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A BINARY REPRESENTATION FOR FINITE BOOLEAN FUNCTIONS

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A BINARY REPRESENTATION FOR FINITE BOOLEAN FUNCTIONS

CHAPTER I

INTRODUCTION

Finite Boolean algebras have been studied intensively since the discovery by Shannon [21] in 1938 that Boolean algebra can be applied to the analysis and synthesis of switching circuits. Many different mathematical methods have been used for representing Boolean functions. Among these methods are algebraic [2], [10], [12], [16], [17], and [18]; topological [19] and [23]; diagramatic [9] and [24]; and real analytic [3] and [14]. Harrison [6] provides an excellent exposition on all of the above methods. Descriptions of all of the methods except the last are given by Hu [7] and Miller [11].

In Chapter II of this paper, it is shown that the set of all Boolean functions of a finite Boolean algebra has the structure of a vector space over the field of integers modulo two, and that by a suitable change of basis, many structural properties of Boolean functions may be determined easily from the coordinates of functions. The structural properties of functions which are studied include symmetry, independence of a variable, self-duality, anti-self-duality, and linearity.

From an engineering viewpoint it is natural to call two Boolean functions "equivalent" if the two functions can be mechanized with the same switching circuit of a given type. Several different definitions

of equivalence are commonly used depending upon the type of switching circuits being considered. In Chapter III it is shown that a canonical representative may be assigned to each equivalence class of functions based on the coordinates introduced in Chapter II. For each of six definitions of equivalence, algorithms are developed for finding the canonical representatives of an arbitrary Boolean function.

The following definitions and theorems or their equivalents may be found in any of the references [4], [6], [7], [11], [14], and [22]. The theorems are stated here without proof. A Boolean algebra $\langle B, \vee, \cdot, ', 0, 1\rangle$ is a set B with binary operations $\vee$ (logical sum, disjunction, or, join) and $\cdot$ (logical product, conjunction, and, meet), a unary operation $'$ (complement, negation, not), and distinguished elements 0 and 1 satisfying the following postulates essentially due to Huntington [8]:

(1) For all $a,b \in B$, $a \vee b = b \vee a$.

(2) For all $a,b \in B$, $ab = ba$.

(3) For all $a \in B$, $a \vee 0 = a$.

(4) For all $a \in B$, $a1 = a$.

(5) For all $a,b,c \in B$, $a \vee bc = (a \vee b)(a \vee c)$.

(6) For all $a,b,c \in B$, $a(b \vee c) = ab \vee ac$.

(7) For all $a \in B$, $a \vee a' = 1$ and $aa' = 0$.

As usual $a \cdot b$ is written ab, $\cdot$ takes precedence over $\vee$ if parentheses are omitted, and $ab' = a(b')$.

Theorem 1.1. For all $a,b,c \in B$, $(a \vee b) \vee c = a \vee (b \vee c)$ and $(ab)c = a(bc)$.

Theorem 1.2. For all $a \in B$, $a \vee a = a$ and $aa = a$.

Theorem 1.3. For all $a \in B$, $a \vee 1 = 1$ and $a0 = 0$.

Theorem 1.4. For all $a \in B$, $(a')' = a$.

Theorem 1.5. For all $a,b \in B$, $(a \vee b)' = a'b'$ and $(ab)' = a' \vee b'$.

Theorem 1.6. $0' = 1$ and $1' = 0$.

Theorem 1.7. For all $a,b \in B$, $ab \vee ab' = a$.

Let $B_1 = \langle \{0,1\}, \vee, \cdot, ', 0, 1 \rangle$ where the operations are defined by:

$0 \vee 0 = 0$ $\qquad$ $1 \cdot 1 = 1$

$0 \vee 1 = 1 \vee 0 = 1 \vee 1 = 1$ $\qquad$ $0 \cdot 0 = 0 \cdot 1 = 1 \cdot 0 = 0$

$0' = 1$ $\qquad$ $1' = 0$.

The symbol B_1 will be used to identify $\{0,1\}$ as well as the structure $\langle \{0,1\}, \vee, \cdot, ', 0, 1 \rangle$.

Theorem 1.8. B_1 is a Boolean algebra.

Let $B_n = \overset{n}{\underset{i=1}{X}} B_1$ be the n-fold direct product of B_1, i.e., $\vee$, $\cdot$, $'$, 0, and 1 are defined by components on the set $B_n = \overset{n}{\underset{i=1}{X}} B_1$:

$$(a_1,\ldots,a_n) \vee (b_1,\ldots,b_n) = (a_1 \vee b_1,\ldots,a_n \vee b_n)$$

$$(a_1,\ldots,a_n) \cdot (b_1,\ldots,b_n) = (a_1b_1,\ldots,a_nb_n)$$

$$(a_1,\ldots,a_n)' = (a_1',\ldots,a_n')$$

$$0 = (0,\ldots,0)$$

$$1 = (1,\ldots,1) .$$

Theorem 1.9. B_n is a Boolean algebra.

Theorem 1.10. Every finite Boolean algebra is isomorphic to B_n for some natural number n.

Theorem 1.11. B_n contains 2^n elements.

A Boolean function (switching function) of n variables is a mapping from B_n into B_1. Let F_n be the set of all Boolean functions of n variables and let $x = (x_1,\ldots,x_n) \in B_n$. The operations $\vee$, $\cdot$, $'$ and distinguished elements 0 and 1 are defined in F_n by:

$$(f \vee g)(x) = f(x) \vee g(x)$$

$$(fg)(x) = f(x)\cdot g(x)$$

$$f'(x) = (f(x))'$$

$$0(x) = 0$$

$$1(x) = 1 .$$

The symbol F_n will be used to identify $\langle F_n, \vee, \cdot, ', 0, 1\rangle$.

A subset G of a Boolean algebra B is a set of generators for B if every element of B can be expressed as a logical sum of logical products of elements of G or elements whose complements are elements of G. A set G of generators for B is a set of free generators for B if every mapping of G into an arbitrary Boolean algebra C can be extended to a homomorphism of B into C. A Boolean algebra is free if it contains a set of free generators.

Theorem 1.12. F_n is a free Boolean algebra with n generators and 2^{2^n} elements.

Let i be an integer such that $0 \leq i \leq 2^n - 1$ and let $i_n\ldots i_1$ be the binary number equivalent to i. The atom (minimal polynomial, minterm) $a_i \in F_n$ is defined by

$$a_i(x) = \begin{cases} 1 \text{ if } x_1 = i_1,\ldots,x_n = i_n \\ 0 \text{ otherwise.} \end{cases}$$

Theorem 1.13. There are 2^n atoms in F_n.

Theorem 1.14. If $i \neq j$, $a_i a_j = 0$.

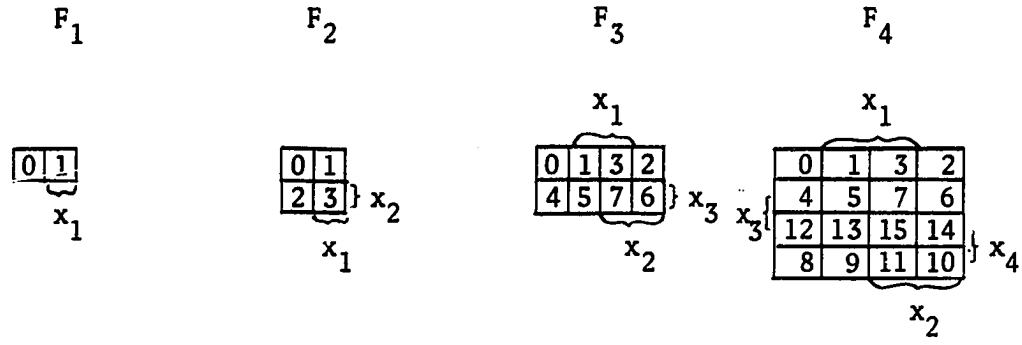
Theorem 1.15. Each $f \in F_n$, can be expressed uniquely as a logical sum of atoms.

If m is an integer such that $1 \leq m \leq n$, the symbol x_m may be interpreted as one of the components (variables) of $x = (x_1,\ldots,x_n)$ or as a function $x_m \in F_n$ defined by $x_m(x) = 1$ if the variable $x_m = 1$ and $x_m(x) = 0$ if the variable $x_m = 0$. The symbol x_m' denotes the complement of the function x_m. The functions x_m and x_m' are called literals. If $i_m \in \{0,1\}$, the literal $x_m^{i_m}$ is defined by $x_m^1 = x_m$ and $x_m^0 = x_m'$. If $\{i_1,\ldots,i_m\} \subset \{1,\ldots,n\}$ and $j_k \in \{0,1\}$ for $k = 1,\ldots,m$, the logical product $x_{i_1}^{j_1}\cdots x_{i_m}^{j_m} \in F_n$ is called a fundamental product. The 1-function is considered to be the fundamental product corresponding to $\{i_1,\ldots,i_m\} = \emptyset$.

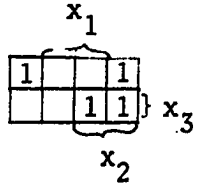
Theorem 1.16. If $a_i = a_{i_n\ldots i_1}$ is an atom of F_n, $a_{i_n\ldots i_1} = x_n^{i_n}\cdots x_1^{i_1}$.

If a_i and a_j are atoms of F_n, $a_{i_n\ldots i_1}$ is adjacent to $a_{j_n\ldots j_1}$, provided there is exactly one integer $m \in \{1,\ldots,n\}$ such that $i_m \neq j_m$. If a_i and a_j are adjacent atoms in F_n, the function $a_i \vee a_j$ can be expressed as a fundamental product of n - 1 literals using Theorems 1.16 and 1.7.

One of the most useful methods of representing Boolean functions utilizes modified Venn diagrams called Veitch diagrams [24] or Karnaugh maps [9]. Veitch diagrams for representing functions in F_1, F_2, F_3, and F_4 are given below:



The Veitch diagram for F_n is divided into 2^n squares corresponding to the 2^n atoms of F_n. The integer i appearing in a square of a diagram identifies that square with the atom a_i. If $x_i \in F_n$, one-half of the atoms of F_n are dominated by x_i and the other half of the atoms are dominated by x_i'. The braces and symbols x_i appearing at the edges of each diagram indicate those atoms dominated by x_i. In view of Theorem 1.15, a Boolean function $f \in F_n$ may be represented by a Veitch diagram in which 1 appears in the square corresponding to a_i iff f dominates the atom a_i, i.e., $fa_i = 1$. For example, the function $f \in F_3$ given by $f = a_0 \vee a_2 \vee a_6 \vee a_7 = a_{000} \vee a_{010} \vee a_{110} \vee a_{111}$ is represented by the Veitch diagram:



A major feature of Veitch diagrams is the connection between physical adjacency of squares and adjacency of atoms defined on p. 5. In the diagram for F_3 consider the left and right edges to be identified so that square 0 is adjacent to square 2 and square 4 is adjacent to square 6. In the diagram for F_4 consider the left and right edges

to be identified and the top and bottom to be identified so that square 0 is adjacent to squares 2 and 8, etc. Then it is easy to see that the atoms a_i and a_j are adjacent iff squares i and j are adjacent. Veitch diagrams are particularly useful in finding minimal algebraic expressions for functions as a logical sum of fundamental products where minimal is defined in the sense of fewest literals. For example, the minimal expression for the function given in the preceding Veitch diagram is $f = x_3'x_1' \vee x_3x_2$.

