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JORDAN-KALUZA-KLEIN TYPE UNIFIED THEORIES OF GAUGE AND GRAVITY FIELDS

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A DISSERTATION

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in partial fulfillment of the requirements for the

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By

TERRY D. BRADFIELD

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JORDAN-KALUZA-KLEIN TYPE UNIFIED THEORIES

OF GAUGE AND GRAVITY FIELDS

A DISSERTATION

APPROVED FOR THE DEPARTMENT OF PHYSICS AND ASTRONOMY

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CHAPTER I

INTRODUCTION

It is now widely accepted that the theory of fiber bundles provides a convenient description of the gauge fields encountered in physics from a geometric viewpoint analogous to that of gravitation in General Relativity. In both theories, the fields which mediate the forces between material particles are assumed to arise from a connection on space time. In General Relativity, the connection is a linear connection which defines covariant differentiation for the tangent space of spacetime and its tensor products. Bundle theory generalizes the idea of a connection to more-or-less arbitrary vector spaces associated with spacetime for which an action of the structure group G of the bundle on the vector space can be defined.

In the bundle formalism, the vector potentials $A^{\alpha}_{\ a}$[1] are analogous to the connection symbols Γ^{a}_{bc} in General Relativity; they are the components of the connection form on spacetime.

The gauge field tensor F^{α}_{ab} is the bundle equivalent of the Riemann tensor $R^{a}_{\ bcd}$, its components are the components of the curvature form of the connection. The linear connection Γ is in fact a special case

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[1]Throughout this thesis I shall use Greek letters to denote internal degrees of freedom and Latin indices to denote those of spacetime.

of bundle connections, that for which the structure group is GL(4,R) and the vector space is the tangent space to spacetime. The standard minimal coupling rule assumed in gauge theories for the interaction of particle and gauge fields is a consequence of the assumption that the change in particle fields from point-to-point in spacetime is measured by the covariant derivative defined by the connection.

The main application of bundle techniques to date has been in the construction of Unified Field Theories for gravitation and gauge fields of the Jordan-Kaluza-Klein type [1-3]. In the original approach outlined by these authors, spacetime was replaced by a five-dimensional space as the arena for physics; the fifth dimension was assumed to have no direct physical significance. The expectation was that electromagnetic effects could then be described by the new components in the connection coefficients introduced into the theory by the extra dimension. As in General Relativity, the connection components were assumed to be calculable from the metric for this space. To give the new components in this metric the correct vector potential structure, it was necessary to impose a "strong cylindricy condition" in the theory, which meant that physical fields were to depend only on four of the five coordinates in the new space.[1]

The bundle formalism makes it possible to sidestep this approach to a geometric interpretation of gauge fields, but generalizations of the Jordan-Kaluza-Klein theory are still of interest in physics. The reasons for this can be outlined as follows.

[1]For more details on these ideas, see Bergmann [4].

Although the vector potentials and the field tensor arise naturally as geometric structures in the bundle formalism, to determine their explicit form in a given situation we must somehow specify field equations for the A^{α}_{a}'s. Of course, the obvious way to do this is to give a Lagrangian for the fields. There is not unfortunately any canonical way to do this, although in geometric theories the most appealing candidates[1] for this object are the invariant forms that can be built from the curvature. In a fiber bundle, the simplest such object is the "square" of the curvature form

$$L = {*\Omega} \wedge \Omega \, . \tag{1.1}$$

This N-Form on the bundle is invariant under coordinate transformations and its value is independent of the internal degrees of freedom in the bundle manifold. Since it turns out to be quadratic in the field tensor F^{α}_{ab}, application of the standard field variation methods to it will produce the correct field equations for general gauge fields in a vacuum.

Interaction of the gauge fields with matter is taken into account in the same way as in General Relativity; kinetic and potential terms for the matter fields present are added to the Lagrangian with ordinary partial derivatives replaced by covariant ones.

Although this procedure produces the correct results for gauge fields in interaction with matter, a bundle theory of this type stands by itself, without any apparent connection with the other successful geometric theory of particle interactions: gravitation. It is therefore natural to seek an extension of the bundle theory

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[1]For a discussion of this topic see Trautman [5].

which will incorporate both gravitational and gauge fields. If, as it is currently believed, both the strong and the electroweak interactions are correctly described by the gauge formalism, such a theory would be the long sought unified[1] theory of physics. The Jordan-Kaluza-Klein type theories are the simplest and most obvious such extensions that would appear to accomplish this objective.

To construct a theory of this type, one begins with a bundle manifold which has spacetime as its base space and a structure group determined by the particle fields being described. If we choose the appropriate basis for the tangent space of the bundle manifold, then a metric for it can be constructed from a metric on spacetime and a metric on the structure group. From this metric we can calculate the associated linear connection and the Ricci scalar built from it. If we adopt the Ricci scalar as the Lagrangian Density of the theory, the resulting field equations will be the standard ones for coupled gravitational and gauge fields. Calculations of this type were first suggested by Trautman [6] and have been carried out by Cho and Freund [7] for a theory based on a torsion free linear connection on the bundle, and by Kopczynski [8] for a theory which includes torsion.

The successes of this approach encourage the idea that bundle space is more than a mere mathematical tool; that it may in fact represent the true arena in which particle interactions take place.

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[1]Unified as a classical theory at least, the resulting gravity theory is Einstein's and therefore presents serious difficulties for quantization.

In this thesis I shall present present arguments for this point of view. To begin I will give a brief review [1] of the fundamental properties of a fiber bundle, and also the definitions and meanings of a connection and covariant differentiation in the bundle formalism. I will then describe the procedure for defining fields on bundle space, and show that spacetime fields and adjoint fields[2] in the Lie algebra of the structure group can be mapped into the tangent space of the bundle in a natural way. This procedure strengthens the case for a Kaluza-Klein type approach to unification, since the linear connection on the bundle is then needed to define covariant differentiation for these fields, as well as the mixed tensors derived from them. I will use this technique to construct the metric for bundle space. The components of this metric that correspond to a metric on the structure group will in general be scalar functions on spacetime. This is the type of theory proposed by Cho and Freund, while Kopczynski has considered only the invariant killing metrics whose components are spacetime constants.

Having introduced scalar fields into the theory in this way, it is natural to wonder if they might be the Higgs fields of the theory. This, however, is apparently not the case; as Cho and Freund have shown, the natural choice of a Lagrangian for the theory does not yield potential terms for these fields which have non-vanishing vacuum

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[1] For a more rigorous treatment of the subject, see Trautman [6].

[2] The so-called standard Higgs Field is an example of a field of this type; see Trautman [5].

expectation values. Thus, the physical role of these fields is unclear.

It should be noted, however, that in constructing this theory we have developed three separate concepts of covariant differentiation. The first one is the linear connection on spacetime M which determines covariant differentiation for objects in $\otimes TM$.[1] The next is the bundle connection itself, which defines gauge covariant differentiation in representations of G. The last is the linear connection on the bundle manifold P, which determines covariant differentiation for objects in $\otimes TP$. Since fields in $\otimes TM$ and fields in $\otimes G'$ can be mapped into $\otimes TP$, consistency of this theory demands that the linear connection of the bundle manifold must produce the same covariant derivatives for these fields as was produced by the spacetime and bundle connections for the original fields. This aspect of the theory has not been considered in previous treatments of the subject. In Chapter III, I shall formulate this consistency requirement precisely and derive the conditions placed on the bundle linear connection by it. I will then show that these restrictions essentially eliminate the components of the group metric tensor as degrees of freedom for the theory, and that these objects can therefore be treated as spacetime constants in at least some coordinate systems (gauges) for the bundle. I will then present a few examples of groups which admit metrics which satisfy these restrictions. Among these are the groups SU(3) and U(1)xSU(2), which are believed to provide descriptions of the strong and electroweak interactions which are consistent with our present knowledge of these processes.

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[1] $\otimes TM$ stands for the tangent space of M, its dual, or tensor products of these spaces.

In Chapter IV, I shall focus attention on the geodesics of the bundle linear connection and the choice of a Lagrangian for the theory. First, I will show that the geodesics of a torsion-free connection which satisfies the consistency restrictions laid down previously will reproduce the expected equation of motion for a classical particle moving under the influence of combined gauge and gravitational fields. In particular, these equations will produce the Lorentz Force Law for a charged particle in an electromagnetic field.

As a Lagrangian for this theory, I shall choose the standard one for an Einstein-type theory on spacetime

$$L = \sqrt{-g}\, R$$

where R is the Ricci scalar built from the components of the bundle linear connection. I will then show that this Lagrangian is in fact the usual one for coupled gravitation and gauge fields, with the possible addition of a cosmological constant term. This constant is not arbitrary in this theory as is the case in the more-or-less standard versions of General Relativity, but instead is determined by the vertical components of the metric on bundle space. To evaluate it I shall first introduce some slight redefinitions of the fundamental objects in the theory to recast it in the standard form utilized by particle physicists, drawing upon results obtained previously by Cho and Freund [7] and Kopczynski [8].

When the theory is rewritten in this form, it is possible to calculate the cosmological term in the Lagrangian in terms of the Planck length and the dimensionless coupling parameters for the gauge fields. The resulting number is far too large to be acceptable unless the coupling parameters are ridiculously small. I will therefore end with a brief discussion of possible methods to circumvent this difficulty.

CHAPTER II

THE BUNDLE FORMALISM

2.1 The Fundamental Definitions

To construct a smooth principle fiber bundle the following objects are required.[1]

1. Differentiable Manifolds P and M
2. A Lie Group G
3. A map π: $P \to M$
4. A map ψ: $P \times G \to P$

P is known as the total bundle space and M as the base space, assumed here to be spacetime. The map π is continuous and surjective and satisfies the requirement that:

$$\forall x \varepsilon M, \ \pi^{-1}(x) \text{ is isomorphic to } G.$$

$\pi^{-1}(x)$ is called the fiber at x and G is the typical fiber.

The map ψ defines the (right) action of G on P (i.e. $\forall a, b \varepsilon G \ \forall p \varepsilon P$ $\psi_a o \psi_b(p) = \psi_{ba}(p)$). It has properties

1. $\psi_a(p) = p$ implies $a = id_G$
2. $\forall p, q \varepsilon P$ with $\pi(p) = \pi(q) \exists a \varepsilon G$ such that $\psi_a(p) = q$
3. ψ_{id} = identity map on P
4. $\pi o \psi_a(p) = \pi(p)$

For convenience the map $\psi_a p$ is frequently written $\psi_a p = pa$.

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[1]For a rigorous treatment of the topics in this chapter, see Trautman [6].

A local cross section of the bundle P is a mapping from an open set $U \subseteq M$

$$\Phi : U \rightarrow P$$

such that $\forall x \in U, \pi \circ \Phi(x) = x$. These local cross sections always exist in principle bundles, but it is not in general possible to construct a global one.

Local cross are used to define coordinates on the bundle in the following way.

Let Φ: $U \rightarrow P$ be a cross section with $\Phi(x)=p$. If $q \in P$ satisfying $\pi(q)=\pi(p)$ then $q=pa$ for some $a \in G$. The point q is assigned coordinates (x,a). This procedure establishes an isomorphism between $\pi^{-1}(U)$ and $U \times G$, but unless the bundle connection is integrable it will not be possible to establish a natural isomorphism in this way; no cross section will be preferred over any other.

In this picture (see Figure 2.1), a change of coordinates corresponds to a change of cross section. To see this let Φ_1 and Φ_2 be two cross-sections on U with $\Phi_1(x)=p$ and $\Phi_2(x)=q$. If $r=pa=q\bar{a}$ then r has coordinates (x,a) in system 1 and coordinates $(x,\bar{a})$ in system 2. Since $q=pb$ for some $b \in G$ it follows that

$$\bar{a} = b^{-1}a$$

For most cases b will be a function of x, in which case it is known as a gauge transformation of the second kind.

2.2 Left Invariant Vector Fields and Lie Algebras

A vector field v is a mapping

$$v : G \rightarrow T(G) \text{ such that } v(g) \in T_g$$

The action of v on a function f on G is defined as $(vf)(g)=v_g(f)$.

