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COVERING REFINEMENTS, COLLECTIONWISE NORMALITY,

AND RELATED PROPERTIES

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COVERING REFINEMENTS, COLLECTIONWISE NORMALITY,
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INTRODUCTION

William Haver [Ha] first introduced the notion of a metric C-space in 1973 at the VPI topology conference at Blacksburg, Virginia. In 1974, John Gresham [Gr] gave a topological definition of a C-space, and he thoroughly investigated such spaces.

In this dissertation, spaces with various covering properties are examined. In particular, weakly C-spaces [1.2], finitely (strongly) screenable spaces [2.2], screenable spaces [2.1], CCC spaces [5.1], Lindelöf spaces, and SCN [1.9] spaces are involved in many of the results obtained herein.

In Chapter One, weakly C-spaces are introduced and many theorems concerning these spaces are stated. However, most proofs are omitted because they require only obvious modifications to Gresham's. Also in Chapter One there are several theorems which are aimed at deciding under what conditions a weakly C-space is a C-space.

In Chapter Two, finitely screenable spaces and finitely strongly screenable spaces are introduced, and it is shown how these new spaces are related to certain well-known spaces. The main result proved is that every finitely screenable, normal, T_1 space is finitely strongly screenable. The results in Chapter Two are summarized in an implication diagram [2.18].

Many examples are examined in the third chapter, including the tangent disc space [3.7], the Hilbert cube [3.8], and Bing's example G [3.9]. A special type of subset, called a CIC-subset [3.3], is defined and this definition is shown to be useful in investigating spaces for such properties as screenability and being collectionwise Hausdorff. The main results of Chapter Three are summarized by means of a containment diagram [3.14].

In Chapter Four, Fleissner's example, called "George" [4.1], is reconstructed and it is shown that George is neither countable dimensional nor screenable. The main result of this dissertation is the construction derived in this chapter, which details how to build a normal, collectionwise Hausdorff space which is not collectionwise normal by starting with a space that is sufficiently like Bing's example G.

In Chapter Five, the concept of a CIC-subset is revisited and results are obtained which relate this concept to CCC spaces, Lindelöf spaces, and screenable spaces.

<ins>Notation</ins>:

(1) If P is a property, then a space X will be called a $\sigma - P$ space if $X = \bigcup_1^\infty X_n$ where each X_n has property P. For example, a space X is σ-compact if $X = \bigcup_1^\infty X_n$ where X_n is compact.

(2) The letter N will denote the set of natural numbers.

(3) If F is a collection of sets, then F^* will denote the union of the elements of F.

(4) If A is a subset of a space X, then $Bd_X A$ will denote

the boundary of A in X. When no confusion is likely, the notation $\mathrm{Bd}A$ will be used.

(5) If X is a space and F is a collection of subsets of X, then for each $x \in X$, $\mathrm{St}(x,F)$ will denote the set $\bigcup\{f \in F: x \in f\}$.

(6) If M is a subset of a space X and F is a collection of subsets of X, then $\{f \cap M: f \in F\}$ is called the trace of F on M and is denoted by $\mathrm{Tr}_M F$.

Ordinal spaces are assumed to have the usual (order) topology. Since the open ordinal space $[0,\omega_1)$, also called the space of countable ordinals, is used extensively in this dissertation, a few facts concerning this space will now be listed.

(A) Every discrete closed subset of $[0,\omega_1)$ is finite.

(B) If H and K are disjoint closed subsets of $[0,\omega_1)$, then either H or K is bounded.

(C) If A is a countable subset of $[0,\omega_1)$, then $\sup A = \sup\{\alpha: \alpha \in A\} < \omega_1$. In particular, if $\{\alpha_n\}_1^\infty$ is a sequence in $[0,\omega_1)$, then $\sup\alpha_n < \omega_1$.

(D) <u>The Pressing Down Lemma</u>. If $f: \omega_1 \to \omega_1$ is a function such that $f(\lambda) < \lambda$ for each λ in some subset of $[0,\omega_1)$ which intersects every closed unbounded set in $[0,\omega_1)$, then there exists an ordinal $\beta \in [0,\omega_1)$ and a cofinal subset Γ of $[0,\omega_1)$ such that $f(\gamma) = \beta$ for each $\gamma \in \Gamma$.

Note: The symbol χ denotes aleph.

CHAPTER ONE

WEAKLY C-SPACES

1.1 Definition: A space X is a C-space if for every sequence $\{C_n\}_1^\infty$ of open covers of X there is a sequence $\{U_n\}_1^\infty$ of pairwise disjoint open collections such that, for each n, U_n weakly refines C_n (U_n refines C_n, but U_n need not be a cover), and $\overset{\infty}{\underset{1}{\cup}} U_n$ is a cover of X. The collection $\{U_n\}_1^\infty$ is called à C-refinement of $\{C_n\}_1^\infty$.

1.2 Definition: A space X is a weakly C-space if every sequence $\{C_n\}_1^\infty$ of finite open covers has a C-refinement.

1.3 Definition: A space X is a strongly C-space if every sequence $\{C_n\}_1^\infty$ of open covers has a C-refinement $\{U_n\}_1^\infty$ such that, for some natural number k, $\overset{\infty}{\underset{j=k}{\cup}} U_j = \phi$.

1.4 Theorem: If X is a weakly C-space and $\{C_n\}_1^\infty$ is a sequence of finite open covers of X, then there exists a C-refinement $\{U_n\}_1^\infty$ of $\{C_n\}_1^\infty$ such that each U_n is finite.

1.5 Definition: A family $\{F_a : a \in A\}$ of subsets of a space X is discrete if each point of X has a neighborhood which intersects at most one member of the family. It is well-known that $\{F_a : a \in A\}$ is discrete if and only if $\{\bar{F}_a : a \in A\}$ is pairwise disjoint and, for every subset B of A, the set $\cup\{\bar{F}_b : b \in B\}$ is closed.

1.6 Definition: A space X is collectionwise normal (CWN) if for every discrete family $\{F_a : a \in A\}$ in X there exists a pairwise disjoint open family $\{U_a : a \in A\}$ in X such that $F_a \subset U_a$ for each $a \in A$.

1.7 Definition: A pair $\{A,B\}$ of subsets of space X is separated if $\bar{A} \cap B = A \cap \bar{B} = \phi$. A family $\{F_a : a \in A\}$ is separated if, for each $a \in A$, $\bar{F}_a \cap (\bigcup_{b \neq a} F_b) = F_a \cap (\overline{\bigcup_{b \neq a} F_b}) = \phi$.

1.8 Definition: A space X is completely normal if for every separated pair $\{A,B\}$ of subsets of X there exist disjoint open sets U and V in X containing A and B, respectively.

1.9 Definition: A space X is strongly completely normal (SCN) if for every separated family $\{F_a : a \in A\}$ in X there exists a pairwise disjoint open family $\{U_a : a \in A\}$ in X such that $F_a \subset U_a$ for each $a \in A$.

1.10 Definition: A collection V of subsets of a space X is of order α at y if y is contained in precisely α members of V. This will be denoted by $\operatorname{ord}(y,V) = \alpha$.

1.11 Definition: A space X has dimension $\leq n$, denoted $\dim X \leq n$, if every finite open cover of X has an open refinement of order $\leq n + 1$. Also, $\dim \phi = -1$.

1.12 Definition: A space X has strong covering dimension $\leq n$, denoted $\operatorname{sdim} X \leq n$, if every open cover has an open refinement of order $\leq n + 1$. Again, $\operatorname{sdim} \phi = -1$.

1.13 Definition: A space X is countable dimensional (CD) if $X = \bigcup_{1}^{\infty} X_n$, where $\operatorname{sdim} X_n = 0$ for each n.

1.14 Theorem: (A) A space is SCN if and only if it is hereditarily CWN.

(B) A space is completely normal if and only if it is hereditarily normal.

(C) If X is completely normal and $\{F_1, F_2, \ldots, F_n\}$ is a finite separated family in X, then there exists a pairwise disjoint open family $\{U_1, U_2, \ldots, U_n\}$ such that $F_i \subset U_i$ for $i = 1, 2, \ldots, n$.