Veitch diagrams may be extended readily to functions of more than four variables by using the appropriate number of copies of the diagram for four variables. For example, the diagram for functions in F_5 consists of one copy of the F_4 diagram with squares labeled 0 through 15 and a second copy of the F_4 diagram with squares labeled 16 through 31. In the case of F_5, squares in the same relative position of the two F_4 diagrams are considered to be adjacent.

All theorems, corollaries, lemmas, and algorithms appearing in Chapters II and III are believed to be original with two exceptions. Theorem 2.1 is essentially the same as problem 6, p. 61 of Harrison's book [6], and Corollary 2.9 is a well-known result (p. 173, [6]).

CHAPTER II

THE FUNDAMENTAL REPRESENTATION OF BOOLEAN FUNCTIONS

In $B_1 = \{0,1\}$, the binary operation $\oplus$ (modulo two addition, exclusive or) is defined by $0 \oplus 0 = 1 \oplus 1 = 0$ and $0 \oplus 1 = 1 \oplus 0 = 1$. Obviously $\langle B_1, \oplus, 0\rangle$ is an abelian group with identity 0. If $\oplus$ is extended in the standard manner to B_n and F_n, it is clear that $\langle B_n, \oplus, 0\rangle$ and $\langle F_n, \oplus, 0\rangle$ are abelian groups with identities $0 = (0,\ldots,0)$ and the 0-function, respectively. Let Z_2 be the field of integers modulo two and define scalar product by $0f = 0$ and $1f = f$ for all $f \in F_n$. Then clearly F_n has the structure of a vector space over Z_2. Note that $0f$ and $1f$ may be interpreted as scalar multiples of f or as logical products of the 0-function with f and the 1-function with f, respectively. However, with either interpretation $0f = 0$ and $1f = f$.

Each of the 2^n atoms of F_n may be represented as a 2^n-tuple with a single 1 and the remaining positions 0. For $i = 0,\ldots,2^n - 1$, let $a_i = a_{i_n \ldots i_1}$ be the 2^n-tuple with 1 in position $i + 1$ from the left so that $a_0 = (1,0,\ldots,0)$, $a_1 = (0,1,0,\ldots,0)$, $\ldots$, $a_{2^n-1} = (0,\ldots,0,1)$. Then the ordered set $A_n = \{a_i : i = 0,\ldots,2^n-1\}$ is the standard basis for F_n as a vector space over Z_2 so F_n has dimension 2^n. If $g_1, g_2, \ldots, g_m \in F_n$, let

$$\sum_{i=1}^{m} g_i = g_1 \vee g_2 \vee \ldots \vee g_m$$

and let

$$\bigoplus_{i=1}^{m} g_i = g_1 \oplus g_2 \oplus \ldots \oplus g_m .$$

By Theorem 1.15, each $f \in F_n$ has unique representation $f = \sum_{i=0}^{2^n-1} f_i a_i$ where $f_i = fa_i$. By Theorem 1.14, $a_i a_j = 0$ if $i \neq j$. Hence

$$a_i \oplus a_j = a_i \vee a_j \text{ if } i \neq j.$$

Hence

$$f = \sum_{i=0}^{2^n-1} f_i a_i = \bigoplus_{i=0}^{2^n-1} f_i a_i .$$

The ordered set, set A_n will be called the <u>atomic basis</u> for F_n,

$$\bigoplus_{i=0}^{2^n-1} f_i a_i$$

will be called the <u>atomic representation</u> of f, and f_i, $i = 0,\ldots,2^n-1$ will be called the <u>atomic coordinates</u> of f. The atomic representation

$$\bigoplus_{i=0}^{2^n-1} f_i a_i$$

of f also may be written as the vector $(f_i) = (f_0,\ldots,f_{2^n-1})$. If $f = (f_i)$ and $g = (g_i)$, then $f \oplus g = (f_i \oplus g_i)$ and $f' = 1 \oplus f = (1) \oplus (f_i) = (1 \oplus f_i)$.

Let $a_i = a_{i_n\ldots i_1}$ be a fixed atom and let $c_j^i = c_{j_n\ldots j_1}^{i_n\ldots i_1}$ be the fundamental product $(x_n)_{j_n}^{i_n} \ldots (x_1)_{j_1}^{i_1}$ where

$$(x_k)_0^{i_k} = 1, \ (x_k)_1^0 = x_k', \text{ and } (x_k)_1^1 = x_k .$$

Then $C_n^i = \{c_j^i : j = 0,\ldots,2^n-1\}$ is the set of all fundamental products which include the atom a_i. Let $a_k = a_{k_n\ldots k_1}$ be an atom. Then

$$c_{j_n\ldots j_1}^{i_n\ldots i_1} = 1$$

for the atom a_k iff for each m = 1,...,n either $j_m = 0$ or $k_m = i_m$. Let

$$c^{i}_{j,k} = c^{i_n \cdots i_1}_{j_n \cdots j_1, k_n \cdots k_1}$$

be the k'th atomic coordinate of c^{i}_{j}. Then $c^{i}_{j,k} = 1$ iff for each m = 1,...,n either $j_m = 0$ or $k_m = i_m$. For example C^5_3 is the set of all fundamental products of three variables which include the atom $a_5 = a_{101} = (0,0,0,0,0,1,0,0)$. The elements of C^5_3 are:

$$c^5_0 = c^{101}_{000} = 1 = (1,1,1,1,1,1,1,1)$$

$$c^5_1 = c^{101}_{001} = x_1 = (0,1,0,1,0,1,0,1)$$

$$c^5_2 = c^{101}_{010} = x_2' = (1,1,0,0,1,1,0,0)$$

$$c^5_3 = c^{101}_{011} = x_2'x_1 = (0,1,0,0,0,1,0,0)$$

$$c^5_4 = c^{101}_{100} = x_3 = (0,0,0,0,1,1,1,1)$$

$$c^5_5 = c^{101}_{101} = x_3x_1 = (0,0,0,0,0,1,0,1)$$

$$c^5_6 = c^{101}_{110} = x_3x_2' = (0,0,0,0,1,1,0,0)$$

$$c^5_7 = c^{101}_{111} = x_3x_2'x_1 = (0,0,0,0,0,1,0,0)$$

Theorem 2.1. The ordered set $C^{2^n-1}_n = \{c^{2^n-1}_j : j = 0,1,\ldots,2^n-1\}$ is a basis for F_n. Let T_n be the matrix of the coordinate transformation from the $C^{2^n-1}_n$ basis to the A_n basis. Then

$$T_n^{-1} = T_n = \begin{bmatrix} T_{n-1} & T_{n-1} \\ 0 & T_{n-1} \end{bmatrix}$$

where $T_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

Proof. That $C_n^{2^n-1}$ is a basis for F_n will follow immediately if T_n can be shown to have an inverse. Let $T_n = [t_{ij}]$. Then t_{ij} is the j'th coefficient in the atomic representation of $c_i^{2^n-1}$. The structure of T_n can be shown by induction. The ordered set

$$C_1^1 = \{1, x_1\} = \{(1,1), (0,1)\}.$$

Hence

$$T_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

and

$$T_1 T_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

so

$$T_1^{-1} = T_1 .$$

Assume that $T_k^{-1} = T_k$. From the definition of

$$C_{k+1}^{2^{k+1}-1} = \{c_j^{2^{k+1}-1} : j = j_{k+1} \cdots j_1 = 0, \ldots, 2^{k+1}-1\},$$

the first 2^k elements of $C_{k+1}^{2^{k+1}-1}$ do not contain x_{k+1} or x_{k+1}' as a factor since $j_{k+1} = 0$. Hence the first 2^k coordinates in atomic representation for each $c_j^{2^{k+1}-1}$, $j = 0, \ldots, 2^k-1$, are the same as the coordinates of $c_j^{2^k-1}$. The second 2^k coordinates of $c_j^{2^{k+1}-1}$ are ordered duplicates of the first 2^k coordinates. Hence

$$T_{k+1} = \begin{bmatrix} T_k & T_k \\ A & B \end{bmatrix}$$

where the second 2^k rows of T_{k+1} (the blocks A and B) have yet to be determined. For $j = 2^k, \ldots, 2^{k+1}-1$, $j_{k+1} = 1$ so $c_j^{2^{k+1}-1}$ has x_{k+1} as a factor and the first 2^k coordinates of $c_j^{2^{k+1}-1}$ are 0. The second 2^k

coordinates of $c_j^{2^{k+1}-1}$ are the coordinates of $c_j^{2^k-1}$. Hence the blocks A and B are 0 and T_k, respectively. Finally,

$$T_{k+1}T_{k+1} = \begin{bmatrix} T_k & T_k \\ 0 & T_k \end{bmatrix} \cdot \begin{bmatrix} T_k & T_k \\ 0 & T_k \end{bmatrix} = \begin{bmatrix} T_kT_k & T_kT_k \oplus T_kT_k \\ 0 & T_kT_k \end{bmatrix} .$$

By the induction hypothesis $T_kT_k = I$, and $I \oplus I = 0$. Hence $T_{k+1}^{-1} = T_{k+1}$.

Let $a_i = a_{i_n \ldots i_1}$ be an atom and let P_i be the $2^n \times 2^n$ permutation matrix obtained by interchanging rows of the $2^n \times 2^n$ identity matrix according to the following rule: row $k = k_n \ldots k_1$ is interchanged with row $h = h_n \ldots h_1$ provided $k_m = h_m$ if $i_m = 1$ and $k_m = h_m'$ if $i_m = 0$ for $m = 1, \ldots, n$.

Theorem 2.2. Let $a_i = a_{i_n \ldots i_1}$ be an atom. The ordered set $C_n^i = \{c_j^i : j = 0, \ldots, 2^n-1\}$ is a basis for F_n and the matrix T_n^i of the coordinate transformation from the C_n^i basis to the A_n basis is T_nP_i.

Proof. Let $T_n^i = [t_{jk}^i]$. Then t_{jk}^i is the k'th coordinate of c_j^i expressed relative to the A_n basis. Recall that $c_{j_n \ldots j_1}^{i_n \ldots i_1} = 1$ for the atom $a_{k_n \ldots k_1}$ iff for each $m = 1, \ldots, n$ $j_m = 0$ or $k_m = i_m$. Hence the atomic coordinates of $c_{j_n \ldots j_1}^{i_n \ldots i_1}$ are a permutation of the atomic coordinates of $c_{j_n \ldots j_1}^{1 \ldots 1}$ according to the following rule: the coordinate in position $k_n \ldots k_1$ of $c_{j_n \ldots j_1}^{i_n \ldots i_1}$ is interchanged with the coordinate in position $h_n \ldots h_1$ provided $k_m = h_m$ if $i_m = 1$ and $k_m = h_m'$ if $i_m = 0$ for $m = 1, \ldots, n$. By the definition of P_i, multiplication of T_n on the right by P_i effects an interchange of columns of $T_n = T_n^{2^n-1}$ according to the preceding rule. Since $|T_n| = 1$ and $|P_i| = 1$, it follows that $|T_n^i| = |T_n| \cdot |P_i| = 1$ so C_n^i is a basis for F_n.

Each $f \in F_n$ may be represented as

$$f = \bigoplus_{j=0}^{2^n-1} f_j^i \, c_j^i$$

or as a vector (f_j^i) of length 2^n relative to the ordered basis C_n^i. The ordered basis C_n^i will be called the fundamental basis for F_n relative to a_i and the f_j^i will be called the fundamental coordinates (coefficients) of f relative to a_i.