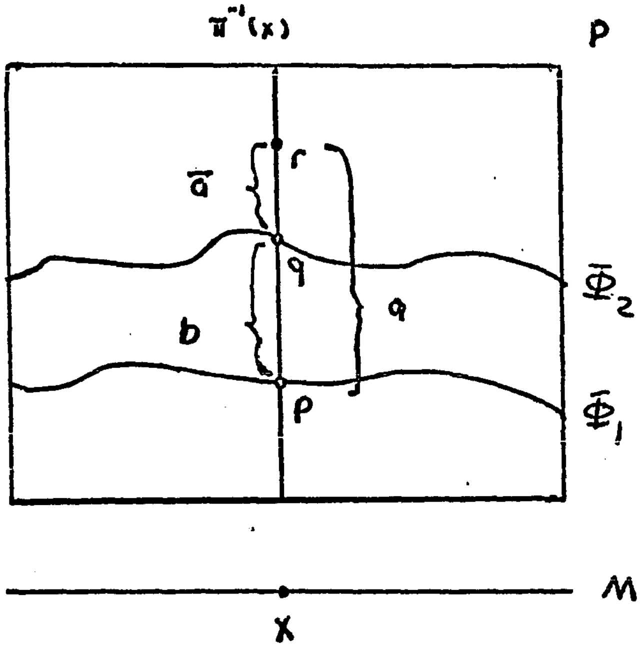

Figure 2.1

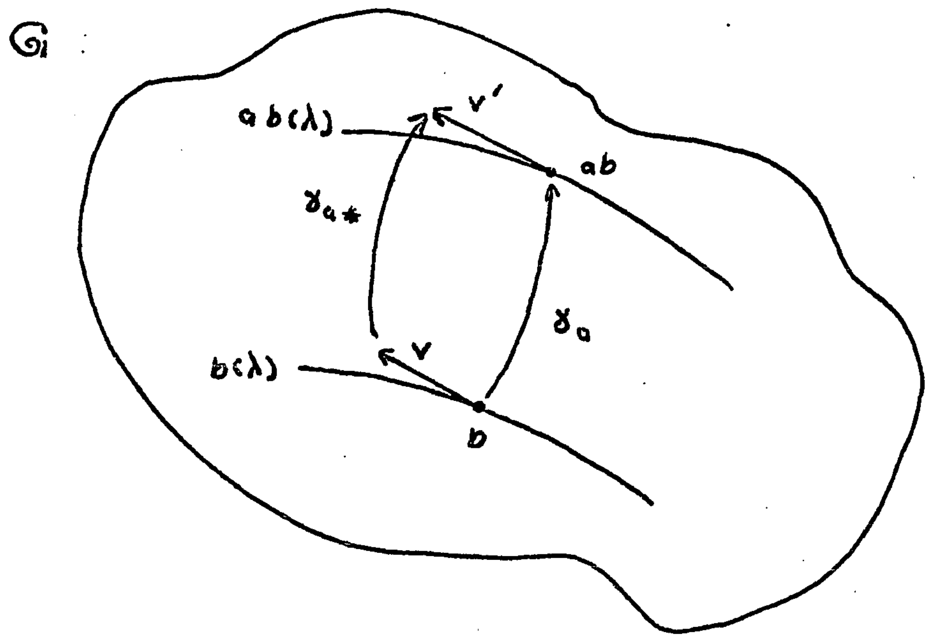

Figure 2.2

The left (right) translation of G by $a\varepsilon G$ is denoted by $\gamma_a(\delta_a)$. Associated to these maps are the maps

$$\gamma_{a*}\colon \; T_b \to T_{ab}$$

$$\delta_{a*}\colon \; T_b \to T_{ba}$$

which are defined as follows. If $v\varepsilon T_b$ there exists a curve $b(\lambda)$: $R \to G$ $(b(0)=b)$ whose tangent is v. Left (right) multiplication of this curve by a generates another curve $ab(\lambda)$ $(b(\lambda)a)$. γ_{a*} (δ_{a*}) carries v to the vector $v'\varepsilon T_{ab}(T_{ba})$ which is the tangent to the curve $ab(\lambda)$ $(b(\lambda)a)$ at ab (ba). (See Figure 2.2.)

A vector field χ on G is called left (right) invariant if $\forall_{a,b}$ G

$$\gamma_{a*}\chi_b = \chi_{ab} \; (\delta_{a*}\chi_b = \chi_{ba})$$

Vector fields on G will be left (right) invariant if

$$\chi_b = \gamma_{b*}\chi_e \; (\chi_b = \delta_{b*}\chi_e) \forall b\varepsilon G$$

Obviously there are N (N=dimG) linearly independent such fields, and moreover they are uniquely determined by their value in T_e. The set of all left (right) invariant vector fields is isomorphic to the Lie algebra (usually denoted by G') of G. In addition to the usual operations of addition and scalar multiplication there is a vector product operation defined for these fields, called the Lie Bracket

$$[X_\alpha, X_\beta] = X_\alpha X_\beta - X_\beta X_\alpha$$

G' is closed under this operation hence

$$[X_\alpha, X_\beta] = C^\gamma_{\alpha\beta} X_\gamma$$

The quantities $C^\gamma_{\alpha\beta}$ are independent of the point of evaluation; they are known as the structure constants of the Lie algebra.

Vector fields on the bundle can be defined in terms of curves (1-parameter subgroups) on G that pass through id_G. If $a(\lambda)$ is such a curve with $a(0)=e$, the map $\psi_{a(\lambda)}(p)$ carries this curve into a curve $p(\lambda)$ on P and ψ_{a*} carries the tangents of $a(\lambda)$ into the tangents $\psi_{a(\lambda)}p$. In particular, the tangent to $a(\lambda)$ at $\lambda=0$ maps into T_p. Since the map ψ is defined for all points p, this procedure defines a vector field on P. A vector field defined in this way is known as a Killing field. There being N 1-parameter subgroups of G at e whose tangents are linearly independent, we can construct from them N linearly independent vector fields on P.

Although the tangents to 1-parameter subgroups are nominally only defined along a curve, they may be easily extended (via the γ_* operation) into a left invariant vector field on G. These vector fields provide a convenient representation of the Killing fields on P relative to a coordinate system. If the point p has coordinates (x,b) relative to some cross section Φ and L(a) is the left invariant vector field generated by the tangent to $a(\lambda)$ at $\lambda=0$, then the curve $\psi_{a(\lambda)}(p)$ has coordinates $(X,ba(\lambda))$ and therefore the tangent $\ell(p)$ to $\psi_{a(\lambda)}(p)$ at p can be written as

$$\gamma_{b*}L(e) = L(b) \ .$$

If $\Phi_1(x)$ and $\Phi_2(x)$ are two cross sections with $\Phi_2(x) = \Phi_1(x)b$ then coordinates relative to these cross sections change by

$$a_2 = b^{-1}a_1$$

and a Killing field written as $L(a_1)$ in system 1 goes to $\gamma_{b^{-1}*}L(a_1) = L(a_2)$.

2.3 A Connection on a Fiber Bundle

The simplest way to motivate the need for a connection is to examine the construction of the tangent space TN to a differentiable manifold N. To each point $n \epsilon N$ there is an associated a vector space T_n called the tangent space at n. This vector space has elements which are the tangents to curves which pass through n. They are also the direction derivative operators on functions on N at n. These tangent spaces are also isomorphic to a single vector space v, but nothing in the structure of N determines a natural choice of isomorphisms at each point. Without this there is no way to compare vectors in different tangent spaces. The need for such a relation becomes obvious when one attempts to compute derivatives of tensor fields. To do this, one must define some additional structure, called a connection, on the manifold N. With this connection, we will be able to compute covariant derivatives of tensor fields on N. A covariant derivative of a tensor field is assumed to measure the "true" rate of change of a tensor field at a point in some direction, with spurious changes due to the change of basis subtracted out.

A connection for the tangent space to a manifold is known as a linear, or affine connection. Recently, however, there has arisen in physics a need to associate other vector spaces with points in a differentiable manifold; specifically, particle fields are today frequently described in terms of vector fields whose elements lie in an abstract vector space acted on by some Lie group G. A generalization of the concept of a connection is required to handle these cases. This generalization is realized by defining a connection on a principle fiber bundle P whose structure group is G and whose base space is spacetime. This connection determines covariant differentiation for any vector space on which an action by the structure group G is defined.

To accomplish this goal fields on the bundle, the connection form, and the covariant derivative are all defined in such a way as to reproduce the linear connection on the base space when P is the frame bundle and V the tangent space of the base space. The definitions are then assumed to be correct for all principle bundles and the associated vector spaces. The results for non-Abelian gauge fields then reproduce the classical results obtained in Yang-Mills theories, thus justifying the validity of this approach in physics.

To define the connection on a bundle P, we introduce the concept of vertical and horizontal vectors in TP.

A vector v in TP is called vertical if it satisfies the condition

$$\pi_* v = 0 \ .$$

Since P has dimension 4+N and M has dimension 4 there will be N linearly independent vectors satisfying this condition in each tangent space T_p. The Killing fields on P are vertical vectors. From their definition they are tangent to curves $pa(\lambda)$ at $\lambda=0$, and these curves satisfy $\pi(pa(\lambda) = \pi(p)$ for all λ, therefore, π_* maps their tangents to the zero vector in $TM_{\pi(p)}$. Since there are N of them, the Killing fields provide a convenient basis for vertical vectors in TP.

A horizontal subspace of TP_p is a set of vectors in T_p chosen so that

$$\pi_* H_p \to TM_{\pi(p)}$$

is a vector space isomorphism. These vectors are distinct from V_p, the vertical subspace at p since if $w \varepsilon H_p \cap V_p$ then $\pi_* w=0$ and therefore $w=0$ since π_* is a vector space isomorphism. Vectors in T_p can be decomposed uniquely into vertical and horizontal parts once H_p is specified.

The difficulty is that although vertical vectors can be determined uniquely in terms of the Killing fields, no such determination exists for horizontal vectors. To remedy this one defines a connection on P as follows.

A connection on a principle fiber bundle is the differentiable assignment to each $p \varepsilon P$ a subspace $H_p \subseteq TP_p$ which has the properties

(i) $\pi_* H_p \to TM_{\pi(p)}$ is a vector space isomorphism

(ii) $\psi_{a*} H_p = H_{\psi_{a(p)}}$

Given a connection on P, it can be described by a G'-valued 1-form $\omega(X) = \psi^{-1}_{p*}$ (ver X) for all $X \varepsilon TP$. We call ω the connection form on P. It has properties

(1) $x_r \varepsilon H_r$ iff $\omega_r (x) = 0$

(2) $\omega(\psi'_{p*} v) = v \; \forall \; v \varepsilon G'$

(3) $\omega_{pa}(\psi_{a*} v_p) = Ad_{a^{-1}} \omega_p (v_p)$

Properties (1) and (2) are obviously consequences of the definition of ω. For a proof of (3), see Trautman [1]. The claim that this is a connection will be justified by defining a covariant differentiation associated with it.

At this point it is appropriate to develop an explicit expression for ω. To do this we define a basis for TP in terms of a basis for TM and G'. This we can do because for $U \subseteq M$ open $\pi^{-1}(u)$ is isomorphic to $U \times G$.

To proceed we note that a basis e_A ($A = 1,2,3,\ldots 4+N$) of TP can be split into two subspaces by whether or not

$$\pi_* e_A = 0$$

is true for a given e_A. As we have seen previously, vectors of TP in the kernal of π_* are vertical vectors and can be expanded in terms of the Killing fields in T_p. We choose N of the e_A so that

$$e_\alpha(p) = \ell_\alpha(p) \stackrel{*}{=} L_\alpha(b) \text{ when } p \stackrel{*}{=} (x,b)$$

The remaining 4 vectors e_a are chosen so that

$$\pi_*(e_i(p)) \stackrel{*}{=} \frac{\partial}{\partial x_i} \quad - \frac{\partial}{\partial x_i} \in TM_{\pi(p)}$$

where the $\partial/\partial x^i$'s form a natural basis for TU relative to a coordinate system on U In a coordinate system a vector in TP_p can thus be written in terms of the pair $(\frac{\partial}{\partial x_i}, L_\alpha(b))$. Once a connection is specified on P, the horizontal vectors in $TP_{\pi^{-1}(u)}$ can be written as

$$h_i \stackrel{*}{=} \partial_i - (Ad_{a^{-1}})^\alpha{}_\beta A^\beta{}_i(x) L_\alpha(a) \ .$$

Since the h_i and L_α are linearly independent, they also represent a basis for TP_p. This basis turns out to be the preferred one for calculations.

Denote by dx^i and Φ^α the duals to ∂_i and L_α respectively. A dual basis to the vectors h_i and L_α is given by

$$\phi^i \stackrel{*}{=} dx^i$$
$$\phi^\alpha \stackrel{*}{=} \Phi^\alpha + (Ad_{a^{-1}})^\alpha{}_\beta A^\beta{}_i(x) dx^i \ .$$

To see that horizontal vectors written in this way satisfy (ii), note that in a coordinate system $\psi_a = \delta_a$. The vector $(Ad_{a^{-1}})^\beta{}_\alpha L_\beta(a)$ $= \gamma_{a*^{-1}} \delta_{a*} L_\beta(a)$. Now

$$\begin{aligned}
\delta_{b*} \gamma_{a*^{-1}} \delta_{*a} L_\beta(a) &= \delta_{b*} \delta_{a*} L_\beta(e) \\
&= \delta_{ab*} L_\beta(e) \\
&= \gamma_{(ab)^{-1}*} \gamma_{ab*} \delta_{ab*} L_\beta(e) \\
&= \gamma_{(ab)^{-1}*} \delta_{ab*} L_\beta(ab) \\
&- (Ad_{(ab)^{-1}})^\beta{}_\alpha L_\beta(ab)
\end{aligned}$$

thus $(Ad_{a^{-1}})^\beta{}_\alpha L_\beta$ is a right invariant vector field.

In a coordinate system we write the connection form as

$$\omega(a) \stackrel{*}{=} L_\alpha(e)\ \phi^\alpha(a)$$

Properties (1) and (2) can be verified for this basis by applying ω to vectors h_a and v ($v \in G'$: $v = v^\alpha L_\alpha(e)$). To verify (3), note that if p has coordinates (x,b) in some system, then a Killing basis vector can be written as $\ell_\alpha \stackrel{*}{=} L_\alpha(b)$.

Now

$$\delta_{a*} L_\alpha(b) = \gamma_{ba*}(\gamma_{a^{-1}*}\ \delta_{a*}\ L_\alpha(e))$$

and

$$\omega_{ba}(\delta_{a*} L_\alpha(b) = \gamma_{a^{-1}*}\ \delta_{a*}\ L_\alpha(e) = (Ad_{a^{-1}}) L_\alpha(e) \text{ from (ii).}$$

Associated to the connection form u is the curvature form

$$\Omega = \text{hor}^1 d\omega = \text{hor} L_\alpha(e) d\phi^\alpha = +\frac{1}{2} L_\alpha(e) (Ad_{a^{-1}})^\alpha{}_\beta F^\beta{}_{cb} dx^c \wedge d_x{}^b$$

[1]The horizontal part of a form α is defined as hor $\alpha(v,w\ldots)$ $= \alpha(\text{hor } v, \text{hor } w \ldots)$.

$$F^{\alpha}{}_{ab} = \partial_a A^{\alpha}{}_b - \partial_b A^{\alpha}{}_a + C^{\alpha}{}_{\pi\lambda} A^{\pi}{}_a A^{\lambda}{}_b \ .$$

The connection ω is integrable if Ω=0. Also, since

$$[h_a, h_b] = -(Ad_{a^{-1}})^{\alpha}{}_{\beta} F^{\beta}{}_{ab} L_{\alpha}$$

it follows that horizontal vectors form tangents to surfaces in P if the connection ω is integrable. In the case where they form surfaces, we can define a cross-section everywhere in P as just one of these surfaces, thus identifying P with the trivial bundle M x G. In this coordinate system $A^{\alpha}{}_a(s) = 0$ everywhere.