By using part (C) of this theorem along with Theorem 1.4, many new theorems related to weakly C-spaces can be obtained from the results found in the first three chapters of John Gresham's dissertation [Gr]. In particular, the theorems in [Gr] can be modified by replacing the expressions "C-space" and "SCN" by "weakly C-space" and "completely normal," respectively. For example, Gresham's proposition 3.1.6 states: Let X be an SCN space such that $X = \bigcup_{1}^{\infty} X_n$ and each X_n is a

C-space (in the relative topology). Then X is a C-space. Modifying this as indicated previously, the following theorem is obtained.

1.15 Theorem: Let X be a completely normal space such that $X = \bigcup_1^\infty X_n$ and each X_n is a weakly C-space (in the relative topology). Then X is a weakly C-space.

Proof: Let $\{C_n\}_1^\infty$ be a sequence of finite open covers of X. Let $g\colon N \times N \to N$ be a bijection and let $C_{(i,j)}$ denote the cover $C_{g(i,j)}$. Fix $i \in N$, and consider $\{C_{(i,j)}\}_{j=1}^\infty$, a sequence of finite open covers of X_i. Since X_i is weakly C, Theorem 1.4 implies that there exists a C-refinement $\{U_{(i,j)}\}_{j=1}^\infty$ of $\{C_{(i,j)}\}_{j=1}^\infty$ relative to X_i such that each $U_{(i,j)}$ is a finite collection. Each $U_{(i,j)}$ is a finite pairwise disjoint collection of relatively open subsets of X_i, and therefore each $U_{(i,j)}$ is a finite separated family in X. Applying Theorem 1.14(C), there is a finite pairwise disjoint family $\{F(U)\colon U \in U_{(i,j)}\}$ of open subsets of X such that $U \subset F(U)$ for each $U \in U_{(i,j)}$. Also, for each $U \in U_{(i,j)}$, there is a set $G(U)$ in $C_{(i,j)}$ such that $U \subset G(U)$. Let $V_{(i,j)} = \{F(U) \cap G(U)\colon U \in U_{(i,j)}\}$, a finite family of open subsets of X. Setting $W_n = V_{(i,j)}$ where $g(i,j) = n$, it is easily checked that $\{W_n\}_1^\infty$ is a C-refinement of $\{C_n\}_1^\infty$.

Many theorems which can be proved essentially as in [Gr] will now be listed, and some of them will be proved.

1.16 Theorem: If X is weakly C, then every closed subspace of X is weakly C.

1.17 Theorem: If every G_δ-subspace of X is weakly C, then X is hereditarily weakly C.

1.18 Theorem: If X is a completely normal space and $\{X_n\}_1^\infty$ is a sequence of weakly C-subspaces of X, then $\bigcup_1^\infty X_n$ is a weakly C-subspace of X.

1.18.1 Corollary: If X is a completely normal weakly C-space, then every F_σ-subspace of X is weakly C.

1.18.2 Corollary: If X is a perfectly normal weakly C-space, then every open subspace of X is weakly C.

Proof: Every perfectly normal space is completely normal by VII 4.4 in [Du]. Every open subspace of a perfectly normal space is known to be an F_σ-subspace. The result now follows from Corollary 1.18.1.

1.19 Theorem: If $\dim X = 0$, then X is weakly C.

Proof: Since $\dim X = 0$, every finite open cover of X has a pairwise disjoint open refinement; i.e., an open refinement of order 1. The result is now obvious.

1.20 Theorem: Every countable dimensional completely normal space is weakly C.

1.20.1 Corollary: Every finite (or countable) dimensional metric space is weakly C. In fact, it is shown in [Gr] that such spaces are actually C-spaces.

1.21 Theorem: A disjoint union (topological sum) of weakly C-spaces is a weakly C-space.

1.21.1 Corollary: If $\{F_a : a \varepsilon A\}$ is a separated family of weakly C-subspaces of X, then $\bigcup_{a\varepsilon A} F_a$ is a weakly C-subspace of X.

1.21.2 Corollary: If $\{F_a : a \varepsilon A\}$ is a discrete family of weakly C-subspaces of X, then $\bigcup_{a\varepsilon A} F_a$ is a weakly C-subspace of X.

1.22 Theorem: Let X be a completely normal space such that for each sequence $\{C_n\}_1^\infty$ of finite open covers of X there is a subset N' of N and a collection $\{U_n\}_{n\varepsilon N'}$ such that:

(1) U_n is a pairwise disjoint family of open sets for each $n \varepsilon N'$.

(2) For each $n \varepsilon N'$, U_n weakly refines C_n.

(3) $X - \bigcup_{n\varepsilon N'} U_n$ is contained in some weakly C-subspace of X.

(4) $N - N'$ is infinite.

Then X is a weakly C-space.

1.23 Theorem: Suppose X is completely normal. Then X is Lindelöf and weakly C if and only if every open cover of X has a countable refinement by open sets whose boundaries are weakly C.

Proof: Suppose every open cover of X has a countable refinement by open sets whose boundaries are weakly C. Then X is obviously Lindelöf. We now prove that X is weakly C. Let $\{C_i\}_1^\infty$ be a sequence of finite open covers of X. Let $\{u_j\}_0^\infty$ be a countable refinement of C_1 such that $\mathrm{Bd}u_j$ is weakly C for each j. Let $V_0 = u_0$ and let $V_m = u_m - \bigcup_{j<m} \bar{u}_j$ for $m \geq 1$. Then $\{V_j\}_1^\infty$ is a pairwise disjoint open weak refinement of C_1 such that $X - \bigcup_0^\infty V_j \subset \bigcup_0^\infty \mathrm{Bd}u_j$. To see this last containment, suppose $x \notin \mathrm{Bd}u_j$ for each j. Then either $x \varepsilon u_j$ or

$x \notin \bar{u}_j$ because u_j is open. Let j_0 be the least whole number such that $x \in u_{j_0}$. Hence for $n < j_0$, $x \notin \bar{u}_n$. Therefore, $x \in u_{j_0} - \bigcup_{n<j_0} \bar{u}_n = V_{j_0}$. It follows that $x \notin X - \bigcup_0^\infty V_j$. This proves the desired containment. By Theorem 1.18, $\bigcup_0^\infty B\dot{d}u_j$ is weakly C. The hypothesis of Theorem 1.22 is now satisfied, using $N' = \{1\}$. Therefore, X is weakly C. The converse is trivial.

1.24 Theorem: If X is a completely normal Lindelöf space, then the following are equivalent:

(a) X is a weakly C-space.

(b) X has a basis of open sets whose boundaries are weakly C.

(c) Each point of X has a local base of open sets whose boundaries are weakly C.

(d) Every open cover of X has a refinement consisting of open sets with weakly C boundaries.

1.25 Theorem: Let X be a completely normal paracompact space. Then X is weakly C if and only if every point of X has a weakly C neighborhood.

1.26 Theorem: If X is a weakly C-space, and Y is a completely normal, paracompact, locally compact subspace of X, then Y is weakly C.

1.26.1 Corollary: If X is a locally compact, hereditarily paracompact, weakly C-space and Y is the intersection of an open subspace of X and a closed subspace of X, then Y is weakly C.

1.27 Theorem: If X is completely normal, paracompact, and has a locally countable cover by weakly C-subspaces, then X is weakly C.

1.27.1 Corollary: In hereditarily paracompact spaces, a locally countable union of weakly C-subspaces is a weakly C-subspace.

1.28 Theorem: If X is a regular completely normal space, then the following are equivalent:

(a) X is paracompact and weakly C.

(b) Every open cover of X has a σ-discrete refinement by open sets with weakly C boundaries.

(c) Every open cover of X has a σ-locally finite refinement by open sets with weakly C boundaries.

(d) Every open cover of X has a locally finite refinement by open sets with weakly C boundaries.

1.29 Theorem: For any space X, the following are equivalent:

(a) X is metrizable and weakly C.

(b) X is T_1, regular, and the topology has a σ-locally finite base of open sets with weakly C boundaries.

(c) X is T_1, regular, and the topology has a σ-discrete basis of open sets with weakly C boundaries.

1.30 Theorem: A metric space X is weakly C if and only if for every disjoint pair $\{A,B\}$ of closed sets in X there exists an open set U such that $A \subset U \subset X - B$ and BdU is weakly C.

