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FORMULATION AND OPTIMIZATION OF CONVENTIONAL AND NON-CONVENTIONAL PRODUCTION INVENTORY MODELS

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DISSERTATION COMMITTEE

Dedicated to

My Parents: Dharamdas and Devi Nasta

Who Made My Education Possible

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CHAPTER I

INTRODUCTION

A non-linear production inventory system is one in which either the demand function or the production function or both are non-linear. Many inventory situations exist in actual practice in which the behavior of either the demand function or the production function is non-linear with time. For example, the demand function becomes non-linear in situations where the resources in inventory are subject to physical depletion, direct spoilage, and deterioration. The influence of keen competition, advertising, and loss of goodwill due to stockout may also make the demand function non-linear.

The effects of physical depletion, direct spoilage, and deterioration on the demand function have been investigated by Ghare and Shrader.[1] They state that:

[1] P. M. Ghare and G. F. Shrader, "A Model for an Exponentially Decaying Inventory," Journal of Industrial Engineering, September-October, 1963, pp. 238-243.

Depletion by direct spoilage is commonly encountered in fruit and other food stuffs of a highly perishable nature, and is of major concern to organizations dealing with commodities of this type. Physical depletion over time often occurs in certain inventory situations involving electronic components, radioactive substances, blood banks, photographic film, grain and other materials in which a gradual loss of potential or utility is associated with the passage of time.

The influence of competition and advertising on the non-linearity of the demand function would be appreciable if the commodity under consideration is just beginning to penetrate the market. Loss in goodwill due to commodity stockout causes dissatisfaction of the customer, influencing him to change his order policy in the future. A group of dissatisfied customers would thus influence the demand curve for the subsequent inventory cycles. After the number of cycles becomes sufficiently large, a steady state is reached with a stationary non-linear demand function.

The production function could be non-linear when dealing with the manufacture of pharmaceuticals, rare metals, or with products which during the production period are subject to physical depletion or gradual loss of potential and utility. A production system with either limited resources or limited budgets could also give rise to non-linear production function. The following are three examples of situations having a non-linear production function. The first two examples are from D. B. Smith and

H. E. Shweyer.[2]

1. A pharmaceutical is produced by chemical reaction at a rate of $100e^{-kt}$ pounds per day, where k=0.1 $(\text{days})^{-1}$. The production time required for the reaction is 30 days. Demand for the product is constant throughout the year at 10 lbs/day.

2. In a refining operation of a precious metal where the demand rate is 10 lbs/day, the metal is produced at the constant rate of 20 lbs/day for 30 days. The production rate then decreases and is governed by the equation $m=20e^{(1-t/30)}$. The production follows this rate for 30 additional days.

3. Nuclear weapons are subject to physical depletion due to radioactive decay and gradual loss of potential or utility due to psychological effects and obsolescence. The effect of these parameters on the production function of nuclear weapons would be non-linear.

The above discussion shows the importance of non-linear production-demand functions in practical production-inventory situations. This problem, however, has received little attention in the literature. The following three models represent the most prominent work in this area. These models will be summarized here as a background for

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[2]D. B. Smith and H. E. Schweyer, "Inventory Analysis of Repetitive Operations," Engineering Progress at The University of Florida, Vol. XVIII, No. 10, October, 1964.

the research to be carried out in this dissertation.

I. Ghare and Shrader's Model[3]

Ghare and Shrader have developed a model of a system in which the inventory is subject to decay due to direct spoilage as in fruit, physical depletion as in highly volatile liquids, or deterioration as in electronic components or grain. The input to the system is assumed to be instantaneous and the output is assumed to be exponential due to the effect of "decay" on demand.

In the development of the mathematical model, the authors first derive from fundamental principles a differential equation representing the level of inventory in the system at time t. The equation is

$$\frac{dI_x}{dx} + I_x\theta = -D(x)$$

where I_x is the inventory at the end of time x, $D(x)$ is some regular demand function, and θ a fraction representing the rate of decay per time period. The above differential equation is then solved and an expression for loss due to decay is derived. The total cost is then expressed as a function of time.

The authors have shown that the effect of nonlinearity is substantial. To exemplify this effect, the

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[3]Ghare and Shrader, *op. cit*.

particular inventory problem considered calls for determining the economic lot size and the re-order point. The lot size and the re-order point for the case with decay is found to be 76 units and 7 days, respectively, whereas that for the case without decay is found to be 1000 units and 100 days.

The disadvantage of the above model is that its application is limited to cases in which the decay is exponential, the production is instantaneous, no shortages are allowed, the costs cannot vary with time, and the approximation by deleting the second order terms is allowable.

II. Fabrycky and Banks Model[4]

Fabrycky and Banks consider the analysis of an inventory system in which the demand and production functions are linearly dependent on time. The various costs allowed for in development of the mathematical model are shortage cost, holding cost, procurement cost, and item cost. Procurement quantity and procurement level were chosen as the decision variables for the optimization, and the following assumptions were made to develop the model:

1. Demand is increasing.

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[4] W. J. Fabrycky and Jerry Banks, Procurement and Inventory Systems: Theory and Analysis, (New York: Reinhold Publishing Corporation, 1967), p. 64-68.

2. Demand as a function of time can be represented by $d(t) = a + bt$ where a and b are constants.

3. Replenishment rate is constant with respect to time.

4. Replenishment rate R is greater than the demand rate D.

5. Unsatisfied demand is backlogged.

The inventory cycle of the system considered by Fabrycky and Banks is shown in Figure 1.

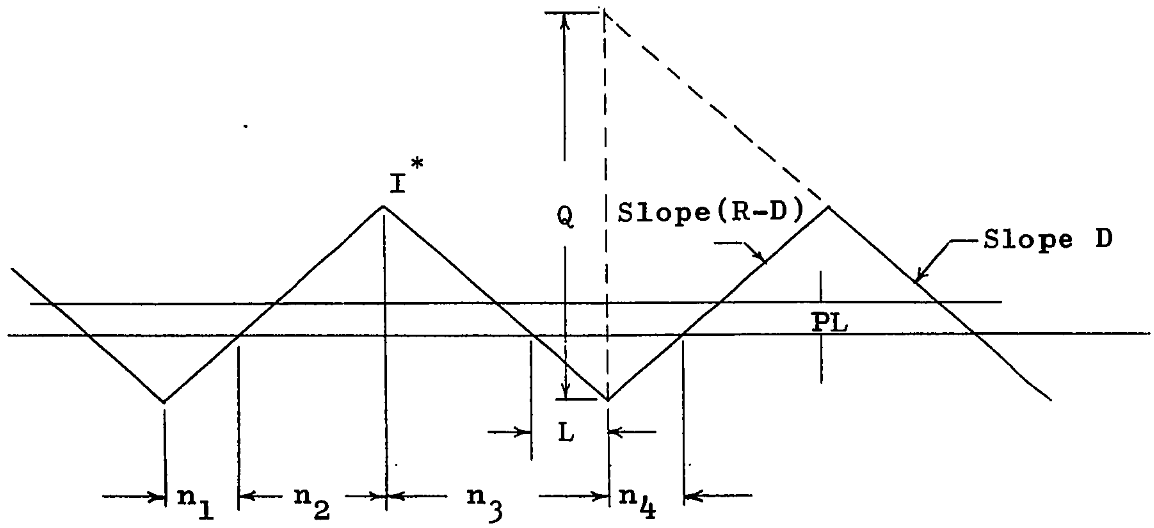

Figure 1. Linear Production Inventory Cycle

For ease of reference we summarize the nomenclature as used by Fabrycky and Banks in developing their model:

Q	=	Procurement quantity.
PL	=	Procurement level.
L	=	Lead time.
I*	=	Maximum inventory.
R	=	Replenishment rate.
D	=	Demand rate.
n_2+n_3	=	Holding period.
n_1+n_4	=	Shortage period.
$n_1+n_2+n_3+n_4$	=	The number of periods per cycle.

With the help of geometrical properties of Figure 1, the total cost of the inventory system was expressed as a function of the decision variables PL and PQ. The values of PL and PQ were then found with the help of calculus to minimize the total cost of the system.

The drawback of the above approach is that its application is limited to analysis of linear systems only, because the approach is based on the geometrical properties of the inventory cycle.

III. Smith and Schweyer's Model[5]

This paper deals with the application of inventory analysis of repetitive operations in the chemical process industries. The input to the various inventory situations studied corresponds to the production of products such as

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[5]Smith and Schweyer, op. cit.

pharmaceuticals, rare metals, or radioactive isotopes. As most of these products are produced by chemical reactions, or are subject to decay with time in the production period, the different inventory situations analyzed correspond to non-linear production functions. The output is assumed to be linear. The only cost considered for the development of total cost expression is the storage cost. The authors have not given any attention to the minimization of the cost, but they are concerned with the problem of determining the times at which peak inventories occur and the average inventory carried. To achieve this, Smith and Schweyer have developed generalized relations of inventory time curves for the cycle.

The general equation of inventory at time t in the system developed by Smith and Schweyer is given by

$$I = \int_0^t m\,dt - \int_0^t w\,dt + I_o$$

where, I = inventory at any time, t

I_o = inventory at time = 0

w = withdrawal rate, which will be considered constant

m = make rate.

Seven types of repetitive operations are considered:

1. Constant make rate.
2. Increasing make rate (initial make rate less

than withdrawal rate).

3. Increasing make rate (initial rate greater than withdrawal rate).

4. Decreasing make rate (rate at the end of make time less than withdrawal rate).

5. Decreasing make rate (rate at the end of make time greater than withdrawal rate).

6. Batch case.

7. Composite case (two (two or more production curves in one cycle).

Smith and Schweyer do not consider optimization, and do not allow shortages. Furthermore, their approach is limited to linear withdrawal rate.

IV. Objective of the Dissertation

The three articles referred to above have dealt with various aspects of linear and non-linear production inventory systems. Table 1 gives a summary of the different characteristics of the above three models as compared to the more general model proposed here.

This comparison indicates that none of the three models take into account the inventory situation in which the demand and production functions are both non-linear and the cost parameters are time dependent. Furthermore, the three models deal with specific inventory situations which do not consider one or more of the following

TABLE 1

Summary and Comparison of the Characteristics of Different Inventory Models

Author	Type of production function	Type of demand function	Independent variables chosen for optimization	Optimization technique used	Variable Costs Considered				Can the costs be time varying	Can production or demand curve be more than one in one cycle
					Holding	Shortage	Item	Procurement		
Ghare and Schrader	Instantaneous	Exponential	Time	Calculus	Yes	No	Yes	Yes	No	No
Smith and Schweyer	Non-linear	Uniform rate	Time	None	Yes	No	Yes	No	No	Yes
Fabrycky and Banks	Uniform rate	Uniform rate	1. Procurement quantity. 2. Procurement level.	Calculus	Yes	Yes	Yes	Yes	No	No
This presentation	Non-linear	Non-linear	Time	(i) Calculus. (ii) Numerical analysis.	Yes	Yes	Yes	Yes	Yes	Yes

situations: 1. Shortages, 2. Optimization, 3. Composite demand, 4. Composite production.

This dissertation proposes a more general solution to the non-linear problem which will include these models as special cases. The specific objective of the analysis is to find the values of time periods at which the production process should be stopped and restarted to operate the inventory system at minimum total cost.

V. Plan of Dissertation

Chapter II develops a general model for a non-linear inventory system. Chapter III presents numerical examples of non-linear production inventory systems. Then, the solution of the non-linear system is compared with the corresponding approximated linear system. In Chapter IV, an approach involving calculus of variation will be applied to study the non-conventional production inventory system.

CHAPTER II

CONVENTIONAL PRODUCTION-INVENTORY SYSTEM

In this chapter a general mathematical model for the production-inventory system is developed and a method for optimizing such a system is presented.

This chapter is organized into three sections: section I gives the assumptions and the nomenclature used in the development of the model; section II derives the general mathematical model; section III then gives a method for optimizing the model.

I. Assumptions and Nomenclature

The basic assumptions of the model are the following:

1. The relevant costs are holding cost, shortage cost, procurement cost or set-up cost, and item cost.

2. The estimates of the relevant costs per unit per unit time are known with certainty.

3. The production and demand functions are known with certainty.

4. The system deals with a single item.

The symbols used in this chapter including those referred to in the Figures are defined as follows:

t_1 = time at which the production is stopped.

t_2 = time at which the inventory level is zero.

t_3 = time at which the production restarts.

t_4 = time at which the inventory is again zero or the number of time units per cycle.

KM = number of units produced till time t.

$P(t)$ = the total number of units produced by the system till time t (Figure 2).

$D(t)$ = the total number of units demanded from the system till time t (Figure 3).

$H(t)$ = holding cost per unit per unit time at time t.

$S(t)$ = shortage cost per unit per unit time at time t.

PC = procurement cost per cycle.

$i(t)$ = item cost at time t.

$I_1(t)$ = total number of units in the system at time t, $(0 \leq t \leq t_1)$, in the growth period (Figure 4).

$I_2(t)$ = total number of units in the system at time t, $(t_1 \leq t \leq t_3)$, in the decay period.

$I_3(t)$ = total number of units in the system at time t, $(t_3 \leq t \leq t_4)$.

