           ANY SUBROUTINES NCT SPECIFICALLY LCCATED IN THE OVERLAY TREE
C           ARE AUTOMATICALLY LUMPED INTO THE MAIN 'TRUNK'.
C
C       ******************************************************************
C       ******************************************************************
C       ******************************************************************
C
C                          INPUT DATA FCRMAT INSTRUCTIONS
C       ..................................................................
C
C       TITLE IDENTIFICATION CARD (1 CARC).                   FCRMAT (18A4)
C         COLUMN                        DESCRIPTICN
C
C          01-80  ANY ALPHAMERIC DATA TC SERVE AS A PAGE HEADING TO
C                 IDENTIFY THE ANAYSIS BEING RUN. (CARD MAY BE LEFT BLANK
C                 BUT MUST BE PRESENT)
C       ..................................................................
C
C       GENERAL PARAMETER CARD (1 CARD).                      FORMAT (20I4)
C         COLUMN                        DESCRIPTICN
C
C          01-04  THE NUMBER OF ELEMENTS IN THE ANALYSIS.  (9 MAXIMUM)
C          05-08  THE NUMBER CF NODES IN THE ANALYSIS. (16 MAXIMUM)
C          09-12  THE NUMBER CF LOWER EIGENVALUES (CTHER THAN ZERO) FOR
C                 WHICH THE MODAL DEFLECTICNS, W, U, AND V, WILL BE
C                 DETERMINED AT 25 LOCATICNS (5X5 GRID) ON EACH ELEMENT.
C                 ALL NUMBERS MUST BE 'RIGHT JUSTIFIED' AND NO DECIMALS
C                 ARE TO BE INCLUDED.
C       ..................................................................
C
C       MATERIAL DATA CARD (1 CARD).                          FORMAT (6F9.C)
C         COLUMN                        DESCRIPTICN
C
C          01-09  MAJOR MODULUS OF ELASTICITY (PSI)
C          10-18  MINOR MODULUS OF ELASTICITY (PSI)
C          19-27  SHEAR MODULUS OF ELASTICITY (PSI)
C          28-36  MAJOR POISSON'S RATIC (NONDIMENSIONAL)
C          37-45  MINOR POISSON'S RATIC (NONDIMENSIONAL)
C          46-54  DENSITY OF MATERIAL (LB/CU-IN)
C       ..................................................................
```

```
C
C     FLAG PRINT-OUT CARD (1 CARD).                          FORMAT (8F9.0)
C       COLUMN                           DESCRIPTION
C
C         01-09  FLAG 1: IF EQUAL TO 1, ELEMENT CURVATURES WILL BE
C                        PRINTED.  IF NOT DESIRED, LEAVE BLANK.
C
C         10-18  FLAG 2: IF EQUAL TO +1, THE 3X3 A,B, AND D SUB-MATRICES
C                        OF THE CONSTITUTIVE RELATIONSHIP FOR ELEMENT 1
C                        ONLY ARE PRINTED.  IF EQUAL TO -1, THESE
C                        MATRICES FOR ALL ELEMENTS ARE PRINTED.  IF NONE
C                        DESIRED, LEAVE BLANK.
C
C         19-27  FLAG 3: IF EQUAL TO +1, THE 8X8 R MATRIX DEFINED IN THE
C                        THEORY WILL BE PRINTED FOR ELEMENT 1 ONLY.  IF
C                        EQUAL TO -1, THIS MATRIX FOR EACH ELEMENT WILL
C                        BE PRINTED.  IF NOT DESIRED, LEAVE BLANK.
C
C         28-36  FLAG 4: IF EQUAL TO +1, THE 20X20 INVERSE OF THE AK
C                        MATRIX AS DEFINED IN THE THEORY WILL BE PRINTED
C                        FOR ELEMENT 1 ONLY.  IF EQUAL TO -1, THIS
C                        MATRIX FOR EACH ELEMENT WILL BE PRINTED.  IF NOT
C                        DESIRED, LEAVE BLANK.
C
C         37-45  FLAG 5: IF EQUAL TO +1, THE FINAL STIFFNESS MATRIX WILL
C                        BE PRINTED FOR ELEMENT 1 ONLY.  IF EQUAL TO -1,
C                        THIS MATRIX FOR EACH ELEMENT WILL BE PRINTED.
C                        IF NOT DESIRED, LEAVE BLANK.
C
C         46-54  FLAG 6: IF EQUAL TO +1, THE FINAL MASS MATRIX WILL BE
C                        PRINTED FOR ELEMENT 1 ONLY.  IF EQUAL TO -1,
C                        THIS MATRIX FOR EACH ELEMENT WILL BE PRINTED.
C                        IF NOT DESIRED, LEAVE BLANK.
C
C         55-63  FLAG 7: IF EQUAL TO +1, THE FINAL GENERAL STIFFNESS AND
C                        MASS MATRICES WILL BE PRINTED, BOTH BEFORE AND
C                        AFTER IMPOSING NODAL CONSTRAINTS.  IF NOT
C                        DESIRED, LEAVE BLANK.
C
C         64-72  FLAG 8: IF EQUAL TO +1, THE EIGENVALUES FOR THE
C                        GENERAL MASS MATRIX WILL BE PRINTED.
C
C                NOTE: IF ALL FLAGS ARE BLANK(ZERO), ONLY THE INPUT DATA,
C                      THE EIGENVALUES, THE FREQUENCIES, AND THE
C                      NORMALIZED EIGENVECTORS(MODE SHAPES) WILL BE
C                      PRINTED.
C     ...................................................................
C
C     NODAL X-COORDINATE CARD(S) (2 CARDS MAXIMUM).          FORMAT (8F9.0)
```

```
C         COLUMN                          DESCRIPTION
C
C          01-09  THE X-COORDINATE OF NODE NO. 1
C          10-18  ETC. UP TO THE NUMBER OF NODES.
C
C                 NOTE° ONLY 8 VALUES PER CARD MAXIMUM.
C       ..................................................................
C
C       NODAL Y-COORDINATE CARD(S) (2 CARDS MAXIMUM).          FORMAT (8F9.0)
C         COLUMN                          DESCRIPTION
C
C          01-09  THE Y-COORDINATE OF NODE NO. 1
C          10-18  ETC. UP TO THE NUMBER OF NODES
C
C                 NOTE° ONLY 8 VALUES PER CARD MAXIMUM.
C       ..................................................................
C
C       NODAL CONSTRAINT CARD(S) (2 CARDS MAXIMUM).             FORMAT (20I4)
C         COLUMN                          DESCRIPTION
C
C          01-04  THE NODAL CONSTRAINT VALUE FOR NODE 1
C          05-08  ETC. UP TO THE NUMBER OF NODES
C
C                 NOTE° 20 VALUES PER CARD MAXIMUM.
C                       ALL VALUES MUST BE 'RIGHT JUSTIFIED' WITH NO
C                       DECIMAL POINTS ALLOWED.
C                       32 POSSIBILITIES EXIST - SEE NEXT SECTION OF
C                       COMMENTS FOR INSTRUCTIONS.
C       ..................................................................
C
C       ELEMENT-NODAL CARD(S) (2 CARDS MAXIMUM).                FORMAT (20I4)
C         COLUMN                          DESCRIPTION
C
C          01-04  NUMBER OF NODE IN THE THIRD QUADRANT OF ELEMENT 1
C          05-08  NUMBER OF NODE IN THE SECOND QUADRANT OF ELEMENT 1
C          09-12  NUMBER OF NODE IN THE FOURTH QUADRANT OF ELEMENT 1
C          13-16  NUMBER OF NODE IN THE FIRST QUADRANT OF ELEMENT 1
C          17-20  KEY TO DETERMINE IF THIS ELEMENT IS THE SAME AS THE
C                 PREVIOUS ELEMENT.  IF ZERO, THE ELEMENT IS DIFFERENT
C                 AND NEW ELEMENT STIFFNESS AND MASS MATRICES ARE
C                 DETERMINED.
C          21-24  ETC. FOR ELEMENTS IN CONSECUTIVE ORDER.
C
C                 NOTE° 20 VALUES PER CARD MAXIMUM.
C                       ALL VALUES MUST BE 'RIGHT JUSTIFIED' WITH NO
C                       DECIMAL POINTS ALLOWED.
C                       NODES MUST BE PUT IN THE QUADRANT ORDER SPECIFIED
C                       FOR ELEMENT 1 ABOVE.
C       ..................................................................
```

```
C
C      AVERAGE SIZE OF ELEMENT IN X-DIRECTION CARD(S)       FORMAT (8F9.0)
C        COLUMN                          DESCRIPTION
C
C         01-09  AVERAGE SIZE OF ELEMENT 1 IN THE X DIRECTION
C         10-18  ETC. UP TO THE NUMBER OF ELEMENTS
C
C                NOTE  ONLY 8 VALUES PER CARD  2 CARDS MAXIMUM
C      ...................................................................
C
C      AVERAGE SIZE OF ELEMENT IN Y-DIRECTION CARD(S)       FORMAT (8F9.0)
C        COLUMN                          DESCRIPTION
C
C         01-09  AVERAGE SIZE OF ELEMENT 1 IN THE Y DIRECTION
C         10-18  ETC. UP TO THE NUMBER OF ELEMENTS
C
C                NOTE  ONLY 8 VALUES PER CARD  2 CARDS MAXIMUM
C      ...................................................................
C
C      NUMBER OF LAMINATES IN EACH ELEMENT CARD(S).         FORMAT (8F9.0)
C        COLUMN                          DESCRIPTION
C
C         01-09  THE NUMBER OF LAMINATES IN ELEMENT 1
C         10-18  ETC. UP TO THE NUMBER OF ELEMENTS.
C
C                NOTE° 8 VALUES PER CARD; 2 CARDS MAXIMUM.
C                      MAXIMUM NUMBER OF LAMINATES IS 10.
C      ...................................................................
C
C      THE LAMINATE THICKNESS CARD(S) FOR ELEMENT 1.        FORMAT (8F9.0)
C        COLUMN                          DESCRIPTION
C
C         01-09  DISTANCE FROM MID-PLANE OF ELEMENT TO THE OUTSIDE EDGE
C                OF UPPER-MOST LAMINATE.
C         10-18  DISTANCE FROM MID-PLANE OF ELEMENT TO THE OUTSIDE EDGE
C                OF THE NEXT LAMINATE.
C         19-27  REPEAT FOR EACH LAMINATE UNTIL THE LAST VALUE IS THE
C                DISTANCE FROM THE MID-PLANE OF ELEMENT TO THE OUTSIDE
C                EDGE OF LOWER-MOST LAMINATE.
C
C                NOTE° THERE WILL ALWAYS BE ONE MORE OF THESE VALUES THAN
C                      THERE ARE LAMINATES.  IF A ONE LAMINATE ELEMENT IS
C                      USED (SUCH AS FOR ISOTROPIC METALS) OF THICKNESS
C                      T, THE 2 VALUES USED WILL BE +T/2 AND -T/2 IN THAT
C                      ORDER.  MAXIMUM NUMBER OF VALUES IS 11 WITH A
C                      MAXIMUM OF 2 CARDS.
C      ...................................................................
C
C      THE LAMINATE ANGLE CARD(S) FOR ELEMENT 1.            FORMAT (8F9.0)
```

```
C       COLUMN                         DESCRIPTION
C
C        01-09  THE ANGLE BETWEEN THE MAJOR MATERIAL AXIS OF THE UPPER-
C               MOST LAMINATE AND THE LOCAL X-AXIS OF THE ELEMENT
C        10-18  THE SIMILAR ANGLE FOR THE NEXT LAMINATE.
C        19-27  REPEAT FOR EACH LAMINATE IN THE ELEMENT.
C
C               NOTE: FOR ISOTROPIC MATERIALS ENTER A SINGLE VALUE OF 0.
C                     MAXIMUM NUMBER OF VALUES IS 10 WITH A MAXIMUM OF
C                     2 CARDS.
C     ..................................................................
C
C     REPEAT LAST TWO DATA SETS FOR EACH REMAINING ELEMENT.
C     ..................................................................
C
C     ADDITIONAL PROBLEM DATA DECKS MAY BE STACKED.
C     ..................................................................
C
C
C            INPUT CODING INSTRUCTIONS FOR THE NODAL CONSTRAINTS
C
C     DISCUSSION:
C          THERE ARE THREE DISPLACEMENTS AND TWO ROTATIONS POSSIBLE AT
C       EACH NODE; THESE ARE W, DW/DX, DW/DY, U AND V.  TO FIX, I.E. TO
C       SET EQUAL TO ZERO, ANY ONE OR COMBINATION OF THESE, ENTER THE
C       APPROPRIATE ICODE NUMBER SELECTED FROM BELOW PER THE INPUT
C       INSTRUCTIONS.
C
C        ICODE      W      DW/DX    DW/DY      U        V
C
C            1
C            2      0
C            3             0
C            4                      0
C            5                                 0
C            6                                          0
C            7      0      0
C            8      0               0
C            9      0                          0
C           10      0                                   0
C           11             0        0
C           12             0                   0
C           13             0                            0
C           14                      0          0
C           15                      0                   0
C           16                                 0        0
C           17      0      0        0
C           18      0      0                   0
C           19      0      0                            0
```

```
      IF(F1)54,54,53
53    CALL PRIN2(I,ZX,ZY,ZXY)
54    NOLAM = ALAM(I)
      DO 71 M=1,4
71    IJKL(M)=IJK(I,M)
C      THE I ELEMENT STIFFNESS MATRIX IS FORMED IN SUBROUTINE STIF1
      IF(IJK(I,5))73,72,73
72    CALL STIF1(I)
C      THE I ELEMENT MASS MATRIX IS FORMED IN SUBROUTINE AMASS
      CALL AMASS(I)
C      THE I ELEMENT STIFFNESS AND MASS MATRICES WILL NOW BE 'PLACED'
C      IN THE GENERAL SHELL STIFFNESS AND MASS MATRICES BY SUBROUTINE
C       AJKLM.
      DO 721 J=1,20
      DO 721 K=1,20
      IF(DABS(S(K,J))-(1.0D-5))11,11,721
11    S(K,J)=0.D0
721   CONTINUE
      DO 722 J=1,20
      DO 722 K=1,20
      IF(DABS(F(K,J))-(1.0D-12))12,12,722
12    F(K,J)=0.D0
722   CONTINUE
73    CALL AJKLM(AKG,IJKL,S)
      CALL AJKLM(AMG,IJKL,F)
C     AKG IS THE GENERAL SHELL STIFFNESS MATRIX.
C     AMG IS THE GENERAL SHELL MASS MATRIX
51    CONTINUE
      KS=KN*5
      KM=KS+1
      IF(F7)6,6,5
5     CALL PI6J(AMG,KS,KS,80,80,HD,DATE,KM,16,1)
      CALL PI6J(AKG,KS,KS,80,80,HD,DATE,KM,17,1)
C     KS = ORDER OF GENERAL SHELL MASS AND STIFFNESS MATRICES AND IS
C          REDUCED ACCORDING TO THE IMPOSED NODAL EDGE CONSTRAINTS.
C     THE NECESSARY ROWS AND COLUMNS WILL NOW BE DELETED FROM THE
C     GENERAL SHELL MASS AND STIFFNESS MATRICES IN COMPLIANCE WITH THE
C     IMPOSED NODAL EDGE CONSTRAINTS BY SUBROUTINE EDCON.
6     CALL EDCON(AKG,KN,KEC,KS)
      KS=KN*5
      CALL EDCON(AMG,KN,KEC,KS)
      KM=KS+1
      IF(F7)8,8,7
7     CALL PI6J(AMG,KS,KS,80,80,HD,DATE,KM,18,1)
      CALL PI6J(AKG,KS,KS,80,80,HD,DATE,KM,19,1)
C      AKG AND AMG WILL NOW BE CONVERTED TO SINGLE SUBSCRIPTED ARRAYS
8     L=0
      DO 1 J=1,KS
      DO 1 K=1,KS
```

