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OSCILLATION OF HIGHER ORDER DELAY AND
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CONTENTS

OSCILLATION OF HIGHER ORDER DELAY AND FUNCTIONAL DIFFERENTIAL EQUATIONS

SECTION 0

INTRODUCTION

In this paper, we shall give some oscillation criteria for delay and functional differential equations and also for damped differential equations of arbitrary order $n \geq 2$. Our theorems extend some results in [3]-[8], [11], [15], [17], [19], [20], [22], [23], [27]-[29], and [33].

A nontrivial solution of a differential equation which exists on the half line $[t_0,\infty)$ is said to be oscillatory if it has a zero on the interval $[t,\infty)$ for every $t > t_0$; otherwise it is said to be non-oscillatory on $[t_0,\infty)$. Throughout this paper, only nontrivial solutions of the corresponding differential equation which are extensible infinitely to the right of zero on the real line are considered.

We denote by $C(X,Y)$ the class of continuous functions with domain X and range in Y. The set of all real numbers is denoted by $\mathcal{R}$. We simply write $C(X)$ for $C(X,\mathcal{R})$. If $X = [a,b]$, we write $C[a,b]$ for $C([a,b])$. Similarly for X being open or half open intervals, we write $C(a,b)$, $C[a,b)$ and $C(a,b]$ respectively for $C((a,b))$, $C([a,b))$ and $C((a,b])$.

SECTION I

Consider the n'th order delay differential equations of the form

$$(1.1) \qquad y^{(n)}(t) + (-1)^{n+1} \sum_{i=1}^{m} F_i(t,y(t),y(g_i(t))) = 0 ,$$

where the functions F_i and g_i ($i = 1,2,\ldots,m$) satisfy the following conditions:

(I_1) $F_i \in C([0,\infty) \times \mathbb{R}^2)$, $i = 1,2,\ldots,m$, <u>and for some index</u> k, $1 \leq k \leq m$, $F_k(t,u,v_k)$ <u>is nondecreasing in</u> u <u>and in</u> v_k <u>for each fixed</u> t;

(I_2) $F_i(t,u,v_i)$ <u>has the same sign as that of</u> u <u>and</u> v_i <u>for</u> $i = 1,2,\ldots,m$, <u>if</u> $uv_i > 0$;

(I_3) $g_i \in C[0,\infty)$, $g_i(t) \to \infty$ <u>as</u> $t \to \infty$ <u>for</u> $i = 1,2,\ldots,m$.

For convenience, we shall employ the following notation.

S = the set of all solutions of the given differential equation which are continuable indefinitely to the right.

For an integer ℓ, $0 \leq \ell \leq n-1$,

$$S_\ell^{+\infty} = \{y \in S: \lim_{t\to\infty} y^{(j)}(t) = \infty,\ j = 0,1,\ldots,\ell\},$$

$$S_\ell^{-\infty} = \{y \in S: -y \in S_\ell^{+\infty}\},$$

$$S^{e} = \{y \in S: 0 < \lim_{t\to\infty} y(t) < \infty \text{ and } \lim_{t\to\infty} y^{(j)}(t) = 0,\ j = 1,2,\ldots,n-1\},$$

$$S^{-e} = \{y \in S: -y \in S^{e}\} ,$$

$$S^0 = \{y \in S: \lim_{t\to\infty} y^{(j)}(t) = 0 \text{ monotonically}, j = 0,1,\ldots,n-1\},$$

$$S^{\sim} = \{y \in S: y(t) \text{ is oscillatory}\} .$$

In this section and the next we shall classify solutions of certain differential equations and inequalities in terms of the sets defined above. This will be done under certain assumptions concerning the convergence, or divergence, of an improper integral over $[0,\infty)$ where the integrand involves certain coefficient functions. Previous related results are numerous in the literature. Much of our work relates directly to that of Ladas, Ladde, Lakshmikantham and Papadakis, [17], [22], [23].

We shall provide three preliminary lemmas, the first two are those of Kiguradze [14].

LEMMA 1.1 (Lemma 1 in [14]). _If y is a function, which together with its derivatives of order up to $(n-1)$ inclusive, is absolutely continuous and of constant sign on the interval $[t_0,\infty)$ and $y^{(n)}(t)y(t) \le 0$ on $[t_0,\infty)$, then there is an integer ℓ, $0 \le \ell \le n-1$, which is odd when n is even and even when n is odd, such that_

$$y^{(j)}(t)\; y(t) \ge 0, \; j = 0,1,\ldots,\ell, \tag{1.2}$$

$$(-1)^{n+j-1} y^{(j)}(t)\; y(t) \ge 0, \; j = \ell+1,\ldots,n,$$

and

$$|y(t)| \ge \frac{(t - t_0)^{n-1}}{(n-1)(n-2)\ldots(n-\ell)}\, |y^{(n-1)}(2^{n-\ell-1}\, t)| \tag{1.3}$$

for $t \in [t_0,\infty)$.

LEMMA 1.2 (Lemma 2 in [14]). *If* y *is a function, which together with its derivatives of order up to* $(n-1)$ *inclusive, is absolutely continuous and of constant sign on the interval* $[t_0,\infty)$ *and* $y^{(n)}(t)\, y(t) \geq 0$ *on* $[t_0,\infty)$, *then either*

$$y^{(j)}(t)\, y(t) \geq 0, \quad j = 0,1,\ldots,n, \tag{1.4}$$

or there is an integer ℓ, $0 \leq \ell \leq n-2$, *which is even when* n *is even and odd when* n *is odd, such that*

$$\begin{aligned} &y^{(j)}(t)\, y(t) \geq 0, \quad j = 0,1,\ldots,\ell \\ &(-1)^{n+j}\, y^{(j)}(t)\, y(t) \geq 0, \quad j = \ell+1,\ldots,n, \end{aligned} \tag{1.5}$$

and the inequality (1.3) *holds for* $t \in [t_0,\infty)$.

LEMMA 1.3 *If* y *is as in* Lemma 1.1 (*or* Lemma 1.2), *and if for some* $0 \leq j \leq n-1$,

$$\lim_{t\to\infty} y^{(j)}(t) = c, \qquad c \in \mathbb{R} \;,$$

then

$$\lim_{t\to\infty} y^{(j+m)}(t) = 0, \quad m = 1,2,\ldots,n-j-1.$$

Proof. If y is as in Lemma 1.1, the proof is referred to that of Lemma 2 in [24]. A similar proof applies to the case of y being as in Lemma 1.2.

The following result generalizes Ladde's result [23, Theorem 2.1] to the arbitrary order delay differential equation (1.1). A related theorem for (1.1) with odd $n \geq 3$ is provided by Theorem 1.3. Further related results for equation (1.1'), which appears after Theorem 1.4, are provided by Theorems 1.5 and 1.7. For a comparison of the results of this section it is natural to compare Theorem 1.1 to Theorem 1.7 and Theorem 1.3 to Theorem 1.5.

THEOREM 1.1. For $n \geq 2$ even, assume that the equation (1.1) satisfies

$$(1.6) \qquad \int^{\infty} F_k(t,\gamma t,\gamma g_k(t))\, dt = \pm\infty \text{ for each constant } \gamma \neq 0,$$

Then
$$S = S_{n-1}^{+\infty} \cup S_{n-1}^{-\infty} \cup S^{e} \cup S^{-e} \cup S^{0} \cup S^{\sim} .$$

Proof. Let $y \in S - S^{\sim}$. Then there is a $T_0 > 0$ such that $y(t) > 0$ on $[T_0,\infty)$ or $y(t) < 0$ on $[T_0,\infty)$.

Case 1. Let $y(t) > 0$ on $[T_0,\infty)$. Since $g_i(t) \to \infty$ as $t \to \infty$, there is a $T_1 \geq T_0$ such that $g_i(t) \geq T_0$ for $t \geq T_1$ $(i = 1,2,\ldots,m)$. By (1.1) and (I_2), $y^{(n)}(t) > 0$ for $t \geq T_1$. There is a $T_2 \geq T_1$ such that each $y^{(j)}(t)$, $j = 0,1,\ldots,n-1$, is of constant sign for $t \geq T_2$.

Let $y'(t) > 0$ for $t \geq T_2$. Then by Lemma 1.2, $y''(t) > 0$, and then $y'(t)$ is increasing on $[T_2,\infty)$. This implies that $0 < \lim_{t\to\infty} y'(t) = y'(\infty) \leq \infty$. Since $y(t) > 0$ and $y'(t) \geq y'(T_2) > 0$ for $t \geq T_2$, it follows that $y(t) \to \infty$ as $t \to \infty$ and we have

$$\lim_{t\to\infty} \frac{y(t)}{t} = \lim_{t\to\infty} \frac{y(t) - y(T_2)}{t - T_2} \geq y'(T_2) > 0 .$$

Thus, for $\beta = \frac{1}{2} y'(T_2) > 0$, there is a $T_3 \geq T_2$ such that $\frac{1}{t} y(t) > \beta$ for $t \geq T_3$. There is a $T_4 \geq T_3$ such that $g_k(t) \geq T_3$ for $t \geq T_4$. Thus, we have

(1.7) $\quad y(t) > \beta t$ and $y(g_k(t)) > \beta g_k(t)$ for $t \geq T_4$.

Integrating (1.1) from T_4 to $t > T_4$ and using (1.7), (I_1) and (1.6), we have

$$y^{(n-1)}(t) = y^{(n-1)}(T_4) + \int_{T_4}^{t} \sum_{i=1}^{m} F_i(s, y(s), y(g_i(s)))\, ds$$

$$\geq y^{(n-1)}(T_4) + \int_{T_4}^{t} F_k(s, y(s), y(g_k(s)))\, ds$$

$$\geq y^{(n-1)}(T_4) + \int_{T_4}^{t} F_k(s, \beta s, \beta g_k(s))\, ds \to \infty \text{ as } t \to \infty.$$

This says that (1.4) holds, since (1.5) cannot hold for $j = n-1$. Thus $y^{(j)}(t) \to \infty$ as $t \to \infty$ for all $0 \leq j \leq n-1$. Consequently $y \in S^{\infty}_{n-1}$.

Let $y'(t) < 0$ for $t \geq T_2$. Then $y(t)$ is decreasing. Since $y(t) > 0$ for $t \geq T_2$, $\lim_{t\to\infty} y(t)$ exists and is nonnegative. Moreover, by Lemma 1.3, $y'(\infty) = y''(\infty) = \ldots = y^{(n-1)}(\infty) = 0$. That is, $y \in S^e \cup S^o$.

Case 2. Let $y(t) < 0$ for $t \geq T_0$. Then the proof follows similar to the case 1 and we have the conclusion that $S = S^{-\infty}_{n-1} \cup S^{-e} \cup S^o$ for this case.

We can modify the proof of Theorem 1.1 to show that Theorem 1.1 still holds for the equation

$$y^{(n)}(t) - \sum_{i=1}^{m} F_i(t, y(t), y(g_i(t)), y'(t), y'(g_i(t))) = 0$$

if we replace conditions (I_1), (I_2) and (1.6) respectively by

(I_1') $F_i \in C([0,\infty) \times R^4)$, $i = 1,2,\ldots,m$, and for some index k, $1 \leq k \leq m$, $F_k(t,u,u_k,v,v_k)$ is nondecreasing in u and in u_k, and is also

<u>nondecreasing in</u> v <u>and in</u> v_k <u>when</u> $uu_k > 0$ <u>and</u> $vv_k > 0$,

(I_2') $F_i(t,u,u_i,v,v_i)$ <u>has the same sign as that of</u> u <u>and</u> u_i <u>for</u> $i = 1,2,\ldots,m$, <u>if</u> $uu_i > 0$, and

$$(1.6') \qquad \int^{\infty} F_k(t,\gamma t,\gamma g_k(t),\alpha,\alpha) = \pm\infty \quad \text{if } \alpha\gamma > 0.$$

The next result generalizes Theorem 2.1 of Ladas, Ladde and Papadakis [17] to equation (1.1) for even n. It also includes Theorem 3.1 of Ladas, Lakshmikantham and Papadakis [22]. The proof of our theorem is obtained by modifying that of the previous mentioned result where only linear equations are considered. At the conclusion of Section II we have further comments relating to Theorem 3.1 of [22] and the result to follow. Further related results for equation (1.1) with odd $n \geq 3$ and for equation (1.1') are provided by Theorems 1.4, 1.6 and 1.8. It is natural to compare the conclusion of Theorem 1.2 to Theorem 1.8 and Theorem 1.4 to Theorem 1.6, though we point out that the assumptions of Theorems 1.2 and 1.4 involve condition (1.8) rather than condition (1.14) used in the other results.

THEOREM 1.2. <u>For</u> $n \geq 2$ <u>even, assume that all hypotheses of</u> Theorem 1.1 <u>hold. Furthermore, assume that</u>

$$(1.8) \qquad F_k(t,\gamma w,\gamma g_k(w)) \begin{cases} \leq wF_k(t,\gamma,\gamma) & \text{for any } \gamma > 0, \\ \\ \geq wF_k(t,\gamma,\gamma) & \text{for any } \gamma < 0, \end{cases}$$

<u>for all sufficiently large</u> t <u>and</u> $w > 0$, <u>where</u> F_k <u>is as in</u> (I_1). <u>Then</u>

$$S = S_{n-1}^{+\infty} \cup S_{n-1}^{-\infty} \cup S^0 \cup S^{\sim} .$$

Proof. Let $y \in S - \tilde{S}$. As in the proof of Theorem 1.1, we have that each $y^{(j)}(t)$, $j = 0,1,\ldots,n-1$, is of constant sign for $t \geq T_2$.

Case 1. Let $y(t) > 0$ for $t \geq T_0$.

If $y'(t) > 0$ for $t \geq T_2$, then it follows from the proof of Theorem 1.1 that $y \in S_{n-1}^{+\infty}$.

If $y'(t) < 0$ for $t \geq T_2$, then $y(t)$ is decreasing on $[T_2,\infty)$ and then $\lim_{t\to\infty} y(t)$ exists and is nonnegative. We claim that $y(t) \to 0$ as $t \to \infty$. Otherwise suppose that $y(t) \to 2\beta > 0$ as $t \to \infty$. By Lemma 1.3, $y'(\infty) = y''(\infty) = \ldots = y^{(n-1)}(\infty) = 0$. Since $y'(t) < 0$ for $t \geq T_2$, by (1.5) of Lemma 1.2, $(-1)^j y^{(j)}(t) > 0$, $j = 0,1,\ldots,n$, for $t \geq T_2$. There is a $T_3 \geq T_2$ such that

$$y(g_k(t)) \geq \beta \text{ and } y(t) \geq \beta \text{ for } t \geq T_3. \tag{1.9}$$

Integrating (1.1) from T_3 to $t > T_3$, we have

$$y^{(n-1)}(t) = y^{(n-1)}(T_3) + \int_{T_3}^{t} \sum_{i=1}^{m} F_i(s,y(s),y(g_i(s)))\, ds. \tag{1.10}$$

Let $t \to \infty$, we obtain

$$y^{(n-1)}(T_3) = - \int_{T_3}^{\infty} \sum_{i=1}^{m} F_i(s,y(s),y(g_i(s)))\, ds. \tag{1.11}$$

Integrating (1.10) from T_3 to $t > T_3$ and using (I_2), (I_3), (1.11), (1.9) and (1.10), we have

$$y^{(n-2)}(t) = y^{(n-2)}(T_3) + (t - T_3)\, y^{(n-1)}(T_3)$$

$$+ \int_{T_3}^{t} (t - s) \sum_{i=1}^{m} F_i(s,y(s),y(g_i(s)))\, ds$$

$$= y^{(n-2)}(T_3) + \int_{T_3}^{t} (T_3 - s) \sum_{i=1}^{m} F_i(s,y(s),y(g_i(s)))\, ds$$

$$- (t - T_3) \int_{t}^{\infty} \sum_{i=1}^{m} F_i(s,y(s),y(g_i(s)))\, ds$$

$$\leq y^{(n-2)}(T_3) + \int_{T_3}^{t} (T_3 - s) \sum_{i=1}^{m} F_i(s,y(s),y(g_i(s)))\, ds$$

$$= y^{(n-2)}(T_3) + T_3 \int_{T_3}^{t} y^{(n)}(s)\, ds - \int_{T_3}^{t} s \sum_{i=1}^{m} F_i(s,y(s),y(g_i(s)))\, ds$$

$$= y^{(n-2)}(T_3) + T_3[y^{(n-1)}(y) - y^{(n-1)}(T_3)]$$

$$- \int_{T_3}^{t} s \sum_{i=1}^{m} F_i(s,y(s),y(g_i(s)))\, ds$$

$$\leq y^{(n-2)}(T_3) - T_3\, y^{(n-1)}(T_3) - \int_{T_3}^{t} s\, F_k(s,y(s),y(g_k(s)))\, ds$$

$$\leq y^{(n-2)}(T_3) - T_3\, y^{(n-1)}(T_3) - \int_{T_3}^{t} s\, F_k(s,\beta,\beta)\, ds$$

$$\leq y^{(n-2)}(T_3) - T_3\, y^{(n-1)}(T_3) - \int_{T_3}^{t} F_k(s,\beta s,\beta g_k(s))\, ds.$$

By (1.6), $y^{(n-2)}(t)$ would be negative for all sufficiently large t. This is a contradiction since $y^{(n-2)}(t) > 0$ for $t \geq T_2$. Therefore, $y(t) \to 0$ as $t \to \infty$. Moreover, by Lemma 1.3, $y \in S^0$.

