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Abstract

In this dissertation, we study the *Nash problem* over terminal singularities in dimension 3, or *3-fold terminal singularities*. In particular, for 3-fold terminal singularities, we propose two conjectures that aim to characterize their induced *Nash valuations*.

(A) Any prime divisor with discrepancy bounded by 1 induces a Nash valuation.

(B) Any prime divisor with minimal discrepancy induces a Nash valuation.

We prove that Conjecture A holds for toric terminal singularities in arbitrary dimension, and Conjecture B holds for 3-fold terminal singularities of type cAx/2. In the Gorenstein cases, we provide a partial result with examples for both of the conjectures.

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I would like to conclude with my favorite sentence that appears in the novel *Kino no Tabi-the Beautiful World*, written by Keiichi Sigsawa.

The world is not beautiful. Therefore, it is.

Keng-Hung Steven Lin, in Fort. Collins, Colorado

Chapter 1

Introduction

The aim of this dissertation is to study Nash valuations over 3-fold terminal singularities.

Singularities play a crucial role in the study of algebraic varieties [Ish18], and they often provide useful characterizations of the varieties under consideration.

In birational geometry, singularities can tell us some important information in birational classifications of algebraic varieties. By applying certain birational maps, one can study how singularities behave and transform. A natural and fundamental tool in this context is the resolution of singularities.

Definition 1.0.1 (Resolution of Singularities)**.** Let X be an algebraic variety over a field k. A *resolution of singularities* over X, or a *resolution of* X for short, is a proper birational morphism $f : Y \to X$ such that Y is non-singular.

If the ground field k is of characteristic 0, the existence is confirmed in [Hir64] by H. Hironaka,

Theorem 1.0.2 (Existence of Resolution of Singularities, [Hir64])**.** *Let X be a variety over a field k of characteristic 0, then X has a resolution of singularities.*

For a resolution $f : Y \to X$, the preimage of the singular locus $f^{-1}(X_{\text{sing}})$ is called the *exceptional locus* of f. Since an exceptional locus arises from resolving singularities, it is expected to encode some information about the singularities we study. In particular, we are interested in the *exceptional divisors* with the associated *valuations* induced from the exceptional locus of a resolution. However, resolutions of singularities are generally not unique, so it is natural to seek a birational invariant that connects exceptional loci among all resolutions. This induces the concept of *essential exceptional divisors*.

Definition 1.0.3 (Essential Exceptional Divisors)**.** Let X be a variety over k. An exceptional prime divisor is *essential* over X if for any resolution of singularities $f : Y \to X$, its center is an irreducible component of $f^{-1}(X_{\text{sing}}) \subseteq Y$

The valuation associated with an essential divisor over X is called an *essential valuation* of X. Such valuations can be defined independently from the choice of resolutions, making them natural candidates for birational invariants associated with the given singularities.

However, it is technically difficult to check whether a given valuation of X is essential over X or not, so we need some intrinsic ways to detect it. In [Nas95], J.F. Nash proposed that we can use the irreducible families of arcs based on the singularities in X to characterize essential valuations. In other words, the correspondence

$$\mathcal{N}_X : \{\text{Nash valuations of } X\} \to \{\text{essential valuations of } X\}$$

is a well-defined map and an injection.

The map $\mathcal{N}_X$ is called the *Nash map* of X. It is natural to ask if a Nash map is surjective, i.e., can any essential valuation of X be obtained from a valuation induced from some irreducible family of arcs based at X_{sing}?

Because of this, we are interested in characterizing the images of Nash maps. In [dFD16], de Fernex and Docampo use the *relatively minimal model program (rMMP)* over X to characterize part of the image of a Nash map. More precisely, they introduce the concept of *terminal valuations* of X and prove that they are Nash valautions of X.

$$\{\text{terminal valuations of } X\} \subseteq \{\text{Nash valuations of } X\}.$$

However, this characterization becomes useless when X has only terminal singularities. In particular, there will be no terminal valuations of X when X has only terminal singularities, so we need other characterizations for these cases.

In this dissertation, we try to use *discrepancies* to characterize Nash valuations of X when X has only terminal singularities. We obtain the following two main theorems.

Theorem 1.0.4. *Let $X(\Delta)$ be a $\mathbb{Q}$-Gorenstein toric variety associated with a fan $\Delta \subseteq N_{\mathbb{R}}$ of dimension at least 3 having only terminal singularities, then any exceptional prime divisor E over $X(\Delta)$ with discrepancy $a(E, X(\Delta)) \leq 1$ induces a Nash valuation of $X(\Delta)$.*

Motivated by this result and the ones of [IK03, Che23], we are led to propose the following conjectures.

Conjecture A. *Let (X, p) be a germ of 3-fold terminal singularity with the Gorenstein index $m \geq 1$, then every exceptional prime divisor E with $a(E, X) \leq 1$ induces a Nash valuation of X.*

Conjecture B. *Let (X, p) be a germ of 3-fold terminal singularity with the Gorenstein index $m \geq 1$, then every exceptional prime divisor E with minimal discrepancy (we have $a(E, X) = \frac{1}{m}$ by [Mar96, Kaw93]) induces a Nash valuation of X.*

We prove that Conjecture B is true for a 3-fold terminal singularity of type cAx/2.

Theorem 1.0.5. *Let (X, p) be a germ of 3-fold terminal singularity of type cAx/2, then any exceptional prime divisor E over X with minimal discrepancy ($a(E, X) = \frac{1}{2}$ in this case) induces a Nash valuation of X.*

In other words, our results show that Conjecture A holds for toric terminal singularities in any dimension at least 3, and Conjecture B holds for 3-fold terminal singularities of type cAx/2.

The structure of this dissertation is organized as follows.

Chapter 2 reviews some fundamental notions and tools in algebraic geometry, including singularities, resolutions, and divisors (especially the relationships between Weil divisors and Cartier divisors with the concept of a divisor is $\mathbb{Q}$-Cartier), and canonical divisors.

Chapter 3 offers a brief introduction to birational geometry, providing the background for the development of our theory. We begin with a short overview of the minimal model program (MMP), then define terminal singularities and minimal models. In particular, our focus is on the 3-fold cases. To study these singularities explicitly, we introduce their classifications along with the classical tool used for their characterization: divisorial contraction with minimal discrepancy.

Chapter 4 introduces the main objects of this dissertation: arc spaces and the Nash problem. We go through some concepts of the arc space of a variety, including its definition and examples. Then the Nash problem is formulated in the language of valuations, and we review some known results and progress that motivate our work.

Chapter 5 briefly recalls the tools from toric algebraic geometry that are essential for the development of the next chapter. This chapter does not provide any detailed proof or explanation.

Chapters 6 through 8 present our results on the Nash problem over 3-fold terminal singularities. In particular, Chapter 6 focuses on Nash valuations over toric terminal singularities of dimension at least 3, and introduces the two conjectures motivated by the observation in this chapter. Chapter 7 deals with Nash valuations over 3-fold terminal singularities of type cAx/2. Finally, Chapter 8 provides a partial result on the Nash problem over 3-fold Gorenstein terminal singularities with some examples.

Chapter 2

Preliminaries

In this chapter, we recall and introduce some fundamental definitions, concepts, notation, and results in algebraic geometry. More details about them can be found in [Har77, Laz04a, Laz04b, Cut18].

In this dissertation, a *variety* X means an integral (reduced and irreducible) and separated scheme of finite type over $\mathbb{C}$. We put the Zariski topology on X.

Remark 2.0.1. The ground field $\mathbb{C}$ can be replaced by any field k that is algebraically closed and of characteristic 0 (for example, let $k = \bar{\mathbb{Q}}$, the field of algebraic numbers).

2.1 Singularities of a Variety

For a variety X, we can ask what is the algebraic condition for characterizing if it can be parametrized by local coordinates (or the condition that X can be a manifold). The following definition tells us that this can be described by the localization of its structure sheaf $\mathcal{O}_X$.

Definition 2.1.1. Let X be a variety. A point $p \in X$ is called a *non-singular point* (or a *regular point*) if the localization of the structure sheaf $\mathcal{O}_{X,p}$ is a regular local ring. Otherwise, p is called a *singularity* of X.

X is called *non-singular* (or *regular*) if all points on X are non-singular.

Remark 2.1.2. Notice that for the varieties in our setting, smooth points and non-singular points are the same concepts (see [Har77, Chapter III, Corollary 10.7]). We will use the terminology *smooth points* to refer to non-singular points, and *smooth variety* for non-singular variety.

Remark 2.1.3. If $X = V(f_1, \ldots, f_m) \subseteq \mathbb{C}^n$ is an affine variety, then the definition 2.1.1 for $p \in X$ being smooth is equivalent to the Jacobian matrix

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

has rank equal to $\operatorname{codim}(X, \mathbb{C}^n)$ at p. This definition agrees with the notion that X is a manifold at p in differential geometry, which follows from the implicit function theorem (see [Har77, p. 32]). In particular, when X is a complete intersection, this condition is equivalent to its Jacobian matrix being of full rank at p.

The set of all singular points on X is denoted by X_{sing}, and it is called the *singular locus* of X. It is a closed set in X (see [Har77, Theorem 5.3, p. 33]). Its complement $X_{\text{sm}} := X \setminus X_{\text{sing}}$ is called the *smooth locus* of X.

We begin by recalling the notion of *birational equivalence* between two varieties. Roughly speaking, two varieties are birational if they are isomorphic on dense open subsets.

Definition 2.1.4 (Birational Varieties/Birational Morphisms)**.** Let X, Y be varieties. We say X is *birational* to Y if there are Zariski open sets U and V, with $U \subseteq X$ and

$V \subseteq Y$, such that U is isomorphic to V. Furthermore, a *birational morphism* $f : X \to Y$ is a morphism such that the restriction $f|_U : U \to V$ is an isomorphism for some Zariski open sets U and V, with $U \subseteq X$ and $V \subseteq Y$.

In birational geometry, it is interesting to understand the behaviors of the singular loci of varieties under birational morphisms. Among such morphisms, one of the most intuitive and important candidates is *resolution of singularities*.

Definition 2.1.5 (Resolution of Singularities)**.** Let X be a variety. A *resolution of singularities* of X, or a *resolution of* X for short, is a proper birational morphism $f : Y \to X$ such that Y is smooth. A resolution $f : Y \to X$ is called a *Hironaka resolution* if, furthermore, f induces an isomorphism between $Y \smallsetminus f^{-1}(X_{\mathrm{sing}})$ and $X \smallsetminus X_{\mathrm{sing}}$. A Hironaka resolution $f : Y \to X$ is called *good* if $f : Y \to X$ can be obtained by finitely many blowing-ups along smooth centers.

The result of existence for characteristic 0 is proved by H. Hironaka.

Theorem 2.1.6 (Existence of Resolution of Singularities [Hir64])**.** *Good Hironaka resolutions of singularities exist for varieties over any field of characteristic 0. In particular, resolutions of singularities exist.*

2.2 Divisors of a Variety

A fundamental tool for the algebraic study of varieties is the concept of *divisors*, which encodes some geometric information of a given variety.

Definition 2.2.1. Let X be a variety. X is *regular in codimension one* if every local ring $\mathcal{O}_{X,p}$ with $\dim(\mathcal{O}_{X,p}) = 1$ is regular.

Example 2.2.2.

- Smooth varieties are regular in codimension one.
- Normal varieties are regular in codimension one.

Assumption 2.2.3. We assume that our varieties are normal (for a variety X, every local ring $\mathcal{O}_{X,p}$ for every $p \in X$ is integrally closed in the field of fractions) in the rest of this dissertation. In particular, they are regular in codimension one.

Definition 2.2.4 (Divisors/$\mathbb{Q}$-Divisors)**.** Let X be a variety. A *(Weil) prime divisor* E is a subvariety of X with codimension $\operatorname{codim}(E, X) = 1$. A *(Weil) divisor* D is a finite formal sum $D = \sum_{i \in I} a_i E_i$ where the E_i are prime divisors and $a_i \in \mathbb{Z}$. We denote the free abelian group generated by prime divisors of X by $\operatorname{WDiv}(X)$. That is,

$$\operatorname{WDiv}(X) := \bigoplus_{E:\text{prime divisor of } X} \mathbb{Z}E.$$

A $\mathbb{Q}$-*divisor* is an element $D \in \operatorname{WDiv}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. An *effective* divisor (or $\mathbb{Q}$-divisor) is a divisor $D = \sum_{i=1}^{m} a_i E_i$ with all $a_i \geq 0$.

Let E be a prime divisor on X, and let η be the generic point of E. Then the local ring $\mathcal{O}_{X,\eta}$ is a discrete valuation ring, which induces a discrete valuation on the function field $\mathbb{C}(X)$. We call it the *valuation of* E, and denote it by val_E.

Remark 2.2.5. Notice that for a variety X, E is uniquely determined by its valuation since X is separated (see [Har77, Exercise 4.5, p. 106]).

Definition 2.2.6 (Divisors over X)**.** Let X be a variety. A prime divisor E is called a *divisor over* X if there is a proper birational morphism $f : Y \to X$ such that $E \subseteq Y$ is a prime divisor. Such E is called an *exceptional divisor over* X if f is not an isomorphism at the generic point of E. The Zariski closure of $f(E) \subseteq X$ is called *the center of* E in X, and it is denoted by c_X(E).

Definition 2.2.7 (Principal Divisors)**.** Let X be a variety, and let $f \in \mathbb{C}(X)^\times$ be a unit element in the function field of X. The *principal divisor* induced by f in X is defined to be

$$\mathrm{div}_X(f) := \sum \mathrm{val}_E(f)E,$$

where E runs over all prime divisors of X.

Remark 2.2.8. Notice that we have $\mathrm{val}_E(f) = 0$ for all but finitely many prime divisors in X (see [Har77, Lemma 6.1, p. 131]). Therefore, we have $\mathrm{div}_X(f) \in \mathrm{WDiv}(X)$.

It is common to use divisors on X to describe the geometry of X.

By comparing to the usual (Weil) divisors, there is another type of divisors that comes from invertible sheaves, which is called *Cartier divisors*.

Definition 2.2.9 (Cartier Divisors)**.** Let X be a variety. A *Cartier divisor D* consists of the data $\{(U_i, \phi_i)\}_{i\in I}$, where $\{U_i\}_{i\in I}$ is an open cover of X and $\phi_i \in \Gamma(U_i, \mathcal{K}_X^\times)$ which satisfies $\frac{\phi_i}{\phi_j} \in \Gamma(U_i \cap U_j, \mathcal{O}_X^\times)$ for all $i, j \in I$. Let $\{(U_i, \phi_i)\}_{i\in I}$ and $\{(V_j, \psi_j)\}_{j\in J}$ be two Cartier divisors. We define their sum as

$$[\{(U_i, \phi_i)\}_{i\in I}] + [\{(V_j, \psi_j)\}_{j\in J}] := \{(U_i \cap V_j, \phi_i\psi_j)\}_{(i,j)\in I\times J}.$$

The family of Cartier divisors of X with this sum forms an abelian group, and we denote it by $\mathrm{CDiv(X)}$.

Now we relate the Cartier divisors to the (Weil) divisors in the following way. Let $\Phi = \{(U_i, \phi_i)\}_{i\in I}$ be a Cartier divisor. Let E be a prime divisor in X, and we define

$$\mathrm{val}_E(\Phi) := \mathrm{val}_E(\phi_i)$$

for any $i \in I$ such that $c_X(E) \cap U_i \neq \varnothing$ (notice that $\mathrm{val}_E(\phi_i)$ is independent of U_i). Consider the following correspondence

$$\mathrm{cyc} : \mathrm{CDiv}(X) \longrightarrow \mathrm{WDiv}(X)$$
$$\Phi = \{(U_i, \phi_i)\}_{i \in I} \mapsto \sum_{E:\text{ prime divisor of X}} \mathrm{val}_E(\Phi)E.$$

This correspondence is injective because X is normal (see [Ful98, Example 2.1.1]). Therefore, we can identify a Cartier divisor of X with the corresponding (Weil) divisor of X via cyc.

