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MATHEMATICAL MODEL OF FLOW OF FLUID IN POROUS MEDIA

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ABSTRACT

In this dissertation, we investigate the flow of fluid in porous media. The mathematical drainage models are developed in a domain that is partitioned into a saturated portion and a unsaturated portion. The boundary value problem of elliptic equation and parabolic equation on the time dependent domain which represents the model of pressure in the saturated domain were posted with given inflow and outflow measurements. By change of variables, we formulate the problem with time dependent coefficients on a fixed domain at every time t. We prove the existence and uniqueness of the weak solution of the problem by the variational method. The continuity of the solution with respect to permeability was also discussed. In order to estimate the permeability of existing pavement system, we pose a minimization problem to find the permeability k minimizing the error between calculated outflow and observed outflow. Furthermore, the numerical analysis and computer simulation results are presented for both elliptic and parabolic problems.

CHAPTER ONE

Introduction

We consider a rectangular solid region Ω that is partitioned into two subdomains U and W that represent unsaturated and saturated domains, see Figure 1. It is further assumed unrealistically that there is no fluid in the domain U and that the domain W has saturation 1. Fluid is assumed to enter Ω through the upper boundary and essentially to pass immediately into W. Fluid may leave through a portion of a boundary. As a result of the inflow and outflow of fluid through Ω, the domains U and W are time dependent. We determine a differential equation that relates these domains to the measured inflow and outflow. Within W we assume that fluid flows according to Darcy's law. Our models consist of (1) an initial value problem governing the domain and (2) a partial differential equation governing flow within W. The partial differential equations are either elliptic or parabolic depending on the inclusion of the compressibility term. In either case a change of variables determines boundary value problems on a fixed domain but with time dependent coefficients.

To validate our model we consider roadbed drainage field data. Our approach is to pose a parameter identification problem to estimate permeability so as to fit the observed inflow and outflow data. To carry out this program requires an analysis of properties of the partial differential equations, numerical solution of these equations, and an analysis of minimization problems associated with the identification problem.

The theory of boundary value problem of elliptic equations and parabolic equations with initial condition can be found in Hormander, Lions and Magenes and Ladyzenskaya. The numerical solution of resulting system by finite difference technique is studied in [3]. In the book by Chavent and Jaffre [2], the finite element method is applied to the porous media problems. In the paper [11] by Luther White and M. Zaman, a simplified model and some simulation results in two dimension are presented.

In Chapter Two, we construct the mathematical model of the height function and the models in the saturated domain for both incompressible and compressible fluid. In Chapter Three and

Four we use the change of variable to formulate the problem as a differential equation with time dependent coefficients on a fixed domain at every time t, and then reformulate the problem as a variational problem and furthermore, prove the existence an uniqueness of solution. The continuity of the solution with respect to permeability is also discussed. In order to estimate the permeability of existing roadbed system, we establish the detail analysis of the minimization problem in Chapter Five to estimate the permeability k minimizing the error between calculated outflow and observed outflow. A comprehensive numerical analysis is discussed in Chapter Six. We use finite element method to solve the partial differential equation and use this together with gradient decent method to iteratively solve the minimization problem. All posed problems are solved on 3 dimensional domain, and the results are presented.

The problem considered in this work is motivated by the very practical need to model the drainage in roadbeds of high performance highways and the effort to explain data accumulated over years by the Oklahoma Department of Transportation. Historically, the rational modeling of flow of fluid in porous media has always been an important consideration in environment, energy, construction and other related fields. Drainage of water was an important consideration in old construction practices in ancient Rome, Greece, and Egypt. In 18th and early 19th centuries, French and English engineer addressed this problem. P.M.J. Tresaquet, the introducer of the "French drain" in France, and Thomas Telford and John L. McAdam in England made significant contributions in this regard. In the early 20th century, The mechanization of handling, processing and placing of materials resulted in major changes in roadway designed for drainage of internal water. Usually these roads were not designed for drainage of internal water. With increase in traffic volume and load, however, these types of roads are no longer successful. As a result, drainage has drawn more attention of the pavement designer and theoretical treatment becomes more important. Determination of drainage properties, estimate permeability of the media, and asset the degradation of water become more important factors for the construction designer.

CHAPTER TWO

Development of Mathematical Models

The models of flow in a porous medium serve a variety of purposes and will vary in degrees of precision. It is important to realize that generally porous media models are complicated nonlinear problems. Without simplifying assumptions they may be difficult to analyze and may require substantial computational effort. In this research, we consider a domain as shown in Figure 1, denoted by Ω, which is defined as follow: Let G be given by $G = (0, L_1) \times (0, L_2)$, and $H_0, H_1 : G \longrightarrow R$ be real valued functions defined on G, where $H_0(x,y) < H_1(x,y)$ for any $(x,y) \in G$. Define the domain

$$\Omega = \{(x,y,z) : (x,y) \in G, H_0(x,y) < z < H_1(x,y)\}$$

Denote the upper boundary of Ω by

$$\Gamma_a = \{(x,y,z) : (x,y) \in G, z = H_1(x,y)\}$$

Suppose that $H : G \times (0,T) \longmapsto R$ is a real valued function defined on $G \times (0,T)$ such that $H_0(x,y) < H(x,y,y) < H_1(x,y)$ which represents the height of surface Γ separating fully saturated domain $W(t)$ and unsaturated domain $U(t)$

$$\Gamma(t) = \{(x,y,H(x,y,t)) : (x,y) \in G\}$$

$$W(t) = \{(x,y,z) : (x,y) \in G, H_0(x,y) < z < H(x,y,t)\}$$

$$U(t) = \{(x,y,z) : (x,y) \in G, H(x,y,t) < z < H_1(x,y)\}$$

$$\Gamma_0 = \{(x,y,z) : x_1 < x < x_2, y = y_0, H_0(x,y_0) < z < H(x,y,t)\}$$

Without loss of generality, we assume $H_0(x,y) = 0$, $H_1(x,y) = z_0$. Let $\alpha(t)$ represent the outflow leaving Γ_0 and $\mu(t)$ represent the inflow of fluid into the domain from Γ_a, ϕ denote the porosity, s denote fluid saturation, and ρ denote mass density of the fluid.

Mathematical model of height function

The behavior of the fully saturated domain $W(t)$ as time evolves is one of the issues when drainage is of importance. The saturation in $W(t)$ is equal to 1. As time varies, the boundary of $W(t)$ determined by height $H(x,y,t)$ will vary. Let $\rho = \rho_0(1+cp)$, where p is pressure. ρ_0 is a constant and c is compressibility. In order to develop the model in the fully saturated region $W(t)$ with upper boundary $\Gamma(t)$, we suppose that $\mu(x,y,t)$ is the fluid flux entering $W(t)$ through $\Gamma(t)$, and $o(t)$ is the fluid flux leaving $W(t)$ through $\Gamma_0(t)$, so the mass of fluid entering $W(t)$ through $\Gamma(t)$ is expressed in term of integral

$$\int_{\Gamma(t)} \rho\mu(x,y,t)\,dxdy \tag{2.1}$$

and the mass leaving $W(t)$ through Γ_0 is given by integral

$$\int_{\Gamma_0} \rho o(t)\,dxdz.$$

Let $V(t)$ denote the volume of fluid in $W(t)$, then

$$V(t) = \int_{W(t)} o(x,y,z)\,dxdydz = \int_G \int_0^{H(x,y,t)} o(x,y,z)\,dzdxdy$$

Hence, the change in mass of fluid in $W(t)$ per unit of time is

$$\begin{aligned} \frac{d}{dt}(\rho V(t)) &= \rho\frac{d}{dt}\int_G \int_0^{H(x,y,t)} o(x,y,z)\,dzdxdy \\ &= \rho\int_G o(x,y,H(x,y,t))\frac{\partial}{\partial t}H(x,y,t)\,dxdy. \end{aligned} \tag{2.3}$$

On the other hand, from (2.1) and (2.2) we have

$$\begin{aligned} \frac{d}{dt}V(t) &= \int_G \mu(x,y,t)\,dxdy - \int_{\Gamma_0} o(t)\,dzdx \\ &= \int_G \mu(x,y,t)\,dxdy - o(t)\int_{x_1}^{x_2}\int_0^{H(x,y_0,t)} dzdx \\ &= \int_G \mu(x,y,t)\,dxdy - o(t)\int_{x_1}^{x_2} H(x,y_0,t)\,dx \end{aligned} \tag{2.4}$$

Combining (2.3) and (2.4) we obtain a relation satisfied by the height function H for each time

t.

Proposition 2.1 Let $\rho \approx \rho_0$, then

$$\int_G \varphi(x,y,H(x,y,t))\frac{\partial}{\partial t}H(x,y,t)dxdy = \int_G \mu(x,y,t) - o(t)\int_{x_1}^{x_2} H(x,y_0,t)dx$$

If we consider the domain is relatively thin, $\varphi = \varphi(x,y)$, we have the following corollary:

Corollary 2.2 Suppose that $\varphi = \varphi(x,y)$, $H(x,y,t) = \beta(x,y)h(t)$, and $\mu(x,y,t) = \alpha(x,y)r(t)$, then the height function $h(t)$ satisfies the following ordinary differential equation

$$\varphi_0\frac{d}{dt}h(t) = ar(t) - bo(t)h(t) \tag{2.5}$$

where

$$\begin{aligned}
\varphi_0 &= \int_G \phi(x,y)\beta(x,y)dxdy \\
a &= \int_G \alpha(x,y)dxdy \\
b &= \int_{x_1}^{x_2} \beta(x,y_0)dx
\end{aligned}$$

The solution of (2.5) will give the height of surface $\Gamma(t)$ which determines the time dependent domain $W(t)$. Next, we seek to develop the model describing the pressure field within the saturated domain $W(t)$.

Mathematical models in fully saturated domain

Assume that pressure is zero, actually a constant in unsaturated domain $\Omega\setminus W(t)$, and that fluid entering $W(t)$ is incompressible. Suppose that $H(x,y,t)$, the solution from (2.5), satisfies $0 < H(x,y,t) < z_0$, for anytime t. Then the mass in $W(t)$ can be expressed in terms of integral

$$\int_{W(t)} \varphi\rho \cdot dv$$

So, the change of mass at time t, derivative with respect to t is

$$\frac{d}{dt}\int_{W(t)} \varphi\rho \cdot dv$$

So, the change of mass at time t, derivative with respect to t is

$$\frac{d}{dt}\int_{W(t)} \phi\rho \cdot dv$$

on the other hand, the change of mass is difference between of water entering and leaving $W(t)$. It can be calculated by integral of $\rho dv \cdot n$ along the boundary, where n is outward normal along the boundary. Thus, we have

$$\frac{d}{dt}\int_{W(t)} \phi\rho \cdot dv = -\int_{\partial W(t)} \rho v \cdot n d\sigma \tag{2.6}$$

from the divergence theorem,

$$\int_{\partial W(t)} \phi\rho v \cdot n d\sigma = -\int_{W(t)} \nabla(\rho v) dv \tag{2.7}$$

combine (2.6) and (2.7)

$$\int_{W(t)} \phi\frac{d}{dt}\rho \cdot dv = -\int_{W(t)} \nabla(\rho v) dv$$

we obtain the equation for mass conservation within $W(t)$

$$\phi\frac{\partial}{\partial t}\rho = -\nabla \cdot (\rho v) \tag{2.8}$$

Since $\rho \approx \rho_0$, $c = 0$, and from Darcy's law $v = -k\nabla\Phi$, we thus obtain an elliptic equation

$$\nabla(k\nabla\Phi) = 0$$

in $W(t)$, where $\Phi = p - \rho g z$.

In addition, we suppose that pressure is equal to zero, or constant on top surface Γ and side outlet Γ_0, and there is no flow on the part of boundary $\tilde{\Gamma} = \partial W(t) \backslash \Gamma_0(t) \cup \Gamma(t)$, i.e.

$$k\nabla\Phi \cdot n = 0$$

and also suppose that there is the additional outflow condition on Γ_0

$$k\nabla\Phi \cdot n = o(t),$$

we can conclude the following.

Proposition 2.3 Suppose that the fluid is incompressible, ρ_0 is density of the fluid, then the pressure p is the solution of the elliptic boundary problem

$$\nabla \cdot (k\nabla\Phi) = 0 \qquad \text{in } W(t) \tag{2.9}$$

$$p = 0 \qquad \text{on } \Gamma \cup \Gamma_0 \tag{2.10}$$

$$k\nabla\Phi = 0 \qquad \text{on } \partial W(t)/\Gamma \cup \Gamma_0 \tag{2.11}$$

with additional boundary condition

$$k\nabla\Phi \cdot n = o(t) \qquad \text{on } \Gamma_0 \tag{2.12}$$

Remark 2.4 This elliptic equation is time independent, but the boundary is time varying determined by height function H.

If we assume that the fluid is compressible with the compressibility c, from (2.8) we will derive that pressure p satisfies a parabolic equation with same boundary condition as (2.10), (2.11) and (2.12).

Proposition 2.5 Suppose that the fluid is compressible with compressibility c, ρ_0 is density of the fluid, then the pressure p is the solution of the parabolic boundary value problem

$$co\frac{\partial p}{\partial t} = \nabla \cdot (k\nabla\Phi) \qquad \text{in } W(t) \tag{2.13}$$

$$p = 0 \qquad \text{on } \Gamma \cup \Gamma_0 \tag{2.14}$$

$$k\nabla\Phi = 0 \qquad \text{on } \partial W(t)/\Gamma \cup \Gamma_0 \tag{2.15}$$

with initial condition

$$p(0) = p^0 \tag{2.16}$$

and additional boundary condition

$$k\nabla\Phi \cdot n = o(t) \tag{2.17}$$

CHAPTER THREE

Boundary Value Problem Of Elliptic Equation With A Time Dependent Domain

In Chapter Two we developed the mathematical models in fully saturated domain and posed the elliptic boundary value problem (2.9)-(2.11) on accretive domain when assuming fluid is incompressible. In this chapter, we will define the change of variables and reformulate the problem as a variational problem on a fixed domain for every time t. Then we prove the existence of an unique solution of the variational problem and discuss the continuous dependence of solution with respect to the permeability k of the porous media.

Define $\Pi(t) : R^3 \longrightarrow R^3$ map (x, y, z) to (ξ, η, ζ) by

$$\begin{cases} \xi = x \\ \eta = y \\ \zeta = \frac{z}{H(x,y,t)} \end{cases} \tag{3.1}$$

then $\Pi(t)$ will map domain $W^*(t)$ to W_0^*, where $W_0^* = \{(\xi, \eta, \zeta) : 0 < \xi < L_1, 0 < \eta < L_2, 0 < \zeta < 1\}$ (see Figure-2).

Remark 3.1 The map F is invertible, because the Jacobian determinant can be simply calculated as

$$J = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{H} \end{vmatrix} = \frac{1}{H} \neq 0, \text{ if } H \neq 0.$$

Define $u(\xi, \eta, \zeta, t)$ as pressure function under change of variables

$$u(\xi, \eta, \zeta, t) = p(x, y, z, t).$$

then

$$\begin{aligned} p_x &= u_\xi - \frac{\zeta H_x}{H} \cdot u_\zeta, \\ p_y &= u_\eta - \frac{\zeta H_y}{H} \cdot u_\zeta, \\ p_z &= \frac{1}{H} \cdot u_\zeta, \\ p_t &= \frac{\zeta H_t}{H} u_\zeta. \end{aligned}$$

To simplify the notation, we define, through this dissertation,

$$\begin{aligned} a_1 &= \frac{\zeta H_x}{H} \\ a_2 &= \frac{\zeta H_y}{H} \\ a_3 &= \frac{1}{H} \\ a_t &= \frac{\zeta H_t}{H} \end{aligned}$$

then in terms of new variables, the equation can be expressed as

$$\left(\frac{\partial}{\partial\xi} - a_1\frac{\partial}{\partial\zeta}, \frac{\partial}{\partial\eta} - a_2\frac{\partial}{\partial\zeta}, a_3\frac{\partial}{\partial\zeta}\right)\cdot k\left(\frac{\partial\Phi}{\partial\xi} - a_1\frac{\partial\Phi}{\partial\zeta}, \frac{\partial\Phi}{\partial\eta} - a_2\frac{\partial\Phi}{\partial\zeta}, a_3\frac{\partial\Phi}{\partial\zeta}\right) = 0 \qquad (3.2)$$

where

$$\Phi = u + \rho g\zeta H$$

Existence of solution

Let V be Hilbert space defined by

$$V = \left\{\varphi \in H^1(\Omega), \varphi = 0 \quad \text{on } \Gamma \cup \Gamma_0\right\} \qquad (3.3)$$

with inner product defined by $(u, v) = \int_{W} uv$. Unless otherwise specified, let $\|\cdot\|$ denote the norm of Hilbert space V. We seek the solution of variational problem and utilize the famous Lax- Milgram Lemma quoted below to obtain the existence of solution.

Lax-Milgram Lemma: Let V be a Hilbert space, $a(\cdot,\cdot) : V \times V \longrightarrow R$ be a continuous V-elliptic bilinear form, and let $f : V \longrightarrow R$ be a continuous linear form. Then the abstract variational problem: Find an element u such that

$$u \in V \qquad and \qquad \forall v \in V, \qquad a(u, v) = f(v)$$

has one and only one solution.