Theorem 2.3. Let $f = (f_j^i)$ and let $g = (g_j^i)$. Then $f \oplus g = (f_j^i \oplus g_j^i)$ and $f' = (1 \oplus f_0^i, f_1^i, \ldots, f_{2^n-1}^i)$.

Proof.

$$f \oplus g = \bigoplus_{j=0}^{2^n-1} f_j^i \, c_j^i \oplus \bigoplus_{j=0}^{2^n-1} g_j^i \, c_j^i = \bigoplus_{j=0}^{2^n-1} (f_j^i \oplus g_j^i) \, c_j^i \, .$$

Note that $f' = 1 \oplus f$ and $c_0^i = 1$. Hence

$$f' = 1 \oplus f = 1 \oplus \bigoplus_{j=0}^{2^n-1} f_j^i \, c_j^i = 1 \oplus f_0^i \, c_0^i \oplus \bigoplus_{j=1}^{2^n-1} f_j^i \, c_j^i =$$

$$(1 \oplus f_0^i) \, c_0^i \oplus \bigoplus_{j=1}^{2^n-1} f_j^i \, c_j^i \, .$$

Suppose f is given in atomic coordinates by $f = (f_j)$ and in fundamental coordinates relative to a_i by $f = (f_j^i)$. Then $(f_j) = (f_j^i) T_n^i$ and

$$(f_j^i) = (f_j)[T_n^i]^{-1} = (f_j)[T_n P_i]^{-1} = (f_j) \, P_i^{-1} \, T_n .$$

The group of all permutation matrices of dimension 2^n under multiplication is isomophic to the symmetric group S_{2^n} [1]. Since P_i is obtained

from I by simple interchanges of rows, the image of P_i in S_{2^n} is a product of disjoint cycles of length two. Hence $P_i^{-1} = P_i$ and $(f_j^i) = (f_j)\, P_i T_n$. Hence $[T_n^i]^{-1}$ may be obtained from T_n by interchanging row $k_n \ldots k_1$ with row $h_n \ldots h_1$ provided $k_m = h_m$ if $i_m = 1$ and $k_m = h_m'$ if $i_m = 0$ for $m = 1,\ldots,n$. It is important to note that $[T_n^i]^{-1} = T_n^i$ and that T_n^i has a very convenient block form only if $i = 2^n - 1$. In particular, $(f_j) = (f_j^{2^n-1})T_n$ and $(f_j^{2^n-1}) = (f_j)T_n$. Let $T_n = [t_{jk}]$. Then

$$t_{jk} = c_{j,k}^{2^n-1}$$

which is the k'th atomic coordinate of $c_j^{2^n-1}$. Hence $t_{jk} = 1$ iff $j_m = 0$ or $k_m = 1$. Hence

$$f_k^{2^n-1} = \sum_{j=0}^{2^n-1} f_j\, t_{jk} = \sum_{j \in J_k} f_j$$

where

$$J_k = \{j = j_n \ldots j_1 : j_m = 0 \text{ or } k_m = 1,\ m = 1,\ldots,n\}\ .$$

For example in case n = 3:

$$J_{000} = \{000\} \qquad f_0^7 = f_{000}$$

$$J_{001} = \{000,001\} \qquad f_1^7 = f_{000} \oplus f_{001}$$

$$J_{010} = \{000,010\} \qquad f_2^7 = f_{000} \oplus f_{010}$$

$$J_{011} = \{000,001,010,011\} \qquad f_3^7 = f_{000} \oplus f_{001} \oplus f_{010} \oplus f_{011}$$

$$J_{100} = \{000;100\} \qquad f_4^7 = f_{000} \oplus f_{100}$$

$$J_{101} = \{000,001,100,101\} \qquad f_5^7 = f_{000} \oplus f_{001} \oplus f_{100} \oplus f_{101}$$

$$J_{110} = \{000,010,100,110\} \qquad f_6^7 = f_{000} \oplus f_{010} \oplus f_{100} \oplus f_{110}$$

$$J_{111} = \{000,001,010,011,\ 100,101,110,111\} \qquad f_7^7 = \sum_{j=0}^{7} f_j$$

Let $[T_n^i]^{-1} = [u_{jk}^i] = P_i[t_{jk}]$. Then $u_{jk}^i = 1$ iff $j_m = i_m'$ or $k_m = 1$ for $m = 1,\ldots,n$. Hence

$$f_k^i = \sum_{j=0}^{2^n-1} f_j \, u_{jk}^i = \sum_{j \in J_k^i} f_j$$

where

$$J_k^i = \{j = j_n \cdots j_1 : j_m = i_m' \text{ or } k_m = 1,\ m = 1,\ldots,n\}.$$

Note that

$$J_k = J_k^{2^n-1} .$$

Let

$$M_k^i = \sum_{j \in J_k^i} a_j .$$

Then $f_k^i = 0(1)$ iff $f = 1$ for an even (odd) number of atoms in M_k^i. From the structure of J_k^i it can be seen that M_k^i may be written as the fundamental product

$$(x_n)_{k_n'}^{i_n'} \cdots (x_1)_{k_1'}^{i_1'} ,$$

where

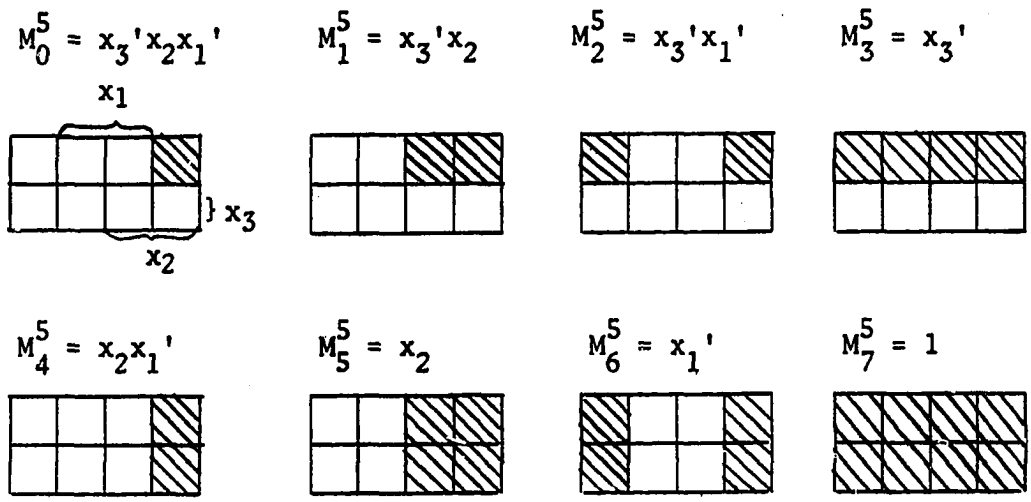
$$(x_m)_{1'}^{i_m'} = (x_m)_0^{i_m'} = 1, \quad (x_m)_{0'}^{0'} = (x_m)_1^1 = x_m ,$$

and

$$(x_m)_{0'}^{1'} = (x_m)_1^0 = x_m' \text{ for } m = 1,\ldots,n.$$

Note that $\{M_k^i : k = 0,\ldots,2^n-1\}$ is the set of all fundamental products which include the atom $a_{i_n'\ldots i_1'}$. In terms of Veitch diagrams, each M_k^i can be drawn as a mask with shading in those areas for which $M_k^i = 1$. If f is given by a Veitch diagram, $f_k^i = 1$ iff f has an odd number of "ones" in the shaded region of M_k^i.

For example let n = 3 and i = 5. The masks M_k^5 are:



Let $f(x_3,x_2,x_1) = x_1'(x_2 \vee x_3') \vee x_3x_2x_1$. As a sum of minimal polynomials $f(x_3,x_2,x_1) = x_3'x_2'x_1' \vee x_3'x_2x_1' \vee x_3x_2x_1' \vee x_3x_2x_1$ and in atomic form $f = a_{000} \oplus a_{010} \oplus a_{110} \oplus a_{111}$ so $(f_j) = (1,0,1,0,0,0,1,1)$. The Veitch diagram for f is:

1			1
		1	1

Comparing the Veitch diagram for f to the masks M_0^5 through M_7^5 yields $(f_k^5) = (1,1,0,0,0,1,1,0)$. The fundamental coordinates of f relative to a_5 also could be obtained by multiplying (f_j) on the right by $[T_3^5]^{-1} = P_5T_3$:

$$[T_3^5]^{-1} = \begin{bmatrix} 0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0 \end{bmatrix} \cdot \begin{bmatrix} 1&1&1&1&1&1&1&1 \\ 0&1&0&1&0&1&0&1 \\ 0&0&1&1&0&0&1&1 \\ 0&0&0&1&0&0&0&1 \\ 0&0&0&0&1&1&1&1 \\ 0&0&0&0&0&1&0&1 \\ 0&0&0&0&0&0&1&1 \\ 0&0&0&0&0&0&0&1 \end{bmatrix} = \begin{bmatrix} 0&0&1&1&0&0&1&1 \\ 0&0&0&1&0&0&0&1 \\ 1&1&1&1&1&1&1&1 \\ 0&1&0&1&0&1&0&1 \\ 0&0&0&0&0&0&1&1 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&1&1&1&1 \\ 0&0&0&0&0&1&0&1 \end{bmatrix}$$

$$(f_k^5) = (1,0,1,0,0,0,1,1) \begin{bmatrix} 0&0&1&1&0&0&1&1 \\ 0&0&0&1&0&0&0&1 \\ 1&1&1&1&1&1&1&1 \\ 0&1&0&1&0&1&0&1 \\ 0&0&0&0&0&0&1&1 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&1&1&1&1 \\ 0&0&0&0&0&1&0&1 \end{bmatrix} = (1,1,0,0,0,1,1,0)$$

The rows of $[T_3^5]^{-1}$ are the fundamental coordinates of the atoms relative to a_5. Since f is the modulo two sum of the atoms dominated by f, Theorem 2.3 may be used to compute (f_k^5) as follows:

$$\begin{aligned} (f_k^5) &= (0,0,1,1,0,0,1,1) \\ &\oplus (1,1,1,1,1,1,1,1) \\ &\oplus (0,0,0,0,1,1,1,1) \\ &\oplus (0,0,0,0,0,1,0,1) \\ &= (1,1,0,0,0,1,1,0). \end{aligned}$$

Also by Theorem 2.3, the fundamental coordinates of f' relative to a_5 are given by $(f_k^5)' = (0,1,0,0,0,1,1,0)$.

Theorem 2.4. Let $\tau_m : F_n \to F_n$ be defined by

$$\tau_m(f(x_n,\ldots,x_m,\ldots,x_1)) = f(x_n,\ldots,x_m',\ldots,x_1),$$

i.e., τ_m complements variable x_m of f. Then

$$(\tau_m(f))_k^i = \begin{cases} f_k^i \text{ if } k_m = 1 \\ f_k^i \oplus f^i_{k_n\ldots k_m'\ldots k_1} \text{ if } k_m = 0. \end{cases}$$

Proof. Complementing x_m has the effect of changing the fundamental basis from

$$C_n^{i_n \cdots i_m \cdots i_1} \text{ to } C_n^{i_n \cdots i_m' \cdots i_1} .$$

Hence

$$(\tau_m(f))_k^{i_n \cdots i_m \cdots i_1} = f_k^{i_n \cdots i_m' \cdots i_1} .$$

Recall that $f_k^i = 1$ iff $f = 1$ for an odd number of atoms of

$$M_k^i = (x_n)_{k_n}^{i_n'} \cdots (x_1)_{k_1}^{i_1'}$$

where

$$(x_m)_{1'}^{i_m'} = 1, \ (x_m)_{0'}^{0'} = x_m ,$$

and

$$(x_m)_{0'}^{1'} = x_m' .$$

Hence $(\tau_m(f))_k^i = 1$ iff $f = 1$ for an odd number of atoms of $M_k^{i_n \cdots i_m' \cdots i_1}$.