The mathematical assumptions made to develop the general model are:

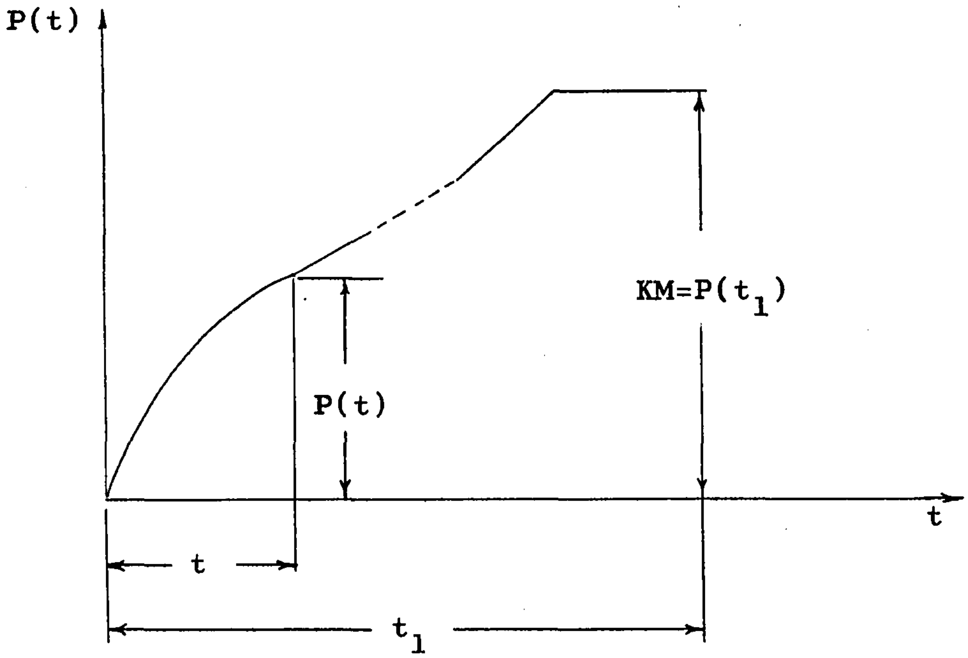

Figure 2. Graphical Representation of the Production Function

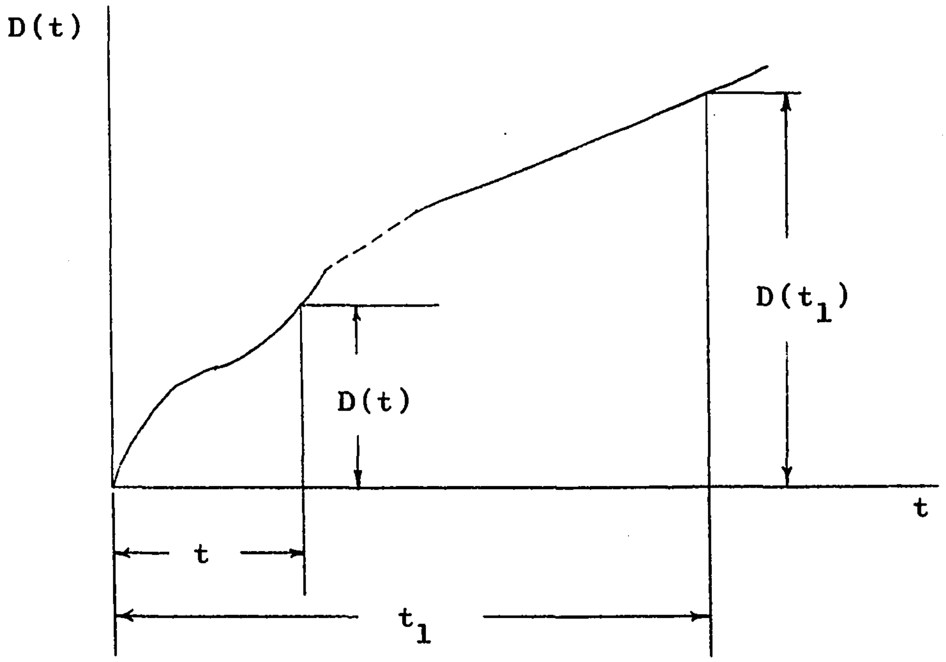

Figure 3. Graphical Representation of the Demand Function

1. The functions $P(t)$, $D(t)$, $H(t)$, $S(t)$, and $i(t)$ are stationary and piecewise continuous on a finite interval. In addition, $H(t)$ is convex.

II. Construction of the General Model

The derivation of the general model for the production inventory system can be broken up into the following steps:

1. Defining a measure of effectiveness.
2. Choosing the decision variables.
3. Developing an expression for the measure of effectiveness in terms of decision variables.
4. Deriving the constraints on the decision variables.

The total cost per unit time is chosen as a measure of effectiveness. An optimization of this measure of effectiveness should yield the optimal values of the decision variables t_1, t_2, t_3 and t_4 defined above.

We now derive the expressions for the four relevant costs of the model defined above. The total cost expression is given by the sum of these four costs:

1. Holding Cost (HC).

Referring to Figure 4, the holding cost for the time interval [t, t+dt] is:

$$I_1(t)H(t)dt, \qquad 0 \leq t \leq t_1$$

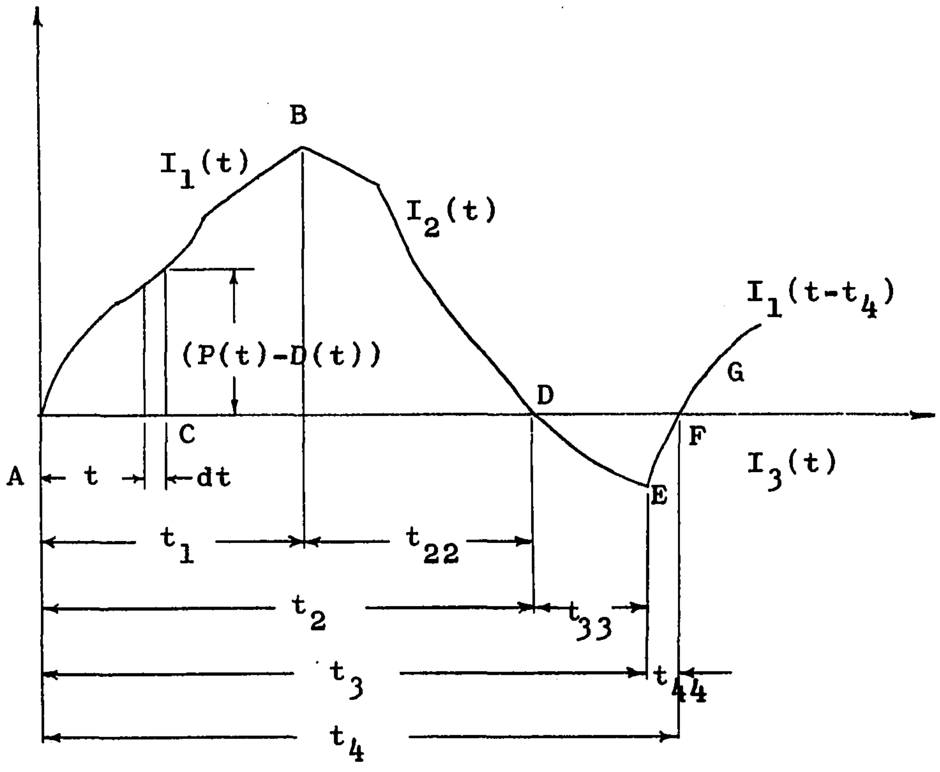

Figure 4. Graphical Representation of the Inventory Cycle

and,

$$I_2(t)H(t)dt, \qquad t_1 \leq t \leq t_2$$

therefore,

$$\text{holding cost per cycle} = \int_0^{t_1} I_1(t)H(t)dt + \int_{t_1}^{t_2} I_2(t)H(t)dt \qquad (2.1)$$

but,

$$I_1(t) = P(t) - D(t),$$

and,

$$I_2(t) = KM - D(t),$$

therefore, substituting in (2.1), the holding cost per cycle is given by,

$$HC = \int_{o}^{t_1} [P(t)-D(t)]H(t)dt + \int_{t_1}^{t_2} [(KM-D(t)]H(t)dt \quad .$$

Therefore, holding cost per unit time is given by,

$$HC/unit\ time = \frac{1}{t_4}[\int_{o}^{t_1} (P(t)-D(t))H(t)dt + \int_{t_1}^{t_2} (KM-D(t))H(t)dt] \qquad (2.2)$$

2. Shortage Cost (SC).

Applying the same reasoning as in the holding cost case to obtain the shortage cost, SC/unit time, this gives:

$$SC/unit\ time = -[\frac{1}{t_4}\int_{t_2}^{t_3} I_2(t)S(t)dt + \frac{1}{t_4}\int_{t_3}^{t_4} I_3(t)S(t)dt] \qquad (2.3)$$

Now, $I_3(t)$ is an inventory curve passing through t_4. Since $D(t)$ and $P(t)$ are stationary with time, therefore, $I_3(t)$ is given by,

$$I_3(t) = I_1(t-t_4).$$

Substituting for $I_3(t)$ in terms of $I_1(t)$ in (3) gives:

$$SC/unit\ time = -[\frac{1}{t_4}\int_{t_2}^{t_3} I_2(t)S(t)dt + \frac{1}{t_4}\int_{t_3}^{t_4} I_1(t-t_4)S(t)dt] \qquad (2.4)$$

$$I_3(t) = I_1(t-t_4)$$

$$= P(t-t_4) - D(t-t_4).$$

Substituting value of $I_3(t)$ in (4) gives:

$$\text{SC/unit time} = -\frac{1}{t_4}\int_{t_2}^{t_3} [(KM - D(t))S(t)]dt$$

$$-\frac{1}{t_4}\int_{t_3}^{t_4} [(P(t-t_4) - D(t-t_4))S(t)]dt \qquad (2.5)$$

3. Procurement Cost (PC).

Procurement cost, PC, is the total cost associated with a series of actions beginning with the initiation of procurement action and ending with the receipt of replenishment stock.

$$\text{PC/unit time} = \frac{PC}{t_4} \qquad (2.6)$$

4. Item Cost (IC).

Item cost, $i(t)$, is the price of the item in dollars per unit at time t.

$$\text{IC/unit time} = \frac{1}{t_4}\int_{o}^{t_4} [i(t)\frac{d}{dt}(D(t))]dt \qquad (2.7)$$

Therefore, the total cost per unit of time is given by:

$$\text{TC/unit time} = \frac{1}{t_4}\int_{o}^{t_1} [P(t) - D(t)]\, H(t)dt$$

$$+\frac{1}{t_4}\int_{t_1}^{t_2} [(KM - D(t))\, H(t)]dt$$

$$- \frac{1}{t_4} \int_{t_2}^{t_3} [(KM-D(t))S(t)]dt - \frac{1}{t_4} \int_{t_3}^{t_4} [(P(t-t_4)-D(t-t_4))S(t)]dt$$

$$+ \frac{PC}{t_4} + \frac{1}{t_4} \int_{o}^{t_4} [i(t) \frac{d}{dt} (D(t))]dt \qquad (2.8)$$

KM, as defined earlier, is the number of units produced till time t_1. But the number of units produced until time t_1 is also given by $P(t_1)$; therefore,

$$KM = P(t_1) \qquad (2.9)$$

Substituting the value of KM in (2.8), we get:

$$\text{TC/unit time} = \frac{1}{t_4} \int_{o}^{t_1} (P(t) - D(t)) H(t)dt$$

$$+ \frac{1}{t_4} \int_{t_1}^{t_2} [(P(t_1) - D(t)) H(t)]dt$$

$$- \frac{1}{t_4} \int_{t_2}^{t_3} [(P(t_1)-D(t))S(t)]dt - \frac{1}{t_4} \int_{t_3}^{t_4} [(P(t-t_4)-D(t-t_4))S(t)]dt$$

$$+ \frac{PC}{t_4} + \frac{1}{t_4} \int_{o}^{t_4} [i(t) \frac{d}{dt}(D(t))]dt \qquad (2.10)$$

The decision variables t_1, t_2, t_3, and t_4 are subject to two constraints. Referring to Figure 3, it is evident that the first constraint corresponds to the fact that the level of inventory at point D is zero, and the second constraint corresponds to the fact that the value of the shortage amount at time t_3 as computed by substituting $t=t_3$ in $I_2(t)$, and $t=t_3$ in $I_1(t-t_4)$ should be the same. The two constraints on t_1, t_2, t_3, and t_4 can thus be symbolically written as:

$$I_2(t) = 0 \text{ for } t=t_2, \tag{2.11}$$

and
$$I_2(t_3) = I_1(t_3-t_4), \tag{2.12}$$

but,
$$I_2(t) = KM-D(t), \text{ and by (2.9)}$$

$$KM = P(t_1).$$

Therefore, for $t = t_2$ (2.11) gives:

$$I_2(t_2) = P(t_1) - D(t_2) = 0,$$

and the first constraint is

$$P(t_1) - D(t_2) = 0.$$

Now,

$$I_2(t) = KM - D(t),$$

$$I_2(t_3) = KM - D(t_3), \text{ and by (2.9)}$$

$$I_2(t_3) = P(t_1) - D(t_3). \tag{2.13}$$

Also,

$$I_1(t_1) = P(t_1) - D(t_1), \text{ and}$$

$$I_1(t_3-t_4) = P(t_3-t_4) - D(t_3-t_4).$$

Substituting these values in (2.12) and (2.13), one obtains:

$$P(t_1)-D(t_3) = P(t_3-t_4) - D(t_3-t_4).$$

Therefore, the second constraint is

$$P(t_1) - P(t_3-t_4) - D(t_3) + D(t_3-t_4) = 0 \quad .$$

III. Optimization of the General Model

The optimization problem consists of minimization of the total cost subject to two constraints. Total cost expression and the constraints are derived in the previous discussion. The problem of optimization can be stated as:

