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EVALUATION OF GNMA PASS-THROUGH SECURITIES

The University of Oklahoma PH.D. 1984

University
Microfilms
International 300 N. Zeeb Road, Ann Arbor, MI 48106

THE UNIVERSITY OF OKLAHOMA

GRADUATE COLLEGE

EVALUATION OF GNMA PASS-THROUGH SECURITIES

A DISSERTATION

SUBMITTED TO THE GRADUATE FACULTY

in partial fulfillment of the requirements for the

degree of

DOCTOR OF PHILOSOPHY

by

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Norman, Oklahoma

1984

EVALUATION OF GNMA

PASS-THROUGH SECURITIES

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TABLE OF CONTENTS

Evaluation of GNMA Pass-Through Securities

1. INTRODUCTION

The Government National Mortgage Association (GNMA) first issued modified pass-through securities backed by a pool of government-guaranteed mortgages in February, 1970. The purpose of this new security was to make Federal Housing Administration (FHA) and Veterans Administration (VA) Mortgage more marketable, and so less risky, resulting in a reduction in the spread between the yield on Treasury securities and government-guaranteed mortgage debt.

The new security improved the homogeneity of mortgage investments by guaranteeing the prompt payment of interest and principle in the case of a default. The eventual payment of interest and principle had been guaranteed on government-backed mortgages before the emergence of the GNMA modified pass-through securities. However, the investor would suffer a loss from a default when his claim for remaining principle was not satisfied promptly. This led to increased transaction costs, as the buyer of the mortgage had to evaluate each individual mortgage purchase. Black, Garbade, and Silber (1981) have shown that the objective of the GNMA modified pass-through security has, to a major extent, been achieved.

The GNMA modified pass-through program did not reduce another source of risk to the investor in mortgaged-backed securities: the mortgager is able to make non-scheduled repayment of principle. This necessitates the creation of an expected schedule of repayment for pools of mortgage based on the experience of lenders. The actual repayment behavior of borrowers will depart from this expected repayment schedule because principle repayments may be made earlier or later than expected. This leads to reinvestment and market risk as the pass-through holder will have to make unscheduled purchases or sales of securities at a future date at yields not yet determined. It is this problem that this paper addresses. The problem is made more complicated by the fact that individuals do not always act in their own economic best interests, at least as defined by a particular economic model. It is common observation that an individual will sell his home to another who does not wish to assume a below-market interest rate existing mortgage. Some individuals will make a suboptimal capital repayment, while others will not refinance a mortgage when it is profitable for them to do so.

GNMA modified pass-through securities have become a significant investment instrument since they were introduced in 1970. At the end of 1982 the total nominal value of GNMA pools was $118,940,000,000. While this is only 8.2% of the total mortgage debt outstanding, it does represent a substantial amount in absolute terms. The ability to evaluate callable bonds where the holder of the call acts irrationally is clearly of economic importance. The theory in this paper has been developed under the assumption that there is no interruption of the cash flow to the investor when a default occurs as in a GNMA. The model may also be used to evaluate non-GNMA mortgage pools and trusts guaranteed by other governmental agencies, ($97,714,000,000) and the substantial portfolios of commercial banks ($301,742,000,000), mutual savings banks ($94,452,000,000), savings and loans ($482,414,000,000), and life insurance companies ($141,919,000,000), under the assumption that the cash flow delay resulting from a default is insignificant, and that the mortgages are adequately insured against default. (Numerical Data from the Federal Reserve Bulletin (F.R.B.), June 1983).

This paper deals primarily with the valuation of traditional GNMA pass-through securities. In developing the theory, a model is derived that will evaluate any default-free bond. The newer non-fixed rate mortgages may also be evaluated using the theory developed here. For instance, if the mortgage has an interest rate that is adjustable every six months to some interest rate that is a function of a short-term interest rate, then the specification of the boundary and policy conditions become somewhat harder, but this increase in difficulty is of a bookkeeping nature rather than requiring additional theory.

Some Federal agencies actually hold mortgages themselves, and issue their own bonds in the market. For instance, the Federal National Mortgage Association holds $71,814,000,000 of mortgages and the Federal Land Banks hold $50,350,000,000. The theory developed may be used to determine under what terms they should buy mortgages and under what terms they should issue bonds to the public to achieve particular risk-related goals. (FRB, June 1983)

The model developed has the following advantages:

1. The irrational behavior of holders of options is correctly analyzed.

2. A multi-dimensional interest rate process is used with a functional form that should be flexible enough to adequately model the empirical process.

3. The state vector is in terms of bond prices rather than interest rates.

This draws attention to the importance of the coupon rate on the bond being used to generate the interest rate observations. This formulation also permits direct numerical solution without truncation and transformation.

4. The market price of risk does not appear in the final model.

5. The drift term of the generating stochastic process does not appear in the final model.

6. The instantaneous rate of interest does not appear in the model. All the state variables are then relevant to the problem being analyzed.

7. A flexible, efficient and theoretically correct method for estimating the parameters of the interest rate process is provided.

The stochastic calculus is not easy to learn without some strong motivation and good textbooks. Some of the textbooks used are listed in the bibliography. The most useful ones found were the ones authored by Gihman and Skorohod referred to in the bibliography. Two new textbooks designed for the finance community have recently been published, "Stochastic Methods in Economics and Finance" by A.G. Malliaris contains a wealth of relevant theorms together with some applications. The second book is "Option Pricing, Theory and Applications" edited by Menachem Brenner, and contains a number of papers with a stress on application.

1.1. Historical Development

The most widely used model for evaluation of mortgage-backed securities is to treat the security as if it were a regular bond without a call feature. An early repayment date is assumed, the most common being 12 years for mortgages with a remaining scheduled life substantially longer than 12 years. The origin of this model has not been determined.

While this model does have simplicity, it ignores the fact that yields-to-maturity are stochastic and that the exercise of the call feature on a mortgage will depend upon interest rates. It also ignores the fact that each mortgage pool appears to have its own rate of early repayments, and that using a measure of only the distribution of early repayment's central tendency will introduce substantial bias into the model. The Financial Publishing Company's (F.P.C.) method of valuation of GNMA pass-through securities does not use a single expected termination date (F.P.C. Publication 715). Rather it uses the expected proportion of mortgages that will be retired each year based on FHA experience of all insured mortgages. Each mortgage pool's history of early retirement is compared with the history of early retirement on all pools. Each pool is then assigned a percentage history of FHA experience." On the assumption that the percentage history of FHA experience will continue on each pool, a schedule of the expected proportion of early retirements from the pool may be calculated. The pass-through security is then evaluated using conventional bond valuation theory by treating the security as if it were a portfolio of regular bonds. This method does have the advantages of also modeling the distribution of early repayment by thirty probability densities rather than a single measure of central tendency.

The Curley and Guttentag (1974) (1977) model attempted to determine the annual termination rate of mortgages in year t as a function of policy year, maturity, contract rate prevailing during year t, the discount prevailing in year t and the contract rate on mortgages in the cohort. Their paper has the advantage that an attempt is made to make explicit the determinates of the termination probabilities of individual pools of mortgages rather than relying on the historical percentage of FHA experience. A regression analysis was used to estimate the distribution of early repayments. A major problem with this method is that at an earlier time, we do not know what the prevailing discount rate and contract rate will be in a later year. The model is then only useful, without substantial extensions, in conditional valuation.

In order to value a mortgage satisfactorily, the call feature must be treated correctly. The only satisfactory way of modeling options is through the use of contingent claims theory. Vasicek (1977) and Brennan and Schwartz (1977) models provide us with contingent claims models for the valuation of bonds. Both of these models had a single state vector, the short-term rate of interest.

The Brennan and Schwartz paper was developed under the assumption that the value of any bond is a function of only the instantaneous spot rate of interest and the term to maturity, and that the expectations hypothesis regarding the future interest rates holds. As a result of this final assumption, the market price of risk becomes identically zero. There is no question that investors will require a premium for carrying risk. The Vasicek model includes the market price of risk but is otherwise very similar. The Brennan and Schwartz paper develops the boundary conditions to be placed on the partial differential equation that describes the value of any bond. The resulting model enables us to place a value on a wide range of bonds with put and call features.

The major disadvantage of the above models, even if the better features of each are assembled together in a single model, are:

1. The state vector contains only the instantaneous rate of interest, and this does not adequately describe the behavior of long-term bonds.

2. The functional form of the market price of risk has not been determined.

3. The functional form given to the interest rate

process is too simple.

4. The model will evaluate only those bonds that have option features that are exercised in an optimal manner.

In a later paper by Brennan and Schwartz (1980), a model is developed that has a state vector that contains two processes: the spot instantaneous rate of interest on a risk-free bond, and the spot rate on a government consol bond. This model was found by the authors to be inadequate, and they recommended that a three state vector model should be developed. Their model was, in fact, misspecified and this may have the cause of the poor empirical fit they observed. At the stage where they attempt to eliminate one of the two market prices of risk functions, they treat the two dimensional process as if it were a single dimensional process. The result is that a covariance is excluded from their analysis. The two dimensional model developed retains the market price of risk function associated with the instantaneous rate of interest.

The problems with this model are that

1. The model is not developed in a sufficiently general framework that permits higher dimensional processes to be used as the state vector.

2. The functional form given to the interest rate process is more flexible but is still too simple.

3. The market price of risk is retained in the model.

4. The boundary conditions of the resulting valuation partial differential equation are not analyzed in any detail.

5. The irrational exercise of any option features is not analyzed.

6. The drift term of the generating process is retained in the model.

Cox, Ingersoll, and Ross (1981) provide a multi-dimensional model of the bond process. Their development is at a somewhat more abstract level. The final model is a partial differential equation that includes a vector of market price of risk functions of the same dimension as the state vector. The primary purpose of that paper was not to produce a bond valuation model, but rather to examine the consistency of various term structure of interest theories within a contingent claim framework. It is not criticism of their paper to indicate that the coupon on the bonds used to generate the state vector is implicitly set to zero. Their theory then applies to pure discount bonds but could be extended to more general types of bonds with comparative ease.

The fact that the rate of interest on any bond used as a component of the state vector will be dependent on the coupon rate on that bond requires addressing. Clearly, the coupon rate will affect the duration of the bond, and hence, the interest rate of the bond itself. In the Brennan and Schwartz models, the coupon on the bonds that generate the interest rate process is irrelevant because of the structure of their models. The very short-term rate and the consol rate are limiting cases of the more general one where the interest rate is generated by a bond with finite life and pays a coupon. This problem is addressed in the model developed in this paper.

One type of bond which has an option feature which is exercised irrationally is analyzed by Dunn and McConnell (1981). The bond they analyze is a GNMA pass-through security. Their model is based on a one dimensional interest rate process that uses the spot instantaneous risk-free rate of interest as the state variable. What makes their model interesting is the introduction of a jump or poisson process to describe early retirement of the underlying mortgage. The probability of early refinancing is made a function of the interest rate process and the remaining term to maturity on the mortgage. Their model has several serious faults:

1. The spot instantaneous interest rate is used as the only state variable. Mortgages are essentially long-term securities, and one would rather use a longer term interest rate to describe the world if only one state variable is used.

2. While it is possible to generalize the Dunn and McConnell model to make the interest rate process multi-dimensional, it would still contain a serious misspecification. If the optimal date to refinance is t*, then there are two categories of suboptimal refinancing of the underlying mortgage: The householder might refinance before t* or the householder might refinance after t*. The former behavior is modeled by the Dunn and McConnell model, but the latter is not. The problem is very real because the householder does not have access to a method by which he may value a (rational) call bond. A mortgage is just a callable bond issued by the householder. Householders are likely to act in an irrational manner when exercising their call

option through ignorance.

3. Obtaining a functional form for the parameter of the jump process used to determine suboptimal refinancing is not addressed.

4. The market price of risk function is given a functional form on the basis of its being reasonable despite the fact that a market price of risk is implicit in the model itself. This will lead to possible model inconsistency.

5. The drift term of the generating function is given a functional form, but in fact should not occur at all in a correctly developed model. This will also lead to possible model inconsistency.

6. Given that a closed form solution was not obtained, there is no reason why the interest rate process could not have been specified in a more general manner and the parameters estimated. The amount of computational work involved would not increase substantially.

7. A factor in determining whether a householder will refinance or not at any particular time is the front-end financing costs of obtaining a new mortgage (points) and the third party transaction costs. These factors are not

considered in the Dunn and McConnell model.