```
      L=L+1
1     Q1(L)=AKG(K,J)
      L=0
      DO 2 J=1,KS
      DO 2 K=1,KS
      L=L+1
2     Q2(L)=AMG(K,J)
      DO 3 J=1,6400
3     EIGVEC(J)=0.D0
C      THE EIGENVALUES AND EIGENVECTORS WILL NOW BE DETERMINED BY
C      SUBROUTINE NROOT
      CALL NROOT(KS,Q1,Q2,EIGVAL,EIGVEC,F8)
C      THE FREQUENCIES IN HERTZ WILL NOW BE COMPUTED
      DO 9 J=1,KS
      IF(EIGVAL(J,1))90,91,91
90    EIGVAL(J,1)=-EIGVAL(J,1)
91    EIGVAL(J,2)=(DSQRT(EIGVAL(J,1)))/(PI*2.D0)
9     CONTINUE
C      THE EIGENVECTORS IN THE SINGLE SUBSCRIPTED ARRAY EIGVEC WILL BE
C      CONVERTED TO THE DOUBLE SUBSCRIPTED ARRAY AKG TO FACILITATE
C      NORMALIZATION AND RE-IMPOSING THE NODAL CONSTRAINTS
      L=0
      DO 10 J=1,KS
      DO 10 K=1,KS
      L=L+1
10    AKG(K,J)=EIGVEC(L)
C      THE EIGENVECTORS WILL BE NORMALIZED BY SUBROUTINE VNORM
      CALL VNORM(AKG,KS)
C      ZEROS WILL BE INSERTED IN EIGENVECTORS FOR APPROPRIATE NODAL
C      CONSTRAINTS BY SUBROUTINE EDCONI
      KA=KS
      CALL EDCONI(AKG,KN,KEC,KS)
C      THE EIGENVALUES, FREQUENCIES, AND EIGENVECTORS WILL NOW BE
C      PRINTED BY SUBROUTINES PI5J AND PI6J.
      CALL PI5J(EIGVAL,KA,2,80,2,HD,DATE,KM,2)
      CALL PI6J(AKG,KS,KA,80,80,HD,DATE,KM,3,1)
C      THE LOWEST KB FREQUENCIES AND EIGENVECTORS (OTHER THAN ZERO)
C      WILL BE SET ASIDE.
      IF(KB)1970,101,1970
1970  K=1
      KA=KS
40    IF(K-KB)41,41,100
41    E(K)=EIGVAL(KA,2)
      DO 44 J=1,KS
44    EI(J,K)=AKG(J,KA)
      K=K+1
      KA=KA-1
      GO TO 40
C      THE MODE SHAPES FOR THE LOWEST KB FREQUENCIES WILL BE
```

```
C       DETERMINED BY SUBROUTINE MODAL.
100   CALL MODAL(E,EI,KS,KE,EK,EJ,HD,DATE,IJK,KB)
101   GO TO 99
      END
```

```
C
C     .................................................................
C
C        SUBROUTINE INITL
C
C        PURPOSE
C           INITIALIZE (SET EQUAL TO ZERO) ALL INPUT VALUES PRIOR TO
C           READING IN DATA
C
C        USAGE
C           CALL INITL
C
C        DESCRIPTION OF PARAMETERS
C                     - IDENTIFIED IN MAIN PROGRAM COMMENTS
C
C        REMARKS
C           THIS SUBROUTINE SETS ALL THE PERTINENT INPUT VALUES EQUAL TO
C           ZERO TO ASSURE THAT IF THEY ARE NOT DEFINED, ERRONEOUS
C           EXTRANEOUS VALUES WILL NOT BE ASSUMED AND USED BY THE
C           PROGRAM.  INITL IS CALLED BY MAIN; DO NOT CONFUSE WITH INTIL
C
C        SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
C           NONE
C
C     .................................................................
C
      SUBROUTINE INITL
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON H(16,11),THETA(16,11),S(20,20),AK(20,20),F(20,20)
      COMMON XE(4),YE(4),HE(11),THE(11)
      COMMON EK(16),EJ(16),X(25),Y(25),ALAM(16),HD(18),DATE(3)
      COMMON       F1,F2,F3,F4,F5,F6,F7,F8,ZX,ZY,ZXY,E1,E2,G1,P1,P2
      COMMON EPK,EPJ,HET,RHOH,RHOHT
      COMMON IJK(16,5),IJKL(4),KEC(25)
      COMMON KE,KN,NOLAM,KB
      DO 1 I=1,16
      EK(I)=0.D0
      EJ(I)=0.D0
1     ALAM(I)=0.D0
      DO 2 I=1,25
      X(I)=0.D0
2     Y(I)=0.D0
      DO 3 I=1,16
      DO 3 J=1,11
      H(I,J)=0.D0
3     THETA(I,J)=0.D0
      E1=0.D0
      E2=0.D0
      G1=0.D0
```

```
P1=0.D0
P2=0.D0
F1=0.D0
F2=0.D0
F3=0.D0
F4=0.D0
F5=0.D0
F6=0.D0
F7=0.D0
F8=0.D0
RHOH=0.D0
RETURN
END
```

```
C
C       ............................................................
C
C          SUBROUTINE INTIL
C
C          PURPOSE
C             INITIALIZE (OR ZERO) A MATRIX
C
C          USAGE
C             CALL INTIL(AR,N,M)
C
C          DESCRIPTION OF PARAMETERS
C                       AR - GENERAL MATRIX TO BE INITIALIZED
C                        N - NUMBER OF ROWS IN MATRIX
C                        M - NUMBER OF COLUMNS IN MATRIX
C
C          REMARKS
C             THIS SUBROUTINE IS CALLED THROUGHOUT THE PROGRAM
C
C          SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
C             NONE
C
C       ............................................................
C
      SUBROUTINE INTIL(AR,N,M)
      DOUBLE PRECISION AR(N,M)
      DO 1 J=1,N
      DO 1 K=1,M
1     AR(J,K)=0.D0
      RETURN
      END
```

```
C
C     ...............................................................
C
C        SUBROUTINE AMAT
C
C        PURPOSE
C           FORM THE 20 X 20 INVERSE OF THE MATRIX WHICH TRANSFORMS
C           DISPLACEMENTS AND ROTATIONS AT THE NODES OF AN ELEMENT TO
C           COEFFICIENTS OF THE POLYNOMIAL DISPLACEMENT FUNCTIONS.
C
C        USAGE
C           CALL AMAT(AK,EPK,EPJ)
C
C        DESCRIPTION OF PARAMETERS
C                  AK - IDENTIFIED IN MAIN PROGRAM COMMENTS
C                 EPK - IDENTIFIED IN MAIN PROGRAM COMMENTS
C                 EPJ - IDENTIFIED IN MAIN PROGRAM COMMENTS
C
C        REMARKS
C           THE AK MATRIX IS FIRST FORMED PER THEORY.  THE INVERSE IS
C           THEN FORMED AND IS DEFINED AS AK WHICH DESTROYS THE ORIGINAL
C           MATRIX.  AMAT IS CALLED BY STIF1 AND MODAL.
C
C        SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
C           INTIL, INVRSR
C
C        METHOD
C           THE FIRST UPPER FOURTH PARTITION IS FORMED.  THE OTHER THREE
C           PARTITIONS ARE EQUATED TO THE FIRST(THE MAGNITUDES ARE THE
C           SAME); THEN THE SIGN DIFFERENCES ARE ASSIGNED.  FINALLY, THE
C           MATRIX IS CONVERTED TO VECTOR ARRAY; THE INVERSE FOUND; AND
C           THE RESULT RE-CONVERTED BACK TO MATRIX AK.
C
C     ...............................................................
C
      SUBROUTINE AMAT(AK,EPK,EPJ)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION AK(20,20),Q(400)
      DIMENSION HD(18),DATE(3)
      CALL INTIL(AK,20,20)
C     DEFINE W AND W'' TERMS FOR FIRST PARTITION
      AK(1,1)=1.D0
      AK(1,2)=-EPK*.5D0
      AK(1,3)=-EPJ*.5D0
      AK(1,4)=EPK*EPK*.25D0
      AK(1,5)=EPK*EPJ*.25D0
      AK(1,6)=EPJ*EPJ*.25D0
      AK(1,7)=-EPK*EPK*EPK*.125D0
      AK(1,8)=-EPK*EPK*EPJ*.125D0
```

```
*****************************************************************

   SUBROUTINE INTIL

   PURPOSE
      INITIALIZE (OR ZERO) A MATRIX

   USAGE
      CALL INTIL(AR,N,M)

   DESCRIPTION OF PARAMETERS
                AR - GENERAL MATRIX TO BE INITIALIZED
                 N - NUMBER OF ROWS IN MATRIX
                 M - NUMBER OF COLUMNS IN MATRIX

   REMARKS
      THIS SUBROUTINE IS CALLED THROUGHOUT THE PROGRAM

   SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
      NONE

*****************************************************************

SUBROUTINE INTIL(AR,N,M)
DOUBLE PRECISION AR(N,M)
DO 1 J=1,N
DO 1 K=1,M
AR(J,K)=0.D0
RETURN
END
```

```
C
C     .....................................................................
C
C        SUBROUTINE AMAT
C
C        PURPOSE
C           FORM THE 20 X 20 INVERSE OF THE MATRIX WHICH TRANSFORMS
C           DISPLACEMENTS AND ROTATIONS AT THE NODES OF AN ELEMENT TO
C           COEFFICIENTS OF THE POLYNOMIAL DISPLACEMENT FUNCTIONS.
C
C        USAGE
C           CALL AMAT(AK,EPK,EPJ)
C
C        DESCRIPTION OF PARAMETERS
C                   AK - IDENTIFIED IN MAIN PROGRAM COMMENTS
C                  EPK - IDENTIFIED IN MAIN PROGRAM COMMENTS
C                  EPJ - IDENTIFIED IN MAIN PROGRAM COMMENTS
C
C        REMARKS
C           THE AK MATRIX IS FIRST FORMED PER THEORY.  THE INVERSE IS
C           THEN FORMED AND IS DEFINED AS AK WHICH DESTROYS THE ORIGINAL
C           MATRIX.  AMAT IS CALLED BY STIFI AND MODAL.
C
C        SUBROUTINES AND F[redacted]GRAMS REQUIRED
C           INTIL, INVRS[redacted]
C
C        METHOD
C           THE FIRST [redacted]IS FORMED.  THE OTHER THREE
C           PARTITION[redacted]T(THE MAGNITUDES ARE THE
C           SAME); TH[redacted]RE ASSIGNED.  FINALLY, THE
C           MATRIX IS[redacted]Y; THE INVERSE FOUND; AND
C           THE RESUL[redacted]ATRIX AK.
C
C     ...............[redacted]...........................
C
      SUBROUTINE AMAT(AK,[redacted]
      IMPLICIT REAL*8(A-H,O-[redacted]
      DIMENSION AK(20,20),Q(400)
      DIMENSION HD(18),DATE(3)
      CALL INTIL(AK,20,20)
C     DEFINE W AND W'' TERMS FOR FIRST PARTITION
      AK(1,1)=1.D0
      AK(1,2)=-EPK*.5D0
      AK(1,3)=-EPJ*.5D0
      AK(1,4)=EPK*EPK*.25D0
      AK(1,5)=EPK*EPJ*.25D0
      AK(1,6)=EPJ*EPJ*.25D0
      AK(1,7)=-EPK*EPK*EPK*.125D0
      AK(1,8)=-EPK*EPK*EPJ*.125D0
```

```
      AK(1,9)=-EPK*EPJ*EPJ*.125D0
      AK(1,10)=-EPJ*EPJ*EPJ*.125D0
      AK(1,11)=EPJ*(EPK**3)*.0625D0
      AK(1,12)=EPK*(EPJ**3)*.0625D0
      AK(2,2)=-1.D0
      AK(2,4)=EPK
      AK(2,5)=EPJ*.5D0
      AK(2,7)=-EPK*EPK*.75D0
      AK(2,8)=-EPK*EPJ*.5D0
      AK(2,9)=-EPJ*EPJ*.25D0
      AK(2,11)=EPK*EPK*EPJ*.375D0
      AK(2,12)=EPJ*EPJ*EPJ*.125D0
      AK(3,3)=-1.D0
      AK(3,5)=EPK*.5D0
      AK(3,6)=EPJ
      AK(3,8)=-EPK*EPK*.25D0
      AK(3,9)=-EPK*EPJ*.5D0
      AK(3,10)=-EPJ*EPJ*.75D0
      AK(3,11)=EPK*EPK*EPK*.125D0
      AK(3,12)=EPK*EPJ*EPJ*.375D0
C     EQUATE W TERMS FOR OTHER THREE PARTITIONS
      DO 5 K=1,3
      IK=K+5
      IJ=K+10
      IL=K+15
      DO 5 J=1,12
      AK(IJ,J)=AK(K,J)
      AK(IK,J)=AK(K,J)
5     AK(IL,J)=DABS(AK(K,J))
      DO 6 J=1,12
      AK(17,J)=-AK(17,J)
6     AK(18,J)=-AK(18,J)
C     DETERMINE W TERMS FOR SECOND QUARTER PARTITION
      AK(6,3)=-AK(6,3)
      AK(6,5)=-AK(6,5)
      AK(6,8)=-AK(6,8)
      AK(6,10)=-AK(6,10)
      AK(6,11)=-AK(6,11)
      AK(6,12)=-AK(6,12)
      AK(7,5)=-AK(7,5)
      AK(7,8)=-AK(7,8)
      AK(7,11)=-AK(7,11)
      AK(7,12)=-AK(7,12)
      AK(8,6)=-AK(8,6)
      AK(8,9)=-AK(8,9)
C     DETERMINE W TERMS FOR THIRD QUARTER PARTITION
      AK(11,2)=-AK(11,2)
      AK(11,5)=-AK(11,5)
      AK(11,7)=-AK(11,7)
```

```
      AK(11,9)=-AK(11,9)
      AK(11,11)=-AK(11,11)
      AK(11,12)=-AK(11,12)
      AK(12,4)=-AK(12,4)
      AK(12,8)=-AK(12,8)
      AK(13,5)=-AK(13,5)
      AK(13,9)=-AK(13,9)
      AK(13,11)=-AK(13,11)
      AK(13,12)=-AK(13,12)
C        DETERMINE U TERMS FOR FIRST PARTITION
      AK(4,13)=1.D0
      AK(4,14)=-EPK*.5D0
      AK(4,15)=-EPJ*.5D0
      AK(4,16)=EPK*EPJ*.25D0
C      EQUATE OTHER THREE NODES
      DO 2 J=13,16
      AK(9,J)=AK(4,J)
      AK(14,J)=AK(4,J)
2     AK(19,J)=DABS(AK(4,J))
C        DEFINE DIFFERENCES
      AK(9,15)=-AK(9,15)
      AK(9,16)=-AK(9,16)
      AK(14,14)=-AK(14,14)
      AK(14,16)=-AK(14,16)
C        DETERMINE V TERMS FOR ALL FOUR PARTITIONS
      DO 8 J=13,16
      JJ=J+4
      AK(5,JJ)=AK(4,J)
      AK(10,JJ)=AK(9,J)
      AK(15,JJ)=AK(14,J)
8     AK(20,JJ)=AK(19,J)
      L=0
      DO 40 K=1,20
      DO 40 J=1,20
      L=L+1
40    Q(L)=AK(J,K)
      CALL INVRSR(Q,20,20)
      L=0
      DO 41 K=1,20
      DO 41 J=1,20
      L=L+1
41    AK(J,K)=Q(L)
      RETURN
      END
```

```
C
C     ...................................................................
C
C        SUBROUTINE PI6J
C
C        PURPOSE
C           PRINT A GENERAL MATRIX IN SIX COLUMNS PER PAGE
C
C        USAGE
C           CALL PI6J(A,M,N,IM,JN,HD,DATE,NC,JM,IN)
C
C        DESCRIPTION OF PARAMETERS
C                     A - THE MATRIX TO BE PRINTED
C                     M - THE NUMBER OF ROWS TO BE PRINTED
C                     N - THE NUMBER OF COLUMNS TO BE PRINTED
C                    IM - THE TOTAL NUMBER OF ROWS IN MATRIX A
C                    JN - THE TOTAL NUMBER OF COLUMNS IN MATRIX A
C                    HD - IDENTIFIED IN MAIN PROGRAM COMMENTS
C                  DATE - IDENTIFIED IN MAIN PROGRAM COMMENTS
C                    NC - A VALUE WHICH IS TESTED TO DETERMINE WHEN A
C                         NEW PAGE IS REQUIRED.  IF NC IS GREATER THAN
C                         47, A NEW PAGE IS STARTED WITH THE DATE AND
C                         PROBLEM HEADING PRINTED ON IT.
C                    JM - A KEY WHOSE VALUE DETERMINES WHICH HEADING
C                         FORMAT IS PRINTED (TO IDENTIFY THE MATRIX) IN
C                         SUBROUTINE FOMATI.
C                    IN - THE NUMBER OF THE WORKING ELEMENT
C
C        REMARKS
C           SIX COLUMNS PER PAGE ARE PRINTED IN D20.11 FORMAT UP TO 50
C           ROWS PER PAGE.  IF MORE ROWS ARE REQUIRED, THEY ARE
C           CONTINUED ON THE NEXT PAGE.  ALL ROWS AND COLUMNS ARE
C           NUMBERED FOR EASY IDENTIFICATION.
C
C        SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
C           FOMATI
C
C     ...................................................................
C
      SUBROUTINE PI6J (A,M,N,IM,JN,HD,DATE,NC,JM,IN)
      DOUBLE PRECISION A(IM,JN),HD(18),DATE(3)
    1 FORMAT (1H ,'   I/J ',5(I2,18X),I2)
    2 FORMAT (1H ,I4,6D20.11)
      K = 0
    3 L = K + 1
      K = K + 6
      CALL FOMATI (HD,DATE,M,NC,JM,IN)
      IF (N - K) 4,4,6
4     IF(N-50)40,40,41
```