If $k_m = 1$,

$$M_k^{i_n \cdots i_m' \cdots i_1} = M_k^{i_n \cdots i_m \cdots i_1}$$

so

$$(\tau_m(f))_k^i = f_k^i .$$

If $k_m = 0$,

$$M_{k_n \cdots k_m \cdots k_1}^{i_n \cdots i_m' \cdots i_1} = (x_n)_{k_n}^{i_n'} \cdots (x_m)_{0'}^{i_m} \cdots (x_1)_{k_1}^{i_1'}$$

$$= (x_n)_{k_n}^{i_n'} \cdots (x_m)_{0'}^{i_m'} \cdots (x_1)_{k_1}^{i_1'}$$

$$\oplus (x_n)_{k_n}^{i_n'} \cdots (x_m)_{1'}^{i_m'} \cdots (x_1)_{k_1}^{i_1'}$$

$$= M^{i_n \ldots i_m \ldots i_1}_{k_n \ldots k_m \ldots k_1} \oplus M^{i_n \ldots i_m \ldots i_1}_{k_n \ldots k_m' \ldots k_1} .$$

Hence $(\tau_m(f))^i_k = f^i_k \oplus f^i_{k_n \ldots k_m' \ldots k_1}$.

As an example of the application of Theorem 2.4, consider the function of the previous example:

$$(f^5_k) = (f^{101}_{000}, f^{101}_{001}, f^{101}_{010}, f^{101}_{011}, f^{101}_{100}, f^{101}_{101}, f^{101}_{110}, f^{101}_{111})$$

$$= (1 , 1 , 0 , 0 , 0 , 1 , 1 , 0) .$$

Complementing x_1 produces

$$(\tau_1(f))^{1\,0\,1}_{k_3 k_2 1} = f^{1\,0\,1}_{k_3 k_2 1} \text{ and } (\tau_1(f))^{1\,0\,1}_{k_3 k_2 0} = f^{1\,0\,1}_{k_3 k_2 0} \oplus f^{1\,0\,1}_{k_3 k_2 1} .$$

Hence the fundamental coordinates of $\tau_1(f)$ relative to a_5 are $((\tau_1(f))^5_k = (0,1,0,0,1,1,1,0)$. Note that $(\tau_1(f))^5_k = f^4_k$.

A function $f \in F_n$ is said to be <u>independent of x_m</u> if there is an algebraic expression for f as logical sum of fundamental products which include neither x_m nor x_m' as a factor.

<u>Theorem 2.5</u>. The function f is independent of x_m iff $f^i_k = 0$ for all $k = k_n \ldots k_m \ldots k_1$ with $k_m = 1$.

Proof. Clearly f is independent of x_m iff $\tau_m(f) = f$. Hence f is independent of x_m iff $(\tau_m(f))^i_k = f^i_k$ for $k = 0,\ldots,2^n-1$. Hence by Theorem 2.4 f is independent of x_m iff $f^i_k = 0$ for all k such that $k_m = 1$.

The <u>dual</u> f^D of a function f is defined by $f^D(x_1,\ldots,x_n) = f'(x_1',\ldots,x_n')$. A function f is said to be <u>self-dual</u> if $f^D = f$ and <u>anti-self-dual</u> if $f^D = f'$.

Theorem 2.6. The function f is anti-self-dual iff $f^0_k = f^{2^n-1}_k$ for $k = 0,\ldots,2^n-1$; f is self-dual iff $f^0_0 = 1 \oplus f^{2^n-1}_0$ and $f^0_k = f^{2^n-1}_k$ for $k = 1,\ldots,2^n-1$.

Proof. Complementing all variables has the effect of changing the fundamental basis C^0_n to $C^{2^n-1}_n$. Hence f is anti-self-dual iff $f^0_k = f^{2^n-1}_k$ for all k. By Theorem 2.3, complementing f changes the fundamental coordinates of f relative to a_i only by changing f^i_0 to $1 \oplus f^i_0$. Hence f is self-dual iff $f^0_0 = 1 \oplus f^{2^n-1}_0$ and $f^0_k = f^{2^n-1}_k$ for $k = 1,\ldots,2^n-1$.

Let σ be a permutation on $\{1,\ldots,n\}$. Then σ may also be considered to be a permutation of the digits $k_n\ldots k_1$, or as a mapping of F_n into F_n which permutes the variables of a function f. As indicated by the following theorem, the fundamental bases relative to a_0 and a_{2^n-1} are especially convenient for considering the effects of permuting the variables of a function.

Theorem 2.7. If σ is a permutation on $\{1,\ldots,n\}$ then

$$(\sigma(f))^0_{k_n\ldots k_1} = f^0_{\sigma(k_n\ldots k_1)} \quad \text{and} \quad (\sigma(f))^{2^n-1}_{k_n\ldots k_1} = f^{2^n-1}_{\sigma(k_n\ldots k_1)}.$$

Proof. The elements of the fundamental basis C^0_n are given by $c^0_k = (x_n)^0_{k_n}\ldots(x_1)^0_{k_1}$ where $(x_m)^0_0 = 1$ and $(x_m)^0_1 = x_m'$. Hence each variable which appears in the fundamental product c^0_k is complemented. Hence permuting the variables of f by σ simply reorders the elements of C^0_n by mapping

$$c^0_{k_n\ldots k_1} \text{ to } c^0_{\sigma(k_n\ldots k_1)}.$$

Hence

$$(\sigma f)^0_{k_n \ldots k_1} = f^0_{\sigma(k_n \ldots k_1)}.$$

Similarly, each variable which appears in the fundamental product $c_k^{2^n-1}$ is uncomplemented so

$$(\sigma f)^{2^n-1}_{k_n \ldots k_1} = f^{2^n-1}_{\sigma(k_n \ldots k_1)} .$$

The binary number $k_n \ldots k_1$ is said to be of <u>length</u> m if exactly m of the binary digits $k_n, \ldots, k_1$ are 1. The fundamental product c_k^i or the coordinate f_k^i is of length m provided $k = k_n \ldots k_1$ is of length m. Evidently there are $\binom{n}{m}$ fundamental products or coordinates of length m for $m = 0, \ldots, n$. In investigating the structure of functions it is convenient to arrange the c_k^i and the f_k^i in descending order of length, and within groups of fundamental products or coordinates of the same length, in decreasing values of k. For example the elements of C_3^5 are ordered as follows:

$$c^{101}_{111} \quad c^{101}_{110} \quad c^{101}_{101} \quad c^{101}_{011} \quad c^{101}_{100} \quad c^{101}_{010} \quad c^{101}_{001} \quad c^{101}_{000} .$$

A function $f \in F_n$ is said to be <u>symmetric</u> if f is invariant under all permutations of variables.

<u>Theorem 2.8</u>. The function f is symmetric iff all coordinates f_k^0 of the same length have the same value; f is symmetric iff all $f_k^{2^n-1}$ of the same length have the same value.

Proof. Each permutation σ on $\{1, \ldots, n\}$ when applied to $k_n \ldots k_1$ does not change the length of $k_n \ldots k_1$. Hence by Theorem 2.7, f is symmetric iff all f_k^0 (or $f_k^{2^n-1}$) of the same length have the same value.

Theorem 2.8 provides a simple proof of a well-known result [6] on the number symmetric functions:

Corollary 2.9. There are 2^{n+1} symmetric functions of n variables.

Proof. For n variables there are n + 1 possible lengths for k (0,1,...,n) and all coordinates of each length may be either 0 or 1 for a symmetric function.

A function $f \in F_n$ is said to be linear if it is of the form

$$f = e_0 \oplus e_1x_1 \oplus \ldots \oplus e_nx_n = \sum_{m=0}^{n} \oplus e_mx_m$$

where $e_m = 0$ or 1 for $m = 0,1,\ldots,n$ and $x_0 = 1$. A linear function f is said to be even (odd) if $e_0 = 0$ (1).

Theorem 2.10. A function is linear iff $f_k^0 = 0$ ($f_k^{2^n-1} = 0$) for all k of length greater than one. A linear function is even iff $f_0^{2^n-1} = 0$.

Proof. The fundamental coordinates of the 0 function are all 0 relative to any atom, i.e., $0_k^i = 0$ for $k = 0,\ldots,2^n-1$. Hence by Theorem 2.3 the fundamental coordinates of the 1 function are $1_0^i = 1$ and $1_k^i = 0$ for $k = 1,\ldots,2^n-1$. Hence $(e_0)_k^i = 0$ for $k = 1,\ldots,2^n-1$. Note that $c_{2^{m-1}}^0 = x_m'$ and $c_{2^{m-1}}^{2^n-1} = x_m$ for $m = 1,\ldots,n$. Hence the fundamental coordinates of the function e_mx_m are

$$(e_mx_m)_k^0 = \begin{cases} e_m \text{ if } k = 0 \text{ or } 2^{m-1} \\ 0 \text{ if } k \neq 0 \text{ and } k \neq 2^{m-1} \end{cases}$$

and

$$(e_mx_m)_k^{2^n-1} = \begin{cases} e_m \text{ if } k = 2^{m-1} \\ 0 \text{ if } k \neq 2^{m-1} \end{cases} \quad \text{for } m = 1,\ldots,n.$$

By Theorem 2.3, the fundamental coordinates of the linear function

$$f = \sum_{m=0}^{n} e_m x_m$$

are given by

$$f_k^0 = \sum_{m=0}^{n} (e_m x_m)_k^0 \quad \text{and} \quad f_k^{2^n-1} = \sum_{m=0}^{n} (e_m x_m)_k^{2^n-1} .$$

If $k = 0$, k has length 0, and $k = 2^{m-1}$ has length 1. Hence

$$(e_m x_m)_k^0 = (e_m x_m)_k^{2^n-1} = 0$$

provided k has length greater than one. Hence the conditions $f_k^0 = 0$ and $f_k^{2^n-1} = 0$ for all k of length greater than one are necessary for linearity. From the definition of a linear function it is clear that there are 2^{n+1} linear functions of n variables. Since there are $n + 1$ coordinate positions of length 0 or 1, all linear functions are accounted for under the described conditions so these conditions are also sufficient for linearity. Finally, $(e_m x_m)_0^{2^n-1} = 1$ iff $m = 0$ and $e_0 = 1$. Hence a linear function f is even iff $f_0^{2^n-1} = 0$.

CHAPTER III

EQUIVALENCE CLASSES OF FUNCTIONS AND CANONICAL REPRESENTATIVES

There are 2^{2^n} Boolean functions of n variables which may be partitioned into equivalence classes with respect to several important equivalence relations on F_n. The following equivalence relations are discussed by Harrison [6] : Assume $f,g \in F_n$, then

(1) f is <u>P-equivalent</u> to g if there is a permutation σ on $\{x_1,\ldots,x_n\}$ such that $f(x_1,\ldots,x_n) = g(\sigma(x_1,\ldots,x_n))$;

(2) f is <u>N-equivalent</u> to g if there is a complementation operator τ on $\{x_1,\ldots,x_n\}$ (τ complements each variable in some subset of $\{x_1,\ldots,x_n\}$) such that $f(x_1,\ldots,x_n) = g(\tau(x_1,\ldots,x_n))$;

(3) f is <u>PN-equivalent</u> to g if f is P-equivalent to some $h \in F_n$ such that h is N-equivalent to g;

(4) f is <u>NP-equivalent</u> to g if f is P-equivalent to g or g';

(5) f is <u>NN-equivalent</u> to g if f is N-equivalent to g or g';

(6) f is <u>NPN-equivalent</u> to g if f is PN-equivalent to g or g'.

