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VORTICITY-BASED ESTIMATION OF VERTICAL VELOCITIES FROM RADAR DATA: ACCURACY AND SENSITIVITY

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VORTICITY-BASED ESTIMATION OF VERTICAL VELOCITIES FROM RADAR DATA: ACCURACY AND SENSITIVITY

A DISSERTATION APPROVED FOR THE DEPARTMENT OF MATHEMATICS

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ABSTRACT

In this thesis, we make use of the weak formulation of the vorticity equation and the mass continuity equation to calculate the vertical component, w, of the wind field $\vec{V}$, given the two horizontal components u and v. Since w will be depending on u and v and its approximation, w^N, will be depending on approximations of u and v and the number of points in some mesh, we will examine the effect of the latter on w^N as well as the effect of small perturbations of u and v on w and w^N. Theoretically, the sensitivity of w to u and v can be investigated by studying the existence of the $Fréchet\ differential$ of w with respect to the couple (u, v). We develop methods that significantly reduce the computational time when using the finite element method in both linear and cubic approximations. The mentioned methods are the origin of an idea, presented in Chapter VIII, that can be applied to any variational problem in order to reduce the cost of the computation of the stiffness matrix and the load vector. We set the problem such that we obtain an estimation of w from estimations of u and v based on radar data. The first and obvious finding is that the finer the mesh, the more accurate the approximation; the second states that more accurate results are obtained when using both the vorticity equation and the mass continuity equation than when working with only one of them. And the third one tells us that the coarser the mesh, the less sensitive the approximation; of course, the latter is a result of using a smaller number of points; because when we work with the same values of u and v and the same number of mesh points but in a larger or smaller domain, the sensitivity of w^N remains the same.

CHAPTER I

INTRODUCTION

The synthesis of three dimensional vector wind fields from Doppler radar data is an important part of mesoscale research and operational meteorology, with particularly vital applications in hazard warning and nowcasting (e.g., tornado detection and prediction), and in numerical weather prediction. But accurate retrieval of the vertical velocity field in dual-Doppler analyses is an ongoing problem in radar meteorology. Reliable synthesis of the vertical velocity field is desirable not only in studies of tornadogenesis (Dowell and Bluestein 1997) and in initialization for numerical weather prediction models (Lin et al. 1993) but also in many situations for which dual-Doppler analyses are commonly applied, such as in studies of the microphysical (Ziegler et al. 1983), kinematic (Brandes 1977), dynamic (Parsons and Kropfli 1990), and thermodynamics (Gal-Chen and Kropfli 1984) structures of thunderstorms.

The vertical velocity field is typically the most difficult component of the wind field to accurately synthesize in dual-Doppler analyses (Clark et al. 1980; Ray et al. 1980; Testud and Chong 1983). Assuming the terminal velocity of the scatterers is a known quantity, anelastic mass continuity is often incorporated into dual-Doppler analyses as a means of bringing closure to a system that otherwise contains two equations in three unknowns.

In practice, errors in specifying the vertical velocity boundary condition on the upper or lower data surfaces and cumulative errors arising from integration of the mass continuity equation in the presence of horizontal divergence errors make reliable synthesis of the vertical velocity field difficult.

Shapiro and Mewes (2002) introduced the anelastic vertical vorticity equation as an additional constraint in dual-Doppler analyses in order to improve retrievals of vertical velocities. Their approach bears little resemblance to that of Protat and Zawadzki (2000), where the vertical vorticity equation is applied to improve retrievals of the perturbation pressure and temperature fields in multiple-Doppler analyses. In the first part of their work, Shapiro and Mewes presented an exact solution of the vertical vorticity equation; but, unfortunately, it can be shown that small errors in the horizontal components can lead to significant errors in the calculated solution. In the second part, they formulated new techniques using concepts from calculus of variations (Sasaki 1970; Courant and Hilbert 1953). In these new techniques, the anelastic vertical vorticity equation was introduced as a weak contraint with the goal of obtaining the boundary condition(s) required when applying mass continuity in dual-Doppler analyses. Two main cases were investigated; in the first case, the mass continuity equation was used as a strong constraint and the vorticity equation was used as a weak contraint. And in the second case, both equations were used as weak constraints.

The focus here is on the second case where a cost functional which is equal to the integral, over some volume V, of the sum of the square of the left hand side of the vorticity equation, multiplied by a weighting function, and the square of the left hand side of the mass continuity equation is minimized. The latter method is similar to technique B of Matejka and Bartlets (1998) which was shown to be more robust in the presence of errors. This suggests that the vorticity equation applied with the second method might provide substantial benefits in datasets that exhibit nonrandom errors.

It is interesting to note that if both equations are written in Cartesian coordinates, $\bar{x} = (x, y, z)^T \in \Re^3$, then there is no partial derivative with respect to z of the vertical wind field in the vorticity equation while it is the only partial derivative of w that appears in the mass continuity equation. From one point of view, the two equations complete each other. Especially that the formulation of the problem, Chapter II, introduces a term that contains a partial derivative with respect to z; and no matter how small that term can be made, it may tremendously influence the computational results in case when only the vorticity equation is included.

Chapter II mainly discusses the formulation of the problem and the dependence of the solution on the horizontal components of the wind field. Given the two horizontal components, u and v, the vertical component, w, will be the minimizer of a functional defined on a Hilbert space V. One term of the functional involves regularization/well-posedness and is the sum of $\int_\Omega \epsilon\,(\nabla w \cdot \nabla w)\,d\bar{x}$ and $\int_\Omega \epsilon_0\,(w - w_p)^2\,d\bar{x}$ where w_p is some predicted vertical wind field that does not depend on u and v. The presence of the latter term is crucial to the existence and the uniqueness of the solution; but we prove that if w_p is not reliable then ϵ_0 can costlessly be taken equal to zero in most cases. If we have a case in which we cannot drop ϵ_0 then we can take $w_p = 0$ and give ϵ_0 a very very small value; and we have some numerical results, in Chapter VII, that show that a very small ϵ_0 has little influence, if any, on the numerical approximation of the solution. Since any perturbations in u and v will generate changes in w, we show that the latter, as a function of (u, v), is *Fréchet differentiable*.

In Chapter III, we talk about the utilization of the finite element method in the search for the approximation, w_0^N, of the solution w_0. We prove existence and uniqueness of w_0^N and we formulate its stability as a function of the approximations of u and v.

Chapter IV discusses the use of linear splines in the finite element method and Chapter V discusses the sensitivity and the dependence on the approximations of u and v of the approximation, w_0^N, obtained from the previously mentioned finite element method. Since the values of u and v come from radar data, they are defined on some mesh points x_i, $i = 1, \cdots, N$. We follow the classical finite element method in the calculation of the approximation, w_0^N, of w. The methods we follow to construct the "*stiffness matrix*", M, and the "*load vector*", B, in chapters four and six are some of the major ideas of this work. They consist of writing M, (resp. B) as the sum of matrices (resp. vectors) each of which whose dependences on u and v can easily be determined. The methods give us the continuity of w_0^N as a function of the approximations of u and v and also allow us to write fast algorithms.

Since the natural way of doing this kind of work is to provide some numerical results, the latters, presented in Chapter VII, reveal five main outcomes. The first and obvious one is that the finer the mesh the more accurate the approximation; the second states that both equations have to be used in order to obtain more accurate results. The third one says that the coarser the mesh the more stable the numerical solution; and this is a direct result of using a smaller number of points. The fourth result shows that the size of the domain does not affect the sensitivity of the numerical approximation as long as the number of the mesh points remains unchanged. The fifth and last result in this chapter shows that cubic B-splines gives more accurate results than linear splines when working witha coarse mesh. This may not be true if the solution is not smooth enough.

In Chapter VIII, we develop the idea behind the methods presented in Chapter IV and Chapter VI that reduce the amount of time needed to compute the stiffness matrix and the load vector. The numerical experiments show that, indeed, the computational time can be 400 times less than traditional integration methods.

The results of this work are:

1. Rigorous formulation of the estimation of w form information on u and v and the continuous dependence of w on the latters.

2. Assessment of the effect of errors in the measurements of u and v on the unique approximation of w and the stability of the latter.

3. Accuracy of the numerical solution, referred to as the approximation, and its sensitivity.

We will use the words sensitivity and stability interchangeably.

CHAPTER II

FORMULATION OF THE PROBLEM AND DIFFERENTIABILITY OF THE SOLUTION

2.1 Definitions and Properties

Consider a right-handed Cartesian x, y, z coordinate system in which the $z-axis$ points in the opposite direction of the gravity vector. The position vector of an analysis point with respect to the origin, $(0,0,0)^T$ -the superscript T is the transpose operation, of this coordinate system is given by $\bar{x}=(x,y,z)^T=x\vec{i}+y\vec{j}+z\vec{k}$.

If $\gamma=(\gamma_1,\gamma_2,\gamma_3)$ is an 3-tuple of non negative integers γ_i. We call γ a multi-index and we define the operator $D^\gamma=\frac{\partial^{|\gamma|}}{\partial^{\gamma_1}x\,\partial^{\gamma_2}y\,\partial^{\gamma_3}z}$, where $|\gamma|=\gamma_1+\gamma_2+\gamma_3$.

Let Ω be an open domain in $\Re^3$; For a function $g:\Omega\longrightarrow\Re$, we define:

- $$\|g\|_p=\left(\int_\Omega |g(\bar{x})|^p\,d\bar{x}\right)^{1/p} \tag{2.1}$$

- $$\|g\|_\infty=ess\ sup\left\{|g(\bar{x})|\ \ ;\ \bar{x}\in\Omega\right\} \tag{2.2}$$

  where *ess sup* is the essential supremum.

- The functional $\|.\|_{m,p}$, where m is a non negative integer and $1\le p<\infty$, as follows:

$$\|g\|_{m,p}=\left(\sum_{0\le|\gamma|\le m}\|D^\gamma g\|_p^p\right)^{1/p}\qquad for\quad 1\le p<\infty \tag{2.3}$$

- $$\|g\|_{m,\infty} = Max\,\{\|D^\gamma g\|_\infty \;\; ; 0 \le |\gamma| \le m\} \tag{2.4}$$

  (2.2), (2.3) and (2.4) define norms for any function g for which the right hand sides make sense.

Now, for the non negative integer, m, and $1 \le p \le \infty$ define the spaces:

- $L^p(\Omega) = \Big\{g : \Omega \longrightarrow \Re; \|g\|_p < \infty\Big\}$.
- $H^{m,p}(\Omega) =$ the completion of $\Big\{g \in C^m(\Omega) : \|g\|_{m,p} < \infty\Big\}$ with respect to the norm $\|.\|_{m,p}$.
- $W^{m,p}(\Omega) = \{g \in L^p(\Omega) : D^\gamma g \in L^p(\Omega) \;\; for \;\; 0 \le |\gamma| \le m\}$, where $D^\gamma g$ is the weak (or distributional) partial derivative.
- $W_0^{m,p}(\Omega) =$ the closure of $C_0^\infty(\Omega)$ in the space $W^{m,p}(\Omega)$.
- $W^{0,p}(\Omega) = L^p(\Omega) = W_0^{0,p}(\Omega)$
- $H^{m,p}(\Omega) = W^{m,p}(\Omega)$
- $W^{m,p}(\Omega)$ is a Banach space.
- $H^{m,2}(\Omega)$ is a Hilbert space, denoted $H^m(\Omega)$, with inner product:

  $(g_1, g_2)_m = \sum_{0\le\gamma\le m}(D^\gamma g_1, D^\gamma g_2)$, where $(g_1, g_2) = \int_\Omega g_1(\bar{x})g_2(\bar{x})d\bar{x}$ is the inner product on $L^2(\Omega)$.
- In $H^1(\Omega)$:

  The inner product $(g_1, g_2)_E = \epsilon_0(g_1, g_2) + \epsilon\sum_{|\gamma|=1}(D^\gamma g_1, D^\gamma g_2)$, where ϵ and ϵ_0 are two positive real constants.

  The semi-norm $|g|_{m,2} = \Big[\sum_{|\gamma|=m}(D^\gamma g, D^\gamma g)\Big]^{1/2}$.

  And the norm $\|g\|_E = [(g,g)_E]^{1/2}$ which is equivalent to $\|\cdot\|_{1,2}$.

- $$V = H^1(\Omega) \tag{2.5}$$

- $$W^{3,\infty}(\Omega \times (0,\infty)) = \left\{ \begin{array}{l} g : \Omega \times (0,\infty) \longrightarrow \Re \; ; \; \|D^\gamma g\|_\infty < \infty \; , \; \forall |\gamma| \leq 3 \\ \text{with } \gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4) \end{array} \right\} \tag{2.6}$$

2.2 The vorticity equation

Let's define the wind field $\vec{V}$ by

$$\vec{V} = u \, \vec{i} + v \, \vec{j} + w \, \vec{k} \tag{2.7}$$

The approximate equations of the horizontal one-dimensional forms of Newton's second law of motion, which states that the net acceleration is equal to the net force per unit mass, applied to the atmosphere are:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = fv - \frac{1}{\rho}\frac{\partial p}{\partial x} + \tau\nabla^2 u + \tilde{F}x \tag{2.8}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -fu - \frac{1}{\rho}\frac{\partial p}{\partial y} + \tau\nabla^2 v + \tilde{F}y \tag{2.9}$$

where p is the pressure, ρ is the air density, assumed to be constant since we restrict attention to incompressible flows, $\vec{\nabla} \cdot \vec{V} = 0$ with $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})^T$ and $\cdot$ is the dot product in $\Re^3$, $\tilde{F}x$ and $\tilde{F}y$ are the components of the friction force in the x and the y directions, t is time, f is the Coriolis parameter and τ is the air' s kinematic viscosity, also constant.

Since the vorticity at a point is given by the *curl* (rotational part) of the velocity

$\vec{V}$ we have

$$curl\left(\vec{V}\right) = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\vec{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\vec{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k}. \tag{2.10}$$

We denote by

$$\xi = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \tag{2.11}$$

the vertical vorticity which is normal to the earth's surface.

Therefore, by differentiating (2.8) with respect to y, substracting it from the derivative of (2.9) with respect to x, considering f constant and $\frac{\partial \tilde{F}x}{\partial y} - \frac{\partial \tilde{F}y}{\partial x} = 0$, we obtain the vorticity equation, VE, short for the vertical vorticity equation

$$\frac{\partial \xi}{\partial t} + u\frac{\partial \xi}{\partial x} + v\frac{\partial \xi}{\partial y} + (\xi + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - \tau\nabla^2\xi + \frac{\partial \xi}{\partial z}w + \frac{\partial v}{\partial z}\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\frac{\partial w}{\partial y} = 0 \tag{2.12}$$

In order to simplify the writing of VE, we will write the partial derivative of a function g with respect to a variable t, $\frac{\partial g}{\partial t}$, as g_t; and let's also define the following functions

- $e = u_x + v_y$
- $a = \xi_t + u\xi_x + v\xi_y + e(\xi + f) - \tau\nabla^2\xi$
- $b = v_{zx} - u_{zy}$ (2.13)
- $c = v_z$
- $d = -u_z$

Using this notation the VE becomes

$$a + bw + cw_x + dw_y = 0 \tag{2.14}$$

and the mass continuity equation, MCE, which is $\nabla \cdot (\rho\ \vec{V}) = 0$, and this time consider $\rho = \rho(z)$, is written as

$$\rho e + \rho_z w + \rho w_z = 0 \tag{2.15}$$

2.3 Origin of data

Consider the situation in which there are two radars, 1 and 2, whose positions with respect to the origin are $\bar{x}_1 = (x_1, y_1, z_1)^T$ and $\bar{x}_2 = (x_2, y_2, z_2)^T$, respectively. The position vector of an analysis point $\bar{x}$ with respect to the radar i is

$$\vec{R}_i(\bar{x}) = \bar{x} - \bar{x}_i = R_i(\bar{x})\vec{e}_i(\bar{x}) \tag{2.16}$$

with $R_i(\bar{x}) = \sqrt{(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2}$

The velocity $\vec{v}_s$ of the scattering particles in a volume sampled by the radars is related to the air velocity $\vec{V}$ and the terminal velocity of the scatterers W_t by $\vec{v}_s = \vec{V} - W_t\vec{k}$, W_t is considered positive. Assuming the radar beams are straight, the radial velocity observed by the i^{th} radar is $v_{ri}(\bar{x}) = \vec{R}_i(\bar{x}) \cdot \vec{v}_s$. Assuming geopotential surfaces are flat v_{ri} can be written as

$$v_{ri} = (x - x_i)u + (y - y_i)v + (z - z_i)(w - W_t) \tag{2.17}$$

Shapiro and Mewes, 1999, developed three formulations for dual-Doppler wind analysis; one formulation consists of minimizing a functional that is equal to the integral of the sum of the anelastic MCE multipied by a weighting factor and $\sum_{i=1}^{2} \alpha_i^2 (R_i(\bar{x})v_{ri} - (x - x_i)u + (y - y_i)v + (z - z_i)(w - W_t))^2$, where the α_i s are weighting functions. The analysis reduces to solving a second-order linear partial differential equation whose solution is unique. The new formulations are well posed when, for each point in the analysis domain, radial velocity data from both radars are available along the coplanar azimuthal coordinate line running through that point and extending to lower and upper boundaries on which vertical velocity (or coplane azimuthal velocity) data are specified. This imposes a restriction on the shape of the domains in which the problems are well posed. Regions of the domain that fail to satisfy the well-posed condition are associated with functions of integration or coefficients that cannot be evaluated. Such points are "orphaned" by these procedures.

White and Shapiro, 2005, presented a Hilbert space method for wind field retrieval. The retrieved wind field is the solution of a minimization problem on the Hilbert space $\mathcal{H}$ of the functional $\mathcal{J}$, where

$$\begin{aligned}\mathcal{J}(V) = \; & \tfrac{\epsilon}{2}((V,V)) + \tfrac{\mathcal{K}}{2}(V,V)_1 + \tfrac{\epsilon\mathcal{K}_0}{2}\int_\Omega \left\{\phi_0(\bar{x})\left[v_r(\bar{x},\bar{x}_0) - v_{r0}(\bar{x})\right]^2\right\} d\bar{x} \\ & + \tfrac{\mathcal{K}_1}{2}\int_\Omega \left\{\sum_{i=1}^n \phi_i(\bar{x})\left[v_r(\bar{x},\bar{x}_i) - v_{ri}(\bar{x})\right]^2\right\} d\bar{x}\end{aligned} \tag{2.18}$$

with $V = (u, v, w)^T$ is a column vector,

$v_r(\bar{x}, \bar{x}_i) = \vec{R}_i \cdot \vec{v}_s$ is the actual radial velocity observed at the point $\bar{x}$ (assuming the point is within the coverage set associated with the i^{th} radar) from the i^{th} located at the point $\bar{x}_i$,

$\vec{v}_s = V - w_t$ (w_t is the terminal velocity of the scatterers),

$v_{ri}(\bar{x})$ is an observation,

ϕ_i is a function that models the coverage area of the i^{th} radar,

$(V,V)_1 = \int_\Omega [\nabla \cdot V]^2 \, d\bar{x}$ is the integral of the square of the anelastic MCE $\left(\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})^T\right)$,

$((V,V)) = \int_\Omega \left\{[\nabla u \cdot \nabla u] + [\nabla v \cdot \nabla v]^2 + [\nabla w \cdot \nabla w]^2\right\} d\bar{x}$

n is the number of radars minus 1 (there is a radar whose index is zero),

ϵ, $\mathcal{K}$, $\mathcal{K}_0$ and $\mathcal{K}_1$ are positive constants,

and the Hilbert space $\mathcal{H} = H^1(\Omega, \Re^3)$

Remark concerning the problem

If the latter method gives a solution $V = (u, v, w)^T$ why do we need to calculate w again?

To understand the neediness of this work, suppose we want to minimize a simple function $f(x, y) = (\alpha(x - x_m) + \beta(y - y_m))^2$, where α and β are two weighting factors. Of course, the minimizer is (x_m, y_m) but let' s say that we want to do it numerically by just approximating (x_m, y_m). We start at a point (x_0, y_0) and we look for the numerical minimizer (x_{mn}, y_{mn}) using some random search or a gradient directed search or any other method such that $x_n = x_{n-1} \pm \delta stp$ and $y_n = y_{n-1} \pm \delta stp$ with stp is the fixed mesh step and δ is 0 or 1. The search stops when $f(x_n, y_n) \leq \epsilon_s$. If we make α large and β very small then the search will more likely stop when x_{mn} is very close to x_m and y_{mn} is still at a "large" distance from y_m. Thus the weight of each variable in the function f plays a very big role in how close the corresponding component in the numerical solution is to that in the exact solution.

In White and Shapiro' s formulation, ϵ is very small compared to $\mathcal{K}$, $\mathcal{K}_0$ and $\mathcal{K}_1$, so the regularization term $((V, V))$, which tells us how close the approximation is to the exact solution, is not given an important weight in the computational part. Therefore, the choice of a mesh and a piecewise linear or cubic spline approximation is similar to fixing stp and ϵ_s and it is the terms multiplied by $\mathcal{K}$, $\mathcal{K}_0$ and $\mathcal{K}_1$ that really tell us how accurate the approximation is. Besides, the objective in that work is to determine a Hilbert space formulation and a scanning strategy, accuracy is not top priority.

For each radar i let $\mathcal{Z}$ be the azimuth angle measured clockwise with zero being north and $\mathcal{E}_i$ be the elevation angle and $\mathcal{Z}_i$ be a shift of $\mathcal{Z}$ that determines the position of the radar before we start scanning. We have

$$\begin{aligned} v_r(\bar{x}, \bar{x}_i) &= \vec{R}_i \cdot \vec{v}_s \\ &= ucos(\mathcal{E}_i)cos(\mathcal{Z}_i + \mathcal{Z}) + vcos(\mathcal{E}_i)sin(\mathcal{Z}_i + \mathcal{Z}) + (w - w_t)sin(\mathcal{E}_i) \end{aligned} \tag{2.19}$$

for $i = 0, \cdots, n$

Since the elevation angles $\mathcal{E}_i$ are normally between 0 and 4 degrees then $0 \leq$

$sin(\mathcal{E}_i) \leq 0.07$ and $0.9975 \leq cos(\mathcal{E}_i)$; thus in the terms

$$\int_\Omega \left\{ \sum_{i=1}^{n} \phi_i(\bar{x}) \left[v_r(\bar{x}, \bar{x}_i) - v_{ri}(\bar{x}) \right]^2 \right\} d\bar{x}, \ i = 0, \cdots, n, \tag{2.20}$$

the vertical velocity, w, has a very small weight compared to the horizontal velocities u and v which means that the w part of the numerical approximation will not be as accurate as the u part and the v part. And one can have an idea on how inaccurate it will be by looking at the ratio $\frac{0.9975}{0.07} = 14.25$.

White and Shapiro [2006] studied radar placement based on a geometric uncertainty multiplier reduction criterion. And using an algebraic approach, if two radars observe a point then, under the assumption that the vertical velocity is zero, a set of equations can be obtained estimating the horizontal components in terms of the angle between the radar beams and the observed radial components.

In this work, the formulation is based on data which are the u and the v that come from the minimization of $\mathcal{J}$ incorporated with White and Shapiro [2006] by assuming $w = 0$ then improved by using the Kalman filter or other filtering techniques. And we make use of these improved data along with the VE and the MCE in oder to have a better approximation of w.

2.4 Formulation of the problem

Suppose that we have an ordinary differential equation that we are trying to solve in some given interval. One way of looking at the problem is by studying the existence, uniqueness... of a function that minimizes a functional which is the integral of the square of the left hand side of the ODE over the given interval. This idea is inspired from some of John Neuberger' s work.

Suppose $\mathcal{X}$ is a Banach space with the property that if f is a continuous linear function from $\mathcal{X}$ to $\Re$, then, given $c > 0$, there is a unique element $h \in \mathcal{X}$ so that

$$Sup_{\{g\in\mathcal{X},\|g\|_{\mathcal{X}}=c\}} fg = fh \tag{2.21}$$

Such space $\mathcal{X}$ includes all Hilbert spaces as well as many Sobolev spaces which are not Hilbert spaces. If ϕ is a C^1 functional on the Sobolev space $\mathcal{X}$ with the above property, then the Sobolev gradient of ϕ at $x \in \mathcal{X}$ is the elemnt $(\nabla\phi)(x)$ so that

$$Sup_{\{g\in\mathcal{X},\|g\|_{\mathcal{X}}=|\phi'(x)|\}} \phi'(x)g = \phi'(x)(\nabla\phi(x)) \tag{2.22}$$

Given a function ϕ as above, one seeks a critical point of ϕ by either continuous or discrete steepest descent. Our interest is in continuous steepest descent and functions ϕ for which there is a function $\mathcal{F}$ form $\mathcal{X}$ to a second Banach space $\mathcal{Y}$ such that

$$\phi(x) = \frac{\|\mathcal{F}(x)\|_{\mathcal{Y}}^2}{2}\ ,\ x \in \mathcal{X} \tag{2.23}$$

We think of $\mathcal{F}$ as being so that a zero x of $\mathcal{F}$ is a solution to a system of partial differential equations. It is quite common that a critical point of ϕ is actually a zero of $\mathcal{F}$, something not so common for general optimization problems. Functionals ϕ of this type are generally used only when one has a system of partial differential equations for which a conventional cariational principle is not available.

The method followed in the formulation of this problem is based on the above theory with the VE and the MCE forming the system of partial differential equations, $\mathcal{X} = H^1(\Omega)$, $\mathcal{Y} = L^2(\Omega) \times L^2(\Omega)$ and $\|g = (g_1, g_2)\|_{\mathcal{Y}} = \sqrt{\alpha_v^2 \|g_1\|_2^2 + \alpha_m^2 \|g_2\|_2^2}$. Thus it is similar to the method of Shapiro and Mewes (2002) except for the fact that each equation, VE and MC, is multiplied by a weighting function, α_v or α_m, (in this study the weighting functions will be constant).

Because the functions u and v can have any values, in order to have well posedness of the problem and some kind of regularization of the solution, another term -which will not play a significant role in the computational part- will be added to the integral.

Remark 2.1 We would like to stress that, in this study, time t is fixed.

Mathematically speaking, let α_v and α_m be the weighting constants of the VE and the MCE, respectively. The goal is to minimize the functional

$$Min_{w \in V} \int_\Omega \alpha_v^2 (a + bw + cw_x + dw_y)^2 + \alpha_m^2 (\rho e + \rho_z w + \rho w_z)^2 + \epsilon_0 (w - w_p)^2 \, d\bar{x} \quad (2.24)$$

The term multiplied by α_v^2 is the square of the VE, the one multiplied by α_m^2 is the square of the MCE, ϵ_0 is a positive real number, and w_p is some predicted vertical wind field that does not depend on u and v.

After expansion, (2.24) becomes

$$Min_{w \in V} \left\{ \begin{array}{c} \int_\Omega \alpha_v^2 (bw + cw_x + dw_y)^2 + \alpha_m^2 (\rho_z w + \rho w_z)^2 + \epsilon_0 w^2 d\bar{x} \\ +2 \int_\Omega \alpha_v^2 a (bw + cw_x + dw_y) + \alpha_m^2 \rho e (\rho_z w + \rho w_z) - \epsilon_0 w_p w d\bar{x} \\ + \int_\Omega \left(\alpha_v^2 \, a^2 + \alpha_m^2 \rho^2 \, e^2 + \epsilon_0 w_p^2 \right) d\bar{x} \end{array} \right\} \quad (2.25)$$

Since u, v, ρ, α_v, α_m and w_p are supposed to be given, the third term,

$$\int_\Omega \left(\alpha_v^2 \, a^2 + \alpha_m^2 \rho^2 \, e^2 + \epsilon_0 w_p^2 \right) d\bar{x} \quad (2.26)$$

is not a function of w so it can be dropped. Therefore (2.25) is equivalent to

$$Min_{w\in V} J_1(w) = \left\{ \begin{array}{c} \int_\Omega \alpha_v^2(bw + cw_x + dw_y)^2 + \alpha_m^2(\rho_z w + \rho w_z)^2 + \epsilon_0 w^2 d\bar{x} \\ +2\int_\Omega \alpha_v^2 a(bw + cw_x + dw_y) + \alpha_m^2 \rho e(\rho_z w + \rho w_z) - 2w_p w d\bar{x} \end{array} \right\} \tag{2.27}$$

To insure existence and uniqueness of the solution, we will add $\epsilon\left(w_x^2 + w_y^2 + w_z^2\right)$ to the functional J_1.

$\|w\|_E^2$ is called the regularization/well-posedness term.

Besides insuring existence and uniqueness, the term $\epsilon\left(w_x^2 + w_y^2 + w_z^2\right)$ forces the solution to have some kind of smoothness, depending on the values of α_v, α_m. ϵ and ϵ_0.

As we already mentioned, in case w_p is not given or is not reliable then we will take it to be zero and we will prove that in most cases we can make $\epsilon_0 = 0$ and still have existence and uniqueness of the solution.

Discussion

We already mentioned that the functions u and v come from the minimization of $\mathcal{J}$ (2.18) so they are in $H^1(\Omega)$ and by adding assumptions on the coverage functions of the radars and the observations v_{r_i} we can have u and v in $H^2(\Omega)$. But for our problem to be well posed, u and v can be considered in $W^{3,\infty}(\Omega \times (0,\infty))$ -we can weaken this consideration. To make this happen, there are many methods one can follow. One of the methods is to interpolate $u(\bar{x}, t_i)$ and $v(\bar{x}, t_i)$ at each fixed time t_i by cubic or B-cubic spline, interpolation is only in $\bar{x}$, then, for fixed $\bar{x}$, to interpolate $u(\bar{x}, t)$ and $v(\bar{x}, t)$ by linear splines between each two different moments t_{i-1} and t_i, this interpolation is in t only.

Suppose that the function $u \in W^{3,\infty}(\Omega \times (0,\infty))$, the function $v \in W^{3,\infty}(\Omega \times (0,\infty))$, $\rho \in L^\infty(\Omega)$, $\rho_z \in L^\infty(\Omega)$, the Coriolis parameter $f \in L^2(\Omega)$ and $w_p \in L^2(\Omega)$.

The above conditions on u and v can be costlessly weakened.

Define

- $$a_v(w_1, w_2) = \int_\Omega \alpha_v^2 (bw_1 + cw_{1x} + dw_{1y})(bw_2 + cw_{2x} + dw_{2y}) d\bar{x} \qquad (2.28)$$

- $$a_m(w_1, w_2) = \int_\Omega \alpha_m^2 (\rho_z w_1 + \rho w_{1z})(\rho_z w_2 + \rho w_{2z}) d\bar{x} \qquad (2.29)$$

- $$a(w_1, w_2) = a_v(w_1, w_2) + a_m(w_1, w_2) + (w_1, w_2)_E \qquad (2.30)$$

  $a(\cdot, \cdot)$ is a real symmetric bilinear form defined on $V \times V$.

- $$f_v(w) = -\int_\Omega \alpha_v^2 \, a(bw + cw_x + dw_y) d\bar{x} \qquad (2.31)$$

- $$f_m(w) = -\int_\Omega \alpha_m^2 \rho \, e(\rho_z w + \rho w_z) d\bar{x} \qquad (2.32)$$

- $$f_{\epsilon_0}(w) = \int_\Omega \epsilon_0 w_p w d\bar{x} \qquad (2.33)$$

- $$f(w) = f_v(w) + f_m(w) + f_{\epsilon_0}(w) \qquad (2.34)$$

  $f(\cdot)$ is a real linear functional defined on V .

- $$J(w) = J_1(w) + \|w\|_E^2 \qquad (2.35)$$

- We have

$$J(w) = a\,(w, w) - 2f(w) \qquad (2.36)$$

$J(\cdot)$ is a real functional defined on V.

$$\boxed{\begin{array}{l}\textbf{The problem is formulated in the following form:}\\ \textbf{Find } w_0 \in V \textbf{ such that } J(w_0) = Inf\left\{J(w); w \in V\right\}\end{array}} \tag{2.37}$$

Let

$$M_a = Max\left\{\|u\|_{u,t}, \|v\|_{v,t}, \|\rho_z\|_\infty, \|\rho\|_\infty, \|f\|_2, \epsilon, \epsilon_0, \epsilon_0 \|w_p\|_2\right\}, \tag{2.38}$$

$$m_a = Min\left\{\epsilon, \epsilon_0\right\}, \tag{2.39}$$

and

$$C_a = Max\left\{\alpha_v^2, \alpha_m^2, 1\right\} \tag{2.40}$$

For w_1 and w_2 in the space V, we have

$$|a(w_1, w_2)| \leq C_a M_a^2 \begin{pmatrix} 6\|w_1\|_2\|w_2\|_2 + \|w_1\|_2\|w_{2x}\|_2 + \|w_1\|_2\|w_{2y}\|_2 \\ +\|w_{1x}\|_2\|w_2\|_2 + 2\|w_{1x}\|_2\|w_{2x}\|_2 + \|w_{1x}\|_2\|w_{2y}\|_2 \\ +\|w_{1y}\|_2\|w_2\|_2 + \|w_{1y}\|_2\|w_{2x}\|_2 + 2\|w_{1y}\|_2\|w_{2y}\|_2 \\ +\|w_1\|_2\|w_{2z}\|_2 + \|w_{1z}\|_2\|w_2\|_2 + 2\|w_{1z}\|_2\|w_{2z}\|_2 \end{pmatrix} \tag{2.41}$$

the term between the parentheses in the left hand side of (2.41) is

$$\leq 6\left(\|w_1\|_2 + \|w_{1x}\|_2 + \|w_{1y}\|_2 + \|w_{1z}\|_2\right)\left(\|w_2\|_2 + \|w_{2x}\|_2 + \|w_{2y}\|_2 + \|w_{2z}\|_2\right) \tag{2.42}$$

using the inequality

$$a_1 + a_2 + a_3 + a_4 \leq 2\sqrt{a_1^2 + a_2^2 + a_3^2 + a_4^2} \tag{2.43}$$

where a_i, $i = 1, \cdots, 4$, are non negative numbers.

we obtain

$$|a(w_1, w_2)| \leq 24C_a M_a^2 \|w_1\|_{1,2} \|w_2\|_{1,2} \ , \forall w_1, w_2 \in V \tag{2.44}$$

which means that $a(\cdot, \cdot)$ is bounded.

and we have

$$m_a \|w\|_{1,2}^2 \leq a(w, w) \ , \forall w \in V \tag{2.45}$$

which means that $a(\cdot, \cdot)$ is coercive.

So, $a(\cdot, \cdot) : V \times V \longrightarrow \Re$ is a bounded, coercive and symmetric bilinear form.

Remark 2.2 In fact, the coercivity of $a(\cdot, \cdot)$ on $V = H^1(\Omega)$ is obtained because of the existence of the term $\epsilon_0(g_1, g_2)$ in the inner product $(\cdot, \cdot)_E$.

Theorem 2.2 [16] Let Ω be a connected bounded open subset of $\Re^3$ with boundary Γ, $\Gamma_0 \subseteq \Gamma$ and $V_0 = \{g \in H^1(\Omega); \quad g = 0 \quad on \quad \Gamma_0\}$. Then the space V_0 is a closed subspace of $H^1(\Omega)$.

If the $ds - measure$ of Γ_0 is strictly positive, the semi-norm $|\cdot|_{1,2}$ is a norm over the space V_0 equivalent to the norm $\|\cdot\|_{1,2}$.

Remark 2.3 Since, sometimes, part of the boundary of our domain Ω will be the ground surface, the roof of a building, a mountain etc, it will contain a perfectly horizontal surface Γ_0 on which w will be zero. Then, we can make $\epsilon_0 = 0$

and, thanks to Theorem (2.2), $a(\cdot,\cdot)$ will still be coercive on the space V_0. Therefore, the problem will remain in the form of (2.37) but with $V = V_0$ and $\epsilon_0 = 0$.

Remark 2.4 In the computational part of this work, ϵ_0 will be taken very small compared to the other terms of $a(\cdot,\cdot)$ and very often zero if the boundary, Γ, of our domain Ω contains some Γ_0 whose area is non zero such that $w(\bar{x}) = 0 \quad \forall \bar{x} \in \Gamma_0$; or if the matrix M, defined later, is invertible.

We also have for $w \in V$

$$|f(w)| \leq C_a \left(20M_a^3 + (8 + cte_1)M_a^2\right) \sqrt{\text{volume of } \Omega} \left(\|w\|_2 + \|w_x\|_2 + \|w_y\|_2 + \|w_z\|_2\right) \tag{2.46}$$

where cte_1 is a constant that depends on the volume of Ω.

Using (2.43) we obtain

$$|f(w)| \leq C_a \left(40M_a^3 + (16 + 2cte_1)M_a^2\right) \sqrt{\text{volume of } \Omega} \, \|w\|_{1,2} \ , \forall w \in V \tag{2.47}$$

which means that $f(\cdot)$ is bounded.

So, $f(\cdot) : V \to \Re$ is a bounded linear functional on V .

Theorem 2.3 Assume that $a(\cdot,\cdot)$ is a symmetric, positive definite bilinear form and that f is a bounded linear form on the Hilbert space V. If $J(w) = a(w,w) - 2f(w)$ then $J(w_0) \leq J(w) \ \ \forall w \in V$ if and only if $a(w_0, w) = f(w) \ \ \forall w \in V$

Theorem 2.4 [16]($Lax - Milgram\ Lemma$) If the bilinear form $a(\cdot,\cdot)$ is bounded and coercive in the Hilbert space V, and f is a bounded linear form in V, then there exists a unique vector $w_0 \in V$ such that $a(w_0, w) = f(w) \ \ \forall w \in V$. Moreover, the energy estimate $\|w_0\|_V \leq \frac{1}{m_a} \|f\|_{V'}$ holds, where V' is the dual space of V.

2.5 The regularization/well-posedness term

In case w_p in unknown or we want to make it equal to zero, the presence of the term $\epsilon_0(w_1, w_2)$ in the bilinear form a makes us wonder how it will affect the numerical results. In fact, in most cases we can take $\epsilon_0 = 0$ and still obtain a unique solution. The term in question can be dropped if the derivative of the vertical vorticity with respect to z or the derivative of the air density, ρ, with respect to z is non zero on some subset of Ω of positive measure. But if $\frac{\partial \xi}{\partial z} = 0$ and $\frac{\partial \rho}{\partial z} = 0$ almost everywhere on Ω then we cannot even guarantee the existence of a solution unless we keep ϵ_0. It is possible to have many numerical solutions if the theoretical solution does not exist.

Let $L^{m,p}(\Omega)$ denote the space of distributions on Ω with derivatives of order m in the space $L^p(\Omega)$. We equip $L^{m,p}(\Omega)$ with the seminorm

$$\left|\nabla_p^m g\right| = \left[\int_\Omega \left(\sum_{|\gamma|=m} |D^\gamma g(\bar{x})|^2\right)^{p/2} d\bar{x}\right]^{1/p} \tag{2.48}$$

Claim 2.1 The seminorm $\left|\nabla_p^m \cdot\right|$ is equivalent to the seminorm

$$|g|_{m,p} = \left[\int_\Omega \sum_{|\gamma|=m} |D^\gamma g(\bar{x})|^p \, d\bar{x}\right]^{1/p} \tag{2.49}$$

Proof

From Jensen' s inequality, we have for $1 \leq p < \infty$

$$\left(\sum_{i=1}^k y_i^2\right)^p \leq k^{p-1} \sum_{i=1}^k y_i^{2p} \qquad \text{for all } y_1 \geq 0, \cdots, y_k \geq 0.$$

$$\Rightarrow \quad \left(\sum_{i=1}^{k} y_i^2\right)^p \leq k^{p-1}\sum_{i=1}^{k} y_i^{2p} \qquad \text{for all } y_1 \geq 0, \cdots, y_k \geq 0.$$

$$\Rightarrow \quad \left(\sum_{i=1}^{k} y_i^2\right)^{p/2} \leq k^{p/2-1/2}\left(\sum_{i=1}^{k} y_i^{2p}\right)^{1/2} \leq k^{p/2-1/2}\sum_{i=1}^{k} y_i^p.$$

We also have $y_i^p = y_i^{2p/2} \leq \left(\sum_{i=1}^{k} y_i^2\right)^{p/2}$, thus

$$\sum_{i=1}^{k} y_i^p \leq k\left(\sum_{i=1}^{k} y_i^2\right)^{p/2}.$$

Therefore

$$\sum_{i=1}^{k} y_i^p \leq k\left(\sum_{i=1}^{k} y_i^2\right)^{p/2} \leq k^{p/2+1/2}\sum_{i=1}^{k} y_i^p. \tag{2.50}$$

The equivalence of the two norms follows immediately from (2.50).

■

We introduce the space $W_p^m(\Omega) = L^{m,p}(\Omega) \cap L^p(\Omega)$ equipped with the norm

$$\|g\|_{W_p^m(\Omega)} = \left|\nabla_p^m g\right| + \|g\|_p$$

Definition 2.1 A domain $\Omega \subset \Re^d$ possesses the cone property if each point of Ω is the vertex of a cone contained in Ω along with its closure, the cone being represented by the inequalities $x_1^2 + \cdots + x_{d-1}^2 < constant_1 x_d^2$, $0 < x_d < constant_2$ in some Cartesian coordiante system.

Corollary 2.1 The spaces $W^{m,p}(\Omega)$, $W_p^m(\Omega)$ and $L^{m,p}(\Omega)$ coincide for Ω having the cone property.

Theorem 2.5 [19] Let Ω be a bounded domain such that $L^{m,p}(\Omega) \subset L^p(\Omega)$ (for example, Ω has the cone property). Let $\mathcal{F}(g)$ be a continuous functional in $W_p^m(\Omega)$ such that $\mathcal{F}(\prod_{m-1}) \neq 0$ for any nonzero polynomial $\prod_{m-1}$ of degree not higher than $m-1$. Then the norm

$$\left|\nabla_p^m g\right| + \mathcal{F}(g)$$

is equivalent to the norm in $W_p^m(\Omega)$.

$\blacksquare$

Claim 2.2 Suppose that $\epsilon_0 = 0$. If $\int_\Omega \left(\frac{\partial \xi}{\partial z}\right)^2 d\bar{x} > 0$ or $\int_\Omega \left(\frac{\partial \rho}{\partial z}\right)^2 d\bar{x} > 0$ then there exists a unique solution to the problem (2.37).

Proof

Since Ω satsfies the cone property, we have

$$V = H^1(\Omega) = W^{1,2}(\Omega) = W_2^1(\Omega).$$

Suppose that $\epsilon_0 = 0$ and let $\prod_0$ be a nonzero constant polynomial, we have

$$a_v\left(\prod\nolimits_0, \prod\nolimits_0\right) + a_m\left(\prod\nolimits_0, \prod\nolimits_0\right) = \left(\prod\nolimits_0\right)^2 \left[\int_\Omega \left(\frac{\partial \xi}{\partial z}\right)^2 d\bar{x} + \int_\Omega \left(\frac{\partial \rho}{\partial z}\right)^2 d\bar{x}\right] > 0,$$

thus, by Theorem 2.5, there exists a constant $m_a' > 0$ such that

$$a(w,w) = a_v(w,w) + a_m(w,w) + \epsilon\, |w|_{1,2}^2 \geq m_a' \, \|w\|_{1,2}^2 \quad \forall w \in V.$$

Therefore, the bilinear form a is coercive which allows us to conclude that there exists a unique solution.

2.6 From a discussion with Dr. Alan Shapiro

We have

$$
\begin{aligned}
J(w) = \; & \int_\Omega \{ \; \alpha_v^2 \left[a + bw + cw_x + dw_y\right]^2 + \alpha_m^2 \left[e\rho + \rho w_z + \rho_z w\right]^2 \\
& + \epsilon \left(w_x^2 + w_y^2 + w_z^2\right) + \epsilon_0 (w - w_p)^2 \; \} \, d\bar{x}
\end{aligned}
\tag{2.51}
$$

so

$$
\begin{aligned}
\delta J = \; & \int_\Omega \{ \; 2\alpha_v^2 \left[a + bw + cw_x + dw_y\right] \left[b\delta w + c\delta w_x + d\delta w_y\right] \\
& + 2\alpha_m^2 \left[e\rho + \rho w_z + \rho_z w\right] \left[\rho\delta w_z + \rho_z \delta w\right] \\
& + 2\epsilon \left(w_x + w_y + w_z\right) \left(\delta w_x + \delta w_y + \delta w_z\right) + 2\epsilon_0 (w - w_p) \delta w \; \} \, d\bar{x}
\end{aligned}
\tag{2.52}
$$

suppose that any small variation δw is equal to zero on the boundary of Ω, after integrating by parts, we obtain

$$
\begin{aligned}
\delta J = \; & 2\alpha_v^2 \int_\Omega \{ \; b \left[a + bw + cw_x + dw_y\right] - \frac{\partial}{\partial x} \left[c \left(a + bw + cw_x + dw_y\right)\right] \\
& - \frac{\partial}{\partial y} \left[d \left(a + bw + cw_x + dw_y\right)\right] \; \} \, \delta w d\bar{x} \\
& + 2\alpha_m^2 \int_\Omega \{ \; \rho_z \left[e\rho + \rho w_z + \rho_z w\right] - \frac{\partial}{\partial z} \left[\rho \left(e\rho + \rho w_z + \rho_z w\right)\right] \; \} \, \delta w d\bar{x} \\
& + 2 \int_\Omega \{ \; - \epsilon \left(w_{xx} + w_{yy} + w_{zz}\right) + \epsilon_0 (w - w_p) \; \} \, \delta w d\bar{x}
\end{aligned}
\tag{2.53}
$$

Optimality condition $\Longrightarrow \delta J = 0 \; \forall \delta w$

so

$$
\begin{aligned}
& \alpha_v^2 \{ \; b \left[a + bw + cw_x + dw_y\right] - \frac{\partial}{\partial x} \left[c \left(a + bw + cw_x + dw_y\right)\right] \\
& - \frac{\partial}{\partial y} \left[d \left(a + bw + cw_x + dw_y\right)\right] \; \} \\
& + \alpha_m^2 \{ \; \rho_z \left[e\rho + \rho w_z + \rho_z w\right] - \frac{\partial}{\partial z} \left[\rho \left(e\rho + \rho w_z + \rho_z w\right)\right] \; \} \\
& + \{ \; - \epsilon \left(w_{xx} + w_{yy} + w_{zz}\right) + \epsilon_0 (w - w_p) \; \} = 0
\end{aligned}
\tag{2.54}
$$

which gives

$$\begin{aligned}
Eq(w) = \quad & -\alpha_v^2(c^2+\epsilon)w_{xx} - \alpha_v^2(d^2+\epsilon)w_{yy} - \alpha_m^2(\rho^2+\epsilon)w_{zz} - 2\alpha_v^2 cdw_{xy} \\
& -\alpha_v^2\left[\left((c^2)_x + (cd)_y\right)w_x + \left((d^2)_y + (cd)_x\right)w_y\right] - \alpha_m^2(\rho^2)_z w_z \\
& + \left[\alpha_v^2\left(b^2 - (bc)_x - (bd)_y\right) - \alpha_m^2\rho\rho_{zz} + \epsilon_0\right]w \\
& + \alpha_v^2\left[ab - (ac)_x - (ad)_y\right] - \alpha_m^2\rho(e\rho)_z - \epsilon_0 w_p \\
= \quad & 0.
\end{aligned} \tag{2.55}$$

Now suppose that there exsits an $\epsilon_0 > 0$ such that (2.55) has two classical solutions w_1 and w_2. Here we consider that the values of w_1, w_2 and their first and second partial derivatives are finite on the boundary of Ω. Let $\phi(\bar{x}) \in C_0^\infty(\Omega)$, we have $Eq(w_1)\phi = 0$ and $Eq(w_2)\phi = 0$. Which gives $\int_\Omega Eq(w_1)\phi d\bar{x} = 0$ and $\int_\Omega Eq(w_2)\phi d\bar{x} = 0$.

