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PLEATING COORDINATES FOR A SLICE OF THE DEFORMATION SPACE OF A HYPERBOLIC 3-MANIFOLD WITH COMPRESSIBLE BOUNDARY

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PLEATING COORDINATES FOR A SLICE OF
THE DEFORMATION SPACE OF A HYPERBOLIC
3-MANIFOLD WITH COMPRESSIBLE BOUNDARY

A DISSERTATION
APPROVED FOR THE DEPARTMENT OF MATHEMATICS

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Dedicated

to

my parents

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1 Introduction

A Kleinian group G is a discrete subgroup of $PSL(2, \mathbf{C})$. It acts as a group of isometries on hyperbolic space $\mathbf{H}^3$ and as a group of conformal automorphisms on $\hat{\mathbf{C}}$. The regular set $\Omega(G)$ is the maximal subset of $\hat{\mathbf{C}}$ on which G acts properly discontinuously and its complement $\Lambda(G)$ is called the limit set.

An important object of study in the theory of hyperbolic 3-manifolds is the boundary ∂C in $\mathbf{H}^3$ of the convex hull of $\Lambda(G)$. It is a surface which is invariant under G and consists of totally geodesic two-dimensional pieces which meet along a collection of geodesics invariant under the group. With this additional structure, it is called a pleated surface. The image of the geodesics in $\partial C/G$ is a disjoint union of geodesics called the bending lamination. Topologically but not conformally, the components of $\partial C/G$ are equivalent to the components of $\Omega(G)/G$.

Our objective is to study a slice of the deformation space of the group G on four parabolic generators T_1, T_2, E_1, E_2 satisfying the following conditions

(i) $[T_1, T_2] = 1 = [E_1, E_2]$

(ii) Every parabolic in G is conjugate into either $\langle T_1, T_2\rangle$ or $\langle E_1, E_2\rangle$.

Up to conjugation by an element of $PSL(2, \mathbf{C})$ the generators have the form

$$T_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \; T_2 = \begin{pmatrix} 1 & \tau_2 \\ 0 & 1 \end{pmatrix}, \; E_1 = \begin{pmatrix} 1 & 0 \\ \sigma_1 & 1 \end{pmatrix}, \; E_2 = \begin{pmatrix} 1 & 0 \\ \sigma_2 & 1 \end{pmatrix}$$

The quotient $\mathbf{H}^3 \cup \Omega(G)/G$ is a 3-manifold with two cusps and $\Omega(G)/G$ is a surface of genus 2. Figure 1 shows two ways to visualize this surface. By a theorem of Bers (Theorem 2.4 below), the quasiconformal deformation space of G is isomorphic to the Teichmüller space $\mathcal{T}(S)$ of the genus two surface modulo the action of the group $\langle\gamma\rangle$ where γ is the commutator $[T_1, T_2]$. Thus the deformation space is a six-dimensional space. A two-dimensional slice can be obtained by fixing the two cusp structures. If we give both the cusps a standard structure then the generators after renaming have the form

$$T_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \; T_i = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}, \; E_\rho = \begin{pmatrix} 1 & 0 \\ \rho & 1 \end{pmatrix}, \; E_{i\rho} = \begin{pmatrix} 1 & 0 \\ i\rho & 1 \end{pmatrix}$$

where $\rho \in \mathbf{C}$. Let G_ρ denote the group with presentation

$$\langle T_1, T_i, E_\rho, E_{i\rho} \mid [T_1, T_i] = 1, [E_\rho, E_{i\rho}] = 1\rangle$$

and let

$$\mathcal{S} = \{\rho \in \mathbf{C} : \Omega(G_\rho)/G_\rho \text{ is a genus two surface} \ .$$

Using Bers' theorem, we show in Proposition 3.2 that $\mathcal{S}$ is topologically an open annulus. The remainder of our work is directed toward producing coordinates on the slice $\mathcal{S}$ that reflect the geometry of the pleated surfaces $\partial C_\rho/G_\rho$. In order to do that we first need to identify those laminations that arise as bending laminations. We do this by observing that all the groups in $\mathcal{S}$ are normalized by $R_{\pi/2}$, the rotation about the origin through through an angle of $\pi/2$. Consequently, $R_{\pi/2}$ preserves the limit set of each G_ρ, so preserves C_ρ and the bending lamination. Now if we quotient out the genus two surface $S_\rho = \Omega(G_\rho)/G_\rho$ by the action of $R_{\pi/2}$, we get an orbifold $\mathcal{O}$ whose underlying topological space is the 2-sphere, and which contains two order 2 cone points and two order 4 cone points. Any lamination that is invariant under $R_{\pi/2}$ projects to a lamination on $\mathcal{O}$. We show that the set of geodesic laminations that are invariant under $R_{\pi/2}$ can be identified with $\mathbf{R} \cup \{\infty\}$, and that all except the one corresponding to ∞ occur as bending laminations.

Our coordinates on $\mathcal{S}$ as $\mathbf{R}/8\mathbf{Z} \times \mathbf{R}_{>0}$ are given by Theorem 11.2. The first coordinate represents the bending lamination and the second the normalized length of the bending lamination. The advantages of our coordinates are that they reflect the geometry of the pleated surface $\partial C_\rho/G_\rho$ and they can be computed directly from the generators of the group. The set of groups whose bending lamination corresponds to a fixed $\lambda \in \mathbf{R}$ coincides with a branch of the real locus of an analytic function. For $\lambda \in \mathbf{Q}$, the lamination consists of closed geodesics, and the analytic function corresponds to the traces of the group elements that represent these closed curves. All these branches are disjoint and non-singular and it can be shown that the rational branches are dense in $\mathcal{S}$ by interpolating them with rays along which the pleating locus is a lamination containing nonclosed geodesics.

This work was motivated by the papers of L. Keen and C. Series [10] [11], in which they introduced pleating coordinates on the Maskit embedding of the Teichmüller space of a punctured torus and on the Riley slice of Schottky space. For the groups which they consider, the set of possible bending laminations coincides with the set of all possible laminations on $\Omega(G)/G$. All simple closed geodesics determine the same structure at infinity, that of a thrice punctured sphere. In our slice $\mathcal{S}$, the set of bending laminations is a proper subset of the space of all possible laminations. Bending laminations having compact leaves determine one of two possible structures at infinity, a four times punctured sphere or two once-punctured tori.

The presentation is organized as follows. In section 2 we review the relevant theory of the quasiconformal deformations of a Kleinian group. Most of the results here are from Ahlfors [1] and Bers [4]. In section 3 we define the slice $\mathcal{S}$, give a rough description of its shape, and prove that it is an open annulus. The orbifold $\mathcal{O}$ is introduced in section 4. We show here that the measured laminations on S_ρ invariant under $R_{\pi/2}$ can be identified with the circle S^1. In section 5 we identify each $R_{\pi/2}$-invariant lamination on S_ρ that consists of simple closed geodesics with a rational number p/q, defined modulo 8. We also find words in the conjugacy class in G_ρ that represent the closed geodesics corresponding to p/q. In section 6 we define F-peripheral groups and use them to characterize pleating rays in terms of the combinatorics of the limit set. In section 7 we single out special branches of the hyperbolic loci of the trace polynomials by identifying them with the rational pleating rays. Section 8 deals with real pleating rays. In section 9 we define the complex length functions $L_{p/q}(\rho)$ and show that they form a normal family. In section 10 we characterize the normalized length functions by using the theory of measured laminations. We use this characterization to prove the uniqueness of the limit functions of the normal family $\{L_{p/q}(\rho)\}$. Finally in section 11 we prove that the laminations and their normalized lengths give coordinates for $\mathcal{S}$.

2 Quasiconformal deformations

In this section we will provide basic definitions and results and prove that there exist local sections from the quasiconformal deformation space $QD(G)$ into the space of quasiconformal maps.

Recall that a Kleinian group is *Fuchsian* if it keeps invariant some circular disk. The limit set of a Fuchsian group is contained in the boundary of its invariant disk.

A *Beltrami coefficient* on a Kleinian group G is a function $\mu \in L^\infty(\mathbf{C})$, having norm less than 1 and satisfying the conditions

$$\mu(\gamma(z))\overline{\gamma'(z)}/\gamma'(z) = \mu(z) \ \ \forall \gamma \in G$$

$$\text{and} \ \ \mu|_\Lambda = 0 \ .$$

Let Bel(G) denote the space of Beltrami coefficients on G. According to the measurable Riemann Mapping Theorem, for every μ in Bel(G) there exists a unique quasiconformal homeomorphism f_μ of $\hat{\mathbf{C}}$ which has Beltrami coefficient μ and which keeps the points 0, 1, and ∞ fixed. Moreover $f_\mu G f_\mu{}^{-1}$ is again a Kleinian group and is called a *quasiconformal deformation* of G.

Two quasiconformal deformations of G are equivalent if they differ by a conformal map. The *deformation space* $QD(G)$ of G is defined to be the set of equivalence classes of quasiconformal deformations.

Let F be a Fuchsian group which leaves the upper half-plane $\mathbf{H}$ invariant. Suppose $\mu \in L^{\infty}(\mathbf{H})$ satisfies

$$\mu(\gamma(z))\overline{\gamma'(z)}/\gamma'(z) = \mu(z) \ \ \forall \gamma \in F, \ \ \forall z \in \mathbf{H} \ .$$

Extend μ to the lower half plane by symmetry. Then there exists a corresponding f_{μ} which maps $\mathbf{H}$ on itself. Two Beltrami coefficients $\mu, \mu' \in \mathrm{Bel}(F)$ are said to be equivalent if $f_{\mu} = f_{\mu'}$ on $\mathbf{R}$. The *Teichmüller* space $\mathcal{T}(F)$ of F is defined to be the set of equivalence classes under the above relation. For $\mu \in \mathrm{Bel}(F)$, let f^{μ} denote the unique quasiconformal homeomorphism normalized to fix 0, 1, and ∞ with Beltrami coefficient μ in the upper half plane and 0 in the lower half plane.

Lemma 2.1 *$f_{\mu} = f_{\nu}$ on the real axis if and only if $f^{\mu} = f^{\nu}$ on the real axis.*

Proof: Let A^{μ} be the unique conformal mapping from $f^{\mu}(\mathbf{H})$ to $\mathbf{H}$ normalized to fix 0, 1, and ∞. Since $A^\mu \circ f^\mu|_{\mathbf{H}}$ has the same Beltrami coefficient as f_μ, is normalized at 0, 1, and ∞, and maps $\mathbf{H}$ to $\mathbf{H}$, we have $f_\mu(z) = A^\mu \circ f^\mu(z)$ for all $z \in \mathbf{H}$. If $f^\mu(x) = f^\nu(x)$ for all $x \in \mathbf{R}$, then $A^\mu(x) = A^\nu(x)$ for all $x \in \mathbf{R}$ and thus $f_\mu(x) = f_\nu(x)$. If $f_\mu(x) = f_\nu(x)$ for all $x \in \mathbf{R}$, then $h = f_\nu^{-1} \circ f_\mu$ is the identity on $\mathbf{R}$ so it can be extended to a quasiconformal mapping of the whole plane by putting $h(z) = z$ in $\mathbf{H}^*$, the lower half plane. Consider the quasiconformal map $A = f^\nu \circ h \circ (f^\mu)^{-1}$. In $f_\mu(\mathbf{H}^*)$, $A = f^\nu \circ (f^\mu)^{-1}$ is conformal because both f^ν and f^μ are conformal on $\mathbf{H}^*$. In $f^\mu(\mathbf{H})$, $A = f^\nu \circ (f_\nu)^{-1} \circ f_\mu \circ (f^\mu)^{-1}$ is conformal. Thus A is a quasiconformal map that is conformal almost everywhere. So by definition it is conformal. Hence it is a Möbius transformation and the normalization makes it the identity. Therefore $f^\mu = f^\nu$ on $\mathbf{H}^*$.

Definition 2.2 *The Schwarzian derivative of f is defined to be*

$$\{f, z\} = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 .$$

It is easy to verify that the Schwarzian of a Möbius transformation is 0.

Consider the behavior of the Schwarzian under composition:

$$F(z) = f(\zeta(z))$$

$$F'(z) = f'(\zeta(z))\zeta'(z)$$

$$\frac{F''}{F'} = \frac{f''(\zeta)}{f'(\zeta)}\zeta' + \frac{\zeta''}{\zeta'}$$

$$\frac{F'''}{F'} - \left(\frac{F''}{F'}\right)^2 = \left(\frac{f'''(\zeta)}{f'(\zeta)} - \left(\frac{f''(\zeta)}{f'(\zeta)}\right)^2\right)(\zeta')^2 + \frac{f''(\zeta)}{f'(\zeta)}\zeta'' + \frac{\zeta'''}{\zeta'} - \left(\frac{\zeta''}{\zeta'}\right)^2$$

$$\{F, z\} = \{f, \zeta\}\zeta'(z)^2 + \{\zeta, z\}$$

Lemma 2.3 *If $\mu \in \mathrm{Bel}(F)$ then the Schwarzian of f^μ, $\{f^\mu, z\} = \phi$ is a holomorphic quadratic differential for F in the lower half plane $\mathbf{H}^*$.*

Proof: From the calculation above, we have

$$\{f \circ g, z\} = \{f, g(z)\}(g'(z))^2 + \{g, z\} .$$

There are two special cases.

(i) For $f = A$ a Möbius transformation,

$$\{A \circ g, z\} = \{A, g(z)\}(g'(z))^2 + \{g, z\} = \{g, z\} .$$

(ii) For $g = A$ a Möbius transformation,

$$\{f \circ A, z\} = \{f, Az\}(A'(z))^2 .$$

Therefore if $\phi = \{f^\mu, z\}$, we have

$$\phi \circ A = \{f^\mu, z\} \circ A = \{f^\mu, Az\} ,$$

the second equality following from the definition of the Schwarzian, and

$$(\phi \circ A)(A')^2 = (\{f^\mu, Az\})(A')^2 = \{f^\mu \circ A, z\} = \{f^\mu \circ A \circ (f^\mu)^{-1} \circ f^\mu, z\}$$

using (ii) for the middle equality. Since $\mu \in \mathrm{Bel}(F)$, $f^\mu \circ A \circ f^{\mu^{-1}}$ is a Möbius transformation, say B, and

$$(\phi \circ A)(A')^2 = \{B \circ f^\mu, z\} = \{f^\mu, z\} = \phi$$

using (i) for the middle inequality. Therefore $(\phi \circ A)(A')^2 = \phi$ and so ϕ is a quadratic differential for F.

Denote by $Q(F)$ the space of holomorphic quadratic differentials for F. A theorem of Nehari-Kraus states that if f is holomorphic and univalent in the lower half plane, then $|\{f, z\}y^2| \leq 3/2$. Defining a norm on $Q(F)$ by

$\|\phi\| = sup\,|\phi(z)|y^2$, this becomes $\|\phi\| \leq 3/2$. Define

$$\Phi: \mathcal{T}(F) \to Q(F) \text{ by}$$

$$[\mu] \mapsto \{f^\mu, z\} .$$

To verify that Φ is well-defined, observe first that if $\mu \in [\mu']$ then $f_\mu = f_{\mu'}$ on $\mathbf{R}$. Lemma 2.1 then implies that $f^\mu = f^{\mu'}$ on $\mathbf{R}$ and hence on $\mathbf{H}^*$. Therefore $\{f^\mu, z\} = \{f^{\mu'}, z\}$. By reversing the steps one shows that Φ is injective.

The Ahlfors-Weil Lemma states that if $\phi \in Q(F)$ satisfies $\|\phi\| \leq 1/2$, then the function μ defined by

$$\mu(z) = \begin{cases} -y^2\phi(\overline{z}) & \text{if } z \in \mathbf{H} \\ 0 & \text{if } z \in \mathbf{H}^* \end{cases}$$

satisfies $\{f^\mu, z\} = \phi$ and $\mu \in \text{Bel}(F)$. Thus the image of Φ is contained in the disk of radius 3/2 and contains the disk of radius 1/2.

Let G be a finitely generated nonelementary Kleinian group. Let Ω be the region of discontinuity of G and Δ a component of Ω. Since G is nonelementary there exists a holomorphic universal covering $h: \mathbf{H} \to \Delta$. Let F be the set of all $\alpha \in PSL(2, \mathbf{R})$ for which there is a $\gamma \in G$ satisfying $\gamma(\Delta) = \Delta$ and

$h \circ \alpha = \gamma \circ h$. Then F is a Fuchsian group called the *Fuchsian equivalent* of G over Δ. It is determined by G and Δ up to conjugation in $PSL(2, \mathbf{R})$.