This paper develops a multi-dimensional interest rate model which does not include a residual market price of risk function. It treats both kinds of irrational behavior involved in the refinancing decision. Transactional costs are included in the model. The interest rate process is be specified in such a manner that it is flexible enough to accommodate any reasonable interest rate process. In addition, a method of estimating the parameters of the interest rate process is described.

II. DEVELOPMENT OF A NEW GENERAL THEORY OF BOND EVALUATION.

The purpose of this chapter is to develop a contingent claims theory for irrational bonds subject to transaction costs.

Let us first define an irrational bond. All irrational bonds contain an option feature. The controller of the option feature may not act in accordance to a rational economic model that appears to contain all the variables necessary to come to an optimal decision on when to exercise his rights. If this occurs, then the bond is said to be irrational. A bond may be irrational because the controller of the option feature does not understand his own best interest, or it may be that the model does not take into account all the variables necessary to make a correct decision.

An example of an irrational bond is a callable bond. The company that issued the bond may call the bond at an earlier time than indicated by the model. There could be tax or cash flow considerations not included in the "rational" model. In addition, the treasurer may falsely believe that he has correct intuitions about the future direction of interest rates. Action based on this false belief would lead to a suboptimal call, making the bond an irrational bond.

A second example of an irrational bond is the home mortgage. The home mortgage is a callable bond. The issuer of the bond (the house owner), does not have access to capital markets, has objectives that are less easy to analyze, and is somewhat less sophisticated than an industrial or governmental borrower. One may then expect mortgages to be irrational bonds.

The author's basic objective is to take a step towards developing a model that may be used with confidence to evaluate all bonds. The theoretical concepts could have been illustrated with somewhat simpler models than those developed, and their functional forms could have been left open. While this would have made the paper marginally easier to read, it would have also provided a model of less utility.

Where it was found practical, only quantities that are measurable are included in the model. The instantaneous rate of interest has not been shown in the literature to be measured by any of the surrogates used to measure it. It has therefore been eliminated from the model. Where a quality may be measured by either a quantity which may be directly measured or by some transformation of the measurable quantity, then the directly measurable quantity has been used in the model. An example of this is our use of the bond-price-process rather than the yield-to-maturity in the state vector.

There is no chance of being able to obtain a closed form solution to a sufficiently interesting model. The final model is a system of partial differential equations. Excepting certain specific cases there is no known analytical solution to a particular partial differential equation together with its boundary conditions. If a numerical solution must be obtained, it then makes little difference to the modeler whether the system is small and simple, or large and complex. In this paper the system was allowed to grow quite large. The size of the model may be reduced at the parameter estimation stage.

A parallel approach was taken to that sometimes taken when modeling using regression analysis. The exact functional form may not be known, but one does have some idea of what the functional form should look like. A functional form is then postulated but with a sufficient number of parameters left open to be estimated later from the data. The result is a model with a substantially larger number of parameters than traditionally found. One must clearly avoid the many problems that are associated with over-parameterization of models.

The final model does have a certain beauty which may not be directly apparent. Given the complexity of what is being done, the model is extraordinarily simple. For example, there is no need to determine the functional forms of the market price of risk or the drift term, as they are eliminated from the final equation. The elimination is entire and does not require a complicated substitution as seen in the earlier models reviewed in the last chapter. (The reason why the market price of risk and the drift term can be eliminated entirely is because bond processes rather than interest rate processes are used as state variables).

Having created a model with a large number of open parameters, it becomes necessary to find a way to estimate them. This chapter provides three methods by which parameters may be estimated. The first has been used in communications theory. The second is, to the author's knowledge, new. The third was independently developed by the author but was later found to be a generalization of a method used to estimate ordinary differential equations in the biological sciences. It is believed that the estimation schemes are important contributions to contingent claims theory.

2.1. The Interest Rate Process

This section develops a model of the bond process. 2.1.1. discusses the quantities that should be included in the state vector. 2.1.2. develops the equations that may be used with appropriate constraint equation to evaluate any type of rational bond. 2.1.3. suggests the functional form of an interest rate process.

2.1.1. The State Vector

The most common components of the state vector used in the bond valuation literature are interest rates. The interest rate process is one example of a class of processes we shall call bond processes. The advantage of using interest rates alone in the state vector is that the time between observations may be made very short. The major objection to using a pure interest rate state vector is that the relationships between interest rates and macro-economic theory are not incorporated in the model.

The major disadvantage of a macro-economic based model is that the observations may be sampled with a much lower frequency when compared with the observation frequency of the interest rate process. This is because interest rates are obtainable on an hourly basis, whereas other possible state variables are often reported only on a monthly basis. In addition, the observation error is probably substantially less when measuring bond prices (and so indirectly, yields-to-maturity) in comparison with the error in other macro-economic variables such as inflation, savings, capital formation or consumption.

Having decided to use a pure bond process as the state vector, the next step is to determine what measures of the bond process should be used in the state vector. The two most obvious are the bond prices and the yields-to-maturity of selected bonds.

When a bond is bought or sold, what is observed is a price. A transformation of a current bond price is the current market yield-to-maturity. This yield-to-maturity has the advantage of providing a standardized, easy to understand, index. As there is a simple transformation from a pure discount bond's price to its yield-to-maturity, there is little to choose between them from a theoretical vantage.

The major advantages that bond prices have over yield-to-maturities are that the boundary conditions of the derived partial differential equations may be set so that no approximation nor unnatural transformation is necessary, the state vector is directly observable, and that bonds other than special cases may be used in forming the state vector because the coupons may be treated explicitly in the model's derivation. Another major advantage is that the instantaneous drift function of the process may be shown to be irrelevant and so need not be specified. These advantages are significant and override the disadvantage that the yield-to-maturity has traditionally been used in the literature.

The relevant transformations between price and yield-to-maturity on a pure discount bond are

$$P = P^{*}(1+r)^{-N}$$

$$P = P^{*}e^{-r'N}$$

where N is the remaining life of the standard bond in years, P^{*} is the par value of the standard bond, P is the current price of the bond, r is the yield-to-maturity, and r' is the continuous yield-to-maturity.

It should be noticed that neither transformation has an inverse when N = 0. The yield-to-maturity of a bond that is maturing must be described in terms of a limiting process as N approaches 0. The instantaneous interest rate on a period basis, r_0, and continuous basis, r'_0 are then defined to be:

$$r_0 = \lim_{N \to 0} (P/P^*)^{-1/N} - 1$$

$$r'_0 = \lim_{N \to 0} (((\ln(P^*/P))/N)$$

Some serious problems however remain:

The first problem has to do with transaction costs. The shorter the time to maturity, the more important transaction cost become. For very short maturities, the underlying process is masked by the transaction costs involved, making direct observation of the process impractical.

A second problem has to do with the limiting behavior of the instantaneous interest rate. Because of the problems connected with observing an instantaneous interest rate, one might be tempted to select a very short-term rate as a surrogate for the instantaneous rate of return. However, the yield curve suggests that yields-to-maturity typically are very sensitive to maturity as the maturity becomes shorter. Whether the overnight federal funds rate, or the rate on short-term repurchase agreements, are good surrogates for the instantaneous rate of return should have been investigated before being used in valuation models found in the literature. A related problem is whether federal funds and repurchase agreements are in the same default and risk class as Treasury bills, and if not, what impact does the risk premium have on their yield-to-maturity.

We are then left with the choice of using a state variable, r, that cannot be directly observed or selecting a short-term bond measure that may be observed. If the unobservable instantaneous interest rate is retained as a variable, and bond prices are used for the other state variables, then this would destroy the symmetry of the model. There is no bond price that corresponds to the instantaneous rate of interest r_0 (or r_0') that may serve as a state variable, as the value of such a bond would be non-stochastic in the limit as $N \to \infty$. Accordingly, the bond price processes used in the state vector will have a maturity greater than zero, but finite. Consuls have been excluded because they are not available to investors in U.S. government securities and would detract from the formulation of a model which is both powerful and may be expressed in a compact notation.

The problem of using the instantaneous interest rate has not been addressed in the literature, yet all received contingent claim models contain the instantaneous risk-free rate of interest. In some models, such as some versions of the Black-Scholes option pricing model, it is non-stochastic, and in others, such as the Brennan and Schwartz bond pricing models, it is stochastic. If one is willing to select a particular observable short-term interest rate, and use it as the instantaneous risk-free rate of interest, then the problem is assumed away. There is no received empirical study that supports such an assumption.

One of the advantages of the procedure defined in this paper is that the instantaneous risk-free interest rate, r_0, is not used. This advantage is purchased with the disadvantage that the development of the model is somewhat more complicated. However, this disadvantage is outweighed by the advantage that the final model does not contain the drift function and the market price of risk function, resulting in a final model which is simpler than it would have been.

The next thing to be determined is what type of bonds should be used in the state vector. The securities that are being evaluated are all default free. We shall then restrict our attention to Federal Government bonds. Different government bonds are subject to different tax provisions. Thus, some components of a bond returns are subject to long-term capital gain provisions, while other returns are subject to short-term capital gains and ordinary income provisions. This differential treatment of returns for tax purposes results in different yield curves for otherwise similar classes of securities. Securities that are trading at a large premium or discount to par will also have different yield curves partly due to the tax clientele effect and partly due to durational effects.

The tax effect on bond prices is excluded from the present model as its inclusion would make the model much more complicated. Should the yield curve be seriously discontinuous for bonds with maturities of a about a year then the model will have to be respecified to include taxes.

Bonds pay interest at discrete times and not continuously. The mathematics becomes intractable if we consider real bonds. Thus, the bonds we consider pay their interest continuously. This should be considered when collection of data is undertaken, as the basis bonds must be adjusted for payment of dividends.

Long-term Government bonds tend to carry a call provision. This makes such bonds less valuable to investors, and hence they demand higher yields on them. It would be very difficult to use such securities as state variables. We are thus restricted to a wide range of securities with maturities between one day and about ten years as our state variables and a narrow range of securities with lives greater than ten years. This should not be too serious a constraint because the majority of the characteristic shape of the yield curve occurs within the former range.

The final consideration is the number of state variables to include. While it makes little mathematical difference (one just moves into vector notation when the scalar equations become cumbersome), it is better to use as small a number of state variables as adequately describe the system. Traditionally, one or two state variables have been used. Three state variables should be more than adequate, but the theory is developed in general.

To summarize, we have made the following restrictions on the state variables of the mode being developed:

1. Bond prices alone are used in the state vector. The bonds used as state variables are not consols, are not repaid in the next instant of time, and do not carry option features.

2. Taxes are not treated.

3. Bonds pay interest continuously and not at discrete times.

4. There is no limit on the number of state variables to be included in the model, but a reasonable number is two or three.

2.1.2. Model Development

The reader is referred to the appendix to Chapter 2 where a one dimensional model and a two dimensional model are developed if contingent claim analysis is new to him before this section is read. The one dimensional model was developed after the following section was developed, and may prove to be clearer on two grounds. First, the algebra is simpler, as it is in one dimension. Second, some new terminology is introduced that, once understood, makes the development less verbal. The old notation has been retained here because it provides an alternative to that found in the appendix.

2.1.2.1. General Comments

In this section, an n-dimensional state variable model is developed, where the state vector contains only continuous coupon bond processes, for the valuation of any type of risk-free bond not containing conversion privileges to equities or commodities.

A bond may be defined as a security with a deterministic payout function $k(t)$. The payout function $k(t)$ is the remaining undiscounted total cash flow from the bond from any time t to ∞ .

The value of a bond may be expressed in terms of the basis bond state vector $P(t)$ and the term to maturity τ. Thus, we have that:

$\beta(P,\tau)$

There are some technical advantages in using bond price processes rather than interest rate processes. A discussion on the issues involved is to be found in Section 2.1.1. We shall now define the basis bond price process P.

A basis bond price process, P, may be defined as a vector of bond prices $[P_i]$ that adequately describes the price of all bonds of interest to us. Each component, P_i of the stochastic state vector, P, is not itself a bond. Rather it is the derived process defined below.

Let us consider bonds with finite maturities. We further restrict the bonds examined to those with cash flows from t to T that are identical except for a time shift. That is, the set of all bonds at time s that have a payout function k_s such that for all t, 0 < = t < T:

$$k_s(s+t) = k_0(t).$$

. Such a set of bonds is searched for a particular maturity τ, and the price of that security $B_\tau(t)$ at time t is defined to be $P_i(t)$.