Definition 2.2.10. A (Weil) $\mathbb{Q}$-divisor D is $\mathbb{Q}$-*Cartier* if rD is Cartier for some $r \in \mathbb{Z}_{>0}$.

Remark 2.2.11. For a smooth variety, divisors and Cartier divisors coincide. For a singular variety, a divisor is not necessarily Cartier. See [Ful98, Example 2.1.1].

Recall that we can define the intersection product $D.C$ for a Cartier divisor D and a curve C of X (see [Har77, Ful98]). For a $\mathbb{Q}$-Cariter $\mathbb{Q}$-divisor D and a curve C of X, we define $D.C := \frac{1}{m}(mD.C)$ where $m \in \mathbb{Z}_{>0}$ makes mD be Cartier.

Definition 2.2.12 (Nef Divisors)**.** Let D be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. D is *nef* if $D.C \geq 0$ for all curves $C \subseteq X$.

Among all divisor of X, we usually use a *canonical divisor* to characterize geometric properties of a variety in birational geometry.

Let X be a smooth variety, and let $\Omega_{X/k}$ be the sheaf of differential on X. Consider its canonical sheaf (the sheaf of top forms) $\omega_X := \bigwedge^{\dim X} \Omega_{X/k}$, which is an invertible sheaf (line bundle) on X. Hence, $\omega_X \cong \mathcal{O}_X(K_X)$ for some Cartier divisor K_X. This K_X is not unique, but any two such divisors are linearly equivalent (i.e., their difference is a principal divisor).

Definition 2.2.13 (Canonical Divisor)**.** Such K_X is called a *canonical divisor* of X.

Remark 2.2.14. For a normal variety, we can still have a concept of canonical divisors, which is generalized from the smooth one. We will introduce it in the next chapter. Furthermore, we will use canonical divisors to define *terminal singularities*.

Chapter 3

Concepts and Tools in Birational Geometry

In this section, we recall some important results and tools in birational geometry which will be used throughout our theorems and arguments. More precisely, we will introduce the concepts of *terminal singularities* and the tool used to study them in 3-dimensional cases, which is *divisorial contraction with minimal discrepancy*.

Most of the results that we discuss are treated in detail in [KM98, Mat02, Laz04a, Laz04b, Hay99, Hay00]. We work over the field of complex numbers $\mathbb{C}$, unless indicated otherwise.

In mathematics, a common approach to understanding a certain geometry is to classify the geometric objects which we care about via some suitable equivalence. For example, in differential geometry, manifolds are studied through classifications via homeomorphisms, diffeomorphisms, isometries, conformal maps, homotopies, and other morphisms, depending on the context.

In algebraic geometry, we are interested in studying algebraic varieties up to *birational equivalence* rather than bi-regular isomorphism, and it forms the topic called *birational geometry*. The principal goal of birational geometry is to birationally classify algebraic varieties with suitable representatives, and to study their associated birational invariants.

The initial step of birational classification is to find suitable representatives in a given birational equivalence class, and then study their properties. Recently, people have tried developing a certain algorithmic procedure to find such representatives by treating algebraic varieties as input, and this leads to the formulation of the so-called *minimal model program (MMP)*.

Generally speaking, MMP aims to produce a suitable representative for a fixed birational equivalence class as output by iteratively simplifying geometric structures within each fixed birational class.

3.1 An Overview of the MMP

MMP for Algebraic Curves (1-Dimensional Cases)

Let $[C]$ be the birational equivalence of algebraic curves represented by some curve C. Notice that there are many birational algebraic curves, either singular or smooth, in $[C]$. However, we can consider the normalization $\tilde{C} \to C$. Notice that $\tilde{C}$ is smooth and unique in $[C]$, so we can take $\tilde{C}$ as our suitable representative for $[C]$ (see [Ful89, Section 7.5] for details).

MMP for Algebraic Surfaces (2-Dimensional Cases)

For algebraic surfaces, the scenario becomes more interesting. Let $[S]$ be a birational equivalence of surfaces represented by some surface S. Similarly to the approach for curves, we can take a resolution of singularities of S, say $\tilde{S} \to S$, and $\tilde{S}$ is still in the same class $[S]$ since resolutions are birational morphisms.

Potentially, we could select $\tilde{S}$ as our suitable representative for $[S]$. However, resolutions of singularities are generally not unique for a fixed surface S, which means that there are many choices for $\tilde{S}$. In fact, given a smooth surface T we can apply a *blowing-up* along a point on T and obtain a birational morphism $T_1 \to T$, then T_1 is still smooth. Therefore, we can obtain infinitely many smooth models from resolutions of S by repeating this process, which are far from being our candidates for a suitable representative of $[S]$.

Nevertheless, we are still able to select one as our suitable representative among these smooth models. Notice that we can apply blowing-ups at points on a smooth surface to produce another smooth one, so it is natural to consider something that performs an inverse operation of blowing-ups, or called a *blowing-down*.

A blowing-down exists due to *Castelnuovo's Contractibility Criterion*.

Theorem 3.1.1 (Castelnuovo's Contractibility Criterion). *Let S be a smooth projective surface, and let $C \subset S$ be a (-1)-curve on S (that is, an algebraic curve isomorphic to $\mathbb{P}^1$ with self intersection $C.C = -1$). Then there is a morphism $\pi_C : S \to S_1$ for some smooth projective surface S_1 such that*

- *$\pi_C(C) = p \in S_1$ is a point on S_1,*
- *The restriction $\pi_C|_{(S \setminus C)} : S \setminus C \to S_1 \setminus \{p\}$ is an isomorphism.*

Here we have an algorithm: Take a resolution $\tilde{S}$ of S, then there could be some (-1)-curves on $\tilde{S}$, so we can start applying a blowing-down $\tilde{S} \to S_1$ to contract them to points and obtain a "smaller" smooth surface with fewer (-1)-curves. We continue this process to obtain a sequence of morphisms $\tilde{S} \to S_1 \to S_2 \to \dots$. After several blowing-downs, this process terminates, and we can obtain a smooth surface that is either a *minimal surface* or a *ruled surface* as an output. They will be the suitable representative we want for a given birational equivalence (see [Mat02]).

MMP for 3-Folds

For 3-folds, the situation becomes more complicated, but the directions are similar to the surface cases. Given a birational equivalence class $[X]$ for 3-folds, we apply a resolution of singularities of X, say $\pi : \tilde{X} \to X$. Then like the case for surfaces, we apply the *cone theorem* and the *contraction theorem* to obtain a *divisorial contraction* of $\tilde{X}$ to eliminate K_X-negative curves (i.e., the curves with $K_X.C < 0$) on $\tilde{X}$ and get a new 3-fold with fewer K_X-negative curves. In some steps, we need to apply *flipping contractions* if we obtain a small contraction, then we do *flips* to obtain a new variety to continue doing contractions. However, the new 3-fold after the contraction might no longer be smooth, and the singularities on it are called *terminal singularities*.

By adding terminal singularities to the MMP for 3-folds, the whole algorithm still works as the surface cases and produces an output 3-fold which is either a *minimal model* or a *Mori fiber space*.

Unlike surface case, we have many minimal models in a fixed birational equivalence class. However, it has been proved that these minimal models are connected by finite numbers of *flops* in the same birational equivalence class. On the other hand, Mori fiber spaces are also classified in a fixed birational equivalence class. The whole MMP works for 3-folds (see [KM98, Mat02] for details).

3.2 Terminal Singularities and Minimal Models

In this section, we introduce some tools and concepts for developing our theory. In particular, we will rigorously define terminal singularities and minimal models.

First, let us define the concept of canonical divisors for singular varieties (normal varieties, more precisely). Let X be a normal variety, not necessarily smooth. Consider its smooth locus X_{sm}, and we can define a canonical divisor $K_{X_{\mathrm{sm}}}$ of X_{sm} as in the previous chapter. Recall that such $K_{X_{\mathrm{sm}}}$ is not unique and that any two of them are linearly equivalent.

Definition 3.2.1 (Canonical Divisors for Normal Varieties)**.** Let X be a normal variety. A *canonical divisor* of X is defined to be the Zariski closure of $K_{X_{\mathrm{sm}}}$. More precisely, if $K_{X_{\mathrm{sm}}} = \sum_{i\in I} b_i E_i^\circ$ where the E_i° are prime divisors in X_{sm}, then $K_X \coloneqq \sum_{i\in I} b_i E_i$ where each E_i is the Zariski closure of E_i°.

Remark 3.2.2. Like in the smooth case, the K_X defined here is also not unique, and any two of them are linearly equivalent.

Remark 3.2.3. It immediately follows from the definition that if $f\colon Y \to X$ is a proper birational map between normal varieties, and K_Y is a canonical divisor on Y, then $f_*(K_Y)$ is a canonical divisor on X.

Now we can use resolutions to compare a canonical divisor in X and a canonical divisor on a smooth model of X. Let X be a normal variety such that K_X is $\mathbb{Q}$-Cartier (we will need the condition that K_X is $\mathbb{Q}$-Cartier to do a pullback of it). Take a resolution of singularities $f : Y \to X$ and choose K_X and K_Y such that $f_*(K_Y) = K_X$. Define the *relatively canonical divisor* as $K_{Y/X} \coloneqq K_Y - f^*(K_X)$ on Y where $f^*(K_X) \coloneqq \frac{1}{r} f^*(rK_X)$ for any $r \in \mathbb{Z}_{>0}$ such that rK_X is Cartier. Notice that such $K_{Y/X}$ is independent of the choice of K_Y. Also, $K_{Y/X}$ is supported on the exceptional locus of f. Let $\{E_1, \ldots, E_m\}$ be the exceptional prime divisors of f for some $m \in \mathbb{Z}_{>0}$, so we have

$$K_{Y/X} = \sum_{i=1}^{m} a_i E_i$$

for some $a_i \in \mathbb{Q}$. We denote $a(E_i, X) \coloneqq a_i$.

Definition 3.2.4 (Discrepancies)**.** Let X be a normal variety such that K_X is $\mathbb{Q}$-Cartier (or X is called $\mathbb{Q}$-Gorenstein), and let E be a prime divisor over X. The number $a(E, X) \in \mathbb{Q}$ is called the *discrepancy* of E over X. The *minimal discrepancy* of X is defined as the infimum of the discrepancies $a(E, X)$, where E runs through all exceptional divisors over X.

Remark 3.2.5. Recall that a prime divisor E over X induces a valuation val_E. If two divisors E and E' over X induce the same valuation ($\mathrm{val}_E = \mathrm{val}_{E'}$), then we have $a(E, X) = \mathrm{val}_E(K_{Y/X}) = \mathrm{val}_{E'}(K_{Y'/X}) = a(E', X)$ where Y and Y' are the corresponding models. Therefore, in this sense, the discrepancy $a(E, X)$ is independent of the choice of resolution.

Discrepancies in birational geometry encode the information of X_{sing}. In particular, the minimal discrepancy is an important birational invariant of X_{sing}.

Here, we define the concept of terminal singularities.

Definition 3.2.6 (Terminal Singularities)**.** Let X be a $\mathbb{Q}$-Gorenstein normal variety. We say X has only *terminal singularities* if $a(E, X) > 0$ for all exceptional prime divisors E over X.

With the concept of terminal singularities, we can define minimal models / relatively minimal models in higher dimensions.

Definition 3.2.7 (Minimal Model)**.** Let X be a $\mathbb{Q}$-Gorenstein normal variety. X is called a *minimal model* if

- X has only terminal singularities,
- A canonical divisor K_X is nef.

Definition 3.2.8 (Relatively Minimal Model)**.** Let X be a variety, and let Y be a $\mathbb{Q}$-Gorenstein normal variety . Y is called a *relatively minimal model* over X if

- there is a projective birational morphism $f : Y \to X$,
- Y has only terminal singularities,
- A canonical divisor K_Y is f-nef. That is, $K_Y.C \geq 0$ for all f-exceptional curves $C \subseteq Y$.

Here is an important theorem that applies to varieties of general type (we will not define this term here).

Theorem 3.2.9 ([BCHM10])**.** *Let X be a variety of general type, then X is always birational to a minimal model.*

Terminal singularities appear in many research topics in algebraic geometry. In particular, people have tried classifying these kinds of singularities. For 3-folds, this has been done in [Dan82, MS84, Mor85, Rei83]. Notice that for X being terminal, its singular locus X_{sing} has the following lower bound for its codimension (see [KM98, Corollary 5.18, p. 159]).

$$\operatorname{codim}(X_{\text{sing}}, X) \geq 3.$$

Therefore, the singular locus X_{sing} of a terminal 3 -fold X is a set of isolated points.

3.3 3-Fold Terminal Singularities

Definition 3.3.1 (Gorenstein index)**.** Let X be a $\mathbb{Q}$-Gorenstein normal variety. The *Gorenstein index* of X is the smallest $m \in \mathbb{Z}_{>0}$ such that mK_X is Cartier.

For a germ of 3-dimensional terminal singularities (X, p), the classification for the Gorenstein index equal to 1 (or *Gorenstein terminal*, shortly) are done by M. Reid in [Rei83].

Theorem 3.3.2 ([Rei83]). *A germ of 3-fold Gorenstein singularity is terminal if and only if it is an isolated cDV point.*

For the cases of Gorenstein index greater than 1, the complete classification is treated in [Dan82, MS84, Mor85].

Let us introduce the classic notation to describe the classification. Denote the coordinates of the complex space $\mathbb{C}^4$ by (x, y, z, u). Let $\mathbb{Z}_m$ be a cyclic group of order m, and consider the action $\tau : \mathbb{Z}_m \times \mathbb{C}^4 \to \mathbb{C}^4$ by $\tau(x) = \zeta^{\alpha_1} x, \tau(y) = \zeta^{\alpha_2} y, \tau(z) = \zeta^{\alpha_3} z, \tau(u) = \zeta^{\alpha_4} u$ where ζ is a primitive m-root of unity and $\alpha_i \in \mathbb{Z}$ for $i = 1, 2, 3, 4$. The quotient variety of $\mathbb{C}^4$ with respect to this action is denoted by

$$\mathbb{C}^4 / \frac{1}{m}(\alpha_1, \alpha_2, \alpha_3, \alpha_4).$$

Its ring of regular functions agrees with the ring of invariant regular functions with respect to $\frac{1}{m}(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ (see [CLS11, Chapter 5]).

Denote the ring of the complex convergent power series in the variables x, y, z, u by $\mathbb{C}\{x, y, z, u\}$. Let $\phi \in \mathbb{C}\{x, y, z, u\}$ be a $\mathbb{Z}_m$-semi-invariant with respect to the previous action determined by the vector $\frac{1}{m}(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$. Then we obtain an induced action on the germ of hypersurface $(\phi = 0) \subseteq \mathbb{C}^4$. It induces a germ of subvariety of $\mathbb{C}^4 / \frac{1}{m}(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$, and it is denoted by $(\phi = 0) \subseteq \mathbb{C}^4 / \frac{1}{m}(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$. It is called a hyperquotient singularity.