By the Ahlfors Finiteness Theorem, G has a finite number, say n, of inequivalent components, and every Fuchsian equivalent of G is finitely generated and of the first kind. Fix n inequivalent components $\Delta_1, \Delta_2, \ldots, \Delta_n$ and n universal coverings $h_j\colon \mathbf{H} \to \Delta_j$. The corresponding set of Fuchsian equivalents $\{F_1, F_2, \ldots, F_n\}$ is called the Fuchsian model of G. Denote by $\mathrm{Bel}(\mathbf{H}, F)$ the elements of norm less than 1 that satisfy the condition

$$\mu(f(z))\overline{f'(z)}/f(z) = \mu(z),\ \forall z \in \mathbf{H},\ f \in F\ .$$

Define

$$h^*\colon \mathrm{Bel}(G) \to \mathrm{Bel}(\mathbf{H}, F_1) \times \cdots \times \mathrm{Bel}(\mathbf{H}, F_n) \text{ by}$$

$$\mu \mapsto (\mu_1, \cdots, \mu_n) \text{ where}$$

$$\mu_j(z) = \mu(h_j(z))\overline{h_j'(z)}/h_j'(z)\ .$$

We will verify that $\mu_j \in \mathrm{Bel}(\mathbf{H}, F_j)$. Let $f \in F_j$, then there exists $\gamma \in G$ such that $\gamma(\Delta_j) = \Delta_j$ and $h_j \circ f = \gamma \circ h_j$. Then $h_j'(f(z))f'(z) = \gamma'(h_j(z))h_j'(z)$ and

$$\mu_j(f(z))\overline{f'(z)}/f'(z) \quad = \quad \mu(h_j(f(z)))\overline{h_j'(f(z))}/h_j'(f(z)) \cdot \overline{f'(z)}/f'(z)$$

$$= \mu(\gamma(h_j(z)))\overline{\gamma'(h_j(z))}/\gamma'(h_j(z)) \cdot \overline{h_j'(z)}/h_j'(z) \ .$$

Since μ is a Beltrami coefficient for G, the latter equals $\mu(h_j(z))\overline{h_j'(z)}/h_j'(z)$ and this equals $\mu_j(z)$. Next we will show that h^* is injective. Suppose that $(\mu_1, \ldots, \mu_n) = (\nu_1, \ldots, \nu_n)$. Then $\mu_j(z) = \nu_j(z)$ for $z \in \mathbf{H}$, and $j = 1, \ldots, n$, and hence by definition of h^*, $\mu(h_j(z)) = \nu(h_j(z))$. Since $h_j: \mathbf{H} \to \Delta_j$ is surjective this implies that $\mu(w) = \nu(w)$ for all w in Δ_j and hence in Ω. Since $\mu|_\Lambda = 0 = \nu|_\Lambda$ we get $\mu = \nu$ and hence h^* is injective. We will now show that h^* is surjective. Given $(\mu_1, \ldots, \mu_n)$, we need to find a corresponding μ. Define $\mu|_\Lambda = 0$. For $z \in \Delta_j$, define

$$\mu(z) = \mu(h_j(w)) = \mu_j(w)h_j'(w)/\overline{h_j'(w)} \ .$$

If Δ_j' is equivalent to Δ_j, say $\Delta_j' = \gamma\Delta_j$, $\gamma \in G$, define for $z' \in \Delta_j'$,

$$\mu(z') = \mu(\gamma(z)) = \mu_j(z)\gamma'(z)/\overline{\gamma'(z)} \ .$$

We must check that μ is independent of the choice of γ. Suppose that $z' = \gamma_1(z_1)$ and $z' = \gamma_2(z_2)$. Then $\mu(z') = \mu(z_1)\gamma_1'(z_1)/\overline{\gamma_1'(z_1)}$ and $\mu(z') = \mu(z_2)\gamma_2'(z_2)/\overline{\gamma_2'(z_2)}$. Put $\gamma = \gamma_2^{-1}\gamma_1$, so that $\gamma(z_1) = z_2$. We calculate

$$\begin{aligned} \mu(z_2)\gamma_2'(z_2)/\overline{\gamma_2'(z_2)} &= \mu(\gamma(z_1))\ \gamma_2'(\gamma(z_1))/\overline{\gamma_2'(\gamma(z_1))} \\ &= \mu(z_1)\ \gamma_2'(\gamma(z_1))/\overline{\gamma_2'(\gamma(z_1))} \cdot \gamma'(z_1)/\overline{\gamma'(z_1)} \end{aligned}$$

$$= \mu(z_1)\,(\gamma_2 \circ \gamma)'(z_1)/\overline{(\gamma_2 \circ \gamma)'(z_1)}$$
$$= \mu(z_1)\gamma_1'(z_1)/\overline{\gamma_1'(z_1)}\,.$$

Finally we will show that, $\mu \to h^*(\mu)$ is $\mathbf{C}$-linear and hence holomorphic. Let $a, b \in \mathbf{C}$ and $\mu, \nu \in B(G)$. Then

$$h^*(a\mu + b\nu) = ((a\mu + b\nu)_1, \ldots, (a\mu + b\nu)_n)$$

where $(a\mu + b\nu)_j(z) = (a\mu + b\nu)(h_j(z))\overline{h_j'(z)}/h_j'(z)$. Since the latter equals $(a\mu(h_j(z)) + b\mu(h_j(z)))\overline{h_j'(z)}/h_j'(z)$ we get $h^*(a\mu + b\nu) = ah^*(\mu) + bh^*(\nu)$ proving that it is linear. Thus h^* is a holomorphic bijection.

Let t denote the Teichmüller extremal map; then $t\colon \mathcal{T}F \to \mathrm{Bel}(F)$ is an injective map and $\phi \circ t = id$ where

$$\phi\colon \mathrm{Bel}(F) \to \mathcal{T}F$$
$$\mu \mapsto [\mu]\,.$$

Define

$$\tilde{t}\colon \mathcal{T}F_1 \times \cdots \times \mathcal{T}F_n \to \mathrm{Bel}(G)$$
$$([\mu_1], \ldots, [\mu_n]) \mapsto (h^*)^{-1}(t[\mu_1], \ldots, t[\mu_n])$$

and

$$\hat{\phi}: \mathrm{Bel}(G) \to QD(G)$$
$$\mu \mapsto [f_\mu G f_\mu{}^{-1}] \ .$$

The next result is due to Bers [4].

Theorem 2.4 *The mapping* $\Psi = \hat{\phi} \circ \tilde{t}$, $\Psi: \mathcal{T}F_1 \times \ldots \times \mathcal{T}F_n \to QD(G)$ *is a covering map.*

We thank Dick Canary for providing us with the next result, which assures that there exist local sections from $QD(G)$ into the space of quasiconformal maps.

Lemma 2.5 *Let* G *be a finitely generated Kleinian group and let* $\{F_1, \ldots F_n\}$ *be a Fuchsian model of* G. *Let* $[fGf^{-1}] \in QD(G)$, *and let* $([\mu_f^1], \ldots, [\mu_f^n]) \in \mathcal{T}F_1 \times \cdots \times \mathcal{T}F_n$ *with* $\Psi([\mu_f^1], \ldots, [\mu_f^n]) = [fGf^{-1}]$. *Then there exists a neighborhood* U *of* $[fGf^{-1}]$ *in* $QD(G)$ *and a continuous map* $s: U \to \mathrm{Bel}(G)$ *such that for each* $x \in U$, $[f_{s(x)} G f_{s(x)}{}^{-1}] = x$.

Proof: By Theorem 2.4 we may choose a neighborhood U of $[fGf^{-1}]$ in $QD(G)$ and a neighborhood V of $([\mu_f^1], \ldots, [\mu_f^n])$ in $\mathcal{T}F_1 \times \cdots \times \mathcal{T}F_n$ such that $\Psi|_V$ is a homeomorphism from V to U. Let $x \in U$ and $([\mu_x^1], \ldots, [\mu_x^n])$ be its preimage in V.

Define $s(x)$ to be $\tilde{t}([\mu_f^1], \ldots, [\mu_f^n])$. Then $s(x)$ is a Beltrami coefficient for G. Let $f_{s(x)}$ be the unique quasiconformal homeomorphism with Beltrami coefficient $s(x)$ and which fixes 0, 1, and ∞. Then $[f_{s(x)}Gf_{s(x)}^{-1}] = \hat{\phi}(s(x))$. Since the latter equals $\Psi([\mu_x^1], \ldots, [\mu_x^n])$ which equals x, we are done. Next we prove a result that will be useful in the later sections.

Next we prove a result that will be useful in the later sections.

Definition 2.6 *Let G be a finitely generated Kleinian group and D a domain in $\mathbf{C}$. For $\rho \in D$ let f_ρ be a quasiconformal homeomorphism of $\hat{\mathbf{C}}$. The family $\{f_\rho G f_\rho'\}$ of groups is said to be an analytic family of groups based on D if for every $z \in \hat{\mathbf{C}}$ the family $\{f_\rho(z)\}$ is analytic as a function of ρ.*

Proposition 2.7 *Let G be a finitely generated Kleinian group and Y a simply connected subset of $QD(G)$. There exists an analytic family of groups* *based on Y.*

Proof: The inclusion map i from Y to $QD(G)$ is analytic. Let $\{F_1, \ldots F_n\}$ be a Fuchsian model of G. By covering space theory there exists a lift $\tilde{i}: Y \to \mathcal{T}F_1 \times \cdots \times \mathcal{T}F_n$ so that $p \circ \tilde{i} = i$. Since p and i are analytic, so is $\tilde{i}$. Now the map $\tilde{t}: \mathcal{T}F_1 \times \cdots \times \mathcal{T}F_n \to \text{Bel}(G)$ defined earlier is an analytic map. According to the Measurable Riemann Mapping Theorem [2] there exists a unique quasiconformal homeomorphism $f_\rho: \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ with Beltrami coefficient $t \circ \tilde{i}(\rho)$ that is normalized to fix 0, 1, and ∞. Moreover f_ρ changes analytically with $t \circ \tilde{i}(\rho)$. Thus the family $\{f_\rho(z)\}$ is analytic as a function of ρ and hence $\{f_\rho G f_\rho'\}$ is an analytic family of groups based on Y.

3 The slice $\mathcal{S}$

Our objective is to study a slice of the deformation space of the group G on four parabolic generators T_1, T_2, E_1, E_2 satisfying the following conditions:

(i) $[T_1, T_2] = 1 = [E_1, E_2]$.

(ii) Every parabolic in G is conjugate into either $\langle T_1, T_2\rangle$ or $\langle E_1, E_2\rangle$.

Let a be the fixed point of T_1. Since $[T_1, T_2] = 1$, $T_1 = T_2T_1T_2^{-1}$ which implies that the fixed point of T_1 is $T_2(a)$, so T_1 and T_2 have a common fixed point. If we normalize G so that the fixed point of E_1 and E_2 is 0, and the fixed point of T_1 and T_2 is ∞, and so that T_1 equals translation by 1, then the generators have the form

$$T_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \; T_2 = \begin{pmatrix} 1 & \tau_2 \\ 0 & 1 \end{pmatrix}, \; E_1 = \begin{pmatrix} 1 & 0 \\ \sigma_1 & 1 \end{pmatrix}, \; E_2 = \begin{pmatrix} 1 & 0 \\ \sigma_2 & 1 \end{pmatrix}.$$

The quotient $(\mathbf{H}^3 \cup \Omega(G))/G$ is a 3-manifold with two cusps and $\Omega(G)/G$ is a surface of genus 2. Figure 1 shows two ways to visualize this 3-manifold. By Theorem 2.4, the quasiconformal deformation space of G is isomorphic to the Teichmüller space $\mathcal{T}(S)$ of the genus two surface modulo the action of the group $\langle\gamma\rangle$ where γ is the commutator $[T_1, T_2]$. Thus the deformation space is a six-dimensional space. A two-dimensional slice can be obtained by fixing the two cusp structures. If we give both the cusps a standard structure then the generators after renaming have the form

$$T_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},\ T_i = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix},\ E_\rho = \begin{pmatrix} 1 & 0 \\ \rho & 1 \end{pmatrix},\ E_{i\rho} = \begin{pmatrix} 1 & 0 \\ i\rho & 1 \end{pmatrix}$$

where $\rho \in \mathbf{C}$. Let G_ρ denote the group with presentation

$$\langle T_1, T_i, E_\rho, E_{i\rho} \mid [T_1, T_i] = 1, [E_\rho, E_{i\rho}] = 1\rangle\ .$$

Definition 3.1 *The standard slice $\mathcal{S}$ is defined as $\mathcal{S} = \{\rho \in \mathbf{C} : \Omega(G_\rho)/G_\rho$ is a genus two surface$\}$.*

Before initiating our more detailed study of $\mathcal{S}$ in later sections, we will obtain some rough bounds on the shape of $\mathcal{S}$, and prove that $\mathcal{S}$ is an annulus.

There is a classical result due to Shimizu and Leutbecher [8], which says that if X is an element of G_ρ, the radius of the isometric circle of X must be less than 1, since G_ρ is discrete and contains translations by 1. This implies that if $|\rho| < 1$ the group G_ρ cannot be discrete. Therefore $\mathcal{S}$ is contained in the exterior of the unit circle.

Let $R_{\pi/2}$ denote rotation through $\pi/2$ about the origin. $R_{\pi/2}T_1R_{\pi/2}^{-1} = T_i$ and $R_{\pi/2}E_\rho R_{\pi/2}^{-1} = E_{-i\rho}$ and so $R_{\pi/2}$ normalizes the group G_ρ. Therefore if $\rho \in \mathcal{S}$ then $i\rho$ is also in $\mathcal{S}$.

The isometric circles of E_ρ and E_ρ^{-1} are defined by the equations $|\rho z + 1| = 1$ and $|\rho z - 1| = 1$ respectively. Writing $\rho = |\rho|e^{i\theta}$, the isometric circles of E_ρ touch the lines $\Re z = 1/2$ when $\frac{1}{|\rho|}\cos(-\theta)+\frac{1}{|\rho|} = \pm\frac{1}{2}$. The latter is equivalent to $|\rho| = \pm 2(1+\cos\theta)$. Similarly, the isometric circles touch the lines $\Im z = \pm 1/2$ when $\frac{1}{|\rho|}\sin(-\theta)+\frac{1}{|\rho|} = \pm 1/2$, which is equivalent to $\pm 2(1-\sin\theta) = |\rho|$. For ρ in the region exterior to both of the cardioids $|\rho| = \pm 2(1+\cos\theta)$ and $|\rho| = \pm 2(1-\sin\theta)$, the isometric circles of E_ρ, E_ρ^{-1}, $E_{i\rho}$, $E_{i\rho}^{-1}$ are inside the square formed by the lines $\Re z = \pm 1/2$ and $\Im z = \pm 1/2$. In this case the region inside the squares and outside the circles is a fundamental domain $\mathcal{F}_\rho$ for G_ρ. For such ρ, the regular set $\Omega(G_\rho)$ is connected and the limit set is a Cantor set. Since the latter properties are preserved under quasiconformal conjugacy, the same is true for all groups in $\mathcal{S}$.

If $\rho = -4$, then the isometric circles of E_ρ and E_ρ^{-1} touch the vertical lines $\Re z = \pm 1/2$ and the element T_1E_ρ is parabolic. Since T_1E_ρ is not conjugate into either $\langle T_1, T_2\rangle$ or $\langle E_1, E_2\rangle$. G_{-4} does not belong to $\mathcal{S}$. Since $G_{-4-\epsilon}$ is in $\mathcal{S}$ for all $\epsilon > 0$, G_{-4} is in the boundary of $\mathcal{S}$. The symmetry of the slice implies that the groups corresponding to $\rho = 4$ and $\rho = \pm 4i$ also lie in the boundary of $\mathcal{S}$.

Proposition 3.2 *$\mathcal{S}$ is homeomorphic to an open annulus.*

Proof: Let $\tilde{G}_\rho$ be the group with presentation

$$\langle T_1, T_i, E_\rho, E_{i\rho}, R_{\pi/2} \mid [T_1, T_i] = 1, [E_\rho, E_{i\rho}] = 1, R_{\pi/2}^4 = 1, R_{\pi/2}T_1R_{\pi/2}^{-1} = T_i,$$
$$R_{\pi/2}T_iR_{\pi/2}^{-1} = T_1^{-1}, R_{\pi/2}E_\rho R_{\pi/2}^{-1} = E_{i\rho}^{-1}, R_{\pi/2}E_{i\rho}R_{\pi/2}^{-1} = E_\rho^{-1}\rangle .$$

$\mathbf{H}^3 \cup \Omega/\tilde{G}_\rho$ is a 3 orbifold which is topologically a ball with two interior points removed, whose boundary is an orbifold $\mathcal{O}$ having two cone points of order 4 and two cone points of order 2. The orbifold $\mathcal{O}$ is described more thoroughly in the next section.

Applying Theorem 2.4, one shows that the deformation space of $\tilde{G}_\rho$ is isomorphic to the Teichmüller space $\mathcal{T}(\mathcal{O})$ of the orbifold modulo the action of the group $\langle\gamma(\infty)\rangle$, where $\gamma(\infty)$ is the commutator of T_1 and T_2. Now

$\mathcal{T}(\mathcal{O})$, according to Bers[4], is biholomorphic to the Teichmüller space of the four times punctured sphere. Hence it has complex dimension 1 and so $\mathcal{T}(\mathcal{O})/\langle\gamma(\infty)\rangle$ is topologically an annulus.