After integrating by parts we obtain

$$a(w_1, \phi) = f(\phi) \tag{2.56}$$

and

$$a(w_2, \phi) = f(\phi) \tag{2.57}$$

thus

$a(w_1 - w_2, \phi) = 0 \ \forall \phi \in C_0^\infty(\Omega)$ and since $C_0^\infty(\Omega)$ is dense in $L^2(\Omega)$ then

$a(w_1 - w_2, \phi) = 0 \ \forall \phi \in L^2(\Omega)$.

Since the values of w_1 and w_2 are finite on the boundary and both functions are classical solutions - so continuous- then they are in $L^2(\Omega)$. Thus $a(w_1 - w_2, w_1 - w_2) = 0$ which gives

$$\int_\Omega (w_1 - w_2)^2 d\bar{x} = 0. \tag{2.58}$$

$\Longrightarrow w_1 - w_2 = 0$ almost everywhere in Ω and since the functions are continuous, we have $w_1(\bar{x}) = w_2(\bar{x}) \ \forall \bar{x} \in \Omega$.

2.7 Regularity of the solution

Theorem 2.6 Let Ω be bounded and open in $\Re^n$ and suppose its boundary is a $C^2 - manifold$ of dimension $n-1$. Let $a_{ij} \in C^1(\Omega)$, $1 \leq i, j \leq n$, and $a_j \in C^1(\Omega)$, $0 \leq j \leq n$, all have bounded derivatives and assume that the sesquilinear form defined by

$$a(\phi, \psi) = \int_\Omega \left\{ \sum_{i,j=1}^{n} a_{ij} \partial_i \phi \bar{\partial_j} \psi + \sum_{j=0}^{n} a_i \partial_j \phi \bar{\psi} \right\} dx, \qquad \phi, \psi \in H^1(\Omega) \tag{2.59}$$

is strongly elliptic. Let $F \in L^2(\Omega)$ and suppose $u \in H^1(\Omega)$ satisfies

$$a(u, v) = \int_\Omega F \bar{v} dx, \qquad v \in H^1(\Omega) \tag{2.60}$$

Then $u \in H^2(\Omega)$.

■

We already know that our solution is in $H^1(\Omega)$; but in order to have more regularity we have to have a C^2 boundary. Since our domain Ω will be some parallelepiped then we can smoothen the boundary and, numerically, keep the value of the volume almost the same; but the coefficients in the bilinear form are functions of u, v and their partial derivatives of first and second order and all we can say about the latters come from **Discussion 2.1**.

Which gives us $a \in L^\infty(\Omega)$, $b \in C^0(\Omega)$, $c \in C^1(\Omega)$, and $d \in C^1(\Omega)$. Plus being in $W^{3,\infty}(\Omega, (0, \infty))$, u and v have to be in $C^3(\Omega)$ in order to have $w_0 \in H^2(\omega)$.

2.8 Differentiability of the solution w_0 with respect to (u, v)

In this section we investigate the differentiability of the solution w_0 with respect to u and v. In fact the bilinear form $a(\cdot, \cdot)$ and the linear functional $f(\cdot)$ are functions of u and v and so will be the solution w_0. Thus, the "correct" writing of the solution is of the form

$$a_{(u,v)}(w_0, w) = f_{(u,v)}(w) \ , \forall w \in V. \tag{2.61}$$

There is a mapping, denoted W_0 for simplicity

$$\begin{aligned} W_0 : \qquad V_t &\longrightarrow V \\ (u,v) &\longrightarrow w_0 = w_0(u,v) \end{aligned} \tag{2.62}$$

where $V_t = W^{3,\infty}(\Omega, (0,\infty)) \times W^{3,\infty}(\Omega, (0,\infty))$ and
$\|(u,v)\|_{V_t} = \sqrt{\|u\|_{3,\infty}^2 + \|v\|_{3,\infty}^2}$.

Proposition 2.1 [7] let X be a vector space, Y a normed space, and T a (possibly nonlinear) transformation defined on an open set $D \subset X$ and having range $R \subset Y$. If the transformation T has a *Fréchet differential* at $x \in D$, then T is continuous at x.

2.8.1 Differentiability of the bilinear form $a_{(u,v)}$ with respect to (u,v)

Let $(u', v') \in V_t$, for all ψ_1 and $\psi_2 \in V$ define

$$\delta_\eta^a(u', v')(\psi_1, \psi_2) = \frac{1}{\eta}\left[a_{(u+\eta u', v+\eta v')}(\psi_1, \psi_2) - a_{(u,v)}(\psi_1, \psi_2)\right] \tag{2.63}$$

where η is a non-zero real number.
It can, easily, be shown that

$$
\begin{aligned}
\left[Da(u',v')\right](\psi_1,\psi_2) &= lim_{\eta\to 0}\tfrac{1}{\eta}\delta^a_\eta(u',v')(\psi_1,\psi_2)\\
&= -\textstyle\int_\Omega \alpha_v^2\left(u'_{zy}\psi_1 + u'_y\psi_{1y}\right)\left[b_{(u,v)}\psi_2 + c_{(u,v)}\psi_{2x} + d_{(u,v)}\psi_{2y}\right]d\bar{x}\\
&\quad -\textstyle\int_\Omega \alpha_v^2\left(u'_{zy}\psi_2 + u'_y\psi_{2y}\right)\left[b_{(u,v)}\psi_1 + c_{(u,v)}\psi_{1x} + d_{(u,v)}\psi_{1y}\right]d\bar{x}\\
&\quad +\textstyle\int_\Omega \alpha_v^2\left(v'_{zx}\psi_1 + v'_x\psi_{1y}\right)\left[b_{(u,v)}\psi_2 + c_{(u,v)}\psi_{2x} + d_{(u,v)}\psi_{2y}\right]d\bar{x}\\
&\quad +\textstyle\int_\Omega \alpha_v^2\left(v'_{zx}\psi_2 + v'_x\psi_{2y}\right)\left[b_{(u,v)}\psi_1 + c_{(u,v)}\psi_{1x} + d_{(u,v)}\psi_{1y}\right]d\bar{x}
\end{aligned}
\tag{2.64}
$$

Where $a_{(u,v)}$, $b_{(u,v)}$, $c_{(u,v)}$, $d_{(u,v)}$ and $e_{(u,v)}$ are the functions a, b, c, d and e, defined in (2.13), evaluated at (u,v)

Therefore, $a_{(u,v)}$ is *Gateaux differentiable*

Remark 2.5 Equation (2.64) is obtained by simple calculus operations; we do not need Lebesgue's dominated convergence theorem because we do not have to interchange the integral and the limit. η can be put out of the integral which facilitates the calculation of the limit.

$\left[Da(u',v')\right](\cdot,\cdot)$ is bilinear on $V\times V$.

We have

$$
\begin{aligned}
\left|\left[Da(u',v')\right](\psi_1,\psi_2)\right| &\le 6C_aM_a\left[\|u'\|_t + \|v'\|_t\right]\|\psi_1\|_{1,2}\|\psi_2\|_{1,2}\\
&\le 6\sqrt{2}C_aM_a\|(u',v')\|_t\|\psi_1\|_{1,2}\|\psi_2\|_{1,2}\\
&\qquad\qquad \forall(\psi_1,\psi_2)\in V\times V
\end{aligned}
\tag{2.65}
$$

Since $\left[Da(u',v')\right](\cdot.\cdot)$ is bilinear, (2.65) says that $Da(u',v')$ is a bounded bilinear form on $V\times V$. For r_1 and $r_2\in\Re$. We have

$$
\begin{aligned}
[Da(r_1(u_1', v_1') + r_2(u_2', v_2'))]\,(\psi_1, \psi_2) = \\
r_1\,[Da(u_1', v_1')]\,(\psi_1, \psi_2) + r_2\,[Da(u_2', v_2')]\,(\psi_1, \psi_2) \\
\forall(\psi_1, \psi_2) \in V \times V
\end{aligned} \tag{2.66}
$$

so

$$Da(r_1(u_1', v_1') + r_2(u_2', v_2')) = r_1 Da(u_1', v_1') + r_2 Da(u_2', v_2')$$

which means that the operator $Da(\cdot, \cdot)$ is linear on V_t .

From (2.65) we have

$$lim_{(u',v')\to(0,0)} Da(u', v') = 0 \tag{2.67}$$

so $Da(\cdot, \cdot)$ is continuous at $(0, 0) \in V_t$ and therefore it is continuous on V_t.

So

$$
\begin{aligned}
Da: \quad V_t &\longrightarrow \text{space of bilinear functionals on } V \times V \\
(u', v') &\longrightarrow Da(u', v')
\end{aligned} \tag{2.68}
$$

is continuous and linear. Therefore, the mapping

$$
\begin{aligned}
A: \quad V_t &\longrightarrow \text{space of bilinear functionals on } V \times V \\
(u, v) &\longrightarrow A(u, v)
\end{aligned} \tag{2.69}
$$

where $[A(u, v)]\,(\psi_1, \psi_2) = a_{(u,v)}(\psi_1, \psi_2)$, $\forall(\psi_1, \psi_2) \in V \times V$, is $Fréchet\ differentiable$ and so it is continuous.

2.8.2 Differentiability of the linear functional $f_{(u,v)}$ with respect to (u,v)

Let $(u',v') \in V_t$, for all $\psi \in V$ define

$$\delta_\eta^f(u',v')(\psi) = \frac{1}{\eta}\left[f_{(u+\eta u',v+\eta v')}(\psi) - f_{(u,v)}(\psi)\right] \tag{2.70}$$

where η is a non-zero real number.

It can, easily, be shown that

$$\begin{aligned}
\left[Df(u',v')\right](\psi) &= lim_{\eta\to 0}\tfrac{1}{\eta}\delta_\eta^f(u',v')(\psi) \\
&= \int_\Omega \alpha_v^2 a_{(u,v)}\left[\left(v'_{zx} - u'_{zy}\right)\psi + v'_x\psi_x - u'_y\psi_y\right]d\bar{x} \\
&\quad + \int_\Omega \alpha_v^2\left(\delta a_{(u,v)}\right)\left[b_{(u,v)}\psi + c_{(u,v)}\psi_x + d_{(u,v)}\psi_y\right]d\bar{x} \\
&\quad + \int_\Omega \alpha_m^2\left(u'_x + v'_y\right)\rho\left(\rho_z\psi + \rho\psi_z\right)d\bar{x}
\end{aligned} \tag{2.71}$$

where

$$\begin{aligned}
\delta a_{(u,v)} = \;& v'_{tx} - u'_{ty} + u'\xi_x + v'\xi_y + u\left(v'_{xx} - u'_{xy}\right) + v\left(v'_{xy} - u'_{yy}\right) \\
&+ \left(u'_x + v'_y\right)(\xi + f) + e_{(u,v)}\left(v'_x - u'_y\right) - \tau\left(\nabla^2 v'_x - \nabla^2 u'_y\right)
\end{aligned} \tag{2.72}$$

So $f_{(u,v)}$ is *Gateaux differentiable* -no need for the Lebesgue bounded convergence theorem to calculate the limit- and $\left[Df(u',v')\right](\cdot)$ is linear on V .

We have

$$\begin{aligned}
\left|\left[Df(u',v')\right](\psi)\right| &\leq 12C_a\left(M_a + 9M_a^2 + cte_2 M_a\right)\sqrt{\text{volume of }\Omega}\left(\|u'\|_t + \|v'\|_t\right)\|\psi\|_{1,2} \\
&\leq 12\sqrt{2}C_a\left(M_a + 9M_a^2 + cte_2 M_a\right)\sqrt{\text{volume of }\Omega}\,\|(u',v')\|_t\,\|\psi\|_{1,2} \\
&\qquad \forall\psi \in V
\end{aligned} \tag{2.73}$$

with cte_2 is a constant that depends on the volume of Ω4.

Since $\left[Df(u',v')\right](\cdot)$ is linear, (2.73) says that $Df(u',v')$ is in the dual space of

V, denoted V', and that

$$\|Df(u',v')\|_{V'} \leq 12\sqrt{2}C_a\left(M_a + 9M_a^2 + cte_2 M_a\right)\sqrt{\text{volume of } \Omega}\,\|(u,v)\|_t \tag{2.74}$$

we also have

$$\begin{aligned} &[Df(r_1(u_1'+v_1')+r_2(u_2'+v_2'))]\,(\psi) = \\ &\qquad r_1\,[Df(u_1',v_1')]\,(\psi) + r_2\,[Df(u_2'+v_2')]\,(\psi)\ ,\ \forall\psi\in V \end{aligned} \tag{2.75}$$

so

$$Df(r_1(u_1'+v_1')+r_2(u_2'+v_2')) = r_1 Df(u_1',v_1') + r_2 Df(u_2',v_2')$$

which means that the operator $Df(\cdot,\cdot)$ is linear on the space V_t .

So

$$\begin{array}{rrcl} Df: & V_t & \longrightarrow & V' \\ & (u',v') & \longrightarrow & Df(u',v') \end{array} \tag{2.76}$$

is continuous and linear. Therefore, the mapping

$$\begin{array}{rrcl} F: & V_t & \longrightarrow & V' \\ & (u,v) & \longrightarrow & F(u,v) \end{array} \tag{2.77}$$

where $[F(u,v)]\,(\psi) = f_{(u,v)}(\psi)\ ,\ \forall\psi\in V$, is *Fréchet differentiable* so it is continuous.

■

Again, let $(u',v') \in V_t$ and η a non-zero real number and define the linear operator

$$\begin{array}{rrcl} F_1: & V & \longrightarrow & \Re \\ & \psi & \longrightarrow & F_1(\psi) = [Df(u',v')]\,(\psi) - [Da(u',v')]\,(w_0(u,v),\psi) \end{array} \tag{2.78}$$

From (2.65) and (2.73) we can conclude that F_1 is bounded. So there exists a unique $\Im(u',v') \in V$ such that

$$a_{(u,v)}\left(\Im(u',v'),\psi\right) = F_1\left(\psi\right) \ \ , \forall\psi \in V \tag{2.79}$$

Let

$$\delta_\eta^{w_0}(u',v') = \frac{w_0(u+\eta u', v+\eta v') - w_0(u,v)}{\eta} \tag{2.80}$$

we have

$$\begin{aligned} a_{(u+\eta u',v+\eta v')}\left(\delta_\eta^{w_0}(u',v') - \Im(u',v'),\psi\right) = \ & \tfrac{f_{(u+\eta u',v+\eta v')}(\psi) - f_{(u,v)}(\psi)}{\eta} \\ & -\tfrac{a_{(u+\eta u',v+\eta v')}(w_0(u,v),\psi) - a_{(u,v)}(w_0(u,v),\psi)}{\eta} \\ & -a_{(u+\eta u',v+\eta v')}\left(\Im(u',v'),\psi\right) \end{aligned} \tag{2.81}$$

Since $a_{(u+\eta u',v+\eta v')}\left(\cdot,\cdot\right)$ is also coercive and has the same coercivity constant as $a_{(u,v)}$, we obtain, from Theorem (2.4), the inequality

$$\left\|\delta_\eta^{w_0}(u',v') - \Im(u',v')\right\|_{1,2} \leq \frac{1}{m_a}\left\| \begin{array}{c} \frac{f_{(u+\eta u',v+\eta v')}(\cdot) - f_{(u,v)}(\cdot)}{\eta} \\ -\frac{a_{(u+\eta u',v+\eta v')}(w_0(u,v),\cdot) - a_{(u,v)}(w_0(u,v),\cdot)}{\eta} \\ -a_{(u+\eta u',v+\eta v')}\left(\Im(u',v'),\cdot\right) \end{array} \right\|_{V'} \tag{2.82}$$

Because A, defined in (2.69), is continuous, the right hand side of (2.82) goes to zero when $\eta \to 0$, so

$$lim_{\eta\to 0}\frac{w_0(u+\eta u', v+\eta v') - w_0(u,v)}{\eta} = \Im(u',v') \tag{2.83}$$

$w_0(u,v)$ is *Gateaux differentiable* .

From (2.79) and Theorem (2.4) we have

$$\|\Im(u',v')\|_{1,2} \leq \|Df(u',v')\|_{V'} + \|Da(u',v')(w_0(u,v),\cdot)\|_{V'} \tag{2.84}$$

And since $Da(\cdot,\cdot)$ and $Df(\cdot,\cdot)$ are continuous, the right hand side of (2.84) goes to zero as $(u', v') \to (0,0)$; so $\Im(u', v')$ is continuous at $(0,0) \in V_t$.

We also have

$$
\begin{aligned}
a_{(u,v)}(\Im(r_1(u_1', v_1') + r_2(u_2', v_2')), \psi) = {} & [Df(r_1(u_1', v_1')) + r_2(u_2', v_2'))] (\psi) \\
& - [Da(r_1(u_1', v_1') + r_2(u_2', v_2'))] (w_0(u,v), \psi)
\end{aligned}
\tag{2.85}
$$

from the linearity of Da and Df, (2.85) becomes

$$
\begin{aligned}
a_{(u,v)}(\Im(r_1(u_1', v_1') + r_2(u_2', v_2')), \psi) = {} & a_{(u,v)}(r_1 \Im(u_1', v_1'), \psi) \\
& + a_{(u,v)}(r_2 \Im(u_2', v_2'), \psi) \\
= {} & a_{(u,v)}(r_1 \Im(u_1', v_1') + r_2 \Im(u_2', v_2'), \psi) \\
& \forall \psi \in V
\end{aligned}
\tag{2.86}
$$

which means that

$$
\Im(r_1(u_1', v_1') + r_2(u_2', v_2')) = r_1 \Im(u_1', v_1') + r_2 \Im(u_2', v_2') \tag{2.87}
$$

Therefore $Im(u', v')$ is linear and since it is continuous at zero then it is continuous on V_t. Thus $w_0(u,v)$ is *Fréchet differentiable* so it is continuous.

■

Conclusion

Since the mapping

$$
\begin{aligned}
W_0 : \quad V_t & \longrightarrow V \\
(u,v) & \longrightarrow w_0 = w_0(u,v)
\end{aligned}
\tag{2.88}
$$

is *Fréchet differentiable* then it is continuous.

CHAPTER III

THE FINITE ELEMENT METHOD

Let Ω denote an observational volume that, for ease, is a rectangular solid of points $\bar{x} = (x, y, z)^T \in \Re^3$ such that

$$\Omega = \{\bar{x} : 0 < x < L_x, 0 < y < L_y, 0 < z < L_z\}$$

Remark 3.1

If the landscape is diverse (existence of hills and mountains for example), then the shape of Ω will be a little bit messy. The theoretical part will remain the same but the computational part will be more complex, especially, the writing of a code.

The approximation of w_0 follows the classical finite element arguments. Approximations may be based on finite elements obtained as tensor products of piecewise linear splines, cubic splines or B-splines defined on partitions of the intervals $(0, L_x)$, $(0, L_y)$ and $(0, L_z)$ into $n_x - 1$, $n_y - 1$ and $n_z - 1$ subintervals, respectively. Hence, n_x, n_y and n_z represent the number of x, y and z elements, respectively. The number of basis elements and mesh points for the 3 spatial dimensional problem is given by

$$N = \begin{cases} n_x \times n_y \times n_z & \text{if we use linear splines} \\ (n_x + 2)(n_y + 2)(n_z + 2) & \text{if we use cubic B-splines} \end{cases} \tag{3.1}$$

We denote the basis elements as

$$\Phi_I = \Phi_I(\bar{x}) = \phi_i(x)\phi_j(y)\phi_k(z) \tag{3.2}$$

where

$$I = \begin{cases} i + n_x(j-1) + n_x n_y(k-1) \text{ if we use linear splines} \\ \text{with } 1 \le i \le n_x, 1 \le j \le n_y \text{ and } 1 \le k \le n_z \\ \\ i + (n_x+2)(j-1) + (n_x+2)(n_y+2)(k-1) \text{ if we use cubic B-splines} \\ \text{with } 1 \le i \le n_x+2, 1 \le j \le n_y+2 \text{ and } 1 \le k \le n_z+2 \end{cases} \tag{3.3}$$

Let $S_N = Span\left\{\Phi_I\right\}_{I=1}^{N}$. We now pose the finite-dimensional problem to find $w_0^N \in S_N$ such that

$$J\left(w_0^N\right) = Inf\left\{J\left(w^N\right);\ w^N \in S_N\right\} \tag{3.4}$$

Note that the projection, w^N, of any $w \in V$ on the space S_N is written as

$w_N(\bar{x}) = \sum_{I=1}^{N} C_I \Phi_I(\bar{x})$ where $C_I = w(\bar{x_I})$ with $\bar{x_I}$ for $I = 1, \cdots, N$ are the mesh points.

For a vector $q = [q_1, \cdots, q_n]^T \in \Re^n$ define

$$\|q\|_{d\infty} = Max_{j=1}^{n} |q_j| \tag{3.5}$$

and

$$\|q\|_d = \sqrt{\sum_{j=1}^{n} q_j^2} \tag{3.6}$$

It is known that

$$\|q\|_{d\infty} \le \|q\|_d \tag{3.7}$$

For a matrix $G \in \Re^{n_1 \times n_2}$

$$\|G\|_m = Sup\left\{\frac{\|G\ q\|_d}{\|q\|_d}\ ;\ q \in \Re^n \text{ and } q \ne 0\right\}. \tag{3.8}$$

and

$$\|G\|_d = \sqrt{\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} |G_{ij}|^2} \tag{3.9}$$

Since we will discuss only uniform partitions of the domain Ω, let $h_x = L_x/(n_x - 1)$, $h_y = L_y/(n_y - 1)$ and $h_z = L_z/(n_z - 1)$.

For $w^N \in S_N$ we have

$$\begin{aligned} J(w^N) &= a\left(\sum_{I=1}^{N} C_I \Phi_I, \sum_{I=1}^{N} C_I \Phi_I\right) - 2f\left(\sum_{I=1}^{N} C_I \Phi_I\right) \\ &= \sum_{I=1}^{N} C_I \left[\sum_{J=1}^{N} C_J a\left(\Phi_I, \Phi_J\right)\right] - 2\sum_{I=1}^{N} C_I f\left(\Phi_I\right) \end{aligned} \tag{3.10}$$

Let $M = [M_{IJ}]$ be an $N \times N$ matrix such that $M_{IJ} = a(\Phi_J, \Phi_I)$ and $B = [B_1, B_2, \cdots, B_N]^T$ be a vector in $\Re^N$ such that $B_I = f(\Phi_I)$. Since $a(\cdot, \cdot)$ is symmetric and positive definite (form coercivity) then M is symmetric, positive definite and of course invertible. And thus

$$J\left(w_N\right) = \langle MC, C\rangle - 2\langle B, C\rangle \tag{3.11}$$

where $C = [C_1, \cdots, C_N]^T \in \Re^N$ and $\langle \cdot, \cdot \rangle$ is the usual inner product in $\Re^N$.

Claim 3.1 There exists a unique $w_0^N = \sum_{I=1}^{N} C_I^0 \Phi_I$ with

$$M\tilde{C} = B \qquad \left(\tilde{C} = \left[C_1^0, \cdots, C_N^0\right]^T \in \Re^N\right) \tag{3.12}$$

and

$$a\left(w_0^N, w^N\right) = f\left(w^N\right) \ , \forall w^N \in S_N \tag{3.13}$$

such that

$$J\left(w_0^N\right) = Inf\left\{J\left(w^N\right);\ \ w^N \in S_N\right\} \tag{3.14}$$

moreover we have

$$\left\|\tilde{C}\right\|_d \leq \frac{1}{\lambda}\left\|B\right\|_d \tag{3.15}$$

where

$\lambda = Min\left\{\lambda_i\,;\ \lambda_i\ eigenvalue\ of\, M,\ \ i = 1, \cdots, N\right\}.$

Proof

Let $E_k = [0, 0, \cdots, \overbrace{1}^{k^{th}position}, 0, 0, \cdots, 0]^T$

$\frac{\partial J\left(w^N\right)}{\partial C_k} = \langle ME_k, C\rangle + \langle MC, E_k\rangle + \langle B, E_k\rangle$

$= \sum_i M_{ik}C_i + \sum_j M_{kj}C_j - 2B_k$

$= 2\sum_j M_{kj}C_j - 2B_k$

Thus, if $\sum_j M_{kj}C_j^0 = B_k$ for $k = 1, \cdots, N$, which means that $M\tilde{C} = B$, then $\tilde{C} = [C_1^0, \cdots, C_N^0]^T$ is a critical point, $\tilde{C}$ is unique because M is invertible. And since the Hessian matrix of J is $Hess(J) = 2M$ and M is positive definite then $\tilde{C}$ is a minimum.

Let $w^N \in S_N$ with $w^N = \sum_{I=1}^N C_I\Phi_I$, $C = [C_1, \cdots, C_N]^T$

We have

$$a\left(w_0^N, w^N\right) = \sum_I \left(C_I^0 \sum_J C_J M_{IJ}\right) = \left\langle \tilde{C}, MC\right\rangle$$

Since M is symmetric

$$\left\langle \tilde{C}, MC\right\rangle = \left\langle M\tilde{C}, C\right\rangle$$

thus

$$a\left(w_0^N, w^N\right) = \left\langle M\tilde{C}, C\right\rangle = \langle B, C\rangle = f\left(w^N\right)$$

We know that the matrix M is symmetric so there exsists an orthonormal basis of eigenvector of M and thus

$\lambda \left\|\tilde{C}\right\|_d^2 \leq \left\langle M\tilde{C}, \tilde{C}\right\rangle = \left\langle B, \tilde{C}\right\rangle \leq \|B\|_d \left\|\tilde{C}\right\|_d$

and thus $\left\|\tilde{C}\right\|_d \leq \frac{1}{\lambda}\|B\|_d$

Theorem 3.1 There exist two constants $C_\Omega^{(1)}$ and $C_\Omega^{(2)}$ such that

$$\left\|w_0 - w_0^N\right\|_{1,2} \leq C_\Omega^{(1)} \left\|w_0 - w^N\right\|_{1,2} \text{ for any } w^N \in S_N \tag{3.16}$$

If $w_0 \in H^2(\Omega)$ then

$$\left\|w_0 - w_0^N\right\|_{1,2} \leq C_\Omega^{(2)} h\,|w_0|_{2,2} \tag{3.17}$$

where $\qquad h = \sqrt{h_x^2 + h_y^2 + h_z^2}$

Proof

Let $w^N \in S_N$, We have

$$\begin{aligned} a(w_0 - w_0^N, w_0 - w_0^N) &= a(w_0 - w_0^N, w_0 - w^N + w^N - w_0^N) \\ &= a(w_0 - w_0^N, w_0 - w^N) + a(w_0 - w_0^N, w^N - w_0^N) \\ &= a(w_0 - w_0^N, w_0 - w^N) + f(w^N - w_0^N) \\ &\qquad -f(w^N - w_0^N) \\ &\leq 24 C_a M_a^2 \left\|w_0 - w_0^N\right\|_{1,2} \left\|w_0 - w^N\right\|_{1,2} \end{aligned}$$

Since $m_a \left\|w_0 - w_0^N\right\|_{1,2}^2 \leq a(w_0 - w_0^N, w_0 - w_0^N)$, we obtain

$\left\|w_0 - w_0^N\right\|_{1,2}^2 \leq \frac{1}{m_a} 24 C_a M_a^2 \left\|w_0 - w_0^N\right\|_{1,2} \left\|w_0 - w^N\right\|_{1,2}$.

Thus

$\left\|w_0 - w_0^N\right\|_{1,2} \leq \frac{1}{m_a} 24 C_a M_a^2 \left\|w_0 - w^N\right\|_{1,2}$

Let $w_1^N = \sum_{I=1}^N w_0(\bar{x}_I)\Phi(\bar{x}_I)$, we know that if $w_0 \in H^2(\Omega)$ then there exists a constant C'_Ω such that

$\left\|w_0 - w_1^N\right\|_{1,2} \leq C'_\Omega h\,|w_0|_{2,2}$

by replacing w^N in (3.16) by w_1^N we obtain (3.17).

3.1 Definitions

Define

- $$\text{The matrix } M_{b^2} \text{ where } \left[M_{b^2}\right]_{IJ} = \left(b\Phi_I, b\Phi_J\right), \tag{3.18}$$

- $$\text{the matrix } M_{bc} \text{ where } \left[M_{bc}\right]_{IJ} = \left(b\Phi_I, c\Phi_{Jx}\right) + \left(c\Phi_{Ix}, b\Phi_J\right), \tag{3.19}$$

- $$\text{The matrix } M_{bd} \text{ where } \left[M_{bd}\right]_{IJ} = \left(b\Phi_I, d\Phi_{Jy}\right) + \left(d\Phi_{Iy}, b\Phi_J\right), \tag{3.20}$$

- $$\text{The matrix } M_{cd} \text{ where } \left[M_{cd}\right]_{IJ} = \left(c\Phi_{Ix}, d\Phi_{Jy}\right) + \left(d\Phi_{Iy}, c\Phi_{Jx}\right), \tag{3.21}$$

- $$\text{The matrix } M_{c^2} \text{ where } \left[M_{c^2}\right]_{IJ} = \left(c\Phi_{Ix}, c\Phi_{Jx}\right), \tag{3.22}$$

- $$\text{The matrix } M_{d^2} \text{ where } \left[M_{d^2}\right]_{IJ} = \left(d\Phi_{Iy}, d\Phi_{Jy}\right), \tag{3.23}$$

- $$\text{The matrix } M_{\rho_z^2} \text{ where } \left[M_{\rho_z^2}\right]_{IJ} = \left(\rho_z\Phi_I, \rho_z\Phi_J\right), \tag{3.24}$$

- $$\text{The matrix } M_{\rho^2} \text{ where } \left[M_{\rho^2}\right]_{IJ} = \left(\rho\Phi_{Iz}, \rho\Phi_{Jz}\right), \tag{3.25}$$

- $$\text{The matrix } M_{\rho\rho_z} \text{ where } [M_{\rho\rho_z}]_{IJ} = (\rho\Phi_{Iz}, \rho_z\Phi_J) + (\rho_z\Phi_I, \rho\Phi_{Jz}), \tag{3.26}$$

- $$\text{The matrix } G \text{ where } [G]_{IJ} = (\Phi_{Ix}, \Phi_{Jx}) + (\Phi_{Iy}, \Phi_{Jy}) + (\Phi_{Iz}, \Phi_{Jz}), \tag{3.27}$$

- $$\text{The matrix } Co \text{ where } [Co]_{IJ} = (\Phi_I, \Phi_J), \tag{3.28}$$

where $(\cdot, \cdot)$ is the inner product on $L^2(\Omega)$,

- $$Matrix_1 \otimes Matrix_2 \text{ as the Kronecker product of } Matrix_1 \text{ and } Matrix_2, \tag{3.29}$$

- $$(g, 1) = \int_\Omega g(\bar{x})d\bar{x} \text{ for } g \in L^2(\Omega), \tag{3.30}$$

- $$\text{the vector } B_b \text{ where } [B_b]_I = (ab\Phi_I, 1), \tag{3.31}$$

- $$\text{the vector } B_c \text{ where } [B_c]_I = (ac\Phi_{Ix}, 1), \tag{3.32}$$

- $$\text{the vector } B_d \text{ where } [B_d]_I = (ad\Phi_{Iy}, 1), \tag{3.33}$$

- $$\text{the vector } B_\rho \text{ where } \left[B_\rho\right]_I = (e\rho^2 \Phi_{Iz}, 1), \tag{3.34}$$

- $$\text{the vector } B_{\rho_z} \text{ where } \left[B_{\rho_z}\right]_I = (e\rho\rho_z \Phi_I, 1) \tag{3.35}$$

- $$\text{and the vector } B_{w_p} \text{ where } \left[B_{w_p}\right]_I = -(w_p \Phi_I, 1). \tag{3.36}$$

■

Using definitions (3.18)...(3.36), the matrix M and the vector B can be written as

$$\begin{aligned} M = \ & \alpha_v^2 \overbrace{\left[M_{b^2} + M_{c^2} + M_{d^2} + M_{bc} + M_{bd} + M_{cd}\right]}^{M_v} \\ & + \alpha_m^2 \overbrace{\left[M_{\rho_z^2} + M_{\rho^2} + M_{\rho\rho_z}\right]}^{M_m} \\ & + \epsilon G + \epsilon_0 Co \end{aligned} \tag{3.37}$$

$$B = -\left[\alpha_v^2 \left(B_b + B_c + B_d\right) + \alpha_m^2 \left(B_\rho + B_{\rho_z}\right) + \epsilon_0 B_{w_p}\right] \tag{3.38}$$

3.2 Stability of the approximation

Suppose we have the linear equation

$$A_1 + L(D)(X) = 0 \tag{3.39}$$

where

$$L(D)(\cdot) : \chi \longrightarrow \chi \text{ is a linear matrix operator depending on the parameter } D. \tag{3.40}$$

We have

$$L(D)(X) = \sum_{i=1}^{p} B_i X C_i \tag{3.41}$$

where the B_is and the C_is are matrices. In fact

$$D := (B_1, C_1, \cdots, B_p, C_p) := (A_2, \cdots, A_r) \text{ with } r = 2p + 1. \tag{3.42}$$

We have

$$A_1 + \delta A_1 + L(D + \delta D)(X + \delta X) = 0 \tag{3.43}$$

We assume that δD is such that $L^{-1}(D + \delta D)$ is invertible, provided that

$$\left\| L^{-1}(D) \circ \left[L(D + \delta D) - L(D) \right] \right\|_m < 1. \tag{3.44}$$

Let

$$L_1(D, \delta D) = L(D + \delta D) - L(D), \tag{3.45}$$

$$\mu = [\mu_2, \cdots, \mu_r] \in \Re_+^{r-1} \text{ be a given vector} \tag{3.46}$$

and

$$\beta = \beta(\mu) = Max \left\{ \| L_1(D, G) \| ; \underbrace{G = (G_2, \cdots, G_r)}_{as\, in} \; and \; \underbrace{\|G_i\| \leq \mu_i}_{i=2,\cdots,r} \right\} \tag{3.47}$$

Since $\beta(0) = 0$ and β is continuous, then μ may be chosen such that

$$\beta(\mu) \left\| L^{-1}(D) \right\|_m < 1 \tag{3.48}$$

If $A_1 \neq 0$ then $X \neq 0$ and we have

$$\frac{\|\delta X\|_d}{\|X\|_d} \leq \frac{\beta \left\| L^{-1}(D) \right\|_m + \frac{\left\| L^{-1}(D) \right\|_m \|\delta A_1\|_d}{\|X\|_d}}{1 - \beta \left\| L^{-1}(D) \right\|_m} \tag{3.49}$$

(3.15) says that $\|M^{-1}\|_m = \lambda$ and we have

$$\beta = \|\delta M\|_m \ . \tag{3.50}$$

Assuming that

$$\|\delta M\|_m < \frac{1}{\lambda} \ , \tag{3.51}$$

we obtain from (3.49)

$$\boxed{\left\|\delta \tilde{C}\right\|_d \leq \frac{\|\delta M\|_m \left\|\tilde{C}\right\|_d + \|\delta B\|_d}{\frac{1}{\lambda} - \|\delta M\|_m}} \tag{3.52}$$

which can be written, using (3.15), as

$$\left\|\delta \tilde{C}\right\|_d \leq \frac{\|\delta M\|_m \frac{\|B\|_d}{\lambda} + \|\delta B\|_d}{\frac{1}{\lambda} - \|\delta M\|_m} \ . \tag{3.53}$$

Remark 3.2

The construction of the matrix M and the vector B is one of the major parts of this work because we have to be able to track $\bar{u}$ and $\bar{v}$ in order to study the sensitivity of the solution.

CHAPTER IV

APPROXIMATION BY PRODUCTS OF PIECEWISE LINEAR SPLINES

The main goal of our approach is to facilitate the study of the sensitivity of w_0 to errors in $\bar{u}$ and $\bar{v}$, and therefore to provide us with a "clear" picture of how M and B are constructed. From (3.37), M is a linear combination of eleven matrices; the idea is to write each of the matrices of m_v and M_m as the sum of eight matrices whose sensitivities to errors in $\bar{u}$ and $\bar{v}$ can be approximated without difficulties. The presence of the function a in the five vectors that define B in (3.38) makes tracking errors in $\bar{u}$ and $\bar{v}$ somewhat complicated; but, by writing each one of them as the sum of eight vectors and with the help of a software like Mathematica, one can estimate the sensitivity of the mentioned five vectors. It is also important to note that most of the computational time is spent in the calculation of M and B and not in solving the linear system; and what was observed was that this method not only allowed us to find an estimate for

$$\|\delta M\|_m = \|M(\bar{u}+\delta\bar{u}, \bar{v}+\delta\bar{v}) - M(\bar{u},\bar{v})\|_m \tag{4.1}$$

and

$$\|\delta B\|_d = \|B(\bar{u}+\delta\bar{u}, \bar{v}+\delta\bar{v}) - B(\bar{u},\bar{v})\|_d \tag{4.2}$$

where $\delta\bar{u}$ and $\delta\bar{v}$ are perturbations of $\bar{u}$ and $\bar{v}$, respectively,

but also tremendously reduced the computational time.

Definition 4.1

Let

$$\phi(x) = \begin{cases} 1+x & if \quad -1 \leq x \leq 0, \\ 1-x & if \quad 0 \leq x \leq 1, \\ 0 & if \quad x \notin [-1,1], \end{cases} \tag{4.3}$$

in the case of a uniform partition with mesh length h and $n-1$ subintervals we define the piecewise linear splines

$$\phi_i(x) = \phi\left(\frac{x}{h} - i + 1\right) \text{ for } i = 1, ..., n \tag{4.4}$$

4.1 Definitions

For a positive integer n and a positive real number h let

- $I(n)$ be the $n \times n$ identity matrix,
- $$A_v(n) = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 & \cdot & \cdot & \cdot & 0 & 0 & 0 \\ & & & & & & & & & \\ & & & & & & & & & \\ & & & & & & & & & \\ 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 1/2 & 1/2 \end{bmatrix}_{(n-1)\times n}, \tag{4.5}$$

  the product of the matrix $A_v(n)$ and an n vector q gives an $n-1$ vector whose i^{th} entry is the average value of the i^{th} and the $i+1^{st}$ entries of q.

- $$D(n,h) = \frac{1}{h} \begin{bmatrix} -1 & 1 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & 0 \\ -1/2 & 0 & 1/2 & 0 & \cdot & \cdot & \cdot & 0 & 0 & 0 \\ 0 & -1/2 & 0 & 1/2 & \cdot & \cdot & \cdot & 0 & 0 & 0 \\ \\ \\ \\ \\ 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & -1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & -1 & 1 \end{bmatrix}_{n \times n}, \tag{4.6}$$

  the product of the matrix $D(n,h)$ and an n vector q gives an n vector whose entries we consider the discrete derivatives of the entries of q.

- $$L(n) = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & 0 \\ \\ \\ \\ \\ 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & 1 \end{bmatrix}_{n \times (n-1)}, \tag{4.7}$$

- $$U(n) = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 & 0 & 0 \\ \\ \\ \\ \\ 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & 0 \end{bmatrix}_{n\times(n-1)}, \tag{4.8}$$

- $$Ldgdg(n,h) = \frac{1}{h}\begin{bmatrix} 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & \cdot & \cdot & \cdot & 0 & 0 & 0 \\ \\ \\ \\ \\ 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & -1 & 1 \end{bmatrix}_{n\times n}, \tag{4.9}$$

$$[Ldgdg(n,h)]_{ij} = \begin{cases} \int_{(i-2)h}^{(i-1)h} \phi_i'(x)\phi_j'(x)dx \text{ for } 2 \leq i \leq n \text{ and } 1 \leq j \leq n \\ 0 \text{ otherwise} \end{cases},$$

- $$Udgdg(n,h) = \frac{1}{h}\begin{bmatrix} 1 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & \cdots & 0 & 0 & 0 \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}_{n\times n}, \qquad (4.10)$$

$$[Udgdg(n,h)]_{ij} = \begin{cases} \int_{(i-1)h}^{ih} \phi_i'(x)\phi_j'(x)dx \text{ for } 1 \leq i \leq n-1 \text{ and } 1 \leq j \leq n \\ 0 \text{ otherwise} \end{cases},$$

- $$dgdg(n,h) = Ldgdg(n,h) + Udgdg(n,h), \qquad (4.11)$$

- $$Ldgg(n) = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 & \cdots & 0 & 0 & 0 \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ 0 & 0 & 0 & 0 & \cdots & 1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1/2 & 1/2 \end{bmatrix}_{n\times n}, \qquad (4.12)$$

$$[Ldgg(n,h)]_{ij} = \begin{cases} \int_{(i-2)h}^{(i-1)h} \phi_i'(x)\phi_j(x)dx \text{ for } 2 \le i \le n \text{ and } 1 \le j \le n \\ 0 \text{ otherwise} \end{cases}, ,$$

- $$Udgg(n) = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & \cdots & 0 & 0 & 0 \\ \\ \\ \\ 0 & 0 & 0 & 0 & \cdots & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}_{n\times n}, \qquad (4.13)$$

$$[Udgg(n,h)]_{ij} = \begin{cases} \int_{(i-1)h}^{ih} \phi_i'(x)\phi_j(x)dx \text{ for } 1 \le i \le n-1 \text{ and } 1 \le j \le n \\ 0 \text{ otherwise} \end{cases},$$

- $$Lgdg(n) = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1/2 & 1/2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1/2 & 1/2 & 0 & \cdots & 0 & 0 & 0 \\ \\ \\ \\ 0 & 0 & 0 & 0 & \cdots & -1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1/2 & 1/2 \end{bmatrix}_{n\times n}, \qquad (4.14)$$

$$[Lgdg(n,h)]_{ij} = \begin{cases} \int_{(i-2)h}^{(i-1)h} \phi_i(x)\phi_j'(x)dx \text{ for } 2 \le i \le n \text{ and } 1 \le j \le n \\ 0 \text{ otherwise} \end{cases}, ,$$

- $$Ugdg(n) = \begin{bmatrix} -1/2 & 1/2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1/2 & 1/2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -1/2 & 1/2 & \cdots & 0 & 0 & 0 \\ \\ \\ \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1/2 & 1/2 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}_{n\times n}, \tag{4.15}$$

$$[Ugdg(n,h)]_{ij} = \begin{cases} \int_{(i-1)h}^{ih} \phi_i(x)\phi_j'(x)dx \text{ for } 1 \le i \le n-1 \text{ and } 1 \le j \le n \\ 0 \text{ otherwise} \end{cases}$$

- $$Lgg(n,h) = h \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1/6 & 1/3 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1/6 & 1/3 & 0 & \cdots & 0 & 0 & 0 \\ \\ \\ \\ 0 & 0 & 0 & 0 & \cdots & 1/6 & 1/3 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1/6 & 1/3 \end{bmatrix}_{n\times n}, \tag{4.16}$$

$$[Lgg(n,h)]_{ij} = \begin{cases} \int_{(i-2)h}^{(i-1)h} \phi_i(x)\phi_j(x)dx \text{ for } 2 \le i \le n \text{ and } 1 \le j \le n \\ 0 \text{ otherwise} \end{cases},$$

- $$Ugg(n,h) = h \begin{bmatrix} 1/3 & 1/6 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1/3 & 1/6 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 1/6 & \cdots & 0 & 0 & 0 \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1/3 & 1/6 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}_{n\times n}, \quad (4.17)$$

$$[Ugg(n,h)]_{ij} = \begin{cases} \int_{(i-1)h}^{ih} \phi_i(x)\phi_j(x)dx \text{ for } 1 \le i \le n-1 \text{ and } 1 \le j \le n \\ 0 \text{ otherwise} \end{cases},$$

- $$gg(n,h) = Lgg(n,h) + Ugg(n,h), \quad (4.18)$$

- $$Ldg(n) = \begin{bmatrix} 0 \\ 1 \\ 1 \\ \\ \\ \\ 1 \end{bmatrix}_{n\times 1}, \quad (4.19)$$

$$[Ldg(n,h)]_i = \begin{cases} \int_{(i-2)h}^{(i-1)h} \phi_i'(x)dx \text{ for } 2 \le i \le n \\ 0 \text{ otherwise} \end{cases},$$

-
$$
Udg(n) = \begin{bmatrix} -1 \\ -1 \\ \\ \\ \\ -1 \\ 0 \end{bmatrix}_{n\times 1}, \tag{4.20}
$$

$$
[Udg(n,h)]_i = \begin{cases} \int_{(i-1)h}^{ih} \phi_i'(x)dx \text{ for } 1 \le i \le n-1 \\ 0 \text{ otherwise} \end{cases},
$$

-
$$
Lg(n,h) = \frac{h}{2} \begin{bmatrix} 0 \\ 1 \\ 1 \\ \\ \\ \\ 1 \end{bmatrix}_{n\times 1}, \tag{4.21}
$$

$$
[Lg(n,h)]_i = \begin{cases} \int_{(i-2)h}^{(i-1)h} \phi_i(x)dx \text{ for } 2 \le i \le n \\ 0 \text{ otherwise} \end{cases},
$$

- $$Ug(n,h) = \frac{h}{2} \begin{bmatrix} 1 \\ 1 \\ \\ \\ \\ 1 \\ 0 \end{bmatrix}_{n \times 1}, \tag{4.22}$$

$$[Ug(n,h)]_i = \begin{cases} \int_{(i-1)h}^{ih} \phi_i(x)dx \text{ for } 1 \le i \le n-1 \\ 0 \text{ otherwise} \end{cases},$$

- $$A_v = A_v(n_z) \otimes A_v(n_y) \otimes A_v(n_x), \tag{4.23}$$

- $$D_x = I(n_z) \otimes I(n_y) \otimes D(n_x, h_x), \tag{4.24}$$

- $$D_y = I(n_z) \otimes D(n_y, h_y) \otimes I(n_x), \tag{4.25}$$

- $$D_x = D(n_z, h_z) \otimes I(n_y) \otimes I(n_x), \tag{4.26}$$

- $$LLL = L(n_z) \otimes L(n_y) \otimes L(n_x) = SM_1, \tag{4.27}$$

- $$LLU = L(n_z) \otimes L(n_y) \otimes U(n_x) = SM_2, \tag{4.28}$$

- $$LUL = L(n_z) \otimes U(n_y) \otimes L(n_x) = SM_3, \tag{4.29}$$

- $$LUU = L(n_z) \otimes U(n_y) \otimes U(n_x) = SM_4, \tag{4.30}$$

- $$ULL = U(n_z) \otimes L(n_y) \otimes L(n_x) = SM_5, \tag{4.31}$$

- $$ULU = U(n_z) \otimes L(n_y) \otimes U(n_x) = SM_6, \tag{4.32}$$

- $$UUL = U(n_z) \otimes U(n_y) \otimes L(n_x) = SM_7 \tag{4.33}$$

- and
$$UUU = U(n_z) \otimes U(n_y) \otimes U(n_x) = SM_8. \tag{4.34}$$

- Let G be a matrix in $\Re^{m\times n}$ and q a vector in $\Re^m$

$$G = \begin{bmatrix} g_{11} & g_{12} & \cdot & \cdot & \cdot & g_{1n} \\ g_{21} & g_{22} & \cdot & \cdot & \cdot & g_{2n} \\ \\ \\ \\ g_{m1} & \cdot & \cdot & \cdot & \cdot & g_{mn} \end{bmatrix}_{m\times n} , \qquad q = \begin{bmatrix} q_1 \\ q_2 \\ \\ \\ \\ q_m \end{bmatrix}_{m\times 1}$$

define the product $\bar{\times}$ by

$$G\bar{\times}q = q\bar{\times}G = \begin{bmatrix} g_{11}q_1 & g_{12}q_1 & \cdot & \cdot & \cdot & g_{1n}q_1 \\ g_{21}q_2 & g_{22}q_2 & \cdot & \cdot & \cdot & g_{2n}q_2 \\ & & & & & \\ & & & & & \\ & & & & & \\ g_{m1}q_m & & \cdot & \cdot & \cdot & g_{mn}q_m \end{bmatrix}_{m\times n} \tag{4.35}$$

Claim 4.1 D_x, D_y and D_z commute in the usual matrix product.