1.31 <u>Definition</u>: A space X is <u>weakly infinite dimensional</u> (<u>wid</u>) if for every sequence $\{(A_n,B_n)\}_1^\infty$ of disjoint pairs of closed sets there exists a sequence $\{U_n\}_1^\infty$ of open sets such that $A_n \subset U_n \subset X - B_n$ for each n and $\bigcap_1^\infty \mathrm{Bd}U_n = \phi$.

1.32 <u>Theorem</u>: Every normal weakly C-space is wid.

No modification of Gresham's proof is needed to prove this last theorem because only binary covers are used in his proof.

1.32.1 <u>Corollary</u>: A metric space X is weakly C if and only if for every sequence $\{(A_n,B_n)\}_1^\infty$ of disjoint pairs of closed sets there exists a sequence $\{U_n\}_1^\infty$ of open sets such that $A_n \subset U_n \subset X - B_n$ for each n, $\bigcap_1^\infty \mathrm{Bd}U_n = \phi$, and $\mathrm{Bd}U_n$ is weakly C for each n.

1.32.2 <u>Corollary</u>: A completely normal countable dimensional space is wid.

1.33 <u>Definition</u>: A space X is <u>subparacompact</u> if every open cover of X has a σ-discrete closed refinement. This definition is due to D.K. Burke [Bu 1].

1.34 <u>Theorem</u>: A completely normal subparacompact space X is hereditarily weakly C if and only if every point of X has an hereditarily weakly C neighborhood.

<u>Proof</u>: Suppose every point of X has an hereditarily weakly C neighborhood. Let U be an open cover of X by sets u each contained in an hereditarily weakly C neighborhood of some point of X. Since X is subparacompact, there exists a σ-discrete closed re-

finement V of U. Thus, each $v \in V$ is weakly C because each $v \in V$ is a subspace of some hereditarily weakly C-subspace of X. Let $V = \bigcup_1^\infty V_n$, where each V_n is a closed discrete family. For each n, V_n^* is weakly C by Corollary 1.21.2. Let A be an arbitrary subspace of X. Then, since each $v \in V$ is actually hereditarily weakly C, it follows that for each n, $\bigcup Tr_A V_n$ is a discrete family of weakly C-subspaces of A. Applying Corollary 1.21.2 again, it follows that $\bigcup Tr_A V_n$ is a weakly C-subspace of A. Since A is completely normal and A can be written as a countable union of weakly C-subspaces of A, namely $A = \bigcup_{n=1}^\infty (\bigcup Tr_A V_n)$, it follows by Theorem 1.15 that A is weakly C. The converse is trivial.

1.34.1 <u>Corollary</u>: If X is a completely normal subparacompact space and $\{F_a : a \in A\}$ is a locally countable cover of X by hereditarily weakly C-subspaces of X, then X is hereditarily weakly C.

<u>Proof</u>: Let $x \in X$. Let $N(x)$ be a neighborhood of x intersecting at most countably many members of $\{F_a : a \in A\}$, say $F_1, F_2, \ldots$. Then $N(x) \subset \bigcup_1^\infty F_n$. The proof will be completed by showing that $\bigcup_1^\infty F_n$ is hereditarily weakly C. Suppose $B \subset \bigcup_1^\infty F_n$. Then $B = \bigcup_1^\infty (B \cap F_n)$, and each $B \cap F_n$ is weakly C because each F_n is hereditarily weakly C. It follows from Theorem 1.18 that B is weakly C.

1.34.2 <u>Corollary</u>: In a completely normal space, every countable union of hereditarily weakly C-subspaces is an hereditarily weakly C-subspace.

1.35 Definition: A map $f: X \to Y$ is perfect if it is a continuous closed surjection and $f^{-1}(y)$ is compact for each $y \in Y$.

1.36 Definition: A subset A of a space X is dense in itself if every point $x \in A$ is a limit point of $A - \{x\}$.

1.37 Theorem: Let X and Y be metric spaces, X hereditarily weakly C, and $f: X \to Y$ a perfect map. Let $Y_1 = \{y \in Y: f^{-1}(y)$ is dense in itself$\}$. If Y_1 is hereditarily weakly C, then Y is hereditarily weakly C.

1.37.1 Corollary: Using the same notation as in Theorem 1.37, if Y_1 is empty (each $f^{-1}(y)$ has an isolated point), then Y is hereditarily weakly C.

1.37.2 Corollary: Using the same notation as in Theorem 1.37, if $f^{-1}(y)$ is topologically discrete for each $y \in Y$, then Y is hereditarily weakly C.

1.38 Theorem: Suppose X is completely normal, Y is a C-space, and $f: X \to Y$ is a closed, continuous, surjective map such that $\dim f^{-1}(y) = 0$ for each $y \in Y$. Then X is a weakly C-space.

1.38.1 Corollary: If Y is regular, compact, $\dim Y = 0$, and X is a C-space such that $X \times Y$ is completely normal, then $X \times Y$ is weakly C.

1.39 Theorem: Let X and Y be metric spaces, X hereditarily weakly C, and $f: X \to Y$ a continuous closed surjection. Let $Y_0 = \{y \in Y: \mathrm{Bd} f^{-1}(y)$ is dense in itself and nonempty$\}$. If Y_0 is hereditarily weakly C, then Y is hereditarily weakly C.

1.40 Theorem: Let X be a metric space and $\{F_a: a \in A\}$ a well-ordered cover of X such that each F_a is weakly C and, for each $a \in A$, $\bigcup_{b<a} F_b$ is closed in X. Then X is weakly C.

1.40.1 Corollary: Let X be a metric space and $\{F_a: a \in A\}$ a closed, hereditarily closure-preserving cover of X by weakly C-subspaces of X. Then X is weakly C.

1.41 Theorem: Let X and Y be metric spaces, X a weakly C-space, and $f: X \to Y$ an open and closed, continuous surjection. Let $Y_2 = \{y \in Y: \mathrm{Bd}f^{-1}(y) = f^{-1}(y)$ and $\mathrm{Bd}f^{-1}(y)$ is dense in itself$\}$. If Y_2 is contained in a weakly C-subspace of Y, then Y is weakly C.

1.41.1 Corollary: If, in Theorem 1.41, both X and Y_2 are hereditarily weakly C, then Y is hereditarily weakly C.

1.41.2 Corollary: Let X, f, and Y be as in Theorem 1.41. If $\mathrm{Bd}f^{-1}(y)$ is not dense in itself for any $y \in Y$, then Y is weakly C.

1.42 Example: If ω_1 denotes (as usual) the first uncountable ordinal, then $[0,\omega_1]$ is a C-space. This is proved in [Gr].

1.43 Example: The open ordinal space $[0,\omega_1)$ is not a C-space since it is not screenable, but, since $\dim [0,\omega_1) = 0$, it follows from Theorem 1.19 that $[0,\omega_1)$ is weakly C.

The remaining theorems and corollaries in this section are related to the problem of deciding when a weakly C-space is actually a

C-space. However, the following question is still unanswered.

1.44 <u>Question</u>: Is every metric weakly C-space a C-space?

This question relates to a famous unsolved problem known as <u>Alexandroff's problem</u> [Al]: Is every compact metric wid space also CD? In particular, if one could answer Question 1.44 negatively, this would provide a negative answer to Alexandroff's problem (in the non-compact case). Note also that a compact metric C-space which is not hereditarily C would provide a negative answer to Alexandroff's problem.

1.45 <u>Theorem</u>: Every locally compact, Lindelöf, SCN, weakly C-space X is a C-space.

<u>Proof</u>: Let B be a basis for X consisting of relatively compact open sets; i.e., open sets whose closures are compact. If $b \in B$, then Bdb is closed in $\bar{b}$. Therefore Bdb is a compact weakly C-subspace of X. Since every compact weakly C-space is obviously a C-space, Bdb is a C-subspace of X. Therefore X has a basis of open sets whose boundaries are C-spaces. It follows from Theorem 3.2.2 of [Gr] that X is a C-space.

1.45.1 <u>Corollary</u>: Every locally compact, σ-compact, SCN, weakly C-space is a C-space.

<u>Proof</u>: See XI 7.2 in [Du] where it is proved that every locally compact σ-compact space is Lindelof.

1.45.2 <u>Corollary</u>: Every paracompact, locally compact, SCN, weakly C-space is a C-space.