Take any $v \in V$. Multiply (3.2) by v and integrate the both sides on W_0, we obtain

$$0 = \int_{W_0}\left(\frac{\partial}{\partial\xi} - a_1\frac{\partial}{\partial\zeta}, \frac{\partial}{\partial\eta} - a_2\frac{\partial}{\partial\zeta}, a_3\frac{\partial}{\partial\zeta}\right)\cdot k\left(\frac{\partial\Phi}{\partial\xi} - a_1\frac{\partial\Phi}{\partial\zeta}, \frac{\partial\Phi}{\partial\eta} - a_2\frac{\partial\Phi}{\partial\zeta}, a_3\frac{\partial\Phi}{\partial\zeta}\right)\cdot\varphi\, d\xi\, d\eta\, d\zeta$$

$$= \int_{\partial W_0} k(\Phi_\xi - a_1\Phi_\zeta, \Phi_\eta - a_2\Phi_\zeta, a_3\Phi_\zeta) v d\sigma$$
$$- \int_{W_0} k(\Phi_\xi - a_1\Phi_\zeta, \Phi_\eta - a_2\Phi_\zeta, a_3\Phi_\zeta)(v_\xi - a_1 v_\zeta, \varphi_\eta - a_2 v_\zeta, a_3 v_\zeta) d\xi d\eta d\zeta$$

Applying the boundary condition, we have

$$\int_{W_0} k(\Phi_\xi - a_1\Phi_\zeta, \Phi_\eta - a_2\Phi_\zeta, a_3\Phi_\zeta)(v_\xi - a_1 v_\zeta, v_\eta - a_2 v_\zeta, a_3 v_\zeta) d\xi d\eta d\zeta = 0$$

since $\Phi = u + \rho g \zeta H$.

$$0 = \int_{W_0} k(u_\xi - a_1 u_\zeta, u_\eta - a_2 u_\zeta, a_3 u_\zeta)(v_\xi - a_1 v_\zeta, v_\eta - a_2 v_\zeta, a_3 v_\zeta) d\xi d\eta d\zeta$$
$$+ \int_{W_0} k\rho g H(a_1 v_\xi + a_2 v_\eta + (a_3^2 - a_1^2 - a_2^2) v_\zeta) d\xi d\eta d\zeta$$

Define operator $a(\cdot,\cdot) : V \times V \longmapsto R$ by

$$a(u, v) = \int_{W_0} k(u_\xi - a_1 u_\zeta, v_\eta - a_2 u_\zeta, a_3 u_\zeta)(v_\xi - a_1 v_\zeta, v_\eta - a_2 v_\zeta, a_3 v_\zeta) d\xi d\eta d\zeta$$

and linear functional $F : V \longmapsto R$ by

$$F(v) = \int_{W_0} k\rho g H(a_1 v_\xi + a_2 v_\eta + (a_3^2 - a_1^2 - a_2^2) v_\zeta) d\xi d\eta d\zeta$$

then represent the boundary value problem (2.9)-(2.11) as a variational problem(VP): Find $u \in V$ such that $a(u, v) = F(v)$ for all $v \in V$.

It is clear that $a(\cdot,\cdot)$ is a bilinear form, and for any $v \in V$,

$$\begin{aligned}
a(v, v) &= \int_{W_0} k((v_\xi - a_1 v_\zeta)^2 + (v_\eta - a_2 v_\zeta)^2 + a_3^2 v_\zeta^2) d\xi d\eta d\zeta \\
&= \int_{W_0} k(v_\xi^2 + v_\eta^2 - 2a_1 v_\xi v_\zeta - 2a_2 v_\eta v_\zeta + (a_1^2 + a_2^2 + a_3^2) v_\zeta^2 \\
&\geq \int_{W_0} k[v_\xi^2 + v_\eta^2 + (a_1^2 + a_2^2 + a_3^2) v_\zeta^2 - \varepsilon v_\xi^2 - \frac{a_1^2}{\varepsilon} v_\zeta - \varepsilon v_\eta^2 - \frac{a_2^2}{\varepsilon} v_\zeta^2] \\
&= \int_{W_0} k[(1 - \varepsilon) v_\xi^2 + (1 - \varepsilon) v_\eta^2 + (a_1^2 + a_2^2 + a_3^2 - \frac{1}{\varepsilon}(a_1^2 - a_2^2) v_\zeta^2] \qquad (3.4)
\end{aligned}$$

We want to choose $\varepsilon (0 < \varepsilon < 1)$ such that

$$a_1^2 + a_2^2 + a_3^2 - \frac{1}{\varepsilon}(a_1^2 + a_2^2)$$

is greater than a positive number. Suppose that

$$a_1^2 + a_2^2 + a_3^2 - \frac{1}{\varepsilon}(a_1^2 + a_2^2) > d_0 > 0 \tag{3.5}$$

then

$$a_1^2 + a_2^2 + a_3^2 - d_0 > \frac{1}{\varepsilon}(a_1^2 + a_2^2)$$

Hence, ε must satisfy the following condition

$$\varepsilon > \frac{a_1^2 + a_2^2}{a_1^2 + a_2^2 + a_3^2 - d_0}$$

and since $a_3 \neq 0$, if we choose $d_0 < a_3^2$, then

$$a_1^2 + a_2^2 < a_1^2 + a_2^2 + a_3^2 - d_0$$

so,

$$\frac{a_1^2 + a_2^2}{a_1^2 + a_2^2 + a_3^2 - d_0} < 1$$

thus, we can choose positive number $\varepsilon < 1$ satisfying (3.5). With ε chosen as above and condition $k > c_0 > 0$, from (3.4) we have

$$a(v, v) > \alpha \|v\|^2 \tag{3.6}$$

for some constant α. On the other hand, for any $v \in V$, assume $k \in L_\infty(\Omega)$, $H \in X$, where X is defined as $X = \{H |\ H > H_0 > 0, H \in C^1(0, T: W_{1,4}(\Omega))$

$$\begin{aligned}
|F(v)| &= \left| \int_{W_0} k\rho g H(a_1 v_\xi + a_2 v_\eta + (a_3^2 - a_1^2 - a_2^2) v_\zeta) d\xi d\eta d\zeta \right| \\
&\leq \rho g \|k\|_\infty (\|H_x\|_2 \cdot \|v_\xi\| + \|H_y\|_2 \cdot \|v_\eta\| + \frac{1}{H_0}(1 + \|H_x\|_4^2 + \|H_y\|_4^2 \quad v_\zeta \\
&\leq C \|v\|
\end{aligned}$$

where C is dependent on norm of k in $L^\infty(\Omega)$ and norm of H in X. Therefore, F is continuous functional and we can conclude the following theorem by applying the Lax-Milgram Theorem.

Theorem 3.2 Suppose that $k \in L^\infty(\Omega)$, $H(x, y, z, t) \in X$ and there exist $c_0 > 0$ such that $k > c_0 > 0$, then the problem (2.9)-(2.11) have unique solution

$$u(t, \cdot) \in V \quad \text{for every } t \in (0, T)$$

such that

$$a(u,v) = F(v), \quad \text{for all } v \in V.$$

Properties of solution of variational problem

We have proved the variational problem has unique solution $u(t,\cdot) \in V$ for every time t and given parameter k. Now, we discuss the continuous dependence of the solution with respect to k. Define a mapping from $k \in L^\infty(\Omega)$ to the corresponding solution $u(k)$

$$g : L^\infty(\Omega) \longmapsto V; g(k) = u(k)$$

Proposition 3.3 Let $\{k_n\}_{n=1}^{\infty}$, be a sequence in $L^\infty(\Omega)$, satisfying $k_n \geq c_0 > 0$, and $k_n \to k_0$ in $L^\infty(\Omega)$, where $k_0 \in L^\infty(\Omega)$, and $k_0 \geq c_0 > 0$. Suppose that $u_n = u(k_n)$ and $u_0 = u(k_0)$ are solutions of (VP)

$$a(u,\varphi) = F(\varphi) \qquad \text{for all } \varphi \in V$$

corresponding to k_n and k_0 respectively, then $\|u_n - u_0\| \to 0$, when $n \to \infty$.

Proof: Since u_n, u_0 are solutions of (VP) corresponding to k_n, k_0, for any $\varphi \in V$.

$$\int_{w_0} k_n \left[(u_{n\xi} - a_1 u_{n\zeta})(\varphi_{,\xi} - a_1\varphi_{,\zeta}) + (u_{n\eta} - a_2 u_{n\zeta})(\varphi_{,\eta} - a_2\varphi_{,\zeta}) + a_3^2 u_{n\zeta}\varphi_{,\zeta}\right] d\xi d\eta d\zeta$$

$$= \int_{w_0} k_n H\rho g \left[a_1(\varphi_{,\xi} - a_1\varphi_{,\zeta}) + a_2(\varphi_{,\eta} - a_2\varphi_{,\zeta}) + a_3^2\varphi_{,\zeta}\right] d\xi d\eta d\zeta \tag{3.7}$$

and

$$\int_{w_0} k_0 \left[(u_{0\xi} - a_1 u_{0\zeta})(\varphi_{,\xi} - a_1\varphi_{,\zeta}) + (u_{0\eta} - a_2 u_{0\zeta})(\varphi_{,\eta} - a_2\varphi_{,\zeta}) + a_3^2 u_{0\zeta}\varphi_{,\zeta}\right] d\xi d\eta d\zeta$$

$$= \int_{w_0} k_0 H\rho g \left[a_1(\varphi_{,\xi} - a_1\varphi_{,\zeta}) + a_2(\varphi_{,\eta} - a_2\varphi_{,\zeta}) + a_3^2\varphi_{,\zeta}\right] d\xi d\eta d\zeta \tag{3.8}$$

Let $\varphi = u_n - u_0$ in (3.7) and (3.8) and take the difference of (3.7) and (3.8), then

$$\int_{w_0} k_n \left[(u_{n\xi} - a_1 u_{n\zeta}) - k_0(u_{0\xi} - a_1 u_{0\zeta})\right]\left[(u_n - u_0)_{,\xi} - a_1(u_n - u_0)_{,\zeta}\right] d\xi d\eta d\zeta$$

$$+ \int_{w_0} k_n \left[(u_{n\eta} - a_2 u_{n\zeta}) - k_0(u_{0\eta} - a_2 u_{0\zeta})\right]\left[(u_n - u_0)_{,\eta} - a_2(u_n - u_0)_{,\zeta}\right] d\xi d\eta d\zeta$$

$$+ \int_{w_0} a_3^2 \left[k_n u_{n\zeta} - k_0 u_{0\zeta}\right](u_n - u_0)_{,\zeta} \tag{3.9}$$

$$= \int_{w0} (k_n - k_0) H\rho g[a_1((u_n - u_0)_{.\xi} - a_1(u_n - u_0)_{.\zeta}) + a_2((u_n - u_0)_{.\eta} - a_2(u_n - u_0)_{.\zeta})$$
$$+ a_3^2(u_n - u_0)_{.\zeta}]$$

For the sake of convenience, define the first integral in the left hand side as I_1, the second one as I_2, and the third one as I_3, and define the right hand side as I_r, then the (3.9) is

$$I_1 + I_2 + I_3 = I_r \tag{3.10}$$

$$\begin{aligned}
I_1 &= \int_{w0} \{(k_n - k_0)(u_{n\xi} - a_1 u_{n\zeta}) + k_0[(u_n - u_0)_{.\xi} - a_1(u_n - u_0)_{.\zeta}]\} \\
&\quad [(u_n - u_0)_{.\xi} - a_1(u_n - u_0)_{.\zeta}]\, d\xi d\eta d\zeta \\
I_1 &= \int_{w0} \{(k_n - k_0)(u_{n\xi} - a_1 u_{n\zeta})[(u_n - u_0)_{.\xi} - a_1(u_n - u_0)_{.\zeta}] \\
&\quad + k_0(u_n - u_0)^2_{.\xi} - 2k_0 a_1(u_n - u_0)_{.\xi}(u_n - u_0)_{.\zeta} + k_0 a_1^2(u_n - u_0)^2_{.\xi}
\end{aligned}$$

$$\begin{aligned}
I_2 &= \int_{w0} \{(k_n - k_0)(u_{n\eta} - a_2 u_{n\zeta}) + k_0[(u_n - u_0)_{.\eta} - a_2(u_n - u_0)_{.\zeta}]\} \\
&\quad [(u_n - u_0)_{.\eta} - a_2(u_n - u_0)_{.\zeta}]\, d\xi d\eta d\zeta \\
I_2 &= \int_{w0} \{(k_n - k_0)(u_{n\eta} - a_2 u_{n\zeta})[(u_n - u_0)_{.\eta} - a_1(u_n - u_0)_{.\zeta}] \\
&\quad + k_0(u_n - u_0)^2_{.\eta} - 2k_0 a_2(u_n - u_0)_{.\eta}(u_n - u_0)_{.\zeta} + k_0 a_2^2(u_n - u_0)^2_{.\eta}
\end{aligned}$$

$$\begin{aligned}
I_3 &= \int_{w0} a_3^2[(k_n - k_0)u_{n\zeta} + k_0(u_n - u_0)_{.\zeta}](u_n - u_0)_{.\zeta} d\xi d\eta d\zeta \\
I_3 &= \int_{w0} a_3^2(k_n - k_0)u_{n\zeta}(u_n - u_0)_{.\zeta} + a_3^2 k_0(u_n - u_0)^2_{.\zeta}
\end{aligned}$$

$$\begin{aligned}
|I_r| &= \left| \int_{w0} (k_n - k_0) H\rho g[a_1((u_n - u_0)_{.\xi} - a_1(u_n - u_0)_{.\zeta}) + a_2((u_n - u_0)_{.\eta} - a_2(u_n - u_0)_{.\zeta}) \right. \\
&\quad \left. + a_3^2(u_n - u_0)_{.\zeta}] \right| \\
&\le \frac{1}{\varepsilon_0}\rho g \|k_n - k_0\|_\infty \frac{1}{H_0^2}(\|H_x\|_2^2 + \|H_y\|_2^2 + \frac{1}{2}) + \varepsilon_0 \|u_n - u_0\|^2
\end{aligned}$$

Substitute I_1, I_2, I_3, I_r into (3.10), then we have

$$\int_{w0} k_0(u_n - u_0)^2_{.\xi} + k_0(u_n - u_0)^2_{.\eta} + k_0 a_1^2(u_n - u_0)^2_{.\xi} + k_0 a_2^2(u_n - u_0)^2_{.\eta} + a_3^2 k_0(u_n - u_0)^2_{.\zeta}$$

$$
\begin{aligned}
\leq\ & \Big|\int_{w_0} 2k_0a_1(u_n-u_0)_{.\xi}(u_n-u_0)_{.\zeta}+2k_0a_2(u_n-u_0)_{.\eta}(u_n-u_0)_{.\zeta}-a_3^2(k_n-k_0)u_{n\zeta}(u_n-u_0)_{.\zeta}\\
& -(k_n-k_0)(u_{n\xi}-a_1u_{n\zeta})[(u_n-u_0)_{.\xi}-a_1(u_n-u_0)_{.\zeta}]\\
& -(k_n-k_0)(u_{n\eta}-a_2u_{n\zeta})[(u_n-u_0)_{.\eta}-a_1(u_n-u_0)_{.\zeta}]+|I_r|\\
\leq\ & \varepsilon\int_{w_0} k_0(u_n-u_0)_{.\xi}^2+\frac{1}{\varepsilon}\int_{w_0} k_0a_1^2(u_n-u_0)_{.\zeta}^2+\varepsilon\int_{w_0} k_0(u_n-u_0)_{.\eta}^2+\frac{1}{\varepsilon}\int_{w_0} k_0a_2^2(u_n-u_0)_{.\zeta}^2\\
& +\frac{1}{\varepsilon_1}\int_{w_0} a_3^2(k_n-k_0)^2u_{n\zeta}^2+\varepsilon\int_{w_0} a_3^2(u_n-u_0)_{.\zeta}^2\\
& +\varepsilon_2\int_{w_0}(u_n-u_0)_{.\xi}^2+\varepsilon_2\int_{w_0} a_1^2(u_n-u_0)_{.\zeta}^2+\frac{1}{\varepsilon_2}\int_{w_0}(k_n-k_0)^2(u_{n\xi}^2+a_1^2u_{n\zeta}^2)\\
& +\varepsilon_3\int_{w_0}(u_n-u_0)_{.\eta}^2+\varepsilon_3\int_{w_0} a_2^2(u_n-u_0)_{.\zeta}^2+\frac{1}{\varepsilon_3}\int_{w_0}(k_n-k_0)^2(u_{n\eta}^2+a_2^2u_{n\zeta}^2)\\
& +\frac{1}{\varepsilon_0}\rho g\,\|k_n-k_0\|_\infty\frac{1}{H_0^2}(\|H_x\|_2^2+\|H_y\|_2^2+\frac{1}{2})+\varepsilon_0\,\|u_n-u_0\|^2
\end{aligned}
$$

$$
\begin{aligned}
& (1-\varepsilon-\varepsilon_1)\int_{w_0} k_0(u_n-u_0)_{.\xi}^2+(1-\varepsilon-\varepsilon_2)\int_{w_0} k_0(u_n-u_0)_{.\eta}^2\\
& +[a_1^2-a_2^2+a_3^2-\frac{1}{\varepsilon}(a_1^2+a_2^2)-\varepsilon_2a_1^2-\varepsilon_3a_2^2-\varepsilon_1a_3^2]\int_{w_0} k_0(u_n-u_0)_{.\zeta}^2\\
\leq\ & \frac{1}{\varepsilon_1}\|k_n-k_0\|_\infty^2\,\|a_3\|_4^2\,\|u\|^2+\frac{1}{\varepsilon_1}\|k_n-k_0\|_\infty^2\,(1+\|a_1\|_2^2)\,\|u\|^2\\
& +\frac{1}{\varepsilon_0}\rho g\,\|k_n-k_0\|_\infty\frac{1}{H_0^2}(\|H_x\|_2^2+\|H_y\|_2^2+\frac{1}{2})+\varepsilon_0\,\|u_n-u_0\|^2
\end{aligned}
$$

Choose ε satisfying (3.5) and $\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3$ small enough, and note that $k_0 > c_0 > 0$, we therefore have

$$\|u_n-u_0\|^2 \leq C(\|H\|_X, \|u\|, \rho, g)\,\|k_n-k_0\|_\infty$$

where $C(\|H\|_X, \|u\|, \rho, g)$ is a number dependent on $\|H\|_X$, $\|u\|$, ρ and g. Thus, $\|u_n-u_0\| \to 0$, as $n \to \infty$ if $\|k_n-k_0\|_\infty \to 0$. ■

Remark 3.4 Similar to The Propesition 3.3, we can discuss the continuous dependence of solution u with respect to t. Let $t_n \to t_0$ in $[0, T]$, $u(t_n)$ and $u(t_0)$ be solutions corresponding to t_n and t_0. If $H \doteq \beta(x, y)h(t)$, and $h(t) \in C^1[0, T]$, then we can show that $\|u(t_n)-u(t_0)\| \to 0$ as $t_n \to t_0$, with the similar procesure in the proof of Proposition 3.3.