Case 2. Let $y(t) < 0$ for $t \geq T_0$. Then the proof follows similar to the case 1 and it concludes that $y \in S_{n-1}^{-\infty} \cup S^0$ for this case.

THEOREM 1.3. For $n \geq 3$ odd, assume that the equation (1.1) satisfies (1.6). Then $S = S^{e} \cup S^{-e} \cup S^{o} \cup S^{\sim}$.

Proof. Let $y \in S - S^{\sim}$. As in the proof of Theorem 1.1, we discuss the following two cases.

Case 1. If $y(t) > 0$ for $t \geq T_0$. Since $g_i(t) \to \infty$ as $t \to \infty$, there is a $T_1 \geq T_0$ such that $g_i(t) \geq T_0$ for $t \geq T_1$ $(i = 1,2,\ldots,m)$. By (1.1) and (I_2), $y^{(n)}(t) < 0$ for $t \geq T_1$. There is a $T_2 \geq T_1$ such that each $y^{(j)}(t)$, $j = 0,1,\ldots,n-1$, is of constant sign for $t \geq T_2$.

If $y'(t) > 0$ for $t \geq T_2$, then $\ell \geq 2$ in (1.2) of Lemma 1.1, and $y'(t)$ is monotone increasing and positive for $t \geq T_2$. Thus, as in the proof of Theorem 1.1 there is a $\beta > 0$ and a $T_3 \geq T_2$ such that

$$\frac{1}{t}\, y(t) \geq \beta \text{ for } t \geq T_3.$$

Since $g_k(t) \to \infty$ as $t \to \infty$, there is a $T_4 \geq T_3$ such that

$$g_k(t) \geq T_3 \text{ for } t \geq T_4.$$

Thus we have

$$(1.12) \qquad y(t) \geq \beta t \text{ and } y(g_k(t)) \geq \beta g_k(t) \text{ for } t \geq T_4.$$

Integrating (1.1) from T_4 to $t \geq T_4$ and using (1.12) and (I_1), we have

$$y^{(n-1)}(t) = y^{(n-1)}(T_4) - \int_{T_4}^{t} \sum_{i=1}^{m} F_i(s,y(s),y(g_i(s)))\, ds$$

$$\leq y^{(n-1)}(T_4) - \int_{T_4}^{t} F_k(s,y(s),y(g_k(s)))\, ds$$

$$\leq y^{(n-1)}(T_4) - \int_{T_4}^{t} F_k(s,\beta s,\beta g_k(s))\, ds.$$

By (1.6), $y^{(n-1)}(t)$ would be negative for all sufficiently large t. This is a contradiction since $y^{(n-1)}(t) > 0$ for $t \geq T_2$. Therefore, $y'(t) < 0$ for $t \geq T_2$.

It follows that $\lim_{t\to\infty} y(t)$ exists and is nonnegative. Moreover, by Lemma 1.3, $y^{(j)}(t) \to 0$ as $t \to \infty$, $j = 1,2,\ldots,n-1$. That is, $y \in S^e \cup S^o$.

Case 2. If $y(t) < 0$ for $t \geq T_o$, then the proof follows similar to the case 1 that $y \in S^{-e} \cup S^o$.

THEOREM 1.4. For $n \geq 3$ odd, assume that all hypotheses of Theorem 1.3 hold. Furthermore, assume that (1.8) holds. Then $S = S^o \cup S^{\sim}$.

Proof. Let $y \in S - S^{\sim}$. As in the proof of Theorem 1.1, we have the following two cases.

Case 1. If $y(t) > 0$ for $t \geq T_o$. Then as in the proof of Theorem 1.3, we have the conclusion that $y'(t) < 0$ for $t \geq T_2$. Then $\lim_{t\to\infty} y(t)$ exists and is nonnegative. Moreover, by Lemma 1.3, $y^{(j)}(t) \to 0$ as $t \to \infty$, $j = 1,2,\ldots,n-1$. We claim that $y(t) \to 0$ as $t \to \infty$. Otherwise suppose that $y(t) \to 2\beta > 0$ as $t \to \infty$. Then there exists a $T_3 \geq T_2$ such that

$$y(g_k(t)) > \beta \text{ and } y(t) > \beta \text{ for } t \geq T_3.$$

Since $y'(t) < 0$ for $t \geq T_2$, by (1.2) of Lemma 1.1, $(-1)^j\, y^{(j)}(t) > 0$, $j = 0,1,\ldots,n$, for $t \geq T_2$. Integrating (1.1) from T_3 to $t > T_3$, we have

$$(1.13) \qquad y^{(n-1)}(t) = y^{(n-1)}(T_3) - \int_{T_3}^{t} \sum_{i=1}^{m} F_i(s,y(s),y(g_i(s)))\, ds.$$

Let $t \to \infty$, we obtain

$$y^{(n-1)}(T_3) = \int_{T_3}^{\infty} \sum_{i=1}^{m} F_i(s,y(s),y(g_i(s)))\, ds.$$

Integrating (1.13) from T_3 to $t > T_3$ and using the same procedures as in the proof of Theorem 1.2, we have

$$y^{(n-2)}(t) \geq y^{(n-2)}(T_3) - T_3\, y^{(n-1)}(T_3) + \int_{T_3}^{t} F_k(s,\beta s,\beta g_k(s))\, ds$$

which would be positive for all sufficiently large t. This is a contradiction since $y^{(n-2)}(t) < 0$ for $t \geq T_2$. Therefore, $y \in S^o$.

Case 2. If $y(t) < 0$ for $t \geq T_0$, the the proof follows similar to the case 1.

We now consider the equation

$$(1.1') \qquad y^{(n)}(t) + (-1)^n \textstyle\sum_{i=1}^m F_i(t,y(t),y(g_i(t))) = 0,$$

where F_i, g_i ($i = 1,2,\ldots,m$) satisfy (I_1), (I_2) and (I_3). We define the following subsets of S:

$$S^E = \{y \in S:\ 0 < \lim_{t\to\infty} y(t) \leq \infty \text{ and } \lim_{t\to\infty} y^{(j)}(t) = 0,\ j = 1,2,\ldots,n-1\},$$

$$S^{-E} = \{y \in S:\ -y \in S^E\}.$$

Clearly, $S^e \subseteq S^E$ and $S^{-e} \subseteq S^{-E}$ hold.

We have the following theorem corresponding to Theorem 1.1.

THEOREM 1.5. _Assume_ n _is even and that_ (1.1') _satisfies_ (1.6). _Then_ $S = S^E \cup S^{-E} \cup S^{\sim}$.

Proof. Let $y \in S - S^{\sim}$. Then there is a $T_0 > 0$ such that $y(t) > 0$ on $[T_0,\infty)$ or $y(t) < 0$ on $[T_0,\infty)$.

Case 1. If $y(t) > 0$ on $[T_0,\infty)$. Since $g_i(t) \to \infty$ as $t \to \infty$, there is a $T_1 \geq T_0$ such that $g_i(t) \geq T_0$ for $t \geq T_1$ ($i = 1,2,\ldots,m$). By (1.1') and (I_2), $y^{(n)}(t) < 0$ for $t \geq T_1$. There is a $T_2 \geq T_1$ such that each

$y^{(j)}(t)$, $j = 0,1,\ldots,n-1$, is of constant sign for $t \geq T_2$. By Lemma 1.1, $y'(t) > 0$ for $t \geq T_2$. It follows that $0 < \lim_{t\to\infty} y(t) \leq \infty$. Now either $y''(t) > 0$ for $t \geq T_2$ or $y''(t) < 0$ for $t \geq T_2$ and $0 \leq \lim_{t\to\infty} y'(t) \leq \infty$. In either case, either $y'(t) \to 0$ as $t \to \infty$ and we are done, or $\lim_{t\to\infty} y'(t) > 0$. The remainder of the argument is much as in the proof of Theorems 1.1 and 1.3. That is, $y \in S^E$.

Case 2. Let $y(t) < 0$ for $t \geq T_0$. Then the proof follows similar to the case 1 and we have the conclusion that $S = S^{-E}$ for this case.

THEOREM 1.6. <u>Assume</u> n <u>is even and that</u> (1.1') <u>satisfies</u>

$$\int^{\infty} F_k(t,\gamma,\gamma)\, dt = \pm\infty \quad \text{\underline{for each} } \gamma \neq 0. \tag{1.14}$$

<u>Then</u> $S = S^{\sim}$.

<u>Proof</u>. Let y be a nonoscillatory solution of (1.1'). We can assume that $y(t) > 0$ for $t \geq T_0$. As in the proof of Theorem 1.5, there is a $T_2 \geq T_0$ such that each $y^{(j)}(t)$, $j = 0,1,\ldots,n-1$, is of constant sign and $y'(t) > 0$ for $t \geq T_2$. It follows that $y(t) > \beta$ for some $\beta > 0$ and for each $t \geq T_2$. Since $g_k(t) \to \infty$ as $t \to \infty$, there is a $T_3 \geq T_2$ such that $y(g_k(t)) > \beta$ for $t \geq T_3$. Integrating (1.1') from T_3 to $t > T_3$, we have

$$\begin{aligned} y^{(n-1)}(t) &= y^{(n-1)}(T_3) - \int_{T_3}^{t} \textstyle\sum_{i=1}^{m} F_i(s,y(s),y(g_i(s)))\, ds \\ &\leq y^{(n-1)}(T_3) - \int_{T_3}^{t} F_k(s,y(s),y(g_k(s)))\, ds \\ &\leq y^{(n-1)}(T_3) - \int_{T_3}^{t} F_k(s,\beta,\beta)\, ds. \end{aligned}$$

It follows from (1.22) that $y^{(n-1)}(t)$ would be negative for all sufficiently large t. This is a contradiction since by Lemma 1.1, $y^{(n-1)}(t) > 0$ for $t \geq T_2$.

THEOREM 1.7. <u>For</u> $n \geq 3$ <u>odd</u>, <u>assume that</u> (1.1') <u>satisfies</u> (1.6). <u>Then</u> $S = S_{n-1}^{+\infty} \cup S_{n-1}^{-\infty} \cup S^{E} \cup S^{-E} \cup S^{\sim}$.

<u>Proof</u>. Let $y \in S - S^{\sim}$, then there is a $T_0 > 0$ such that $y(t) > 0$ or $y(t) < 0$ on $[T_0,\infty)$.

Case 1. If $y(t) > 0$ on $[T_0,\infty)$. Since $g_i(t) \to \infty$ as $t \to \infty$, there is a $T_1 \geq T_0$ such that $g_i(t) \geq T_0$, $i = 1,2,\ldots,m$, for $t \geq T_1$. Now $y^{(n)}(t) > 0$ on $[T_1,\infty)$. There is a $T_2 \geq T_1$ such that each $y^{(j)}(t)$, $j = 0,1,\ldots,n-1$, is of constant sign on $[T_2,\infty)$. By Lemma 1.2, either (1.4) holds or there is an odd integer ℓ such that (1.5) holds. If (1.4) holds, then by an argument similar to a portion of the proof of Theorem 1.1 it follows that $y^{(n-1)}(t) \to \infty$ as $t \to \infty$ so $y \in S_{n-1}^{+\infty}$. If (1.5) holds, then $y'(t) > 0$ and either $y''(t) > 0$ or $y''(t) < 0$ on $[T_2,\infty)$. It follows that $0 \leq \lim_{t\to\infty} y'(t) \leq \infty$. If $y'(t) \to 0$ as $t \to \infty$, then by Lemma 1.3, $y \in S^{E}$. If $0 < \lim_{t\to\infty} y'(t) \leq \infty$, then as previously we have $y^{(n-1)}(t) > 0$ for all sufficiently large t, a contradiction. Thus, $y \in S_{n-1}^{+\infty} \cup S^{E}$.

Case 2. If $y(t) < 0$ on $[T_0,\infty)$. Then by the similar argument as that of case 1, we can show that $y \in S_{n-1}^{-\infty} \cup S^{-E}$.

THEOREM 1.8. <u>For</u> $n \geq 3$ <u>odd</u>, <u>assume that</u> (1.1') <u>satisfies</u> (1.14). <u>Then</u> $S = S_{n-1}^{+\infty} \cup S_{n-1}^{-\infty} \cup S^{\sim}$.

<u>Proof</u>. Let $y \in S - S^{\sim}$. As in the proof of Theorem 1.7, we have $y \in S_{n-1}^{+\infty}$ or $y'(t) > 0$ and $y^{(n-1)}(t) < 0$ on $[T_2,\infty)$ for case 1. If

$y'(t) > 0$ on $[T_2,\infty)$, then there is a $T_3 \geq T_2$ and a $\beta > 0$ such that $y(t) > \beta$ and $y(g_k(t)) > \beta$ for $t \geq T_3$. It follows that

$$\begin{aligned} y^{(n-1)}(t) &= y^{(n-1)}(T_3) + \int_{T_3}^{t} \textstyle\sum_{i=1}^{m} F_i(s,y(s),y(g_i(s)))\, ds \\ &\geq y^{(n-1)}(T_3) + \int_{T_3}^{t} F_k(s,y(s),y(g_k(s)))\, ds \\ &\geq y^{(n-1)}(T_3) + \int_{T_3}^{t} F_k(s,\beta,\beta)\, ds. \end{aligned}$$

By (1.22), $y^{(n-1)}(t)$ would be positive for all sufficiently large t, a contradiction. Thus $y \in S_{n-1}^{+\infty}$.

Similarly, we have $y \in S_{n-1}^{-\infty}$ for case 2.

SECTION II

Consider the following deviating argument differential inequalities of the forms

(2.1) $$y(t)\ [y^{(n)}(t) + F(t,y(t),y(g(t)))] \leq 0$$

and

(2.2) $$y(t)\ [y^{(n)}(t) - F(t,y(t),y(g(t)))] \geq 0,$$

where $F(t,u,v)$ satisfies (I_1), (I_2) for $F = F_k$ and $g(t) \to \infty$ as $t \to \infty$.