Proof: It follows from Theorem 7.3 in [Du] that such a space is a free union of locally compact, σ-compact, SCN spaces. Each member of this free union, being closed in the space, is weakly C by Theorem 1.16. By the previous corollary, each member of this free union is a C-space. The result follows due to Gresham's proof that a free union of C-spaces is a C-space.

1.45.3 Corollary: A locally compact metric space is a C-space if and only if it is a weakly C-space.

1.46 Theorem: Every σ-compact, SCN, weakly C, Hausdorff space is a C-space.

1.46.1 Corollary: A σ-compact metric space is a C-space if and only if it is a weakly C-space.

CHAPTER TWO

SCREENABLE SPACES

2.1 Definition: A space is screenable (strongly screenable) if every open cover has a σ-disjoint (σ-discrete) open refinement.

2.2 Definition: A space X is finitely screenable (finitely strongly screenable) if for every open cover U of X there is a refinement $W = \bigcup_1^k W_n$ such that $k \in N$ and each W_n is a disjoint (discrete) open collection. The obvious abbreviations FS and FSS will be used.

2.3 Theorem:

(a) Every FS space is metacompact and screenable.

(b) Every FSS space is paracompact and strongly screenable.

(c) Every FSS space is an FS space.

(d) Every strongly C-space is an FS space.

(e) Every compact space is an FSS space.

2.4 Theorem: If X is paracompact, T_2, and FS, then X is FSS.

Proof: Let U be an open cover of X. Since X is FS, there exists a refinement $W' = \bigcup_1^k W'_n$ of U such that each W'_n is a disjoint open collection. By paracompactness, W' has a precise locally finite open refinement $W = \bigcup_1^k W_n$ such that each W_n is a discrete collection. Therefore W is the desired refinement of U, showing that X is FSS.

2.4.1 <u>Corollary</u>: If X is normal, T_2, and FS, then X is FSS.

<u>Proof</u>: Since X is normal, T_2, and FS, it is a regular screenable metacompact space. Therefore, by Theorem 6 of [Nm 2], X is paracompact.

2.4.2 <u>Corollary</u>: Every finite dimensional metric space X is FSS.

<u>Proof</u>: By proposition 3.6.3 of [Gr], X is a strongly C-space. Therefore X is FS, normal, and T_2.

2.5 <u>Definition</u>: A property P of a space X is <u>weakly hereditary</u> if every closed subspace of X also has property P.

2.6 <u>Theorem</u>: Each of the following properties is weakly hereditary: screenability, strong screenability, FS, and FSS.

2.7 <u>Theorem</u>: If every open subspace of X is screenable, strongly screenable, FS or FSS, then every subspace of X is screenable, strongly screenable, FS or FSS, respectively.

In [Gr] it is proved that every SCN space which is the countable union of closed C-subspaces is a C-space. Such a countable sum theorem for FS or FSS spaces does not hold (consider the disjoint union $\bigcup_1^\infty X_n$ where each X_n is a metric space of dimension n).

However, the next theorem shows that a finite sum theorem does hold.

2.8 Theorem: If X is CWN and $X = \bigcup_{1}^{k} X_n$ where each X_n is closed and FSS, then X is FSS.

Proof: Let U be an open cover of X. Let $Q^n = \bigcup_{i=1}^{j(n)} Q_i^n$ be a refinement of $Tr_{X_n} U$ such that each Q_i^n is a discrete open collection in X_n. Since X_n is closed, Q_i^n is discrete in X also. The refinement of U given by $\bigcup_{n=1}^{k} Q^n$ is a refinement consisting of $\sum_{n=1}^{k} j(n)$ discrete collections. By using collectionwise normality these collections can be expanded to disjoint open collections in X. This proves that X is FS, and since X is normal, it follows by Corollary 2.4.1 that X is FSS.

The following theorem and its corollary are due to J.A. French [Fr], although he was not aware of the present terminology.

2.9 Theorem: If X is CWN and sdim $X \leq n$, then X is FSS.

2.9.1 Corollary: If X is FSS, then sdim $X \leq n$ if and only if every open cover U of X has an open refinement $W = \bigcup_{1}^{k} W_n$, where each W_n is discrete, and for each $w \in W$ there exists $u \in U$ such that $\bar{w} \subset u$.

2.10 Definition: Let X be a topological space, U an open cover of X, and $V = \bigcup_{1}^{\infty} V_n$ an open refinement of U. Consider the following conditions of these covers:

(1) For each $x \in X$, there is a natural number $n(x)$ such that $0 < \mathrm{ord}(x, V_{n(x)}) \leq \omega$.

(2) For each $x \in X$, there is a natural number $n(x)$ such that $0 < \mathrm{ord}(x, V_{n(x)}) < \omega$.

(3) For each $n \in N$, V_n covers X.

If every open cover U of X has a refinement $V = \bigcup_1^\infty V_n$ satisfying (1), then the space is called <u>weakly δθ-refinable</u>. If V satisfies (1) and (3), then X is called <u>δθ-refinable</u>. If V satisfies (2), X is called <u>weakly θ-refinable</u>. If V satisfies (2) and (3), X is called <u>θ-refinable</u>. These terms are found in [WW2].

2.11 <u>Theorem</u>: If X is screenable, then X is weakly θ-refinable.

<u>Proof</u>: Let U be an open cover of X and let $W = \bigcup_1^\infty W_n$ be a σ-disjoint open refinement of U. Since W is a cover of X, for each $x \in X$ there is at least one natural number $n(x)$ such that $\mathrm{ord}(x, W_n) = 1$. Hence, X is weakly θ-refinable.

2.11.1 <u>Corollary</u>: A screenable space is compact if and only if it is countably compact.

<u>Proof</u>: It is proved in [WW2] that a weakly θ-refinable space is compact if and only if it is countably compact. This corollary is also proved directly in [He].

2.11.2 <u>Corollary</u>: A C-space is compact if and only if it is countably compact.

<u>Proof</u>: Every C-space is screenable. This is easily established by iterating any given open cover U; i.e., let $C_n = U$ for each $n \in N$. Then a C-refinement $\{U_n\}_1^\infty$ of $\{C_n\}_1^\infty$ is clearly a σ-disjoint open refinement of U.

2.12 <u>Example</u>: The open ordinal space $[0,\omega_1)$ is a countably compact weakly C-space which is not compact.

2.13 <u>Definition</u>: A space is called <u>perfect</u> if every closed subset of X is a G_δ in X.

2.14 <u>Definition</u>: A space is called F_σ-<u>screenable</u> if every open cover has a σ-discrete closed refinement. This definition is due to McAuley [Mc1].

2.15 <u>Definition</u>: According to Arhangel'skiĭ [Ar], a space X is <u>σ-paracompact</u> if for every open cover U of X there is a sequence $\{U_n\}_1^\infty$ of open covers of X such that if $x \in X$, there is a natural number $n(x)$ and a set $u \in U$ such that $St(x,U_{n(x)}) \subset u$, where $St(x,U_{n(x)}) = \cup\{v \in U_{n(x)}: x \in v\}$.

2.16 <u>Theorem</u>: [Bu 1] The following properties of a space X are equivalent:

(a) X is σ-paracompact.

(b) X is F_σ-screenable.

(c) X is subparacompact.

(d) Every open cover of X has a σ-locally finite closed refinement.

(e) Every open cover of X has a σ-closure preserving closed refinement.

(f) For every open cover U there is a sequence $\{U_n\}_1^\infty$ of open refinements such that if $x \in X$ there is a natural number $n(x)$ such that $\mathrm{ord}(x, u_{n(x)}) = 1$.

2.17 Comments: From (f) it is obvious that every subparacompact space is θ-refinable. Bennett and Lutzer [BL] have proved that every perfect, weakly θ-refinable space is subparacompact. Hodel [Ho] has proved that every countably subparacompact screenable space is subparacompact.

2.18 An implication diagram: This chapter's main results are summarized in the following diagram. In this diagram, a solid line denotes unconditional implication, while a dotted line indicates conditional implication with the needed condition written near the dotted line.

See diagram on following page.