With this notation, we have the following list for the classification:

Theorem 3.3.3. *([Dan82, MS84, Mor85, Hay99, Hay00, Che16]) Let X be a germ of 3-fold terminal singularity with Gorenstein index greater than 1, then it is isomorphic to one* *of the following form:*

- *(type cA/m)*

$$(xy + f(z,u) = 0) \subseteq \mathbb{C}^4/\frac{1}{m}(a,-a,1,0)$$

  where $a, m \in \mathbb{Z}_{>0}$ *with* $\gcd(a,m) = 1$ *and* $f(z,u) \in \mathbb{C}\{z,u\}$ *is a* $\mathbb{Z}_m$*-invariant.*

- *(type cAx/2)*

$$(x^2 + y^2 + f(z,u) = 0) \subseteq \mathbb{C}^4/\frac{1}{2}(0,1,1,1)$$

  where $f(z,u) \in (z,u)^4 \subseteq \mathbb{C}\{z,u\}$ *is a* $\mathbb{Z}_2$*-invariant.*

- *(type cAx/4)*

$$(x^2 + y^2 + f(z,u) = 0) \subseteq \mathbb{C}^4/\frac{1}{4}(1,3,1,2)$$

  where $f(z,u) \in \mathbb{C}\{z,u\}$ *is a* $\mathbb{Z}_4$*-semi-invariant such that* $f(z,u)$ *does not contain* u *as a monomial.*

- *(type cD/2)*

$$(\phi = 0) \subseteq \mathbb{C}^4/\frac{1}{2}(1,1,0,1)$$

  where ϕ *has one of the following forms:*

  - *(cD/2-1)* $\phi = u^2 + xyz + x^{2a} + y^{2b} + z^c$ *where* $a, b \geq 2$*,* $c \geq 3$*.*
  - *(cD/2-2)* $\phi = u^2 + y^2z + \lambda yx^{2a+1} + g(x,z)$ *where* $\lambda \in \mathbb{C}$*,* $a \geq 1$*,* $g(x,z) \in (x^4, x^2z^2, z^3) \subseteq \mathbb{C}\{x,z\}$.

- *(type cD/3)*

$$(\phi = 0) \subseteq \mathbb{C}^4/\frac{1}{3}(1,2,2,0)$$

  where ϕ *has one of the following forms:*

  - *(cD/3-1)* $\phi = x^3 + u^2 + yz(y+z)$.
  - *(cD/3-2)* $\phi = x^3 + u^2 + yz^2 + xy^4\alpha(y^2) + y^6\beta(y^3)$ *where* $\alpha(y^3), \beta(y^3) \in \mathbb{C}\{y^3\}$ *with* $4\alpha^3 + 27\beta^2 \neq 0$.
  - *(cD/3-3)* $\phi = x^3 + u^2 + y^3 + xyz^3\alpha(z^3) + xz^4\beta(z^3) + yz^5\gamma(z^3) + z^6\delta(z^3)$ *where* $\alpha(z^3), \beta(z^3), \gamma(z^3), \delta(z^3) \in \mathbb{C}\{z^3\}$.

- *(type cE/2)*
$$(x^3 + u^2 + g(y,z) + h(y,z) = 0) \subseteq \mathbb{C}^4/\frac{1}{2}(0,1,1,1)$$
where $g(y,z) \in (y,z)^4 \subseteq \mathbb{C}\{y,z\}$, $h(y,z) \in (y,z)^4 \smallsetminus (y,z)^5 \subseteq \mathbb{C}\{y,z\}$.

In particular, three-fold toric terminal singularities are studied in [Dan82] and [MS84]. The classification is completed by the following result in [KSB88].

Theorem 3.3.4 ([KSB88, Theorem 6.5])**.** *Let*
$$X = (\phi(x,y,z,u) = 0) \subseteq \mathbb{C}^4/\frac{1}{m}(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$$
be one of the hyperquotient singularities listed in Theorem 3.3.3. Assume

- $\phi(x,y,z,u) \in \mathbb{C}[x,y,z,u]$ *defines an isolated singularity at* $(0,0,0,0) \in \mathbb{C}^4$*,*
- *The action* $\tau : \mathbb{Z}_m \times \mathbb{C}^4 \to \mathbb{C}^4$ *defined previously is free outside* $(0,0,0,0) \in \mathbb{C}^4$*.*

Then $(X, (0,0,0,0))$ *defines a 3-fold terminal singularity.*

Three-fold terminal singularities have been studied explicitly in many articles [Hay99, Hay00, Hay05, Che16]. One of the important tools for explicitly studying them is *divisorial contraction with minimal discrepancy* (or *divisorial blowing-ups with minimal discrepancy*).

Theorem 3.3.5 ([Kaw93, Mar96])**.** *For a 3-fold terminal singularity with Gorenstein index* $r \geq 1$*, its minimal discrepancy is exactly* $\frac{1}{r}$*.*

Definition 3.3.6 ([Hay99] and [Hay00])**.** Let X be a germ of 3-fold terminal singularity with Gorenstein index $m \geq 2$. A *divisorial contraction with minimal discrepancy* to X is a projective birational morphism $\pi : \overline{X} \to X$ such that

- $\overline{X}$ has only terminal singularities,
- the exceptional locus of π is a prime divisor E over X,
- and $K_{\overline{X}/X} = \frac{1}{m}E$, i.e. E computes the minimal discrepancy $a(E,X) = \frac{1}{m}$.

Remark 3.3.7. In the definition 3.3.6 of the divisorial contraction, we see that $-K_{\overline{X}}$ is π-ample.

The existence of such morphisms is proved by T. Hayakawa.

Theorem 3.3.8 ([Hay99, Hay00]). *Let X be a germ of 3-fold terminal singularity such that its Gorenstein index is greater than 1. Then there is a divisorial contraction with minimal discrepancy to X.*

Here is a rough idea of the proof from T. Hayakawa. Notice that 3-fold terminal singularities with Gorenstein indices $m > 1$ have been classified completely in Theorem 3.3.3. For any such singularity, Hayakawa selects some particular 4-tuple $\sigma = \frac{1}{m}(a, b, c, d) \in \mathbb{C}^4$ and applies the weighted blowing-up to the singularity with respect to such weight σ. Then he shows that such weighted blowing-up provides the divisorial contraction by establishing a correspondence between certain equivalences of maximal weighted pseudo-valuations with the parameter $\frac{1}{m}$ and the set of isomorphism classes of divisorial contraction with minimal discrepancy to X.

Chapter 4

Arc Spaces and the Nash Problem

In this chapter, we introduce some important concepts about arc spaces of a given variety X. Especially, we will formulate the Nash problem over given singularities. The readers can check the details in [IK03, CLNS18, dFD16, dF18].

4.1 Arc Spaces of a Variety

Nash first introduces the concept of arc spaces to study the singularities of a variety X in [Nas95]. Generally speaking, an arc of a variety X is a formal analytic curve in X, and the arc space X_∞ is the space which parametrizes all formal analytic curves in X. Here we rigorously define arcs of X in the language of the functor of points.

Definition 4.1.1. Let X be a scheme over $\mathbb{C}$, and let $\mathbb{K}$ be a $\mathbb{C}$-algebra. A $\mathbb{K}$-*arc* in X is a $\mathbb{K}[[t]]$-valued point of X. In other words, a $\mathbb{K}$-arc on X is a morphism $\alpha \in \mathrm{Hom}_{\mathrm{Schemes}/\mathbb{C}}(\mathrm{Spec}(\mathbb{K}[[t]]), X)$.

Theorem 4.1.2 ([CLNS18, Corollary 3.3.7] and [Bha16, Remark 4.6]). *The functor*

$$X_\infty(-) := \mathrm{Hom}_{\mathrm{Schemes}/\mathbb{C}}(\mathrm{Spec}(-[[t]]), X) : (\mathbb{C}\text{–Algebras}) \to (\mathrm{Sets})$$

is representable by a scheme over $\mathbb{C}$. *That is, there is a scheme* Y *over* $\mathbb{C}$ *such that*

$$\mathrm{Hom}_{\mathrm{Schemes}/\mathbb{C}}(\mathrm{Spec}(\mathbb{K}[[t]]), X) \cong \mathrm{Hom}_{\mathrm{Schemes}/\mathbb{C}}(\mathrm{Spec}(\mathbb{K}), Y).$$

By the Yoneda lemma, such a representative Y is unique up to isomorphism, and we denote it by X_∞.

Definition 4.1.3. This representative X_∞ is called the *arc space* of X.

Remark 4.1.4. Notice that X_∞ is just a scheme, not a variety.

Remark 4.1.5. In fact, this is not the original definition for the arc space of X. Originally, it is defined as the inverse limit of the jet schemes X_m of X for $m \in \mathbb{Z}_{\geq 0}$.

$$X_\infty := \varprojlim X_m.$$

From [Bha16], our definition is equivalent to the original one.

In many scenarios, we simply call a $\mathbb{K}$-arc an *arc* of X if it is unnecessary to specify the base $\mathbb{K}$.

Notice that when $\mathbb{K}$ is a field, $\mathbb{K}[[t]]$ is a discrete valuation ring (DVR), so it has a unique maximal ideal $(t) \subset \mathbb{K}[[t]]$, associated with the unique closed point in $\mathrm{Spec}(\mathbb{K}[[t]])$. Denote this closed point of $\mathrm{Spec}(\mathbb{K}[[t]])$ by 0 and the generic point by η. In fact, we have $\mathrm{Spec}(\mathbb{K}[[t]]) = \{0, \eta\}$.

Example 4.1.6. Let $X = \mathrm{Spec}(\mathbb{C}[x,y]/(y^2 - x^3))$, then an arc is a morphsim of schemes $\phi : \mathrm{Spec}(\mathbb{K}[[t]]) \to \mathrm{Spec}(\mathbb{C}[x,y]/(y^2 - x^3))$, which corresponds to a $\mathbb{C}$-algebra homomorphism $\phi^\# : \mathbb{C}[x,y]/(y^2 - x^3) \to \mathbb{K}[[t]]$ and it is determined by $\phi^\#(x) = \alpha(t)$,

$\phi^{\#}(y) = \beta(t)$ for some $\alpha(t), \beta(t) \in \mathbb{K}[[t]]$. Notice that we have $\beta(t)^2 - \alpha(t)^3 = 0$ since $\phi^{\#}$ is a $\mathbb{C}$-algebra homomorphism. Therefore, we have

$$X_\infty(\mathbb{K}) = \{(\alpha(t), \beta(t)) \in \mathbb{K}[[t]] \mid \beta(t)^2 - \alpha(t)^3 = 0\}.$$

For the characterization of X_∞, let $(\alpha(t), \beta(t)) \in X_\infty$ and write $\alpha(t) = a_0 + a_1 t + a_2 t^2 + \ldots$ and $\beta(t) = b_0 + b_1 t + b_2 t^2 + \ldots$, then we obtain

$$(b_0 + b_1 t + b_2 t^2 + \ldots)^2 - (a_0 + a_1 t + a_2 t^2 + \ldots)^3 = 0.$$

Comparing the coefficients of t^i for $i = 0, 1, 2\ldots$ in this equation, we have the following relations:

$$\begin{gathered} b_0^2 - a_0^3 = 0, \\ 2b_0 b_1 - 3a_0^2 a_1 = 0, \\ b_1^2 + 2b_0 b_2 - 3a_0(a_1)^2 - 3a_0^2 a_2 = 0, \\ \vdots \end{gathered}$$

Therefore, we have the following expression for X_∞.

$$X_\infty = \left\{ (a_0, a_1, a_2, \ldots, b_0, b_1, b_2, \ldots) \;\middle|\; \begin{gathered} b_0^2 - a_0^3 = 0, \\ 2b_0 b_1 - 3a_0^2 a_1 = 0, \\ b_1^2 + 2b_0 b_2 - 3a_0(a_1)^2 - 3a_0^2 a_2 = 0, \\ \vdots \end{gathered} \right\}.$$

Or equivalently,

$$X_\infty = \operatorname{Spec}\left(\frac{\mathbb{C}[a_0, a_1, a_2, \ldots, b_0, b_1, b_2, \ldots]}{(\, b_0^2 - a_0^3, 2b_0 b_1 - 3a_0^2 a_1, b_1^2 + 2b_0 b_2 - 3a_0(a_1)^2 - 3a_0^2 a_2, \ldots)} \right).$$

It is not hard to see that X_∞ is a scheme over $\mathbb{C}$, not a variety.

Example 4.1.7. In general, if $X = \operatorname{Spec}(k[x_0, \ldots, x_n]/(f_0, \ldots, f_m))$ is an affine variety, then an arc is a formal power series solution in $\mathbb{K}[[t]]$ of the system of equations $\{f_0 = 0, \ldots, f_m = 0\}$, and the arc space X_∞ collects all such formal power series solutions. In other words,

$$X_\infty(\mathbb{K}) = \{\, \alpha(t) \in (\mathbb{K}[[t]])^{n+1} \mid f_0(\alpha(t)) = 0, \ldots, f_m(\alpha(t)) = 0 \,\}.$$

Consider the canonical map $\mathbb{K}[[t]] \to \mathbb{K}$ defined by $t \mapsto 0$, which induces the map $\operatorname{Spec}(\mathbb{K}) \to \operatorname{Spec}(\mathbb{K}[[t]])$. Then we have

$$\operatorname{Hom}_{\text{Schemes}/k}(\operatorname{Spec}(\mathbb{K}[[t]]), X) \to \operatorname{Hom}_{\text{Schemes}/k}(\operatorname{Spec}(\mathbb{K}), X),$$

and it induces the canonical projection.

$$\pi_X : X_\infty \longrightarrow X$$
$$\alpha \longmapsto \alpha(0).$$

Let $S \subseteq X$, then $\pi_X^{-1}(S)$ is the set of all arcs passing through S.

$$\pi_X^{-1}(S) = \{\, \alpha \in X_\infty \mid \alpha(0) \in S \,\}.$$

If $p \in X$ is a non-singular point, then we have $\pi_X^{-1}(\{p\}) \cong \{p\} \times \mathbb{A}_k^\infty$. If $p \in X$ is singular, the situation will become complicated. This leads to the point of view of J.F. Nash [Nas95] in the next section:

What information can arcs passing through X_{sing} tell us about resolutions of X?

4.2 The Nash Problem: Formulation

In this section, we will formulate the so-called *Nash problem*. This is an interesting observation proposed by J. F. Nash in [Nas95]. From his point of view, the set of all arcs passing through the singular locus of X (or $\pi_X^{-1}(X_{\text{sing}})$) can help us study resolutions of singularities of X.

Essential Divisors / Essential Valuations

We need some notions and facts to formulate Nash's observation. Let X be a variety, and we introduce the concept that a prime divisor over X "appears" in other models over X.

Definition 4.2.1 (Essential Divisors / Essential Valuations)**.** Let E be an exceptional prime divisor over X. E is called *essential* over X if any resolution $f : Y \to X$ is given, the center of val_E in Y is an irreducible component of $f^{-1}(X_{\mathrm{sing}})$. We say E *appears* in Y if the center of val_E is a divisor in Y. The divisorial valuation induced by an essential exceptional prime divisor of X is called an essential valuation of X.

For examples / non-examples of essential divisors / valuations, see [IK03, Example 2.4,2.5,2.6].

Nash Components / Nash Valuations

For the remainder of this chapter, we let $\mathbb{K}$ denote a field extension of $\mathbb{C}$.

Let X be a topological space, and let $x, y \in X$. We say y is a *generalization* of x (or x is a *specialization* of y) if $x \in \overline{\{y\}}$ in X, where $\overline{\{\ \}}$ denotes the closure in X.

Proposition 4.2.2 ([IK03])**.** *Let X be a variety, and let $\alpha : \mathrm{Spec}(\mathbb{K}[[t]]) \to X$ be a $\mathbb{K}$-arc such that $\alpha(0) \in X_{\mathrm{sing}}$. Then α has a generalization $\beta \in X_\infty$ such that $\beta(0) \in X_{\mathrm{sing}}$ and $\beta(\eta) \notin X_{\mathrm{sing}}$.*

Proof. See [IK03, Lemma 2.12] or [CLNS18, Chapter 3, Proposition 4.3.2]. □

Remark 4.2.3. The proof of this proposition actually needs the condition that X is over a field k of characteristic zero.