Minimize: $$\text{TC/unit time} = \frac{1}{t_4} \int_{o}^{t_1} [P(t)-D(t)]H(t)dt$$

$$+ \frac{1}{t_4} \int_{t_1}^{t_2} [(KM-D(t))H(t)]dt - \frac{1}{t_4} \int_{t_2}^{t_3} [(P(t_1)-D(t))S(t)]dt$$

$$- \frac{1}{t_4} \int_{t_3}^{t_4} [(P(t-t_4) - D(t-t_4))S(t)]dt + \frac{PC}{t_4} \qquad (2.14)$$

$$+ \frac{1}{t_4} \int_{o}^{t_4} [i(t) \frac{d}{dt} (D(t))]dt,$$

subject to constraints

$$P(t_1) - D(t_2) = 0, \quad \text{and}$$

$$P(t_1) - P(t_3-t_4) - D(t_3) + D(t_3-t_4) = 0 \quad .$$

Again, (2.14) can be rewritten as:

$$\text{TC/unit time} = \frac{1}{t_4} (HC + SC + PC + IC) \quad . \qquad (2.15)$$

The upper and lower limits of integration of HC, SC, PC, and IC will now be converted to 0 and 1, respectively, by applying a suitable linear transformation to each one of the integrals. The linear transformation necessary to convert the limits A and B of $\int_A^B F(t)dt$ to 0 and 1, respectively, is $t = (B-A)T+A$ and the transformed integral is given by

$$\int_A^B F(t)dt = (B-A) \int_0^1 F[(B-A)T+A]dt \qquad (2.16)$$

Applying (2.16) to HC, SC, PC, and IC, we get

$$HC = \int_0^1 [P(t_1T+0) - D(t_1T+0)]H(t_1T+0)\ (t_1-0)dt$$

$$+ (t_2-t_1) \int_0^1 [(P(t_1) - D((t_2-t_1)T+t_1))H(t_2-t_1)T+t_1)]dt \qquad (2.17)$$

$$SC = (t_3-t_2) \int_0^1 [(P(t_1) - D((t_3-t_2)T+t_2))S((t_3-t_2)T+t_2)]dt$$

$$+ (t_4-t_3) \int_0^1 [(P((t_4-t_3)T+t_3-t_4)$$

$$- D((t_4-t_3)T+t_3-t_4))S((t_4-t_3)T+t_3)]dt \qquad (2.18)$$

$$IC = t_4 \int_0^1 I(t_4T+0) \frac{d}{dt} [D(t_4T)]dt \qquad (2.19)$$

It is evident from (2.15), (2.17), (2.18) and (2.19) that

$$\text{TC/unit time} = \frac{1}{t_4} \int_o^1 \Phi(t_1,t_2,t_3,t_4,t)dt,$$

where $\frac{1}{t_4} \int_o^1 \Phi(t_1,t_2,t_3,t_4,t)dt = \frac{HC+SC+IC+PC}{t_4}$.

Let $$F(t_1,t_2,t_3,t_4) = \frac{1}{t_4} \int_o^1 \Phi(t_1,t_2,t_3,t_4,t)dt,$$

$$G(t_1,t_2) = P(t_1) - D(t_2) = 0,$$

$$H(t_1,t_2,t_3,t_4) = P(t_1)-P(t_3-t_4)-D(t_3)+D(t_3-t_4) = 0 .$$

The optimization problem reduces to:

Minimize: $$\text{TC/unit time} = F(t_1,t_2,t_3,t_4) . \qquad (2.20)$$

subject to constraints

$$\begin{aligned} G(t_1,t_2) &= 0, \text{ and} \\ H(t_1,t_2,t_3,t_4) &= 0 . \end{aligned} \qquad (2.21)$$

Due to the complex nature of (2.20) and (2.21), analytical methods do not insure an explicit optimization of (2.20) subject to constraints (2.21). However, for such cases numerical methods can be used.

The application of numerical methods for optimization of (2.20) subject to constraints (2.21) consists of trying various combinations of t_1,t_2,t_3, and t_4 and finding the one that gives a minimum total cost and also satisfies the constraints. The method of searching the optimum values of t_1, t_2, t_3, and t_4 is as follows:

A function of four variables t_1, t_2, t_3, and t_4 is to be optimized subject to two 'equality' constraints; which therefore permits only two variables to take arbitrary values. Before proceeding to the search technique and the choice of the variables that will be given arbitrary values, the optimizing function is modified in the following manner.

It is evident from Figure (4) that

$$t_2 = t_1 + t_{22}$$

$$t_3 = t_2 + t_{33} = t_1 + t_{22} + t_{33}$$

$$t_4 = t_3 + t_{44} = t_1 + t_{22} + t_{33} + t_{44}$$

Hence, $F(t_1, t_2, t_3, t_4)$ can be rewritten as a function of t_1, t_{22}, t_{33}, and t_{44}, say $F'(t_1, t_{22}, t_{33}, t_{44})$. Similarly, $G(t_1, t_2)$ becomes $G'(t_1, t_{22})$ and $H(t_1, t_2, t_3, t_4)$ becomes $H'(t_1, t_{22}, t_{33}, t_{44})$. Therefore, the modified optimization problem is:

Minimize: $$F'(t_1, t_{22}, t_{33}, t_{44}), \tag{2.22}$$

subject to $$G'(t_1, t_{22}) = 0, \tag{2.23}$$

and

$$H'(t_1, t_{22}, t_{33}, t_{44}) = 0, \tag{2.24}$$

where t_1, t_{22}, t_{33}, and t_{44} are independent of each other. This implies that the number of independent variables that can take arbitrary values is two. After showing that $F'(t_1, t_{22}, t_{33}, t_{44})$ is a convex function and that $G'(t_1, t_{22})$ and $H'(t_1, t_{22}, t_{33}, t_{44})$ are convex functions satisfying the regularity conditions over the feasible region, then the following procedure is used to determine t_1^*, t_2^*, t_3^*, and t_4^* for an optimal solution.

1. A certain value is given to t_1.
2. t_{22} is computed by solving (2.23).
3. A certain value is given to t_{44}.
4. t_{33} is computed by solving (2.24).
5. Having known t_1, t_{22}, t_{33}, and t_{44}, the value of $F'(t_1, t_{22}, t_{33}, t_{44})$ is computed.
6. This process is repeated until it converges.
7. The proposed optimal solution is checked with the help of Kuhn-Tucker conditions.

Details about the properties of concave and convex functions, and the Kuhn-Tucker conditions, are given in Appendix A.

CHAPTER III

APPLICATION TO TWO TYPES OF NON-LINEAR INVENTORY SITUATIONS

The purpose of this chapter is two-fold: (1) to illustrate the application of the general model developed in Chapter II by using two types of non-linear inventory systems; (2) to study the effect of approximating the above non-linear systems with an equivalent linear system, and to compare the results of two systems.

This chapter is organized into three sections: section I defines broadly the types of non-linear inventory systems that will be dealt with in this chapter; section II derives the two inventory models corresponding to the above inventory situations using the general model developed in Chapter II; in section III numerical examples corresponding to five inventory situations are analyzed and their solution is compared with that of the approximated linear models.

I. Types of Non-Linear Inventory Systems Considered

The two types of numerical examples illustrated in this chapter correspond to inventory situations in which the production and demand functions are defined as follows:

1. The production rate and the demand rate are directly dependent on the number of units produced and the number of units demanded, respectively. Such inventory situations may occur as a result of one or more of the non-linear factors described in Chapter I. The differential equations describing the production and demand functions corresponding to the above inventory situation are the following:

$$\frac{dP}{dt} = b_1' + b_2' P \text{ , } P(0) = b_o' \qquad (3.1)$$

and

$$\frac{dD}{dt} = d_1' + d_2' D \text{ , } D(0) = d_o' \qquad (3.2)$$

where b_1', b_2', b_o', d_1', d_2', and d_o' are constants determined from the characteristics of the system under consideration.

2. The production and the demand rates vary directly with time. This yields the following differential equations for the production and demand functions:

$$\frac{dP}{dt} = b_1'' + b_2'' t \text{ , } P(0) = b_o'' \qquad (3.3)$$

$$\frac{dD}{dt} = d_1'' + d_2'' t \text{ , } D(0) = d_o'' \qquad (3.4)$$

where b_1'', b_2'', b_o'', d_1'', d_2'', and d_o'' are constants which again can be specified depending on the characteristics of the system.

The production and demand can be expressed as functions of time by solving the differential equations described above. Thus for the first type, by solving (1) we get the production function:

$$P(t) = \frac{1}{b_2'} e^{b_2'(t+K_1)} - \frac{b_1'}{b_2'} \tag{3.5}$$

where K_1 is the constant of integration. From the initial condition, we get:

$$K_1 = \frac{1}{b_2'} \ln(b_o' b_2' + b_1')$$

and hence, after simplification,

$$P(t) = b_3' \, e^{b_2' t} + b_4' \tag{3.6}$$

where,

$$b_3' = b_o' + \frac{b_1'}{b_2'} ,$$

and

$$b_4' = - \frac{b_1'}{b_2'} .$$

In an exactly similar manner, by solving (2) we get the demand function.

$$D(t) = d_3' \, e^{d_2' t} + d_4' \tag{3.7}$$

where,

$$d_3' = d_o' + \frac{d_1'}{d_2'}$$

and

$$d_4' = -\frac{d_1'}{d_2'} \quad .$$

For the second type of problem, the demand and production functions are also derived using the above procedure. This yields:

$$P(t) = b_1''t + \frac{b_2''t^2}{2} + b_o'', \tag{3.8}$$

and

$$D(t) = d_1''t + \frac{d_2''t^2}{2} + d_o'' \tag{3.9}$$

The two types of problems studied in this chapter correspond to inventory situations in which the production and demand functions are defined by (3.6), (3.7), (3.8), and (3.9), respectively.

II. Derivation of the Two Inventory Models

The mathematical models corresponding to the two types of inventory situations described in the previous section will be derived by substituting the above demand and production functions in the general inventory model discussed in Chapter II. It is assumed that $H(t)$, $S(t)$, and $i(t)$ are constants denoted by H_c, S_c, and i_c, respectively. The cost, i_c, of manufacturing an item, per unit, is given by

$$i_c = \frac{K N^n}{n+1} (dl)(1+fb) + dm$$

where,

K = number of direct labor hours required to produce the initial item,

N = number of items that are required,

n = $\frac{\log \Phi}{\log 2}$,

= the per cent improvement in decimal form,

dl = direct labor hourly rate,

dm = direct material cost per unit,

fb = factory burden rate expressed as a percentage of direct labor hourly rate.[6]

<u>Model I</u>.

Model I corresponds to the first type of inventory situation described in section I. By substituting the production and demand functions described by equations (3.6) and (3.7), in equations (2.22), (2.23), and (2.24), as well as denoting the corresponding objective function by $F_I'(t_1, t_{22}, t_{33}, t_{44})$, this model becomes:

Minimize $$F_I'(t_1, t_{22}, t_{33}, t_{44}) \tag{3.10}$$

---

[6]Fabrycky and Banks, <u>op. cit</u>.

subject to

$$b_3' e^{b_2' t_1} - d_3' e^{d_2'(t_1+t_{22})} + b_4' - d_4' = 0 ,$$

and (3.11)

$$b_3' e^{b_2' t_1} - b_3' e^{-b_2' t_{44}} - d_3' e^{d_2'(t_1+t_{22}+t_{33})} + d_3' e^{-d_3 t_{44}} = 0 .$$

As mentioned in Chapter II, due to the complex nature of the problem, analytical methods do not insure an explicit optimization of (3.10) subject to constraints (3.11). However, for such cases, numerical methods can be used. The method of searching the optimum values of t_1, t_{22}, t_{33}, and t_{44} for the above model is the same as the general procedure given in Chapter II.

<u>Model II</u>.

This model corresponds to the second type of inventory situation described in section I. Again, by substituting the production and demand functions described by equations (3.8) and (3.9) in equations (2.22), (2.23), and (2.24), as well as denoting the corresponding objective function by $F_{11}'(t_1,t_{22},t_{33},t_{44})$, this model becomes:

Minimize $\quad F_{II}'(t_1,t_{22},t_{33},t_{44})$ (3.12)

subject to

$$d_2'' t_{22}^2 + (d_2''-b_2''-b_1'')t_1^2 + 2d_2'' t_1 t_{22}$$

$$+ d_2'' t_{22} + d_1'' t_1 + d_o'' - b_o'' = 0 .$$

$$(d_2''-b_1'')t_{44}^2 + (b_2''-d_1'')t_{44} + (b_1''+b_2'')t_1^2 \qquad (3.13)$$

$$- d_2''(t_2+t_{33})^2 - d_1''(t_2+t_{33}) = 0 \quad .$$

The procedure for searching the optimum values of t_1, t_{22}, t_{33}, and t_{44} differs in one respect from the one described in Chapter II. In this case, t_{33} takes arbitrary values and t_{44} is computed from constraint (3.13).

III. Numerical Examples

In this section numerical examples corresponding to the two types of non-linear inventory situations discussed in section II are presented. The inventory cycle, demand function, and the production function corresponding to the inventory situation of the first type are shown in Figures 5, 6, 7, and 8, and that for the second type is the same as Figure 8 of the first type.