It can be checked by simple linear algebra operations.

Claim 4.2 If G^r, $r = 1, \cdots, p$, are matrices in $\Re^{m\times n}$ and q^r, $r = 1, \cdots, p$, are vectors in $\Re^m$ then

$$\left\|\sum_{r=1}^{p} G^r \bar{\times} q^r\right\|_m \leq \left\|\sum_{r=1}^{p} G^r\right\|_m (Max_{r=1}^p \|q^r\|_{d\infty}) \tag{4.36}$$

Proof

$$\begin{aligned} \left\|\left(\sum_{r=1}^p G^r \bar{\times} q^r\right) s\right\|_d &= \sqrt{\sum_{i=1}^m \left(\sum_{j=1}^n \sum_{r=1}^p g_{ij}^r q_i^r s_j\right)^2} \\ &\leq \sqrt{\sum_{i=1}^m \left(\sum_{j=1}^n \left(\sum_{r=1}^p g_{ij}^r\right) s_j\right)^2} Max_{r=1}^p \|q^r\|_{d\infty} \\ &\leq \left\|\left(\sum_{r=1}^p G^r\right) s\right\|_d Max_{r=1}^p \|q^r\|_{d\infty} \\ &\leq \left\|\sum G^r\right\|_m \|s\|_d Max_{r=1}^p \|q^r\|_{d\infty} \end{aligned}$$

so for $0 \neq \vartheta \in \Re^n$ we have

$$\frac{\|(\sum G^r \bar{\times} q^r)\vartheta\|_d}{\|\vartheta\|_d} \leq Max_r \|q\|_{d\infty} \frac{\|(\sum G^r)\vartheta\|_d}{\|\vartheta\|_d}$$

$$\leq Max_r \|q\|_{d\infty} Sup_{s\neq 0}\frac{\|(\sum_r G^r)s\|_d}{\|s\|_d} = Max_r \|q\|_{d\infty} \|\sum_r G^r\|_m$$

hence

$$\|\sum_r G^r \bar{\times} q^r\|_m = Sup_{\vartheta\neq 0}\frac{\|(\sum_r G^r \bar{\times} q^r)\vartheta\|_d}{\|\vartheta\|_d} \leq \|\sum_r G^r\|_m Max_r \|q^r\|_{d\infty}$$

Claim 4.3 If $g = [g_1, \cdots, g_m]^T$ and $q = [q_1, \cdots, q_m]^T$ are both in $\Re^m$ then we have

$$\|g\bar{\times}q\| \leq \|g\|_\infty \|q\| \leq \|g\| \|q\| \tag{4.37}$$

where $\|g\|_\infty = Max\{|g_i|;\ \ i = 1, \cdots, m\}$.

Proof

$$\text{since} \quad g = \begin{bmatrix} g_1 \\ g_2 \\ . \\ . \\ . \\ g_m \end{bmatrix} \quad \text{then} \quad g\bar{\times}q = \begin{bmatrix} g_1q_1 \\ g_2q_2 \\ . \\ . \\ . \\ g_mq_m \end{bmatrix}$$

so $\|g\bar{\times}q\| = \sqrt{\sum_{i=1}^m g_i^2q_i^2} \leq Max_i |g_i| \sqrt{\sum_{i=1}^m q_i^2} = \|g\|_\infty \|q\|$

and we have $Max_i |g_i| \sqrt{\sum_{i=1}^m q_i^2} \leq \sqrt{\sum_{i=1}^m g_i^2}\sqrt{\sum_{i=1}^m q_i^2} = \|g\| \|q\|$

4.2 The one-dimensional case

Since the functions u and v are defined only at the mesh points, we will take their average values when calculating the elements of the matrix M and the vector B. Let $0 = x_1 < x_2 < \cdots < x_n = 1$ be a uniform partition of the interval $[0, 1]$ with $h = 1/(n-1)$ Let's suppose that we have the following ordinary differential equation

$$-[a(x)u(x)']' + a'(x)u(x) = f(x) \text{ for } 0 < x < 1 \text{ where } a'(x) = \tfrac{da(x)}{dx}.$$

The variational formulation is

Find $u \in H^1(0,1)$ such that $a(u,v) = (f,v)$, $\forall v \in h^1(0,1)$. Where $a(u,v) = \int_0^1 (au'v' + a'uv)dx$ and $(f,v) = \int_0^1 fv\, dx$ And suppose that $a(x)$, $a'(x)$ and $f(x)$ are such that the solution exists and is unique.

But, in real life, $a(x)$ and $f(x)$ are only defined at the mesh points, x_i. So in the calculation of the stiffness matrix, M, and the load vector, B, we will consider the averages of the functions, $a(x)$, $a'(x)$ and $f(x)$ over each interval $[x_i, x_{i+1}]$; for example $a(x) = [a(x_{i+1}) + a(x_i)]/2 \quad \forall x \in (x_i, x_{i+1})$. And $a'(x_i)$ will be taken equal to $[a(x_{i+1}) - a(x_{i-1})]/2h$ if $x_i \neq 0, 1$ with $a'(0) = [a(x_2) - a(0)]/h$ and $a'(1) = [a(1) - a(x_{n-1})]/h$.

So if $a_d = [a_1 = a(x_1), a_2 = a(x_2), \cdots, a_n = a(1)]^T$ and $f_d = [f_1 = f(x_1),\ f_2 = f(x_2), \cdots, f_n = f(1)]^T$ represent the values of $a(x)$ and $f(x)$ at the mesh points, respectively, then a_d and f_d can be averaged over the subintervals of the mesh using the $(n-1) \times n$ matrix $A_v(n)$.

We will obtain the average vector by the usual matrix-vector product $a_v = A_v(n) \cdot a_d$.

The vector, a'_d, that represents the discrete derivative of a_d is

$a'_d = D(n,h) \cdot a_d$

The stiffness matrix and the load vector are calculated as follows

$$\begin{aligned}M = Ldgdg(n,h)\bar{\times}\,[L(n)\cdot A_v(n)\cdot a_d]\\ +Udgdg(n,h)\bar{\times}\,[U(n)\cdot A_v(n)\cdot a_d]\\ +Ugg(n,h)\bar{\times}\,[U(n)\cdot A_v(n)\cdot D(n,h)\cdot a_d]\\ +Ugg(n,h)\bar{\times}\,[U(n)\cdot A_v(n)\cdot D(n,h)\cdot a_d]\end{aligned} \tag{4.38}$$

$$B = Lg(n,h)\bar{\times}\,[L(n)\cdot A_v(n)\cdot f_d] + Ug(n,h)\bar{\times}\,[U(n)\cdot A_v(n)\cdot f_d] \tag{4.39}$$

4.3 The three-dimensional case

Since the functions u and v are defined only at the mesh points, we will take their average values on the sub-volumes formed by our partition of the domain Ω when calculating the elements of the matrix M and the vector B. In the three-dimensional case, which is our case, the nature of the bilinear form $a(\cdot,\cdot)$ and the linear functional $f(\cdot)$ requires the use of the matrices (4.23)...(4.34).

Let $\bar{u} = [u_1,\cdots,u_N]^T$ and $\bar{v} = [v_1,\cdots,v_N]^T$ be the values of the two horizontal components, u and v, of the wind field at the mesh points and $\bar{u}_t = [u_{1t},\cdots,u_{Nt}]^T$ and $\bar{v}_t = [v_{1t},\cdots,v_{Nt}]^T$ be the values of their derivatives with respect to time, respectively. And let $\rho_d = [\rho_1,\cdots,\rho_N]^T$, $w_p = [w_{p_1},...,w_{p_N}]^T$ and $\bar{f} = [f_1,\cdots,f_N]^T$ be the values of the air density ρ, the predicted vertical wind field w_p and the Coriolis factor f at the mesh points, respectively.

4.3.1 The values of the functions a, b, c, d and e

Here we introduce the vectors $\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{d}$, $\bar{e}$, $\bar{\rho}$, $\bar{\rho}_z$ and $\bar{f}$ that represent the average values of the functions a, b, c, d, e, ρ, ρ_z and f, respectively, on the $(n_x-1)(n_y-1)(n_z-1)$ sub-volumes we have. τ is considered constant.

For given $\bar{u}$, $\bar{v}$, $\bar{u}_t$, $\bar{v}_t$, $\bar{\rho}$ and $\bar{f}$ let' s define

- $$\begin{aligned}\bar{a} = \; & A_v[D_x\bar{v}_t - D_y\bar{u}_t + \bar{u}\bar{\times}\left(D_xD_x\bar{v} - D_xD_y\bar{u}\right) \\ & +\bar{v}\bar{\times}\left(D_yD_x\bar{v} - D_yD_y\bar{u}\right) + \left(D_x\bar{u} + D_y\bar{v}\right)\bar{\times}\left(D_x\bar{v} - D_y\bar{u} + \bar{f}\right) \\ & -\tau\left(D_x^2 + D_y^2 + D_z^2\right)\left(D_x\bar{v} - D_y\bar{u}\right)],\end{aligned} \tag{4.40}$$

- $$\bar{b} = A_v[D_z(D_x\bar{v} - D_y\bar{u})], \tag{4.41}$$

- $$\bar{c} = A_vD_z\bar{v}, \tag{4.42}$$

- $$\bar{d} = -A_vD_z\bar{u}, \tag{4.43}$$

- $$\bar{e} = A_v\left[D_x\bar{u} + D_x\bar{u}\right], \tag{4.44}$$

- $$\bar{\rho} = A_v\rho_d, \tag{4.45}$$

- $$\bar{\rho_z} = A_vD_z\rho_d, \tag{4.46}$$

- and
$$\bar{w}_p = A_vw_p. \tag{4.47}$$

4.3.2 Construction of the *"stiffness matrix"* M

Let

- $$\bar{b}^{(2)} = \bar{b}\bar{\times}\bar{b}, \tag{4.48}$$

- $$\bar{bc} = \bar{b}\bar{\times}\bar{c}, \tag{4.49}$$

- $$\bar{bd} = \bar{b}\bar{\times}\bar{d}, \tag{4.50}$$

- $$\bar{cd} = \bar{c}\bar{\times}\bar{d}, \tag{4.51}$$

- $$\bar{c}^{(2)} = \bar{c}\bar{\times}\bar{c}, \tag{4.52}$$

- $$\bar{d}^{(2)} = \bar{d}\bar{\times}\bar{d}, \tag{4.53}$$

- $$\bar{\rho_z}^{(2)} = \bar{\rho}_z\bar{\times}\bar{\rho}_z, \tag{4.54}$$

- $$\bar{\rho}^{(2)} = \bar{\rho}\bar{\times}\bar{\rho} \tag{4.55}$$

- and
$$\bar{\rho\rho_z} = \bar{\rho}\bar{\times}\bar{\rho}_z. \tag{4.56}$$

The matrices defined in (3.37) are calculated as follows

$$\begin{aligned} G = \; & gg(n_z, h_z) \otimes gg(n_y, h_y) \otimes dgdg(n_x, h_x) \\ & + gg(n_z, h_z) \otimes dgdg(n_y, h_y) \otimes gg(n_x, h_x) \\ & + dgdg(n_z, h_z) \otimes gg(n_y, h_y) \otimes gg(n_x, h_x), \end{aligned} \tag{4.57}$$

$$Co = gg(n_z, h_z) \otimes gg(n_y, h_y) \otimes gg(n_x, h_x), \tag{4.58}$$

$$
\begin{aligned}
M_{b^2} = M_{\bar{b}^{(2)}} = \;& \overbrace{\left[Lgg(n_z,h_z) \otimes Lgg(n_y,h_y) \otimes Lgg(n_x,h_x)\right]}^{\Xi_1^b} \,\bar{\times}\, \left(LLL \cdot \bar{b}^{(2)}\right) \\
& + \overbrace{\left[Lgg(n_z,h_z) \otimes Lgg(n_y,h_y) \otimes Ugg(n_x,h_x)\right]}^{\Xi_2^b} \,\bar{\times}\, \left(LLU \cdot \bar{b}^{(2)}\right) \\
& + \overbrace{\left[Lgg(n_z,h_z) \otimes Ugg(n_y,h_y) \otimes Lgg(n_x,h_x)\right]}^{\Xi_3^b} \,\bar{\times}\, \left(LUL \cdot \bar{b}^{(2)}\right) \\
& + \overbrace{\left[Lgg(n_z,h_z) \otimes Ugg(n_y,h_y) \otimes Ugg(n_x,h_x)\right]}^{\Xi_4^b} \,\bar{\times}\, \left(LUU \cdot \bar{b}^{(2)}\right) \\
& + \overbrace{\left[Ugg(n_z,h_z) \otimes Lgg(n_y,h_y) \otimes Lgg(n_x,h_x)\right]}^{\Xi_5^b} \,\bar{\times}\, \left(ULL \cdot \bar{b}^{(2)}\right) \\
& + \overbrace{\left[Ugg(n_z,h_z) \otimes Lgg(n_y,h_y) \otimes Ugg(n_x,h_x)\right]}^{\Xi_6^b} \,\bar{\times}\, \left(ULU \cdot \bar{b}^{(2)}\right) \\
& + \overbrace{\left[Ugg(n_z,h_z) \otimes Ugg(n_y,h_y) \otimes Lgg(n_x,h_x)\right]}^{\Xi_7^b} \,\bar{\times}\, \left(UUL \cdot \bar{b}^{(2)}\right) \\
& + \overbrace{\left[Ugg(n_z,h_z) \otimes Ugg(n_y,h_y) \otimes Ugg(n_x,h_x)\right]}^{\Xi_8^b} \,\bar{\times}\, \left(UUU \cdot \bar{b}^{(2)}\right),
\end{aligned}
\tag{4.59}
$$

$$
\begin{aligned}
& M_{bc} = M_{\bar{bc}} = \\
& \overbrace{\left[Lgg(n_z,h_z) \otimes Lgg(n_y,h_y) \otimes \left(Lgdg(n_x) + Ldgg(n_x)\right)\right]}^{\Xi_1^{bc}} \,\bar{\times}\, \left(LLL \cdot \bar{bc}\right) \\
& + \overbrace{\left[Lgg(n_z,h_z) \otimes Lgg(n_y,h_y) \otimes \left(Ugdg(n_x) + Udgg(n_x)\right)\right]}^{\Xi_2^{bc}} \,\bar{\times}\, \left(LLU \cdot \bar{bc}\right) \\
& + \overbrace{\left[Lgg(n_z,h_z) \otimes Ugg(n_y,h_y) \otimes \left(Lgdg(n_x) + Ldgg(n_x)\right)\right]}^{\Xi_3^{bc}} \,\bar{\times}\, \left(LUL \cdot \bar{bc}\right) \\
& + \overbrace{\left[Lgg(n_z,h_z) \otimes Ugg(n_y,h_y) \otimes \left(Ugdg(n_x) + Udgg(n_x)\right)\right]}^{\Xi_4^{bc}} \,\bar{\times}\, \left(LUU \cdot \bar{bc}\right) \\
& + \overbrace{\left[Ugg(n_z,h_z) \otimes Lgg(n_y,h_y) \otimes \left(Lgdg(n_x) + Ldgg(n_x)\right)\right]}^{\Xi_5^{bc}} \,\bar{\times}\, \left(ULL \cdot \bar{bc}\right) \\
& + \overbrace{\left[Ugg(n_z,h_z) \otimes Lgg(n_y,h_y) \otimes \left(Ugdg(n_x) + Udgg(n_x)\right)\right]}^{\Xi_6^{bc}} \,\bar{\times}\, \left(ULU \cdot \bar{bc}\right) \\
& + \overbrace{\left[Ugg(n_z,h_z) \otimes Ugg(n_y,h_y) \otimes \left(Lgdg(n_x) + Ldgg(n_x)\right)\right]}^{\Xi_7^{bc}} \,\bar{\times}\, \left(UUL \cdot \bar{bc}\right) \\
& + \overbrace{\left[Ugg(n_z,h_z) \otimes Ugg(n_y,h_y) \otimes \left(Ugdg(n_x) + Udgg(n_x)\right)\right]}^{\Xi_8^{bc}} \,\bar{\times}\, \left(UUU \cdot \bar{bc}\right),
\end{aligned}
\tag{4.60}
$$

$$
\begin{aligned}
&M_{bd} = M_{\bar{bd}} = \\
&\overbrace{\left[Lgg(n_z,h_z) \otimes (Lgdg(n_y) + Ldgg(n_y)) \otimes Lgg(n_x,h_x)\right]}^{\Xi_1^{bd}} \bar{\times} \left(LLL \cdot \bar{bd}\right) \\
&+ \overbrace{\left[Lgg(n_z,h_z) \otimes (Lgdg(n_y) + Ldgg(n_y)) \otimes Ugg(n_x,h_x)\right]}^{\Xi_2^{bd}} \bar{\times} \left(LLU \cdot \bar{bd}\right) \\
&+ \overbrace{\left[Lgg(n_z,h_z) \otimes (Ugdg(n_y) + Udgg(n_y)) \otimes Lgg(n_x,h_x)\right]}^{\Xi_3^{bd}} \bar{\times} \left(LUL \cdot \bar{bd}\right) \\
&+ \overbrace{\left[Lgg(n_z,h_z) \otimes (Ugdg(n_y) + Udgg(n_y)) \otimes Ugg(n_x,h_x)\right]}^{\Xi_4^{bd}} \bar{\times} \left(LUU \cdot \bar{bd}\right) \\
&+ \overbrace{\left[Ugg(n_z,h_z) \otimes (Lgdg(n_y) + Ldgg(n_y)) \otimes Lgg(n_x,h_x)\right]}^{\Xi_5^{bd}} \bar{\times} \left(ULL \cdot \bar{bd}\right) \\
&+ \overbrace{\left[Ugg(n_z,h_z) \otimes (Lgdg(n_y) + Ldgg(n_y)) \otimes Ugg(n_x,h_x)\right]}^{\Xi_6^{bd}} \bar{\times} \left(ULU \cdot \bar{bd}\right) \\
&+ \overbrace{\left[Ugg(n_z,h_z) \otimes (Ugdg(n_y) + Udgg(n_y)) \otimes Lgg(n_x,h_x)\right]}^{\Xi_7^{bd}} \bar{\times} \left(UUL \cdot \bar{bd}\right) \\
&+ \overbrace{\left[Ugg(n_z,h_z) \otimes (Ugdg(n_y) + Udgg(n_y)) \otimes Ugg(n_x,h_x)\right]}^{\Xi_8^{bd}} \bar{\times} \left(UUU \cdot \bar{bd}\right),
\end{aligned}
\tag{4.61}
$$

$$
\begin{aligned}
M_{c^2} = M_{\bar{c}^{(2)}} = \; & \overbrace{\left[Lgg(n_z,h_z) \otimes Lgg(n_y,h_y) \otimes Ldgdg(n_x,h_x)\right]}^{\Xi_1^{c}} \bar{\times} \left(LLL \cdot \bar{c}^{(2)}\right) \\
&+ \overbrace{\left[Lgg(n_z,h_z) \otimes Lgg(n_y,h_y) \otimes Udgdg(n_x,h_x)\right]}^{\Xi_2^{c}} \bar{\times} \left(LLU \cdot \bar{c}^{(2)}\right) \\
&+ \overbrace{\left[Lgg(n_z,h_z) \otimes Ugg(n_y,h_y) \otimes Ldgdg(n_x,h_x)\right]}^{\Xi_3^{c}} \bar{\times} \left(LUL \cdot \bar{c}^{(2)}\right) \\
&+ \overbrace{\left[Lgg(n_z,h_z) \otimes Ugg(n_y,h_y) \otimes Udgdg(n_x,h_x)\right]}^{\Xi_4^{c}} \bar{\times} \left(LUU \cdot \bar{c}^{(2)}\right) \\
&+ \overbrace{\left[Ugg(n_z,h_z) \otimes Lgg(n_y,h_y) \otimes Ldgdg(n_x,h_x)\right]}^{\Xi_5^{c}} \bar{\times} \left(ULL \cdot \bar{c}^{(2)}\right) \\
&+ \overbrace{\left[Ugg(n_z,h_z) \otimes Lgg(n_y,h_y) \otimes Udgdg(n_x,h_x)\right]}^{\Xi_6^{c}} \bar{\times} \left(ULU \cdot \bar{c}^{(2)}\right) \\
&+ \overbrace{\left[Ugg(n_z,h_z) \otimes Ugg(n_y,h_y) \otimes Ldgdg(n_x,h_x)\right]}^{\Xi_7^{c}} \bar{\times} \left(UUL \cdot \bar{c}^{(2)}\right) \\
&+ \overbrace{\left[Ugg(n_z,h_z) \otimes Ugg(n_y,h_y) \otimes Udgdg(n_x,h_x)\right]}^{\Xi_8^{c}} \bar{\times} \left(UUU \cdot \bar{c}^{(2)}\right),
\end{aligned}
\tag{4.62}
$$

$$
\begin{aligned}
&M_{cd} = M_{c\bar{d}} = \\
&\overbrace{\begin{pmatrix} Lgg(n_z,h_z)\otimes[Lgdg(n_y)\otimes Ldgg(n_x) \\ +Ldgg(n_y)\otimes Lgdg(n_x)] \end{pmatrix}}^{\Xi_1^{cd}} \bar{\times}\left(LLL\cdot c\bar{d}\right) \\
&+\overbrace{\begin{pmatrix} Lgg(n_z,h_z)\otimes[Lgdg(n_y)\otimes Udgg(n_x) \\ +Ldgg(n_y)\otimes Ugdg(n_x)] \end{pmatrix}}^{\Xi_2^{cd}} \bar{\times}\left(LLU\cdot c\bar{d}\right) \\
&+\overbrace{\begin{pmatrix} Lgg(n_z,h_z)\otimes[Ugdg(n_y)\otimes Ldgg(n_x) \\ +Udgg(n_y)\otimes Lgdg(n_x)] \end{pmatrix}}^{\Xi_3^{cd}} \bar{\times}\left(LUL\cdot c\bar{d}\right) \\
&+\overbrace{\begin{pmatrix} Lgg(n_z,h_z)\otimes[Ugdg(n_y)\otimes Udgg(n_x) \\ +Udgg(n_y)\otimes Ugdg(n_x)] \end{pmatrix}}^{\Xi_4^{cd}} \bar{\times}\left(LUU\cdot c\bar{d}\right) \\
&+\overbrace{\begin{pmatrix} Ugg(n_z,h_z)\otimes[Lgdg(n_y)\otimes Ldgg(n_x) \\ +Ldgg(n_y)\otimes Lgdg(n_x)] \end{pmatrix}}^{\Xi_5^{cd}} \bar{\times}\left(ULL\cdot c\bar{d}\right) \\
&+\overbrace{\begin{pmatrix} Ugg(n_z,h_z)\otimes[Lgdg(n_y)\otimes Udgg(n_x) \\ +Ldgg(n_y)\otimes Ugdg(n_x)] \end{pmatrix}}^{\Xi_6^{cd}} \bar{\times}\left(ULU\cdot c\bar{d}\right) \\
&+\overbrace{\begin{pmatrix} Ugg(n_z,h_z)\otimes[Ugdg(n_y)\otimes Ldgg(n_x) \\ +Udgg(n_y)\otimes Lgdg(n_x)] \end{pmatrix}}^{\Xi_7^{cd}} \bar{\times}\left(UUL\cdot c\bar{d}\right) \\
&+\overbrace{\begin{pmatrix} Ugg(n_z,h_z)\otimes[Ugdg(n_y)\otimes Udgg(n_x) \\ +Udgg(n_y)\otimes Ugdg(n_x)] \end{pmatrix}}^{\Xi_8^{cd}} \bar{\times}\left(UUU\cdot c\bar{d}\right),
\end{aligned}
\tag{4.63}
$$

$$
\begin{aligned}
M_{d^2} = M_{\bar{d}^{(2)}} = \; & \overbrace{\left[Lgg(n_z,h_z)\otimes Ldgdg(n_y,h_y)\otimes Lgg(n_x,h_x)\right]}^{\Xi_1^d} \bar{\times} \left(LLL\cdot \bar{d}^{(2)}\right) \\
& + \overbrace{\left[Lgg(n_z,h_z)\otimes Ldgdg(n_y,h_y)\otimes Ugg(n_x,h_x)\right]}^{\Xi_2^d} \bar{\times} \left(LLU\cdot \bar{d}^{(2)}\right) \\
& + \overbrace{\left[Lgg(n_z,h_z)\otimes Udgdg(n_y,h_y)\otimes Lgg(n_x,h_x)\right]}^{\Xi_3^d} \bar{\times} \left(LUL\cdot \bar{d}^{(2)}\right) \\
& + \overbrace{\left[Lgg(n_z,h_z)\otimes Udgdg(n_y,h_y)\otimes Ugg(n_x,h_x)\right]}^{\Xi_4^d} \bar{\times} \left(LUU\cdot \bar{d}^{(2)}\right) \\
& + \overbrace{\left[Ugg(n_z,h_z)\otimes Ldgdg(n_y,h_y)\otimes Lgg(n_x,h_x)\right]}^{\Xi_5^d} \bar{\times} \left(ULL\cdot \bar{d}^{(2)}\right) \\
& + \overbrace{\left[Ugg(n_z,h_z)\otimes Ldgdg(n_y,h_y)\otimes Ugg(n_x,h_x)\right]}^{\Xi_6^d} \bar{\times} \left(ULU\cdot \bar{d}^{(2)}\right) \\
& + \overbrace{\left[Ugg(n_z,h_z)\otimes Udgdg(n_y,h_y)\otimes Lgg(n_x,h_x)\right]}^{\Xi_7^d} \bar{\times} \left(UUL\cdot \bar{d}^{(2)}\right) \\
& + \overbrace{\left[Ugg(n_z,h_z)\otimes Udgdg(n_y,h_y)\otimes Ugg(n_x,h_x)\right]}^{\Xi_8^d} \bar{\times} \left(UUU\cdot \bar{d}^{(2)}\right),
\end{aligned}
\tag{4.64}
$$

$$
\begin{aligned}
M_{\rho_z^2} = M_{\bar{\rho_z}^{(2)}} = \; & \overbrace{\left[Lgg(n_z,h_z)\otimes Lgg(n_y,h_y)\otimes Lgg(n_x,h_x)\right]}^{\Xi_1^{\rho z}} \bar{\times} \left(LLL\cdot \bar{\rho_z}^{(2)}\right) \\
& + \overbrace{\left[Lgg(n_z,h_z)\otimes Lgg(n_y,h_y)\otimes Ugg(n_x,h_x)\right]}^{\Xi_2^{\rho z}} \bar{\times} \left(LLU\cdot \bar{\rho_z}^{(2)}\right) \\
& + \overbrace{\left[Lgg(n_z,h_z)\otimes Ugg(n_y,h_y)\otimes Lgg(n_x,h_x)\right]}^{\Xi_3^{\rho z}} \bar{\times} \left(LUL\cdot \bar{\rho_z}^{(2)}\right) \\
& + \overbrace{\left[Lgg(n_z,h_z)\otimes Ugg(n_y,h_y)\otimes Ugg(n_x,h_x)\right]}^{\Xi_4^{\rho z}} \bar{\times} \left(LUU\cdot \bar{\rho_z}^{(2)}\right) \\
& + \overbrace{\left[Ugg(n_z,h_z)\otimes Lgg(n_y,h_y)\otimes Lgg(n_x,h_x)\right]}^{\Xi_5^{\rho z}} \bar{\times} \left(ULL\cdot \bar{\rho_z}^{(2)}\right) \\
& + \overbrace{\left[Ugg(n_z,h_z)\otimes Lgg(n_y,h_y)\otimes Ugg(n_x,h_x)\right]}^{\Xi_6^{\rho z}} \bar{\times} \left(ULU\cdot \bar{\rho_z}^{(2)}\right) \\
& + \overbrace{\left[Ugg(n_z,h_z)\otimes Ugg(n_y,h_y)\otimes Lgg(n_x,h_x)\right]}^{\Xi_7^{\rho z}} \bar{\times} \left(UUL\cdot \bar{\rho_z}^{(2)}\right) \\
& + \overbrace{\left[Ugg(n_z,h_z)\otimes Ugg(n_y,h_y)\otimes Ugg(n_x,h_x)\right]}^{\Xi_8^{\rho z}} \bar{\times} \left(UUU\cdot \bar{\rho_z}^{(2)}\right),
\end{aligned}
\tag{4.65}
$$

$$
\begin{aligned}
M_{\rho^2} = M_{\bar{\rho}^{(2)}} = \; & \overbrace{[Ldgdg(n_z,h_z) \otimes Lgg(n_y,h_y) \otimes Lgg(n_x,h_x)]}^{\Xi_1^{\rho}} \,\bar{\times}\, \left(LLL \cdot \bar{\rho}^{(2)}\right) \\
& + \overbrace{[Ldgdg(n_z,h_z) \otimes Lgg(n_y,h_y) \otimes Ugg(n_x,h_x)]}^{\Xi_2^{\rho}} \,\bar{\times}\, \left(LLU \cdot \bar{\rho}^{(2)}\right) \\
& + \overbrace{[Ldgdg(n_z,h_z) \otimes Ugg(n_y,h_y) \otimes Lgg(n_x,h_x)]}^{\Xi_3^{\rho}} \,\bar{\times}\, \left(LUL \cdot \bar{\rho}^{(2)}\right) \\
& + \overbrace{[Ldgdg(n_z,h_z) \otimes Ugg(n_y,h_y) \otimes Ugg(n_x,h_x)]}^{\Xi_4^{\rho}} \,\bar{\times}\, \left(LUU \cdot \bar{\rho}^{(2)}\right) \\
& + \overbrace{[Udgdg(n_z,h_z) \otimes Lgg(n_y,h_y) \otimes Lgg(n_x,h_x)]}^{\Xi_5^{\rho}} \,\bar{\times}\, \left(ULL \cdot \bar{\rho}^{(2)}\right) \\
& + \overbrace{[Udgdg(n_z,h_z) \otimes Lgg(n_y,h_y) \otimes Ugg(n_x,h_x)]}^{\Xi_6^{\rho}} \,\bar{\times}\, \left(ULU \cdot \bar{\rho}^{(2)}\right) \\
& + \overbrace{[Udgdg(n_z,h_z) \otimes Ugg(n_y,h_y) \otimes Lgg(n_x,h_x)]}^{\Xi_7^{\rho}} \,\bar{\times}\, \left(UUL \cdot \bar{\rho}^{(2)}\right) \\
& + \overbrace{[Udgdg(n_z,h_z) \otimes Ugg(n_y,h_y) \otimes Ugg(n_x,h_x)]}^{\Xi_8^{\rho}} \,\bar{\times}\, \left(UUU \cdot \bar{\rho}^{(2)}\right)
\end{aligned} \tag{4.66}
$$

and

$$
\begin{aligned}
& M_{\rho\rho_z} = M_{\rho\bar{\rho}_z} = \\
& \overbrace{[(Ldgg(n_z) + Lgdg(n_z)) \otimes Lgg(n_y,h_y) \otimes Lgg(n_x,h_x)]}^{\Xi_1^{\rho\rho z}} \,\bar{\times}\, (LLL \cdot \rho\bar{\rho}_z) \\
& + \overbrace{[(Ldgg(n_z) + Lgdg(n_z)) \otimes Lgg(n_y,h_y) \otimes Ugg(n_x,h_x)]}^{\Xi_2^{\rho\rho z}} \,\bar{\times}\, (LLU \cdot \rho\bar{\rho}_z) \\
& + \overbrace{[(Ldgg(n_z) + Lgdg(n_z)) \otimes Ugg(n_y,h_y) \otimes Lgg(n_x,h_x)]}^{\Xi_3^{\rho\rho z}} \,\bar{\times}\, (LUL \cdot \rho\bar{\rho}_z) \\
& + \overbrace{[(Ldgg(n_z) + Lgdg(n_z)) \otimes Ugg(n_y,h_y) \otimes Ugg(n_x,h_x)]}^{\Xi_4^{\rho\rho z}} \,\bar{\times}\, (LUU \cdot \rho\bar{\rho}_z) \\
& + \overbrace{[(Udgg(n_z) + Ugdg(n_z)) \otimes Lgg(n_y,h_y) \otimes Lgg(n_x,h_x)]}^{\Xi_5^{\rho\rho z}} \,\bar{\times}\, (ULL \cdot \rho\bar{\rho}_z) \\
& + \overbrace{[(Udgg(n_z) + Ugdg(n_z)) \otimes Lgg(n_y,h_y) \otimes Ugg(n_x,h_x)]}^{\Xi_6^{\rho\rho z}} \,\bar{\times}\, (ULU \cdot \rho\bar{\rho}_z) \\
& + \overbrace{[(Udgg(n_z) + Ugdg(n_z)) \otimes Ugg(n_y,h_y) \otimes Lgg(n_x,h_x)]}^{\Xi_7^{\rho\rho z}} \,\bar{\times}\, (UUL \cdot \rho\bar{\rho}_z) \\
& + \overbrace{[(Udgg(n_z) + Ugdg(n_z)) \otimes Ugg(n_y,h_y) \otimes Ugg(n_x,h_x)]}^{\Xi_8^{\rho\rho z}} \,\bar{\times}\, (UUU \cdot \rho\bar{\rho}_z) \,.
\end{aligned} \tag{4.67}
$$

4.3.3 Construction of the *"load vector"* B

Let

- $$\bar{ab} = \bar{a}\bar{\times}\bar{b}, \tag{4.68}$$

- $$\bar{ac} = \bar{a}\bar{\times}\bar{c}, \tag{4.69}$$

- $$\bar{ad} = \bar{a}\bar{\times}\bar{d}, \tag{4.70}$$

- $$\bar{e\rho} = \bar{e}\bar{\times}\bar{\rho}^{(2)} \tag{4.71}$$

- and
$$e\bar{\rho}_z = \bar{e}\bar{\times}\rho\bar{\rho}_z. \tag{4.72}$$

The vectors defined in (3.38) are calculated as follows

$$\begin{aligned}
B_{ab} = B_{\bar{ab}} = \ & \overbrace{[Lg(n_z,h_z)\otimes Lg(n_y,h_y)\otimes Lg(n_x,h_x)]}^{\Theta_1^{ab}}\bar{\times}\left(LLL\cdot\bar{ab}\right)\\
&+\overbrace{[Lg(n_z,h_z)\otimes Lg(n_y,h_y)\otimes Ug(n_x,h_x)]}^{\Theta_2^{ab}}\bar{\times}\left(LLU\cdot\bar{ab}\right)\\
&+\overbrace{[Lg(n_z,h_z)\otimes Ug(n_y,h_y)\otimes Lg(n_x,h_x)]}^{\Theta_3^{ab}}\bar{\times}\left(LUL\cdot\bar{ab}\right)\\
&+\overbrace{[Lg(n_z,h_z)\otimes Ug(n_y,h_y)\otimes Ug(n_x,h_x)]}^{\Theta_4^{ab}}\bar{\times}\left(LUU\cdot\bar{ab}\right)\\
&+\overbrace{[Ug(n_z,h_z)\otimes Lg(n_y,h_y)\otimes Lg(n_x,h_x)]}^{\Theta_5^{ab}}\bar{\times}\left(ULL\cdot\bar{ab}\right)\\
&+\overbrace{[Ug(n_z,h_z)\otimes Lg(n_y,h_y)\otimes Ug(n_x,h_x)]}^{\Theta_6^{ab}}\bar{\times}\left(ULU\cdot\bar{ab}\right)\\
&+\overbrace{[Ug(n_z,h_z)\otimes Ug(n_y,h_y)\otimes Lg(n_x,h_x)]}^{\Theta_7^{ab}}\bar{\times}\left(UUL\cdot\bar{ab}\right)\\
&+\overbrace{[Ug(n_z,h_z)\otimes Ug(n_y,h_y)\otimes Ug(n_x,h_x)]}^{\Theta_8^{ab}}\bar{\times}\left(UUU\cdot\bar{ab}\right),
\end{aligned} \tag{4.73}$$

$$
\begin{aligned}
B_{ac} = B_{\bar{a}\bar{c}} = \quad & \overbrace{[Lg(n_z,h_z) \otimes Lg(n_y,h_y) \otimes Ldg(n_x)]}^{\Theta_1^{ac}} \,\bar{\times}\, (LLL \cdot \bar{a}\bar{c}) \\
& + \overbrace{[Lg(n_z,h_z) \otimes Lg(n_y,h_y) \otimes Udg(n_x)]}^{\Theta_2^{ac}} \,\bar{\times}\, (LLU \cdot \bar{a}\bar{c}) \\
& + \overbrace{[Lg(n_z,h_z) \otimes Ug(n_y,h_y) \otimes Ldg(n_x)]}^{\Theta_3^{ac}} \,\bar{\times}\, (LUL \cdot \bar{a}\bar{c}) \\
& + \overbrace{[Lg(n_z,h_z) \otimes Ug(n_y,h_y) \otimes Udg(n_x)]}^{\Theta_4^{ac}} \,\bar{\times}\, (LUU \cdot \bar{a}\bar{c}) \\
& + \overbrace{[Ug(n_z,h_z) \otimes Lg(n_y,h_y) \otimes Ldg(n_x)]}^{\Theta_5^{ac}} \,\bar{\times}\, (ULL \cdot \bar{a}\bar{c}) \\
& + \overbrace{[Ug(n_z,h_z) \otimes Lg(n_y,h_y) \otimes Udg(n_x)]}^{\Theta_6^{ac}} \,\bar{\times}\, (ULU \cdot \bar{a}\bar{c}) \\
& + \overbrace{[Ug(n_z,h_z) \otimes Ug(n_y,h_y) \otimes Ldg(n_x)]}^{\Theta_7^{ac}} \,\bar{\times} (UUL \cdot \bar{a}\bar{c} \\
& + \overbrace{[Ug(n_z,h_z) \otimes Ug(n_y,h_y) \otimes Udg(n_x)]}^{\Theta_8^{ac}} \,\bar{\times}\, (UUU \cdot \bar{a}\bar{c})\,,
\end{aligned}
\tag{4.74}
$$

$$
\begin{aligned}
B_{ad} = B_{\bar{a}\bar{d}} = \quad & \overbrace{[Lg(n_z,h_z) \otimes Ldg(n_y) \otimes Lg(n_x,h_x)]}^{\Theta_1^{ad}} \,\bar{\times}\, \left(LLL \cdot \bar{a}\bar{d}\right) \\
& + \overbrace{[Lg(n_z,h_z) \otimes Ldg(n_y) \otimes Ug(n_x,h_x)]}^{\Theta_2^{ad}} \,\bar{\times}\, \left(LLU \cdot \bar{a}\bar{d}\right) \\
& + \overbrace{[Lg(n_z,h_z) \otimes Udg(n_y) \otimes Lg(n_x,h_x)]}^{\Theta_3^{ad}} \,\bar{\times}\, \left(LUL \cdot \bar{a}\bar{d}\right) \\
& + \overbrace{[Lg(n_z,h_z) \otimes Udg(n_y) \otimes Ug(n_x,h_x)]}^{\Theta_4^{ad}} \,\bar{\times}\, \left(LUU \cdot \bar{a}\bar{d}\right) \\
& + \overbrace{[Ug(n_z,h_z) \otimes Ldg(n_y) \otimes Lg(n_x,h_x)]}^{\Theta_5^{ad}} \,\bar{\times}\, \left(ULL \cdot \bar{a}\bar{d}\right) \\
& + \overbrace{[Ug(n_z,h_z) \otimes Ldg(n_y) \otimes Ug(n_x,h_x)]}^{\Theta_6^{ad}} \,\bar{\times}\, \left(ULU \cdot \bar{a}\bar{d}\right) \\
& + \overbrace{[Ug(n_z,h_z) \otimes Udg(n_y) \otimes Lg(n_x,h_x)]}^{\Theta_7^{ad}} \,\bar{\times}\, \left(UUL \cdot \bar{a}\bar{d}\right) \\
& + \overbrace{[Ug(n_z,h_z) \otimes Udg(n_y) \otimes Ug(n_x,h_x)]}^{\Theta_8^{ad}} \,\bar{\times}\, \left(UUU \cdot \bar{a}\bar{d}\right)\,,
\end{aligned}
\tag{4.75}
$$

$$
\begin{aligned}
B_{e\rho} = B_{\bar{e\rho}} = \quad & \overbrace{[Ldg(n_z) \otimes Lg(n_y,h_y) \otimes Lg(n_x,h_x)]}^{\Theta_1^{e\rho}} \bar{\times} (LLL \cdot \bar{e\rho}) \\
& + \overbrace{[Ldg(n_z) \otimes Lg(n_y,h_y) \otimes Ug(n_x,h_x)]}^{\Theta_2^{e\rho}} \bar{\times} (LLU \cdot \bar{e\rho}) \\
& + \overbrace{[Ldg(n_z) \otimes Ug(n_y,h_y) \otimes Lg(n_x,h_x)]}^{\Theta_3^{e\rho}} \bar{\times} (LUL \cdot \bar{e\rho}) \\
& + \overbrace{[Ldg(n_z) \otimes Ug(n_y,h_y) \otimes Ug(n_x,h_x)]}^{\Theta_4^{e\rho}} \bar{\times} (LUU \cdot \bar{e\rho}) \\
& + \overbrace{[Udg(n_z) \otimes Lg(n_y,h_y) \otimes Lg(n_x,h_x)]}^{\Theta_5^{e\rho}} \bar{\times} (ULL \cdot \bar{e\rho}) \\
& + \overbrace{[Udg(n_z) \otimes Lg(n_y,h_y) \otimes Ug(n_x,h_x)]}^{\Theta_6^{e\rho}} \bar{\times} (ULU \cdot \bar{e\rho}) \\
& + \overbrace{[Udg(n_z) \otimes Ug(n_y,h_y) \otimes Lg(n_x,h_x)]}^{\Theta_7^{e\rho}} \bar{\times} (UUL \cdot \bar{e\rho}) \\
& + \overbrace{[Udg(n_z) \otimes Ug(n_y,h_y) \otimes Ug(n_x,h_x)]}^{\Theta_8^{e\rho}} \bar{\times} (UUU \cdot \bar{e\rho})
\end{aligned}
\tag{4.76}
$$

$$
\begin{aligned}
B_{e\rho_z} = B_{e\bar{\rho}_z} = \quad & \overbrace{[Lg(n_z,h_z) \otimes Lg(n_y,h_y) \otimes Lg(n_x,h_x)]}^{\Theta_1^{e\rho z}} \bar{\times} (LLL \cdot e\bar{\rho}_z) \\
& + \overbrace{[Lg(n_z,h_z) \otimes Lg(n_y,h_y) \otimes Ug(n_x,h_x)]}^{\Theta_2^{e\rho z}} \bar{\times} (LLU \cdot e\bar{\rho}_z) \\
& + \overbrace{[Lg(n_z,h_z) \otimes Ug(n_y,h_y) \otimes Lg(n_x,h_x)]}^{\Theta_3^{e\rho z}} \bar{\times} (LUL \cdot e\bar{\rho}_z) \\
& + \overbrace{[Lg(n_z,h_z) \otimes Ug(n_y,h_y) \otimes Ug(n_x,h_x)]}^{\Theta_4^{e\rho z}} \bar{\times} (LUU \cdot e\bar{\rho}_z) \\
& + \overbrace{[Ug(n_z,h_z) \otimes Lg(n_y,h_y) \otimes Lg(n_x,h_x)]}^{\Theta_5^{e\rho z}} \bar{\times} (ULL \cdot e\bar{\rho}_z) \\
& + \overbrace{[Ug(n_z,h_z) \otimes Lg(n_y,h_y) \otimes Ug(n_x,h_x)]}^{\Theta_6^{e\rho z}} \bar{\times} (ULU \cdot e\bar{\rho}_z) \\
& + \overbrace{[Ug(n_z,h_z) \otimes Ug(n_y,h_y) \otimes Lg(n_x,h_x)]}^{\Theta_7^{e\rho z}} \bar{\times} (UUL \cdot e\bar{\rho}_z) \\
& + \overbrace{[Ug(n_z,h_z) \otimes Ug(n_y,h_y) \otimes Ug(n_x,h_x)]}^{\Theta_8^{e\rho z}} \bar{\times} (UUU \cdot e\bar{\rho}_z)\,,
\end{aligned}
\tag{4.77}
$$

and

$$
\begin{aligned}
B_{w_p} = B_{\bar{w}_p} = \quad & \overbrace{[Lg(n_z,h_z) \otimes Lg(n_y,h_y) \otimes Lg(n_x,h_x)]}^{\Theta_1^{w_p}} \bar{\times} (LLL \cdot \bar{w}_p) \\
& + \overbrace{[Lg(n_z,h_z) \otimes Lg(n_y,h_y) \otimes Ug(n_x,h_x)]}^{\Theta_2^{w_p}} \bar{\times} (LLU \cdot \bar{w}_p) \\
& + \overbrace{[Lg(n_z,h_z) \otimes Ug(n_y,h_y) \otimes Lg(n_x,h_x)]}^{\Theta_3^{w_p}} \bar{\times} (LUL \cdot \bar{w}_p) \\
& + \overbrace{[Lg(n_z,h_z) \otimes Ug(n_y,h_y) \otimes Ug(n_x,h_x)]}^{\Theta_4^{w_p}} \bar{\times} (LUU \cdot \bar{w}_p) \\
& + \overbrace{[Ug(n_z,h_z) \otimes Lg(n_y,h_y) \otimes Lg(n_x,h_x)]}^{\Theta_5^{w_p}} \bar{\times} (ULL \cdot \bar{w}_p) \\
& + \overbrace{[Ug(n_z,h_z) \otimes Lg(n_y,h_y) \otimes Ug(n_x,h_x)]}^{\Theta_6^{w_p}} \bar{\times} (ULU \cdot \bar{w}_p) \\
& + \overbrace{[Ug(n_z,h_z) \otimes Ug(n_y,h_y) \otimes Lg(n_x,h_x)]}^{\Theta_7^{w_p}} \bar{\times} (UUL \cdot \bar{w}_p) \\
& + \overbrace{[Ug(n_z,h_z) \otimes Ug(n_y,h_y) \otimes Ug(n_x,h_x)]}^{\Theta_8^{w_p}} \bar{\times} (UUU \cdot \bar{w}_p) \,.
\end{aligned} \tag{4.78}
$$

CHAPTER V

SENSITIVITY OF THE PIECEWISE LINEAR APPROXIMATION

We already know that w_0^N is represented by the vector $\tilde{C}$. Suppose that $\delta\bar{u} = [\delta u_1, \cdots, \delta u_N]$ and $\delta\bar{v} = [\delta v_1, \cdots, \delta v_N]$ represent the errors in $\bar{u}$ and $\bar{v}$, respectively. This means that $"real"\bar{u} = \bar{u} \pm \delta\bar{u}$ and $"real"\bar{v} = \bar{v} \pm \delta\bar{v}$. We would like to find a relation between $\delta\tilde{C}$ on one hand and $\bar{u}$, $\bar{v}$, $\delta\bar{u}$ and $\delta\bar{v}$ on the other hand if there is any.

