8004393

MOORE, JEANNA BETH

CLASSIFICATION OF THE NORMAL SUBGROUPS OF GENERAL LINEAR GROUP

The University of Oklahoma PH.D. 1979

University Microfilms International 300 N. Zeeb Road, Ann Arbor, MI 48106 18 Bedford Row, London WC1R 4EJ, England

THE UNIVERSITY OF OKLAHOMA

GRADUATE COLLEGE

CLASSIFICATION OF THE NORMAL SUBGROUPS OF $GL_n(R)$

A DISSERTATION

SUBMITTED TO THE GRADUATE FACULTY

in partial fulfillment of the requirements for the

degree of

DOCTOR OF PHILOSOPHY

BY

JEANNA BETH MOORE

Norman, Oklahoma

1979

CLASSIFICATION OF THE NORMAL SUBGROUPS OF $GL_n(R)$

APPROVED BY

Paul Rubel

B L Foote

John W [illegible]

A B Schwarzkopf

Harold Huneke

DISSERTATION COMMITTEE

ACKNOWLEDGEMENTS

I would like to express my appreciation to Dr. Bernard R. McDonald for his guidance, encouragement, and patience during the writing of this dissertation. I would also like to thank my family whose support and encouragement were my inspiration.

TABLE OF CONTENTS

CLASSIFICATION OF THE NORMAL SUBGROUPS OF $GL_n(R)$

CHAPTER I

THE NORMAL SUBGROUP PROBLEM

Let R be a ring and n a positive integer. Denote by $(R)_n$ the ring of all $n \times n$ matrices with elements in R. Let $GL_n(R)$, the <u>general linear group</u>, be the group of all invertible elements of $(R)_n$. The elementary matrix units E_{ij} are those elements of $(R)_n$ with 1 in the i^{th} row and j^{th} column and 0 in all other positions. From these we form the elementary transvections $T_{ij}(\lambda) = I + \lambda E_{ij}$ where I is the $n \times n$ identity matrix, λ is in R, and $i \neq j$. The group generated by $\{T_{ij}(\lambda) \mid i \neq j, 1 \leq i, j \leq n, \lambda \text{ in } R\}$ is denoted by $E_n(R)$ and is called the <u>group of elementary transvections</u>. For any element σ of $(R)_n$ we denote by σ_{ij} the element in the i^{th} row and j^{th} column of σ.

Let A be a two-sided ideal of R. The surjective ring morphism $\lambda: R \to R/A$ induces a natural group morphism $\lambda_A: GL_n(R) \to GL_n(R/A)$. The normal subgroup <u>of</u> $E_n(R)$ generated by all elementary transvections in the kernel of λ_A is denoted by $E_n(R,A)$ and is called the <u>elementary congruence subgroup of level</u> A. The <u>general congruence subgroup of level</u> A, denoted by $GL_n(R,A)$, is the inverse image under λ_A of the center of $GL_n(R/A)$.

A normal subgroup H of $GL_n(R)$ is said to be <u>standard normal</u> if there is a two-sided ideal A such that

$$E_n(R,A) \leq H \leq GL_n(R,A).$$

The "normal subgroup problem" is then one of determining when normal subgroups of $GL_n(R)$ are standard normal.

If R is a field, then noncentral normal subgroups of $GL_n(R)$ where $n \geq 3$ contain the special linear group (see Artin [1], Dieudonné [8]). Brenner ([4],[5],[6]) was the first to examine the problem for a ring which was not a field. He showed that if $n \geq 3$, the normal subgroups of $GL_n(R)$ are standard normal where R is the ring of rational integers or its residue class rings. In 1961, Klingenberg [9] characterized the normal subgroups of $GL_n(R)$ as standard normal if $n \geq 3$ and R is a commutative local ring with 2 a unit. Later Dennin and McQuillan [7] were to show this for semilocal rings. In 1964 Bass [3] showed that for any ring R, normal subgroups of $GL_n(R)$ are standard normal if $n > \max\{2, \text{stable range of } R\}$. He also showed that for any ring R, normal subgroups of $GL(R) = \bigcup_n GL_n(R)$ are standard normal. If F is a free R-module of dimension n, then $GL_n(R)$ is isomorphic to $\{\sigma : F \to F \mid \sigma \text{ is an invertible linear map}\}$. Thus if V is free of infinite dimension, we denote $\{\sigma : V \to V \mid \sigma \text{ is an invertible linear map}\}$ by GL(V). Some work has also been done [10] on the "normal subgroup problem" for normal subgroups of GL(V).

A significant result by J.S. Wilson [15] in 1970 shows that for <u>any commutative ring</u> R, normal subgroups of $GL_n(R)$ are standard normal if $n \geq 4$.

Suppose $n \geq 4$ and let $A = [a_{ij}]$ be an $n \times n$ invertible matrix over an arbitrary ring R. The matrix A is invertible means there is a matrix B with AB = BA = I. Let $A^{-1} = [b_{ij}]$. Consider the system of equations

$$\left.\begin{array}{l} b_{21} X_1 + b_{23} X_3 + \ldots + b_{2n} X_n = a_{21} \\ b_{31} X_1 + b_{33} X_3 + \ldots + b_{3n} X_n = 0 \\ \quad\vdots \\ b_{n1} X_1 + b_{n3} X_3 + \ldots + b_{nn} X_n = 0 \end{array}\right)$$

that is, a system whose coefficient matrix is the submatrix of A^{-1} obtained by deleting the first row and second column of A^{-1}. Denote the above system by $S(\langle 1,2\rangle,A)$. We may construct $S(\langle i,j\rangle,A)$ similarly.

Wilson's argument depends on the solvability of the system $S(\langle i,j\rangle,A)$ for all distinct i and j and all invertible A when R is a commutative ring. An analysis of this proof in II shows that for any ring R (not necessarily commutative) the solvability of such systems is sufficient to show that normal subgroups of $GL_n(R)$ are standard normal when $n \geq 4$.

The case $n = 2$ has always been more difficult to investigate. Dieudonné [8] showed that for R a field, the normal subgroups of $GL_2(R)$ are standard normal. The ring of integers, however, was shown by Mennicke [12] to give pathological results to the normal subgroup problem. Klingenberg [9] was able to show that if R is a local ring with 2 a unit, the normal subgroups of $GL_2(R)$ are again standard normal.

Thus, it appears that the structure of $GL_2(R)$ depends on the number of units of R, since fields and local rings, rings with many units, have standard normal subgroups of $GL_2(R)$, while the ring of integers, a ring with few units, does not. Hence, McDonald [11] shows that normal subgroups of $GL_2(R)$ are standard normal when R is a ring "with many units."

CHAPTER II

CLASSIFICATION OF NORMAL SUBGROUPS

Let R be a ring and suppose that the system $S(\langle i,j\rangle, A)$, $i\neq j$, described in the previous chapter has a solution. We will show that all normal subgroups H of $GL_n(R)$ are standard normal for $n\geq 4$.

Let $\sigma = [\sigma_{ij}]$ be in $GL_n(R)$. Define the <u>order of</u> σ, denoted by $0(\sigma)$, to be the two-sided ideal generated by $\{\sigma_{ij} \mid 1\leq i,j\leq n,\ i\neq j\}$ together with $\{\sigma_{ii}-\sigma_{jj} \mid 1\leq i,j\leq n, i\neq j\}$. For H a subgroup of $GL_n(R)$, define $0(H)$, the <u>order of</u> H, by $0(H) = \sum_{\sigma \text{ in } H} 0(\sigma)$.

<u>Lemma II.1</u> Let σ be in $GL_n(R)$ and $\beta = \sigma^{-1}$. Then $0(\sigma) = 0(\beta)$.

<u>Proof</u> Let $\overline{\sigma} = \lambda_{0(\sigma)}(\sigma)$ and $\overline{\beta} = \lambda_{0(\sigma)}(\beta)$ (see page 1). Then $I = \lambda_{0(\sigma)}(I) = \lambda_{0(\sigma)}(\sigma\beta) = \lambda_{0(\sigma)}(\sigma)\ \lambda_{0(\sigma)}(\beta) = \overline{\sigma}\,\overline{\beta}$. But $\overline{\sigma} = \alpha I$ for some α in $R/0(\sigma)$. Thus, $I = (\alpha I)\overline{\beta} = \alpha\overline{\beta}$. Then

for every i, $1 \leq i \leq n$, $\alpha\overline{\beta}_{ii} = 1$ in $R/0(\sigma)$. Thus α is a unit in $R/0(\sigma)$. Hence $\overline{\beta} = \alpha^{-1}I$. Thus $0(\beta) = 0(\sigma)$.

<u>Theorem II.2</u> Let H be a subgroup of $GL_n(R)$ normalized by $E_n(R)$ with $n \geq 4$. Let $B = \Sigma 0(\sigma)$ where the sum extends over all σ in H differing from the identity by at most two columns. Then 0(H) is contained in B.

<u>Proof</u> We show first that for any σ in H, all off-diagonal positions of σ are in B. This is done by constructing, for each σ_{ij}, an element q of $GL_n(R)$ whose order, 0(q), contains σ_{ij} and is contained in B.