The cost parameters, the parameters defining the production function and demand function for the two types of inventory situations, are given in Table 2 and Table 4, respectively. The above mentioned parameters corresponding to the approximated linear inventory systems is given in Table 3 and Table 4, respectively. The optimum values of the decision variables t_1, t_2, t_3, t_4, and the

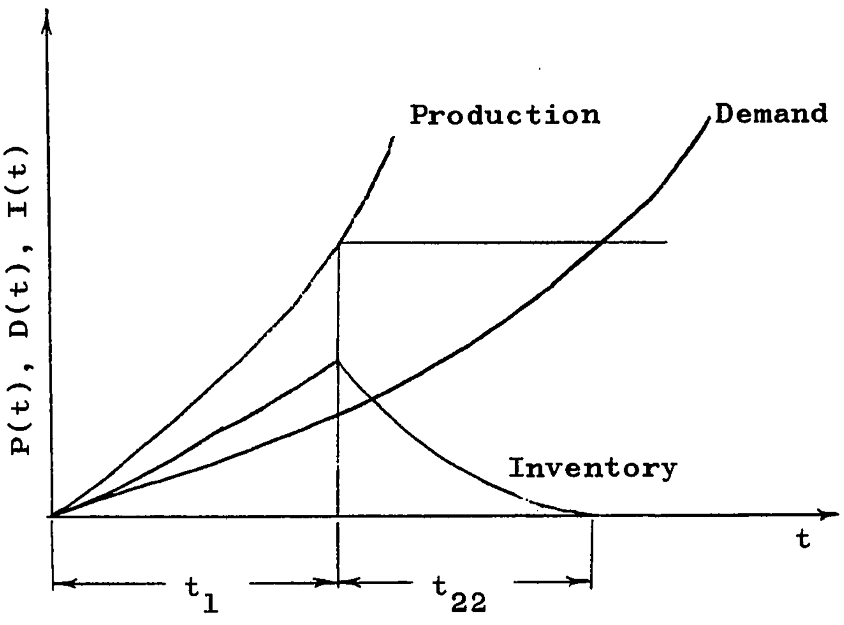

Figure 5. Graphical Representation of Example 1

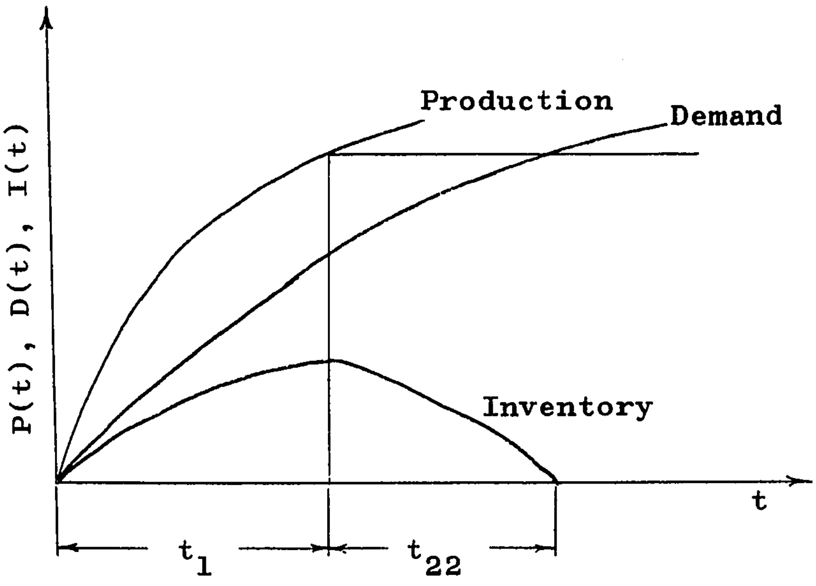

Figure 6. Graphical Representation of Example 2

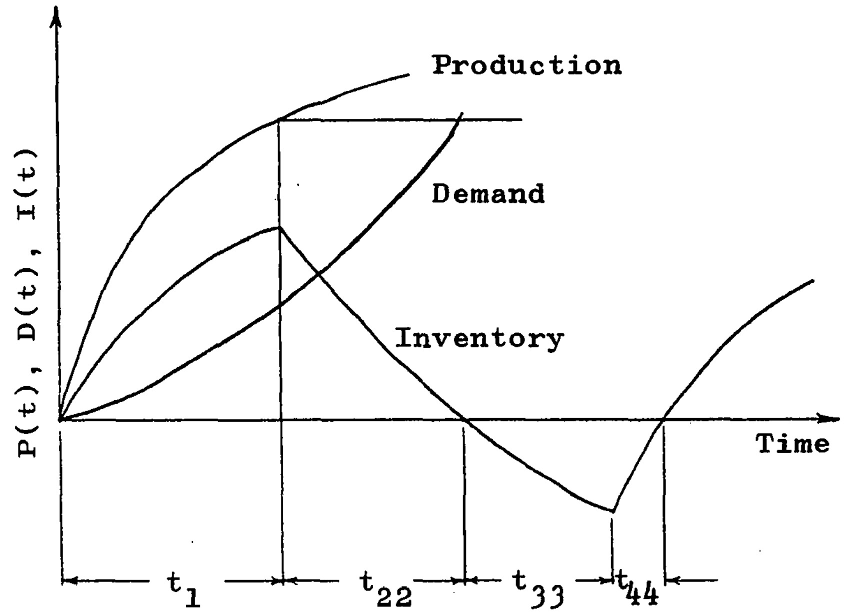

Figure 7. Graphical Representation of Example 3

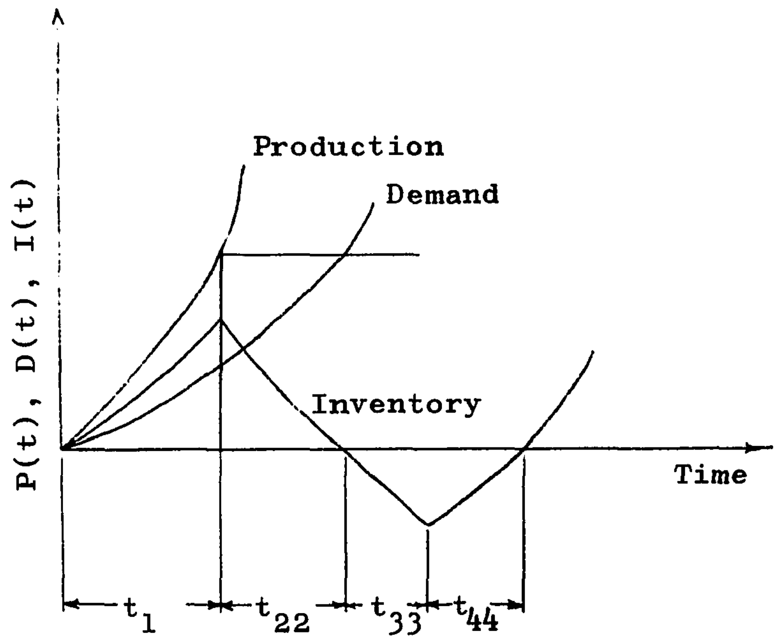

Figure 8. Graphical Representation of Examples 4 and 5

TABLE 2

Parameters of the Actual Inventory Situations of the First Type

Example No.	Parameters defining the production function			Parameters defining the demand function			Cost Parameters								
	b_1'	b_2'	b_o'	d_1'	d_2'	d_o'	H_c	S_c	PC	lr	fb	dm	N	Φ	K
1	10	0.1	0	2.5	0.05	0	0.2	∞	16	3	0.8	2.5	3000	0.8	4
2	10	-0.1	0	2.5	-0.05	0	0.2	∞	16	3	0.8	2.5	3000	0.8	4
3	10	-0.1	0	2.5	0.05	0	0.2	0.4	16	3	0.8	2.5	3000	0.8	4
4	-2	0.01	586.75	-3.37	0.01	586.75	0.2	0.4	16	3	0.8	2.5	3000	0.8	4

TABLE 3

Parameters of the Approximated Inventory Situations of the First Type

Example No.	Parameters defining the linearly approximated production function			Parameters defining the linearly approximated demand function			Cost Parameters								
	b_1''	b_2''	b_o'	d_1''	d_2''	d_o''	H_c	S_c	PC	lr	fb	dm	N	Φ	K
1	0	10	0	0	2.5	0	0.2	∞	16	3	0.8	2.5	3000	0.8	4
2	0	10	0	0	2.5	0	0.2	∞	16	3	0.8	2.5	3000	0.8	4
3	0	10	0	0	2.5	0	0.2	0.4	16	3	0.8	2.5	3000	0.8	4
4	0	3.5	585.0	0	2.5	585.0	0.2	0.4	16	3	0.8	2.5	3000	0.8	4

TABLE 4

Parameters of the Actual and Approximated* Inventory Situation of the Second Type

Example No.	Parameters defining the production function			Parameters defining the demand function			Cost Parameters								
	a_1	a_2	a_3	a_4	a_5	a_6	H_c	S_c	P_c	lr	fb	dm	N	Φ	K
5	0.3	30	0	0.1	15	0	0.2	0.4	100	3	0.8	2.4	3000	0.8	4
6*	0.0	30	0	0.0	15	0	0.2	0.4	100	3	0.8	2.4	3000	0.8	4

total cost per unit time corresponding to the actual and approximated inventory systems of the first type and the second type are given in Table 4 and Table 5, respectively.

The approximated linear system is derived by neglecting terms of the second order in t of the production and demand functions. The optimum values of the decision variables and the total cost per unit time is computed using IBM 360/40. The flow chart for the computer program for the optimization of the inventory system is given in Appendix B.

Table 5 summarizes the optimum values of the decision variables t_1, t_2, t_3, t_4, and the objective function for the actual models and the corresponding linearly approximated models. The comparison of the results of the actual and the approximated models shows a substantial difference in the optimum solution of the two models.

In conclusion, Table 5 shows that a non-linear production inventory system cannot always be represented by a linear system and, hence, explains the importance of the study of non-linear production inventory system.

A discussion of one other new avenue of research will be presented in the next chapter.

TABLE 5

Summary and Comparison of the Results of Actual and Approximated Situations

Example No.	Optimum Values for the Actual Models					Optimum Values for the Linearly Approximated Models				
	t_1	t_2	t_3	t_4	TC per unit time	t_1	t_2	t_3	t_4	TC per unit time
1	1.349	5.078	5.078	5.078	18.266	0.3	0.525	0.525	0.525	31.0497
2	4.999	30.922	30.922	30.922	9.153	0.3	0.525	0.525	0.525	31.0497
3	0.025	0.099	4.565	6.065	31.868	1.0	4.00	10.00	11.333	14.740
4	1.75	2.718	35.559	135.309	75.076	4.50	6.30	46.8	143.85	51.145
5	2.00	3.925	23.425	50.465	89.273	1.00	2.00	7.500	9.999	92.348

CHAPTER IV

THE NON-CONVENTIONAL PRODUCTION-INVENTORY SYSTEM

The production-inventory system presented in this chapter differs from the conventional system presented in Chapter II. In the conventional problem it is assumed that the production function, the demand function, and the relevant cost parameters are known, while the objective is to find the values of the decision variables t_1, t_2, t_3, and t_4 which will minimize the relevant total costs per unit time. The production-inventory system presented in this chapter assumes that the decision variables and the relevant cost parameters are given, and the objective is to determine the production function and the demand function that will minimize the total costs considered in the conventional model, and, in addition, the cost of controlling the production and demand functions. These functions are controlled by controlling input parameters like man hours, resources, machine hours, advertising, etc., so as to minimize the total cost. Calculus of variations is used to solve the system.

This chapter is organized into four sections:

section I gives the nomenclature and the assumptions used in the development of the models; section II develops models for two types of single-product, single-stage systems; section III develops the constrained problem of model II; section IV develops a model for a single-product, two-stage system using the model presented in section II; section IV presents numerical examples illustrating the application of these models.

I. Nomenclature and Assumptions

The symbols used for the i^{th} stage (i=1,2), including those referred to in the Figures are defined as follows:

t_{1i} = time at which the production is stopped.

t_{2i} = time at which the inventory level is zero.

$D_{1i}(t)$ = cumulative number of units demanded from the system at time t, $0 \leq t \leq t_{1i}$.

$D_{2i}(t)$ = cumulative number of units demanded from the system at time t, $t_{1i} \leq t \leq t_{2i}$.

$P_{1i}(t)$ = cumulative number of units produced by the system at time t, $0 \leq t \leq t_{1i}$.

F_{1i} = functional[7] representing the holding cost per unit of time at time t, $0 \leq t \leq t_{2i}$.

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[7] Functional is a kind of function where the independent variable is itself a function (or curve). For example, the independent variables for the functional defined in this chapter are $\dot{P}_{1i}$, P_{1i}, $\dot{D}_{ji}$, D_{ji} and t.

F_{2i} = functional representing the cost controlling the demand and/or production function per unit time at time t, $0 \leq t \leq t_{2i}$.

The basic assumptions of the models are:

1. The relevant costs are holding cost and the cost of controlling the production and/or the demand function. This implies that shortages are not allowed.

2. Delivery time is negligible.

3. The functions $\dot{P}_{1i}(t)$, $P_{1i}(t)$, $\dot{D}_{ji}(t)$ and $D_{ji}(t)$ are piecewise smooth on the interval $0 \leq t \leq t_{2i}$ (The dot represents the first derivative of the function with respect to t).