In order to do this, let's look at some properties of some of what was previously defined.

5.1 Properties of some of the previously defined matrices

- $$\|A_v\|_m \leq 1 \tag{5.1}$$

- $$\|D_x\|_m \leq \sqrt{\frac{5}{2}}\,\frac{1}{h_x} \tag{5.2}$$

- $$\|D_y\|_m \leq \sqrt{\frac{5}{2}}\,\frac{1}{h_y} \tag{5.3}$$

- $$\|D_z\|_m \leq \sqrt{\frac{5}{2}}\,\frac{1}{h_z} \tag{5.4}$$

- $$\|LLL = SM_1\|_m = \|LLU = SM_2\|_m = \cdots = \|UUU = SM_8\|_m = 1 \tag{5.5}$$

- $$\|SM_i vector\|_{d\infty} = \|vector\|_{d\infty} \text{ for } i = 1, \cdots, 8 \tag{5.6}$$

- $$\left\| \sum_{i=1}^{8} \Xi_i^b \right\|_m \leq h_x h_y h_z \tag{5.7}$$

- $$\left\| \sum_{i=1}^{8} \Xi_i^{bc} \right\|_m \leq h_y h_z \tag{5.8}$$

- $$\left\| \sum_{i=1}^{8} \Xi_i^{bd} \right\|_m \leq h_x h_z \tag{5.9}$$

- $$\left\| \sum_{i=1}^{8} \Xi_i^{cd} \right\|_m \leq 2h_z \tag{5.10}$$

- $$\left\| \sum_{i=1}^{8} \Xi_i^{c} \right\|_m \leq \frac{4h_y h_z}{h_x} \tag{5.11}$$

- $$\left\| \sum_{i=1}^{8} \Xi_i^{d} \right\|_m \leq \frac{4h_x h_z}{h_y} \tag{5.12}$$

- $$\left\| \sum_{i=1}^{8} \Theta_i^{ab} \bar{\times} \left(SM_i \cdot vector\right) \right\|_d \leq h_x h_y h_z \left\|vector\right\|_d \tag{5.13}$$

- $$\left\| \sum_{i=1}^{8} \Theta_i^{ac} \bar{\times} \left(SM_i \cdot vector\right) \right\|_d \leq 2h_y h_z \left\|vector\right\|_d \tag{5.14}$$

- $$\left\| \sum_{i=1}^{8} \Theta_i^{ad} \bar{\times} \left(SM_i \cdot vector\right) \right\|_d \leq 2h_x h_z \left\|vector\right\|_d \tag{5.15}$$

- $$\left\| \sum_{i=1}^{8} \Theta_i^{e\rho} \bar{\times} \left(SM_i \cdot vector\right) \right\|_d \leq 2h_x h_y \left\|vector\right\|_d \tag{5.16}$$

- $$\left\| \sum_{i=1}^{8} \Theta_i^{e\rho_z} \bar{\times} \left(SM_i \cdot vector\right) \right\|_d \leq h_x h_y h_z \left\|vector\right\|_d \tag{5.17}$$

From claims 4.2 and 4.3 and (5.1)...(5.17) we can obtain, easily, the following results

5.2 Bound for the perturbation of the *"stiffness matrix"*

In order to have a bound for the norm of δM, we will proceed as follows

- The error, $\delta\bar{b}^{(2)}$, in $\bar{b}^{(2)}$ is

$$\begin{aligned}
\delta\bar{b}^{(2)} &= \bar{b}^{(2)}(\bar{u}+\delta\bar{u}, \bar{v}+\delta\bar{v}) - \bar{b}^{(2)}(\bar{u}, \bar{v}) \\
&= \bar{b}^{(2)}(\bar{u}+\delta\bar{u}, \bar{v}+\delta\bar{v}) - \bar{b}^{(2)}(\bar{u}, \bar{v}+\delta\bar{v}) + \bar{b}^{(2)}(\bar{u}, \bar{v}+\delta\bar{v}) - \bar{b}^{(2)}(\bar{u}, \bar{v}) \\
&= A_v \left\{D_z[D_x(\bar{v}+\delta\bar{v}) - D_y(\bar{u}+\delta\bar{u})]\right\} \\
&\qquad\qquad \bar{\times} A_v \left\{D_z[D_x(\bar{v}+\delta\bar{v}) - D_y(\bar{u}+\delta\bar{u})]\right\} \\
&\quad - A_v \left\{D_z[D_x(\bar{v}+\delta\bar{v}) - D_y\bar{u}]\right\} \\
&\qquad\qquad \bar{\times} A_v \left\{D_z[D_x(\bar{v}+\delta\bar{v}) - D_y\bar{u}]\right\} \\
&\quad + A_v \left\{D_z[D_x(\bar{v}+\delta\bar{v}) - D_y\bar{u}]\right\} \\
&\qquad\qquad \bar{\times} A_v \left\{D_z[D_x(\bar{v}+\delta\bar{v}) - D_y\bar{u}]\right\} \\
&\quad - A_v \left\{D_z[D_x\bar{v} - D_y\bar{u}]\right\} \bar{\times} A_v \left\{D_z[D_x\bar{v} - D_y\bar{u}]\right\}
\end{aligned}$$

after simplification we obtain

$$\delta\bar{b}^{(2)} = \{A_v D_z[D_x(2\bar{v}+\delta\bar{v}) - D_y(2\bar{u}+\delta\bar{u})]\} \,\bar{\times}\, \{A_v D_z[D_x\delta\bar{v} - D_y\delta\bar{u}]\}$$

so

$$\begin{aligned}\left\|\delta\bar{b}^{(2)}\right\|_d \leq & \left\{\|A_v\|_m \|D_z\|_m \left[\|D_x\|_m \|2\bar{v}+\delta\bar{v}\|_d + \|D_y\|_m \|2\bar{u}+\delta\bar{u}\|_d\right]\right\} * \\ & * \left\{\|A_v\|_m \|D_z\|_m \left[\|D_x\|_m \|\delta\bar{v}\|_d + \|D_y\|_m \|\delta\bar{u}\|_d\right]\right\}\end{aligned}$$

$$\left\|\delta\bar{b}^{(2)}\right\|_d \leq \underbrace{\frac{25}{4h_z^2}\left[\frac{\|2\bar{v}+\delta\bar{v}\|_d}{h_x} + \frac{\|2\bar{u}+\delta\bar{u}\|_d}{h_y}\right]\left[\frac{\|\delta\bar{v}\|_d}{h_x} + \frac{\|\delta\bar{u}\|_d}{h_y}\right]}_{(\delta b^2-estimate)} \tag{5.18}$$

We also have

$$\begin{aligned}\delta M_{\bar{b}^{(2)}} &= M_{\bar{b}^{(2)}}(\bar{u}+\delta\bar{u}, \bar{v}+\delta\bar{v}) - M_{\bar{b}^{(2)}}(\bar{u}, \bar{v}) \\ &= \textstyle\sum_{i=1}^{8}\left[\Xi_i^b \,\bar{\times}\left(SM_i\delta\bar{b}^{(2)}\right)\right]\end{aligned}$$

so

$$\|\delta M_{\bar{b}^{(2)}}\|_m \leq \left\|\sum_i \Xi_i^b\right\|_m \left\|\delta\bar{b}^{(2)}\right\|_{d\infty} \leq h_x h_y h_z \left\|\delta\bar{b}^{(2)}\right\|_d \tag{5.19}$$

and thus

$$\|\delta M_{\bar{b}^{(2)}}\|_m \leq \frac{25h_x h_y}{4h_z}\left[\frac{\|2\bar{v}+\delta\bar{v}\|_d}{h_x} + \frac{\|2\bar{u}+\delta\bar{u}\|_d}{h_y}\right]\left[\frac{\|\delta\bar{v}\|_d}{h_x} + \frac{\|\delta\bar{u}\|_d}{h_y}\right] \tag{5.20}$$

Using the same technique we obtain

- $$\left\|\delta\bar{bc}\right\|_d \leq \underbrace{\frac{5}{2h_z^2}\sqrt{\frac{5}{2}}\left(\begin{array}{c}\left[\frac{\|\bar{v}+\delta\bar{v}\|_d}{h_x} + \frac{\|\bar{u}+\delta\bar{u}\|_d}{h_y}\right]\|\delta\bar{v}\|_d \\ +\left[\frac{\|\delta\bar{v}\|_d}{h_x} + \frac{\|\delta\bar{u}\|_d}{h_y}\right]\|\bar{v}\|_d\end{array}\right)}_{\delta bc-estimate}, \tag{5.21}$$

$$\|\delta M_{\bar{bc}}\|_m \leq \frac{5h_y}{2h_z}\sqrt{\frac{5}{2}}\begin{pmatrix} \left[\frac{\|\bar{v}+\delta\bar{v}\|_d}{h_x}+\frac{\|\bar{u}+\delta\bar{u}\|_d}{h_y}\right]\|\delta\bar{v}\|_d \\ +\left[\frac{\|\delta\bar{v}\|_d}{h_x}+\frac{\|\delta\bar{u}\|_d}{h_y}\right]\|\bar{v}\|_d \end{pmatrix}, \tag{5.22}$$

- $$\left\|\delta\bar{bd}\right\|_d \leq \underbrace{\frac{5}{2h_z^2}\sqrt{\frac{5}{2}}\begin{pmatrix} \left[\frac{\|\bar{v}+\delta\bar{v}\|_d}{h_x}+\frac{\|\bar{u}+\delta\bar{u}\|_d}{h_y}\right]\|\delta\bar{u}\|_d \\ +\left[\frac{\|\delta\bar{v}\|_d}{h_x}+\frac{\|\delta\bar{u}\|_d}{h_y}\right]\|\bar{u}\|_d \end{pmatrix}}_{\delta bd-estimate}, \tag{5.23}$$

$$\|\delta M_{\bar{bd}}\|_m \leq \frac{5h_x}{2h_z}\sqrt{\frac{5}{2}}\begin{pmatrix} \left[\frac{\|\bar{v}+\delta\bar{v}\|_d}{h_x}+\frac{\|\bar{u}+\delta\bar{u}\|_d}{h_y}\right]\|\delta\bar{u}\|_d \\ +\left[\frac{\|\delta\bar{v}\|_d}{h_x}+\frac{\|\delta\bar{u}\|_d}{h_y}\right]\|\bar{u}\|_d \end{pmatrix}, \tag{5.24}$$

- $$\left\|\delta\bar{cd}\right\|_d \leq \underbrace{\frac{5}{2h_z^2}\left[\|\bar{u}\|_d\|\delta\bar{v}\|_d+\|\bar{v}+\delta\bar{v}\|_d\|\delta\bar{u}\|_d\right]}_{\delta cd-estimate}, \tag{5.25}$$

$$\|\delta M_{\bar{cd}}\|_m \leq \frac{5}{h_z}\left[\|\bar{u}\|\,\|\delta\bar{v}\|_d+\|\bar{v}+\delta\bar{v}\|_d\|\delta\bar{u}\|_d\right], \tag{5.26}$$

- $$\left\|\delta\bar{c}^{(2)}\right\|_d \leq \underbrace{\frac{5}{2h_z^2}\|2\bar{v}+\delta\bar{v}\|_d\|\delta\bar{v}\|_d}_{\delta c^2-estimate}, \tag{5.27}$$

$$\|\delta M_{\bar{c}^{(2)}}\|_m \leq \frac{10h_y}{h_xh_z}\|2\bar{v}+\delta\bar{v}\|_d\|\delta\bar{v}\|_d, \tag{5.28}$$

- $$\left\|\delta\bar{d}^{(2)}\right\|_d \leq \underbrace{\frac{5}{2h_z^2}\|2\bar{u}+\delta\bar{u}\|_d\|\delta\bar{u}\|_d}_{\delta d^2-estimate}, \tag{5.29}$$

  and

$$\|\delta M_{\bar{d}^{(2)}}\|_m \leq \frac{10h_x}{h_yh_z}\|2\bar{u}+\delta\bar{u}\|_d\|\delta\bar{u}\|_d. \tag{5.30}$$

Suppose there exists $\mathcal{K}_m \in \Re_+$ that does not depend on the mesh such that

$$\mathcal{K}_m = Max\left\{\|\bar{u}\|_d, \|\bar{v}\|_d, \|\bar{u}+\delta\bar{u}\|_d, \|\bar{v}+\delta\bar{v}\|_d\right\} \tag{5.31}$$

(we can say this because the wind field is bounded, it cannot be more than large value on our planet Earth, and we are not going to consider meshes with number of points less than some given number)

Since

$$\delta M = \alpha_v^2 \left[\delta M_{\bar{b}^{(2)}} + \delta M_{\bar{c}^{(2)}} + \delta M_{\bar{d}^{(2)}} + \delta M_{\bar{bc}} + \delta M_{\bar{bd}} + \delta M_{\bar{cd}}\right] \tag{5.32}$$

then

$$\|\delta M\|_m \leq \frac{5\alpha_v^2 \mathcal{K}_m}{2h_z} \left(\begin{array}{c} \left[\frac{h_x}{h_y}\left(13+\sqrt{10}\right) + \left(5+\sqrt{10}\right)\right] \|\delta \bar{u}\|_d \\ + \left[\frac{h_y}{h_x}\left(13+\sqrt{10}\right) + \left(5+\sqrt{10}\right)\right] \|\delta \bar{v}\|_d \end{array} \right) \tag{5.33}$$

which gives

$$\boxed{\begin{array}{rcl} \|\delta M\|_d & \longrightarrow & 0 \\ \left(\|\delta \bar{u}\|_d, \|\delta \bar{v}\|_d\right) & \longrightarrow & (0,0) \end{array}} \tag{5.34}$$

5.3 Bound for the perturbation of the *"load vector"*

In order to obtain a bound for the norm of δB we proceed as follows.

It is easy to see that

- $$\left\|\delta \bar{b}\right\|_d \leq \underbrace{\frac{5}{2h_z}\left[\frac{\|\delta \bar{v}\|_d}{h_x} + \frac{\|\delta \bar{u}\|_d}{h_y}\right],}_{\delta b-estimate} \tag{5.35}$$

- $$\|\delta \bar{c}\|_d \leq \underbrace{\sqrt{\frac{5}{2}}\frac{1}{h_z}\|\delta \bar{v}\|_d\,,}_{\delta c-estimate} \tag{5.36}$$

- $$\left\|\delta\bar{d}\right\|_d \leq \underbrace{\sqrt{\frac{5}{2}}\frac{1}{h_z}\left\|\delta\bar{u}\right\|_d,}_{\delta d-estimate} \tag{5.37}$$

- $$\left\|\delta\bar{e}\right\|_d \leq \underbrace{\sqrt{\frac{5}{2}}\left[\frac{\left\|\delta\bar{v}\right\|_d}{h_y} + \frac{\left\|\delta\bar{u}\right\|_d}{h_x}\right]}_{\delta e-estimate} \tag{5.38}$$

- and

$$\left\|\delta\bar{a}\right\|_d \leq \underbrace{\left(\begin{array}{c} \frac{\sqrt{5}\|\delta\bar{v_t}\|_d}{\sqrt{2}h_x} + \frac{\sqrt{5}\|\delta\bar{u_t}\|_d}{\sqrt{2}h_y} \\ + \left[\frac{5\|2\bar{v}+\delta\bar{v}\|_d}{h_x h_y} + \frac{5\|\bar{u}\|_d}{h_x^2} + \frac{5\|\bar{u}+\delta\bar{u}\|_d}{h_y^2} + \frac{\sqrt{5}\left\|\bar{f}\right\|_d}{\sqrt{2}h_y}\right]\left\|\delta\bar{v}\right\|_d \\ + \left[\frac{5\|2\bar{u}+\delta\bar{u}\|_d}{h_x h_y} + \frac{5\|\bar{v}\|_d}{h_y^2} + \frac{5\|\bar{v}+\delta\bar{v}\|_d}{h_x^2} + \frac{\sqrt{5}\left\|\bar{f}\right\|_d}{\sqrt{2}h_x}\right]\left\|\delta\bar{u}\right\|_d \\ + \frac{5}{2}\sqrt{\frac{5}{2}}\tau\left[\frac{1}{h_x^2} + \frac{1}{h_y^2} + \frac{1}{h_z^2}\right]\left(\frac{\|\delta\bar{v}\|_d}{h_x} + \frac{\|\delta\bar{u}\|_d}{h_y}\right) \end{array}\right)}_{\delta a-estimate} \tag{5.39}$$

Since $\bar{ab} = \bar{a}\bar{\times}\bar{b}$ then

$$\begin{aligned} \delta\bar{ab} &= \bar{a}(\bar{u}+\delta\bar{u},\bar{v}+\delta\bar{v})\bar{\times}\bar{b}(\bar{u}+\delta\bar{u},\bar{v}+\delta\bar{v}) - \bar{a}(\bar{u},\bar{v})\bar{\times}\bar{b}(\bar{u},\bar{v}) \\ &\quad \pm\bar{a}(\bar{u},\bar{v})\bar{\times}\bar{b}(\bar{u}+\delta\bar{u},\bar{v}+\delta\bar{v}) \\ &= \delta\bar{a}\bar{\times}\bar{b}(\bar{u}+\delta\bar{u},\bar{v}+\delta\bar{v}) + \bar{a}(\bar{u},\bar{v})\bar{\times}\delta\bar{b} \end{aligned}$$

therefore

- $$\left\|\delta\bar{ab}\right\|_d \leq \left\|\delta\bar{a}\right\|_d\left\|\bar{b}(\bar{u}+\delta\bar{u},\bar{v}+\delta\bar{v})\right\|_d + \left\|\bar{a}(\bar{u},\bar{v})\right\|_d\left\|\delta\bar{b}\right\|_d \tag{5.40}$$

  Similarly, we obtain

- $$\left\|\delta\bar{ac}\right\|_d \leq \left\|\delta\bar{a}\right\|_d\left\|\bar{c}(\bar{u}+\delta\bar{u},\bar{v}+\delta\bar{v})\right\|_d + \left\|\bar{a}(\bar{u},\bar{v})\right\|_d\left\|\delta\bar{c}\right\|_d \tag{5.41}$$

- $$\left\|\delta \bar{ad}\right\|_d \leq \left\|\delta\bar{a}\right\|_d \left\|\bar{d}(\bar{u}+\delta\bar{u},\bar{v}+\delta\bar{v})\right\|_d + \left\|\bar{a}(\bar{u},\bar{v})\right\|_d \left\|\delta\bar{d}\right\|_d \tag{5.42}$$

- $$\left\|\delta\bar{e\rho}\right\|_d \leq \left\|\bar{\rho}^{(2)}\right\|_d \left\|\delta\bar{e}\right\|_d \leq \left\|\rho_d\right\|_d^2 \left\|\delta\bar{e}\right\|_d \tag{5.43}$$

- $$\left\|\delta e\bar{\rho}_z\right\|_d \leq \left\|\rho\bar{\rho}_z\right\|_d \left\|\delta\bar{e}\right\|_d \leq \sqrt{\frac{5}{2}}\frac{\left\|\rho_d\right\|_d^2}{h_z} \left\|\delta\bar{e}\right\|_d \tag{5.44}$$

In order to put the latter terms in a "nice" looking form let' s suppose that there exists a positive real function $\eta = \eta(t)$ such that

$$\begin{aligned} \left\|\bar{u}_t\right\|_d &\leq \eta \left\|\bar{u}\right\|_d, \\ \left\|\bar{v}_t\right\|_d &\leq \eta \left\|\bar{v}\right\|_d, \\ \left\|\delta\bar{u}_t\right\|_d &\leq \eta \left\|\delta\bar{u}\right\|_d, \\ \text{and} \quad \left\|\delta\bar{v}_t\right\|_d &\leq \eta \left\|\delta\bar{v}\right\|_d. \end{aligned} \tag{5.45}$$

We obtain

- $$\underbrace{\left\|\bar{a}\right\|_d \leq 5\mathcal{K}_m^2\left(\frac{1}{h_x}+\frac{1}{h_y}\right)^2 + \sqrt{\frac{5}{2}}\mathcal{K}_m\left[\eta + \left\|\bar{f}\right\|_{d\infty} + \frac{5}{2}\tau\left(\frac{1}{h_x^2}+\frac{1}{h_y^2}+\frac{1}{h_z^2}\right)\right]\left(\frac{1}{h_x}+\frac{1}{h_y}\right),}_{This\ inequality\ is\ valid\ for\ both\ \bar{a}(\bar{u},\bar{v})\ and\ \bar{a}(\bar{u}+\delta\bar{u},\bar{v}+\delta\bar{v})} \tag{5.46}$$

$$\left\|\delta\bar{a}\right\|_d \leq \left(\begin{array}{c} \left\{5\mathcal{K}_m\left[\frac{1}{h_x}+\frac{1}{h_y}\right]^2 + \sqrt{\frac{5}{2}}\left[\frac{\eta}{h_y}+\frac{\left\|\bar{f}\right\|_{d\infty}}{h_x}\right]\right\}\left\|\delta\bar{u}\right\|_d \\ +\left\{5\mathcal{K}_m\left[\frac{1}{h_x}+\frac{1}{h_y}\right]^2 + \sqrt{\frac{5}{2}}\left[\frac{\eta}{h_x}+\frac{\left\|\bar{f}\right\|_{d\infty}}{h_y}\right]\right\}\left\|\delta\bar{v}\right\|_d \\ +\frac{5}{2}\sqrt{\frac{5}{2}}\tau\left[\frac{1}{h_x^2}+\frac{1}{h_y^2}+\frac{1}{h_z^2}\right]\left(\frac{\left\|\delta\bar{v}\right\|_d}{h_x}+\frac{\left\|\delta\bar{u}\right\|_d}{h_y}\right) \end{array}\right), \tag{5.47}$$

- $$\underbrace{\left\|\bar{b}\right\|_d \leq \frac{5\mathcal{K}_m}{2h_z}\left[\frac{1}{h_x}+\frac{1}{h_y}\right]}_{This\, inequality\, is\, valid\, for\, both\, \bar{b}(\bar{u},\bar{v})\, and\, \bar{b}(\bar{u}+\delta\bar{u},\bar{v}+\delta\bar{v})}, \tag{5.48}$$

$$\left\|\delta\bar{b}\right\| \leq \frac{5}{2h_z}\left[\frac{\|\delta\bar{u}\|_d}{h_y}+\frac{\|\delta\bar{v}\|_d}{h_x}\right], \tag{5.49}$$

- $$\underbrace{\|\bar{c}\|_d \leq \sqrt{\frac{5}{2}}\frac{\mathcal{K}_m}{h_z}}_{This\, inequality\, is\, valid\, for\, both\, \bar{c}(\bar{u},\bar{v})\, and\, \bar{c}(\bar{u}+\delta\bar{u},\bar{v}+\delta\bar{v})}, \tag{5.50}$$

$$\|\delta\bar{c}\|_d \leq \sqrt{\frac{5}{2}}\frac{\|\delta\bar{v}\|_d}{h_z}, \tag{5.51}$$

- $$\underbrace{\left\|\bar{d}\right\|_d \leq \sqrt{\frac{5}{2}}\frac{\mathcal{K}_m}{h_z}}_{This\, inequality\, is\, valid\, for\, both\, \bar{d}(\bar{u},\bar{v})\, and\, \bar{d}(\bar{u}+\delta\bar{u},\bar{v}+\delta\bar{v})}, \tag{5.52}$$

$$\left\|\delta\bar{d}\right\|_d \leq \sqrt{\frac{5}{2}}\frac{\|\delta\bar{u}\|_d}{h_z} \tag{5.53}$$

  and

- $$\|\delta\bar{e}\|_d \leq \sqrt{\frac{5}{2}}\left[\frac{\|\delta\bar{u}\|_d}{h_x}+\frac{\|\delta\bar{v}\|_d}{h_y}\right]. \tag{5.54}$$

Since

$$\delta B = -\left[\alpha_v^2\left(\delta B_{\bar{a}\bar{b}}+\delta B_{\bar{a}\bar{c}}+\delta B_{\bar{a}\bar{d}}\right)+\alpha_m^2\left(\delta B_{\bar{e}\bar{\rho}}+\delta B_{e\bar{\rho}_z}\right)\right] \tag{5.55}$$

and

$$\delta B_\times = \sum_{i=1}^{8}\Theta_i^\times\bar{\times}\left(SM_i\delta\times\right), \tag{5.56}$$

then using (5.13)...(5.17) we obtain

$$\|\delta B\|_d \leq \begin{pmatrix} \alpha_v^2 \left[h_x h_y h_z \left\|\delta \bar{a}\bar{b}\right\|_d + 2h_y h_z \left\|\delta \bar{a}\bar{c}\right\|_d + 2h_x h_z \left\|\delta \bar{a}\bar{d}\right\|_d\right] \\ +\alpha_m^2 \left[2h_x h_y \left\|\delta \bar{e}\bar{\rho}\right\|_d + h_x h_y h_z \left\|\delta \bar{e}\bar{\rho}_z\right\|_d\right] \end{pmatrix}. \tag{5.57}$$

Now let

$$\Lambda_{0u} = \left(2 + \sqrt{\frac{5}{2}}\right) \begin{pmatrix} 5\alpha_v^2 \mathcal{K}_m \left[\left\|\bar{f}\right\|_{d\infty} \left(1 + \frac{h_x}{2h_y} + \frac{h_y}{2h_x}\right) + \eta \left(1 + \frac{h_x}{h_y}\right)\right] \\ +\alpha_m^2 \sqrt{\frac{5}{2}} h_y \left\|\rho_d\right\|_d^2 \end{pmatrix}, \tag{5.58}$$

$$\Lambda_u = \alpha_v^2 \frac{5(5 + 2\sqrt{10})(h_x + h_y)^2 (2h_x + h_y)}{2h_x^2 h_y^2} \mathcal{K}_m^2, \tag{5.59}$$

$$\Lambda_{\tau u} = \alpha_v^2 \frac{25}{4h_z} \left(\frac{1}{h_x^2} + \frac{1}{h_y^2} + \frac{1}{h_z^2}\right) \left[\frac{1}{h_x} + \frac{3}{h_y} + \sqrt{\frac{5}{2}} \frac{2}{h_y} \left(\frac{1}{h_x} + \frac{1}{h_y}\right)\right] \mathcal{K}_m \tau, \tag{5.60}$$

$$\Lambda_{0v} = \left(2 + \sqrt{\frac{5}{2}}\right) \begin{pmatrix} 5\alpha_v^2 \mathcal{K}_m \left[\left\|\bar{f}\right\|_{d\infty} \left(1 + \frac{h_x}{2h_y} + \frac{h_y}{2h_x}\right) + \eta \left(1 + \frac{h_y}{h_x}\right)\right] \\ +\alpha_m^2 \sqrt{\frac{5}{2}} h_x \left\|\rho_d\right\|_d^2 \end{pmatrix}, \tag{5.61}$$

$$\Lambda_v = \alpha_v^2 \frac{5(5 + 2\sqrt{10})(h_x + h_y)^2 (h_x + 2h_y)}{2h_x^2 h_y^2} \mathcal{K}_m^2, \tag{5.62}$$

and

$$\Lambda_{\tau v} = \alpha_v^2 \frac{25}{4h_z} \left(\frac{1}{h_x^2} + \frac{1}{h_y^2} + \frac{1}{h_z^2}\right) \left[\frac{3}{h_x} + \frac{1}{h_y} + \sqrt{\frac{5}{2}} \frac{2}{h_x} \left(\frac{1}{h_x} + \frac{1}{h_y}\right)\right] \mathcal{K}_m \tau \tag{5.63}$$

We obtain the estimate

$$\|\delta B\|_d \leq (\Lambda_{ou} + \Lambda_u + \Lambda_{\tau u}) \|\delta \bar{u}\|_d + (\Lambda_{ov} + \Lambda_v + \Lambda_{\tau v}) \|\delta \bar{v}\|_d \tag{5.64}$$

which gives

$$\boxed{\begin{array}{rcl} \|\delta B\|_d & \longrightarrow & 0 \\ \left(\|\delta \bar{u}\|_d , \|\delta \bar{v}\|_d\right) & \longrightarrow & (0,0) \end{array}} \tag{5.65}$$

Since in most cases we take

$$h_x = h_y = h \tag{5.66}$$

then we obtain

$$\|\delta M\|_m \leq \frac{5\left(9+\sqrt{10}\right)\alpha_v^2 \mathcal{K}_m}{h_z}\left[\|\delta \bar{u}\|_d + \|\delta \bar{v}\|_d\right], \tag{5.67}$$

$$\Lambda_{0u} = \Lambda_{0v} = \left(2+\sqrt{\frac{5}{2}}\right)\left[10\alpha_v^2 \mathcal{K}_m \left(\|\bar{f}\|_{d\infty} + \eta\right) + \alpha_m^2 \sqrt{\frac{5}{2}} h \|\rho_d\|_d^2\right], \tag{5.68}$$

$$\Lambda_u = \Lambda_v = 30\left(5+2\sqrt{10}\right)\frac{\alpha_v^2 \mathcal{K}_m^2}{h}\ . \tag{5.69}$$

and

$$\Lambda_{\tau u} = \Lambda_{\tau v} = \alpha_v^2 \frac{25}{h_z}\left(\frac{2}{h^2} + \frac{1}{h_z^2}\right)\left(\frac{1}{h} + \sqrt{\frac{5}{2}\frac{1}{h^2}}\right). \tag{5.70}$$

The assumption (3.51) says that

$$\frac{5\alpha_v^2 \mathcal{K}_m}{2h_z}\left(\begin{array}{c} \left[\frac{h_x}{h_y}\left(13+\sqrt{10}\right) + \left(5+\sqrt{10}\right)\right]\|\delta \bar{u}\|_d \\ + \left[\frac{h_y}{h_x}\left(13+\sqrt{10}\right) + \left(5+\sqrt{10}\right)\right]\|\delta \bar{v}\|_d \end{array}\right) < \frac{1}{\lambda} \tag{5.71}$$

and under (5.66) means that

$$\|\delta \bar{u}\|_d + \|\delta \bar{v}\|_d < \frac{h_z}{5\left(9+\sqrt{10}\right)\alpha_v^2 \mathcal{K}_m \lambda} \tag{5.72}$$

5.4 Stability vs mesh finess

As we will see in the computational part, the coarser the mesh the less sensitive w_0^N. To study this relation let' s assume that

$$h_x = h_y = h \ , \ h_z = \kappa h \tag{5.73}$$

and there exsists $\Delta_m \in \Re_+$ that does not depend on the mesh such that

$$\begin{aligned} \|\delta \bar{u}\|_d &\leq \Delta_m \\ and \quad \|\delta \bar{v}\|_d &\leq \Delta_m \end{aligned} \tag{5.74}$$

Then we have

$$\|B\|_d \leq \left(\begin{array}{c} \alpha_v^2 \left[25 \left(2 + \sqrt{\frac{5}{2}} \right) \left(2 + \frac{1}{\kappa^2} \right) \tau \frac{\mathcal{K}_m^2}{h^2} + \left(100 + 40\sqrt{10} \right) \frac{\mathcal{K}_m^3}{h} \right] \\ + \alpha_v^2 \left(20 + 5\sqrt{10} \right) \left(\eta + \left\| \bar{f} \right\|_{d\infty} \right) \mathcal{K}_m^2 + \alpha_m^2 \left(5 + \sqrt{10} \right) h \mathcal{K}_m \left\| \rho_d \right\|_d^2 \end{array} \right), \tag{5.75}$$

$$\|\delta M\|_m \frac{\|B\|_d}{\lambda} \leq \left(\begin{array}{c} \left(\frac{2}{\kappa} + \frac{1}{\kappa^3} \right) \left[125 \left(2 + \sqrt{\frac{5}{2}} \right) \left(9 + \sqrt{10} \right) \tau \mathcal{K}_m^3 \right] \frac{\alpha_v^4}{\lambda h^3} \\ + \frac{1}{\kappa} \left[100 \left(65 + 23\sqrt{10} \right) \mathcal{K}_m^4 \right] \frac{\alpha_v^4}{\lambda h^2} \\ + \frac{1}{\kappa} \left[25 \left(46 + 13\sqrt{10} \right) \left(\eta + \left\| \bar{f} \right\|_{d\infty} \right) \mathcal{K}_m^3 \right] \frac{\alpha_v^4}{\lambda h} \\ + \frac{1}{\kappa} \left[5 \left(55 + 14\sqrt{10} \right) \left\| \rho_d \right\|_d^2 \mathcal{K}_m^2 \right] \frac{\alpha_v^2 \alpha_m^2}{\lambda} \end{array} \right) (2\Delta_m) \tag{5.76}$$

Since the u_ks, the v_ks and the ρ_ks involved in the calculation of some element M_{ij} of the matrix M are measurements of u, v and ρ, respectively, at mesh points that belong to adjacent subvolumes, we can claim, after neglecting the second term of the regularization/well-posedness term, that

$$M_{ij} = \alpha_v^2 \frac{G_v^{ij}(\bar{u}, \bar{v})}{\kappa h} + \alpha_m^2 \frac{G_m^{ij}(\bar{\rho}) h}{\kappa} + \epsilon \sigma^{ij} \left(2\kappa + 1 \right) h^2 \tag{5.77}$$

where σ^{ij}s are known real numbers,

$$G_v^{ij}(\bar{u},\bar{v}) = \sum_{k\in I_u^{ij}} (\sigma_v^{ij})_k u_k^2 + \sum_{k\in I_u^{ij}} \sum_{s\in I_v^{ij}} (\sigma_{u,v}^{ij})_{k,s} u_k v_s + \sum_{s\in I_v^{ij}} (\sigma_v^{ij})_s v_s^2 \tag{5.78}$$

and

$$G_m^{ij}(\bar{\rho}) = \sum_{k\in I_\rho^{ij}} (\sigma_\rho^{ij})_k \rho_k^2 \tag{5.79}$$

with I_u^{ij}, I_v^{ij} and I_ρ^{ij} are sets of positive integers that are not larger than N and $(\sigma_u^{ij})_k$, $(\sigma_v^{ij})_s$, $(\sigma_{u,v}^{ij})_{k,s}$ and $(\sigma_\rho^{ij})_k$ are known real numbers. All of the mentioned sets and real numbers depend on i and j but do not depend on the mesh or the values of u, v, ρ and f.

The stability of w_0^N will depend on whether we are giving more weight to the VE then to the MCE or the opposite. So let' s investigate both cases.

5.4.1 Giving more weight to the vorticity equation

Let' s fix α_v; as we make h larger and larger we obtain from (5.67)

$$\begin{aligned} \|\delta M\|_m &\longrightarrow 0 \\ h &\longrightarrow +\infty \end{aligned} \tag{5.80}$$

and from (5.69) and (5.70) we have

$$\begin{aligned} \Lambda_u = \Lambda_v &\longrightarrow 0 \\ h &\longrightarrow +\infty \end{aligned} \tag{5.81}$$

and

$$\begin{aligned} \Lambda_{\tau u} = \Lambda_{\tau v} &\longrightarrow 0 \\ h &\longrightarrow +\infty \end{aligned} \tag{5.82}$$

Assuming that we take $\alpha_m^2 = \frac{\epsilon_m}{h^{1+\eta_m}}$ and $\epsilon = \frac{\epsilon'_m}{h^{2+\eta'_m}}$ where $\epsilon_m \approx 0$, $\epsilon'_m \approx 0$,

$\eta_m > 0$ and $\eta_m' \geq 0$, and that the values of u_{ij}, v_{ij} and ρ_{ij} do not change much, we obtain, from (5.77), $\lambda \approx \frac{cc_1}{h}$ where cc_1 is some constant.

Thus

$$\begin{array}{rcl} \frac{1}{\lambda} - \|\delta M\|_m & \longrightarrow & +\infty \\ h & \longrightarrow & +\infty \end{array}, \tag{5.83}$$

$\left(\frac{2}{\kappa} + \frac{1}{\kappa^3}\right)\left[125\left(2+\sqrt{\frac{5}{2}}\right)\left(9+\sqrt{10}\right)\tau\mathcal{K}_m^3\right]\frac{\alpha_v^4}{\lambda h^3} \longrightarrow 0$

$\frac{1}{\kappa}\left[100\left(65+23\sqrt{10}\right)\mathcal{K}_m^4\right]\frac{\alpha_v^4}{\lambda h^2} \longrightarrow 0$

and

$\frac{1}{\kappa}\left[5\left(55+14\sqrt{10}\right)\|\rho_d\|_d^2\mathcal{K}_m^2\right]\frac{\alpha_v^2\alpha_m^2}{\lambda} \longrightarrow 0$

and if we neglect the derivatives of u_y and v_x with respect to time and the Coriolis factor, we obtain

$$\Lambda_{ou} = \Lambda_{ov} \approx 0. \tag{5.84}$$

Thus

$$\begin{array}{rcl} \|\delta M\|_m \frac{\|B\|_d}{\lambda} & \longrightarrow & 0 \\ h & \longrightarrow & +\infty \end{array} \tag{5.85}$$

and

$$\begin{array}{rcl} \|\delta B\|_d & \longrightarrow & 0 \\ h & \longrightarrow & +\infty \end{array}. \tag{5.86}$$

Therefore, from (3.53), we conclude that

$$\begin{array}{rcl} \left\|\delta\tilde{C}\right\|_d & \longrightarrow & 0 \\ h & \longrightarrow & +\infty \end{array}. \tag{5.87}$$

5.4.2 Giving more weight to the mass continuity equation

Let' s fix α_m; as we make h smaller and smaller and assuming that we take $\alpha_v^2 = \epsilon_v h^{1.5+\eta_v}$ and $\epsilon = \frac{\epsilon_v'}{h^{2-\eta_v'}}$ where $\epsilon_v \approx 0$, $\epsilon_v' \approx 0$, $\eta_v > 0$ and $\eta_v' \geq 0$, and that the values of u_{ij}, v_{ij} and ρ_{ij} do not change much, we obtain, from (5.77), $\lambda \approx cc_2 h$ where cc_2

is some constant.

$$\begin{array}{rcl} \lambda \left\| \delta M \right\|_m & \longrightarrow & 0 \\ h & \longrightarrow & 0 \end{array}, \tag{5.88}$$

$$\begin{array}{rcl} \lambda \Lambda_u = \lambda \Lambda_v & \longrightarrow & 0 \\ h & \longrightarrow & 0 \end{array} \tag{5.89}$$

$$\begin{array}{rcl} \lambda \Lambda_{u0} = \lambda \Lambda_{v0} & \longrightarrow & 0 \\ h & \longrightarrow & 0 \end{array} \tag{5.90}$$

$$\begin{array}{rcl} \lambda \Lambda_{\tau u} = \lambda \Lambda_{\tau v} & \longrightarrow & 0 \\ h & \longrightarrow & 0 \end{array} \tag{5.91}$$

and

$$\begin{array}{rcl} \left\| \delta M \right\|_m \left\| B \right\|_d & \longrightarrow & 0 \\ h & \longrightarrow & 0 \end{array}. \tag{5.92}$$

Since

$$\left\| \delta C \right\|_d \leq \frac{\left\| \delta M \right\|_m \left\| B \right\|_d + \lambda \left\| \delta B \right\|_d}{1 - \lambda \left\| \delta M \right\|_m} \tag{5.93}$$

then

$$\begin{array}{rcl} \left\| \delta C \right\|_d & \longrightarrow & 0 \\ h & \longrightarrow & 0 \; . \end{array} \tag{5.94}$$

CHAPTER VI

APPROXIMATION BY PRODUCTS OF CUBIC "B-SPLINES"

In the problem formulation we said that it is preferable to approximate the functions u and v by products of piecewise cubic B-splines; so a fair game would be to use the latters in approximating the solution w_0. Again, the presence of u and v in the bilinear form $a(\cdot,\cdot)$ and the linear functional $f(\cdot)$ makes the calculation of the matrix M and the vector B last a very long time in case we do it element by element. This is due to the fact that we have to integrate the products of the splines in each element calculation. The goal of the first part of this chapter is to develop a method that will facilitate and accelerate the computation of M and B by utilizing the kronecker products of matrices whose elememnts are predefined integrals instead of integrating the sums of the products of the splines at each element calculation; and will help us track $\bar{u}$ and $\bar{v}$ in order to study the sensitivity of the solution' s approximation to errors in $\bar{u}$ and $\bar{v}$.

Definition 6.1

Let the function

$$\phi(x) = \begin{cases} \frac{(2-x)^3}{24} - \frac{(1-x)^3}{6} - \frac{x^3}{4} + \frac{(1+x)^3}{6} & , \quad -2 \le x \le -1 \\ \frac{(2-x)^3}{24} - \frac{(1-x)^3}{6} - \frac{x^3}{4} & , \quad -1 \le x \le 0 \\ \frac{(2-x)^3}{24} - \frac{(1-x)^3}{6} & , \quad 0 \le x \le 1 \\ \frac{(2-x)^3}{24} & , \quad 1 \le x \le 2 \\ 0 & , \quad x \notin [-2,2] \end{cases} \tag{6.1}$$

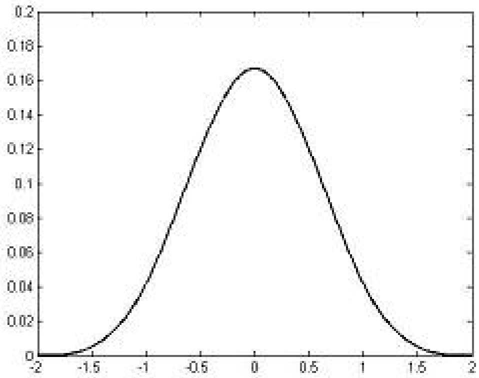

Figure 1: graph of ϕ

In the case of a uniform partition with mesh length h and number of subintervals n, we define the cubic B-splines

$$\phi_i(x) = \phi\left(\frac{x}{h} - i + 2\right) \text{ for } 1 \leq i \leq n+3 \tag{6.2}$$

Definition 6.2

For $1 \leq i \leq n_x + 2$, $1 \leq j \leq n_y + 2$ and $1 \leq k \leq n_z + 2$ let

$$\Phi_I(x, y, z) = \phi_i(x)\phi_j(y)\phi_k(z) \tag{6.3}$$

where $I = i + (j-1) * (n_x + 2) + (k-1) * (n_x + 2)(n_y + 2)$.