```
40    WRITE (3,1) (J,J = L,N)
      DO 5 I = 1,M
    5 WRITE (3,2) I, (A(I,J),J = L,N)
      RETURN
41    WRITE(3,1)(J,J=L,N)
      DO 50 I=1,50
50    WRITE(3,2)I,(A(I,J),J=L,N)
      CALL FCMATI (HD,DATE,M,NC,JM,IN)
      WRITE (3,1) (J,J = L,N)
      DO 51 I=51,M
51    WRITE(3,2)I,(A(I,J),J=L,N)
      RETURN
6     IF(N-50)60,60,61
60    WRITE (3,1) (J,J = L,K)
      DO 7 I = 1,M
    7 WRITE (3,2) I, (A(I,J),J = L,K)
      GO TO 3
61    WRITE (3,1) (J,J = L,K)
      DO 70 I=1,50
70    WRITE (3,2) I, (A(I,J),J = L,K)
      CALL FCMATI (HD,DATE,M,NC,JM,IN)
      WRITE (3,1) (J,J = L,K)
      DO 71 I=51,M
71    WRITE (3,2) I, (A(I,J),J = L,K)
      GO TO 3
      END
```

```
C
C     ..................................................................
C
C        SUBROUTINE PI5J
C
C        PURPOSE
C           PRINT A GENERAL MATRIX IN FIVE COLUMNS PER PAGE
C
C        USAGE
C           CALL PI5J(A,M,N,IM,JH,HD,DATE,NC,JM)
C
C        DESCRIPTION OF PARAMETERS
C                       A - THE MATRIX TO BE PRINTED
C                       M - THE NUMBER OF ROWS TO BE PRINTED
C                       N - THE NUMBER OF COLUMNS TO BE PRINTED
C                      IM - THE TOTAL NUMBER OF ROWS IN MATRIX A
C                      JN - THE TOTAL NUMBER OF COLUMNS IN MATRIX A
C                      HD - IDENTIFIED IN MAIN PROGRAM COMMENTS
C                    DATE - IDENTIFIED IN MAIN PROGRAM COMMENTS
C                      NC - A VALUE WHICH IS TESTED TO DETERMINE WHEN A
C                           NEW PAGE IS REQUIRED.  IF NC IS GREATER THAN
C                           47, A NEW PAGE IS STARTED WITH THE DATE AND
C                           PROBLEM HEADING PRINTED ON IT.
C                      JM - A KEY WHOSE VALUE DETERMINES WHICH HEADING
C                           FORMAT IS PRINTED (TO IDENTIFY THE MATRIX) IN
C                           SUBROUTINE FOMATI.
C
C        REMARKS
C           FIVE COLUMNS PER PAGE ARE PRINTED IN D24.15 FORMAT UP TO 50
C           ROWS PER PAGE.  IF MORE ROWS ARE REQUIRED, THEY ARE
C           CONTINUED ON THE NEXT PAGE.  ALL ROWS AND COLUMNS ARE
C           NUMBERED FOR EASY IDENTIFICATION.
C
C        SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
C           FOMATI
C
C     ..................................................................
C
      SUBROUTINE PI5J (A,M,N,IM,JN,HD,DATE,NC,JM)
      DOUBLE PRECISION A(IM,JN)
      DOUBLE PRECISION HD(18),DATE(3)
    1 FORMAT (1H ,'   I/J ',4(I2,22X),I2)
    2 FORMAT (1H ,I4,5D24.15)
      K = 0
    3 L = K + 1
      K = K + 5
      CALL FOMATI (HD,DATE,M,NC,JM,1)
      IF (N - K) 4,4,6
4     IF(N-50)40,40,41
```

```
40     WRITE (3,1) (J,J = L,N)
       DO 5 I = 1,M
     5 WRITE (3,2) I, (A(I,J),J = L,N)
       RETURN
41     WRITE(3,1)(J,J=L,N)
       DO 50 I=1,50
50     WRITE(3,2)I,(A(I,J),J=L,N)
       CALL FCMATI (HD,DATE,M,NC,JM,1)
       WRITE (3,1) (J,J = L,N)
       DO 51 I=51,M
51     WRITE(3,2)I,(A(I,J),J=L,N)
       RETURN
6      IF(N-50)60,60,61
60     WRITE (3,1) (J,J = L,K)
       DO 7 I = 1,M
     7 WRITE (3,2) I, (A(I,J),J = L,K)
       GO TO 3
61     WRITE (3,1) (J,J = L,K)
       DO 70 I=1,50
70     WRITE (3,2) I, (A(I,J),J = L,K)
       CALL FOMATI (HD,DATE,M,NC,JM,1)
       WRITE (3,1) (J,J = L,K)
       DO 71 I=51,M
71     WRITE (3,2) I, (A(I,J),J = L,K)
       GO TO 3
       END
```

```
C
C      ..............................................................
C
C         SUBROUTINE FOMATI
C
C         PURPOSE
C            SELECTS THE APPROPRIATE TITLE TO BE PRINTED AS A HEADING
C            PRIOR TO A MATRIX PRINT-OUT TO IDENTIFY THE MATRIX
C
C         USAGE
C            CALL FOMATI(HD,DATE,M,NC,JM,I)
C
C         DESCRIPTION OF PARAMETERS
C                        HD - IDENTIFIED IN MAIN PROGRAM COMMENTS
C                      DATE - IDENTIFIED IN MAIN PROGRAM COMMENTS
C                         M - THE ORDER OF THE MATRIX
C                        NC - A VALUE WHICH IS TESTED TO DETERMINE IF A NEW
C                             PAGE OF PRINT IS REQUIRED. IF NC IS GREATER
C                             THAN 49 A NEW PAGE IS STARTED WITH THE DATE
C                             AND PROBLEM HEADING PRINTED ON IT.
C                        JM - THE KEY WHOSE VALUE DETERMINES WHICH HEADING
C                             FORMAT IS PRINTED TO IDENTIFY THE MATRIX
C                         I - THE NUMBER OF THE WORKING ELEMENT
C
C         REMARKS
C            THIS SUBROUTINE IS USUALLY CALLED BY PI5J OR PI6J
C
C         SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
C            NONE
C
C      ..............................................................
C
      SUBROUTINE FOMATI (HD,DATE,M,NC,JM,I)
      DOUBLE PRECISION HD(18),DATE(3)
    1 FORMAT (1H1,T60,A2,'-',A2,'-',A2)
    2 FORMAT (1H ,3X,18A4)
    4 FORMAT (1H0,3X,'INPUT MATRIX PRINTOUT')
    5 FORMAT (1H0,12X,'EIGENVALUES',12X,'FREQUENCIES (CPS)')
    6 FORMAT (1H0,3X,'NORMALIZED EIGENVECTORS')
    7 FORMAT (1H0,3X,'CHECK MATRIX = X(INVERSE) * A * X - D = 0')
   16 FORMAT (1H0,3X,'INVERSE OF EIGENVECTORS = X(INVERSE)')
30    FORMAT (1H0,3X,'THE AK MATRIX FOR ELEMENT NO.',I4)
31    FORMAT (1H0,3X,'THE INVERSE OF THE MIDDLE OR LOWER ( 4 X 4 ) PARTI
     1TION OF THE AK MATRIX FOR ELEMENT NO.',I4)
32    FORMAT (1H0,3X,'THE INVERSE OF THE UPPER (12 X 12) PARTITION OF TH
     1E AK MATRIX FOR ELEMENT NO.',I4)
33    FORMAT (1H0,3X,'THE COMPLETE INVERSE OF THE AK MATRIX FOR ELEMENT
     1NO.',I4)
34    FORMAT (1H0,3X,'THE X MATRIX FOR ELEMENT NO.',I4)
```

```
35    FORMAT (1H0,3X,'THE S MATRIX = ((INVERSE AK)TRANSPOSE)(X)(INVERSE
     1AK), FOR ELEMENT NO.',I4)
36    FORMAT (1H0,3X,'THE Y MATRIX FOR ELEMENT NO.',I4)
37    FORMAT (1H0,3X,'THE F MATRIX = ((INVERSE AK)TRANSPOSE)(Y)(INVERSE
     1AK), FOR ELEMENT NO.',I4)
38    FORMAT(1H0,3X,'THE TOTAL MASS MATRIX BEFORE IMPOSING CONSTRAINTS')
39    FORMAT(1H0,3X,'THE TOTAL STIFFNESS MATRIX BEFORE IMPOSING CONSTRAI
     1NTS')
40    FORMAT(1H0,3X,'THE TOTAL MASS MATRIX AFTER IMPOSING CONSTRAINTS')
41    FORMAT(1H0,3X,'THE TOTAL STIFFNESS MATRIX AFTER IMPOSING CONSTRAIN
     1TS')
42    FORMAT (1H0,3X,'THE W MODE SHAPE')
43    FORMAT (1H0,3X,'THE U MODE SHAPE')
44    FORMAT (1H0,3X,'THE V MODE SHAPE')
      NC = NC + M + 2
      IF (50 - NC) 8,8,9
    8 WRITE (3,1) DATE
      WRITE (3,2) HD
      NC = M + 6
9     J=JM
      GO TO (10,11,12,13,14,15,17,18,19,20,21,22,23,24,25,26,27,28,29,50
     1,51,52),J
   10 WRITE (3,4)
      GO TO 14
   11 WRITE (3,5)
      GO TO 14
   12 WRITE (3,6)
      GO TO 14
   13 WRITE (3,7)
      GO TO 14
   17 WRITE (3,16)
      GO TO 14
18    WRITE(3,30)I
      GO TO 14
19    WRITE(3,31)I
      GO TO 14
20    WRITE(3,32)I
      GO TO 14
21    WRITE(3,33)I
      GO TO 14
22    WRITE(3,34)I
      GO TO 14
23    WRITE(3,35)I
      GO TO 14
24    WRITE(3,36)I
      GO TO 14
25    WRITE(3,37)I
      GO TO 14
26    WRITE(3,38)
```

```
      GO TO 14
27    WRITE(3,39)
      GO TO 14
28    WRITE(3,40)
      GO TO 14
29    WRITE(3,41)
      GO TO 14
50    WRITE(3,42)
      GO TO 14
51    WRITE(3,43)
      GO TO 14
52    WRITE(3,44)
      GO TO 14
   14 NC = NC + 1
   15 RETURN
      END
```

```
C
C     ..................................................................
C
C        SUBROUTINE AMATM
C
C        PURPOSE
C           MULTIPLY MATRIX A TIMES MATRIX B TO OBTAIN RESULT, MATRIX C
C
C        USAGE
C           CALL AMATM(A,N,M,B,L,C)
C
C        DESCRIPTION OF PARAMATERS
C                    A - FIRST MATRIX OF ORDER N X M
C                    N - THE NUMBER OF ROWS IN A AND C
C                    M - THE NUMBER OF COLUMNS IN A AND ROWS IN B
C                    B - SECOND MATRIX OF ORDER M X L
C                    L - THE NUMBER OF ROWS IN B AND COLUMNS IN C
C                    C - THE RESULTING N X L MATRIX
C
C        REMARKS
C           THIS SUBROUTINE IS CALLED THOUGHOUT THE PROGRAM
C
C        SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
C           NONE
C
C     ..................................................................
C
      SUBROUTINE AMATM(A,N,M,B,L,C)
      DOUBLE PRECISION A(N,M),B(M,L),C(N,L)
      DO 5 I=1,N
      DO 5 J=1,L
      C(I,J)=0.D0
      DO 5 K=1,M
    5 C(I,J)=C(I,J)+A(I,K)*B(K,J)
      RETURN
      END
```

```
C
C     ...............................................................
C
C        SUBROUTINE INVRSR
C
C        PURPOSE
C           INVERT A MATRIX
C
C        USAGE
C           CALL INVRSR(A,N,D,L,M)
C
C        DESCRIPTION OF PARAMETERS
C           A - INPUT MATRIX, DESTROYED IN COMPUTATION AND REPLACED BY
C               RESULTANT INVERSE.
C           N - ORDER OF MATRIX A
C           D - RESULTANT DETERMINANT
C           L - WORK VECTOR OF LENGTH N
C           M - WORK VECTOR OF LENGTH N
C
C        REMARKS
C           MATRIX A MUST BE A GENERAL MATRIX.  THIS IS FROM THE IBM
C           SSP GROUP IN WHICH IT IS IDENTIFIED AS MINV. INVRSR IS
C           CALLED BY AMAT.
C
C        SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
C           NONE
C
C        METHOD
C           THE STANDARD GAUSS-JORDAN METHOD IS USED. THE DETERMINANT
C           IS ALSO CALCULATED. A DETERMINANT OF ZERO INDICATES THAT
C           THE MATRIX IS SINGULAR.
C
C     ...............................................................
C
C
C        ............................................................
C
C        IF A DOUBLE PRECISION VERSION OF THIS ROUTINE IS DESIRED, THE
C        C IN COLUMN 1 SHOULD BE REMOVED FROM THE DOUBLE PRECISION
C        STATEMENT WHICH FOLLOWS.
C
C
C        THE C MUST ALSO BE REMOVED FROM DOUBLE PRECISION STATEMENTS
C        APPEARING IN OTHER ROUTINES USED IN CONJUNCTION WITH THIS
C        ROUTINE.
C
C        THE DOUBLE PRECISION VERSION OF THIS SUBROUTINE MUST ALSO
C        CONTAIN DOUBLE PRECISION FORTRAN FUNCTIONS.  ABS IN STATEMENT
C        10 MUST BE CHANGED TO DABS.
```

```
C        ..............................................................
C
C          SEARCH FOR LARGEST ELEMENT
C
      SUBROUTINE INVRSR (A,N,IM)
      DIMENSION A(1),L(20),M(20)
      DOUBLE PRECISION A,D,BIGA,HOLD
      D=1.0
      NK=-N
      DO 80 K=1,N
      NK=NK+N
      L(K)=K
      M(K)=K
      KK=NK+K
      BIGA=A(KK)
      DO 20 J=K,N
      IZ=N*(J-1)
      DO 20 I=K,N
      IJ=IZ+I
   10 IF(DABS(BIGA)-DABS(A(IJ))) 15,20,20
   15 BIGA=A(IJ)
      L(K)=I
      M(K)=J
   20 CONTINUE
C
C          INTERCHANGE ROWS
C
      J=L(K)
      IF(J-K) 35,35,25
   25 KI=K-N
      DO 30 I=1,N
      KI=KI+N
      HOLD=-A(KI)
      JI=KI-K+J
      A(KI)=A(JI)
   30 A(JI) =HOLD
C
C          INTERCHANGE COLUMNS
C
   35 I=M(K)
      IF(I-K) 45,45,38
   38 JP=N*(I-1)
      DO 40 J=1,N
      JK=NK+J
      JI=JP+J
      HOLD=-A(JK)
      A(JK)=A(JI)
   40 A(JI) =HOLD
C
```