As pointed out previously by Cho and Freund [2] the Jacobi Identity for horizontal vectors

$$[h_a, [h_b, h_c]] + [h_c, [h_a, h_b]] + [h_b,]h_c, h_a]] = 0$$

leads to the restriction

$$h_a(F^{\alpha}{}_{bc}) + h_c(F^{\alpha}{}_{ab}) + h_b(F^{\alpha}{}_{ca}) = 0$$

which is the Bianchi identity for a curvature tensor.

2.4 Fields on the Bundle

Let V be a vector space and ρ: GxV→V satisfying

$$\rho(ab, v) = \rho_{ab}(v) = \rho_a(\rho_b(v))$$

and

$$\rho_a(\alpha v_1 + \beta v_2) = \alpha\rho_a(v_1) + \beta\rho_a(v_2) \ .$$

The map ρ is thus a homomorphism of G into GL(V), called the (left) action of G on V. A mapping χ: P→V is said to be a vector field of type ρ if

$$\chi(pa) = \rho_{a^{-1}}\chi(p) \quad \text{[Fig. 2.3]}$$

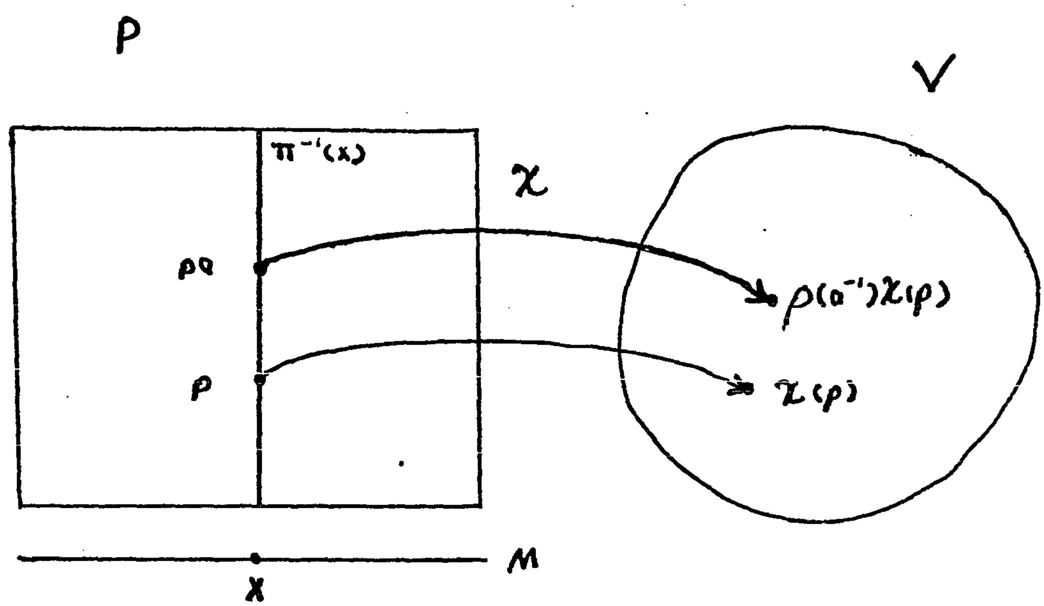

Figure 2.3

χ is well defined by this procedure since

$$\chi(pab) = \rho_{(ab)^{-1}} \chi(p)$$

$$= \rho_{b^{-1}a^{-1}} \chi(p)$$

$$= \rho_{b^{-1}} \chi(pa) \ .$$

A vector field like this is uniquely determined by its value at a single point in each fiber of P. Relative to a basis e_A of V the field χ can be written

$$\chi(p) = \chi^A(p) \quad (A=1,2,\ldots\dim(V))$$

and

$$\chi^B(pa) = (\rho_{a^{-1}})^B{}_A \chi^A(p)$$

where $(\rho_a)^B{}_A$ is the element of GL(V) corresponding to a.

Fields on the bundle are defined in this way with the frame bundle in mind. In this bundle, the structure group is GL(4,R) and the base space is spacetime. The points in a fiber $\pi^{-1}(x)$ can be identified with the frames of TM_x. The group action ρ on the points $p(e_c)$ in the bundle thus corresponds to a change of frames for $TM_{\pi(e_c)}$.

Now suppose that $V=R^4$ which is isomorphic to TM_x. The components $\chi^A(e_C)$ of χ can then be regarded as the components of a vector in $TM_{\pi(p)}$ relative to the frame e_C. If e'_C is another frame of $TM_{\pi(p)}$, then $e'_C = e_D a^D{}_C$ for some $a^D{}_C \in GL(4,R)$. Since $\chi^A(e'_C) = (\rho_{a^{-1}})^A{}_B \chi^B(e_C)$, the values of χ along a fiber at x can be interpreted as the components of a single vector in TM_x. These ideas are easily generalized to tensor products of TM and TM*. This picture will break down for general principle bundles, however, since for these cases no association can be made between points in the bundle and bases for a vector space V.

There is one additional structure that will be needed for later calculations. This is the map

$$\rho': G' \to \mathrm{End}(V)$$

associated to ρ which can be used to calculate the action of elements of G' on fields whose image is in V. Since ENDGL(V) and GL(V) both have matrix representations, this enables us to calculate the action of elements of G' on vectors in v by matrix multiplication.

Formally, we would write the map ρ' as

$$\rho'(L^{(e)}) = \frac{d}{d\lambda}\rho_{a(\lambda)}\Big|_{\lambda=0}$$

where $a(\lambda)$ is the 1-parameter subgroup generated by $L^{(e)}$.

For simply connected Lie Groups, it is equally valid and frequently more practical to first define the map ρ' and then construct the map ρ by means of the exponential map. This we can do because the diagram

$$\begin{array}{ccc} G & \xrightarrow{\rho} & GL(V) \\ {\scriptstyle \exp(\lambda L)}\uparrow & & \uparrow{\scriptstyle \exp(\lambda E)} \\ G' & \xrightarrow[\rho']{} & \mathrm{End}\ GL(V) \end{array}$$

commutes, and $\exp(\lambda L)$ is onto for simply connected groups.

2.5 Covariant Derivatives

Given a vector field $\chi: P \to v$ we can calculate the 1-form field $d\chi: TP \to v$. The gauge covariant derivative of χ is defined as

$$D\chi(X) = d\chi(\text{hor } X) \ .$$

In words, the gauge covariant derivative of χ is a 1-form which acts on a vector in TP to return the directional derivative of χ in the direction of hor v. With this definition of covariant differentiation, it is clear why ω is called a connection, since it is the form on P which determines which vectors are horizontal. Relative to a basis e_A of V the gauge covariant derivative can be expressed as

$$D\chi^A = \text{hor } d\chi^A \tag{2.5.1}$$

the horizontal part of $d\chi^A$ can be determined by using the canonical connection form ω

$$D\chi^A = d\chi^A + \chi^B(\rho' \circ \omega)^A{}_B \tag{2.5.2}$$

Equivalence of (2.5.1) and (2.5.2) can be verified as follows.

Relative to a basis of TP given in Section 2.3 the exterior derivative of χ can be written as

$$d\chi^A = h_a(\chi^A)dx^a + \ell_\alpha(\chi^A)\phi^\alpha$$

If $v_p \in TP_p$ then $v_p = h_p + \ell_p$ and

$$\begin{aligned} d\chi^A(\text{ver } v_p) &= \ell_p(\chi^A) \\ &= [\frac{d}{d\lambda}\psi_{a(\lambda)}(p)]\Big|_{\lambda=0}(\chi^A(p)) \\ &= \frac{d}{d\lambda}\chi^A(pa)\Big|_{\lambda=0} \\ &= \frac{d}{d\lambda}(\rho_{a^{-1}}{}^A{}_B)\Big|_{\lambda=0}\chi^B(p) \\ &= -\rho'(L(e))^A{}_B\,\chi^B(p) \end{aligned}$$

Now $\rho \circ \omega(\ell_p) = \rho'(L^{(e)})^A{}_B$ so that the latter term in (2.5.2) does indeed cancel the vertical part of $d\chi^A$. Normally in physical calculations we work in a cross section of the bundle whose tangents are of the form $v^a\partial_a$, calculation of $D\chi^A$ in these directions is simplified somewhat if we use (2.5.2).

That the D operation is linear follows immediately from the properties of d, ρ' and ω. It remains to verify that D satisfies the other requirement of covariant derivatives

$$D(\chi(pa)(\psi_{a*}v_p) = \rho_{a^{-1}} D\chi(v_p)$$

that is $D\chi$ is a vector field of the same type as χ. To verify this, note that

$$D(\chi(pa)(\psi_{a*}v_p) = D(\chi(pa))(\psi_{a*}h_p)$$

since the ver v_p is carried into another vertical vector at pa by ψ_{a*}. Now

$$D(\chi(pa)(\psi_{a*}h_p) = \psi_{a*}h_p(\psi(pa))$$

$$= \frac{d}{d\lambda}\,\psi(h(\lambda)a)\Big|_p$$

where $h(\lambda)$ is the curve through p whose tangent at p is h. Now

$$\frac{d}{d\lambda}\,\psi(h(\lambda)a)\Big|_p = \frac{d}{d\lambda}\,(\rho_{a^{-1}}\chi(h(\lambda))\Big|_p$$

$$= \rho_{a^{-1}}\frac{d}{d\lambda}\,(\chi(h(\lambda))\Big|_p$$

$$= \rho_{a^{-1}}h\;(\chi(p)) = \rho_{a^{-1}}D\chi(p)(v_p)$$

That the D operation is a reasonable definition of covariant differentiation can be seen by appealing again to the frame bundle case. Here the change in a field in vertical directions can be interpreted as due to the change of frames as we move from point to point in the fiber. A covariant derivative of a vector field is the measure of the "intrinsic" change in the vector field, that is the change in the field that remains when the change due to a shift in the basis is removed. In the frame bundle, this intrinsic change is obviously just the horizontal part of a directional derivative.

2.6 Example: Adjoint Fields

To illustrate how the procedures laid out in Sec. 2.4 and 2.5 work out in practice, consider the case where $V=G'$, the set of left invariant vector fields on G'.

For G', the action $\rho=\mathrm{Ad}$, the adjoint action of the group G on G'. To construct ρ, define first the map ρ' as

$$\rho'(L(e))^{\gamma}{}_{\beta} = \Phi^{\gamma}_{(e)}([L(e), L_{\beta}(e)])$$

$\Phi^{\gamma}_{(e)}$ is an element of a basis for G'^{*}. From this it follows that

$$\rho'(L_{\alpha}(e))^{\gamma}{}_{\beta} = C^{\gamma}{}_{\alpha\beta}$$

ρ' so defined preserves the commutation relations of G' since

$$\begin{aligned}\rho'([L_{\alpha}(e),\, L_{\beta}(e)])^{\pi}{}_{\lambda} &= \rho'(C^{\gamma}_{\alpha\beta}L_{\gamma}(e))^{\pi}{}_{\lambda}\\ &= \Phi^{\pi}{}_{(e)}([C^{\gamma}_{\alpha\beta}L_{\gamma}(e), L_{\lambda}(e)]) = C^{\gamma}_{\alpha\beta}C_{\gamma}{}^{\pi}{}_{\lambda}\\ &= C^{\gamma}{}_{\lambda\beta}C^{\pi}{}_{\gamma\alpha} - C^{\gamma}{}_{\lambda\alpha}C^{\pi}{}_{\gamma\beta} = \rho'(L_{\alpha}(e))^{\pi}{}_{\gamma}\\ &\qquad \rho'(L_{\beta}(e))^{\gamma}{}_{\lambda} - \rho'(L_{\beta}(e))^{\pi}{}_{\gamma}\, \rho'(L_{\alpha}(e))^{\gamma}{}_{\lambda}\end{aligned}$$

A general vector in G' can be written as

$$(L(e))^{\pi}{}_{\lambda} = b^{\gamma}C^{\pi}{}_{\gamma\lambda}$$

The action ρ can then be written as

$$\rho^{\alpha}{}_{\beta} = \exp(b^{\gamma}C_{\gamma}{}^{\pi}{}_{\lambda})^{\alpha}{}_{\beta}$$

Relative to a coordinate system for P and a basis of G' a vector field of type Ad can be written as

$$\chi^{\alpha}(p) = \chi^{\alpha}(x,b) = (Ad_{b^{-1}})^{\alpha}{}_{\beta}\chi^{\beta}(x)$$

$$= \exp(-b^{\gamma}C_{\gamma\pi}{}^{\lambda})^{\alpha}{}_{\beta}\,\chi^{\beta}(x)$$

The gauge covariant derivative of χ is

$$D\chi^{\alpha} = x_A(\chi^{\alpha})\phi^A + (Ad'\circ\omega(X))^{\alpha}{}_{\beta}\,\chi^{\beta}$$

where x_A and ϕ^A are bases for TP and TP*. Suppose that $x_a = \partial_a$, $x_\alpha = \ell_\alpha$, then

$$D_i\chi^{\alpha}_{(p)} = \partial_i\chi^{\alpha}(p) + (Ad'\circ\omega_p(\partial_i))^{\alpha}{}_{\beta}\chi^{\beta}(p)$$

$$= (Ad_{a^{-1}})^{\alpha}{}_{\beta}\partial_i\chi^{\beta}(x) + Ad'\{(Ad_{a^{-1}})^{\gamma}{}_{\pi}A^{\pi}_{i}\,L_{\gamma}(e)\}^{\alpha}{}_{\beta}Ad^{\beta}{}_{a^{-1}\lambda}\chi^{\lambda}(x)$$

$$= (Ad_{a^{-1}})^{\alpha}{}_{\beta}\partial_i\chi^{\beta}(x) + C^{\alpha}_{\gamma\beta}(Ad_{a^{-1}})^{\gamma}{}_{\pi}(Ad_{a^{-1}})^{\beta}{}_{\gamma}A^{\pi}{}_{i}(x)\chi^{\lambda}(x)\ .$$

The second term may be simplified using

$$C^{\alpha}{}_{\gamma\beta}(Ad_{a^{-1}})^{\gamma}{}_{\pi}(Ad_{a^{-1}})^{\beta}{}_{\lambda} = (Ad_{a^{-1}})^{\alpha}{}_{u}(Ad_{a})^{u}{}_{\sigma}C^{\sigma}{}_{\gamma\beta}(Ad_{a^{-1}})^{\gamma}{}_{\pi}(Ad_{a^{-1}})^{\beta}{}_{\lambda}$$

$$= (Ad_{a^{-1}})^{\alpha}{}_{\sigma}C^{\sigma}{}_{\pi\lambda}$$

since the structure constants are Ad invariant.