4. All functionals are deterministic.

II. Single Stage Models

Models corresponding to two types of production inventory system are presented in this section. The two costs associated with both these systems as mentioned earlier are the cost of controlling the demand and/or the production function and the holding cost. Shortage cost is not included since shortages are not allowed.

At any particular time t, the cost of controlling the production and/or demand function per unit of time involves the costs of changing the production and/or the demand rates, the cumulative number of units produced by the system, and the cumulative number of units demanded from the system. Hence, at time t the cost of controlling these functions will depend upon the values of $\dot{P}_{1i}(t)$, $P_{1i}(t)$, $\dot{D}_{1i}(t)$, $\dot{D}_{ji}(t)$, and $D_{ji}(t)$. It will be assumed in the development of the models that the cost of controlling the production and/or demand function is a functional F_{2i}. It should be noted that $P_{1i}(t)$ and $D_{1i}(t)$ are both subject to control in the time interval $[0,t_{1i}]$, whereas only $D_{2i}(t)$ is subject to control in the time interval $[t_{1i},t_{2i}]$. This is due to the fact that the production process is shut down during this interval of time.

In general, at time t the holding cost per unit time will depend upon the values of $\dot{P}_{1i}(t)$, $P_{1i}(t)$, $\dot{D}_{ji}(t)$ and $D_{ji}(t)$. For the systems where the cost of transportation and handling of the products to and in the storage area are significant, $\dot{P}_{1i}(t)$ and $\dot{D}_{ji}(t)$ will influence the holding cost per unit time. Hence, in the development of the models considered in this chapter, it is assumed that the holding cost per unit time is a functional F_{1i}.

The two models presented in this section are:

Model I.

It is assumed that the production process starts with no units in the system and is stopped after a prescribed time interval t_{1i}, regardless of the total number of units produced in that time. As shortages are not allowed, this implies the demand at beginning of the process is also zero.

The optimum functions $P_{11}(t)$, $D_{11}(t)$, and $D_{21}(t)$ for prescribed values of t_{11} and t_{21} are to be determined by minimizing the total cost of the system. As stated earlier, the total cost includes cost of controlling $P_{11}(t)$, $D_{11}(t)$, $D_{21}(t)$, and the holding cost. Before considering the elements of the total cost, it is necessary to define the feasibility constraints of the system.

As $P_{11}(t)$, $D_{11}(t)$ and $D_{21}(t)$ are cumulative (or increasing) functions and they are always positive; therefore, they should satisfy the following conditions: For the interval $[0,t_{11}]$ the feasibility constraint is

$$\dot{P}_{11}(t) \geq \dot{D}_{11}(t) \geq 0 \qquad (4.1)$$

and for the interval $[t_{11},t_{21}]$, it is

$$\dot{D}_{21}(t) \geq 0 \ . \qquad (4.2)$$

1. Holding Cost (HC).

The holding cost for the interval $[t,t+dt]$ is given by the $F_{11} \cdot dt$ where

$$F_{11} = \begin{cases} F'_{11}[\dot{P}_{11}(t),\ P_{11}(t),\ \dot{D}_{11}(t),\ D_{11}(t),\ t] & 0 \leq t \leq t_{11} \\ F''_{11}[\dot{D}_{21}(t),\ D_{21}(t),\ t] & t_{11} \leq t \leq t_{21} \end{cases} \qquad (4.3)$$

As mentioned earlier F''_{11} is independent of $\dot{P}_{11}(t)$ and $P_{11}(t)$ because the production process is stopped at time t_{11}. Hence, the holding cost per cycle is given by

$$HC = \int_{0}^{t_{11}} F'_{11}dt + \int_{t_{11}}^{t_{21}} F''_{11}dt \quad . \tag{4.4}$$

2. Cost of Controlling $P_{11}(t)$, $D_{11}(t)$, and $D_{21}(t)$ (PDC).

Applying the same reasoning as in the holding cost case, to obtain the cost of controlling production and demand functions, PDC/cycle, this gives:

$$PDC = \int_{0}^{t_{11}} F'_{21}dt + \int_{t_{11}}^{t_{21}} F''_{21}dt \ , \tag{4.5}$$

where

$$F_{21} = \begin{cases} F'_{21}[\dot{P}_{11}(t),\ P_{11}(t),\ \dot{D}_{11}(t),\ D_{11}(t),\ t] & 0 \le t \le t_{11} \\ \\ F''_{21}[\dot{D}_{21}(t),\ D_{21}(t),\ t] & t_{11} \le t \le t_{21} \quad . \end{cases} \tag{4.6}$$

The model thus becomes:

$$\text{Minimize} \quad TC/cycle = \int_{0}^{t_{11}} (F'_{11}+F'_{21})dt + \int_{t_{11}}^{t_{21}} (F''_{11}+F''_{21})dt$$

subject to (4.1) and (4.2) and given that

$$P_{11}(0) = D_{11}(0), \tag{4.7}$$

$$P_{11}(t_{11}) = D_{21}(t_{22}),$$

while the point $[t_{11},\ P_{11}(t_{11}),\ D_{11}(t_{11})]$ can vary in the plane $t=t_{11}$. Figure 9 shows the three dimensional representation of Model I. The state of the system at any time t is represented by $(t,\ P_{11}(t),\ D_{11}(t))$.

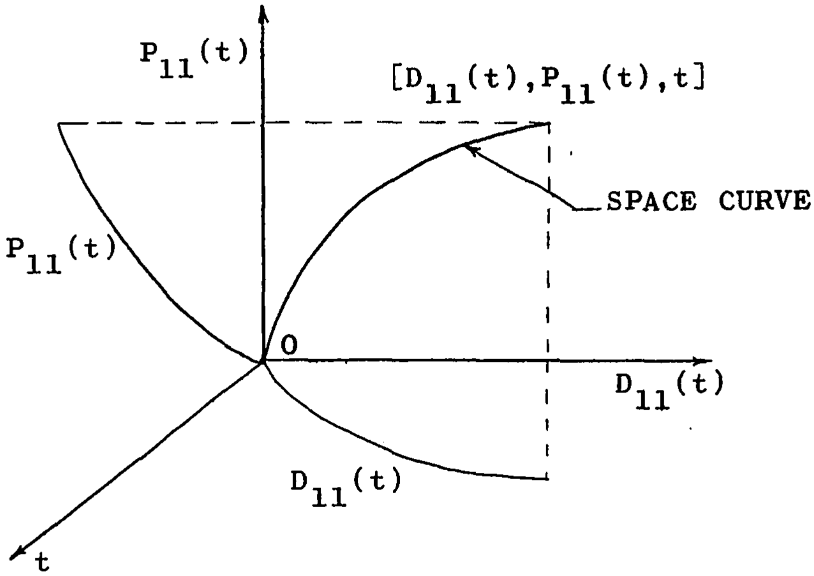

Figure 9. Graphical Representation of Model I

The optimization problem defined by (4.7) can be broken up into two problems. This is possible because $D_{21}(t)$ is independent of $P_{11}(t)$ and $D_{11}(t)$. The first problem is:

$$\text{Minimize} \qquad TC_1/\text{cycle} = \int_0^{t_{11}} (F'_{11}+F'_{21})dt$$

subject to (4.1) and given that

$$P_{11}(0) = D_{11}(0) \tag{4.8}$$

while the point $[t_{11}, P_{11}(t_{11}), D_{11}(t_{11})]$ can vary in the plane $t=t_{11}$.

This is an optimization problem in calculus of variations where one of the end points is fixed and the other is variable. The necessary and sufficient conditions for $P_{11}^*(t)$ and $D_{11}^*(t)$ to be minimizing arcs (extremals) are the following:

1. The extremals, $P_{11}^*(t)$ and $D_{11}^*(t)$ satisfy the Euler's equations

$$[F_1']_{P_{11}} - \frac{d}{dt}[(F_1')_{\dot{P}_{11}}] = 0 \qquad (4.9)$$

$$[F_1']_{D_{11}} - \frac{d}{dt}[(F_1')_{\dot{D}_{11}}] = 0 \qquad (4.10)$$

where

$$F_1' = F_{11}' + F_{21}' \quad .$$

2. The extremals satisfy the transversality conditions (one end variable):

$$[(F_1')_{\dot{P}_{11}}]_{t=t_{11}} = 0 \qquad (4.11)$$

$$[(F_1')_{\dot{D}_{11}}]_{t=t_{11}} = 0 \qquad (4.12)$$

3. The extremals should satisfy the feasibility constraints defined by (4.1) and (4.2).

4. Along the extremals, the determinants

$$L_1 = [F_1']_{\dot{P}_{11}\dot{P}_{11}} > 0 \qquad (4.13)$$

$$L_2 = \begin{vmatrix} [F_1']_{\dot{P}_{11}\dot{P}_{11}} & [F_1']_{\dot{P}_{11}\dot{D}_{11}} \\ [F_{11}']_{\dot{D}_{11}\dot{P}_{11}} & [F_{11}']_{\dot{D}_{11}\dot{D}_{11}} \end{vmatrix} > 0 \qquad (4.14)$$

(the strengthened Legendre conditions)

5. The interval $[0, t_{11}]$ contains no points conjugate[8] to the point 0 (the strengthened Jacobi conditions). (4.15)

The two Jacobi differential equations are:

$$[(F_1')_{P_{11}P_{11}} - \frac{d}{dt}(F_1')_{P_{11}\dot{P}_{11}}]u_1 - \frac{d}{dt}[(F_1')_{\dot{P}_{11}\dot{P}_{11}}\dot{u}_1] = 0$$

and (4.16)

$$u_1(0) = 0, \quad \dot{u}_1(0) = 1$$

$$[(F_1')_{D_{11}D_{11}} - \frac{d}{dt}(F_1')_{D_{11}\dot{D}_{11}}]u_2 - \frac{d}{dt}[(F_1')_{\dot{D}_{11}\dot{D}_{11}}\dot{u}_2] = 0$$

and (4.17)

$$u_2(0) = 0, \quad \dot{u}_2(0) = 1$$

6. $[F_1'[\dot{P}_{11}^*(t), P_{11}^*(t), \dot{D}_{11}^*(t), D_{11}^*(t), t]]_{t=t_{11}} = 0$

(4.18)

This condition is due to the variable boundary condition of the problem.

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[8] The point $C(\neq 0)$ is said to be conjugate to the point 0 if the equations (4.16) and (4.17) have solutions which vanish for t=C and t=0 but are not identically zero.

The optimizing functions $P_{11}^{*}(t)$ and $D_{11}^{*}(t)$ determined by using Equations (4.9) to (4.17) will be used to solve the second optimizing problem which is

$$\text{minimize} \qquad TC_2/\text{cycle} = \int_{t_{11}}^{t_{21}} (F_{11}''+F_{21}'')dt$$

subject to (4.2) and given that

$$D_{21}(t_{11}) = D_{11}^{*}(t_{11}) \tag{4.19}$$

$$D_{21}(t_{22}) = P_{11}^{*}(t_{11})$$

This is an optimization problem in calculus of variations where both the end points are fixed. Again, the necessary and sufficient conditions for $D_{21}^{*}(t)$ to be an extremal are defined by (4.2), (4.10), (4.13), and (4.17). This determines the optimizing function $D_{21}^{*}(t)$ for the second problem defined by (4.19).

Model II.

In this it will be assumed that the production process starts with no units in the systems and is stopped after it has produced K_{11} units in time t_{11}. As no shortages are allowed, this implies that:

$$P_{11}(0) = D_{11}(0) = 0 \tag{4.20}$$

and

$$P_{11}(t_{11}) = D_{21}(t_{21}) = K_{11} \tag{4.21}$$

It is further assumed that a fixed number of units, K_{21}, is demanded from the system in time t_{11} which gives:

$$D_{11}(t_{11}) = K_{21} \tag{4.22}$$

Applying the same reasoning as in the Model I, this gives:

minimize

$$TC/cycle = \int_0^{t_{11}} (F'_{11}+F'_{21})dt + \int_{t_{11}}^{t_{21}} (F''_{11}+F''_{21})dt$$

given the boundary conditions defined by (4.20), (4.21) and (4.22), respectively

$$P_{11}(0) = D_{11}(0) = 0 \ , \tag{4.23}$$

$$P_{11}(t_{11}) = D_{21}(t_{21}) = K_{11} \ ,$$

$$D_{11}(t_{11}) = K_{21} \quad .$$

This problem of optimization can again be broken up into two parts each of which is a problem in calculus of variations with fixed end conditions. The two problems are as follows:

1. minimize $\quad TC_1/cycle = \int_0^{t_{11}} (F'_{11}+F'_{21})dt$

given, $\quad P_{11}(0) = D_{11}(0) = 0 \tag{4.24}$

and

$$P_{11}(t_{11}) = K_{11}, \ D_{11}(t_{11}) = K_{21} \ ;$$

2. minimize $\quad TC_2/\text{cycle} = \int_{t_{11}}^{t_{21}} (F''_{11}+F''_{21})dt$

given, $\quad D_{21}(t_{11}) = K_{21}$ (4.25)

and

$$D_{21}(t_{21}) = K_{11} \quad .$$

The procedure to optimize the above problems is exactly similar to one described to solve the second part of Model I.

III. The Constrained Problem

Model III.