We shall identify this deformation space with our slice $\mathcal{S}$, thus showing that $\mathcal{S}$ is an annulus.

Consider a deformation $\phi\tilde{G}_{\rho_0}\phi^{-1}$ of $\tilde{G}_{\rho_0}$. Then $\phi : \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ is a quasiconformal homeomorphism normalized to fix 0, 1 and ∞ and the isomorphism $\tilde{G}_{\rho_0} \to \phi\tilde{G}_{\rho_0}\phi^{-1}$ is type preserving, that is, it preserves the trace of the elements. Now $(\phi R_{\pi/2}\phi^{-1})^4 = 1$, so $\phi R_{\pi/2}\phi^{-1}$ is a Möbius transformation of order 4 that fixes 0 and ∞. Hence it must represent $R^{\pm 1}_{\pi/2}$. Since ϕ is orientation-preserving and fixes 0 and ∞, $\phi R_{\pi/2}\phi^{-1} = R_{\pi/2}$.

Since ϕ fixes ∞, $\phi T_a\phi^{-1} = T_{a'}$, and since it fixes 0, and 1, $\phi T_1\phi^{-1}(0) = \phi T_1(0) = \phi(1) = 1$. Therefore $\phi T_1\phi^{-1} = T_1$. From the relation that $R_{\pi/2}T_1R^{-1}_{\pi/2} = T_i$, we have $\phi T_i\phi^{-1} = T_i$.

Next, since $\phi E_{\rho_0}\phi^{-1}$ is parabolic and fixes 0, it must equal E_ρ for some ρ. From the relation that $R_{\pi/2}E_{\rho_0}R^{-1}_{\pi/2} = E^{-1}_{i\rho_0}$, we have $\phi E_{i\rho_0}\phi^{-1} = E^{-1}_{\rho}$.

Putting these together shows that $\phi\tilde{G}_{\rho_0}\phi^{-1} = \tilde{G}_\rho$ for some ρ. Moreover. $\phi G_{\rho_0}\phi^{-1} = G_\rho$. Define Θ: $QD(\tilde{G}_{\rho_0}) \to \mathcal{S}$ by sending $[\tilde{G}_\rho]$ to ρ. We claim that this is a well-defined homeomorphism.

If $\tilde{G}_{\rho_1}$ and $\tilde{G}_{\rho_2}$ represent the same point in $QD(\tilde{G}_{\rho_0})$, then they are conjugates of each other by a Möbius transformation. Since conjugation preserves trace, and since the trace of T_1E_ρ is $2+\rho$, $\tilde{G}_{\rho_1}$ and $\tilde{G}_{\rho_2}$ can be the same point of $QD(\tilde{G}_{\rho_0})$ only when $\rho_1 = \rho_2$.

We show next that Θ is surjective. Suppose $\rho \in \mathcal{S}$. Choose a quasiconformal map $\mathcal{O}_{\rho_0} \to \mathcal{O}_\rho$. This lifts to a quasiconformal map of $\overline{\mathbf{C}}$ which conjugates $\tilde{G}_{\rho_0}$ to $\tilde{G}_\rho$, and $\Theta(\tilde{G}_\rho) = G_\rho$. It is easy to verify that Θ is continuous and an open map. Hence it is a homeomorphism.

4 The orbifold $\mathcal{O}$

The boundary $\partial C(G_\rho)$ of the hyperbolic convex hull of $\Lambda(G_\rho)$ in $\mathbf{H}^3$ is a pleated surface, that is, it consists of totally geodesic two dimensional pieces which meet along a collection of geodesics invariant under the group. The image of these geodesics in the surface $\partial C(G_\rho)/G_\rho$ is a disjoint union of geodesics called the bending lamination. The topological frontier ∂C_ρ is a 2-manifold which carries a natural hyperbolic metric and induces a hyperbolic structure on $\partial C_\rho/G_\rho$. For bending laminations that consist of closed geodesics this structure is easily seen by cutting the surface along the bending lines.

There is a natural retraction map from Ω_ρ to ∂C_ρ which maps $z \in \Omega$ to the point where an expanding set of horospheres based at z first hits ∂C_ρ. This map is G-equivariant and is homotopic to a homeomorphism between $\partial C_\rho/G_\rho$ and Ω_ρ/G_ρ. Thus $\partial C_\rho/G_\rho$ is a genus-two surface pleated along some geodesic lamination.

Our aim is to study the bending lamination and use it to coordinatize the slice $\mathcal{S}$. To understand the possible laminations that may arise, we will exploit the 4-fold symmetry of $\partial C_\rho/G_\rho$ induced by conjugation by $R_{\pi/2}$.

Recall that for values of ρ sufficiently far from the origin, the region $\mathcal{F}_\rho$ outside the isometric circles of E_ρ and $E_{i\rho}$ and inside the square formed by the lines $\Re z = \pm 1/2, \Im z = \pm 1/2$ is a fundamental region for the action of G_ρ. Figure 2 illustrates $\mathcal{F}_\rho$. Clearly $\mathcal{F}_\rho$ is invariant under $R_{\pi/2}$. $\mathcal{F}_\rho/G_\rho$ is a genus-two surface S_ρ and $S_\rho/R_{\pi/2}$ is an orbifold $\mathcal{O}$ whose underlying topological space is the 2-sphere. This orbifold has two order 2 cone points and two order 4 cone points and has S_ρ as a 4-fold orbifold covering.

Definition 4.1 *α is said to be a 1-suborbifold of $\mathcal{O}$ if its preimage $\tilde{\alpha}$ in S_ρ is a 1-submanifold of S_ρ.*

Proposition 4.2 *A connected 1-suborbifold of $\mathcal{O}$ is either a simple closed curve that does not pass through any of the cone points or a simple arc that ends at the order 2 cone points.*

Proof: Let α and β be short arcs that have one endpoint a regular point and the other endpoint at an order 4 or order 2 cone point respectively, as shown in Figure 3. Let $\tilde{\alpha}$ and $\tilde{\beta}$ be their lifts to S_ρ. Since S_ρ is a manifold, $\tilde{\alpha}$ consists of two transverse arcs that intersect in the point which is the lift of the order 4 cone point, and $\tilde{\beta}$ consists of two arcs that contain the order 2 cone points in their interiors. Since a 1-submanifold of S_ρ cannot have a transverse self intersection, α cannot be part of a 1-suborbifold. Since the underlying topological space of a 1-suborbifold of $\mathcal{O}$ is a 1-manifold, any connected suborbifold containing β must be topologically an arc that connects the two order 2 cone points.

The next Proposition restricts the closed geodesics which can occur in the bending lamination.

Proposition 4.3 *Any simple closed $R_{\pi/2}$-invariant geodesic in S_ρ projects either to an arc connecting the order 2 cone points or to a simple closed curve that separates $\mathcal{O}$ into two disks each with an order 2 and order 4 cone point.*

Proof: By Proposition 4.2, we need only show that any simple closed curve γ in $\mathcal{O}$ which lifts to a geodesic in S_ρ separates $\mathcal{O}$ into two disks, each with an order 2 and order 4 cone point. Let $\overline{\gamma}$ be a closed geodesic on S_ρ that projects to γ. Now γ separates $\mathcal{O}$ into two disks D and D'. If D contains no cone points it is contractible in $\mathcal{O}$. The preimage of D in S_ρ consists of 4 disks, one of which has $\overline{\gamma}$ as a boundary. But a geodesic cannot bound a disk and so D must contain a cone point. If D contains one cone point, then the preimage of D consists of one disk if the cone point is of order 4 and two disks if the cone point is of order 2. In either case $\overline{\gamma}$ bounds a disk so it cannot belong to a lamination. For the same reason, D' cannot contain fewer than 2 cone points, so each of D and D' contains exactly 2 cone points. Suppose for contradiction that one of the discs, say D, has two cone points order 2. Let β be an arc in D joining the order 2 cone points. Then D deformation retracts onto β. Thus D lifts to two annuli $\widetilde{D}_1$ and $\widetilde{D}_2$ containing $\widetilde{\beta_1}$ and $\widetilde{\beta_2}$ respectively and the preimage of γ consists of the boundary of two annuli. This is a contradiction since distinct geodesics in S_ρ cannot be parallel. Thus D must contain a cone point of order 2 and a cone point of order 4.

Proposition 4.4 *Let D be a disk in $\mathcal{O}$ containing exactly one order 2 cone point and order 4 cone point. Then $\widetilde{D}$, the lift of D to S_ρ, is a punctured torus.*

Proof: Let γ be an arc in D joining the order 2 and order 4 cone point. D has orbifold Euler characteristic $-\frac{1}{4}$. Since S_ρ is a 4-fold cover of $\mathcal{O}$, $\widetilde{D}$ has Euler characteristic -1. Now $\tilde{\gamma}$ consists of two simple closed curves through the order 2 cone points which intersect transversely in the order 4 cone point. The only two orientable surfaces with $\chi = -1$ are a punctured torus and a disk with two holes. But a disk with two holes cannot contain transverse intersecting curves. So $\widetilde{D}$ must be a punctured torus.

The preceding propositions show that in the deformation space of G_ρ we could see two kinds of bending along closed geodesics depending on whether the projection of the bending lamination is a closed curve or an arc. In the former case bending would be along a simple closed curve that divides S_ρ into two one-holed tori. In the latter case it would consist of two curves which cut S_ρ into a four-holed sphere.

Proposition 4.5 *The space of all measured laminations on S_ρ invariant under $R_{\pi/2}$ can be identified with the circle S^1, in such a way that the simple closed geodesics and geodesic arcs connecting the order two cone points correspond to $\mathbf{Q} \cup \{\infty\}$.*

Proof: There is a two-fold covering of hyperbolic orbifolds $\tilde{\mathcal{O}} \to \mathcal{O}$ where $\tilde{\mathcal{O}}$ is the 2-sphere with four order 4 cone points. It has two order 2 branch points, each of which maps to an order 2 cone point of $\mathcal{O}$. One can construct $\tilde{\mathcal{O}}$ by letting β be the geodesic arc in $\mathcal{O}$ that connects the order 2 cone points, splitting $\mathcal{O}$ along β, and gluing two copies of the split $\mathcal{O}$ together along their boundary geodesics. The measured laminations in $\tilde{\mathcal{O}}$ cannot pass through the order 4 cone points, so correspond to the measured laminations on a four times punctured sphere. As in [11], these are classified by a slope parameter in $\mathbf{R} \cup \{\infty\}$, where the slope of the preimage circle of β is 0, and rational slope parameters correspond to simple closed geodesics. The covering transformation preserves each measured lamination, thus lifting of measured laminations in $\mathcal{O}$ to measured laminations in $\tilde{\mathcal{O}}$ gives a homeomorphism of the spaces of measured laminations.

5 A Planar Covering of $\mathcal{O}$

In this section we will show how to identify each $R_{\pi/2}$-invariant lamination in S_ρ that consists of simple closed geodesics with a rational number $p/q \in \mathbf{Q}$. We will find elements of G_ρ that represent the closed geodesics corresponding to p/q. Throughout this section, all rational numbers are assumed to be written in the lowest form.

Figure 2 shows the fundamental domain $\mathcal{F}_\rho$ of G_ρ for ρ real sufficiently far from the origin, which under the side pairings of the generators T_1, T_i, E_ρ, and $E_{i\rho}$ yields the surface S_ρ. For $W \in \{T_1^{\pm 1}, T_i^{\pm 1}, E_\rho^{\pm 1}, E_{i\rho}^{\pm 1}\}$, a label of W on a side of $\mathcal{F}_\rho$ indicates that the side is carried by W to the side with label W^{-1}. Let T denote the projection of the sides labelled $T_1^{\pm 1}$ and $T_i^{\pm 1}$ to $\mathcal{O}$, and E denote the projection of $E_\rho^{\pm 1}$ and $E_{i\rho}^{\pm 1}$ to $\mathcal{O}$. The sides labelled T_1 and T_i each contain a lift of an order 2 cone point, and they intersect in a lift of an order 4 cone point. So T is an arc connecting the corresponding order 2 and order 4 cone points. Similarly E is an arc connecting the remaining pair of cone points in $\mathcal{O}$. The arcs in $\mathcal{F}_\rho$ labelled B', B'', B''', and B'''' connect the lifts of the order 2 cone points, while the arcs labelled R', R'', R''', and

R'''' connect the lifts of the order 4 cone points. Let B and R respectively denote their projections on $\mathcal{O}$.

In order to make the connection with the rational numbers we consider a new covering of $\mathcal{O}$. In the plane, let $\mathcal{L}$ be the lattice of points of the form $\{m + ni : m, n \in \mathbf{Z}\}$, with the sublattice $\mathcal{L}' = \{m + 2ni : m, n \in \mathbf{Z}\}$, as shown in Figure 4. Let X be the orbifold which is the plane with order 2 cone points at the sublattice $\mathcal{L}'$. Let M be the square with vertices 0, i, $1+i$ and 1 and let Γ be the group of Euclidean isometries which is generated by reflections in the sides of M. Γ is isomorphic to $(\mathbf{Z}_2 * \mathbf{Z}_2) \times (\mathbf{Z}_2 * \mathbf{Z}_2)$. Let Γ_0 be the orientation-preserving subgroup of Γ of index 2. The set $M \cup M'$ is a fundamental domain for the action of Γ_0 on X and thus $X/\Gamma_0 = \mathcal{O}$, where M' is the square with vertices i, $1+i$, $1+2i$ and $2i$. The order 4 cone points lift to $\mathcal{L}'$ and the order 2 cone points lift to $\mathcal{L} - \mathcal{L}'$. Label the edges of the integral squares on X by T, E, R, or B according to the label of their projections to $\mathcal{O}$. Note that the horizontal lines through the vertices of $\mathcal{L}'$ project to R and the horizontal lines through the vertices of $\mathcal{L} - \mathcal{L}'$ project to B.

Let $l_{p/q}$ be a line of slope p/q through a vertex of $\mathcal{L} - \mathcal{L}'$ if p is even and passing through a point of $\mathcal{L}'$ if p is odd, and let $\gamma_{p/q}$ denote its projection on $\mathcal{O}$. If p is even, $\gamma_{p/q}$ is an arc connecting the order 2 cone points, and if p is odd, $\gamma_{p/q}$ is an arc connecting an order 2 and an order 4 cone point. Let $\tilde{\gamma}_{p/q}$ be the lift of $\gamma_{p/q}$ to S_ρ. From the previous section we know that $\tilde{\gamma}_{p/q}$ consists of two curves $\gamma'_{p/q}$ and $\gamma''_{p/q}$ that are either the generators of the fundamental group of a punctured torus or are two simple closed curves through the order two branch points. Let $V_{p/q}$ and $V'_{p/q}$ be the words representing $\tilde{\gamma}_{p/q} = \{\gamma'_{p/q}, \gamma''_{p/q}\}$. Figures 5a and 5b illustrate the curves $\gamma_{p/q}$ for $p/q = 1/2$ and $p/q = 2/3$ on the orbifold $\mathcal{O}$ and one of their lifts $\gamma'_{p/q}$ to $\mathcal{F}_\rho$. $\gamma'_{p/q}$ occurs as a collection of arcs on $\mathcal{F}_\rho$ which under the side pairings of $\mathcal{F}_\rho$ gives us a simple closed curve on S_ρ. Tracing along these arcs one obtains the words $V_{1/2} = T_i^{-1}T_1^{-1}E_\rho T_1^{-1}E_\rho$ and $V_{2/3} = T_1^{-1}E_\rho T_i E_{i\rho} T_i E_\rho$.

For $z \in \mathbf{C}$ we let $T_z = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$ and $E_z = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}$. Note that $T_{-z} = T_z^{-1}$ and $E_{-z} = E_z^{-1}$.

Proposition 5.1 *For $p/q \in \mathbf{Q}$, $V_{p/q}$ is a word of the form*

$$T_{\epsilon_q} E_{\eta_q} \cdots T_{\epsilon_1} E_{\eta_1} \quad \textit{if } p \textit{ is even}$$

$$T_{-1-i}E_{\eta_q}\cdots T_{\epsilon_1}E_{\eta_1} \quad \textit{if } p \textit{ is odd}$$

where $\epsilon_1,\ldots,\epsilon_q$ *are* ± 1 *or* $\pm i$*, and* $\eta_1,\ldots,\eta_q$ *are* $\pm\rho$ *or* $\pm i\rho$.