Proposition 4.2.4 ([IK03]). *$\pi_X^{-1}(X_{\text{sing}})$ has the following decomposition*

$$\pi_X^{-1}(X_{\text{sing}}) = \bigcup_{i \in I} C_i,$$

where the C_i are the irreducible components of $\pi_X^{-1}(X_{\text{sing}})$, and each C_i contains arcs $\alpha \in X_\infty$ such that $\alpha(\eta) \notin X_{\text{sing}}$.

Proof. See [IK03, Lemma 2.12]. □

Definition 4.2.5 (Nash Components of X)**.** These C_i are called the *Nash components* of X.

A $\mathbb{K}$-arc $\alpha : \operatorname{Spec}(\mathbb{K}[[t]]) \to X$ induces a discrete semi-valuation

$$\operatorname{val}_\alpha(f) := \operatorname{ord}_t(f(\alpha(t)))$$

on the local ring $\mathcal{O}_{X,\pi_X(\alpha)}$, defined by taking composition with the t-adic valuation.

In general, this semi-valuation cannot be extended to a valuation defined on the function field $\mathbb{C}(X)$. This will happen if $\mathbb{C}(X) \subseteq \mathbb{K}((t))$ is a field extension, which is the same as the image of α being dense in X. This leads to the concept of *fat arcs*.

Definition 4.2.6 (Thin / Fat Arcs)**.** An arc $\alpha : \operatorname{Spec}(\mathbb{K}[[t]]) \to X$ is called *thin* (or *algebraically thin*) if the image of α is contained in a proper closed subscheme of X. Otherwise, α is called *fat* (or *algebraically fat*). Equivalently, α is fat if and only if $\operatorname{val}_\alpha$ is a valuation on $\mathbb{C}(X)$.

Now we define the semi-valuations of X induced by the Nash components of X, which will be shown that they are actually valuations of X. Let C_i be a Nash component, and let $\alpha_i \in C_i$ be the generic point. By Proposition 4.2.2, we have a generalization $\tilde{\alpha}_i \in \pi_X^{-1}(X_{\text{sing}})$ such that $\tilde{\alpha}_i(\eta) \notin X_{\text{sing}}$ and $\tilde{\alpha}_i(0) \in X_{\text{sing}}$. Hence, we have $C_i = \overline{\{\alpha_i\}} \subseteq \overline{\{\tilde{\alpha}_i\}}$. Since C_i is an irreducible component of $\pi_X^{-1}(X_{\text{sing}})$, we get

$\tilde{\alpha}_i$ is also the generic point of C_i (i.e., $\tilde{\alpha}_i = \alpha_i$, and this also implies $C_i \nsubseteq \pi_X^{-1}(X_{\mathrm{sing}})$). We obtain the semi-valuation

$$\begin{aligned} \mathrm{val}_{C_i} : \mathcal{O}_{X,\pi_X(\alpha_i)} &\to \mathbb{Z} \\ f &\mapsto \mathrm{ord}_t[f(\alpha_i(t))] \end{aligned}$$

We now explain how arcs of X with $\alpha(0) \in X_{\mathrm{sing}}$ can be lifted to arcs on a resolution of X.

Lemma 4.2.7. *Let $f : Y \to X$ be a resolution of singularities of X, and assume that it induces an isomorphism between $X \setminus Z$ and $Y \setminus f^{-1}(Z)$ for some proper closed subset $Z \subsetneqq X$. Let C_i° denote the open subset of a given Nash component C_i consisting of arcs $\alpha : Spec(\mathbb{K}[[t]]) \to X$ such that $\alpha(\eta) \notin Z$. Then for every $\alpha \in C_i^{\circ}$, the arc α can be uniquely lifted to an arc $\tilde{\alpha} : \mathrm{Spec}(\mathbb{K}[[t]]) \to Y$.*

Proof. Notice that f is isomorphic outside Z, so we have the following commutative diagram.

$$\begin{tikzcd} \mathrm{Spec}(\mathbb{K}((t))) \arrow[r] \arrow[d] & Y \arrow[d, "f"] \\ \mathrm{Spec}(\mathbb{K}[[t]]) \arrow[r, "\alpha"] & X \end{tikzcd}$$

where the downward arrow on the left is induced by the canonical inclusion to the fraction field $\mathbb{K}[[t]] \hookrightarrow \mathbb{K}((t))$, and the top arrow $\mathrm{Spec}(\mathbb{K}((t))) \to Y$ is obtained by the condition $\alpha(\eta) \notin Z$. Notice that f is a proper morphism and $\mathbb{K}[[t]]$ is a valuation ring, so we obtain a unique lifting $\tilde{\alpha} : \mathrm{Spec}(\mathbb{K}[[t]]) \to Y$ by the valuative criterion of properness.

$$\begin{tikzcd} \mathrm{Spec}(\mathbb{K}((t))) \arrow[r] \arrow[d] & Y \arrow[d, "f"] \\ \mathrm{Spec}(\mathbb{K}[[t]]) \arrow[r, "\alpha"] \arrow[ur, dashed, "\exists\tilde{\alpha}"] & X \end{tikzcd}$$

$\square$

Remark 4.2.8. If $f : Y \to X$ is a Hironaka resolution of X, we have $Z = X_{\text{sing}}$.

The Nash Maps / The Nash Problem

Theorem 4.2.9 ([Nas95, IK03, CLNS18]). *Let X be a singular variety with the singular locus X_{sing}, and let C be a Nash component of X. Then there is an unique essential exceptional prime divisor E over X such that*

$$\text{val}_C = \text{val}_E.$$

In particular, val_C *is a valuation and it is essential. Also, there are only finitely many Nash components of X.*

Proof. Let $f : Y \to X$ be a resolution of singularities of X. Without loss of generality, we can assume that f is a Hironaka resolution of X. We have a morphism of arc spaces $f_\infty : Y_\infty \to X_\infty$ induced by f (see [CLNS18, 3.1.2 in Chapter 3]). Consider the exceptional locus $f^{-1}(X_{\text{sing}}) = Z_1 \cup \cdots \cup Z_m$ with $\{Z_i\}_{i=1}^m$ be the family of irreducible components of $f^{-1}(X_{\text{sing}})$.

f_∞ induces the following continuous bijection.

$$\tilde{f}_\infty : \pi_Y^{-1}(f^{-1}(X_{\text{sing}})) \smallsetminus (f^{-1}(X_{\text{sing}}))_\infty \to \pi_X^{-1}(X_{\text{sing}}) \smallsetminus (X_{\text{sing}})_\infty$$

where $\tilde{f}_\infty := f_\infty|_{\pi_Y^{-1}(f^{-1}(X_{\text{sing}})) \smallsetminus (f^{-1}(X_{\text{sing}}))_\infty}$ (see [CLNS18, Proposition 4.4.2 in Chapter 3]). As Y is smooth, $\{\pi_Y^{-1}(Z_i)\}_{i=1}^m$ are the irreducible components of $\pi_Y^{-1}(f^{-1}(X_{\text{sing}}))$

Since C is an irreducible component of $\pi_X^{-1}(X_{\text{sing}})$, there is a unique $\pi_Y^{-1}(Z_j)$ for some $j \in \{1, \dots, m\}$ such that $\tilde{f}_\infty$ restricts to the following continuous bijection.

$$\pi_Y^{-1}(Z_j) \smallsetminus (f^{-1}(X_{\text{sing}}))_\infty \xrightarrow{\tilde{f}_\infty} C \smallsetminus (X_{\text{sing}})_\infty,$$

and we have $\tilde{f}_\infty(\beta) = \alpha$, where α is the generic point of C and β is the generic point of $\pi_Y^{-1}(Z_j)$. Notice that β is a fat arc since Y is smooth, therefore α is also fat, which means that $\text{val}_C = \text{val}_\alpha$ is a valuation of X.

Take the blowing-up $\pi : \tilde{Y} \to Y$ along Z_j and let $E \subseteq \pi^{-1}(Z_j)$ be the unique exceptional prime divisor dominating Z_j, then we have

$$\mathrm{val}_C = \mathrm{val}_E$$

because $(f_\infty \circ \pi_\infty)(\pi_{\tilde{Y}}^{-1}(E)) = C$, which implies that α is the image of the generic point of $\pi_{\tilde{Y}}^{-1}(E)$.

The whole argument does not depend on the choice of resolution $f : Y \to X$, so we have that val_E is an essential valuation of X. $\square$

Definition 4.2.10 (Nash Valuations of X)**.** Let $C_1, \ldots, C_\ell$ be the Nash components of X, then the induced valuations $\mathrm{val}_{C_1}, \ldots, \mathrm{val}_{C_\ell}$ are called the *Nash valuations* of X.

In other words, we have the following correspondence

$$\begin{aligned} \mathcal{N}_X : \{\text{Nash components of } X\} &\to \{\text{essential valuations of } X\} \\ C &\longmapsto \mathrm{val}_C = \mathrm{val}_E \end{aligned}$$

is injective. $\mathcal{N}_X$ is called the *Nash map* of X. Therefore, we have

$$\{\text{Nash valuations of } X\} \subseteq \{\text{essential valuations of } X\}.$$

Consequently, we obtain an intrinsic method to identify an essential valuation of X without requiring to consider all resolutions of singularities of X. It is natural to ask if any essential valuation can be identified in this way:

Is the Nash map $\mathcal{N}_X$ surjective?

Now we can formulate *the Nash Problem of* X:

Understand the image of the Nash map N_X.

4.3 The Nash Problem: Known Results

Unfortunately, it is known that the Nash maps are in general not surjective for $\dim(X) \geq 3$. However, we still have several important cases for which the Nash maps are surjective.

Nash maps are surjective for the following cases:

- Curve Singularities
- Surface Singularities [FdBP12]
- Toric Singularities in arbitrary dimension [IK03]
- Quasi-Ordinary Hypersurface Singularities in arbitrary dimension [GPH09]
- Non-Rational Normal Varieties with Torus Action of Complexity 1 in arbitrary dimension [BLM25]

There are some examples that show that the Nash maps need not be surjective.

Theorem 4.3.1 ([IK03, Proposition 4.5]). *Consider the isolated hypersurface singularity given by*

$$X = (x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^6 = 0) \subseteq \mathbb{A}_k^5$$

over an algebraically closed field k with $\operatorname{char}(k) \neq 2, 3$. Then the Nash map is not surjective. In fact, it has only one Nash valuation and exactly two essential valuations.

Theorem 4.3.2 ([dF13, Theorem 6.1]). *Let*

$$X = ((x_2^2 + x_3^2)x_4 + x_1^3 + x_2^3 + x_3^3 + x_4^5 + x_4^6 = 0) \subseteq \mathbb{A}_{\mathbb{C}}^4.$$

Then the Nash map is not surjective.

4.4 The Nash Problem: Further Developments

We know that a Nash map $\mathcal{N}_X$ is in general not surjective for $\dim(X) \geq 3$, so the next step is to investigate how to characterize the image of $\mathcal{N}_X$.

Here is a result for non-uniruled divisors. Recall that a prime divisor E is *uniruled* if there is a variety E' with $\dim(E') = \dim(X) - 1$ and a dominant rational map $\phi : E' \times \mathbb{P}^1_k \dashrightarrow E$.

Theorem 4.4.1 ([LJR12, Theorem 3.3])**.** *Let X be a singular variety, and let E be a non-uniruled prime divisor over X. Then its divisorial valuation val_E is a Nash valuation of X.*

Proof. See [LJR12]. □

Recently, T. de Fernex and R. Docampo show that we can use a relatively minimal model to characterize it in [dFD16].

Definition 4.4.2 (Terminal Valuations of X, [dFD16])**.** Let X be a variety. A valuation val_E of X is called a *terminal valuation* of X if E is an exceptional prime divisor on a relatively minimal model over X.

Theorem 4.4.3 ([dFD16, Theorem 1.1])**.** *Let X be a variety, then every terminal valuation of X is a Nash valuation of X.*

Proof. See [dFD16]. □

Remark 4.4.4*.* Notice that two relatively minimal models over X are connected by sequence of flops ([BCHM10, Corollary 1.1.3]), which are isomorphisms in codimension 1. In particular, this implies that terminal valuations of X appear on every relatively minimal model over X. Thus, it is enough to consider any one such model to identify them.

Based on this result, we have the following relations of valuations.

$$\{\text{terminal vals. of } X\} \subseteq \{\text{Nash vals. of } X\} \subseteq \{\text{essential vals. of } X\}$$

Notice that this theorem alternatively proves that Nash maps are surjective for surface singularities. Indeed, every relatively minimal model is non-singular for surfaces, so we have

$$\{\text{essential vals. of } X\} \subseteq \{\text{terminal vals. of } X\} \qquad \text{when } \dim X \leq 2.$$

This approach is useful for all types of singularities except for the following two cases:

(1) X has only terminal singularities.

(2) X has small resolutions (i.e., resolutions with exceptional locus having codimension at least 2).

There is no terminal valuations in both cases, and this motivates the whole work of this dissertation. In other words,

find an alternative characterization for the image of a Nash map over terminal singularities.

In [JK13], J. Kollár and J.M. Johnson provide some characterizations of the singularities of type cA in arbitrary dimensions. Recently, there is a result proven by H.-K. Chen [Che23] for the 3-fold terminal singularity of type cA/m.

Theorem 4.4.5 ([Che23, Theorem 1.1])**.** *Let (X, p) be a germ of 3-fold terminal singularity of type cA/m for some $m \in \mathbb{Z}_{>0}$, then there is an one-to-one correspondence between Nash valuations over X and exceptional prime divisors over X whose discrepancies are less than or equal to one.*

On the other hand, the example in [IK03, Proposition 4.5] also mentions that its Nash valuation of X has minimal discrepancy, which is 1.

These observations serve as evidence that discrepancies might provide a useful criterion to identify whether a given valuation of X is Nash or not. In particular, we would like to study the valuations val_E with $a(E, X) \leq 1$, and this condition is also motivated by our result in Chapter 6.

Chapter 5

Some Toric Geometry

In this chapter, we will introduce some fundamental notions and tools in toric algebraic geometry that will be used in the next chapter. Most of their details can be found in [CLS11, Mat02].

From a practical perspective, toric varieties serve as important candidates for testing hypothesis or conjectures in algebraic geometry.

5.1 Definitions, Results, and Notations

Definition 5.1.1 (Toric Variety)**.** A *toric variety* X is a variety that contains an algebraic torus $T_N \cong (\mathbb{C}^*)^n$ as a Zariski open subset such that the action of T_N on itself extends to an algebraic action of T_N on X (i.e., an action $T_N \times X \to X$ given by a morphism).

It is well-known that toric varieties admit a combinatorial construction via fans and lattices, which makes them suitable objects to proceed computations. Here we briefly recall the construction.

Let $N \cong \mathbb{Z}^n$ be a lattice of rank n and $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ be its dual lattice. Denoted $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$, $N_{\mathbb{Q}} := N \otimes_{\mathbb{Z}} \mathbb{Q}$, $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$, and $M_{\mathbb{Q}} := M \otimes_{\mathbb{Z}} \mathbb{Q}$.

Definition 5.1.2 (Fans)**.** A *fan* Δ in $N_{\mathbb{R}}$ is a finite collection of cones $\sigma \subseteq N_{\mathbb{R}}$ such that

- Every $\sigma \in \Delta$ is a strongly convex rational polyhedral cone,
- For every $\sigma \in \Delta$, all its faces are also in Δ,
- For every $\sigma_1, \sigma_2 \in \Delta$, their intersection $\sigma_1 \cap \sigma_2$ is a face of σ_1 and σ_2 (this implies $\sigma_1 \cap \sigma_2 \in \Delta$).

A toric variety $X(\Delta)$ associated with a fan Δ is obtained by gluing the affine toric varieties

$$U_\sigma := \mathrm{Spec}(\mathbb{C}[\sigma^\vee \cap M])$$

for all $\sigma \in \Delta$ in the following way: Two affine patches U_{σ_1} and U_{σ_2} are glued together via $U_{\sigma_1 \cap \sigma_2}$, which is an affine toric variety associated with their intersecting face.