The above models may be more realistic if certain constraints describing functional dependence between the decision variables are specified. Let the constraints be defined as

$$G'_{11}[\dot{P}_{11}(t),\ P_{11}(t),\ \dot{D}_{11}(t),\ D_{11}(t),\ t] \le 0 \qquad (4.26)$$

and

$$G''_{11}[\dot{D}_{21}(t),\ D_{21}(t),\ t] \le 0 \qquad (4.27)$$

In an actual system the above differential equations may represent the limitation of the system to control the demand and production function due to limitations of the available resources. When these constraints are active, the problem becomes more complicated, and to solve the system, the Lagrange method of multipliers has to be used. This consists of constructing auxiliary functions

$$TC_1^* = \int_0^{t_{11}} [F_{11}' + F_{21}' + g_1' G_{11}']dt \tag{4.28}$$

for the first part of the problem, and

$$TC_2^* = \int_{t_{11}}^{t_{22}} [F_{11}'' + F_{21}'' + g_1'' G_{11}'']dt \tag{4.29}$$

for the second part of the problem where $g_1'(t)$ and $g_1''(t)$ are the Lagrange multipliers (functions of t).

Due to constraints (4.26) and (4.27), only three boundary conditions need to be specified as compared to five in the case of Model II. The above is known as the Lagrange problem with nonholonomic side conditions. The first part of the problem is to minimize (4.28) subject to (4.26) with boundary conditions

$$P_{11}(0) = D_{11}(0) = 0$$

and

$$D_{11}(t_{11}) = K_{21} \text{ or } P_{11}(t_{11}) = K_{11}$$

and the second part of the problem is to minimize (4.29) subject to (4.27) with no boundary conditions. The two additional boundary conditions are:

$$[(F_{11}' + F_{21}' + g_1' G_{11}')_{\dot{D}_{11}}]_{t=t_{11}} = 0 \tag{4.29}$$

and

$$[(F''_{11} + F''_{21} + g''_1 G''_{11})_{\dot{D}_{21}}]_{t=t_{21}} = 0 \qquad (4.30)$$

The optimizing procedure for the above problem is explained with the help of a numerical example given in section IV.

IV. Two Stage Model

The two-stage production system is depicted schematically in Figure 10.

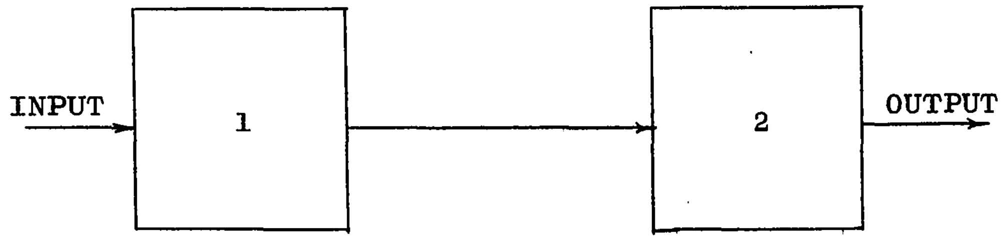

Figure 10. Schematic Representation of the Two Stage Model

This consists of two serial stages where the product moves between the stages in a serial fashion. The production cycle starts at Stage 1. The output from Stage 1 is then fed into Stage 2. Hence, the input for Stage 2 acts as a demand for Stage 1. The interrelationship of the two stages is schematically described in Figure 11:

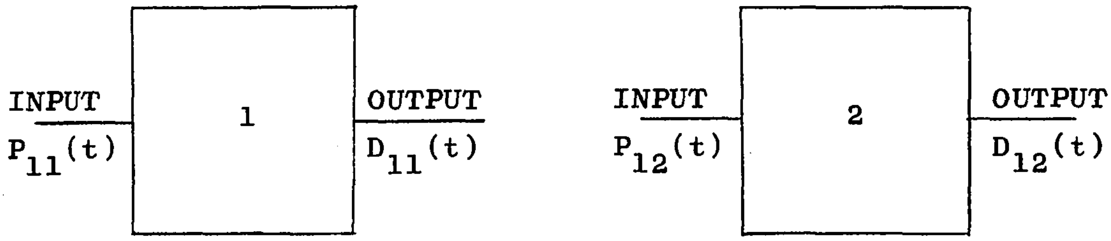

Figure 11. Schematic Representation of the Two Stage Model as Two Single Stage Models

where

$$D_{11}(t) \equiv P_{12}(t) \ .$$

Model IV.

In this model, Model II is used with additional assumptions to consider the two stage model. The additional assumptions are due to the relationships that must exist as a result of coupling of the two stages. The additional assumptions are:

1. The first stage repeats with a time lag of t_L units and the second stage has no time lag between cycles.

2. All stages start the production cycles at the same time.

The above assumptions along with basic assumptions made earlier imply that:

$$P_{1i}(t_{1i}) = D_{2i}(t_{2i}) = P_{2i+1}(t_{1i+1}) = K_{11} \quad i=1,2 \quad (4.32)$$

$$t_{21} + t_L = t_{22} \tag{4.33}$$

$$t_{21} = t_{12} \tag{4.34}$$

Figure 12 shows two cycles of the inventory level variation for the two stages of the above system satisfying the above relationships.

Applying the same reasoning as in Model II to both the stages of the above model separately, beginning from first stage and going forward, gives:

1. First Stage

$$TC^1/\text{cycle} = \int_0^{t_{11}} (F'_{11} + F'_{21})dt + \int_{t_{11}}^{t_{21}} (F''_{11} + F''_{21})dt$$

with given fixed end conditions similar to those in Model II as:

$$P_{11}(0) = D_{11}(0) = 0$$

$$P_{11}(t_{11}) = D_{21}(t_{21}) = K_{11} ,$$

$$D_{11}(t_{11}) = K_{21} .$$

2. Second Stage

It is evident from Figure 12 that the cost of controlling the production function of the second stage has already been incorporated in the first stage; therefore, the holding cost and the control cost for the second stage will be certain functional of $\dot{D}_{12}(t)$, $D_{12}(t)$ and t. It

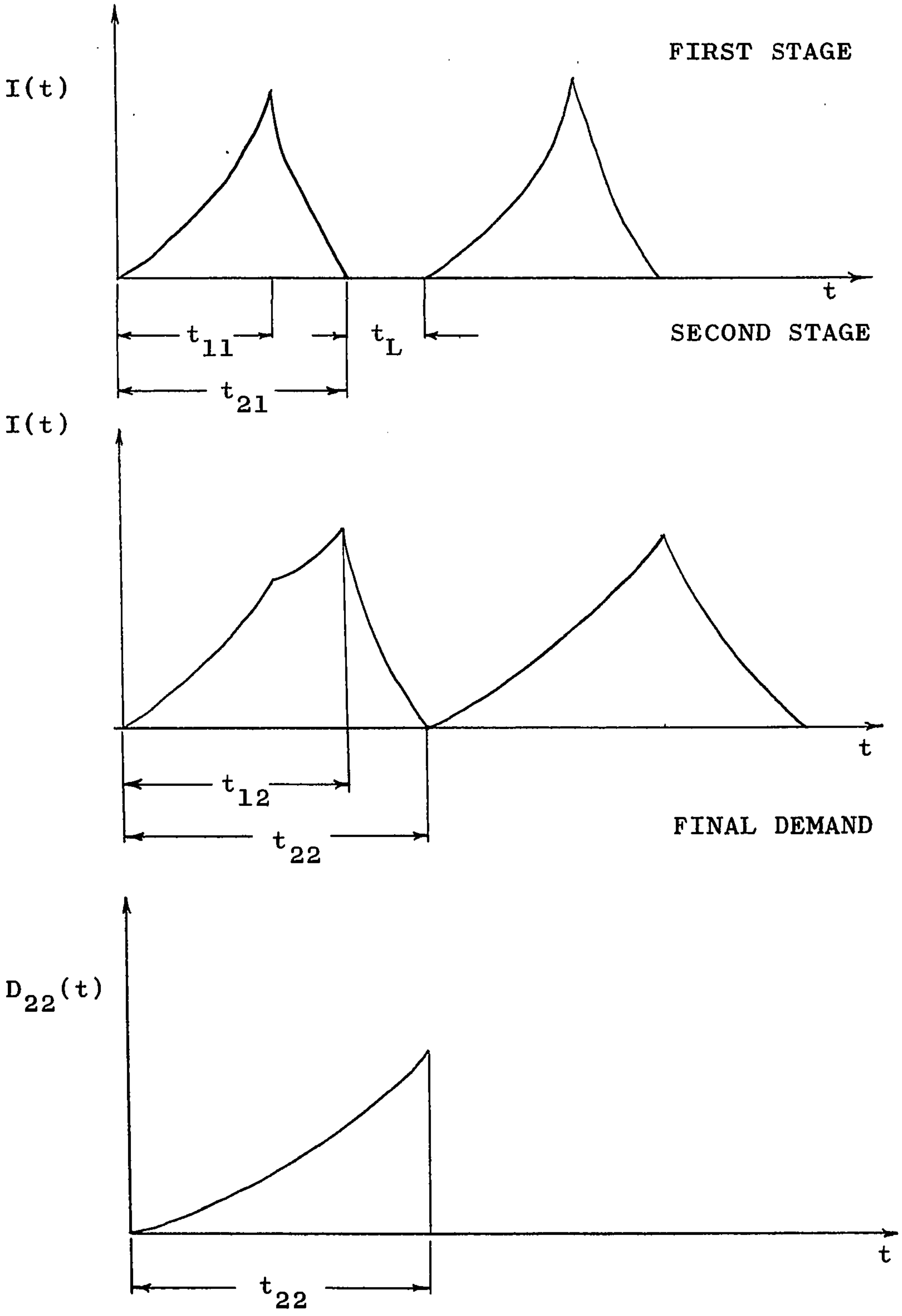

Figure 12. Inventory Variation for Two Cycles of the Two Stage Model With Time Lag

should be noted that $D_{12}(t)$ is continuous over $[0,t_{22}]$; therefore,

$$F'_{12}[\dot{D}_{12}(t),\ D_{12}(t),\ t] \equiv F'_{22}[\dot{D}_{22}(t),\ D_{22}(t),\ t]$$

and

$$F''_{12}[\dot{D}_{12}(t),\ D_{12}(t),\ t] \equiv F''_{22}[\dot{D}_{22}(t),\ D_{22}(t),\ t] \quad .$$

Hence, the model corresponding to this stage is:

$$\text{minimize} \qquad TC^2/\text{cycle} = \int_0^{t_{22}} (F'_{12} + F''_{12})dt$$

with given fixed end conditions defined by (4.32) are

$$D_{22}(0) = 0 \ ,$$

and

$$D_{22}(t_{22}) = K_{11} \quad .$$

The procedure to optimize each of the above two problems is exactly similar to the one described to solve Model II. Also, if each of the above two stages are subject to constraints, then the Lagrange multipliers method outlined in section III can be used to optimize each stage and hence find the optimal solution of Model IV.

V. Numerical Examples

This section presents examples to illustrate the optimizing procedure for the models developed in sections II, III, and IV. In general, the following steps will be followed to solve the examples:

1. Solving Euler's Equations (4.9) and (4.10) to find the extremals.

2. Determining the constants of Integration from the given boundary conditions.

3. Checking if the feasibility constraints (4.1) and (4.2) are satisfied by the extremals.

4. Checking for the Strengthened Legendre's conditions (4.13) and (4.14).

5. Checking for the Strengthened Jacobi conditions (4.15) by solving (4.16) and (4.17).

6. Checking if (4.18) is satisfied for problems with one variable boundary condition.

Example I.

Consider an inventory situation described in Model I with the following data specified:

$$F'_{11} = \dot{P}^2_{11}(t) - 5P_{11}(t) + 2\dot{D}^2_{11}(t) - 4D_{11}(t) + 46$$

$$F'_{21} = 2P_{11}(t) + 2D_{11}(t)$$

$$F''_{11} = 2\dot{D}_{21}{}^2(t) - 4D_{21}(t)$$

$$F''_{21} = 2D_{21}(t) + 9$$

$$t_{11} = 4 \text{ and } t_{21} = 6$$

The objective is to find the extremals $P^*_{11}(t)$,

$D^*_{11}(t)$, and $D^*_{21}(t)$ by solving the system defined by (4.7). The procedure described above will be used to solve the examples.

As mentioned in Model I the problem is broken up into two problems similar to (4.8) and (4.19). The solution of the first problem (4.8) is as follows:

1. Euler's Equation (4.9) gives

$$[F_1']_{P_{11}} - \frac{d}{dt}[(F_1')_{\dot{P}_{11}}] = 0 ,$$

where

$$F_1' = F_{11}' + F_{21}' , \tag{4.35}$$

which gives the differential equation:

$$-3 - \frac{d}{dt}[2\dot{P}_{11}(t)] = 0 . \tag{4.36}$$

Solving (4.35), we get

$$P^*_{11}(t) = -\frac{3}{2}\frac{t^2}{2} + C_1 t + C_2 ,$$

where C_1 and C_2 are constants of integration.