Definition 6.3

On the interval $[0, 1]$ let' s define the functions

$$\begin{aligned}
\psi_1(x) &= \tfrac{(4-x)^3}{24} - \tfrac{(3-x)^3}{6} - \tfrac{(-2+x)^3}{4} + \tfrac{(-1+x)^3}{6} = \tfrac{x^3}{24} \\
\psi_2(x) &= \tfrac{(3-x)^3}{24} - \tfrac{(2-x)^3}{6} - \tfrac{(-1+x)^3}{4} \\
\psi_3(x) &= \tfrac{(2-x)^3}{24} - \tfrac{(1-x)^3}{6} \\
\psi_4(x) &= \tfrac{(1-x)^3}{24}
\end{aligned} \tag{6.4}$$

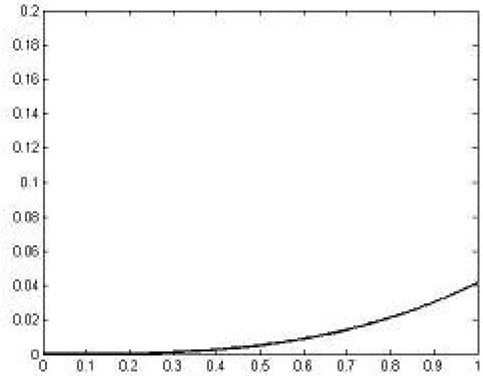

Figure 2: graph of ψ_1

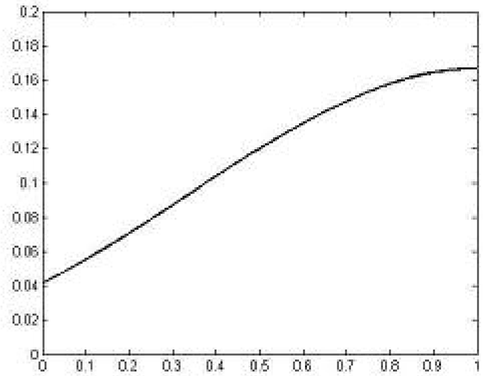

Figure 3: graph of ψ_2

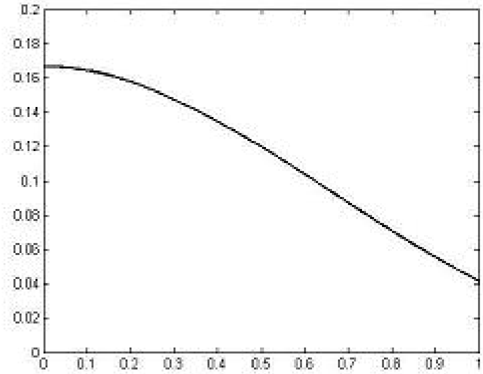

Figure 4: graph of ψ_3

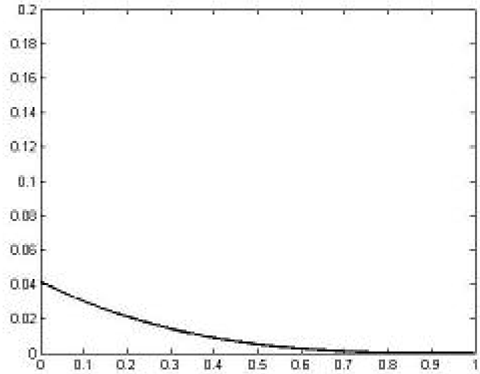

Figure 5: graph of ψ_4

6.1 Definitions

Let' s define the following $4 \times 4 \times 4 \times 4$ matrices

- *gggg* where

$$[gggg]_{klij} = \int_0^1 \psi_k(x)\psi_l(x)\psi_i(x)\psi_j(x)dx \ , \ 1 \leq i, j, k, l \leq 4 \ , \tag{6.5}$$

- *gggdg* where

$$[gggdg]_{klij} = \int_0^1 \psi_k(x)\psi_l(x)\psi_i(x)\psi_j'(x)dx \ , \ 1 \leq i, j, k, l \leq 4 \ , \tag{6.6}$$

- *ggdgg* where

$$[ggdgg]_{klij} = \int_0^1 \psi_k(x)\psi_l(x)\psi_i'(x)\psi_j(x)dx \ , \ 1 \leq i, j, k, l \leq 4 \ , \tag{6.7}$$

- *ggdgdg* where

$$[ggdgdg]_{klij} = \int_0^1 \psi_k(x)\psi_l(x)\psi_i'(x)\psi_j'(x)dx \ , \ 1 \leq i, j, k, l \leq 4 \ , \tag{6.8}$$

- $gdggg$ where

$$[gdggg]_{klij} = \int_0^1 \psi_k(x)\psi_l'(x)\psi_i(x)\psi_j(x)dx \ , \ 1 \le i,j,k,l \le 4 \ , \tag{6.9}$$

- $dgggg$ where

$$[dgggg]_{klij} = \int_0^1 \psi_k'(x)\psi_l(x)\psi_i(x)\psi_j(x)dx \ , \ 1 \le i,j,k,l \le 4 \ , \tag{6.10}$$

- $dgdggg$ where

$$[dgdggg]_{klij} = \int_0^1 \psi_k'(x)\psi_l'(x)\psi_i(x)\psi_j(x)dx \ , \ 1 \le i,j,k,l \le 4 \ , \tag{6.11}$$

- $gdgdgg$ where

$$[gdgdgg]_{klij} = \int_0^1 \psi_k(x)\psi_l'(x)\psi_i'(x)\psi_j(x)dx \ , \ 1 \le i,j,k,l \le 4 \tag{6.12}$$

- $dgggdg$ where

$$[dgggdg]_{klij} = \int_0^1 \psi_k'(x)\psi_l(x)\psi_i(x)\psi_j'(x)dx \ , \ 1 \le i,j,k,l \le 4 \ , \tag{6.13}$$

- $dggdgg$ where

$$[dggdgg]_{klij} = \int_0^1 \psi_k'(x)\psi_l(x)\psi_i'(x)\psi_j(x)dx \ , \ 1 \le i,j,k,l \le 4 \ , \tag{6.14}$$

- $dgdgdgg$ where

$$[dgdgdgg]_{klij} = \int_0^1 \psi_k'(x)\psi_l'(x)\psi_i'(x)\psi_j(x)dx \ , \ 1 \le i,j,k,l \le 4 \ , \tag{6.15}$$

- $dgdggdg$ where

$$[dgdggdg]_{klij} = \int_0^1 \psi_k'(x)\psi_l'(x)\psi_i(x)\psi_j'(x)dx \ , \ 1 \leq i,j,k,l \leq 4 \ , \tag{6.16}$$

- $gdggdg$ where

$$[gdggdg]_{klij} = \int_0^1 \psi_k(x)\psi_l'(x)\psi_i(x)\psi_j'(x)dx \ , \ 1 \leq i,j,k,l \leq 4 \ , \tag{6.17}$$

- gd^2ggg where

$$[gd^2ggg]_{klij} = \int_0^1 \psi_k(x)\psi_l''(x)\psi_i(x)\psi_j(x)dx \ , \ 1 \leq i,j,k,l \leq 4 \ , \tag{6.18}$$

- gd^2ggdg where

$$[gd^2ggdg]_{klij} = \int_0^1 \psi_k(x)\psi_l''(x)\psi_i(x)\psi_j'(x)dx \ , \ 1 \leq i,j,k,l \leq 4 \tag{6.19}$$

- and gd^2gdgg where

$$[gd^2gdgg]_{klij} = \int_0^1 \psi_k(x)\psi_l''(x)\psi_i'(x)\psi_j(x)dx \ , \ 1 \leq i,j,k,l \leq 4 \ . \tag{6.20}$$

And let' s define the following $4 \times 4 \times 4$ matrices

- ggg where

$$[ggg]_{kli} = \int_0^1 \psi_k(x)\psi_l(x)\psi_i(x)dx \ , \ 1 \leq i,k,l \leq 4 \ , \tag{6.21}$$

- $ggdg$ where

$$[ggdg]_{kli} = \int_0^1 \psi_k(x)\psi_l(x)\psi_i'(x)dx \ , \ 1 \leq i,k,l \leq 4 \ , \tag{6.22}$$

- $gdgg$ where

$$[gdgg]_{kli} = \int_0^1 \psi_k(x)\psi_l'(x)\psi_i(x)dx \ , \ 1 \leq i,k,l \leq 4 \ , \tag{6.23}$$

- $dggg$ where

$$[dggg]_{kli} = \int_0^1 \psi_k'(x)\psi_l(x)\psi_i(x)dx \ , \ 1 \leq i,k,l \leq 4 \ , \tag{6.24}$$

- $dggdg$ where

$$[dggdg]_{kli} = \int_0^1 \psi_k'(x)\psi_l(x)\psi_i'(x)dx \ , \ 1 \leq i,k,l \leq 4 \tag{6.25}$$

- $dgdgg$ where

$$[dgdgg]_{kli} = \int_0^1 \psi_k'(x)\psi_l'(x)\psi_i(x)dx \ , \ 1 \leq i,k,l \leq 4 \ , \tag{6.26}$$

- d^2ggg where

$$[d^2ggg]_{kli} = \int_0^1 \psi_k''(x)\psi_l(x)\psi_i(x)dx \ , \ 1 \leq i,k,l \leq 4 \ , \tag{6.27}$$

- d^2gdgg where

$$[d^2gdgg]_{kli} = \int_0^1 \psi_k''(x)\psi_l'(x)\psi_i(x)dx \ , \ 1 \leq i,k,l \leq 4 \ , \tag{6.28}$$

- d^3ggg where

$$[d^3ggg]_{kli} = \int_0^1 \psi_k'''(x)\psi_l(x)\psi_i(x)dx \ , \ 1 \leq i,k,l \leq 4 \ , \tag{6.29}$$

- and d^3gdgg where

$$[d^3gdgg]_{kli} = \int_0^1 \psi_k'''(x)\psi_l'(x)\psi_i(x)dx \ , \ 1 \le i,k,l \le 4 \ . \tag{6.30}$$

Also define the 4×4 matrix gg where

$$[gg]_{kl} = \int_0^1 \psi_k(x)\psi_l(x)dx \tag{6.31}$$

For two given integers k and l with $-3 \le k \le 3$ and $Max\{-3, k-3\} \le l \le Min\{3, k+3\}$, a positive real number h and a positive integer n define the following $n \times n$ matrices

- $GGGG^{(k,l)}(n,h)$ where

$$\left[GGGG^{(k,l)}(n,h)\right]_{ij} = \begin{cases} h\sum_{r=Max\{5,Max\{k+i,l+i,i,j\}+1\}}^{Min\{n+1,Min\{k+i,l+i,i,j\}+4\}} [gggg]_{r-k-i,r-l-i,r-i,r-j} \text{ if} \\ \qquad Max\{1,1-k,1-l\} \le i \le Min\{n,n-k,n-l\} \text{ and} \\ \qquad Max\{1,Max\{k+i,l+i,i\}-3\} \le j \le Min\{n,Min\{k+i,l+i,i\}+3\} \\ \\ 0 \qquad \text{otherwise,} \end{cases} \tag{6.32}$$

the element $\left[GGGG^{(k,l)}(n,h)\right]_{ij} = \int_0^{(n-3)h} \phi_k(x)\phi_l(x)\phi_i(x)\phi_j(x)dx$,

- $GGGdG^{(k,l)}(n,h)$ where

$$\left[GGGdG^{(k,l)}(n,h)\right]_{ij} = \begin{cases} \sum_{r=Max\{5,Max\{k+i,l+i,i,j\}+1\}}^{Min\{n+1,Min\{k+i,l+i,i,j\}+4\}} [gggdg]_{r-k-i,r-l-i,r-i,r-j} \text{ if} \\ \qquad Max\{1,1-k,1-l\} \le i \le Min\{n,n-k,n-l\} \text{ and} \\ \qquad Max\{1,Max\{k+i,l+i,i\}-3\} \le j \le Min\{n,Min\{k+i,l+i,i\}+3\} \\ \\ 0 \qquad \text{otherwise,} \end{cases} \tag{6.33}$$

the element $\left[GGGdG^{(k,l)}(n,h)\right]_{ij} = \int_0^{(n-3)h} \phi_k(x)\phi_l(x)\phi_i(x)\phi_j'(x)dx$,

- $GGdGG^{(k,l)}(n,h)$ where

$$\left[GGdGG^{(k,l)}(n,h)\right]_{ij}=\begin{cases}\sum_{r=Max\{5,Max\{k+i,l+i,i,j\}+1\}}^{Min\{n+1,Min\{k+i,l+i,i,j\}+4\}}[ggdgg]_{r-k-i,r-l-i,r-i,r-j} & \text{if}\\ \quad Max\{1,1-k,1-l\}\leq i\leq Min\{n,n-k,n-l\} \text{ and}\\ \quad Max\{1,Max\{k+i,l+i,i\}-3\}\leq j\leq Min\{n,Min\{k+i,l+i,i\}+3\}\\ \\ 0 & \text{otherwise,}\end{cases} \tag{6.34}$$

the element $\left[GGdGG^{(k,l)}(n,h)\right]_{ij}=\int_0^{(n-3)h}\phi_k(x)\phi_l(x)\phi_i'(x)\phi_j(x)dx$,

- $GGdGdG^{(k,l)}(n,h)$ where

$$\left[GGdGdG^{(k,l)}(n,h)\right]_{ij}=\begin{cases}\frac{1}{h}\sum_{r=Max\{5,Max\{k+i,l+i,i,j\}+1\}}^{Min\{n+1,Min\{k+i,l+i,i,j\}+4\}}[ggdgdg]_{r-k-i,r-l-i,r-i,r-j} & \text{if}\\ \quad Max\{1,1-k,1-l\}\leq i\leq Min\{n,n-k,n-l\} \text{ and}\\ \quad Max\{1,Max\{k+i,l+i,i\}-3\}\leq j\leq Min\{n,Min\{k+i,l+i,i\}+3\}\\ \\ 0 & \text{otherwise,}\end{cases} \tag{6.35}$$

the element $\left[GGdGdG^{(k,l)}(n,h)\right]_{ij}=\int_0^{(n-3)h}\phi_k(x)\phi_l(x)\phi_i'(x)\phi_j'(x)dx$,

- $GdGGG^{(k,l)}(n,h)$ where

$$\left[GdGGG^{(k,l)}(n,h)\right]_{ij}=\begin{cases}\sum_{r=Max\{5,Max\{k+i,l+i,i,j\}+1\}}^{Min\{n+1,Min\{k+i,l+i,i,j\}+4\}}[gdggg]_{r-k-i,r-l-i,r-i,r-j} & \text{if}\\ \quad Max\{1,1-k,1-l\}\leq i\leq Min\{n,n-k,n-l\} \text{ and}\\ \quad Max\{1,Max\{k+i,l+i,i\}-3\}\leq j\leq Min\{n,Min\{k+i,l+i,i\}+3\}\\ \\ 0 & \text{otherwise,}\end{cases} \tag{6.36}$$

the element $\left[GdGGG^{(k,l)}(n,h)\right]_{ij}=\int_0^{(n-3)h}\phi_k(x)\phi_l'(x)\phi_i(x)\phi_j(x)dx$,

- $dGGGG^{(k,l)}(n,h)$ where

$$\left[dGGGG^{(k,l)}(n,h)\right]_{ij}=\begin{cases}\sum_{r=Max\{5,Max\{k+i,l+i,i,j\}+1\}}^{Min\{n+1,Min\{k+i,l+i,i,j\}+4\}}[dgggg]_{r-k-i,r-l-i,r-i,r-j} \text{ if} \\ \quad Max\{1,1-k,1-l\}\leq i\leq Min\{n,n-k,n-l\} \text{ and} \\ \quad Max\{1,Max\{k+i,l+i,i\}-3\}\leq j\leq Min\{n,Min\{k+i,l+i,i\}+3\} \\ 0 \qquad \text{otherwise,}\end{cases} \tag{6.37}$$

the element $\left[dGGGG^{(k,l)}(n,h)\right]_{ij}=\int_0^{(n-3)h}\phi_k'(x)\phi_l(x)\phi_i(x)\phi_j(x)dx$,

- $dGdGGG^{(k,l)}(n,h)$ where

$$\left[dGdGGG^{(k,l)}(n,h)\right]_{ij}=\begin{cases}\frac{1}{h}\sum_{r=Max\{5,Max\{k+i,l+i,i,j\}+1\}}^{Min\{n+1,Min\{k+i,l+i,i,j\}+4\}}[dgdggg]_{r-k-i,r-l-i,r-i,r-j} \text{ if} \\ \quad Max\{1,1-k,1-l\}\leq i\leq Min\{n,n-k,n-l\} \text{ and} \\ \quad Max\{1,Max\{k+i,l+i,i\}-3\}\leq j\leq Min\{n,Min\{k+i,l+i,i\}+3\} \\ 0 \qquad \text{otherwise,}\end{cases} \tag{6.38}$$

the element $\left[dGdGGG^{(k,l)}(n,h)\right]_{ij}=\int_0^{(n-3)h}\phi_k'(x)\phi_l'(x)\phi_i(x)\phi_j(x)dx$,

- $GdGdGG^{(k,l)}(n,h)$ where

$$\left[GdGdGG^{(k,l)}(n,h)\right]_{ij}=\begin{cases}\frac{1}{h}\sum_{r=Max\{5,Max\{k+i,l+i,i,j\}+1\}}^{Min\{n+1,Min\{k+i,l+i,i,j\}+4\}}[gggg]_{r-k-i,r-l-i,r-i,r-j} \text{ if} \\ \quad Max\{1,1-k,1-l\}\leq i\leq Min\{n,n-k,n-l\} \text{ and} \\ \quad Max\{1,Max\{k+i,l+i,i\}-3\}\leq j\leq Min\{n,Min\{k+i,l+i,i\}+3\} \\ 0 \qquad \text{otherwise,}\end{cases} \tag{6.39}$$

the element $\left[GdGdGG^{(k,l)}(n,h)\right]_{ij}=\int_0^{(n-3)h}\phi_k(x)\phi_l'(x)\phi_i'(x)\phi_j(x)dx$,

- $dGGGdG^{(k,l)}(n,h)$ where

$$\left[dGGGdG^{(k,l)}(n,h)\right]_{ij} = \begin{cases} \frac{1}{h}\sum_{r=Max\{5,Max\{k+i,l+i,i,j\}+1\}}^{Min\{n+1,Min\{k+i,l+i,i,j\}+4\}} [dgggdg]_{r-k-i,r-l-i,r-i,r-j} \text{ if} \\ \quad Max\{1,1-k,1-l\} \le i \le Min\{n,n-k,n-l\} \text{ and} \\ \quad Max\{1,Max\{k+i,l+i,i\}-3\} \le j \le Min\{n,Min\{k+i,l+i,i\}+3\} \\ \\ 0 \qquad \text{otherwise,} \end{cases} \tag{6.40}$$

the element $\left[dGGGdG^{(k,l)}(n,h)\right]_{ij} = \int_0^{(n-3)h} \phi_k'(x)\phi_l(x)\phi_i(x)\phi_j'(x)dx,$

Remark 6.1

Each of the above summations is equal to zero when its superscript is less than its subscript. These matrices do not depend on the mesh and if we store these matrices, we will never need to integrate in order to compute M and B.

- $$GG(n,h) = h\begin{bmatrix} \frac{1}{4032} & \frac{43}{26880} & \frac{1}{1344} & \frac{1}{80640} & 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 & 0 \\ \frac{43}{26880} & \frac{151}{10080} & \frac{59}{4480} & \frac{1}{672} & \frac{1}{80640} & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ \frac{1}{1344} & \frac{59}{4480} & \frac{599}{20160} & \frac{397}{26880} & \frac{1}{672} & \frac{1}{80640} & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ \frac{1}{80640} & \frac{1}{672} & \frac{397}{26880} & \frac{151}{5040} & \frac{397}{26880} & \frac{1}{672} & \frac{1}{80640} & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & \frac{1}{80640} & \frac{1}{672} & \frac{397}{26880} & \frac{151}{5040} & \frac{397}{26880} & \frac{1}{672} & \frac{1}{80640} & 0 & \cdot & \cdot & 0 \\ \\ 0 & \cdot & \cdot & 0 & \frac{1}{80640} & \frac{1}{672} & \frac{397}{26880} & \frac{151}{5040} & \frac{397}{26880} & \frac{1}{672} & \frac{1}{80640} & 0 \\ 0 & 0 & \cdot & \cdot & 0 & \frac{1}{80640} & \frac{1}{672} & \frac{397}{26880} & \frac{151}{5040} & \frac{397}{26880} & \frac{1}{672} & \frac{1}{80640} \\ 0 & 0 & \cdot & \cdot & 0 & 0 & \frac{1}{80640} & \frac{1}{672} & \frac{397}{26880} & \frac{599}{20160} & \frac{59}{4480} & \frac{1}{1344} \\ 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & \frac{1}{80640} & \frac{1}{672} & \frac{59}{4480} & \frac{151}{10080} & \frac{43}{26880} \\ 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & 0 & \frac{1}{80640} & \frac{1}{1344} & \frac{43}{26880} & \frac{1}{4032} \end{bmatrix}_{n\times n} \tag{6.41}$$

- $$dGdG(n,h) = \frac{1}{h}\begin{bmatrix} \frac{1}{320} & \frac{7}{1920} & -\frac{1}{160} & -\frac{1}{1920} & 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 & 0 \\ \frac{7}{1920} & \frac{1}{48} & -\frac{11}{960} & -\frac{1}{80} & -\frac{1}{1920} & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ -\frac{1}{160} & -\frac{11}{960} & \frac{37}{960} & -\frac{1}{128} & -\frac{1}{80} & -\frac{1}{1920} & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ -\frac{1}{1920} & -\frac{1}{80} & -\frac{1}{128} & \frac{1}{24} & -\frac{1}{128} & -\frac{1}{80} & -\frac{1}{1920} & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & -\frac{1}{1920} & -\frac{1}{80} & -\frac{1}{128} & \frac{1}{24} & -\frac{1}{128} & -\frac{1}{80} & -\frac{1}{1920} & 0 & \cdot & \cdot & 0 \\ \\ \\ 0 & \cdot & \cdot & 0 & -\frac{1}{1920} & -\frac{1}{80} & -\frac{1}{128} & \frac{1}{24} & -\frac{1}{128} & -\frac{1}{80} & -\frac{1}{1920} & 0 \\ 0 & 0 & \cdot & \cdot & 0 & -\frac{1}{1920} & -\frac{1}{80} & -\frac{1}{128} & \frac{1}{24} & -\frac{1}{128} & -\frac{1}{80} & -\frac{1}{1920} \\ 0 & 0 & \cdot & \cdot & 0 & 0 & -\frac{1}{1920} & -\frac{1}{80} & -\frac{1}{128} & \frac{37}{960} & -\frac{11}{960} & -\frac{1}{160} \\ 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & -\frac{1}{1920} & -\frac{1}{80} & -\frac{11}{960} & \frac{1}{48} & \frac{7}{1920} \\ 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & 0 & -\frac{1}{1920} & -\frac{1}{160} & \frac{7}{1920} & \frac{1}{320} \end{bmatrix}_{n \times n} \tag{6.42}$$

- $$L(n) = \begin{bmatrix} 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 \\ 1 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & 1 & 0 & \cdot & \cdot & \cdot & \cdot & 0 & 0 \\ \\ \\ 0 & 0 & \cdot & \cdot & \cdot & 0 & 1 & 0 & 0 \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 & 1 & 0 \end{bmatrix}_{n \times n}, \tag{6.43}$$

- $$U(n) = \begin{bmatrix} 0 & 1 & 0 & \cdot & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ \\ \\ 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & 1 & 0 \\ 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & 0 & 1 \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 & 0 & 0 \end{bmatrix}_{n \times n}, \tag{6.44}$$

- $$T(n,k) = \begin{cases} L^{|k|}(n) & if \quad k<0 \\ I(n) & if \quad k=0 \text{ where } I(n) \text{ is the } n\times n \text{ identity matrix} \\ U^k(n) & if \quad k>0, \end{cases} \tag{6.45}$$

For three given integers k, l and s with $-3 \le k \le 3$, $Max\{-3, k-3\} \le l \le Min\{3, k+3\}$ and $Max\{-3, k-3, l-3\} \le s \le Min\{3, k+3, l+3\}$, a positive real number h and a positive integer n define the following $n \times 1$ vectors

- $GGGG^{(k,l,s)}(n,h)$ where

$$\left[GGGG^{(k,l,s)}(n,h)\right]_i = \begin{cases} h\sum_{r=Max\{5,Max\{k+i,l+i,s+i,i\}+1\}}^{Min\{n+1,Min\{k+i,l+i,s+i,i\}+4\}}[gggg]_{r-k-i,r-l-i,r-s-i,r-i} \text{ if} \\ \quad Max\{1,1-k,1-l,1-s\} \le i \le Min\{n,n-k,n-l,n-s\} \\ 0 \qquad \text{otherwise,} \end{cases} \tag{6.46}$$

the element $\left[GGGG^{(k,l,s)}(n,h)\right]_i = \int_0^{(n-3)h} \phi_k(x)\phi_l(x)\phi_s(x)\phi_i(x)dx$,

- $GGGdG^{(k,l,s)}(n,h)$ where

$$\left[GGGdG^{(k,l,s)}(n,h)\right]_i = \begin{cases} \sum_{r=Max\{5,Max\{k+i,l+i,s+i,i\}+1\}}^{Min\{n+1,Min\{k+i,l+i,s+i,i\}+4\}}[gggdg]_{r-k-i,r-l-i,r-s-i,r-i} \text{ if} \\ \quad Max\{1,1-k,1-l,1-s\} \le i \le Min\{n,n-k,n-l,n-s\} \\ 0 \qquad \text{otherwise,} \end{cases} \tag{6.47}$$

the element $\left[GGGdG^{(k,l,s)}(n,h)\right]_i = \int_0^{(n-3)h} \phi_k(x)\phi_l(x)\phi_s(x)\phi_i'(x)dx$,

- $GGdGG^{(k,l,s)}(n,h)$ where

$$\left[GGdGG^{(k,l,s)}(n,h)\right]_i = \begin{cases} \sum_{r=Max\{5,Max\{k+i,l+i,s+i,i\}+1\}}^{Min\{n+1,Min\{k+i,l+i,s+i,i\}+4\}}[ggdgg]_{r-k-i,r-l-i,r-s-i,r-i} \text{ if} \\ \quad Max\{1,1-k,1-l,1-s\} \le i \le Min\{n,n-k,n-l,n-s\} \\ 0 \qquad \text{otherwise,} \end{cases} \tag{6.48}$$

the element $\left[GGdGG^{(k,l,s)}(n,h)\right]_i = \int_0^{(n-3)h} \phi_k(x)\phi_l(x)\phi_s'(x)\phi_i(x)dx$,

- $GdGGG^{(k,l,s)}(n,h)$ where

$$\left[GdGGG^{(k,l,s)}(n,h)\right]_i = \begin{cases} \sum_{r=Max\{5,Max\{k+i,l+i,s+i,i\}+1\}}^{Min\{n+1,Min\{k+i,l+i,s+i,i\}+4\}} [gdggg]_{r-k-i,r-l-i,r-s-i,r-i} \text{ if} \\ \quad Max\{1,1-k,1-l,1-s\} \leq i \leq Min\{n,n-k,n-l,n-s\} \\ \\ 0 \qquad \text{otherwise,} \end{cases} \tag{6.49}$$

the element $\left[GdGGG^{(k,l,s)}(n,h)\right]_i = \int_0^{(n-3)h} \phi_k(x)\phi_l'(x)\phi_s(x)\phi_i(x)dx$,

- $dGGGG^{(k,l,s)}(n,h)$ where

$$\left[dGGGG^{(k,l,s)}(n,h)\right]_i = \begin{cases} \sum_{r=Max\{5,Max\{k+i,l+i,s+i,i\}+1\}}^{Min\{n+1,Min\{k+i,l+i,s+i,i\}+4\}} [dgggg]_{r-k-i,r-l-i,r-s-i,r-i} \text{ if} \\ \quad Max\{1,1-k,1-l,1-s\} \leq i \leq Min\{n,n-k,n-l,n-s\} \\ \\ 0 \qquad \text{otherwise,} \end{cases} \tag{6.50}$$

the element $\left[dGGGG^{(k,l,s)}(n,h)\right]_i = \int_0^{(n-3)h} \phi_k'(x)\phi_l(x)\phi_s(x)\phi_i(x)dx$,

- $dGdGGG^{(k,l,s)}(n,h)$ where

$$\left[dGdGGG^{(k,l,s)}(n,h)\right]_i = \begin{cases} \frac{1}{h}\sum_{r=Max\{5,Max\{k+i,l+i,s+i,i\}+1\}}^{Min\{n+1,Min\{k+i,l+i,s+i,i\}+4\}} [dgdggg]_{r-k-i,r-l-i,r-s-i,r-i} \text{ if} \\ \quad Max\{1,1-k,1-l,1-s\} \leq i \leq Min\{n,n-k,n-l,n-s\} \\ \\ 0 \qquad \text{otherwise,} \end{cases} \tag{6.51}$$

the element $\left[dGdGGG^{(k,l,s)}(n,h)\right]_i = \int_0^{(n-3)h} \phi_k'(x)\phi_l'(x)\phi_s(x)\phi_i(x)dx$,

- $GdGdGG^{(k,l,s)}(n,h)$ where

$$\left[GdGdGG^{(k,l,s)}(n,h)\right]_i = \begin{cases} \frac{1}{h}\sum_{r=Max\{5,Max\{k+i,l+i,s+i,i\}+1\}}^{Min\{n+1,Min\{k+i,l+i,s+i,i\}+4\}} [gdgdgg]_{r-k-i,r-l-i,r-s-i,r-i} \text{ if} \\ \quad Max\{1,1-k,1-l,1-s\} \leq i \leq Min\{n,n-k,n-l,n-s\} \\ \\ 0 \qquad \text{otherwise,} \end{cases} \tag{6.52}$$

the element $\left[GdGdGG^{(k,l,s)}(n,h)\right]_i = \int_0^{(n-3)h} \phi_k(x)\phi_l'(x)\phi_s'(x)\phi_i(x)dx$,

- $dGGGdG^{(k,l,s)}(n,h)$ where

$$\left[dGGGdG^{(k,l,s)}(n,h)\right]_i = \begin{cases} \frac{1}{h}\sum_{r=Max\{5,Max\{k+i,l+i,s+i,i\}+1\}}^{Min\{n+1,Min\{k+i,l+i,s+i,i\}+4\}}[dgggdg]_{r-k-i,r-l-i,r-s-i,r-i} \text{ if} \\ \quad Max\{1,1-k,1-l,1-s\} \leq i \leq Min\{n,n-k,n-l,n-s\} \\ \\ 0 \qquad \text{otherwise,} \end{cases} \tag{6.53}$$

the element $\left[dGGGdG^{(k,l,s)}(n,h)\right]_i = \int_0^{(n-3)h} \phi'_k(x)\phi_l(x)\phi_s(x)\phi'_i(x)dx$,

- $dGGdGG^{(k,l,s)}(n,h)$ where

$$\left[dGGdGG^{(k,l,s)}(n,h)\right]_i = \begin{cases} \frac{1}{h}\sum_{r=Max\{5,Max\{k+i,l+i,s+i,i\}+1\}}^{Min\{n+1,Min\{k+i,l+i,s+i,i\}+4\}}[dggdgg]_{r-k-i,r-l-i,r-s-i,r-i} \text{ if} \\ \quad Max\{1,1-k,1-l,1-s\} \leq i \leq Min\{n,n-k,n-l,n-s\} \\ \\ 0 \qquad \text{otherwise,} \end{cases} \tag{6.54}$$

the element $\left[dGGdGG^{(k,l,s)}(n,h)\right]_i = \int_0^{(n-3)h} \phi'_k(x)\phi_l(x)\phi'_s(x)\phi_i(x)dx$,

- $GdGGdG^{(k,l,s)}(n,h)$ where

$$\left[GdGGdG^{(k,l,s)}(n,h)\right]_i = \begin{cases} \frac{1}{h}\sum_{r=Max\{5,Max\{k+i,l+i,s+i,i\}+1\}}^{Min\{n+1,Min\{k+i,l+i,s+i,i\}+4\}}[gdggdg]_{r-k-i,r-l-i,r-s-i,r-i} \text{ if} \\ \quad Max\{1,1-k,1-l,1-s\} \leq i \leq Min\{n,n-k,n-l,n-s\} \\ \\ 0 \qquad \text{otherwise,} \end{cases} \tag{6.55}$$

the element $\left[GdGGdG^{(k,l,s)}(n,h)\right]_i = \int_0^{(n-3)h} \phi_k(x)\phi'_l(x)\phi_s(x)\phi'_i(x)dx$,

- $dGdGdGG^{(k,l,s)}(n,h)$ where

$$\left[dGdGdGG^{(k,l,s)}(n,h)\right]_i = \begin{cases} \frac{1}{h^2}\sum_{r=Max\{5,Max\{k+i,l+i,s+i,i\}+1\}}^{Min\{n+1,Min\{k+i,l+i,s+i,i\}+4\}}[dgdgdgg]_{r-k-i,r-l-i,r-s-i,r-i} \text{ if} \\ \quad Max\{1,1-k,1-l,1-s\} \leq i \leq Min\{n,n-k,n-l,n-s\} \\ \\ 0 \qquad \text{otherwise,} \end{cases} \tag{6.56}$$

the element $\left[dGdGdGG^{(k,l,s)}(n,h)\right]_i = \int_0^{(n-3)h} \phi'_k(x)\phi'_l(x)\phi'_s(x)\phi_i(x)dx$,

- $dGdGGdG^{(k,l,s)}(n,h)$ where

$$\left[dGdGGdG^{(k,l,s)}(n,h)\right]_i = \begin{cases} \frac{1}{h^2}\sum_{r=Max\{5,Max\{k+i,l+i,s+i,i\}+1\}}^{Min\{n+1,Min\{k+i,l+i,s+i,i\}+4\}}[dgdggdg]_{r-k-i,r-l-i,r-s-i,r-i} \text{ if} \\ \quad Max\{1,1-k,1-l,1-s\} \leq i \leq Min\{n,n-k,n-l,n-s\} \\ \\ 0 \qquad \text{otherwise,} \end{cases} \tag{6.57}$$

the element $\left[dGdGGdG^{(k,l,s)}(n,h)\right]_i = \int_0^{(n-3)h} \phi_k'(x)\phi_l'(x)\phi_s(x)\phi_i'(x)dx$,

- $Gd^2GGG^{(k,l,s)}(n,h)$ where

$$\left[Gd^2GGG^{(k,l,s)}(n,h)\right]_i = \begin{cases} \frac{1}{h}\sum_{r=Max\{5,Max\{k+i,l+i,s+i,i\}+1\}}^{Min\{n+1,Min\{k+i,l+i,s+i,i\}+4\}}[gd^2ggg]_{r-k-i,r-l-i,r-s-i,r-i} \text{ if} \\ \quad Max\{1,1-k,1-l,1-s\} \leq i \leq Min\{n,n-k,n-l,n-s\} \\ \\ 0 \qquad \text{otherwise,} \end{cases} \tag{6.58}$$

the element $\left[Gd^2GGG^{(k,l,s)}(n,h)\right]_i = \int_0^{(n-3)h} \phi_k(x)\phi_l''(x)\phi_s(x)\phi_i(x)dx$,

- $Gd^2GGdG^{(k,l,s)}(n,h)$ where

$$\left[Gd^2GGdG^{(k,l,s)}(n,h)\right]_i = \begin{cases} \frac{1}{h^2}\sum_{r=Max\{5,Max\{k+i,l+i,s+i,i\}+1\}}^{Min\{n+1,Min\{k+i,l+i,s+i,i\}+4\}}[gd^2ggdg]_{r-k-i,r-l-i,r-s-i,r-i} \text{ if} \\ \quad Max\{1,1-k,1-l,1-s\} \leq i \leq Min\{n,n-k,n-l,n-s\} \\ \\ 0 \qquad \text{otherwise,} \end{cases} \tag{6.59}$$

the element $\left[Gd^2GGdG^{(k,l,s)}(n,h)\right]_i = \int_0^{(n-3)h} \phi_k(x)\phi_l''(x)\phi_s(x)\phi_i'(x)dx$,

- $Gd^2GdGG^{(k,l,s)}(n,h)$ where

$$\left[Gd^2GdGG^{(k,l,s)}(n,h)\right]_i = \begin{cases} \frac{1}{h^2}\sum_{r=Max\{5,Max\{k+i,l+i,s+i,i\}+1\}}^{Min\{n+1,Min\{k+i,l+i,s+i,i\}+4\}}[gd^2gdgg]_{r-k-i,r-l-i,r-s-i,r-i} \text{ if} \\ \quad Max\{1,1-k,1-l,1-s\} \leq i \leq Min\{n,n-k,n-l,n-s\} \\ \\ 0 \qquad \text{otherwise,} \end{cases} \tag{6.60}$$

the element $\left[Gd^2GdGG^{(k,l,s)}(n,h)\right]_i = \int_0^{(n-3)h} \phi_k(x)\phi_l''(x)\phi_s'(x)\phi_i(x)dx$,

- $GGG^{(k,l)}(n,h)$ where

$$\left[GGG^{(k,l)}(n,h)\right]_i = \begin{cases} h\sum_{r=Max\{5,Max\{k+i,l+i,i\}+1\}}^{Min\{n+1,Min\{k+i,l+i,i\}+4\}}[ggg]_{r-k-i,r-l-i,r-i} \text{ if} \\ \quad Max\{1,1-k,1-l\} \leq i \leq Min\{n,n-k,n-l\} \\ \\ 0 \qquad \text{otherwise,} \end{cases} \tag{6.61}$$

the element $\left[GGG^{(k,l)}(n,h)\right]_i = \int_0^{(n-3)h} \phi_k(x)\phi_l(x)\phi_i(x)dx$,

- $GGdG^{(k,l)}(n,h)$ where

$$\left[GGdG^{(k,l)}(n,h)\right]_i = \begin{cases} \sum_{r=Max\{5,Max\{k+i,l+i,i\}+1\}}^{Min\{n+1,Min\{k+i,l+i,i\}+4\}}[ggdg]_{r-k-i,r-l-i,r-i} \text{ if} \\ \quad Max\{1,1-k,1-l\} \leq i \leq Min\{n,n-k,n-l\} \\ \\ 0 \qquad \text{otherwise,} \end{cases} \tag{6.62}$$

the element $\left[GGdG^{(k,l)}(n,h)\right]_i = \int_0^{(n-3)h} \phi_k(x)\phi_l(x)\phi_i'(x)dx$,

- $GdGG^{(k,l)}(n,h)$ where

$$\left[GdGG^{(k,l)}(n,h)\right]_i = \begin{cases} \sum_{r=Max\{5,Max\{k+i,l+i,i\}+1\}}^{Min\{n+1,Min\{k+i,l+i,i\}+4\}}[gdgg]_{r-k-i,r-l-i,r-i} \text{ if} \\ \quad Max\{1,1-k,1-l\} \leq i \leq Min\{n,n-k,n-l\} \\ \\ 0 \qquad \text{otherwise,} \end{cases} \tag{6.63}$$

the element $\left[GdGG^{(k,l)}(n,h)\right]_i = \int_0^{(n-3)h} \phi_k(x)\phi_l'(x)\phi_i(x)dx$,

- $dGGG^{(k,l)}(n,h)$ where

$$\left[dGGG^{(k,l)}(n,h)\right]_i = \begin{cases} \sum_{r=Max\{5,Max\{k+i,l+i,i\}+1\}}^{Min\{n+1,Min\{k+i,l+i,i\}+4\}}[dggg]_{r-k-i,r-l-i,r-i} \text{ if} \\ \quad Max\{1,1-k,1-l\} \leq i \leq Min\{n,n-k,n-l\} \\ \\ 0 \qquad \text{otherwise,} \end{cases} \tag{6.64}$$

the element $\left[dGGG^{(k,l)}(n,h)\right]_i = \int_0^{(n-3)h} \phi_k'(x)\phi_l(x)\phi_i(x)dx$,

- $dGGdG^{(k,l)}(n,h)$ where

$$\left[dGGdG^{(k,l)}(n,h)\right]_i = \begin{cases} \frac{1}{h}\sum_{r=Max\{5,Max\{k+i,l+i,i\}+1\}}^{Min\{n+1,Min\{k+i,l+i,i\}+4\}}[dggdg]_{r-k-i,r-l-i,r-i} \text{ if} \\ \quad Max\{1,1-k,1-l\} \leq i \leq Min\{n,n-k,n-l\} \\ \\ 0 \qquad \text{otherwise,} \end{cases} \tag{6.65}$$

the element $\left[dGGdG^{(k,l)}(n,h)\right]_i = \int_0^{(n-3)h} \phi_k'(x)\phi_l(x)\phi_i'(x)dx$,

- $dGdGG^{(k,l)}(n,h)$ where

$$\left[dGdGG^{(k,l)}(n,h)\right]_i = \begin{cases} \frac{1}{h}\sum_{r=Max\{5,Max\{k+i,l+i,i\}+1\}}^{Min\{n+1,Min\{k+i,l+i,i\}+4\}}[dgdgg]_{r-k-i,r-l-i,r-i} \text{ if} \\ \quad Max\{1,1-k,1-l\} \leq i \leq Min\{n,n-k,n-l\} \\ \\ 0 \qquad \text{otherwise,} \end{cases} \tag{6.66}$$

the element $\left[dGdGG^{(k,l)}(n,h)\right]_i = \int_0^{(n-3)h} \phi_k'(x)\phi_l'(x)\phi_i(x)dx$.

- $d^2GGG^{(k,l)}(n,h)$ where

$$\left[d^2GGG^{(k,l)}(n,h)\right]_i = \begin{cases} \frac{1}{h}\sum_{r=Max\{5,Max\{k+i,l+i,i\}+1\}}^{Min\{n+1,Min\{k+i,l+i,i\}+4\}}[d^2ggg]_{r-k-i,r-l-i,r-i} \text{ if} \\ \quad Max\{1,1-k,1-l\} \leq i \leq Min\{n,n-k,n-l\} \\ \\ 0 \qquad \text{otherwise,} \end{cases} \tag{6.67}$$

the element $\left[d^2GGG^{(k,l)}(n,h)\right]_i = \int_0^{(n-3)h} \phi_k''(x)\phi_l(x)\phi_i(x)dx$,

- $d^2GdGG^{(k,l)}(n,h)$ where

$$\left[d^2GdGG^{(k,l)}(n,h)\right]_i = \begin{cases} \frac{1}{h^2}\sum_{r=Max\{5,Max\{k+i,l+i,i\}+1\}}^{Min\{n+1,Min\{k+i,l+i,i\}+4\}}[d^2gdgg]_{r-k-i,r-l-i,r-i} \text{ if} \\ \quad Max\{1,1-k,1-l\} \leq i \leq Min\{n,n-k,n-l\} \\ \\ 0 \qquad \text{otherwise,} \end{cases} \tag{6.68}$$

the element $\left[d^2GdGG^{(k,l)}(n,h)\right]_i = \int_0^{(n-3)h} \phi_k''(x)\phi_l'(x)\phi_i(x)dx$,

- $d^3GGG^{(k,l)}(n,h)$ where

$$\left[d^3GGG^{(k,l)}(n,h)\right]_i = \begin{cases} \frac{1}{h^2}\sum_{r=Max\{5,Max\{k+i,l+i,i\}+1\}}^{Min\{n+1,Min\{k+i,l+i,i\}+4\}}[d^3ggg]_{r-k-i,r-l-i,r-i} \text{ if} \\ \quad Max\{1,1-k,1-l\} \leq i \leq Min\{n,n-k,n-l\} \\ \\ 0 \qquad \text{otherwise,} \end{cases} \tag{6.69}$$

the element $\left[d^3GGG^{(k,l)}(n,h)\right]_i = \int_0^{(n-3)h} \phi_k'''(x)\phi_l(x)\phi_i(x)dx$,

- $d^3GdGG^{(k,l)}(n,h)$ where

$$\left[d^3GdGG^{(k,l)}(n,h)\right]_i = \begin{cases} \frac{1}{h^3}\sum_{r=Max\{5,Max\{k+i,l+i,i\}+1\}}^{Min\{n+1,Min\{k+i,l+i,i\}+4\}}[d^3gdgg]_{r-k-i,r-l-i,r-i} \text{ if} \\ \quad Max\{1,1-k,1-l\} \leq i \leq Min\{n,n-k,n-l\} \\ \\ 0 \qquad \text{otherwise,} \end{cases} \tag{6.70}$$

the element $\left[d^3GdGG^{(k,l)}(n,h)\right]_i = \int_0^{(n-3)h} \phi_k'''(x)\phi_l'(x)\phi_i(x)dx$,

- and $GG^{(k)}(n,h)$ where

$$\left[GG^{(k)}(n,h)\right]_i = \begin{cases} h\sum_{r=Max\{5,Max\{k+i,i\}+1\}}^{Min\{n+1,Min\{k+i,i\}+4\}}[gg]_{r-k-i,r-i} \text{ if} \\ \quad Max\{1,1-k\} \leq i \leq Min\{n,n-k\} \\ \\ 0 \qquad \text{otherwise,} \end{cases} \tag{6.71}$$

the element $\left[GG^{(k)}(n,h)\right]_i = \int_0^{(n-3)h} \phi_k(x)\phi_i(x)dx$,

6.2 Construction of the "*stiffness matrix*" M

Let $m_x = n_x + 2$, $m_y = n_y + 2$ and $m_z = n_z + 2$ and suppose that

$$u(x,y,z) = \sum_{I=1}^{N} C_I^u \Phi_I(x,y,z), \tag{6.72}$$

$$v(x,y,z) = \sum_{I=1}^{N} C_I^v \Phi_I(x,y,z), \tag{6.73}$$

$$\frac{du}{dt}(x,y,z) = \sum_{I=1}^{N} C_I^{u_t} \Phi_I(x,y,z), \tag{6.74}$$

$$\frac{dv}{dt}(x,y,z) = \sum_{I=1}^{N} C_I^{v_t} \Phi_I(x,y,z), \tag{6.75}$$

$$\rho(x,y,z) = \sum_{I=1}^{N} C_I^{\rho} \Phi_I(x,y,z), \tag{6.76}$$

the Coriolis factor

$$f(x,y,z) = \sum_{I=1}^{N} C_I^{f} \Phi_I(x,y,z), \tag{6.77}$$

and the predicted vertical wind field

$$w_p(x,y,z) = \sum_{I=1}^{N} C_I^{w_p} \Phi_I(x,y,z). \tag{6.78}$$

(see (6.3))

Let

$$C^u = \begin{bmatrix} C_1^u \\ . \\ . \\ . \\ C_N^u \end{bmatrix}, \quad C^v = \begin{bmatrix} C_1^v \\ . \\ . \\ . \\ C_N^v \end{bmatrix}, \quad C_t^u = \begin{bmatrix} C_1^{u_t} \\ . \\ . \\ . \\ C_N^{u_t} \end{bmatrix}, \quad C_t^v = \begin{bmatrix} C_1^{v_t} \\ . \\ . \\ . \\ C_N^{v_t} \end{bmatrix},$$
$$C^\rho = \begin{bmatrix} C_1^\rho \\ . \\ . \\ . \\ C_N^\rho \end{bmatrix}, \quad C^f = \begin{bmatrix} C_1^f \\ . \\ . \\ . \\ C_N^f \end{bmatrix} \quad \text{and} \quad C^{w_p} = \begin{bmatrix} C_1^{w_p} \\ . \\ . \\ . \\ C_N^{w_p} \end{bmatrix}, \tag{6.79}$$

for $-3 \leq k_x \leq 3$, $Max\{-3, k_x - 3\} \leq l_x \leq Min\{3, k_x + 3\}$, $-3 \leq k_y \leq 3$,

$Max\{-3, k_y-3\} \leq l_y \leq Min\{3, k_y+3\}$, $-3 \leq k_z \leq 3$ and $Max\{-3, k_z-3\} \leq l_z \leq Min\{3, k_z+3\}$ define the following $(m_x m_y m_z) \times (m_x m_y m_z)$ matrices

- $$M_{c^2}^{(k_z,l_z,k_y,l_y,k_x,l_x)} = \left\{ \begin{array}{c} \left[dGdGGG^{(k_z,l_z)}(m_z,h_z) \otimes GGGG^{(k_y,l_y)}(m_y,h_y) \otimes GGdGdG^{(k_x,l_x)}(m_x,h_x)\right] \\ \bar{\times} \\ \left[\begin{array}{c} \{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^v\} \\ \bar{\times} \\ \{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^v\} \end{array} \right] \end{array} \right\} \tag{6.80}$$

- $$M_{d^2}^{(k_z,l_z,k_y,l_y,k_x,l_x)} = \left\{ \begin{array}{c} \left[dGdGGG^{(k_z,l_z)}(m_z,h_z) \otimes GGdGdG^{(k_y,l_y)}(m_y,h_y) \otimes GGGG^{(k_x,l_x)}(m_x,h_x)\right] \\ \bar{\times} \\ \left[\begin{array}{c} \{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^u\} \\ \bar{\times} \\ \{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^u\} \end{array} \right] \end{array} \right\} \tag{6.81}$$

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\begin{aligned}
&M_{b^2}^{(k_z,l_z,k_y,l_y,k_x,l_x)} = \\
&\left\{ \begin{array}{c} \left[dGdGGG^{(k_z,l_z)}(m_z,h_z) \otimes GGGG^{(k_y,l_y)}(m_y,h_y) \otimes dGdGGG^{(k_x,l_x)}(m_x,h_x)\right] \\ \bar{\times} \\ \left[\begin{array}{c} \{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^v\} \\ \bar{\times} \\ \{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^v\} \end{array} \right] \end{array} \right\} \\
&+ \left\{ \begin{array}{c} \left[dGdGGG^{(k_z,l_z)}(m_z,h_z) \otimes dGdGGG^{(k_y,l_y)}(m_y,h_y) \otimes GGGG^{(k_x,l_x)}(m_x,h_x)\right] \\ \bar{\times} \\ \left[\begin{array}{c} \{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^u\} \\ \bar{\times} \\ \{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^u\} \end{array} \right] \end{array} \right\} \\
&- \left\{ \begin{array}{c} \left[dGdGGG^{(k_z,l_z)}(m_z,h_z) \otimes GdGGG^{(k_y,l_y)}(m_y,h_y) \otimes dGGGG^{(k_x,l_x)}(m_x,h_x)\right] \\ \bar{\times} \\ \left[\begin{array}{c} \{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^v\} \\ \bar{\times} \\ \{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^u\} \end{array} \right] \end{array} \right\} \\
&- \left\{ \begin{array}{c} \left[dGdGGG^{(k_z,l_z)}(m_z,h_z) \otimes dGGGG^{(k_y,l_y)}(m_y,h_y) \otimes GdGGG^{(k_x,l_x)}(m_x,h_x)\right] \\ \bar{\times} \\ \left[\begin{array}{c} \{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^u\} \\ \bar{\times} \\ \{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^v\} \end{array} \right] \end{array} \right\}
\end{aligned} \tag{6.82}
$$