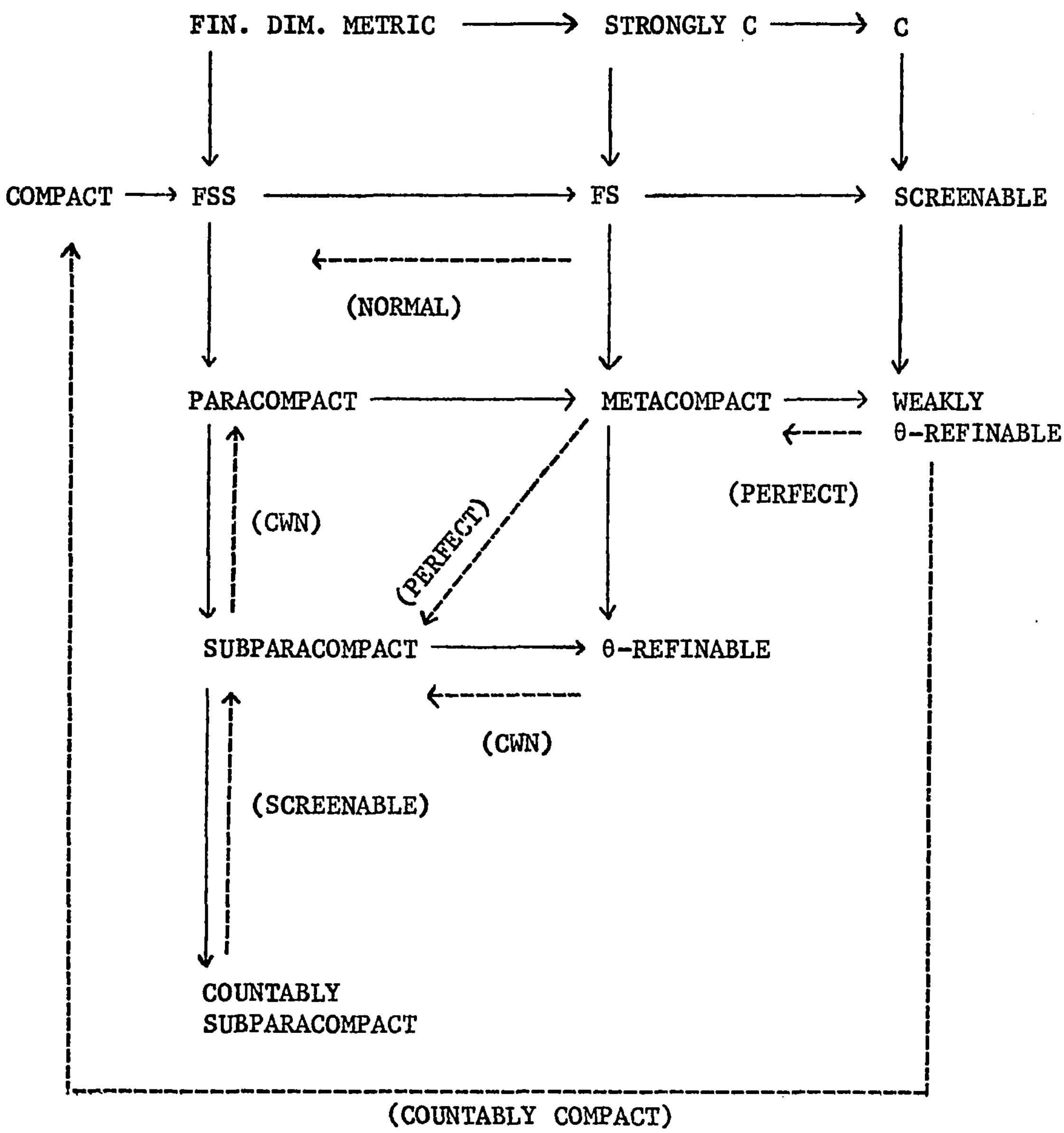

FIN. DIM. METRIC
STRONGLY C
C
COMPACT
FSS
FS
SCREENABLE
(NORMAL)
PARACOMPACT
METACOMPACT
WEAKLY θ-REFINABLE
(PERFECT)
(CWN)
(PERFECT)
SUBPARACOMPACT
θ-REFINABLE
(CWN)
(SCREENABLE)
COUNTABLY SUBPARACOMPACT
(COUNTABLY COMPACT)

Figure 2.1

CHAPTER THREE

EXAMPLES AND COUNTEREXAMPLES

3.1 Burke's Example: [Bu 1]. Let ω_1 (resp. ω_2) be the first ordinal with cardinality χ_1 (resp. χ_2).

Let $X = [0,\omega_2) \times [0,\omega_2)$. For $\alpha \in (0,\omega_2)$, let $L_{1,\alpha} = \{(\alpha,\beta) \in X: \beta \in [0,\omega_2)\}$ and let $L_{2,\alpha} = \{(\beta,\alpha) \in X: \beta \in [0,\omega_2)\}$. Also let $L_{1,0} = L_{2,0} = \{(0,0)\}$. Define an open base for a topology on X as follows: For each $(\alpha,\beta) \in X$ where $\alpha \neq 0$ and $\beta \neq 0$, the singleton set $\{(\alpha,\beta)\}$ is in the base. For $i = 1,2$ and $\alpha \in [0,\omega_2)$ the base includes all subsets of $L_{i,\alpha}$ which have finite complement relative to $L_{i,\alpha}$. Note that $\{(0,0)\}$ is also in the base.

3.1.1 X is a strongly C-space.

Proof: Let $\{C_n\}_1^\infty$ be a sequence of open covers of X. Let $U_1 = \{(\alpha,\beta): \alpha \neq 0$ and $\beta \neq 0$ or $\alpha = \beta = 0\}$. Clearly, U_1 weakly refines C_1. Since C_2 is an open cover of X, there is, for each $\alpha \in (0,\omega_2)$, a basic open set $B_{\alpha,0} \subset L_{1,\alpha}$ such that $(\alpha,0) \in B_{\alpha,0}$ and $B_{\alpha,0}$ is contained in some member of C_2. Let $U_2 = \{B_{\alpha,0}: \alpha \in (0,\omega_2)\}$. Then U_2 weakly refines C_2. Since C_3 is an open cover of X, there is, for each $\beta \in (0,\omega_2)$, a basic open set $B_{0,\beta} \subset L_{2,\beta}$ such that $(0,\beta) \in B_{0,\beta}$ and $B_{0,\beta}$ is contained in some member of C_3. Let $U_3 = \{B_{0,\beta}: \beta \in (0,\omega_2)$. Then U_3 weakly refines

C_3. Let $U_n = \phi$ for $n \geq 4$. Clearly, $\{U_n\}_1^\infty$ is a C-refinement of $\{C_n\}_1^\infty$. Therefore X is a strongly C-space.

3.1.2 X is screenable, metacompact, θ-refinable and FS.

Proof: Every strongly C-space has these properties.

3.1.3 X is CD.

Proof: $X = \{(\alpha,\beta): \alpha \neq 0, \beta \neq 0\} \cup \{(\alpha,0): \alpha \in [0,\omega_2)\} \cup \{(0,\alpha): \alpha \in [0,\omega_2)\}$, and each of these sets has strong covering dimension zero, because each is discrete in the relative topology.

3.1.4 X is not normal.

Proof: Let $A = \{(\alpha,0): \alpha \in (0,\omega_1)\}$ and $B = \{(0,\beta): \beta \in (0,\omega_2)\}$. Then A and B are disjoint closed subsets of X. Suppose U and V are disjoint open sets in X with $A \subset U$ and $B \subset V$. For each $(0,\beta) \in B$ there is a basic open set $B_{(0,\beta)} \subset L_{2,\beta}$ such that $(0,\beta) \in B_{(0,\beta)} \subset V$. Note that each set $B_{(0,\beta)}$ contains points with first coordinate in $(0,\omega_1)$. But, since $U \cap V = \phi$, for each $\alpha \in (0,\omega_1)$ there can be only finitely many such sets $B_{(0,\beta)}$ in V having first coordinate α. This implies that there are at most χ_1 such sets $B_{(0,\beta)}$, which is a contradiction. (There are χ_2 such sets.)

3.1.5 X is not FSS.

Proof: Recall that every FSS space is paracompact. Since X is obviously Hausdorff, if it were FSS then it would be normal, which is false. So X is not FSS.

3.1.6 X is regular.

Proof: Suppose A is a closed subset of X and $p \in X - A$.