Similarly, by solving (4.10), we get:

$$D^*_{11}(t) = \frac{-2}{4}\frac{t^2}{2} + C_1' t + C_2' .$$

2. The boundary conditions defined by (4.7), (4.11) and (4.12), give:

$$P^*_{11}(0) = D^*_{11}(0) = 0 .$$

Therefore,

$$C_2 = 0 \quad \text{and} \quad C_2' = 0$$

and

$$[(F_{11}' + F_{21}')_{\dot{P}_{11}}]_{t=4} = 0 \quad .$$

Therefore,

$$[2\,\dot{P}_{11}(t)]_{t=4} = 0 \quad .$$

Hence,

$$C_1 = +\,6 \quad .$$

Similarly, by using (4.12), we get

$$C_1' = 2 \quad .$$

Therefore,

$$P_{11}^*(t) = \frac{-3t^2}{4} + 6t \;, \tag{4.37}$$

and

$$D_{11}^*(t) = \frac{-t^2}{4} + 2t \;. \tag{4.38}$$

3. The feasibility constraints defined by (4.1) are:

$$\dot{P}_{11}^*(t) \geq \dot{D}_{11}^*(t) \geq 0 \qquad 0 \leq t \leq 4 \;,$$

that is,

$$\frac{-3t}{2} + 6 \geq -\,\frac{t}{2} + 2 \geq 0 \qquad 0 \leq t \leq 4 \;.$$

Hence, the extremals satisfy the feasibility constraints.

4. The strengthened Legendre's conditions defined by

(4.13) and (4.14) are also satisfied because

$$L_1 = 2$$

and

$$L_2 = \begin{vmatrix} 2 & 0 \\ 0 & 4 \end{vmatrix} = 8 \quad .$$

5. The Jacobi differential equation for this system, (4.16), gives:

$$\frac{d}{dt}[2\dot{u}_1] = 0 \quad .$$

Therefore,

$$u_1(t) = C_1''t + C_2'' \quad ,$$

where C_1'' and C_2'' are constants of integration. By using initial conditions (4.16), we get:

$$u_1(t) = t \quad .$$

Therefore, there is no point in the interval [0,4] that is conjugate to 0. Similarly, it can be shown that the strengthened Jacobi condition is also satisfied by the extremal $D_{11}^*(t)$.

6. Finally, as the above problem has one variable boundary condition; therefore, (4.18) should be satisfied, that is:

$$F_1'[\dot{P}_{11}^*(t),\ P_{11}^*(t),\ \dot{D}_{11}^*(t),\ D_{11}^*(t)]_{t_{11}=4} \neq 0 \quad . \qquad (4.39)$$

Substituting the optimal functions in (4.35), we get

$$[F_1']_{t_{11}=4} = -3(12) - 2(4) + 46$$

$$= 2 \quad .$$

Hence, the above condition is satisfied. Therefore, the extremals for the first problem are:

$$P_{11}^*(t) = \frac{-3t^2}{4} + 6t \quad ,$$

$$D_{11}^*(t) = \frac{-t^2}{4} + 2t \quad .$$

For the second problem the functional F_1'' is given by

$$F_1'' = F_{11}'' + F_{21}'' \; .$$

Therefore,

$$F_1'' = 2\,\dot{D}_{21}^{\,2}(t) - 2\,D_{21}(t) + 9$$

Following the procedure exactly similar to the one followed in the first problem, we get:

$$D_{21}^*(t) = \frac{-t^2}{4} + C_1't + C_2' \quad . \qquad (4.40)$$

The boundary conditions for this problem are:

$$D_{21}^*(6) = P_{11}^*(4) = 12 \quad , \qquad (4.41)$$

and

$$D_{21}^*(4) = D_{11}^*(4) = 4 \quad . \qquad (4.42)$$

Substituting (4.41) and (4.42) in (4.40), we get:

$$D^*_{21}(4) = -4 + c'_1 4 + c'_2 = 4 \qquad (4.43)$$

and

$$D^*_{21}(6) = -9 + c'_1 6 + c'_2 = 12 \quad . \qquad (4.44)$$

Solving (4.43) and (4.44), the extremal for the second problem is

$$D^*_{21}(t) = \frac{-t^2}{4} + 6.5t - 18 \qquad (4.45)$$

Hence, the optimal decision variables are given by (4.37), (4.38), and (4.45) are shown graphically by Figure 13. Furthermore, the above solution also suggests that values of $P^*_{11}(t)$ and $D^*_{11}(t)$ at $t_{11}=4$ are 12 and 4, respectively.

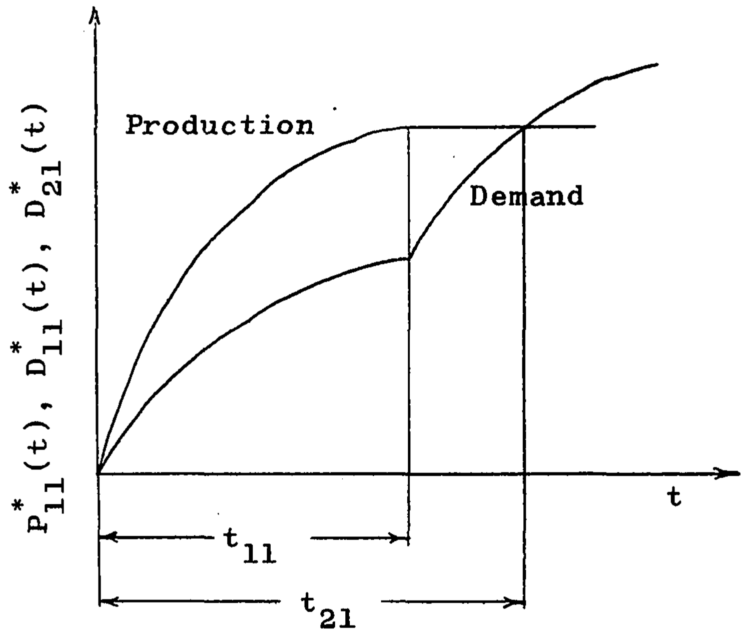

Figure 13. Graphical Representation of the Optimal Solution of Example 1

Example II.

Consider an inventory situation described in Model II with the following data specified:

$$
\begin{aligned}
F_{11}' &= a_1^2 \dot{P}_{11}^2(t) + a_2^2 \dot{D}_{11}^2(t) + a_3 \dot{D}_{11}(t) \dot{P}_{11}(t) \\
F_{21}' &= a_4 \dot{P}_{11}(t) + a_5 \dot{D}_{11}(t) \\
F_{11}'' &= a_2^2 \dot{D}_{21}^2(t) \\
F_{21}'' &= a_5 \dot{D}_{21}(t) ,
\end{aligned}
\tag{4.46}
$$

where a_i's, $i=1,2,3,4,5$, are all known constants.

Again, the objective is to find the extremals $P_{11}^*(t)$, $D_{11}^*(t)$ and $D_{21}^*(t)$ by solving the system defined by (4.23). The solution of the first problem for the above system defined by (4.24) is as follows:

1. Euler's Equation (4.9) gives

$$[F_1']_{P_{11}} - \frac{d}{dt} [(F_1')_{\dot{P}_{11}}] = 0 , \tag{4.47}$$

where

$$F_1' = F_{11}' + F_{21}' .$$

Therefore, (4.47) gives:

$$\frac{d}{dt} [2a_1^2 \dot{P}_{11}(t)] = 0 . \tag{4.48}$$

Hence, solving (4.48), we get:

$$P^*_{11}(t) = C_1 t + C_2 \ , \tag{4.49}$$

where C_1 and C_2 are constants of integration.

Similarly, by solving (4.10), we get

$$D^*_{11}(t) = C_1' t + C_2' \tag{4.50}$$

2. The boundary conditions defined by (4.24) give

$$P^*_{11}(0) = D^*_{11}(0) = 0 \quad .$$

Therefore,

$$C_2 = 0 \text{ and } C_2' = 0 \quad .$$

Also,

$$P_{11}(t_{11}) = K_{11},\ D_{11}(t_{11}) = K_{21} \ , \tag{4.51}$$

and

$$K_{11} > K_{21}$$

Therefore, substituting (4.51) in (4.49) and (4.50), we get

$$C_1 = \frac{K_{11}}{t_{11}},\ C_1' = \frac{K_{21}}{t_{11}} \quad .$$

Therefore,

$$P^*_{11}(t) = \frac{K_{11}}{t_{11}} t \quad ,$$

and (4.52)

$$D^*_{11}(t) = \frac{K_{21}}{t_{11}} t \quad .$$

3. The feasibility constraints defined by (4.1) are satisfied because

$$\dot{P}^*_{11}(t) \geq \dot{D}^*_{11}(t) \geq 0 \qquad 0 \leq t \leq t_{11}$$

along the extremals (4.46).

4. The strengthened Legendre's conditions defined by (4.13) and (4.14) are also satisfied because:

$$L_1 = 2a_1{}^2$$

and

$$L_2 = \begin{vmatrix} 2a_1{}^2 & 0 \\ 0 & 2a_2{}^2 \end{vmatrix} = 4a_1{}^2 a_2{}^2 \quad .$$

5. The Jacobi differential Equation (4.16) for this system gives:

$$\frac{d}{dt}[2\dot{u}_1] = 0 \quad .$$

This is the same as that for the first example. Hence, by the reasoning similar to that in the previous example, it can be shown that both the extremals satisfy the strength Jacobi condition (4.15).

Hence, the extremals for the first problem are

$$P^*_{11}(t) = \frac{K_{11}}{t_{11}} t \quad , \tag{4.53}$$

and

$$D^*_{11}(t) = \frac{K_{21}}{t_{11}} t \quad . \tag{4.54}$$

For the second problem, the functional F_1'' is given by

$$F_1'' = a_2\dot{D}_{21}^{\,2}(t) + a_5\dot{D}_{21}(t) \quad .$$

Following the procedure exactly similar to the one followed in the first problem, we get

$$D_{21}^*(t) = C_1 t + C_2 \quad . \tag{4.55}$$

The boundary conditions for this problem are:

$$D_{21}^*(t_{21}) = P_{11}^*(t_{11}) = K_{11} \quad , \tag{4.56}$$

and

$$D_{21}^*(t_{11}) = D_{11}^*(t_{11}) = K_{21} \quad . \tag{4.57}$$

substituting (4.56) and (4.57) in (4.55), we get

$$C_1 t_{11} + C_2 = K_{21} \quad ,$$

and

$$C_1 t_{21} + C_2 = K_{11} \quad .$$

solving the simultaneous equation, we get:

$$C_1 = \frac{K_{11} - K_{21}}{t_{21} - t_{11}} \quad , \tag{4.58}$$

$$C_2 = \frac{K_{21}t_{21} - K_{11}t_{11}}{t_{21} - t_{11}} \quad . \tag{4.59}$$

Therefore,

$$D_{21}^*(t) = \left(\frac{K_{11}-K_{21}}{t_{21}-t_{11}}\right)t + \left(\frac{K_{21}t_{21}-K_{11}t_{11}}{t_{21}-t_{11}}\right) \quad . \tag{4.60}$$

It should be noted that if

$$K_{21}t_{21} = K_{11}t_{11}$$

the extremal $D^*_{21}(t)$ for the second problem is the same as $D^*_{11}(t)$.

Hence, the optimum demand and production functions are given by (4.53), (4.54) and (4.60) and shown graphically in Figure 14. It should be noted that in the above system the solution is independent of the values of a_i's. Therefore, a system in which the relevant costs are of the form described in (4.46), then the optimum condition is to maintain uniform rate of demand and production in the system.

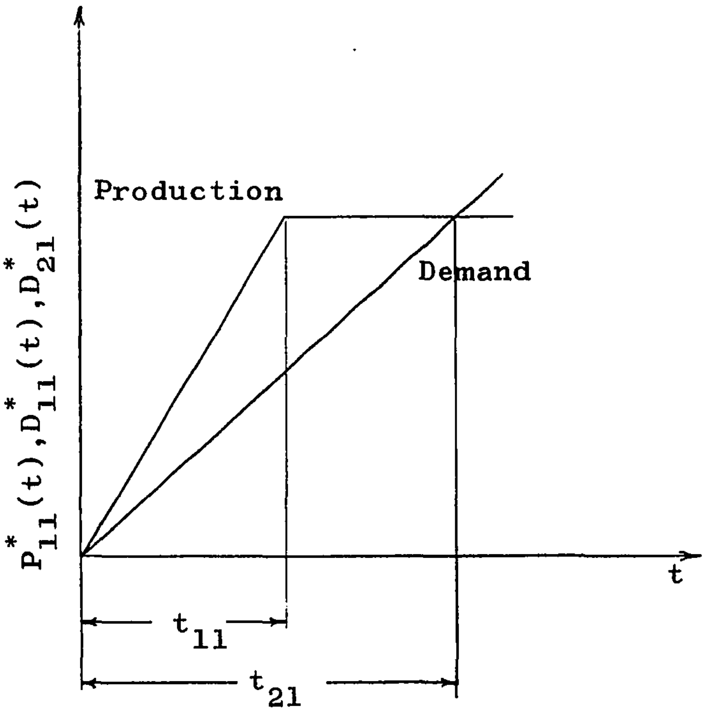

Figure 14. Graphical Representation of the Optimal Solution of Example 2

Example III.