- $$M_{bc}^{(k_z,l_z,k_y,l_y,k_x,l_x)} = \left\{ \begin{array}{c} \left[dGdGGG^{(k_z,l_z)}(m_z,h_z) \otimes GGGG^{(k_y,l_y)}(m_y,h_y) \otimes dGGGdG^{(k_x,l_x)}(m_x,h_x) \right] \\ \bar{\times} \\ \left[\begin{array}{c} \{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^v\} \\ \bar{\times} \\ \{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^v\} \end{array} \right] \end{array} \right\}$$
$$+ \left\{ \begin{array}{c} \left[dGdGGG^{(k_z,l_z)}(m_z,h_z) \otimes GGGG^{(k_y,l_y)}(m_y,h_y) \otimes GdGdGG^{(k_x,l_x)}(m_x,h_x) \right] \\ \bar{\times} \\ \left[\begin{array}{c} \{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^v\} \\ \bar{\times} \\ \{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^v\} \end{array} \right] \end{array} \right\}$$
$$- \left\{ \begin{array}{c} \left[dGdGGG^{(k_z,l_z)}(m_z,h_z) \otimes dGGGG^{(k_y,l_y)}(m_y,h_y) \otimes GGGdG^{(k_x,l_x)}(m_x,h_x) \right] \\ \bar{\times} \\ \left[\begin{array}{c} \{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^u\} \\ \bar{\times} \\ \{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^v\} \end{array} \right] \end{array} \right\}$$
$$- \left\{ \begin{array}{c} \left[dGdGGG^{(k_z,l_z)}(m_z,h_z) \otimes GdGGG^{(k_y,l_y)}(m_y,h_y) \otimes GGdGG^{(k_x,l_x)}(m_x,h_x) \right] \\ \bar{\times} \\ \left[\begin{array}{c} \{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^v\} \\ \bar{\times} \\ \{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^u\} \end{array} \right] \end{array} \right\} \tag{6.83}$$

- $$M_{\rho^2}^{(k_z,l_z,k_y,l_y,k_x,l_x)} = \left\{ \begin{array}{c} \left[GGdGdG^{(k_z,l_z)}(m_z,h_z) \otimes GGGG^{(k_y,l_y)}(m_y,h_y) \otimes GGGG^{(k_x,l_x)}(m_x,h_x) \right] \\ \bar{\times} \\ \left[\begin{array}{c} \{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^\rho\} \\ \bar{\times} \\ \{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^\rho\} \end{array} \right] \end{array} \right\} \tag{6.84}$$

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$$M_{bd}^{(k_z,l_z,k_y,l_y,k_x,l_x)} =$$

$$\left\{ \begin{array}{c} \left[dGdGGG^{(k_z,l_z)}(m_z,h_z) \otimes dGGGdG^{(k_y,l_y)}(m_y,h_y) \otimes GGGG^{(k_x,l_x)}(m_x,h_x)\right] \\ \bar{\times} \\ \left[\begin{array}{c} \{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^u\} \\ \bar{\times} \\ \{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^u\} \end{array} \right] \end{array} \right\}$$
$$+ \left\{ \begin{array}{c} \left[dGdGGG^{(k_z,l_z)}(m_z,h_z) \otimes GdGdGG^{(k_y,l_y)}(m_y,h_y) \otimes GGGG^{(k_x,l_x)}(m_x,h_x)\right] \\ \bar{\times} \\ \left[\begin{array}{c} \{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^u\} \\ \bar{\times} \\ \{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^u\} \end{array} \right] \end{array} \right\}$$
$$- \left\{ \begin{array}{c} \left[dGdGGG^{(k_z,l_z)}(m_z,h_z) \otimes GGGdG^{(k_y,l_y)}(m_y,h_y) \otimes dGGGG^{(k_x,l_x)}(m_x,h_x)\right] \\ \bar{\times} \\ \left[\begin{array}{c} \{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^v\} \\ \bar{\times} \\ \{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^u\} \end{array} \right] \end{array} \right\}$$
$$- \left\{ \begin{array}{c} \left[dGdGGG^{(k_z,l_z)}(m_z,h_z) \otimes GGdGG^{(k_y,l_y)}(m_y,h_y) \otimes GdGGG^{(k_x,l_x)}(m_x,h_x)\right] \\ \bar{\times} \\ \left[\begin{array}{c} \{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^u\} \\ \bar{\times} \\ \{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^v\} \end{array} \right] \end{array} \right\} \tag{6.85}$$

-

$$M_{\rho_z^2}^{(k_z,l_z,k_y,l_y,k_x,l_x)} =$$

$$\left\{ \begin{array}{c} \left[dGdGGG^{(k_z,l_z)}(m_z,h_z) \otimes GGGG^{(k_y,l_y)}(m_y,h_y) \otimes GGGG^{(k_x,l_x)}(m_x,h_x)\right] \\ \bar{\times} \\ \left[\begin{array}{c} \{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^\rho\} \\ \bar{\times} \\ \{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^\rho\} \end{array} \right] \end{array} \right\} \tag{6.86}$$

-

$$
M_{cd}^{(k_z,l_z,k_y,l_y,k_x,l_x)} = \\
-\left\{ \begin{array}{c} \left[dGdGGG^{(k_z,l_z)}(m_z,h_z) \otimes GGGdG^{(k_y,l_y)}(m_y,h_y) \otimes GGdGG^{(k_x,l_x)}(m_x,h_x) \right] \\ \bar{\times} \\ \left[\begin{array}{c} \{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^v\} \\ \bar{\times} \\ \{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^u\} \end{array} \right] \end{array} \right\} \\
-\left\{ \begin{array}{c} \left[dGdGGG^{(k_z,l_z)}(m_z,h_z) \otimes GGdGG^{(k_y,l_y)}(m_y,h_y) \otimes GGGdG^{(k_x,l_x)}(m_x,h_x) \right] \\ \bar{\times} \\ \left[\begin{array}{c} \{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^u\} \\ \bar{\times} \\ \{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^v\} \end{array} \right] \end{array} \right\} \tag{6.87}
$$

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$$
M_{\rho\rho_z}^{(k_z,l_z,k_y,l_y,k_x,l_x)} = \\
\left\{ \begin{array}{c} \left[\begin{array}{c} \left(GdGdGG^{(k_z,l_z)}(m_z,h_z) + dGGGdG^{(k_z,l_z)}(m_z,h_z) \right) \\ \otimes \\ GGGG^{(k_y,l_y)}(m_y,h_y) \otimes GGGG^{(k_x,l_x)}(m_x,h_x) \end{array} \right] \\ \bar{\times} \\ \left[\begin{array}{c} \{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^\rho\} \\ \bar{\times} \\ \{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^\rho\} \end{array} \right] \end{array} \right\} \tag{6.88}
$$

From definitions (6.80)...(6.88) we obtain

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$$
M_{b^2} = \sum_{k_z=-3}^{3} \sum_{l_z=Max\{-3,k_z-3\}}^{Min\{3,k_z+3\}} \sum_{k_y=-3}^{3} \sum_{l_y=Max\{-3,k_y-3\}}^{Min\{3,k_y+3\}} \sum_{k_x=-3}^{3} \sum_{l_x=Max\{-3,k_x-3\}}^{Min\{3,k_x+3\}} M_{b^2}^{(k_z,l_z,k_y,l_y,k_x,l_x)} \tag{6.89}
$$

-

$$
M_{bc} = \sum_{k_z=-3}^{3} \sum_{l_z=Max\{-3,k_z-3\}}^{Min\{3,k_z+3\}} \sum_{k_y=-3}^{3} \sum_{l_y=Max\{-3,k_y-3\}}^{Min\{3,k_y+3\}} \sum_{k_x=-3}^{3} \sum_{l_x=Max\{-3,k_x-3\}}^{Min\{3,k_x+3\}} M_{bc}^{(k_z,l_z,k_y,l_y,k_x,l_x)} \tag{6.90}
$$

- $$M_{bd} = \sum_{k_z=-3}^{3} \sum_{l_z=Max\{-3,k_z-3\}}^{Min\{3,k_z+3\}} \sum_{k_y=-3}^{3} \sum_{l_y=Max\{-3,k_y-3\}}^{Min\{3,k_y+3\}} \sum_{k_x=-3}^{3} \sum_{l_x=Max\{-3,k_x-3\}}^{Min\{3,k_x+3\}} M_{bd}^{(k_z,l_z,k_y,l_y,k_x,l_x)} \tag{6.91}$$

- $$M_{c^2} = \sum_{k_z=-3}^{3} \sum_{l_z=Max\{-3,k_z-3\}}^{Min\{3,k_z+3\}} \sum_{k_y=-3}^{3} \sum_{l_y=Max\{-3,k_y-3\}}^{Min\{3,k_y+3\}} \sum_{k_x=-3}^{3} \sum_{l_x=Max\{-3,k_x-3\}}^{Min\{3,k_x+3\}} M_{c^2}^{(k_z,l_z,k_y,l_y,k_x,l_x)} \tag{6.92}$$

- $$M_{d^2} = \sum_{k_z=-3}^{3} \sum_{l_z=Max\{-3,k_z-3\}}^{Min\{3,k_z+3\}} \sum_{k_y=-3}^{3} \sum_{l_y=Max\{-3,k_y-3\}}^{Min\{3,k_y+3\}} \sum_{k_x=-3}^{3} \sum_{l_x=Max\{-3,k_x-3\}}^{Min\{3,k_x+3\}} M_{d^2}^{(k_z,l_z,k_y,l_y,k_x,l_x)} \tag{6.93}$$

- $$M_{cd} = \sum_{k_z=-3}^{3} \sum_{l_z=Max\{-3,k_z-3\}}^{Min\{3,k_z+3\}} \sum_{k_y=-3}^{3} \sum_{l_y=Max\{-3,k_y-3\}}^{Min\{3,k_y+3\}} \sum_{k_x=-3}^{3} \sum_{l_x=Max\{-3,k_x-3\}}^{Min\{3,k_x+3\}} M_{cd}^{(k_z,l_z,k_y,l_y,k_x,l_x)} \tag{6.94}$$

- $$M_{\rho^2} = \sum_{k_z=-3}^{3} \sum_{l_z=Max\{-3,k_z-3\}}^{Min\{3,k_z+3\}} \sum_{k_y=-3}^{3} \sum_{l_y=Max\{-3,k_y-3\}}^{Min\{3,k_y+3\}} \sum_{k_x=-3}^{3} \sum_{l_x=Max\{-3,k_x-3\}}^{Min\{3,k_x+3\}} M_{\rho^2}^{(k_z,l_z,k_y,l_y,k_x,l_x)} \tag{6.95}$$

- $$M_{\rho_z^2} = \sum_{k_z=-3}^{3} \sum_{l_z=Max\{-3,k_z-3\}}^{Min\{3,k_z+3\}} \sum_{k_y=-3}^{3} \sum_{l_y=Max\{-3,k_y-3\}}^{Min\{3,k_y+3\}} \sum_{k_x=-3}^{3} \sum_{l_x=Max\{-3,k_x-3\}}^{Min\{3,k_x+3\}} M_{\rho_z^2}^{(k_z,l_z,k_y,l_y,k_x,l_x)} \tag{6.96}$$

- $$M_{\rho\rho_z} = \sum_{k_z=-3}^{3} \sum_{l_z=Max\{-3,k_z-3\}}^{Min\{3,k_z+3\}} \sum_{k_y=-3}^{3} \sum_{l_y=Max\{-3,k_y-3\}}^{Min\{3,k_y+3\}} \sum_{k_x=-3}^{3} \sum_{l_x=Max\{-3,k_x-3\}}^{Min\{3,k_x+3\}} M_{\rho\rho_z}^{(k_z,l_z,k_y,l_y,k_x,l_x)} \tag{6.97}$$

- $$\begin{aligned} G = \quad & GG(m_z,h_z) \otimes GG(m_y,h_y) \otimes dGdG(m_x,h_x) \\ & +GG(m_z,h_z) \otimes dGdG(m_y,h_y) \otimes GG(m_x,h_x) \\ & +dGdG(m_z,h_z) \otimes GG(m_y,h_y) \otimes GG(m_x,h_x), \end{aligned} \tag{6.98}$$

and

$$Co = GG(m_z,h_z)\otimes GG(m_y,h_y)\otimes GG(m_x,h_x). \tag{6.99}$$

6.3 Construction of the *"load vector"* B

Again, let $m_x = n_x+2$, $m_y = n_y+2$ and $m_z = n_z+2$, for $-3 \le k_x \le 3$, $Max\{-3, k_x-3\} \le l_x \le Min\{3, k_x+3\}$, $Max\{-3, k_x-3, l_x-3\} \le s_x \le Min\{3, k_x+3, l_x+3\}$, $-3 \le k_y \le 3$, $Max\{-3, k_y-3\} \le l_y \le Min\{3, k_y+3\}$, $Max\{-3, k_y-3, l_y-3\} \le s_y \le Min\{3, k_y+3, l_y+3\}$, $-3 \le k_z \le 3$, $Max\{-3, k_z-3\} \le l_z \le Min\{3, k_z+3\}$ and $Max\{-3, k_z-3, l_z-3\} \le s_z \le Min\{3, k_z+3, l_z+3\}$ define the following $(m_x m_y m_z)\times 1$ vectors. To simplify the writing we will write $GGG^{(k_z,l_z)}$ instead of $GGG^{(k_z,l_z)}(m_z,h_z)$...

-

$$\begin{aligned}
&B_{ab(1)}^{(k_z,l_z,s_z,k_y,l_y,s_y,k_x,l_x,s_x)} = \\
&\left\{ \begin{array}{c} \left[GdGG^{(k_z,l_z)}\otimes GGG^{(k_y,l_y)}\otimes dGdGG^{(k_x,l_x)}\right] \\ \bar{\times}\left[\begin{array}{c} \{(T(m_z,k_z)\otimes T(m_y,k_y)\otimes T(m_x,k_x))\cdot C_t^v\} \\ \bar{\times}\,\{(T(m_z,l_z)\otimes T(m_y,l_y)\otimes T(m_x,l_x))\cdot C^v\} \end{array}\right] \end{array} \right\} \\
&- \left\{ \begin{array}{c} \left[GdGG^{(k_z,l_z)}\otimes GdGG^{(k_y,l_y)}\otimes dGGG^{(k_x,l_x)}\right] \\ \bar{\times}\left[\begin{array}{c} \{(T(m_z,k_z)\otimes T(m_y,k_y)\otimes T(m_x,k_x))\cdot C_t^v\} \\ \bar{\times}\,\{(T(m_z,l_z)\otimes T(m_y,l_y)\otimes T(m_x,l_x))\cdot C^u\} \end{array}\right] \end{array} \right\} \\
&- \left\{ \begin{array}{c} \left[GdGG^{(k_z,l_z)}\otimes dGGG^{(k_y,l_y)}\otimes GdGG^{(k_x,l_x)}\right] \\ \bar{\times}\left[\begin{array}{c} \{(T(m_z,k_z)\otimes T(m_y,k_y)\otimes T(m_x,k_x))\cdot C_t^u\} \\ \bar{\times}\,\{(T(m_z,l_z)\otimes T(m_y,l_y)\otimes T(m_x,l_x))\cdot C^v\} \end{array}\right] \end{array} \right\} \\
&+ \left\{ \begin{array}{c} \left[GdGG^{(k_z,l_z)}\otimes dGdGG^{(k_y,l_y)}\otimes GGG^{(k_x,l_x)}\right] \\ \bar{\times}\left[\begin{array}{c} \{(T(m_z,k_z)\otimes T(m_y,k_y)\otimes T(m_x,k_x))\cdot C_t^u\} \\ \bar{\times}\,\{(T(m_z,l_z)\otimes T(m_y,l_y)\otimes T(m_x,l_x))\cdot C^u\} \end{array}\right] \end{array} \right\}
\end{aligned} \tag{6.100}$$

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$$
\begin{aligned}
&B_{ab(2)}^{(k_z,l_z,s_z,k_y,l_y,s_y,k_x,l_x,s_x)} = \\
&\left\{ \begin{array}{c}
\left[GGdGG^{(k_z,l_z,s_z)} \otimes GGGG^{(k_y,l_y,s_y)} \otimes Gd^2GdGG^{(k_x,l_x,s_x)} \right] \\
\bar{\times} \\
\left[\begin{array}{c}
\{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^u\} \\
\bar{\times} \{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^v\} \\
\bar{\times} \{(T(m_z,s_z) \otimes T(m_y,s_y) \otimes T(m_x,s_x)) \cdot C^v\}
\end{array} \right]
\end{array} \right\} \\
&- \left\{ \begin{array}{c}
\left[GGdGG^{(k_z,l_z,s_z)} \otimes GGdGG^{(k_y,l_y,s_y)} \otimes Gd^2GGG^{(k_x,l_x,s_x)} \right] \\
\bar{\times} \\
\left[\begin{array}{c}
\{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^u\} \\
\bar{\times} \{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^v\} \\
\bar{\times} \{(T(m_z,s_z) \otimes T(m_y,s_y) \otimes T(m_x,s_x)) \cdot C^u\}
\end{array} \right]
\end{array} \right\} \\
&- \left\{ \begin{array}{c}
\left[GGdGG^{(k_z,l_z,s_z)} \otimes GdGGG^{(k_y,l_y,s_y)} \otimes GdGdGG^{(k_x,l_x,s_x)} \right] \\
\bar{\times} \\
\left[\begin{array}{c}
\{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^u\} \\
\bar{\times} \{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^u\} \\
\bar{\times} \{(T(m_z,s_z) \otimes T(m_y,s_y) \otimes T(m_x,s_x)) \cdot C^v\}
\end{array} \right]
\end{array} \right\} \\
&+ \left\{ \begin{array}{c}
\left[GGdGG^{(k_z,l_z,s_z)} \otimes GdGdGG^{(k_y,l_y,s_y)} \otimes GdGGG^{(k_x,l_x,s_x)} \right] \\
\bar{\times} \\
\left[\begin{array}{c}
\{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^u\} \\
\bar{\times} \{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^u\} \\
\bar{\times} \{(T(m_z,s_z) \otimes T(m_y,s_y) \otimes T(m_x,s_x)) \cdot C^u\}
\end{array} \right]
\end{array} \right\}
\end{aligned}
\tag{6.101}
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\begin{aligned}
&B_{ab(3)}^{(k_z,l_z,s_z,k_y,l_y,s_y,k_x,l_x,s_x)} = \\
&\left\{ \begin{array}{c}
\left[GGdGG^{(k_z,l_z,s_z)} \otimes GdGGG^{(k_y,l_y,s_y)} \otimes GdGdGG^{(k_x,l_x,s_x)} \right] \\
\bar{\times} \\
\left[\begin{array}{c}
\{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^v\} \\
\bar{\times} \{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^v\} \\
\bar{\times} \{(T(m_z,s_z) \otimes T(m_y,s_y) \otimes T(m_x,s_x)) \cdot C^v\}
\end{array} \right]
\end{array} \right\} \\
&- \left\{ \begin{array}{c}
\left[GGdGG^{(k_z,l_z,s_z)} \otimes GdGdGG^{(k_y,l_y,s_y)} \otimes GdGGG^{(k_x,l_x,s_x)} \right] \\
\bar{\times} \\
\left[\begin{array}{c}
\{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^v\} \\
\bar{\times} \{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^v\} \\
\bar{\times} \{(T(m_z,s_z) \otimes T(m_y,s_y) \otimes T(m_x,s_x)) \cdot C^u\}
\end{array} \right]
\end{array} \right\} \\
&- \left\{ \begin{array}{c}
\left[GGdGG^{(k_z,l_z,s_z)} \otimes Gd^2GGG^{(k_y,l_y,s_y)} \otimes GGdGG^{(k_x,l_x,s_x)} \right] \\
\bar{\times} \\
\left[\begin{array}{c}
\{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^v\} \\
\bar{\times} \{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^u\} \\
\bar{\times} \{(T(m_z,s_z) \otimes T(m_y,s_y) \otimes T(m_x,s_x)) \cdot C^v\}
\end{array} \right]
\end{array} \right\} \\
&+ \left\{ \begin{array}{c}
\left[GGdGG^{(k_z,l_z,s_z)} \otimes Gd^2GdGG^{(k_y,l_y,s_y)} \otimes GGGG^{(k_x,l_x,s_x)} \right] \\
\bar{\times} \\
\left[\begin{array}{c}
\{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^v\} \\
\bar{\times} \{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^u\} \\
\bar{\times} \{(T(m_z,s_z) \otimes T(m_y,s_y) \otimes T(m_x,s_x)) \cdot C^u\}
\end{array} \right]
\end{array} \right\}
\end{aligned}
\tag{6.102}
$$

- $$
\begin{aligned}
&B_{ab(4)}^{(k_z,l_z,s_z,k_y,l_y,s_y,k_x,l_x,s_x)} = \\
&\left\{ \begin{array}{c}
\left[GGdGG^{(k_z,l_z,s_z)} \otimes GGGG^{(k_y,l_y,s_y)} \otimes dGdGdGG^{(k_x,l_x,s_x)} \right] \\
\bar{\times} \\
\left[\begin{array}{c}
\{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^u\} \\
\bar{\times} \{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^v\} \\
\bar{\times} \{(T(m_z,s_z) \otimes T(m_y,s_y) \otimes T(m_x,s_x)) \cdot C^v\}
\end{array} \right]
\end{array} \right\} \\
&- \left\{ \begin{array}{c}
\left[GGdGG^{(k_z,l_z,s_z)} \otimes GGdGG^{(k_y,l_y,s_y)} \otimes dGdGGG^{(k_x,l_x,s_x)} \right] \\
\bar{\times} \\
\left[\begin{array}{c}
\{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^u\} \\
\bar{\times} \{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^v\} \\
\bar{\times} \{(T(m_z,s_z) \otimes T(m_y,s_y) \otimes T(m_x,s_x)) \cdot C^u\}
\end{array} \right]
\end{array} \right\} \\
&- \left\{ \begin{array}{c}
\left[GGdGG^{(k_z,l_z,s_z)} \otimes GdGGG^{(k_y,l_y,s_y)} \otimes dGGdGG^{(k_x,l_x,s_x)} \right] \\
\bar{\times} \\
\left[\begin{array}{c}
\{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^u\} \\
\bar{\times} \{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^u\} \\
\bar{\times} \{(T(m_z,s_z) \otimes T(m_y,s_y) \otimes T(m_x,s_x)) \cdot C^v\}
\end{array} \right]
\end{array} \right\} \\
&+ \left\{ \begin{array}{c}
\left[GGdGG^{(k_z,l_z,s_z)} \otimes GdGdGG^{(k_y,l_y,s_y)} \otimes dGGGG^{(k_x,l_x,s_x)} \right] \\
\bar{\times} \\
\left[\begin{array}{c}
\{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^u\} \\
\bar{\times} \{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^u\} \\
\bar{\times} \{(T(m_z,s_z) \otimes T(m_y,s_y) \otimes T(m_x,s_x)) \cdot C^u\}
\end{array} \right]
\end{array} \right\} \\
&+ \left\{ \begin{array}{c}
\left[GGdGG^{(k_z,l_z,s_z)} \otimes GGGG^{(k_y,l_y,s_y)} \otimes dGGdGG^{(k_x,l_x,s_x)} \right] \\
\bar{\times} \\
\left[\begin{array}{c}
\{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^u\} \\
\bar{\times} \left\{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^f\right\} \\
\bar{\times} \{(T(m_z,s_z) \otimes T(m_y,s_y) \otimes T(m_x,s_x)) \cdot C^v\}
\end{array} \right]
\end{array} \right\} \\
&- \left\{ \begin{array}{c}
\left[GGdGG^{(k_z,l_z,s_z)} \otimes GGGG^{(k_y,l_y,s_y)} \otimes dGGGG^{(k_x,l_x,s_x)} \right] \\
\bar{\times} \\
\left[\begin{array}{c}
\{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^u\} \\
\bar{\times} \left\{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^f\right\} \\
\bar{\times} \{(T(m_z,s_z) \otimes T(m_y,s_y) \otimes T(m_x,s_x)) \cdot C^u\}
\end{array} \right]
\end{array} \right\}
\end{aligned} \tag{6.103}
$$

- $$
\begin{aligned}
&B_{ab(5)}^{(k_z,l_z,s_z,k_y,l_y,s_y,k_x,l_x,s_x)} = \\
&\left\{ \begin{array}{c}
\left[GGdGG^{(k_z,l_z,s_z)} \otimes dGGGG^{(k_y,l_y,s_y)} \otimes GdGdGG^{(k_x,l_x,s_x)} \right] \\
\bar{\times} \\
\left[\begin{array}{c}
\{ (T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^v \} \\
\bar{\times} \{ (T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^v \} \\
\bar{\times} \{ (T(m_z,s_z) \otimes T(m_y,s_y) \otimes T(m_x,s_x)) \cdot C^v \}
\end{array} \right]
\end{array} \right\} \\
&- \left\{ \begin{array}{c}
\left[GGdGG^{(k_z,l_z,s_z)} \otimes dGGdGG^{(k_y,l_y,s_y)} \otimes GdGGG^{(k_x,l_x,s_x)} \right] \\
\bar{\times} \\
\left[\begin{array}{c}
\{ (T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^v \} \\
\bar{\times} \{ (T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^v \} \\
\bar{\times} \{ (T(m_z,s_z) \otimes T(m_y,s_y) \otimes T(m_x,s_x)) \cdot C^u \}
\end{array} \right]
\end{array} \right\} \\
&- \left\{ \begin{array}{c}
\left[GGdGG^{(k_z,l_z,s_z)} \otimes dGdGGG^{(k_y,l_y,s_y)} \otimes GGdGG^{(k_x,l_x,s_x)} \right] \\
\bar{\times} \\
\left[\begin{array}{c}
\{ (T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^v \} \\
\bar{\times} \{ (T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^u \} \\
\bar{\times} \{ (T(m_z,s_z) \otimes T(m_y,s_y) \otimes T(m_x,s_x)) \cdot C^v \}
\end{array} \right]
\end{array} \right\} \\
&+ \left\{ \begin{array}{c}
\left[GGdGG^{(k_z,l_z,s_z)} \otimes dGdGdGG^{(k_y,l_y,s_y)} \otimes GGGG^{(k_x,l_x,s_x)} \right] \\
\bar{\times} \\
\left[\begin{array}{c}
\{ (T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^v \} \\
\bar{\times} \{ (T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^u \} \\
\bar{\times} \{ (T(m_z,s_z) \otimes T(m_y,s_y) \otimes T(m_x,s_x)) \cdot C^u \}
\end{array} \right]
\end{array} \right\} \\
&+ \left\{ \begin{array}{c}
\left[GGdGG^{(k_z,l_z,s_z)} \otimes dGGGG^{(k_y,l_y,s_y)} \otimes GGdGG^{(k_x,l_x,s_x)} \right] \\
\bar{\times} \\
\left[\begin{array}{c}
\{ (T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^v \} \\
\bar{\times} \left\{ (T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^f \right\} \\
\bar{\times} \{ (T(m_z,s_z) \otimes T(m_y,s_y) \otimes T(m_x,s_x)) \cdot C^v \}
\end{array} \right]
\end{array} \right\} \\
&- \left\{ \begin{array}{c}
\left[GGdGG^{(k_z,l_z,s_z)} \otimes dGGdGG^{(k_y,l_y,s_y)} \otimes GGGG^{(k_x,l_x,s_x)} \right] \\
\bar{\times} \\
\left[\begin{array}{c}
\{ (T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^v \} \\
\bar{\times} \left\{ (T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^f \right\} \\
\bar{\times} \{ (T(m_z,s_z) \otimes T(m_y,s_y) \otimes T(m_x,s_x)) \cdot C^u \}
\end{array} \right]
\end{array} \right\}
\end{aligned}
\tag{6.104}
$$

- $$B_{ab(6)}^{(k_z,l_z,s_z,k_y,l_y,s_y,k_x,l_x,s_x)} = -\tau \left(\begin{array}{c} \left\{ \begin{array}{c} \left[GdGG^{(k_z,l_z)} \otimes GGG^{(k_y,l_y)} \otimes d^3GdGG^{(k_x,l_x)}\right] \\ \bar{\times} \left[\begin{array}{c} \{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^v\} \\ \bar{\times} \{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^v\} \end{array} \right] \end{array} \right\} \\ - \left\{ \begin{array}{c} \left[GdGG^{(k_z,l_z)} \otimes GdGG^{(k_y,l_y)} \otimes d^3GGG^{(k_x,l_x)}\right] \\ \bar{\times} \left[\begin{array}{c} \{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^v\} \\ \bar{\times} \{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^u\} \end{array} \right] \end{array} \right\} \\ + \left\{ \begin{array}{c} \left[GdGG^{(k_z,l_z)} \otimes d^2GGG^{(k_y,l_y)} \otimes dGdGG^{(k_x,l_x)}\right] \\ \bar{\times} \left[\begin{array}{c} \{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^v\} \\ \bar{\times} \{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^v\} \end{array} \right] \end{array} \right\} \\ - \left\{ \begin{array}{c} \left[GdGG^{(k_z,l_z)} \otimes d^2GdGG^{(k_y,l_y)} \otimes dGGG^{(k_x,l_x)}\right] \\ \bar{\times} \left[\begin{array}{c} \{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^v\} \\ \bar{\times} \{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^u\} \end{array} \right] \end{array} \right\} \\ + \left\{ \begin{array}{c} \left[d^2GdGG^{(k_z,l_z)} \otimes GGG^{(k_y,l_y)} \otimes dGdGG^{(k_x,l_x)}\right] \\ \bar{\times} \left[\begin{array}{c} \{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^v\} \\ \bar{\times} \{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^v\} \end{array} \right] \end{array} \right\} \\ - \left\{ \begin{array}{c} \left[d^2GdGG^{(k_z,l_z)} \otimes GdGG^{(k_y,l_y)} \otimes dGGG^{(k_x,l_x)}\right] \\ \bar{\times} \left[\begin{array}{c} \{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^v\} \\ \bar{\times} \{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^u\} \end{array} \right] \end{array} \right\} \end{array} \right) \tag{6.105}$$

- $$B_{ab(7)}^{(k_z,l_z,s_z,k_y,l_y,s_y,k_x,l_x,s_x)} = -\tau \left(\begin{array}{l} -\left\{ \begin{array}{c} \left[GdGG^{(k_z,l_z)} \otimes dGGG^{(k_y,l_y)} \otimes d^2GdGG^{(k_x,l_x)}\right] \\ \bar{\times} \left[\begin{array}{c} \{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^u\} \\ \bar{\times}\{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^v\} \end{array} \right] \end{array} \right\} \\ +\left\{ \begin{array}{c} \left[GdGG^{(k_z,l_z)} \otimes dGdGG^{(k_y,l_y)} \otimes d^2GGG^{(k_x,l_x)}\right] \\ \bar{\times} \left[\begin{array}{c} \{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^u\} \\ \bar{\times}\{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^u\} \end{array} \right] \end{array} \right\} \\ -\left\{ \begin{array}{c} \left[GdGG^{(k_z,l_z)} \otimes d^3GGG^{(k_y,l_y)} \otimes GdGG^{(k_x,l_x)}\right] \\ \bar{\times} \left[\begin{array}{c} \{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^u\} \\ \bar{\times}\{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^v\} \end{array} \right] \end{array} \right\} \\ +\left\{ \begin{array}{c} \left[GdGG^{(k_z,l_z)} \otimes d^3GdGG^{(k_y,l_y)} \otimes GGG^{(k_x,l_x)}\right] \\ \bar{\times} \left[\begin{array}{c} \{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^u\} \\ \bar{\times}\{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^u\} \end{array} \right] \end{array} \right\} \\ -\left\{ \begin{array}{c} \left[d^2GdGG^{(k_z,l_z)} \otimes dGGG^{(k_y,l_y)} \otimes GdGG^{(k_x,l_x)}\right] \\ \bar{\times} \left[\begin{array}{c} \{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^u\} \\ \bar{\times}\{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^v\} \end{array} \right] \end{array} \right\} \\ +\left\{ \begin{array}{c} \left[d^2GdGG^{(k_z,l_z)} \otimes dGdGG^{(k_y,l_y)} \otimes GGG^{(k_x,l_x)}\right] \\ \bar{\times} \left[\begin{array}{c} \{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^u\} \\ \bar{\times}\{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^u\} \end{array} \right] \end{array} \right\} \end{array} \right) \tag{6.106}$$

- $$B_{ab}^{(k_z,l_z,s_z,k_y,l_y,s_y,k_x,l_x,s_x)} = \sum_{I=1}^{7} B_{ab(I)}^{(k_z,l_z,s_z,k_y,l_y,s_y,k_x,l_x,s_x)} \tag{6.107}$$

- $$B_{ac(1)}^{(k_z,l_z,s_z,k_y,l_y,s_y,k_x,l_x,s_x)} = \begin{array}{l} \left\{ \begin{array}{c} \left[GdGG^{(k_z,l_z)} \otimes GGG^{(k_y,l_y)} \otimes dGGdG^{(k_x,l_x)}\right] \\ \bar{\times} \left[\begin{array}{c} \{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C_t^v\} \\ \bar{\times}\{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^v\} \end{array} \right] \end{array} \right\} \\ -\left\{ \begin{array}{c} \left[GdGG^{(k_z,l_z)} \otimes dGGG^{(k_y,l_y)} \otimes GGdG^{(k_x,l_x)}\right] \\ \bar{\times} \left[\begin{array}{c} \{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C_t^u\} \\ \bar{\times}\{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^v\} \end{array} \right] \end{array} \right\} \end{array} \tag{6.108}$$

- $$
B_{ac(2)}^{(k_z,l_z,s_z,k_y,l_y,s_y,k_x,l_x,s_x)} =
\left\{ \begin{array}{c}
\left[GGdGG^{(k_z,l_z,s_z)} \otimes GGGG^{(k_y,l_y,s_y)} \otimes Gd^2GGdG^{(k_x,l_x,s_x)}\right] \\
\bar{\times} \left[\begin{array}{c}
\{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^u\} \\
\bar{\times}\,\{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^v\} \\
\bar{\times}\,\{(T(m_z,s_z) \otimes T(m_y,s_y) \otimes T(m_x,s_x)) \cdot C^v\}
\end{array} \right]
\end{array} \right\}
- \left\{ \begin{array}{c}
\left[GGdGG^{(k_z,l_z,s_z)} \otimes GdGGG^{(k_y,l_y,s_y)} \otimes GdGGdG^{(k_x,l_x,s_x)}\right] \\
\bar{\times} \left[\begin{array}{c}
\{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^u\} \\
\bar{\times}\,\{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^u\} \\
\bar{\times}\,\{(T(m_z,s_z) \otimes T(m_y,s_y) \otimes T(m_x,s_x)) \cdot C^v\}
\end{array} \right]
\end{array} \right\} \tag{6.109}
$$

- $$
B_{ac(3)}^{(k_z,l_z,s_z,k_y,l_y,s_y,k_x,l_x,s_x)} =
\left\{ \begin{array}{c}
\left[GGdGG^{(k_z,l_z,s_z)} \otimes GdGGG^{(k_y,l_y,s_y)} \otimes GdGGdG^{(k_x,l_x,s_x)}\right] \\
\bar{\times} \left[\begin{array}{c}
\{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^v\} \\
\bar{\times}\,\{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^v\} \\
\bar{\times}\,\{(T(m_z,s_z) \otimes T(m_y,s_y) \otimes T(m_x,s_x)) \cdot C^v\}
\end{array} \right]
\end{array} \right\}
- \left\{ \begin{array}{c}
\left[GGdGG^{(k_z,l_z,s_z)} \otimes Gd^2GGG^{(k_y,l_y,s_y)} \otimes GGGdG^{(k_x,l_x,s_x)}\right] \\
\bar{\times} \left[\begin{array}{c}
\{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^v\} \\
\bar{\times}\,\{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^u\} \\
\bar{\times}\,\{(T(m_z,s_z) \otimes T(m_y,s_y) \otimes T(m_x,s_x)) \cdot C^v\}
\end{array} \right]
\end{array} \right\} \tag{6.110}
$$

- $$B_{ac(4)}^{(k_z,l_z,s_z,k_y,l_y,s_y,k_x,l_x,s_x)} = \left\{ \begin{array}{c} \left[GGdGG^{(k_z,l_z,s_z)} \otimes GGGG^{(k_y,l_y,s_y)} \otimes dGdGGdG^{(k_x,l_x,s_x)}\right] \\ \bar{\times} \left[\begin{array}{c} \{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^u\} \\ \bar{\times}\,\{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^v\} \\ \bar{\times}\,\{(T(m_z,s_z) \otimes T(m_y,s_y) \otimes T(m_x,s_x)) \cdot C^v\} \end{array} \right] \end{array} \right\} - \left\{ \begin{array}{c} \left[GGdGG^{(k_z,l_z,s_z)} \otimes GdGGG^{(k_y,l_y,s_y)} \otimes dGGGdG^{(k_x,l_x,s_x)}\right] \\ \bar{\times} \left[\begin{array}{c} \{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^u\} \\ \bar{\times}\,\{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^u\} \\ \bar{\times}\,\{(T(m_z,s_z) \otimes T(m_y,s_y) \otimes T(m_x,s_x)) \cdot C^v\} \end{array} \right] \end{array} \right\} + \left\{ \begin{array}{c} \left[GGdGG^{(k_z,l_z,s_z)} \otimes GGGG^{(k_y,l_y,s_y)} \otimes dGGGdG^{(k_x,l_x,s_x)}\right] \\ \bar{\times} \left[\begin{array}{c} \{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^u\} \\ \bar{\times}\,\left\{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^f\right\} \\ \bar{\times}\,\{(T(m_z,s_z) \otimes T(m_y,s_y) \otimes T(m_x,s_x)) \cdot C^v\} \end{array} \right] \end{array} \right\} \quad (6.111)$$

- $$B_{ac(5)}^{(k_z,l_z,s_z,k_y,l_y,s_y,k_x,l_x,s_x)} = \left\{ \begin{array}{c} \left[GGdGG^{(k_z,l_z,s_z)} \otimes dGGGG^{(k_y,l_y,s_y)} \otimes GdGGdG^{(k_x,l_x,s_x)}\right] \\ \bar{\times} \left[\begin{array}{c} \{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^v\} \\ \bar{\times}\,\{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^v\} \\ \bar{\times}\,\{(T(m_z,s_z) \otimes T(m_y,s_y) \otimes T(m_x,s_x)) \cdot C^v\} \end{array} \right] \end{array} \right\} - \left\{ \begin{array}{c} \left[GGdGG^{(k_z,l_z,s_z)} \otimes dGdGGG^{(k_y,l_y,s_y)} \otimes GGGdG^{(k_x,l_x,s_x)}\right] \\ \bar{\times} \left[\begin{array}{c} \{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^v\} \\ \bar{\times}\,\{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^u\} \\ \bar{\times}\,\{(T(m_z,s_z) \otimes T(m_y,s_y) \otimes T(m_x,s_x)) \cdot C^v\} \end{array} \right] \end{array} \right\} + \left\{ \begin{array}{c} \left[GGdGG^{(k_z,l_z,s_z)} \otimes dGGGG^{(k_y,l_y,s_y)} \otimes GGGdG^{(k_x,l_x,s_x)}\right] \\ \bar{\times} \left[\begin{array}{c} \{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^v\} \\ \bar{\times}\,\left\{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^f\right\} \\ \bar{\times}\,\{(T(m_z,s_z) \otimes T(m_y,s_y) \otimes T(m_x,s_x)) \cdot C^v\} \end{array} \right] \end{array} \right\} \quad (6.112)$$

- $$B_{ac(6)}^{(k_z,l_z,s_z,k_y,l_y,s_y,k_x,l_x,s_x)} = -\tau \left(\begin{array}{l} \left\{ \begin{array}{c} \left[GdGG^{(k_z,l_z)} \otimes GGG^{(k_y,l_y)} \otimes d^3GGG^{(k_x,l_x)}\right] \\ \bar{\times} \left[\begin{array}{c} \{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^v\} \\ \bar{\times} \{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^v\} \end{array} \right] \end{array} \right\} \\ + \left\{ \begin{array}{c} \left[GdGG^{(k_z,l_z)} \otimes d^2GGG^{(k_y,l_y)} \otimes dGGG^{(k_x,l_x)}\right] \\ \bar{\times} \left[\begin{array}{c} \{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^v\} \\ \bar{\times} \{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^v\} \end{array} \right] \end{array} \right\} \\ + \left\{ \begin{array}{c} \left[d^2GdGG^{(k_z,l_z)} \otimes GGG^{(k_y,l_y)} \otimes dGGG^{(k_x,l_x)}\right] \\ \bar{\times} \left[\begin{array}{c} \{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^v\} \\ \bar{\times} \{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^v\} \end{array} \right] \end{array} \right\} \\ - \left\{ \begin{array}{c} \left[GdGG^{(k_z,l_z)} \otimes dGGG^{(k_y,l_y)} \otimes d^2GGG^{(k_x,l_x)}\right] \\ \bar{\times} \left[\begin{array}{c} \{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^u\} \\ \bar{\times} \{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^v\} \end{array} \right] \end{array} \right\} \\ - \left\{ \begin{array}{c} \left[GdGG^{(k_z,l_z)} \otimes d^3GGG^{(k_y,l_y)} \otimes GGG^{(k_x,l_x)}\right] \\ \bar{\times} \left[\begin{array}{c} \{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^u\} \\ \bar{\times} \{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^v\} \end{array} \right] \end{array} \right\} \\ - \left\{ \begin{array}{c} \left[d^2GdGG^{(k_z,l_z)} \otimes dGGG^{(k_y,l_y)} \otimes GGG^{(k_x,l_x)}\right] \\ \bar{\times} \left[\begin{array}{c} \{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^u\} \\ \bar{\times} \{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^v\} \end{array} \right] \end{array} \right\} \end{array} \right) \tag{6.113}$$

- $$B_{ac}^{(k_z,l_z,s_z,k_y,l_y,s_y,k_x,l_x,s_x)} = \sum_{I=1}^{6} B_{ac(I)}^{(k_z,l_z,s_z,k_y,l_y,s_y,k_x,l_x,s_x)} \tag{6.114}$$

- $$B_{ad(1)}^{(k_z,l_z,s_z,k_y,l_y,s_y,k_x,l_x,s_x)} = \begin{array}{l} - \left\{ \begin{array}{c} \left[GdGG^{(k_z,l_z)} \otimes GGdG^{(k_y,l_y)} \otimes dGGG^{(k_x,l_x)}\right] \\ \bar{\times} \left[\begin{array}{c} \{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C_t^v\} \\ \bar{\times} \{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^u\} \end{array} \right] \end{array} \right\} \\ + \left\{ \begin{array}{c} \left[GdGG^{(k_z,l_z)} \otimes dGGdG^{(k_y,l_y)} \otimes GGG^{(k_x,l_x)}\right] \\ \bar{\times} \left[\begin{array}{c} \{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C_t^u\} \\ \bar{\times} \{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^u\} \end{array} \right] \end{array} \right\} \end{array} \tag{6.115}$$

- $$B_{ad(2)}^{(k_z,l_z,s_z,k_y,l_y,s_y,k_x,l_x,s_x)} =$$
$$-\left\{\begin{array}{c}\left[GGdGG^{(k_z,l_z,s_z)} \otimes GGGdG^{(k_y,l_y,s_y)} \otimes Gd^2GGG^{(k_x,l_x,s_x)}\right] \\ \bar{\times}\left[\begin{array}{c}\{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^u\} \\ \bar{\times}\{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^v\} \\ \bar{\times}\{(T(m_z,s_z) \otimes T(m_y,s_y) \otimes T(m_x,s_x)) \cdot C^u\}\end{array}\right]\end{array}\right\}$$
$$+\left\{\begin{array}{c}\left[GGdGG^{(k_z,l_z,s_z)} \otimes GdGGdG^{(k_y,l_y,s_y)} \otimes GdGGG^{(k_x,l_x,s_x)}\right] \\ \bar{\times}\left[\begin{array}{c}\{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^u\} \\ \bar{\times}\{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^u\} \\ \bar{\times}\{(T(m_z,s_z) \otimes T(m_y,s_y) \otimes T(m_x,s_x)) \cdot C^u\}\end{array}\right]\end{array}\right\} \tag{6.116}$$

- $$B_{ad(3)}^{(k_z,l_z,s_z,k_y,l_y,s_y,k_x,l_x,s_x)} =$$
$$-\left\{\begin{array}{c}\left[GGdGG^{(k_z,l_z,s_z)} \otimes GdGGdG^{(k_y,l_y,s_y)} \otimes GdGGG^{(k_x,l_x,s_x)}\right] \\ \bar{\times}\left[\begin{array}{c}\{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^v\} \\ \bar{\times}\{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^v\} \\ \bar{\times}\{(T(m_z,s_z) \otimes T(m_y,s_y) \otimes T(m_x,s_x)) \cdot C^u\}\end{array}\right]\end{array}\right\}$$
$$+\left\{\begin{array}{c}\left[GGdGG^{(k_z,l_z,s_z)} \otimes Gd^2GGdG^{(k_y,l_y,s_y)} \otimes GGGG^{(k_x,l_x,s_x)}\right] \\ \bar{\times}\left[\begin{array}{c}\{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^v\} \\ \bar{\times}\{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^u\} \\ \bar{\times}\{(T(m_z,s_z) \otimes T(m_y,s_y) \otimes T(m_x,s_x)) \cdot C^u\}\end{array}\right]\end{array}\right\} \tag{6.117}$$