One may object that in practice $P_\tau(t)$ will only be rarely observed because there will only rarely be a satisfactory $B_\tau(t)$ in the set of all physical bonds. This objection is valid, but a good measure of $P_\tau(t)$ may be found using interpolation. More advanced and theoretically sound methods are available to measure the $P_i(t)$ indirectly through continously observable bonds. Bond indices might also be considered if due consideration is taken of the representative cash flow of the underlying bonds.

An important distinction between the physical bond process and the basis bond price process is that the trajectory of the former will drift towards the terminal value, while the trajectory of the latter will not do so. A basis bond is a theoretical construct designed to always have the same remaining time to maturity, whatever the current time. The remaining time to maturity on a physical bond will become shorter as time proceeds.

It would have been operationally simpler to work with state vectors which were physical bonds. The problem would arise of how, when bonds mature, to phase in and out new real bonds being used as the basis set without causing disruption. One could however produce a set of equations expressing the basis bond price process in terms of a more observable bond price process.

In the next subsection a partial differential equation is obtained that may be used to evaluate any security that is a function of maturity and our state vector only. The derivation is somewhat novel in that the instantaneous risk-free rate of interest is not one of the state variables and the state variables are bond processes.

2.1.2.2. Mathematical Development: n-Dimensional State Vector

A very straightforward generalization of Appendix B to 2.1, equation [10], results in the valuation equations:

$$-B_{\tau} + \frac{1}{2}\,\mathrm{tr}\,(\beta\beta^{t}B_{pp}) + c - B_{p}(C-G_{\tau}) - r(B-B_{p}P) = 0\,. \quad [1]$$

$$C = (C_1,\ldots,C_n)^{\tau}\,,\; G_{\tau} = (G_{1\tau}\ldots,G_{n\tau})^{\tau}.$$

The subscripts, other than i's refere to partial differentials. The i's refere to the ith state vector. B is the security being evaluated and is a function of the state vector P and maturity, τ, only. c is the coupon rate on the security B, C is the vector of coupon rates associated with the security vector B and G_i is the current bond that corresponds to the ith component of the state vector.

It is necessary that the bond processes making up the state vector have τ_i that are different from one another if the

k_i are of each bond truely different. To be truely different from one another, a bond's cash flow must be different from a linear combination of the cash flows of the other bonds. We shall later obtain a term structure of interest equation. For example, three bonds each with a maturity of 3 years, each paying a par value of $1,000 at maturity, but the first paying no coupon (pure discount bond), the second paying $100 per year (continuously), and the third paying a continuous coupon at instantaneous annual rate of τ x $100, would be sufficient to obtain a term structure of interest. To obtain insight into this possibly counter-intuitive result, one may think of a coupon bearing bond as being made up of possibly an infinite number of component "bondlings". Each bondling is a pure discount bond which, when summed with the correct weight, gives a cash flow equivalent to the bond's. The fact that each bondling has its own maturity accounts for the time structure of interest being contained in the theory.

Equation [i] is independent of the drift term of the generating process and also of investor preferences. The bond valuation model then enjoys the same properties enjoyed by the Black-Scholes Model. This is an important result.

The specification of drift and market price of risk functions may be the cause of the poor empirical results obtained using the Brennan and Schwartz [1979] model.

A new state variable r, has been introduced into the system. For the reasons discussed earlier, r is not to be included as a state variable. A method of eliminating it is required. This element of the paper accomplishes this objective.

The short-term rate of interest, r, may be defined in terms of maturing risk-free pure discount bonds:

$$r = \lim_{\tau \to 0} \left(\frac{\ln(B_0) - \ln(B_\tau)}{\tau} \right)$$

Note that the limit is determined by the functional form of

β_τ, which is a function of τ. As $\tau \to 0$, $B_0/B_\tau \to 1$, so we are unable to make any *a priori* judgements about the value of this limit. Assuming that such a limit exists, then for a sufficiently small τ^*, a further reduction in τ^* will have an insignificant effect on the value of r. (The behavior of the limit will be investigated further in a subsequent paper. However, as B_τ is given by the partial differential equation [1], the problem is non-trivial). Let the par value of a maturing pure discount bond, B^* be $B^*_0 = 1$.

Then we may write for this bond with maturity τ^*:

$$r = - \ell n(B_{\tau}{}^*)/\tau^* \qquad [2]$$

For notational simplicity, write B_{τ^*} as B^*. Then for any bond, its value B is obtained by solving the following set of simultaneous equations:

$$-B^*_{\tau} + \frac{1}{2}\mathrm{tr}\ (\beta\beta^T\ B^*_{pp}) + -B^*_{p} \cdot C + B^*_{p} \cdot G^*_{r} -$$

$$r^*(B^* - P\ B_p) = 0 \qquad [3]$$

$$-B_{\tau} + \frac{1}{2}\ \mathrm{tr}\ (\beta\beta^T\ B_{pp}) + c\text{-}B_{p} \cdot C + C_{p} \cdot G_{\tau} -$$

$$r^*(B - p\ B_p) = 0$$

$$r^* = - \ell n(B^*(P,\tau^*))/\tau^*$$

The first equation gives the value of a bond that has a maturity of τ^*, where τ^* is sufficiently small, and the second evaluates the value of any bond given the value of the first bond. The instantaneous rate of interest is thus not a state variable in this model.

This system will require initial and boundary conditions on B* and B. These are discussed later in this paper. One should note however that the conditions on B will depend upon the specific bond being evaluated, but the conditions on B* are standard.

2.1.3. Function Specification

Equation 3.1.2.2 [3] provides us with a means to value any kind of bond, given the basic bond process is of the form

$$dP = \alpha(P)dt + \beta(P)\ dz(t) \qquad [1]$$

$$\beta(P) = \beta^{*}\ \psi^{1/2}$$

$$dz(t) = \psi^{-1/2}\ du(t)$$

The α, β, β^* and ψ have not been given any form. It is the purpose of this section to specify the form of these entities.

The specification of the functional form of the model is somewhat subjective. There are so many factors that may or may not be important. Yet all possible factors cannot be included. As there is almost no literature on the theoretical form of the interest rate transitional probabilities, the approach to be taken in this paper is that the selected model's functional form will include all the important features of the more common models as approximations. The resulting model will be a substantial super-set of approximations to existing models. The increased generality will be at the cost of a substantial number of parameters that will have to be estimated.

The process we develop should have certain properties. Clearly, the value of a bond may never be negative, and may never be greater than the undiscounted remaining cash flow, k(t). Elastic random walks are often used to describe the behavior of the implied yield-to-maturity, whereby interest rates are pulled towards some ideal value. The ideal value will depend to some extent upon the yield-to-maturity of adjacent basis bonds. However, as we have been able to eliminate the drift form from our model, there is little point in providing it with a functional form. An exception to this would be if we were to estimate the parameters from the generating process itself. This paper will provide an estimation procedure through the final model.

Linking the behavior of the long- and short-term interest rates is important in order that the system models observed behavior. If the long-term interest rate rises, the short-term rates will tend to also. There are two natural ways of introducing this cross linking of interest rates into the system. The first involves the use of the "force" term and the second the noise term.

In their paper on a two- dimensional process, Brennan and Schwartz (1979) used the "drift" term to cross-link the interest rates. They postulate that one of the yields-to-maturity is being pulled towards the other yield-to-maturity. In our model the drift term has no influence whatsoever. The cross linking behavior must be modeled in a different manner.

The other way of linking the various interest rates is to use the noise term. Consider the two independent processes dz_1 and dz_2 defined in 2.1.2.2. The short-term bond process and the long-term bond process are both functions of these independent processes. Thus the cross-linkage is already an integral part of the model. Correct specification of the functional form of $\beta(P)$ will provide the necessary cross-linkage. It is not an easy matter to observe ψ, so we shall work with β rather than β^*.

The form of β defined in the appendix to this chapter is adequate for our purposes, and will be adopted here. Note that the noise functions associated with the change in the state variable P_i depend only upon that state vector. This restriction may be removed if appropriate.

The final model then is a straightforward generalization of the one on and two dimensional models given in the appendix to this section. Subject to consistent and appropriate initial, boundary and policy conditions, we have the model:

$$-B_\tau + \tfrac{1}{2}tr\,(\beta\beta^T B_{pp}) + c - B_p \cdot C + B_p \cdot G_\tau - r^*(B - P\,B_p) = 0 \qquad [2]$$

$$r^* = -\ln(B^*(P,\tau^*))/\tau^*$$

$$\beta_{ij} = a_{ij}\,\left(1 - \frac{P_i}{K_i}\right)^{b_{ij}}\left(\frac{P_i}{K_i}\right)^{d_{ij}}$$

where B^* also satisfies the first equation. The natural boundary conditions are just

$$-B_{\tau}(K,\tau) + c - B_p(\quad C \quad - G \; _{\tau}) = 0$$

$$B(\underline{0},\tau) \quad - \quad P \quad B_p \quad = 0$$

It should be observed that these conditions are a direct consequence of the differential equation itself and no additional assumptions are required to obtain them.

The above restraints may be simplified to the equivalent form:

$$-B(K,\tau) + c = 0$$

$$B(\underline{0},\tau) = 0$$

2.1.4. Specific Restrictive Conditions on a Callable Bond.

A mortgage is just a callable bond. The only major difference between the more common callable bonds and a standard mortgage is that the former is more likely to have its call feature exercised rationally, and that the residual value of a mortgage at the end of its life is zero.

We have already seen that the valuation equation must satisfy certain natural boundary conditions. The forms given below do so.

The initial conditions are just:

$B(P,0) \;= 0 \quad , \quad 0 < P < \underline{K},$

$B^{*}(P,0) = 1 \quad , \quad 0 < P < \underline{K}.$

These conditions are straightforward and just reflect the fact that the value of a mortgage at the end of its life is zero, while the value of a pure discount bond at the end of its life is (in our case) one dollar.

The left-hand boundary conditions area:

$B(\underline{0},\ \tau) = 0, \quad 0 < \tau$

$B^{*}(0,\tau) = 0, \quad 0 < \tau$

In words, the value of any bond must be zero when the interest rate is infinite, or which is the same thing when the bond process is zero. Note that there is a discontinuity between this boundary condition and the initial condition in the case of the pure discount bond.

The right hand boundary conditions are

$B(\underline{k},\ \tau) \;= c\tau$

$B^{*}(\underline{k}$This isthe same thing as saying that the value of a mortgage is just the remaining scheduled cash flow if the interest rate is zero (ie. P=K), and that the value of a pure discount bond is equal its remaining cash flow. These boundary conditions hold if there is no call provision. The conditions require modification if there is a call provision. We must then examine the policy conditions.

The policy conditions on a mortgage are that, if the value of the mortgage bond rises above its remaining principle Q, then the call will be exercised. That is, if B^+ represents the value of a mortgage with a call feature (a real mortgage), and B represents an otherwise identical mortgage without a call feature, then,

$$B^+ = \mathrm{MIN}[Q,\ B]$$
$$Q = \frac{c}{s'}\left(1 - e^{-s'\tau}\right)$$

where s' is obtained from solving

$$\frac{c}{s'}\left(1 - e^{-sN}\right) = 1,$$

and c is the coupon and N is the original life of the mortgage. Details are provided in Appendix A to Chapter 2.

Note that the policy conditions also apply to the boundary conditions. Thus, to evaluate a mortgage, use the equation 2.1.3 [2] to evaluate a bond, together with the conditions:

$$B\ (P,0) = 0,$$

$$B^*(P,0) = 1,$$

$$B\ (\underline{0},\tau) = 0,$$

$$B^*(\underline{0},\tau) = 0, \qquad 0 \neq \tau,$$

$$B(\underline{K},\tau) = c\tau$$

$$B^*(\underline{K},\tau) = 1$$

2.2.1. APPENDIX A TO 2.1: A BOND VALUATION MODEL WITH A SINGLE STATE VARIABLE.

1. INTRODUCTION

This section develops a bond valuation model that may be used for a wide variety of bonds. With modification, the model may be used to evaluate any bond.

The advantage of examining a single state variable model is that the model's development is relatively simple, but yet provides the path by which more advanced models may be constructed. The model developed has some clear advantages over earlier models:

(1) The final model does not include the drift term of the stochastic process used to describe the state variable. We see a similar result in the Black-Scholes model for evaluating stock options. The omission of the drift term results not from simplifying assumptions, but from the main development of the model.

(2) There is no market price of risk term in the model.