```
C          DIVIDE COLUMN BY MINUS PIVOT (VALUE OF PIVOT ELEMENT IS
C          CONTAINED IN BIGA)
C
   45 IF(BIGA) 48,46,48
   46 D=0.0
4     FORMAT(//,10X,'SINGULARITY EXISTS IN THE MATRIX ',//)
      WRITE(3,4)
      CALL EXIT
      RETURN
   48 DO 55 I=1,N
      IF(I-K) 50,55,50
   50 IK=NK+I
      A(IK)=A(IK)/(-BIGA)
   55 CONTINUE
C
C          REDUCE MATRIX
C
      DO 65 I=1,N
      IK=NK+I
      HOLD=A(IK)
      IJ=I-N
      DO 65 J=1,N
      IJ=IJ+N
      IF(I-K) 60,65,60
   60 IF(J-K) 62,65,62
   62 KJ=IJ-I+K
      A(IJ)=HOLD*A(KJ)+A(IJ)
   65 CONTINUE
C
C          DIVIDE ROW BY PIVOT
C
      KJ=K-N
      DO 75 J=1,N
      KJ=KJ+N
      IF(J-K) 70,75,70
   70 A(KJ)=A(KJ)/BIGA
   75 CONTINUE
C
C          PRODUCT OF PIVOTS
C
      D=D*BIGA
C
C          REPLACE PIVOT BY RECIPROCAL
C
      A(KK)=1.0/BIGA
   80 CONTINUE
C
C          FINAL ROW AND COLUMN INTERCHANGE
C
```

```
      K=N
  100 K=(K-1)
      IF(K) 150,150,105
  105 I=L(K)
      IF(I-K) 120,120,108
  108 JQ=N*(K-1)
      JR=N*(I-1)
      DO 110 J=1,N
      JK=JQ+J
      HOLD=A(JK)
      JI=JR+J
      A(JK)=-A(JI)
  110 A(JI) =HOLD
  120 J=M(K)
      IF(J-K) 100,100,125
  125 KI=K-N
      DO 130 I=1,N
      KI=KI+N
      HOLD=A(KI)
      JI=KI-K+J
      A(KI)=-A(JI)
  130 A(JI) =HOLD
      GO TO 100
150   RETURN
      END
```

```
C
C      ..............................................................
C
C         SUBROUTINE INPUT
C
C         PURPOSE
C            READ IN REQUIRED DATA AND PRINT IT OUT FOR CHECKING
C
C         USAGE
C            CALL INPUT
C
C         DESCRIPTION OF PARAMETERS
C                      - ALL IDENTIFIED IN MAIN PROGRAM COMMENTS
C
C         REMARKS
C            IN GENERAL, THE DATA IS PRINTED OUT IN THE ORDER IT'S READ.
C            THE EXCEPTION IS THAT EK AND EJ ARE CALCULATED IN THE
C            INTERIM AND INCLUDED IN THE PRINTOUT.  INPUT IS CALLED BY
C            MAIN.
C
C         SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
C            NONE
C
C      ..............................................................
C
      SUBROUTINE INPUT
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON H(16,11),THETA(16,11),S(20,20),AK(20,20),F(20,20)
      COMMON XE(4),YE(4),HE(11),THE(11)
      COMMON EK(16),EJ(16),X(25),Y(25),ALAM(16),HD(18),DATE(3)
      COMMON        F1,F2,F3,F4,F5,F6,F7,F8,ZX,ZY,ZXY,E1,E2,G1,P1,P2
      COMMON EPK,EPJ,HET,RHOH,RHOHT
      COMMON IJK(16,5),IJKL(4),KEC(25)
      COMMON KE,KN,NOLAM,KB
1     FORMAT(20I4)
2     FORMAT(8F9.0)
3     FORMAT(18A4)
5     FORMAT(1H1,T60,A2,'-',A2,'-',A2)
6     FORMAT(1H ,3X,18A4)
7     FORMAT(//,3X,'NO. OF ELEMENTS =',I4,10X,'NO. OF NODES =',I4)
72    FORMAT(/,3X,'THE NUMBER OF LOWER EIGENVALUES FOR WHICH')
73    FORMAT(3X,'ADDITIONAL MODAL DEFLECTIONS ARE DESIRED =',I4)
8     FORMAT(//,3X,'E1 =',F11.0,' PSI',3X,'E2 =',F11.0,' PSI',3X,'G =',F
     111.0,' PSI')
81    FORMAT(/,3X,'NU1 =',F9.6,3X,'NU2 =',F9.6,3X,'RHO =',F9.6,1X,'LB/CU
     1BIC INCH')
9     FORMAT(/,3X,'F1 =',F3.0,3X,'F2 =',F3.0,3X,'F3 =',F3.0,3X,'F4 =',F3
     1.0,3X,'F5 =',F3.0,3X,'F6 =',F3.0,3X,'F7 =',F3.0,3X,'F8 =',F3.0,//)
10    FORMAT(3X,'ELEM.',10X,'EK',12X,'EJ',12X,'NO. OF',5X,'THE FOUR NODE
```

```
      1 NUMBERS')
11    FORMAT(4X,'NO.',8X,'(INCHES)',6X,'(INCHES)',7X,'LAMINATES',7X,'OF
     1THE ELEMENT',/)
12    FORMAT(3X,I4,7X,F10.5,4X,F10.5,8X,F4.0,6X,I4,2X,I4,2X,I4,2X,I4,/)
13    FORMAT(//,3X,'NODE',5X,'X-COORD.',4X,'Y-COORD.',10X,'NODE')
14    FORMAT(4X,'NO.',5X,'(INCHES)',4X,'(INCHES)',7X,'CONSTRAINT',/)
15    FORMAT(3X,I4,4X,F10.4,2X,F10.4,8X,I4)
16    FORMAT(//,3X,'ELEM.',6X,'NO. OF',8X,'H',11X,'THETA',6X,'KEY')
17    FORMAT(4X,'NO.',5X,'LAMINATES',4X,'(INCHES)',5X,'(DEGREES)',/)
18    FORMAT(3X,I4,7X,I4,33X,I2)
19    FORMAT(23X,F10.6,5X,F9.4)
20    FORMAT(23X,F10.6)
21    FORMAT(1H1)
      RAD=57.2958279D0
      READ (1,3,END=60)HD
      READ(1,1)KE,KN,KB
      READ(1,2)E1,E2,G1,P1,P2,RHCH
      READ(1,2)F1,F2,F3,F4,F5,F6,F7,F8
C     KE = NO. OF ELEMENTAL SECTIONS
C     KN = NO. OF NODES
      WRITE(3,5)DATE
      WRITE(3,6)HD
      WRITE(3,7)KE,KN
      WRITE(3,72)
      WRITE(3,73)KB
      WRITE(3,8)E1,E2,G1
      WRITE(3,81)P1,P2,RHCH
      WRITE(3,9)F1,F2,F3,F4,F5,F6,F7,F8
      READ(1,2)(X(I),I=1,KN)
      READ(1,2)(Y(I),I=1,KN)
C     X(I) = X GLOBAL COORDINATE OF I-TH NODE
C     Y(I) = Y GLOBAL COORDINATE OF I-TH NODE
      READ(1,1)(KEC(I),I=1,KN)
C     KEC(I) DENOTES THE EDGE CONSTRAINT OF THE I-TH NODE.
      READ(1,1)((IJK(I,J),J=1,5),I=1,KE)
C     IJK(I,J) IS A FIVE COLUMN ARRAY OF NUMBERS, EACH LINE OF WHICH
C              CONTAINS THE FOUR NODE NUMBERS FOR THE I-TH ELEMENT AND THE
C              FIFTH NUMBER IS TO KEY WHETHER THE ELEMENT IS THE SAME AS
C              THE PREVIOUS ELEMENT.
      READ(1,2)(EK(I),I=1,KE)
      READ(1,2)(EJ(I),I=1,KE)
C     EJ(I) = LENGTH OF I-TH ELEMENT IN LOCAL Y-DIRECTION
C     EK(I) = LENGTH OF I-TH ELEMENT IN LOCAL X-DIRECTION
      READ(1,2)(ALAM(I),I=1,KE)
C     ALAM(I) = NUMBER OF LAMINATES IN THE I-TH ELEMENT
      WRITE(3,10)
      WRITE(3,11)
      DO 30 I=1,KE
30    WRITE(3,12)I,EK(I),EJ(I),ALAM(I),IJK(I,1),IJK(I,2),IJK(I,3),IJK(I,
```

```
     14)
      WRITE(3,13)
      WRITE(3,14)
      DO 31 I=1,KN
31    WRITE(3,15)I,X(I),Y(I),KEC(I)
      WRITE(3,5)DATE
      WRITE(3,6)HD
      WRITE(3,16)
      WRITE(3,17)
      DO 50 I=1,KF
      NOLAM = ALAM(I)
      WRITE(3,18)I,NOLAM,IJK(I,5)
      ILAM=NOLAM+1
      READ(1,2)(H(I,J),J=1,ILAM)
      READ(1,2)(THETA(I,J),J=1,NOLAM)
      DO 49 J=1,NOLAM
      WRITE(3,19)H(I,J),THETA(I,J)
49    THETA(I,J)=THETA(I,J)/RAD
      WRITE(3,20)H(I,ILAM)
50    CONTINUE
C     H(I,J) = THE DISTANCE FROM THE CENTER PLANE OF THE I-TH ELEMENT TO
C              THE OUTER EDGE OF J-TH LAMINATE.  NOTE THE TOP LAMINATE
C              IS NUMBER ONE, I.E. J=1.
C     THETA(I,J) = THE ANGLE ORIENTATION OF THE J-TH LAMINATE IN THE
C                  I-TH ELEMENT.  THETA IS MEASURED WITH RESPECT TO
C                  THE LOCAL X-AXIS.
      WRITE(3,21)
      RETURN
60    CALL EXIT
      END
```

```
C
C      ..................................................................
C
C         SUBROUTINE SURFU
C
C         PURPOSE
C            DEFINE SHELL CURVATURES
C
C         USAGE
C            CALL SURFU(I,XE,YE,ZX,ZY,ZXY)
C
C         DESCRIPTION OF PARAMETERS
C                    I - THE NUMBER OF THE WORKING ELEMENT
C                   XE - IDENTIFIED IN MAIN PROGRAM COMMENTS
C                   YE - IDENTIFIED IN MAIN PROGRAM COMMENTS
C                   ZX - IDENTIFIED IN MAIN PROGRAM COMMENTS
C                   ZY - IDENTIFIED IN MAIN PROGRAM COMMENTS
C                  ZXY - IDENTIFIED IN MAIN PROGRAM COMMENTS
C
C         REMARKS
C            THE CONTENT OF THIS SUBROUTINE WILL VARY ACCORDING TO THE
C            SURFACE INVOLVED IN THE PROBLEM BEING ANALYZED.  NOTE THAT
C            ZX, ZY, AND ZXY CAN BE FUNCTIONS OF:
C                           1.  XE AND YE FOR EACH ELEMENT      OR
C                           2.  THE ELEMENT NUMBER I
C            THIS SUBROUTINE IS LEFT FLEXIBLE FOR THE USER'S CONVENIENCE.
C            NOTE THAT FOR DIFFERENT PROBLEMS THIS SUBROUTINE HAS TO BE
C            RE-COMPILED.  CONSEQUENTLY, PROBLEM CASES SELDOM WILL BE
C            ABLE TO BE STACKED.  SURFU IS CALLED BY MAIN.
C
C         SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
C            NORMALLY, NONE
C
C      ..................................................................
C
      SUBROUTINE SURFU(I,XE,YE,ZX,ZY,ZXY)
      DOUBLE PRECISION XE(4),YE(4),ZX,ZY,ZXY
C        VALUES FOR ZERO CURVATURE.
      ZX=0.D0
      ZY=0.D0
      ZXY=0.D0
      RETURN
      END
```

```
C
C     ..................................................................
C
C        SUBROUTINE PRIN2
C
C        PURPOSE
C           PRINT THE WORKING ELEMENT CURVATURES, IF DESIRED
C
C        USAGE
C           CALL PRIN2(I,ZX,ZY,ZXY)
C
C        DESCRIPTION OF PARAMETERS
C                      I - THE NUMBER OF THE WORKING ELEMENT
C                     ZX - IDENTIFIED IN MAIN PROGRAM COMMENTS
C                     ZY - IDENTIFIED IN MAIN PROGRAM COMMENTS
C                    ZXY - IDENTIFIED IN MAIN PROGRAM COMMENTS
C
C        REMARKS
C           THE CURVATURES FOR THE ELEMENTS CAN BE PRINTED AT THE USER'S
C           OPTION, DEPENDING ON VALUE ASSIGNED TO FLAG 1.  IF FLAG 1
C           IS POSITIVE, CURVATURES ARE PRINTED.  IF NOT DESIRED, LEAVE
C           BLANK (ZERO).  PRIN2 IS CALLED BY MAIN.
C
C        SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
C           NONE
C
C     ..................................................................
C
      SUBROUTINE PRIN2(I,ZX,ZY,ZXY)
      DOUBLE PRECISION ZX,ZY,ZXY
1     FORMAT(10X,'ELEM. NO.',9X,'Z,XX',9X,'Z,YY',9X,'Z,XY',//)
2     FORMAT(11X,I4,10X,F9.3,4X,F9.3,4X,F9.3,///)
      WRITE(3,1)
      WRITE(3,2)I,ZX,ZY,ZXY
      RETURN
      END
```

```
C
C     ...................................................................
C
C        SUBROUTINE STIF1
C
C        PURPOSE
C           FORM THE WORKING ELEMENT 20 X 20 STIFFNESS MATRIX S
C
C        USAGE
C           CALL STIF1(I)
C
C        DESCRIPTION OF PARAMETERS
C                    I - THE NUMBER OF THE WORKING ELEMENT
C                      - ALL COMMON PARAMETERS DEFINED IN MAIN PROGRAM
C                    A - THE USUAL CONSTITUTIVE A MATRIX FOR LAMINATES
C                    B - THE USUAL CONSTITUTIVE B MATRIX FOR LAMINATES
C                    D - THE USUAL CONSTITUTIVE D MATRIX FOR LAMINATES
C                    R - THE 8 X 8 SYMMETRIC MATRIX DEFINED IN THE
C                        THEORY
C                   S1 - A 20 X 20 WORKING MATRIX
C                   AT - THE TRANSPOSE OF THE AK MATRIX.
C
C        REMARKS
C           FLAGS 3 AND 4 CAN BE USED FOR INTERMEDIATE PRINTOUT
C           STIF1 IS CALLED BY MAIN.
C
C        SUBROUTINES AND FUNCTION PROGRAMS REQUIRED
C           DIRECTLY CALLED FROM STIF1
C              STIFF, PI6J, AMATR, AMAT, AMATM, AMATX
C           OTHERS CALLED BY THE ABOVE SUBROUTINES
C              IHTIL, FOMATI, INVRSR
C
C        METHOD
C           THE METHOD IS TOO COMPLEX FOR SIMPLE EXPLANATION.  SEE THE
C           THEORY FOR DETAILS.
C
C     ...................................................................
C
C
      SUBROUTINE STIF1(I)
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON H(16,11),THETA(16,11),S(20,20),AK(20,20),F(20,20)
      COMMON XE(4),YE(4),HE(11),THE(11)
      COMMON EK(16),EJ(16),X(25),Y(25),ALAM(16),HD(18),DATE(3)
      COMMON       F1,F2,F3,F4,F5,F6,F7,F8,ZX,ZY,ZXY,E1,E2,G1,P1,P2
      COMMON EPK,EPJ,HET,RHOH,RHOHT
      COMMON IJK(16,5),IJKL(4),KEC(25)
      COMMON KE,KN,NOLAM,KB
      DIMENSION A(3,3),B(3,3),D(3,3),R(8,8),S1(20,20),AT(20,20)
```