- $$B_{ad(4)}^{(k_z,l_z,s_z,k_y,l_y,s_y,k_x,l_x,s_x)} =$$
$$-\left\{\begin{array}{c}\left[GGdGG^{(k_z,l_z,s_z)} \otimes GGGdG^{(k_y,l_y,s_y)} \otimes dGdGGG^{(k_x,l_x,s_x)}\right] \\ \bar{\times}\left[\begin{array}{c}\{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^u\} \\ \bar{\times}\{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^v\} \\ \bar{\times}\{(T(m_z,s_z) \otimes T(m_y,s_y) \otimes T(m_x,s_x)) \cdot C^u\}\end{array}\right]\end{array}\right\}$$
$$+\left\{\begin{array}{c}\left[GGdGG^{(k_z,l_z,s_z)} \otimes GdGGdG^{(k_y,l_y,s_y)} \otimes dGGGG^{(k_x,l_x,s_x)}\right] \\ \bar{\times}\left[\begin{array}{c}\{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^u\} \\ \bar{\times}\{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^u\} \\ \bar{\times}\{(T(m_z,s_z) \otimes T(m_y,s_y) \otimes T(m_x,s_x)) \cdot C^u\}\end{array}\right]\end{array}\right\} \quad (6.118)$$
$$-\left\{\begin{array}{c}\left[GGdGG^{(k_z,l_z,s_z)} \otimes GGGdG^{(k_y,l_y,s_y)} \otimes dGGGG^{(k_x,l_x,s_x)}\right] \\ \bar{\times}\left[\begin{array}{c}\{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^u\} \\ \bar{\times}\left\{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^f\right\} \\ \bar{\times}\{(T(m_z,s_z) \otimes T(m_y,s_y) \otimes T(m_x,s_x)) \cdot C^u\}\end{array}\right]\end{array}\right\}$$

- $$B_{ad(5)}^{(k_z,l_z,s_z,k_y,l_y,s_y,k_x,l_x,s_x)} =$$
$$-\left\{\begin{array}{c}\left[GGdGG^{(k_z,l_z,s_z)} \otimes dGGGdG^{(k_y,l_y,s_y)} \otimes GdGGG^{(k_x,l_x,s_x)}\right] \\ \bar{\times}\left[\begin{array}{c}\{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^v\} \\ \bar{\times}\{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^v\} \\ \bar{\times}\{(T(m_z,s_z) \otimes T(m_y,s_y) \otimes T(m_x,s_x)) \cdot C^u\}\end{array}\right]\end{array}\right\}$$
$$+\left\{\begin{array}{c}\left[GGdGG^{(k_z,l_z,s_z)} \otimes dGdGGdG^{(k_y,l_y,s_y)} \otimes GGGG^{(k_x,l_x,s_x)}\right] \\ \bar{\times}\left[\begin{array}{c}\{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^v\} \\ \bar{\times}\{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^u\} \\ \bar{\times}\{(T(m_z,s_z) \otimes T(m_y,s_y) \otimes T(m_x,s_x)) \cdot C^u\}\end{array}\right]\end{array}\right\} \quad (6.119)$$
$$-\left\{\begin{array}{c}\left[GGdGG^{(k_z,l_z,s_z)} \otimes dGGGdG^{(k_y,l_y,s_y)} \otimes GGGG^{(k_x,l_x,s_x)}\right] \\ \bar{\times}\left[\begin{array}{c}\{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^v\} \\ \bar{\times}\left\{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^f\right\} \\ \bar{\times}\{(T(m_z,s_z) \otimes T(m_y,s_y) \otimes T(m_x,s_x)) \cdot C^u\}\end{array}\right]\end{array}\right\}$$

- $$B_{ad(6)}^{(k_z,l_z,s_z,k_y,l_y,s_y,k_x,l_x,s_x)} = -\tau \left(\begin{array}{c} -\left\{ \begin{array}{c} \left[GdGG^{(k_z,l_z)} \otimes GGG^{(k_y,l_y)} \otimes d^3GGG^{(k_x,l_x)}\right] \\ \bar{\times} \left[\begin{array}{c} \{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^v\} \\ \bar{\times} \{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^u\} \end{array} \right] \end{array} \right\} \\ -\left\{ \begin{array}{c} \left[GdGG^{(k_z,l_z)} \otimes d^2GGG^{(k_y,l_y)} \otimes dGGG^{(k_x,l_x)}\right] \\ \bar{\times} \left[\begin{array}{c} \{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^v\} \\ \bar{\times} \{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^u\} \end{array} \right] \end{array} \right\} \\ -\left\{ \begin{array}{c} \left[d^2GdGG^{(k_z,l_z)} \otimes GGG^{(k_y,l_y)} \otimes dGGG^{(k_x,l_x)}\right] \\ \bar{\times} \left[\begin{array}{c} \{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^v\} \\ \bar{\times} \{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^u\} \end{array} \right] \end{array} \right\} \\ +\left\{ \begin{array}{c} \left[GdGG^{(k_z,l_z)} \otimes dGGG^{(k_y,l_y)} \otimes d^2GGG^{(k_x,l_x)}\right] \\ \bar{\times} \left[\begin{array}{c} \{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^u\} \\ \bar{\times} \{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^u\} \end{array} \right] \end{array} \right\} \\ +\left\{ \begin{array}{c} \left[GdGG^{(k_z,l_z)} \otimes d^3GGG^{(k_y,l_y)} \otimes GGG^{(k_x,l_x)}\right] \\ \bar{\times} \left[\begin{array}{c} \{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^u\} \\ \bar{\times} \{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^u\} \end{array} \right] \end{array} \right\} \\ +\left\{ \begin{array}{c} \left[d^2GdGG^{(k_z,l_z)} \otimes dGGG^{(k_y,l_y)} \otimes GGG^{(k_x,l_x)}\right] \\ \bar{\times} \left[\begin{array}{c} \{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^u\} \\ \bar{\times} \{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^u\} \end{array} \right] \end{array} \right\} \end{array} \right) \quad (6.120)$$

- $$B_{ad}^{(k_z,l_z,s_z,k_y,l_y,s_y,k_x,l_x,s_x)} = \sum_{I=1}^{6} B_{ad(I)}^{(k_z,l_z,s_z,k_y,l_y,s_y,k_x,l_x,s_x)} \quad (6.121)$$

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$$B_{e\rho}^{(k_z,l_z,s_z,k_y,l_y,s_y,k_x,l_x,s_x)} = \left\{ \begin{array}{c} \left[GGGdG^{(k_z,l_z,s_z)} \otimes GGGG^{(k_y,l_y,s_y)} \otimes dGGGG^{(k_x,l_x,s_x)}\right] \\ \bar{\times} \left[\begin{array}{c} \{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^u\} \\ \bar{\times}\, \{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^\rho\} \\ \bar{\times}\, \{(T(m_z,s_z) \otimes T(m_y,s_y) \otimes T(m_x,s_x)) \cdot C^\rho\} \end{array} \right] \end{array} \right\} + \left\{ \begin{array}{c} \left[GGGdG^{(k_z,l_z,s_z)} \otimes dGGGG^{(k_y,l_y,s_y)} \otimes GGGG^{(k_x,l_x,s_x)}\right] \\ \bar{\times} \left[\begin{array}{c} \{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^v\} \\ \bar{\times}\, \{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^\rho\} \\ \bar{\times}\, \{(T(m_z,s_z) \otimes T(m_y,s_y) \otimes T(m_x,s_x)) \cdot C^\rho\} \end{array} \right] \end{array} \right\} \tag{6.122}$$

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$$B_{e\rho_z}^{(k_z,l_z,s_z,k_y,l_y,s_y,k_x,l_x,s_x)} = \left\{ \begin{array}{c} \left[GGdGG^{(k_z,l_z,s_z)} \otimes GGGG^{(k_y,l_y,s_y)} \otimes dGGGG^{(k_x,l_x,s_x)}\right] \\ \bar{\times} \left[\begin{array}{c} \{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^u\} \\ \bar{\times}\, \{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^\rho\} \\ \bar{\times}\, \{(T(m_z,s_z) \otimes T(m_y,s_y) \otimes T(m_x,s_x)) \cdot C^\rho\} \end{array} \right] \end{array} \right\} + \left\{ \begin{array}{c} \left[GGdGG^{(k_z,l_z,s_z)} \otimes dGGGG^{(k_y,l_y,s_y)} \otimes GGGG^{(k_x,l_x,s_x)}\right] \\ \bar{\times} \left[\begin{array}{c} \{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^v\} \\ \bar{\times}\, \{(T(m_z,l_z) \otimes T(m_y,l_y) \otimes T(m_x,l_x)) \cdot C^\rho\} \\ \bar{\times}\, \{(T(m_z,s_z) \otimes T(m_y,s_y) \otimes T(m_x,s_x)) \cdot C^\rho\} \end{array} \right] \end{array} \right\} \tag{6.123}$$

-
$$B_{w_p}^{(k_z,k_y,k_x)} = \left[GG^{(k_z)} \otimes GG^{(k_y)} \otimes GG^{(k_x)}\right] \bar{\times} \{(T(m_z,k_z) \otimes T(m_y,k_y) \otimes T(m_x,k_x)) \cdot C^{w_p}\} \tag{6.124}$$

- let' s use the notation

$$\sum_{k=-3}^{3} \sum_{l=Max\{-3,k-3\}}^{Min\{3,k+3\}} \sum_{s=Max\{-3,k-3,l-3\}}^{Min\{3,k+3,l+3\}} = \sum_{(k,l,s)} \tag{6.125}$$

As in the construction of the matrix M we have

- $$B_{ab} = \sum_{(k_z,l_z,s_z)} \sum_{(k_y,l_y,s_y)} \sum_{(k_x,l_x,s_x)} B_{ab}^{(k_z,l_z,s_z,k_y,l_y,s_y,k_x,l_x,s_x)}, \tag{6.126}$$

- $$B_{ac} = \sum_{(k_z,l_z,s_z)} \sum_{(k_y,l_y,s_y)} \sum_{(k_x,l_x,s_x)} B_{ac}^{(k_z,l_z,s_z,k_y,l_y,s_y,k_x,l_x,s_x)}, \tag{6.127}$$

- $$B_{ad} = \sum_{(k_z,l_z,s_z)} \sum_{(k_y,l_y,s_y)} \sum_{(k_x,l_x,s_x)} B_{ad}^{(k_z,l_z,s_z,k_y,l_y,s_y,k_x,l_x,s_x)}, \tag{6.128}$$

- $$B_{e\rho} = \sum_{(k_z,l_z,s_z)} \sum_{(k_y,l_y,s_y)} \sum_{(k_x,l_x,s_x)} B_{e\rho}^{(k_z,l_z,s_z,k_y,l_y,s_y,k_x,l_x,s_x)} \tag{6.129}$$

- $$B_{e\rho_z} = \sum_{(k_z,l_z,s_z)} \sum_{(k_y,l_y,s_y)} \sum_{(k_x,l_x,s_x)} B_{e\rho_z}^{(k_z,l_z,s_z,k_y,l_y,s_y,k_x,l_x,s_x)} \tag{6.130}$$

- and
$$B_{w_p} = \sum_{k_z=-3}^{3} \sum_{k_y=-3}^{3} \sum_{k_x=-3}^{3} B_{w_p}^{(k_z,k_y,k_x)}. \tag{6.131}$$

6.4 Sensitivity of the approximation to errors in u and v

Similar results to those obtained in chapter V can be achieved by finding upper bounds to the $\|\cdot\|_m$ of the matrices $dGdGGG^{(k_z,l_z)}(m_z,h_z) \otimes GGGG^{(k_y,l_y)}(m_y,h_y) \otimes GGdGdG^{(k_x,l_x)}(m_x,h_x)$...
...$dGGGdG^{(k_z,l_z)}(m_z,h_z) \otimes GGGG^{(k_y,l_y)}(m_y,h_y) \otimes GGGG^{(k_x,l_x)}(m_x,h_x)$ defined in (6.80)...(6.88) and to the $\|\cdot\|_d$ of the vectors
$GdGG^{(k_z,l_z)}(m_z,h_z) \otimes GGG^{(k_y,l_y)}(m_y,h_y) \otimes dGdGG^{(k_x,l_x)}(m_x,h_x)$...
...$GGdGG^{(k_z,l_z)}(m_z,h_z) \otimes dGGGG^{(k_y,l_y)}(m_y,h_y) \otimes GGGG^{(k_x,l_x)}(m_x,h_x)$ defined in (6.100)...(6.124).

CHAPTER VII

NUMERICAL EXPERIMENTS FOR THE APPROXIMATION BY LINEAR SPLINES

We use an exact analytic solution of the Navier-Stokes equations to test the accuracy of w_0^N. The exact solution is a viscously decaying extension of a Beltrami flow used in previous studies of thunderstorm rotation, and consists of a periodic array of counterrotating updrafts and downdrafts. This flow is noteworthy in that it is three-dimensional, free of singularities, and satisfies the Navier-Stokes equations with nontrivial (i.e. nonvanishing) inertial terms. The simple form of the analytic solution and its provision for arbitrarily large spacial gradients suggest its potential utility in validating numerical flow models and in testing the relative merits of various numerical solution algorithms. In a viscous Beltrami flow the velocity, $\vec{V}$, and the vorticity vector, $\vec{\xi}$, are everywhere aligned; that is, $\vec{\xi} = \sigma \vec{V}$. Where σ is a constant, often referred to as the abnormality.

We can write the exact viscous solution as:

$$u_B = -\frac{A}{l_x^2 + l_y^2} \begin{pmatrix} \lambda_0 l_y cos(l_x x) sin(l_y y) sin(l_z z) \\ + \\ l_x l_z sin(l_x x) cos(l_y y) cos(l_z z) \end{pmatrix} e^{-\tau \lambda_0^2 t}, \tag{7.1}$$

$$v_B = -\frac{A}{l_x^2 + l_y^2} \begin{pmatrix} \lambda_0 l_x sin(l_x x) cos(l_y y) sin(l_z z) \\ - \\ l_y l_z cos(l_x x) sin(l_y y) cos(l_z z) \end{pmatrix} e^{-\tau \lambda_0^2 t} \tag{7.2}$$

and

$$w_B = Acos(l_x x)cos(l_y y)sin(l_z z)e^{-\tau\lambda_0^2 t} \tag{7.3}$$

where $\quad \lambda_0^2 = l_x^2 + l_y^2 + l_z^2$

Equations (7.1), (7.2) and (7.3) can be used to test not only the integrity of a model (i.e. check that the code is error free) but also the relative merits of various numerical schemes when applied to flows with large spatial momentum gradients.

In order that (7.1), (7.2) and (7.3) satisfy the mass continuity equation and the vorticity equation we have to have $\rho(z) = constante$ -= 1 for simplicity- and we must take the Coriolis factor f equal to zero. The study focuses on a time-independent flow, $\tau = 0$, with $w_p = 0$. But before we we will try approximation by cubic B-splines, we will see some cases where, assuming the air temperature is between $60^o F$ and $70^o F$, we consider $\tau = 1.6 \times 10^{-4}$.

We will take $L_x = 150\ km$, $L_y = 150\ km$, $L_z = 3\ km$, $n_x = 16$, $n_y = 16$, $n_z = 4$, and $\epsilon_0 = 10^{-18}$. So $N = n_x n_y n_z = 2^{10}$, $h_x = 10\ km$, $h_y = 10\ km$ and $h_z = 1\ km$.

We will examine two major cases. In the first one, (A), the amplitude of the horizontal velocity will be 70 miles per hour, meaning $max\sqrt{u_B^2 + v_B^2} \approx 70\ miles/hour$. In the second case, (B), we will look at small velocities, so we will take $max\sqrt{u_B^2 + v_B^2} \approx 7\ miles/hour$.

Let $\bar{w_B} = [w_B(\bar{x_1}), \cdots, w_B(\bar{x_N})]^T$ be the vector that represents the values of the function w_B, defined in (6.3), at the mesh points $\bar{x_i}$.

Define:

- The relative error $\quad RE = \frac{\left\|\bar{w_B} - \tilde{C}\right\|_d}{\|\bar{w_B}\|_d}$,
- (i_M, j_M) = the indices of the element $(M_v)_M = V_M$ such that $|(M_v)_M| =$

$Max_{i,j}\left|(M_v)_{ij}\right|$. If the maximum is reached at many positions in the matrix M_v, we just pick the smallest i and then the smallest j. And let $(M_m)_{i_M j_M} = M_M$ be the element in the matrix M_m whose indices are (i_M, j_M) .

- (i_m, j_m) = the indices of the element $(M_v)_m = V_m$ such that $|(M_v)_m| = Min_{i,j}\left\{\left|(M_v)_{ij}\right|; (M_v)_{ij} \neq 0\right\}$. If the minimum is reached at many positions in the matrix M_v, we just pick the smallest i and then the smallest j. And let $(M_m)_{i_m j_m} = M_m$ be the element in the matrix M_m whose indices are (i_m, j_m) .

- Let $b\alpha_v^2$, $b\alpha_m^2$ and $b\epsilon$ be, respectively, the α_v^2, the α_m^2 and the ϵ that give us the smallest RE.

We have $\|M_m\|_d \approx 336.87$ and $\|G\|_d \approx 336.9$.

To simulate a mesh that is becoming finer and finer we will be decreasing the values of l_x, l_y and l_z (see (7.1)-(7.3)).

7.1 Accuracy

For differente values of l_x, l_y and l_z we obtain the following results.

7.1.1 Case(A) ($70\,miles/hour$)

Expmnt	l_x	l_y	l_z	RE when $\epsilon_0 = 10^{-18}$	RE when $\epsilon_0 = 0$
A-1	$2\pi/75$	$2\pi/75$	$\pi/3$	25.98×10^{-2}	26.96×10^{-2}
A-2	$\pi/75$	$\pi/75$	$\pi/3$	12.73×10^{-2}	12.65×10^{-2}
A-3	$\pi/150$	$\pi/150$	$\pi/3$	97.27×10^{-3}	97.47×10^{-3}
A-4	$\pi/300$	$\pi/300$	$\pi/15$	43.08×10^{-3}	41.4×10^{-3}
A-5	$\pi/1500$	$\pi/1500$	$\pi/15$	84.01×10^{-4}	85.45×10^{-4}
A-6	$\pi/1500$	$\pi/1500$	$\pi/30$	28.93×10^{-4}	29.09×10^{-4}
A-7	$\pi/7500$	$\pi/7500$	$\pi/30$	89.71×10^{-5}	13.69×10^{-4}

Table(1.A.1)

	$\epsilon_0 = 10^{-18}$			$\epsilon_0 = 0$		
Expmnt	$b\alpha_v^2$	$b\alpha_m^2$	$b\epsilon$	$b\alpha_v^2$	$b\alpha_m^2$	$b\epsilon$
A-1	10^5	10^{14}	10^{-3}	1	10^9	10^{-7}
A-2	10^3	10^{11}	10^{-2}	0.1	10^7	10^{-6}
A-3	10^3	10^{11}	10^{-2}	0.1	10^7	10^{-6}
A-4	10^5	10^{11}	10^{-4}	0.1	10^5	10^{-8}
A-5	10^7	10^{11}	1	10^2	10^6	10^{-5}
A-6	10^7	10^{11}	0.1	10^3	10^6	10^{-5}
A-7	10^4	10^7	1	10^3	10^6	10^{-3}

Table(1.A.2)

Expmt	$\|A_v\|_d$	$\|B_v\|_d$	$\|B_m\|_d$	$\|V_M\|$	$\|M_M\|$	$\|V_m\|$	$\|M_m\|$
A-1	61.88×10^{-5}	48.54×10^{-7}	3.36	37.16×10^{-5}	88.88	51.15×10^{-37}	2.777
A-2	10.58×10^{-4}	14.94×10^{-7}	2.42	50.82×10^{-5}	88.88	16.04×10^{-37}	2.777
A-3	11.7×10^{-4}	31.23×10^{-8}	1.32	54.47×10^{-5}	88.88	40.5×10^{-38}	2.777
A-4	10.71×10^{-5}	19.57×10^{-9}	0.615	36.03×10^{-6}	88.88	47.69×10^{-12}	2.777
A-5	14.15×10^{-6}	68.15×10^{-11}	0.24	56.73×10^{-7}	88.88	11.15×10^{-13}	5.555
A-6	30.63×10^{-7}	19.54×10^{-11}	0.253	14.38×10^{-7}	88.88	28.74×10^{-14}	5.555
A-7	13.52×10^{-8}	19.26×10^{-13}	0.0523	64.33×10^{-9}	88.88	10.71×10^{-15}	5.555

Table(1.A.3)

	$\epsilon_0 = 10^{-18}$		$\epsilon_0 = 0$	
Expmnt	$\frac{b\alpha_m^2\|M_m\|_d}{b\alpha_v^2\|M_v\|_d}$	$\frac{b\alpha_m^2\|B_m\|_d}{b\alpha_v^2\|B_v\|_d}$	$\frac{b\alpha_m^2\|M_m\|_d}{b\alpha_v^2\|M_v\|_d}$	$\frac{b\alpha_m^2\|B_m\|_d}{b\alpha_v^2\|B_v\|_d}$
A-1	54.43×10^{13}	69.22×10^{13}	54.43×10^{13}	69.22×10^{13}
A-2	31.84×10^{12}	16.19×10^{13}	31.84×10^{12}	16.19×10^{13}
A-3	28.79×10^{12}	42.26×10^{13}	28.79×10^{12}	42.26×10^{13}
A-4	31.45×10^{11}	31.42×10^{12}	31.45×10^{11}	31.42×10^{12}
A-5	23.8×10^{10}	35.21×10^{11}	23.8×10^{10}	35.21×10^{11}
A-6	10.99×10^{11}	12.94×10^{12}	10.99×10^{10}	12.94×10^{11}
A-7	24.91×10^{11}	27.15×10^{12}	24.91×10^{11}	27.15×10^{12}

Table(1.A.4)

	$\epsilon_0 = 10^{-18}$		$\epsilon_0 = 0$	
Expmnt	$\frac{b\alpha_v^2\|M_v\|_d}{b\epsilon\|G\|_d} \times 10^{14}$	$\frac{b\alpha_m^2\|M_m\|_d}{b\epsilon\|G\|_d}$	$\frac{b\alpha_v^2\|M_v\|_d}{b\epsilon\|G\|_d} \times 10^{14}$	$\frac{b\alpha_m^2\|M_m\|_d}{b\epsilon\|G\|_d}$
A-1	18.36×10^{15}	99.99×10^{15}	18.36×10^{14}	99.99×10^{14}
A-2	31.4×10^{12}	99.99×10^{11}	31.4×10^{12}	99.99×10^{11}
A-3	34.72×10^{12}	99.99×10^{11}	34.72×10^{12}	99.99×10^{11}
A-4	31.78×10^{15}	99.99×10^{13}	31.78×10^{13}	99.99×10^{11}
A-5	42×10^{12}	99.99×10^{9}	42×10^{12}	99.99×10^{9}
A-6	90.91×10^{12}	99.99×10^{10}	90.91×10^{12}	99.99×10^{9}
A-7	40.13×10^{7}	99.99×10^{5}	40.13×10^{9}	99.99×10^{7}

Table(1.A.5)

For Experiment A-2, Figures 7 through 12 give the $u_B - v_B$ vector field with contours of the exact vertical wind field, w_B, and its approximation, w^N, refered to as the numerical. wind field, at $z = 0km$, $z = 1km$, $z = 2km$ and $z = 3km$. Since, in Experiment A-2, the Beltrami solution $w_B = 0km/s$ for $z = 0km$ and $z = 3km$ but the numerical solution gives a maximal error that is in the range of $3 \times 10^{-4}km/s$. Figures 6 and 13 give the distributions of the values of w^N at the mesh points for $z = 0km$ and $z = 3km$, respectively. Those distributions show that most of the values are between $-1.5 \times 10^{-4}km/s$ and $1.5 \times 10^{-4}km/s$. We also have for each z

z	$Max \left\| w_B - w^N \right\| \times 10^{-4}km/s$	$Max\{w_B\} \times 10^{-4}km/s$	$Min\{w_B\} \times 10^{-4}km/s$
$0km$	2.966	0	0
$1km$	3.561	21.52	-21.05
$2km$	3.561	21.52	-21.05
$3km$	3.105	theoretically=0	theoretically=0

Table (1.A.6)

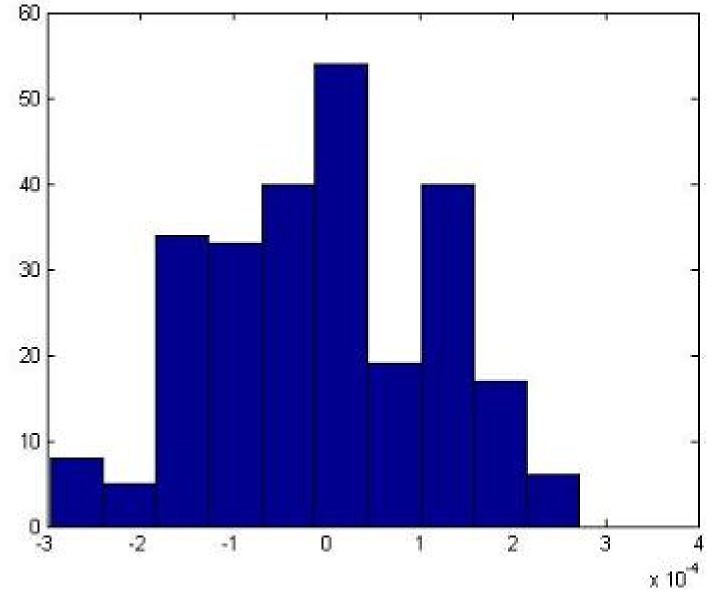

Figure 6: Distribution of the values of w^N at $z = 0km$

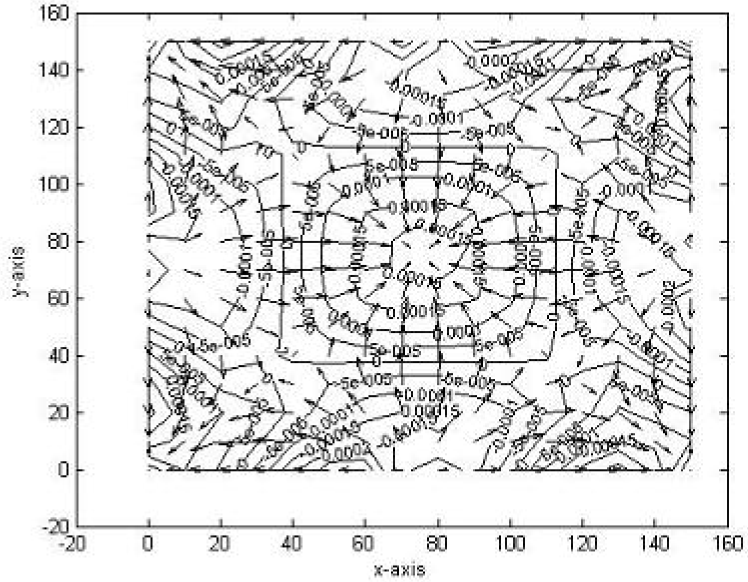

Figure 7: u-v wind field and contours of w_N at $z = 0km$

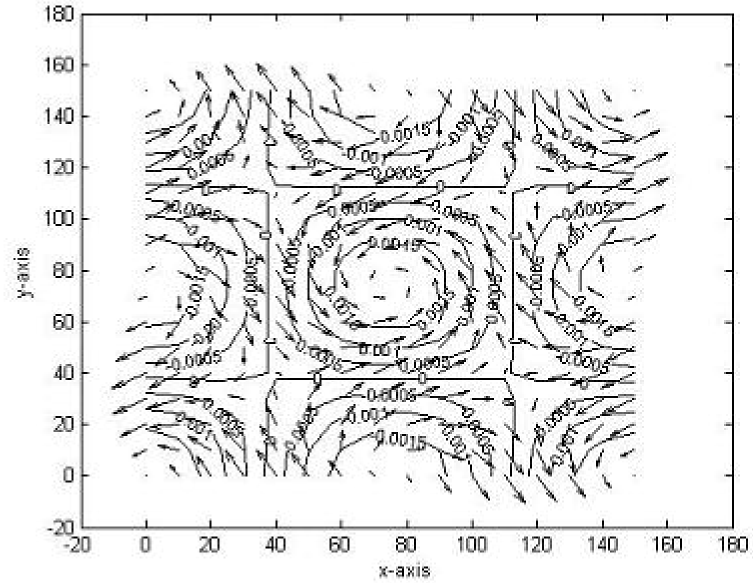

Figure 8: u-v wind field and contours of w^N at $z = 1km$

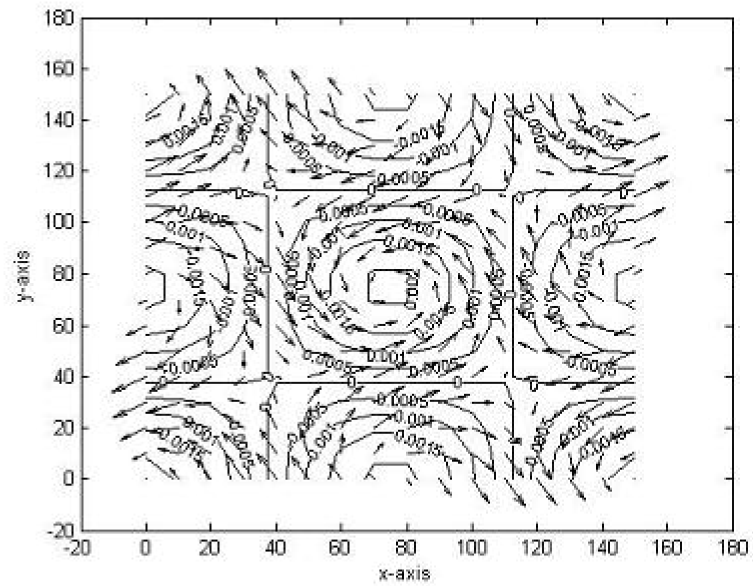

Figure 9: u-v wind field and contours of w_B at $z = 1km$

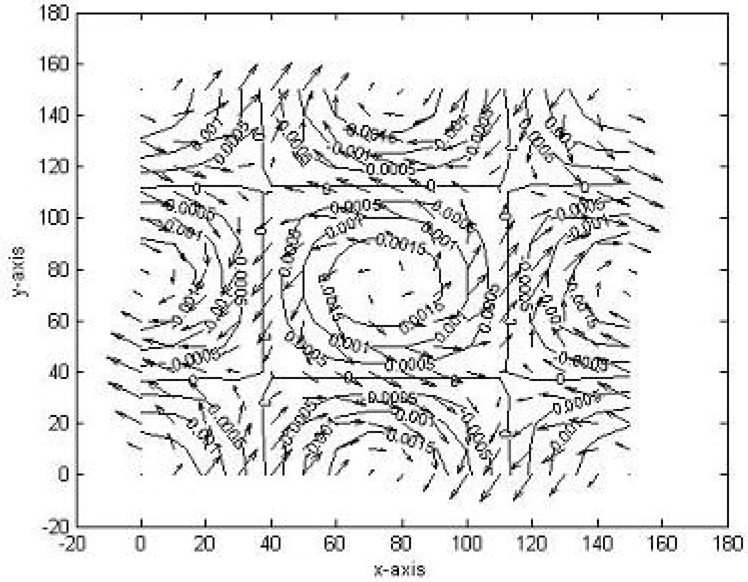

Figure 10: u-v wind field and contours of w^N at $z = 2km$

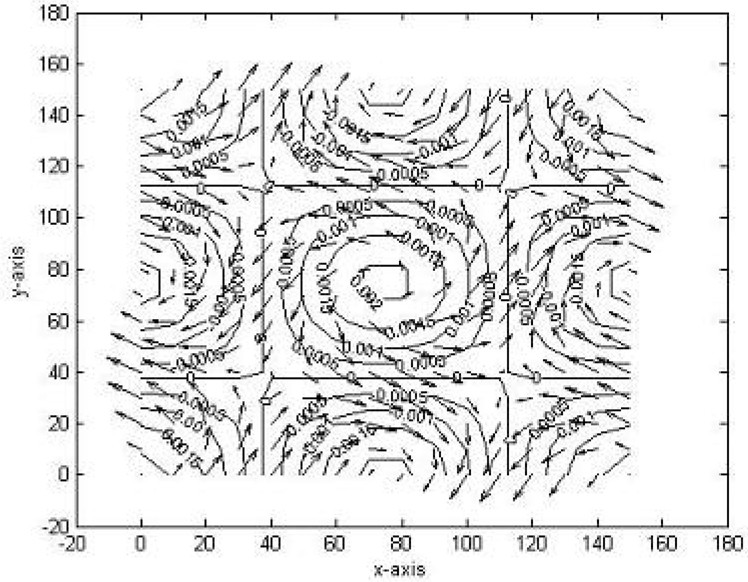

Figure 11: u-v wind field and contours of w_B at $z = 2km$

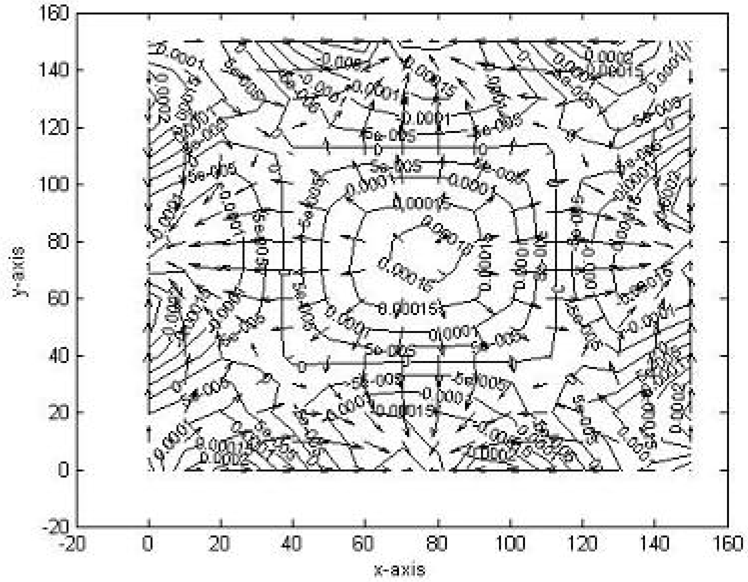

Figure 12: u-v wind field and contours of w^N at $z = 3km$

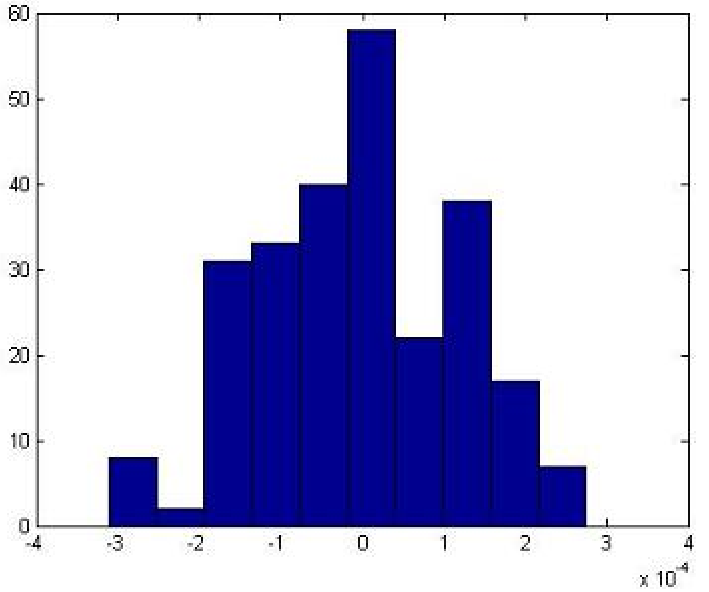

Figure 13: Distribution of the values of w^N at $z = 3km$

7.1.2 Case(B) ($7 \, miles/hour$)

Expmnt	l_x	l_y	l_z	RE when $\epsilon_0 = 10^{-18}$	RE when $\epsilon_0 = 0$
B-1	$2\pi/75$	$2\pi/75$	$\pi/3$	27.26×10^{-2}	26.88×10^{-2}
B-2	$\pi/75$	$\pi/75$	$\pi/3$	12.53×10^{-2}	12.7×10^{-2}
B-3	$\pi/150$	$\pi/150$	$\pi/3$	94.66×10^{-3}	92.08×10^{-3}
B-4	$\pi/300$	$\pi/300$	$\pi/15$	43.56×10^{-3}	41.71×10^{-3}
B-5	$\pi/1500$	$\pi/1500$	$\pi/15$	84.46×10^{-4}	86.48×10^{-4}
B-6	$\pi/1500$	$\pi/1500$	$\pi/30$	27.87×10^{-4}	29.17×10^{-4}
B-7	$\pi/7500$	$\pi/7500$	$\pi/30$	96.07×10^{-5}	89.99×10^{-5}

Table(1.B.1)

	$\epsilon_0 = 10^{-18}$			$\epsilon_0 = 0$		
Expmnt	$b\alpha_v^2$	$b\alpha_m^2$	$b\epsilon$	$b\alpha_v^2$	$b\alpha_m^2$	$b\epsilon$
B-1	10^6	10^{12}	10^{-4}	10^3	10^{10}	10^{-6}
B-2	10^8	10^{14}	10	10	10^7	10^{-6}
B-3	10^9	10^{15}	100	10	10^7	10^{-6}
B-4	10^9	10^{12}	1	10	10^5	10^{-8}
B-5	10^9	10^{11}	1	10^4	10^6	10^{-5}
B-6	10^9	10^{11}	0.1	10^5	10^6	10^{-5}
B-7	10^9	10^{10}	10^3	10^6	10^7	1

Table(1.B.2)

Expmnt	$\|A_v\|_d$	$\|B_v\|_d$	$\|B_m\|_d$	$\lvert V_M\rvert$	$\lvert M_M\rvert$	$\lvert V_m\rvert$	$\lvert M_m\rvert$
B-1	61.88×10^{-7}	48.54×10^{-10}	0.336	37.16×10^{-7}	88.88	46.8×10^{-39}	2.777
B-2	10.58×10^{-6}	14.94×10^{-10}	0.242	50.82×10^{-7}	88.88	16.06×10^{-39}	2.777
B-3	11.7×10^{-6}	31.23×10^{-11}	0.132	54.47×10^{-7}	88.88	43.47×10^{-40}	2.777
B-4	10.71×10^{-7}	19.57×10^{-12}	0.0615	36.03×10^{-8}	88.88	47.69×10^{-14}	2.777
B-5	14.15×10^{-8}	68.15×10^{-14}	0.024	56.73×10^{-9}	88.88	11.15×10^{-15}	5.555
B-6	30.63×10^{-9}	19.54×10^{-14}	0.0253	14.38×10^{-9}	88.88	28.74×10^{-16}	5.555
B-7	13.52×10^{-10}	19.26×10^{-16}	0.00523	64.33×10^{-11}	88.88	10.71×10^{-17}	5.555

Table(1.B.3)

	$\epsilon_0 = 10^{-18}$		$\epsilon_0 = 0$	
Expmnt	$\frac{b\alpha_m^2\|M_m\|_d}{b\alpha_v^2\|M_v\|_d}$	$\frac{b\alpha_m^2\|B_m\|_d}{b\alpha_v^2\|B_v\|_d}$	$\frac{b\alpha_m^2\|M_m\|_d}{b\alpha_v^2\|M_v\|_d}$	$\frac{b\alpha_m^2\|B_m\|_d}{b\alpha_v^2\|B_v\|_d}$
B-1	54.43×10^{12}	69.22×10^{12}	54.43×10^{13}	69.22×10^{13}
B-2	31.84×10^{12}	16.19×10^{13}	31.84×10^{12}	16.19×10^{13}
B-3	28.79×10^{12}	42.26×10^{13}	28.79×10^{12}	42.26×10^{13}
B-4	31.45×10^{10}	31.42×10^{11}	31.45×10^{11}	31.42×10^{12}
B-5	23.8×10^{10}	35.21×10^{11}	23.8×10^{10}	35.21×10^{11}
B-6	10.99×10^{11}	12.94×10^{12}	10.99×10^{10}	12.94×10^{11}
B-7	24.91×10^{11}	27.15×10^{12}	24.91×10^{11}	27.15×10^{12}

Table(1.B.4)

	$\epsilon_0 = 10^{-18}$		$\epsilon_0 = 0$	
Expmnt	$\frac{b\alpha_v^2\|M_v\|_d}{b\epsilon\|G\|_d} \times 10^{14}$	$\frac{b\alpha_m^2\|M_m\|_d}{b\epsilon\|G\|_d}$	$\frac{b\alpha_v^2\|M_v\|_d}{b\epsilon\|G\|_d} \times 10^{14}$	$\frac{b\alpha_m^2\|M_m\|_d}{b\epsilon\|G\|_d}$
B-1	18.36×10^{15}	99.99×10^{14}	18.36×10^{14}	99.99×10^{14}
B-2	31.4×10^{12}	99.99×10^{11}	31.4×10^{12}	99.99×10^{11}
B-3	34.72×10^{12}	99.99×10^{11}	34.72×10^{12}	99.99×10^{11}
B-4	31.78×10^{13}	99.99×10^{10}	31.78×10^{13}	99.99×10^{11}
B-5	42×10^{12}	99.99×10^{9}	42×10^{12}	99.99×10^{9}
B-6	90.91×10^{12}	99.99×10^{10}	90.91×10^{12}	99.99×10^{9}
B-7	40.13×10^{7}	99.99×10^{5}	40.13×10^{7}	99.99×10^{5}

Table(1.B.5)

From tables (1.A.1) and (1.B.1), it is clear that the finer the mesh the more accurate the solution. And by examining tables (1.A.4), (1.A.5), (1.B.4) and (1.B.5), in this type of wind fields, one can have a general idea of how $b\alpha_v^2$, $b\alpha_m^2$ and $b\epsilon$ should be chosen. Since the search for the latters was based on a mesh whose points coordinates are integer powers of 10, the relations among them are not accurate and can be summarized in saying that $\frac{b\alpha_m^2\|M_m\|_d}{b\alpha_v^2\|M_v\|_d}$ and $\frac{b\alpha_v^2\|M_v\|_d}{b\epsilon\|G\|_d}$ should be of the order of 10^{13} and 0.1, respectively. But of course one can study these relations for each specific class of wind fields using more accurate search techniques. Tables (1.A.3) and (1.B.3) do not show any pattern that can be followed in order to determine $b\alpha_v$, $b\alpha_m$ and ϵ. By saying a class of wind fields, we mean certain ranges of the values of u, v and their first and second derivatives. Such thing can be done by working with different models and data from given measurements.

Remark 7.1

From the value of $\|M_m\|_2$ and the values of $\|M_v\|_2$, α_v^2 and α_m^2 one concludes that in order to obtain the most accurate approximation we have to give the MCE more weight than the VE. So we ask ourselves two questions:

1) What if we solve only the MCE?

2) What kind of result we will have if we solve only the VE?

7.2 Using either the vorticity eq. or the mass continuity eq.

Even though we made the mesh finer by increasing n_z from 4 to 5, tables (2.A.1), (2.B.1), (2.A.2) and (2.B.2) show us that when we take $\alpha_v = 0$ we obtain an approximation that is almost as accurate as if we take $\alpha_m = 0$; which means that the VE, alone, gives almost the same accuracy as the MCE alone. But in both cases, VE alone or MCE alone, the approximation is much less accurate that when both equations are used.

7.2.1 The vorticity equation alone ($\alpha_m = 0$) with $n_z = 5$

We take $\epsilon_0 = 0$ in experiments (B-5), (B-6) and (B-7) and $\epsilon_0 = 10^{-18}$ in the other experiments.

Case A: 70 miles per hour				
Expmnt	RE	$b\alpha_v^2$	$b\epsilon$	$\|M_v\|_2$
A-1	48.77×10^{-2}	10^8	1	67.17×10^{-5}
A-2	47.01×10^{-2}	10^8	1	10.96×10^{-4}
A-3	50.96×10^{-2}	10^4	10^{-4}	12.07×10^{-4}
A-4	10.57×10^{-2}	10^{13}	10^{-10}	85.2×10^{-6}
A-5	17.07×10^{-2}	10^{12}	0.1	11.56×10^{-6}
A-6	9.29×10^{-2}	10	10^{-12}	26.29×10^{-7}
A-7	9.38×10^{-2}	10^2	10^{-13}	11.6×10^{-8}

Table(2.A.1)

Case B: 7 miles per hour				
Expmnt	RE	α_v^2	ϵ	$\|M_v\|_2$
B-1	48.77×10^{-2}	10^6	10^{-4}	67.17×10^{-5}
B-2	47.01×10^{-2}	10^9	0.1	10.96×10^{-4}
B-3	50.96×10^{-2}	10^6	10^{-4}	12.07×10^{-4}
B-4	10.57×10^{-2}	10^{17}	10^{-8}	85.2×10^{-6}
B-5	17.07×10^{-2}	10^{-3}	10^{-18}	11.56×10^{-6}
B-6	9.29×10^{-2}	10^3	10^{-12}	26.29×10^{-7}
B-7	9.38×10^{-2}	10^6	10^{-11}	11.6×10^{-8}

Table(2.B.1)

7.2.2 The mass continuity equation alone ($\alpha_v = 0$) with $n_z = 5$

We take $\epsilon_0 = 0$.

Case A: 70 miles per hour				
Expmnt	RE	$b\alpha_v^2$	$b\epsilon$	$\|B_m\|_2$
A-1	92.03×10^{-2}	10^4	10^3	3.51
A-2	61.03×10^{-2}	10^{10}	10^{-3}	2.53
A-3	33.65×10^{-2}	10^{16}	10^3	1.38
A-4	53.57×10^{-2}	10^{14}	10^9	61.33×10^{-2}
A-5	9.4×10^{-2}	10^{14}	10^9	23.97×10^{-2}
A-6	8.14×10^{-2}	10^{14}	10^9	25.35×10^{-2}
A-7	6.62×10^{-2}	10^{12}	1	5.23×10^{-2}

Table(2.A.2)

Case B: 7 miles per hour				
Expmnt	RE	α_v^2	ϵ	$\|B_m\|_2$
B-1	91.53×10^{-2}	10^{10}	10^{-3}	0.351
B-2	65.74×10^{-2}	10^{10}	10^{-3}	0.253
B-3	29.29×10^{-2}	10^{16}	10^3	0.138
B-4	51.25×10^{-2}	10^{14}	10^9	6.133×10^{-2}
B-5	2.06×10^{-2}	10^{14}	10^9	2.397×10^{-2}
B-6	4.56×10^{-2}	10^{23}	10^{13}	2.535×10^{-2}
B-7	0.8×10^{-2}	10^{23}	10^{15}	0.523×10^{-2}

Table(2.B.2)

7.3 Sensitivity of the solution to u and v

In this part, we perturb u and v by δu and δv, respectively, considering three rates of perturbation:

- rate(I): The perturbation of each u_i is between $\pm 0 \times u_i$ and $\pm 0.03 u_i$

  The perturbation of each v_i is between $\pm 0 \times v_i$ and $\pm 0.03 v_i$

- rate(II): The perturbation of each u_i is between $\pm 0.03u_i$ and $\pm 0.07u_i$

  The perturbation of each v_i is between $\pm 0.03v_i$ and $\pm 0.07v_i$

- rate(III): The perturbation of each u_i is between $\pm 0.07u_i$ and $\pm 0.12u_i$

  The perturbation of each v_i is between $\pm 0.07v_i$ and $\pm 0.12v_i$

And in each experiment, the values of α_v^2, α_m^2 and ϵ are $b\alpha_v^2$, $b\alpha_m^2$ and $b\epsilon$, respectively.