Equivalence in F_n was first studied systematically by Pólya [15]. Using the theory of group characters, Pólya was able to determine the number of PN-equivalence classes for 1, 2, 3, and 4 variables (there are 402 classes for 4 variables). Pólya's method was refined by Slepian

[10] who determined the number of PN-equivalence classes for 5 and 6 variables. The problem of counting the number of classes for equivalence relations other than PN has been studied by Ninomiya [13], Harrison [5], and others. Harrison [6] lists the number of equivalence classes for the six types of equivalence defined above for $n \leq 6$ variables.

Golomb [3] and Ninomiya [14] independently developed methods for assigning a canonical representative function to each NPN-equivalence class and procedures for determining the canonical representative corresponding to an arbitrary function. Using either method requires the assignment of an ordered set of 2^n integers to each function or equivalence class of functions of n variables. These integers are called "invariants" by Golomb and "coordinates" by Ninomiya. Ninomiya [14] lists tables of coordinates for the canonical representatives of the 14 NPN-equivalence classes of functions of 3 variables and the 222 NPN-equivalence classes of functions of 4 variables. Ninomiya [14] has also developed tables of minimal switching networks including minimal relay networks, diode circuits, vacuum tube circuits, and transistor circuits for the 402 PN-equivalance classes of functions of 4 variables. The minimal diode circuits are given algebraically as minimal logical sums of fundamental products. Harrison [6] uses tables of the "invariants" of Golomb with Ninomiya's tables of minimal switching networks to reduce the problem of optimum design to a table look-up procedure for functions of up to 4 variables.

It is the purpose of this chapter to show that the fundamental coordinates for functions developed in Chapter II may be used in place of Golomb's "invariants" and Ninomiya's "coordinates" for the purposes

described above. The use of fundamental coordinates has the following advantages over the other methods:

(1) The fundamental coordinates of a function of n variables are 2^n binary digits instead of 2^n integers. (Each "invariant" is a non-negative integer $\leq 2^n$, and each "coordinate" is an integer with absolute value $\leq 2^{n-1}$.) Hence fundamental coordinates provide a more compact notation better suited for automatic computation.

(2) Every ordered set of 2^n binary digits form the fundamental coordinates of some function of n variables. Golomb [3] was not able to determine sufficient conditions for an ordered set of 2^n integers to be the "invariants" of a function. Ninomiya [14] gives a test requiring 2^n steps which will determine whether or not a given ordered set of 2^n integers form the "coordinates" of a function.

(3) The algorithms for determining the canonical representative corresponding to an arbitrary given function are, in general, easier to apply using fundamental coordinates than either of the other methods.

The fundamental coordinates $(f_k^{2^n-1}) = (f_{k_n \ldots k_1}^{1 \ldots 1})$ of a function $f \in F_n$ will be ordered as indicated in the discussion following Theorem 2.7, i.e., $f_{k_n \ldots k_1}^{1 \ldots 1}$ precedes $f_{h_n \ldots h_1}^{1 \ldots 1}$ if length k > length h or if length k = length h, $f_k^{2^n-1}$ precedes $f_h^{2^n-1}$ if $k > h$. Note that fundamental coordinates relative to the atom a_{2^n-1} are to be used. For convenience, $f_{2^n-1}^{2^n-1}$ will be relabeled f_0 and if $k \neq 2^n-1$, $f_{k_n \ldots k_1}^{2^n-1}$ will be relabeled

$$f_{i_1 i_2 \ldots i_m}$$

where i_1 is the least integer m such that $k_m = 0$, i_2 is the second least integer m such that $k_m = 0$, etc. For example the coordinates

$$(f^{2^n-1}_{k_3 k_2 k_1})$$

are ordered and relabeled as follows:

$$f^{111}_{111} \quad f^{111}_{110} \quad f^{111}_{101} \quad f^{111}_{011} \quad f^{111}_{100} \quad f^{111}_{010} \quad f^{111}_{001} \quad f^{111}_{000}$$

$$f_0 \quad f_1 \quad f_2 \quad f_3 \quad f_{12} \quad f_{13} \quad f_{23} \quad f_{123}$$

Note that the length of $f_{i_1 i_2 \ldots i_m}$ is n - m. From the definition of M^i_k in Chapter II,

$$M^{2^n-1}_k = (x_n')_{k_n'} \ldots (x_1')_{k_1'}$$

where $(x_m')_{1'} = 1$ and $(x_m')_{0'} = x_m'$. Hence

$$f_{i_1 i_2 \ldots i_m} = 1$$

iff f = 1 for an odd number of atoms in the fundamental product $(x_{i_1}')(x_{i_2}') \ldots (x_{i_m}')$ and $f_0 = 1$ iff f = 1 for an odd number of atoms. One simple method of obtaining the fundamental coordinates of a function f is to form the modulo two sum of the fundamental coordinates of the atoms dominated by f (see Theorem 2.3). The fundamental coordinates of the atoms for n = 1,2,3, and 4 are given in Table 3.1.

Table 3.1. Fundamental Coordinates of Atoms.

n = 1		f_0	f_1
	a_0	1	1
	a_1	1	0

n = 2		f_0	$f_1 f_2$	f_{12}
	a_0	1	1 1	1
	a_1	1	0 1	0
	a_2	1	1 0	0
	a_3	1	0 0	0

n = 3		f_0	$f_1 f_2 f_3$	$f_{12} f_{13} f_{23}$	f_{123}
	a_0	1	1 1 1	1 1 1	1
	a_1	1	0 1 1	0 0 1	0
	a_2	1	1 0 1	0 1 0	0
	a_3	1	0 0 1	0 0 0	0
	a_4	1	1 1 0	1 0 0	0
	a_5	1	0 1 0	0 0 0	0
	a_6	1	1 0 0	0 0 0	0
	a_7	1	0 0 0	0 0 0	0

n = 4		0	1	2	3	4	1 2	1 3	1 4	2 3	2 4	3 4	1 2 3	1 2 4	1 3 4	2 3 4	1 2 3 4
	a_0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	a_1	1	0	1	1	1	0	0	0	1	1	1	0	0	0	1	0
	a_2	1	1	0	1	1	0	1	1	0	0	1	0	0	1	0	0
	a_3	1	0	0	1	1	0	0	0	0	0	1	0	0	0	0	0
	a_4	1	1	1	0	1	1	0	1	0	1	0	0	1	0	0	0
	a_5	1	0	1	0	1	0	0	0	0	1	0	0	0	0	0	0
	a_6	1	1	0	0	1	0	0	1	0	0	0	0	0	0	0	0
	a_7	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
	a_8	1	1	1	1	0	1	1	0	1	0	0	1	0	0	0	0
	a_9	1	0	1	1	0	0	0	0	1	0	0	0	0	0	0	0
	a_{10}	1	1	0	1	0	0	1	0	0	0	0	0	0	0	0	0
	a_{11}	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
	a_{12}	1	1	1	0	0	1	0	0	0	0	0	0	0	0	0	0
	a_{13}	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
	a_{14}	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	a_{15}	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Let R be one of the six equivalence relations defined above. If $f \in F_n$, let $[f]_R$ be the R-equivalence class containing f. The canonical representative f^R of $[f]_R$ is defined to be that function $f^R \in [f]_R$ whose ordered fundamental coordinates $(f^R_{i_1 \ldots i_m})$ are miminum as a binary number.

The fundamental coordinates of all functions of 1,2, and 3 variables are listed in Tables 3.2, 3.3, and 3.4, respectively. A Veitch diagram is given for each function and each function is paired with its complement. The functions are grouped in PN-equivalence classes with a space between PN classes. Within each PN class, the functions are grouped in P-equivalence classes. If f dominates r atoms, then f' dominates 2^n-r atoms. Each equivalence class $[f]_R$ of functions whose members dominate $<2^{n-1}$ atoms is assigned an integer m and the symbol m' is assigned to the class $[f']_R$. If $n \leq 3$ and f dominates exactly 2^{n-1} atoms, $f' \in [f]_N$ and $f' \in [f]_{PN}$, but $f' \notin [f]_P$ (see Ninomiya [14]). Hence if f dominates 2^{n-1} atoms, an integer is assigned to $[f]_N = [f']_N$ and an integer is assigned to $[f]_{PN} = [f']_{PN}$. If f dominates 2^{n-1} atoms and $f_{1 \ldots n} = 0$, an integer m is assigned to $[f]_P$ and m' is assigned to $[f']_P$. The equivalence class number of the canonical representative of each class is underlined and a double underline denotes the representative of an NP, NN, or NPN class.

Algorithms now may be developed for determining the canonical representative for an arbitrary function. Let S be the set of all permutations σ on $\{1,\ldots,n\}$. The ordered fundamental coordinates

$$f_{12\ldots m}\, f_{13\ldots(m+1)} \cdots f_{(n-m+1)(n-m+2)\ldots n}$$

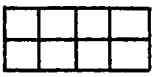
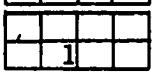
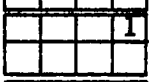
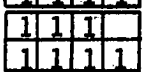
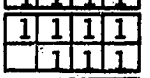
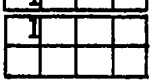
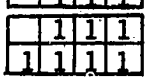
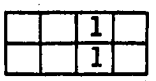
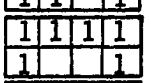
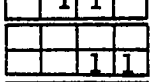
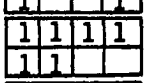
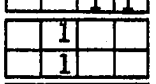
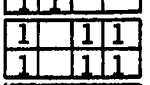
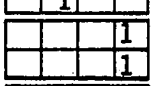
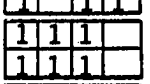
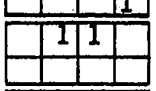
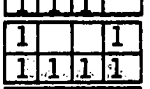
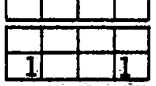
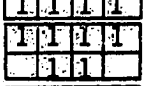
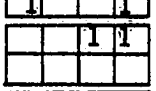
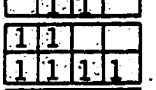
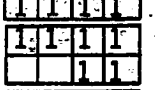
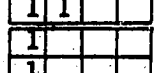
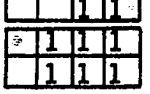
Table 3.2. Functions of One Variable.

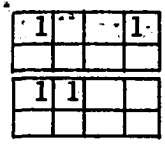
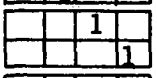
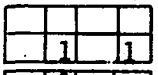
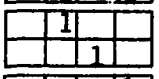
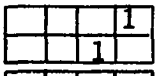
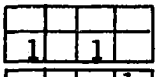
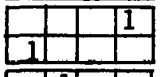
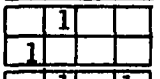
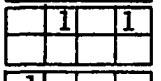
Veitch Diagram	Fundamental Coordinates		Equivalence Class								
$\boxed{0}\boxed{1}$ x_1	f_0	f_1	P	N	PN		f_0	f_1	P	N	PN
$\boxed{\ }\boxed{\ }$	0	0	$\underline{1}$	$\underline{1}$	$\underline{1}$	$\boxed{1}\boxed{1}$	0	1	$\underline{1}'$	$\underline{1}'$	$\underline{1}'$
$\boxed{\ }\boxed{1}$	1	0	$\underline{2}$	$\underline{2}$	$\underline{2}$	$\boxed{1}\boxed{\ }$	1	1	$\underline{2}'$	2	2

Table 3.3. Functions of Two Variables.