Two functions that have been selected with greater or less degrees of arbitrariness have thus been eliminated when evaluating bonds. A new function has been introduced that must be specified. This function has a straightforward empirical interpretation, and may be specified with increasing confidence as the number of state variables is increased.

(3) The state variable selected is observable, and is unambigious.

It is usual to select a rate of interest as the state variable. Except in the two simplest cases of a maturing bond and a consul, it is easy to forget that the yield-to-maturity on two bonds in equilibrium with the market, but with different ratios of coupon rate to par, will in general be different, even when the maturity dates of each are the same. Our language must be specific to avoid the danger of not distinguishing between these two yields if one of the bonds is used as the generator of the yield-to-maturity used as the state variable. The present model is constructed so as to make such an error impossible.

(4) The state variable may be selected so that it is the most appropriate for the valuation being performed.

The single state variable models developed in the literature use the instantaneous interest rate, r, as the state variable. r is a somewhat eccentric selection of a state variable for evaluation of bonds with a medium or long-term maturity. If a single state variable must be used, it would be better to use one that has characteristics as close to the security being evaluated as possible. The importance of this consideration becomes less as the number of state variables is increased.

The model is based on the following key assumptions:

(1) The bond pays a continuous coupon $c(t)$, which does not have to be constant, and has a terminal lumpsum payment, k. One would normally have that c, k > = 0, but this is not a requirement.

(2) The price of a bond, B, is a function of the state variable and the maturity of the bond. All other characteristics of the bond are modeled by the boundary, initial and policy conditions. That is:

$$B(P,\tau) \qquad [1]$$

(3) The state variable, P, follows a stationary Itô process with a wiener noise $dz(t)$:

$$dP = \alpha(P)dt + \beta(P)\,dz(t) \qquad [2]$$

(4) The state variable is an E-bond.

An E-bond is a construct that has been implicit in the analysis of the time value of money. The following definitions make the construct explicit:

An equivalence class of bonds, E, is defined as the set of all bonds such that: from a specified time before maturity, μ^*, to maturity, $\mu=0$, the ratio of the coupon rate to par value is equal for all bonds in the class at all times to maturity μ, $0 <= \mu <= \mu^*$.

An example will help clarify the definition. Consider the two bonds A and B:

A: Pay a coupon as follows:

[1983, 1984) \$100/year continuously

[1984, 1985) \$120/year continuously

[1985, 1986) \$110/year continuously

and has a par value of \$1000 payable on the last day of 1986.

B: Pays a coupon as follows:

[1983, 1984) \$ 8/year continuously

[1984, 1985) \$ 9/year continuously

[1985, 1986) \$12/year continuously

[1986, 1987) \$11/year continuously

and has a par value of \$100 payable on the last day of 1987.

If μ^* is 2 years, then both bonds belong to the same equivalence class, while if μ^* is greater than two years they are not.

An E-bond is a process defined on a particular E-class of bonds so that the maturity is always a constant, μ. Any E-class may be used to generate any number of E-bonds, each of which will have a different constant nominal maturity, μ. We will denote a specific E-bond process (E-bond for short) with a nominal maturity μ by:

$$P_{E,\mu}(t) \qquad [3]$$

The yield-to-maturity, R(t), at any time t, may be determined for any bond in the equivalence class that has a remaining maturity of μ, $0 < \mu <= \mu^*$. At some time later, say $(t + \delta t)$ we may determine the yield-to-maturity $R(t + \delta t)$ of another bond in the equivalence class that has a remaining maturity of μ. It should be observed that the bond selected at $(t + \delta t)$ was not the one selected at t, for at that time it had a remaining maturity of $(\mu + \delta t)$. The process R(.) so defined is the interest rate process corresponding to the E_μ-bond process. At any time, t, any E-class of bonds defines a yield curve.

For reasons that will become clear in the analysis of this section, we will use the price of bonds in an E-class as our state variable rather than the yield-to-maturity process specified above.

We may drop the subscripts from $P_{E,\mu}$ in the case of the one dimensional model where the equivalence class and the nominal maturity are understood from the context. Process [3] is required to satisfy the stochastic differential equation [2] which leads us to practical questions of observing P when the E-Bond process is not dense. This will not be discussed here, because it will be assumed that the same type of pragmatic solution is applied to observing the E-bond process as is applied when observing the yield-to-maturity process. Note that once one of the processes is determined then so is the other.

(5) There are no profitable risk-free arbitrage opportunities to be found in the bond market.

(6) The partials of the price of a bond $B(P, \tau)$ exist whenever they are specified in the model. In particular, pB/pt, pB/pP and ppB/pPP exist.

(7) There are no trading costs in the bond market.

2. Mathematical Development

Ito's lemma applied to the price of a bond, $B(P, \tau)$

given that P has the process specified by [2] results in:

$$dB = (-B_\tau + \alpha B_p + \tfrac{1}{2}\beta^2 B_{pp})dt + \beta B_p dz(t) \qquad [4]$$

While E-Bond processes are not actually traded on organized exchanges, they do exist in the form of variable rate notes, Indexed Bonds, and mortgages (adjustable rate mortgages, for example). It is not necessary for our development that they exist, it is only necessary that they could exist. They are the theoretical construct that enables us to obtain a consistent model, and they do not form part of the final model. We shall consider a particular E-Bond process, the state variable P.

If we purchased \$W1 of the bond $B(P, \tau)$, (that is $W1/B(P, \tau)$ units), and purchased \$W2 of P (that is $W2/S(P,\mu)$ units), and the total value of our portfolio is \$1, then the proportional change in value of the portfolio, V, in the next instant of time would be

$$\frac{dV}{V} = W_1 \left(\frac{dB}{B} + \frac{c}{B}\right) + W_2 \left(\frac{dP}{P} + \frac{c}{P} - G_\tau\right) \qquad [5]$$

$$W_1 + W_2 = 1$$

where c is the coupon of the bond B, C is the coupon paid on the equivalence class of bonds that define P, dB and dP are given by [4] and [2] respectively and finally, $-G_\tau$ is the capital gain resulting from a reduction in maturity of the bond G that currently has the same maturity as P's nominal maturity.

The risk term of [5] is then

$$\left(\frac{W_1 \beta B_p}{B} + \frac{W_2 \beta}{P}\right) dz(t) \qquad [6]$$

By hypothesis pB/pP exists. If B, P $\neq$ 0 we may select W1 (and hence W2) such that the numerator of dz(t) is identically zero. For this to occur W1 or W2 may be negative.

$$\frac{W_1}{B} \beta B_p + \frac{W_2}{P} \beta = 0 \qquad [7]$$

This selection of W1 (and so W2 = 1 - W1) makes the portfolio instantaneously risk free. We have yet to show that the risk-free nature of the portfolio may be maintained. How this might be done is explained as a note to this section.

The portfolio is now instantaneously risk free. By the assumption of no profitable risk-free arbitrages in the bond market, we have that:

$$\frac{dV}{V} = rdt \qquad [8]$$

where r is the yield-to-maturity on a maturing default-free bond.

Thus,

$$\frac{W_1}{B}\left[-B_\tau + \alpha B_p + \tfrac{1}{2}\beta^2 B_{pp} + c - rB\right] + \frac{W_2}{P}\left[\alpha - rp + C - G_\tau\right] = 0 \quad [9]$$

Equations [7] and [9] form a singular system. In forming these equations, τ was arbitrary. Thus,

$$-B_\tau + \alpha B_p + \frac{1}{2}\beta^2 B_{pp} + c - rB = \lambda\beta B_p \qquad [10]$$

and

$$\alpha - rp + C - G_\tau = \lambda\beta \qquad [11]$$

and where $\lambda(P)$ is a function of P at most. [11] may be used in [10] to obtain the model:

$$-B_\tau + \frac{1}{2}\beta^2 B_{pp} + c - C B_p - B_p G_\tau + r(P B_p - B) = 0 \qquad [12]$$

This equation is a generalization of the Black-Scholes model for evaluating options:

$$Z_t + \frac{1}{2}\sigma^2 Z_{ss} + r(SZ_s - Z) = 0,$$

where:

Z is the value of the option

σ^2 is the variance of the generating security

S is the generating security price process

t is time.

Jarrow and Rudd (1983) report a generalization attributed to Y. Bergman of the standard Black-Scholes model (equation 8-12):

$$Z_t + \frac{1}{2}\sigma^2 Z_{ss} + r(SZ_s - Z) - DSZ_s = 0 .$$

We have again made the generating security process more general than that reported in order to make comparison more straight forward. The notation is identical excepting that D is the dividend yield on the security.

In our equation [12], another term associated with a cash flow occurs, the coupon c. This term does not occur in the model reported by Jarrow and Rudd because options do not pay a coupon.

Equation [12] has most of the desired properties. It does, however, contain r(P), and no functional form of r has been specified:

Any bond satisfies equation [12]. Thus maturing default free pure discount bonds also satisfy this equation. We already know that the value of such a bond with a par value of \$1, B* is, in the limit as $\tau \to 0$, \$1. In the limit, the price of a maturing bond is non-stochastic, but the associated interest rate process is stochastic. Let us define r to be

$$r = - \lim_{\tau \to 0} (\ln(B^*(P,\tau))/\tau) \qquad [13]$$

[12] and [13] imply that B* must satisfy:

$$\lim_{\tau\to 0} [-B_\tau + \frac{1}{2} \beta^2 B_{pp} + c - C B_p + B_p G_\tau$$

$$- (\ln(B^*(P,\tau))/\tau) (PB_p - B)] = 0$$

We do not in fact want B* (we already know its value) we want the ratio [13]. The numerator and denominator are both zero in the limit as $\tau \to 0$,so numerically [13] is undefined. However, if we select a τ small enough, we can obtain as good an approximation as we desire, limited only by the precision of the computer environment.

Let τ^* be a number such that the limit in [13] is almost achieved. We may substitute

$$(\ln(B^*(P,\tau^*))/\tau^*)$$

for every occurance of r. The yield-to-maturity $\tilde{r}$ on the bond G is defined by the relationship

$$P=G = \frac{c}{\tilde{r}}(1 - e^{-\tilde{r}\tau}) + G_0 e^{-\tilde{r}\tau} \quad [14]$$

In the classical case $\tilde{r}$ is independent of τ, and so we can easily prove that:

$$G_\tau = C - G\tilde{r}; \quad \tilde{r}=r. \qquad [14']$$

This is a useful result as it explains the price behaviour of bonds as they approach maturity.

[14'] and [12] imply that

$$-B_\tau + \frac{1}{2}\beta^2 B_{pp} + c - rB = 0 \qquad [14"]$$

We observe that in the classical case $\beta = 0$, and the solution to [14"] is then just the bond valuation formula or alternatively the definition of yield-to-maturity equation [14].

In the stochastic case, $\tilde{r}$ is not only a function of τ, it is a stochastic function of τ. Stochastic calculus must then be used to obtain an analog of [14'], and this will result in a stochastic variable being reintroduced into the valuation formula. If $\tilde{r}_\tau$ and $\tilde{r}_{pp}$ are small, then the error introduced will not be significant. From casual observation these conditions tend to be met when interest rates are longer term.

We should next observe that equation [12] does not have a solution when the bond being evaluated is the current bond from the equivalence set that is used to define the bond process. This is because the left-hand side is identically zero. We require the value of G_τ, however. We should also observe that G_τ is a function of P, and this makes the equation [12] non-quasi-linear. The equation may be quasi-linearized taking [14'] as our clue:

Let $G_\tau \approx c' - D\, P\tilde{r}(P)$

where c' is a constant that may not equal c, D is a constant that may not equal 1, and $\hat{r}$ is obtained from [14] using a numerical method. Quasi-linearization will enable one to use standard PDE packages.

Another method is to introduce one or two new bonds into the system for solution that have maturities close to the nominal maturity of the state vector. G_τ will be just a polynomial in P. Any occurance of τ will be as a constant.

In case the above discussion gives the impression that another assumption in the guise of an assumed functional form has been introduced into the model, the following point is made. The model is completely specified without the quasi-linearization suggested above. It may be solved to any specified degree of accuracy. The full specification of the model will lead to a doubling of the size of the system of partial differential equations, and this is avoided for clarity.