Remark 3.5 Let u be the solution of (2.9)-(2.11), from Remark 3.4, if $H \in X$, we actually have

$u \in C^0(0, T; V)$.

CHAPTER FOUR

Boundary Value Problem of Parabolic Equation

In this chapter, we consider that the fluid is compressible with compressibility c. The mathematical model in the fully saturated domain $W(t)$ was developed in Chapter Two. The pressure p is the solution of a parabolic equation with boundary conditions (2.13) - (2.15) and initial condition (2.16). The additional boundary condition is given in (2.17). Under the same mapping $\Pi(t)$ as in Chapter Three for change of variables, we reformulate the problem (2.13)-(2.15) as a variational problem on a fixed domain for every time t and prove the existence of an unique solution of the variational problem and discuss the continuous dependence of the solution with respect to the permeability k.

Existence of solution

Recall $\Pi(t) : R^3 \longrightarrow R^3$ defined in (3.1) which maps domain $W(t)$ to W_0, and define $u(\xi, \eta, \zeta, t)$ as pressure function under change of variables

$$u(\xi, \eta, \zeta, t) = p(x, y, z, t)$$

then in terms of new variables, the equation (2.13) can be expressed as

$$c\left(\frac{\partial\Phi}{\partial t} - a_t\frac{\partial\Phi}{\partial\zeta}\right) - \left(\frac{\partial}{\partial\xi} - a_1\frac{\partial}{\partial\zeta}, \frac{\partial}{\partial\eta} - a_2\frac{\partial}{\partial\zeta}, a_3\frac{\partial}{\partial\zeta}\right)\cdot k\left(\frac{\partial\Phi}{\partial\xi} - a_1\frac{\partial\Phi}{\partial\zeta}, \frac{\partial\Phi}{\partial\eta} - a_2\frac{\partial\Phi}{\partial\zeta}, a_3\frac{\partial\Phi}{\partial\zeta}\right) = 0 \tag{4.1}$$

Let V be Hilbert space defined by in (3.3), and take any $v \in V$, multiply (4.1) by v and integrate the both sides on W_0, and apply the boundary condition we then obtain

$$c\int_{W_0}\frac{\partial\Phi}{\partial t}v - c\int_{W_0}a_t\frac{\partial\Phi}{\partial\zeta}v$$

$$= \int_{W_0} k(\Phi_\xi - a_1\Phi_\zeta, \Phi_\eta - a_2\Phi_\zeta, a_3\Phi_\zeta)(v_\xi - a_1 v_\zeta, v_\eta - a_2 v_\zeta, a_3 v_\zeta)\,d\xi d\eta d\zeta$$

since $\Phi = u + \rho g\zeta H$.

$$c\int_{W_0}\frac{\partial u}{\partial t}v - c\int_{W_0}a_t\frac{\partial u}{\partial\zeta}v - \int_{W_0}\rho g a_t H v$$

$$= \int_{W_0} k(u_\xi - a_1 u_\zeta, u_\eta - a_2 u_\zeta, a_3 u_\zeta)(v_\xi - a_1 v_\zeta, v_\eta - a_2 v_\zeta, a_3 v_\zeta)d\xi d\eta d\zeta$$
$$+ \int_{W_0} k\rho g H(a_1 v_\xi + a_2 v_\eta + (a_3^2 - a_1^2 - a_2^2)v_\zeta)d\xi d\eta d\zeta$$

Define operator $a(\cdot,\cdot) : V \times V \longmapsto R$ by

$$a(u,v) = \int_{W_0} k(u_\xi - a_1 u_\zeta, u_\eta - a_2 u_\zeta, a_3 u_\zeta)(v_\xi - a_1 v_\zeta, v_\eta - a_2 v_\zeta, a_3 v_\zeta)d\xi d\eta d\zeta)$$

and linear functional $F : V \longmapsto R$ by

$$F(v) = \int_{W_0} k\rho g H(a_1 v_\xi + a_2 v_\eta + (a_3^2 - a_1^2 - a_2^2)v_\zeta)d\xi d\eta d\zeta + \int_{W_0} \rho g a_t H v$$

From (3.6) we have $a(\cdot,\cdot)$ is a bilinear elliptic operator. Define an operator $a_1(\cdot,\cdot) : V \times V \longmapsto R$ by

$$a_1(u,v) = a(u,v) - c\int_{W_0} a_t u_\zeta v$$

and impose the condition on H such that $H \in X_1$, where $X_1 = \{H > H_0 > 0, H \in C^1(0,T; L_\infty) \cap C^0(0,T; W_{1,4}(\Omega))\}$ then

$$a_1(u,u) = a(u,u) - c\int_{W_0} a_t u_\zeta u \geq \alpha \|u\|^2 - c\|a_t\|_\infty (\varepsilon\|u\|^2 - \frac{1}{\varepsilon}\|u\|_{L_2}^2)$$
$$a_1(u,u) + \varepsilon' \|u\|_{L_2}^2 \geq \alpha \|u\|^2 - c\|a_t\|_\infty \varepsilon \|u\|^2$$

Choose the ε such that $(\alpha - c\|a_t\|_\infty \varepsilon) \geq \alpha' > 0$, therefore, we have

$$a_1(u,u) + \varepsilon' \|u\|_{L_2}^2 \geq \alpha' \|u\|^2 \tag{4.2}$$

On the other hand, for any $v \in V$, if $k \in L_\infty(\Omega)$, $H \in X_1$

$$\begin{aligned} |F(v)| &= \left|\int_{W_0} k\rho g H(a_1 v_\xi + a_2 v_\eta + (a_3^2 - a_1^2 - a_2^2)v_\zeta)d\xi d\eta d\zeta\right| + \left|\int_{W_0} \rho g a_t H v\right| \\ &\leq \rho g \|k\|_\infty (\|H_x\|_2 \cdot \|v_\xi\| + \|H_y\|_2 \cdot \|v_\eta\| + \frac{1}{H_0}(1 + \|H_x\|_4^2 + \|H_y\|_4^2)\|v_\zeta\|) \\ &\quad + \rho g \|a_t\|_\infty \|H\|_2 \|v\|_2 \\ &\leq C\|v\| \end{aligned}$$

where C is dependent on norm of k in $L^\infty(\Omega)$ and the norm of H in X_1. Therefore, F is continuous functional and we can conclude the following theorem:

Theorem 4.1 Suppose that $k \in L^\infty(\Omega)$, $H(x,y,z,t) \in X_1$ for every $t \in (0,T)$, and there exist

$c_0 > 0$ such that $k > c_0 > 0$, then there exists an unique solution[9]

$$u(t,\cdot) \in V \quad \text{for every } t \in (0,T)$$

such that

$$\int cu_t v + a_1(u,v) = F(v), \quad \text{for all } v \in V \tag{4.3}$$

Corollary 4.2 Let $u(t,\cdot)$ be a solution of (4.3), then $u(t,\cdot) \in C^0(0,T;V)$ if the condition $H \in X_1$ is satisfied.

Proof: Take $v = u$ in (4.3)

$$\int_{w_0} cu_t u + a_1(u,u) = F(u)$$

$$\frac{1}{2}\frac{d}{dt}\int_{w_0} c\,u^2 + a_1(u,u) \le C\,\|u\|^2$$

from (4.2)

$$a_1(u,u) \ge \alpha'\,\|u\|^2 - \varepsilon'\,\|u\|^2_{L_2}$$

so

$$\frac{1}{2}\frac{d}{dt}\int_{w_0} c\,u^2 + \alpha'\,\|u\|^2 \le \varepsilon'\,\|u\|^2_{L_2} + C\,\|u\|^2 \tag{4.4}$$

Integrate (4.4) from 0 to t, for every $t \in [0,T]$

$$\frac{1}{2}c\,\|u(t)\|^2 - \frac{1}{2}c\,\|u(0)\|^2 + \int_0^t \alpha'\,\|u\|^2 \le \varepsilon'\int_0^t \|u\|^2_{L_2} + C\int_0^t \|u\|^2$$

Suppose the initial condition $u(0) = u^0$, then

$$\begin{aligned} \|u(t)\|^2 + \int_0^t \|u\|^2 &\le \|u^0\|^2 + \varepsilon'\int_0^t \|u\|^2_{L_2} + C\int_0^t \|u\|^2 \\ &\le \|u^0\|^2 + C'\int_0^t \|u\|^2 \end{aligned}$$

Define

$$w'(t) = \|u\|^2 + \int_0^t \|u\|^2 \tag{4.5}$$

then integrate (4.5) with respect to τ from 0 to t,

$$w(t) - \|u^0\|^2 = \int_0^t \|u(s)\|^2 ds + \int_0^t \left(\int_0^\tau \|u(s)\|^2 ds\right) d\tau$$

$$= \int_0^t \|u(s)\|^2 ds + \int_0^t (t-\tau) \|u(\tau)\|^2 d\tau$$
$$\geq \int_0^t \|u(s)\|^2 ds$$

so

$$\begin{aligned} \psi'(t) &= \|u\|^2 + \int_0^t \|u\|^2 \leq \|u^0\|^2 + C' \int_0^t \|u\|^2 \leq \|u_0\|^2 + C'(\psi(t) - \|u^0\|^2) \\ &\leq (1-C') \|u^0\|^2 + C' \psi(t) \end{aligned} \tag{4.6}$$

Define

$$\Theta(t) = (1-C') \|u^0\|^2$$

then the (4.6) can be expressed as

$$\psi'(t) \leq \Theta(t) + C' \psi(t) \tag{4.7}$$

Solve (4.7), we obtain

$$\psi'(t) \leq \Theta(t) + C' e^{C'} \int_0^t \Theta(s) ds$$

since $\Theta(t) = (1-C') \|u^0\|^2$ is independent of time t, then

$$\psi'(t) \leq (1-C') \|u^0\|^2 + Ce^{C'} T(1-C') \|u_0\|^2$$

thus,

$$\|u\|^2 \leq (1-C') \|u^0\|^2 + Ce^{C'} T(1-C') \|u_0\|^2$$

where C is a constant dependent on $\|H\|_{X_1}$. Therefore, $u(t,\cdot) \in C^0(0,T;V)$. Thus, we complete the proof. ■

Properties of solution of variational problem

We have proved the variational problem has unique solution $u(t,\cdot) \in C^0(0,T;V)$ for every given parameter k. Therefore, in order to discuss the continuous dependence of the solution with respect to k, we can define a mapping from every $k \in L^\infty(\Omega)$ to the corresponding solution $u(k)$,

$$g : L^\infty(\Omega) \longmapsto V : g(k) = u(k)$$

Proposition: Let $\{k_n\}_{n=1}^\infty$ be a sequence in $L^\infty(\Omega)$ satisfying $k_n \geq c_0 > 0$, and $k_n \to k_0$ in

$L^\infty(\Omega)$, where $k_0 \in L^\infty(\Omega)$, and $k_0 \geq c_0 > 0$. Suppose that u_n and u_0 are solutions of (4.3)

$$\int cu_t\varphi + a_1(u,\varphi) = F(\varphi), \qquad \text{for all } \varphi \in V$$

corresponding to k_n and k_0 respectively, then $\|u_n - u_0\| \longmapsto 0$.

Proof: Since u_n, u_0 are solutions of (4.3) corresponding k_n, k_0,

$$\begin{aligned} \int cu_{nt}\varphi + a_1(u_n,\varphi) &= F(\varphi) \\ \int cu_{0t}\varphi + a_1(u_0,\varphi) &= F(\varphi) \end{aligned}$$

similar to the proof in Chapter Three for the elliptic (VP), Let $\varphi = u_n - u_0$. then the difference between above equations is as

$$\int c_0(u_{nt} - u_{0t})(u_n - u_0) + \int a_t(u_{n\zeta} - u_{0\zeta})(u_n - u_0) + I_1 + I_2 + I_3 = I_r$$

where I_1, I_2, I_3, I_r are same as defined in Chapter Three.

$$\begin{aligned} \frac{1}{2}c_0\frac{d}{dt}\int (u_n - u_0)^2 + I_1 + I_2 + I_3 &\leq |I_r| + \left|\int a_t(u_{n\zeta} - u_{0\zeta})(u_n - u_0)\right| \\ &\leq |I_r| + \frac{1}{2}\|a_t\|_\infty(\|u_{n\zeta} - u_{0\zeta}\|_2^2 + \|(u_n - u_0)\|_2^2) \\ &\leq |I_r| + \|a_t\|_\infty \|u_n - u_0\|^2 \qquad (4.8) \end{aligned}$$

substitute the estimate of I_1, I_2, I_3, I_r from Chapter Three into (4.8) then we have

$$\frac{1}{2}c_0\frac{d}{dt}\|u_n - u_0\|^2 + \|u_n - u_0\|^2 \leq C(\|H\|_{X_1}, \rho, g)\,\|k_n - k_0\|_\infty + \|a_t\|_\infty \|u_n - u_0\|^2 \quad (4.9)$$

where C is a number dependent on $\|H\|_{X_1}$, ρ, and g. For every t, integrate (4.9) from 0 to t.

$$\frac{1}{2}c_0\|u_n - u_0\|^2 + \int_0^t \|u_n - u_0\|^2 \leq tC(\|H\|_{X_1}, \rho, g)\,\|k_n - k_0\|_\infty + \int_0^t \|a_t\|_\infty \|u_n - u_0\|^2$$

since $H \in X_1$, there exist constant, denoted by C again for convenience such that

$$\|u_n - u_0\|^2 + \int_0^t \|u_n - u_0\|^2 \leq tC(\|H\|_{X_1}, \rho, g)\,\|k_n - k_0\|_\infty + C\int_0^t \|u_n - u_0\|^2$$

let

$$w'(t) = \|u_n - u_0\|^2 + \int_0^t \|u_n - u_0\|^2 \qquad (4.10)$$

so, integrate (4.10) from 0 to t, then

$$\begin{aligned} v(t) &= \int_0^t \|u_n - u_0\|^2 + \int_0^t \left(\int_0 \tau \|u_n(s) - u_0(s)\|^2 ds\right) d\tau \\ &= \int_0^t \|u_n - u_0\|^2 + \int_0^t (t-\tau)\|u_n(\tau) - u_0(\tau)\|^2 \\ &\geq \int_0^t \|u_n - u_0\|^2 \end{aligned}$$

and let

$$\Theta(t) = tC(\|H\|_{X_1}, \rho, g)\|k_n - k_0\|_\infty$$

then

$$\begin{aligned} v'(t) &= \|u_n - u_0\|^2 + \int_0^t \|u_n - u_0\|^2 \leq \Theta(t) + C\int_0^t \|u_n - u_0\|^2 \\ &\leq \Theta(t) + Cv(t) \end{aligned} \tag{4.11}$$

by solving (4.11),

$$v(t) \leq \int_0^t e^{C(t-s)}\Theta(s)ds$$

then

$$v'(t) \leq \Theta(t) + Ce^{Ct}\int_0^t \Theta(s)ds$$

therefore,

$$\|u_n - u_0\|^2 \leq t(1 - \frac{1}{2}t)C\ \|k_n - k_0\|_\infty \leq T(1 + \frac{1}{2}T)C\ \|k_n - k_0\|_\infty$$

we thus can conclude that $u_n \longmapsto u_0$, in $C^0(0,T;V)$ when $k_n \longmapsto k_0$ in $L_\infty(\Omega)$. ■

CHAPTER FIVE

Optimal Estimation Of Permeability

Formulation of Optimization Problem

In Chapter Two and Three we presented the boundary value problems to solve the pressure in fully saturated domain with given permeability. Based on the analysis of the existence of solution, a mapping from permeability k to the pressure u has been established. For any given permeability k, we can obtain an unique $u \in V$ by solving the variational problem. Let Z be a given Hilbert space serving as the space of observation. Let $C : V \longmapsto Z$ be a bounded linear operator and z be the desired target. Let $Q = H^2(\Omega)$, $Q_{ad} = \{k \in Q, k \geq c_0 > 0\}$, and consider the following optimization problem(OP): Find $k_0 \in Q_{ad}$, such that

$$J(k_0) = d = \inf\{J(k)|k \in Q_{ad}\}$$

with

$$J(k) = \int_0^T \|Cu(k) - z\|_Z^2 + \beta \|k\|_Q^2 \tag{5.1}$$

where β is a constant.

Remark 5.1 In this research, we consider the target functional $C : C(0, T; V) \longmapsto R^T$ as

$$C(u) = \int_{\Gamma_0} k\nabla u \cdot n d\sigma$$

which represents the volume of fluid leaving Γ_0 at time t. Other target functionals can also be considered as necessary for various application problems.