Veitch Diagram	Fundamental Coordinates			Equivalence Class				Fundamental Coordinates			Equivalence Class		
x_2 (0 1 / 2 3), x_1	f_0	$f_1 f_2$	f_{12}	P	N	PN		f_0	$f_1 f_2$	f_{12}	P	N	PN
	0	0 0	0	1	1	1		0	0 0	1	1'	1'	1'
	1	0 0	0	2	2	2		1	0 0	1	2'	2'	2'
	1	0 1	0	3	2	2		1	0 1	1	3'	2'	2'
	1	1 0	0	3	2	2		1	1 0	1	3'	2'	2'
	1	1 1	1	4	2	2		1	1 1	0	4'	2'	2'
	0	0 1	0	5	3	3		0	0 1	1	5'	3	3
	0	1 0	0	5	4	3		0	1 0	1	5'	4	3
	0	1 1	0	6	5	4		0	1 1	1	6'	5	4

TABLE 3.4. FUNCTIONS OF THREE VARIABLES

Veitch Diagram	Fundamental Coordinates				Equivalence Class			Veitch Diagram	Fundamental Coordinates				Equivalence Class		
	0	123	112 233	1 2 3	P	N	PN		0	123	112 233	1 2 3	P	N	PN
	0	000	000	0	1	1	1		0	000	000	1	1'	1'	1'
	1	000	000	0	2	2	2		1	000	000	1	2'	2'	2'
	1	001	000	0	3	2	2		1	001	000	1	3'	2'	2'
	1	010	000	0	3	2	2		1	010	000	1	3'	2'	2'
	1	100	000	0	3	2	2		1	100	000	1	3'	2'	2'
	1	011	001	0	4	2	2		1	011	001	1	4'	2'	2'
	1	101	010	0	4	2	2		1	101	010	1	4'	2'	2'
	1	110	100	0	4	2	2		1	110	100	1	4'	2'	2'
	1	111	111	1	5	2	2		1	111	111	0	5'	2'	2'
	0	001	000	0	6	3	3		0	001	000	1	6'	3'	3'
	0	010	000	0	6	4	3		0	010	000	1	6'	4'	3'
	0	100	000	0	6	5	3		0	100	000	1	6'	5'	3'
	0	001	001	0	7	3	3		0	001	001	1	7'	3'	3'
	0	001	010	0	7	3	3		0	001	010	1	7'	3'	3'
	0	010	001	0	7	4	3		0	010	001	1	7'	4'	3'
	0	010	100	0	7	4	3		0	010	100	1	7'	4'	3'
	0	100	010	0	7	5	3		0	100	010	1	7'	5'	3'
	0	100	100	0	7	5	3		0	100	100	1	7'	5'	3'
	0	001	011	1	8	3	3		0	001	011	0	8'	3'	3'

	0	123	112 233	1 2 3	P	N	PN
	0	010	101	1	8	4	3
	0	100	110	1	8	5	3
	0	011	000	0	9	6	4
	0	101	000	0	9	7	4
	0	110	000	0	9	8	4
	0	011	001	0	10	6	4
	0	101	010	0	10	7	4
	0	110	100	0	10	8	4
	0	011	110	0	11	6	4
	0	101	101	0	11	7	4
	0	110	011	0	11	8	4
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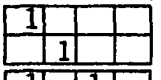
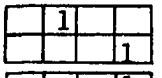
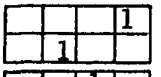
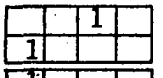
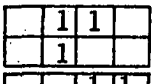
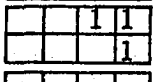
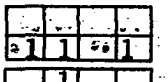
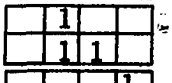
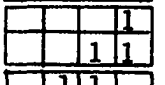
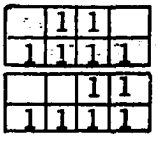
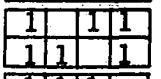
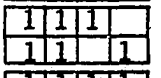
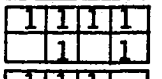
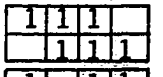
	0	101	111	1	12	7	4
	0	110	111	1	12	8	4
	0	111	001	0	13	9	5
	0	111	010	0	13	9	5
	0	111	100	0	13	9	5
	0	111	111	1	14	9	5
	1	000	001	0	15	10	6
	1	000	010	0	15	11	6
	1	000	100	0	15	12	6
	1	001	001	0	16	10	6
	1	001	010	0	16	11	6
	1	010	001	0	16	10	6

	0	123	112 233	1 2 3	P	N	PN
	0	010	101	0	8'	4	3'
	0	100	110	0	8'	5'	3'
	0	011	000	1	9'	6'	4'
	0	101	000	1	9'	7'	4'
	0	110	000	1	9'	8'	4'
	0	011	001	1	10'	6'	4'
	0	101	010	1	10'	7'	4'
	0	110	100	1	10'	8'	4'
	0	011	110	1	11'	6'	4'
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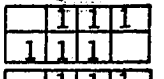
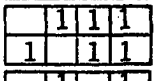
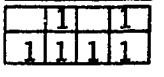
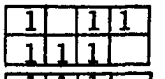
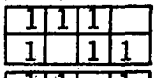
	0	011	111	0	12'	6'	4'
	0	101	111	0	12'	7'	4'
	0	110	111	0	12'	8'	4'
	0	111	001	1	13'	9'	5'
	0	111	010	1	13'	9'	5'
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	1	000	010	1	15'	11'	6'
	1	000	100	1	15'	12'	6'
	1	001	001	1	16'	10'	6'
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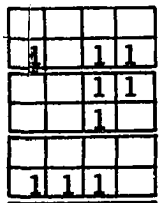	1	010	100	0	16	12	6
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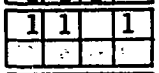	1	001	100	1	17	12	6
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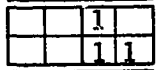	1	101	000	0	18	11	6
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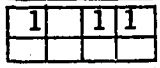
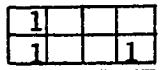
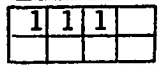
	1	011	101	1	19	12	6
	1	101	011	1	19	10	6
	1	101	110	1	19	12	6
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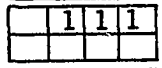
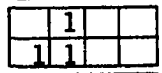
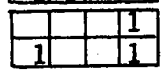
	1	111	011	0	20	12	6
	1	111	101	0	20	11	6
	1	111	110	0	20	10	6
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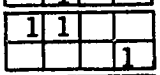	1	000	110	1	21	15	7
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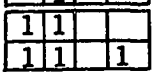
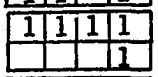
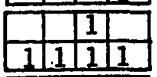
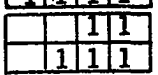
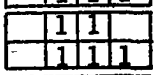
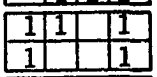
	1	100	010	1	16'	11'	6'
	1	100	100	1	16'	12'	6'
	1	001	100	0	17'	12'	6'
	1	010	010	0	17'	11'	6'
	1	100	001	0	17'	10'	6'
	1	011	000	1	18'	10'	6'
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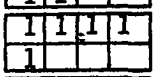
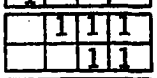
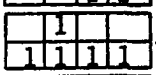
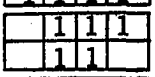
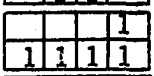
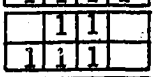
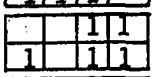
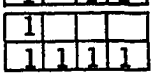
	1	110	000	1	18'	12'	6'
	1	011	011	0	19'	11'	6'
	1	011	101	0	19'	12'	6'
	1	101	011	0	19'	10'	6'
	1	101	110	0	19'	12'	6'
	1	110	101	0	19'	10'	6'
	1	110	110	0	19'	11'	6'
	1	111	011	1	20'	12'	6'
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	1	000	101	0	21'	14'	7'
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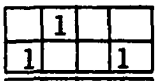
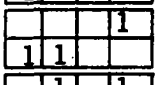
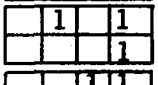
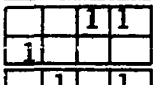
	1	001	101	0	23	14	7
	1	001	110	0	23	15	7
	1	010	011	0	23	13	7
	1	010	110	0	23	15	7
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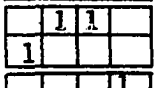
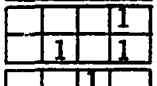
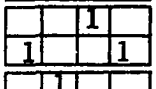
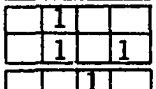
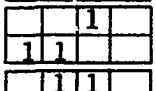
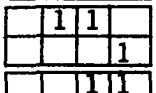
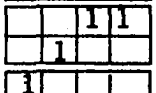
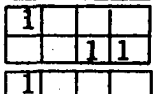
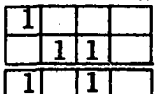
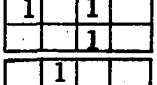
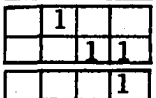
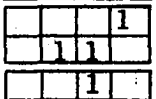
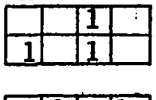
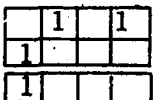
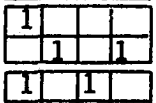
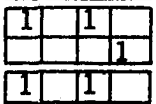
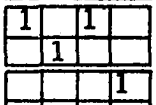
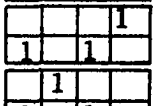
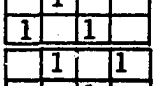
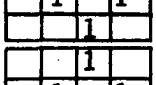
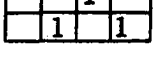
	1	100	101	0	23	14	7
	1	011	010	0	24	13	7
	1	011	100	0	24	14	7
	1	101	001	0	24	13	7
	1	101	100	0	24	15	7
	1	110	001	0	24	14	7
	1	110	010	0	24	15	7
	1	011	111	1	25	15	7
	1	101	111	1	25	14	7
	1	110	111	1	25	13	7
	1	111	001	0	26	15	7
	1	111	010	0	26	14	7
	1	111	100	0	26	13	7
	1	000	111	0	27	16	8
	1	001	111	1	28	16	8
	1	010	111	1	28	16	8
	1	100	111	1	28	16	8
	1	011	110	0	29	16	8
	1	101	101	0	29	16	8
	1	110	011	0	29	16	8
	1	111	000	0	30	16	8