If $p = (\alpha,\beta)$ where $\alpha \neq 0$ and $\beta \neq 0$, or if $p = (0,0)$, then $\{p\}$ is clopen and separates p from A. Suppose $p = (0,\beta)$ for some $\beta \in [0,\omega_2)$. Without loss of generality, we may also assume that $A \subset \{0\} \times [0,\omega_2)$. Then $L_{2,\beta}$ is a clopen set containing p and separating p from A. Other cases are handled similarly.

3.1.7 X is not subparacompact.

Proof: See Burke's proof in [Bu 1].

3.1.8 X is not countably subparacompact.

Proof: Hodel [Ho] has proved that every screenable countably subparacompact space is subparacompact. Since X is screenable and not subparacompact, it is not countably subparacompact.

3.1.9 X is locally compact.

Proof: The specified basis consists entirely of compact subsets of X.

3.1.10 Summary: This example demonstrates that an FS space which is not normal may fail to be FSS even though it has many additional properties.

3.2 Example: A nonmetrizable paracompact space which is not FS.

Construction: For each $n \in N$, let $Z_n = [0,\omega_1] \times I^n$. It follows by Lemma 18-2 in [Nm 2] that each Z_n has dimension n. Let Z denote the disjoint union, $Z = \bigcup_1^\infty Z_n$. Let U_n be an open cover of Z_n such that no refinement of U_n has order less than $n + 1$. Let $U = \bigcup_1^\infty U_n$. Then U is an open cover of Z. Suppose $W = \bigcup_1^K W_j$ refines U, where each W_j is a discrete open collection in Z. Then, for each $n \in N$, $Tr_{X_n} W$ is an open refinement of U_n having order K. Since K is fixed, a contradiction arises when $K < n + 1$. Therefore, Z is not FSS.

Before considering the next example, a useful definition will be introduced, and a theorem involving this definition will be established as a tool to be used in deciding that certain of the examples to follow are not screenable.

3.3 Definition: A subset A of a space X is called a CIC-subset if every pairwise disjoint open collection in X contains at most countably many members which intersect A. The letters CIC indicate a "countable intersection condition."

3.4 Theorem: If X contains a closed, uncountable, discrete, CIC-subset K, then X is not screenable.

Proof: For each $k \in K$, let $V(k)$ be an open set containing k and no other point of K. Let $C = \{V(k): k \in K\} \cup \{X - K\}$. Then C is an open cover of X. Suppose $\{U_n\}$ is a sequence of pairwise disjoint open collections in X. Since K is a CIC-subset of X, each U_n contains at most countably many members which intersect K. Therefore, $\overset{\infty}{\underset{1}{\cup}} U_n$ contains at most countably many members which intersect K. If each U_n weakly refines C, then $\overset{\infty}{\underset{1}{\cup}} U_n$ does not cover K, because K is uncountable. Therefore X is not screenable.

3.5 Definition: A space is collectionwise Hausdorff (CWH) if for every closed discrete subset $\{x_\alpha: \alpha \in A\}$ in X, there exists a pairwise disjoint open collection $\{V_\alpha: \alpha \in A\}$ in X such that

$x_\alpha \in V_\alpha$ for each $\alpha \in A$.

From the definitions involved, the following result is clear.

3.6 Theorem: If X contains a closed, uncountable, discrete, CIC-subset, then X is not CWH.

3.7 Example: The Tangent Disc Space [SS]. Let X be the points on or above the x-axis in the xy-plane. Points above the x-axis have their usual neighborhood base consisting of the interiors of discs centered at the point, while a point t on the x-axis has a neighborhood base consisting of t along with interiors of discs centered above t and tangent to the x-axis at t. From the definition of the topology on X and the fact that the plane is separable it follows that the x-axis is a closed, uncountable, discrete, CIC-subset of X. By Theorems 3.4 and 3.6, X is neither screenable nor CWH.

The tangent disc space is a well-known example of a Moore space (a regular developable space). McAuley [Mc 1] proved that every developable space is subparacompact. Thus, the tangent disc space illustrates the fact that a subparacompact space need not be screenable.

3.8 The Hilbert Cube, I^ω. This classical example of a compact metric space has been studied for years by analysts and topologists, and it will continue fascinating mathematicians for generations. Since it is compact, it is FSS, but it is not a C-space. This fact is proved in [Gr] by first proving that every metric

C-space is wid and then quoting Nagata's proof in [Nt 2] that I^ω is not wid. William Haver proved in [Ha] that I^ω is not a C-space by first proving that every compact, locally contractible, metric C-space is an ANR; then he embedded Borsuk's example [Bo] of a compact, locally contractible metric space which is not an ANR into I^ω.

Since a direct proof that I^ω is not a C-space seems desirable, it is given below. For this purpose, I^ω will be considered as the product of countably many copies of the closed unit interval $I = [-1,1]$. For each $n \in N$. let $A_n = \pi_n^{-1}[-1,\frac{1}{2})$ and $B_n = \pi_n^{-1}(-\frac{1}{2},1]$, where π_n denotes the projection of I^ω onto the nth factor.

Let $C_n = \{A_n,B_n\}$. Then $\{C_n\}_1^\infty$ is a sequence of (finite) open covers of I^ω. Suppose $\{U_n\}_1^\infty$ is a sequence of pairwise disjoint open collections such that U_n weakly refines C_n for each $n \in N$. Without loss of generality suppose $U_n = \{U_n^1,U_n^2\}$, where $U_n^1 \subset A_n$ and $U_n^2 \subset B_n$. Let $V_n = \{V_n^1,V_n^2\}$, where $V_n^1 = \pi_n^{-1}[-1,-\frac{1}{4}) \cup U_n^1$ and $V_n^2 = \pi_n^{-1}(\frac{1}{4},1] \cup U_n^2$. If it can be shown that $\bigcup_1^\infty V_n$ is not a cover of I^ω, it will follow that $\bigcup_1^\infty U_n$ is not a cover of I^ω, and consequently, $\{C_n\}_1^\infty$ has no C-refinement. For each n, $I^\omega - \mathrm{Bd}V_n^1 = V_n^1 \cup (I^\omega - \bar{V}_n^1)$. Also, V_n^1 and $I^\omega - \bar{V}_n^1$ are disjoint open sets in $I^\omega - \mathrm{Bd}V_n^1$ such that $\pi_n^{-1}(-1) \subset V_n^1$ and $\pi_n^{-1}(1) \subset I^\omega - \bar{V}_n^1$. Thus, $\mathrm{Bd}V_n^1$ separates $\pi_n^{-1}(-1)$ and $\pi_n^{-1}(1)$. It will now be proved that $\bigcap_1^\infty \mathrm{Bd}V_n^1 \neq \phi$. This will be done by first showing that $\bigcap_1^i \mathrm{Bd}V_n^1 \neq \phi$ for each $i \in N$, and then using the compactness of I^ω. The fact that $\bigcap_1^i \mathrm{Bd}V_n^1 \neq \phi$ follows from the fact that $\bigcap_1^i \mathrm{Bd}V_n^1 \cap I^i \neq \phi$, where I^i is the i-dimensional cube embedded in I^ω in the first i coordinates (see Theorem D, page 40 of [HW]). Let $x \in \bigcap_1^\infty \mathrm{Bd}V_n^1$. Then x is not in any member of any V_n, because x is clearly not in V_n^1 for any n, and if $x \in V_n^2$ for some n, then $V_n^1 \cap V_n^2$ would not be empty. So $\bigcup_1^\infty V_n$ is not a cover of I^ω. This completes the argument that I^ω is not a C-space. Note that I^ω also serves as an example of a screenable space which is not a C-space.

3.9 Bing's Example G. This space was first introduced in 1951 in Bing's famous metrization article [Bi] as an example of a normal Hausdorff space that is not CWN. His proof, however, shows that it is not even CWH. Over twenty years later William Fleissner [Fl] constructed a normal space which is CWH but not CWN. He called his example "George." It will be examined in detail later and a simpler construction will be given.

It has been proved in [Gr] that every countable dimensional SCN space is a C-space, but the question of whether every countable dimensional normal space is a C-space remained unanswered. It will be shown later that Bing's space provides a negative answer to this question. In fact, it will be shown that Bing's example G is countable dimensional but is not even screenable. In doing this, Fleissner's notation will be used because it is somewhat simpler than

Bing's original notation and, of course, it is the same notation used in describing George.