Let $\sigma = [\sigma_{ij}]$ be in H and let $\sigma^{-1} = \beta = [\beta_{ij}]$. Consider the system of equations $S(\langle 1,2\rangle,\sigma)$:

$$\begin{aligned}
\beta_{21}X_1+\beta_{23}X_3+\beta_{24}X_4+\dots+\beta_{2n}X_n &= \sigma_{21}\\
\beta_{31}X_1+\beta_{33}X_3+\beta_{34}X_4+\dots+\beta_{3n}X_n &= 0\\
&\vdots\\
\beta_{n1}X_1+\beta_{n3}X_3+\beta_{n4}X_4+\dots+\beta_{nn}X_n &= 0
\end{aligned}$$

By hypothesis this system has a solution. Then let

$$g = \begin{bmatrix} 1 & X_1 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & X_3 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & X_n & 0 & \dots & 1 \end{bmatrix}.$$

Now

$$g = I + \sum_{r\neq 2} X_r E_{r2}$$

$$= \prod_{r\neq 2} (I+X_r E_{r2})$$

$$= \prod_{r\neq 2} T_{r2}(X_r).$$

Hence g is in $E_n(R)$.

Now let $p = [\sigma,g] = \sigma^{-1}g^{-1}\sigma g$. Then

$$p = \beta(I-\sum_{r\neq 2} X_r E_{r2})\sigma(I+\sum_{s\neq 2} X_s E_{s2})$$

$$= I + \sum_{s\neq 2} X_s E_{s2} - \sum_{r\neq 2} \beta X_r E_{r2}\sigma - \sum_{\substack{r\neq 2\\ s\neq 2}} (\beta X_r E_{r2}\sigma) X_s E_{s2}.$$

But

$$\sum_{r\neq 2} \beta X_r E_{r2}\sigma = \begin{bmatrix} 0 & \beta_{11}X_1+\beta_{13}X_3+\ldots+\beta_{1n}X_n & 0 & \ldots & 0 \\ 0 & \beta_{21}X_1+\beta_{23}X_3+\ldots+\beta_{2n}X_n & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & \beta_{n1}X_1+\beta_{n3}X_3+\ldots+\beta_{nn}X_n & 0 & \ldots & 0 \end{bmatrix} \sigma$$

$$= \begin{bmatrix} \sum_{r\neq 2} \beta_{1r}X_r \\ \sum_{r\neq 2} \beta_{2r}X_r \\ \vdots \\ \sum_{r\neq 2} \beta_{nr}X_r \end{bmatrix} [\sigma_{21},\sigma_{22},\ldots,\sigma_{2n}]$$

$$= \begin{bmatrix} \sum_{r\neq 2} \beta_{1r} X_r \\ \sigma_{21} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad [\sigma_{21}, \sigma_{22}, \ldots, \sigma_{2n}]$$

$$= \begin{bmatrix} * & * & \cdots & * \\ \sigma_{21}^2 & \sigma_{21}\sigma_{22} & \cdots & \sigma_{21}\sigma_{2n} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Then

$$p-I= \begin{bmatrix} 0 & X_1 & 0\ldots 0 \\ 0 & 0 & 0\ldots 0 \\ 0 & X_3 & 0\ldots 0 \\ \vdots & \vdots & \vdots \\ 0 & X_n & 0\ldots 0 \end{bmatrix} - \begin{bmatrix} * & * & \cdots & * \\ \sigma_{21}^2 & \sigma_{21}\sigma_{22} & \cdots & \sigma_{21}\sigma_{2n} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} - \sum_{s\neq 2} \begin{bmatrix} * & * & \cdots & * \\ \sigma_{21}^2 & \sigma_{21}\sigma_{22} & \cdots & \sigma_{21}\sigma_{2n} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & 0 \end{bmatrix} X_s E_{s2}.$$

Thus for $i>2$ and $j\neq 2$,

$$(p-I)_{ij} = 0. \qquad (1)$$

Let $\hat{0}(p)$ be the two-sided ideal generated by $\{(p-I)_{2j} \mid 1\leq j\leq n\}$. We note that since p_{2j} is in $0(p)$ for $j\neq 2$ and $p_{22}-1 = p_{22} - p_{33}$ is in $0(p)$, then $\hat{0}(p)$ is contained in $0(p)$.

Now $(p-I)_{2j} = -\sigma_{21}(\sigma_{2j}+\sum_{s\neq 2}\sigma_{2s}X_s\delta_{j2})$ where

$$\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j. \end{cases}$$

Thus for $j\neq 2$, $(p-I)_{2j}=-\sigma_{21}\sigma_{2j}$ and hence $-\sigma_{21}\sigma_{2j}$ is in $0(\hat{p})$ for $j\neq 2$.

Then

$$(p-I)_{22} = -\sigma_{21}(\sigma_{22}+\sum_{s\neq 2}\sigma_{2s}X_s)$$

$$= -\sigma_{21}\sigma_{22}+\sum_{s\neq 2}(-\sigma_{21}\sigma_{2s})X_s \quad .$$

So $\sigma_{21}\sigma_{22} = \sum_{s\neq 2}(-\sigma_{21}\sigma_{2s})X_s-(p-I)_{22}$.

But from above, each of the coefficients of X_s is in $o(\hat{p})$ and by definition of $o(\hat{p})$, $(p-I)_{22}$ is in $0(\hat{p})$. Thus since $o(\hat{p})$ is an ideal, $\sigma_{21}\sigma_{22}$ is in $o(\hat{p})$.

Since $\sigma\beta = I$,

$$\begin{aligned} \sigma_{21} &= \sigma_{21}\cdot 1 \\ &= \sigma_{21}(\sum_{j=1}^{n}\sigma_{2j}\beta_{j2}) \\ &= \sum_{j=1}^{n}(\sigma_{21}\sigma_{2j})\beta_{j2}. \end{aligned}$$

But as shown above, the coefficient $\sigma_{21}\sigma_{2j}$ is in $0(\hat{p})$ for $1\leq j\leq n$. Thus σ_{21} is in $0(\hat{p})$.

Now choose distinct r and s with $r,s \le n$ and $s>2$. If $q = p^{-1}$, let

$$
\begin{aligned}
b &= [p, I + E_{rs}] \\
&= q(I-E_{rs})p(I+E_{rs}) \\
&= I - q\,E_{rs}p\,E_{rs} - q\,E_{rs}p + E_{rs}.
\end{aligned}
$$

Then $(b-I)_{ij} = \delta_{ir}\delta_{js} - q_{ir}p_{sj} - \delta_{js}q_{ir}p_{sr}$. But since $s>2$, then by (1),

$$
p_{sj} = \begin{cases} \delta_{sj} & \text{if } j\neq 2 \\ p_{s2} & \text{if } j=2 \end{cases} \quad \text{and} \quad p_{sr} = \begin{cases} \delta_{sr}=0 & \text{if } r\neq 2 \\ p_{s2} & \text{if } r=2 \end{cases} .
$$

Thus

$$
\begin{aligned}
(b-I)_{ij} &= \delta_{ir}\delta_{js} - q_{ir}\delta_{sj}(1-\delta_{2j}) - q_{ir}p_{s2}\delta_{2j} - \delta_{2r}\delta_{js}q_{ir}p_{s2} \\
&= \delta_{ir}\delta_{js} - q_{ir}\delta_{sj} - q_{ir}p_{s2}\delta_{2j} - \delta_{2r}\delta_{js}q_{ir}p_{s2}. \qquad (2)
\end{aligned}
$$

Then $(b-I)_{ij} = 0$ if $j\neq 2$ and $j\neq s$; that is, b differs from the identity matrix only in the second and the s^{th} column.

We now show that for all distinct i and r, q_{ir} is in B and $q_{ii} - q_{rr}$ is in B and hence $0(q)$ is contained in B. Since $n\ge 4$, then given i and r we may choose s with $s>2$ and $s\neq r$ and thus obtain a matrix b as above. Then from (2),

$$
\begin{aligned}
(b-I)_{i2} &= \delta_{ir}\delta_{2s} - q_{ir}\delta_{s2} - q_{ir}p_{s2}\delta_{22} - \delta_{2r}\delta_{2s}q_{ir}p_{s2} \\
&= - q_{ir}p_{s2}
\end{aligned}
$$

and

$$(b-I)_{is} = \delta_{ir}\delta_{ss} - q_{ir}\delta_{ss} - q_{ir}p_{s2}\delta_{2s} - \delta_{2r}\delta_{ss}q_{ir}p_{s2}$$

$$= \delta_{ir} - q_{ir} - \delta_{2r}q_{ir}p_{s2} \quad .$$

Thus

$$b-I = \begin{array}{c} \\ \\ \\ \\ \\ (r) \\ \\ \\ \\ \\ \end{array}\begin{array}{c} \begin{array}{ccccccc} & & & & (s) & & \end{array} \\ \begin{bmatrix} 0 & -q_{1r}p_{s2} & 0. & . & .0 & -q_{1r}(1+\delta_{2r}p_{s2}) & 0. \;. \;.0 \\ 0 & -q_{2r}p_{s2} & 0. & . & .0 & -q_{2r}(1+\delta_{2r}p_{s2}) & 0. \;. \;.0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & -q_{rr}p_{s2} & 0. & . & .0 & 1-q_{rr}(1+\delta_{2r}p_{s2}) & 0. \;. \;.0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & -q_{nr}p_{s2} & 0. & . & .0 & -q_{nr}(1+\delta_{2r}p_{s2}) & 0. \;. \;.0 \end{bmatrix} \end{array} .$$