Proof: A simple closed curve γ on S_ρ appears as a collection of pairwise disjoint arcs running between the sides of $\mathcal{F}_\rho$. We can read off an expression for γ as a product of elements of G_ρ by reading off along γ the sequence of labels of sides cut by γ and by adopting the convention of writing the elements from right to left. Such a sequence is called a *cutting sequence* for γ. The sequence of sides cut by $\gamma_{p/q}$ on $\mathcal{O}$ is alternating in E and T, therefore the cutting sequence of $\tilde{\gamma}_{p/q}$ is also alternating in E_{η_j} and T_{ϵ_j}. $\tilde{\gamma}_{p/q}$ occurs as a collection of arcs on $\mathcal{F}_\rho$ which under the side pairings of $\mathcal{F}_\rho$ form two simple closed curves each of which double covers $\gamma_{p/q}$. Let $\bar{l}(a,b)$ denote a line segment of slope a/b whose projection along the y axis is a units and whose projection along the x axis is b units and which starts at a vertex of $\mathcal{L}'$ if p is odd and at a vertex of $\mathcal{L}-\mathcal{L}'$ otherwise. $\bar{l}(2p,2q)$ is a double cover of $\gamma_{p/q}$. Since its projection along the x axis has length $2q$ its cutting sequence is a word of length $2q$ alternating in T and E. Thus $V_{p/q}$ is a word of the form $T_{\epsilon_q}E_{\eta_q}\cdots T_{\epsilon_1}E_{\eta_1}$, where ϵ_j and η_j, $j=1,\ldots,q$ are determined from the cutting sequence of $\tilde{\gamma}_{p/q}$ on $\mathcal{F}_\rho$. For p odd $\tilde{\gamma}_{p/q}$ passes through the order 4 branch points. Thus to determine ϵ_q one can choose a curve homotopic to $\tilde{\gamma}_{p/q}$ that does not pass through the order 4 branch points. As indicated in Figure 5a, such a curve would give us the sequence $T_1^{-1}T_i^{-1}$. Thus $V_{p/q} = T_{-1-i}E_{\eta_q} \cdots T_{\epsilon_1}E_{\eta_1}$ for p odd.

Observe that the fundamental domain $\mathcal{F}_\rho$ can be subdivided into eight polygonal regions, each region corresponding to a unit integral square on X. Hence the cutting sequence of $\tilde{\gamma}_{p/q}$ is the same as that of $\gamma_{8+p/q}$. So $V_{8+p/q} = V_{p/q}$.

Proposition 5.2 $\mathrm{Tr}(T_{a_1}E_{b_1\rho} \cdots T_{a_n}E_{b_n\rho}) = a_1b_1 \cdots a_nb_n\rho^n + \cdots + 2.$

Proof: Let $c_i(\rho^n)$ denote a polynomial in ρ of degree at most n. We will show by induction that

$$T_{a_1}E_{b_1\rho} \cdots T_{a_n}E_{b_n\rho} = \begin{pmatrix} (1+a_1b_1\rho)\cdots(1+a_nb_n\rho) + c_1(\rho^{n-1}) & c_2(\rho^{n-1}) \\ c_3(\rho^n) & c_4(\rho^{n-1}) \end{pmatrix}.$$

Note first that $T_aE_{b\rho} = \begin{pmatrix} 1+ab\rho & a \\ b\rho & 1 \end{pmatrix}$. We calculate

$$T_{a_1}E_{b_1\rho} \cdots T_{a_n}E_{b_n\rho} =$$

$$\begin{pmatrix} (1+a_1b_1\rho)\cdots(1+a_{n-1}b_{n-1}\rho)+c_1(\rho^{n-2}) & c_2(\rho^{n-2}) \\ c_3(\rho^{n-1}) & c_4(\rho^{n-2}) \end{pmatrix} \begin{pmatrix} 1+a_nb_n\rho & a_n \\ b_n\rho & 1 \end{pmatrix}$$
$$= \begin{pmatrix} (1+a_1b_1\rho)\cdots(1+a_nb_n\rho)+c_1(\rho^{n-1}) & c_2(\rho^{n-1}) \\ c_3(\rho^{n}) & c_4(\rho^{n-1}) \end{pmatrix} .$$

It follows that

$$\mathbf{Tr}(T_{a_1}\cdots E_{b_n\rho}) = a_1b_1\cdots a_nb_n\rho^n + \cdots + c .$$

For $\rho = 0$ we have

$$c = \mathbf{Tr}(T_{a_1}T_{a_2}\ldots T_{a_n}) = \mathbf{Tr}\begin{pmatrix} 1 & a_1+a_2+\cdots+a_n \\ 0 & 1 \end{pmatrix} = 2 .$$

Notice that $\bar{l}(p,q)$ and $\bar{l}(2p,2q)$ both cover $\gamma_{p/q}$, and recall that $\bar{l}(2p,2q)$ is a double cover of $\gamma_{p/q}$. Therefore a cutting sequence of $\bar{l}(2p,2q)$ can be obtained by adjoining to the cutting sequence of $\bar{l}(p,q)$ on the left, the mirror image of the cutting sequence of $\bar{l}(p,q)$. Then, whenever $\bar{l}(2p,2q)$ crosses a lift of R between an E_{η_j} and an T_{ϵ_j}, it has to contain a corresponding crossing of a lift of R between $T_{\epsilon_{k-1}}$ and E_{η_k}, and vice versa. That is, exactly half the crossings of lifts of R occur as $\bar{l}(2p,2q)$ is going from an E to a T, and the remaining occur when it is going from a T to an E.

Proposition 5.3

$$\mathbf{Tr}(V_{p/q}) = \begin{cases} \pm i^{p/2}\rho^q + \cdots + 2 & \textit{if } p \textit{ is even} \\ \pm(1+i)i^{p-1/2}\rho^q + \cdots + 2 & \textit{if } p \textit{ is odd} \end{cases}$$

Proof: From the previous proposition we have $\mathbf{Tr}V_{p/q} = b_q a_q \cdots b_1 a_1 \rho^q$. where each a_j and b_j is either 1, -1, i, or $-i$. We must determine the product $b_q a_q \cdots b_1 a_1$ up to multiplication by ± 1. It is convenient to think of this product in the form

$$(b_q a_q)(b_{q-1} a_{q-1}) \cdots (b_1 a_1) \ .$$

For a given j, if both b_j and a_j are real, or both are imaginary, then $b_j a_j$ contributes only a factor of ± 1 to the product $b_q a_q \cdots b_1 a_1$, and consequently can be ignored since we are only determining the product up to sign.

The region in $\mathcal{F}_\rho$ bounded by the sides R', T_1, R'' and E_ρ is a fundamental region for the action of $R_{\pi/2}$ on $\mathcal{F}_\rho$. The subscripts of all its sides are of the form ± 1 or $\pm\rho$, according as they have T or E labels. The subscripts of the fundamental domain bounded by R''', T_1^{-1}, R'''' and E_ρ^{-1} are also of this form. The adjacent fundamental domains have sides with subscripts of the form $\pm i$ or $\pm i\rho$, according as they have T or E labels. Thus whenever we cross a lift of an R on $\mathcal{F}_\rho$ we change from a fundamental region with subscripts ± 1 and $\pm\rho$ to a fundamental region with subscripts $\pm i$ and $\pm i\rho$, or vice versa.

Consider the crossings of lifts of R that occur between $E_{b_j\rho}$ and T_{a_j}. If there are an odd number of such crossings, then $b_j a_j$ is $\pm i$, while if there are an even number, then it is ± 1. Hence for purposes of calculation up to sign, we can regard each crossing of a lift of R as contributing $\pm i$ to the product. On the other hand, the crossings of lifts of R that occur between T_{a_j} and $E_{b_{j-1}\rho}$ do not affect the product up to sign.

We consider two cases.

Case (i): p is even. We have observed that half the crossings of $\bar{l}(2p, 2q)$ over a lift of R occur between an $E_{b_j\rho}$ and a T_{a_j}. These produce $p/2$ factors of $\pm i$. Since all other factors are ± 1, the leading coefficient of $\mathbf{Tr}(V_{p/q})$ is $\pm i^{p/2}$.

Case (ii): p is odd. In this case $\bar{l}(2p, 2q)$ crosses $p-1$ lifts of R, so there are exactly $(p-1)/2$ factors of $\pm i$ coming from crossing of lifts of R that occur between $E_{b_j\rho}$ and T_{a_j}. Since $V_{p/q} = T_{a_q} E_{b_q\rho} \dots T_{a_1} E_{b_1\rho}$, where $a_q = -(1+i)$, a_q contributes $\pm(1+i)$ making the leading coefficient $\pm(1+i)i^{(p-1)/2}$.

6 F-peripheral groups

Definition 6.1 *A subgroup F of G is called F-peripheral if it is Fuchsian and if one of its invariant open disks does not contain any points of $\Lambda(G)$. Such a disk is called a peripheral disk of F.*

Definition 6.2 *A surface S_1 imbedded in S is said to be incompressible if for each component G of S_1 the homomorphism $\pi_1(G) \to \pi_1(S)$ is injective. A collection of subgroups $\{H_i\}$ of G is said to be geometric if each H_i is a fundamental group of a component of an incompressible surface S_1 in S. such that no component of S_1 is a disc or annulus.*

We select a set $\widetilde{W}_{p/q}$ of group elements representing $\tilde{\gamma}_{p/q}$. If p is odd then we take $\widetilde{W}_{p/q} = \{[V_{p/q}, V_{p/q}']\}$, and if p is even then $\widetilde{W}_{p/q} = \{V_{p/q}, V_{p/q}'\}$. Define $\mathcal{U}_{p/q}$ to be the collection of geometric subgroups where each boundary circle represents an element conjugate to an element of $\widetilde{W}_{p/q}$.

For the next lemma, suppose that F_{ρ_0} is a finitely generated F-peripheral subgroup of G_{ρ_0} with peripheral disk Δ_0. Suppose also that $\overline{\Delta}_0 \cap \Lambda(G_{\rho_0}) =$

$\Lambda(F_{\rho_0})$. Let $\rho(t)$, $0 \leq t \leq \epsilon$, be a path in $\mathcal{S}$ with $\rho(0) = \rho_0$, and denote $G_{\rho(t)}$ by G_t. After possibly making ϵ smaller, use Lemma 2.5 to select a corresponding path f_t, $0 \leq t \leq \epsilon$ of quasiconformal homeomorphisms such that $G_t = f_t G_{\rho_0} f_t^{-1}$. Define $F_t = f_t F_{\rho_0} f_t^{-1}$.

Proposition 6.3 *If F_t is Fuchsian, $0 \leq t \leq \epsilon$, then there exists $\delta > 0$ such that F_t is also F-peripheral for $0 \leq t \leq \delta$.*

Proof: Without loss of generality we may assume that $\infty \notin \overline{\Delta}_0$. Let R_0 be a fundamental domain for the action of F_{ρ_0} on Δ_0. Since F_t is Fuchsian, $\Lambda(F_t)$ is contained in the circle bounding the disk $\Delta_t = f_t(\Delta_0)$. A fundamental domain for the action of F_t on Δ_t is $R_t = f_t(R_0)$. The map sending (t, z) to $f_t(z)$ is continuous and hence the limit set $\Lambda(G_t)$ and the fundamental domain R_t change continuously with t. Given $\eta > 0$, there exists $\delta > 0$ such that if $0 \leq t \leq \delta$ then $dist(\Lambda(F_t), \Lambda(F_0)) < \eta$ and $dist(R_t, R_0) < \eta$. Now R_0 may contain subintervals of the intervals of discontinuity of F_0 in $\partial\Delta_0$. Since $\Lambda(G_0) \cap \overline{\Delta}_0 = \Lambda(F_0)$ there are no limit points of G_0 in $\overline{R}_0$. Hence $dist(\Lambda(G_0), \overline{R}_0) = d > 0$. Choose δ small enough to ensure that $\eta < d/4$ then $dist(\overline{R}_t, \Lambda(G_t)) \geq d/2$ and so there are no limit points of G_t in R_t. Since translates of R_t fill up Δ_t, there are no limit points of G_t in Δ_t. Thus F_t is F-peripheral.

Lemma 6.4 *Let f be a Möbius transformation that fixes -1 and 1, f not equal to the identity, and $g = R_{\frac{\pi}{2}} f R_{\frac{\pi}{2}}^{-1}$. Then $\mathbf{Tr}[f, g] = \pm 2$ implies that $\mathbf{Tr} f = \pm 2\sqrt{2}$ or $\mathbf{Tr} f = 0$.*

Proof: Since f fixes 1 and -1, we can write f and g as

$$f = \begin{pmatrix} a & b \\ b & a \end{pmatrix}, \; g = \begin{pmatrix} a & ib \\ -ib & a \end{pmatrix}$$

where $a^2 - b^2 = 1$.

One calculates that $\mathbf{Tr}[f, g] = 2(a^4 - 2a^2b^2 - b^4)$. Thus if $\mathbf{Tr}[f, g] = -2$ then $a^4 - 2a^2b^2 - b^4 = -1$, which implies that $a = \pm\sqrt{2}$ or $a = 0$, and hence that $\mathbf{Tr} f = \pm 2\sqrt{2}$ or 0. If $\mathbf{Tr}[f, g] = 2$ then $b = 0$ and $a = \pm 1$ which makes f the identity transformation.

Define

$$\widetilde{\mathbf{H}}_{p/q} = \{\rho \in \mathbf{C} : \Im \mathbf{Tr} V_{p/q} = 0, \; |\Re \mathbf{Tr} V_{p/q}| > 2 \;\; \textit{if } p \textit{ is even}, \;\; \textit{and } |\Re \mathbf{Tr} V_{p/q}| > 2\sqrt{2} \;\; \textit{if } p \textit{ is odd} \}.$$

Lemma 6.5 *Let $F_\rho \in \mathcal{U}_{p/q}$. Then F_ρ is Fuchsian if and only if $\rho \in \widetilde{\mathbf{H}}_{p/q}$.*

Proof: $F_\rho \in \mathcal{U}_{p/q}$ implies that $F_\rho = \pi_1(S_1)$ where S_1 is imbedded in S and ∂S_1 represents $\widetilde{W}_{p/q}$. Any parabolic in G_ρ is conjugate into either $\langle T_1, T_i \rangle$ or $\langle E_\rho, E_{i\rho} \rangle$. Since the elements of $\widetilde{W}_{p/q}$ alternate in T and E, none of them are parabolics.

Suppose that F_ρ is Fuchsian. Then $\mathbf{Tr}\widetilde{W}_{p/q} \in \mathbf{R} - (-2, 2)$. We consider two cases:

Case I: p is even. In this case $\widetilde{W}_{p/q} = \{V_{p/q}, V_{p/q}'\}$ with $\mathbf{Tr}V_{p/q} \in \mathbf{R} - (-2, 2)$.

Case II: p is odd. In this case $\widetilde{W}_{p/q} = \{[V_{p/q}, V_{p/q}']\}$ and F_ρ is $\langle V_{p/q}, V_{p/q}' \rangle$. By using Lemma 6.4 we get $\mathbf{Tr}V_{p/q} \in \mathbf{R} - (-2\sqrt{2}, 2\sqrt{2})$ or $\mathbf{Tr}V_{p/q} = 0$. Now an element with trace equal to 0 has finite order. Since $V_{p/q}$ has infinite order its trace cannot be 0. Therefore $\rho \in \widetilde{\mathbf{H}}_{p/q}$.

Conversely suppose that $\rho \in \widetilde{\mathbf{H}}_{p/q}$. Since $F_\rho \in \mathcal{U}_{p/q}$, it is the fundamental group of an imbedded surface whose boundary is represented by elements conjugate to elements of $\widetilde{W}_{p/q}$. For p odd, $\widetilde{W}_{p/q} = \{[V_{p/q}, V_{p/q}']\}$ and S_1 is a punctured torus, so we may assume that $F_\rho = \langle V_{p/q}, V_{p/q}' \rangle$. If p is even then $\widetilde{W}_{p/q} = \{V_{p/q}, V_{p/q}'\}$ and the imbedded surface S_1 is a four holed sphere with the boundary circles identified in pairs. Recall that $V_{p/q}$ and $V_{p/q}'$ represent the curves $\gamma'_{p/q}$ and $\gamma''_{p/q}$ that are the lifts of $\gamma_{p/q}$ on the orbifold $\mathcal{O}$. Hence they are conjugates of each other under $R_{\frac{\pi}{2}}$. Thus the fundamental group $\pi_1(S_1)$ is conjugate to a subgroup of the form $\langle V_{p/q}, V_{p/q}', V_{p/q}'', V_{p/q}''' \rangle$ where $V_{p/q}' = R_{\frac{\pi}{2}} V_{p/q} R_{\frac{\pi}{2}}^{-1}$, $V_{p/q}'' = R_{\frac{\pi}{2}} V_{p/q}' R_{\frac{\pi}{2}}^{-1}$, $V_{p/q}''' = R_{\frac{\pi}{2}} V_{p/q}'' R_{\frac{\pi}{2}}^{-1}$, and $V_{p/q} = R_{\frac{\pi}{2}} V_{p/q}''' R_{\frac{\pi}{2}}^{-1}$. So the fixed points of the generators of F_ρ lie on some circle C that is invariant under $R_{\frac{\pi}{2}}$. Since $\rho \in \widetilde{H}_{p/q}$, the traces of the generators are real, so the generators preserve C. Therefore every element in F_ρ preserves C, and F_ρ is Fuchsian.