The solution of [12] will then proceed in two stages:

$$- B^*_\tau + \frac{1}{2} \beta^2 B^*_{pp} + 0 - C B_p + B_p G_\tau + r (PB^*_p - B^*)=0 \quad [15]$$

$$- B_\tau + \frac{1}{2} \beta^2 B_{pp} + c - C B_p + B_p G_\tau + r (PB_p - B)=0$$

where

$$r = -\mathrm{Ln}\ (B^*(P,\ \tau^*))/\tau^*$$

There is no need to find a functional form for the drift parameter α of equation [2] as it does not appear in the final model. We require a functional form for the noise parameter, β. Our assumption is that when the interest rate is 0, $(P = K)$, then $\beta = 0$. When interest rates are infinite, $(P = 0)$, then $\beta = 0$. One function that has this property is

$$\beta = d\left(\frac{P}{K}\right)^{f}\left(1 - \frac{P}{K}\right)^{g} \qquad [16]$$

where d, f and g are positive constants.

The model is now complete, except for the specification of the boundary, initial, and policy conditions. As our prime interest is in mortgage backed securities, we address the problem of callable bonds with no residual par value. The coupon is constant and established using classical analysis.

3. Initial, Boundary and Policy Conditions.

The boundary and initial conditions on the maturing pure-discount bond are found as follows. There being no option feature, there are no binding policy conditions.

*Initial Conditions (Maturing Bond):

At $\tau = 0$, the security will be valued at its par value, \$1. Thus:

$$B^*(P, 0) = 1$$

*Boundary Conditions (Maturing Pure Discount Bond):

When $P = 0$, then $\beta = 0$, and equation [15] becomes:

$$B^*_\tau = 0 \qquad [17]$$

on the assumption that P and B* have the same order of asymptotic behavior. The condition

$$B^*(0, \tau) = 0, \quad \tau \neq 0 \qquad [18]$$

is consistent with [17]

When $P = K$, equation [15] becomes

$$B^*_{pp} = 0 \qquad [19]$$

The condition

$$B^*(K, \tau) = 1 \qquad [20]$$

is consistent with [19] and has the required physical properties.

We will now analyse the conditions on the mortgage backed bond, B:

*Initial Conditions (Mortgage Bond):

Let us assume that the coupon for one dollar of debt at the start of the contract is c. Let there be N years to maturity.

At $\tau = 0$, the security will have no value because it will have liquidated itself.

$$B(P, 0) = 0 \qquad [21]$$

*Boundary Conditions (Mortgage Bond):

From equation [15], when P = K we have that

$$- B_\tau + c = 0. \qquad [22]$$

$$B (K, \tau) = c\tau \qquad [23]$$

is consistent with the initial condition and the natural boundary condition [22].

When P = 0, equation [15] implies that

$$B_{pp} = 0 \qquad [24]$$

A solution that satisfies this natural boundary condition and has the required physical conditions is:

$$B(0, \tau) = c\tau \qquad [25]$$

*Policy Condition (Mortgage bond):

The bond will be called when the value of a bond without the call feature is greater than the remaining loan balance. The remaining loan balance may be calculated as follows:

From conventional theory we have that

$$L = \frac{c}{s}\,(1 - e^{-sN}) = 1 \qquad [26]$$

where s is the yield-to-maturity on the mortgage bond given no call, and L is the amount borrowed, in our case 1.

We know c and N, and e is the base of the natural log, so we may in principle obtain value of s from [26]. Call it s'.

From traditional theory, the remaining principle, Q, outstanding at any time τ to maturity is

$$Q = \frac{c}{s'}\,(1 - e^{-s'\tau}) \qquad [27]$$

The policy condition is then that the value of a bond B^+ is the minimum of Q and B, where B is the value of the bond without the call feature.

$$B^+ = \mathrm{MIN}\,[Q,\ B]. \qquad [34]$$

It would be useful to summarize the above model. The value of a mortgage bond, B, at any time τ, $0 <= \tau <= T$ is given by the solution of the system of equations:

$$- B^*_\tau + \frac{1}{2}\,\beta^2\,B^*_{pp} - C\,B^*_p + B^*_p\,G^*_\tau + r\,(PB^*_p - B^*) = 0 \qquad [28]$$

$$- B_\tau + \frac{1}{2}\,\beta^2\,B_{pp} + c - C\,B_p + B_p\,G_\tau + r\,(PB_p - B) = 0$$

$$r = -\ln\,(B^*(P,\ \tau^*))/\tau^*$$

$$\beta = a(1 - \frac{P}{K})^b\,(\frac{P}{K})^d$$

subject to:

$$B^*\,(P,\ 0) = 1$$

$B\ (P, 0) = 0$

$B^*\ (0, \tau) = 0 \qquad \tau \neq 0$

$B\ (0, \tau) = 0$

$B^*\ (K, \tau) = 1$

$B\ (K, \tau) = c\tau$

$B^+ = \mathrm{MIN}\ (Q(\tau),\ B(P, \tau))$

where

$Q = c(1 - e^{-s\tau})/s$

and s is the solution to

$$1 = c(1 - e^{-sN})/s \qquad [35]$$

The parameters of the system are to be estimated from empirical data, but must be constrained as follows:

$a > 0$

$b >= 0$

$d > 0$ (stochastic case)

$f > 0$

$g >= 0$

$o < \tau^* <$ (a τ such that convergence to the risk-free rate is satisfactory)

In the model summary, B* denotes the price of a maturing default-free pure discount bond, and is calculated as a by-product of the value of the mortgage, B^+. B is the value of an equivalent mortgage to B^+, excepting that it may not be called. The coupon on the mortgage is c per \$1 of the principle when the maturity was equal to the original life of the mortgage and is paid continuously. The mortgage had an initial life of N years. The system is a function of a single state variable P which is the price of a standard E-bond and which has a nominal coupon of $c^{(p)}$ and a par value of P_0. The remaining nominal cash flow $k = c^{(p)}\tau^{(p)} + P_0$ where $\tau^{(p)} = \mu$ the nominal maturity of the state vector.

4. A Note on Maintenance of a Risk-Free Hedge.

There remains one point that requires discussion. Implicit in the construction of the model is the assumption that the risk-free hedge can be maintained from the current time to the maturity date of the security being evaluated. At any time, a portfolio may be constructed such that it is instantaneously risk-free. The cost of adjusting the portfolio is \$0, so that the portfolio may be adjusted to make it instantaneously risk free (at as many times as one would required, in any interval) without penalty. The problem remains that the portfolio is still not risk-free at all times in the interval.

Let us divide any given interval into M equal sub-intervals, $I(i)$, $0 < i <= M$, i integer. Let S be a given probability close to one. Define $J(i)$ to be the quantity:

$$J(i) = \int_{I(i)} (W\beta\, B_p/B + (1-W)\beta\, S_p/S)\, dz(t) \qquad [36]$$

If we adjust the portfolio at the beginning of each interval, then at the beginning of the interval, the integrand is by hypothesis zero. However, at any time later (in the interval) the integrand will be non-zero with probability one. For an arbitrary ε we may select an M such that the largest $J(i)$, $J(*)$, is less than ε, with a probability S. The integral over the whole interval must be less than $M.J(*)$ with a probability greater than S.

If it could be shown that in the limit as $M \to \infty$, the quantity $M.J(I)$ is 0, then we will have shown that a risk-free portfolio can be maintained with a given probability. If further, this could be shown to be true for any S strictly less than one, then the required result follows.

In establishing the above result, we will probably have to make use of the fact that the quantity:

$$\beta\, B_p/B$$

is continuous with probability one. In establishing the result one may also have to use the Fokker-Planck equation.

While a proof that the hedge can be maintained is highly desirable, we could restrict our bonds to those that satisfy the hedge. It is then an empirical matter to determine what class of bonds satisfy the final model. In the case of stock options, there is sufficient evidence to show that a similar risk-free hedge can be maintained. Any such test is, of course, a joint test of all the assumptions of the model.

The present author is conducting joint research with the objective of proving that a hedge may be maintained, and under what conditions.

2.2.2. Appendix B to 2.1

A BOND VALUATION MODEL WITH TWO STATE VARIABLES.

1. INTRODUCTION

The purpose of this section is to extend the single state variable model to include a second state variable. The resulting model provides a more adequate check of the multidimensional model.

The same assumptions are made as in the one dimensional case, except that the state vector has two components.

2. Development of the Model Equation

Let the value of a bond B be a function of two E-Bond processes P_1 and P_2, $P_1 \neq P_2$, and its term to maturity, τ. It will be further assumed that the processes P_1 and P_2 are generated by Itô equations with wiener noise:

$$dP_1 = \alpha_1\, dt + \beta_{11}\, dZ_1(t) + \beta_{12}\, dZ_2(t) \qquad [1]$$

$$dP_2 = \alpha_2\, dt + \beta_{21}\, dZ_1(t) + \beta_{22}\, dZ_2(t)$$

where α and β one functions of P and not explicit function of t.

The Itô lemma provides us with the following result:

$$dB = \gamma dt + \delta dZ(t) \qquad [2]$$

where

$$\gamma = (- B_\tau + B_p\alpha + \frac{1}{2}\,\mathrm{tr}\,(\beta\beta^T B_{pp}))$$

$$\delta = (B_p\beta)$$

It should be noticed that γ is a scalar and δ is a 2 element row vector. B_p is the vector

$$(B_{p_1}, B_{p_2})$$

and B_{pp} the matrix

$$\begin{pmatrix} B_{p_1p_1} & B_{p_1p_2} \\ B_{p_1p_2} & B_{p_2p_2} \end{pmatrix} .$$

tr is the trace function on a matrix.

Purchase, long or short, $\$W_0$ of B $\$W_1$ of P_1 and $\$W_2$ of P_2 so that $W_0 + W_1 + W_2 = 1$. This restriction on W is not necessary, but makes the development somewhat simpler. As a result, we shall have purchased W_0/B units of B, W_1/P_1 of P_1 and W_2/P_2 of P_2. It is be assumed that B pays a continuous dividend c, which we assume to be constant over time. Denote the portfolio by V. By assumption, V=1. The change in the portfolio will be given by:

$$dV = \frac{W_0}{B}(\gamma dt + \delta_1\, dZ_1(t) + \delta_2\, dZ_2(t) + c dt) \qquad [3]$$

$$+ \frac{W_1}{P_1}(\alpha_1\, dt + \beta_{11}\, dZ_1(t) + \beta_{12}\, dZ_2(t))$$

$$+ \frac{W_2}{P_2}(\alpha_2\, dt + \beta_{21}\, dZ_1(t) + \beta_{22}\, dZ_2(t))$$

Select W so that the sum of the risk terms is zero in the next instant of time.

$$\frac{W_0}{B}\,\delta_1 + \frac{W_1}{P_1}\,\beta_{11} + \frac{W_2}{P_2}\,\beta_{21} = 0 \qquad [4A]$$

$$\frac{W_0}{B}\,\delta_2 + \frac{W_1}{P_1}\,\beta_{12} + \frac{W_2}{P_2}\,\beta_{22} = 0$$

Under these conditions

$$dV = \left(\frac{W_0}{B}(\gamma+c) + \frac{W_1}{P_1}\,\alpha_1 + \frac{W_2}{P_2}\,\alpha_2\right) dt$$

The portfolio is riskless, so to avoid arbitrage it must be equal to the return on a maturing pure discount bond, r

$$dV = r\, dt.$$

Recalling that $(W_0 + W_1 + W_2) = 1$ we obtain:

$$\frac{W_0}{B}(\gamma+c-rB) + \frac{W_1}{P_1}(\alpha_1+C_1-G_{1_\tau}-rP_1) + \frac{W_2}{P_2}(\alpha_2+C_2-G_{2_\tau}-rP_2)=0 \qquad [4C]$$

Equations [4A], [4B] and [4C] form a singular system, so that the following matrix:

$$\begin{pmatrix} \gamma + c - rB & \alpha_1 + C_1 - G_{1_\tau} - rP_1 & \alpha_2 + C_2 - G_{2_\tau} - rP_2 \\ \delta_1 & \beta_{11} & \beta_{21} \\ \delta_2 & \beta_{12} & \beta_{22} \end{pmatrix}$$

has rows that are linear combinations of one another. The combination function $(\lambda_1, \lambda_2)^T$ is not a function of τ as this is arbitrary, and so, is at most a function of the state vector P. This function has been called the market price of risk, but is of little concern to us, as it is eliminated from the model.