Hence the off-diagonal positions of b contain the elements $-q_{ir}p_{s2}$ where $i\neq 2$, $-q_{ir}(1+\delta_{2r}p_{s2})$ where $i\neq r$ and $i\neq s$, and $1-q_{rr}(1+\delta_{2r}p_{s2})$. The differences of diagonal positions of b are given by $q_{2r}p_{s2}$, $q_{sr}(1+\delta_{2r}p_{s2})$, and $q_{2r}p_{s2} - q_{sr}(1+\delta_{2r}p_{s2})$. From this we see that for all i, $q_{ir}p_{s2}$ and $\delta_{ir}-q_{ir}(1+\delta_{2r}p_{s2})$ are in $0(b)$. It follows that if $r=2$, then for all i, $[q_{ir}p_{s2}] + [\delta_{ir}-q_{ir}(1+\delta_{2r}p_{s2})] = q_{i2}p_{s2} + \delta_{i2} - q_{i2}(1+\delta_{22}p_{s2}) = \delta_{i2}-q_{i2} = \delta_{ir}-q_{ir}$ is in $0(b)$. Also, if $r\neq 2$, then for all i, $\delta_{ir}-q_{ir}(1+\delta_{2r}p_{s2}) = \delta_{ir} - q_{ir}$ is in $0(b)$.

Thus, for every i and r, there exists a matrix b in H which differs from the identity in at most two columns with

$\delta_{ir}-q_{ir}$ in $0(b)$. Hence, for every i and r, $\delta_{ir}-q_{ir}$ is in B. In particular, if $i \neq r$, q_{ir} is in B and $q_{ii}-q_{rr} = (1-q_{rr}) - (1-q_{ii})$, as the sum of elements of B, is in B. Since all the generators of $0(q)$ are of these forms, $0(q)$ is contained in B.

Now σ_{21} is in $\hat{0}(p)$ which is contained in $0(p)$ and since $q = p^{-1}$, $0(p) = 0(q)$ by II.1. Thus σ_{21} is contained in $o(q)$ and hence in B.

Similarly, we can show σ_{ij} is in B for all i and j with $i \neq j$.

Next we show that all differences of diagonal positions of elements of H are in B. Let

$$\begin{aligned} t &= (I + E_{12})^{-1}\sigma(I + E_{12}) \\ &= (I - E_{12})\ \ \sigma(I + E_{12}) \\ &= \sigma - E_{12}\sigma + \sigma E_{12} - E_{12}\sigma E_{12} \end{aligned}$$

$$= \begin{bmatrix} \sigma_{11} & \sigma_{12}\cdots\sigma_{1n} \\ \sigma_{21} & \sigma_{22}\cdots\sigma_{2n} \\ \vdots & \vdots \\ \sigma_{n1} & \sigma_{n2}\cdots\sigma_{nn} \end{bmatrix} - \begin{bmatrix} \sigma_{21} & \sigma_{22}\cdots\sigma_{2n} \\ 0 & 0 \ \cdots \ 0 \\ \vdots & \vdots \\ 0 & 0 \ \cdots \ 0 \end{bmatrix} + \begin{bmatrix} 0 & \sigma_{11} & 0\cdots 0 \\ 0 & \sigma_{21} & 0\cdots 0 \\ \vdots & \vdots & \vdots \\ 0 & \sigma_{n1} & 0\cdots 0 \end{bmatrix}$$

$$
- \begin{bmatrix} 0 & \sigma_{21} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}
$$

$$
= \begin{bmatrix} \sigma_{11}-\sigma_{21} & \sigma_{12}-\sigma_{22}+\sigma_{11}-\sigma_{21} & \sigma_{13}-\sigma_{23} & \cdots & \sigma_{1n}-\sigma_{2n} \\ \sigma_{21} & \sigma_{22}+\sigma_{21} & \sigma_{23} & \cdots & \sigma_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ \sigma_{n1} & \sigma_{n2}+\sigma_{n1} & \sigma_{n3} & \cdots & \sigma_{nn} \end{bmatrix} .
$$

Since H is normalized by $E_n(R)$, t is in H. By the first part of this proof, all off-diagonal positions of elements of H are in B. Thus $t_{12} = \sigma_{12}-\sigma_{22}+\sigma_{11}-\sigma_{21}$ is in B. But again by the first part of this proof, σ_{21} and σ_{12} are in B. Hence $(\sigma_{12}-\sigma_{22}+\sigma_{11}-\sigma_{21}) + \sigma_{21} - \sigma_{12} = \sigma_{11} - \sigma_{22}$ is in B.

A similar argument shows $\sigma_{ii} - \sigma_{jj}$ is in B for all i and j with $i \neq j$.

Thus all elements of the forms σ_{ij} and $\sigma_{ii}-\sigma_{jj}$ with $i \neq j$ are in B. Hence $0(\sigma)$ is contained in B.

Since σ was arbitrary in H, $0(H)$ is contained in B and the proof is complete.

<u>Theorem II.3</u> Let H be a subgroup of $GL_n(R)$ normalized by $E_n(R)$ where $n \geq 4$. Then $E_n(R, 0(H))$ is a subgroup of H.

<u>Proof</u> Let $\bar{B} = \sum_h 0(h)$ where the summation extends over all h with h differing from the identity in only one row.

We show first that $\bar{B} = 0(H)$. Let h be in H with h differing from the identity by at most two columns. We may assume without loss of generality that h differs from the identity in the first two columns, that is,

$$h = \begin{bmatrix} * & * & 0 & \cdots & 0 \\ * & * & 0 & \cdots & 0 \\ * & * & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ * & * & 0 & \cdots & 1 \end{bmatrix}.$$

Choose r and s with $r \neq s$ and $r > 2$ and form

$$\begin{aligned} p &= [h, I+E_{rs}] \\ &= h^{-1}(I-E_{rs})h(I+E_{rs}) \\ &= I-h^{-1}E_{rs}h+E_{rs}-h^{-1}E_{rs}hE_{rs}. \end{aligned}$$

By direct computation, we find that $(h^{-1})_{ij} = \delta_{ij}$ for $j>2$.

Then, since $h^{-1}E_{rs}hE_{rs} = 0$,

$$\begin{aligned} p - I &= E_{rs} - h^{-1}E_{rs}h \\ &= E_{rs} - E_{rs}h \end{aligned}$$

$$= \begin{cases} E_{rs} - \begin{array}{cc} & \begin{array}{ccccccccc} \phantom{h_{s1}} & \phantom{h_{s2}} & & & & (s) & & & \end{array} \\ \begin{array}{c} \\ \\ \\ (r) \\ \\ \\ \\ \end{array} & \left[\begin{array}{ccccccccc} 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ h_{s1} & h_{s2} & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{array}\right] \end{array} & \text{if } s>2 \\ \\ E_{rs} - \begin{array}{cc} \begin{array}{c} \\ \\ \\ (r) \\ \\ \\ \\ \end{array} & \left[\begin{array}{ccccc} 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ h_{s1} & h_{s2} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{array}\right] \end{array} & \text{if } s\leq 2 \end{cases}$$

So $(p-I)_{ij} = \delta_{ir}\delta_{js} - \delta_{ir}h_{sj}$. Now p differs from the identity only in the r^{th} row. So p_{rj} with $j\neq r$ and $p_{rr}-p_{11}$ are in $\overline{B}$. But if $j\neq r$, $p_{rj} = \delta_{js}-h_{sj}$ and $p_{rr}-p_{11} = \delta_{sr}-h_{sr}$. Hence $\delta_{js}-h_{sj}$ is in $\overline{B}$ for all j.

Now let j and s be arbitrary with $j\neq s$. Then since $n\geq 4$, we may choose r with $r\neq s$ and $r>2$ and form p as above. By the above, $\delta_{sj}-h_{sj} = -h_{sj}$ is in $\overline{B}$ and $\delta_{ss}-h_{ss} = 1-h_{ss}$ is in $\overline{B}$. An analogous argument shows that $\delta_{jj} - h_{jj} = 1-h_{jj}$ is in $\overline{B}$. Thus, $h_{ss}-h_{jj} = (1-h_{jj}) - (1-h_{ss})$ is in $\overline{B}$. Hence $0(h)$ is contained in $\overline{B}$. Since h was an arbitrary element of H differing from the identity in at most two columns, B is contained in $\overline{B}$ where B is defined as in II.2. But by II.2, $0(H)$ is contained in B. Hence $0(H)$ is contained in $\overline{B}$. But $\overline{B}$ is contained in $0(H)$. Hence $0(H) = \overline{B}$.