Theorem 5.1.3. *Let Δ be a fan in $N_{\mathbb{R}}$, then $X(\Delta)$ is a normal separated toric variety.*

Proof. See [CLS11, Theorem 3.1.5, p. 107]. □

With this characterization, we can express the geometric properties of toric varieties in combinatoric ways. Here we list some of them without providing proofs. The readers can check their proofs in [CLS11, Mat02]

Definition 5.1.4. Let $\sigma \subseteq N_{\mathbb{R}}$ be a strongly convex rational polyhedral cone.

- σ is called *smooth* if its minimal generators form part of a $\mathbb{Z}$-basis of N.
- σ is called *singular* if it is not smooth.
- σ is called *simplicial* if its minimal generators are linearly independent over $\mathbb{R}$.

Definition 5.1.5. Let Δ be a fan in $N_{\mathbb{R}}$.

- Δ is *smooth* (or *regular*) if every cone $\sigma \in \Delta$ is smooth (or regular).
- Δ is simplicial if every cone $\sigma \in N_{\mathbb{R}}$ is simplicial.
- Δ is *complete* if its support $|\Delta| := \bigcup_{\sigma \in \Delta} \sigma = N_{\mathbb{R}}$.

Theorem 5.1.6. *Let $X(\Delta)$ be a toric variety associated with a fan Δ in $N_{\mathbb{R}}$, then*

(1) $X(\Delta)$ is a smooth (or non-singular) variety if and only if the fan Δ is smooth.

(2) $X(\Delta)$ is factorial (every Weil divisor is Cartier) if and only if Δ is smooth.

(3) $X(\Delta)$ is $\mathbb{Q}$-factorial (every Weil divisor is $\mathbb{Q}$-Cartier) if and only if Δ is simplicial.

(4) $X(\Delta)$ is compact in the classical topology if and only if Δ is complete.

Proof. For (1) and (4), see [CLS11, Theorem 3.1.19, p. 113]. For (2), see [CLS11, Proposition 4.2.6, p. 179]. For (3), see [CLS11, Proposition 4.2.7, p. 180]. □

5.2 Some Birational Geometry of Toric Varieties

Here we recall some characterizations for $\mathbb{Q}$-Gorenstein toric varieties with terminal singularities. One can check details from [CLS11] and [Mat02].

Proposition 5.2.1. *Let $X(\Delta)$ be toric variety associtated with a fan Δ.*

(1) Its canonical divisor $K_{X(\Delta)}$ is $\mathbb{Q}$-Cartier (i.e. $X(\Delta)$ is $\mathbb{Q}$-Gorenstein) if and only if for any cone $\sigma \in \Delta$, there exists an $m_\sigma \in M_{\mathbb{Q}}$ such that

$$\langle m_\sigma, v_i \rangle = 1$$

for every primitive element $v_i \in N$ with $\langle v_i \rangle \in \Delta$ and $\langle v_i \rangle \subset \sigma$.

(2) Assume $K_{X(\Delta)}$ is $\mathbb{Q}$-Cartier. For a singular cone $\sigma \in \Delta$, the discrepancy can be expressed as follows.

$$a(E, X(\Delta)) = \langle m_\sigma, u_E \rangle - 1$$

where $u_E \in \sigma \cap N$ is the primitive vector corresponding to the divisor E over X.

(3) *Assume $K_{X(\Delta)}$ is $\mathbb{Q}$-Cartier, then $X(\Delta)$ has terminal singularities if and only if*

$$\{w \in \sigma \cap N \mid \langle m_\sigma, w\rangle \leq 1\} = \{0\} \cup \{v_i \in N \mid v_i : \textit{primitive}, \langle v_i\rangle \in \Delta \textit{ and } \langle v_i\rangle \subset \sigma\}$$

for every $\sigma \in \Delta$. In particular, we have

$$\langle m_\sigma, u\rangle > 1$$

for a non-zero $u \in (\sigma \cap N) \smallsetminus \{v_i \in N \mid v_i : \textit{primitive}, \langle v_i\rangle \in \Delta \textit{ and } \langle v_i\rangle \subset \sigma\}$.

Proof. See [Mat02, Proposition 14-3-1, p. 442], [CLS11, Lemma 11.4.10, p. 550], and [CLS11, Proposition 11.4.12, p. 550]. □

Chapter 6

Nash Valuations over Toric Terminal Singularities

It is natural to consider the Nash problem over toric terminal singularities as our initial cases. In this chapter, we present our results for the Nash problem over toric terminal singularities in any dimension. In particular, we prove that any exceptional prime divisor E over toric terminal singularities with discrepancy

$$a(E, X) \leq 1$$

will induce a Nash valuation.

Let Δ be a fan in $N_{\mathbb{R}}$. Given a cone $\sigma \in \Delta$, we consider the following partial order in $\sigma \cap N$.

Definition 6.0.1. Let σ be a strongly convex polyhedral cone, and let $u_1, u_2 \in \sigma \cap N$. We define $u_1 \leq u_2$ if $u_2 \in u_1 + \sigma$. i.e., there exists a $u' \in \sigma \cap N$ such that $u_2 = u_1 + u'$. For a subset $A \subset \sigma \cap N$, $u \in A$ is called minimal in A if $v \in A$ with $v \leq u$ implies $u = v$.

Consider

$$S_\sigma := \Big(\bigcup_{\tau\le\sigma,\tau:\text{singular}} \overset{\circ}{\tau}\Big) \cap N$$

for a given cone $\sigma \in \Delta$. $\tau \le \sigma$ means that τ is a face of σ, and $\overset{\circ}{\tau}$ denotes the relative interior of τ.

6.1 Discrepancy bounded by 1 induces Nash Valuations

Here we provide a sufficient condition for being a Nash valuation over toric terminal singularities in arbitrary dimensions.

Proposition 6.1.1. *Let Δ be a fan associated to a toric variety $X(\Delta)$ of dimension at least 3, and let $\sigma \in \Delta$ be a singular cone which defines a terminal singularity. A primitive element $w \in \overset{\circ}{\sigma} \cap N$ is minimal in S_σ if $\langle m_\sigma, w\rangle \le 2$.*

Proof. We prove its contrapositive statement. Let $w \in \overset{\circ}{\sigma} \cap N$ which is not minimal in S_σ, so there exists a non-zero element $u \in S_\sigma$ such that $u \le w$. i.e. $w = u + u'$ for some non-zero $u' \in \sigma \cap N$. Notice that $\langle m_\sigma, v_i\rangle = 1$ because $K_{X(\Delta)}$ is $\mathbb{Q}$-Cartier, and

$$\langle m_\sigma, u\rangle > 1$$

for a non-zero $u \in (\sigma \cap N) \smallsetminus \{v_i \in N \mid v_i : \text{primitive}, \langle v_i\rangle \in \Delta \text{ and } \langle v_i\rangle \subset \sigma\}$ since the U_σ has a terminal singularity.

We have the following two cases for u':

Case 1 If $u' \in (\sigma \cap N) \smallsetminus (\cup_i \langle v_i\rangle)$, then we have $\langle m_\sigma, u'\rangle > 1$, and

$$\langle m_\sigma, w\rangle = \langle m_\sigma, u\rangle + \langle m_\sigma, u'\rangle > 1 + 1 = 2.$$

Case 2 If $u' \in \cup_i \langle v_i \rangle$, then we have $u' = \alpha v_j$ for some v_j and some $\alpha \in \mathbb{Z}_{>0}$, and we get $\langle .m_\sigma, u' \rangle = \alpha \langle m_\sigma, v_j \rangle = \alpha \cdot 1 \geq 1$. So

$$\langle m_\sigma, w \rangle = \langle m_\sigma, u \rangle + \langle m_\sigma, u' \rangle > 1 + 1 = 2.$$

Therefore, we obtain

$$\langle m_\sigma, w \rangle > 2.$$

□

Theorem 6.1.2 ([IK03, 3. The Nash problem for toric singularities])**.** *Let $X(\Delta)$ be an affine toric variety in any dimension, then its Nash map is bijective. Moreover, every Nash valuation of $X(\Delta)$ (also being an essential valuation of $X(\Delta)$) is one-to-one corresponded to a minimal element in S_σ for some cone $\sigma \in \Delta$.*

By combining Proposition 6.1.1 and Theorem 6.1.2, we obtain the following Theorem.

Theorem 6.1.3. *Let $X(\Delta)$ be a toric variety with $\mathbb{Q}$-Cartier $K_{X(\Delta)}$ of dimension at least 3 having only terminal singularities (we have $a(E, X(\Delta)) > 0$), and let E be an exceptional prime divisor over $X(\Delta)$. If its discrepancy $a(E, X(\Delta)) \leq 1$, then the induced valuation val_E is a Nash valuation of $X(\Delta)$.*

Proof. Recall that for every exceptional prime divisor E over a toric variety, its discrepancy is $a(E, X(\Delta)) = \langle m_\sigma, w \rangle - 1$, where $w \in \overset{\circ}{\sigma} \cap N$ is the primitive element of a ray $\rho_E := \langle w \rangle \subset \overset{\circ}{\sigma}$ corresponding to E.

By assumption, since $a(E, X(\Delta)) = \langle m_\sigma, w \rangle - 1 \leq 1$, we get $\langle m_\sigma, w \rangle \leq 2$, and we obtain $w \in \overset{\circ}{\sigma} \cap N$ is minimal in S_σ by Proposition 6.1.1. Therefore, the induced valuation val_E is a Nash valuation of $X(\Delta)$ according to Theorem 6.1.2. □

6.2 Nash Valuations have Discrepancy bounded by 1

Next, we show that the converse of Theorem 6.1.3 is also true if $X(\Delta)$ is a toric terminal 3-fold. To prove this, we need the notion of heights of elements in the Hilbert basis for a cone.

Recall that for a given polyhedral cone σ, there exists a set $\mathcal{H}_N(\sigma) \subset \sigma \cap N$ such that

(1) Every $v \in \sigma \cap N$ can be expressed as a linear combination of elements in $\mathcal{H}_N(\sigma)$ over $\mathbb{Z}_{\geq 0}$, i.e. $v \in \sum_{h \in H_N(\sigma)} \mathbb{Z}_{\geq 0} h$,

(2) $\mathcal{H}_N(\sigma)$ has minimal cardinality with respect to all subsets of $\sigma \cap N$ for which (1) holds.

This set $\mathcal{H}_N(\sigma)$ is called the *Hilbert basis* for σ. Moreover, if σ is a pointed cone (i.e., $\sigma \cap -\sigma = \{0\}$), then $\mathcal{H}_N(\sigma)$ is uniquely determined by the following expression.

$\mathcal{H}_N(\sigma) = \{v \in (\sigma \cap N) \setminus \{0\} \mid v$ is not the sum of two other vectors in $(\sigma \cap N) \setminus \{0\}\}$.

Definition 6.2.1. Let $\sigma = \sum_{i=1}^{m} \mathbb{R}_{\geq 0} v_i \subset N_{\mathbb{R}}$ be a pointed cone, and let $u \in \mathcal{H}_N(\sigma)$. The number defined to be

$$g_\sigma(u) := \min\{\sum_{i=1}^{m} \lambda_i \mid u = \sum_{i=1}^{m} \lambda_i v_i, \lambda_i \geq 0, 1 \leq i \leq m\}$$

is called the *height* of u.

Proposition 6.2.2 ([HW97, LTZ93, EW91])**.** *Let $\sigma \subset N_{\mathbb{R}} \cong \mathbb{R}^d$ be a pointed cone, and let $u \in \mathcal{H}_N(\sigma)$. We have an upper bound for $g_\sigma(u)$:*

$$g_\sigma(u) < d - 1$$

for $d \geq 2$.

Corollary 6.2.3. *Let $X(\Delta)$ be a toric terminal 3-fold, then a valuation val_E associated to an exceptional prime divisor E over $X(\Delta)$ is a Nash valuation of $X(\Delta)$ if and only if*

$$a(E, X(\Delta)) \leq 1.$$

Proof. For the sufficient condition, apply Theorem 6.1.3. For the necessary condition, let E be an exceptional prime divisor over $X(\Delta)$ inducing a Nash valuation val_E. By [IK03], val_E is corresponded to a $w \in \overset{\circ}{\sigma} \cap N$ which is minimal in S_σ for some $\sigma \in \Delta$ being singular.

We have the following two cases for w:

Case 1. If $w \notin \mathcal{H}_N(\sigma)$, we have $w = v_1 + v_2$ for some primitive elements $v_1, v_2 \in \sigma \cap N$ such that $\langle v_1 \rangle$, $\langle v_2 \rangle \in \Delta$ and $\langle v_1 \rangle$, $\langle v_2 \rangle \subset \sigma$. Then we obtain

$$\langle m_\sigma, w \rangle = \langle m_\sigma, v_1 \rangle + \langle m_\sigma, v_2 \rangle = 1 + 1 = 2.$$

Notice that we use $\langle m_\sigma, v_i \rangle = 1$ for all $i = 1, 2$ since $X(\Delta)$ is $\mathbb{Q}$-Gorenstein ($X(\Delta)$ is terminal).

Case 2. If $w \in \mathcal{H}_N(\sigma)$, then we can express $w = \sum_{i=1}^m \lambda_i v_i$, where $\sigma = \sum_{i=1}^m \mathbb{R}_{\geq 0} v_i$. Then we get

$$\langle m_\sigma, w \rangle = \sum_{i=1}^m \lambda_i \langle m_\sigma, v_i \rangle = \sum_{i=1}^m \lambda_i.$$

Notice that we use $\langle m_\sigma, v_i \rangle = 1$ for all $i = 1 \ldots m$ since $X(\Delta)$ is $\mathbb{Q}$-Gorenstein. Now we consider $w = \sum_{i=1}^m \lambda_i v_i$ minimizing $\sum_{i=1}^m \lambda_i$, then we get

$$\langle m_\sigma, w \rangle = g_\sigma(w).$$

We obtain $\langle m_\sigma, w \rangle = g_\sigma(w) < 3 - 1 = 2$ by Proposition 6.2.2 (notice that $X(\Delta)$ is a 3-fold).

Therefore, we have $\langle m_\sigma, w \rangle \leq 2$ by Case 1 and Case 2, which implies that

$$a(E, X(\Delta)) = \langle m_\sigma, w \rangle - 1 \leq 2 - 1 = 1. \qquad \square$$

Remark 6.2.4. Notice that in the proof of the necessary condition of Corollary 6.2.3, we do not need the assumption $X(\Delta)$ is terminal. In fact, we use the condition that $X(\Delta)$ is $\mathbb{Q}$-Gorenstein.

6.3 Main Result and Conjectures

By combining the previous results, the condition $a(E, X(\Delta)) \leq 1$ fully characterizes Nash valuations over 3-fold toric terminal singularities.

Theorem 6.3.1. *Let $X(\Delta)$ be a toric terminal 3-fold, and val_E be a valuation associated to an exceptional prime divisor E over $X(\Delta)$. Then the following statements are equivalent:*

(1) val_E *is essential over* $X(\Delta)$*.*

(2) val_E *is Nash over* $X(\Delta)$*.*

(3) $a(E, X(\Delta)) \leq 1$*.*

Proof. For the equivalence (1)⇔(2) apply Theorem 6.1.2, and for (2)⇔(3) apply Corollary 6.2.3. □

Theorem 6.1.3 (or Theorem 6.3.1) provides a potential approach that we can try using discrepancies to characterize Nash valuations over terminal singularities. In particular, we propose the following conjecture.

Conjecture 6.3.2 (Conjecture A)**.** *Let (X, p) be a germ of 3-fold terminal with the Gorenstein index $m \geq 1$, then every exceptional prime divisor E with discrepancy $a(X, E) \leq 1$ induces a Nash valuation over (X, p).*

We propose a weaker version of Conjecture A.