We shall investigate the oscillatory behavior of solutions of inequalities (2.1) and (2.2) via the oscillatory behavior of solutions of a class of differential inequalities

(2.3) $$x\ [x^{(n)} + p(t)N(x)] \leq 0$$

and

(2.4) $$x\ [x^{(n)} - p(t)N(x)] \geq 0,$$

where $N \in C(-\infty,\infty)$ is nondecreasing on $[0,\infty)$ and $x\,N(x) > 0$ for $x \neq 0$. Corollary 2.1 and Corollary 2.2, under weaker conditions, extend Teufel's results for bounded solutions in [28] to arbitrary even order functional differential equations. Corollary 2.3 reduces to Theorem 2.7 [8] and also improves Theorem 3.1 [11] and the sufficient part of Theorem 3.2 [11] in the case $n = 2$. Corollary 2.5 [11] and Theorem 2.1 [8] are special cases of Corollary 2.3 for $n = 2$ and $F(t,y(t),y(g(t))) = p(t)f(y(t),y(g(t)))$.

For $n = 2$ and $g(t) = t$, Corollary 2.3 also includes the sufficient part of Theorem 1 [33].

THEOREM 2.1. Let x be a nonoscillatory solution of (2.3), ℓ satisfies (1.2) and

$$\int^{\infty} t^{n-\ell}\, p(t)\, N(\gamma t^{\ell-1})\, dt = \pm\infty \text{ for each } \gamma \neq 0. \tag{2.5}$$

Then, if n is even, $x^{(\ell-1)}(t)$ tends to infinity monotonically. If n is odd, either $x(t)$ tends to a limit or $x^{(\ell-1)}(t)$ tends to infinity monotonically.

Proof. Suppose that $x(t) > 0$ on $[T_0,\infty)$. It follows that

$$x^{(n)}(t) + p(t)\, N(x(t)) \leq 0 \text{ for } t \in [T_0,\infty). \tag{2.6}$$

Thus $x^{(n)}(t) \leq 0$ on $[T_0,\infty)$ follows also. There is a $T_1 \geq T_0$ such that each $x^{(j)}(t)$, $j = 0,1,\dots,n-1$, is of constant sign on $[T_1,\infty)$. If n is even, then by Lemma 1.1, there is an odd integer ℓ such that (1.2) holds. Since $x^{(\ell-1)}(t) > 0$ and $x^{(\ell)}(t) > 0$ on $[T_1,\infty)$, there is an $L > 0$ such that $x^{(\ell-1)}(t) > L$ for $t \geq T_1$. Integrating this repeatedly and using (1.2), we have

$$x(t) > \gamma t^{\ell-1} \quad \text{for some } \gamma > 0 \text{ and for each } t \geq T_2 \geq T_1. \tag{2.7}$$

Integrating (2.6) from $s \geq T_2$ to $t > s$, we obtain

$$0 \leq x^{(n-1)}(t) \leq x^{(n-1)}(s) - \int_s^t p(u)\, N(x(u))\, du$$

or

$$x^{(n-1)}(\sigma) \geq \int_\sigma^t p(u)\, N(x(u))\, du \quad \text{for } t > \sigma \geq T_2.$$

Integrating this from $s \geq T_2$ to $t > s$, we have

$$0 \geq x^{(n-2)}(t) \geq x^{(n-2)}(s) + \int_s^t \left[\int_\sigma^t p(u)\, N(x(u))\, du\right] d\sigma$$

$$= x^{(n-2)}(s) + \int_s^t (u - s)\, p(u)\, N(x(u))\, du$$

or

$$x^{(n-2)}(\sigma) \leq - \int_\sigma^t (u - \sigma)\, p(u)\, N(x(u))\, du \quad \text{for } t > \sigma \geq T_2 .$$

Continuing this process, we get

$$x^{(\ell)}(\sigma) \geq \int_\sigma^t \frac{(u - \sigma)^{n-\ell-1}}{(n-\ell-1)!}\, p(u)\, N(x(u))\, du \quad \text{for } t > \sigma \geq T_2.$$

Integrating this from T_2 to $t > T_2$ and using (2.7), we obtain

$$x^{(\ell-1)}(t) \geq x^{(\ell-1)}(T_2) + \int_{T_2}^t \frac{(u - T_2)^{n-\ell}}{(n - \ell)!}\, p(u)\, N(\gamma u^{\ell-1})\, du.$$

It follows from (2.5) that $x^{(\ell-1)}(t) \to \infty$ as $t \to \infty$.

If n is odd, then ℓ is an even integer satisfying (1.2). If $\ell = 0$, then $x'(t) < 0$ for all sufficiently large t and then $x(t)$ tends to a limit. If $\ell \geq 2$, then the proof for n even go through.

Let x be a nonoscillatory solution of (2.3) with n even. If x is bounded and assume that $x(t) > 0$ on $[T_0,\infty)$, then $\ell = 1$ in Lemma 1.1. Thus we have the following

COROLLARY 2.1. *If (2.5) holds for $\ell = 1$, then every bounded solution of (2.3) with n even is oscillatory.*

THEOREM 2.2. *Let* $\liminf_{t\to\infty} \frac{g(t)}{t} > 0$. *Assume that* y *is a non-oscillatory solution of* (2.1), ℓ *satisfies* (1.2) *and* (2.5) *holds. Furthermore, assume that*

$$(2.8) \qquad u\,F(t,u,u) \geq u\,p(t)\,N(u) \text{ for } u \neq 0.$$

Then, if n *is even,* $y^{(\ell-1)}(t)$ *tends to infinity monotonically. If* n *is odd, either* $y(t)$ *tends to a limit or* $y^{(\ell-1)}(t)$ *tends to infinity monotonically.*

Proof. As in the proof of Theorem 2.1, we only consider the case when n is even. Without loss of generality, we can assume that $y(t) > 0$ on $[T_0,\infty)$. Since $g(t) \to \infty$ as $t \to \infty$, there is a $T_1 \geq T_0$ such that $y(g(t)) > 0$ for $t \geq T_1$. It follows from (2.1) that $y^{(n)}(t) < 0$ on $[T_1,\infty)$. There is a $T_2 \geq T_1$ such that each $y^{(j)}(t)$, $j = 0,1,\ldots,n-1$, is of constant sign on $[T_2,\infty)$. Let ℓ be as in Lemma 1.1 and assume that $y^{(\ell-1)}$ is bounded. Then $\infty > \lim_{t\to\infty} y^{(\ell-1)}(t) = L > 0$. There is a $T_3 \geq T_2$ such that

$$L \geq y^{(\ell-1)}(t) > \frac{L}{2} \quad \text{for } t \geq T_3$$

Integrating this repeatedly and using estimation, we obtain $Ht^{\ell-1} \geq y(t) \geq Kt^{\ell-1}$ for some $H > 0$, $K > 0$ and for all sufficiently large t.
Thus we have

$$y(g(t)) \geq K(g(t))^{\ell-1} \geq K\left(\frac{g(t)}{t}\right)^{\ell-1} t^{\ell-1} \text{ for all sufficiently large } t.$$

Let $\liminf_{t\to\infty} \frac{g(t)}{t} = \alpha > 0$, then we have

$$(2.9) \quad y(g(t)) \geq K\left(\frac{\alpha}{2}\right)^{\ell-1} t^{\ell-1} \geq \frac{K}{H}\left(\frac{\alpha}{2}\right)^{\ell-1} y(t) \text{ for all sufficiently large } t.$$

Let $M = \min\{\frac{K}{H}\left(\frac{\alpha}{2}\right)^{\ell-1}, 1\}$, then

$y(g(t)) \geq M\, y(t)$ for all sufficiently large t.

It follows that

$$0 \geq y^{(n)}(t) + F(t,y(t),y(g(t))) \geq y^{(n)}(t) + F(t,y(t),My(t))$$

$$\geq y^{(n)}(t) + F(t,My(t),My(t)) \geq y^{(n)}(t) + p(t)\, N(My(t)).$$

Replacing $N(x)$ by $N(Mx)$ in Theorem 2.1, we get a contradiction.

Similar to Corollary 2.1, we have the following

COROLLARY 2.2. *If all the hypotheses of* Theorem 2.2 *hold and* $\ell = 1$ *in* (2.5), *then every bounded solution of* (2.1) *with* n *even is oscillatory*.

In Corollary 2.2, we can remove the condition $\liminf_{t\to\infty} \frac{g(t)}{t} > 0$ and only require that $\lim_{t\to\infty} g(t) = \infty$. That is because in the proof of Theorem 2.2, we have $y(t) \to L$ as $t \to \infty$ and then corresponding to (2.9) we have $y(g(t)) \geq \frac{L}{2} \geq \frac{1}{2}\, y(t)$ for all sufficiently large t. The proof of Theorem 2.2 goes through if we replace M by $\frac{1}{2}$.

THEOREM 2.3. *Let* $\liminf_{t\to\infty} \frac{g(t)}{t} > 0$. *Assume that* y *is a non-oscillatory solution of* (2.1), ℓ *satisfies* (1.2) *and*

(2.10) $$\int^{\infty} t^{n-\ell}\, F(t,\gamma t^{\ell-1},\gamma t^{\ell-1})\, dt = \pm\infty \text{ for each } \gamma \neq 0.$$

Then, if n *is even,* $y^{(\ell-1)}(t)$ *tends to infinity monotonically*. *If* n *is odd, either* y(t) *tends to a limit or* $y^{(\ell-1)}(t)$ *tends to infinity monotonically*.

Proof. We can assume that $y(t) > 0$ for all sufficiently large t. As in the proof of Theorem 2.1, we only consider n even. Then there is a $T_1 > 0$ such that each $y^{(j)}(t)$, $j = 0,1,\dots,n-1$, is of constant sign on $[T_1,\infty)$ and an odd integer ℓ satisfying (1.2). Since $y^{(\ell-1)}(t) > 0$ and $y^{(\ell)}(t) > 0$ for $t \geq T_1$, $0 < \lim_{t\to\infty} y^{(\ell-1)}(t) \leq \infty$. If $y^{(\ell-1)}(t) \to \infty$ as $t \to \infty$, then our proof is done. Let $0 < \lim_{t\to\infty} y^{(\ell-1)}(t) = L < \infty$. There is a $T_2 \geq T_1$ such that

$$\frac{L}{2}\, t^{\ell-1} < y(t) < L\, t^{\ell-1} \text{ for } t \geq T_2 .$$

Let $\liminf_{t\to\infty} \frac{g(t)}{t} = \alpha > 0$. There is a $T_3 \geq T_2$ such that

$$y(g(t)) > \frac{L}{2}\, [g(t)]^{\ell-1} = \frac{L}{2}\left[\frac{g(t)}{t}\right]^{\ell-1} t^{\ell-1} \geq \frac{1}{2}\, L\alpha^{\ell-1}\, t^{\ell-1} \text{ for } t \geq T_3.$$

Let $p(t) = \frac{1}{Lt^{\ell-1}} F(t, \frac{1}{2} L\alpha^{\ell-1} t^{\ell-1}, \frac{1}{2} L\alpha^{\ell-1} t^{\ell-1})$ and let $N(x) \equiv x$ in Theorem 2.1, then

$$y^{(n)}(t) + p(t)N(y(t)) \leq y^{(n)}(t) + \frac{1}{Lt^{\ell-1}} F(t, \frac{1}{2} L\alpha^{\ell-1} t^{\ell-1}, \frac{1}{2} L\alpha^{\ell-1} t^{\ell-1})Lt^{\ell-1}$$

$$= y^{(n)}(t) + F(t, \frac{1}{2} L\alpha^{\ell-1} t^{\ell-1}, \frac{1}{2} L\alpha^{\ell-1} t^{\ell-1})$$

$$\leq y^{(n)}(t) + F(t,y(t),y(g(t))) \leq 0.$$

Thus $y(t)$ satisfies (2.3). Moreover

$$\int^{\infty} t^{n-\ell}\, p(t)\, N(Lt^{\ell-1})\, dt = \int^{\infty} t^{n-\ell}\, F(t, \frac{1}{2} L\alpha^{\ell-1} t^{\ell-1}, \frac{1}{2} L\alpha^{\ell-1} t^{\ell-1})\, dt = \infty.$$

By Theorem 2.1, we get a contradiction.

Similar to Corollary 2.2, we have the following.

COROLLARY 2.3. *If $\int^{\infty} t^{n-1} F(t,\gamma,\gamma)\, dt = \pm\infty$ for each $\gamma \neq 0$, then every bounded solution of (2.1) with n even is oscillatory.*

Apply the same argument as in the proofs of Theorem 2.1, Theorem 2.2 and Theorem 2.3, we have the following theorems.

THEOREM 2.4. Assume x is a nonoscillatory solution of (2.4), ℓ satisfies (1.5) with $\ell < n$ and (2.5) holds. Then, if n is odd, $x^{(\ell-1)}(t)$ tends to infinity monotonically. If n is even, either $x(t)$ tends to a limit or $x^{(\ell-1)}(t)$ tends to infinity monotonically.

THEOREM 2.5. Let $\liminf_{t\to\infty} \frac{g(t)}{t} > 0$. Assume that y is a nonoscillatory solution of (2.2), ℓ satisfies (1.5) with $\ell < n$ and (2.5) holds. Furthermore, assume that (2.8) holds. Then, if n is odd, $y^{(\ell-1)}(t)$ tends to infinity monotonically. If n is even, either $y(t)$ tends to a limit or $y^{(\ell-1)}(t)$ tends to infinity monotonically.

THEOREM 2.6. Let $\liminf_{t\to\infty} \frac{g(t)}{t} > 0$. Assume that y is a nonoscillatory solution of (2.2), ℓ satisfies (1.5) with $\ell < n$ and (2.10) holds. Then, if n is odd, $y^{(\ell-1)}(t)$ tends to infinity monotonically. If n is even, either $y(t)$ tends to a limit or $y^{(\ell-1)}(t)$ tends to infinity monotonically.

Equations (1.1) and (1.1') in Section I are special cases of (2.1) and (2.2). In fact, most results of Section I could have been stated in terms of differential inequalities. One reason for not doing so is to present the readers a variety of settings in which the computations can be made. If we define as in [22], $S^{+\infty} = \bigcup_{\ell=1}^{n-1} S_\ell^{+\infty}$ and $S^{-\infty} = \bigcup_{\ell=1}^{n-1} S_\ell^{-\infty}$, the decomposition in Theorem 3.1 [22] is $S = S^{+\infty} \cup S^{-\infty} \cup S^0 \cup S^{\sim}$. Our Theorem 2.6 ($n$ even) has the same decomposition, where we give weaker conditions on $g(t)$ and replace (3.1) [22] by (2.10).

Since, when $F(t,y(t),y(g(t))) = p(t)\, y(g(t))$, (2.10) becomes

$$(2.11) \qquad \int^{\infty} t^{n-1}\, p(t)\, dt = \infty,$$

(2.10) is also weaker than (3.1) [22]. If N has certain growth conditions, for instance, if $\frac{N(x)}{x}$ is monotone increasing, then (2.5) also may be replaced by (2.11).

SECTION III

We shall consider the delay differential equations of the form

$$y^{(n)}(t) + y(t - \tau(t))\, f(t,y(t - \tau(t))) = 0, \tag{3.1}$$

where $0 \leq \tau(t) \leq \tau_0$ and $f(t,u)$ satisfies the following hypotheses:

(III_1) $f \in C([0,\infty) \times \mathcal{R})$ <u>and</u> $f(t,v) \leq f(t,u)$ <u>for</u> $t \in [0,\infty)$, $0 < v < u$; <u>and</u>

(III_2) $f(t,u) > 0$ <u>for each</u> $t \in [0,\infty)$ <u>and</u> $u > 0$.