```
1       FORMAT(1H1,'   PRINTOUT OF THE A MATRIX FOR ELEMENT NO.',I4)
2       FORMAT(/,'   PRINTOUT OF THE B MATRIX FOR ELEMENT NO.',I4)
3       FORMAT(/,'   PRINTOUT OF THE D MATRIX FOR ELEMENT NO.',I4)
4       FORMAT(/,'   PRINTOUT OF THE R MATRIX FOR ELEMENT NO.',I4)
        DO 55 L=1,NOLAM
        HE(L)=H(I,L)
55      THE(L)=THETA(I,L)
        ILAM=NOLAM+1
        HE(ILAM)=H(I,ILAM)
        CALL STIFF(I,A,B,D,THE,HE,NOLAM,E1,E2,P1,P2,G1)
        IF(F2)10,11,12
12      F2=0.D0
10      WRITE(3,1)I
        CALL PI6J(A,3,3,3,3,HD,DATE,6,1,I)
        WRITE(3,2)I
        CALL PI6J(B,3,3,3,3,HD,DATE,6,1,I)
        WRITE(3,3)I
        CALL PI6J(D,3,3,3,3,HD,DATE,6,1,I)
C       NOW FORM R MATRIX
11      CALL AMATR(I,A,B,D,R,ZX,ZY,ZXY)
        IF(F3)13,14,15
15      F3=0.D0
13      WRITE(3,4)I
        CALL PI6J(R,8,8,8,8,HD,DATE,16,1,I)
14      EPK=EK(I)
        EPJ=EJ(I)
        CALL AMAT(AK,EPK,EPJ)
        IF(F4)19,20,21
21      F4=0.D0
19      CALL PI6J(AK,20,20,20,20,HD,DATE,40,11,I)
20      CALL AMATX(I,S,EPK,EPJ,R)
        CALL AMATM(S,20,20,AK,20,S1)
        DO 70 M=1,20
        DO 70 N=1,20
70      AT(N,M)=AK(M,N)
        CALL AMATM(AT,20,20,S1,20,S)
        IF(F5)25,26,27
27      F5=0.D0
25      CALL PI6J(S,20,20,20,20,HD,DATE,40,13,I)
26      RETURN
        END
```

```
C
C     ******************************************************************
C
C        SUBROUTINE STIFF
C
C        PURPOSE
C           FORM THE A, B, AND D CONSTITUTIVE 3 X 3 MATRICES FOR THE
C           WORKING ELEMENT
C
C        USAGE
C           CALL STIFF(I,A,B,D,THE,HE,NOLAM,E1,E2,P1,P2,G1)
C
C        DESCRIPTION OF PARAMETERS
C                     I - THE NUMBER OF THE WORKING ELEMENT
C                     A - THE USUAL CONSTITUTIVE A MATRIX FOR LAMINATES
C                     B - THE USUAL CONSTITUTIVE B MATRIX FOR LAMINATES
C                     D - THE USUAL CONSTITUTIVE D MATRIX FOR LAMINATES
C                   THE - IDENTIFIED IN MAIN PROGRAM COMMENTS
C                    HE - IDENTIFIED IN MAIN PROGRAM COMMENTS
C                 NOLAM - IDENTIFIED IN MAIN PROGRAM COMMENTS
C                    E1 - IDENTIFIED IN MAIN PROGRAM COMMENTS
C                    E2 - IDENTIFIED IN MAIN PROGRAM COMMENTS
C                    P1 - IDENTIFIED IN MAIN PROGRAM COMMENTS
C                    P2 - IDENTIFIED IN MAIN PROGRAM COMMENTS
C                    G1 - IDENTIFIED IN MAIN PROGRAM COMMENTS
C
C        REMARKS
C           STIFF IS CALLED BY STIF1
C
C        SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
C           INTIL
C
C        METHOD
C           SEE 'PRIMER ON COMPOSITE MATERIALS' BY ASHTON, ET AL,
C           EQUATIONS (3-25), (3-26), AND (3-32).
C
C     ******************************************************************
C
      SUBROUTINE STIFF(I,A,B,D,THE,HE,NOLAM,E1,E2,P1,P2,G1)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION A(3,3),B(3,3),D(3,3),R(8,8),C(3,3),THE(11),HE(11)
      Q1=E1/(1.D0-P1*P2)
      Q2=E2/(1.D0-P1*P2)
      Q3=Q1*P2
      Q6=G1
      U1=.125D0*((3.D0*Q1)+(3.D0*Q2)+(2.D0*Q3)+(4.D0*Q6))
      U2=.5D0*(Q1-Q2)
      U3=.125D0*(Q1+Q2-(2.D0*Q3)-(4.D0*Q6))
      U4=.125D0*(Q1+Q2+(6.D0*Q3)-(4.D0*Q6))
```

```
      U5=.125DC*(Q1+Q2-(2.D0*Q3)+(4.D0*C6))
      CALL INTIL (A,3,3)
      CALL INTIL (B,3,3)
      CALL INTIL (D,3,3)
      DO 1 L=1,NOLAM
      T1=2.D0*THE(L)
      T2=4.D0*THE(L)
      C2=DCOS(T1)*U2
      C4=DCOS(T2)*U3
      C1=DSIN(T1)*U2*.5D0
      C3=DSIN(T2)*U3
      Q(1,1)=U1+C2+C4
      Q(2,2)=U1-C2+C4
      Q(1,2)=U4-C4
      Q(3,3)=U5-C4
      Q(1,3)=-C1-C3
      Q(2,3)=-C1+C3
      Q(2,1)=Q(1,2)
      Q(3,1)=Q(1,3)
      Q(3,2)=Q(2,3)
      DO 1 J=1,3
      DO 1 K=1,3
      A(J,K)=A(J,K)+Q(J,K)*(HE(L)-HE(L+1))
      B(J,K)=B(J,K)+(Q(J,K)*((HE(L)**2)-(HE(L+1)**2))*.5D0)
1     D(J,K)=D(J,K)+((Q(J,K)*((HE(L)**3)-(HE(L+1)**3)))/3.DC)
      RETURN
      END
```

```
C
C     ...............................................................
C
C        SUBROUTINE AMATR
C
C        PURPOSE
C           FORM THE 8 X 8 R MATRIX
C
C        USAGE
C           CALL AMATR(I,A,B,D,R,ZX,ZY,ZXY)
C
C        DESCRIPTION OF PARAMETERS
C                     I - THE NUMBER OF THE WORKING ELEMENT
C                     A - THE USUAL CONSTITUTIVE A MATRIX FOR LAMINATES
C                     B - THE USUAL CONSTITUTIVE B MATRIX FOR LAMINATES
C                     D - THE USUAL CONSTITUTIVE D MATRIX FOR LAMINATES
C                     R - THE MATRIX TO BE FORMED
C                    ZX - IDENTIFIED IN MAIN PROGRAM COMMENTS
C                    ZY - IDENTIFIED IN MAIN PROGRAM COMMENTS
C                   ZXY - IDENTIFIED IN MAIN PROGRAM COMMENTS
C
C        REMARKS
C           AMATR IS CALLED BY STIF1.  R IS SYMMETRIC AND IS DEFINED IN
C           THE THEORY AS MATRIX R.
C
C        SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
C           INTIL
C
C     ...............................................................
C
      SUBROUTINE AMATR(I,A,B,D,R,ZX,ZY,ZXY)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION A(3,3),B(3,3),D(3,3),R(8,8)
      CALL INTIL(R,8,8)
      R(1,1)=(A(1,1)*ZX*ZX*.5D0)+(A(1,2)*ZX*ZY)+(A(2,2)*ZY*ZY*.5D0)+(A(3
     1,3)*ZXY*ZXY*2.D0)+(A(1,3)*ZX*ZXY*2.D0)+(A(2,3)*ZY*ZXY*2.D0)
      R(1,2)=(((B(1,1)*ZX)+(B(1,2)*ZY))*.5D0)+(B(1,3)*ZXY)
      R(1,3)=(((B(1,2)*ZX)+(B(2,2)*ZY))*.5D0)+(B(2,3)*ZXY)
      R(1,4)=(((B(1,3)*ZX)+(B(2,3)*ZY))*.5D0)+(B(3,3)*ZXY)
      R(1,5)=(((-A(1,1)*ZX)-(A(1,2)*ZY))*.5D0)-(A(1,3)*ZXY)
      R(1,6)=(((-A(1,3)*ZX)-(A(2,3)*ZY))*.5D0)-(A(3,3)*ZXY)
      R(1,7)=R(1,6)
      R(1,8)=(((-A(1,2)*ZX)-(A(2,2)*ZY))*.5D0)-(A(2,3)*ZXY)
      R(2,2)=D(1,1)*.5D0
      R(2,3)=D(1,2)*.5D0
      R(2,4)=D(1,3)*.5D0
      R(2,5)=-B(1,1)*.5D0
      R(2,6)=-B(1,3)*.5D0
      R(2,7)=R(2,6)
```

```
      R(2,8)=-B(1,2)*.5D0
      R(3,3)=D(2,2)*.5D0
      R(3,4)=D(2,3)*.5D0
      R(3,5)=R(2,8)
      R(3,6)=-B(2,3)*.5D0
      R(3,7)=R(3,6)
      R(3,8)=-B(2,2)*.5D0
      R(4,4)=D(3,3)*.5D0
      R(4,5)=R(2,6)
      R(4,6)=-B(3,3)*.5D0
      R(4,7)=R(4,6)
      R(4,8)=R(3,7)
      R(5,5)=A(1,1)*.5D0
      R(5,6)=A(1,3)*.5D0
      R(5,7)=R(5,6)
      R(5,8)=A(1,2)*.5D0
      R(6,6)=A(3,3)*.5D0
      R(6,7)=R(6,6)
      R(6,8)=A(2,3)*.5D0
      R(7,7)=R(6,6)
      R(7,8)=R(6,8)
      R(8,8)=A(2,2)*.5D0
      DO 1 K=1,8
      DO 1 J=1,8
1     R(J,K)=R(K,J)
      RETURN
      END
```

```
C
C     ..................................................................
C
C        SUBROUTINE AMATX
C
C        PURPOSE
C           FORM THE INTEGRATION OF THE MATRIX (X).(R).(X) OVER THE
C           WORKING ELEMENT.  IN THE THEORY THIS IS MATRIX C.
C
C        USAGE
C           CALL AMATX(I,S,EPK,EPJ,R)
C
C        DESCRIPTION OF PARAMETERS
C                       I - THE NUMBER OF THE WORKING ELEMENT
C                       S - THE RESULTING 20 X 20 MATRIX
C                     EPK - IDENTIFIED IN MAIN PROGRAM COMMENTS
C                     EPJ - IDENTIFIED IN MAIN PROGRAM COMMENTS
C                       R - DEFINED IN STIF1 COMMENTS
C
C        REMARKS
C           Z, G, XT, AND P ARE WORKING MATRICES.  AMATX IS CALLED BY
C           STIF1.
C
C        SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
C           INTIL, AMATM
C
C        METHOD
C           WEDDLE'S METHOD OF DOUBLE INTEGRATION IS USED WHICH REQUIRES
C           THE ELEMENT TO BE DIVIDED INTO 6 X 6 SUBDIVISIONS.
C           REFERENCE°  "NUMERICAL MATHEMATICAL ANALYSIS - FIFTH
C                       EDITION" BY J.B. SCARBOROUGH, THE JOHN HOPKINS
C                       PRESS, 1962
C
C     ..................................................................
C
      SUBROUTINE AMATX(I,S,EPK,EPJ,R)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION Z(8,20),G(8,20),XT(20,8),P(20,20),R(8,8),S(20,20)
      CALL INTIL (S,20,20)
      E=EPK/6.D0
      F=EPJ/6.D0
      DO 1 K=1,7
      QK=K
      XK=(((QK-1.D0)/6.D0)-.5D0)*EPK
      GO TO (11,12,11,13,11,12,11),K
11    C1=.3D0*E
      GO TO 5
12    C1=1.5D0*E
      GO TO 5
```

```
13     C1=1.8D0*F
5      DO 1 J=1,7
       CALL INTIL (Z,8,20)
       QJ=J
       YJ=(((QJ-1.D0)/6.D0)-.5D0)*EPJ
       GO TO (14,15,14,16,14,15,14),J
14     C2=.3D0*F
       GO TO 6
15     C2=1.5D0*F
       GO TO 6
16     C2=1.8D0*F
6      DEL=C1*C2
       Z(1,1)=1.D0
       Z(1,2)=XK
       Z(1,3)=YJ
       Z(1,4)=XK*XK
       Z(1,5)=XK*YJ
       Z(1,6)=YJ*YJ
       Z(1,7)=XK**3
       Z(1,8)=XK*XK*YJ
       Z(1,9)=XK*YJ*YJ
       Z(1,10)=YJ**3
       Z(1,11)=YJ*(XK**3)
       Z(1,12)=XK*(YJ**3)
       Z(2,4)=2.D0
       Z(2,7)=XK*6.D0
       Z(2,8)=YJ*2.D0
       Z(2,11)=XK*YJ*6.D0
       Z(3,6)=2.D0
       Z(3,9)=XK*2.D0
       Z(3,10)=YJ*6.D0
       Z(3,12)=XK*YJ*6.D0
       Z(4,5)=1.D0
       Z(4,8)=XK*2.D0
       Z(4,9)=YJ*2.D0
       Z(4,11)=XK*XK*3.D0
       Z(4,12)=YJ*YJ*3.D0
       Z(5,14)=1.D0
       Z(5,16)=YJ
       Z(6,15)=1.D0
       Z(6,16)=XK
       Z(7,18)=1.D0
       Z(7,20)=YJ
       Z(8,19)=1.D0
       Z(8,20)=XK
       CALL INTIL(XT,20,8)
       DO 3 M=1,8
       DO 3 N=1,20
3      XT(N,M)=Z(M,N)
```

```
      CALL INTIL(G,8,20)
      CALL AMATM(R,8,8,Z,20,G)
      CALL INTIL(P,20,20)
      CALL AMATM(XT,20,8,G,20,P)
      DO 4 M=1,20
      DO 4 N=1,20
4     S(M,N)=S(M,N)+(P(M,N)*DEL)
1     CONTINUE
      RETURN
      END
```

```
C
C     ..................................................................
C
C        SUBROUTINE AMASS
C
C        PURPOSE
C           FORM THE WORKING ELEMENT MASS MATRIX
C
C        USAGE
C           CALL AMASS (I)
C
C        DESCRIPTION OF PARAMETERS
C                       I - THE NUMBER OF THE WORKING ELEMENT
C
C        REMARKS
C           THE COMMON VARIABLES ARE IDENTIFIED IN MAIN PROGRAM COMMENTS
C           THE DENSITY OF ALL LAMINATES IN ALL ELEMENTS IS ASSUMED TO
C           BE THE SAME.  THE AK MATRIX FROM AMAT IS REQUIRED HERE.
C           AMASS IS CALLED BY MAIN.
C
C        SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
C           AMATF, AMATM, PI6J
C
C        METHOD
C           SEE DETAILED THEORY.
C
C     ..................................................................
C
      SUBROUTINE AMASS(I)
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON H(16,11),THETA(16,11),S(20,20),AK(20,20),F(20,20)
      COMMON XE(4),YE(4),HE(11),THE(11)
      COMMON EK(16),EJ(16),X(25),Y(25),ALAM(16),HD(18),DATE(3)
      COMMON        F1,F2,F3,F4,F5,F6,F7,F8,ZX,ZY,ZXY,E1,E2,G1,P1,P2
      COMMON EPK,EPJ,HET,RHOH,RHOHT
      COMMON IJK(16,5),IJKL(4),KEC(25)
      COMMON KE,KN,NOLAM,KB
      DIMENSION S1(20,20),AT(20,20)
      ILAM=NOLAM+1
      RHOHT=RHOH*(H(I,1)-H(I,ILAM))/(386.04DO)
      CALL AMATF(I,F,EPK,EPJ,RHOHT)
      CALL AMATM(F,20,20,AK,20,S1)
      DO 74 M=1,20
      DO 74 N=1,20
74    AT(N,M)=AK(M,N)
      CALL AMATM(AT,20,20,S1,20,F)
      IF(F6)5,6,7
7     F6=0.DO
5     CALL PI6J(F,20,20,20,20,HD,DATE,40,15,I)
```