The answer then takes the form

$$D_i\chi^{\alpha} = (Ad_{a^{-1}})^{\alpha}{}_{\beta}(\partial_i\chi^{\beta}(x) - C^{\beta}{}_{\gamma\lambda}A^{\lambda}{}_{i}(x)\chi^{\gamma}(x))$$

This is identical to the answer obtained from

$D\chi^{\alpha}(h_i)$ since hor $\partial_i = h_i$.

2.7 Gauge Transformations

It is now time to determine how fields on the bundle will change under a change of fiber coordinates. There are two cases to consider; one is the transformation law for fields of type ρ, and the other is the rule for the connection components $A^{\alpha}{}_{a}$.

Turning first to fields of type ρ, recall that relative to a basis e_α of V a vector field of type ρ has the form

$$v(p) = v^{\alpha}(p)e_{\alpha} .$$

In a coordinate system where $p=(x,a)$ this can be written as

$$v(p) = (\rho_{a^{-1}})^{\alpha}{}_{\pi} v^{\pi}(x) e_{\alpha} .$$

In a system where $p=(x,\bar{a})$ this becomes

$$v(p) = (\rho_{\bar{a}^{-1}})^{\alpha}{}_{\pi} \bar{v}^{\pi}(x) e_{\alpha} ,$$

so it must be true that

$$(\rho_{a^{-1}})^{\alpha}{}_{\pi} v^{\pi}(x) = (\rho_{\bar{a}^{-1}})^{\alpha}{}_{\pi} \bar{v}^{\pi}(x) .$$

Now $\bar{a} = b^{-1}a$ where b may be a function of x; that is, a gauge transformation of the second kind. Therefore

$$(\rho_{\bar{a}^{-1}})^{\alpha}{}_{\pi} = (\rho_{a^{-1}})^{\alpha}{}_{\lambda} (\rho_{b})^{\lambda}{}_{\pi} ,$$

and hence $v^{\pi}(x) = (\rho_b)^{\pi}{}_{\lambda}\, \bar{v}^{\lambda}(x)$. Usually we would express the new functions $\bar{v}^{\alpha}$ in terms of the old functions v^{α}. This is obviously

$$\bar{v}^{\lambda}(x) = (\rho_{b^{-1}})^{\lambda}{}_{\pi}\, v^{\pi}(x) .$$

Thus for a field in G' on P the transformation rule is

$$\bar{v}^{\alpha}(x) = (Ad_{b^{-1}})^{\alpha}{}_{\pi} v^{\pi}(x) \tag{2.7.1}$$

and for a field in G'* on P the transformation rule is

$$\bar{f}_{\alpha}(x) = (Ad_b)^{\pi}{}_{\alpha}\, f_{\pi}(x) = (Ad_b{}^{T} f)_{\alpha} \tag{2.7.2}$$

In infinitisimal form these become

$$\bar{v}^{\alpha}(x) = (\delta^{\alpha}_{\pi} + b^{\gamma}(x)\, C^{\alpha}_{\pi\gamma})v^{\pi}(x)$$

$$\bar{f}_{\alpha}(x) = (\delta^{\alpha}_{\pi} - b^{\gamma}(x)\, C^{\alpha}_{\pi\gamma})v^{\pi}(x) \ .$$

Obtaining the transformation rule for the $A^{\alpha}{}_{a}$'s is more difficult. To do this we look at the pullback of the canonical connection form ω to the base space by a cross section Φ_1

$$(\Phi_1^{*}\omega)(x) = \omega(\Phi_{1*}x) \qquad x \in TM$$

Since $\Phi_1(x) \stackrel{*}{=} (x,e)$ by definition $\Phi_{1*}\partial_a \stackrel{*}{=} \partial_a$ in this coordinate system. Therefore

$$\Phi_1{}^{*}(\omega)(\partial_a) = \omega(\Phi_{1*}\partial_a) = A^{\alpha}{}_{a}(x)L_{\alpha}(e).$$

If Φ_2 is another cross section, then

$$(\Phi_2{}^{*}\omega)(\partial_a) = \bar{A}^{\alpha}{}_{a}(x)L_{\alpha}(e) \ .$$

Now $\Phi_2 = \psi_{b(x)} \circ \Phi_1$ so

$$(\Phi_2{}^{*}\omega)(\partial_a) = \omega((\psi_{b(x)} \circ \Phi_1)_{*}\partial_a) = \omega(\psi_{b(x)*}\partial_a)$$

in the Φ_1 coordinate system. This is equivalent to

$$(\Phi_2{}^{*}\omega)(\partial_a) = L_{\alpha}(e)\phi^{\alpha}(b)(\psi_{b(x)*}\partial_a) \tag{2.7.3}$$

$$= L_{\alpha}(e)[\Phi^{\alpha}(b) + Ad_{b^{-1}}{}^{\alpha}{}_{\beta}A^{\beta}{}_{c}(x)dx^{c}](\partial_a + \frac{\partial b^{\pi}}{\partial x^{a}}\, \phi^{\lambda}{}_{(b)}\ (\frac{\partial}{\partial b^{\pi}})L_{\lambda}(b))$$

where the b^{π}'s are the coordinates of b in some coordinate system for G. Therefore 2.7.3 yields the result

$$(\Phi_2{}^{*}\ )(\partial_a) = L_{\alpha}(e)[(Ad_{b^{-1}})^{\alpha}{}_{\beta}A^{\beta}{}_{a}(x) + \frac{\partial b^{\pi}}{\partial x^{a}}\, \phi^{\alpha}(\frac{\partial}{\partial b^{\pi}})]$$

which says that

$$\bar{A}^{\alpha}{}_{a}(x) = (Ad_{b^{-1}})^{\alpha}{}_{\beta}A^{\beta}{}_{a}(x) + \frac{\partial b^{\pi}}{\partial x^{a}}\Phi^{\alpha}\left(\frac{\partial}{\partial b^{\pi}}\right) \tag{2.7.4}$$

To give meaning to $(\partial b/\partial x^{a})^{\pi}$ recall that b can be written $b = \exp(b^{\gamma}L_{\gamma})$. Therefore in lowest order Eq. (2.7.4) is

$$\bar{A}^{\alpha}{}_{a}(x) = A^{\alpha}{}_{a}(x) + C^{\alpha}{}_{\beta\lambda}A^{\beta}{}_{a}(x)b^{\lambda} + \frac{\partial b^{\alpha}}{\partial x^{a}},$$

which is recognizable as the correct formula for an infinitesimal gauge transformation for a general vector potential.

CHAPTER III

PHYSICS ON THE BUNDLE

3.1 Lifting Horizontal Fields and Sliding Adjoint Fields

In the previous chapter I have tried to make clear the basic properties of a principle fiber bundle and the procedure for defining a connection on it. The next step in the construction of a theory of the Kaluza-Klein type is to construct a linear connection on the bundle space regarding it as a differentiable manifold. To make this useful, however, there must be fields defined within the tangent space for this connection to act on. At this stage, the physical fields of interest all reside either in the tangent space of the base manifold, spacetime, or in some representation of the structure group G. To remedy this we define the "lift" of a spacetime field and a new object, the "slide" of an adjoint field as follows.

A general tensor field T(x) on spacetime may be lifted directly into a tensor field T'(p) on the bundle by the prescription

$$T(x)^{ab\ldots}{}_{cd\ldots}\ \partial_a \otimes \partial_{b\ldots}\ dx^c \otimes dx^d \ldots \rightarrow$$

$$T'(p) = T(\pi(p))^{ab\ldots}{}_{cd\ldots}\ h_a \otimes h_{b\ldots}\ dx^c \otimes dx^d \ \ldots$$

This lifting is obviously well defined in the region $\pi^{-1}(U)$ $(U \subseteq M)$; moreover, the lifted field is invariant under the group action ψ, since horizontal vectors and their duals are mapped into themselves by this operation. In a coordinate system this invariance is equivalent to right invariance.

To construct an adjoint vector field in bundle tangent space, we make use of the fundamental map ψ' which maps elements of G' into the Killing fields in TP. A vector field in G' can be written in terms of some basis of G' as

$$v(p) = v^{\alpha}(p) L_{\alpha}(e) \ .$$

The "slide" of this field is defined as the field

$$v(p) = v^{\alpha}(p) \ell_{\alpha}(p) \qquad (\ell_{\alpha}(p) = \psi'(p) L_{\alpha}(e)) \ .$$

As we have seen previously, in terms of a coordinate system we can write this as

$$v(p) = v(x,a) = v^{\alpha}(x,a) L_{\alpha}(a) \ .$$

If we apply the right translation operator ψ_{b*} to this field, we obtain using the transformation law for a field of type Ad

$$\begin{aligned} \psi_{b*} v^{\alpha}(x,a) L_{\alpha}(a) &= \psi_{b*} v^{\beta}(x,e) (\mathrm{Ad}_{a^{-1}})^{\alpha}{}_{\beta} \, L_{\alpha}(a) \\ &= v^{\beta}(x,e) (\mathrm{Ad}_{(ab)^{-1}})^{\alpha}{}_{\beta} L_{\alpha}(ab) \\ &= v^{\alpha}(x,ab) L_{\alpha}(ab) \ . \end{aligned}$$

Thus, the slide of an adjoint field is also a right invariant field.

A dual vector field in verTP* can be defined as the pullback of a field on G'* using the canonical connection form ω:

$$F(X) = f(\omega(x)) \qquad x \in TP \qquad f \in G'^{*} \ .$$

A form field f in G'* on P is a field of type $\mathrm{Ad}^{-1\,T}$, where the transpose of the group action on a vector space is defined as

$$(\mathrm{Ad}^T f)(v) = f(\mathrm{Ad}v) \ .$$

Since $\mathrm{Ad}^{-1}_{a^{-1}} = \mathrm{Ad}_a$ a form field of type $(\mathrm{Ad}^{-1})^T$ in G'* obeys the relation

$$f(pa) = (\mathrm{Ad}_a)^T f(p) \ .$$

Such a field is well defined since

$$\begin{aligned} f(pab) &= (\mathrm{Ad}_{ab})^T f(p) \\ &= (\mathrm{Ad}_b)^T (\mathrm{Ad}_a)^T f(p) \\ &= (\mathrm{Ad}_b)^T f(pa) \ . \end{aligned}$$

Relative to a basis $\Phi^\alpha(e)$, f is written as

$$f(p) = f_\alpha(p)\Phi^\alpha(e) \ .$$

To calculate the components F_α of the field relative to the basis $\phi^\alpha(p)$ we use the relation

$$\begin{aligned} F_\alpha(p) &= F_\beta(p)\phi^\beta(p)(\ell_\alpha(p)) \\ &= f_\beta(p)\Phi^\beta(e)(L_\gamma(e)\phi^\gamma(p)(\ell_\alpha(p))) \\ &= f_\beta(p)\Phi^\beta(e)(L_\alpha(e)) = f_\alpha(p) \ . \end{aligned}$$

To check right invariance we express F(p) in a coordinate system

$$F(p) = f_\alpha(x)(\mathrm{Ad}_a)^\alpha{}_\beta(\Phi^\beta(a) + (\mathrm{Ad}_{a^{-1}})^\beta{}_\gamma A^\gamma{}_a(x)dx^a) \ .$$

Right translation for forms is determined by the relation

$$\delta^*{}_a\Phi(ba) = \Phi'(b) \ .$$

This expression is consistent with the rule $(f^*\Phi)(v) \equiv \Phi(f_* v)$. From this it follows that

$$\psi^*_{b^{-1}}F(p) = f_\alpha(x)(\psi^*_{b^{-1}}(Ad_a)^\alpha{}_\beta\Phi^\beta(a) + A^\alpha{}_a(x)dx^a)$$

Now

$$\psi^*_{b^{-1}}(Ad_a)^\alpha{}_\beta\Phi^\beta(a) = \delta^*_{b^{-1}}\gamma^*_a\delta^*_{a^{-1}}\Phi^\alpha(a)$$

$$= \delta^*_{b^{-1}}\delta^*_{a^{-1}}\Phi^\alpha(e)$$

$$= \gamma^*_{ab}\delta^*_{(ab)^{-1}}\Phi^\alpha(ab)$$

$$= (Ad_a)^\alpha{}_\beta\Phi^\beta(ab) \ .$$

Therefore $\psi^*_{b^{-1}}F(p) = f_\alpha(x)(Ad_{ab})^\alpha{}_\beta[\Phi^\beta(ab) + (Ad_{ab})^\alpha{}_\beta A^\beta{}_a(x)dx^a]$

$$= F(pb) \ ,$$

so a form field in G'* also maps into a right invariant field on the bundle. Its gauge covariant derivative is the object

$$\nabla_i f_\alpha = (\partial_i - (Ad_{a^{-1}})^\pi{}_\lambda A^\lambda{}_i(x)L_\pi(a))[(Ad_a)^\beta{}_\alpha f_\beta(x)]$$

$$= (Ad_a)^\beta{}_\alpha(\partial_i f_\beta(x) + C^\lambda{}_{\beta\pi}A^\pi{}_i(x)f_\lambda(x))$$

where $L_\pi(a)[(Ad_a)^\beta{}_\alpha]$ is calculated from

$$L_\pi(a)[\delta^\alpha{}_\beta] = L_\pi(a)[(Ad_a)^\gamma{}_\lambda(Ad_{a^{-1}})^\lambda{}_\beta] = 0$$

using the Liebnizt rule.