Proposition 5.2 There is a solution k_0 of optimization problem.

Proof: Let $d = \inf\{J(k)|k \in Q_{ad}\}$ and k_n be a decreasing sequence in Q_{ad} such that

$$J(k_n) \rightarrow d$$

so, there exist $M > 0$ such that $\|k_n\| < M$. Consequently, there is a subsequence k_{n_j} weakly converges to k_0 in H^2. Since Q_{ad} is closed, $k_0 \in Q_{ad}$. By the continuous dependence of u on

k, we have

$$\|u(k_{n_j}) - u(k_0)\| \to 0$$

Hence,

$$\lim_{j\to\infty} \inf J(k_{n_j}) \geq J(k_0)$$

it follows that

$$J(k_0) = d.$$

Thus, k_0 is a solution of the optimization problem. ■

Furthermore, in order to analyze more properties of solution, we define a functional

$$G : H^2 \longmapsto H^r, \quad G(k) = c_0 - k$$

where r is a constant satisfying $\frac{3}{2} < r < 2$, and reformulate the optimization problem as follows: Find k_0 which minimize

$$\begin{aligned} J(k) &= \int_0^T (Cu(k) - o(t))^2 + \beta \|k\|^2 \\ \text{subject to } G(k) &\leq 0 \end{aligned}$$

To obtain optimization condition, we apply the generalized Kuhn-Tucker theorem[5] which is quoted here.

Definition 5.3 Let X be a vector space and Z be a normed space with positive cone P having nonempty interior. Let G be a mapping $G : X \longmapsto Z$. A point $x_0 \in X$ is said to be a regular point of inequality $G(X) \leq 0$ if $G(x_0) \leq 0$, and there is $h \in X$ such that $G(x_0) + \delta G(x_0; h) < 0$.

Theorem 5.4(Generalized Kuhn-Tucker Theorem) Let X be a vector space and Z a normed space having positive cone P which has nonempty interior. Let f be a Gateaux differentiable mapping from X into Z. Assume that the Gateaux differentials are linear in their increments. Suppose X_0 is regular point of $G(x) \leq 0$. Then there is a $Z_0^* \in Z^*, Z_0^* \geq 0$ such that the Lagrangian

$$f(x) + \langle G(x), Z_0^* \rangle$$

is stationary at X_0, furthermore, $\langle G(x_0), Z_0^*\rangle = 0$.

Lemma 5.5 If k_0 is a solution of (OP), let $Z = H^r$ then there exists $\lambda \in Z^*$ such that the Lagrangian

$$F(k,\lambda) = J(k) + \langle \lambda, G(k)\rangle \tag{5.2}$$

is stationary at k_0.

Proof: For any $k \in Q_{ad}$ and any positive $h \in H^2$.

$$G(k) = c_0 - k \leq 0$$

and

$$G(k) + DG(k)h = c_0 - k - h \leq 0$$

so, regular point condition is satisfied. However, space C^0 has positive cone[5] and H^r imbeds into C^0, from the generalized Kuhn-Tucker Theorem there exist a positive $\lambda \in (H^r)^*$ such that solution k_0 of (OP) is a stationary point of F. ∎

From lemma 5.5, we obtain the necessary condition of k_0

$$DF(k_0,\lambda)h = DJ(k_0)h - \langle \lambda, h\rangle = 0 \tag{5.3}$$

Since

$$DJ(k_0)h = 2\int_0^T (Cu(t,k) - o(t)) \cdot Du(k_0,h) + 2\beta(k_0,h)_Q$$

then (5.3) can be written as

$$2\int_0^T (Cu(k) - o(t)) \cdot Du(k_0,h) + 2\beta(k_0,h)_Q - \langle \lambda, h\rangle = 0 \tag{5.4}$$

Lemma 5.6 Let u be the solution of (2.9), then the Gateaux differentials of u with respect to k exists and let $v = Du(k)$, then v is a solution of the following problem

$$\begin{aligned} -\nabla(k\nabla v) &= \nabla(h\nabla\Phi) \\ v &= 0 \quad \text{on } \Gamma \\ k\nabla v\cdot n &= 0 \quad \text{on } \partial W/\Gamma \end{aligned}$$

Proof: Let u be the solution of (2.9),

$$\nabla(k\nabla\Phi) = 0 \tag{5.5}$$

for any $\varepsilon > 0$ and increment $h \in L_\infty$. let $u(k+\varepsilon h)$ be the solution of (2.9) corresponding $k+\varepsilon h$.

$$\nabla((k+\varepsilon h)\nabla\Phi(k+\varepsilon h)) = 0 \tag{5.6}$$

Subtract (5.5) from (5.6),

$$\begin{aligned}\nabla((k+\varepsilon h)\nabla\Phi(k+\varepsilon h)) - \nabla(k\nabla\Phi) &= \nabla((k+\varepsilon h)(\nabla\Phi(k+\varepsilon h) - \nabla\Phi(k)) \\ &\quad +\nabla((k+\varepsilon h - k)\nabla\Phi(k))\end{aligned}$$

divide the equation by ε

$$\frac{\nabla((k+\varepsilon h)(\nabla u(k+\varepsilon h) - \nabla u(k))}{\varepsilon} = \nabla(h\nabla\Phi(k))$$

Let $v_\varepsilon = \frac{u(k+\varepsilon h)-u(k)}{\varepsilon}$.

$$\|\nabla((k-\varepsilon h)\nabla v_\varepsilon)\| = \|\nabla(h\nabla\Phi(k))\|$$

since the right hand side is independent of ε. let $\varepsilon \to 0$. then $\|\nabla(k\nabla v_\varepsilon)\|$ is bounded. therefore. $v_\varepsilon \rightharpoonup v$ weekly. and furthermore.

$$\nabla(k\nabla v) = \nabla(h\nabla\Phi(k)) \tag{5.7}$$

■

We next introduce the adjoint equation in order to obtain the regularity of the parameter. Let p be the adjoint. multiply p to both side of (5.7) and integrate over W:

$$\int_W -\nabla(k\nabla v)p = \int_W \nabla(h\nabla\Phi)p$$

Applying integration by part and boundary condition

$$\begin{aligned}\int_W -\nabla(k\nabla v)p &= -\int_{\partial W} k\nabla vp\cdot n d\sigma + \int_W k\nabla v\nabla p \\ &= -\int_\Gamma k\nabla vp\cdot n d\sigma + \int_{\partial W} vk\nabla p\cdot n d\sigma - \int_W v\nabla(k\nabla p) \\ &= -\int_\Gamma k\nabla vp\cdot n d\sigma + \int_{\partial W\setminus\Gamma} vk\nabla p\cdot n d\sigma - \int_W v\nabla(k\nabla p)\end{aligned}$$

$$\begin{aligned}\int_W \nabla(h\nabla\Phi)p &= \int_{\partial W} h\nabla\Phi p\cdot n d\sigma - \int_W h\nabla\Phi\nabla p \\ &= \int_{\Gamma\cup\Gamma_0} h\nabla\Phi p\cdot n d\sigma - \int_W h\nabla\Phi\nabla p\end{aligned}$$

impose the boundary condition

$$\begin{aligned} p &= 0 \quad \text{on } \Gamma \\ k\nabla p &= 0 \quad \text{on } \partial W/\Gamma\cup\Gamma_0\end{aligned}$$

then we can conclude that p satisfies

$$\int_{\Gamma_0} vk\nabla p\cdot n - \int_W v\cdot\nabla(k\nabla p) = -\int_W h\nabla\Phi\cdot\nabla p + \int_{\Gamma_0} h\nabla\Phi p\cdot n \tag{5.8}$$

We define the adjoint problem

$$\begin{aligned}\nabla(k\nabla p) &= 0 \\ k\nabla p\cdot n &= 0 \quad \text{on } \partial w/\Gamma\cup\Gamma_0 \\ p &= 0 \quad \text{on } \Gamma \\ k\nabla p\cdot n &= \nabla u(k_0) - z\end{aligned}$$

The existence of the solution of adjoint problem can be obtained similar to (VP). Now, if p is a solution of the adjoint equation, from (5.8),

$$\int_{\Gamma_0} (\nabla u(k_0) - z)v\cdot n = \int_{\Gamma_0} k\nabla pv\cdot n = -\int_W h\nabla\Phi\cdot\nabla p$$

therefore,

$$DJ(k_0)h = -2\int_0^T\int_W h\nabla\Phi\cdot\nabla p + 2\beta\langle k_0, h\rangle_Q$$

and

$$DF(k_0)h = -2\int_0^T\int_W h\nabla\Phi\cdot\nabla p + 2\beta\langle k_0, h\rangle_Q - \langle\lambda, h\rangle = 0$$

$$\lambda\in(H^r)' = \dot{H}^{-r} \text{ for } r > \frac{3}{2}$$

Since $p, u\in H^1$, $\nabla u\cdot\nabla p\in L^1$, so, for $r>1$, $\nabla u\cdot\nabla p\in H^{-r}$. We rewrite the equation as

$$\langle k_0, h\rangle_Q = \frac{1}{\beta}\int h\nabla u\cdot\nabla p + \frac{1}{2\beta}\langle\lambda, h\rangle \tag{5.9}$$

Let $\tilde{\lambda} = \frac{1}{2\beta}\lambda + \frac{1}{\beta}\nabla u \cdot \nabla p$. then $\tilde{\lambda} \in H^{-r}$, and the equation (5.9) becomes

$$(k_0, h)_Q = \langle \tilde{\lambda}, h \rangle \tag{5.10}$$

Introduce some basic properties of the intermediate spaces which can be used to show the result of regularity of the parameter. Let $V = H^2$, $W = L^2$, then the relation between V, W and their dual space can be given as

$$V \subset W \equiv W' \subset V'$$

and for any r, satisfying $\frac{3}{2} < r < 2$,

$$V \subset H^r \subset W \equiv W' \subset (H^r)' \subset V'$$

Define operator $A : H^2 \longrightarrow (H^r)'$ by

$$(Au, v) = (u, v)_Q$$

then the operator A has the following properties:

1°. Operator A has countably many positive eigenvalues $\lambda_i = w_i^2$, $i = 1, 2, \cdots$ such that

$$0 < \lambda_1 < \lambda_2 < \cdots.$$

and choose the corresponding eigenvectors e_i such that they form an orthonormal basis in W and orthogonal basis in V and V'.

2°. Let $v = \sum_{i=1}^{\infty} v_i e_i$, then $v \in V$ if and only if

$$\sum_{i=1}^{\infty} w_i^2 v_i^2 < +\infty$$

3°. Let $v = \sum_{i=1}^{\infty} v_i e_i$, define the fractional power of operator A, A^{α} by

$$A^{\alpha} v = \sum_{i=1}^{\infty} w_i^{2\alpha} v_i e_i$$

and

$$D(A^{\alpha}) = \{v = \sum_{i=1}^{\infty} v_i e_i | \sum_{i=1}^{\infty} w_i^{4\alpha} v_i^2 < +\infty\}$$

with the inner product $(u,v)_{D(A^\alpha)} = \sum_{i=1}^{\infty} w_i^{4\alpha} u_i v_i$, then

$$D(A^\alpha) \subset D(A^\beta) \quad \text{for } \alpha > \beta$$

with dense and compact imbedding.

4°. In our problem, we supposed $V = H^2, W = L^2$

$$\begin{aligned} D(A^{\frac{1}{2}}) &= \{v = \sum_{i=1}^{\infty} v_i e_i | \sum_{i=1}^{\infty} w_i^2 v_i^2 < +\infty\} = H^2 = V \\ D(A^0) &= \{v = \sum_{i=1}^{\infty} v_i e_i | \sum_{i=1}^{\infty} {}_i^2 v_i^2 < +\infty\} = L^2 = W \end{aligned}$$

Theorem 5.7 Let $\tilde{\lambda} \in (H^r)'$, with $\frac{3}{2} < r < 2$ and

$$(k_0, h)_Q = (\tilde{\lambda}, h)$$

then

$$k_0 \in H^{2-s}, \quad 0 \leq s < \frac{1}{2}$$

Proof: Let V, W and A^α be same as defined above, introduce the intermediate spaces for $0 \leq \theta \leq 1$

$$[V, W]_\theta = [H^2, H^0]_\theta = H^{(1-\theta)0+\theta 2} = H^{2\theta}$$

and

$$H^{2\theta} = D(A^{\frac{\theta}{2}})$$

so, we have

$$\begin{aligned} D(A^{\frac{1}{2}}) &= H^2 \\ D(A^{\frac{1}{4}}) &= H^1 \\ D(A^{\frac{3}{8}}) &= H^{\frac{3}{2}} \end{aligned}$$

Using property 3, for $\frac{3}{2} < r \leq 2$,

$$\begin{aligned} V &\subseteq H^r \subset H^{\frac{3}{2}} \\ D(A^{\frac{1}{2}}) &\subseteq H^r \subset D(A^{\frac{3}{8}}) \end{aligned}$$

therefore, there exist η, satisfying $\frac{3}{8} < \eta \leq \frac{1}{2}$, such that

$$D(A^{\eta}) = H^{r} \quad \text{for } \frac{3}{2} < r \leq 2,$$

However,

$$D(A^{-\eta}) = (H^{r})'$$

so,

$$\widetilde{\lambda} \in (H^{r})' = D(A^{-\eta})$$

From the definition of operator A,

$$\langle Ak_0, h\rangle = (k_0, h)_Q = \langle \widetilde{\lambda}, h\rangle$$

we have

$$Ak_0 \in D(A^{-\eta})$$

and from the Property 2,

$$Ak_0 = \sum_{i=1}^{\infty} w_i^2 k_i e_i \in D(A^{-\eta})$$

if and only if

$$\sum_{i=1}^{\infty} w_i^{-4\eta} w_i^4 k_i^2 < +\infty$$

that is,

$$\sum_{i=1}^{\infty} w_i^{4(1-\eta)} k_i^2 < +\infty$$

Therefore,

$$k_0 \in D(A^{1-\eta})$$

Let $\tau = 1 - \eta$, then τ satisfies $\frac{1}{2} \leq \tau < \frac{5}{8}$, from property 3 again, we obtain

$$D(A^{\frac{5}{8}}) \subset D(A^{\tau}) \subseteq D(A^{\frac{1}{2}})$$

and

$$H^{\frac{5}{2}} \subset D(A^{\tau}) \subseteq H^2$$

thus, for s satisfying $0 \leq s < \frac{1}{2}$,

$$k_0 \in H^{2+s}$$

Stability of parameter with respect to observation

Suppose original data is z_0, and let z be perturbed data, k_0 is the solution of Optimization Problem (OP) and k be the solutions of Optimization Problem with respect to perturbation $z(OP_z)$. We discuss the stability in the sense defined as follows[1]

Definition 5.8 The family of problems, (OP_z) is said to be stable at z_0, if there exist real number $r > 0, \alpha > 0$. and neighborhood V of z_0 such that for each $z \in V$,

(a) For any sequence $\{k_n\} \subset Q_r$with property $\lim_{n\to\infty} J(k_n, z) = e_r(z)$, it follows that $k_n \in intB(k_0, r)$ for sufficiently large n.

(b) If there exists a point $k_z \in Q_r$ with $e_r(z) = J(k_n, z)$, then $k_n \in intB(k_0, r)$ and

$$\|k_z - k_0\|_{H^2} \leq \alpha \|z - z_0\|^{\beta}$$

Proposition 5.9 For the Lagrangian defined in (5.2), there exists a Λ, such that

$$D^2F(k, \lambda)(h, h) \geq \Lambda \|h\|^2 \tag{5.11}$$

if

$$\beta - \|Cu(k) - z\|_Z \|D^2u\| > 0$$

Proof: Let $v_1 = Du(k)h$, $v_2 = D^2u(k)(h_1, h_2)$, and recall (5.2)

$$F(k, \lambda) = J(k) + \langle \lambda, G(k) \rangle$$

and

$$DJ(k_0)h = 2\int_0^T (Cu(t, k) - o(t)) \cdot Du(k_0, h) + 2\beta(k_0, h)_Q$$

so

$$DF(k, \lambda)h = 2\int_0^T (Cu(t, k) - o(t)) \cdot Du(k_0, h) + 2\beta(k_0, h)_Q - \langle \lambda, h \rangle$$

then the second order derivative of the Lagrangian F is given by

$$\begin{aligned} D^2F(k, \lambda)(h, h) &= 2\int (CDu)^2 + 2\int (Cu(k) - z)D^2u + 2\beta \|h\|_Q^2 \\ &\geq 2\int (Cu(k) - z)D^2u + 2\beta \|h\|_Q^2 \end{aligned}$$

$$\geq 2\beta \|h\|_Q^2 - 2\|(Cu(k) - z)\| \cdot \|D^2u\| \cdot \|h\|^2$$
$$= 2(\beta - \|(Cu(k) - z)\| \cdot \|D^2u\|) \|h\|^2$$

so, if $\beta - \|(Cu(k) - z)\| \cdot \|D^2u\| > 0$, let $\Lambda = 2(\beta - \|(Cu(k) - z)\| \cdot \|D^2u\|)$

$$D^2F(k,\lambda)(h,h) \leq \Lambda \|h\|^2$$

■

Applying the result from above Proposition into Theorem 5.6 in [6], we obtain the following theorem.

Theorem 5.10 If (5.11) hold at k_0, then there is $\alpha > 0$. $\rho > 0$. such that

$$J(k) \geq J(k_0) + \alpha \|k - k_0\|^2 \quad \text{for } k \in Q_{ad}$$

for all k satisfying $\|k - k_0\| \leq \rho$.