Thus:

$$\begin{aligned} \lambda + c - rB &= \lambda_1 \delta_1 + \lambda_2 \delta_2 \\ \alpha_1 + C_1 - G_{1_\tau} - rP_1 &= \lambda_1 \beta_{11} + \lambda_2 \beta_{12} \\ \alpha_2 + C_2 - G_{2_\tau} - rP_2 &= \lambda_1 \beta_{21} + \lambda_2 \beta_{22} \end{aligned} \qquad [5]$$

The second and third equations imply that

$$\lambda = \beta^{-1} \begin{pmatrix} \alpha_1 + C_1 - G_{1_\tau} - rP_1 \\ \alpha_2 + C_2 - G_{2_\tau} - rP_2 \end{pmatrix} \qquad [6]$$

The first equation of [5] is just:

$$\gamma + c - rB = \delta\lambda \qquad [7]$$

Recall from [2] that

$$\delta = (B_p \beta) \qquad [8]$$

Substitution of [6] and [8] into [7]

$$\gamma + c - rB = B_p \beta \beta^{-1} \begin{pmatrix} \alpha_1 + C_1 - G_{1\tau} - rP_1 \\ \alpha_2 + C_2 - G_{2\tau} - rP_2 \end{pmatrix} \qquad [9]$$

Substituting for γ from [2] into [9] results in the model equation for evaluating bonds:

$$- B_\tau + \frac{1}{2} \operatorname{tr} (\beta\beta^T B_{pp}) + c - B_p (C_1 , C_2)^T + B_p (G_{1\tau} , G_{2\tau})^T - r(B - B_{p_1} P_1 - B_{p_2} P_2) = 0 \qquad [10]$$

3. Constraints on the Model Equation.

These are a direct generaliztion of the one dimensional case.

III. ESTIMATION OF THE PARAMETERS

In the last section, the valuation equation for any kind of bond was derived (equation 2.1.3 [2]). In order to use it, it requires estimates of the parameters and initial and boundary conditions. This section develops an estimation procedure for the parameters given the initial and boundary conditions on the basis bond.

The current author proposes three methods by which the parameters of the system can be found. All three will be discussed, but only the most practical (number 3) will be analyzed in detail. They are:

1. Direct estimation using non-linear filter theory.
2. Direct estimation using the Fokker-Planc equation.
3. Indirect estimation using equation 2.1.3 [2] of the last section.

3.1. Estimation Using Non-Linear Filter Theory

The development of digital filter theory used in signal processing and control theory is quite advanced. This subsection describes a procedure by which this theory may be used to estimate the parameters of the original system of stochastic equations. The theory is set out in substantial detail in Sage and Melsa's (S&M) book "Estimation Theory with Applications to Communications and Control", Chapter 9, "Nonlinear Estimation". The treatment is somewhat long and involved and is not repeated in detail here.

A useful feature of the text is that various algorithms are written in a form that makes the generation computer code reasonably easy to implement. In our case, the true state variables of the system are directly observable and so do not require estimation. In the class of models considered by the S&M text, the state variables are observed through a noisy observation model. The S&M algorithm may be made to suit our purposes by thinking of the parameters of our mode as state variables. These new "state variables" are now observable only through a noisy observation model. It was found that using a maximum á posteriori estimation (MAP) discrete algorithm the estimation of the parameters resulted in instability in the sense that as each new observation was fed into the model, the estimates of some of the parameters moved off to infinity. The effect was particularly pronounced after a large jump in the observed interest rate.

There is an additional algorithm that may be incorporated into the MAP algorithm used above called a discrete fixed-lag smoothing algorithm. Although it was not implemented, it is the present author's view that it may result in satisfactory estimations.

One advantage of the above type of algorithm is that it provides estimates of the parameters at the time of each new observation. One is then able to obtain some idea of the stationarity of the parameters.

Another advantage of this class of algorithms is that the drift vector of the generating process may be estimated. While this vector has no value in our final model, it may have uses in other models.

The procedure was abandoned for higher dimensional models because of the large number of parameters to be estimated by such an on-line filter, and because no confidence statistics were provided by the algorithm. The algorithm is however economical with computer time and should prove useful to a practical implementation of the valuation algorithm if only it can be made stable.

3.2. Direct Estimation using the Fokker-Planc Equation.

The Fokker-Planc equation may be used to convert the

Itô stochastic equation of a process into a parabolic partial differential equation. The solution of this equation is a conditional probability density function. The initial conditions of this equation is a Dirac delta function with the singularity at the current bond price. Consistent with this are the requirements that the probability density function is identically zero for all t at $P_i = 0$ and $P_i = K_i$, for all i.

It is clear then, that given any particular parameterization q', the conditional probability density function may be found as a solution to the partial differential equation. Consider any observation of the state vector and the corresponding observation one period later. This is a sample from the theoretical conditional probability density function. This observation may be mapped onto the theoretical model's cumulative conditional probability density function with parameterization q'. The cumulative probability of the point may then be read off, given that the theoretical model and its current parameterization are true.

The multivariate cumulative probabilities should come from a uniform distribution on [0,1]. If a sufficiently powerful non-parametric test statistic is selected, one should be able to "fit" the data to the uniform distribution by the optimal selection of the parameterization, q*.

Using this method, one is able to obtain confidence intervals for the parameters using well developed statistical methods.

The method involves a very large number of solutions to partial differential equations which would involve substantial computer time. By careful use of interpolation, however, the number of solutions of the pde can be reduced substantially.

An advantage of the method is that only the basis bonds are used. In the indirect method, to be discussed next, additional non-basis bonds must be used. The present method makes fewer assumptions. There is no assumption about efficient markets, for example.

3.3. Indirect Estimation.

From a conceptual point of view, the method of estimation to be described below is very straightforward. In addition, the subroutine modules necessary to estimate the parameters are available. A system of partial differential equations (pde) is nothing but a system of hard equations to solve. For some parameterization, q' equation 2.1.3 [2] is solved in two steps. First B* is solved using the first equation of the system and the results substituted into the second equation. Then the equation in B is solved. The equations could, of course, be solved simultaneously, but this would place a further restriction on the sources of pre-written computer code. There is no possibility of being able to solve the equations analytically today, so a numerical procedure is necessary. The numerical solution of the equation depends upon the parameters of the equation, q, as well as upon the current value of the state vector P. The solution is the price of a bond, and the correct initial and boundary conditions of the bond type being considered must also be specified.

If the bond B is selected so that it corresponds to a member of a set of observable bonds, then we have a regression problem. The only differences between the current problem and a regular regression analysis is that the equation is not explicit. Because derivatives of B with respect to the components of q, the parameter set, are not available, we should look for a derivative-free non-linear regression code. Thus, given actual values of P and B, together with other knowns, a route exists to estimate the parameters of the system.

Two views heard frequently in pde circles are that one should not generally write one's own code to solve pde's and that the finite element method is superior to more conventional methods. The only code located in the market place to solve our system was written by Bill Schiesser of Lehigh University called DSS/2, but it uses the method of lines. For the record, ACM algorithm 540, PDECOL (written by Madsen and Sincovec) uses the finite element method but is restricted to systems of equations with one space variable, and our system has been developed for a higher number of state variables.

An appropriate regression code called BMDPAR was written by Mary Ralston of BMDP. The use of this code to estimate ordinary differential equations has been recognized in BMDP technical reports 46, 54, 58 and 70. The present author independently recognized the use of this code to estimate partial differential equations. The theory of the code is described in Ralston and Jennrich (1977), and use a very efficient secant pseudo-Gauss-Newton algorithm that minimizes the expensive functional evaluations.

A practical note should be included at this point. When estimating a higher dimensional bond process, one should use a wide range of different bond types with widely different maturities. If this is not done, then the confidence with which one can estimate some of the parameters suffers. For instance, if only long-term bonds are used, then we would find that confidence statistics of the parameters associated with the short-term basis bond processes would be poor.

A further point of interest is that we may examine the stationarity of the parameters over time by collecting bond prices over a short time, a day or a week. Given sufficient observations the parameters may be estimated with a high degree of confidence. The process may be repeated in subsequent periods, and the stationarity of the parameters observed.

To summarize, a practical way to estimate the parameter of the bases bond process is to use BMDPAR coupled with DSS/2 or PDECOL. An advantage of the Ralston code is that statistics are provided. Once the parameters of the system have been estimated, any rational bond may be valued. Irrational bonds may also be valued, but the estimated model must be first embedded in another model.

IV.BOND VALUATIONS WITH THIRD PARTY TRANSACTION COSTS.

It is convenient to analyze the world into three sets of persons. They are the issuer, the investor, and third parties. The analysis already provided enables us to analyze all default-free rational bonds in the absence of any cash flow to third parties. For instance, if a callable bond has a call premium, this may be considered to be part of the exercise price. This premium is just part of the cash flow between the issuer and the investor. The problem arises when third party transaction costs are incurred. This is because the cash flow associated with the bond of the issuer is not identical to the negative of the cash flow of the investor. This results in a separation of the yield curve into two curves, one for the investor and one for the issuer. For instance, an investor's required rate of return on a pure discount bond will be lower than the cost of capital to the issuer. This is a result of the flotation charges and the cost of administrating the bond issue.

We have been able to avoid the problem associated with creation of an investment planning horizon because of the contingent claims setting of our model. Elton and Gruber's (1975) dynamic programming model requires a time horizon because there is no arbitrage analysis in their model. In our model, interest rates are all in equilibrium, so that given that a bond exists with a certain maturity equal to the investment planning horizon, then any rational bond will be correctly valued by our model. If the correct transition probabilities are specified, then the two approaches will give identical results.

One way to avoid treatment of the planning horizon is to recognize that the owner of the option feature will include all his costs in making an option decision. Let us illustrate the impact of this by analyzing a callable bond with third party transaction costs.

It is be assumed that on a particular mortgage there is a remaining balance of $Q(\tau)$, and a penalty of $U(B,(P,\tau), \tau)$ payable to the mortgage bond holder. The third party premium required to call in the old mortgage and to issue a replacement mortgage is $M(B(P,\tau), \tau)$. The issuer's effective exercise price is $Q + U + M$. Any bond issue may be used to replace the mortgage as the market is in equilibrium. We will use a bond without a call feature but otherwise identical to the mortgage bond. This reduces the complications of the issue of a second callable bond.

Note that $B(P,\tau)$ and τ are known. The functions M and U should be determined by analysis of investment banking practices.

In the valuation of GNMA pass-through securities, we must set U equal to zero as there is no early repayment fee. The correct value of the third party costs, M, is harder to determine. For some charges, there can be no doubt that they should be included in M. Any payments to institutions other than the new lender would certainly be included in M. The problem arises where lending institutions allow the borrower to buy-down the interest rate or allocate front end expenses to increase the interest rate coupon. We have to ask to what extent are "points", for example, a recovery of bank expenses, and to what extent a buy-down of interest rates.

Careful analysis then enables us to include third party transaction costs into the valuation of bonds. We do have to take great care to ensure that we separate these costs from the market rates. This will sometimes require the use of critical financial judgement.

To summarize, we use Q + M in place of Q in the model 2.1.3 [2].

V. IRRATIONAL BONDS WITH TRANSACTION COSTS.

In this section we analyze the irrational callable bond. The analysis may be extended quite simply to include any type of irrational bond, but it is be simpler to analyze GNMA pass-through securities rather than a generalized bond. GNMA pass-through securities are just portfolios of mortgages, which in turn are just a specific type of callable bond.

There are several ways to analyze irrational behavior. Dunn and McConnell [1981a,b] attempted to use a jump process in deriving the valuation equation. As explained in Chapter 1, this method has several drawbacks not least of which is the fact that the type of suboptimal behavior whereby a householder holds onto a mortgage that should rationally be refinanced is not modeled. It is not possible to make a straightforward modification to their model.

At any time, a callable bond can be given a rational value, $B(P,\tau)$, using the theory developed in this chapter. The remaining balance, Q, of the bond may be determined using classical theory together with any premium that must be contractually paid. If there are any third party transaction costs, M, associated with refinancing, then these may also be determined. Thus, at any time, the opportunity cost of refinancing or not refinancing may be determined. Let us define the refinancing variable, S to be:

$$S = B(P,\tau) - Q - M$$

If $S < 0$, then the mortgage should optimally be refinanced now

If $S > 0$, then the mortgage should have been refinanced at some past time.

The above definition of S enables us to find the actual distribution of S when mortgages are repaid.

If the distribution of S is stationary and the variance is not too large, we may use it to estimate the value of an irrational bond. The equation 2.1.3 [2] requires the further modification whereby for every occurance of the symbol Q, a replacement string Q + M + S is inserted.

$$[1] \quad \bar{B} = \int_{S} B(P,\tau;s)\, dP(s)$$

If it turns out that the distribution of S is unsatisfactory, then we may make s a function of a set of relevant independent variables and perform a regression analysis. The resulting estimated functional form of S together with the residual error analysis may be used in an analogous manner to that just described.