7.3.1 Case(A) (70 $miles/hour$)

Expmnt	$\lVert\bar{u}\rVert_d$	$\lVert\bar{v}\rVert_d$	Rate	$\frac{\lVert\delta\bar{u}\rVert_d+\lVert\delta\bar{v}\rVert_d}{\lVert\bar{u}\rVert_d+\lVert\bar{v}\rVert_d}$	$\frac{\lVert\delta\tilde{C}\rVert_d}{\lVert\tilde{C}\rVert_d}$
(A-1)	$49.48\ 10^{-2}$	$49.48\ 10^{-2}$	(I)	$17.75\ 10^{-3}$	$90.46\ 10^{-3}$
			(II)	$51.07\ 10^{-3}$	$12.29\ 10^{-2}$
			(III)	$96.55\ 10^{-3}$	$13.7\ 10^{-2}$
(A-2)	$49.63\ 10^{-2}$	$49.63\ 10^{-2}$	(I)	$17.4\ 10^{-3}$	$25.63\ 10^{-3}$
			(II)	$51.46\ 10^{-3}$	$60.88\ 10^{-3}$
			(III)	$96.18\ 10^{-3}$	$12.49\ 10^{-2}$
(A-3)	$49.67\ 10^{-2}$	$49.67\ 10^{-2}$	(I)	$17.25\ 10^{-3}$	$50.21\ 10^{-3}$
			(II)	$52.08\ 10^{-3}$	$11.4\ 10^{-2}$
			(III)	$95.85\ 10^{-3}$	$21.43\ 10^{-2}$
(A-4)	$54.09\ 10^{-2}$	$44.81\ 10^{-2}$	(I)	$17.26\ 10^{-3}$	$97.7\ 10^{-3}$
			(II)	$51.77\ 10^{-3}$	$27.7\ 10^{-2}$
			(III)	$96.35\ 10^{-3}$	$69.25\ 10^{-2}$
(A-5)	$20.94\ 10^{-2}$	$14.11\ 10^{-2}$	(I)	$17.71\ 10^{-3}$	$76.84\ 10^{-3}$
			(II)	$51.62\ 10^{-3}$	$41.88\ 10^{-2}$
			(III)	$96.11\ 10^{-3}$	$60.12\ 10^{-2}$
(A-6)	$19.68\ 10^{-2}$	$15.82\ 10^{-2}$	(I)	$17.07\ 10^{-3}$	$17.74\ 10^{-2}$
			(II)	$51.95\ 10^{-3}$	$61.58\ 10^{-2}$
			(III)	$96.71\ 10^{-3}$	$83.29\ 10^{-2}$
(A-7)	$40.47\ 10^{-3}$	$32.43\ 10^{-3}$	(I)	$16.93\ 10^{-3}$	$16.81\ 10^{-2}$
			(II)	$51.59\ 10^{-3}$	$37.26\ 10^{-2}$
			(III)	$96.07\ 10^{-3}$	$66.93\ 10^{-2}$

Table(2.A)

7.3.2 Case(B) ($7\,miles/hour$)

Expmnt	$\lVert\bar{u}\rVert$	$\lVert\bar{v}\rVert$	Rate	$\frac{\lVert\delta\bar{u}\rVert+\lVert\delta\bar{v}\rVert}{\lVert\bar{u}\rVert+\lVert\bar{v}\rVert}$	$\frac{\lVert\delta\tilde{C}\rVert}{\lVert\tilde{C}\rVert}$
(B-1)	$49.48\ 10^{-3}$	$49.48\ 10^{-3}$	(I)	$17.24\ 10^{-3}$	$23.32\ 10^{-3}$
			(II)	$50.72\ 10^{-3}$	$44.75\ 10^{-3}$
			(III)	$95.7\ 10^{-3}$	$90.82\ 10^{-3}$
(B-2)	$49.63\ 10^{-3}$	$49.63\ 10^{-3}$	(I)	$17.25\ 10^{-3}$	$29.16\ 10^{-3}$
			(II)	$50.47\ 10^{-3}$	$63.9\ 10^{-3}$
			(III)	$94.93\ 10^{-3}$	$12.69\ 10^{-2}$
(B-3)	$49.67\ 10^{-3}$	$49.67\ 10^{-3}$	(I)	$17.01\ 10^{-3}$	$52.07\ 10^{-3}$
			(II)	$51.01\ 10^{-3}$	$13.65\ 10^{-2}$
			(III)	$95.66\ 10^{-3}$	$21.36\ 10^{-2}$
(B-4)	$54.09\ 10^{-3}$	$44.81\ 10^{-3}$	(I)	$17.06\ 10^{-3}$	$10.92\ 10^{-2}$
			(II)	$51.6\ 10^{-3}$	$35.8\ 10^{-2}$
			(III)	$96.18\ 10^{-3}$	$57.28\ 10^{-2}$
(B-5)	$20.94\ 10^{-3}$	$14.11\ 10^{-3}$	(I)	$17.25\ 10^{-3}$	$72.18\ 10^{-3}$
			(II)	$51.39\ 10^{-3}$	$25.65\ 10^{-2}$
			(III)	$95.9\ 10^{-3}$	$63.18\ 10^{-2}$
(B-6)	$19.68\ 10^{-3}$	$15.82\ 10^{-3}$	(I)	$17.23\ 10^{-3}$	$14.19\ 10^{-2}$
			(II)	$51.11\ 10^{-3}$	$50.27\ 10^{-2}$
			(III)	$96.07\ 10^{-3}$	$95.2\ 10^{-2}$
(B-7)	$40.47\ 10^{-4}$	$32.43\ 10^{-4}$	(I)	$17.34\ 10^{-3}$	$85.05\ 10^{-3}$
			(II)	$51.17\ 10^{-3}$	$42.81\ 10^{-2}$
			(III)	$96.63\ 10^{-3}$	$30.04\ 10^{-2}$

Table(2.B)

From tables (1.A.1) and (1.B.1) one can conclude that working with a very fine mesh is a good idea because, obviously, a finer mesh gives a lower RE. But if we look at tables (2.A) and (2.B) we see that a finer mesh produces a less stable result. One thing that can be done to show this is to redo experiments (A-5), (A-6), (B-5) and (B-6) but this time with $n_x = 5$, $n_y = 5$ and $n_z = 3$.

For $n_x = 5$, $n_y = 5$ and $n_z = 3$ the results are:

(In each experiment, we keep the same l_x, l_y and l_z that were previously used.)

Expt	RE	$b\alpha_v^2$	$b\alpha_m^2$	$b\epsilon$	Rate	$\frac{\|\delta\bar{u}\|+\|\delta\bar{v}\|}{\|\bar{u}\|+\|\bar{v}\|}$	$\frac{\|\delta\tilde{C}\|}{\|\tilde{C}\|}$
(A-5)	$11.9\ 10^{-3}$	10^5	10^{10}	0.1	(I)	$17.21\ 10^{-3}$	$24.94\ 10^{-3}$
					(II)	$49.82\ 10^{-3}$	$69.49\ 10^{-3}$
					(III)	$96.86\ 10^{-3}$	$19.32\ 10^{-2}$
(A-6)	$10.62\ 10^{-3}$	10	10^5	10^{-7}	(I)	$17.21\ 10^{-3}$	$80.46\ 10^{-3}$
					(II)	$49.63\ 10^{-3}$	$17.15\ 10^{-2}$
					(III)	$99.15\ 10^{-3}$	$39.84\ 10^{-2}$
(B-5)	$12.9\ 10^{-3}$	10^5	10^8	10^{-3}	(I)	$16.63\ 10^{-3}$	$22.4\ 10^{-3}$
					(II)	$49.53\ 10^{-3}$	$11.51\ 10^{-2}$
					(III)	$97.18\ 10^{-3}$	$16.37\ 10^{-2}$
(B-6)	$10.02\ 10^{-3}$	10^3	10^5	10^{-7}	(I)	$17.58\ 10^{-3}$	$42.39\ 10^{-3}$
					(II)	$52.39\ 10^{-3}$	$23.39\ 10^{-2}$
					(III)	$93.56\ 10^{-3}$	$56.58\ 10^{-2}$

Table(C-1)

It is clear that $\frac{\|\delta\tilde{C}\|}{\|\tilde{C}\|}$ decreases which is not in contradiction with what we claimed.

One possible explanation is that as we make the mesh coarser we allow w_0^N and *perturbed* w_0^N to move farther from w_0 and *perturbed* w_0, respectively. And it is more likely that they, w_0^N and *perturbed* w_0^N, get close to each other; but this does not make much sense because they may also drift away from each other. The other explanation is that, in a coarser mesh, we use a smaller number of points and thus we are dealing with perturbed matrix of a smaller size.

7.4 A time-dependent Beltrami flow

In this section, we will give accuracy results for a beltrami flow that depends on time; which means that $\tau \neq 0$. Here $\tau = 1.6 \times 10^{-4}$.

Case 1 For $t = 9$ seconds and $\Delta t = 4$ seconds, which means $t_{i-1} = 5$ and $t_i = 9$, we obtain the tables

Experiment	$b\alpha_v^2$	$b\alpha_m^2$	$b\epsilon$	RE	$E\delta$
A-2	0.1	10^7	10^{-6}	12.65×10^{-2}	0.31
A-4	0.1	10^5	10^{-8}	39.04×10^{-3}	0.78
A-6	10^2	10^4	10^{-6}	31.15×10^{-4}	0.81

Table D-1

Experiment	$b\alpha_v^2$	$b\alpha_m^2$	$b\epsilon$	RE	$E\delta$
B-2	10	10^7	10^{-6}	17.05×10^{-2}	0.31
B-4	1	10^4	10^{-9}	51.53×10^{-3}	0.78
B-6	10^2	10^4	10^{-6}	17.46×10^{-3}	0.81

Table D-2

where $E\delta = \frac{\left\| A_v[(D_x\bar{v}_t - D_y\bar{u}_t) - \tau(D_x^2 + D_y^2 + D_z^2)(D_x\bar{v} - D_y\bar{u})] \right\|_d}{Min\{\|A_v[(D_x\bar{v}_t - D_y\bar{u}_t)]\|_d, \tau \left\| A_v[(D_x^2 + D_y^2 + D_z^2)(D_x\bar{v} - D_y\bar{u})] \right\|_d\}}$

We can see that these results are as accurate as those of $\tau = 0$ eventhough $E\delta$ is of the order of 50%.

Case 2 For $t = 60$ seconds and $\Delta t = 60$ seconds, which means $t_{i-1} = 0$ and $t_i = 60$, the results are

Experiment	$b\alpha_v^2$	$b\alpha_m^2$	$b\epsilon$	RE	$E\delta$
A-2	0.1	10^7	10^{-6}	12.75×10^{-2}	0.31
A-4	1	10^6	10^{-10}	39.09×10^{-3}	0.78
A-6	10^5	10^8	10^{-3}	20.84×10^{-4}	0.81

Table D-3

Experiment	$b\alpha_v^2$	$b\alpha_m^2$	$b\epsilon$	RE	$E\delta$
B-2	10	10^7	10^{-6}	16.82×10^{-2}	0.29
B-4	10^4	10^8	10^{-5}	48.7×10^{-3}	0.78
B-6	10^2	10^4	10^{-6}	17.49×10^{-3}	0.81

Table D-4

Case 3 For $t = 300$ seconds and $\Delta t = 300$ seconds, which means $t_{i-1} = 0$ and $t_i = 300$, the results are

Experiment	$b\alpha_v^2$	$b\alpha_m^2$	$b\epsilon$	RE	$E\delta$
A-2	0.1	10^7	10^{-6}	12.71×10^{-2}	0.29
A-4	1	10^6	10^{-10}	39.11×10^{-3}	0.78
A-6	10^5	10^8	10^{-3}	20.83×10^{-4}	0.81

Table D-5

Experiment	$b\alpha_v^2$	$b\alpha_m^2$	$b\epsilon$	RE	$E\delta$
B-2	10	10^7	10^{-6}	15.32×10^{-2}	0.29
B-4	10^4	10^8	10^{-5}	48.58×10^{-3}	0.78
B-6	10^2	10^4	10^{-6}	17.43×10^{-3}	0.81

Table D-6

Remark 7.2 For A-6 and B-6 the numerical approximations are not as close to w_B as the ones obtained for $\tau = 0$. We think that it is due to the errors in the approximations of the derivatives of u and v with respect to time by $\frac{\Delta u}{\Delta t}$ and $\frac{\Delta v}{\Delta t}$, respectively.

7.5 Using cubic B-splines

We will consider the same mesh, $L_x = 150km$, $L_y = 150km$, $L_z = 3km$, $h_x = 10km$, $h_y = 10km$, and $h_z = 3km$ and we will study only the case $\epsilon_0 = 0$.

When we compare the accuracies of the approximations obtained by the linear splines against those from the cubic B-splines, we see that the latters are almost as accurate as the first ones except for the experiments A-1, A-2, A3, B-1, B-2 and B-3 where the cubic showed that the relative error, RE, can be significantly reduced. One obvious reason behind this improvment is that u_B, v_B and w_B are C^∞; so, in a coarse mesh, the cubic B-splines will approximate better. The relative errors are shown in the following table

Experiment	RE	Experiment	RE
A-1	82.37×10^{-3}	B-1	81.23×10^{-3}
A-2	60.21×10^{-3}	B-2	61.44×10^{-3}
A-3	47.04×10^{-3}	B-3	43.75×10^{-3}
A-4	38.93×10^{-3}	B-4	36.08×10^{-3}
A-5	94.1×10^{-4}	B-5	79.88×10^{-4}
A-6	29.71×10^{-4}	B-6	35.06×10^{-4}

Table E-1

CHAPTER VIII

A TECHNIQUE FOR AVOIDING INTEGRATION IN THE FINITE ELEMENT METHOD

In this chapter we develop a technique, semilar to the one in Chapter VI, that reduces the amount of time spent in the computation of the *stiffness matrix* and the *load vector* which is the most expensive part in the finite element method. Unlike this method, that calculates the exact values of the integrals that give the elements of the matrix and the vector, many techniques try to approximate the mentioned integrals which may produce a singular matrix.

In Chapter VI we target a particular problem, for which we use specific splines, while in this chapter we will deal with the general case. A general case means any variational formulation of a problem whose domain, $\Omega_1 = (0, L_1) \times (0, L_2) \times ... \times (0, L_d)$, is bounded in $\Re^d$ and whose solution is a real valued function. All the notation in this chapter are not related to the notation in the previous chapters except for the notation of the Kronecker product $\otimes$ and the product $\bar{\times}$.

Suppose that we want to approximate $u \in H(\Omega_1) = H\ (\ \Omega_1 \subset \Re^d\ )$ with $a(u, v) = f(v)\ \forall v \in H$ using the finite element method; where H is a Hilbert space whose elements are functions from $\Re^d$ to $\Re^s$ with $s = 1$ (similar method can be developed for $s > 1$), $a(\cdot, \cdot)$ is a bilinear form on H and $f(\cdot)$ is a linear functional on H. The bilinear form and the linear functional can be written as

$$a(u, v) = \sum_{ni=1}^{na} a_{ni}(u, v) \tag{8.1}$$

and

$$f(v) = \sum_{ni=1}^{nf} f_{ni}(v) \tag{8.2}$$

where the $a_{ni}(\cdot, \cdot)$ s are bilinear forms and the $f_{ni}(\cdot)$ s are linear functionals. The *siffness matrix* M and *load vector* B can be written as

$$M = \sum_{ni=1}^{na} M_{ni} \tag{8.3}$$

and

$$B = \sum_{ni=1}^{nf} B_{ni} \tag{8.4}$$

where $[M_{ni}]_{IJ} = a_{ni}(\Phi_I, \Phi_J)$ and $[B_{ni}]_I = f_{ni}(\Phi_I)$.

Here we consider

- $$\begin{aligned} &\Phi_I(x) = \textstyle\prod_{r=1}^{d} \phi_{i_r,h_r}(x_r) \text{ for all } x = (x_1, ..., x_d) \in \Omega \subset \Re^d \\ &\text{with } 1 \le i_1 \le m_1 \ , \ \cdots \ , 1 \le i_r \le m_r \text{ and } h_1 > 0 \ , \ \cdots \ , h_d > 0, \end{aligned} \tag{8.5}$$

- $\phi_{i,h}(t) = \phi(\frac{t}{h} + \alpha i + \beta)$ with $\alpha = \pm 1$, β an integer and $\phi : \Re \longrightarrow \Re$ a spline of degree p that is non-zero on q intervals whose lengths are all equal to 1. Which means that

  $$\phi(t) = \varphi_i(t) \text{ for } i-1 \le t \le i \ , \ i = 1, ..., q, \tag{8.6}$$

  we will say that the spline ϕ has q intervals,

- m_r is the number of splines, ϕ_{i_r,h_r}, in the r^{th} dimension,

- h_r is the mesh step in the r^{th} dimension,

- $$I = i_1 + (i_2 - 1)m_1 + (i_3 - 1)m_1m_2 + ... + (i_k - 1)\prod_{r=1}^{k-1} m_r + ... + (i_d - 1)\prod_{r=1}^{d-1} m_r$$
  for $i_1 = 1, ..., m_1, ...i_d = 1, ..., m_d$ (8.7)

- and $N = \prod_{r=1}^{d} m_r$

Let' s define, on the interval $[0, 1]$, the functions

$$\psi_i(t) = \varphi_i(t + i - sc) \text{ for } i = 1, ..., q \tag{8.8}$$

with the integer sc is fixed and chosen such that the graph of $\psi_i(t)$ on the interval $[0, 1]$ is the same as that of $\phi(t)$ on its i^{th} interval, and consider the notation $\frac{d^k\psi_i(t)}{dt^k} = \psi_i^{(k)}(t)$.

8.1 The stiffness matrix

For ni_o with $1 \leq ni_o \leq na$ suppose that

$$a_{ni_o}(u, v) = constant \int_{\Omega_1} \left[\prod_{j=1}^{R_{ni_o}} D^{\gamma_{(ni_o,j)}} F_{(ni_o,j)}(x)\right] D^{\gamma_{(ni_o,o)}} u(x) D^{\gamma_{(ni_o,oo)}} v(x) dx \tag{8.9}$$

where $\gamma_{(ni_o,j)} = \left(\gamma^{(1)}_{(ni_o,j)}, \gamma^{(2)}_{(ni_o,j)}, \cdots, \gamma^{(d)}_{(ni_o,j)}\right)$, with $\gamma^{(r)}_{(ni_o,j)} \geq 0$, $r = 1, ..., d$, for $1 \leq j \leq R_{ni_o}$,

$\gamma_{(ni_o,o)} = \left(\gamma^{(1)}_{(ni_o,o)}, \gamma^{(2)}_{(ni_o,o)}, \cdots, \gamma^{(d)}_{(ni_o,o)}\right)$, with $\leq \gamma^{(r)}_{(ni_o,o)} \leq p$, $r = 1, ..., d$,

and $\gamma_{(ni_o,oo)} = \left(\gamma^{(1)}_{(ni_o,oo)}, \gamma^{(2)}_{(ni_o,oo)}, \cdots, \gamma^{(d)}_{(ni_o,oo)}\right)$, with $0 \leq \gamma^{(r)}_{(ni_o,oo)} \leq p$, $r = 1, ..., d$.

Here, all the $D^{\gamma_{(ni_o,j)}} F_{(ni_o,j)}(x)$ s are non-constant functions. If one of them is constant then it will be taken out of the integral and dumpped in the *constant*.

The $D^{\gamma_{(ni_o,j)}} F_{(ni_o,j)}(x)$ can be classifed into three categories:

1. The ones whose explicit expressions are known.

2. The ones for which we only have measurements at the mesh points.

3. The ones for which we do not have measurements but we have measurements of their $F_{(ni_o,j)}(x)$ s at the mesh points.

Each one from the first or the second category will be approximated, directly, by the Φ_I s. For the ones from the third category, we will approximate the measurements of the $F_{(ni_o,j)}(x)$ s. So that we will write them in the finite element method as

$$D^{\gamma_{(ni_o,j)}} F_{(ni_o,j)}(x) = \begin{cases} \sum_{k=1}^{N} C_k^{(ni_o,j)} \Phi_k(x) \text{ for functions from category 1 or 2} \\ \sum_{k=1}^{N} C_k^{(ni_o,j)} D^{\gamma_{(ni_o,j)}} \Phi_k(x) \text{ for functions from category 3} \end{cases} \tag{8.10}$$

Remark 8.1

For the functions that belong to the third category we have to have $0 \leq \gamma_{(ni_o,j)}^{(r)} \leq p$, $r = 1, ..., d$, otherwise we will need splines of a higher order.

Now, let' s rearrange our functions such that the $\prod_{j=1}^{R_{ni_o}} D^{\gamma_{(ni_o,j)}} F_{(ni_o,j)}(x)$ can be written as the product of two terms. The first term is

$$\prod_{j=1}^{R_{ni_1}} D^{\gamma_{(ni_o,j)}} F_{(ni_o,j)}(x) \tag{8.11}$$

where the $D^{\gamma_{(ni_o,j)}} F_{(ni_o,j)}(x)$ s belong to either the first or the second category, and the second term is

$$\prod_{j=R_{ni_1}+1}^{R_{ni_o}} D^{\gamma_{(ni_o,j)}} F_{(ni_o,j)}(x) \tag{8.12}$$

where the $D^{\gamma_{(ni_o,j)}} F_{(ni_o,j)}(x)$ s belong to the third category.

Having this done, we will have

$$a_{ni_o}(u,v) = constant \int_{\Omega_1} \left\{ \begin{array}{c} \left[\prod_{j=1}^{R_{ni_1}} D^{\gamma_{(ni_o,j)}} F_{(ni_o,j)}(x)\right] \\ \times \left[\prod_{j=R_{ni_1}+1}^{R_{ni_o}} D^{\gamma_{(ni_o,r)}} F_{(ni_o,r)}(x)\right] \\ \times \left[D^{\gamma_{(ni_o,o)}} u(x) D^{\gamma_{(ni_o,oo)}} v(x)\right] dx \end{array} \right\} \tag{8.13}$$

thus

$$[M_{ni_o}]_{IJ} = constant \int_{\Omega_1} \left\{ \begin{array}{c} \left[\prod_{j=1}^{R_{ni_1}} \left(\sum_{K=1}^{N} C_K^{(ni_o,j)} \Phi_K(x)\right)\right] \\ \times \left[\prod_{j=R_{ni_1}+1}^{R_{ni_o}} \left(\sum_{K=1}^{N} C_K^{(ni_o,j)} D^{\gamma_{(ni_o,j)}} \Phi_K(x)\right)\right] \\ \times \left[D^{\gamma_{(ni_o,o)}} \Phi_I(x) D^{\gamma_{(ni_o,oo)}} \Phi_J(x)\right] dx \end{array} \right\}. \tag{8.14}$$

Let

$$\pi(ni_o, r) = \left(\gamma^{(r)}_{(ni_o,R_{ni_1}+1)}, \gamma^{(r)}_{(ni_o,R_{ni_1}+2)}, \cdots, \gamma^{(r)}_{(ni_o,R_{ni_o})}, \gamma^{(r)}_{(ni_o,o)}, \gamma^{(r)}_{(ni_o,oo)}\right), \tag{8.15}$$

$$|\pi(ni_o, r)| = \gamma^{(r)}_{(ni_o,R_{ni_1}+1)} + \gamma^{(r)}_{(ni_o,R_{ni_1}+2)} + \cdots + \gamma^{(r)}_{(ni_o,R_{ni_o})} + \gamma^{(r)}_{(ni_o,o)} + \gamma^{(r)}_{(ni_o,oo)} \tag{8.16}$$

and define the $\underbrace{q \times q \times q \times \cdots \times q}_{R_{ni_o} \text{ times}} \times q \times q$ matrices $G(\pi(ni_o, r))$ where

$$[G(\pi(ni_o, r))]_{i_1, i_2, \ldots, i_{R_{ni_o}}, s_1, s_2} =$$

$$\int_0^1 \left[\prod_{j=1}^{R_{ni_1}} \psi_{i_j}(t)\right] \left[\prod_{j=R_{ni_1}+1}^{R_{ni_o}} \psi_{i_j}^{\left(\gamma^{(r)}_{(ni_o,r)}\right)}(t)\right] \psi_{s_1}^{\left(\gamma^{(r)}_{(ni_o,o)}\right)}(t) \psi_{s_2}^{\left(\gamma^{(r)}_{(ni_o,oo)}\right)}(t) dt$$

$$i_j = 1, ..., q \text{ for } j = 1, ..., R_{ni_o}\ ,\ s_1 = 1, ..., q \text{ and } s_2 = 1, ..., q \tag{8.17}$$

where $r = 1, ..., d$. The elements of the matrix mabove are the coefficients that do not depend on any mesh; and once stored, these elements make the computation of M and B quick and easy.

Let k_1, k_2,..., and $k_{R_{ni_o}}$ be R_{ni_o} integers such that

$$\begin{array}{rcl} -q_o \leq & k_1 & \leq q_o \\ Max\{-q_o, k_1 - q_o\} \leq & k_2 & \leq Min\{q_o, k_1 + q_o\} \\ & \vdots & \\ Max\{-q_o, k_1 - q_o, ..., k_{j-1} - q_o\} \leq & k_j & \leq Min\{q_o, k_1 + q_o, ..., k_{j-1} + q_o\} \\ & \vdots & \\ Max\{-q_o, k_1 - q_o, ..., k_{R_{ni_o}-1} - q_o\} \leq & k_{R_{ni_o}} & \leq Min\{q_o, k_1 + q_o, ..., k_{R_{ni_o}-1} + q_o\} \end{array} \tag{8.18}$$

where $q_o = q - 1$, and

$$\mathcal{S}_{ni_o} = \left\{ k = (k_1, ..., k_{R_{ni_o}}) \text{ ; such that } k_1, ..., k_{R_{ni_o}} \text{ satisfy condition (8.18)} \right\}. \tag{8.19}$$

For a positive integer n, a positive real number h, all k s in the set $\mathcal{S}_{ni_o}$ and the $\pi(ni_o, r)$, $r = 1, ..., d$, define the $n \times n$ matrices $G^k\left(\pi(ni_o, r), n, h\right)$ such that

$$\left[G^k\left(\pi(ni_o, r), n, h\right)\right]_{ij} = \begin{cases} h^{1-|\pi(ni_o,r)|} \sum_{s=lb(k,i,j)}^{ub(k,i,j)} \left[G\left(\pi(ni_o, r)\right)\right]_{s-k_1-i, s-k_2-i, ..., s-k_{R_{ni_o}}, s-i, s-j} \text{ if} \\ \qquad lbi(k) \leq i \leq ubi(k) \text{ and} \\ \qquad lbj(k, i) \leq j \leq ubj(k, i) \text{ where} \\ \qquad lbi(k) = Max\{1, 1 - k_1, ..., 1 - k_{R_{ni_o}}\}, \\ \qquad ubi(k) = Min\{n, n - k_1, n - k_2, ..., n - k_{R_{ni_o}}\}, \\ \qquad lbj(k, i) = Max\{1, Max\{k_1, k_2, ..., k_{R_{ni_o}}, 0\} + i - q_o\}, \\ \qquad ubj(k, i) = Min\{n, Min\{k_1, k_2, ..., k_{R_{ni_o}}, 0\} + i + q_o\}, \\ \qquad lb(k, i, j) = Max\{q + 1, Max\{k_1, k_2, ..., k_{R_{ni_o}}, 0, j - i\} + i + 1\} \text{ and} \\ \qquad ub(k, i, j) = Min\{n + 1, Min\{k_1, k_2, ..., k_{R_{ni_o}}, 0, j - i\} + i + q\}. \\ \\ 0 \qquad \text{otherwise,} \end{cases} \tag{8.20}$$

the element

$$
\begin{aligned}
&\left[G^k\left(\pi(ni_o, r), n, h\right)\right]_{ij} = \\
&\quad \int_0^{(n-q_o)h} \left(\prod_{s=1}^{R_{ni_1}} \phi_s(t)\right) \left(\prod_{s=R_{ni_1}+1}^{R_{ni_o}} \phi_s^{\left(\gamma^{(r)}_{(ni_o,r)}\right)}(t)\right) \phi_i^{\left(\gamma^{(r)}_{(ni_o,o)}\right)}(t) \phi_j^{\left(\gamma^{(r)}_{(ni_o,oo)}\right)}(t).
\end{aligned} \tag{8.21}
$$

Let $\mathcal{S}_{ni_o,r} = \mathcal{S}_{ni_o}$, $r = 1, ..., d$, be copies of the set $\mathcal{S}_{ni_o}$; we denote one element from the set $\mathcal{S}_{ni_o,r}$ by $k^{(r)} = \left(k_1^{(r)}, ..., k_{R_{ni_o}}^{(r)}\right)$. For some $k^{(1)} \in \mathcal{S}_{ni_o,1}$,...,$k^{(d)} \in \mathcal{S}_{ni_o,d}$, some or all of the $k^{(d)}$ s may be equal, define, for $j = 1, ..., R_{ni_o}$, the matrices

$$
\mathcal{T}(j, k^{(d)}, ..., k^{(1)}) = T\left(m_1, k_j^{(d)}\right) \otimes T\left(m_{d-1}, k_j^{(d)}\right) \otimes \cdots \otimes T\left(m_1, k_j^{(1)}\right), \tag{8.22}
$$

where T is defined in (6.45), the matrix

$$
G\left(k^{(d)}, ..., k^{(1)}\right) = \left\{
\begin{array}{c}
G^{k^{(1)}}\left(\pi(ni_o, d), m_d, h_d\right) \\
\otimes \\
G^{k^{(d-1)}}\left(\pi(ni_o, d-1), m_{d-1}, h_{d-1}\right) \\
\otimes \\
\vdots \\
\otimes \\
G^{k^{(1)}}\left(\pi(ni_o, 1), m_1, h_1\right)
\end{array}
\right\} \tag{8.23}
$$

and the vector

$$
C^{ni_o, k^{(d)}, ..., k^{(1)}} = \left\{
\begin{array}{c}
\left[\mathcal{T}(1, k^{(d)}, ..., k^{(1)}) \cdot C^{(ni_o,1)}\right] \\
\bar{\times} \\
\left[\mathcal{T}(2, k^{(d)}, ..., k^{(1)}) \cdot C^{(ni_o,2)}\right] \\
\bar{\times} \\
\vdots \\
\bar{\times} \\
\left[\mathcal{T}(R_{ni_o}, k^{(d)}, ..., k^{(1)}) \cdot C^{(ni_o,R_{ni_o})}\right]
\end{array}
\right\} \tag{8.24}
$$

where $C^{(ni_o,1)} = \begin{bmatrix} C_1^{(ni_o,1)} \\ C_2^{(ni_o,1)} \\ \vdots \\ C_N^{(ni_o,1)} \end{bmatrix}, \ldots, C^{(ni_o,R_{ni_o})} = \begin{bmatrix} C_1^{(ni_o,R_{ni_o})} \\ C_2^{(ni_o,R_{ni_o})} \\ \vdots \\ C_N^{(ni_o,R_{ni_o})} \end{bmatrix}$.

Thus we obtain

$$M_{ni_o} = \sum_{ni_o} G\left(k^{(1)}, ..., k^{(d)}\right) \bar{\times} C^{ni_o,k^{(1)},...,k^{(d)}} \tag{8.25}$$

where

$$\sum_{ni_o} = \sum_{k^{(d)} \in \mathcal{S}_{ni_o,d}} \; \sum_{k^{(d-1)} \in \mathcal{S}_{ni_o,d-1}} \cdots \sum_{k^{(2)} \in \mathcal{S}_{ni_o,2}} \; \sum_{k^{(1)} \in \mathcal{S}_{ni_o,1}} \tag{8.26}$$

8.2 The load vector

The load vector is obtained by means of the same technique where, if for some $1 \leq ni_o \leq n_f$ we have

$$f_{ni_o}(v) = constant \int_{\Omega_1} \left\{ \begin{array}{c} \left[\prod_{j=1}^{R_{ni_1}} D^{\gamma_{(ni_o,j)}} F_{(ni_o,j)}(x)\right] \\ \times \left[\prod_{j=R_{ni_1}+1}^{R_{ni_o}} D^{\gamma_{(ni_o,r)}} F_{(ni_o,r)}(x)\right] \\ \times \left[D^{\gamma_{(ni_o,o)}} v(x)\right] dx \end{array} \right\} \tag{8.27}$$

which gives

$$[B_{ni_o}]_I = constant \int_{\Omega_1} \left\{ \begin{array}{c} \left[\prod_{j=1}^{R_{ni_1}} \left(\sum_{K=1}^{N} C_K^{(ni_o,j)} \Phi_K(x)\right)\right] \\ \times \left[\prod_{j=R_{ni_1}+1}^{R_{ni_o}} \left(\sum_{K=1}^{N} C_K^{(ni_o,j)} D^{\gamma_{(ni_o,j)}} \Phi_K(x)\right)\right] \\ \times \left[D^{\gamma_{(ni_o,o)}} \Phi_I(x)\right] dx \end{array} \right\}, \tag{8.28}$$

then we will put

$$\pi(ni_o, r) = \left(\gamma^{(r)}_{(ni_o,R_{ni_1}+1)}, \gamma^{(r)}_{(ni_o,R_{ni_1}+2)}, \cdots, \gamma^{(r)}_{(ni_o,R_{ni_o})}, \gamma^{(r)}_{(ni_o,o)}\right) \tag{8.29}$$

and

$$|\pi(ni_o,r)| = \gamma^{(r)}_{(ni_o,R_{ni_1}+1)} + \gamma^{(r)}_{(ni_o,R_{ni_1}+2)} + \cdots + \gamma^{(r)}_{(ni_o,R_{ni_o})} + \gamma^{(r)}_{(ni_o,o)}; \tag{8.30}$$

and we will define the the $\underbrace{q \times q \times \cdots \times q}_{R_{ni_o} \text{ times}} \times q$ matrices $G\left(\pi(ni_o,r)\right)$ where

$$[G\left(\pi(ni_o,r)\right)]_{i_1,i_2,\ldots,i_{R_{ni_o}},s} =$$

$$\int_0^1 \left[\prod_{j=1}^{R_{ni_1}} \psi_{i_j}(t)\right] \left[\prod_{j=R_{ni_1}+1}^{R_{ni_o}} \psi_{i_j}^{\left(\gamma^{(r)}_{(ni_o,r)}\right)}(t)\right] \psi_s^{\left(\gamma^{(r)}_{(ni_o,o)}\right)}(t)dt \tag{8.31}$$

$i_j = 1,...,q$ for $j = 1,...,R_{ni_o}$ and $s = 1,...,q$

where $r = 1,...,d$.

Here we will also define the space $\mathcal{S}_{ni_o}$ and the vectors $B^k\left(\pi(ni_o,r),n,h\right)$, $r = 1,...,d$, such that

$$\left[B^k\left(\pi(ni_o,r),n,h\right)\right]_i =$$
$$\begin{cases} h^{1-|\pi(ni_o,r)|} \sum_{s=lb(k,i)}^{ub(k,i)} [G\left(\pi(ni_o,r)\right)]_{s-k_1-i,s-k_2-i,\ldots,s-k_{R_{ni_o}},s-i} \text{ if} \\ \qquad lbi(k) \le i \le ubi(k) \text{ where} \\ \qquad lbi(k) = Max\{1, 1-k_1,...,1-k_{R_{ni_o}}\}, \\ \qquad ubi(k) = Min\{n, n-k_1, n-k_2,...,n-k_{R_{ni_o}}\}, \\ \qquad lb(k,i) = Max\{q+1, Max\{k_1,k_2,...,k_{R_{ni_o}},0\}+i+1\} \text{ and} \\ \qquad ub(k,i) = Min\{n+1, Min\{k_1,k_2,...,k_{R_{ni_o}},0\}+i+q\}. \\ \\ 0 \qquad \text{otherwise,} \end{cases} \tag{8.32}$$

and we will obtain

$$B_{ni_o} = \sum_{ni_o} B\left(k^{(1)}, ..., k^{(d)}\right) \bar{\times} C^{ni_o, k^{(1)}, ..., k^{(d)}} \tag{8.33}$$

where

$$B\left(k^{(d)}, ..., k^{(1)}\right) = \left\{ \begin{array}{c} B^{k^{(1)}}\left(\pi(ni_o, d), m_d, h_d\right) \\ \otimes \\ B^{k^{(d-1)}}\left(\pi(ni_o, d-1), m_{d-1}, h_{d-1}\right) \\ \otimes \\ \vdots \\ \otimes \\ B^{k^{(1)}}\left(\pi(ni_o, 1), m_1, h_1\right) \end{array} \right\} \tag{8.34}$$

8.3 How does it avoid integration?

For a particular problem, for which we use a specific kind of splines, once all the matrices $G\left(\pi(ni_o, r)\right)$ are computed we will store and never computed again because they do not depend on any mesh step or mesh size. It is clear that, once we have the mentioned matrices, the calculatiom of M and B does not require any integral computation.

The method is much faster than the element-by-element calculation of M and B by the Gaussian quadratures that use $p+1$ points and weighting coefficients in order to give the exact value of the integral of a polynomial of degree p. The technique can be even faster if m_1,...,m_d,h_1,...,and h_d are fixed by storing the matrices $\left[G^k\left(\pi(ni_o, r), m_r, h_r\right)\right]$ and the vectors $\left[B^k\left(\pi(ni_o, r), m_r, h_r\right)\right]$ or going even one more step, if the size of the memory allows it, and storing the $G\left(k^{(d)}, ..., k^{(1)}\right)$ s and $B\left(k^{(d)}, ..., k^{(1)}\right)$ s.

8.4 Numerical comparison

In this section we will compare this technique with the Gaussian quadrature, that uses 13 points to calculate the exact value of a polynomial of degree 12, by measuring the amount of time spent by each method in the computation of a simple stiffness matrix for different sizes of meshes. We will use the B-cubic splines from Chapter VI and we will investigate the one-dimensional case and the two-dimensional case.

8.4.1 The one-dimensional case

We take $h = 2$; suppose that the elements of the matrix M are

$$M_{ij} = \int \left(\sum_{k=1}^{n} u_k \phi_k(x) \right) \left(\sum_{s=1}^{n} v_s \phi_s(x) \right) \phi_i(x) \phi_j(x) dx \tag{8.35}$$

where the u_i s and the v_i s are random numbers.

Remark 8.2

When working with the Gaussian quadrature, we integrate only on the intervals on which the product $\left(\sum_{k=1}^{n} u_k \phi_k(x)\right) \left(\sum_{s=1}^{n} u_s \phi_s(x)\right) \phi_i(x) \phi_j(x)$ is not a zero function and we pick only the k s and the s s for which the $(\phi_k(x)\phi_s(x)\phi_i(x)\phi_j(x))$ s are also not zero functions. We denote by n the number of splines, by $t(n)$ the amount of time spent by our technique in the calculation of the matrix, $ts(n)$ the amount of time spent if we consider a fixed h and we store the matrices $G\left(k^{(d)}, ..., k^{(1)}\right)$ and by $tg(n)$ the amount of time spent by the element-by-element calculation using the Gaussian method, all three amounts of time are in seconds. For a machine that has a 3GHZ processor and a 1GB RAM, the results are

n	t(n)	ts(n)	tg(n)	n	t(n)	ts(n)	tg(n)
10	0.047	0.016	3.172	200	0.515	0.469	94.218
20	0.063	0.031	7.937	300	1.203	1.234	
30	0.079	0.032	12.719	400	1.906	2.047	190
40	0.094	0.047	17.515	500	2.796	3.032	
50	0.109	0.063	22.282	600	3.906	4.266	285.89
60	0.125	0.078	27.078	700	5.234	5.75	
70	0.156	0.093	31.859	800	6.687	7.532	
80	0.172	0.109	36.641	900	8.36	9.203	
90	0.172	0.14	41.437	1000	10.016	11.328	476.828
100	0.219	0.156	46.203				

Table F-1

n	$b_m = \frac{tg(n)}{Min\{t(n),ts(n)\}}$	n	$b_m = \frac{tg(n)}{Min\{t(n),ts(n)\}}$
10	198.25	80	336.16
20	256.03	90	295.98
30	397.47	100	296.17
40	372.66	200	200.89
50	353.68	400	99.68
60	347.15	600	73.19
70	342.57	1000	47.61

Table F-2

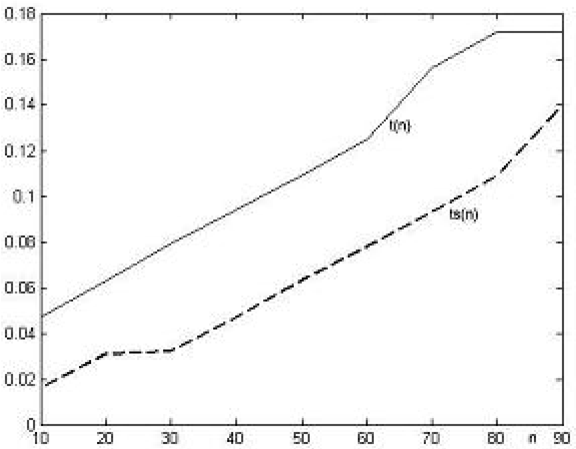

Figure 14: $10 \leq n \leq 100$

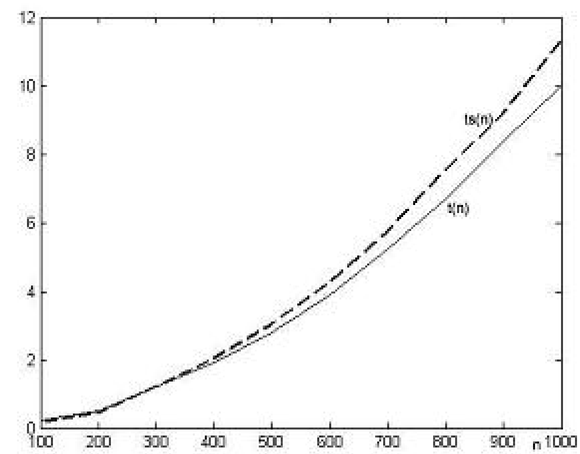

Figure 15: $200 \leq n \leq 1000$

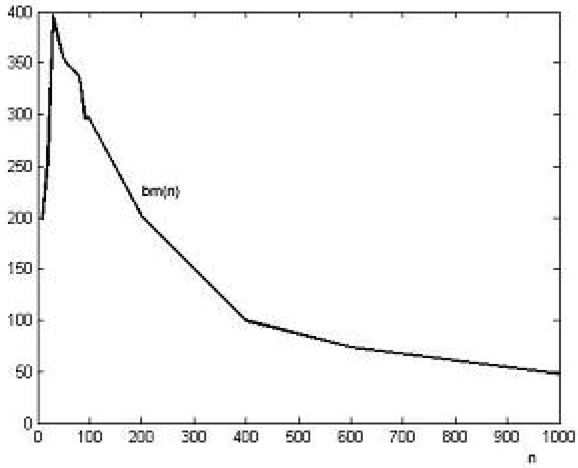

Figure 16: b_m as a function of n

8.4.2 The two-dimensional case

We will take $h_1 = 2$, $h_2 = 3$ and the same form that gives the matrix. For different values of m_1 and m_2 we obtain the following amounts of time, t_{m_1,m_2}, spent in the computation of M, using our technique.

		m_2				
		8	12	16	30	50
m_1	10	5.703	9.672	15.062		
	15	9.468	17.969	27.797		
	20	14.5	27.406	45.953		
	30				305.453	
	50					2182.391

Table F-3

Let $tg_1(i_1, i_2)$ be the function that gives the amount of time needed to compute the non-zero elements of the I^{th} row of the matrix M, where $I = i_1 + (i_2 - 1) * m_1$, using the element-by-element Gaussian integration.

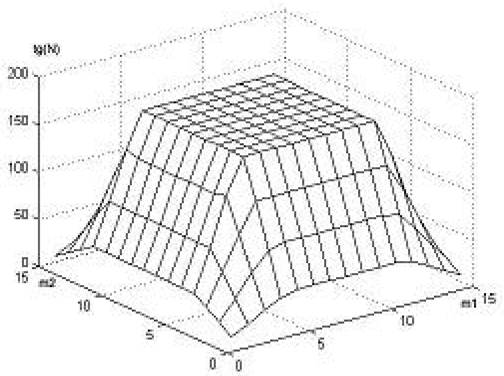

Figure 17: Graph of $tg_1(i, j)$

From this graph, we can calculate the amount of time needed to compute all the elements of the matrix M which is equal to $TG(m_1, m_2) \approx 165m_1m_2 - 23(m_1 + m_2) + 2833$ seconds, if w use the previously mentioned machine. If we calculate the ratio $\frac{TG(m_1,m_2)}{t_{m_1,m_2}}$, we obtain the table

		m_2				
		8	12	16	30	50
m_1	10	517.62	319.89	214.84		
	15	326.78	189.27	133.39		
	20	223.17	135.29	90.96		
	30				490.92	
	50					189.25

Table F-4

CHAPTER IX

CONCLUSIONS AND FURTHER WORK

In this thesis, we investigated the use of a variational fromulation of the vorticity equation and the mass continuity equation in the estimation of the vertical wind velocity. We showed that the theoretical solution exists, is unique and, as a function of the vertical components of the wind field, is *Fréchet differentiable*. We then write a method that allows us to prove that the numerical approximation, as a function of the measurements of the horizontal velocities, can be made less sensitive by choosing a mesh that has s smaller number of points. The numerical results tell us that the approximation can be made as accurate as we want; but if we choose a certain degree of accuracy then the horizontal velocities should almost have the same degree of accuracy other wise the error in w will be higher. They also reveal that the errors when working with both equations are smaller than when working with only one of them. One other numerical finding is that the method we develop in the computation of the stiffness matrix and the load vector is much faster than traditional integration methods. When we used linear splines, the amount of time needed to calculate the matrix and the vector was almost 45 minutes, using the Gaussian quadrature that requires the evaluation at $27 = 3 \times 3 \times 3$ points when integrating on each rectangular prism of the mesh. But when using our method it takes only 19 seconds to perform the task. The numerical results from using cubic B-splines showed significant improvement in the accuracy of the solution for coarse meshes; but this may not be always realizable, as the solution may not be C^2.

We also develop a technique that reduces the cost of the computation of the stiffness matrix and the load vector in a variational problem. The method is based on calculating some coefficients that do not depend on the n and the h of the mesh. And once those coefficients stored, the user will not need to integrate any other function. Numerically, this technique proved to be much more efficient than traditional Gaussian integration methods. For cubic B-splines, the technique was almost 300 times fater.

Though this thesis gives most of the theoretical analysis, there are other issues that have to be addressed:

1. Since the number of the mesh points determines how sensitive the numerical solution is, one wants to find a mesh that provide us with solutions that satisfie some predefined accuracy and stability conditions.

2. Because the values of weights of the equations play a significant role in the accuracy of the solution, one important question is: given a mesh and measurements of the horizontal wind field, which weights give the most accurate solution? We think that the answer to this question will be based on a certain classification of wind fields such that each class contains horizontal wind fields whose values are in some and the values of their derivatives are in some set.

3. The technique we presented that reduces the computational time of the stiffness matrix and the load vector is for a domains that has the shape of a rectangular prism with a uniform mesh. Future work in this area will be adapting the technique to different shapes and non-uniform meshes and developing a method that uses a different kind of splines in each dimension.

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