In family of optimization problem (OP_z), let

$$\begin{aligned} G(k,z) &= G(k) \\ J(k,z) &= \int_0^T (Cu(k) - z)^2 + \beta \|k\|^2 \end{aligned}$$

and K is closed convex cone defined by

$$K = \{G(k) | k \geq c_0\}$$

Definition 5.11 k_0 is called regular if $G(\cdot, z)$ is Gateaux-differentiable in a neighborhood of k for each z. G and DG are continuous at (k_0, z_0), and $0 \in int\{G(k_0, z_0) + D_kG(k_0, z_0)X - K\}$. where $X = H^2$.

Lemma 5.12 The optimal solution k_0 is regular.

Proof: All conditions can be easily verified. ■

Lemma 5.13 Let J, k, k_0, z, and z_0 be as defined in problem (OP) and (OP_z). Then there exists a constant $L > 0$ such that

$$|J(k,z) - J(k_0, z_0)| \leq L(\|k - k_0\|_Q^2 + \|z - z_0\|_Z^2)$$

Proof:

$$\begin{aligned}
|J(k,z) - J(k_0, z_0)| &= |\int_0^T \|Cu(k) - z\|_Z^2 + \beta\|k\|_Q^2 - \|Cu(k_0) - z_0\|_Z^2 + \beta\|k_0\|_Q^2 | \\
&\leq \int_0^T \|Cu(k) - z - Cu(k_0) + z_0\| + \beta\|k - k_0\|_Q^2 \\
&\leq \int_0^T \|Cu(k) - Cu(k_0)\|^2 + \|z - z_0\|^2 + \beta\|k - k_0\|_Q^2 \\
&\leq \|Cu\|^2 \cdot \|k - k_0\|^2 + \|z - z_0\|^2 + \beta\|k - k_0\|_Q^2 \\
&\leq L(\|k - k_0\|^2 + \|z - z_0\|^2)
\end{aligned}$$

where $L = (1 + \beta + \int_0^T \|Cu\|^2)$. ■

With the regularity of k_0 and the result of above lemma, all conditions of the theorem 5.6 in [6] are satisfied. Thus, we obtain the following theorem.

Theorem 5.14 The family of optimization problem (OP_z) is stable at k_0 in the sense defined in Definition 5.8.

CHAPTER SIX

Numerical Analysis

In this chapter, we present the numerical analysis of solving pressure from the variational problem discussed in previous chapters. The finite element method is introduced to find the approximation of the solution. From resulting pressure, Volume of fluid leaving Γ_0 can be calculated. Comparing the calculated outflow with given observed data, we define the error criteria and then use the Gradient Descent Method to estimate the parameter. In later of this chapter, the simulation results are also presented.

Approximation Theory

In general, approximation must be carried out for both the state function and parameter. To formulate corresponding discrete minimization problems for (OP), it is desirable to relate the admissible set for the discrete problem with that of (OP).

Let $\{\varphi_i\}_{i=1}^{M}$ be a linearly independent set of function in $Q = H^2(\Omega)$ and that span a subspace Q^M of Q. Assume that $I^M : Q \longrightarrow Q^M$ is interpolation operator with $I^M(k) = k^M$, then I^M has the following properties[8]:

1° For any $\varphi \in Q$,

$$\|I^M\varphi - \varphi\|_Q \le C\,\|\varphi\|_Q \tag{6.1}$$

2° For any $\varphi \in H^4$,

$$\|I^M\varphi - \varphi\|_Q \le \alpha(M)\,\|\varphi\|_{H^4} \tag{6.2}$$

where $\alpha(M)$ is a function independent of φ and $\alpha(M) \to 0$ as $M \to \infty$.

From the interpolation of linear operators [4], we conclude that for any $\varphi \in H^r(\Omega)$ with $r \in [2,4]$

$$\|I^M\varphi - \varphi\|_{H^2} \le \hat{\alpha}(M)\,\|\varphi\|_r \tag{6.3}$$

where $\hat{\alpha}(M) = C^{(4-r)/2}\alpha(M)^{(r-2)/2}$ for $r \in [2,4]$. Applying this discussion to the optimal coefficient k_0, we see from (6.3) and the theorem of regularity that

$$\left\|I^M k_0 - k_0\right\|_{H^2} \leq \hat{a}(M)\, \|k_0\|_r$$

from the embedding of H^2 into L^∞

$$|I^M k_0 - k_0| \leq c\left\|I^M k_0 - k_0\right\|_{H^2} \leq c\hat{a}(M)\, \|k_0\|_r$$

Setting $\varepsilon_M = c\hat{a}(M)\, \|k_0\|_r$, we define M sufficiently large such that $c - \varepsilon_M > 0$.

$$Q_{ad}^M = \{k \in Q^M : k \geq c - \varepsilon_M\}$$

we note that $I^M k_0 \in Q_{ad}^M$.

For the approximation of state, we let $\{B_l\}_{l=1}^N$ be a set of linearly independent functions in V, and $V^N = span\{B_l : l = 1, \ldots, N\}$. We assume that $V^N \subset V^{N-1} \subset \cdots V$ and that $cl(\bigcup_{N=1}^{\infty} V^N) = V$. Hence, for given $\varphi \in V$, there is sequence $\varphi_N \rightarrow \varphi$ as $N \rightarrow \infty$. We consider the following problem:

Find $u^N \in Q^N$ such that

$$a(u^N, B_l) = F(B_l) \qquad \text{for } l = 1, \ldots, N \tag{6.4}$$

and introduce the family of minimization problems as follows:

Find $k_0^{M,N} \in Q_{ad}$ such that

$$J^N(k_0^{M,N}) = \inf\{J^N(k) : k \in Q_{ad}^M\} \tag{6.5}$$

where

$$J^N(k) = \int_0^T (Cu^N(k) - o(t))^2 + \beta\, \|k\|^2$$

Remark 6.1 The existence of solution to problem (6.4) and (6.5) are straightforward.

Theorem 6.2 Let $\{k^{M_i}\} \in Q_{ad}^M$ with $k^{M_i} \rightarrow k$ weakly in Q and strongly in L^∞ as $M_i \rightarrow \infty$. Then the sequence of solutions $\{u^N(k^{M_i})\}$ of equation (6.4) has the property that $u^N(k^{M_i}) \rightarrow u(k)$ weakly in V.

Proof: Since $u^N(k^{M_i})$ is a solution of (6.4), we have estimate from (3.7)

$$\|u^N(k^{M_i})\| \leq \frac{1}{c-\varepsilon_M}\rho g C$$

where C is dependent on $\|H\|_X$. Therefore, $\{u^N(k^{M_i})\}$ is a bounded set in V. Then there exists a subsequence $\{u^{N_j}(k^{M_j})\}$ such that $u^{N_j}(k^{M_j}) \to u$ weakly in V. Since $\{k^{M_i}\}$ converges to a $k \in Q_{ad}$, it follows that for $\varphi \in V^N \subset V^{N+1} \subset ... \subset V$, we have

$$a(u^{N_j}, \varphi) = F(\varphi).$$

By uniqueness, we see that $u = u(k)$ and $u^N(k^{M_i}) \to u(k)$ as $N \to \infty$, and $M_i \to \infty$. ■

Theorem 6.3 Let $k^{M,N}$ be a solution of problem (6.5), and denote the set of solutions of (6.5) by $A^{M,N}$. $A = \cup_{M,N=1}^{\infty} A^{M,N}$, then any weak limit point of the set A is the solution of original (OP).

Proof: Suppose that $k_0^{M,N}$ is a solution of (6.5) and k_0 is the solution of original (OP), from the regularity results and the approximation assumption (6.1) and (6.2) and $I^M k_0 \to k_0$ in Q and $I^M k_0 \in Q_{ad}^M$, then

$$J^N(I^M k_0) \geq J^N(k_0^{M,N}).$$

From above theorem, we have

$$J^N(I^M k_0) \to J(k_0) \qquad \text{as } N, M \to \infty$$

therefore, the set A is bounded in Q. From theorem 6.2, A is closed and furthermore, compact. So, any sequence $\{k_0^{M_i,N_i}\}$ has a weak cluster point, say $\tilde{k}_0$. From Theorem 6.2, $u^{N_i}(k_0^{M_i,N_i}) \to u(\tilde{k}_0)$ as $i \to \infty$. Since function J is continuous, it follows that

$$\liminf J^{N_i}(k_0^{M_i,N_i}) \geq J(\tilde{k}_0),$$

therefore $J(k_0) \geq J(\tilde{k}_0)$. Since $\tilde{k}_0 \in Q$, then $\tilde{k}_0$ is a solution of original (OP). ■

Approximation of elliptic problem

From previous chapter, we posed the elliptic boundary value problem with minimization problem when assuming fluid is incompressible. In order to solve this problem numerically, we approximate the solution of the boundary value problem (2.9)-(2.11) for a given k by means of linear combinations of known functions. We must then update k by means of minimization procedure so as to decrease the functional $J(k)$ given in (5.1). We accomplish this in two cases. For the purpose of validating our models, in fact we take k to be much simpler that in the previous sections.

In the first case, we suppose that parameter k is a constant. Recall boundary value problem on $W(t)$

$$-\nabla(k\nabla\Phi) = 0$$

with boundary condition

$$\begin{aligned} u &= 0 \quad \text{on } \Gamma_0 \cup \Gamma \\ k\nabla\Phi &= 0 \quad \text{on } \partial W/\Gamma_0 \cup \Gamma. \end{aligned}$$

Multiplying equation by $\varphi \in V(t)$ and integrate over $W(t)$

$$\int_{W(t)} -\nabla(k\nabla\Phi)\varphi = 0$$

then integrate by part and apply the boundary condition, we obtain

$$\int_{W(t)} k\nabla u\nabla\varphi = -\rho g\int_{W(t)} k\varphi_z. \tag{6.6}$$

Since k is constant,

$$\int_{W(t)} \nabla u\nabla\varphi = -\rho g\int_{W(t)} \varphi_z.$$

Let $\{\varphi_i\}$ be set of linearly independent functions and express u as linear combination

$$u = \sum_{i=1}^{M} \alpha_i\varphi_i,$$

for $j = 1,\cdots,M$, we have M equations

$$\int_{W(t)} \sum_{i=1}^{M} \alpha_i\nabla\varphi_i \cdot \nabla\varphi_j = -\rho g\int_{W(t)} \varphi_{jz}. \tag{6.7}$$

Define $M \times M$ matrix

$$(G_0)_{ij} = \int_{W(t)} \nabla\varphi_i \cdot \nabla\varphi_j$$

and the M dimensional vector

$$(F_0)_j = -\rho g \int_{W(t)} \varphi_{jz},$$

then the discrete form of problem is given by

$$G_0 \alpha = F_0 \tag{6.8}$$

where $\alpha = (\alpha_1, \cdots, \alpha_M)^T$ is M dimension coefficient vector. By solving (6.8) we obtain α, then the approximation of pressure is given by

$$u = \sum_{i=1}^{M} \alpha_i \varphi_i.$$

To find the k which is assumed constant, we calculate the volume of fluid leaving Γ_0 at time t

$$\int_{\Gamma_0} k\nabla\Phi = \int_{\Gamma_0} k\nabla u = \int_{\Gamma_0} k \sum_{i=1}^{M} \alpha_i \nabla\varphi_i = k \sum_{i=1}^{M} \alpha_i \int_{\Gamma_0} \varphi_{yi}$$

and compare it with the given measurement $o(t)$. For the simplicity, let $f_1 = \sum_{i=1}^{M} \alpha_i \int_{\Gamma_0} \varphi_{yi}$, $f_2 = o(t)h(t)$. The parameter k is estimated by selecting the value of k that minimize the functional

$$J(k) = \int_0^T (kf_1 + f_2)^2.$$

and the solution can be obtained by

$$k = -\frac{\int_0^T f_1 f_2}{\int_0^T f_1^2} \tag{6.9}$$

In the second case, we Suppose that permeability k is a piecewise constant function.

$$k = \begin{cases} k_1 & z \leq h_0 \\ k_2 & z > h_0 \end{cases}$$

where h_0 is a positive number satisfying $0 < h_0 < z_0$. Without loss of generality, we suppose $h(t) \leq h_0$ for $t = 1, \cdots, T_1$ where $T_1 < T$. With the same procedure as that in the first case, we can estimate the k_1 by (6.9). Then take an initial guess for the value of k_2, write the both integral of (6.6) as sum of two integral in each domain the k is constant

$$k_1 \int_G \int_0^{h_0} \nabla u \nabla\varphi + k_2 \int_G \int_{h_0}^{z_0} \nabla u \nabla\varphi = -\rho g \int_G \int_0^{h_0} k_1 \varphi_z - -\rho g \int_G \int_{h_0}^{z_0} k_2 \varphi_z. \tag{6.10}$$

Let $\{\varphi_i\}$ be the set of linearly independent functions and set the linear combination

$$u^N = \sum_{i=1}^{N} \alpha_i(t)\varphi_i$$

$i = 1, \cdots, N$, to approximate u. Substituting u^N for u in equation (6.10) and replacing φ by φ_i for $i = 1, \dots, N$, we obtain M equations

$$\begin{aligned} & k_1 \sum_{i=1}^{M} \alpha_i(t) \int_G \int_0^{h_0} \nabla\varphi_i \nabla\varphi_j + k_2 \sum_{i=1}^{M} \alpha_i(t) \int_G \int_{h_0}^{z_0} \nabla\varphi_i \nabla\varphi_j \qquad (6.11) \\ = & -\rho g k_1 \int_G \int_0^{h_0} \varphi_{jz} - \rho g k_2 \int_G \int_{h_0}^{z_0} \varphi_{jz} \end{aligned}$$

Define $M \times M$ matrices

$$\begin{aligned} (G_1)_{ij} &= \int_G \int_0^{h_0} \nabla\varphi_i \nabla\varphi_j \\ (G_2)_{ij} &= \int_G \int_{h_0}^{z_0} \nabla\varphi_i \nabla\varphi_j \end{aligned}$$

and vectors

$$\begin{aligned} (F_1)_j &= -\rho g \int_G \int_0^{h_0} \varphi_{jz} \\ (F_2)_j &= -\rho g k_2 \int_G \int_{h_0}^{z_0} \varphi_{jz} \end{aligned}$$

and let $\alpha(t) = (\alpha_1(t), \alpha_2(t), \cdots, \alpha_M(t))^T$, then the (6.11) can be expressed by

$$(k_1 G_1 + k_2 G_2)\alpha(t) = k_1 F_1 + k_2 F_2 \qquad (6.12)$$

After solving (6.12) for $\alpha(t)$, the approximation of pressure is given by

$$u = \sum_{i=1}^{M} \alpha_i(t)\varphi_i.$$

Calculate the volume of fluid leaving Γ_0 at time t

$$\int_{\Gamma_0} k\nabla\Phi = \int_{\Gamma_0} k\nabla u = \int_{\Gamma_0} k \sum_{i=1}^{M} \alpha_i \nabla\varphi_i = k_1 \sum_{i=1}^{M} \alpha_i \int_0^{L_1} \int_0^{h_0} \varphi_{yi} + k_2 \sum_{i=1}^{M} \alpha_i \int_0^{L_1} \int_{h_0}^{z_0} \varphi_{yi}$$

and compare it with the given measurement $o(t)$, and then construct the minimization problem:

Find k_2 minimize the

$$J(k_2) = \int_0^T (k_1 \sum_{i=1}^{M} \alpha_i \int_0^{L_1} \int_0^{h_0} \varphi_{yi} + k_2 \sum_{i=1}^{M} \alpha_i \int_0^{L_1} \int_{h_0}^{z_0} \varphi_{yi}) + o(t)h(t))^2 dt. \quad (6.13)$$

Let

$$\begin{aligned} \eta_1 &= \int_0^{L_1} \int_0^{h_0} \varphi_{yi} \\ \eta_2 &= \int_0^{L_1} \int_{h_0}^{z_0} \varphi_{yi} \end{aligned}$$

then the equation (6.13) can be written in vector form as

$$J(k_2) = \int_0^T (k_1 \eta_1^T \alpha(t) + k_2 \eta_2^T \alpha(t) + o(t)h(t))^2$$

Proposition 6.4 The derivative of $J(k_2)$ with respect to k_2 can be calculated as follow

$$DJ(k_2) = 2 \int_0^T (k_1 \eta_1^T \alpha(t) + k_2 \eta_2^T \alpha(t) + o(t)h(t))[(k_1 \eta_1^T + k_2 \eta_2^T) D\alpha(t) + \eta_2^T \alpha] \quad (6.14)$$

and $D\alpha(t)$ can be obtained by solving the equation

$$(k_1 G_1 + k_2 G_2) D\alpha + G_2 \alpha = F_2 \quad (6.15)$$

Now, we summarize the algorithm used to solve the numerical problem.

Algorithm−1

- Step-1 Select all the data such that height $h(t) < h_0$, and Compute k_1 from (6.9) described in the first case.
- Step-2 Take an initial guess of k_2 denoted by $k_2^{(0)}$, and compute the $\alpha(k_2^{(0)})$ as the solution of (6.12).
- Step-3 Compute $D\alpha(t)$ by solving equation (6.15) for $t = 1, \cdots, T$.
- Step-4 Compute $DJ(k_2^{(0)})$ from (6.14).
- Step-5 Update k_2, define $k_2^{(1)} = k_2^{(0)} - \mu DJ(k_2^{(0)})$, where μ is chosen so that $J(k_2^{(1)}) \leq J(k_2^{(0)})$.
- Step-6 If $J(k_2^{(1)})$ is less than the predefined threshold, stop; if it is not, goto step 2.