Conjecture 6.3.3 (Conjecture B). *Let (X,p) be a germ of 3-fold terminal with the Gorenstein index $m \geq 1$, then every exceptional prime divisor E with discrepancy $a(X,E) = \frac{1}{m}$ (i.e. the minimal discrepancy of X) induces a Nash valuation over (X,p).*

In this chapter, we prove that Conjecture A holds for toric terminal singularities for dimensions at least 3. In the next chapter, we will prove that for (X,p) is a germ of 3-fold terminal singularity of type cAx/2, then Conjecture B is true.

Chapter 7

Nash Valuations over 3-Fold Terminal Singularities of Type cAx/2

In this chapter, we will prove that the exceptional prime divisors with minimal discrepancy (it is $\frac{1}{2}$ in this case) will induce Nash valuations over 3-fold terminal singularities X of type cAx/2.

Let (X, p) be a germ of 3-fold terminal singularity of type cAx/2. It admits an embedding

$$(x^2 + y^2 + f(z, u) = 0) \subseteq \mathbb{C}^4 / \frac{1}{2}(0, 1, 1, 1)$$

such that $p = (0, 0, 0, 0) \in \mathbb{C}^4/\frac{1}{2}(0, 1, 1, 1)$ and $f(z, u) \in (z, u)^4 \subseteq \mathbb{C}\{z, u\}$ is a $\mathbb{Z}_2$-invariant.

Here is our main Theorem in this chapter.

Theorem 7.0.1. *Let (X, p) be a germ of 3-fold terminal singularity of type cAx/2, then any prime divisor E over (X, p) with the minimal discrepancy (it is $\frac{1}{2}$ in this case) induces a Nash valuation val_E over (X, p).*

In other words, we prove that Conjecture B holds for the case (X, p) being a 3-fold terminal singularity of type cAx/2.

7.1 Candidates for Nash Valuations

Let (X, p) be a germ of terminal singularity. It makes sense to choose a prime divisor E over (X, p) with $a(E, X) \leq 1$ to be a candidate to produce a Nash valuation of X because val_E is, in fact, an essential valuation of X.

Proposition 7.1.1 ([JK13, Proposition 24])**.** *Let (X, p) be a germ of canonical singularity (see [KM98, Mat02]), and let E be an exceptional prime divisor over X centered at p and*

$$a(E, X) < 1 + \textit{the minimal discrepancy of } (X, p).$$

Then val_E is an essential valuation of X.

Corollary 7.1.2. *Let (X, p) be a 3-fold terminal singularity with Gorenstein index $m \geq 1$, then any exceptional prime divisor E over (X, p) such that $a(E, X) \leq 1$ is essential. In particular, any exceptional divisor over (X, p) with minimal discrepancy (in this case it is $\frac{1}{m}$) is essential.*

7.2 Setup for the Proof

Define a partial order among the valuations over (X, p):

$$\mathrm{val}_E \leq_X \mathrm{val}_F \quad \text{if} \quad \mathrm{val}_E(f) \leq \mathrm{val}_F(f) \quad \text{for all} \quad f \in \mathcal{O}_{X,p}$$

Lemma 7.2.1. *Let X be a singular variety, and let E be an exceptional prime divisor over X such that val_E is non-Nash over X. Then there exists another prime divisor F over X*

such that

$$\mathrm{val}_F <_X \mathrm{val}_E.$$

Proof. Define

$$N_G := \overline{\{f_\infty(\pi_Y^{-1}(G))\}} = \overline{\{\alpha \in X_\infty \mid \mathrm{val}_\alpha = \mathrm{val}_G\}}$$

for a prime divisor G in some Y such that $f : Y \to X$ is a resolution of X, and $\overline{\{\quad\}}$ denotes the Zariski closure in X_∞.

Let E be an exceptional prime divisor over X such that val_E is non-Nash over X. Since E is non-Nash, we have that N_E is contained in a Nash component of X. Let F be the prime divisor such that val_F is the Nash valuation of X induced by such the Nash component, then we have

$$N_E \subsetneq N_F,$$

and this implies $\mathrm{val}_E <_X \mathrm{val}_F$. □

Let (X, p) be a 3-fold terminal singularity with the Gorenstein index $r \geq 1$, and let $\pi : \overline{X} \to X$ be a divisorial contraction with minimal discrepancy in Definition 3.3.6. Denote $E = \pi^{-1}(p)$ to be the $\mathbb{Q}$-Cartier exceptional prime divisor in $\overline{X}$ obtained by π. Let F be an essential exceptional prime divisor over X centered at p. Here we derive an equation relating val_E and val_F.

Since $\pi : \overline{X} \to X$ is a divisorial contraction, $\overline{X}$ has only terminal singularities, which are isolated points (notice that $\mathrm{codim}(\overline{X}_{\mathrm{sing}}, \overline{X}) \geq 3$ since $\overline{X}$ is terminal). Let $\phi : Y \to \overline{X}$ be a resolution. Notice that val_F is essential over X, so F appears in Y. Hence, F is centered at some terminal singular point $q \in \overline{X}$.

Let $g \in m_{\overline{X},q} \setminus \mathcal{I}_{E,q}$ be irreducible, where $m_{\overline{X},q} \subseteq \mathcal{O}_{\overline{X},q}$ is the maximal ideal, and $\mathcal{I}_E$ is the sheaf of ideals associated with E. Let $G = \mathrm{div}_{\overline{X}}(g) \subseteq \overline{X}$ be the principal divisor induced by g, which is a prime divisor in $\overline{X}$.

Let $h \in \mathcal{I}_{\pi_* G,p}$, where $\pi_* G$ is the push-forward of G via π in X, which is a Weil divisor containing p, and $\mathcal{I}_{\pi_* G}$ is the sheaf of ideals associated with $\pi_* G$. Let $H = \mathrm{div}_X(h) \subseteq X$ be the principal divisor induced by h, which is Cartier. Then

$$\pi^* H = aG + bE + A$$

where $a, b \in \mathbb{Z}_{\geq 1}$ and A is some $\mathbb{Q}$-Cartier divisor in $\overline{X}$ without components from G or E (notice that here E is $\mathbb{Q}$-Cartier, $a \geq 1$ since $h \in \mathcal{I}_{\pi_* G}$, and $b \geq 1$ since $x \in \pi_* G$).

Applying ϕ^* to $\pi^* H$, we get

$$\phi^* \pi^* H = \phi^*(aG + bE + A) = a\tilde{G} + b\tilde{E} + (aa_1 + bc + d)F + (\text{others}) \tag{\#}$$

where $\phi^* E = \tilde{E} + cF + (\text{others})$, $\phi^* G = \tilde{G} + a_1 F + (\text{others})$, $\phi^* A = dF + \tilde{A} + (\text{others})$, with $a_1 \in \mathbb{Z}_{\geq 1}$, $c \in \mathbb{Q}_{>0}, d \in \mathbb{Q}_{\geq 0}$. "Others" means the divisors whose supports are disjoint from the shown divisors in the corresponding equations. In addition, $\tilde{G}$, $\tilde{E}$, and $\tilde{A}$ denote the strict transforms of G, E, and A to Y.

Here we can see that

$$\mathrm{val}_E(h) = b, \qquad \mathrm{val}_F(h) = aa_1 + bc + d.$$

In other words,

$$\mathrm{val}_F(h) = aa_1 + c\,\mathrm{val}_E(h) + d.$$

Our goal is to find some particular $h \in \mathcal{I}_{\pi_* G,p}$ such that $\mathrm{val}_E(h) < \mathrm{val}_F(h)$.To obtain this inequality, it suffices to construct an $h \in \mathcal{I}_{\pi_* G,p}$ satisfying either (1) $b = 1$ or (2) $c \geq 1$.

Remark 7.2.2. In the following proof, we will construct an $h \in \mathcal{I}_{\pi_* G,p}$ such that $b = 1$.

7.3 Weighted Blowing-Ups

In [Hay99, Hay00], we know that any divisorial contraction with minimal discrepancy is obtained by a *weighted blowing-up* with some weight $\sigma = \frac{1}{m}(a,b,c,d) \in \frac{1}{m}(\mathbb{Z}_{>0})^4$. In this section, we introduce the notation and tools that will be used in the proof of our main theorem in this chapter. The details can be found in [Hay99, Hay00, Che16, Che23]

The cyclic quotient singularity $Y = \mathbb{C}^4/\frac{1}{m}(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$ is the toric variety corresponding to the lattice $\overline{N} := \mathbb{Z}e_1 + \mathbb{Z}e_2 + \mathbb{Z}e_3 + \mathbb{Z}e_4 + \mathbb{Z}e = N + \mathbb{Z}e$ and the cone $C = \mathbb{R}_{\geq 0}e_1 + \mathbb{R}_{\geq 0}e_2 + \mathbb{R}_{\geq 0}e_3 + \mathbb{R}_{\geq 0}e_4$, where e_i, $i = 1,2,3,4$, are the standard basis vectors for $\mathbb{Z}^4$, and $e = \frac{1}{m}(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \in \frac{1}{m}\mathbb{Z}^4$. $\overline{N}$ and its fan Δ consist of all faces of C. In other words,

$$Y = \mathbb{C}^4/\frac{1}{m}(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \cong \operatorname{Spec}(\mathbb{C}[C^\vee \cap \overline{M}])$$

where $\overline{M} := \operatorname{Hom}_{\mathbb{Z}}(\overline{N},\mathbb{Z})$.

Let us give a description of weighted blowing-ups for $\mathbb{C}^4/\frac{1}{m}(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$. Let $\sigma = \frac{1}{m}(a,b,c,d) \in \overline{N}$ with $a,b,c,d > 0$, and assume e_1,e_2,e_3,e_4,σ generate the lattice $\overline{N}$. This σ will be called a *weight*.

We construct the *weighted blowing-up* $\bar{\pi}_\sigma : \overline{Y} \to Y = \mathbb{C}^4/\frac{1}{m}(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$ with weight σ as follows.

We divide the cone C by adding 1-dimensional cone $\mathbb{R}_{\geq 0}\sigma$, and we get a subdivision of C into 4 cones:

$$C_1 = \mathbb{R}_{\geq 0}\sigma + \mathbb{R}_{\geq 0}e_2 + \mathbb{R}_{\geq 0}e_3 + \mathbb{R}_{\geq 0}e_4,$$

$$C_2 = \mathbb{R}_{\geq 0}e_1 + \mathbb{R}_{\geq 0}\sigma + \mathbb{R}_{\geq 0}e_3 + \mathbb{R}_{\geq 0}e_4,$$

$$C_3 = \mathbb{R}_{\geq 0}e_1 + \mathbb{R}_{\geq 0}e_2 + \mathbb{R}_{\geq 0}\sigma + \mathbb{R}_{\geq 0}e_4,$$

$$C_4 = \mathbb{R}_{\geq 0}e_1 + \mathbb{R}_{\geq 0}e_2 + \mathbb{R}_{\geq 0}e_3 + \mathbb{R}_{\geq 0}\sigma.$$

Let Δ' be the fan consisting of all faces of C_1, C_2, C_3, C_4. Then the *weighted blowing-up* $\overline{Y}$ of Y is defined as the toric variety corresponding to the lattice $\overline{N}$ and the fan Δ', and $\bar{\pi}_\sigma : \overline{Y} \to Y$ is the morphsim induced from the natural map of fans $(\overline{N}, \Delta') \to (\overline{N}, \Delta)$.

The toric variety $\overline{Y}$ is covered by 4 affine open sets $\overline{U}_1, \overline{U}_2, \overline{U}_3, \overline{U}_4$ corresponding to the cones C_1, C_2, C_3, C_4, respectively. These affine open sets are as follows:

$$\begin{aligned}
\overline{U}_1 &= \mathbb{C}^4/\frac{1}{a}(m, -b, -c, -d) \cong \operatorname{Spec}(\mathbb{C}[C_1^\vee \cap \overline{M}]),\\
\overline{U}_2 &= \mathbb{C}^4/\frac{1}{b}(-a, m, -c, -d) \cong \operatorname{Spec}(\mathbb{C}[C_2^\vee \cap \overline{M}]),\\
\overline{U}_3 &= \mathbb{C}^4/\frac{1}{c}(-a, -b, m, -d) \cong \operatorname{Spec}(\mathbb{C}[C_3^\vee \cap \overline{M}]),\\
\overline{U}_4 &= \mathbb{C}^4/\frac{1}{d}(-a, -b, -c, m) \cong \operatorname{Spec}(\mathbb{C}[C_4^\vee \cap \overline{M}]).
\end{aligned}$$

And $\bar{\pi}_\sigma$ are formulated as follows:

$$\begin{aligned}
&\bar{\pi}_\sigma|_{\overline{U}_1} : \overline{U}_1 \longrightarrow Y \qquad (x, y, z, u) \mapsto (x^{\frac{a}{m}}, x^{\frac{b}{m}} y, x^{\frac{c}{m}} z, x^{\frac{d}{m}} u),\\
&\bar{\pi}_\sigma|_{\overline{U}_2} : \overline{U}_2 \longrightarrow Y \qquad (x, y, z, u) \mapsto (y^{\frac{a}{m}} x, y^{\frac{b}{m}}, y^{\frac{c}{m}} z, y^{\frac{d}{m}} u),\\
&\bar{\pi}_\sigma|_{\overline{U}_3} : \overline{U}_3 \longrightarrow Y \qquad (x, y, z, u) \mapsto (z^{\frac{a}{m}} x, z^{\frac{b}{m}} y, z^{\frac{c}{m}}, z^{\frac{d}{m}} u),\\
&\bar{\pi}_\sigma|_{\overline{U}_4} : \overline{U}_4 \longrightarrow Y \qquad (x, y, z, u) \mapsto (u^{\frac{a}{m}} x, u^{\frac{b}{m}} y, u^{\frac{c}{m}} z, u^{\frac{d}{m}}).
\end{aligned}$$

In these expressions, x, y, z, u denote coordinates in the ambient $\mathbb{C}^4$. For example, the map $\bar{\pi}_\sigma|_{\overline{U}_1}$ is determined by a map $\mathbb{C}^4 \longrightarrow \mathbb{C}^4$ which, at the level of coordinate rings, is given by

$$\pi_\sigma^\#(x) = x^{\frac{a}{m}}, \quad \pi_\sigma^\#(y) = x^{\frac{b}{m}} y, \quad \pi_\sigma^\#(z) = x^{\frac{c}{m}} z, \quad \pi_\sigma^\#(u) = x^{\frac{d}{m}} u.$$

The exceptional divisor $\overline{E}$ of $\bar{\pi}_\sigma$ is isomorphic to the weighted projective space $\mathbb{P}_{\mathbb{C}}(a, b, c, d)$.

Let us formulate weighted blowing-ups for a subvariety defined by a semi-invariant in $\mathbb{C}^4/\frac{1}{m}(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$. Let $\phi \in \mathbb{C}\{x, y, z, u\}$ be a $\mathbb{Z}_m$-semi-invariant, and it defines a hyperquotient singularity $X = (\phi = 0) \subseteq \mathbb{C}^4/\frac{1}{m}(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$. Let $\overline{X} = (\bar{\pi}_\sigma)^{-1}(X)$ be the strict transform of X via $\bar{\pi}_\sigma$ and let $\pi_\sigma = \bar{\pi}_\sigma|_{\overline{X}}$ be the restriction of $\bar{\pi}_\sigma$ in $\overline{X}$. Then $\pi_\sigma : \overline{X} \to X$ us called the *weighted blowing-up with weight* $\sigma = \frac{1}{m}(a, b, c, d)$ (or briefly σ-blowing-up). Let $U_i := \overline{U}_i \cap \overline{X}$ for $i = 1, 2, 3, 4$, then each U_i is a hyperquotient singularity in $\overline{U}_i$ and $\overline{X}$ is covered by 4 charts U_1, U_2, U_3, U_4. We denote $E := \overline{E} \cap \overline{X} \subseteq \mathbb{P}_{\mathbb{C}}(a, b, c, d)$, and notice that E is not necessarily irreducible in $\overline{X}$.