We wish to show now that $E_n(R,0(H))$ is contained in H. But since $0(H) = \overline{B}$, $E_n(R,0(H))$ is generated by elements of the form $I + \lambda E_{ij}$ where $i\neq j$ and λ is in $\overline{B}$, hence by elements of the form $I + \lambda E_{ij}$ where $i\neq j$ and λ is in $0(k)$ for some k in H differing from the identity in only one row. In fact, then, $E_n(R,0(H))$ is generated by elements of the form $I + (k_{rt}-\delta_{rt})E_{ij}$, $i\neq j$, where k is an element of H differing from the identity in only the r^{th} row. We show that all such elements are contained in H.

Let k be in H with k differing from I in only the r^{th} row. Let i and j with $i \neq j$ be given. Since $n \geq 4$, we may choose s with $s \neq i$, $s \neq r$, and $s \neq t$. Form $y = [I+E_{ts}, k^{-1}]$ which is in H since k is in H and H is normalized by $E_n(R)$. Then $y = I - E_{ts} + k\, E_{ts} k^{-1} + E_{ts} k\, E_{ts} k^{-1}$.

By direct computation we find that k^{-1} also differs from the identity in only the r^{th} row. Now

$$k\,E_{ts} = \begin{cases} \begin{array}{cc} (s) & \\ \left[\begin{array}{c} 1 \\ k_{rt} \end{array}\right] & \begin{array}{c} (t) \\ (r) \end{array} \end{array} & \text{if } t \neq r \\[2ex] \begin{array}{cc} (s) & \\ \left[\begin{array}{c} k_{rt} \end{array}\right] & (t=r) \end{array} & \text{if } t = r \end{cases}$$

where zeros occur in all unmarked positions of these matrices. It is easy to show then that

$$k\,E_{ts} k^{-1} = \begin{cases} \begin{array}{cc} (s) & \\ \left[\begin{array}{c} 1 \\ k_{rt} \end{array}\right] & \begin{array}{c} (t) \\ (r) \end{array} \end{array} & \text{if } r \neq t \\[2ex] \begin{array}{cc} (s) & \\ \left[\begin{array}{c} k_{rt} \end{array}\right] & (t=r) \end{array} & \text{if } r = t \end{cases}$$

$$= \begin{cases} E_{ts} + k_{rt}E_{rs} & \text{if} \quad r\neq t \\ \quad k_{rt}\, E_{rs} & \text{if} \quad r=t \end{cases}$$

$$= \quad k_{rt}\, E_{rs} + (1-\delta_{rt})\, E_{ts}.$$

Also $E_{ts}\ k\ E_{ts}\ k^{-1} = 0.$

Thus

$$y = I - E_{ts} + k_{rt}\, E_{rs} + (1-\delta_{tr})E_{ts}$$

$$= I + k_{rt}\, E_{rs} - \delta_{tr}\, E_{ts}.$$

Hence if $r=t$, $y=I + (k_{rr}-1)E_{rs}$ and if $r\neq t$, $y = I+k_{rt}E_{rs}$.

This says $y = I + (k_{rt} - \delta_{rt})\, E_{rs}$ which is in H.

Suppose $j\neq r$. If $j=s$, this gives $y = I + (k_{rt}-\delta_{rt})E_{rj}$ in H. If $j\neq s$, $[y,I+E_{sj}] = [I+(k_{rt}-\delta_{rt})E_{rs},I+E_{sj}] = I + (k_{rt}-\delta_{rt})E_{rj}$ is in H since H is normalized by $E_n(R)$. Hence $I + (k_{rt}-\delta_{rt})E_{rj}$ is in H for all j with $j\neq r$. If $i\neq r$ and $i\neq j$, $[I+E_{ir},I+(k_{rt}-\delta_{rt})E_{rj}] = I + (k_{rt}-\delta_{rt})E_{ij}$ is in H since H is normalized by $E_n(R)$. Thus $I + (k_{rt}-\delta_{rt})E_{ij}$ is in H for all i and j with $i\neq j$. Since this is an arbitrary generator of $E_n(R,O(H))$, $E_n(R,O(H))$ is contained in H and the proof is complete.

Theorem II.4 Let H be a subgroup of $GL_n(R)$ normalized by $E_n(R)$ and suppose $E_n(R,O(H))$ is contained in H. Then $\lambda_{O(H)}(H)$ is contained in the center of $GL_n(R/O(H))$.

Proof Recall the natural group morphism $\lambda_{O(H)}:GL_n(R) \to GL_n(R/O(H))$. Let $\bar{H} = \lambda_{O(H)}(H)$. Clearly $\bar{H} = \{\bar{\alpha}I \mid \bar{\alpha}$ is a unit in $R/O(H)\}$. Now since H is normalized by $E_n(R)$ and $\lambda_{O(H)}:E_n(R) \to E_n(R/O(H))$ is surjective, $\bar{H}$ is normalized by $E_n(R/O(H))$.

We show now that $\bar{H}$ is contained in the center of $GL_n(R/O(H))$. Note that the center of $GL_n(R/O(H))$ = $\{\lambda I \mid \lambda$ is in the center of $R/O(H)\}$. Then let $\bar{\alpha}I$ be in $\bar{H}$ and let $\bar{\beta}$ be in $R/O(H)$. Since $\bar{H}$ is normalized by $E_n(R/O(H))$, $y = (I+\bar{\beta}E_{12})(\bar{\alpha}I)(I-\bar{\beta}E_{12})$ is in $\bar{H}$. But

$$\begin{aligned} y &= (\bar{\alpha}+\bar{\beta}\,\bar{\alpha}\,E_{12})(I-\bar{\beta}E_{12}) \\ &= \bar{\alpha}I+\bar{\beta}\,\bar{\alpha}\,E_{12}-\bar{\alpha}\,\bar{\beta}\,E_{12} \\ &= \bar{\alpha}I+(\bar{\beta}\,\bar{\alpha}-\bar{\alpha}\,\bar{\beta})E_{12}. \end{aligned}$$

Since y is in $\bar{H}$, $\bar{\beta}\,\bar{\alpha} - \bar{\alpha}\,\bar{\beta} = 0$, that is, $\bar{\beta}\,\bar{\alpha} = \bar{\alpha}\,\bar{\beta}$. Thus $\bar{\alpha}$ lies in the center of $R/O(H)$. Hence $\bar{\alpha}I$ lies in the center of $GL_n(R/O(H))$. Thus $\bar{H}$ is contained in the center of $GL_n(R/O(H))$.

The following theorem gives the desired result that the normal subgroups of $GL_n(R)$ are standard normal if $n \geq 4$.

<u>Theorem II.5</u> Let H be a subgroup of $GL_n(R)$ normalized by $E_n(R)$ with $n \geq 4$. Then $E_n(R,O(H)) \leq H \leq GL_n(R,O(H))$.

<u>Proof</u> By II.3, $E_n(R,O(H))$ is contained in H. Then by II.4, $\lambda_{O(H)}(H)$ lies in the center of $GL_n(R/O(H))$. But $GL_n(R,O(H))$ is the inverse image under $\lambda_{O(H)}$ of the center of $GL_n(R/O(H))$. Hence H is contained in $GL_n(R,O(H))$.

This answers, in a sense, half of the normal subgroup problem. The normal subgroups are shown to be sandwiched between congruence subgroups of certain types. It remains to be shown that $E_n(R)$ and $E_n(R,A)$ are themselves normal subgroups. If R is commutative, the normality of $E_n(R)$ and $E_n(R,A)$ was proven by Suslin in a letter to H. Bass in 1975 (A short proof of this occurs in the appendix of B.R. McDonald, "Automorphisms of $GL_n(R)$," <u>Trans. A.M.S.</u> 246 (1978), 170-171.). At the writing of this paper the question remains open for R noncommutative.

CHAPTER III

SOLUTIONS OF $S(<i,j>,A)$

In Chapter II we showed that for any ring R in which we have solutions to all systems $S(<i,j>,A)$, $i\neq j$, the normal subgroups of $GL_n(R)$ are standard normal when $n\geq 4$. In this chapter we examine various rings in which the solvability of the $S(<i,j>,A)$ occurs.

(a) Commutative Rings

Let R be a commutative ring and A an element of $GL_n(R)$. Consider the system $S(<1,2>,A)$. If $A = [a_{ij}]$ and $A^{-1} = B = [b_{ij}]$, then $S(<1,2>,A)$ is the system

$$\begin{aligned} b_{21}X_1 + b_{23}X_3 + \ldots + b_{2n}X_n &= a_{21} \\ b_{31}X_1 + b_{33}X_3 + \ldots + b_{3n}X_n &= 0 \\ &\vdots \\ b_{n1}X_1 + b_{n3}X_3 + \ldots + b_{nn}X_n &= 0 \end{aligned} \quad .$$

Let M be the coefficient matrix of this system.

Denote the cofactor of b_{2i} in M by M_{2i}. Then by Laplace expansion, $\sum_{r\neq 2} b_{ir}M_{2r} = 0$ for $i>2$ and $\sum_{r\neq 2} b_{2r}M_{2r} = \det M$.