We note that if $\rho \in \mathcal{S}$, the element(s) $\widetilde{W}_{p/q}(\rho)$ must be loxodromic. We denote the corresponding geodesic(s) in $\mathbf{H}^3/G_\rho$ by $\hat{\gamma}_{p/q}$. We denote the pleating locus, that is, the union of the bending lines of $\partial C_\rho / G_\rho$, by $pl(\rho)$ and let

$$P_{p/q} = \{\rho \in \mathcal{S} \mid pl(\rho) = \hat{\gamma}_{p/q}\}.$$

We call the sets $P_{p/q}$ rational pleating rays. Since $V_{p/q} = V_{8+p/q}$, it follows that $\hat{\gamma}_{p/q} = \hat{\gamma}_{8+p/q}$ and therefore $P_{p/q} = P_{8+p/q}$.

Now we will investigate the precise connection between the pleating locus of $\partial C_\rho / G_\rho$ being $\hat{\gamma}_{p/q}$ and the element(s) of $\widetilde{W}_{p/q}(\rho)$ having real trace.

Proposition 6.6 *$\rho \in P_{p/q}$ if and only if there exists a collection of F-peripheral subgroups of G_ρ which are the fundamental groups of the components of an incompressible surface S_1 in S_ρ with $\pi_1(S_1) \in \mathcal{U}_{p/q}$ and $\chi(S_1) = \chi(S_\rho)$.*

Proof: $\rho \in P_{p/q}$ implies that $pl(\rho) = \hat{\gamma}_{p/q}$. Let S_1 be the surface obtained by removing a small open regular neighborhood of $\hat{\gamma}_{p/q}$ from S_ρ. Then $\pi_1(S_1) \in \mathcal{U}_{p/q}$. Let Σ_i denote the components of S_1 and $\widetilde{\Sigma_i}$ denote a connected component of a lift of Σ_i to $\mathbf{H}^3$. Since $pl(\rho) = \hat{\gamma}_{p/q}$, $\widetilde{\Sigma_i}$ is totally geodesic and hence contained in some half plane H_i. Let F_i be the stabilizer of $\widetilde{\Sigma_i}$, then F_i also stabilizes H_i and hence is Fuchsian. Since $\widetilde{\Sigma_i} \subset \partial C_\rho$, the plane H_i separates $\mathbf{H}^3$ into two half spaces and the convex hull of $\Lambda(G_\rho)$ lies entirely on one side of H_i. Hence one of the disks cut out by H on $\hat{\mathbf{C}}$ does not contain any points of $\Lambda(G_\rho)$ and thus F_i is F-peripheral.

Conversely let S_1 be an incompressible surface in S_ρ with $\chi(S_1) = \chi(S_\rho)$, and let F_i be the fundamental groups of the components of S_1, where each F_i is F-peripheral. Let H_i be the geodesic plane in $\mathbf{H}^3$ whose closure contains $\Lambda(F_i)$, and let N_i be the Nielsen convex hull for the action of F_i on H_i. Since F_i is F-peripheral, $N_i \subset \partial C_\rho$. Since the surface S_1 is imbedded in S_ρ, F_i is the stabilizer of N_i in G_ρ and hence $N_i/F_i = N_i/G_\rho$ is a maximal totally geodesic piece of $\partial C_\rho/G_\rho$. Since $\pi_1(S_1) \in \mathcal{U}_{p/q}$ the boundary of N_i/G_ρ is represented by $\widetilde{W}_{p/q}$. For p odd this implies that the boundary circle is represented by $[V_{p/q}, V_{p/q}']$ and hence N_i/G_ρ is a one holed torus. Since $\chi(S_1) = \chi(S)$ there must be two such components glued together along $\hat{\gamma}_{p/q}$. For p even the boundary curves are represented by $V_{p/q}$ and $V_{p/q}'$. Since N_i/G_ρ is totally geodesic its boundary must consist of geodesics in the conjugacy class of $V_{p/q}$ and $V_{p/q}'$. Again the Euler characteristic implies that in this case N_i/G_ρ is a four holed sphere with the boundary circles glued together along $\hat{\gamma}_{p/q}$. This implies that $\rho \in P_{p/q}$.

Theorem 6.7 *The pleating rays $P_{p/q}$ are unions of connected components of the hyperbolic locus $\widetilde{\mathbf{H}}_{p/q}$.*

Proof: By definition, if $\rho \in P_{p/q}$ then $pl(\rho) = \hat{\gamma}_{p/q}$. By Proposition 6.6

there exist an incompressible surface S_1 in S_ρ, with $\pi_1(S_1) \in \mathcal{U}_{p/q}$, and F-peripheral subgroups F_i, one corresponding to each component of S_1. Since the F_i are Fuchsian and belong to $\mathcal{U}_{p/q}$, Lemma 6.5 shows that $\rho \in \widetilde{\mathbf{H}}_{p/q}$ and thus $P_{p/q} \subseteq \widetilde{\mathbf{H}}_{p/q}$. We will show that $P_{p/q}$ is open and closed in $\widetilde{\mathbf{H}}_{p/q}$.

Pick a $\rho_0 \in P_{p/q}$. Let K be a connected component of $\widetilde{\mathbf{H}}_{p/q}$ containing ρ_0. Since K is open in $\widetilde{\mathbf{H}}_{p/q}$ and since $\widetilde{\mathbf{H}}_{p/q}$ is open in the real locus of an analytic function, we can find an open arc $\alpha \subset K$, $\rho_0 \in \alpha$. Let $F_i(\rho_0)$ be the F-peripheral subgroups of G_{ρ_0} obtained by Proposition 6.6. For ρ sufficiently close to ρ_0 in α, define $F_i(\rho)$ as in the paragraph before Lemma 6.3. Since $\alpha \subset \widetilde{\mathbf{H}}_{p/q}$, Lemma 6.5 implies that the $F_i(\rho)$ are Fuchsian. By Proposition 6.3, $F_i(\rho)$ is F-peripheral for all $\rho \in \alpha$ and sufficiently close to ρ_0. By Proposition 6.6, $\rho \in P_{p/q}$. This shows that $P_{p/q}$ is open in $\widetilde{\mathbf{H}}_{p/q}$.

Let ρ_n be a sequence in $P_{p/q}$ converging to ρ_∞ in $\widetilde{\mathbf{H}}_{p/q}$. Let $S_i(\rho_n)$, $1 \leq n \leq \infty$, be the incompressible surfaces obtained by removing a small regular neighborhood of $\tilde{\gamma}_{p/q}$ from $S(\rho_n)$ and let $F_i(\rho_n) = \pi_1(S_i(\rho_n))$. The elements $\widetilde{W}_{p/q}(\rho_n)$ lie in $F_i(\rho_n)$ hence $F_i(\rho_n) \in \mathcal{U}_{p/q}$. Since $\rho_n \in P_{p/q}$ for $1 \leq n < \infty$, by Proposition 6.6 each $F_i(\rho_n)$ is F-peripheral. Let $\Delta_i(\rho_n)$ be the peripheral disk corresponding to ρ_n. Since $\rho_\infty \in \widetilde{\mathbf{H}}_{p/q}$, Lemma 6.5 implies that $F_i(\rho_\infty)$ is Fuchsian. Let $\Delta_i(\rho_\infty)$ be the disk whose boundary contains the limit set of $F_i(\rho_\infty)$. As $\rho_n \to \rho_\infty$, $\Delta_i(\rho_n) \to \Delta_i(\rho_\infty)$. Since $\Delta_i(\rho_n)$ contains no limit points of G_{ρ_n} and since the limit set changes continuously with ρ, $\Delta_i(\rho_\infty)$ does not contain any limit points of G_{ρ_∞}. Therefore $F_i(\rho_\infty)$ is F-peripheral and $\rho_\infty \in P_{p/q}$. Hence $P_{p/q}$ is closed.

7 Rational Pleating Rays

In this section we single out special branches of the hyperbolic loci of the trace polynomials by identifying them with the rational pleating rays. By Proposition 6.7 we know that the pleating ray $P_{p/q}$ is a union of connected components of the hyperbolic locus $\widetilde{\mathbf{H}}_{p/q}$. It also follows from the definition that $P_{p/q} \cap P_{r/s} = \emptyset$ if $p/q \neq r/s$.

First we will identify some specific pleating rays, which will furnish the starting point for our later induction.

Proposition 7.1 *The pleating ray $P_{0/1}$ is $\{\rho \in \mathbf{C} : \rho > 4\}$*

Proof: The methods of section 5 yield that $V_{0/1} = T_1^{-1} E_\rho$ and so $\mathrm{Tr} V_{0/1} = 2-\rho$. Thus the hyperbolic locus $\widetilde{\mathbf{H}}_{0/1}$ is $(-\infty, 0) \cup (4, \infty)$. Suppose that $\rho > 4$. Then the fundamental domain of G_ρ looks as in Figure 6. We wish to show that $\rho \in P_{0/1}$. We are looking for a pattern of overlapping circles in the limit set and Fuchsian groups corresponding to the circles such that if $\Omega(F)$ is the region of discontinuity of F then $\Omega(F)/F$ is a surface embedded in $\Omega(G)/G$.

Let $F = \langle T_1^{-1}E_\rho, T_iE_{i\rho}, T_iT_1^{-1}E_\rho T_i^{-1}, T_1^{-1}T_iE_{i\rho}T_1\rangle$. By calculation, the fixed points of the generators lie on a circle C and all the generators have trace greater than 2, hence are hyperbolics. Therefore they leave invariant the circle C and both the disks bounded by C. Hence every element of F leaves the disk D bounded by C invariant and so F is a Fuchsian group. Translates of D intersect D in the axes of the generators of F and their translates, and so the generators represent curves in the bending lamination. $\Omega(F)/F$ is a four-holed sphere with the boundary circles corresponding to the generators. Note that $T_1^{-1}E_\rho$ and $T_iT_1^{-1}E_\rho T_i^{-1}$ belong to the same orbit in G and hence represent the same curve in the bending lamination. Similarly the other two elements give us the other curve in the bending lamination. Gluing the holes under G gives us our genus-two surface pleated along $\tilde{\gamma}_{0/1}$. This proves that $\rho \in P_{0/1}$. In section 3 we showed that if $|\rho| < 1$ the group G_ρ cannot be discrete. Since $P_{0/1}$ is a connected component of $\widetilde{\mathbf{H}}_{0/1}$ and cannot contain groups which are not discrete, it must equal $(4, \infty)$.

Proposition 7.2 *The pleating ray $P_{1/1}$ is $\{\rho \in \mathbf{C} : \rho = re^{-i\pi/4},\ r > 2+\sqrt{2}\}$.*

Proof: From section 5, $V_{1/1} = T_i^{-1}T_1^{-1}E_\rho$ (see Figure 7) and so $\mathbf{Tr}V_{1/1} =$

$2-(1+i)\rho$. Thus $\mathbf{Tr}V_{1/1}$ is real if $\rho = re^{-i\pi/4}$ and the hyperbolic locus $\widetilde{\mathbf{H}}_{1/1} = (-\infty, \sqrt{2}-2)e^{-i\pi/4} \cup (2+\sqrt{2}, \infty)e^{-i\pi/4}$. Since $P_{1/1}$ cannot contain any groups which are indiscrete, it cannot contain $(-\infty, \sqrt{2}-2)e^{-i\pi/4}$. Let $\rho = re^{-i\pi/4}$, $r > 2+\sqrt{2}$. Then $\mathbf{Tr}V_{1/1} > 2\sqrt{2}$. The fundamental domain of G_ρ looks as in Figure 7. Let $V'_{1/1} = T_i^{-1}E_{i\rho}^{-1}T_1$ and $F_1 = \langle V_{1/1}, V'_{1/1}\rangle$. Note that $V'_{1/1} = T_1^{-1}R_{\pi/2}V_{1/1}R_{\pi/2}^{-1}T_1$ and so $V_{1/1}$ and $V'_{1/1}$ have the same trace and their fixed points lie on the boundary of the invariant disk D_1. The generators of F_1 are hyperbolic. Therefore they leave invariant the disk D_1 and the disk complement to it. Hence every element of F_1 leaves the disk D_1 invariant and so F_1 is Fuchsian. The axes of $V_{1/1}$ and $V'_{1/1}$ intersect each other transversely and $\Omega(F_1)/F_1$ is a torus with one hole. Let $F_2 = \langle U, U'\rangle$, where $U = E_{i\rho}T_i^{-1}E_\rho$ and $U' = E_\rho^{-1}E_{i\rho}^{-1}T_1$. By using the argument above one can show that F_2 is a Fuchsian group with invariant disk D_2 and $\Omega(F_2)/F_2$ is a torus with one hole. Translates of D_1 and D_2 fill up the region of dicontinuity of G and their intersection gives us curves which correspond to the commutator of $V_{1/1}$ and $V'_{1/1}$. This implies that $\rho \in P_{1/1}$.

By very similar arguments, one can check the following.

Proposition 7.3 *The following are pleating rays.*

1. $P_{-3/1} = \{\rho \in \mathbf{C} : \rho = re^{3i\pi/4}, r > 2 + \sqrt{2}\}$.

2. $P_{-2/1} = \{i\rho \in \mathbf{C} : \rho > 4\}$.

3. $P_{-1/1} = \{\rho \in \mathbf{C} : \rho = re^{i\pi/4}, r > 2 + \sqrt{2}\}$.

4. $P_{2/1} = \{-i\rho \in \mathbf{C} : \rho > 4\}$.

5. $P_{3/1} = \{\rho \in \mathbf{C} : \rho = re^{-3i\pi/4}, r > 2 + \sqrt{2}\}$.

6. $P_{4/1} = P_{-4/1} = \{-\rho \in \mathbf{C} : \rho > 4\}$.

We are now ready to identity all rational pleating rays. Recall that $P_{p/q} = P_{8+p/q}$. Throughout the remainder of this section, all rational numbers that we use are assumed to lie between -4 and 4.

From Proposition 5.3, we know the leading coefficient of $\mathbf{Tr}(V_{p/q})$ up to sign, so we can determine the asymptotic slope of the branches of $\widetilde{\mathbf{H}}_{p/q}$. If $\rho = re^{i\theta}$, $i^{p/2}\rho^q = r^q e^{i\frac{\pi}{2}\frac{p}{2}} e^{iq\theta} = r^q e^{i(\frac{\pi p}{4}+q\theta)}$. Therefore if $\theta = \frac{-\pi}{4}\frac{p}{q}$ then $i^{\frac{p}{2}}\rho^q$ is real. Similarly $(1+i)i^{\frac{p-1}{2}}\rho^q = r^q e^{i\frac{\pi}{4}} e^{i\frac{\pi}{2}\frac{p-1}{2}} e^{iq\theta} = r^q e^{i\frac{\pi(p-1)}{4}+\frac{\pi}{4}+q\theta}$ which implies that $(1+i)i^{\frac{p-1}{2}}\rho^q$ is real if $\theta = \frac{-\pi}{4}\frac{p}{q}$. Therefore $\widetilde{\mathbf{H}}_{p/q}$ has $2q$ branches with asymptotic slope $\frac{-\pi}{4}\frac{p}{q} + n\frac{\pi}{q}$, $n = 0, \ldots, 2q-1$. Let $\mathbf{H}_{p/q}$ denote the branch of $\widetilde{\mathbf{H}}_{p/q}$ with asymptotic slope $\frac{-\pi}{4}\frac{p}{q}$.

Theorem 7.4 *For $p/q \in \mathbf{Q}$, the pleating rays $P_{p/q}$ coincide with the branch $\mathbf{H}_{p/q}$ of $\widetilde{\mathbf{H}}_{p/q}$. This branch contains no singularities, in fact $P_{p/q}$ is a properly imbedded line in $\mathcal{S}$ whose complement is an open disc. Moreover, $P_{p/q} = P_{8+p/q}$.*

Proof: We will work only in the region between $P_{0/1}$ and $P_{1/1}$. The proofs for the other seven pie slices of $\mathcal{S}$ are analogous.

We know that $P_{0/1} = (4, \infty)$ and $P_{1/1} = re^{-i\frac{\pi}{4}}$, $r > 2 + \sqrt{2}$. We will use Farey induction to establish the theorem for all rational numbers between 0 and 1.

Assume the result is true for $P_{p/q}$ and $P_{r/s}$, where p/q and r/s lie between 0 and 1 and are Farey neighbors, i. e. $ps - rq = 1$ (so $r/s < p/q$). Consider $\mathbf{H}_{\frac{p+q}{r+s}}$. Let $\mathcal{B}$ denote the component of $\mathcal{S} - (\mathbf{H}_{p/q} \cup \mathbf{H}_{r/s})$ not intersecting the negative real axis.