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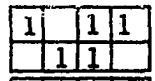
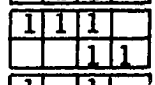
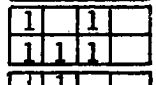
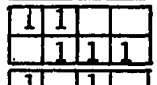
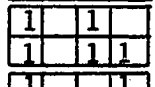
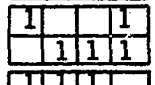
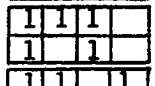
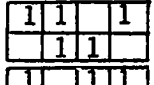
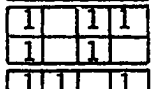
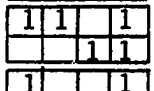
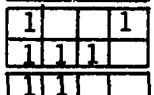
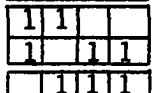
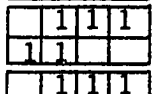
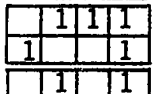
	1	001	101	1	23'	14'	7'
	1	001	110	1	23'	15'	7'
	1	010	011	1	23'	13'	7'
	1	010	110	1	23'	15'	7'
	1	100	011	1	23'	13'	7'
	1	100	101	1	23'	14'	7'
	1	011	010	1	24'	13'	7'
	1	011	100	1	24'	14'	7'
	1	101	001	1	24'	13'	7'
	1	101	100	1	24'	15'	7'
	1	110	001	1	24'	14'	7'
	1	110	010	1	24'	15'	7'
	1	011	111	0	25'	15'	7'
	1	101	111	0	25'	14'	7'
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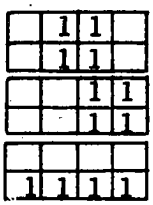	0	000	001	0	31	17	9
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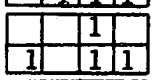	0	011	100	0	32	20	10
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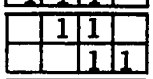
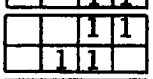
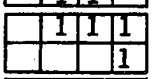
	0	110	001	0	32	22	10
	0	110	010	0	32	22	10
	0	011	011	0	33	20	10
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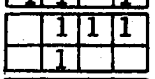
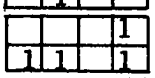
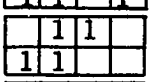
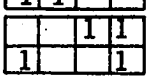
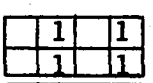
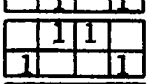
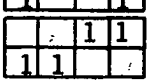
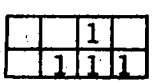
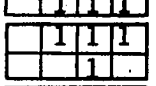
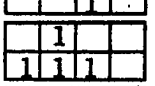
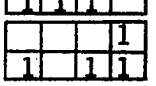
	0	101	011	0	33	21	10
	0	101	110	0	33	21	10
	0	110	101	0	33	22	10
	0	110	110	0	33	22	10
	0	000	011	0	34	23	11
	0	000	101	0	34	24	11
	0	000	110	0	34	25	11
	0	111	000	0	35	26	12
	0	111	011	0	36	26	12
	0	111	101	0	36	26	12
	0	111	110	0	36	26	12

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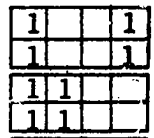	0	000	001	1	31'	17	9
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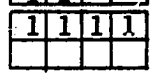	0	000	011	1	31'	19	9
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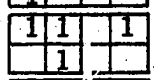
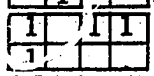
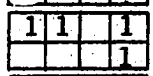
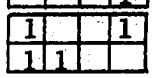
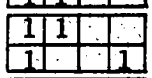
	0	011	100	1	32'	20	10
	0	101	001	1	32'	21	10
	0	101	100	1	32'	21	10
	0	110	001	1	32'	22	10
	0	110	010	1	32'	22	10
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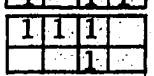
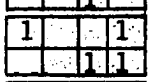
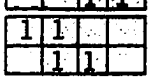
	0	101	011	1	33'	21	10
	0	110	101	1	33'	22	10
	0	110	110	1	33'	22	10
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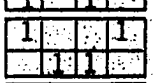
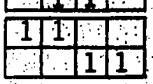
	0	000	101	1	34'	24	11
	0	000	110	1	34'	25	11
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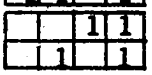	0	010	010	0	37	28	13
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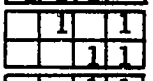
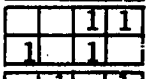
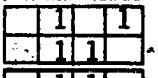
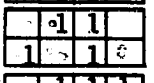
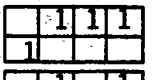
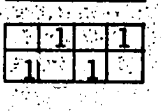
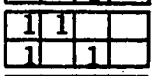
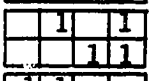
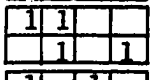
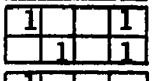
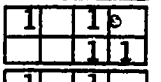
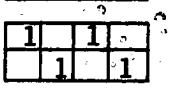
	0	001	110	0	38	27	13
	0	010	011	0	38	28	13
	0	010	110	0	38	28	13
	0	100	011	0	38	29	13
	0	100	101	0	38	29	13
	0	001	111	0	39	27	13
	0	010	111	0	39	28	13
	0	100	111	0	39	29	13
	0	000	111	0	40	30	14

	0	123	233	3	P	N	PN
	0	001	100	1	37'	27	13
	0	010	010	1	37'	28	13
	0	100	001	1	37'	29	13
	0	001	101	1	38'	27	13
	0	001	110	1	38'	27	13
	0	010	011	1	38'	28	13
	0	010	110	1	38'	28	13
	0	100	011	1	38'	29	13
	0	100	101	1	38'	29	13
	0	001	111	1	39'	27	13
	0	010	111	1	39'	28	13
	0	100	111	1	39'	29	13
	0	000	111	1	40'	30	14

of length n - m may be considered as a binary number m_f. Each $\sigma \in S$ may be considered as a permutation of the digits of m_f which maps the digit in position $f_{i_1 i_2 \ldots i_m}$ to position $f_{\sigma(i_1)\sigma(i_2)\ldots\sigma(i_m)}$. A sequence $S_1,\ldots,S_{n-1}$ of subsets of S may be defined recursively as follows:

(1) $S_1 = \{\sigma \in S : \sigma(1_f) = \sigma(f_1 \ldots f_n) \text{ is minimum}\}$.

(2) $S_m = \{\sigma \in S_{m-1} : \sigma(m_f) \text{ is minimum}\}$.

Algorithm 3.1. Let f^P be the canonical representative of the P-equivalence class containing the function f. The fundamental coordinates of f^P are given by f_0, $\sigma(f_1 \ldots f_n)$, $\sigma(f_{12} \ldots f_{(n-1)n}),\ldots,f_{1\ldots n}$ where σ is any permutation in S_{n-1}.

Proof. By Theorem 2.7 and the definition of length following the proof of this theorem, it is evident that each $\sigma \in S$ permutes only coordinates of the same length. The choice of the sequence $S_1,\ldots,S_{n-1}$ insures that the coordinates of f^P will be minimum as required for the canonical representative. A symmetry of f (and $[f]_P$) is indicated by the presence of more than one element in S_{n-1}.

As an example of the application of Algorithm 3.1, consider the function given in Chapter II: $f = a_0 \oplus a_2 \oplus a_6 \oplus a_7$. The fundamental coordinates of f may be calculated from Table 3.1 using Theorem 2.3:

$$\begin{aligned} f &= (1,111,111,1) \\ &\oplus (1,101,010,0) \\ &\oplus (1,100,000,0) \\ &\oplus (1,000,000,0) \\ &= (0,110,101,1) . \end{aligned}$$

The fundamental coordinates of length two form the binary number $f_1f_2f_3 = 110$. Hence $S_1 = \{(13), (12)(13)\}$. In tabular form:

								1	
				1	1	2		2	
0		1	2	3	2	3	3	3	
0		1	1	0	1	0	1	1	$\sigma_1 = (13)$
0		0	1	1	1	0	1	1	
0		0	1	1	0	1	1	1	$\sigma_2 = (12)(13)$

From the above table, $S_2 = \{(12)(13)\}$ so the fundamental coordinates of f^P are (0, 011, 011, 1). Note that $f^P = \sigma_2(f)$ so in atomic form $f^P = \sigma_2(a_{000} \oplus a_{010} \oplus a_{110} \oplus a_{111}) = a_{000} \oplus a_{100} \oplus a_{101} \oplus a_{111}$ $(a_k = a_{k_3k_2k_1})$.

If $\{i_1,\ldots,i_m\} \subset \{1,\ldots,n\}$, the complementation operator $[i_1,\ldots,i_m]$ is defined to be a mapping from F_n into F_n which complements variables $i_1,\ldots,i_m$ and leaves the other variables fixed. Clearly

$$[i_j,i_k](f) = ([i_j][i_k])(f) = [i_j]([i_k](f)) = [i_k]([i_j](f)) .$$

If $f \in F_n^*$, the problem of finding the canonical representative f^N of $[f]_N$ consists of finding a complementation operator $[i_1,\ldots,i_m]$ such that the binary number formed from the fundamental coordinates of $[i_1,\ldots,i_m](f)$ is minimum. If $i_1\ldots i_m$ is the subscript of a fundamental coordinate and $i_p \in \{i_1,\ldots,i_m\}$, let $(i_1\ldots i_m)*i_p$ be the subscript obtained by deleting i_p from $i_1\ldots i_m$ with the convention that $(i_m)*i_m = 0$. Then Theorem 2.4 may be stated in terms of the relabeled

fundamental coordinates by the following equation:

$$([i_p](f))_{i_1\ldots i_m} = \begin{cases} f_{i_1\ldots i_m} & \text{if } i_p \notin \{i_1,\ldots,i_m\} \\ f_{i_1\ldots i_m} \oplus f_{(i_1\ldots i_m)*i_p} & \text{if } i_p \in \{i_1,\ldots,i_m\}. \end{cases} \tag{3.1}$$

If $f_0 = 1$, the fundamental coordinates of f^N are very easy to determine. Clearly $([i_1,\ldots,i_m](f))_0 = 1$ for every complementation operator $[i_1,\ldots,i_m]$. By Eq. (3.1), the fundamental coordinates of $[i_p](f)$ of length n-1 are given by

$$([i_p](f))_{i_m} = \begin{cases} f_{i_m} & \text{if } i_p \neq i_m \\ f_{i_m} \oplus 1 & \text{if } i_p = i_m \end{cases}$$

Let $[j_1,\ldots,j_q]$ be the complementation operator obtained by including $j_p \in \{j_1,\ldots,j_q\}$ iff $f_{j_p} = 1$. Then all coordinates of length n - 1 of $[j_1,\ldots,j_q](f)$ are zero and if $[i_1,\ldots,i_m] \neq [j_1,\ldots,j_q]$, at least one coordinate of length n - 1 of $[i_1,\ldots,i_m](f)$ is one. Hence

$$f^N = [j_1,\ldots,j_q](f),\ f_0^N = 1,\ f_1^N = f_2^N = \ldots = f_n^N = 0\ ,$$

and the remaining coordinates of f^N may be found by application of Eq. (3.1).