3.2 A Metric for Bundle Space

Armed with the techniques described in the previous section, we are now ready to define a metric for the bundle. Before doing so, a brief description of notation is appropriate. To denote the basis of TP I shall use

$$X_A \quad A = 1,2\ldots4+N$$

$$X_i = h_i \quad i = 1,2,3,4 \tag{3.2.1a}$$

$$X_\alpha = \ell_\alpha \quad \alpha = 5\ldots4+N \tag{3.2.1b}$$

The dual to this basis is Θ^A where

$$\Theta^i = dx^i \qquad \Theta^\alpha = \phi^\alpha$$

At a point $p \in P$ with coordinates (x,a) these objects may be written

$$h_i(p) = \frac{\partial}{\partial x^i} - A^\alpha{}_i (Ad_{a^{-1}})^\beta{}_\alpha\, L_\beta(a)$$

$$\ell_\alpha(p) = L_\alpha(a)$$

$$\phi^\alpha = \Phi^\alpha(a) + A^\alpha{}_i (Ad_{a^{-1}})^\beta{}_\alpha\, dx^i \ .$$

A metric is a symmetric 2-form on P

$$g(p) = g_{AB}(p)\ \Theta^A \otimes \Theta^B$$

which defines the scalar product of vectors in TP.

The horizontal part of this metric is chosen to be the lift of the metric on space time

$$g_{ab}(p) = g_{ab}{}^{ST}\ (\pi(p)) \ .$$

The vertical part of the metric is chosen to be a field of type Ad on $G'^* \otimes G'^*$, it therefore obeys the condition

$$g_{\alpha\beta}(pa) = g_{\pi\lambda}(p)(Ad_a)^\pi{}_\alpha (Ad_a)^\lambda{}_\beta \ .$$

In a coordinate system this metric can be written

$$g_{\alpha\beta}(x,a) = g_{\pi\lambda}(x)(Ad_a)^\pi{}_\alpha (Ad_a)^\lambda{}_\beta \ .$$

The vertical part of the metric is a scalar function on space time and its dependence on group coordinates is determined by its transformation law.

To justify this approach, recall that what we seek is a theory that incorporates both the gravitational and gauge notions of covariant differentiation. Since the connection for this is determined (at least partly) by the metric, we must require that the horizontal parts of this metric project into the spacetime metric. For the vertical part it is not immediately clear what will be necessary to accomplish the desired unification; so initially we assume only that it is a field whose gauge covariant derivative is defined. There is a potential problem with this, since it is not clear what physical role such fields will have. I will address this question in later sections.

3.3 The Linear Connection the Bundle

To construct the linear connection on P, I shall use the traditional approach from differential geometry. In this formalism, we assume that the covariant derivative of a basis vector field x_B in the direction of a basis vector x_A is given by

$$\nabla_{x_A} x_B = \Gamma_B{}^C{}_A \, x_C \ .$$

The connection is determined by the coefficients Γ^C_{BA}. The covariant derivative of a basis form field is

$$\nabla_{x_A} \phi^B = -\Gamma_C{}^B{}_A \, \phi^C \ .$$

The covariant derivative of a general vector field in an arbitrary direction is then

$$\nabla_V \, w = (v^A x_A(w^B) + v^A w^c \Gamma_c{}^B{}_A) x_B$$

and the corresponding result for a form field is

$$\nabla_v F = (v^A x_A(F_B) - v^A F_c \Gamma_B{}^c{}_A)\phi^B \ .$$

The basis vectors in these equations may be either holonomic or non-holonomic.

Defining the connection in this way makes it clear that there is no inherent relationship between a connection and a metric; such relationships originate in restrictions imposed for physical and/or aesthetic reasons. One such restriction can be obtained by consideration of the geodesics of a connection.

The geodesics of a connection on a manifold are the curves whose tangents satisfy the equation

$$\nabla_v v = ((v^B x_B(v^A) + v^B v^C \Gamma^A_{BC})x_A = 0 \ . \qquad (3.3.2)$$

Such curves are the straight lines of the connection Γ. They can be parameterized by an "interval" along the curve; this parameter is known as the affine parameter of the connection. In relativity theory, this parameter is adopted as the measure of proper distance along a geodesic.

A metric on a manifold also defines a concept of proper distance for the space. In physics we assume that these two notions coincide along geodesics, so that proper distance in space is a uniquely defined concept. To obtain this, it is sufficient to adopt the restriction

$$\nabla g = 0 \ . \qquad (3.3.3)$$

Equation (3.3.3) is the usual statement of compatibility of a metric and a connection. When it holds, covariant differentiation commutes with the raising and lowering operations on tensor indices.

To see how (3.3.3) guarantees the equivalence of metric and geodesic distance, we proceed as follows.

Let s be the metric interval along a geodesic and λ the affine interval. If

$$\frac{ds}{d\lambda} = \text{constant} \tag{3.3.4}$$

then $s = A\lambda+B$ and therefore s and λ are equivalent, since affine parameters are unique cnly up to a linear transformation. Now Equation (3.3.4) implies that

$$\frac{d^2s}{d\lambda^2} = 0 \ .$$

Since $(ds/d\lambda)^2 = g_{AB}v^Av^B$ it follows that

$$\frac{d}{d\lambda}\,(g_{AB}v^Av^B) \equiv v^C x_C(g_{AB}v^Av^B) = 0$$

if (3.3.4) is to hold for arbitrary geodesics.

Bearing in mind that v satisfies the geodesic equation, we can write this as

$$v^Av^Bv^C[x_C(g_{AB}) - g_{AD}\Gamma_B{}^D{}_c - g_{BD}\Gamma_A{}^D{}_c] = 0 \ ,$$

which is true if g satisfies (3.2).

From the compatibility condition we obtain a formula for the connection coefficients

$$\Gamma^C_{(BD)} = g^{CA}[\tfrac{1}{2}\,(X_D(g_{AB}) + X_B(g_{DA}) - X_A(g_{BD})) - g_{BE}\Gamma^E_{[AD]} - g_{DE}\Gamma^E_{[AB]}$$

$$\Gamma^c_{BD} = \Gamma^C_{(BD)} + \Gamma^C_{[BD]} \ .$$

In the Einstein theory, the strong principle of equivalence demands that a coordinate system exist for which the Γ's vanish at a point.

For this to hold requires

$$\Gamma^{C}_{[BD]} = -\frac{1}{2} C^{C}_{BD} \ .$$

In a more general theory we would have

$$\Gamma^{C}_{[BD]} = Q^{C}_{BD} - \frac{1}{2} C^{C}_{BD} \ .$$

The object Q^{C}_{BD} is a tensor, known as the torsion of the connection Γ.

I shall consider primarily the torsion-free connection on P built from the metric constructed in Sec. 3.2. In terms of the basis (3.2.1) the connection coefficients are

$$\Gamma^{C}_{BD} = \frac{1}{2}\{g^{CA}[x_D(g_{AB}) + x_B(g_{DA}) - x_A(g_{BD})) + g_{BE}C^{E}{}_{AD} + g_{DE}C^{E}{}_{AB}] - C^{C}_{BD}\} \tag{3.2.2}$$

where

$$C^{\gamma}_{ab} = -F^{\gamma}_{ab}$$

$$C^{\gamma}_{\alpha\beta} = \text{structure constants of } G' \ .$$

Explicitly, the various components of the connection are:

$$\Gamma^{\gamma}_{\alpha\beta} = g^{\gamma\pi} g_{\lambda(\alpha} C^{\lambda}_{\beta)\pi} - \frac{1}{2} C^{\gamma}{}_{\alpha\beta} \tag{3.2.3a}$$

$$\Gamma^{c}_{\alpha\beta} = \frac{1}{2} g^{cd}\, h_d(g_{\alpha\beta}) \tag{3.2.3b}$$

$$\Gamma^{c}{}_{a\beta} = \Gamma^{c}{}_{\beta a} = -\frac{1}{2} g^{cd}\, g_{\beta\pi}\, F^{\pi}{}_{da} \tag{3.2.3c}$$

$$\Gamma^{\gamma}_{ab} = +\frac{1}{2} F^{\gamma}{}_{ab} \tag{3.2.3d}$$

$$\Gamma^{\gamma}_{a\beta} = \Gamma^{\gamma}_{\beta a} = \frac{1}{2} g^{\gamma\pi} h_a(g_{\beta\pi}) \tag{3.2.4e}$$

$$\Gamma^{c}_{ab} = \frac{1}{2} g^{cd}[g_{bd,a} + g_{ad,b} - g_{ab,d}] \tag{3.2.4f}$$

Having constructed the connection in TP, we are now able to calculate covariant derivatives of admissible fields in bundle in all directions. The meaning of this covariant derivative is the topic of the next section.

3.4 Compatibility of the Connections

I would now like to turn to a problem which has not been addressed previously. As I stated in the introduction, the motive for construction of a linear connection on the bundle is the unification of gravity and gauge theories within a single framework. Doing this will enable us to specify field equations for the gravitation and gauge fields in the same manner as in the Einstein theory. It will also make it possible to assign a meaning to covariant differentiation of tensors which have both spacetime and internal degrees of freedom. The curvature form is an example of such a field.

It is important to realize, however, that among the fields defined on the bundle are the horizontal lifts of spacetime tensors and the slides of adjoint tensors. The covariant derivatives of these fields are already defined by the gravitational and gauge covariant derivatives, respectively. This gives us three candidates for the covariant derivative of a tensor. In what follows I shall denote them as ∇, D, and Δ for bundle, gauge, and gravitational covariant derivatives respectively. In view of this problem it is natural to ask that the appropriately comparable components of these derivatives agree. Specifically, consistency of these derivatives is the demand that the horizontal lift of the gravitational covariant derivative of a spacetime tensor be equal to the horizontal part of the bundle covariant derivative in a horizontal direction of the lift of such a field, that is

$$\mathrm{Lift}[\Delta_a(t^{a\cdots}_{b\cdots}(x)\partial_a \otimes \ldots dx^b \ldots)]$$

$$= \mathrm{hor}[\nabla_a(t^{a\cdots}_{b\cdots}(\pi(p))h_a \otimes \ldots dx^b \ldots)] \tag{3.4.1}$$

In other words, there should be commutativity of the diagram

$$\begin{tikzcd}
\mathrm{hor} \otimes TP \arrow[r, "\nabla_a"] & \otimes TP \arrow[d, "\pi"] \\
\otimes TM \arrow[u, "\mathrm{Lift}"] \arrow[r, "\Delta_a"'] & \otimes TM
\end{tikzcd}$$

For an adjoint field, the analogous requirement is that the slide of the gauge covariant derivative of such a field be equal to the bundle covariant derivative in a horizontal direction of the slide of the field.

$$\mathrm{slide}[D_a(T^{\alpha\cdots}_{\beta\cdots}(p)L_\alpha \otimes \ldots \Phi^\alpha \ldots)]$$

$$= \mathrm{ver}[\nabla_a(T^{\alpha\cdots}_{\beta\cdots}(p)\ell_\alpha \otimes \ldots \phi^\alpha \ldots)] \tag{3.4.2}$$

so that there is commutativity of the diagram

$$\begin{tikzcd}
\mathrm{ver} \otimes TP \arrow[r, "\nabla_a"] & \otimes TP \arrow[d, "\omega"] \\
\otimes G' \arrow[u, "\mathrm{slide}"] \arrow[r, "D_a"'] & \otimes G'
\end{tikzcd}$$

To check this out consider for simplicity what happens for vector fields. For a spacetime vector we have

$$\nabla_{h_a}(v^b h_b) = h_a(v^b)h_b + v^C \Gamma_C{}^B{}_a X_B$$

$$= (h_a(v^b) + v^c\Gamma^b_{ca})h_b + v^c\Gamma^\beta_{ca}\ell_\beta$$

Now $\pi_*\nabla_{h_a}(v^b h_b) = (\partial_a(v^b) + v^c\Gamma^b_{ca})\partial_b$ and $\Gamma^c_{ba} = \Gamma_{st}{}^c{}_{ba}$, so compatibility of bundle and spacetime covariant derivatives is guaranteed without further restrictions on the bundle metric.