We claim that $\mathbf{H}_{\frac{p+r}{q+s}}$ is the only branch of $\widetilde{\mathbf{H}}_{\frac{p+r}{q+s}}$ that can be contained in $\mathcal{B}$. We calculate $\frac{p+r}{q+s} - \frac{p}{q} = \frac{1}{q(q+s)}$ and $\frac{-\pi}{4}\frac{p+r}{q+s} + \frac{\pi}{4}\frac{p}{q} = \frac{-\pi}{4q(q+s)} > \frac{-\pi}{q+s}$ which implies that $\frac{-\pi}{4}\frac{p+r}{q+s} + \frac{\pi}{q+s} > \frac{-\pi}{4}\frac{p}{q}$. Similarly $\frac{-\pi}{4}\frac{p+r}{q+s} - \frac{\pi}{q+s} < \frac{-\pi}{4}\frac{r}{s}$. Consideration of asymptotic slopes shows that any other branch of $\widetilde{\mathbf{H}}_{\frac{p+r}{q+s}}$ would eventually move out of $\mathcal{B}$.

Pick ρ_1 and ρ_2 on $P_{p/q}$ and $P_{r/s}$ respectively. Choose an arc α in $\mathcal{B}$ connecting ρ_1 and ρ_2. L. Keen and C. Series showed in [9] that if we have a holomorphic family of groups based on a domain D then the pleating locus changes continuously on D. This result along with Proposition 2.7 implies that the pleating locus changes continuously on $\mathcal{B}$. Thus there exists a point $\rho \in \alpha$ such that $pl(\rho) \in P_{\frac{p+r}{q+s}}$. The continuity also implies that $P_{\frac{p+r}{q+s}}$ is contained in $\mathcal{B}$. If, for example, there were a point in $P_{\frac{p+r}{q+s}}$ in the region $\mathcal{S}'$ of $\mathcal{S}$ between $P_{p/q}$ and $P_{4/1}$ (that is, in the lower half plane but not between $P_{0/1}$ and $P_{p/q}$), then since $\frac{p+r}{q+s} < p/q < 4$, on an arc in $\mathcal{S}'$ running from $P_{p/q}$ to $P_{4/1}$, there would have to be a point whose pleating locus is $\hat{\gamma}_{p/q}$. But by induction, $P_{p/q}$ has only one component.

Next we show that no component of $\widetilde{\mathbf{H}}_{\frac{p+r}{q+s}}$ in $P_{\frac{p+r}{q+s}}$ contains a singularity.

Suppose for the sake of contradiction that $P_{\frac{p+r}{q+s}}$ contains a component of $\widetilde{\mathbf{H}}_{\frac{p+r}{q+s}}$ that has a singularity at ρ_0. The local coordinate at a branch point of order k can be written in the form $\rho - \rho_0 = z^k$, $k > 1$. Thus the preimage of a line through ρ_0 has $2k$ branches meeting at $z = 0$. According to Keen and Series in [10], starting at a singularity one can find at least two branches along which the trace is increasing and goes to ∞. Since the asymptotic slope of any other branch of $\widetilde{\mathbf{H}}_{\frac{p+r}{q+s}}$ is greater than $\frac{-\pi}{4}\frac{p}{q}$ and less than $\frac{-\pi}{4}\frac{r}{s}$, one of these branches would eventually have to leave $\mathcal{B}$ while remaining in $\mathcal{S}$ since $P_{p/q} \subset \mathcal{S}$. This would create a point of intersection of $P_{p/q}$ and $P_{\frac{p+r}{q+s}}$ or $P_{r/s}$ and $P_{\frac{p+r}{q+s}}$. But $P_l \cap P_m = \emptyset$ if $l \neq m$.

Since there are no singularities, the function $\mathbf{Tr}V_{p/q}(\rho)$ has no maximum or minimum on any component of $P_{p/q}$. For at a maximum or a minimum, $\mathbf{Tr}V_{p/q}(\rho)$ would have derivative zero in the direction along $\widetilde{\mathbf{H}}_{p/q}$.

We show now that $\mathbf{Tr}V_{p/q}(\rho)$ is unbounded on every connected component of $P_{p/q}$. Suppose that K is a connected component of $P_{p/q}$ on which $\mathbf{Tr}V_{p/q}(\rho)$ is bounded. On K, $\mathbf{Tr}V_{p/q}(\rho)$ is a real valued polynomial, so K lies in a bounded region of the plane. Therefore the closure $\overline{K}$ is compact. Either

vertices of one of the 4-gons. By Proposition 2.7 we can obtain an analytic family $\{f_\rho G_{\rho_0} f_\rho'\}$ of groups based on $\mathcal{S}\backslash P_0$. The points $v_i(\rho) = f_\rho(v_i)$ vary analytically with ρ, therefore their cross ratio also varies analytically. Define the analytic function F_0 on $\mathcal{S}\backslash P_0$ by putting $F_0(\rho)$ equal to the cross ratio $[v_1(\rho), v_2(\rho), v_3(\rho), v_4(\rho)]$. The cross ratio is real only when all the $v_i(\rho)$ lie on the boundary of the same support plane. Therefore P_λ is contained in the real locus of F_0.

Similarly by starting with $\mathcal{S}\backslash P_1$ one can obtain another analytic family $\{g_\rho G_{\rho_0} g_\rho'\}$ of groups and an analytic function F_1 whose real locus contains P_λ. If $\rho \in \mathcal{S}\backslash P_0 \cap \mathcal{S}\backslash P_1$ then $\{f_\rho G_{\rho_0} f_\rho'\}$ and $\{g_\rho G_{\rho_0} g_\rho'\}$ represent the same point in $QD(G_{\rho_0})$ and so the maps f and g differ by a conformal map c, that is $f = cg$. Since conformal maps preserve cross ratio, $F_0(\rho) = F_1(\rho)$ and thus P_λ is contained in the real locus of an analytic function defined on $\mathcal{S}$.

The next result shows that the rational rays are dense in $\mathcal{S}$.

Lemma 8.3 *Suppose that $\rho_0 \in P_\lambda$ and that $p_n/q_n \in \mathbf{Q}$, $p_n/q_n \to \lambda$ as $n \to \infty$. Then there exists $\rho_n \in P_{p_n/q_n}$ with $\rho_n \to \rho_0$.*

Proof: Without loss of generality we may assume that the sequence p_n/q_n is increasing with limit λ. Pick $\rho_1 \in P_{p_1/q_1}$. Join ρ_1 to ρ_0 by a path α transversal to P_λ so that $\alpha \cap P_\lambda = \{\rho_0\}$. L. Keen and C. Series showed in [9] that if we have an analytic family of groups based on a domain D then the pleating locus changes continuously on D. This result along with Proposition 2.7 implies that the pleating locus changes continuously on α. Since $p_1/q_1 < p_n/q_n < \lambda$. α must intersect all the pleating rays P_{p_n/q_n} at points $\rho_n = \alpha(t_n)$. We may clearly assume that $\ldots < t_n < t_{n+1} < \ldots$ so that $\{\rho_n\}$ has a limit $\alpha(t_\infty) \in \alpha$. Again by the continuity of the pleating locus. $pl(\alpha(t_\infty)) = \lim_{n\to\infty} p_n/q_n = \lambda$ so that $\alpha(t_\infty) \in P_\lambda$. Since by construction $\alpha \cap P_\lambda = \{\rho_0\}$, we have $\alpha(t_\infty) = \rho_0$ which implies that $\rho_n \to \rho_0$.

9 Normalized Traces

Given an element A in $PSL(2, \mathbf{C})$ define the complex length, $L(A)$, as

$$2\cosh\left(\frac{L(A)}{2}\right) = \mathbf{Tr}A$$

and requiring that $\Re(L(A)) \geq 0$. Since the trace of an element of $PSL(2, \mathbf{C})$ is only defined up to a choice of sign, the complex length is only defined up to addition of $i\pi$. We write $L(A) = l(A) + i\phi(A)$ where $l(A)$ is the translation length of A along its axis. Thus $\phi(A)$ is defined only modulo π when $A \in PSL(2, \mathbf{C})$ but for some definite choice of lift of A to $SL(2, \mathbf{C})$ it is defined modulo 2π.

Consider the function $f(z) = \cosh(z)$, $\Re z > 0$. The image of f is $\mathbf{C} - [-1, 1]$. f is periodic with period $2\pi i$ and the restriction of f to any horizontal strip of width less than 2π is injective. Thus f^{-1} is a multiple-valued analytic function. Let $L(V_{p/q})$ denote a single-valued analytic branch of $f^{-1}(-\mathbf{Tr}V_{p/q})$ which is analytic on $\mathcal{S}$ and is real on $\mathcal{P}_{p/q}$. Recall that $\widetilde{W}_{p/q} = \{[V_{p/q}, V'_{p/q}]\}$ for p odd and $\widetilde{W}_{p/q} = \{V_{p/q}, V'_{p/q}\}$ for p even. Define

$$L_{p/q}(\rho) = \begin{cases} \frac{1}{8q}L([V_{p/q}, V'_{p/q}]) & \text{if } p \text{ is odd} \\ \frac{1}{4q}(L(V_{p/q}) + L(V'_{p/q})) & \text{if } p \text{ is even.} \end{cases}$$

For a matrix $A = (a_{ij})$, define $\|A\|$ to be the maximum of the $|a_{ij}|$.

Lemma 9.1 *Let A_k, $1 \leq k \leq n$ be 2×2 matrices such that $\|A_k\| \leq 2$. Then $\| \prod_{k=1}^{n} A_k\| \leq 2^{2n-1}$.*

Proof: Write $A_k = (a_{ij}^k)$ and $\prod_{k=1}^{r} A_k = (d_{ij}^r)$. The lemma is obviously true for $n = 1$. Inductively, we have

$$\delta_{ij}^{r+1} = d_{i1}^r a_{1j}^{r+1} + d_{i2}^r a_{2j}^{r+1}$$

and hence

$$|d_{ij}^{r+1}| \leq 2(|d_{i1}^r| + |d_{i2}^r|) \leq 4 \cdot 2^{2r-1} = 2^{2r+1}.$$

Lemma 9.2 *Let $V(\rho)$ be a word of length $2q$ in T_a and $E_{b\rho}$, $a, b \in \{\pm 1, \pm i\}$. Write $\mathbf{Tr}(V(\rho)) = c_q\rho^q + \cdots + c_0$. Then $max_{i=0,\ldots,q}|c_i| \leq 4^q$.*

Proof: The $2q$ matrices all have entries satisfying $|a_{ij}| \leq 2$ on the circle $|\rho| = 1$. By the lemma above, the entries of $V(\rho)$ satisfy $|d_{ij}(\rho)| \leq 2^{4q-1}$ so on the unit circle $\rho = 1$,

$$\mathbf{Tr}(V(\rho)) = |d_{11}(\rho) + d_{22}(\rho)| \leq 2(2^{4q-1}) = 2^{4q} = 16^q\ .$$

Now think of $\mathbf{Tr}(V(e^{i\theta}))$ as a Fourier series on the unit circle. Recall Parseval's identity

$$\frac{1}{2\pi}\int_{|\rho|=1}|c_q\rho^q+\cdots+c_0|^2=\Sigma_{i=0}^q|c_i|^2\,.$$

It implies that $\Sigma_{i=0}^q|c_i|^2\leq 16^q$.

Corollary 9.3 *For $p/q\in\mathbf{Q}$ and $V\in\tilde{W}_{p/q}$, the coefficients of $\mathbf{Tr}V(\rho)$ are bounded by 4^{4q}.*

Proof: Since V a product at most $8q$ matrices as in Lemma 9.2, the result follows.

Thus the family $L_{p/q}(\rho)$ is uniformly bounded on compact subsets of $\mathcal{S}$ and hence is a normal family. The limit functions L_λ of this family, taken as $p/q\in\mathbf{Q}\to\lambda\in\mathbf{R}$, are the normalized length functions that we require. We shall prove the existence and uniqueness of these functions by using the theory of measured laminations.

10 Pleating measure and pleating length

A transverse measure ν on a geodesic lamination L on a hyperbolic surface X of finite area is an assignment of a regular countably additive measure to every interval transversal to L in such a way that these measures are preserved by any isotopy mapping one transversal to another and preserving the leaves of the lamination. We call the pair (L, ν) a measured lamination. By abuse of terminology we usually refer to ν as a measured lamination and write $|\nu|$ for the underlying set L.

We denote by $\mathcal{ML}(X)$ the space of measured laminations on X. The weak topology on measures gives a natural topology on $\mathcal{ML}(X)$.

For $\nu \in \mathcal{ML}(X)$, the lamination length of ν, $\ell(\nu)$, is the total mass of the measure on X that is locally the product of the measure ν on transversals to $|\nu|$ and hyperbolic distance along the leaves of $|\nu|$. Note that if γ is a simple closed geodesic then $\ell(\delta_\gamma)$ is exactly the hyperbolic length of γ, where δ_γ is the measured lamination whose leaves consist of γ and the measure is the atomic unit mass on γ. It is a well known fact in the theory of measured laminations that $\ell : \mathcal{ML}(X) \to \mathbf{R}^+$ is continuous.

If $\nu \in \mathcal{ML}(X)$ and if γ is a simple closed geodesic on X, the intersection number $i(\gamma, \nu)$ is the minimal measure given by ν to a curve isotopic to γ. In particular, if $\nu = \delta_\alpha$ for some simple geodesic α, then $i(\gamma, \delta_\alpha)$ is just the geometric intersection number in the usual sense. It is known that the map $i_\gamma : \mathcal{ML}(X) \to \mathbf{R}$, $i_\gamma(\nu) = i(\gamma, \nu)$ is continuous [5].

The fundamental domain $\mathcal{F}_\rho$ is topologically an annulus, so contains (up to isotopy) a unique noncontractible simple closed loop. Let $\tilde{\gamma}(\infty)$ be the unique geodesic in S_ρ which is homotopic in S_ρ to the image of this loop. This is the unique simple geodesic in S_ρ that bounds a disk in $(\mathbf{H}^3 \cup \Omega(\rho))/G_\rho$. Let $\gamma(\infty)$ denote the projection of $\tilde{\gamma}(\infty)$ to the orbifold $\mathcal{O}$, also a simple geodesic since it is invariant under $R_{\pi/2}$. The next proposition shows that if $\rho \in P_{p/q}$ then the complex translation length $L_{p/q}(\rho)$ is the ratio of the length of the pleating locus to the intersection number of the pleating locus with $\tilde{\gamma}(\infty)$.

In order to simplify our notation we shall write $\delta_{p/q}$ for $\delta_{\tilde{\gamma}_{p/q}}$.

Proposition 10.1 *With the notation as above, if $\rho \in P_{p/q}$ then*

$$L_{p/q}(\rho) = \ell_\rho(\delta_{p/q})/i(\tilde{\gamma}(\infty), \delta_{p/q})$$

where ℓ_ρ denotes the lamination length on the surface $S_\rho = \partial C/G_\rho$.

Proof: Since the pleating locus of ρ is $\hat{\gamma}_{p/q}$, the length of $\hat{\gamma}_{p/q}$ in the hyperbolic surface S_ρ coincides with its length in the 3-manifold $\mathbf{H}^3/G_\rho$. As $\rho \in P_{p/q}$, the traces of the elements of $\widetilde{W}_{p/q}$ are real. So $\ell_\rho(\delta_{p/q}) = L([V_{p/q}, V'_{p/q}])$ for p odd and $L(V_{p/q}) + L(V'_{p/q})$ for p even.

It remains therefore to show that $i(\tilde{\gamma}(\infty), \delta_{p/q})$ is $8q$ for p odd, and $4q$ for p even. Recall from section 5 that the line segment $l(p,q)$ covers the arc $\gamma_{p/q}$ exactly once and therefore $\gamma_{p/q}$ has exactly q intersections with $\gamma(\infty)$. Since S_ρ is a four fold cover of $\mathcal{O}$, $\tilde{\gamma}_{p/q}$ has $4q$ intersections with $\tilde{\gamma}(\infty)$. For p odd this implies that the commutator $\hat{\gamma}_{p/q}$ of $\gamma'_{p/q}$ and $\gamma''_{p/q}$ has $8q$ intersections with $\tilde{\gamma}(\infty)$. Hence $i(\tilde{\gamma}(\infty), \delta_{p/q})$ is $8q$ for p odd, and $4q$ for p even.

Recall that the bending measure $\beta(\rho)$ of $pl(\rho)$ is a natural transverse measure that measures the total angle through which support planes of the convex hull are bent when moving along a transversal to $pl(\rho)$.

Define the normalized bending lamination $\beta_n(\rho)$ to be

$$\beta_n(\rho) = (|\beta(\rho)|, \beta(\rho)/i(\tilde{\gamma}(\infty), \beta(\rho)))$$

and the pleating length $PL(\rho) = \ell_\rho(\beta_n(\rho))$.

Proposition 10.2 *The function $PL: S \to \mathbf{R}$ is continuous, and if $\rho \in P_{p/q}$ then $PL(\rho) = L_{p/q}(\rho)$.*

Proof: The continuity of ℓ and of $i_{\tilde{\gamma}(\infty)}$ were mentioned above. The continuity of the bending lamination follows from [9]. The fact that $PL(\rho) = L_{p/q}(\rho)$ follows from Lemma 10.1.

We shall use the above characterization of the normalized length functions to prove the uniqueness of the limit functions of the normal family $\{L_{p/q}(\rho)\}$.