If $f_0 = 0$, finding the coordinates of f^N is usually more difficult. If f is the 0-function or the 1-function, $f^N = f$. If f is not the 0-function or the 1-function, let p be the least integer m such that the coordinates $f_{i_1\ldots i_m}$ of length n-m are not all zero. Then

$1 \le p \le n - 1$. Let $[i_1,\ldots,i_m]$ be any complementation operator. By Eq. (3.1), all coordinates of lengths $n,\ldots,n-p$ of $[i_1,\ldots,i_m](f)$ are the same as the corresponding coordinates of f. Hence the coordinates of f^N of length $n - p$ are the same as the corresponding coordinates of f. Let T_p be the set of all $i_1\ldots i_p$ such that $f_{i_1\ldots i_p} = 1$. By the definition of p, $T_p \neq \emptyset$. Each coordinate of f of length $n - p - 1$ may be labeled $f_{i_1\ldots i_{p+1}}$. There are $t = \binom{n}{n-p-1}$ such coordinates labeled in order:

$$f_{12\ldots(p+1)}, f_{13\ldots(p+2)},\ldots,f_{(n-p)\ldots n}.$$

Let

$$f^r_{i_1\ldots i_{p+1}}$$

be the coordinate in position r from the left so $r = 1,\ldots,t$. The coordinates of $f^N = [i_1,\ldots,i_m](f)$ may be determined using an iterative procedure. A set of functions B_r is associated with each $r = 1,\ldots,t$, and to each $g \in B_r$ is assigned a set of possible complementation operators S^g_r. Let $B_0 = \{f\}$ and let $S^f_0 = \{1,\ldots,n\}$. For each $g \in B_{r-1}$ there are three possibilities:

Case 1. $g^r_{i_1\ldots i_{p+1}} = 0$. Let $B^{g1}_r = \{g\} \cup \{[j_1,\ldots,j_q]g : q$ is even and $(i_1\ldots i_{p+1})*j_s \in T_p$ for $s = 1,\ldots,q\}$. Let

$$S^g_r = S^{[j_1,\ldots,j_q]g}_r = \{m \in S^g_{r-1} : (i_1\ldots i_{p+1})*m \notin T_p\}.$$

Case 2. $g^r_{i_1\ldots i_{p+1}} = 1$ and for at least one $m \in S^g_{r-1}$, $(i_1\ldots i_{p+1})*m \in T_p$. Let $B^{g2}_r = \{[j_i,\ldots,j_q]g : q$ is odd and $(i_1\ldots i_{p+1})*j_s \in T_p$ for $s = i,\ldots,q\}$. Let

$$S_r^{[j_1,\ldots,j_q]g} = \{m \in S_{r-1}^g : (i_1,\ldots,i_{p+1})*m \notin T_p\}.$$

Case 3. $g^r_{i_1\ldots i_{p+1}} = 1$ and for all $m \in S^g_{r-1}$, $(i_1\ldots i_{p+1})*m$ is not defined or $(i_1\ldots i_{p+1})*m \notin T_p$. Let $B_r^{g^3} = \{g\}$ and let $S_r^g = S_{r-1}^g$. Let

$$B_r^a = \bigcup_{g \in B_{r-1}} (B_r^{g^1} \cup B_r^{g^2})$$

and let

$$B_r^b = \bigcup_{g \in B_{r-1}} B_r^{g^3} .$$

Let

$$B_r = \begin{cases} B_r^a \text{ if } B_r^a \neq \emptyset \\ B_r^b \text{ if } B_r^a = \emptyset \end{cases} .$$

In case 1, $h^r_{i_1\ldots i_{p+1}} = 0$ for each $h \in B_r^{g^1}$ and S_r^h contains only complementation operators m such that $h^s_{i_1\ldots i_{p+1}} = 0$ for $s \leq r$ implies

$$[m]h^s_{i_1\ldots i_{p+1}} = 0 .$$

Similarly, each $h \in B_r^{g^2}$ has the properties attributed to each $h \in B_r^{g^1}$. If $h \in B_r^{g^3}$, there is no complementation operator m such that

$$[m]h^r_{i_1\ldots i_{p+1}} = 0 .$$

Hence if there are functions $g \in B_{r-1}$ satisfying the conditions of cases 1 or 2, functions satisfying the conditions of case 3 need not be considered. If $S_r^g = \emptyset$ for each $g \in B_r$, f^N is that $g \in B_r$ whose fundamental coordinates are minimum. If B_t contains a function g such that $S_t^g \neq \emptyset$,

then f^N is the function in $B_t \cup \{[i_1,\ldots,i_m]g : g \in B_t,\ i_s \in S_t^g$ for $s = 1,\ldots,m\}$ whose coordinates are minimum. The procedure for finding f^N is summarized below.

Algorithm 3.2. If $f \in F_n$, the fundamental coordinates of f^N are obtained as follows:

(1) If $f_0 = 1$, $f^N = [j_1,\ldots,j_q]f$ where $j_p \in \{j_1,\ldots,j_q\}$ iff $f_{j_p} = 1$. Then $f_0^N = 1$, $f_1^N = f_2^N = \ldots = f_n^N = 0$, and the remaining coordinates of f^N are found by applying Eq. (3.1) to $([j_1]\ldots[j_m])f$.

(2) If all coordinates of f are 0, $f = 0 = f^N$.

(3) If only the last coordinate of f is 1, $f = 1 = f^N$.

(4) If $f_0 = 0$, $f \neq 0$, and $f \neq 1$, form the sets T_p, B_r, and S_r^g for $r = 1,\ldots,t = \binom{n}{p-1}$. The coordinates of each $g \in B_r$ are computed using Eq. (3.1). Then f^N is the function in $B_t \cup \{[i_1,\ldots,i_m]g : g \in B_t,\ i_s \in S_t^g$ for $s = 1,\ldots,m\}$ whose coordinates are minimum.

As an example of the application of Algorithm 3.2, let f be the function of four variables given in atomic form by $f = a_0 \oplus a_5 \oplus a_{10} \oplus a_{13} \oplus a_{14} \oplus a_{15}$. From Table 3.1 and Theorem 2.3 the fundamental coordinates of f are:

$$
\begin{aligned}
f = {} & (1,\ 1111,\ 111111,\ 1111,\ 1) \\
& \oplus (1,\ 0101,\ 000010,\ 0000,\ 0) \\
& \oplus (1,\ 1010,\ 010000,\ 0000,\ 0) \\
& \oplus (1,\ 0100,\ 000000,\ 0000,\ 0) \\
& \oplus (1,\ 1000,\ 000000,\ 0000,\ 0) \\
& \oplus (1,\ 0000,\ 000000,\ 0000,\ 0) \\
& = (0,\ 1100,\ 101101,\ 1111,\ 1) \quad .
\end{aligned}
$$

Hence f falls under part 4 of Algorithm 3.2 and the fundamental coordinates of f^N may be calculated with a table.

					1		
				1112	2		
			111223	2233	3		
	0	1234	234344	3444	4		
f	0	1100	101101	1111	1	$B_0 = \{f\}$	$S_o^f = \{1,2,3,4\}$
						$B_1 = \{[1]f,[2]f\}$	$S_1^{[1]f} = S_1^{[2]f} = \{3,4\}$
[1]f	0	1100	001101	0101	0		
[2]f	0	1100	001101	1010	0		
						$B_2 = \{[1]f,[2]f\}$	$S_2^{[1]f} = S_2^{[2]f} = \{4\}$
						$B_3 = \{[4,1]f,[4,2]f\}$	$S_3^{[4,1]f} = S_3^{[4,2]f} = \emptyset$
[4,1]f	0	1100	000111	0100	0		
[4,2]f	0	1100	000111	1011	1		

$T_p = \{1,2\}$

Since $S_3^{[4,1]f} = S_3^{[4,2]f} = \emptyset$ and [4,1]f has minimum coordinates $f^N = [4,1]f$.

The following Lemma is of some assistance in calculating the coordinates of f^{PN}.

Lemma 3.3. Let $f \in F_n$. If $f_0 = 1$ or $n \leq 3$, then $f^{PN} = (f^N)^P$.

Proof. Suppose $f \in F_n$ and $f_0 = 1$. As indicated in the discussion preceding Algorithm 3.2, there is a unique complementation operator $[i_1,\ldots,i_m]$ such that all coordinates of $[i_1,\ldots,i_m]f$ of length n - 1 are zero. By Theorem 2.7, any permutation of the variables of f does not change the number of coordinates of length n - 1 which are one. Hence f^{PN} may be obtained by finding f^N (Algorithm 3.2) and then finding the permutation σ such that the coordinates of $\sigma(f^N)$ are minimum (Algorithm

3.1). Tables 3.2, 3.3, and 3.4 list all the functions of $n \leq 3$ variables. It may be seen from the tables that whenever f and g belong to the same PN-equivalence class, f^N and g^N belong to the same P-equivalence class. Hence if $n \leq 3$, $f^{PN} = (f^N)^P$.

Unfortunately, there are functions f of four or more variables with $f_0 = 0$ such that $f^{PN} \neq (f^N)^P$. For example consider the function of four variables whose coordinates are given in the table below:

					1
				1112	2
			111223	2233	3
	0	1234	234344	3444	4
$f = f^N$	0	0111	011000	0000	0
$(24)f$	0	0111	110000	0000	0
$((24)f)^N$	0	0111	001000	0000	0

Application of Algorithm 3.2 shows that $f = f^N$ and that $((24)f)^N = [1]((24)f)$. Clearly $f^{PN} = ((24)f)^N \neq (f^N)^P = f$. From Table 3.4 it is easy to find examples of functions $f \in F_3$ such that $(f^P)^N \neq f^{PN}$.

Algorithm 3.4. If $f \in F_n$, the fundamental coordinates of f^{PN} are obtained as follows:

(1) If $f_0 = 1$ or $n \leq 3$, apply Algorithms 3.2 and 3.1 to obtain $(f^N)^P = f^{PN}$.

(2) If all coordinates of f are 0, $f = 0 = f^{PN}$.

(3) If only the last coordinate of f is 1, $f = 1 = f^{PN}$.

(4) If $f_0 = 0$, $n \geq 4$, $f \neq 0$, and $f \neq 1$, let p be the least integer m such that the coordinates $f_{i_1 \ldots i_m}$ of length $n - m$ are not all 0. Then $1 \leq p \leq n - 1$. The ordered fundamental coordinates of

length n - p form a binary number

$$p_f = f_{12\ldots p}\, f_{13\ldots(p+1)} \cdots f_{(n-p+1)(n-p)\ldots n} .$$

Let S be the set of all permutations σ on $\{1,\ldots,n\}$ such that $\sigma(p_f)$ is minimum. For each $\sigma \in S$, find $(\sigma(f))^N$ using Algorithm 3.2. Then f^{PN} is the function in $\{(\sigma(f))^N : \sigma \in S\}$ whose coordinates are minimum.

Proof. Case (1) is an immediate consequence of Lemma 3.3. Cases (2) and (3) are obvious. In case (4) no complementation operator can alter the coordinates of length n - p according to Theorem 2.4. Hence $\{\sigma(f) : \sigma \in S\}$ contains all possible candidates for f^{PN} with respect to permutation of variables. Hence f^{PN} is the function in $\{(\sigma(f))^N : \sigma \in S\}$ whose coordinates are minimum.

Algorithm 3.5. The fundamental coordinates of f^{NN}, f^{NP}, and f^{NPN} may be obtained from the coordinates of f^N, f^P, and f^{PN} respectively, by making the last coordinate 0.

Proof. By Theorem 2.3, all coordinates of f are the same as the corresponding coordinates of f' except that the last coordinates satisfy the condition $f'_{1\ldots n} = 1 \oplus f_{1\ldots n}$. Since the canonical representative of an equivalence class is defined to be the class member whose coordinates form the minimum binary number, the algorithm is obvious.

Ninomiya [14] and Harrison [6] provide tables listing the atomic coordinates of one representative function for each of the 222 NPN equivalence classes in F_4. A computer program could be written based on Algorithm 3.5 and the above tables to determine canonical representatives for the NPN classes in F_4.

Future research may extend the results of this paper to include investigations of necessary and sufficient conditions for a Boolean function to be unate, k-monotonic, k-assumable, or to possess other properties associated with the study of threshold logic. Winder [25] gives a recent analysis of the literature on threshold logic.

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