```
6       RETURN
        END
```

```
C
C     ..................................................................
C
C        SUBROUTINE AMATF
C
C        PURPOSE
C           FORM THE INTEGRATION OF THE MATRIX RHO.H.(Y).(Y) OVER THE
C           WORKING ELEMENT.  IN THE THEORY THIS IS MATRIX F.
C
C        USAGE
C           CALL AMATF(I,F,EPK,EPJ,RHOHT)
C
C        DESCRIPTION OF PARAMETERS
C                    I - THE NUMBER OF THE WORKING ELEMENT
C                    F - THE RESULTING 20 X 20 MATRIX
C                  EPK - IDENTIFIED IN MAIN PROGRAM COMMENTS
C                  EPJ - IDENTIFIED IN MAIN PROGRAM COMMENTS
C                RHOHT - DENSITY PER UNIT AREA OF ELEMENT (LB/IN-SQ)
C
C        REMARKS
C           Z, P, AND ZI ARE WORKING MATRICES.  AMATF IS CALLED BY AMASS
C
C        SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
C           INTIL, AMATM
C
C        METHOD
C           WEDDLE'S METHOD OF DOUBLE INTEGRATION IS USED WHICH REQUIRES
C           THE ELEMENT TO BE DIVIDED INTO 6 X 6 SUBDIVISIONS.
C           REFERENCE°  "NUMERICAL MATHEMATICAL ANALYSIS - FIFTH
C                       EDITION" BY J.B. SCARBOROUGH, THE JOHN HOPKINS
C                       PRESS, 1962
C
C     ..................................................................
C
      SUBROUTINE AMATF(I,F,EPK,EPJ,RHOHT)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION Z(3,20),P(20,20),ZI(20,3),F(20,20)
      CALL INTIL(F,20,20)
      E=EPK/6.D0
      D=EPJ/6.D0
      DO 1 K=1,7
      QK=K
      XK=(((QK-1.D0)/6.D0)-.5D0)*EPK
      GO TO (11,12,11,13,11,12,11),K
11    C1=.3D0*E
      GO TO 5
12    C1=1.5D0*E
      GO TO 5
13    C1=1.8D0*E
```

```
5       DO 1 J=1,7
        CALL INTIL (Z,3,20)
        CALL INTIL(P,20,20)
        QJ=J
        YJ=(((QJ-1.D0)/6.D0)-.5D0)*EPJ
        GO TO (14,15,14,16,14,15,14),J
14      C2=.3D0*D
        GO TO 6
15      C2=1.5D0*D
        GO TO 6
16      C2=1.8D0*D
6       DEL=C1*C2
        Z(1,1)=1.D0
        Z(1,2)=XK
        Z(1,3)=YJ
        Z(1,4)=XK*XK
        Z(1,5)=XK*YJ
        Z(1,6)=YJ*YJ
        Z(1,7)=XK**3
        Z(1,8)=XK*XK*YJ
        Z(1,9)=XK*YJ*YJ
        Z(1,10)=YJ**3
        Z(1,11)=YJ*(XK**3)
        Z(1,12)=XK*(YJ**3)
        Z(2,13)=1.D0
        Z(2,14)=XK
        Z(2,15)=YJ
        Z(2,16)=XK*YJ
        Z(3,17)=1.DC
        Z(3,18)=XK
        Z(3,19)=YJ
        Z(3,2C)=XK*YJ
        CALL INTIL(ZI,20,3)
        DO 3 M=1,3
        DO 3 N=1,20
3       ZI(N,M)=Z(M,N)
        CALL AMATM(ZI,2C,3,Z,20,P)
        DC 4 M=1,20
        DO 4 N=1,20
4       F(M,N)=F(M,N)+(P(M,N)*DEL*RHCHT)
1       CONTINUE
        RETURN
        END
```

```
C
C     ..................................................................
C
C        SUBROUTINE AJKLM
C
C        PURPOSE
C           DIVIDES THE 20 X 20 ELEMENT STIFFNESS OR MASS MATRIX INTO
C           SIXTEEN 5 X 5 SUB-MATRICES AND PLACES THEM IN THEIR
C           APPROPRIATE LOCATIONS IN THE GENERAL STIFFNESS OR MASS
C           MATRICES ACCORDING TO THE ASSOCIATED ELEMENT NODES.
C
C        USAGE
C           CALL AJKLM(A,IJKL,B)
C
C        DESCRIPTION OF PARAMETERS
C                       A - THE GENERAL STIFFNESS OR MASS MATRIX, CAN BE
C                           AS LARGE AS 80 X 80
C                    IJKL - THE FOUR TERM VECTOR CONTAINING THE ELEMENT
C                           NODE NUMBERS
C                       B - THE 20 X 20 ELEMENT STIFFNESS OR MASS MATRIX
C
C        REMARKS
C           THE SUBROUTINE IS GENERAL AND IS USED FOR BOTH STIFFNESSS
C           AND MASS MATRIX FORMULATION PRIOR TO IMPOSING NODAL
C           CONSTRAINTS.  AJKLM IS CALLED BY MAIN.
C
C        SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
C           NONE
C
C     ..................................................................
C
      SUBROUTINE AJKLM(A,IJKL,B)
      DOUBLE PRECISION A(80,80),B(20,20)
      DIMENSION IJKL(4)
      J=IJKL(1)
      K=IJKL(2)
      L=IJKL(3)
      M=IJKL(4)
      JJ=5*J-5
      KK=5*K-5
      LL=5*L-5
      MM=5*M-5
C        INSERT (1,1) PARTITION OF B MATRIX
      J1=JJ
      J2=JJ
      DO 1 N=1,5
      J1=J1+1
      DO 11 NN=1,5
      J2=J2+1
```

```
11      A(J1,J2)=B(N,NN)+A(J1,J2)
1       J2=JJ
C          INSERT (2,2) PARTITION OF B MATRIX
        K1=KK
        K2=KK
        DO 2 N=6,10
        K1=K1+1
        DO 20 NN=6,10
        K2=K2+1
20      A(K1,K2)=B(N,NN)+A(K1,K2)
2       K2=KK
C          INSERT (3,3) PARTITION OF B MATRIX
        L1=LL
        L2=LL
        DO 3 N=11,15
        L1=L1+1
        DO 30 NN=11,15
        L2=L2+1
30      A(L1,L2)=B(N,NN)+A(L1,L2)
3       L2=LL
C          INSERT (4,4) PARTITION OF B MATRIX
        M1=MM
        M2=MM
        DO 4 N=16,20
        M1=M1+1
        DO 40 NN=16,20
        M2=M2+1
40      A(M1,M2)=B(N,NN)+A(M1,M2)
4       M2=MM
C          INSERT (1,2) AND (2,1) PARTITIONS OF B MATRIX
        J1=JJ
        K1=KK
        DO 5 N=1,5
        J1=J1+1
        DO 50 NN=6,10
        K1=K1+1
        A(K1,J1)=B(NN,N)+A(K1,J1)
50      A(J1,K1)=B(N,NN)+A(J1,K1)
5       K1=KK
C          INSERT (1,3) AND (3,1) PARTITIONS OF B MATRIX
        J1=JJ
        L1=LL
        DO 6 N=1,5
        J1=J1+1
        DO 60 NN=11,15
        L1=L1+1
        A(L1,J1)=B(NN,N)+A(L1,J1)
60      A(J1,L1)=B(N,NN)+A(J1,L1)
6       L1=LL
```

```
C        INSERT (1,4) AND (4,1) PARTITICNS OF B MATRIX
      J1=JJ
      M1=MM
      DO 7 N=1,5
      J1=J1+1
      DO 70 NN=16,20
      M1=M1+1
      A(M1,J1)=B(NN,N)+A(M1,J1)
70    A(J1,M1)=B(N,NN)+A(J1,M1)
7     M1=MM
C        INSERT (2,3) AND (3,2) PARTITICNS OF B MATRIX
      K1=KK
      L1=LL
      DO 8 N=6,10
      K1=K1+1
      DO 80 NN=11,15
      L1=L1+1
      A(L1,K1)=B(NN,N)+A(L1,K1)
80    A(K1,L1)=B(N,NN)+A(K1,L1)
8     L1=LL
C        INSERT (2,4) AND (4,2) PARTITICNS OF B MATRIX
      K1=KK
      M1=MM
      DO 9 N=6,10
      K1=K1+1
      DO 90 NN=16,20
      M1=M1+1
      A(M1,K1)=B(NN,N)+A(M1,K1)
90    A(K1,M1)=B(N,NN)+A(K1,M1)
9     M1=MM
C        INSERT (3,4) AND (4,3) PARTITICNS OF B MATRIX
      L1=LL
      M1=MM
      DO 10 N=11,15
      L1=L1+1
      DO 100 NN=16,20
      M1=M1+1
      A(M1,L1)=B(NN,N)+A(M1,L1)
100   A(L1,M1)=B(N,NN)+A(L1,M1)
10    M1=MM
      RETURN
      END
```

```
C
C     ..................................................................
C
C        SUBROUTINE EDCON
C
C        PURPOSE
C           IMPOSES THE SELECTED NODAL CONSTRAINTS (32 POSSIBILITIES,
C           SEE MAIN PROGRAM COMMENTS FOR DESCRIPTION) ON THE GENERAL
C           STIFFNESS OR MASS MATRICES.
C
C        USAGE
C           CALL EDCON(A,KN,KEC,KS)
C
C        DESCRIPTION OF PARAMETERS
C                       A - THE GENERAL STIFFNESS OR MASS MATRIX, CAN BE
C                           AS LARGE AS 80 X 80
C                      KN - THE NUMBER OF NODES IN THE PROBLEM
C                     KEC - THE VECTOR ARRAY OF NODAL CONSTRAINT CODES
C                      KS - THE ORDER OF MATRIX A, FOR THE PARTICULAR
C                           PROBLEM INVOLVED
C
C        REMARKS
C           KS IS REDUCED ACCORDING TO THE ZERO CONSTRAINTS IMPOSED.
C           EDCON IS CALLED BY MAIN.
C        SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
C           NONE
C
C        METHOD
C           THE APPROPRIATE ROWS AND COLUMNS ARE DELETED FROM MATRIX A
C           ACCORDING TO THE CONSTRAINT CODES SELECTED.  KS IS REDUCED
C           ALSO AND REMAINS THE ORDER OF MATRIX A.
C
C     ..................................................................
C
      SUBROUTINE EDCON(A,KN,KEC,KS)
      DOUBLE PRECISION A(80,80)
      DIMENSION KEC(25)
      LE=0
      DO 100 JN=1,KN
      L=5*(JN-1)-LE
      K=KEC(JN)
      GO TO (100,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,
     123,24,25,26,27,28,29,30,31,32),K
2     L=L+2
      GO TO 50
3     L=L+3
      GO TO 50
4     L=L+4
      GO TO 50
```

```
5       L=L+5
        GO TO 50
6       L=L+6
50      L1=L-2
        LE=LE+1
        IF(L1)5011,5011,5010
5010    DO 501 M=1,L1
        DO 501 N=L,KS
        A(M,N-1)=A(M,N)
501     A(N-1,M)=A(N,M)
5011    DO 502 M=L,KS
        DO 502 N=L,KS
502     A(M-1,N-1)=A(M,N)
        KS=KS-1
        GO TO 100
7       L=L+3
        L2=L-3
        GO TO 51
8       L1=L+4
        IF(L)78,78,77
77      DO 79 M=1,L
        A(L+1,M)=A(L+2,M)
79      A(M,L+1)=A(M,L+2)
78      L2=L
        A(L+1,L+1)=A(L+2,L+2)
        DO 80 M=L1,KS
        A(L+1,M-2)=A(L+2,M)
80      A(M-2,L+1)=A(M,L+2)
        L=L+4
        GO TO 51
9       L1=L+5
        IF(L)88,88,87
87      DO 89 M=1,L
        A(L+1,M)=A(L+2,M)
        A(L+2,M)=A(L+3,M)
        A(M,L+1)=A(M,L+2)
89      A(M,L+2)=A(M,L+3)
88      L2=L
        A(L+1,L+1)=A(L+2,L+2)
        A(L+1,L+2)=A(L+2,L+3)
        A(L+2,L+1)=A(L+3,L+2)
        A(L+2,L+2)=A(L+3,L+3)
        DO 90 M=L1,KS
        A(L+1,M-2)=A(L+2,M)
        A(L+2,M-2)=A(L+3,M)
        A(M-2,L+1)=A(M,L+2)
90      A(M-2,L+2)=A(M,L+3)
        L=L+5
        GO TO 51
```

```
10    L1=L+1
      L3=L+3
      IF(L)105,105,104
104   DO 103 M=1,L
      DO 103 N=L1,L3
      A(N,M)=A(N+1,M)
103   A(M,N)=A(M,N+1)
105   L2=L
      DO 102 M=L1,L3
      DO 102 N=L1,L3
      A(M,N)=A(M+1,N+1)
102   A(N,M)=A(N+1,M+1)
      L1=L+6
      DO 101 M=L1,KS
      A(L+1,M-2)=A(L+2,M)
      A(L+2,M-2)=A(L+3,M)
      A(L+3,M-2)=A(L+4,M)
      A(M-2,L+1)=A(M,L+2)
      A(M-2,L+2)=A(M,L+3)
101   A(M-2,L+3)=A(M,L+4)
      L=L+6
      GO TO 51
11    L=L+4
      L2=L-3
      GO TO 51
12    L1=L+5
      L2=L+1
      DO 119 M=1,L2
      A(L+2,M)=A(L+3,M)
119   A(M,L+2)=A(M,L+3)
      A(L+2,L+2)=A(L+3,L+3)
      DO 120 M=L1,KS
      A(L+2,M-2)=A(L+3,M)
120   A(M-2,L+2)=A(M,L+3)
      L=L+5
      GO TO 51
13    L1=L+6
      L2=L+1
      DO 129 M=1,L2
      A(L+2,M)=A(L+3,M)
      A(L+3,M)=A(L+4,M)
      A(M,L+2)=A(M,L+3)
129   A(M,L+3)=A(M,L+4)
      A(L+2,L+2)=A(L+3,L+3)
      A(L+2,L+3)=A(L+3,L+4)
      A(L+3,L+2)=A(L+4,L+3)
      A(L+3,L+3)=A(L+4,L+4)
      DO 130 M=L1,KS
      A(L+2,M-2)=A(L+3,M)
```

```
      A(L+3,M-2)=A(L+4,M)
      A(M-2,L+2)=A(M,L+3)
130   A(M-2,L+3)=A(M,L+4)
      L=L+6
      GO TO 51
14    L=L+5
      L2=L-3
      GO TO 51
15    L1=L+6
      L2=L+2
      DO 149 M=1,L2
      A(L+3,M)=A(L+4,M)
149   A(M,L+3)=A(M,L+4)
      A(L+3,L+3)=A(L+4,L+4)
      DO 151 M=L1,KS
      A(L+3,M-2)=A(L+4,M)
151   A(M-2,L+3)=A(M,L+4)
      L=L+6
      GO TO 51
16    L=L+6
      L2=L-3
      GO TO 51
51    LE=LE+2
      IF(L2)5111,5111,5110
5110  DO 511 M=1,L2
      DO 511 N=L,KS
      A(M,N-2)=A(M,N)
511   A(N-2,M)=A(N,M)
5111  DO 512 M=L,KS
      DO 512 N=L,KS
512   A(M-2,N-2)=A(M,N)
      KS=KS-2
      GO TO 100
17    L=L+4
      L2=L-4
      GO TO 52
18    L1=L+5
      L2=L
      IF(L)178,178,177
177   DO 179 M=1,L2
      A(L+1,M)=A(L+3,M)
179   A(M,L+1)=A(M,L+3)
178   A(L+1,L+1)=A(L+3,L+3)
      DO 180 M=L1,KS
      A(L+1,M-3)=A(L+3,M)
180   A(M-3,L+1)=A(M,L+3)
      L=L+5
      GO TO 52
19    L1=L+6
```

```
      L2=L
      IF(L)188,188,187
187   DO 189 M=1,L2
      A(L+1,M)=A(L+3,M)
      A(L+2,M)=A(L+4,M)
      A(M,L+1)=A(M,L+3)
189   A(M,L+2)=A(M,L+4)
188   A(L+1,L+1)=A(L+3,L+3)
      A(L+1,L+2)=A(L+3,L+4)
      A(L+2,L+1)=A(L+4,L+3)
      A(L+2,L+2)=A(L+4,L+4)
      DO 190 M=L1,KS
      A(L+1,M-3)=A(L+3,M)
      A(L+2,M-3)=A(L+4,M)
      A(M-3,L+1)=A(M,L+3)
190   A(M-3,L+2)=A(M,L+4)
      L=L+6
      GO TO 52
20    L1=L+5
      L2=L
      IF(L)198,198,197
197   DO 199 M=1,L2
      A(L+1,M)=A(L+2,M)
199   A(M,L+1)=A(M,L+2)
198   A(L+1,L+1)=A(L+2,L+2)
      DO 200 M=L1,KS
      A(L+1,M-3)=A(L+2,M)
200   A(M-3,L+1)=A(M,L+2)
      L=L+5
      GO TO 52
21    L1=L+6
      L2=L
      IF(L)208,208,207
207   DO 209 M=1,L2
      A(L+1,M)=A(L+2,M)
      A(L+2,M)=A(L+4,M)
      A(M,L+1)=A(M,L+2)
209   A(M,L+2)=A(M,L+4)
208   A(L+1,L+1)=A(L+2,L+2)
      A(L+1,L+2)=A(L+2,L+4)
      A(L+2,L+1)=A(L+4,L+2)
      A(L+2,L+2)=A(L+4,L+4)
      DO 210 M=L1,KS
      A(L+1,M-3)=A(L+2,M)
      A(L+2,M-3)=A(L+4,M)
      A(M-3,L+1)=A(M,L+2)
210   A(M-3,L+2)=A(M,L+4)
      L=L+6
      GO TO 52
```