For an adjoint vector we have

$$\nabla_{h_a}(v^\alpha \ell_\alpha) = h_a(v^\alpha)\ell_\alpha + v^\gamma \Gamma^B_{\gamma a} x_B$$

$$= (h_a(v^\alpha) + v^\gamma \Gamma^\alpha_{\gamma a})\ell_\alpha + v^\gamma \Gamma^b_{\gamma a} h_b \ .$$

If we apply the connection form ω to this we obtain

$$\omega(\nabla_{h_a}(v^\alpha \ell_\alpha)) = (h_a(v^\alpha) + v^\gamma \Gamma^\alpha_{\gamma a})L_\alpha \neq D_a(v^\alpha)L_\alpha$$

unless $\Gamma^\alpha_{\gamma a}=0$. Now $\Gamma^\alpha{}_{\gamma a} = \frac{1}{2} g^{\alpha\pi} h_a(g_{\gamma\pi})$ in a torsion free theory, so $\Gamma^\alpha_{\gamma a}=0$ implies that

$$h_a(g_{\alpha\beta}) = 0 \ . \tag{3.4.3}$$

Since this is just the components of the slide of $D_a g$, it follows that the necessary and sufficient condition for compatibility of gauge and bundle covariant derivatives is that the group metric be gauge covariantly constant. This is in fact the usual statement of compatibility of a connection and a metric.

This compatibility requirement on the group metric has a particularly interesting consequence for proper distance as determined by the metric for the theory. As we have seen previously, the equation

$$\nabla g = 0$$

guarantees that proper distance as determined by the bundle metric and affine parameters are the same along geodesics. In a theory without torsion, Eq. (3.4.3) implies that proper distance measured by the full bundle metric is equivalent to that measured by the spacetime metric itself. To see this, note that along a geodesic

$$\left(\frac{d\tau}{d\lambda}\right)^2 = g_{ab}v^a v^b$$

where $d\tau$ is the differential of proper distance determined by the spacetime metric. Now

$$2\,\frac{d\tau}{d\lambda}\,\frac{d^2\tau}{d\lambda^2} = v^D x_D(g_{ab}v^a v^b)$$

$$= v^D v^a v^b x_D(g_{ab}) - 2v^D v^b v^E\,\Gamma^a_{ED}\,g_{ab}$$

since $v^D x_D$ is the tangent to a geodesic. The right-hand side may now be written

$$v^D v^a v^b x_D(g_{ab}) - 2v^D v^b v^E\,\Gamma^a_{ED}\,g_{ab} = v^d v^a v^b(\partial_d(g_{ab}) - 2\Gamma^e_{ad}\,g_{eb})$$

$$- 4v^\pi v^b v^a\,\Gamma^e_{a\pi}\,g_{eb}$$

since $\Gamma^a_{\epsilon\pi} = g^{af}h_f(g_{\epsilon\pi}) = 0$ by assumption. Now

$$v^\pi v^b v^a\,\Gamma^e_{a\pi}\,g_{eb} = -\frac{1}{2}\,v_\pi v^b v^a F^\pi{}_{ba} = 0$$

so it follows that

$$2\,\frac{d\tau}{d\lambda}\,\frac{d^2\tau}{d\lambda^2} = v^d v^a v^b(\partial_d g_{ab} - 2g_{be}\,\Gamma^e_{ad}) = 0$$

if the space-time connection is metric compatible. Therefore we conclude that

$$\frac{d^2\tau}{d\lambda^2} = 0$$

which establishes the conjecture. The gravitational part of the theory is then the Einstein version rather than the one proposed by Brans and Dicke [9].

It is now established that the Jordan-Kaluza-Klein approach will in fact unify the standard versions of gravitation and gauge theories without doing violence to either provided that (3.4.3) holds, which in effect requires that the vertical components of the bundle metric cannot be introduced as independent fields in this theory. This rules out the formulation of the theory proposed by Cho and Freund [7]. To check Kopczynski's formulation, we must determine how the compatibility conditions will change in a theory which includes torsion. In a theory of this type the connection coefficients have the form

$$\Gamma^{C}_{BD} = g^{CA}[\tfrac{1}{2}\,(x_D(g_{AB}) + x_B(g_{DA}) - x_A(g_{BD}) - g_{BE}(Q^{E}{}_{AD} - \tfrac{1}{2}\,C^{E}{}_{AD}) - g_{DE}(Q^{E}{}_{AB} - \tfrac{1}{2}\,C^{E}{}_{AB})] + Q^{C}{}_{BD} - \tfrac{1}{2}\,C^{C}{}_{BD}\;. \tag{3.4.4}$$

For the connection coefficients

$$\Gamma^{c}_{bd} = g^{ca}[\tfrac{1}{2}(g_{ab,d} + g_{da,b} - g_{bd,a}) - g_{be}Q^{e}{}_{ad} - g_{de}Q^{e}{}_{ab}] + Q^{c}{}_{bd} = \Gamma_{st}{}^{c}{}_{bd}$$

if we identify the tensor $Q^{a}{}_{bc}$ with the lift of the torsion field on spacetime, so (3.3.1) still holds for this theory. Equation (3.3.2) still requires $\Gamma^{\gamma}{}_{\beta d}=0$. Now in this theory

$$\Gamma^{\gamma}{}_{\beta d} = g^{\gamma\alpha}[\tfrac{1}{2}\,h_d(g_{\alpha\beta}) - g_{\beta\varepsilon}Q^{\varepsilon}{}_{\alpha d} - g_{de}Q^{e}{}_{\alpha\beta}] + Q^{\gamma}{}_{\beta d}\;.$$

Setting this expression equal to zero yields the expression

$$\tfrac{1}{2}\,g^{\gamma\alpha}h_d(g_{\alpha\beta}) = -Q^{\gamma}{}_{\beta d} + g^{\gamma\alpha}(Q_{\beta\alpha d} + Q_{d\alpha\beta})$$

which is equivalent to

$$\tfrac{1}{2}\,h_d(g_{\alpha\beta}) = +2Q_{[\beta\alpha]d} + Q_{d\alpha\beta} \tag{3.4.5}$$

The only way this equation can be satisfied is if

$$h_d(g_{\alpha\beta}) = 0 \tag{3.4.6a}$$

$$Q_{d\alpha\beta} = -2Q_{[\beta\alpha]d} \tag{3.4.6b}$$

since the metric tensor is symmetric in α and β, while the right-hand side of (3.4.5) is antisymmetric in these indices. The theory constructed by Kopczynski [8] satisfies Eq. (3.4.6b) since the mixed components of the torsion with two Greek and 1 Latin index are required to vanish by his assumptions about the nature of the connection. It will also satisfy (3.5.6a) with the appropriate choice of metric.

3.5 The Conditions for an Admissable Bundle Metric

As shown in the last section, compatibility of the bundle and gauge connections leads to the requirement that the group components of the bundle metric be gauge covariantly constant, that is

$$D_a \tilde{g}_{\alpha\beta} = h_a(g_{\alpha\beta}) = 0 \ . \tag{3.5.1}$$

This is a differential equation for the components of $g_{\alpha\beta}$. In a coordinate system it takes the form[1]

$$\partial_a g_{\alpha\beta}(x,a) = 2g_{\pi(\alpha}(x,a)C^{\pi}_{\beta\lambda}A^{\lambda}_a(x,a) \ .$$

This equation will have solutions provided that

$$\partial_{[a}\partial_{b]}g_{\alpha\beta}(x,a) = 0 \ .$$

This integrability condition for (3.5.1) is equivalent to the statement

[1]To simplify the notation in this section, I am using the expression $g_{\alpha\beta}(x,a)$ to represent the object $g_{\pi\lambda}(x)\mathrm{Ad}_{a^{-1}}{}^{\pi}{}_{\alpha}\mathrm{Ad}_{a^{-1}}{}^{\lambda}{}_{\beta}$ where possible.

that

$$[h_a, h_b] g_{\alpha\beta} = 0 \ ,$$

which is easier to calculate. Since $[h_a, h_b] = F^{\pi}{}_{ab} L_{\pi}$ the integrability condition for (3.4.1) is

$$2 g_{\pi(\alpha}(x,a) C^{\pi}_{\beta)\lambda} F^{\lambda}{}_{ab} = 0 \ . \qquad (3.5.2)$$

A metric which satisfies (3.5.1) cannot be a true degree of freedom in the theory, since it is not possible to construct a Lagrangian for it which contains a non-vanishing kinetic energy term. One might suspect, therefore, that a metric of this type would have only a spurious dependence on spacetime coordinates that can be removed by the appropriate choice of gauge. To investigate this, note that if the gauge transformation is given as $\bar{a}=b(x)a$, then the equation to be satisfied is

$$\bar{g}_{\alpha\beta}(x) = g_{\pi\lambda}(x)(Ad_{b^{-1}})^{\pi}{}_{\alpha}(Ad_{b^{-1}})^{\lambda}{}_{\beta} = \text{Constant} \ .$$

These algebraic relations imply the differential relations

$$d\bar{g}_{\alpha\beta}(x) = 0 \ .$$

Explicitly this equation becomes

$$-d(g_{\alpha\beta}(x) = g_{\pi\lambda}(x) d[(Ad_{b^{-1}})^{\pi}{}_{\varepsilon}(Ad_{b^{-1}})^{\lambda}{}_{\omega})](Ad_b)^{\varepsilon}{}_{\alpha}(Ad_b)^{\omega}{}_{\beta} \qquad (3.5.3)$$

The right hand side of Eq. (3.5.3) can be evaluated by using

$$\frac{\partial}{\partial x^a}(Ad_b{}^{\alpha}{}_{\beta}) = \frac{\partial b^{\pi}}{\partial x^a}\frac{\partial}{\partial b^{\pi}}[(Ad_b)^{\alpha}{}_{\beta}]$$

$$(b_* \frac{\partial}{\partial x^a})[(Ad_b)^{\alpha}{}_{\beta}]$$

$$= \Phi^{\gamma}(b_* \frac{\partial}{\partial x^a}) L_{\gamma}[(Ad_b)^{\alpha}{}_{\beta}] = b^*\Phi^{\gamma}(\frac{\partial}{\partial x^a}) L \ [(Ad_b)^{\alpha}{}_{\beta}] \ .$$

The action of L_γ on the matrix Ad is

$$L_\gamma[(Ad_b)^\alpha{}_\beta] = -(Ad_b)^\lambda{}_\beta\, C^\alpha{}_{\gamma\lambda} \;.$$

Using this we can rewrite (3.5.3) as

$$dg_{\alpha\beta}(x) = 2g_{\pi\lambda}(x)(Ad_{b^{-1}})^\rho{}_\varepsilon\, C^\pi{}_{\gamma\rho}\, (Ad_{b^{-1}})^\lambda{}_u (Ad_b)^\varepsilon{}_{(\alpha}(Ad_b)^u{}_{\beta)}\, b^*\phi^\gamma$$

where $b^*\phi^\gamma = b^*\phi^\gamma(\frac{\partial}{\partial x^a})dx^a$. This expression reduces to

$$dg_{\alpha\beta}(x) = -2g_{\pi(\alpha}\, C^\pi{}_{\beta)\gamma}\, b^*\phi^\gamma \;. \tag{3.5.4}$$

Equation (3.5.4) cannot be solved for arbitrary metrics, however, for metrics which satisfy (3.5.1) we can rewrite it as

$$g_{\pi(\alpha}(x)\, C^\pi{}_{\beta)\gamma}\, [b^*\phi^\gamma - A^\gamma] = 0 \tag{3.5.5}$$

with $A^\gamma = A^\gamma{}_a\, dx^a$.

This equation can be solved at any point x_0. Its exterior derivative at x_0 is then

$$g_{\pi(\alpha}(x)\, C^\pi{}_{\beta)\gamma}\, F^\gamma{}_{ab}\, dx^a \wedge dx^b = 0$$

whenever (3.5.1) holds. These conditions are sufficient to construct the functions b(x) via a Taylor series. This establishes the conjecture.

Since a metric of this type is not a true degree of freedom in the theory, we would naturally prefer to do physics in which $\bar{g}_{\alpha\beta}$ = Constant. Having found such a gauge the remaining gauge transformations which preserve it are those satisfying

$$g_{\pi(\alpha}\, C^\pi{}_{\beta)\lambda}\, b^*\bar{\phi}^\lambda = 0 \;.$$

Let us now turn our attention to the selection of metrics which satisfy these conditions.