Our theorem extends for the case $n = 4$ a recent result due to Terry and Wong [27, Theorem 3.2]. When $n = 4$, $\tau(t) \equiv 0$ and $f(t,y(t-\tau(t))) = p(t)$ our result reduces to that of Leighton and Nehari [19, Theorem 11.2]. Our theorem also generalizes a theorem of Moore and Nehari [20, Theorem III] for the case of $n = 2$, $\tau(t) \equiv 0$ and $f(t,y(t-\tau(t))) = p(t)\,[y(t)]^{2m}$, where p is positive and continuous on $[0,\infty)$ and m is a positive integer. In [10], Eliason considers the second order equation of the form $y''(t) + p(t)|y(t-\tau(t))|^{\gamma} \operatorname{sgn} y(t-\tau(t)) = 0$ ($\gamma > 0$) and gives necessary and sufficient conditions for this equation to have a bounded nonoscillatory solution with asymptotic behavior.

LEMMA 3.1. <u>Let</u> y <u>satisfy the hypotheses of</u> Lemma 1.1. <u>Then</u> $\lim_{t\to\infty} (\ell + 2j)!\,t^{-(\ell+2j)}\, y(t) = \lim_{t\to\infty} y^{(\ell+2j)}(t)$ <u>for each</u> $j = 0,1,\ldots,\frac{n-\ell-1}{2}$. <u>Both limits are finite</u>.

Proof. Assume that $y(t) > 0$ on $[T_0,\infty)$ and ℓ satisfies (1.2). By Taylor's formula with integral form of remainder, we have

$$y(t) = y(T_0) + y'(T_0)(t - T_0) + \dots + \frac{y^{(\ell+2j)}(T_0)}{(\ell + 2j)!}(t - T_0)^{\ell+2j}$$

$$+ \frac{1}{(\ell + 2j)!}\int_{T_0}^{t} (t - s)^{\ell+2j}\, y^{(\ell+2j+1)}(s)\, ds \geq y(T_0) + y'(T_0)(t - T_0)$$

$$+ \dots + \frac{y^{(\ell+2j)}(T_0)}{(\ell + 2j)!}(t - T_0)^{\ell+2j} + \frac{(t - T_0)^{\ell+2j}}{(\ell + 2j)!}[y^{(\ell+2j)}(t) - y^{(\ell+2j)}(T_0)]$$

for $t \geq T_0$. Dividing this by $(t - T_0)^{\ell+2j}$, we obtain

$$\frac{y(t)}{(t - T_0)^{\ell+2j}} \geq \frac{y(T_0)}{(t - T_0)^{\ell+2j}} + \frac{y(T_0)}{(t - T_0)^{\ell+2j-1}} + \dots + \frac{y^{(\ell+2j)}(T_0)}{(\ell + 2j)!}$$

$$+ \frac{1}{(\ell + 2j)!}[y^{(\ell+2j)}(t) - y^{(\ell+2j)}(T_0)]$$

or

$$y^{(\ell+2j)} \leq \frac{(\ell+2j)!y(t)}{(t - T_0)^{\ell+2j}} - \frac{(\ell+2j)!y(T_0)}{(t - T_0)^{\ell+2j}} - \frac{(\ell+2j)!y'(T_0)}{(t - T_0)^{\ell+2j-1}} \tag{3.2}$$

$$- \dots - \frac{(\ell+2j)y^{(\ell+2j-1)}(T_0)}{t - T_0}$$

for $t > T_0$.

Since $y^{(\ell+2j)}(t) > 0$ and $y^{(\ell+2j+1)}(t) < 0$ on $[T_0,\infty)$, $\lim_{t\to\infty} y^{(\ell+2j)}(t)$ exists. It follows from (3.2) that

$$\lim_{t\to\infty} y^{(\ell+2j)}(t) \leq \liminf_{t\to\infty} \frac{(\ell+2j)!y(t)}{(t - T_0)^{\ell+2j}} \tag{3.3}$$

To prove the reverse inequality, we choose a σ such that $T_0 < \sigma < t$. Then we have $(t - s)^{\ell+2j} \geq (t - \sigma)^{\ell+2j}$ for $s \in [T_0,\sigma]$. Hence

$$y(t) \leq y(T_0) + y'(T_0)(t - T_0) + \ldots + \frac{y^{(\ell+2j)}(T_0)}{(\ell + 2j)!}(t - T_0)^{\ell+2j}$$

$$+ \frac{(t-\sigma)^{\ell+2j}}{(\ell + 2j)!}\int_{T_0}^{\sigma} y^{(\ell+2j+1)}(s)\, ds = y(T_0) + y'(T_0)(t - T_0)$$

$$+ \ldots + \frac{y^{(\ell+2j)}(T_0)}{(\ell + 2j)!}(t - T_0)^{\ell+2j} + \frac{(t - \sigma)^{\ell+2j}}{(\ell + 2j)!}[y^{(\ell+2j)}(\sigma) - y^{(\ell+2j)}(T_0)]$$

or

$$y^{(\ell+2j)}(t_0) \geq \frac{(\ell + 2j)!}{(t - T_0)^{\ell+2j}} y(t) - \frac{(\ell + 2j)!}{(t - T_0)^{\ell+2j}} y(T_0) - \frac{(\ell + 2j)!}{(t - T_0)^{\ell+2j-1}} y'(T_0)$$

$$- \ldots - \frac{\ell + 2j}{t - T_0} y^{(\ell+2j-1)}(T_0) - \left(\frac{t - \sigma}{t - T_0}\right)^{\ell+2j}[y^{(\ell+2j)}(\sigma) - y^{(\ell+2j)}(T_0)]$$

for $t > T_0$. Let σ be fixed, we have

$$y^{(\ell+2j)}(T_0) \geq \limsup_{t\to\infty} \frac{(\ell + 2j)!}{(t - T_0)^{\ell+2j}} y(t) - y^{(\ell+2j)}(\sigma) + y^{(\ell+2j)}(T_0).$$

It follows that

$$y^{(\ell+2j)}(\sigma) \geq \limsup_{t\to\infty} \frac{(\ell + 2j)!}{(t - T_0)^{\ell+2j}} y(t).$$

Since σ is arbitrary and since $\lim_{t\to\infty} y^{(\ell+2j)}(t)$ exists, we obtain

$$\lim_{t\to\infty} y^{(\ell+2j)}(t) \geq \limsup_{t\to\infty} \frac{(\ell + 2j)!}{(t - T_0)^{\ell+2j}} y(t). \tag{3.4}$$

The proof is then complete by combining (3.3) and (3.4).

In the proof of Theorem 3.1, we only use the case of $j = \frac{n-\ell-1}{2}$ in Lemma 3.1.

THEOREM 3.1. <u>Equation</u> (3.1) <u>has an ultimately positive solution</u> $y(t)$ <u>satisfying</u>

$$\lim_{t\to\infty} t^{-(n-1)}y(t) = K, \quad 0 < K < \infty, \tag{3.5}$$

<u>if, and only if, for some</u> $c > 0$

$$\int^{\infty} t^{n-1} f(t,ct^{n-1})\, dt < \infty. \tag{3.6}$$

<u>Proof</u>. Suppose (3.6) holds for some $c > 0$. Choose a sufficiently large $T_0 \geq \tau_0$ such that

$$\int_{T_0}^{\infty} t^{n-1} f(t,ct^{n-1})\, dt \leq \frac{1}{2}\,. \tag{3.7}$$

We now construct a solution $y(t) = y(t,T_0)$ of (3.1) with the initial conditions $y(T_0) = y'(T_0) =\ldots= y^{(n-2)}(T_0) = 0$, $y^{(n-1)}(T_0) = c$ and $y(t) = 0$ for $T_0 - \tau_0 \leq t \leq T_0$. Since $y^{(n-1)}(T_0) = c > 0$, $y(t,T_0)$ is positive on some open interval whose left end point is T_0. Let $t = T_1$ be the first zero of $y(t,T_0)$ in (T_0,∞), if such exists. Then by Taylor's Theorem, it follows that

$$y^{(n-1)}(\xi) = 0 \quad \text{for some } T_0 < \xi < T_1. \tag{3.8}$$

By Taylor's formula with integral form of remainder, we have for $T_0 \leq t \leq T_1$

$$y(t) = \frac{c}{(n-1)!}(t-T_0)^{n-1} + \frac{1}{(n-1)!}\int_{T_0}^{t}(t-s)^{n-1}y^{(n)}(s)\, ds \tag{3.9}$$

$$= \frac{c}{(n-1)!}(t-T_0)^{n-1} - \frac{1}{(n-1)!}\int_{T_0}^{t}(t-s)^{n-1}y(s-\tau(s))f(s,y(s-\tau(s)))\, ds.$$

By (III_2), it follows that

(3.10) $$y(t) \le \frac{c}{(n-1)!}(t-T_0)^{n-1} \le ct^{n-1} \text{ for } T_0 \le t \le T_1.$$

Since $y(t) = 0$ on $[T_0 - \tau_0, T_0]$, (3.10) holds for $T_0 - \tau_0 \le t \le T_1$. It follows that

(3.11) $$0 \le y(s-\tau(s)) \le cs^{n-1} \text{ for } T_0 \le s \le T_1.$$

Integrating (3.1) from T_0 to t with $T_0 < t < T_1$, and using (3.11) and (3.7), we have

$$y^{(n-1)}(t) = c - \int_{T_0}^{t} y(s-\tau(s))\, f(s,y(s-\tau(s)))\, ds$$

$$\ge c - \int_{T_0}^{t} cs^{n-1}\, f(s,cs^{n-1})\, ds \ge c - \frac{1}{2}c = \frac{1}{2}c > 0,$$

which contradicts (3.8). So we have proved the existence of a positive nonoscillatory solution y of (3.1), since as long as the above and (3.10) hold, the solution may be extended indefinitely to the right.

For this solution $y(t) = y(t,T_0)$, (3.9) and (3.10) hold for $t \ge T_0$. It follows that

$$c(t-T_0)^{n-1} = (n-1)!\, y(t) + \int_{T_0}^{t} (t-s)^{n-1}\, y(s-\tau(s))\, f(s,y(s-\tau(s)))\, ds$$

$$\le (n-1)!\, y(t) + c(t-T_0)^{n-1} \int_{T_0}^{t} s^{n-1}\, f(s,cs^{n-1})\, ds$$

$$\le (n-1)!\, y(t) + \frac{1}{2}c(t-T_0)^{n-1}$$

or

$$\frac{1}{2} c \le (n-1)!\, y(t)(t - T_0)^{-(n-1)}.$$

Combining this with (3.10), we obtain

$$\frac{1}{2} c \le (n-1)!\, y(t)(t - T_0)^{-(n-1)} \le c.$$

Since $y^{(n)}(t) < 0$, $y^{(n-1)}(t)$ is decreasing and then (3.5) follows from Lemma 3.1 with $j = \frac{n-\ell-1}{2}$.

To prove necessity, we let y be an ultimately positive solution of (3.1) satisfying (3.5). We assume that $y(t) > 0$ for $t \ge T_0$. By Lemma 3.1 with $j = \frac{n-\ell-1}{2}$, we have

$$\text{(3.12)} \qquad \int_{T_0}^{\infty} y(s-\tau(s))\, f(s,y(s-\tau(s)))\, ds = - \int_{T_0}^{\infty} y^{(n)}(s)\, ds$$

$$= y^{(n-1)}(T_0) - K(n-1)!\ .$$

It follows from (3.5) that for any $0 < \varepsilon < K$, there is a $T_1 \ge T_0$ such that

$$y(t-\tau(t)) \ge (K - \varepsilon)[t - \tau(t)]^{n-1} \ge (K - \varepsilon)(t - \tau_0)^{n-1}$$

for $t \ge T_1 + \tau_0$. By (III_1), we have for $s \ge T_1 + \tau_0$

$$f(s,y(s-\tau(s))) \ge f(s,(K-\varepsilon)(s-\tau_0)^{n-1}).$$

Substituting T_0 by $T_1 + \tau_0$ in (3.12), we obtain

$$\text{(3.13)} \quad \int_{T_1+\tau_0}^{\infty} (K-\varepsilon)(s - \tau_0)^{n-1} f(s,(K-\varepsilon)(s-\tau_0)^{n-1})\, ds$$

$$\le y^{(n-1)}(T_1+\tau_0) - K(n-1)!$$

For $s \geq 2\tau_0$, it follows that $s \leq 2(s-\tau_0)$ and then by (III_1), we have

$$(3.14)\quad \int^{\infty} s^{n-1} f(s,cs^{n-1})\, ds \leq 2^{n-1} \int^{\infty} (s-\tau_0)^{n-1} f(s,c2^{n-1}(s-\tau_0)^{n-1})\, ds$$

$$= 2^{n-1} \int^{\infty} (s-\tau_0)^{n-1} f(s,(K-\varepsilon)(s-\tau_0)^{n-1})\, ds,$$

where $K-\varepsilon = 2^{n-1}c$.

Combining (3.13) and (3.14), we have

$$\int_{T_1+\tau_0}^{\infty} s^{n-1} f(s,cs^{n-1})\, ds \leq \frac{2^{n-1}}{K-\varepsilon} [y^{(n-1)}(T_1+\tau_0) - K(n-1)!].$$

The conclusion (3.6) is then proved.

SECTION IV

In this section, we are concerned with the delay differential equations of the form

$$(4.1) \qquad y^{(n)}(t) + p(t)\, y(t - \tau) = 0,$$

where $\tau \geq 0$ and p is a continuous nonnegative function on $[0,\infty)$.

For the existence and the uniqueness of solutions of the linear equation (4.1), see §4.1 in [13].

Our theorem extends Kung's Theorem 3 in [15] for equation (4.1).

LEMMA 4.1 (Lemma 2 in [25]). If y is a solution of (4.1) with $y(t) > 0$ and $y'(t) \geq 0$ for every $t \in [T_0,\infty)$, then $y^{(n-1)}$ is nonnegative, nonincreasing on $[T_0+\tau,\infty)$ and

$$y'(t) \geq \frac{(t - T_0)^{n-2}}{(n-2)!}\, y^{(n-1)}(2^{n-2}t) \quad \text{for } t \in [T_0+\tau,\infty).$$

THEOREM 4.1. Assume that

(IV_1) there exists a positive and continuously differentiable function ϕ on $[\alpha,\infty)$ and $\lim_{t\to\infty} \int_\alpha^t \frac{s^{n-2}}{\phi(s)}\, ds = \infty$;

$$(IV_2) \quad K(t,k) = \int_\alpha^t \left(\gamma_k(s)\phi(s)p(s) - \frac{(n-2)!2^{(n-1)(n-2)}}{4(s - T)} \frac{[\phi'(s)]^2}{\phi(s)}\right) ds$$

$$+ \frac{(n-2)!2^{(n-1)(n-2)}}{2(t - T)^{n-2}}\, \phi'(t) \to \infty \text{ as } t \to \infty$$

<u>for every sufficiently large</u> T <u>and every positive</u> k, <u>where</u>

$$\gamma_k(t) = \begin{cases} 1 - \dfrac{k}{\sqrt{t}} & \text{if } n = 2 \\ 1 & \text{if } n > 2. \end{cases}$$

<u>Then, if</u> n <u>is even, every solution of</u> (4.1) <u>is oscillatory; if</u> n <u>is odd, every solution</u> y <u>of</u> (4.1) <u>is either oscillatory or</u> y(t) <u>tends to a limit and</u> $\lim_{t\to\infty} y^{(j)}(t) = 0$, $j = 1,2,\ldots,n-1$.

<u>Proof</u>. For the case of $n = 2$, it is proved in [16]. So we assume that $n \geq 3$ in the following proof.