The cofactor of b_{12} in B is $-\det M$. Since

$A = \frac{1}{\det(B)} \operatorname{adj}(B)$,

$$a_{21} = \frac{1}{\det(B)} (\operatorname{adj}(B))_{21}$$

$$= \frac{1}{\det(B)} (\text{cofactor in } B \text{ of } b_{12})$$

$$= \frac{1}{\det(B)} (-\det M)$$

Thus $\det M = -\det(B)a_{21}$.

Then setting $X_i = -\frac{1}{\det(B)} M_{2i}$ for $i=1,3,4,\ldots,n$, we get a solution to the system $S(\langle 1,2\rangle,A)$.

Similarly we may solve $S(\langle i,j\rangle,A)$ where $1\leq i,j\leq n$ and $i\neq j$. This was Wilson's approach in [15].

(b) Division Rings

Let k be a division ring and let A be an element of $GL_n(k)$. The system $S(\langle i,j\rangle,A)$ is a system of equations with coefficients in k.

To show the solvability of S(<i,j>,A) we make use of the Ore determinant defined in [13]. For an element of $(k)_2$ we define the <u>right-hand determinant of order two</u> by

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}\Bigg\| = a_{11}A_{22} - a_{21}A_{12} \text{ where } a_{12}A_{22}=a_{22}A_{12}.$$

We make the assumption that if $a_{12}\neq 0$ and $a_{22}\neq 0$, then $A_{22}\neq 0$ and $A_{12}\neq 0$; if $a_{12}=0$ and $a_{22}\neq 0$, then $A_{12}=0$ and $A_{22}\neq 0$; if $a_{12}\neq 0$ and $a_{22}=0$, then $A_{12}\neq 0$ and $A_{22}=0$; and if $a_{12}=a_{22}=0$, then $A_{12}=A_{22}=0$. The chosen A_{i2} are not unique but it is easily shown that if a different choice is made for the pair, then the determinants based on the two pairs are multiples of each other, that is $\Delta = \Delta' h$ for some non-zero h. Hence either both determinants are zero or both are non-zero. We say the two determinants are <u>equivalent</u>, denoted by $\Delta \sim \Delta'$.

Similarly, we define the <u>left-hand determinant of</u> <u>of order two</u> by

$$\Bigg\|\begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} = B_{22}b_{11}-B_{12}b_{21} \quad \text{where} \quad B_{22}b_{12}=B_{12}b_{22}.$$

Assumptions analogous to those for a right-hand determinant are made in the case one or both of b_{12} and b_{22} are zero. Likewise, any two left-hand determinants of the same matrix are equivalent.

Call two determinants $\left\|\begin{matrix} a_1 & s \\ a_2 & t \end{matrix}\right\|$ and $\left\|\begin{matrix} b_1 & s \\ b_2 & t \end{matrix}\right\|$ proportional if

$$\left\|\begin{matrix} a_1 & s \\ a_2 & t \end{matrix}\right\| = a_1T-a_2S \quad \text{and} \quad \left\|\begin{matrix} b_1 & s \\ b_2 & t \end{matrix}\right\| = b_1T-b_2S \quad \text{where}$$

$sT = tS$. Likewise we may define when two left-hand determinants are proportional.

For the right-hand system

$$X_1a_{11} + X_2a_{12} = b_1$$

$$X_1a_{21} + X_2a_{22} = b_2$$

we may show that

$$X_1 \left\|\begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{matrix}\right\| = \left\|\begin{matrix} b_1 & a_{12} \\ b_2 & a_{22} \end{matrix}\right\| \quad \text{and } X_2 \left\|\begin{matrix} a_{12} & a_{11} \\ a_{22} & a_{21} \end{matrix}\right\| = \left\|\begin{matrix} b_1 & a_{11} \\ b_2 & a_{21} \end{matrix}\right\|$$

where in each equality the two determinants are proportional.

Let $\left|\begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{matrix}\right|\Bigg| = a_{11}A_{22} - a_{21}A_{12}$ and $\left|\begin{matrix} b_1 & a_{12} \\ b_2 & a_{22} \end{matrix}\right|\Bigg| = b_1A_{22}-b_2A_{12}$

where $a_{12}A_{22} = a_{22}A_{12}$. Multiplying the first equation in the system on the right by A_{22} and the second on the right by A_{12} we get:

$$X_1a_{11}A_{22} + X_2a_{12}A_{22} = b_1A_{22}$$

$$X_1a_{21}A_{12} + X_2a_{22}A_{12} = b_2A_{12}$$

Subtracting gives

$$X_1(a_{11}A_{22}-a_{21}A_{12}) + X_2(a_{12}A_{22}-a_{22}A_{12}) = b_1A_{22}-b_2A_{12},$$

that is,

$$X_1 \left|\begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{matrix}\right|\Bigg| + X_2(0) = \left|\begin{matrix} b_1 & a_{12} \\ b_2 & a_{22} \end{matrix}\right|\Bigg| .$$

Analogously, we may show that $X_2 \left|\begin{matrix} a_{12} & a_{11} \\ a_{22} & a_{21} \end{matrix}\right|\Bigg| = \left|\begin{matrix} b_1 & a_{11} \\ b_2 & a_{21} \end{matrix}\right|\Bigg|$.

Observe that

$\left|\begin{matrix} a_{12} & a_{11} \\ a_{22} & a_{21} \end{matrix}\right|\Bigg| \sim \left|\begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{matrix}\right|\Bigg|$ ([13],p.471) and thus this system has a solution if and only if $\left\|\begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{matrix}\right\| \neq 0.$

Similarly for the left-hand system

$$\left.\begin{aligned} c_{11}X_1 + c_{12}X_2 &= d_1 \\ c_{21}X_1 + c_{22}X_2 &= d_2 \end{aligned}\right)$$

we have

$$\left\|\begin{matrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{matrix}\right| X_1 = \left\|\begin{matrix} d_1 & c_{12} \\ d_2 & c_{22} \end{matrix}\right| \quad \text{and} \quad \left\|\begin{matrix} c_{12} & c_{11} \\ c_{22} & c_{21} \end{matrix}\right| X_2 = \left\|\begin{matrix} d_1 & c_{11} \\ d_2 & c_{21} \end{matrix}\right|$$

where in each equality the two determinants are proportional.

Likewise, this system has a solution if and only if

$$\left\|\begin{matrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{matrix}\right| \neq 0.$$

We define the <u>right-hand determinant of order</u> n inductively. We assume we can solve both right-hand and left-hand systems of order less than n. For $A = [a_{ij}]$ an element of $(k)_n$, let

$$|a_{ij}\| = a_{11}A^{(1)} + a_{21}A^{(2)} + \ldots + a_{n1}A^{(n)}$$

where

$$a_{12}A^{(1)} + a_{22}A^{(2)} +\ldots+ a_{n2}A^{(n)} = 0$$

$$a_{13}A^{(1)} + a_{23}A^{(2)} +\ldots+ a_{n3}A^{(n)} = 0$$

$$\vdots$$

$$a_{1n}A^{(1)} + a_{2n}A^{(2)} +\ldots+ a_{nn}A^{(n)} = 0 \; .$$

To show that the $A^{(i)}$ satisfying this system exist we use the induction hypothesis. Let Δ_i be the left-hand determinant of the matrix obtained by deleting the i^{th} column of

$$\begin{bmatrix} a_{12} & a_{22} & \cdots & a_{n2} \\ a_{13} & a_{23} & \cdots & a_{n3} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix} .$$

If $\Delta_i=0$ for all i, then let $A^{(i)}=0$ for all i. If $\Delta_k\neq 0$ for some k, we may arbitrarily assign a value to $A^{(k)}$ and we can solve the remaining (n-1)x(n-1) system by induction. Of course, each choice of $A^{(k)}$ may give a different solution to the system but each solution may be obtained from a given by right multiplication by a non-zero constant. Thus the resulting determinants may be determined from a given by right multiplication by a non-zero constant and thus are equivalent.

Then for the right-hand system

$$X_1a_{11} + X_2a_{12} + \ldots + X_na_{1n} = b_1$$

$$X_1a_{21} + X_2a_{22} + \ldots + X_na_{2n} = b_2$$

$$\vdots$$

$$X_1a_{n1} + X_2a_{n2} + \ldots + X_na_{nn} = b_n$$

we have

$$X_k \begin{vmatrix} a_{1k} & a_{11} \cdots a_{1,k-1} & a_{1,k+1} \cdots a_{1n} \\ a_{2k} & a_{21} \cdots a_{2,k-1} & a_{2,k+1} \cdots a_{2n} \\ \vdots & \vdots \quad\quad \vdots & \vdots \quad\quad \vdots \\ a_{nk} & a_{n1} \cdots a_{n,k-1} & a_{n,k+1} \cdots a_{nn} \end{vmatrix} =$$

$$\begin{vmatrix} b_1 & a_{11} \cdots a_{1,k-1} & a_{1,k+1} \cdots a_{1n} \\ b_2 & a_{21} \cdots a_{2,k-1} & a_{2,k+1} \cdots a_{2n} \\ \vdots & \vdots \quad\quad \vdots & \vdots \quad\quad \vdots \\ b_n & a_{n1} \cdots a_{n,k-1} & a_{n,k+1} \cdots a_{nn} \end{vmatrix}$$

where the two determinants are proportional. This means we have chosen the same solution to the system

$$a_{11}B^{(1)} + \dots + a_{1,k-1}B^{(k-1)} + a_{1,k+1}B^{(k+1)} + \dots + a_{1n}B^{(n)} = 0$$

$$a_{21}\, B^{(1)} + \dots + a_{2,k-1}B^{(k-1)} + a_{2,k+1}B^{(k+1)} + \dots + a_{2n}B^{(n)} = 0$$

$$\vdots$$

$$a_{n1}B^{(1)} + \dots + a_{n,k-1}B^{(k-1)} + a_{n,k+1}B^{(k+1)} + \dots + a_{nn}B^{(n)} = 0$$

for both determinants.