For arbitrary fields $F^{\lambda}{}_{ab}$ Eq. (3.5.2) will only be satisfied by metrics for which

$$g_{\pi(\alpha}(x)\; C^{\pi}{}_{\beta)\lambda} = 0 \; . \tag{3.5.8}$$

Group metrics satisfying this condition are invariant under the group action Ad since to lowest order

$$\begin{aligned} g_{\alpha\beta}(x,a) &= g_{\pi\lambda}(x)(\delta^{\pi}{}_{\alpha} - b^{\gamma}C^{\pi}{}_{\alpha\gamma})(\delta^{\lambda}{}_{\beta} - b^{\sigma}C^{\lambda}{}_{\beta\sigma}) \\ &= g_{\pi\lambda}(x) - 2g_{\pi(\alpha}C^{\pi}{}_{\beta)\gamma}b^{\gamma} = g_{\pi\lambda}(x) \; . \end{aligned}$$

Such metrics are obviously constant along the fiber as well as gauge invariant, moreover (3.5.1) implies that they are also constant along any cross section. Existence of such metrics depends upon the nature of the group being considered. For example, it follows from (3.5.8) that

$$g^{\alpha\beta}g_{\pi(\alpha}C^{\pi}{}_{\beta)\lambda} = 2C^{\beta}{}_{\beta\lambda} = 0 \; .$$

Groups whose structure constants do not satisfy this relation therefore cannot possess Ad-invariant metrics. For groups which do satisfy it, a possible candidate metric is the object

$$g_{\alpha\beta} = C^{\mu}{}_{\nu\alpha}C^{\nu}{}_{\mu\beta}$$

known as the Killing metric. For this metric (3.5.8) becomes

$$\begin{aligned} C^{\mu}{}_{\nu\pi}C^{\nu}{}_{\mu(\alpha}C^{\pi}{}_{\beta)\lambda} &= \tfrac{1}{2}(C^{\mu}{}_{\nu\pi}C^{\nu}{}_{\mu\alpha}C^{\pi}{}_{\beta\lambda} + C^{\mu}{}_{\nu\pi}C^{\nu}{}_{\mu\beta}C^{\pi}{}_{\alpha\lambda}) \\ &= + \tfrac{1}{2}(-C^{\mu}{}_{\lambda\pi}C^{\nu}{}_{\mu\alpha}C^{\pi}{}_{\nu\beta} - C^{\mu}{}_{\beta\pi}C^{\nu}{}_{\mu\alpha}C^{\pi}{}_{\lambda\nu} \\ &\quad + C^{\mu}{}_{\nu\beta}C^{\nu}{}_{\pi\lambda}C^{\pi}{}_{\mu\alpha} + C^{\mu}{}_{\nu\beta}C^{\nu}{}_{\pi\alpha}C^{\pi}{}_{\lambda\mu}) = 0 \; . \end{aligned}$$

Obviously, any constant times the Killing metric is also Ad-invariant.

Ad-invariant metrics also exist for Abelian Groups and for groups whose Killing metrics are degenerate. The general form of such a metric can be obtained by direct construction using (3.5.8). Let us consider three examples, the Weinberg-Salem electroweak group U(1)xSU(2), the color group SU(3) and the 4-parameter group classified as U3I2 by MacCallum [10].

For U(1)xSU(2) the structure constants are determined by the commutation relations

$$[L_i, L_j] = \varepsilon^k{}_{ij}\, L_k \qquad i,j,k = 1,2,3$$

$$[L_i, L_4] = 0 \ .$$

The independent non-zero components of $g_{\alpha\beta}$ are found by writing out the sum over π in (3.4.8) for fixed values of α and β. The result is

$$g_{11} = g_{22} = g_{33} = A \text{ and}$$

$$g_{44} = B$$

$$g_{\alpha\beta} = 0 \qquad \alpha \neq \beta.$$

So that the metric is the diagonal matrix

$$g = \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & A & 0 & 0 \\ 0 & 0 & A & 0 \\ 0 & 0 & 0 & B \end{pmatrix}$$

with A and B arbitrary.

For SU(3) there are 8 generators whose independent non-zero structure constants are

$$C^3_{12} = 1 \qquad C^7_{14} = \frac{1}{2} \qquad C^6_{15} = -\frac{1}{2}$$

$$C^7_{25} = \frac{1}{2} \qquad C^5_{34} = \frac{1}{2} \qquad C^7_{36} = -\frac{1}{2}$$

$$C^8_{45} = \frac{\sqrt{3}}{2} \qquad C^8_{67} = \frac{\sqrt{3}}{2}$$

The only Ad-invariant metric for this case turns out to be

$$g_{\alpha\beta} = A\delta_{\alpha\beta} = AC^{\mu}{}_{\nu\alpha}C^{\nu}{}_{\mu\beta} .$$

For the U3I2 group the commutation relations are

$$[L_1, L_2] = L_4 \qquad [L_2, L_3] = L_1$$

$$[L_3, L_1] = L_2 \qquad [L_4, L_\alpha] = 0$$

The Killing metric for this group is degenerate but it nevertheless possesses an Ad-invariant metric. In matrix form it is

$$g = \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & A & 0 & 0 \\ 0 & 0 & B & A \\ 0 & 0 & A & 0 \end{pmatrix}$$

The admissible metrics obtained from this procedure for SU(3) and SU(2)xU(1) are easy to recognize, up to a constant factor(s)[1] they are the standard ones adopted when constructing gauge field theories based on the groups. This is a fortunate outcome for the Kaluza-Klein approach to unification, since as we have seen the theory can only be given an unambiguous meaning for these groups when these metrics are adopted. A new feature that has appeared is the possibility of erecting theories based on non-simple Lie Groups.

[1]The meaning of these factors will be discussed in the next chapter.

CHAPTER IV

GEODESICS AND THE LAGRANGIAN

4.1 Geodesics of the Connection Γ

A desirable feature of a unified theory is that classical free-particle trajectories are determined by the geodesics of the connection. In a bundle theory, what is required is that the projection into spacetime of the bundle geodesic be the correct equation of motion for a particle which experiences both gauge and gravitational forces. I shall now proceed to verify this for the torsion-free case.

A geodesic in the bundle is a curve whose tangent v satisfies Eq. (3.3.2)

$$v^B x_B(v^A) + v^B v^C \Gamma^A_{BC} = 0 .$$

Now $v^B x_B = d/d\lambda$ where λ is the affine parameter of the geodesic, so the equation can be written

$$\frac{d}{d\lambda} v^A + v^B v^C \Gamma^A_{BC} = 0 .$$

It is convenient to break this down into equations for the horizontal and vertical parts of v. These are

$$\frac{dv^a}{d\lambda} + 2v^b v^\gamma \Gamma^a_{b\gamma} + v^b v^c \Gamma^a_{bc} + v^\beta v^\gamma \Gamma^a_{\beta\gamma} = 0 , \qquad (4.1.2a)$$

$$\frac{dv^\alpha}{d\lambda} + v^\beta v^\gamma \Gamma^\alpha_{\beta\gamma} + 2v^b v^\gamma \Gamma^\alpha_{b\gamma} + v^b v^c \Gamma^\alpha_{bc} = 0 . \qquad (4.1.2b)$$

The fourth term in (4.1.2a) and the third term in (4.1.2b) vanish since $\Gamma^a_{\beta\gamma} = \Gamma^\alpha_{b\gamma} = 0$. The fourth term in (4.1.2b) also vanishes since $\Gamma^\alpha{}_{bc} = \frac{1}{2} F^\alpha{}_{bc}$ is antisymmetric in b and c. These equations therefore reduce to

$$\frac{dv^a}{d\lambda} + 2v^b v^\gamma \Gamma^a{}_{b\gamma} + v^b v^c \Gamma^a{}_{bc} = 0 \tag{4.1.3a}$$

$$\frac{dv^\alpha}{d\lambda} + v^\beta v^\gamma \Gamma^\alpha{}_{\beta\gamma} = 0 \ . \tag{4.1.3b}$$

Using the expressions for the connection coefficients given in (3.2.3), the first equation may be written as

$$\frac{dv^a}{d\lambda} + v^b v^c \Gamma_{st}{}^a{}_{bc} = v^b v_\pi g^{ad} F^\pi{}_{bd} \ .$$

Since v^a, g^{ad} and $v_\pi F^\pi cd$ are invariant under the group action ψ, this equation for v^a describes a unique path in the base space whose tangent is v^a. For the electromagnetic case (G=U(1)), this equation may be recognized as the Lorentz Force Law for a particle moving in an electromagnetic field if $v_\pi = v_5$ is identified with the charge of the particle.

Equation (4.1.3b) may be written as

$$\frac{dv^\alpha}{d\lambda} + v^\beta v^\gamma g^{\alpha\pi} g_{\lambda(\beta} C^\lambda{}_{\gamma)\pi} = 0$$

which is equivalent to

$$\frac{dv^\alpha}{d\lambda} + v^\gamma v_\lambda g^{\alpha\pi} C^\lambda{}_{\gamma\pi} = 0 \ .$$

This latter equation implies that

$$\frac{dv_\alpha}{d\lambda} = \frac{d}{d\lambda} (g_{\alpha\pi} v^\pi) = 0 \ ,$$

so that the charge v_α is a conserved quantity. Obviously, if the group metric on P is Ad-invariant, then both v^α and v_α are conserved.

4.2 Calculation of the Lagrangian

The final item needed to complete the theory is a Lagrangian for the fields. For linear connection theories, the possible candidates are usually constructed from the curvature tensor $R^A{}_{BCD}$. In this thesis I shall adapt the standard one used in the Einstein version of the theory

$$L = \sqrt{-g}\, R \ . \tag{4.2.1}$$

Lagrangians quadratic in the curvature are also possible, but at present (4.2.1) is the only one which is known for certain to produce the correct field equations for gravitation.

To calculate R, we use the relation

$$R = g^{BD}\, R^A{}_{BAD} \tag{4.2.2}$$

where $R^A{}_{BCD} = x_C(\Gamma^A{}_{BD}) - x_D(\Gamma^A{}_{BC}) + \Gamma^A{}_{MC}\, \Gamma^M{}_{BD} - \Gamma^A{}_{MD}\, \Gamma^M{}_{BA} - \Gamma^A{}_{BM}\, C^M{}_{CD}$.

(4.2.3)

The above expression for $R^A{}_{BCD}$ is valid for both holonomic and non-holonomic bases and for connections with and without torsion. To calculate R, it is convenient to break (4.2.1) into parts

$$g^{BD}\, R^A{}_{BAD} = g^{bd}\, R^a{}_{bad} + g^{bd}\, R^\alpha{}_{b\alpha d}$$

$$+ g^{\beta\pi}\, R^a{}_{\beta a\pi} + g^{\beta\pi}\, R^\alpha{}_{\beta\alpha\pi} \ .$$

The components of $R^A{}_{BCD}$ we need are

$$R^a{}_{bad} = R_{st}{}^a{}_{bad} - \frac{1}{4}\, g_{\alpha\beta}\, g^{am}\, F^\alpha{}_{ba}\, F^\beta{}_{dm}$$

$$R^\alpha{}_{b\alpha d} = + \frac{1}{4}\, g_{\alpha\lambda}\, g^{m\ell}\, F^\alpha{}_{md}\, F^\lambda{}_{\ell b}$$

$$R^a{}_{\beta a\pi} = + \frac{1}{4}\, g_{\beta\varepsilon}\, g_{\pi\lambda}\, g^{a\ell}\, g^{mn}\, F^\varepsilon{}_{\ell m}\, F^\lambda{}_{an}$$

$$R^{\alpha}{}_{\beta\alpha\pi} = g^{\omega\varepsilon} C^{\alpha}{}_{\omega\alpha}\, g_{\lambda(\beta} C^{\lambda}{}_{\pi)\varepsilon} - g^{\alpha\varepsilon} g^{\lambda\kappa} g_{\omega(\lambda} C^{\omega}{}_{\pi)\varepsilon}\, g_{\nu(\beta} C^{\nu}{}_{\alpha)\kappa}$$

$$- \frac{1}{2} g^{\alpha\varepsilon} g_{\omega(\lambda} C^{\omega}{}_{\pi)\alpha} C^{\lambda}{}_{\beta\varepsilon} + \frac{1}{2} g^{\lambda\kappa} g_{\nu(\beta} C^{\nu}{}_{\alpha)\kappa} C^{\alpha}{}_{\lambda\pi}$$

$$- \frac{1}{4} C^{\alpha}_{\omega\beta}\, C^{\omega}_{\alpha\pi} .$$

From this it follows that

$$R = R_{ST} - \frac{1}{4} g_{\alpha\beta} F^{\alpha}{}_{bd} F^{\beta bd} - g^{\beta\pi} C^{\alpha}{}_{\beta\alpha} C^{\gamma}{}_{\pi\gamma}$$

$$- \frac{1}{2} g^{\beta\pi} C^{\alpha}{}_{\lambda\beta} C^{\lambda}{}_{\alpha\pi} + \frac{1}{4} g_{\beta\pi} g^{\alpha\lambda} g^{\mu\varepsilon} C^{\beta}{}_{\mu\lambda} C^{\pi}{}_{\alpha\varepsilon} . \qquad (4.2.4)$$

If the group metric is Ad-invariant, then the last term in (4.2.4) is

$$+ \frac{1}{4} g^{\beta\pi} C^{\alpha}{}_{\lambda\beta} C^{\lambda}{}_{\alpha\pi} .$$

The third term in (4.2.4) also vanishes for Ad-invariant metrics, since $C^{\beta}{}_{\beta\gamma}=0$ in this case. In such a case R will reduce to

$$R = R_{ST} - \frac{1}{4} g_{\alpha\beta} F^{\alpha}{}_{bd} F^{\beta bd} - \frac{1}{4} g^{\beta\pi} C^{\alpha}{}_{\lambda\beta} C^{\lambda}{}_{\alpha\pi} . \qquad (4.2.5)$$

For a theory of coupled electromagnetic and gravitational fields based on the structure group U(1) this Lagrangian is equivalent to that used in the standard Einstein theory, since the last term in (4.2.5) vanishes. For a non-Abelian group, however, this term will play the role of a cosmological constant in gravitational calculations. This is potentially a serious problem for the theory, since it is by no means obvious that a gauge group which produces an acceptable description of observed particle phenomenon will also generate a cosmological constant sufficiently small to match current limits for this quantity derived from astronomical observations.

To examine this point in more detail, it is necessary to recast the theory in the standard form used in field theory calculations.

In the usual formulation of gauge theories the gauge covariant derivative is written in the form

$$D_a = \partial_a - b\, A^\alpha{}_a\, L_\alpha \, .$$

The constant factor b is the coupling constant which determines the strength of the interaction between gauge and particle fields. The notation here is convenient but somewhat misleading, since if the gauge group is the direct product of two or more subgroups, the constant b can be different for each subalgebra of corresponding generators.