This method is a hybrid of contingent claims bond evaluation methodology and the regression methods used by Curley and Guttentag (EER, 1974, 1,1). The independent variables used by them are:

Policy Year

Maturity

Control rate prevailing in year t

Contract rate on mortgages in cohort

Rational opportunity cost of not refinancing

The present author would like to add to these the house price inflation and delete the rational opportunity cost (this is just s in our model). C&G were estimating the annual termination rate in year t).

Note that the irrational behavior of holders of other types of bond-option holders may be modeled in a similar manner. Indeed, the irrational behavior of any option holder may be so analyzed. The value of the option, given that people act irrationally, may next be found. Such an estimate would be more useful to a seller of options than the corresponding rational estimate of the option's value.

VI. Further Research.

1. The initial, boundary and policy conditions of other types of bonds may be simply derived.

2. Estimates of the parameters of the model should be obtained in the manner described in 3.3.

3. The mixed noise problem should be addressed where the underlying state vector contains poisson as well as wiener noise.

4. The distribution of irrational costs, $\dot{S}$ should be determined and investigated, and a pragmatic study made of the mortgage refinancing behavior.

5. The existence of a continuous hedge must be established. It is not sufficient to show that a hedge can be created at any time. This is a major concern in the case of mixed noise models.

6. We should compare the results of the model in the bond context solved with a general structure numerically with other solutions obtained with more restrictive parameter selection using analytic methods. Following Black-Scholes [1973], a solution should exist in terms of the error function and following Dothan [1978] in terms of Bessel functions given suitable selection of parametric form of the free functions of the model.

BIBLIOGRAPHY

AITCHISON, J. AND BROWN J.A.C. (1957) "The Log-normal Distribution with Special Reference to its uses in Economics", Cambridge University Press.

AIVAZIAN AND JEFFREY L. CALLEN, JF (1980) "Future Investment Opportunities and the Value of the Call Provision on a Bond: Comment". Vol. XXXV, No. 4.

ANANTHANARAYANAN, AL. L. AND EDUARDO S. SCHWART, JF (1980) "Retractable and Extendable Bonds: The Canadian Experience". Vol. XXXV, No. 1.

ARDITTIA, FRED D. AND KOSE JOHN, JF (1980) "Spanning the State Space with Options". Vol. XV, No. 1.

ARNOLD, LUDWIG (1974) "Stochastic Processes and Estimation Theory with Applications". English language edition: 1974 John Wiley.

ASSEFI TOURAJ (1979) "Stochastic Processes and Estimation Theory with Applications". Wiley-Interscience.

AZIZ, A.K. (ed) (1972) "The Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations", Academic Press.

BAZARAA, MOKHTAR S. AND C.M. SHETTY (1979) "Non-linear Programming, Theory and Algorithms. John Wiley.

BART, JOHN AND ISIDORE J. MASSE, JF (1981) "Divergence of Opinion and Risk". Vol. XVI, No. 1.

BECK, JAMES V. AND KENNETH J. ARNOLD (1977) "Parameter Estimation in Engineering and Science". John Wiley.

BECKERS, STAN JF (1980) "The Constant Elasticity of Variance Model and Its Implications for Option Pricing". Vol. XXI, No. 3.

BECKERS, STAN, JF (1980) "A Note on Estimating the Parameters of the Diffusion-Jump Model of Stock Returns". Vol. XXI, No. 1.

BELLMAN, RICHARD (JAN. 1961) "Project Rand: Adaptive Control Processes: A Guided Tour". R350. Princeton University Press/Oxford University Press.

BHATTACHARYA, MIHIR, JF (1980) "Empirical Properties of the Black-Scholes Formula". Vol. XV, No. 5.

BHATTACHARYA, SUDIPTO, JF (1981) "Notes on Multiperiod Valuation and the Pricing of Options". Vol. XXXVI, No. 1.

BLACK, DEBORAH G., KENNETH D. GABADE AND WILLIAM L. SILBER, JF (1981) "The Impact of the GNMA Pass-through Program on FHA Mortgage Costs". Vol. XXXVI, No. 2.

BLACK, FISCHER AND JOHN C. COX, JF (1976) "Valuing Corporate Securities: Some Effects of Bond Indenture Provisions". Vol. XXXI, No. 2.

BLACK, FISCHER AND M.J. SCHOLES (1973), The Pricing of Options and Corporate Liabilities, Journal of Political Economy, 81, 637-659.

BODIE, ZVI AND ROBERT A. TAGGART, JR., (1980) "Future Investment Opportunities and the Value of the Call Provision on a Bond: Rely". Vol. XXXV, No. 4.

BOYLE, PHELIM P. AND DAVID EMANUEL, JF (1980) "Discretely Adjusted Option Hedges", pp., 239-282.

BOMBERGER, WILLIAM A. AND WILLIAM J. FRAZER, JR., JF (1981) "Interest Rates, Uncertainty and the Livingston Data". Vol. XXXVI, No. 3.

BOYLE, PHELIM P. JFE (1977) "Options: A Monte Carlo Approach", pp. 323-338.

BRENNAN, MICHAEL J. AND EDUARDO S. SCHWARTZ, JF (1976) "The Pricing of Equity-Linked Life Insurance Policies with an Asset Value Guarantee", pp. 195-213.

BRENNAN, MICHAEL J. AND EDUARDO S. SCHWARTZ, JFE (1977) "Savings Bonds, Retractable Bonds and Callable Bonds", pp. 67-88.

BRENNAN, MICHAEL J. AND EDUARDO S. SCHWARTZ, JF (1980) "Analyzing Convertible Bonds". Vol. XV, No. 4.

BRENNER, MENACHEM (1983) "Option Pricing, Theory and Applications" Lexington Books.

BRETON, A. LE (1976) "On Continuous and Discrete Sampling for Parameter Estimation in Diffusion Type Processes", pp. 124-144.

BROCKETT, ROGER W. (1976) "Parametrically Stochastic Linear Differential Equations", pp. 8-21.

CARGILL, THOMAS F. AND ROBERT A. MEYER, JF (1980), "The Term Structure of Inflationary Expectations and Market Efficiency". Vol. XXXV, No. 1.

CARNAHAN, BRICE, H.A. LUTHER AND JAMES O. WILKES (1969) "Applied Numerical Methods". John Wiley.

CARRIER, GEORGE F. AND CARL E. PEARSON (1976) "Partial Differential Equations. Theory and Technique", Academic Press.

CHIRAS, DONALD P. AND STEVEN MANASTER, JFE (1978) "The Information Content of Option Prices and a Test of Market Efficiency", pp. 213-234.

COHEN, JEROME B. "Session Topic: Portfolio Theory and Security Analysis", JF.

COURTADON, GEORGE JF (1982) "The Pricing of Options on Default-Free Bonds". Vol. XVII, No. 1.

COX, D.R. AND VALERIE ISHAM (1980) "Point Process". Chapman and Hall.

COX, JOHN C., JFE (1976) "The Valuation of Options for Alternative Stochastic Processes", pp. 145-166.

COX, JOHN C., STEPHEN A. ROSS AND MARK RUBINSTEIN, JFE (1979) "Option Pricing: a Simplified Approach", pp. 229-263.

COX, JOHN C., JONATHAN E. INGERSOLL, JR. AND STEPHEN A. ROSS, JF (1980) "An Analysis of Variable Rate Loan Contracts". Vol. XXXV, No. 2.

COX, JOHN C., JONATHAN E. INGERSOLL, JR. AND STEPHEN A. ROSS, JF (1981) "A Re-examination of Traditional Hypotheses about the Term Structure of Interest Rates". Vol. XXXVI, No. 4.

COX, JOHN C., JONATHAN E. INGERSOLL, JR. AND STEPHEN A. ROSS, JFE (1981) "The Relation Between Forward Prices and Futures Prices", pp. 321-346.

COX, JOHN C. AND EDUARDO S. SCHWARTZ, JF (1982) "Option Pricing Theory and its Application". Vol. XXXVII, No. 2.

DESAI, CHANDRAKANT S. (1979) "Elementary Finite Element Method" Prentice-Hall.

DEW, P.M. AND J.E. WALSH (1981) "A Set of Library Routines for Solving Parabolic Equations in One Space Variable". ACM Transactions on Mathematical Software, Vol. 7, No. 3, pp. 295-314.

DOTHAN, L. URI, JFE (1978) "On the Term Structure of Interest Rates", pp. 59-69.

DRAPER, N.R. AND SMITH H. (1981) "Applied Regression Analysis" 2nd Ed. John Wiley.

DUNN, KENNETH B. AND JOHN J. MCCONNELL, JF (1981) "Valuation of GNMA Mortgage-Backed Securities". Vol. XXXVI, No. 3.

DUNN, KENNETH B. AND JOHN J. MCCONNELL, JF (1981) "A Comparison of Alternative Models for pricing GNMA Mortgage-Backed Securities". Vol. XXXVI, No. 2.

EILON, SAMUEL AND TERENCE R. FOWKES, JF "Applications of Management Science in Banking and Finance.

ELLIOT, ROBERT J. (1982) Stochastic Calculus and Applications.

ELLIOT, ROBERT J. WALTER AND J.R. BAIER, JF (1980) "Economic Value of the Call Option". Vol. 27. No. 4: pp. 891-901.

ELTON, EDWIN J. AND MARTIN J. GRUBER (1975) "Finance as a Dynamic Process". Prentice-Hall.

FARKAS KAREN L. AND ROBERT E. HOSKIN, Journal of Economic Theory (1978) "Testing a Valuation Model for American Puts.

FEDERAL RESERVE BULLETIN June 1983.

FINANCIAL PUBLISHING COMPANY. "Pass-Through Yield and Value Tables for GNMA Mortgage-Backed Securities. Publication Number 715, Seventh Edition".

FIGLEWSKI, STEPHEN, JF (1981) "GNMA Pass-Through Securities:". Vol. XXXVI, No. 2.

FLEMING, WENDELL H. (1973) "Optimal Control of Diffusion Processes".

FLEMING, WENDELL H. AND RAYMOND W. RISHEL (1975). "Deterministic and Stochastic Optimal control. Springer-Verlag.

FOGLER, H. RUSSELL, KOSE JOHN AND JAMES TIPTON, JF (1981) "Three Factors, Interest Rate Differentials and Stock Groups". Vol. XXXVI, No. 2.

FRIEDMAN, AVNER (1975) "Stochastic Differential Equations and Applications". Vol. 1. Academic Press.

FREEDMAN, DAVID (1970) "Brownian Motion and Diffusion". Holden-Day.

GALAI, DAN AND RONALD WE. MASULIS, JFE (1976) "The Option Pricing Model and the Risk Factor of Stock", pp. 53-81.

GARMAN, MARK B., JFE (1976) "An Algebra for Evaluating Hedge Portfolios", pp. 403-427.

GARMAN, MARK B., JFE (1978) "The Pricing of Supershares", pp. 3-10.

GARMAN, MARK B. AND JAMES A. OHLSON, JFE (1981) "Valuation of Risky Assets in Arbitrage-free Economies with Transactions Costs", pp. 271-280.

GEMAN, STUART, "A Method of Averaging for Random Differential Equations with Applications to Stability and Stochastic Approximations".

GESKE, R., JFE (1979) "The Valuation of Compound Options", pp. 63-81.

GIHMAN, I.I. AND SKOROKHOD, A.V. (1969) "Introduction to the Theory of Random Processes. English ed.

GIHMAN, I.I. AND A.V. SKOROHOD "The Theory of Stochastic Processes" I (1974), II (1975), III (1979). Springer-Verlag.

GIHMAN, I.I. AND A.V. SKOROHOD "Stochastic Differential Equations" Springer-Verlag, 1972.

GLEIT, ALAN, JFE (1978) "Valuation of General Contingent Claims", pp. 71-87.

GRAUER, ROBERT R., JFE (1978) "Generalized Two Parameter Asset Pricing Models", pp. 11-32.

GUSTAFSON, KARL E. (1980) "Introduction to Partial Differential Equations and Hilbert Space Methods". John Wiley.

HAWAWINI, GABRIEL A., JF (1980) "An Analytical Examination of the Intervaling Effect on Skewness and Other Moments". Vol. XV, No. 5.

HESS, PATRICK J. AND JAMES L. BICKSLER, JFE (1975) "Capital Asset Prices Versus Time Series Models as Predictors of Inflation", pp. 341-360.