Approximation of parabolic problem

Consider the fluid is compressible with compressibility c, we resulted in the parabolic equation with boundary and initial conditions from previous chapters. Also, we posed the minimization problem to estimate the permeability. In order to solve this problem numerically, we approximate the solution of the variational problem (2.11)-(2.13) for a given k by means of linear combinations of known functions. We must then update k by means of minimization procedure so as to decrease the functional $J(k)$ defined in (5.1). we assume that

$$H(x, y, z, t) = h(t)$$

then,

$$a_t = \frac{\zeta h'(t)}{h(t)}, a_1 = 0, a_2 = 0, a_3 = \frac{1}{h} \tag{6.16}$$

and consider two cases.

In the first case, we suppose that parameter k is a constant, $L_1 = L_2 = 1$, and recall parabolic equation (2.13) with boundary condition (2.14)-(2.15) and initial condition (2.16). Under change of variables defined in (3.1),

$$c(\frac{\partial\Phi}{\partial t} - a_t\frac{\partial\Phi}{\partial\zeta}) - (\frac{\partial}{\partial\xi} - a_1\frac{\partial}{\partial\zeta}, \frac{\partial}{\partial\eta} - a_2\frac{\partial}{\partial\zeta}, a_3\frac{\partial}{\partial\zeta}) \cdot k(\frac{\partial\Phi}{\partial\xi} - a_1\frac{\partial\Phi}{\partial\zeta}, \frac{\partial\Phi}{\partial\eta} - a_2\frac{\partial\Phi}{\partial\zeta}, a_3\frac{\partial\Phi}{\partial\zeta}) = 0.$$

Multiplying equation by $\varphi \in V(t)$ and integrating over $W_0(t)$, then applying (6.16)

$$\int_{W_0} c\frac{\partial\Phi}{\partial t}\varphi - \int_{W_0} c\frac{\zeta h'(t)}{h(t)}\frac{\partial\Phi}{\partial\zeta}\varphi - \int_{W_0}(\frac{\partial}{\partial\xi}, \frac{\partial}{\partial\eta}, a_3\frac{\partial}{\partial\zeta}) \cdot k(\frac{\partial\Phi}{\partial\xi}, \frac{\partial\Phi}{\partial\eta}, a_3\frac{\partial\Phi}{\partial\zeta})\varphi = 0.$$

integrate by part and apply the boundary condition, we obtain

$$\int_{W_0} c\Phi_t\varphi - \int_{W_0} c\frac{\zeta h'(t)}{h(t)}\Phi_\zeta\varphi + \int_{W_0} k(\Phi_\xi\varphi_\xi + \Phi_\eta\varphi_\eta + \frac{1}{h^2}\Phi_\zeta\varphi_\zeta) = 0.$$

Since $\Phi = p + \rho g z$,

$$\int_{W_0} cu_t\varphi - \int_{W_0} c\frac{\zeta h'(t)}{h(t)}u_\zeta\varphi + \int_{W_0} k(u_\xi\varphi_\xi + u_\eta\varphi_\eta + \frac{1}{h^2}u_\zeta\varphi_\zeta) + \frac{1}{h}\int_{W_0} k\varphi_\zeta = 0.$$

Let $\{\varphi_i\}$ be set of linearly independent functions and express u as linear combination

$$u = \sum_{i=1}^{M} \alpha_i(t)\varphi_i.$$

For $j = 1, \cdots, M$. we have M equations

$$\int_{W_0} c\sum_{i=1}^{M} \alpha_i'(t)\varphi_i\varphi_j - \int_{W_0} c\frac{\zeta h'(t)}{h(t)}\sum_{i=1}^{M} \alpha_i(t)\varphi_{i\zeta}\varphi_j + \int_{W_0} k\sum_{i=1}^{M} \alpha_i(t)(\varphi_{i\xi}\varphi_{j\xi}$$
$$+\varphi_{i\eta}\varphi_{j\eta} + \frac{1}{h^2}\varphi_{i\zeta}\varphi_{j\zeta}) + \frac{1}{h}\int_{W_0} k\varphi_{j\zeta} = 0.$$

Define $M \times M$ matrices

$$\begin{aligned}
(G_0)_{ij} &= \int_{W_0} \varphi_i\varphi_j \\
(G_1)_{ij} &= -\int_{W_0} \zeta\varphi_{i\zeta}\varphi_j \\
(G_2)_{ij} &= \int_{W_0} (\varphi_{i\xi}\varphi_{j\xi} + \varphi_{i\eta}\varphi_{j\eta}) \\
(G_3)_{ij} &= \int_{W_0} \varphi_{i\zeta}\varphi_{j\zeta}
\end{aligned}$$

and the M dimensional vector

$$(F)_j = -\rho g\int_{W_0} \varphi_{j\zeta}.$$

then the discrete form of problem is given by

$$cG_0\alpha'(t) + c\frac{h'}{h}G_1\alpha(t) + kG_2\alpha(t) + \frac{k}{h^2}G_3\alpha(t) = \frac{k}{h}F. \qquad (6.17)$$

where α is M dimension coefficients vector, $\alpha = (\alpha_1, \cdots, \alpha_M)^T$. Take the Δt as time step. the discrete equation of (6.17) with respect to t can be written as

$$cG_0\frac{\alpha(t+\Delta t) - \alpha(t)}{\Delta t} + (c\frac{h'}{h}G_1 + kG_2 + \frac{k}{h^2}G_3)\frac{\alpha(t+\Delta t) - \alpha(t)}{2} = \frac{k}{h}F \qquad (6.18)$$

Note that the initial condition $u(\cdot, t)|_{t=0} = u_0$. and assume that $I^M(u_0) = \sum_{i=1}^{M} \alpha_{0i}\varphi_i$. therefore. by taking initial value $\alpha_i(0) = \alpha_{i0}$, we can solve $\alpha(t)$ from (6.18) iteratively for $t = 1, \ldots, T$. Then the approximation of pressure is given by

$$u = \sum_{i=1}^{M} \alpha_i(t)\varphi_i, \qquad \text{for } t = 1, \ldots, T.$$

To find the k which is assumed constant, we calculate the volume of fluid leaving Γ_0 at time t

$$\int_{\Gamma_0} k\nabla\Phi = \int_{\Gamma_0} k\nabla u = \int_{\Gamma_0} k\sum_{i=1}^{M} \alpha_i\nabla\varphi_i = k\sum_{i=1}^{M} \alpha_i\int_{\Gamma_0} \varphi_{yi}$$

compare it with the given measurement $o(t)$, and construct the minimization problem: Find k minimizing the

$$J(k) = \int_0^T (k \sum_{i=1}^{M} \alpha_i \int_0^1 \int_0^1 \varphi_{\eta i} + o(t)h(t))^2 dt \tag{6.19}$$

Let

$$\eta = \int_0^1 \int_0^1 \varphi_{\eta i}$$

then the equation (6.19) can be written in vector form as

$$J(k) = \int_0^T (k\eta^T \alpha(t) + o(t)h(t))^2$$

Proposition 6.5 The derivative of $J(k)$ with respect to k can be calculated as

$$DJ(k) = 2 \int_0^T (k\eta^T \alpha(t) + o(t)h(t))[k\eta^T D\alpha(t) + \eta^T \alpha] \tag{6.20}$$

and $D\alpha(t)$ can be obtained by solving the equation iteratively with given initial value of $D\alpha(0)$

$$cG_0 D\alpha'(t) - (c\frac{h'}{h}G_1 + kG_2 + \frac{k}{h^2}G_3)D\alpha = \frac{F}{h} - (G_2 + \frac{1}{h}G_3)\alpha(t). \tag{6.21}$$

Now, we summarize the algorithm used to solve the numerical problem.

Algorithm−2

- Step-1 Initialize the permeability k.
- Step-2 Computer $\alpha(k^{(0)})$ as the solution of (6.18).
- Step-3 Compute $D\alpha(t)$ by solving equation (6.21) for $t = 1, \cdots T$.
- Step-4 Compute $DJ(k^{(0)})$ from (6.20).
- Step-5 Update k. define $k^{(1)} = k^{(0)} - \mu DJ(k^{(0)})$, where μ is chosen so that $J(k^{(1)}) \leq J(k^{(0)})$.
- Step-6 If $J(k^{(1)})$ is less than the predefined threshold, stop; if it is not, go to step 2.

In the second case, we Suppose that permeability k is a piecewise constant function,

$$k = \begin{cases} k_1 & z \leq h_0 \\ k_2 & z > h_0 \end{cases}$$

where h_0 is a positive number satisfying $0 < h_0 < z_0$. Initialize k_1 and k_2, and write the both integral of (6.5) as sum of two integral in each domain the k is constant

$$\int_{w_0} c\sum_{i=1}^{M}\alpha_i'(t)\varphi_i\varphi_j - \int_{w_0} c\frac{\zeta h'(t)}{h(t)}\sum_{i=1}^{M}\alpha_i(t)\varphi_{i\zeta}\varphi_j + \int_G\int_0^{h_1} k_1\sum_{i=1}^{M}\alpha_i(t)(\varphi_{i\xi}\varphi_{j\xi}+\varphi_{i\eta}\varphi_{j\eta}+\frac{1}{h^2}\varphi_{i\zeta}\varphi_{j\zeta})$$

$$+\int_G\int_{h_1}^{1} k_2\sum_{i=1}^{M}\alpha_i(t)(\varphi_{i\xi}\varphi_{j\xi}+\varphi_{i\eta}\varphi_{j\eta}+\frac{1}{h^2}\varphi_{i\zeta}\varphi_{j\zeta})+\frac{1}{h}\int_G\int_0^{h_1} k_1\varphi_{j\zeta}+\frac{1}{h}\int_G\int_{h_1}^{1} k_2\varphi_{j\zeta}=0 \tag{6.22}$$

where $h_1 = \frac{h_0}{h(t)}$.

Define $M \times M$ matrices

$$\begin{aligned}
(G_0)_{ij} &= \int_{w_0} \varphi_i\varphi_j\\
(G_1)_{ij} &= -\int_{w_0} \zeta\varphi_{i\zeta}\varphi_j\\
(G_{21})_{ij} &= \int_G\int_0^{h_1}(\varphi_{i\xi}\varphi_{j\xi}+\varphi_{i\eta}\varphi_{j\eta})\\
(G_{22})_{ij} &= \int_G\int_{h_1}^{1}(\varphi_{i\xi}\varphi_{j\xi}+\varphi_{i\eta}\varphi_{j\eta})\\
(G_{31})_{ij} &= \int_G\int_0^{h_1}\varphi_{i\zeta}\varphi_{j\zeta}\\
(G_{32})_{ij} &= \int_G\int_{h_1}^{1}\varphi_{i\zeta}\varphi_{j\zeta}
\end{aligned}$$

and M dimension vectors

$$\begin{aligned}
F_1 &= -\rho g\int_G\int_0^{h_1}\varphi_{j\zeta}\\
F_2 &= -\rho g\int_G\int_{h_1}^{1}\varphi_{j\zeta}.
\end{aligned}$$

then the (6.22) can be expressed as

$$\begin{aligned}
& cG_0\alpha'(t)+c\frac{h'}{h}G_1\alpha(t)+(k_1G_{21}+k_2G_{22})\alpha(t)+\frac{1}{h^2}(k_1G_{31}+k_2G_{32})\alpha(t)\\
=\ & \frac{1}{h}(k_1F_1+k_2F_2)
\end{aligned} \tag{6.23}$$

Solving (6.23) for $\alpha(t)$ with initial $\alpha(0) = \alpha_0$ same as in the first case, for $t = 1, \ldots, T$. Then the approximation of pressure is given by

$$u = \sum_{i=1}^{M} \alpha_i(t)\varphi_i.$$

To estimate k_1 and k_2, we calculate the volume of fluid leaving Γ_0 at time t

$$\int_{\Gamma_0} k\nabla\Phi = \int_{\Gamma_0} k\nabla u = \int_{\Gamma_0} k\sum_{i=1}^{M} \alpha_i \nabla\varphi_i = k_1 \sum_{i=1}^{M} \alpha_i \int_0^1 \int_0^{h_1} \varphi_{yi} + k_2 \sum_{i=1}^{M} \alpha_i \int_0^1 \int_{h_1}^1 \varphi_{yi}.$$

compare it with the given measurement $o(t)$, and construct the minimization problem: Find k_1, k_2 minimize the

$$J(k_1, k_2) = \int_0^T (k_1 \sum_{i=1}^{M} \alpha_i \int_0^1 \int_0^{h_1} \varphi_{yi} + k_2 \sum_{i=1}^{M} \alpha_i \int_0^1 \int_{h_1}^1 \varphi_{yi}) + o(t)h(t))^2 dt. \quad (6.24)$$

Let

$$\begin{aligned} \eta_1 &= \int_0^1 \int_0^{h_1} \varphi_{yi} \\ \eta_2 &= \int_0^1 \int_{h_1}^1 \varphi_{yi}, \end{aligned}$$

then the equation (6.24) can be written in vector form as

$$J(k_1, k_2) = \int_0^T (k_1\eta_1^T\alpha(t) + k_2\eta_2^T\alpha(t) + o(t)h(t))^2.$$

Proposition 6.6 The derivative of $J(k_1, k_2)$ with respect to k_1 and k_2 can be calculated respectively as follow:

$$\begin{aligned} D_{k_1}J(k_1, k_2) &= 2\int_0^T (k_1\eta_1^T\alpha(t) + k_2\eta_2^T\alpha(t) \quad (6.25)\\ &\quad + o(t)h(t))[(k_1\eta_1^T + k_2\eta_2^T)D_{k_1}\alpha(t) + \eta_1^T\alpha] \\ D_{k_2}J(k_1, k_2) &= 2\int_0^T (k_1\eta_1^T\alpha(t) + k_2\eta_2^T\alpha(t) \quad (6.26)\\ &\quad + o(t)h(t))[(k_1\eta_1^T + k_2\eta_2^T)D_{k_2}\alpha(t) + \eta_2^T\alpha] \end{aligned}$$

and $D_{k_1}\alpha(t)$ and $D_{k_2}\alpha(t)$ can be obtained respectively by solving the equations

$$\begin{aligned} & cG_0 D_{k_1}\alpha'(t) + (c\frac{h'}{h}G_1 + k_1G_{21} + k_2G_{22} + \frac{k_1}{h^2}G_{31} + \frac{k_2}{h^2}G_{32})D_{k_1}\alpha \\ = & \frac{F_1}{h} - (G_{21} - \frac{1}{h}G_{31})\alpha(t) \quad (6.27) \end{aligned}$$

$$
\begin{aligned}
& cG_0 D_{k2}\alpha'(t) + (c\frac{h'}{h}G_1 + k_1 G_{21} + k_2 G_{22} + \frac{k_1}{h^2}G_{31} + \frac{k_2}{h^2}G_{32})D_{k_2}\alpha \\
= & \frac{F_2}{h} - (G_{22} + \frac{1}{h}G_{32})\alpha(t).
\end{aligned} \tag{6.28}
$$

Now, we summarize the algorithm used to solve the numerical problem.

Algorithm−3

- Step-1 Take initial values of k_1 and k_2 denoted by $k_1^{(0)}$ and $k_2^{(0)}$.
- Step-2 Compute the α as the solution of (6.23)
- Step-3 Compute $D_{k_1}\alpha(t)$ and $D_{k_2}\alpha(t)$ by solving equation (6.27) and (6.28) respectively for $t = 1, ..., T$.
- Step-4 Compute $D_{k_1}J(k_1^{(0)}, k_2^{(0)})$ and $D_{k_2}J(k_1^{(0)}, k_2^{(0)})$ from (6.25) and (6.26) respectively.
- Step-5 Define $k_1^{(1)} = k_1^{(0)} - \mu DJ_{k_1}(k_1^{(0)}, k_2^{(0)}), k_2^{(1)} = k_2^{(0)} - \nu D_{k_2}J(k_1^{(0)}, k_2^{(0)})$, where μ and ν are chosen so that $J(k_1^{(1)}, k_2^{(1)}) \le J(k_1^{(0)}, k_2^{(0)})$.
- Step-6 If $J(k_1^{(1)}, k_2^{(1)})$ is less than the predefined threshold, stop; if it is not, goto step 2.

Simulation-1

In this simulation, the rainfall and measurement outflow data are collected for May. 11, 1993[11] as follow:

rainfall	1500.0	0.0	12000.0	0.0	2000.0	500.0	0.0	0.0	0.0	0.0	0.0
outflow	0.0	10.0	40.0	220.0	175.0	130.0	70.0	40.0	35.0	25.0	20.0

Take $a = 0.0001$, $b = 0.002$ in the height equation, and solve the equation with time step 1.0, then we obtain the height

Height	0.0	0.15	0.123	0.91	0.00009	0.073	0.1226	0.0022	0.001	0.00046

When fluid is assumed incompressible, or compressibility is small enough to be ignored, we solve the elliptic equation and optimization problem numerically in two cases. In case-1, assuming permeability is constant, we have the estimated outflow with relative error 0.09, and

$k = 1.9024$. See Observed outflow(O) and Estimated outflow(E) as follow. Also see Figure-3.

O	0.0	0.03	0.098	4.003	0.00034	0.189	0.01712	0.00173	0.00064	0.00022
E	0.0	0.348	0.247	3.968	0.000001	0.095	0.0029	0.00001	0.00001	0.00003

In case−2, when permeability is piecewise constant, we estimate k_1 by (6.9) and take initial value of $k_2 = .1$, follow the Algorithm−1 we have the estimated outflow with relative error 0.03, and $k_1 = 11.3118$, and $k_2 = 3.1833$. See Figure-4.