Let y be a nonoscillatory solution of (4.1). We can assume that $y(t-\tau) > 0$ for $t \geq T_0$. Then $y^{(n)}(t) \leq 0$ on $[T_0,\infty)$. There is a $T_1 \geq T_0$ such that each $y^{(j)}(t)$, $j = 0,1,\ldots,n-1$, is of constant sign on $[T_1,\infty)$. By Lemma 1.1, $y^{(n-1)}(t) > 0$ on $[T_1,\infty)$.

(i) Let $y'(t) < 0$. By Lemma 1.1, this can occur only when n is odd. Then y is decreasing and then tends to a limit as $t \to \infty$. Furthermore, by Lemma 1.3, $\lim_{t\to\infty} y^{(j)}(t) = 0$, $j = 1,2,\ldots,n-1$.

(ii) Let $y'(t) > 0$ on $[T_1,\infty)$. Let

$$q(t) = -\frac{y^{(n-1)}(t)}{y\left(\frac{t}{2^{n-2}}\right)}\,\phi(t).$$

Then

$$q'(t) = -\frac{y\left(\frac{t}{2^{n-2}}\right)[y^{(n)}(t)\phi(t) + y^{(n-1)}(t)\phi'(t)] - \frac{1}{2^{n-2}}\,y'\left(\frac{t}{2^{n-2}}\right)y^{(n-1)}(t)\phi(t)}{y^2\left(\frac{t}{2^{n-2}}\right)}$$

$$= \frac{p(t)y(t-\tau)\phi(t)}{y\left(\frac{t}{2^{n-2}}\right)} - \frac{y^{(n-1)}(t)\phi'(t)}{y\left(\frac{t}{2^{n-2}}\right)} + \frac{y'\left(\frac{t}{2^{n-2}}\right) y^{(n-1)}(t)\phi(t)}{2^{n-2}\, y^2\left(\frac{t}{2^{n-2}}\right)} .$$

By Lemma 4.1, we have

$$y'\left(\frac{t}{2^{n-2}}\right) \geq \frac{\left(\frac{t}{2^{n-2}} - T_1\right)^{n-2}}{(n-2)!}\, y^{(n-1)}(t) \text{ for } \frac{t}{2^{n-2}} \geq T_1 .$$

Let $T_2 = 2^{n-2}\, T_1$. Then

$$y'\left(\frac{t}{2^{n-2}}\right) \geq \frac{(t - T_2)^{n-2}}{(n-2)!\; 2^{(n-1)(n-2)}}\, y^{(n-1)}(t) \text{ for } t \geq T_2 .$$

It follows that

$$q'(t) \geq \frac{p(t)y(t-\tau)\phi(t)}{y\left(\frac{t}{2^{n-2}}\right)} - \frac{y^{(n-1)}(t)\phi'(t)}{y\left(\frac{t}{2^{n-2}}\right)} + \frac{(t-T_2)^{n-2}[y^{(n-1)}(t)]^2\phi(t)}{(n-2)!\; 2^{(n-1)(n-2)} y^2\left(\frac{t}{2^{n-2}}\right)}$$

$$= \frac{1}{\phi(t)}\left\{\left[\sqrt{\frac{(t - T_2)^{n-2}}{(n-2)!\; 2^{(n-1)(n-2)}}}\; \frac{y^{(n-1)}(t)\phi(t)}{y\left(\frac{t}{2^{n-2}}\right)}\right]^2 - \frac{y^{(n-1)}(t)\phi(t)\phi'(t)}{y\left(\frac{t}{2^{n-2}}\right)}\right.$$

$$+ \frac{(n-2)!\; 2^{(n-1)(n-2)}[\phi'(t)]^2}{4(t - T_2)^{n-2}} + \frac{p(t)y(t-\tau)\phi^2(t)}{y\left(\frac{t}{2^{n-2}}\right)}$$

$$\left. - \frac{(n-2)!\; 2^{(n-1)(n-2)}[\phi'(t)]^2}{4(t - T_o)^{n-2}}\right\}$$

$$= \frac{1}{\phi(t)}\left(\sqrt{\frac{(t - T_2)^{n-2}}{(n-2)!\,2^{(n-1)(n-2)}}}\ \frac{y^{(n-1)}(t)\phi(t)}{\phi\left(\frac{t}{2^{n-2}}\right)} - \sqrt{\frac{(n-2)!\,2^{(n-1)(n-2)}}{4(t - T_2)^{n-2}}}\ \phi'(t)\right)^2$$

$$+ \frac{p(t)y(t-\tau)\phi(t)}{y\left(\frac{t}{2^{n-2}}\right)} - \frac{(n-2)!\,2^{(n-1)(n-2)}\,[\phi'(t)]^2}{4(t - T_2)^{n-2}\,\phi(t)}\ .$$

Since $y'(t) > 0$ on $[T_1,\infty)$, $y(t)$ is increasing on $[T_1,\infty)$ and then

$$\frac{y(t - \tau)}{y\left(\frac{t}{2^{n-2}}\right)} \geq 1 \text{ if } t-\tau \geq \frac{t}{2^{n-2}}\ \text{, i.e., if } t \geq \frac{2^{n-2}}{2^{n-2} - 1}\,\tau.$$

Let $T_3 = \dfrac{2^{n-2}}{2^{n-2} - 1}\,\tau$ and let $T = \max\{T_2,T_3\}$. Define

$$h(t) = \int_T^t \left(\frac{p(s)\phi(s)y(t-\tau)}{y\left(\frac{s}{2^{n-2}}\right)} - \frac{(n-2)!\,2^{(n-1)(n-2)}[\phi'(s)]^2}{4(s - T_2)^{n-2}\,\phi(s)}\right) ds$$

and define

$$Z(t) = q(t) - h(t) \text{ for } t \geq T.$$

Then

$$Z'(t) = q'(t) - h'(t) \geq \frac{1}{\phi(t)}\left(\sqrt{\frac{(t - T_2)^{n-2}}{(n-2)!\,2^{(n-1)(n-2)}}}\ \frac{y^{(n-1)}(t)\phi(t)}{\phi\left(\frac{t}{2^{n-2}}\right)}\right.$$

$$\left. - \sqrt{\frac{(n-2)!\,2^{(n-1)(n-2)}}{4(t - T_2)^{n-2}}}\ \phi'(t)\right)^2$$

$$= \frac{1}{\phi(t)}\left[-q(t) - \frac{(n-2)!\,2^{(n-1)(n-2)}}{2(t - T_2)^{n-2}}\ \phi'(t)\right]^2 \frac{(t - T_2)^{n-2}}{(n-2)!\,2^{(n-1)(n-2)}}$$

$$= \frac{(t - T_2)^{n-2}}{(n-2)!\, 2^{(n-1)(n-2)}\, \phi(t)} \left(Z(t) + h(t) + \frac{(n-2)!\, 2^{(n-1)(n-2)}}{2(t - T_2)^{n-2}}\, \phi'(t) \right)^2$$

$$= \frac{(t - T_2)^{n-2}}{(n-2)!\, 2^{(n-1)(n-2)}\, \phi(t)} \left[Z(t) + \int_T^t \left(\frac{p(s)\phi(s)y(s-\tau)}{y\left(\frac{s}{2^{n-2}}\right)} - \frac{(n-2)!\, 2^{(n-1)(n-2)}[\phi'(s)]^2}{4(s - T_2)^{n-2}\, \phi(s)} \right) ds + \frac{(n-2)!\, 2^{(n-1)(n-2)}\, \phi'(t)}{2(t - T_2)^{n-2}} \right]^2$$

for $t \geq T$. We shall estimate the quantity in the bracket as follows.

$$Z(t) + \int_T^t \left(\frac{p(s)\phi(s)y(t-\tau)}{y\left(\frac{s}{2^{n-2}}\right)} - \frac{(n-2)!\, 2^{(n-1)(n-2)}[\phi'(s)]^2}{4(s - T_2)^{n-2}\, \phi(s)} \right) ds$$

$$+ \frac{(n-2)!\, 2^{(n-1)(n-2)}\, \phi'(t)}{2(t - T_2)^{n-2}}$$

$$\geq Z(t) + \int_T^t \left(p(s)\phi(s) - \frac{(n-2)!\, 2^{(n-1)(n-2)}[\phi'(s)]^2}{4(s - T_2)^{n-2}\, \phi(s)} \right) ds$$

$$+ \frac{(n-2)!\, 2^{(n-1)(n-2)}\, \phi'(t)}{2(t - T_2)^{n-2}} = Z(t) + c + K(t,k) \text{ for } t \geq T \text{ and}$$

for some constant c.

Let β be such that $Z(T) + \beta > 0$. By condition (IV_2), there is a $T_4 \geq T$ such that

$$Z(t) + c + K(t,k) \geq Z(t) + \beta \geq Z(T) + \beta > 0 \text{ for } t \geq T_4.$$

Then

$$Z'(t) \geq \frac{(t - T_2)^{n-2}}{(n-2)!\ 2^{(n-1)(n-2)}\phi(t)} [Z(t) + \beta]^2 \quad \text{for } t \geq T_4.$$

Dividing this by $[Z(t) + \beta]^2$ and integrating from T_4 to $t > T_4$, we obtain

$$\frac{1}{Z(T_4) + \beta} > - \frac{1}{Z(t) + \beta} + \frac{1}{Z(T_4) + \beta} \geq \frac{1}{(n-2)!\ 2^{(n-1)(n-2)}} \int_{T_4}^{t} \frac{(s - T_2)^{n-2}}{\phi(s)}\, ds.$$

By (IV_1), we get a contradiction.

From the above proof, we see that Theorem 4.1 still holds if we consider the equation

$$y^{(n)}(t) + p(t)\, y(g(t)) = 0$$

for $n > 2$, where $g(t) \leq t$ and $g(t) \geq \frac{t}{2^{n-2}}$ on $[\alpha,\infty)$ for some large α. Theorem 4.1 also holds if we replace equation (4.1) by differential inequality

$$(4.2) \qquad y(t)\, [y^{(n)}(t) + p(t)\, y(t-\tau)] \leq 0.$$

If the delay function $g(t)$ is such that $t-\tau \leq g(t) \leq t$ and if y is a non-oscillatory solution of $y^{(n)}(t) + p(t)\, y(g(t)) = 0$, $y(t) > 0$ and $y'(t) > 0$, then y satisfies (4.2) and then Theorem 4.1 would apply.

By taking $\phi(t) \equiv 1$, we have the following

COROLLARY 4.1. If $\int^{\infty} p(s)\, ds = \infty$, then the result of Theorem 4.1 holds.

For $\tau = 0$, Corollary 4.1 coincides with Theorem V in [12]. In case of $n = 2$, Corollary 4.1 reduces to Corollary 4 in [15]. For $n = 2$ and $\tau = 0$, Corollary 4.1 coincides with Theorem (i) in [9]. For this case, also see Corollary 2 in [32], the Theorem in [30], Theorem 1.1 in [18], Theorem 1 in [26] and the Theorem in [31].

COROLLARY 4.2. *If* $\liminf_{t\to\infty} p(t) > 0$, *then the result of* Theorem 4.1 *holds*.

SECTION V

Let h be a fixed positive number. For any continuous function $x(t)$ with domain $[t_0-h,\infty)$ and range $\mathcal{R}$, x_t is defined to be the function with domain $[-h,0]$ such that $x_t(s) = x(t+s)$ for $-h \leq s \leq 0$. Moreover, we use the following notation in this section for convenience.

$C = C[-h,0]$

$C^+ = C^+[-h,0] = \{\phi \in C: \phi(s) \geq 0 \text{ for } s \in [-h,0]\}$

$C\uparrow = C\uparrow[-h,0] = \{\phi \in C^+: \phi \text{ is monotone increasing on } [-h,0]\}$.

Parallel definitions are to hold for C^- and $C\downarrow$.

$m(\phi) = \inf_{-h\leq s\leq 0} |\phi(s)|$

$\|\phi\| = \sup_{-h\leq s\leq 0} |\phi(s)|$

$D = \{\phi \in C: \|\phi\| \leq 1\}$.

In Theorems 1, 2 and 3, we consider the functional differential equations of the form

$$(5.1) \qquad y^{(n)}(t) + p(t)\, f(y_t) = 0,$$

where p is a nonnegative, continuous function on $[t_0,\infty)$ and is not identically zero on any subinterval of $[t_0,\infty)$ and f is continuous such that $\operatorname{sgn} f(\phi) = \operatorname{sgn} \phi$ for $\phi \in C^+$ or $\phi \in C^-$.

In Theorems 4, 5 and 6, the following functional differential equation

$$(5.2) \qquad y^{(n)}(t) + p(t)\, f(t,y_t,y_t',\ldots,y_t^{(n-1)}) = 0$$

is considered, where p is as in (5.1) and f is continuous on $[t_0,\infty) \times C \times \ldots \times C$ such that $\operatorname{sgn} f(t,\phi,\ldots) = \operatorname{sgn} \phi$ for all $t \geq t_0$ and $\phi \in C^+$ or $\phi \in C^-$.

The purpose of this section is to extend Burkowski's results in [6] and [7] to arbitrary order functional differential equations (5.1) and (5.2).

THEOREM 5.1. *For n even, assume that*

(V_1) *f satisfies a Lipschitz condition with Lipschitz constant* $K > 0$ *for the domain D.*

Then a necessary and sufficient condition that equation (5.1) *has a bounded nonoscillatory solution is that*

$$(5.3) \qquad \int^{\infty} t^{n-1}\, p(t)\, dt < \infty.$$

Proof. By condition (5.3), the proof of the necessity is similar to that of Theorem 4.1 in [16] and we omit it here.

The proof of sufficiency is essentially contained in Atkinson [1]. In [7], Burkowski also sketches the proof. We shall show that the integral equation

$$y(t) = 1 - \frac{1}{(n-1)!} \int_t^{\infty} (s - t)^{n-1}\, p(s)\, f(y_s)\, ds$$

has a nonnegative solution (also satisfying (5.1)) which is continuous and uniformly bounded as $t \to \infty$ under the properties of p and f and condition (5.3).

Using the Picard method of successive approximation, we define a sequence of functions

$$y_0(t) \equiv 1$$

(5.4)
$$y_{m+1}(t) = \begin{cases} 1 - \frac{1}{(n-1)!} \int_t^\infty (s-t)^{n-1}\, p(s)\, f[(y_m)_s]\, ds, & t \geq T_0 \\ y_{m+1}(T_0), & T_0 - h \leq t \leq T_0, \end{cases}$$

where T_0 is chosen, by (5.3), such that

$$\frac{K}{(n-1)!} \int_t^\infty s^{n-1}\, p(s)\, ds < r < 1$$

for $t \geq T_0$. It follows that

$$\frac{K}{(n-1)!} \int_t^\infty (s-t)^{n-1}\, p(s)\, ds < r < 1$$

for $t \geq T_0$.

If $(y_m)_t \in D \cap C^+$ for $t \geq T_0$, then $(y_{m+1})_t \in D \cap C^+$ for $t \geq T_0$ since

$$\frac{1}{(n-1)!} \int_t^\infty (s-t)^{n-1}\, p(s)\, f[(y_m)_s]\, ds \leq \frac{K}{(n-1)!} \int_t^\infty (s-t)^{n-1}\, p(s)\, ds < r < 1.$$

By (5.4) and (V_1), we have

$$|y_{m+2}(t) - y_{m+1}(t)| = \left| \frac{1}{(n-1)!} \int_t^\infty (s-t)^{n-1}\, p(s)\{f[(y_{m+1})_s] - f[(y_m)_s]\} ds \right|$$

$$\leq \max_{t \leq s < \infty} \|(y_{m+1})_s - (y_m)_s\| \frac{K}{(n-1)!} \int_t^\infty (s-t)^{n-1}\, p(s)\, ds$$

$$< \max_{t \leq s < \infty} \|(y_{m+1})_s - (y_m)_s\|\, r < r^{m+2} \qquad \text{for } t \geq T_0.$$

This established the uniform convergence of the infinite series

$$y_0(t) + [(y_1)(t) - (y_0)(t)] + \ldots + [(y_m)(t) - (y_{m-1})(t)] + \ldots$$

from which we deduce the uniform convergence of the sequence $\{(y_m)(t)\}_{m=0}^{\infty}$ for $t \geq T_0$. The required solution is the limit function $y(t)$ which is continuous, bounded and nonoscillatory since $\lim_{t\to\infty} y(t) = 1$ and $\lim_{t\to\infty} y'(t) = 0$.