For any set S, let ${}^{S}2$ be the set of functions from S into $2 = \{0,1\}$. Let e denote either 0 or 1. For each $a \in S$, let $V(a,e) = \{f \in {}^{S}2\colon f(a) = e\}$. Then $\{V(a,e)\colon a \in S,\ e \in 2\}$ is a subbasis for the usual product topology on ${}^{S}2$. A set is said to be of degree n if it is the nonempty intersection of n subbasis sets; equivalently, it is the result of restricting on n coordinates. There are at most 2^n pairwise disjoint open sets of degree n. Let $P(S)$ denote the power set of S; i.e., $P(S)$ is the set of all subsets of S. For every set S, a topology G_S on ${}^{P(S)}2$ will be constructed so that G_S is finer than the product topology. For each $a \in S$, let $f_{S,a}$ denote the function in ${}^{P(S)}2$ defined so that for $A \in P(S)$, $f_{S,a}(A) = 1$ if and only if $a \in A$. Let $X_S = \{f_{S,a}\colon a \in S\}$. A basis for G_S is $\{B\colon B$ is basic open in ${}^{P(S)}2\} \cup \{\{p\}\colon p \in {}^{P(S)}2 - X_S\}$. Bing's example G is obtained by letting S be an uncountable set. Burke [Bu 4] has examined this space under the assumption that $|S| > c$. Let G denote the topological space $({}^{P(S)}2, G_S)$ where S is uncountable.

It is easily seen that X_S is a closed, discrete set in G because $G - X_S$ is open and $V(\{a\},1) \cap X_S = \{f_{S,a}\}$. Since S is uncountable, so is X_S. To see that G is CD simply note that $G = X_S \cup (G - X_S)$ and both X_S and $G - X_S$ are discrete in the relative topology. If it can be shown that X_S is a CIC-subset of G, it will follow from Theorems 3.4 and 3.6 that G is neither screenable nor CWH. Let F be a pairwise disjoint open collection in G such that for each $f \in F$, $f \cap X_S \neq \phi$. Suppose F is uncountable. Then there exists an uncountable subset A of S such that each member of $\{f_{S,a}: a \in A\}$ belongs to a distinct member f_a of F. Then f_a is open and contains $f_{S,a}$, and so there is a basic open set B_a of degree n_a such that $f_{S,a} \in B_a \subset f_a$. Clearly, $B_a \cap B_c = \phi$ if $a \neq c$. Thus, for some $n \in N$, there are uncountably many members of $\{B_a: a \in A\}$ of degree n. This contradiction implies that X_S is a CIC-subset of G.

Bennett and Lutzer asked in [BL] if Bing's example G is θ-refinable. Burke [Bu 4] proved that the answer is no if $|S| > c$. It follows that G is not subparacompact if $|S| > c$ because subparacompactness implies θ-refinability. However, the question remains open if $|S| = c$.

Having seen that Bing's example G is not screenable, a natural question is whether it can be made screenable. Note that a normal screenable space that is not CWH would be a screenable Dowker space [Ru]; i.e., a normal Hausdorff space that is not countably paracompact. This is because of Nagami's result in [Nm 1] that every normal, screenable, countably paracompact space is paracompact. The author wishes to thank John J. Walsh for the next theorem which proves that Bing's example G cannot be made screenable.

3.10 Theorem: Every normal, screenable Haudorff space X is CWH.

Proof: Let C be a closed discrete subset of X. For each $c \in C$, let U_c be an open set in X such that $c \in U_c \subset X - \{b \in C: b \neq c\}$.

Then $Q = \{U_c : c \in C\} \cup \{X - C\}$ is an open cover of X. Let $V = \overset{\infty}{\underset{1}{\cup}} V_i$ be a σ-disjoint open refinement of Q. Let

$$C_1 = \{x \in C: x \in V_1^*\}, \quad C_2 = \{x \in C: x \in V_2^* - V_1^*\},$$

$$C_3 = \{x \in C: x \in V_3^* - (V_1^* \cup V_2^*)\}, \quad \text{and so on.}$$

Then C is the disjoint union, $C = \overset{\infty}{\underset{1}{\cup}} C_i$. Thus, C_1 is closed and V_1^* is open with $C_1 \subset V_1^*$. By normality, there exists an open set W_1 such that $C_1 \subset W_1 \subset \bar{W}_1 \subset V_1^*$. Likewise, C_2 is closed and $V_2^* - \bar{W}_1$ is open with $C_2 \subset V_2^* - \bar{W}_1$. So there exists an open set W_2 such that $C_2 \subset W_2 \subset \bar{W}_2 \subset V_2^* - \bar{W}_1$. In general, there exist open sets W_i such that, for each i, $C_i \subset W_i \subset \bar{W}_i \subset V_i^* - \underset{j<i}{\cup} \bar{W}_j$. Therefore $\{W_i\}_1^\infty$ is a pairwise disjoint open cover of C. Let $c \in C$. Then there is a unique W_i with $c \in W_i$. Also there exists a unique $v_c^i \in V_i$ such that $c \in v_c^i$. Clearly, the pairwise disjoint open collection $\{v_c^i \cap W_i : c \in C, i \in N\}$ separates the points in C. So X is CWH.

The following questions remain open.

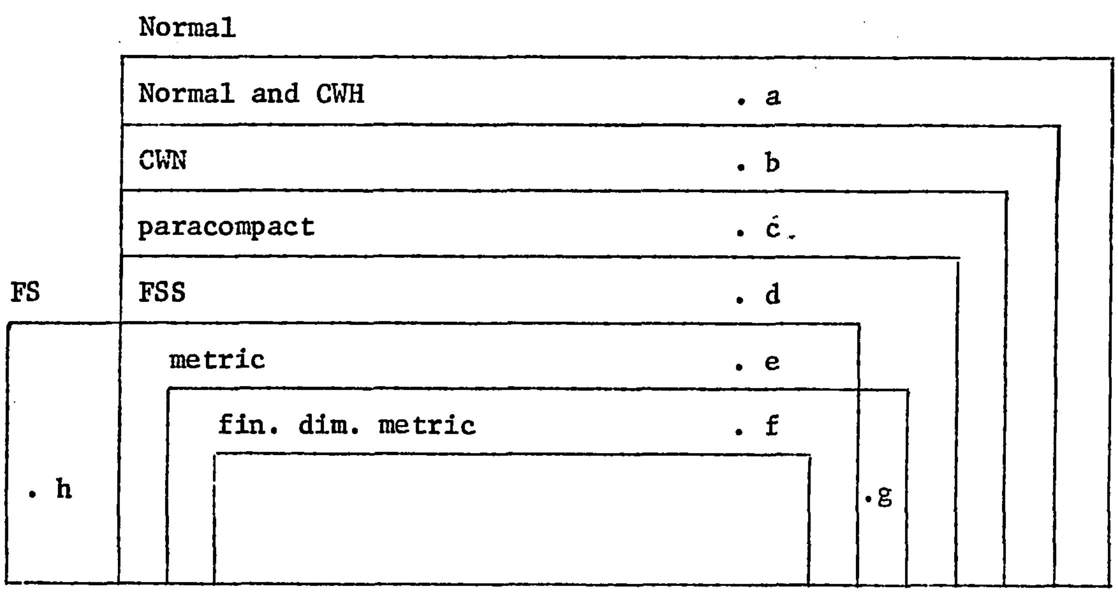
3.11 <u>Question</u>: Is every normal, screenable, Hausdorff space CWN?

3.12 <u>Question</u>: Is every normal, screenable, Hausdorff space paracompact? In other words, is there a screenable Dowker space? If not, then a T_2 space is paracompact if and only if it is normal and screenable.

3.13 Question: If X is normal and screenable and $f: X \to Y$ is a continuous closed mapping of X onto a space Y, then need Y necessarily be screenable?

Note that if this last question could be answered negatively, then the domain space X would be Dowker space because paracompactness is preserved by continuous closed surjections.