It can be shown ([13],p.474) that if two columns of a matrix are interchanged we get equivalent determinants. Hence the above system has a solution if and only if $|a_{ij}|| \neq 0$.

The definition of left-hand determinants and the solution of left-hand systems are analogous to the above.

We can show ([13],p.475) that for any matrix A, $|A|| \sim || A^t|$.

We now use these ideas developed by Ore to show the solvability of S(<i,j>,A). Again we let $A = [a_{ij}]$ and $A^{-1} = B = [b_{ij}]$. First we show that $||B| \neq 0$.

Since AB = I, $\sum_{k=1}^{n} a_{ik}b_{ki} = 1$ and $\sum_{k=1}^{n} a_{ik}b_{kj} = 0$ if $i \neq j$.

Let

$$\|B| = C_1^{(1)}b_{11} + C_1^{(2)}b_{21} + \ldots + C_1^{(n)}b_{n1}$$

where

$$\begin{aligned} C_1^{(1)}b_{12} + C_1^{(2)}b_{22} + \ldots + C_1^{(n)}b_{n2} &= 0 \\ C_1^{(1)}b_{13} + C_1^{(2)}b_{23} + \ldots + C_1^{(n)}b_{n3} &= 0 \\ &\vdots \\ C_1^{(1)}b_{1n} + C_1^{(2)}b_{2n} + \ldots + C_1^{(n)}b_{nn} &= 0 \end{aligned} \quad . \qquad (3)$$

Since

$$\begin{aligned} a_{11}\, b_{12} + a_{12}\, b_{22} + \ldots + a_{1n}b_{n2} &= 0 \\ a_{11}\, b_{13} + a_{12}\, b_{23} + \ldots + a_{1n}b_{n3} &= 0 \\ &\vdots \\ a_{11}\, b_{1n} + a_{12}\, b_{2n} + \ldots + a_{1n}b_{nn} &= 0 \end{aligned} \quad ,$$

then $C_1^{(1)} = a_{11}$, $C_1^{(2)} = a_{12}, \ldots, C_1^{(n)} = a_{1n}$ is a solution to (3). Thus any other solution must be $C_1^{(1)} = a_{11}h$, $C_1^{(2)} = a_{12}h, \ldots, C_1^{(n)} = a_{1n}h$ for $h \neq 0$. Then

$$\begin{aligned} \| B| &= a_{11}b_{11}h + a_{12}b_{21}h + \ldots + a_{1n}b_{n1}h \\ &= (a_{11}b_{11} + a_{12}b_{21} + \ldots + a_{1n}b_{n1})h \\ &= 1 \cdot h \\ &= h \end{aligned}$$

Since $h \neq 0$, $\|B| \neq 0$.

The system $\mathcal{S}(<2,1>,A)$ is

$$\begin{aligned}
b_{12}X_2 + b_{13}X_3 + \; . \; . \; . + b_{1n}X_n &= a_{12} \\
b_{32}X_2 + b_{33}X_3 + \; . \; . \; . + b_{3n}X_n &= 0 \\
&\vdots \\
b_{n2}X_2 + b_{n3}X_3 + \; . \; . \; . + b_{nn}X_n &= 0
\end{aligned}$$

Let

$$N = \begin{bmatrix} b_{12} & b_{13} & \cdots & b_{1n} \\ b_{32} & b_{33} & \cdots & b_{3n} \\ \vdots & \vdots & & \vdots \\ b_{n2} & b_{n3} & \cdots & b_{nn} \end{bmatrix} \text{ and } N^t = \begin{bmatrix} b_{12} & b_{32} & \cdots & b_{n2} \\ b_{13} & b_{33} & \cdots & b_{n3} \\ \vdots & \vdots & & \vdots \\ b_{1n} & b_{3n} & \cdots & b_{nn} \end{bmatrix}.$$

Let $|N^t|| = b_{12}a^{(2)} + b_{13}a^{(3)} + \ldots + b_{1n}a^{(n)}$

where
$$\begin{aligned}
b_{32}a^{(2)} + b_{33}a^{(3)} + \ldots + b_{3n}a^{(n)} &= 0 \\
b_{42}a^{(2)} + b_{43}a^{(3)} + \ldots + b_{4n}a^{(n)} &= 0 \\
&\vdots \\
b_{n2}a^{(2)} + b_{n3}a^{(3)} + \ldots + b_{nn}a^{(n)} &= 0
\end{aligned} \quad . \qquad (4)$$

Right multiply equations (3) by $a^{(2)}, a^{(3)}, \ldots, a^{(n)}$ respectively:

$$C_1^{(1)}b_{12}a^{(2)} + C_1^{(2)}b_{22}a^{(2)} + \dots + C_1^{(n)}b_{n2}a^{(2)} = 0$$

$$C_1^{(1)}b_{13}a^{(3)} + C_1^{(2)}b_{23}a^{(3)} + \dots + C_1^{(n)}b_{n3}a^{(3)} = 0$$

$$\vdots$$

$$C_1^{(1)}b_{1n}a^{(n)} + C_1^{(2)}b_{2n}a^{(n)} + \dots + C_1^{(n)}b_{nn}a^{(n)} = 0$$

Adding these together we get

$$C_1^{(1)}(b_{12}a^{(2)} + b_{13}a^{(3)} + \dots + b_{1n}a^{(n)}) + C_1^{(2)}(b_{22}a^{(2)} + b_{23}a^{(3)} + \dots + b_{2n}a^{(n)})$$

$$+ C_1^{(3)}(b_{32}a^{(2)} + b_{33}a^{(3)} + \dots + b_{3n}a^{(n)}) + \dots +$$

$$C_1^{(n)}(b_{n2}a^{(2)} + b_{n3}a^{(3)} + \dots + b_{n3}a^{(n)}) = 0.$$

The summations in all but the first two terms are zero from (4).

Thus $C_1^{(1)}(\sum_{k=2}^{n} b_{1k}a^{(k)}) + C_1^{(2)}(\sum_{k=2}^{n} b_{2k}a^{(k)}) = 0.$

But $\sum_{k=2}^{n} b_{1k}a^{(k)} = |N^t||$ so that

$$C_1^{(1)}|N^t|| + C_1^{(2)}(\sum_{k=2}^{n} b_{2k}a^{(k)}) = 0. \qquad (5)$$

Let $B_{(i)}$ be the matrix obtained from B by rearranging columns with the i^{th} column of B in the first column position and the other columns in ascending order. For $i \geq 1$, let

$\|B_{(i)}| = \sum_{k=1}^{n} C_i^{(k)}b_{ki}$ where $\sum_{k=1}^{n} C_i^{(k)}b_{kj} = 0$ for $j=1,\dots,n, j \neq i$.

Then

$$CB = \begin{bmatrix} c_1^{(1)} & c_1^{(2)} & \cdots & c_1^{(n)} \\ c_2^{(1)} & c_2^{(2)} & \cdots & c_2^{(n)} \\ \vdots & \vdots & & \vdots \\ c_n^{(1)} & c_n^{(2)} & \cdots & c_n^{(n)} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix} =$$

$$\begin{bmatrix} \|B_{(1)}| & 0 & \cdots & 0 \\ 0 & \|B_{(2)}| & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \|B_{(n)}| \end{bmatrix}.$$

Since $BA = I$,

$$C = \begin{bmatrix} \|B_{(1)}| & 0 & \cdots & 0 \\ 0 & \|B_{(2)}| & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \|B_{(n)}| \end{bmatrix} A.$$

Thus $c_i^{(j)} = \|B_{(i)}|a_{ij}$. So

$$\|B|a_{12} = c_1^{(2)}. \tag{6}$$

Suppose $\sum_{k=2}^{n} b_{2k}a^{(k)} = 0$. Then from (5), either

$c_1^{(1)} = 0$ or $|N^t|| = 0$. The equation $\sum_{k=2}^{n} b_{2k}a^{(k)} = 0$ may

replace any of the equations in (4) and $a^{(2)},\ldots,a^{(n)}$ will

satisfy any of the new systems. Thus

$|N^t|| \sim$ all n-1 order determinants in

$$\left|\begin{matrix} b_{12} & b_{22} & \cdots & b_{n2} \\ b_{13} & b_{23} & \cdots & b_{n3} \\ \vdots & \vdots & & \vdots \\ b_{1n} & b_{2n} & \cdots & b_{nn} \end{matrix}\right\| . \tag{7}$$

So if $|N^t|| = 0$, all n-1 order determinants in

$$\left|\begin{matrix} b_{12} & b_{22} & \cdots & b_{n2} \\ b_{13} & b_{23} & \cdots & b_{n3} \\ \vdots & \vdots & & \vdots \\ b_{1n} & b_{2n} & \cdots & b_{nn} \end{matrix}\right\|$$

are 0. Then $\|B| = 0$ from (3). But $\|B| \neq 0$ so we must have

$c_1^{(1)} = 0$.