O	0	0.03	0.098	4.0028	0.00034	0.189	0.01712	0.00173	0.00064	0.00022
E	0	0.04	0.032	4.0032	0.000001	0.0047	0.01719	0.0005	0.00009	0.00002

When fluid is compressible with compressibility c, we solve the parabolic equation and the optimization problem numerically in also two cases. In the first case, we suppose permeability is constant, using Algorithm−2, we obtain the estimated outflow with relative error 0.25 and permeability is estimated as $k = 0.178$. See Figure-5.

O	0	0.0299	0.108	4.79809	0.00017	0.2773	0.0741	0.02038	0.0118	0.0059
E	0	0.8486	0.072	4.46397	0.00020	0.9984	0.3153	0.19688	0.1106	0.0878

Also with constant permeability, we take the time step as .5, we solve the height and use Algorithm−2, we obtain the estimated outflow with relative error 0.0438 and the permeability is estimated as $k = 1.123$. See Figure-6.

In the second case, we suppose the permeability is piecewise constant, we take initial value of $k_1 = 0.1$, and $k_2 = 0.1$, and apply the Algorithm−3, we estimated the outflow with relative error 0.0021 and the permeability is estimated as $k_1 = .2$, $k_2 = 1.96$. See Figure-7.

O	0	0.03	0.0987	4.0028	0.00034	0.18902	0.0172	0.00173	0.00064	0.0002
E	0	0.1938	0.0188	3.985	0.00008	0.18948	0.0019	0.00176	0.00075	0.00036

Simulation-2

In this simulation, the rainfall and measurement outflow data are collected for April, 14, 1993[11] as follow:

rainfall	0.0	270	200	300	300	380	400	380	280	200	100
outflow	3000	0.0	4000	1000	2000	0.0	7500	750	1000	5000	0.0

Take $a = 0.001$, $b = 0.002$ in the height equation, and solve the equation with time step 1.0, then we obtain the height

Height	0.0	0.000099	0.270100	0.000099	0.15	0.05	0.43	0.000099	0.217	0.14

When fluid is assumed incompressible, or compressibility is small enough to be ignored, we solve the elliptic equation and optimization problem numerically in two cases. In case-1, assuming permeability is constant, we have the estimated outflow with relative error 0.3284, and $k = 5.96054$. Also see Figure-8.

O	0	0	2.16	0.000199	0.6	0	6.45	0.000149	0.4343	0.14
E	0	0.0000006	2.762	0.0000006	1.09	0.146	5.29	0.0000006	1.9854	0.971

In case−2, when permeability is piecewise constant, we take initial value of $k_1 = .1$, $k_2 = .1$, follow the Algorithm−1 we have the estimated outflow with relative error 0.08008, and $k_1 = 18.01$, $k_2 = 75.76$. See Figure 9.

O	0	0	2.16	0.00019	0.6	0.0	6.45	0.0001499	0.4343144	0.14
E	0	0	2.158	0.00000	0.2246	0.0	6.412188	0.00	0.8313454	0.16043

When fluid is compressible with compressibility c, we solve the parabolic equation and the optimization problem numerically in also two cases. In the first case, we suppose permeability is constant, using Algorithm−2, we obtain the estimated outflow with relative error 0.562, permeability $k = 2.21$ when porosity $\phi = 0.1$ and relative error 0.559, permeability $k = 2.16$ when porosity $\phi = 0.05$. see Figure-10,11.

In the second case, we suppose the permeability is piecewise constant, we take initial value of $k_1 = 0.1$, and $k_2 = 0.1$, and apply the Algorithm−3, we estimated the outflow with relative error 0.05419, permeability $k_1 = .158$, $k_2 = 6.05$ when porosity $\phi = 0.1$ and relative error 0.055, permeability $k_1 = 0.165$, $k_2 = 6.095$ when porosity $\phi = 0.05$. See Figure-12,13.

CHAPTER SEVEN
SUMMARY

In this dissertation, we have developed the mathematical models for the fully saturated domain. The surface of the fully saturated domain H satisfies a ordinary differential equation, the solution of this equation with assigned initial condition determines the time dependent saturated domain. The model in saturated domain for the pressure results in a elliptic boundary value problem and a parabolic boundary and initial value problem in terms of the compressibility of the fluid. The existence of solution and continuous dependence of the solution with respect to the parameters have been proved. Also, the optimization problem has been analyzed to approximate the permeability.. The simulation results has been presented for two set of data collected from May, 11 and April, 13, 1996. However, the model discussed in this dissertation has been simplified under some assumption, such as we assume that water entered in the top surface Γ_s will immediately enter the Γ. For more complicated case, we need to consider the time delay, and we need to study on the sensitivity of solution with respect to $\beta(x, y)$ and the initial condition. Moreover, the analysis on the unsaturated domain need to be developed.

References

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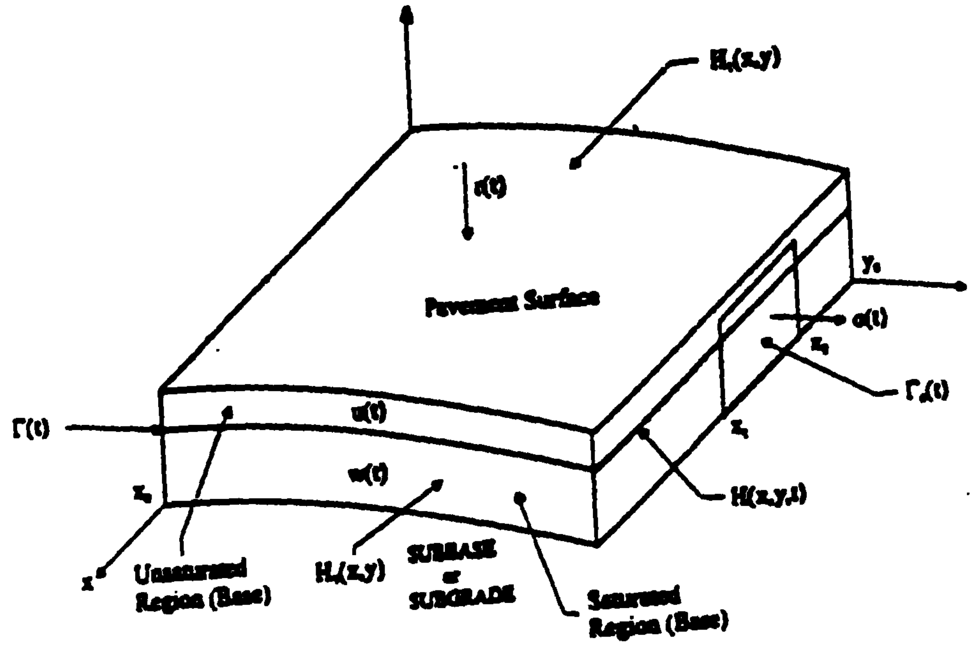


Figure 1: Drainage Model

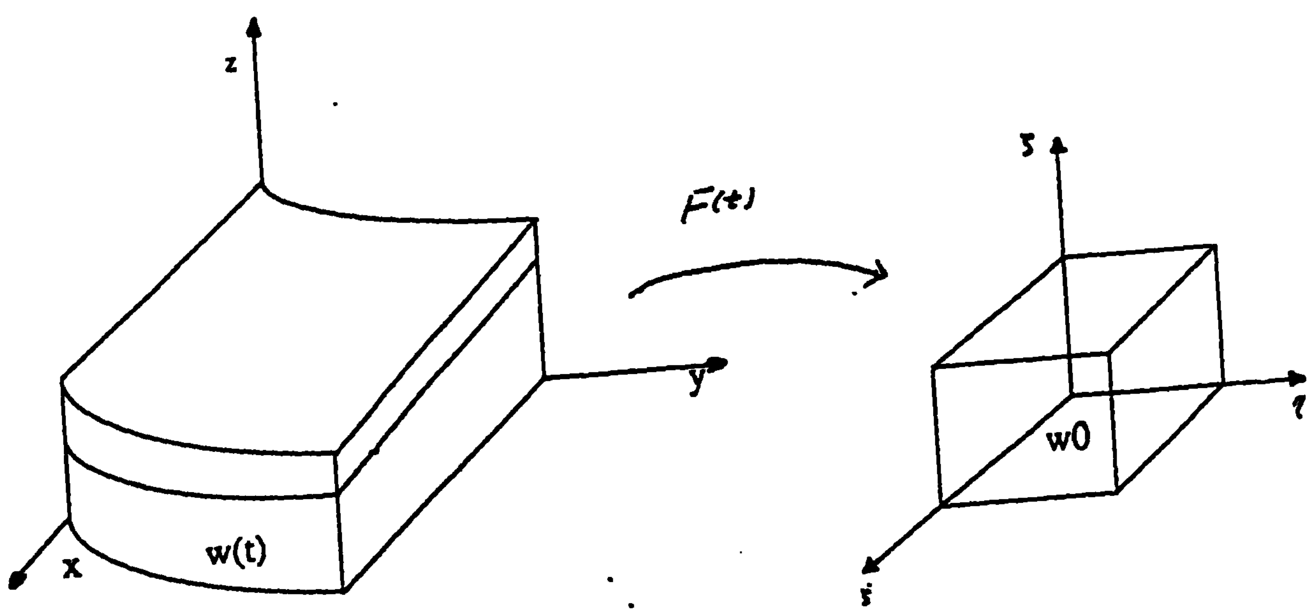


Figure 2: Change of Variables

Figure-3: Estimation of Outflow from Elliptic Problem

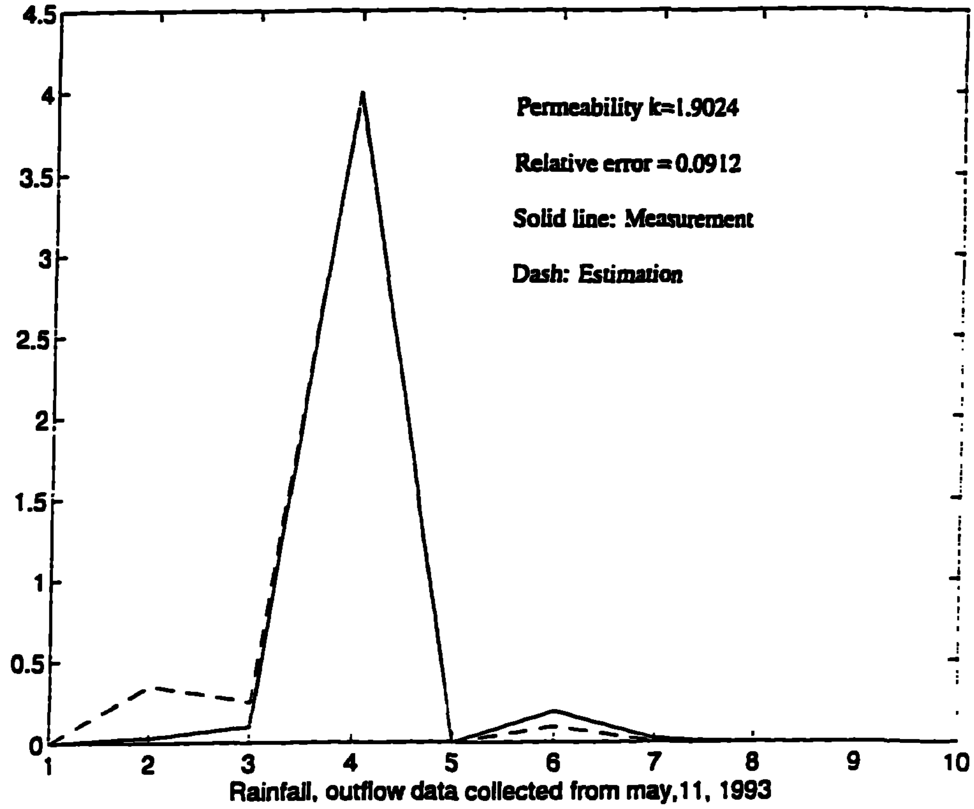


Figure-4: Estimation of Outflow from Elliptic Problem

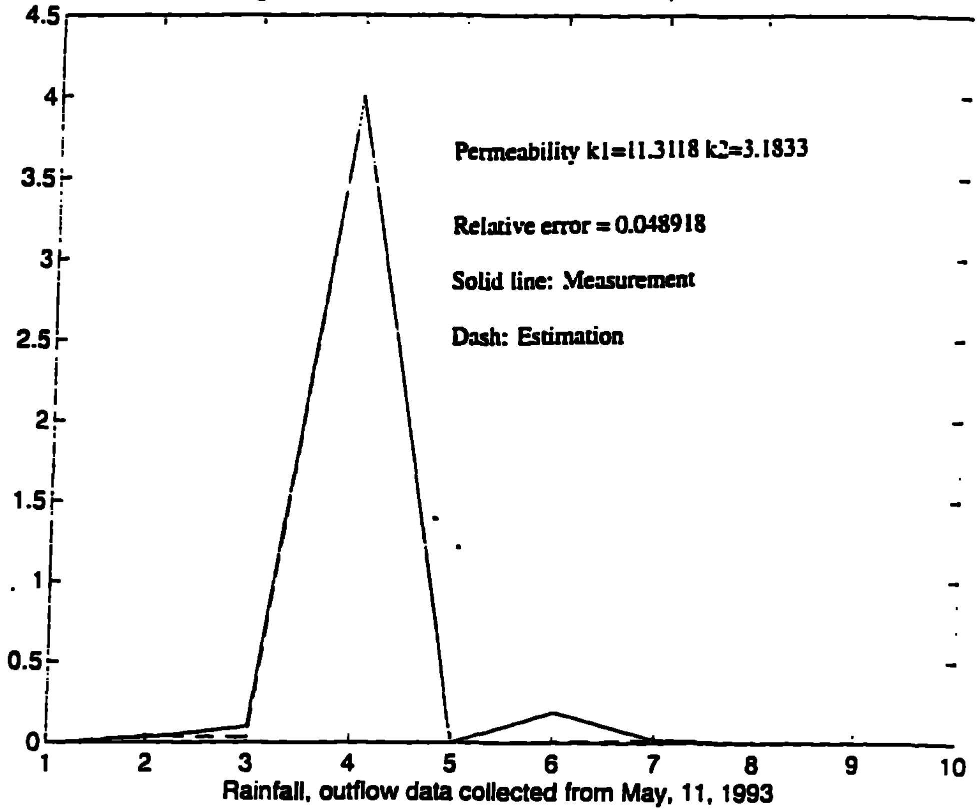
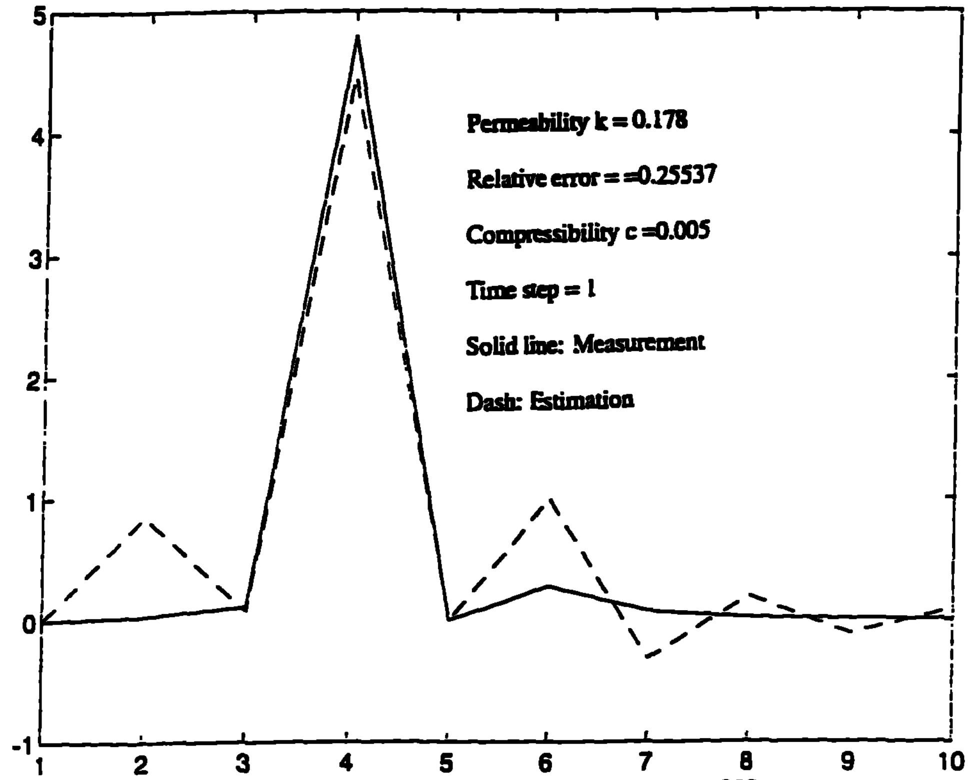
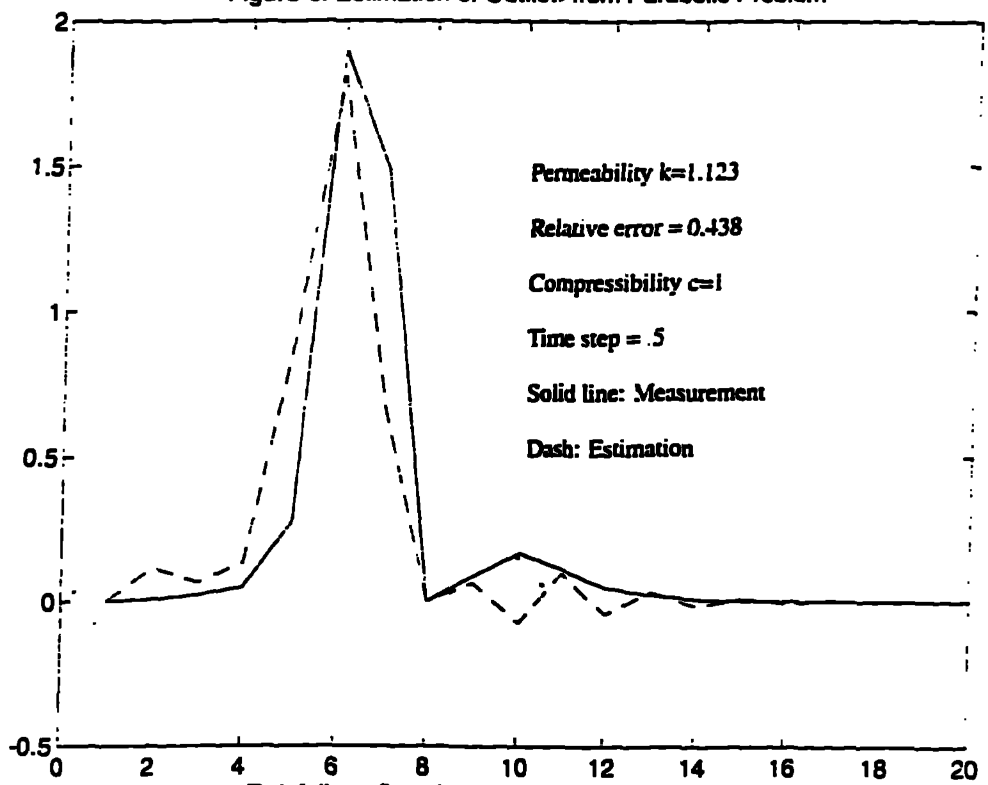