```
22      L1=L+6
        L2=L
        IF(L)218,218,217
217     DO 219 M=1,L2
        A(L+1,M)=A(L+2,M)
        A(L+2,M)=A(L+3,M)
        A(M,L+1)=A(M,L+2)
219     A(M,L+2)=A(M,L+3)
218     A(L+1,L+1)=A(L+2,L+2)
        A(L+1,L+2)=A(L+2,L+3)
        A(L+2,L+1)=A(L+3,L+2)
        A(L+2,L+2)=A(L+3,L+3)
        DO 220 M=L1,KS
        A(L+1,M-3)=A(L+2,M)
        A(L+2,M-3)=A(L+3,M)
        A(M-3,L+1)=A(M,L+2)
220     A(M-3,L+2)=A(M,L+3)
        L=L+6
        GO TO 52
23      L=L+5
        L2=L-4
        GO TO 52
24      L1=L+6
        L2=L+1
        DO 239 M=1,L2
        A(L+2,M)=A(L+4,M)
239     A(M,L+2)=A(M,L+4)
        A(L+2,L+2)=A(L+4,L+4)
        DO 240 M=L1,KS
        A(L+2,M-3)=A(L+4,M)
240     A(M-3,L+2)=A(M,L+4)
        L=L+6
        GO TO 52
25      L1=L+6
        L2=L+1
        DO 249 M=1,L2
        A(L+2,M)=A(L+3,M)
249     A(M,L+2)=A(M,L+3)
        A(L+2,L+2)=A(L+3,L+3)
        DO 250 M=L1,KS
        A(L+2,M-3)=A(L+3,M)
250     A(M-3,L+2)=A(M,L+3)
        L=L+6
        GO TO 52
26      L=L+6
        L2=L-4
        GO TO 52
52      LE=LE+3
        IF(L2)5211,5211,5210
```

```
5210  DO 521 M=1,L2
      DO 521 N=L,KS
      A(M,N-3)=A(M,N)
521   A(N-3,M)=A(N,M)
5211  DO 522 M=L,KS
      DO 522 N=L,KS
522   A(M-3,N-3)=A(M,N)
      KS=KS-3
      GO TO 100
27    L=L+5
      L2=L-5
      GO TO 53
28    L1=L+6
      L2=L
      IF(L)278,278,277
277   DO 279 M=1,L2
      A(L+1,M)=A(L+4,M)
279   A(M,L+1)=A(M,L+4)
278   A(L+1,L+1)=A(L+4,L+4)
      DO 280 M=L1,KS
      A(L+1,M-4)=A(L+4,M)
280   A(M-4,L+1)=A(M,L+4)
      L=L+6
      GO TO 53
29    L1=L+6
      L2=L
      IF(L)288,288,287
287   DO 289 M=1,L2
      A(L+1,M)=A(L+2,M)
289   A(M,L+1)=A(M,L+2)
288   A(L+1,L+1)=A(L+2,L+2)
      DO 290 M=L1,KS
      A(L+1,M-4)=A(L+2,M)
290   A(M-4,L+1)=A(M,L+2)
      L=L+6
      GO TO 53
30    L1=L+6
      L2=L
      IF(L)298,298,297
297   DO 299 M=1,L2
      A(L+1,M)=A(L+3,M)
299   A(M,L+1)=A(M,L+3)
298   A(L+1,L+1)=A(L+3,L+3)
      DO 300 M=L1,KS
      A(L+1,M-4)=A(L+3,M)
300   A(M-4,L+1)=A(M,L+3)
      L=L+6
      GO TO 53
31    L=L+6
```

```
      L2=L-5
      GO TO 53
53    LE=LE+4
      IF(L2)5311,5311,5310
5310  DO 531 M=1,L2
      DO 531 N=L,KS
      A(M,N-4)=A(M,N)
531   A(N-4,M)=A(N,M)
5311  DO 532 M=L,KS
      DO 532 N=L,KS
532   A(M-4,N-4)=A(M,N)
      KS=KS-4
      GO TO 100
32    L=L+6
      LE=LE+5
      L1=L-6
      IF(L1)320,320,319
319   DO 321 M=1,L1
      DO 321 N=L,KS
      A(M,N-5)=A(M,N)
321   A(N-5,M)=A(N,M)
320   DO 322 M=L,KS
      DO 322 N=L,KS
322   A(M-5,N-5)=A(M,N)
      KS=KS-5
100   CONTINUE
      RETURN
      END
```

```
C
C     ..................................................................
C
C        SUBROUTINE NROOT
C
C        PURPOSE
C           COMPUTE EIGENVALUES AND EIGENVECTORS OF A REAL NONSYMMETRIC
C           MATRIX OF THE FORM B-INVERSE TIMES A.
C
C        USAGE
C           CALL NROOT (M,A,B,XL,X,F8)
C
C        DESCRIPTION OF PARAMETERS
C           M  - ORDER OF SQUARE MATRICES A, B, AND X.
C           A  - INPUT MATRIX (M X M).
C           B  - INPUT MATRIX (M X M).
C           XL - OUTPUT VECTOR OF LENGTH M CONTAINING EIGENVALUES OF
C                B-INVERSE TIMES A.
C           X  - OUTPUT MATRIX (M X M) CONTAINING EIGENVECTORS COLUMN-
C                WISE.
C           F8 - KEY FOR PRINTOUT OF THE EIGENVALUES OF MATRIX B.
C                IF F8 IS POSITIVE, VALUES ARE PRINTED.
C
C        REMARKS
C           NONE
C
C        SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
C           EIGEN
C
C        METHOD
C           REFER TO W. W. COOLEY AND P. R. LOHNES, 'MULTIVARIATE PRO-
C           CEDURES FOR THE BEHAVIORAL SCIENCES', JOHN WILEY AND SONS,
C           1962, CHAPTER 3.
C
C     ..................................................................
C
C
C        ...............................................................
C
C        IF A DOUBLE PRECISION VERSION OF THIS ROUTINE IS DESIRED, THE
C        C IN COLUMN 1 SHOULD BE REMOVED FROM THE DOUBLE PRECISION
C        STATEMENT.
C
C
C        THE C MUST ALSO BE REMOVED FROM DOUBLE PRECISION STATEMENTS
C        APPEARING IN OTHER ROUTINES USED IN CONJUNCTION WITH THIS
C        ROUTINE.
C
C        THE DOUBLE PRECISION VERSION OF THIS SUBROUTINE MUST ALSO
```

```
C          CONTAIN DOUBLE PRECISION FORTRAN FUNCTIONS.  SQRT IN STATEMENTS
C          110 AND 175 MUST BE CHANGED TO DSQRT.  ABS IN STATEMENT 110
C          MUST BE CHANGED TO DABS.
C
C          ...............................................................
C
C      COMPUTE EIGENVALUES AND EIGENVECTORS OF B
C
       SUBROUTINE NROOT (M,A,B,XL,X,F8)
       DIMENSION A(1),B(1),XL(1),X(1)
       DOUBLE PRECISION A,B,XL,X,SUMV,F8
1      FORMAT(1H1,'   EIGENVALUES OF MATRIX B',///)
2      FORMAT(6X,D24.15)
       K=1
       DO 100 J=2,M
       L=M*(J-1)
       DO 100 I=1,J
       L=L+1
       K=K+1
  100  B(K)=B(L)
C
C          THE MATRIX B IS A REAL SYMMETRIC MATRIX.
C
       MV=0
       CALL EIGEN (B,X,M,MV)
C
C      FORM RECIPROCALS OF SQUARE ROOT OF EIGENVALUES.  THE RESULTS
C      ARE PREMULTIPLIED BY THE ASSOCIATED EIGENVECTORS.
C
       IF(F8)6,6,5
5      WRITE(3,1)
6      L=0
       DO 110 J=1,M
       L=L+J
       IF(F8)110,110,7
7      WRITE(3,2)B(L)
  110  XL(J)=1.0/DSQRT(DABS(B(L)))
       K=0
       DO 115 J=1,M
       DO 115 I=1,M
       K=K+1
  115  B(K)=X(K)*XL(J)
C
C      FORM (B**(-1/2))PRIME * A * (B**(-1/2))
C
       DO 120 I=1,M
       N2=0
       DO 120 J=1,M
       N1=M*(I-1)
```

```
      L=M*(J-1)+I
      X(L)=0.0
      DO 120 K=1,M
      N1=N1+1
      N2=N2+1
  120 X(L)=X(L)+B(N1)*A(N2)
      L=0
      DO 130 J=1,M
      DO 130 I=1,J
      N1=I-M
      N2=M*(J-1)
      L=L+1
      A(L)=0.0
      DO 130 K=1,M
      N1=N1+M
      N2=N2+1
  130 A(L)=A(L)+X(N1)*B(N2)
C
C     COMPUTE EIGENVALUES AND EIGENVECTORS OF A
C
      CALL EIGEN (A,X,M,MV)
      L=0
      DO 140 I=1,M
      L=L+I
  140 XL(I)=A(L)
C
C     COMPUTE THE NORMALIZED EIGENVECTORS
C
      DO 150 I=1,M
      N2=0
      DO 150 J=1,M
      N1=I-M
      L=M*(J-1)+I
      A(L)=0.0
      DO 150 K=1,M
      N1=N1+M
      N2=N2+1
  150 A(L)=A(L)+B(N1)*X(N2)
      L=0
      K=0
      DO 180 J=1,M
      SUMV=0.0
      DO 170 I=1,M
      L=L+1
  170 SUMV=SUMV+A(L)*A(L)
  175 SUMV=DSQRT(SUMV)
      DO 180 I=1,M
      K=K+1
  180 X(K)=A(K)/SUMV
```

```
RETURN
END
```

```
C
C     ..................................................................
C
C        SUBROUTINE EIGEN
C
C        PURPOSE
C           COMPUTE EIGENVALUES AND EIGENVECTORS OF A REAL SYMMETRIC
C           MATRIX
C
C        USAGE
C           CALL EIGEN(A,R,N,MV)
C
C        DESCRIPTION OF PARAMETERS
C           A - ORIGINAL MATRIX (SYMMETRIC), DESTROYED IN COMPUTATION.
C               RESULTANT EIGENVALUES ARE DEVELOPED IN DIAGONAL OF
C               MATRIX A IN DESCENDING ORDER.
C           R - RESULTANT MATRIX OF EIGENVECTORS (STORED COLUMNWISE,
C               IN SAME SEQUENCE AS EIGENVALUES)
C           N - ORDER OF MATRICES A AND R
C           MV- INPUT CODE
C                   0   COMPUTE EIGENVALUES AND EIGENVECTORS
C                   1   COMPUTE EIGENVALUES ONLY (R NEED NOT BE
C                       DIMENSIONED BUT MUST STILL APPEAR IN CALLING
C                       SEQUENCE)
C
C        REMARKS
C           ORIGINAL MATRIX A MUST BE REAL SYMMETRIC (STORAGE MODE=1)
C           MATRIX A CANNOT BE IN THE SAME LOCATION AS MATRIX R
C
C        SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
C           NONE
C
C        METHOD
C           DIAGONALIZATION METHOD ORIGINATED BY JACOBI AND ADAPTED
C           BY VON NEUMANN FOR LARGE COMPUTERS AS FOUND IN 'MATHEMATICAL
C           METHODS FOR DIGITAL COMPUTERS', EDITED BY A. RALSTON AND
C           H.S. WILF, JOHN WILEY AND SONS, NEW YORK, 1962, CHAPTER 7
C
C     ..................................................................
C
C
C        ...............................................................
C
C        IF A DOUBLE PRECISION VERSION OF THIS ROUTINE IS DESIRED, THE
C        C IN COLUMN 1 SHOULD BE REMOVED FROM THE DOUBLE PRECISION
C        STATEMENT.
C
C
C        THE C MUST ALSO BE REMOVED FROM DOUBLE PRECISION STATEMENTS
```

```
C           APPEARING IN CTHER ROUTINES USED IN CCNJUNCTION WITH THIS
C           ROUTINE.
C
C           THE DOUBLE PRECISION VERSION CF THIS SUBROUTINE MUST ALSC
C           CONTAIN DOUBLE PRECISION FORTRAN FUNCTIONS.  SQRT IN STATEMENTS
C           40, 68, 75, AND 78 MUST BE CHANGED TC DSQRT.  ABS IN STATEMENT
C           62 MUST BE CHANGED TO DABS. THE CONSTANT IN STATEMENT 5 SHOULD
C           BE CHANGED TC 1.0D-12.
C
C           ..............................................................
C
C           GENERATE IDENTITY MATRIX
C
      SUBROUTINE EIGEN(A,R,N,MV)
      DIMENSION A(1),R(1)
      DOUBLE PRECISION A,R,ANORM,ANRMX,THR,X,Y,SINX,SINX2,COSX,
     1                 COSX2,SINCS,RANGE
    5 RANGE=1.0D-12
      IF(MV-1) 10,25,10
   10 IQ=-N
      DO 20 J=1,N
      IQ=IQ+N
      DO 20 I=1,N
      IJ=IQ+I
      R(IJ)=0.0
      IF(I-J) 20,15,20
   15 R(IJ)=1.0
   20 CONTINUE
C
C           COMPUTE INITIAL AND FINAL NORMS (ANORM AND ANORMX)
C
   25 ANORM=0.0
      DO 35 I=1,N
      DO 35 J=I,N
      IF(I-J) 30,35,30
   30 IA=I+(J*J-J)/2
      ANORM=ANORM+A(IA)*A(IA)
   35 CONTINUE
      IF(ANORM) 165,165,40
   40 ANORM=1.414*DSQRT(ANORM)
      ANRMX=ANORM*RANGE/FLOAT(N)
C
C           INITIALIZE INDICATORS AND COMPUTE THRESHOLD, THR
C
      IND=0
      THR=ANORM
   45 THR=THR/FLOAT(N)
   50 L=1
   55 M=L+1
```

```
C
C          COMPUTE SIN AND CCS
C
   60 MQ=(M*M-M)/2
      LQ=(L*L-L)/2
      LM=L+MQ
   62 IF(DABS(A(LM))-THR) 130,65,65
   65 IND=1
      LL=L+LQ
      MM=M+MQ
      X=0.5*(A(LL)-A(MM))
   68 Y=-A(LM)/DSQRT(A(LM)*A(LM)+X*X)
      IF(X) 70,75,75
   70 Y=-Y
   75 SINX=Y/DSQRT(2.0*(1.0+(DSQRT(1.0-Y*Y))))
      SINX2=SINX*SINX
   78 COSX=DSQRT(1.0-SINX2)
      COSX2=COSX*COSX
      SINCS =SINX*COSX
C
C          ROTATE L AND M CCLUMNS
C
      ILQ=N*(L-1)
      IMQ=N*(M-1)
      DO 125 I=1,N
      IQ=(I*I-I)/2
      IF(I-L) 80,115,80
   80 IF(I-M) 85,115,90
   85 IM=I+MQ
      GO TO 95
   90 IM=M+IQ
   95 IF(I-L) 100,105,105
  100 IL=I+LQ
      GO TO 110
  105 IL=L+IQ
  110 X=A(IL)*COSX-A(IM)*SINX
      A(IM)=A(IL)*SINX+A(IM)*COSX
      A(IL)=X
  115 IF(MV-1) 120,125,120
  120 ILR=ILQ+I
      IMR=IMQ+I
      X=R(ILR)*COSX-R(IMR)*SINX
      R(IMR)=R(ILR)*SINX+R(IMR)*COSX
      R(ILR)=X
  125 CONTINUE
      X=2.0*A(LM)*SINCS
      Y=A(LL)*COSX2+A(MM)*SINX2-X
      X=A(LL)*SINX2+A(MM)*COSX2+X
      A(LM)=(A(LL)-A(MM))*SINCS+A(LM)*(COSX2-SINX2)
```