The other difference between the standard form of a gauge theory and the one presented here is that the Lagrangian of the gauge fields is taken to be

$$L = K_{\alpha\beta}\, F^\alpha{}_{ab}\, F^{\beta ab}$$

as opposed to

$$L = g_{\alpha\beta}\, F^\alpha{}_{ab}\, F^{\beta ab} \, .$$

$K_{\alpha\beta}$ is the invariant Killing metric of the gauge group, and $g_{\alpha\beta}$ is any metric satisfying the compatibility condition (3.3.1).

To relate these two structures, recall that for arbitrary gauge fields the only metrics which satisfy (3.3.1) are the invariant constant ones. As has been shown by Kopczynski [3], for the groups currently used in physical calculations the invariant constant metrics are of the form

$$g_{\alpha\beta} = \kappa^2\, K_{\alpha\beta} \tag{4.2.6}$$

where κ^2 is an arbitrary constant.[1]

To replace $g_{\alpha\beta}$ by $K_{\alpha\beta}$, note that rigorously the metric on G' is a symmetric 2-form whose components are determined by the relation

$$g_{\alpha\beta} = g(L_\alpha, L_\beta) = \kappa^2 \, K_{\alpha\beta}$$

where the L_α's are some basis for G'. Relative to a new basis $\bar{L}_\alpha$ defined by

$$L_\alpha = \kappa \, \bar{L}_\alpha$$

the components of the metric become

$$g(\bar{L}_\alpha, \bar{L}_\beta) = K_{\alpha\beta} \, ,$$

which is precisely the desired form. The commutation relations for the new basis are

$$\begin{aligned} [\bar{L}_\alpha, \bar{L}_\beta] &= \kappa^{-2} [L_\alpha, L_\beta] \\ &= \kappa^{-2} \, C^\gamma{}_{\alpha\beta} \, L_\gamma \\ &= \kappa^{-1} \, C^\gamma{}_{\alpha\beta} \, \bar{L}_\gamma \, . \end{aligned}$$

The action of $\bar{L}_\alpha$ on a field of type Ad is obviously

$$\bar{L}_\alpha(v^\beta) = \kappa^{-1} \, v^\pi \, C^\beta{}_{\pi\alpha} \, .$$

Following Cho and Freund [2], it is now convenient to introduce rescaled vector potentials

$$\bar{A}^\alpha{}_a = b^{-1} \, A^\alpha{}_a \, . \tag{4.2.7}$$

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[1]For direct-product groups, κ^2 may again be different for each subalgebra, since $K_{\alpha\beta}$ is then a block-diagonal matrics whose nonzero parts correspond to the Killing metric on each sub-algebra.

In terms of the rescaled vector potentials and Lie Algebra elements, the gauge covariant derivative in a coordinate system now takes the form

$$D_a = \partial_a - b\kappa(Ad_{a^{-1}})^{\alpha}{}_{\beta}\bar{A}^{\beta}{}_{a}\,\bar{L}_{\alpha}(a) \ .$$

The commutator of two horizontal vectors is now

$$[h_a, h_b] = -b\kappa\,\bar{F}^{\alpha}{}_{ab}\,\bar{L}_{\alpha}$$

where $\bar{F}^{\alpha}{}_{ab} = \partial_a \bar{A}^{\alpha}{}_{b} - \partial_b \bar{A}^{\alpha}{}_{a} + b\,C^{\alpha}{}_{\beta\gamma}\,\bar{A}^{\beta}{}_{a}\,\bar{A}^{\gamma}{}_{b}$.

Henceforth I shall for simplicity drop the bar notation and treat the rescaled A's, F's and L's as the primary ones. The Lagrangian (4.2.5)

$$L = \frac{c^4}{16\pi G}\sqrt{-g}\,\left(R_{ST} - \frac{1}{4}b^2\,\kappa^2\,K_{\alpha\beta}\,F^{\alpha}{}_{ab}\,F^{\beta ab} - \frac{K^{\alpha\beta}}{4\kappa^2}\,C^{\pi}{}_{\lambda\alpha}\,C^{\lambda}{}_{\pi\beta}\right)$$

where the factor $c^4/16\pi G$ is introduced so the Lagrangian will have the correct form when matter fields are incorporated into the theory. This Lagrangian will have the usual form assumed for coupled gauge and gravitational fields provided that

$$\kappa^2 = \frac{16\pi G}{c^4 b^2}$$

Now (4.2.7) constitutes in effect a definition of the fundamental unit of charge in the theory; the preceding equation implies that it is ultimately related to the volume of fiber space.

We are now ready to evaluate the cosmological constant. The last term in the Lagrangian (4.2.5) is now

$$-\frac{K^{\alpha\beta}}{4\kappa^2}\,C^{\pi}{}_{\lambda\alpha}\,C^{\lambda}{}_{\pi\beta} = -\frac{1}{4}\sum_i n_i \kappa_i^{-2} \qquad (4.2.7)$$

where the κ_i's are the proportionality factors in (4.2.6) corresponding to each non-Abelian subgroup of the structure group and the n_i's are the dimensions of these subgroups. This result has been derived in a slightly different way by Kopczynski [3].

For a theory based on U(1)xSU(2) there are two coupling factors b_i

$$b = \frac{e}{\hbar c} \sin\theta_W$$

associated to the SU(2) subgroup, and also

$$b' = \frac{e}{\hbar c} \cos\theta_W$$

associated with the U(1) part. θ_W in these expressions is the Weinberg mixing angle. The cosmological constant for this group is

$$\Lambda = -\frac{3}{4}\left[\frac{c^4}{16\pi G}\right]\frac{e^2}{(\hbar c)^2}\sin^{-2}\theta_W$$

$$\geq -\frac{3}{4}\frac{\alpha}{\ell^2}$$

where α is the electromagnetic fine structure constant and ℓ is the Planck length. Numerically this is

$$|\Lambda| \geq 2.8 \times 10^{63}\ \text{cm}^{-2} .$$

This is larger than the maximum possible value by a factor of 10^{120}, which is obviously ridiculous.

Although this result clearly constitutes a serious embarrassment for the theory, it is by no means fatal, since there are ways to remove the cosmological factor (4.2.7) from the Lagrangian. The simplest and most obvious way to do this is to incorporate the "standard" cosmological constant into the Lagrangian (4.2.1), so that the new Lagrangian is

$$L = \sqrt{-g}\,(R + \lambda)\ .$$

The "net" cosmological constant would then be the sum of (4.2.7) and λ, which presumably could fit any observational data whatsoever, since λ is completely arbitrary. However, this modification requires us to accept notion that the difference of these two terms is a residue much smaller in magnitude than either individually; such cancellation would be little short of miraculous. This option is therefore aesthetically unappealing. Another possibility to be considered is to construct theories based on groups which possess invariant metrics but whose Killing forms are degenerate, such as the U3I2 group mentioned earlier. A third possibility is the one adopted by Kopczynski [3], in which the cosmological factor (4.2.7) is removed from the Lagrangian by the inclusion of torsion in the bundle linear connection.

CHAPTER V

CONCLUSIONS

In the preceding chapters, I have tried to illuminate the basic structure of a principle fiber bundle and its associated connection. These concepts would appear to provide a reasonably satisfactory geometric framework for gauge theories of particle interactions. The resulting gauge theory can be combined with a gravitational theory based on a linear connection if due care is exercized in its construction. "Due care" for this procedure can be succinctly stated; the components of the bundle metric corresponding to the internal degrees of freedom in bundle space should not be introduced as independent degrees of freedom in the theory. More precisely, it is required that

$$D_a\, g_{\alpha\beta}(x) = 0 \tag{5.1.1}$$

in this theory. This is true whether or not the bundle linear connection includes torsion. This condition follows from the requirement that the bundle covariant derivative ∇_a be equivalent to the spacetime covariant derivative Δ_a and the gauge covariant derivative D_a. Although at first glance this requirement might seem somewhat arbitrary, it becomes natural when it is realized that spacetime tensor fields and fields in the Lie Algebra of the structure group can also be regarded as fields in the tangent space of the bundle; some such consistency requirement is needed to avoid an ambiguity in the theory, namely, which derivative is the physical one. It is all the more compelling because there is apparently no place for these fields in gauge theories; that they are not likely to be Higgs fields has already been shown by Cho and Freund [2].

Elimination of these degrees of freedom has other interesting consequences also. First, the geodesic equation for the bundle linear connection

$$\nabla_v v = 0$$

implies the relation

$$\frac{dv^a}{d\lambda} + v^b v^c \Gamma^a{}_{bc} = v^b v_\pi g^{ad} F^\pi{}_{bd} . \qquad (5.1.2)$$

If we identify the vertical components of the tangent v to the geodesics with the charge of the particle, this is precisely the expected equation of motion for a charged particle moving under the influence of a gauge field. In particular, for a theory of electromagnetism only this reduces to the usual Lorentz Force Law. If $D_a g_{\alpha\beta} \neq 0$, then the right-hand-side of (5.1.1) would include a term of the form

$$v^\beta v^\gamma g^{ab} h_b(g_{\beta\gamma}) .$$

No evidence of the influence of such a term on the motion of a charged particle has ever been seen.

When Equation (5.1.1) holds, we also obtain from the geodesic equation the relation

$$\frac{dv_\alpha}{d\lambda} = 0 ,$$

which is the statement that the charge of a particle is conserved. If $D_a g_{\alpha\beta} \neq 0$, this simple relationship is destroyed.

It should also be noted that (5.1.1) also guarantees that proper distance s as measured by the full bundle metric and proper distance τ as measured by the spacetime metric are equivalent, that is

$$s = A\tau + B \ .$$

The new theory is therefore consistent with the standard version of General Relativity as proposed by Einstein, in which proper distance is a function of the gravitational potentials g_{ab}, all other fields influence it only indirectly, via the energy momentum tensor T_{ab}. Since this is consistent with all observational evidence to date, the agreement of s and τ in this version of the unified theory must be regarded as an asset.

Fortunately, group metrics which satisfy this compatibility restriction are not difficult to find, for arbitrary gauge fields they are the invariant ones admitted by the group. They can be found explicitly by solving the equations

$$g_{\pi(\alpha} C^{\pi}{}_{\beta)\lambda} = 0 \ ,$$

which is the condition for invariance. Equation (5.1.1) then implies that these metrics must be spacetime constants. These metrics are precisely the ones adopted in the standard formulations of gauge theories.

Finally, the natural choice of a Lagrangian for the theory

$$L = \sqrt{-g}\, g^{BD} R_{BD} \ ,$$

when written explicitly in terms of the fields, becomes

$$L = \frac{c^4}{16\pi G} \sqrt{-g_{ST}} \sqrt{|g_{gr}|} \left(R_{ST} - \frac{1}{4} g_{\alpha\beta} F^{\alpha}{}_{ab} F^{\beta ab} - g^{\beta\pi} C^{\alpha}{}_{\beta\alpha} C^{\gamma}{}_{\pi\gamma}\right.$$

$$- \frac{1}{2} g^{\beta\pi} C^{\alpha}{}_{\lambda\beta} C^{\lambda}{}_{\alpha\pi} + \frac{1}{4} g_{\beta\pi} g^{\alpha\lambda} g^{\mu\varepsilon} C^{\beta}{}_{\mu\lambda} C^{\pi}{}_{\alpha\varepsilon})$$

where the factor $\sqrt{|g_{gr}|}$ is a harmless constant factor which is ultimately determined by the volume of fibers in the total bundle space. For invariant metrics this Lagrangian becomes

$$L = \frac{c^4}{16\pi G} \sqrt{-g_{ST}} \sqrt{|g_{gr}|} \, (R_{ST} - \frac{1}{4} g_{\alpha\beta} F^{\alpha}{}_{ab} F^{\beta ab} - \frac{1}{4} g^{\beta\pi} C^{\alpha}{}_{\lambda\beta} C^{\lambda}{}_{\alpha\pi})$$

which is the standard Einstein theory Lagrangian for gauge and gravitational fields with a cosmological constant term. By a rescaling of vector potentials in terms of the invariant coupling parameters of the group

$$A^{\alpha}{}_{a} \rightarrow b \, A^{\alpha}{}_{a}$$

and a change of basis vectors for the Lie Algebra

$$L_{\alpha} \rightarrow \kappa \, L_{\alpha} \, ,$$

we can rewrite the Lagrangian in the form

$$L = \frac{c^4}{16\pi G} \sqrt{-g_{ST}} \sqrt{|g_{gr}|} \, (R_{ST} - b^2 \kappa^2 K_{\alpha\beta} F^{\alpha}{}_{ab} F^{\beta ab}$$

$$- \frac{1}{4} \kappa^{-2} K^{\beta\pi} C^{\alpha}{}_{\lambda\beta} C^{\lambda}{}_{\alpha\pi}) \, ,$$

from whence it follows that

$$\kappa^2 = \frac{16\pi G}{c^4 b^2} \, .$$

The object $K_{\alpha\beta}$ is the canonical form of the invariant metric of the group. For simple groups and their direct products, this is the Killing form. For these groups the cosmological constant turns out to be

$$\Lambda = - \frac{nb}{4\ell^2}$$

where

$$\ell = 1.62 \times 10^{-33} \text{ cm}^{-2}$$

is the Planck length. The cosmological constant for these groups is therefore much too large unless the coupling parameter(s) b are absurdly small. There are at least two ways in which this difficulty can be avoided. As I have shown, it is possible to construct a self-consistent theory based on groups whose killing forms are degenerate provided that an invariant metric for the group exists. In this case the cosmological factor

$$\Lambda = g^{\alpha\beta} \kappa_{\alpha\beta} = 0 \ .$$

The other possibility is to follow Kopczynski and introduce torsion in the bundle linear connection to cancel the cosmological term. Which, if either, of these methods will be the best approach is a problem for future research.

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