HIEBERT, K.L. (1981) "An Evaluation of Mathematical Software That Solves Non-Liner Least Squares Problems". ACM Transactions on Mathematical Software, Vol. 7, No. 1, pp. 1-16.

HILLIER, FREDERICK S., AND GERALD J. LIEBERMAN (1980) "Introduction to Operations Research", 3rd Ed. Holden-Day.

HIRSCH, MORRIS W. AND STEPHEN SMALE (1974) "Differential Equations, Dynamical Systems and Linear Algebra", Academic Press.

HOEL, PAUL G., SIDNEY C. PORT AND CHARLES J. STONE (1972) "Introduction to Stochastic Processes". Houghton Mifflin.

HORNBECK, ROBERT W. (1975) "Numerical Methods". Quantum Publishes, Inc.

HOWARD, RONALD A. (1960) "Dynamic Programming and Markov Processes, M.I.T.

INGERSOLL, JONATHAN E. JR., JFE (1977) "A Contingent-Claims Valuation of Convertible Securities", pp. 289-322.

ITO, K. (1968) "Lectures on Stochastic Processes" Notes by K. Muralidhara Rao, Tata Institute of Fundamental Research, Bombay. Revised, 1968.

JARROW, ROBERT AND ANDREW RUDD (1983) "Option Pricing" Richard D. Irwin.

JARROW, ROBERT A. AND GEORGE S. OLDFIELD, JFE (1981) "Forward Contracts and Futures Contracts", pp. 373-382.

JOHN, FRITZ (1978) "Partial Differential Equations". 3rd Ed. Springer-Verlag.

JOHN, KOSE, JF (1981) "Efficient Funds in a Financial Market with Options: A New Irrelevance Proposition. Vol. XXXVI, No. 3.

KALYMON, BASIL A., JF (1971) "Bond Refunding with Stochastic Interest Rates", pp. 171-184. Management Science, Vol. 18, No. 3.

KRYLOV, N.V. (1980) "Controlled Diffusion Processes translated by A.B. Aries Springer-Verlag.

KUSHNER, HAROLD J. "Approximation Methods for the Minimum Average Cost per Unit Time Problem with a Diffusion Model".

KUSHNER, HAROLD J. (1967) "Stochastic Stability and Control", Academic Press.

KUSHNER, HAROLD J. (1977) "Probability Methods for Approximations in Stochastic Control and for Elliptic Equations", Academic Press.

LAINIOTIS, D.G. (1971) "Optimal Adaptive Estimation: Structure and Parameter Adapation". IEEE Transactions on Automatic Control, Vol. AC-16, No. 2.

LAINIOTIS, D.G. (1971) "Optimal Non-Linear Estimation" Int. J. Control, Vol. 14, No. 6, pp. 1137-1148.

LATANE, HENRY A. AND RICHARD J. RENDLEMAN, JR., JF (1976) "Standard Deviations of Stock Price Ratios Implied in Option Prices". Vol. XXXI, No. 2.

LAX, MELVIN D. "Obtaining Approximate Solutions of Random Differential Equations by means of the Method of Moments".

LEE, WAYNE , Y., RAMESH K.S. RAO AND J.F.G. AUCHMUTY, JFE (1981) "Option Pricing in a Lognormal Securities Market with Discrete Trading", pp. 75-101.

LEUENBERGER, DAVID G. (1969) "Optimization by Vector Space Methods". John Wiley.

LIPSTER, R.S., A.N. SHIRYAYEV, A.B. ARIES (1977 and 1978), "Statistics of Random Processes I & II".

LIVINGSTON, MILES AND SURESH JAIN, JF (1982) "Flattening of Bond Yield Curves for Long Maturities". Vol. XXXVII, No. 1.

MACBETH, JAMES D. AND LARRY J. MEVILLE, JF (1980) "Tests of the Black-Scholes and Cox Call Option Valuation Models". Vol. XXV, No. 2.

MACHURA, MARKE AND ROLAND A. SWEET (1980) "A Survey of Software for Partial Differential Equations". ACM Transactions on Mathematical Software, Vol. 6, No. 4, pp. 461-488.

MCSHANE, E.L. (1974) "Stochastic Calculus and Stochastic Models".

MALLIARIS, A.G. "Stochastic Methods in Economics and Finance".

MASON, SCOTT P. AND SUDIPTO BHATTACHARYA, JFE (1981) "Risky Debt, Jump Processes, and Safety Covenants", pp. 281-307.

MAYBECK, PETER S. (1979) "Stochastic Models, Estimation, and Control". Vol. 1. Academic Press.

MCSHANE, E.J. (1974) "Stochastic Calculus and Stochastic Models', Academic Press.

MARSH, TERR, JF (1980) "Equilibrium Term Structure Models: Test Methodology". Vol. XXXV, No. 5.

MEDITCH, J.S. (1969) "Stochastic Optimal Linear Estimation and Control". McGraw-Hill.

MELGAARD, DAVID K. AND RICHARD F. SINCOVEC (1981) "Algorithm 565 PDETWO/PSETM/GEARB: Solution of Systems of Two-Dimensional Non-Linear Partial Differential Equations (D3)". ACM Transactions on Mathematical Software, Vol. 7, No. 1, pp. 126-135.

MERTON, ROBERT C. "On the Pricing of Corporate Debt: The Risk Structure of Interest Rates", JF.

MERTON, ROBERT C., JOURNAL OF ECONOMIC THEORY (1971) "Optimum Consumption and Portfolio Rules in a Continuous-Time Model", pp. 373-413.

METON, ROBERT C. JR. (1973) "Theory of Rational Option Pricing". Bell Journal of Economics and Management Science.

MERTON, ROBERT C. JF (1976) "The Impact on Option Pricing of Specification Error in the Underlying Stock Price Returns".

MERTON, ROBERT C., MICHAEL J. BRENNAN AND EDUARDO S. SCHWARTZ, JF (1977), "Options". Vol. XXXII, No. 2.

MERTON, ROBERT C. AND PAUL A. SAMUELSON, JFE (1974) "Fallacy of The Log-Normal Approximation To Optimal Portfolio Decision-Making Over Many Periods", pp. 67-94.

MEYER, PAUL L. (1970) "Introductory Probability and Statistical Applications", 2nd Ed. Addison-Wesley.

MILES, MIKE AND R. STEPHEN SEARS, JF (1981) "An Econometric Approach to the FNMA Free Market System Auction". Vol. XVI, No. 2.

NAGLE, H.T. JR. AND V.P. NELSON (1981) "Digital Filter Implementation on 16-Bit Microcomputers".

NAHI, NASSER E. (1969) "Estimation Theory and Applications". John Wiley.

NEUMAN, SCHLOMO P. JF (1981) "A Eulerian-Lagrangian Numerical Scheme for the Dispersion-Convection Equation Using Conjugate Space - Time Grids". Journal of Computational Physics, Vol. 4, pp. 270-294.

OLDFIELD, GEORGE S. JR., Robert A. Jarrow and Richard J. Rogalski, JFE (1977) "An Autoregressive Jump Process for Common Stock Returns", pp. 389-418.

OLDFIELD, GOERGE S., JR. AND RICHARD J. ROGALSKI, JF (1981) "Treasury Bill Factors and Common Stock Returns". Vol. XXXVI, No. 2.

PARK, SOO-BIN, JFE (1982) "Spot and Forward Rates in the Canadian Treasury Bill Market", pp. 107-114.

PARKINSON, MICHAEL JF "Option Pricing: The American Put". The Journal of Business.

PARZEN, EMANUEL (1962) "Stochastic Processes". Holden-Day.

PERRY, PHILIP R., JF (1982) "The Time-Variance Relationship of Security Returns: Implications for the Return-Generating Stochastic Process". Vol. XXXVII, No. 6.

PESANDO, JAMES E., JF (1980) "On Forecasting Long-Term Interest Rates: Is the Success of the No-Change Prediction Surprising?". Vol. XXXV, No. 4.

PRICE, KELLY, BARBARA PRICE AND TIMOTHY J. NANTELL, JF (1982) "Variance and Lower Partial Moment Measures of Systematic Risk: some Analytical and Empirical Results". Vol. XXXVII, No. 3.

PYE, GORDON "The Value of Call Deferment on A Bond: Some Empirical Results", JF.

RAO, M.M. (1979) "Stochastic Processes and Integration".

REINGANUM, MARC R., JF (1981) "Empirical Tests of Multi-Factor Pricing Model". Vol. XXXVI, No. 2.

RENDLEMAN, RICHARD J. JR., AND BRIT J. BARTTER, JF (1980) "The Pricing of Options on Debt Securities". Vol. XV, No. 1.

RICHARD, SCOTT F., JFE (1978) "An Arbitrage Mode of the Term Structure of Interest Rates", pp. 33-57.

RICHARD, SCOTT F. AND M. SUNDARESAN, JFE (1981) "A Continuous Time Equilibrium Model of Forward Prices and Futures Prices in A Multigood Economy", pp. 347-371.

ROBICHEK, JF (1971) "Business Finance Issues" - Dynamic Programming Applications in Finance". Vol. 26, No. 3: pp. 473-505.

ROLL, RICHARD, JFE (1977) "A Critique of the Asset Pricing Theory's Tests", pp. 129-176.

ROSS, RICHARD JFE (1977) "An Analytic Valuation Formula for Unprotected American Call Options on Stocks with Known Dividends", pp. 251-258.

ROSEN, KENNETH T. AND DAVID E. BLOOM JF (1980) "A Microeconomic Model of Federal Home Loan Mortgage Corporation Activity. Vol. XXXV, No. 4.

SAGE, ANDREW P. (1968) "Optimum Systems Control". Prentice-Hall.

SAGE, ANDREW P. AND GEORGE W. MASTERS (1966) "On-Line Estimation of States and Parameters for Discrete Non-Linear Dynamic Systems". Vol. 12, pp. 677-682.

SAGE, ANDREW P. AND JAMES. L. MELSA (1971) "Estimation Theory with Applications to Communications and Control". McGraw-Hill.

SAGE, ANDREW and Chelsea C. WHITE, III "Optimum Systems Control", 2nd Ed. 1977. Prentice Hall.

SCHALTZ, DONALD G., AND JAMES L. MELSA (1967) "State Functions and Linear Control Systems". McGraw-Hill.

SCHOLES, MYRON, JF (1976) "Taxes and the Pricing of Options".

SCHUSS ZEEV (1980) "Theory and Applications of Stochastic Differential Equations". John Wiley.

SCHWARTZ, EDUARDO S. JFE (1977) "The Valuation of Warrants: Implementing a New Approch", pp. 79-93.

SCHWEPPE, FRED C. (1973) "Uncertain Dynamic System". Prentice-Hall.

SKORCKHOD, A.V. (1965) "Studies in the Theory of Random Processes.

SMITH, CLIFFORD W. JR., JFE (1976) "Option Pricing", pp. 3-51.

MERTON, ROBERT C. (1976) "Option Pricing When Underlying Stock Returns and Discontinuous", pp. 125-144.

STOLL, HANS R., JF (1969) "The Relationship Between Put and Call Option Prices". Vol. XXIV, No. 5.

STRANG, GILBERT AND GEORGE J. FIX (1973) "An Analysis of the Finite Element Method". Prentice-Hall.

TAYLOR, STEPHEN J., JF (1982) "Tests of the Random Walk Hypothesis Against a Price-Trend Hypothesis". Vol. XVII, No. 1.

TUGNAIT, JITENDRA K. (1980) "Identification and Model Approximation for Continuous-Time Systems on Finite Parameter Sets". IEEE Transactions on Automatic Control, Vol. AC-25, No. 6.

VARAIYA, PRAVIN (1973) "Optimal Control of a Partially Observed Stochastic System".

VASICEK, OLDRICH, JFE (1977) "An Equilibrium Characterization of the Term Structure", pp. 177-188.

VASICE, OLDRICH A. AND H. GIFFORD FONG, JF (1982) "Term Structure Modeling Using Exponential Splines". Vol. XXXVII, No. 2.

VERWER, J.G. (1980) "MBRK, An Explicit Time Integrator for Semidiscrete Parabolic Equations". Vol. 6, No. 2, pp. 236-239.

WHALEY, ROBERT E., JFE (1981) "On the Valuation of American Call Options on Stocks with Known Dividends", pp. 207-211.

WHALEY, ROBERT E., JFE (1982) "Valuation of American Call Options on Dividend-Paying Stocks", pp. 29-58.