Figure-5: Estimation of Outflow from Parabolic Problem



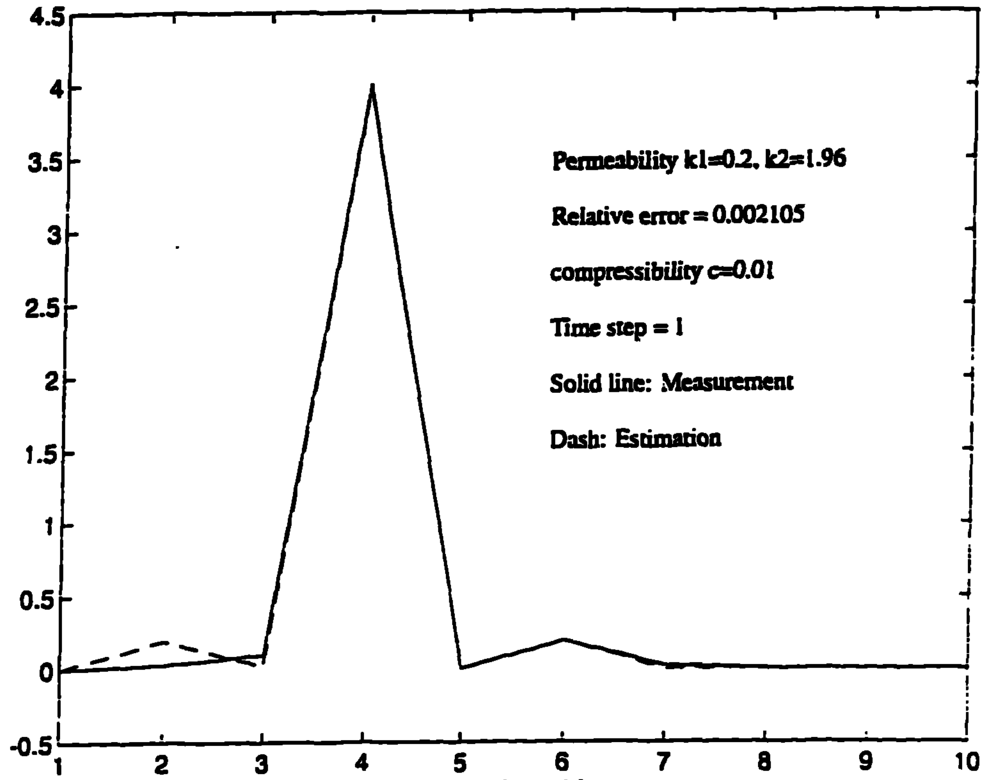
Rainfall, outflow data collected for May, 11, 1993

Figure-6: Estimation of Outflow from Parabolic Problem



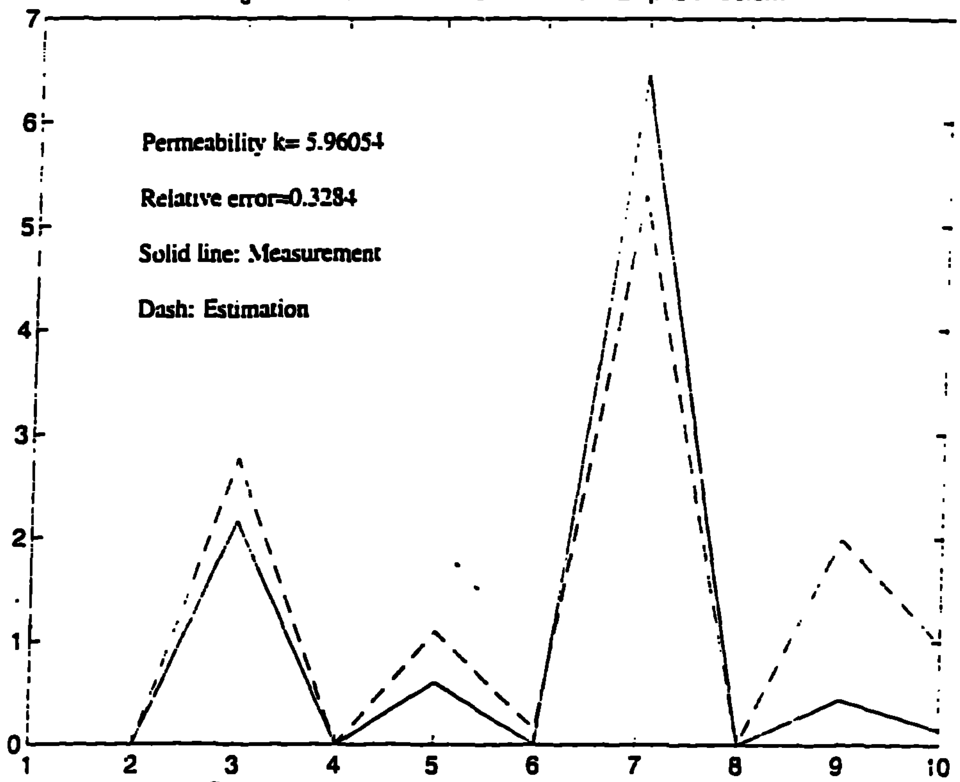
Rainfall, outflow data collected for May, 11, 1993

Figure-7: Estimation of Outflow from Parabolic Problem



Rainfall, outflow data collected from May.11.1993

Figure-8: Estimation of Outflow from Elliptic Problem



Rainfall, outflow data collected from April, 14, 1993

Figure-9: Estimation of Outflow from Elliptic Problem

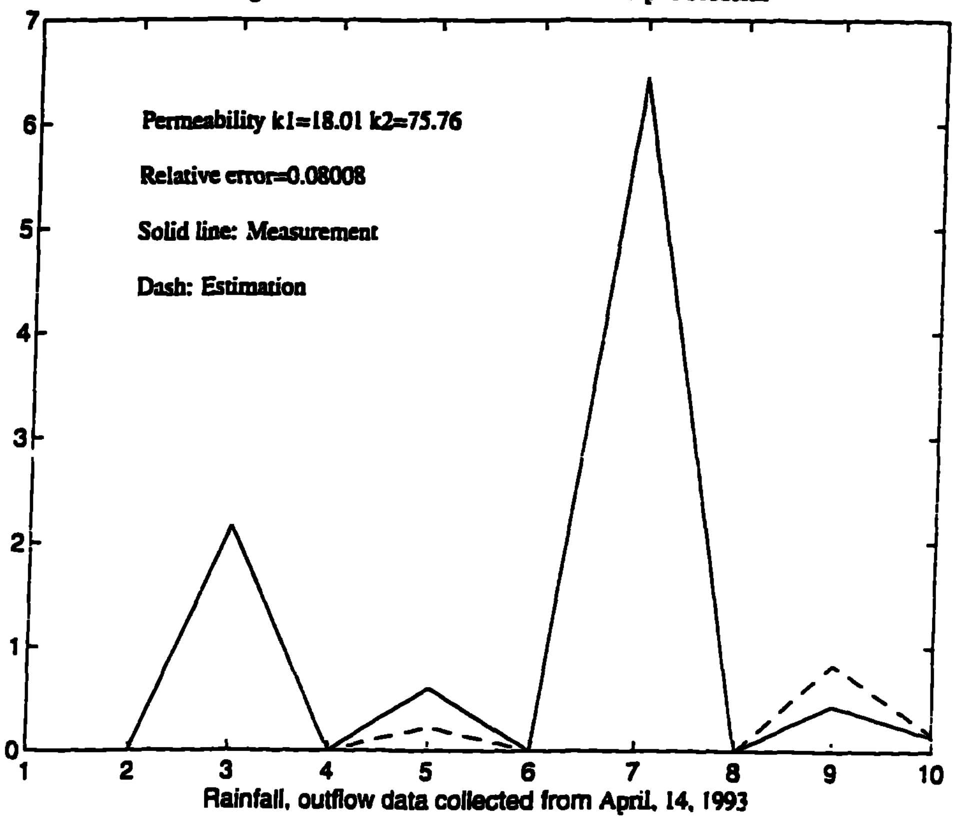


Figure-10: Estimation of Outflow from Parabolic Problem

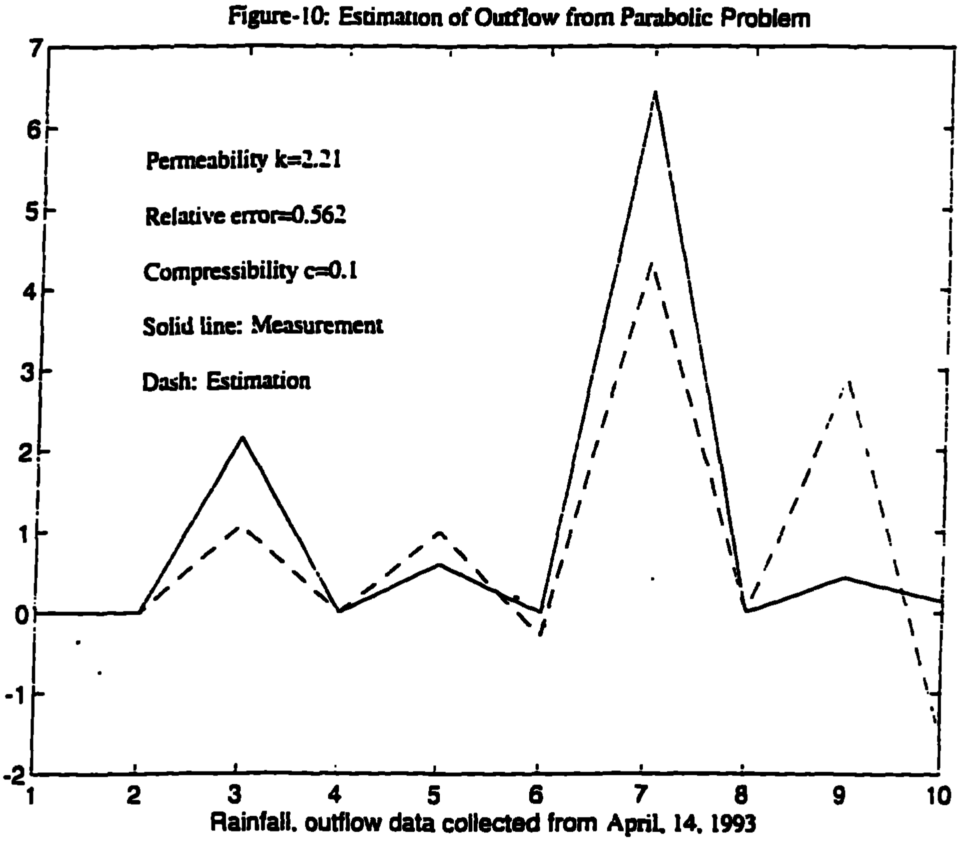


Figure-11: Estimation of Outflow from Parabolic Problem

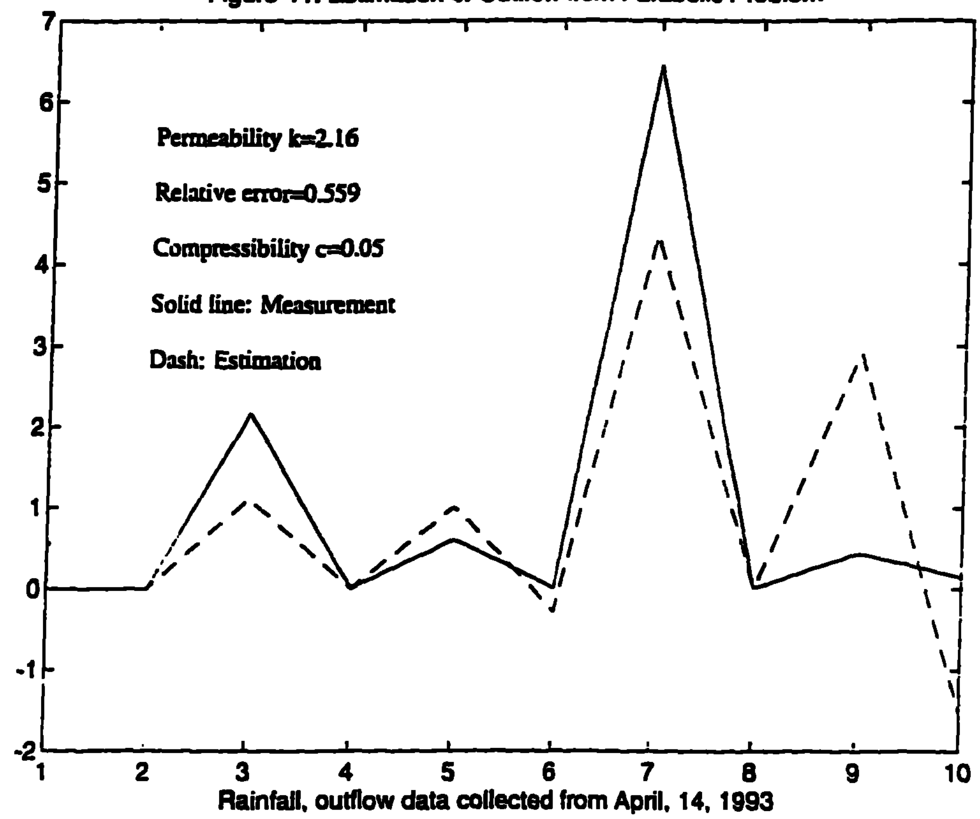


Figure-12: Estimation of Outflow from Parabolic Problem

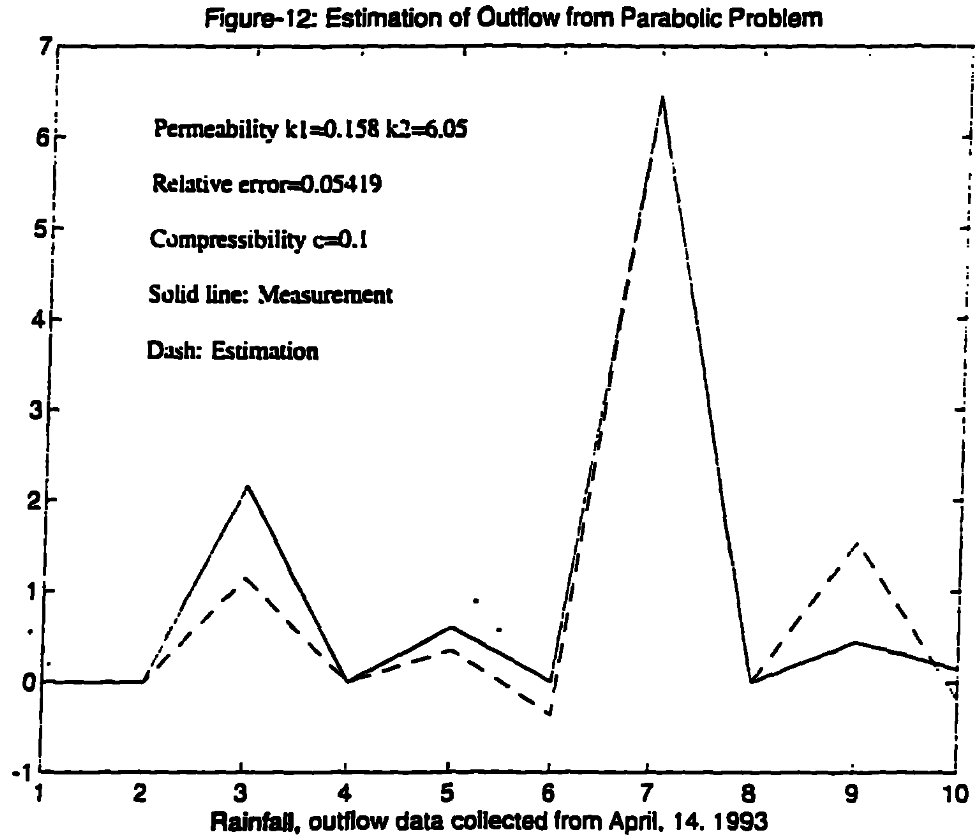


Figure-13: Estimation of Outflow from Parabolic Problem