THEOREM 5.2. For $n \geq 3$ odd, let f satisfy (V_1). Then a necessary and sufficient condition that equation (5.1) has a bounded nonoscillatory solution not tending to zero is that (5.3) holds.

Proof. For necessity, the proof is as stated in the proof of Theorem 5.1.

For sufficiency, we also use the Picard method of successive approximation. We shall show that the integral equation

$$y(t) = \frac{1}{2} + \frac{1}{(n-1)!} \int_t^{\infty} (s-t)^{n-1}\ p(s)\ f(y_s)\ ds$$

has a nonnegative solution (also satisfying (5.1)) which is continuous and uniformly bounded as $t \to \infty$ under the properties of p and f and (5.3).

We define a sequence of functions

$$y_0(t) = \frac{1}{2}$$

$$y_{m+1}(t) = \begin{cases} \frac{1}{2} + \frac{1}{(n-1)!} \int_t^{\infty} (s-t)^{n-1}\ p(s)\ f[(y_m)_s]\ ds, & t \geq T_0 \\ y_{m+1}(T_0), & T_0-h \leq t < T_0, \end{cases}$$

where T_0 is chosen, by (5.3), such that

$$\frac{K}{(n-1)!} \int_t^{\infty} s^{n-1}\ p(s)\ ds < r < \frac{1}{2} \quad \text{for } t \geq T_0.$$

It follows that

$$\frac{K}{(n-1)!}\int_t^\infty (s-t)^{n-1}\, p(s)\, ds < r < \frac{1}{2} \text{ for } t \geq T_0.$$

The proof for the uniform convergence of the sequence $\{(y_m)(t)\}_{m=0}^\infty$ is similar to that of Theorem 5.1 and finally the required solution is the limit function y(t) which is continuous, bounded, nonoscillatory and not tending to zero since $\lim_{t\to\infty} y(t) = \frac{1}{2}$ and $\lim_{t\to\infty} y'(t) = 0$.

THEOREM 5.3. For n even, let f satisfy in addition to (V_1) that for some $q > 1$,

$$\text{(5.5)} \qquad \liminf_{m(\phi)\to\infty} \frac{|f(\phi)|}{|\phi(-h)|^q} > 0,$$

where the limit is taken through elements of either C↑ or C↓ such that $m(\phi)$ increases toward infinity. Then a necessary and sufficient condition that every solution y(t) of equation (5.1) be oscillatory is that $\int^\infty t^{n-1}\, p(t)\, dt = \infty$.

Proof. Assume that $\int^\infty t^{n-1}\, p(t)\, dt < \infty$. Then by the proof of the sufficiency of Theorem 5.1, we get a nonoscillatory solution of (5.1). This proves the necessity of this theorem.

For the sufficiency part, let y be a nonoscillatory solution of (5.1). We can assume that $y(t) > 0$ for all $t \geq T_0$. Similar argument holds for $y(t) < 0$. There is a $T_1 \geq T_0$ such that each $y^{(j)}(t)$, $j = 0, 1, \ldots, n-1$, is of constant sign on $[T_1, \infty)$. By Lemma 1.1, there is an odd integer ℓ such that (1.2) holds. In particular, $y'(t) > 0$ for $t \geq T_1$. Then $\lim_{t\to\infty} y(t) = \infty$ or y(t) is bounded. If y(t) is bounded, then by Theorem 5.1, y is an oscillatory solution of (5.1), a contradiction. Therefore,

$\lim_{t\to\infty} y(t) = \infty$. From (5.1), we get

$$y^{(n-1)}(t) \geq \int_t^\infty p(s)\, f(y_s)\, ds \text{ for } t \geq T_1.$$

Integrating this repeatedly and using (1.2), we obtain

$$y^{(\ell)}(t) \geq \int_t^\infty \int_{s_1}^\infty \cdots \int_{s_{n-\ell-1}}^\infty p(s_{n-\ell})\, f(y_{s_{n-\ell}})\, ds_{n-\ell} ds_{n-\ell-1} \cdots ds_1.$$

By hypothesis (5.5), there is a $T_2 \geq T_1$ such that

$$\frac{f(y_t)}{y^q(t-h)} \geq \frac{1}{2} \liminf_{t\to\infty} \frac{|f(y_t)|}{|y(t-h)|^q} \geq \alpha > 0 \text{ for } t \geq T_2.$$

It follows from Taylor's expansion and (1.2) that $y(s_{n-\ell}-h) \geq \frac{y^{(\ell-1)}(\sigma)}{(\ell - 1)!} (s_{n-\ell} - h - t)^{\ell-1}$ for $s_{n-\ell}-h > t$ and for some $s_{n-\ell}-h-t < \sigma < s_{n-\ell}-h$. Thus we have (Let $L = [(\ell-1)!]^{1-q}$)

$$\frac{y^{(\ell)}(t)}{[y^{(\ell-1)}(t)]^q} \geq \frac{1}{[y^{(\ell-1)}(t)]^q} \int_t^\infty \int_{s_1}^\infty \cdots \int_{s_{n-\ell-1}}^\infty p(s_{n-\ell})\, y^q(s_{n-\ell}-h)$$

$$\times \frac{f(y_{s_{n-\ell}})}{y^q(s_{n-\ell}-h)}\, ds_{n-\ell}\, ds_{n-\ell-1} \cdots ds_1$$

$$\geq \frac{\alpha}{[y^{(\ell-1)}(t)]^q} \int_t^\infty \int_{s_1}^\infty \cdots \int_{s_{n-\ell-1}}^\infty p(s_{n-\ell}) y^q(s_{n-\ell}-h)\, ds_{n-\ell} ds_{n-\ell-1} \cdots ds_1$$

$$\geq \frac{\alpha}{[y^{(\ell-1)}(t)]^q} \int_{t+h}^\infty \int_{s_1}^\infty \cdots \int_{s_{n-\ell-1}}^\infty p(s_{n-\ell}) y^q(s_{n-\ell}-h)\, ds_{n-\ell} ds_{n-\ell-1} \cdots ds_1$$

$$\geq \frac{L\alpha}{(\ell-1)!} \int_{2t+h}^\infty \int_{s_1}^\infty \cdots \int_{s_{n-\ell-1}}^\infty (s_{n-\ell}-h-t)^{q(\ell-1)} p(s_{n-\ell})\, ds_{n-\ell} ds_{n-\ell-1} \cdots ds_1$$

$$\geq \frac{\alpha}{(\ell-1)!}\int_{2t+h}^{\infty}\int_{s_1}^{\infty}\cdots\int_{s_{n-\ell-1}}^{\infty}(s_{n-\ell}-h-t)^{\ell-1}p(s_{n-\ell})\,ds_{n-\ell}ds_{n-\ell-1}\cdots ds_1$$

$$\geq \frac{\alpha}{(\ell-1)!}\int_{2t+h}^{\infty}(t-h-T_2)^{\ell-1}\int_{s_1}^{\infty}\cdots\int_{s_{n-\ell-1}}^{\infty}p(s_{n-\ell})\,ds_{n-\ell}ds_{n-\ell-1}\cdots ds_1$$

for $t \geq T_2$ and $s_{n-\ell} > 2t > 2t-T_2$.

Integrating this inequality from T_2 to $t > T_2$, we obtain

$$\frac{1}{q-1}\left(\frac{1}{[y^{(\ell-1)}(T_2)]^{q-1}} - \frac{1}{[y^{(\ell-1)}(t)]^{q-1}}\right)$$

$$\geq \frac{\alpha}{(\ell-1)!}\int_{T_2}^{t}\int_{2s+h}^{\infty}(s-h-T_2)^{\ell-1}\int_{s_1}^{\infty}\cdots\int_{s_{n-\ell-1}}^{\infty}p(s_{n-\ell})\,ds_{n-\ell}ds_{n-\ell-1}\cdots ds_1ds$$

$$\geq \frac{\alpha}{(\ell-1)!}\int_{2T_2+h}^{2t+h}\int_{T_2}^{\frac{s_1-h}{2}}(s-h-T_2)^{\ell-1}\int_{s_1}^{\infty}\cdots\int_{s_{n-\ell-1}}^{\infty}p(s_{n-\ell})\,ds_{n-\ell}\cdots ds\,ds_1$$

$$\geq \frac{\alpha}{(\ell-1)!}\int_{2T_2+h}^{2t+h}\int_{T_2+h}^{\frac{s_1-h}{2}}(s-h-T_2)^{\ell-1}\int_{s_1}^{\infty}\cdots\int_{s_{n-\ell-1}}^{\infty}p(s_{n-\ell})\,ds_{n-\ell}\cdots ds\;ds_1$$

$$= \frac{\alpha}{2^{\ell}\,\ell!}\int_{2T_2+h}^{2t+h}(s_1-3h-2T_2)^{\ell}\int_{s_1}^{\infty}\cdots\int_{s_{n-\ell-1}}^{\infty}p(s_{n-\ell})\,ds_{n-\ell}\cdots ds_2\;ds_1$$

$$\geq \frac{\alpha}{2^{\ell}\,\ell!}\int_{2T_2+h}^{2t+h}\int_{2T_2+h}^{s_2}(s_1-3h-2T_2)^{\ell}\int_{s_2}^{\infty}\cdots\int_{s_{n-\ell-1}}^{\infty}p(s_{n-\ell})\,ds_{n-\ell}\cdots ds_3ds_1ds_2$$

$$\geq \frac{\alpha}{2^{\ell}\,\ell!}\int_{2T_2+h}^{2t+h}\int_{2T_2+3h}^{s_2}(s_1-3h-2T_2)^{\ell}\int_{s_2}^{\infty}\cdots\int_{s_{n-\ell-1}}^{\infty}p(s_{n-\ell})\,ds_{n-\ell}\cdots ds_3ds_1ds_2$$

$$= \frac{\alpha}{2^{\ell}(\ell+1)!} \int_{2T_2+h}^{2t+h} (s_2-3h-2T_2)^{\ell+1} \int_{s_2}^{\infty} \cdots \int_{s_{n-\ell-1}}^{\infty} p(s_{n-\ell})\, ds_{n-\ell} \cdots ds_3 ds_2 .$$

Continuing this process, we have

$$\frac{1}{q-1}\left(\frac{1}{[y^{(\ell-1)}(T_2)]^{q-1}} - \frac{1}{[y^{(\ell-1)}(t)]^{q-1}}\right) \geq \frac{\alpha}{2^{\ell}(n-1)!} \int_{2T_2+h}^{2t+h} (s-3h-2T_2)^{n-1} p(s)\, ds.$$

By hypothesis, the right hand side would tend to infinity as $t \to \infty$. This is a contradiction since the left hand side is bounded.

If the hypotheses of Theorem 5.3 hold, then by Theorem 5.2, the necessary part of Theorem 5.3 holds for $n \geq 3$ odd. But the sufficient part of Theorem 5.3 for $n \geq 3$ odd will be stated as follows: Every solution y of (5.1) is either oscillatory or tends to a limit monotonically if $\int^{\infty} t^{n-1}\, p(t)\, dt = \infty$.

In the proof of Theorem 5.3, if we do not replace $q(\ell-1)$ by $\ell - 1$, we shall have that: if $1 \leq \ell < n$, $q > 1$ and $\int^{\infty} s^{q(\ell-1)+(n-\ell)} p(s)\, ds = \infty$, then there are no nonoscillatory solutions of (5.1) which satisfy Lemma 1.1 for this given, or any greater, value of ℓ.

THEOREM 5.4. <u>Assume there exist</u> q, $0 < q < 1$, <u>and</u> $A > 0$ <u>such that</u>

(5.6) $\quad |f(t,u,\ldots)| \leq A|u(0)|^q$ <u>for</u> $t \geq t_0$ <u>and</u> $u \in C\uparrow$ <u>or</u> $u \in C\downarrow$.

<u>Then a necessary condition that all solutions of</u> (5.2) <u>be oscillatory is that</u>

(5.7) $$\int^{\infty} t^{(n-1)q}\, p(t)\, dt = \infty.$$

Proof. Assume that $\int^{\infty} t^{(n-1)q}\, p(t)\, dt < \infty$. We shall show the existence of a nonoscillatory solution of (5.2). Let $\int_{t_0}^{\infty} t^{(n-1)q} p(t)\, dt = M$. Choose a positive number β such that $\beta^{1-q} > AM/[(n-1)!]^q$.

Consider the solution y of (5.2) with the following initial conditions:

$$y^{(j)}(t) = 0 \quad \text{for } t_0 - h \le t \le t_0, \quad j = 0,1,\ldots,n-2,$$
$$y^{(n-1)}(t) = \beta \text{ for } t_0 - h \le t \le t_0.$$

We shall show that this solution does not vanish on (t_0,∞).

If $y(t_1) = 0$ with $y(t) > 0$ for $t \in (t_0,t_1)$, then by Taylor's theorem, we have

$$y(t_1) - y(t_0) = y'(t_0)(t_1 - t_0) +\ldots+ \frac{y^{(n-2)}(t_0)}{(n-2)!}(t_1 - t_0)^{n-2}$$
$$+ \frac{y^{(n-1)}(\xi)}{(n-1)!}(t_1 - t_0)^{n-1}$$

for some $t_0 < \xi < t_1$. It follows that $y^{(n-1)}(\xi) = 0$. We can assume $y^{(n-1)}(t) \ge 0$ on $[t_0,\xi]$. By (5.2), $y^{(n)}(t) \le 0$ on $[t_0,\xi]$. Then $y^{(n-1)}$ is monotone decreasing on $[t_0,\xi]$ and then $y^{(n-1)}(t) \le \beta$ on $[t_0,\xi]$. Furthermore, by Taylor's theorem, we have

$$y(t) = y(t) - y(t_0) = \frac{y^{(n-1)}(\zeta)}{(n-1)!}(t - t_0)^{n-1} \le \frac{\beta}{(n-1)!}(t - t_0)^{n-1}$$

for some $t_0 < \zeta < t$, $(t_0 \le t \le \xi)$. By (5.6), we obtain

$$f(t,y_t,y_t',\ldots,y_t^{(n-1)}) \le A[y(t)]^q \le A\left[\frac{\beta}{(n-1)!}(t - t_0)^{n-1}\right]^q.$$

It follows from (5.2) that

$$\beta - y^{(n-1)}(t) = y^{(n-1)}(t_0) - y^{(n-1)}(t) = -\int_{t_0}^{t} y^{(n)}(s)\,ds$$

$$= \int_{t_0}^{t} p(s)\, f(s,y_s,y_s',\ldots,y_s^{(n-1)})\,ds \le \frac{A\beta^q}{[(n-1)!]^q}\int_{t_0}^{t} p(s)[(s-t_0)^{n-1}]^q\,ds$$

$$\le \frac{AM\beta^q}{[(n-1)!]^q}$$

or

$$y^{(n-1)}(t) \ge \beta - \frac{AM\beta^q}{[(n-1)!]^q} > 0 \text{ on } [t_0,\xi].$$

This is a contradiction since $y^{(n-1)}(\xi) = 0$. This shows that our solution y of (5.2) does not vanish on (t_0,∞).