Lemma 10.3 *Suppose that $p_n/q_n \to \lambda \in \mathbf{R}$ and that $\rho_0 \in P_\lambda$. Then $\Re L_{p_n/q_n}(\rho_0) \to PL(\rho_0)$.*

Proof: We shall denote the geodesic lamination representing the elements of $\widetilde{W}_{p/q}$ in the 3-manifold $\mathbf{H}^3/G_{\rho_0}$ by $\hat{\gamma}_{p_n/q_n}$ and on the surface S_{ρ_0} by $\tilde{\gamma}_{p_n/q_n}$.

The pleating locus of $\partial C/G_{\rho_0}$ is $\hat{\gamma}_\lambda$ because $\rho_0 \in P_\lambda$. Since the intersection number changes continuously, by Proposition 10.1 it is enough to show that $\Re L(\widetilde{W}_{p_n/q_n})(\rho_0) \to l_{\rho_0}(\hat{\gamma})$.

A generic leaf l of $\hat{\gamma}_\lambda$ may be approximated arbitrarily closely by leaves l_n of $\hat{\gamma}_{p_n/q_n}$. Let $\tilde{l_n}$ and $\tilde{l}$ be lifts, of l_n and l respectively, to $\mathbf{H}^3$ so that $\tilde{l_n}$ converges to $\tilde{l}$. Since l is contained in the pleating locus its lift $\tilde{l}$ is a geodesic in $\mathbf{H}^3$. Now the lift $\tilde{l_n}$ is invariant under some conjugate g_n of an element of $\widetilde{W}_{p_n/q_n}$. The endpoints of $\tilde{l_n}$ in the limit set $\Lambda(G_{\rho_0})$ are the fixed points of g_n. By our choice of $\tilde{l_n}$, these endpoints converge to the endpoints of $\tilde{l}$. Hence the geodesic axis of g_n converges to $\tilde{l}$ in $\mathbf{H}^3$. This implies that $\hat{\gamma}_{p_n/q_n}$ converges to $\hat{\gamma}_\lambda$ in the 3-manifold. The length of $\hat{\gamma}_{p_n/q_n}$ is given by $\Re L(\widetilde{W}_{p_n/q_n})(\rho_0)$ and the length of $\hat{\gamma}_\lambda$ equals $l_{\rho_0}(\hat{\gamma})$. Hence $\Re L(\widetilde{W}_{p_n/q_n})(\rho_0) \to l_{\rho_0}(\hat{\gamma})$ as required.

Theorem 10.4 *The family $\{L_{p/q}(\rho)\}_{p/q\in\mathbf{Q}}$ extends to a family $\{L_\lambda(\rho)\}_{\lambda\in\mathbf{R}}$ in such a way that $L_\lambda(\rho) = PL(\rho)$ for $\rho \in P_\lambda$, and L_λ depends continuously on λ.*

Proof: We need to show that for any sequence p_n/q_n with $p_n/q_n \to$.

the sequence of functions $\{L_{p_n/q_n}(\rho)\}$ converges on compact subsets of $\mathcal{S}$ to a limit function which is independent of the sequence p_n/q_n and has the asserted value on P_λ.

We saw in section 9 that the family $\{L_{p/q}(\rho)\}_{p/q \in \mathbf{Q}}$ is normal. So every sequence has a convergent subsequence. Pick a convergent subsequence of $\{L_{p_n/q_n}(\rho)\}$ and by abuse of notation write $L_{p_n/q_n} \to L$. Let $\rho \in P_\lambda$. Then $\Re L_{p_n/q_n}(\rho) \to \Re L(\rho)$ and by the previous lemma $\Re L_{p_n/q_n}(\rho) \to PL(\rho)$. Therefore $\Re L|_{P_\lambda} = PL|_{P_\lambda}$. We claim that L is real on P_λ. Assuming this, we have $L|_{P_\lambda} = PL|_{P_\lambda}$. Therefore L is completely determined by its values on P_λ and so is independent of p_n/q_n.

Proof of the claim: Let $\rho_0 \in P_\lambda$. By Lemma 8.3 we can find $\rho_n \in P_{p_n/q_n}$, with ρ_n converging to ρ_0. Since $L_{p_n/q_n} \to L$ uniformly, $L_{p_n/q_n}(\rho_n) \to L(\rho_0)$. Since $L_{p_n/q_n}(\rho_n) \in \mathbf{R}$, $L(\rho_0) \in \mathbf{R}$ as claimed.

11 Pleating Coordinates

In this final section we will introduce coordinates on our slice $\mathcal{S}$. Before we do that there is one more result that we need.

Theorem 11.1 *The real pleating ray is a connected component of the real locus of the complex pleating length L_λ in $\mathcal{S}$. This component contains no singularities and is asymptotic to the line $te^{-\frac{\pi}{4}\lambda}$, with $t > 0$, as $|\rho| \to \infty$.*

Proof: Let $\frac{p}{q}$ and $\frac{r}{s}$ be Farey neighbors such that $\frac{p}{q} < \lambda < \frac{r}{s}$. Let $A_{(\frac{p}{q},\frac{r}{s})} = \{\rho \in \mathcal{S} \mid pl(\rho) < \frac{p}{q}$ or $pl(\rho) > \frac{r}{s}\}$. Since $\mathcal{S}$ is topologically an annulus, $A_{(\frac{p}{q},\frac{r}{s})}$ is a connected, simply connected subspace of $\mathcal{S}$ bounded by the pleating rays $P_{p/q}$ and $P_{r/s}$. Let $A_\lambda = \cup_{(\frac{p}{q},\frac{r}{s})} A_{(\frac{p}{q},\frac{r}{s})}$. Then A_λ is connected and simply connected. We have $P_\lambda \cap A_\lambda = \emptyset$, and $\mathcal{S} - P_\lambda = A_\lambda$. Thus since $\mathcal{S}$ is an annulus and $\mathcal{S} - P_\lambda$ is simply connected, P_λ must be connected.

Now P_λ is part of the real locus of an analytic function and if it contained a singularity, at least three branches would meet there. These branches would separate $\mathcal{S}$ into at least two components contradicting the fact that A_λ is connected.

We now prove our main result.

Theorem 11.2 *The map*

$$\Pi : \mathcal{S} \longrightarrow \mathbf{R}/8\mathbf{Z} \times \mathbf{R}_{>0}$$

defined by $\Pi(\rho) = (pl(\rho), L_{pl(\rho)}(\rho))$ *is a homeomorphism onto its image.*

Proof: Π is continuous because each of its coordinate functions is continuous. Since the pleating rays do not contain any singularities, there cannot exist two different points in $\mathcal{S}$ with the same pleating locus, so Π is injective.

It is sufficient now to show that Π is open. Let $U \subset \mathcal{S}$ be an open disc, $(\lambda, c) \in \Pi(U)$ and let $\rho(\lambda, c) = \Pi^{-1}(\lambda, c)$. Since $pl(\rho)$ is λ, ρ lies in P_λ. Since U is open we can find an $\epsilon > 0$ such that the image of $U \cap P_\lambda$ contains $(\lambda, c \pm \frac{\epsilon}{2})$. Set $c_1 = c + \frac{\epsilon}{2}$ and $c_2 = c - \frac{\epsilon}{2}$. Draw arcs $\sigma_i \subset U$ transversal to P_λ at the points $\Pi^{-1}(\lambda, c_i)$, $i = 1, 2$. We may choose both σ_1 and σ_2 to have endpoints on the rays P_{λ_1} and P_{λ_2} respectively, with $\lambda_1 < \lambda < \lambda_2$ and short enough that $|L_t(\rho) - c_1| < \frac{\epsilon}{8}$ for $t \in \sigma_1$ and $|L_t(\rho) - c_2| < \frac{\epsilon}{8}$ for $t \in \sigma_2$.

Let $W \subset U$ denote the region bounded by the rays P_{λ_i} and σ_j, $i, j = 1, 2$. We claim that $(\lambda_1, \lambda_2) \times (c - \frac{\epsilon}{4}, c + \frac{\epsilon}{4})$ is a basic open set in $\Pi(U)$ containing (λ, c). This will complete the proof of the theorem.

Let $(t, \zeta) \in (\lambda_1, \lambda_2) \times (c - \frac{\epsilon}{4}, c + \frac{\epsilon}{4})$. $\rho_t = \Pi^{-1}(t, \zeta)$ lies on P_t. By the continuity of $pl \circ \sigma_i$, we can see that $P_t \cap \sigma_i \neq \emptyset$, $i = 1, 2$. Now $L_t(\rho) > c + \frac{3\epsilon}{8}$ for $\rho \in \sigma_1$ and $L_t(\rho) < c - \frac{3\epsilon}{8}$ for $\rho \in \sigma_2$. Since $|L_t(\rho) - c| < \frac{\epsilon}{4}$ and L_t is monotone on P_t we can conclude that $\rho_t \in W$ and we are done.