Since $C_1^{(1)} = 0$, then from (3),

$$\|B| \sim \text{ all n-1 order determinants in } \left\| \begin{matrix} b_{21} & b_{22} & \cdots & b_{2n} \\ b_{31} & b_{32} & \cdots & b_{3n} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{matrix} \right| .$$

(Any of the equations in (3) may be left out and we have the same solutions.) In particular,

$$\|B| \sim \left\| \begin{matrix} b_{22} & b_{23} & \cdots & b_{2n} \\ b_{32} & b_{33} & \cdots & b_{3n} \\ \vdots & \vdots & & \vdots \\ b_{n2} & b_{n3} & \cdots & b_{nn} \end{matrix} \right| .$$

From (7) we have that

$$|N^t\| \sim \left| \begin{matrix} b_{22} & b_{32} & \cdots & b_{n2} \\ b_{23} & b_{33} & \cdots & b_{n3} \\ \vdots & \vdots & & \vdots \\ b_{2n} & b_{3n} & \cdots & b_{nn} \end{matrix} \right\| .$$

But since $|M^t|| \sim ||M|$ for any M, then

$$||B| \sim \begin{vmatrix} b_{22} & b_{32} & \cdots & b_{n2} \\ b_{23} & b_{33} & \cdots & b_{n3} \\ \vdots & \vdots & & \vdots \\ b_{2n} & b_{3n} & \cdots & b_{nn} \end{vmatrix} .$$

Hence $||B| \sim |N^t||$. Since $||B| \neq 0, |N^t|| \neq 0$. But $|N^t|| \sim ||N|$. So $||N| \neq 0$

and thus we have a solution to the system S(<2,1>,A).

Now suppose $\sum_{k=2}^{n} b_{2k} a^{(k)} \neq 0$. If in addition, $C_1^{(2)} \neq 0$,

by (5) $|N^t|| \neq 0$. Then $||N| \neq 0$ and again we have a solution

to the system S(<2,1>,A). But if $C_1^{(2)} = 0$, then

$||B|a_{12} = 0$ by (6). Since $||B| \neq 0, a_{12} = 0$. Then the system

S(<1,2>,A) has the trivial solution $X_i = 0$, $i=2,3,\ldots,n$.

An argument analogous to the above shows that a solution exists for all the systems S(<i,j>,A), $1 \leq i,j \leq n, i \neq j$.

(c) Product of Rings Having Solvability of $S(\langle i,j\rangle,A)$

Let $R = \prod_{\lambda \text{ in } \Lambda} R_\lambda$ where for each λ in Λ, $S(\langle i,j\rangle,A_\lambda)$ is solvable for all A_λ in $(R_\lambda)_n$ and $1 \leq i,j \leq n, i \neq j$. Clearly R is a ring if we consider both multiplication and addition to be done component-wise, that is, if $r = \langle r_\lambda\rangle_{\lambda \text{ in } \Lambda}$ and $s = \langle s_\lambda\rangle_{\lambda \text{ in } \Lambda}$, $r+s = \langle r_\lambda + s_\lambda\rangle_{\lambda \text{ in } \Lambda}$ and $rs = \langle r_\lambda s_\lambda\rangle_{\lambda \text{ in } \Lambda}$.

Let $A = [a_{ij}]$ be in $(R)_n$ with $A^{-1} = [b_{ij}]$. Then if

$$[b_{ij}]_{\langle 1,2\rangle} = \begin{bmatrix} b_{21} & b_{23} & \cdots & b_{2n} \\ b_{31} & b_{33} & \cdots & b_{3n} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n3} & \cdots & b_{nn} \end{bmatrix},$$

the system $S(\langle 1,2\rangle,A)$ is

$$[b_{ij}]_{\langle 1,2\rangle} \begin{bmatrix} x_1 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{21} \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Let $b_{ij} = \langle b_{ij}{}^{\lambda}\rangle_{\lambda \text{ in } \Lambda}$, $X_i = \langle X_i{}^{\lambda}\rangle_{\lambda \text{ in } \Lambda}$, and $a_{21} = \langle a_{21}{}^{\lambda}\rangle_{\lambda \text{ in } \Lambda}$.

Then $S(\langle 1,2\rangle, A)$ is equivalent to the family of systems

$$[b_{ij}{}^{\lambda}]_{\langle 1,2\rangle} \begin{bmatrix} X_1{}^{\lambda} \\ X_3{}^{\lambda} \\ \cdot \\ \cdot \\ \cdot \\ X_n{}^{\lambda} \end{bmatrix} = \begin{bmatrix} a_{21} \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}$$

for each λ in Λ.

It is easily shown that if $A_\lambda = [a_{ij}{}^{\lambda}]$ then $A_\lambda{}^{-1} = [b_{ij}{}^{\lambda}]$ in $(R_\lambda)_n$. Thus each system in this family is $S(\langle 1,2\rangle, A_\lambda)$. By hypothesis each of these is solvable. The products $X_1 = \langle X_1{}^{\lambda}\rangle_{\lambda \text{ in } \Lambda}$, $X_3 = \langle X_3{}^{\lambda}\rangle_{\lambda \text{ in } \Lambda}, \ldots, X_n = \langle X_n{}^{\lambda}\rangle_{\lambda \text{ in } \Lambda}$, are a solution to $S(\langle 1,2\rangle, A)$.

An analogous argument shows that $S(\langle i,j\rangle, A)$ is solvable for all distinct i and j with $1 \leq i,j \leq n$.

(d) Direct Limit of Rings Having Solvability of S(<i,j>,A)

Let I be a directed set, that is, a partially ordered set such that for each h and ℓ in I, there exists k in I such that $h \leq k$ and $\ell \leq k$. Let $\{R_\ell\}_{\ell \text{ in } I}$ be a family of rings. For each h and ℓ in I with $h \leq \ell$, let $\mu_{h\ell}: R_h \to R_\ell$ be a ring morphism such that (1) $\mu_{\ell\ell}$ is the identity mapping on R_ℓ for all ℓ in I and (2) $\mu_{hk} = \mu_{\ell k}\mu_{h\ell}$ when $h \leq \ell \leq k$. Form the direct sum $C = \bigoplus \sum_{\ell \text{ in } I} R_\ell$ and let D be the ideal generated by $\{x_h - \mu_{h\ell}(x_h) \mid x_h \text{ in } R_h \text{ and } h \leq \ell\}$. Define R = C/D and let $\mu: C \to C/D = R$ be the canonical map with $\mu_\ell: R_\ell \to R$ the restriction of μ to R_ℓ. Then R is called the direct limit of $\{R_\ell\}_{\ell \text{ in } I}$, written by $R = \varinjlim R_\ell$. Clearly,

$$\mu_h = \mu_\ell \, \mu_{h\ell} \tag{8}$$

for $h \leq \ell$. It can be shown [2] that every element of R can be written in the form $\mu_\ell(r_\ell)$ for some ℓ in I and some r_ℓ in R_ℓ.

Now let $R = \varinjlim R$ where each R_ℓ has solvability of $S(<i,j>,C)$ for all i and j with $i \neq j$ and all C in $GL_n(R_\ell)$. Let $A = [a_{ij}]$ be in $GL_n(R)$ with $A^{-1} = B = [b_{ij}]$ and consider the system $S(<i,j>,A)$.

For each i and j with $1 \leq i,j \leq n$, there exists k_{ij} in I and $r_{k_{ij}}$ in $R_{k_{ij}}$ with $b_{ij} = \mu_{k_{ij}}(r_{k_{ij}})$. Also there exist t_{ij} in I and $r_{t_{ij}}$ in $R_{t_{ij}}$ with $a_{ij} = \mu_{t_{ij}}(r_{t_{ij}})$.

Since $\{k_{ij} | 1 \leq i,j \leq n\} \cup \{t_{ij} | 1 \leq i,j \leq n\}$ is finite and contained in I, a directed set, there exists k in I with $t_{ij} \leq k$ and $k_{ij} \leq k$ for all i and j.

By (8),

$$b_{ij} = \mu_{k_{ij}}(r_{k_{ij}}) = \mu_k(\mu_{k_{ij}k})(r_{k_{ij}})$$

and $$a_{ij} = \mu_{t_{ij}}(r_{t_{ij}}) = \mu_k(\mu_{t_{ij}k})(r_{t_{ij}}).$$

Thus modulo D, A and B are in $GL_n(R_k)$ and hence $S(<i,j>,A)$ is solvable.