THEOREM 5.5. *For* n *even, assume there exist* $0 < \delta < 1$ *and* $B > 0$ *such that*

(5.8) $$|f(t,u,\ldots)| \ge B|u(-h)|^{\delta} \quad \textit{for } t \ge t_0 \textit{ and } u \in C\uparrow \textit{ or } u \in C\downarrow.$$

Then a sufficient condition that all solutions of (5.2) *to be oscillatory is that*

(5.9) $$\int^{\infty} t^{(n-1)\delta}\, p(t)\,dt = \infty.$$

Proof. Let y be a nonoscillatory solution of equation (5.2). We can assume that $y(t) > 0$ for $t \ge t_0$. It follows from (5.2) that $y^{(n)}(t) \le 0$ for $t \ge T_0 \ge t_0 + h$. There is a $T_1 \ge T_0$ such that each $y^{(j)}(t)$, $j = 0,1,\ldots,n-1$, is of constant sign on $[T_1,\infty)$. By Lemma 1.1, $y'(t)$ is positive and $y^{(n-1)}(t)$ is positive and nonincreasing on $[T_1,\infty)$. Multiplying (5.2) by $[y^{(n-1)}(t)]^{-\delta}$, we obtain

$$(5.10)\quad \frac{y^{(n)}(t)}{[y^{(n-1)}(t)]^{\delta}} = -\frac{p(t)\ f(t,y_t,t_t',\ldots,y_t^{(n-1)})}{[y^{(n-1)}(t)]^{\delta}}$$

$$\leq -\frac{p(t)\ f(t,y_t,y_t',\ldots,y_t^{(n-1)})}{[y^{(n-1)}(t-h)]^{\delta}} \quad \text{for } t \geq T_1.$$

By Lemma 1.1, we have

$$\frac{y(2^{n-\ell-1}\ t)}{y^{(n-1)}(2^{n-\ell-1}\ t)} \geq \frac{y(t)}{y^{(n-1)}(2^{n-\ell-1}\ t)} \geq \frac{(t-T_1)^{n-1}}{(n-1)(n-2)\ldots(n-\ell)} \quad \text{for } t \geq T_1.$$

It follows that

$$(5.11)\quad \frac{y(t)}{y^{(n-1)}(t)} \geq Lt^{n-1} \text{ for some constant } L > 0 \text{ and for}$$

$t \geq 2^{n-\ell-1}\ T_1 = T_2$, say.

From (5.10), (5.11) and (5.8), we obtain

$$\frac{y^{(n)}(t)}{[y^{(n-1)}(t)]^{\delta}} \leq -p(t)\ BL^{\delta}(t-h)^{(n-1)\delta} \quad \text{for } t \geq T_2+h = T_3, \text{ say.}$$

Integrating this from T_3 to $t > T_3$, we have

$$\frac{[y^{(n-1)}(t)]^{1-\delta} - [y^{(n-1)}(T_3)]^{1-\delta}}{1-\delta} \leq -BL^{\delta}\int_{T_3}^{t} p(s)(s-h)^{(n-1)\delta}\,ds$$

or

$$0 < [y^{(n-1)}(t)]^{1-\delta} \leq [y^{(n-1)}(T_3)]^{1-\delta} - BL^{\delta}(1-\delta)\int_{T_3}^{t} p(s)(s-h)^{(n-1)\delta}ds.$$

The right hand side will tends to $-\infty$ as $t \to \infty$. We get a contradiction and the proof is then complete.

From Theorem 5.4 and Theorem 5.5, we have the following

COROLLARY 5.1. *For* n *even, let there exist* $A > 0$, $B > 0$ *and* $0 < q < 1$ *such that*

$$B|u(-h)|^q \leq |f(t,u,\ldots)| \leq A|u(0)|^q$$

for $t \geq t_0$ *and for any* $u \in C\uparrow$ *or* $u \in C\downarrow$.

Then a necessary and sufficient condition that all solutions of (5.2) *be oscillatory is that* (5.7) *holds*.

THEOREM 5.6. *For* n *odd, let* (5.8) *and* (5.9) *hold. Then every solution* y *of* (5.2) *is either oscillatory or tends monotonically to a limit and* $\lim_{t\to\infty} y^{(j)}(t) = 0$, $j = 1,2,\ldots,n-1$.

Proof. Let y be a nonoscillatory solution of (5.2). We can assume that $y(t) > 0$ for $t \geq T_0$. As in the proof of Theorem 5.5, there is a $T_1 \geq T_0$ such that $y'(t) > 0$ for $t \geq T_1$ or $y'(t) < 0$ for $t \geq T_1$.

If $y'(t) < 0$ for $t \geq T_1$, then y is decreasing and bounded below. Hence $y(t)$ tends to a limit as $t \to \infty$ and by Lemma 1.3, $\lim_{t\to\infty} y^{(j)}(t) = 0$, $j = 1,2,\ldots,n-1$.

If $y'(t) > 0$ for $t \geq T_1$, then the proof is the same as that of Theorem 5.5 and we omit it here.

SECTION VI

In this section, we shall generalize Bobisud's results in [5] and Baker's results in [3] to arbitrary order damped nonlinear differential equations of the forms

$$(6.1)\quad y^{(n)} + \phi(t,y,y',\dots,y^{(n-1)})y^{(n-1)} + F(t,y,y',\dots,y^{(n-1)}) = 0$$

and

$$(6.2)\quad y^{(n)} + q(t)\psi(y,y',\dots,y^{(n-1)}) + p(t)y^{\alpha}(t)f(y',y'',\dots,y^{(n-1)}) = 0.$$

We shall employ the following assumptions in our theorems.

(VI_1) F <u>is continuous on</u> $[0,\infty) \times \mathbb{R}^n$ <u>with</u> $u\,F(t,u,u_1,\dots,u_{n-1}) > 0$ <u>for</u> $u \neq 0$.

(VI_2) ϕ <u>is continuous on</u> $[0,\infty) \times \mathbb{R}^n$ <u>and there exist continuous nonnegative functions</u> k <u>and</u> m <u>such that</u>

$$-k(t) \le \phi(t,u,u_1,\dots,u_{n-1}) \le m(t)$$

<u>for</u> $t \in [0,\infty)$.

(VI_3) $\psi \in C(\mathbb{R}^n)$ <u>and there is a constant</u> $B > 0$ <u>such that</u>

$$0 < \psi(x,u_1,u_2,\dots,u_{n-1})\, u_{n-1} \le Bu_{n-1}^2$$

<u>for</u> $u_{n-1} \in \mathbb{R} - \{0\}$ <u>and</u> $x \in \mathbb{R}$.

(VI_4) $f \in C(\mathbb{R}^{n-1})$ <u>and</u> $f(u_1,u_2,\dots,u_{n-1}) \ge K > 0$ <u>for</u> $(u_1,u_2,\dots,u_{n-1}) \in \mathbb{R}^{n-1}$.

(VI_5) $p \in C[0,\infty)$ <u>and</u> $p(t) > 0$ <u>on</u> $[0,\infty)$.

(VI_6) $q \in C([0,\infty),[0,\infty))$ _and for fixed_ $T \geq 0$,

$$\int_T^t \exp[-B\int_T^s q(\tau)\, d\tau]\, ds \to \infty \text{ as } t \to \infty.$$

It follows from (VI_3) that $\psi(x,u_1,u_2,\ldots,u_{n-1},0) = 0$. Moreover, by (VI_4) and (VI_5), if $y(t) \not\equiv 0$ on $[T,\infty)$, then $y^{(j)}(t) \not\equiv 0$ on $[T,\infty)$, $j = 1,2,\ldots,n-1$.

THEOREM 6.1. _Let the functions_ F _and_ ϕ _satisfy_ (VI_1) _and_ (VI_2). _Furthermore, assume the following conditions hold._

(i) _Given_ $M > 0$ _and_ $\delta > 0$, _there exist_ $t_{\delta,M} > 0$ _and a function_ $h(t) = h_{\delta,M}(t)$ _defined on_ $[t_{\delta,M},\infty)$ _with_

$$\int_{t_{\delta,M}}^t h(s)\, ds \to \infty \text{ as } t \to \infty$$

and such that $|u_{n-1}| \leq M$, $|u| > \delta$ _and_ $u\, u_{n-1} > 0$ _imply_

$$|F(t,u,u_1,\ldots,u_{n-1})| \geq h(t).$$

(ii) _For any_ $t_o > 0$,

$$\int_{t_o}^\infty k(s)\, ds < \infty \text{ and } \lim_{t\to\infty} \int_{t_o}^t \exp[-\int_{t_o}^s m(\tau)\, d\tau]\, ds = \infty.$$

Then, if n _is even, every solution of equation_ (6.1) _is oscillatory. If_ n _is odd, every solution_ y _of_ (6.1) _is either oscillatory or tends monotonically to a limit and_ $y^{(j)}(t) \to 0$ _as_ $t \to \infty$, $j = 1,2,\ldots,n-1$.

Proof. Let y be a nonoscillatory solution of (6.1). We can assume that $y(t) > 0$ on $[T_o,\infty)$ for some $T_o > 0$. Similar argument holds for $y(t) < 0$. We claim that $y^{(n-1)}(t) > 0$ on $[T_o,\infty)$. Indeed, first suppose $y^{(n-1)}(t_1) < 0$ for some $t_1 \in [T_0,\infty)$, then $y^{(n-1)}(t)$ remains negative on $[t_1,t_2)$ for some $t_2 > t_1$. From (6.1), we have

$$\frac{y^{(n)}(t)}{y^{(n-1)}(t)} + \phi(t,y(t),y'(t),\dots,y^{(n-1)}(t)) > 0 \text{ for } t \in [t_1,t_2).$$

Integrating this from t_1 to t_2, we get

$$(6.3) \quad y^{(n-1)}(t_2) \le y^{(n-1)}(t_1) \exp[-\int_{t_1}^{t_2} \phi(s,y(s),y'(s),\dots,y^{(n-1)}(s))\, ds]$$

$$\le y^{(n-1)}(t_2) \exp[-\int_{t_1}^{t_2} m(s)\, ds] < 0$$

Therefore, $y^{(n-1)}(t) < 0$ for every $t \in [t_1,\infty)$. By Lemma 1.1, $y^{(n-2)}(t) \ge 0$ for $t \in [t_1,\infty)$. Integrating (6.3) from t_1 to $t > t_1$, we obtain

$$y^{(n-2)}(t) \le y^{(n-2)}(t_1) + y^{(n-1)}(t_1) \int_{t_1}^{t} \exp[-\int_{t_1}^{s} m(\tau)\, d\tau]\, ds \to -\infty \text{ as } t \to \infty,$$

a contradiction. Therefore $y^{(n-1)}(t) \ge 0$ on $[T_0,\infty)$.

Suppose there is a $\bar{t} \in [T_0,\infty)$ such that $y^{(n-1)}(\bar{t}) = 0$. Since $y^{(n)}(\bar{t}) = -F(\bar{t},y(\bar{t}),y'(\bar{t}),\dots,y^{(n-1)}(\bar{t})) < 0$, there is an $\varepsilon > 0$ such that $y^{(n-1)}(t) < 0$ on $(\bar{t},\bar{t}+\varepsilon)$, a contradiction. Hence $y^{(n-1)}(t) > 0$ for $t \ge T_0$. There is a $T_1 \ge T_0$ such that each $y^{(j)}(t)$, $j = 0,1,\dots,n-2$, is of constant sign on $[T_1,\infty)$.

By Lemma 1.2, $y'(t) \le 0$ on $[T_1,\infty)$ or $y'(t) \ge 0$ on $[T_1,\infty)$.

The case of $y'(t) \le 0$ on $[T_1,\infty)$ can occur only when n is odd. So in this case $y(t)$ tends monotonically to a limit. Moreover, by Lemma 1.3, $y^{(j)}(t) \to 0$ as $t \to \infty$ for $j = 1,2,\dots,n-1$.

If $y'(t) \ge 0$ on $[T_1,\infty)$, then there exists a $\delta > 0$ such that $y(t) > \delta$ for $t \ge T_1$. Dividing (6.1) by $y^{(n-1)}(t) > 0$, we have

$$\frac{y^{(n)}(t)}{y^{(n-1)}(t)} = -\phi(t,y(t),y'(t),\dots,y^{(n-1)}(t)) - \frac{F(t,y(t),y'(t),\dots,y^{(n-1)}(t))}{y^{(n-1)}(t)}$$

$$< -\phi(t,y(t),y'(t),\dots,y^{(n-1)}(t)) \le k(t) \text{ for } t \ge T_1.$$

Integrating this from T_1 to $t > T_1$, we obtain

$$y^{(n-1)}(t) \le y^{(n-1)}(T_1) \exp\left[\int_{T_1}^{t} k(s)\,ds\right] \le y^{(n-1)}(T_1) \exp\left[\int_{T_1}^{\infty} k(s)\,ds\right]$$

$$\le M \text{ for some } M > 0.$$

Since $y(t) > \delta$, $0 < y^{(n-1)}(t) \le M$ and $y(t)\, y^{(n-1)}(t) > 0$ for $t \ge T_1$, by (i), there exist $t_{\delta,M}$ and $h(t) = h_{\delta,M}(t)$ such that $\int_{t_{\delta,M}}^{t} h(s)\,ds \to \infty$ as $t \to \infty$ and $F(t,y(t),y'(t),\dots,y^{(n-1)}(t)) \ge h(t)$ for $t \ge T_1$.

Let $T = \max\{T_1, T_{\delta,M}\}$. Integrating (6.1) from T to $t > T$, we get

$$\text{(6.4)} \quad y^{(n-1)}(t) = y^{(n-1)}(T) - \int_{T}^{t} \phi(s,y(s),y'(s),\dots,y^{(n-1)}(s))y^{(n-1)}(s)\,ds$$

$$- \int_{T}^{t} F(s,y(s),y'(s),\dots,y^{(n-1)}(s))\,ds$$

$$\le y^{(n-1)}(T) + \int_{T}^{t} k(s)\, y^{(n-1)}(s)\,ds - \int_{T}^{t} h(s)\,ds.$$

Since $y^{(n-1)}(t)$ is bounded, $k(s)$ is integrable and $\int_{T}^{t} h(s)\,ds \to \infty$ as $t \to \infty$, the left hand side of (6.4) tends to $-\infty$ as $t \to \infty$ and then $y^{(n-1)}(t)$ would be negative for all sufficiently large t. This is a contradiction and our proof is then complete.

THEOREM 6.2. _Let the functions_ F _and_ ϕ _satisfy_ (VI_1) _and_ (VI_2). _Furthermore, assume the following two conditions hold._

(i') *given* $\delta > 0$ *there exist* $t_0 > 0$ *and a function* $h(t) = h_{t_0}(t)$ *defined on* $[t_0,\infty)$ *with*

$$\frac{1}{t^{n-1}} \int_{t_0}^{t} (t-s)^{n-1} h(s)\, ds \to \infty \text{ as } t \to \infty$$

and such that $|u| \geq \delta$ *and* $u\, u_{n-1} > 0$ *imply* $|F(t,u,u_1,\ldots,u_{n-1})| \geq h(t)$.

(ii') *for any* $t_1, t_2 > 0$,

$$\frac{1}{t^{n-1}} \int_{t_1}^{t} (t-s)^{n-1} k(s) \exp\left[\int_{t_2}^{s} k(\tau)\, d\tau\right] ds$$

is bounded while

$$\lim_{t\to\infty} \int_{t_1}^{t} \exp\left[-\int_{t_1}^{s} m(\tau)\, d\tau\right] ds = \infty.$$

Then, if n *is even, every solution of equation* (6.1) *is oscillatory. If* n *is odd, every solution* y *of* (6.1) *is either oscillatory or tends monotonically to a limit and* $y^{(j)}(t) \to 0$ *as* $t \to \infty$, $j = 1,2,\ldots,n-1$.