We can explicitly express the defining equations of $\overline{X} \cap U_i$ for $i = 1, 2, 3, 4$. Write a semi-invariant $\phi = \sum_{i,j,k,\ell} a_{i,j,k,\ell} x^{\alpha_i} y^{\beta_j} z^{\gamma_k} u^{\delta_\ell}$, and we define

$$\mathrm{wt}_\sigma(\phi) := \min\{ \frac{a\alpha_i + b\beta_j + c\gamma_k + d\delta_\ell}{m} \mid a_{i,j,k,\ell} \neq 0 \}$$

for a given weight $\sigma = \frac{1}{m}(a, b, c, d) \in \overline{N}$. Notice that $\overline{X} \cap U_i$ for $i = 1, 2, 3, 4$ is defined by

$$\begin{aligned}
\phi_1 &:= \phi(x^{\frac{a}{m}}, x^{\frac{b}{m}} y, x^{\frac{c}{m}} z, x^{\frac{d}{m}} u) x^{-\mathrm{wt}_\sigma(\phi)} \text{ in } U_1, \\
\phi_2 &:= \phi(y^{\frac{a}{m}} x, y^{\frac{b}{m}}, y^{\frac{c}{m}} z, y^{\frac{d}{m}} u) y^{-\mathrm{wt}_\sigma(\phi)} \text{ in } U_2, \\
\phi_3 &:= \phi(z^{\frac{a}{m}} x, z^{\frac{b}{m}} y, z^{\frac{c}{m}}, z^{\frac{d}{m}} u) z^{-\mathrm{wt}_\sigma(\phi)} \text{ in } U_3, \\
\phi_4 &:= \phi(u^{\frac{a}{m}} x, u^{\frac{b}{m}} y, u^{\frac{c}{m}} z, u^{\frac{d}{m}}) u^{-\mathrm{wt}_\sigma(\phi)} \text{ in } U_4.
\end{aligned}$$

Let us characterize the singular locus of $\overline{X}$. Write

$$(\overline{X})_{\mathrm{sing}} = [(\overline{X})_{\mathrm{sing}}]_{\mathrm{ind}=1} \bigcup [(\overline{X})_{\mathrm{sing}}]_{\mathrm{ind}>1}$$

where $[(\overline{X})_{\mathrm{sing}}]_{\mathrm{ind}=1}$ ($[(\overline{X})_{\mathrm{sing}}]_{\mathrm{ind}>1}$, resp.) denotes the singular locus with Gorenstein index $= 1$ (Gorenstein index > 1, resp.). Then we can interpret $(\overline{X})_{\mathrm{sing}}$ as the following.

Proposition 7.3.1 (2.1 in [Che16]). *Keep the notation in this section. Let $\pi_\sigma : \overline{X} \to X$ be a weighted blowing-up of X with respect to the weight σ, then we have*

(1) $[(\overline{X})_{\text{sing}}]_{\text{ind}>1} = \overline{X} \cap (\mathbb{P}_{\mathbb{C}}(a,b,c,d))_{\text{sing}} = E \cap (\mathbb{P}_{\mathbb{C}}(a,b,c,d))_{\text{sing}}$,

(2) *If* $\overline{U}_i \cong \mathbb{C}^4$ *for some* $i \in \{1,2,3,4\}$, *then* $(\overline{X})_{\text{sing}} \cap U_i \subseteq E_{\text{sing}} \cap U_i$,

(3) *If* $\overline{X}$ *is a terminal 3-fold, then* $[(\overline{X})_{\text{sing}}]_{\text{ind}=1} \subseteq E_{\text{sing}}$.

7.4 Proof of Theorem 7.0.1

We follow the notation of Section 7.2. Assume val_E is non-Nash over (X,p), so there is a divisor F over (X,p) inducing a Nash valuation val_F such that

$$\text{val}_F <_X \text{val}_E,$$

and F must be centered at some singular point of $\overline{X}$. It suffices to show that for any $q \in \overline{X}_{\text{sing}}$ such that F is centered at q, there is $f \in \mathcal{O}_{X,p}$ such $\text{val}_F(f) > \text{val}_E(f)$, which contradicts $\text{val}_F <_X \text{val}_E$, so we can conclude that val_E is Nash over (X,p).

We follow the arguments in [Che16, Proposition 18] and [Hay05, 5.2]. Recall that the standard form of this type of 3-fold terminal singularity (X,p) is

$$X = (x^2 + y^2 + f(z,u) = 0) \subseteq \mathbb{C}^4/\frac{1}{2}(0,1,1,1)$$

where $f(z,u) \in (z,u)^4 \subseteq \mathbb{C}\{z,u\}$ is a $\mathbb{Z}_2$-invariant with respect to $\frac{1}{2}(0,1,1,1)$, and $p = (0,0,0,0) \in X$. Define

$$\tau_0 := \min\{\, i+j \mid z^i u^j \in \operatorname{supp}(f(z,u)) \text{ as a monomial } \} \geq 4.$$

Notice that $f(z,u)$ is a $\mathbb{Z}_2$-invariant with respect to $\frac{1}{2}(0,1,1,1)$, so this implies $\tau_0 \in \mathbb{Z}_{>0}$ is even.

Let $f_{\tau_0}(z,u) = \sum_{i+j=\tau_0} a_{ij} z^i u^j$, and express it in the form of the following factorization:

$$f_{\tau_0}(z,u) = \sum_{i+j=\tau_0} a_{ij} z^i u^j = \prod_{t \in T} (a_t z + b_t u)^{m_t}$$

where $\sum_{t \in T} m_t = \tau_0$.

Case 1: $f_{\tau_0}(z,u)$ is not a perfect square

From [Hay99, 8.2], consider the weighted blow-up with the following weight

$$\sigma := \begin{cases} \frac{1}{2}(\frac{\tau_0}{2}, \frac{\tau_0}{2}+1, 1, 1), & \text{if } \frac{\tau_0}{2} \text{ is even,} \\ \frac{1}{2}(\frac{\tau_0}{2}+1, \frac{\tau_0}{2}, 1, 1), & \text{if } \frac{\tau_0}{2} \text{ is odd.} \end{cases}$$

We consider the first case $\sigma = \frac{1}{2}(\frac{\tau_0}{2}, \frac{\tau_0}{2}+1, 1, 1)$, and the second case is followed by an analogous argument. The weighted blowing-up with respect to σ induces a divisorial contraction $\pi_\sigma : \overline{X} \to X$ with the exceptional prime divisor $E = \pi_\sigma^{-1}(p) \subseteq \overline{X}$ with minimal discrepancy (in this case $a(E, X) = \frac{1}{2}$) with respect to (X, p) by Theorem 8.4 in [Hay99]. Notice that $\overline{X}$ is covered by four affine open sets U_1, U_2, U_3, and U_4. By [Che16, Proposition 18] and [Hay05, 5.2], we have that $\overline{X}$ is non-singular on U_1, it has one terminal quotient singularity of index $\frac{\tau_0}{2}+1$ in U_2, denoted by Q_2, and several Gorenstein terminal singularities of type cD in U_3 and U_4.

In U_2, there is a terminal quotient singularity $Q_2 = (0,0,0,0) \in U_2$ with index $\frac{\tau_0}{2}+1$ (see [Che16, Proposition 18 Case 1] and [Hay05, 5.2]). We consider $g = x^2 y^{\frac{\tau_0}{2}-1} + y^{\frac{\tau_0}{2}} + z^2 + u^2 \in \mathcal{O}_{\overline{X},Q_2} \setminus \mathcal{I}_{E,Q_2}$ ($g \notin \mathcal{I}_{E,Q_2}$ since $E = (x^2 + f_{\tau_0}(z,u) = 0)$ in U_2), and let $G = \operatorname{div}_{\overline{X}}(g) \subseteq \overline{X}$. Consider $h = x^2 + y^2 + z^2 + u^2 \in \mathcal{O}_{X,p}$, and let $H = \operatorname{div}_X(h)$.

Here we compute $\pi_\sigma^* H$, the pullback of H via π_σ, in U_2. Notice that π_σ is the weighted blowing-up with respect to σ given by

$$(\pi_\sigma^\#(x), \pi_\sigma^\#(y), \pi_\sigma^\#(z), \pi_\sigma^\#(u)) = (xy^{\frac{\tau_0}{4}}, y^{\frac{\tau_0}{4}+\frac{1}{2}}, zy^{\frac{1}{2}}, uy^{\frac{1}{2}}).$$

Hence,

$$\begin{aligned} \pi_\sigma^\#(h) = \pi_\sigma^\#(x^2+y^2+z^2+u^2) &= x^2 y^{\frac{\tau_0}{2}} + y^{\frac{\tau_0}{2}+1} + z^2 y + u^2 y \\ &= y(x^2 y^{\frac{\tau_0}{2}-1} + y^{\frac{\tau_0}{2}} + z^2 + u^2) = yg. \end{aligned}$$

In other words, we obtain

$$\pi_\sigma^* H = E + G.$$

Notice that $(\pi_\sigma)_*G = H$, so we have $h = x^2 + y^2 + z^2 + u^2 \in \mathcal{I}_{(\pi_\sigma)_*G,p}$. Apply the equation (#) to π_σ, we have $b = \mathrm{val}_E(h) = 1$, so we have $\mathrm{val}_E(h) < \mathrm{val}_F(h)$ for any divisor F over X centered at $Q_2 \in \overline{X}$.

We compute the strict transform of the defining equation of $\overline{X}$ in U_3, and it is given by

$$(\pi_\sigma^\#(x), \pi_\sigma^\#(y), \pi_\sigma^\#(z), \pi_\sigma^\#(u)) = (xz^{\frac{\tau_0}{4}}, yz^{\frac{\tau_0}{4}+\frac{1}{2}}, z^{\frac{1}{2}}, uz^{\frac{1}{2}}).$$

Hence, in U_3,

$$\begin{aligned}
\pi_\sigma^\#(x^2 + y^2 + f(z,u)) &= \pi_\sigma^\#(x^2 + y^2 + f_{\tau_0}(z,u) + f_{>\tau_0}(z,u)) \\
&= x^2 z^{\frac{\tau_0}{2}} + y^2 z^{\frac{\tau_0}{2}+1} + f_{\tau_0}(z^{\frac{1}{2}}, uz^{\frac{1}{2}}) + f_{>\tau_0}(z^{\frac{1}{2}}, uz^{\frac{1}{2}}) \\
&= x^2 z^{\frac{\tau_0}{2}} + y^2 z^{\frac{\tau_0}{2}+1} + \prod_{t\in T}(a_t z^{\frac{1}{2}} + b_t u z^{\frac{1}{2}})^{m_t} + f_{>\tau_0}(z^{\frac{1}{2}}, uz^{\frac{1}{2}}) \\
&= z^{\frac{\tau_0}{2}}(x^2 + y^2 z + \prod_{t\in T}(a_t + b_t u)^{m_t} + \frac{f_{>\tau_0}(z^{\frac{1}{2}}, uz^{\frac{1}{2}})}{z^{\frac{\tau_0}{2}}}).
\end{aligned}$$

We obtain the strict transform of $X = (x^2 + y^2 + f(z,u) = 0) \subseteq \mathbb{C}/\frac{1}{2}(0,1,1,1)$ in U_3 is $(x^2 + y^2 z + \prod_{t\in T}(a_t + b_t u)^{m_t} + \frac{f_{>\tau_0}(z^{\frac{1}{2}}, uz^{\frac{1}{2}})}{z^{\frac{\tau_0}{2}}} = 0) \subseteq \mathbb{C}^4$, which contains some Gorenstein terminal singularities of type cD.

Here we compute $(\overline{X})_{\text{sing}}$ in U_3 by applying the Jacobian criterion to $(x^2 + y^2 z + \prod_{t\in T}(a_t + b_t u)^{m_t} + \frac{f_{>\tau_0}(z^{\frac{1}{2}}, uz^{\frac{1}{2}})}{z^{\frac{\tau_0}{2}}} = 0) \subseteq \mathbb{C}^4$. Let $F = x^2 + y^2 z + \prod_{t\in T}(a_t + b_t u)^{m_t} + \frac{f_{>\tau_0}(z^{\frac{1}{2}}, uz^{\frac{1}{2}})}{z^{\frac{\tau_0}{2}}}$, and we solve the equation $\mathrm{grad}(F) = (0,0,0,0)$ for (x,y,z,u).

$$\begin{aligned}
2x &= 0, \\
2yz &= 0, \\
y^2 + \frac{\partial}{\partial z}(\frac{f_{>\tau_0}(z^{\frac{1}{2}}, uz^{\frac{1}{2}})}{z^{\frac{\tau_0}{2}}}) &= 0, \\
\frac{\partial}{\partial u}(\prod_{t\in T}(a_t + b_t u)^{m_t} + \frac{f_{>\tau_0}(z^{\frac{1}{2}}, uz^{\frac{1}{2}})}{z^{\frac{\tau_0}{2}}}) &= 0.
\end{aligned}$$

Notice that we have $z = 0$ in U_3 because $(\overline{X})_{\text{sing}} \cap U_3 \subseteq E_{\text{sing}} \cap U_3$ by Proposition 7.3.1.

Some direct calculations show that $(\overline{X})_{\text{sing}}$ in U_3 is $\{(0,\beta_t,0,-\frac{a_t}{b_t})\}_{t\in T}\subseteq\mathbb{C}^4$ for some $\beta_t\in\mathbb{C}$.

Consider $h=x^2+y^2-(\frac{a_t}{b_t})^2z^2+u^2$ ($H=\mathrm{div}_X(h)$), and $g=x^2z^{\frac{\tau_0}{2}-1}+y^2z^{\tau_0}+u^2-(\frac{a_t}{b_t})^2$ ($G=\mathrm{div}_{\overline{X}}(g)$). We have

$$\begin{aligned}\pi_\sigma^{\#}(h)&=\pi_\sigma^{\#}(x^2+y^2-(\frac{a_t}{b_t})^2z^2+u^2)=x^2z^{\frac{\tau_0}{2}}+y^2z^{\frac{\tau_0}{2}+1}-(\frac{a_t}{b_t})^2z+u^2z)\\&=z(x^2z^{\frac{\tau_0}{2}-1}+y^2z^{\tau_0}-(\frac{a_t}{b_t})^2+u^2)=zg.\end{aligned}$$

Hence, we have

$$\pi_\sigma^*H=E+G.$$

Applying equation (#) to π_σ, we have $b=\mathrm{val}_E(h)=1$, so we have $\mathrm{val}_E(h)<\mathrm{val}_F(h)$ for any divisor F over X centered at any point in $\{(0,\beta_t,0,-\frac{a_t}{b_t})\}_{t\in T}$.

In U_4, we have an argument similar to the case in U_3. Therefore, val_E induces a Nash valuation of X.

Case 2: $f_{\tau_0}(z,u)$ is a perfect square

If $f_{\tau_0}(z,u)$ is a perfect square, say $f_{\tau_0}(z,u)=(p_{\frac{\tau_0}{2}}(z,u))^2$, we apply the changes of coordinates in [Hay99, 8.6]:

$$x\mapsto x\pm p_{\frac{\tau_0}{2}}(z,u),\quad y\mapsto y,\quad z\mapsto z,\quad u\mapsto u.$$

and (X,p) is rewritten as

$$(x^2\pm 2xp_{\frac{\tau_0}{2}}(z,u)+y^2+f_{\geq\tau_0+1}(z,u)=0)\subseteq\mathbb{C}^4/\frac{1}{2}(0,1,1,1).$$

By [Hay99, Theorem 8.8], we obtain two non-isomorphic divisorial contractions with minimal discrepancy (in this case the minimal discrepancy is $\frac{1}{2}$). We deal with case $X=(x^2+2xp_{\frac{\tau_0}{2}}(z,u)+y^2+f_{\geq\tau_0+1}(z,u)=0)$ here, and case $X=(x^2-2xp_{\frac{\tau_0}{2}}(z,u)+y^2+f_{\geq\tau_0+1}(z,u)=0)$ is similar.