Consider a constraint problem corresponding to the inventory situation described in Model III. The constraint is in the form of a differential equation describing the capacity of the system during the production period to control the rate of demand and production. The functionals and the constraint for the above system are defined as follows:

$$
\begin{aligned}
F'_{11} &= 5\dot{P}_{11}^{\,2}(t) + 5\dot{D}_{11}^{\,2}(t) \\
F'_{21} &= 5\dot{P}_{11}(t) + 2\dot{D}_{11}(t) \\
F''_{11} &= 5\dot{D}_{21}^{\,2}(t) \\
F''_{21} &= 5\dot{D}_{21}(t)
\end{aligned}
\tag{4.61}
$$

and the constraint is

$$
G'_{11} = 2\dot{P}_{11}^{\,2}(t) + \dot{D}_{11}^{\,2}(t) - 225 = 0 \qquad 0 \le t \le t_{11}, \tag{4.62}
$$

where 225 is the maximum man hours available at time t. Also, the fixed boundary conditions are

$$
\begin{aligned}
P_{11}(0) &= D_{11}(0) = 0 \, , \\
P_{11}(2) &= D_{21}(2.5) = 20 \, .
\end{aligned}
\tag{4.63}
$$

The objective is to solve the system defined by (4.23) subject to above specified constraint. The solving of the first part of the problem consists of constructing the auxiliary function F_1^* defined as

$$F_1^* = F_{11}' + F_{21}' + g_1' G_{11}' \tag{4.64}$$

where $g_1' (= g_1'(t))$ is the Lagrangian multiplier.
Substituting for F_{11}', F_{21}', and G_{11}' in (4.63), we get:

$$F_1^* = 5\dot{P}_{11}^2(t) + 5\dot{D}_{11}^2(t) + 5\dot{P}_{11}(t) + 2\dot{D}_{11}(t)$$
$$+ \; g_1'(2\dot{P}_{11}^2(t) + \dot{D}_{21}^2(t) - 225) \tag{4.65}$$

Now the system is solved in the same manner as described in Example II.

1. Euler's Equations (4.9) and (4.10) give

$$\frac{d}{dt}[10\,\dot{P}_{11}(t) + 4\,g_1'(t)\,\dot{P}_{11}(t)] = 0\;, \tag{4.66}$$

$$\frac{d}{dt}[10\,\dot{D}_{11}(t) + 2\,g_1'(t)\,\dot{D}_{11}(t)] = 0\;, \tag{4.67}$$

so

$$[10 + 4\,g_1'(t)]\,\dot{P}_{11}(t) = C_1 \tag{4.68}$$

and

$$[10 + 2\,g_1'(t)]\,\dot{D}_{11}(t) = C_2\;, \tag{4.69}$$

where C_1 and C_2 are constants of integration in equations (4.68) and (4.69).

$$2\dot{P}_{11}^2(t) + \dot{D}_{11}^2(t) = \left[\frac{C_1}{10+4g_1'(t)}\right]^2 + \left[\frac{C_2}{10+2g_1'(t)}\right]^2 \;. \tag{4.70}$$

Therefore, comparing (4.70) with (4.62) gives that $g_1'(t)$ is a constant. Hence, (4.68) gives:

$$\dot{P}_{11}(t) = C_1' \quad .$$

Therefore,

$$P_{11}^*(t) = C_1' t + C_3' \quad . \tag{4.71}$$

Similarly,

$$D_{11}^*(t) = C_1'' t + C_3'' \quad . \tag{4.72}$$

Substituting the boundary conditions (4.63), we get:

$$P_{11}^*(t) = 10\ t \ , \tag{4.73}$$

and

$$D_{11}^*(t) = C_1''\ t \ . \tag{4.74}$$

The additional boundary condition to determine C_1'' as given by (4.30) is:

$$[(F_1^*)_{\dot{D}_{11}}]_{t=2} = 0 \quad .$$

Therefore,

$$[10\ \dot{D}_{11}^* + 2 + 2\ g_1'\ \dot{D}_{11}^*]_{t=2} = 0 \ .$$

Hence,

$$10\ C_1'' + 2 + 2\ g_1'\ C_1'' = 0 \ ,$$

which gives

$$C_1'' = \frac{-2}{10+g_1'} \ . \tag{4.75}$$

Now, substituting (4.73) and (4.74) in (4.62), we get:

$$200 + \left(\frac{2}{10+g_1'}\right)^2 - 225 = 0 \ ,$$

which gives

$$g_1' = -\frac{52}{5} \quad .$$

Hence, (4.75) gives:

$$C_1'' = 5 \quad .$$

Therefore,

$$D_{11}^*(t) = 5t \quad . \tag{4.76}$$

Thus, by the procedure exactly similar to that of Example 2, it can be shown that $P_{11}^*(t)$ and $D_{11}^*(t)$ satisfy the feasibility constraints, strengthened Legendre's condition and strengthened Jacobi's condition. Hence, $P_{11}^*(t)$ and $D_{11}^*(t)$ give the optimum solution for the first optimization problem of the above system.

For the second problem,

$$F_1'' = 5\, \dot{D}_{21}{}^2(t) + 5\, \dot{D}_2(t) \quad .$$

Therefore, Euler's equation gives

$$D_{21}(t) = 0 \quad .$$

Hence,

$$D_{21}(t) = C_1 t + C_2 \quad . \tag{4.77}$$

Using the boundary conditions (4.63), and (4.73) give:

$$D_{21}^*(t) = 8\ t \quad . \tag{4.78}$$

Hence, the optimum demand and production functions to minimize the total cost of the above system are given by (4.73), (4.76), and (4.78), and are shown in Figure 15.

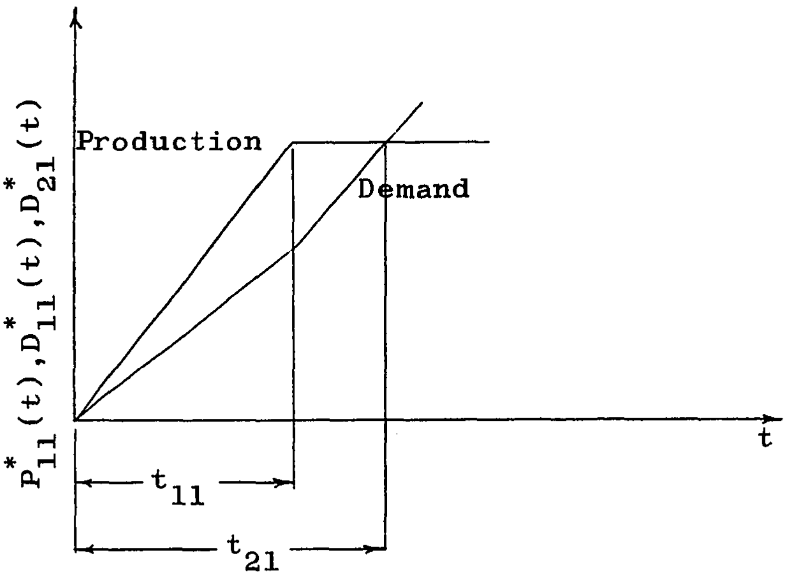

Figure 15. Graphical Representation of the Optimal Solution of Example 3

The optimizing procedure illustrated in the above three examples can be applied to each stage of the two stage model developed in section III and, consequently, no example will be given for this case.

CHAPTER V

CONCLUSIONS AND RECOMMENDATIONS

Conclusions

In this study the formulation and optimization of two kinds of production inventory problems are presented:

The first (or conventional) problem assumes that the demand function, the production function, and the relevant cost parameters are given and are non-linear in time, while the objective is to find the values of the decision variables that will minimize the relevant total costs per unit time. This problem has received very little attention in the literature.

The second (or non-conventional) problem is an outcome of looking at the production inventory system in a non-conventional manner. It assumes that the relevant cost parameters are given while the objective is to determine the production function and/or the demand function which will minimize the total costs considered in the conventional model, and, in addition, the cost of controlling the production and demand functions. This is a new approach of formulating a production inventory system.

The general model presented in Chapter II appears to be a powerful tool to analyze conventional inventory situations having the following properties: (1) Non-linear demand function, (2) Non-linear production function, (3) Cost parameters varying with time, (4) Composite demand, and (5) Composite production. This is possible because of the following characteristics of the general model: (1) Development of the model is independent of the geometry of the inventory cycle. (2) Time is chosen as a decision variable; and (3) It is formulated as constrained problem.

The application of the above models to two types of non-linear inventory situations showed that a non-linear production inventory system cannot always be represented by a linear system and, hence, reveals the significance of the study of non-linear production inventory systems. This result was a consequence of the substantial difference observed in the optimal solution of the actual and linearly approximated models solved in Chapter III.

The new approach of analyzing a production inventory system developed in Chapter IV illustrates the power of using the calculus of variations. It reflects a very high potential of application to many inventory situations that exist in actual practice. Three examples given in Chapter IV demonstrate the application of the new approach.

In the first example, the total cost of the system, is dependent on $\dot{P}_{11}^{2}(t)$, $\dot{D}_{11}^{2}(t)$, $P_{11}(t)$, and $D_{11}(t)$, and the optimal solution shows that: (1) $\dot{P}^*_{11}$ and $\dot{D}^*_{11}(t)$ are both decreasing functions in the interval $[0,t_{11}]$, and (2) $\dot{P}^*_{11}(t_{11})$ and $\dot{D}^*_{11}(t_{11})$ are both equal to zero. (3) $\dot{D}^*_{21}(t)$ is a decreasing function in the interval $[t_{11},t_{21}]$. The above results indicate that the demand and the production rates should decrease as the shut down time approaches.

The optimal solution of the second example indicates that if the system cost is a function of demand and production rates; then, regardless of the system parameters, a_1, a_2, a_3, and a_5, a uniform production and demand rate should be maintained to minimize the total costs. When, to the above inventory situation, a constraint specifying the total capacity (energy) of the system to control $P_{11}(t)$ and $D_{11}(t)$ is added, then, it is also found as shown in Example 3, that uniform production and demand rate should be maintained in the system to minimize the total cost.

Recommendations for Future Work

The present investigation focused on certain aspects of the conventional and non-conventional production inventory system. The possible extensions of the study include consideration of:

Conventional System:

1. Multi-item system with different demand and production functions for each item.
2. Probabilistic demand and production.

3. To develop a method to approximate the non-linear system with a linear system so that the new system is equivalent to the original system.

Non-Conventional System:

1. Multi-stage multi-item production system.
2. To allow shortages.
3. Determination of the functionals defining the cost of control and storage.

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APPENDIX A

NECESSARY AND SUFFICIENT CONDITION FOR A FUNCTION TO BE CONVEX (OR CONCAVE) AND HAVE A MINIMUM (OR MAXIMUM)

1. The necessary and sufficient conditions for the unconstrained function $F(t_1, t_{22}, t_{33}, t_{44})$ to be convex are that the four principal minors of the (4x4) Hessian matrix

$$H = \begin{bmatrix} F_{1111} & F_{1342} & F_{1423} & F_{1234} \\ F_{2341} & F_{2222} & F_{2413} & F_{2314} \\ F_{3421} & F_{3412} & F_{3333} & F_{3124} \\ F_{4231} & F_{4132} & F_{4123} & F_{4444} \end{bmatrix} \tag{A.1}$$

are all positive over the feasible region. This implies that the determinants

$$F_{1111} > 0 \tag{A.2}$$

$$A = \begin{vmatrix} F_{1111} & F_{1324} \\ F_{2341} & F_{2222} \end{vmatrix} > 0 \tag{A.3}$$

$$B = \begin{vmatrix} F_{1111} & F_{1342} & F_{1423} \\ F_{2341} & F_{2222} & F_{2413} \\ F_{3421} & F_{3412} & F_{3333} \end{vmatrix} > 0 \tag{A.4}$$

and

$$C = \begin{vmatrix} F_{1111} & F_{1342} & F_{1423} & F_{1234} \\ F_{2341} & F_{2222} & F_{2413} & F_{2314} \\ F_{3421} & F_{3412} & F_{3333} & F_{3124} \\ F_{4231} & F_{4132} & F_{4123} & F_{4444} \end{vmatrix} > 0$$

The corresponding condition for $F(t_1, t_{22}, t_{33}, t_{44})$ to be concave over the feasible region are

$$F_{1111} < 0, \; A > 0, \; B < 0 \text{ and } C > 0$$

2. The necessary and sufficient conditions as stated by Kuhn and Tucker for the function F under the constraints G' and H' to have a minimum at t_1^*, t_{22}^*, t_{33}^*, and t_{44}^* are

a) $F\,(t_1, t_{22}, t_{33}, t_{44})$ is regular and convex.

b) $G'(t_1, t_{22})$ and $H'(t_1, t_{22}, t_{33}, t_{44})$ are regular and convex.

c) If $t_{jj}^* > 0$, then $\frac{F}{t_{jj}} - u_1 \frac{G'}{t_{jj}} - u_2 \frac{H'}{t_{jj}} = 0$

at $t_{jj} = t_{jj}^*$ for $j = 1,2,3,4$ where $t_{11} = t_1$

and u_1, u_2 are the Lagrange multipliers.

d) If $t_{jj}^* = 0$, then $\frac{F}{t_{jj}} - u_1 \frac{G'}{t_{jj}} - u_2 \frac{H'}{t_{jj}} \geqslant 0$

at $t_{jj} = t_{jj}^*$ for $j = 1,2,3,4$.

e) If $u_1 > 0$, then $G'(t_1^*, t_{22}^*) = 0$ and if $u_2 > 0$, then $H'(t_1^*, t_{22}^*, t_{33}^*, t_{44}^*) = 0$

f) If $u_1 = 0$, then $G'(t_1^*, t_{22}^*) \geqslant 0$, and if $u_2 = 0$, then $H'(t_1^*, t_{22}^*, t_{33}^*, t_{44}^*) \geqslant 0$

g) $t_{jj}^* \geqslant 0 \qquad j = 1,2,3,4$

h) $u_i \geqslant 0 \qquad i = 1,2$

APPENDIX B

FLOW DIAGRAM FOR MINIMIZATION OF TOTAL COST PER UNIT TIME