```
      A(LL)=Y
      A(MM)=X
C
C        TESTS FOR COMPLETION
C
C        TEST FOR M = LAST COLUMN
C
  130 IF(M-N) 135,140,135
  135 M=M+1
      GO TO 60
C
C        TEST FOR L = SECOND FROM LAST COLUMN
C
  140 IF(L-(N-1)) 145,150,145
  145 L=L+1
      GO TO 55
  150 IF(IND-1) 160,155,160
  155 IND=0
      GO TO 50
C
C        COMPARE THRESHOLD WITH FINAL NORM
C
  160 IF(THR-ANRMX) 165,165,45
C
C        SORT EIGENVALUES AND EIGENVECTORS
C
  165 IQ=-N
      DO 185 I=1,N
      IQ=IQ+N
      LL=I+(I*I-I)/2
      JQ=N*(I-2)
      DO 185 J=I,N
      JQ=JQ+N
      MM=J+(J*J-J)/2
      IF(A(LL)-A(MM)) 170,185,185
  170 X=A(LL)
      A(LL)=A(MM)
      A(MM)=X
      IF(MV-1) 175,185,175
  175 DO 180 K=1,N
      ILR=IQ+K
      IMR=JQ+K
      X=R(ILR)
      R(ILR)=R(IMR)
  180 R(IMR)=X
  185 CONTINUE
      RETURN
      END
```

```
C
C     ...................................................................
C
C        SUBROUTINE VNORM
C
C        PURPOSE
C           NORMALIZE COLUMNS OF AN N ORDER MATRIX
C
C        USAGE
C           CALL VNORM(A,N)
C
C        DESCRIPTION OF PARAMETERS
C                    A - THE MATRIX WHOSE COLUMNS ARE TO BE NORMALIZED
C                    N - THE LENGTH AND NUMBER OF THE COLUMNS (80 MAX)
C
C        REMARKS
C           THE SUBROUTINE IS GENERAL; HOWEVER, IT IS ONLY CALLED BY
C           MAIN TO NORMALIZE THE EIGENVECTORS.
C
C        SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
C           NONE
C
C        METHOD
C           A COMPARATIVE SEARCH IS DONE ON EACH VALUE OF A COLUMN TO
C           DETERMINE THE LARGEST ABSOLUTE VALUE, VMAX.  ONCE DETERMINED
C           EACH VALUE IS DIVIDED BY IT SO THAT THE MAX ABSOLUTE VALUE
C           IN THE COLUMN IS 1.
C
C     ...................................................................
C
      SUBROUTINE VNORM(A,N)
      DOUBLE PRECISION A(80,80),Z,VMAX
      DO 1 J=1,N
      VMAX=DABS(A(1,J))
      DO 3 K=2,N
      Z=DABS(A(K,J))
      IF (Z-VMAX)3,3,2
2     VMAX=Z
3     CONTINUE
      DO 1 L=1,N
1     A(L,J)=A(L,J)/VMAX
      RETURN
      END
```

```
C
C     ..................................................................
C
C        SUBROUTINE EDCONI
C
C        PURPOSE
C           PUT THE ZERO ROWS BACK INTO THE EIGENVECTORS ACCORDING
C           TO THE NODAL CONSTRAINTS IMPOSED.
C
C        USAGE
C           CALL EDCONI(A,KN,KEC,KS)
C
C        DESCRIPTION OF PARAMETERS
C                    A - THE MATRIX OF EIGENVECTORS
C                   KN - IDENTIFIED IN MAIN PROGRAM COMMENTS
C                  KEC - IDENTIFIED IN MAIN PROGRAM COMMENTS
C                   KS - IDENTIFIED IN MAIN PROGRAM COMMENTS
C
C        REMARKS
C           EDCONI IS IN A SENSE THE INVERSE OF SUBROUTINE EDCON; EDCONI
C           EXPANDS A MATRIX BY PUTTING THE ZERO ROWS BACK INTO IT
C           WHILE EDCON COLLAPSED A MATRIX BY TAKING THEM OUT.
C           EDCONI IS CALLED BY MAIN.
C
C        SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
C           NONE
C
C     ..................................................................
C
      SUBROUTINE EDCONI(A,KN,KEC,KS)
      DOUBLE PRECISION A(80,80)
      DIMENSION KEC(25)
      KA=KS
      DO 100 JN=1,KN
      L=5*(JN-1)
      K=KEC(JN)
      GO TO (100,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,
     123,24,25,26,27,28,29,30,31,32),K
2     L=L+2
      GO TO 50
3     L=L+3
      GO TO 50
4     L=L+4
      GO TO 50
5     L=L+5
      GO TO 50
6     L=L+6
50    KS=KS+1
      J=KS+1
```

```
      DO 502 M=L,KS
      J=J-1
      DO 502 N=1,KA
502   A(J,N)=A(J-1,N)
      DO 501 N=1,KA
501   A(L-1,N)=0.DC
      GO TO 100
7     L=L+1
      GO TO 51
8     L=L+2
      GO TO 51
9     L=L+3
      GO TO 51
10    L=L+4
      GO TO 51
11    L=L+2
      GO TO 51
12    L=L+3
      GO TO 51
13    L=L+4
      GO TO 51
14    L=L+3
      GO TO 51
15    L=L+4
      GO TO 51
16    L=L+4
      GO TO 51
51    KS=KS+2
      J=KS+1
      L=L+2
      DO 511 M=L,KS
      J=J-1
      DO 511 N=1,KA
511   A(J,N)=A(J-2,N)
      L=5*(JN-1)
      K=K-6
      GO TO (71,81,91,1011,111,121,131,141,1511,161),K
71    L2=L+1
      L3=L+2
      GO TO 510
81    L2=L+1
      L3=L+3
      DO 80 M=1,KA
80    A(L+2,M)=A(L+1,M)
      GO TO 510
91    L2=L+1
      L3=L+4
      DO 90 M=1,KA
      A(L+3,M)=A(L+2,M)
```

```
90      A(L+2,M)=A(L+1,M)
        GO TO 510
1011    L2=L+1
        L3=L+5
        DO 101 M=1,KA
        A(L+4,M)=A(L+3,M)
        A(L+3,M)=A(L+2,M)
101     A(L+2,M)=A(L+1,M)
        GO TO 510
111     L2=L+2
        L3=L+3
        GO TO 510
121     L2=L+2
        L3=L+4
        DO 120 M=1,KA
120     A(L+3,M)=A(L+2,M)
        GO TO 510
131     L2=L+2
        L3=L+5
        DO 130 M=1,KA
        A(L+4,M)=A(L+3,M)
130     A(L+3,M)=A(L+2,M)
        GO TO 510
141     L2=L+3
        L3=L+4
        GO TO 510
1511    L2=L+3
        L3=L+5
        DO 150 M=1,KA
150     A(L+4,M)=A(L+3,M)
        GO TO 510
161     L2=L+4
        L3=L+5
510     DO 5101 M=1,KS
        A(L2,M)=0.D0
5101    A(L3,M)=0.D0
        GO TO 100
17      L=L+1
        GO TO 52
18      L=L+2
        GO TO 52
19      L=L+3
        GO TO 52
20      L=L+2
        GO TO 52
21      L=L+3
        GO TO 52
22      L=L+3
        GO TO 52
```

```
23      L=L+2
        GO TO 52
24      L=L+3
        GO TO 52
25      L=L+3
        GO TO 52
26      L=L+3
        GO TO 52
52      KS=KS+3
        J=KS+1
        L=L+3
        DO 521 M=L,KS
        J=J-1
        DO 521 N=1,KA
521     A(J,N)=A(J-3,N)
        L=5*(JN-1)
        K=K-16
        GO TO (171,181,191,201,211,221,231,241,251,261),K
171     L2=L+1
        L3=L+2
        L4=L+3
        GO TO 520
181     L2=L+1
        L3=L+2
        L4=L+4
        DO 180 M=1,KA
180     A(L+3,M)=A(L+1,M)
        GO TO 520
191     L2=L+1
        L3=L+2
        L4=L+5
        DO 190 M=1,KA
        A(L+4,M)=A(L+2,M)
190     A(L+3,M)=A(L+1,M)
        GO TO 520
201     L2=L+1
        L3=L+3
        L4=L+4
        DO 200 M=1,KA
200     A(L+2,M)=A(L+1,M)
        GO TO 520
211     L2=L+1
        L3=L+3
        L4=L+5
        DO 210 M=1,KA
        A(L+4,M)=A(L+2,M)
210     A(L+2,M)=A(L+1,M)
        GO TO 520
221     L2=L+1
```

```
      L3=L+4
      L4=L+5
      DO 220 M=1,KA
      A(L+3,M)=A(L+2,M)
220   A(L+2,M)=A(L+1,M)
      GO TO 520
231   L2=L+2
      L3=L+3
      L4=L+4
      GO TO 520
241   L2=L+2
      L3=L+3
      L4=L+5
      DO 240 M=1,KA
240   A(L+4,M)=A(L+2,M)
      GO TO 520
251   L2=L+2
      L3=L+4
      L4=L+5
      DO 250 M=1,KA
250   A(L+3,M)=A(L+2,M)
      GO TO 520
261   L2=L+3
      L3=L+4
      L4=L+5
      GO TO 520
520   DO 5201 M=1,KS
      A(L2,M)=0.DO
      A(L3,M)=0.DO
5201  A(L4,M)=0.DO
      GO TO 100
27    L=L+1
      GO TO 53
28    L=L+2
      GO TO 53
29    L=L+2
      GO TO 53
30    L=L+2
      GO TO 53
31    L=L+2
      GO TO 53
53    KS=KS+4
      J=KS+1
      L=L+4
      DO 531 M=L,KS
      J=J-1
      DO 531 N=1,KA
531   A(J,N)=A(J-4,N)
      L=5*(JN-1)
```

```
      K=K-26
      GO TO (271,281,291,301,311),K
271   L2=L+1
      L3=L+2
      L4=L+3
      L5=L+4
      GO TO 530
281   L2=L+1
      L3=L+2
      L4=L+3
      L5=L+5
      DO 280 M=1,KA
280   A(L+4,M)=A(L+1,M)
      GO TO 530
291   L2=L+1
      L3=L+3
      L4=L+4
      L5=L+5
      DO 290 M=1,KA
290   A(L+2,M)=A(L+1,M)
      GO TO 530
301   L2=L+1
      L3=L+2
      L4=L+4
      L5=L+5
      DO 300 M=1,KA
300   A(L+3,M)=A(L+1,M)
      GO TO 530
311   L2=L+2
      L3=L+3
      L4=L+4
      L5=L+5
      GO TO 530
530   DO 5301 M=1,KS
      A(L2,M)=0.D0
      A(L3,M)=0.D0
      A(L4,M)=0.D0
5301  A(L5,M)=0.D0
      GO TO 100
32    L=L+6
      KS=KS+5
      J=KS+1
      DO 321 M=L,KS
      J=J-1
      DO 321 N=1,KA
321   A(J,N)=A(J-5,N)
      L=5*(JN-1)
      L1=L+1
      L2=L+2
```

```
      L3=L+3
      L4=L+4
      L5=L+5
      DO 3220 M=1,KS
      A(L1,M)=0.D0
      A(L2,M)=C.D0
      A(L3,M)=0.DC
      A(L4,M)=0.D0
3220  A(L5,M)=C.D0
10C   CONTINUE
      RETURN
      END
```

```
C
C     ..................................................................
C
C        SUBROUTINE MODAL
C
C        PURPOSE
C           DETERMINE THE VALUES FOR THE COEFFICIENTS OF THE ASSUMED
C           MODAL EXPRESSIONS.  THEN DETERMINE THE MODAL DEFLECTIONS,
C           W, U, AND V, AT 25 LOCATIONS (5X5 GRID) ON EACH ELEMENT FOR
C           A SELECTED NUMBER OF THE LOWER FREQUENCIES (OTHER THAN
C           ZERO FREQUENCIES).
C
C        USAGE
C           CALL MODAL(E,EI,KS,KE,EK,EJ,HD,DATE,IJK,KB)
C
C        DESCRIPTION OF PARAMETERS
C                     E - VECTOR OF LOWEST KB MODAL FREQUENCIES.
C                    EI - MATRIX OF CORRESPONDING EIGENVECTORS
C                    KS - IDENTIFIED IN MAIN PROGRAM COMMENTS
C                    KE - IDENTIFIED IN MAIN PROGRAM COMMENTS
C                    EK - IDENTIFIED IN MAIN PROGRAM COMMENTS
C                    EJ - IDENTIFIED IN MAIN PROGRAM COMMENTS
C                    HD - IDENTIFIED IN MAIN PROGRAM COMMENTS
C                  DATE - IDENTIFIED IN MAIN PROGRAM COMMENTS
C                   IJK - IDENTIFIED IN MAIN PROGRAM COMMENTS
C                    KB - IDENTIFIED IN MAIN PROGRAM COMMENTS
C
C        REMARKS
C           MODAL IS CALLED BY MAIN.
C
C        SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
C           AMAT, AMATM,PI5J
C
C        METHOD
C           THE AK MATRIX IS REDETERMINED FOR EACH ELEMENT.  THEN EACH
C           ELEMENT'S PORTION OF THE WORKING EIGENVECTOR IS SEPARATED
C           OUT AS AVEC(K).  THE CONSTANTS ARE THEN FORMED AS AC(K)=
C           (AK)(AVEC).  SUBSTITUTING THESE VALUES INTO THE ASSUMED
C           MODAL EXPRESSIONS, THE DEFLECTIONS ARE DETERMINED AT 25
C           SUB-POINTS OF EACH ELEMENT AND THE VALUES PRINTED OUT.
C
C     ..................................................................
C
      SUBROUTINE MODAL(E,EI,KS,KE,EK,EJ,HD,DATE,IJK,KB)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION E(80),EI(80,80),EK(16),EJ(16),AK(20,20),HD(18),DATE(3)
      DIMENSION AVEC(20),AC(20),W(5,5),U(5,5),V(5,5),IJK(16,5)
    1 FORMAT(//,3X,'MODAL FREQUENCY =',F12.6,' CPS',9X,'MODE SHAPE FOR E
     1LEMENT NO.',I4,/)
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11    FORMAT(1H1)
      WRITE(3,11)
      NC=0
      DO 60 M=1,KB
      DO 60 I=1,KE
      WRITE(3,1)E(M),I
      NC=NC+4
      EPK=EK(I)
      EPJ=EJ(I)
      IF(IJK(I,5))13,12,13
12    CALL AMAT(AK,EPK,EPJ)
      DO 3 J=1,20
      DO 3 K=1,20
      IF(DABS(AK(J,K))-(1.0D-25))2,2,3
2     AK(J,K)=0.D0
3     CONTINUE
13    L2=0
      DO 61 K=1,4
      L=(5*IJK(I,K))-4
      L1=L+4
      DO 61 J=L,L1
      L2=L2+1
61    AVEC(L2)=EI(J,M)
      DO 5 J=1,20
      IF(DABS(AVEC(J))-(1.0D-25))4,4,5
4     AVEC(J)=0.D0
5     CONTINUE
      CALL AMATM(AK,20,20,AVEC,1,AC)
      DO 62 K=1,5
      QK=K
      XK=(((QK-1.D0)/4.D0)-.5D0)*EPK
      DO 62 J=1,5
      QJ=J
      YJ=(.5D0-((QJ-1.D0)/4.D0))*EPJ
      W(K,J)=AC(1)+(AC(2)*XK)+(AC(3)*YJ)+(AC(4)*XK*XK)+(AC(5)*XK*YJ)+(AC
     1(6)*YJ*YJ)+(AC(7)*(XK**3))+(AC(8)*XK*XK*YJ)+(AC(9)*XK*YJ*YJ)+(AC(1
     20)*(YJ**3))+(AC(11)*(YJ*(XK**3)))+(AC(12)*(XK*(YJ**3)))
      U(K,J)=AC(13)+(AC(14)*XK)+(AC(15)*YJ)+(AC(16)*XK*YJ)
62    V(K,J)=AC(17)+(AC(18)*XK)+(AC(19)*YJ)+(AC(20)*XK*YJ)
      CALL PI5J(W,5,5,5,5,HD,DATE,NC,20)
      CALL PI5J(U,5,5,5,5,HD,DATE,NC,21)
60    CALL PI5J(V,5,5,5,5,HD,DATE,NC,22)
      RETURN
      END
```