From [Hay99, 8.2], consider the weighted blowing-up with respect to the following weight.

$$\sigma := \begin{cases} \frac{1}{2}(\frac{\tau_0}{2}+2, \frac{\tau_0}{2}+1, 1, 1), & \text{if } \frac{\tau_0}{2} \text{ is even,} \\ \frac{1}{2}(\frac{\tau_0}{2}+1, \frac{\tau_0}{2}+2, 1, 1), & \text{if } \frac{\tau_0}{2} \text{ is odd.} \end{cases}$$

We consider the first case $\sigma = \frac{1}{2}(\frac{\tau_0}{2}+2, \frac{\tau_0}{2}+1, 1, 1)$, and the second case $\frac{1}{2}(\frac{\tau_0}{2}+1, \frac{\tau_0}{2}+2, 1, 1)$ will follow to an analogous argument. The weighted blowing-up with respect to σ induces a divisorial contraction $\pi_\sigma : \overline{X} \to X$ with the exceptional prime divisor $E = \pi_\sigma^{-1}(p) \subseteq \overline{X}$ with minimal discrepancy (in this case $a(E, X) = \frac{1}{2}$) of (X, p) by [Hay99, Theorem 8.7].

$\overline{X}$ is non-singular on U_2. In U_1, we have that $\overline{X}$ is singular at $Q_1 \in U_1$ (see [Che16, Proposition 18 Case 2] and [Hay05, 5.2]), which is a terminal quotient singularity of the Gorenstein index $\frac{\tau_0}{2}+2$. We consider $g = x^{\frac{\tau_0}{2}+1} + y^2 x^{\frac{\tau_0}{2}} + z^2 + u^2$, and let $G = \mathrm{div}_{\overline{X}}(g) \subseteq \overline{X}$. Consider $h = x^2 + y^2 + z^2 + u^2$, and let $H = \mathrm{div}_X(h) \subseteq X$.

We compute $\pi_\sigma^* H$, and it is given by

$$(\pi_\sigma^\#(x), \pi_\sigma^\#(y), \pi_\sigma^\#(z), \pi_\sigma^\#(u)) = (x^{\frac{\tau_0}{4}+1}, yx^{\frac{\tau_0}{4}+\frac{1}{2}}, zx^{\frac{1}{2}}, ux^{\frac{1}{2}}).$$

Hence, we have

$$\begin{aligned} \pi_\sigma^\#(h) &= \pi_\sigma^\#(x^2 + y^2 + z^2 + u^2) = x^{\frac{\tau_0}{2}+2} + y^2 x^{\frac{\tau_0}{2}+1} + z^2 x + u^2 x \\ &= x(x^{\frac{\tau_0}{2}+1} + y^2 x^{\frac{\tau_0}{2}} + z^2 + u^2) = xg. \end{aligned}$$

We obtain

$$\pi_\sigma^* H = E + G.$$

Notice that $(\pi_\sigma)_*(G) = H$ (G is the strict transform of H), so we apply the equation (#) to π_σ and we have $b = \mathrm{val}_E(h) = 1$, so we have $\mathrm{val}_E(h) < \mathrm{val}_F(h)$ for any divisor F over X centered at $Q_1 \in U_1 \subseteq \overline{X}$.

We compute the strict transform of the defining equation of $\overline{X}$ in U_3, and it is given by

$$(\pi_\sigma^\#(x), \pi_\sigma^\#(y), \pi_\sigma^\#(z), \pi_\sigma^\#(u)) = (xz^{\frac{\tau_0}{4}+1}, yz^{\frac{\tau_0}{4}+\frac{1}{2}}, z^{\frac{1}{2}}, uz^{\frac{1}{2}}).$$

Hence, in U_3,

$$\begin{aligned}
&\pi_\sigma^\#(x^2 + 2xp_{\frac{\tau_0}{2}}(z,u) + y^2 + f_{>\tau_0+1}(z,u)) \\
&\quad = x^2 z^{\frac{\tau_0}{2}+2} + 2xz^{\frac{\tau_0}{4}+1} p_{\frac{\tau_0}{2}}(z^{\frac{1}{2}}, uz^{\frac{1}{2}}) + y^2 z^{\frac{\tau_0}{2}+1} + f_{>\tau_0+1}(z^{\frac{1}{2}}, uz^{\frac{1}{2}}) \\
&\quad = z^{\frac{\tau_0}{2}+1}(x^2 z + 2xp_{\frac{\tau_0}{2}}(1,u) + y^2 + \frac{f_{>\tau_0+1}(z^{\frac{1}{2}}, uz^{\frac{1}{2}})}{z^{\frac{\tau_0}{2}+1}}).
\end{aligned}$$

We obtain that the strict transform of the defining equation $(x^2+y^2+f(z,u)=0) \subseteq \mathbb{C}/\frac{1}{2}(0,1,1,1)$ in U_3 is $(x^2z + 2xp_{\frac{\tau_0}{2}}(1,u) + y^2 + \frac{f_{>\tau_0+1}(z^{\frac{1}{2}}, uz^{\frac{1}{2}})}{z^{\frac{\tau_0}{2}+1}} = 0) \subseteq \mathbb{C}^4$, which contains Gorenstein terminal singularities at worst type cD (it means either non-singular or of singular of type cD).

Here we compute $(\overline{X})_{\text{sing}}$ in U_3 by applying the Jacobian criterion to $(x^2z + 2xp_{\frac{\tau_0}{2}}(1,u) + y^2 + \frac{f_{>\tau_0+1}(z^{\frac{1}{2}}, uz^{\frac{1}{2}})}{z^{\frac{\tau_0}{2}+1}} = 0) \subseteq \mathbb{C}^4$. Let $F = x^2z + 2xp_{\frac{\tau_0}{2}}(1,u) + y^2 + \frac{f_{>\tau_0+1}(z^{\frac{1}{2}}, uz^{\frac{1}{2}})}{z^{\frac{\tau_0}{2}+1}}$, and we solve equation $\operatorname{grad}(F) = (0,0,0,0)$ for (x,y,z,u).

$$\begin{aligned}
2xz + 2p_{\frac{\tau_0}{2}}(1,u) &= 0, \\
2y &= 0, \\
x^2 + \frac{\partial}{\partial z}(\frac{f_{>\tau_0+1}(z^{\frac{1}{2}}, uz^{\frac{1}{2}})}{z^{\frac{\tau_0}{2}+1}}) &= 0, \\
\frac{\partial}{\partial u}(2xp_{\frac{\tau_0}{2}}(1,u) + \frac{f_{>\tau_0+1}(z^{\frac{1}{2}}, uz^{\frac{1}{2}})}{z^{\frac{\tau_0}{2}+1}}) &= 0.
\end{aligned}$$

Notice that we have $z = 0$ in U_3 because $(\overline{X})_{\text{sing}} \cap U_3 \subseteq E_{\text{sing}} \cap U_3$ by Proposition 7.3.1. Some direct calculations show that $(\overline{X})_{\text{sing}}$ in U_3 is $\{(a_t, 0, 0, b_t)\}_{t \in T}$ for some $a_t, b_t \in \mathbb{C}$. Consider $h = x^2 + y^2 - b_t^2 z^2 + u^2$ ($H = \operatorname{div}_X(h) \subseteq X$), and $g = x^2 z^{\frac{\tau_0}{2}-1} + y^2 z^{\tau_0} +$

$u^2 - b_t^2 \in \mathcal{O}_{\overline{X},(a_t,0,0,b_t)} \setminus \mathcal{I}_{E,(a_t,0,0,b_t)}$ ($G = \mathrm{div}_{\overline{X}}(g) \subseteq \overline{X}$). We have

$$\begin{aligned}\pi_\sigma^\#(h) = \pi_\sigma^\#(x^2 + y^2 - b_t^2 z^2 + u^2) &= x^2 z^{\frac{\tau_0}{2}} + y^2 z^{\frac{\tau_0}{2}+1} - b_t^2 z + u^2 z \\ &= z(x^2 z^{\frac{\tau_0}{2}-1} + y^2 z^{\tau_0} - b_t^2 + u^2) = zg.\end{aligned}$$

Hence we have

$$\pi_\sigma^* H = E + G.$$

Notice that $(\pi_\sigma)_*(G) = H$ (G is the strict transform of H), so we apply the equation (#) to π_σ and we have $b = \mathrm{val}_E(h) = 1$, so we have $\mathrm{val}_E(h) < \mathrm{val}_F(h)$ for any divisor F over X centered at any point in $\{(a_t, 0, 0, b_t)\}_{t \in T}$.

In U_4, we have an argument similar to the case in U_3. Therefore, val_E induces a Nash valuation.

Therefore, we prove that any exceptional prime divisor E with minimal discrepancy (here $a(E, X) = \frac{1}{2}$) induces a Nash valuation val_E over a 3-fold terminal singularity of type cAx/2.

Chapter 8

Nash Valuations over 3-Fold Gorenstein Terminal Singularities

Let (X, p) be a germ of 3-fold Gorenstein terminal singularity. In this final chapter, we establish a partial result in support of Conjectures A and B, and provide some examples as evidence (notice that Conjectures A and B are the same because the minimal discrepancy of a 3-fold Gorenstein terminal singularity is 1 by [Mar96, Theorem 0.1]).

8.1 Main Theorem

We follow the set-up developed in 7.2, then the following result follows directly.

Proposition 8.1.1. *Let (X, p) be a 3-fold Gorenstein terminal singularity, let $\pi : \overline{X} \to X$ be a divisorial contraction with minimal discrepancy (it is 1 in this case), and let $E = \pi^{-1}(p)$ be its exceptional prime divisor. Assume that the singularities of $\overline{X}$ are terminal cyclic quotient or Gorenstein terminal. Then there is an exceptional prime divisor over X*

with discrepancy 1 *that induces a Nash valuation. In other words,*

$$\{F \textit{ exceptional divisor over } X \mid a(F, X) = 1 \textit{ and } \mathrm{val}_F \textit{ is a Nash valuation of } X\} \neq \varnothing.$$

Proof. If val_E is Nash for X, then we are done. If not, then there is an exceptional prime divisor F over X such that val_F is Nash for X, and $\mathrm{val}_F <_X \mathrm{val}_E$ by Lemma 7.2.1.

Let $\phi : Y \to \overline{X}$ be a resolution (notice that F appears in Y since val_F is Nash, which is essential for X). Consider $\phi^* E = \tilde{E} + cF + \sum c_i F_i$ where $\tilde{E}$ denotes the strict transform of E via ϕ. By direct computations, we have

$$a(F, X) = a(F, \overline{X}) + c.$$

Notice that $a(F, X) \in \mathbb{Z}_{>0}$ since (X, p) is Gorenstein.

We claim that this valuation val_F has $a(F, X) = 1$. Since val_F is Nash for X (so val_F is essential for X), F must be centered at some terminal cyclic quotient singularity or some Gorenstein terminal singularity in $\overline{X}$.

If F is centered at any Gorenstein point $q \in \overline{X}$, we have $a(F, \overline{X}) \in \mathbb{Z}_{>0}$. This implies $c = a(F, X) - a(E, \overline{X}) \in \mathbb{Z}_{>0}$ (notice that $c \in \mathbb{Q}_{>0}$ since F is centered at $q \in \overline{X}$). Applying a similar argument as 7.2 to this scenario, we obtain

$$\mathrm{val}_F(h) = a a_1 + c\mathrm{val}_E(h) + d$$

for some $h \in \mathcal{O}_{X,p}$. Since $c \geq 1$, we get $\mathrm{val}_E(h) < \mathrm{val}_F(h)$, contradicting the fact $\mathrm{val}_E >_X \mathrm{val}_F$. Hence, F must be centered at some terminal cyclic quotient singularity in $\overline{X}$. We have

$$0 < a(F, \overline{X}) < 1$$

because this singularity has a resolution such that all exceptional prime divisors have discrepancies less than 1 (see [Hay99, Proposition 5.1], and the corresponding morphism is called *economic resolution*). In this case, we must have $a(F, X) = 1$ because $a(F, X) \geq 2$ implies

$$c = a(E, X) - a(E, \overline{X}) > 2 - 1 = 1,$$

which also provides a $h \in \mathcal{O}_{X,p}$ such that $\mathrm{val}_E(h) < \mathrm{val}_F(h)$. Therefore, there is an exceptional prime divisor over (X, p) with discrepancy equal to 1 such that it induces a Nash valuation over X. $\square$

8.2 Examples

Here we select some examples of 3-fold Gorenstein terminal singularities that satisfy the assumptions of the proposition 8.1.1.

Define $\tau(f) := \min\{j + k \mid z^j u^k \in \mathrm{supp}(f(z,u))$ as a monimial $\}$ for $f \in \mathbb{C}\{z, u\}$. Recall for a Gorenstein terminal singularity (X, p), it is called *type cE* is it has the following expression.

$$(x^2 + y^3 + yg(z,u) + h(z,u) = 0) \subseteq \mathbb{C}^4.$$

for some $g, h \in \mathbb{C}\{z, u\}$ with $\tau(g) \geq 3$ and $\tau(h) \geq 4$.

Furthermore,

(1) it is called type cE_6 if $\tau(h) = 4$ and $\tau(g) \geq 3$.

(2) it is called type cE_7 if $\tau(h) \geq 5$ and $\tau(g) = 3$.

(3) it is called type cE_8 if $\tau(h) = 5$ and $\tau(g) \geq 4$.

Example 8.2.1 ([Che15, Che16]). Let (X, p) be the Gorenstein terminal singularity of type cE_6 which satisfies the conditions in [Che16, Subcase 1-2 of Theorem 34]. After applying the change of variables mentioned in [Che16, Subcase 1-2 of Theorem 34] for (X, p), consider the weighted blowing-up $\pi_\sigma : \overline{X} \to X$ with the weight

$\sigma = (3, 2, 1, 1)$. By the argument in [Che16, Lemma 7 and Theorem 5], π_σ is a divisorial contraction with minimal discrepancy to X (in this case it is 1), and we obtain $(\overline{X})_{\mathrm{sing}} = [(\overline{X})_{\mathrm{sing}}]_{\mathrm{ind}>1} = \{Q_1, Q_2\}$ where Q_1, Q_2 are terminal quotient singularities of the Gorenstein index $3, 2$, respectively. Hence, there is a Nash valuation induced by some exceptional prime divisor over (X, p) with minimal discrepancy due to Proposition 8.1.1. In fact, there are exactly two divisors over (X, p) with discrepancy 1 (see [Che15, Level 1 in 5.4]), so we know that one of them should induce a Nash valuation over (X, p).

Example 8.2.2 ([Che16]). Let (X, p) the Gorenstein terminal singularity of type cE_6 which satisfies the conditions in [Che16, Case 3 of Theorem 34]. Consider the weighted blowing-up $\pi_\sigma : \overline{X} \to X$ with the weight $\sigma = (4, 3, 2, 1)$. By the argument in [Che16, Case 3 of Theorem 34], we have $[(\overline{X})_{\mathrm{sing}}]_{\mathrm{ind}=1}$ are at worst of type cE_6 and $[(\overline{X})_{\mathrm{sing}}]_{\mathrm{ind}>1}$ contains three cyclic quotient singularities. Hence, there is a Nash valuation induced by some exceptional prime divisor over (X, p) with minimal discrepancy due to Proposition 8.1.1.

There are other examples of 3-fold Gorenstein terminal singularities of type cE_7 and cE_8 that satisfy the assumptions in Proposition 8.1.1. See [Che16, Theorem 36 and Theorem 37].

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