(e) The Ring of Cross-Sections of a Sheaf over a Boolean Space

Let X be a topological space. Suppose that for each x in X, there is a ring R_x with zero 0_x and identity 1_x and that $R_x \cap R_y = \emptyset$ for $x \neq y$. Let $\Sigma = \bigcup_{x \text{ in } X} R_x$. Then define $\Pi : \Sigma \to X$ by $\pi(r) = x$ if r is in R_x. Impose a topology on Σ satisfying the following:

(1) For each r in Σ, there exist open sets U in Σ with r in U and N in X with N the homeomorphic image of U under Π.

(2) Let $\Sigma + \Sigma$ be $\{(r,s) \mid \Pi(r) = \Pi(s)\}$ with the topology induced by the product topology in $\Sigma \times \Sigma$. Then the mapping on Σ to Σ given by $r \to -r$ and the mappings on $\Sigma + \Sigma$ to Σ given by $(r,s) \to r+s$ and $(r,s) \to rs$ are continuous.

(3) The mapping $x \to 1_x$ is continuous on X to Σ.

Then Σ is called a sheaf of rings over X. The rings R_x are called stalks of the sheaf Σ.

Let Y be a subset of X. A cross-section of Σ over Y is a continuous mapping σ of Y to Σ such that $\Pi(\sigma(y)) = y$ for all y in Y. The collection of all cross-sections of Σ

over Y is denoted by $\Gamma(Y,\Sigma)$ and is a ring with $(\sigma_1+\sigma_2)(x) = \sigma_1(x) + \sigma_2(x)$, $(\sigma_1\sigma_2)(x) = \sigma_1(x)\sigma_2(x)$, $0(x)=0_x$, and $1(x) = 1_x$ ([14], Lemma 3.2(b)).

Now suppose that X is a Boolean space and let Σ be a sheaf of rings over X each stalk of which has solvability of the systems $S(<i,j>,A_x)$ for all A_x in $GL_n(R_x)$. Let $R = \Gamma(X,\Sigma)$ and let $A = [a_{ij}]$ be an element of $GL_n(R)$ with $A^{-1} = B = [b_{ij}]$. Consider the system $S(<1,2>,A)$:

$$\begin{aligned} b_{21}X_1 + b_{23}X_3 + \dots + b_{2n}X_n &= a_{21} \\ b_{31}X_1 + b_{33}X_3 + \dots + b_{3n}X_n &= 0 \\ &\vdots \\ b_{n1}X_1 + b_{n3}X_3 + \dots + b_{nn}X_n &= 0 \end{aligned}$$

Since $A^{-1} = B$, $\sum_{j=1}^{n} a_{ij}b_{ji} = 1$ for $1\leq i\leq n$ and $\sum_{j=1}^{n} a_{ij}b_{jk} = 0$ for $1\leq i,k\leq n, i\neq k$. Then for each x in X, $\sum_{j=1}^{n} a_{ij}(x)b_{ji}(x)=1_x$ for $1\leq i\leq n$ and $\sum_{j=1}^{n} a_{ij}(x)b_{jk}(x) = 0_x$ for $1\leq i,k\leq n, i\neq k$. Thus if $A_x = [a_{ij}(x)]$, then $B_x = [b_{ij}(x)] = A_x^{-1}$ in $GL_n(R_x)$. The system $S(<1,2>,A_x)$ has a solution by hypothesis, that is, there exist $(X_1)_x, (X_3)_x, \ldots, (X_n)_x$ in R_x with

$$b_{21}(x)(X_1)_x + b_{23}(x)(X_3)_x + \ldots + b_{2n}(x)(X_n)_x = a_{21}(x)$$

$$b_{31}(x)(X_1)_x + b_{33}(x)(X_3)_x + \ldots + b_{3n}(x)(X_n)_x = 0_x$$

$$\vdots$$

$$b_{n1}(x)(X_1)_x + b_{n3}(x)(X_3)x + \ldots + b_{nn}(x)(X_n)_x = 0_x$$

Let i in $\{1,3,4,\ldots,n\}$ be fixed. By Lemma 3.2(c) of [14] we see that for each x in X, there exists a neighborhood N_{ix} of x and a σ_{ix} in $\Gamma(N_{ix},\Sigma)$ such that $\sigma_{ix}(x) = (X_i)_x$.

The set $\{N_{ix} | x \text{ in } X\}$ is an open cover of X and hence, by the partition property of the Boolean space X, there exists a finite set $\{M_1, M_2, \ldots, M_r\}$ of open-closed subsets of X such that

(1) For each j with $j \leq r$, there exists x_j in X with $M_j < N_{ix_j}$.

(2) $M_j \cap M_k = \emptyset$ if $j \neq k$.

(3) $\bigcup_{j=1}^{r} M_j = X$.

By Fact (1), p. 13 of [14], $T_{ix_j} = \sigma_{ix_j}|M_j$ is in $\Gamma(M_j,\Sigma)$ and by Fact (2), p. 13 of [14], we may define T_i in $\Gamma(X,\Sigma)$ by $T_i = T_{ix_1} \cup T_{ix_2} \cup \cdots \cup T_{ix_r}$, that is, $T_i(x) = T_{ix_j}(x)$ if x is in M_j. Then for each x in X, x is in M_j for some j with $1 \leq j \leq r$. Thus

$$(X_i)_x = \sigma_{ix}(x) = \sigma_{ix_j}(x) = T_{ix_j}(x) = T_i(x).$$

Then the system $S(\langle 1,2\rangle, A_x)$ may be written as

$$\begin{aligned}
b_{21}(x)T_1(x) + b_{23}(x)T_3(x) + \ldots + b_{2n}(x)T_n(x) &= a_{21}(x)\\
b_{31}(x)T_1(x) + b_{33}(x)T_3(x) + \ldots + b_{3n}(x)T_n(x) &= 0_x\\
&\vdots\\
b_{n1}(x)T_1(x) + b_{n3}(x)T_3(x) + \ldots + b_{nn}(x)T_n(x) &= 0_x
\end{aligned}$$

Since this is true for all x in X,

$$\begin{aligned}
b_{21}T_1 + b_{23}T_3 + \ldots + b_{2n}T_n &= a_{21}\\
b_{31}T_1 + b_{33}T_3 + \ldots + b_{3n}T_n &= 0\\
&\vdots\\
b_{n1}T_1 + b_{n3}T_3 + \ldots + b_{nn}T_n &= 0
\end{aligned}$$

Thus the system $S(\langle 1,2\rangle,A)$ has a solution.

Solutions of $S(\langle i,j\rangle,A)$ for $1\leq i,\ j\leq n,\ i\neq j$ are found analogously.

Thus we have shown that the systems $S(\langle i,j\rangle,A)$ are solvable for all distinct i and j and A any invertible matrix over a commutative ring, a division ring, a product or direct limit of rings in which the systems $S(\langle i,j\rangle,B)$ are solvable, or the ring of cross-sections of a sheaf over a Boolean space. As shown in Chapter II, this is sufficient to show that for R any of these rings, then all normal subgroups of $GL_n(R)$ are standard normal if $n\geq 4$.

REFERENCES

[1] E. Artin, Geometric Algebra, Wiley-Interscience, New York, 1957.

[2] M. Atiyah and I. Macdonald, Introduction to Commutative Algebra, Addison-Wesley, Reading, Mass., 1969.

[3] H. Bass, "K-theory and stable algebra," Publ. I.H.E.S. 22 (1964), 5-60.

[4] J.L. Brenner, "The linear homogeneous group," Ann. of Math. 39 (1938), 472-493.

[5] J.L. Brenner, "The linear homogeneous group, II," Ann. of Math. 45 (1944), 101-109.

[6] J.L. Brenner, "The linear homogeneous group, III," Ann. of Math. 71 (1960), 210-223.

[7] J. Dennin and D.L. McQuillan, "A note on the classical groups over semilocal rings," Proc. Roy. Irish Acad., Sect. A 68 (1969), 1-4.

[8] J. Dieudonńe, "On the automorphisms of classical groups," Memoirs of A.M.S. 2 (1951).

[9] W. Klingenberg, "Lineare gruppen über lokalen ringen," Amer. J. Math. 83 (1961), 137-153.

[10] G. Maxwell, "Infinite general linear group over rings," Trans. A.M.S. 151 (1970), 371-375.

[11] B.R. McDonald, "GL_2 of rings with many units," to appear, Comm. in Algebra.

[12] J.L. Mennicke, "Finite factor groups of the unimodular group," Ann. of Math. 81 (1965), 31-37.

[13] O. Ore, "Linear equations in non-commutative fields," Ann. of Math. 32 (1931), 463-477.

[14] R.S. Pierce, "Modules over commutative regular rings," Memoirs of A.M.S. 70 (1967).

[15] J.S. Wilson, "The normal and subnormal structure of general linear groups," Proc. Cambridge Philo. Soc. 71 (1972), Part 2, 163-177.