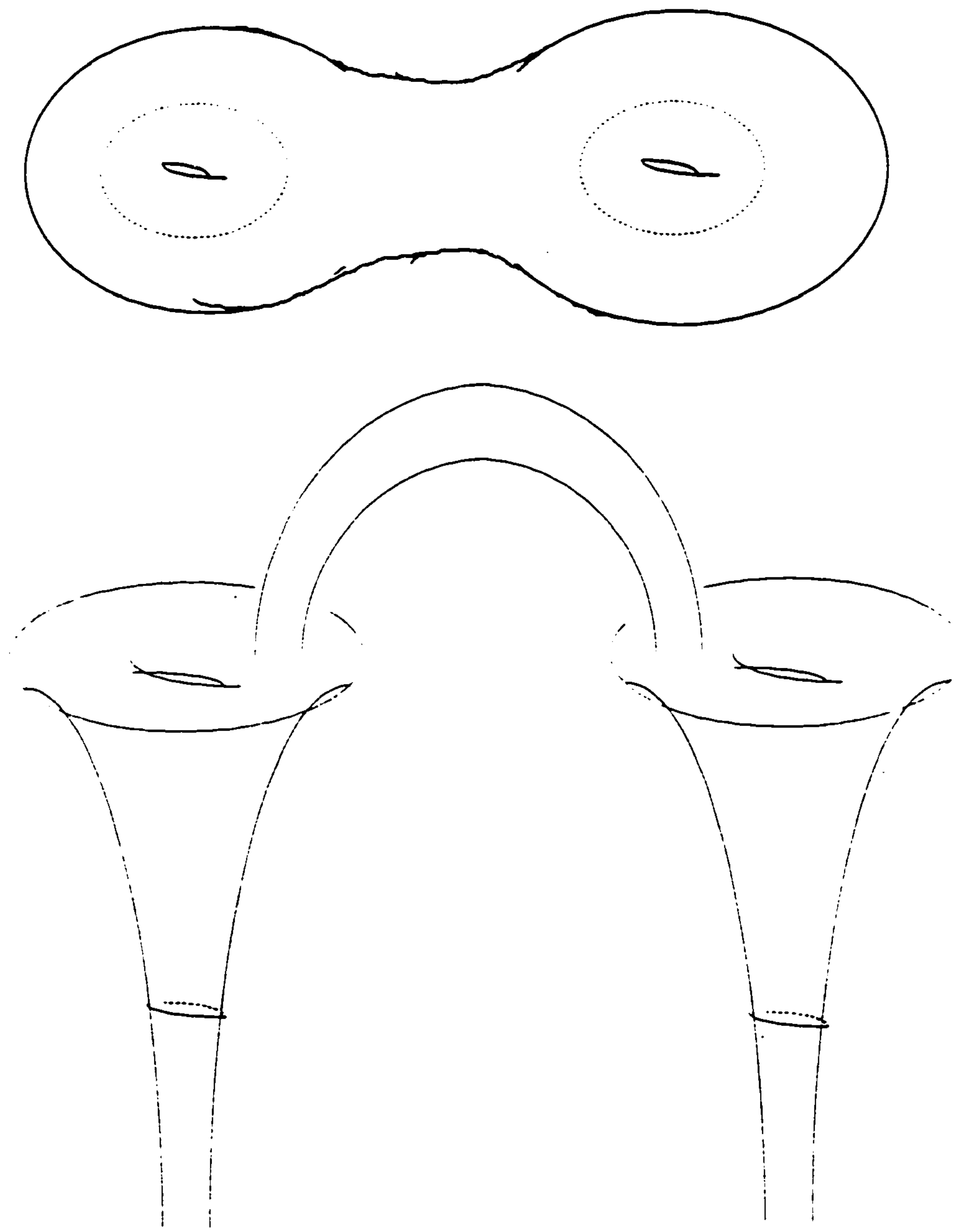

Figure1. The two cusped hyperbolic three manifold

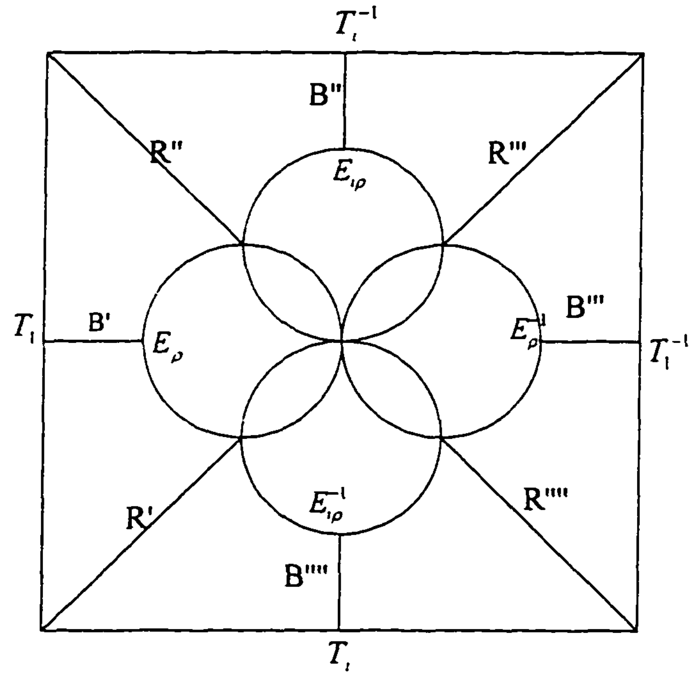

Figure 2. The fundamental domain F_ρ

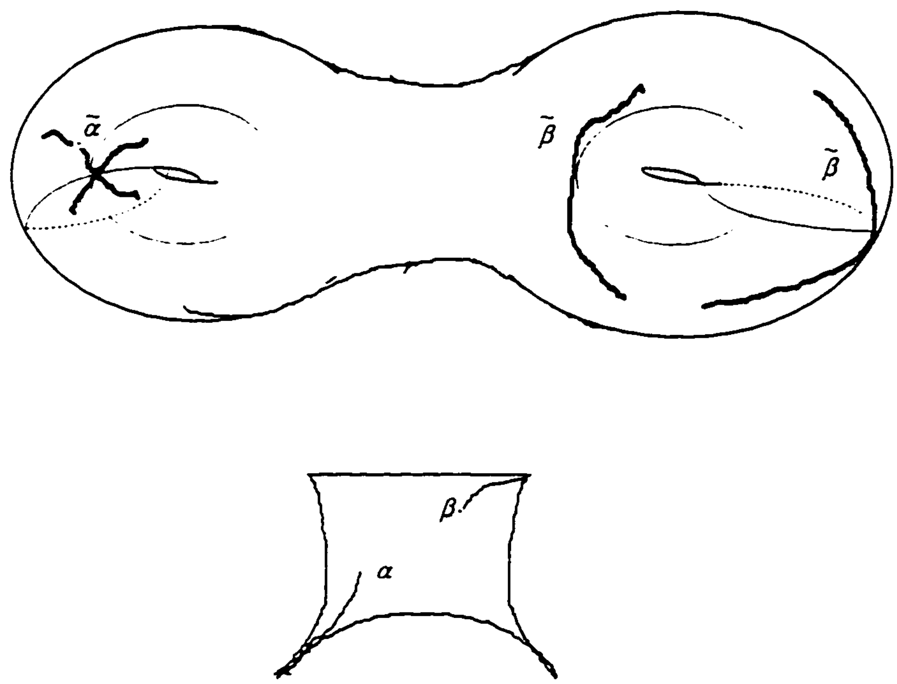

Figure 3

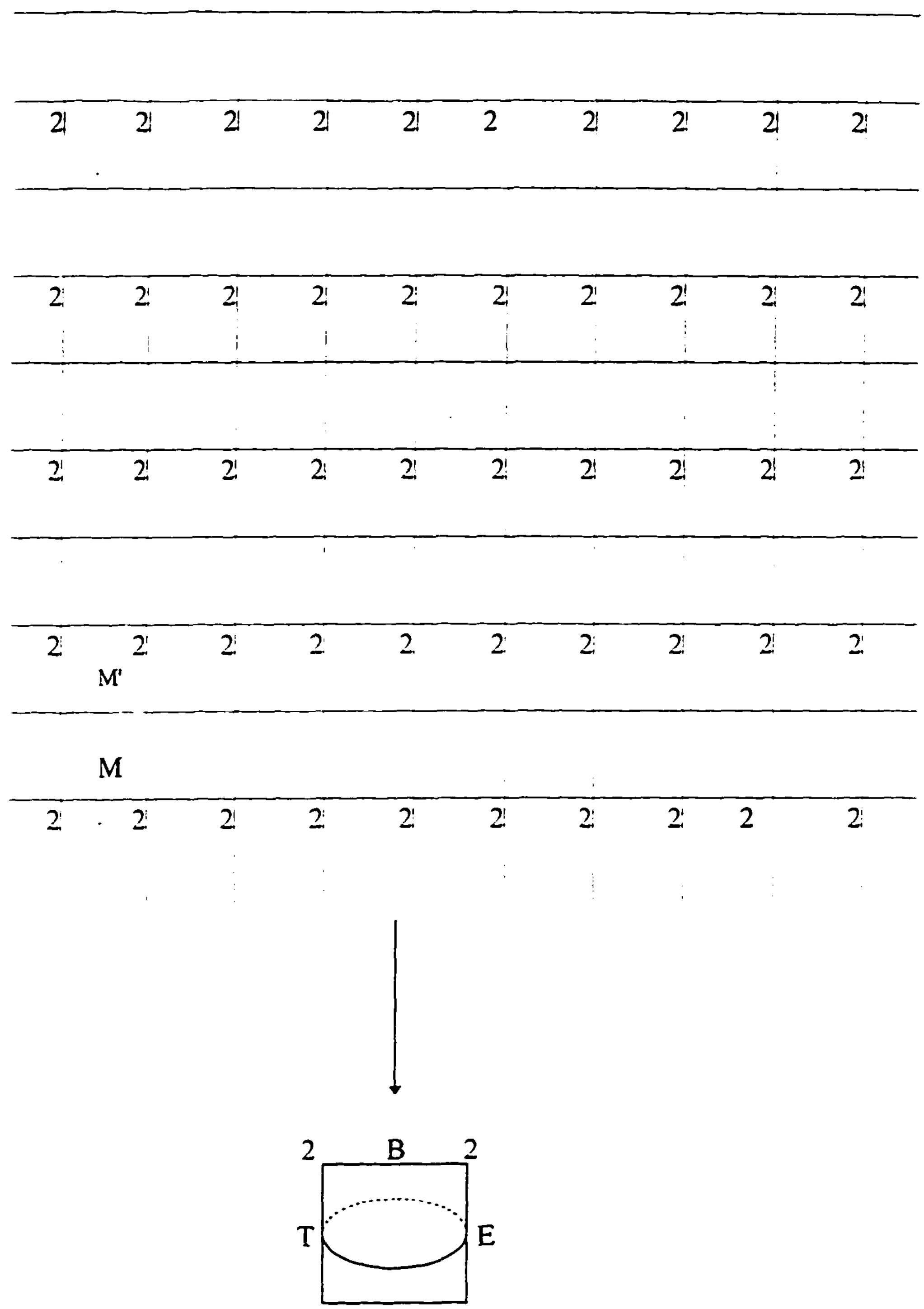

Figure 4. The planar covering of the orbifold O

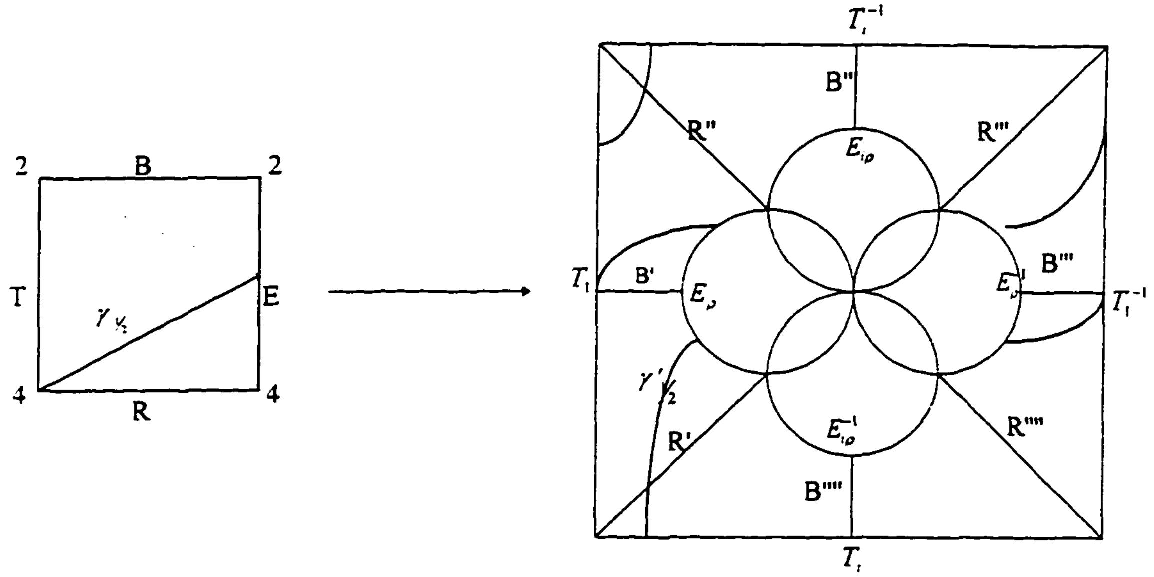

Figure 5a: The curve $\gamma_{1/2}$ on the orbifold O and its lift $\gamma'_{1/2}$ on the fundamental domain F_ρ.

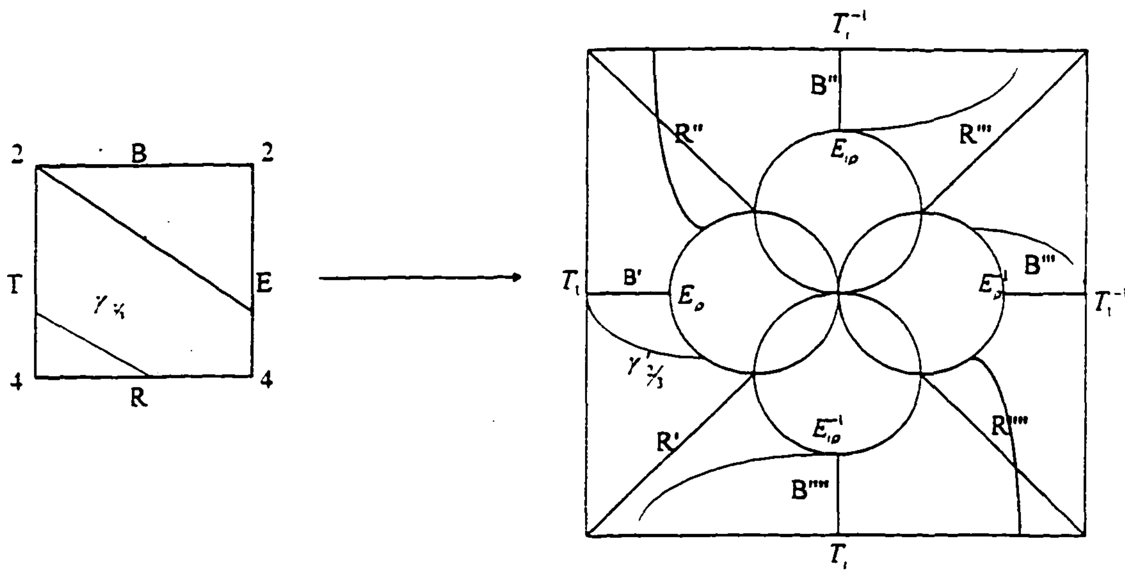

Figure 5b: The curve $\gamma_{3/1}$ on the orbifold O and its lift $\gamma'_{3/3}$ on the fundamental domain F_ρ.

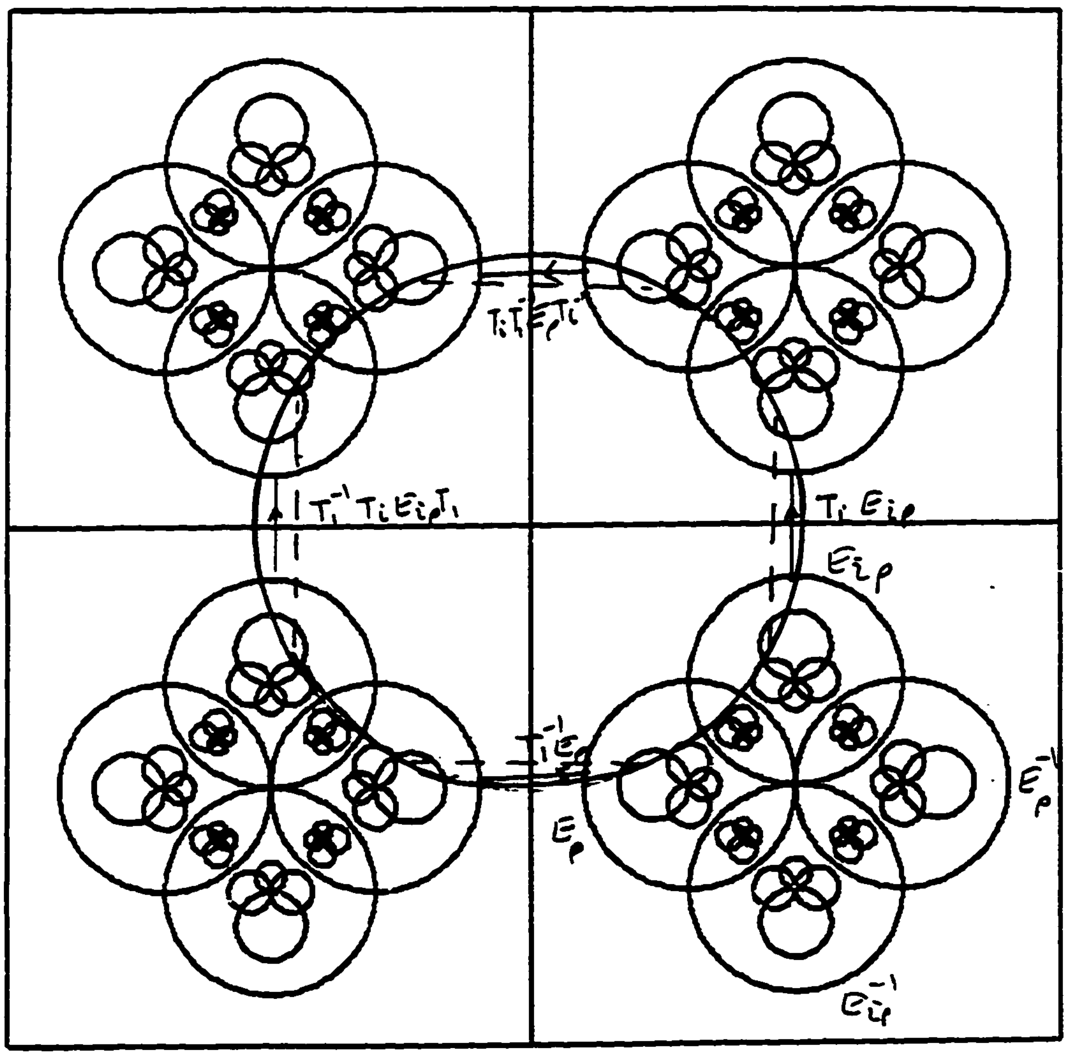

Figure 7. The fundamental domain of G_ρ for ρ real.

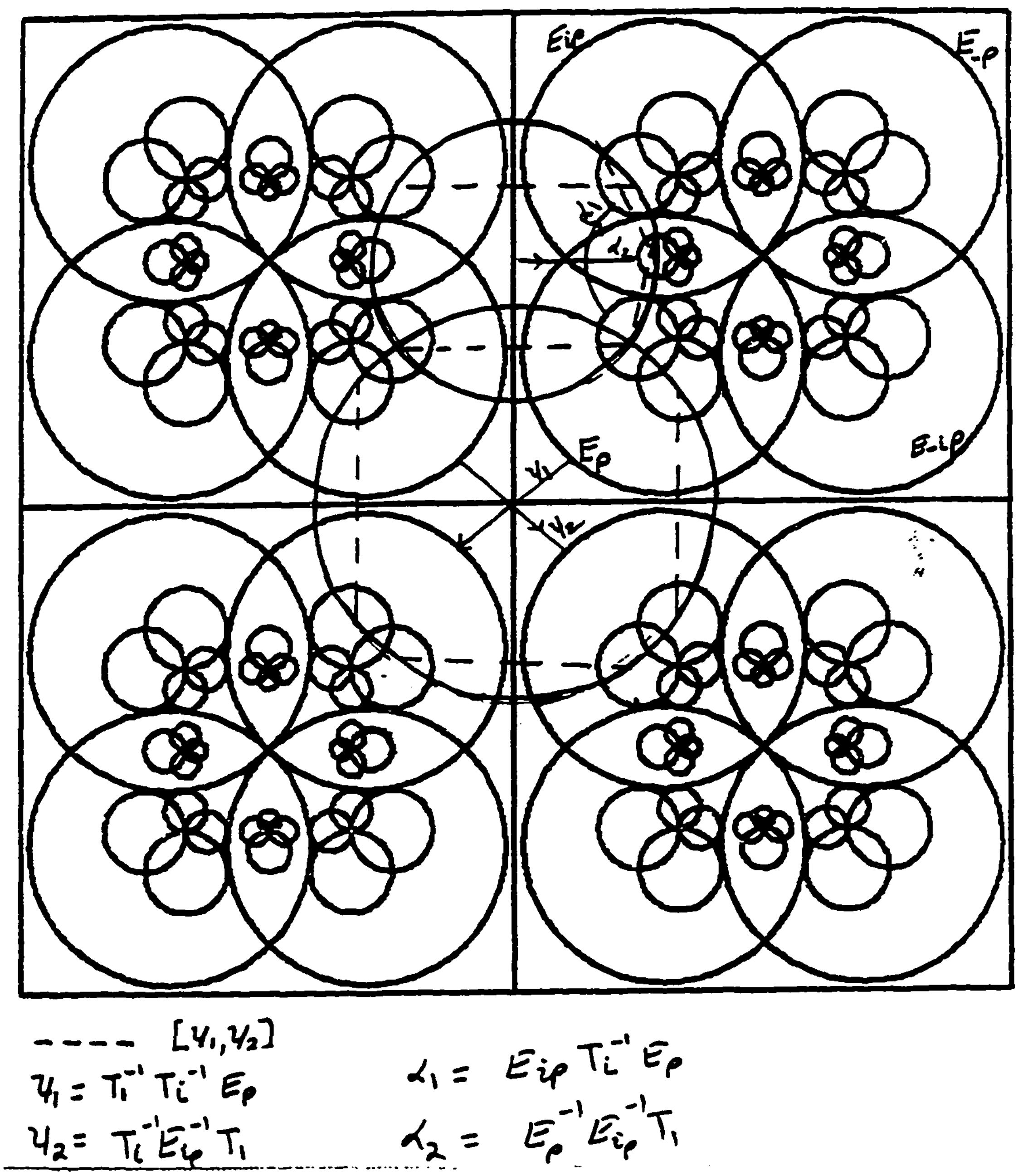

Figure8. The fundamental domain of G_ρ for $\rho = re^{i\pi/4}$, $r > 2 + \sqrt{2}$.

References

[1] L. Ahlfors, *Lectures on Quasiconformal Mappings*, Von Nostrand, 1966.

[2] L. Ahlfors and L. Bers, "Riemann's Mapping Theorem for Variable Metrics," *Annals of Math.* 72 (1960) 385-404.

[3] A. F. Beardon, *The Geometry of Discrete Groups*, Springer, New York, 1983.

[4] L. Bers, "Spaces of Kleinian groups," *1970 Several Complex Variables, I (Proc. Conf., Univ. of Maryland, College Park, Md., 1970*, 9-34.

[5] F. Bonahon, "The geometry of Teichmüller space via geodesic currents," *Invent. Math.* 92 (1988) 139-162.

[6] D. B. A. Epstein and A. Marden, "Convex hulls in hyperbolic spaces, a theorem of Sullivan, and measured pleated surfaces" in *Analytical and Geometrical Aspects of Hyperbolic Spaces*, Cambridge University Press, 1987, 113-253.

[7] F. Gardiner, *Teichmüller Theory and Quadratic Differentials*, John Wiley and Sons, 1987.

[8] H. Shimizu, "On discontinuous groups operating on the product of the upper half planes," *Annals of Math.* (2) 77 (1963), 33-71.

[9] L. Keen and C. Series, "Continuity of convex hull boundaries," *Pacific J. Math.* 168 (1995) 183-206.

[10] L. Keen and C. Series. "Pleating coordinates for the Maskit embedding of the Teichmüller space of punctured tori," *Topology* 32 (1993), 719-749.

[11] L. Keen and C. Series. "Pleating coordinates for the Riley slice of Schottky space," *Proc. London Math. Soc.* 69 (1994), 72-90.

[12] B. Maskit, *Kleinian Groups*. Springer, New York, 1987.

[13] W. Thurston, *The Geometry and Topology of 3-manifolds*, lecture notes.