Proof. As in the proof of Theorem 6.1, let y be a nonoscillatory solution of (6.1) with $y(t) > 0$ on $[T_0,\infty)$ and conclude that $y^{(n-1)}(t) > 0$ and $y^{(n-1)}(t) \leq y^{(n-1)}(T_0) \exp\left[\int_{T_0}^{t} k(s)\, ds\right]$ for $t \geq T_0$.

If $y'(t) \leq 0$ on $[T_0,\infty)$, then as in the proof of Theorem 6.1, the proof is through.

Suppose $y'(t) \geq 0$ on $[T_0,\infty)$, then there exists a $\delta > 0$ such that $y(t) > \delta$ for $t \geq T_0$. By (i'), there exist $T_1 \geq T_0$ and a function h such that $F(t,y(t),y'(t),\ldots,y^{(n-1)}(t)) \geq h(t)$ for $t \geq T_1$ with

$$\frac{1}{t^{n-1}} \int_{T_1}^{t} (t-s)^{n-1} h(s)\, ds \to \infty \text{ as } t \to \infty.$$

Integrating (6.1) from T_1 to $t > T_1$, we obtain

$$y^{(n-1)}(t) = y^{(n-1)}(T_1) - \int_{T_1}^{t} \phi(s,y(s),y'(s),\dots,y^{(n-1)}(s))\, y^{(n-1)}(s)\, ds$$

$$- \int_{T_1}^{t} F(s,y(s),y'(s),\dots,y^{(n-1)}(s))\, ds$$

$$\le y^{(n-1)}(T_1) + \int_{T_1}^{t} k(s)\, y^{(n-1)}(s)\, ds - \int_{T_1}^{t} h(s)\, ds$$

$$\le y^{(n-1)}(T_1) + y^{(n-1)}(T_0) \int_{T_1}^{t} k(s) \exp[\int_{T_0}^{s} k(\tau)\, d\tau]\, ds - \int_{T_1}^{t} h(s)\, ds.$$

Integrating this repeatedly from T_1 to $t > T_1$, we obtain

$$y(t) \le y(T_1) + y'(T_1)(t - T_1) + \dots + y^{(n-1)}(T_1) \frac{(t - T_1)^{n-1}}{(n-1)!}$$

$$+ y^{(n-1)}(T_0) \int_{T_1}^{t} \frac{(t-s)^{n-1}}{(n-1)!} k(s) \exp[\int_{T_0}^{s} k(\tau)\, d\tau]\, ds$$

$$- \int_{T_1}^{t} \frac{(t-s)^{n-1}}{(n-1)!} h(s)\, ds$$

or

$$\frac{y(t)}{t^{n-1}} \le \frac{y(T_1)}{t^{n-1}} + \frac{y'(T_1)}{t^{n-2}} \left(1 - \frac{T_1}{t}\right) + \dots + \frac{y^{(n-1)}(T_1)}{(n-1)!} \left(1 - \frac{T_1}{t}\right)^{n-1}$$

$$+ \frac{y^{(n-1)}(T_0)}{(n-1)!\,t^{n-1}} \int_{T_1}^{t} (t-s)^{n-1} k(s) \exp[\int_{T_0}^{s} k(\tau)\, d\tau]\, ds$$

$$- \frac{1}{(n-1)!\; t^{n-1}} \int_{T_1}^{t} (t-s)^{n-1} h(s)\, ds.$$

By (i') and (ii'), the right hand side tends to $-\infty$ as $t \to \infty$ and then $y(t)$ would be negative for all sufficiently large t. This contradiction proves our theorem.

In the following lemma, we consider an equation which is slightly more general than equation (6.2).

LEMMA 6.1. Consider the differential equation

$$y^{(n)} + q(t)\ \psi(y,y',\ldots,y^{(n-1)}) + p(t)\ \gamma(y)\ f(y',y'',\ldots,y^{(n-1)}) = 0 \tag{6.5}$$

with (VI_3) - (VI_6) holding and let $\gamma \in C(\mathcal{R})$, $x\gamma(x) > 0$ for $x \neq 0$. If y is a nonoscillatory solution of (6.5) on $[0,\infty)$, then there exists a $t_0 \geq 0$ such that

$$y(t)\ y^{(n-1)}(t) > 0 \text{ for } t \geq t_0.$$

Proof. We can assume that $y(t) > 0$ on $[T_0,\infty)$ for some $T_0 \geq 0$. Similar argument holds for $y(t) < 0$. We shall show that $y^{(n-1)}(t) > 0$ on $[T_0,\infty)$. Suppose $y^{(n-1)}(t_1) < 0$ for some $t_1 \in [T_0,\infty)$. Let $y^{(n-1)}(t)$ remain negative on $[t_1,t_2)$. From (6.5), we have

$$y^{(n)}(t) + q(t)\ \psi(y(t),y'(t),\ldots,y^{(n-1)}(t)) \leq 0 \text{ for } t \in [t_1,t_2).$$

By (VI_3), we obtain

$$\frac{y^{(n)}(t)}{y^{(n-1)}(t)} + q(t)\ B \geq \frac{y^{(n)}(t)}{y^{(n-1)}(t)} + q(t)\ \frac{\psi(y(t),y'(t),\ldots,y^{(n-1)}(t))}{y^{(n-1)}(t)} \geq 0$$

for $t \in [t_1,t_2)$. Integrating this from t_1 to t_2, we get

$$y^{(n-1)}(t_2) \leq y^{(n-1)}(t_1) \exp[-B \int_{t_1}^{t_2} q(t)\ dt] < 0. \tag{6.6}$$

Therefore, $y^{(n-1)}(t) < 0$ on $[T_1,\infty)$. Since $y^{(n-2)}$ is decreasing for $t \geq T_1$, by the remark given above Theorem 6.1 and by Lemma 1.1, $y^{(n-2)}(t) > 0$ for all sufficiently large t.

Integrating (6.6) with respect to t_2 from t_1 to $t > t_1$, we obtain

$$y^{(n-1)}(t) \leq y^{(n-1)}(t_1) + y^{(n-1)}(t_1) \int_{t_1}^{t} \exp[-B \int_{t_1}^{s} q(\tau)\, d\tau]\, ds.$$

By (VI_6), $y^{(n-2)}(t)$ would be negative for all sufficiently large t. This contradiction shows that $y^{(n-1)}(t) \geq 0$ for $t \geq T_0$.

Suppose there is a $\bar{t} \in [T_0,\infty)$ such that $y^{(n-1)}(\bar{t}) = 0$. Since $y^{(n)}(\bar{t}) = -p(\bar{t})\, \gamma(y)\, f(y',y'',\ldots,y^{(n-1)}) < 0$, there exists an $\varepsilon > 0$ such that $y^{(n-1)}(t) < 0$ for $t \in (\bar{t},\bar{t}+\varepsilon)$, a contradiction. Therefore, $y^{(n-1)}(t) > 0$ on $[T_0,\infty)$.

THEOREM 6.3. _Let_ $\alpha > 1$ _be the quotient of odd integers and let_ (VI_3) - (VI_6) _be satisfied. Let_

$$\int^{\infty} t^{n-1}\, p(t)\, dt = \infty. \tag{6.7}$$

Then, if n _is even, every solution of_ (6.2) _is oscillatory. If_ n _is odd, every solution_ y _of_ (6.2) _is either oscillatory or tends to a limit monotonically and_ $y^{(j)}(t) \to 0$ _as_ $t \to \infty$, $j = 1,2,\ldots,n-1$.

Proof. Let y be a nonoscillatory solution of (6.2). We can assume that $y(t) > 0$ on $[T_0,\infty)$ for some $T_0 \geq 0$. By Lemma 6.1, $y^{(n-1)}(t) > 0$ for $t \geq T_0$. By (VI_3), $\psi(y(t),y'(t),\ldots,y^{(n-1)}(t)) > 0$ on $[T_0,\infty)$. From (6.2), we see that $y^{(n)}(t) < 0$ on $[T_0,\infty)$. There is a $T_1 \geq T_0$ such that each $y^{(j)}(t)$, $j = 0,1,\ldots,n-1$, is of constant sign on $[T_1,\infty)$. By Lemma 1.1, there is an integer ℓ such that (1.2) holds. The case of

$\ell = 0$ can occur only when n is odd and then y(t) tends monotonically to a limit. Moreover, by Lemma 1.3, $y^{(j)}(t) \to 0$ as $t \to \infty$ for $j = 1,2,\ldots,n-1$.

Apply (VI_4) to (6.2), we have

$$y^{(n)}(t) + p(t)y^{\alpha}(t)K \leq y^{(n)}(t) + p(t)y^{\alpha}(t)f(y'(t),y''(t),\ldots,y^{(n-1)}(t)) \tag{6.8}$$

$$\leq 0 \text{ for } t \geq T_1.$$

Multiplying (6.8) by $t^{n-1}y^{-\alpha}$ and integrating from T_1 to $t > T_1$, we obtain

$$\int_{T_1}^{t} \frac{s^{n-1}\, y^{(n)}(s)}{[y(s)]^{\alpha}}\, ds + K\int_{T_1}^{t} s^{n-1}\, p(s)\, ds \leq 0.$$

Integrating the first integral by parts, we have

$$\left[\frac{s^{n-1}\, y^{(n-1)}(s)}{[y(s)]^{\alpha}}\right]_{T_1}^{t} - (n-1)\int_{T_1}^{t} \frac{s^{n-2}}{[y(s)]^{\alpha}}\, y^{(n-1)}(s)\, ds$$

$$+ \alpha\int_{T_1}^{t} \frac{s^{n-1}\, y'(s)}{[y(s)]^{\alpha+1}}\, y^{(n-1)}(s)\, ds + K\int_{T_1}^{t} s^{n-1}\, p(s)\, ds \leq 0 \text{ for } t \geq T_1.$$

We repeat this procedure on the second term of the above display, etc. From this we obtain

$$P(t) - P(T_1) + Q(t) + K\int_{T_1}^{t} s^{n-1}\, p(s)\, ds \leq 0 \text{ for } t \geq T_1, \tag{6.9}$$

where

$$P(t) = \frac{1}{[y(t)]^{\alpha}}\, [t^{n-1}y^{(n-1)}(t) - (n-1)t^{n-2}y^{(n-2)}(t)$$

$$+ (n-1)(n-2)t^{n-3}y^{(n-3)}(t) - \ldots + (n-1)\ldots(\ell+1)t^{\ell}y^{(\ell)}(t)]$$

$$+ (n-1)(n-2)\ldots\ell\int_{T_1}^{t} \frac{s^{\ell-1}\, y^{(\ell)}(s)}{[y(s)]^{\alpha}}\, ds,$$

and

$$Q(t) = \alpha\int_{T_1}^{t} \frac{y'(s)}{[y(s)]^{\alpha+1}} [s^{n-1}y^{(n-1)}(s) - (n-1)s^{n-2}y^{(n-2)}(s)$$

$$+\ldots+ (n-1)(n-2)\ldots(\ell+1)s^{\ell}y^{(\ell)}(s)]\, ds.$$

It follows from (1.2) that $P(t) > 0$ and $Q(t) > 0$ for $t \geq T_1$. By (6.7), the inequality (6.9) is impossible for all sufficiently large t. The proof is then complete.

LEMMA 6.2. _If_ y _is an_ n'th _continuously differentiable function with_ $y(t) > 0$, $y'(t) > 0$, $y^{(n-1)}(t) > 0$ _and_ $y^{(n)}(t) < 0$ _on_ $[t_0,\infty)$, _then there is a_ $T_0 \geq t_0$ _such that_

$$\frac{y(t)}{y^{(n-1)}(t)} \geq \frac{t^{n-1}}{(n-1)(n-2)\ldots(n-\ell)\, 2^{(n-1)(n-\ell)}}$$

for $t \geq 2^{n-\ell}\, T_0$, _where_ ℓ _is defined in_ Lemma 1.1.

Proof. Since $y^{(n)}(t) < 0$ on $[t_0,\infty)$, there is a $T_0 \geq t_0$ such that each $y^{(j)}(t)$, $j = 0,1,\ldots,n-1$, is of constant sign on $[T_0,\infty)$. By Lemma 1.1, we have

$$\frac{y(t)}{y^{(n-1)}(2^{n-\ell-1}\, t)} \geq \frac{(t - T_0)^{n-1}}{(n-1)(n-2)\ldots(n-\ell)} \quad \text{for } t \geq T_0.$$

Since $y(t)$ is increasing,

$$\frac{y(2^{n-\ell-1}\, t)}{y^{(n-1)}(2^{n-\ell-1}\, t)} \geq \frac{(t - T_0)^{n-1}}{(n-1)(n-2)\ldots(n-\ell)}$$

Let $s = 2^{n-\ell-1}\, t$ and $s_0 = 2^{n-\ell-1}\, T_0$, then

$$\frac{y(s)}{y^{(n-1)}(s)} \geq \frac{(s-s_0)^{n-1}}{(n-1)(n-2)\ldots(n-\ell)\ 2^{(n-1)(n-\ell-1)}} \geq \frac{s^{n-1}}{(n-1)(n-2)\ldots(n-\ell)2^{(n-1)(n-\ell)}}$$

if $s > 2s_0 = 2^{n-\ell}\ T_0$.

THEOREM 6.4. Let $0 < \alpha < 1$ be the quotient of odd integers and let (VI_3) - (VI_6) be satisfied. Let

(6.10) $$\int^{\infty} t^{\alpha(n-1)}\ p(t)\ dt = \infty.$$

Then, if n is even, every solution of (6.2) is oscillatory. If n is odd, every solution y of (6.2) is either oscillatory or tends to a limit monotonically and $y^{(j)}(t) \to 0$ as $t \to \infty$, $j = 1,2,\ldots,n-1$.

Proof. Let y be a nonoscillatory solution of (6.2). We can assume that $y(t) > 0$ on $[T_0,\infty)$ for some $T_0 \geq 0$. By Lemma 6.1, $y^{(n-1)}(t) > 0$ for $t \geq T_0$. It follows from (6.2) that $y^{(n)}(t) < 0$ for $t \geq T_0$. As in the proof of Theorem 6.3, there is an integer ℓ such that (1.2) holds for $t \geq T_1 \geq T_0$. The case of $y'(t) < 0$ on $[T_1,\infty)$ can occur only when n is odd. In this case, y(t) tends monotonically to a limit. Moreover, by Lemma 1.3, $y^{(j)}(t) \to 0$ as $t \to \infty$ for $j = 1,2,\ldots,n-1$.

Also as in the proof of Theorem 6.3, we have inequality (6.8). Dividing (6.8) by $[y^{(n-1)}(t)]^{\alpha}$ and integrating from $T = 2^{n-\ell}\ T_1$ to $t > T$, we have

$$\frac{1}{1-\alpha}\ [y^{(n-1)}(t)]^{1-\alpha} - \frac{1}{1-\alpha}\ [y^{(n-1)}(T)]^{1-\alpha} + K\int_T^t p(s)\left[\frac{y(s)}{y^{(n-1)}(s)}\right]^{\alpha} ds \leq 0.$$

By Lemma 6.2, we obtain

$$\frac{1}{1-\alpha}\,[y^{(n-1)}(t)]^{1-\alpha} - \frac{1}{1-\alpha}\,[y^{(n-1)}(T)]^{1-\alpha}$$

$$+ \frac{K}{[(n-1)(n-2)\ldots(n-\ell)\; 2^{(n-1)(n-\ell)}]^{\alpha}} \int_T^t s^{\alpha(n-1)}\; p(s)\; ds \le 0.$$

Since $y^{(n-1)}(s) > 0$ on $[T,\infty)$ and by (6.10), the above inequality is impossible for all sufficiently large t. This proves Theorem 6.4.

If $q(t) \equiv 0$, then function q satisfies (VI_6). In this case, we have similar results in Theorems 5.5 and 5.6 for functional differential equation (5.2).

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