3.14 A containment diagram: The main results of this chapter are summarized by the following diagram.



a) Bing's example G (3.9)

b) Fleissner's George (4.1)

c) Open ordinal space $[0,\omega_1)$ (1.43)

d) Example 3.2

e) Closed ordinal space $[0,\omega_1]$ (1.42)

f) Hilbert cube (3.8)

g) See the comment following Theorem 2.7

h) Burke's example (3.1)

Figure 3.1

CHAPTER FOUR

FLEISSNER'S GEORGE AND A SIMPLER CONSTRUCTION

In this chapter George will be investigated. In particular, it will be shown that George is neither CD nor screenable. Finally, a general method will be given whereby a space X which is sufficiently like Bing's example G can be used to generate a normal CWH space Y which is not CWN. The space Y will be described in a purely point-set fashion, without all the function spaces which Fleissner used in constructing George. It is felt that the reader will understand the space Y much more easily than George.

4.1 George: The idea in constructing George is to tie together an ω_1-sequence of initial segments of Bing's example G in such a way that each point of the closed discrete set X_S becomes stretched to a (rather large) closed set in George, and the collection of these closed sets is a discrete collection in George which cannot be separated simultaneously by the members of any pairwise disjoint open collection in George.

George is a topology on $\bigcup_{\beta<\omega_1} {}^{P(\beta)}2$. Let $Y_\alpha = \{f_{\beta,\alpha}: \alpha < \beta < \omega_1\}$, and let $X = \bigcup_{\alpha<\omega_1} Y_\alpha = \bigcup_{\beta<\omega_1} X_\beta$. A subbasis for George consists of the following types of sets:

I. $\{p\}$, where $p \in \bigcup_{\beta<\omega_1} {}^{P(\beta)}2 - X$.

II. $V_W = \bigcup_{\beta\in W} {}^{P(\beta)}2$, where W is open in ω_1.

III. $V(A,e) = \{f\colon f \in {}^{P(\beta)}2,\ f(A(\beta)) = e\}$, where

i. $A\colon \omega_1 \to P(\omega_1)$ is a function.

ii. $A(\beta) \subset [0,\beta)$ for each $\beta < \omega_1$.

iii. $H_\alpha = \{\beta\colon \alpha \in A(\beta)\}$ is clopen for each $\alpha < \omega_1$.

It is proved in [F1] that the topology induced on ${}^{P(\beta)}2$ is exactly G_β for each $\beta < \omega_1$, that $Y = \{Y_\alpha\colon \alpha < \omega_1\}$ is a closed discrete collection of closed sets, and that the map taking $f_{\beta,\alpha}$ to β is a homeomorphism of Y_α onto the open ordinal space (α,ω_1) for each $\alpha < \omega_1$.

Note that George is not screenable because each closed subset Y_α of George is not screenable. If George is CD, then each Y_α is CD. But, if Y_α is CD then, since Y_α is SCN, it follows from proposition 3.1.13 in [Gr] that Y_α is a C-space. This implies that Y_α is screenable, which is a contradiction. Therefore, George is not CD. In fact, George fails to have any weakly hereditary property which $[0,\omega_1)$ fails to have. This is due to the fact that George contains closed subspaces (Y_α) which are homeomorphic to $[0,\omega_1)$. For example, George is not metacompact or subparacompact. It is unfortunate, but seemingly unavoidable, that the simpler construction which is to follow also contains closed subsets which are homeomorphic to $[0,\omega_1)$.

Now a general method of constructing a normal CWH space which is not CWN by starting with a certain type of normal non-CWH space will be given. Actually, the method will be derived through a series of attempts that do not quite work.

4.2 Preliminaries: Let X be a space satisfying:

(1) X is normal.

(2) X contains a closed, uncountable, discrete CIC-subset $F = \{f_\alpha : \alpha < \omega_1\}$.

(3) If $p \in X - F$, then $\{p\}$ is open in X.

(4) X has a clopen subbasis B such that if $n \in N$, then there are only finitely many pairwise disjoint open sets in X, each having degree n. A set has degree n if it is the intersection of n subbasis elements.

(5) If H and K are disjoint closed subsets of F, then there is a subbasis element $B(H)$ such that $H \subset B(H) \subset X - K$.

The following notation will be used:

$$X'_\alpha = X - \{f_\beta : \beta \geq \alpha\}$$

$$X_\alpha = X'_\alpha \times \{\alpha\}$$

$$D_\alpha = \{f_\alpha\} \times (\alpha, \omega_1)$$

$$T = \bigcup_{\alpha<\omega_1} D_\alpha$$

$$Y = \bigcup_{\alpha<\omega_1} X_\alpha$$

4.3 <u>First attempt</u>: Noting that $Y \subset X \times [0,\omega_1)$, let Y receive the relative product topology, except that $\{(p,\alpha)\}$ is open in Y if $p \notin F$. Thus, points in Y are of the form (p,α) if $p \notin F$, or of the form (f_α,β) with $\alpha < \beta$. Basic clopen sets are as follows:

(i) $\{(p,\alpha)\}$ with $p \notin F$ and $\alpha < \omega_1$.

(ii) $Y \cap [U \times \{\alpha + 1\}]$ where U is a basic clopen set in X.

(iii) $Y \cap [U \times (\rho,\lambda]]$, where U is a basic clopen set in X, λ is a limit ordinal, and $\rho < \lambda$.

In the sequel, this topology will be referred to as the old topology. Also, if $(f_\alpha,\beta) \in Y$, the set U used in a type (ii) or a type (iii) basic clopen set about (f_α,β) will often be taken (implicitly) so that $f_\alpha \in U \subset X - \{f_\gamma : \gamma \neq \alpha\}$. Also, if $A \subset X_\alpha$, then A can be considered as a subset of either X or X_β with $\alpha < \beta$ in an obvious way. The properties of Y with this topology will now be examined. Those that are obvious or have routine proofs will not be proved.

4.3.1 $\{D_\alpha : \alpha < \omega_1\}$ is a closed discrete collection of closed subsets of Y.

4.3.2 For each $\alpha < \omega_1$, D_α is homeomorphic to the open ordinal space (α,ω_1), via $(f_\alpha,\beta) \to \beta$.

4.3.3 If D is a closed discrete subset of Y, then $D \cap D_\alpha$ is finite for each $\alpha < \omega_1$.

4.3.4 If $\alpha < \omega_1$, the relative topology on X_α is the same as the relative topology on X'_α induced by the topology on X.

4.3.5 Y is a regular T_2 space.

<u>Proof:</u> Y is clearly T_2, and Y has a clopen base.

4.3.6 <u>Lemma:</u> If $H \subset T$ and $H \cap D_\alpha$ is bounded in D_α for each $\alpha < \omega_1$, then there exists a closed unbounded set C in $[0,\omega_1)$ such that $0 \in C$ and, for each $c \in C$, $X_c \cap H = \phi$.

<u>Proof:</u> Let $B = \{\beta$: there exists α such that $(f_\alpha,\beta) \in H\}$. Suppose B intersects every closed unbounded set in $[0,\omega_1)$. For each $\beta \in B$ let α_β be an ordinal such that $(f_{\alpha_\beta},\beta) \in H$. Define the function $g\colon \omega_1 \to \omega_1$ so that $g(\beta) = \alpha_\beta$ if $\beta \in B$ and $g(\beta) = \beta$ if $\beta \notin B$. Then $g(\beta) < \beta$ for each $\beta \in B$. It follows from the pressing down lemma (see Introduction) that there exists a cofinal set Γ in $[0,\omega_1)$ and an ordinal α_γ such that $g(\beta) = \alpha_\gamma$ for each $\beta \in \Gamma$. Then $\{(f_{\alpha_\gamma},\beta)\colon \beta \in \Gamma\} \subset H \cap D_{\alpha_\gamma}$. This implies that $H \cap D_{\alpha_\gamma}$ is not bounded, which is a contradiction. So there exists a closed unbounded set C in $[0,\omega_1)$ such that $B \cap C = \phi$. Since $H \cap X_0 = \phi$, 0 may be included in C. This completes the proof.

4.3.7 <u>Y is CWH.</u>

<u>Proof:</u> Let D be a closed discrete subset of T. Then each set $D \cap D_\alpha$ is finite, hence also bounded, in D_α. By Lemma 4.3.6, there exists a closed unbounded set C in $[0,\omega_1)$ such that $0 \in C$ and for each $c \in C$, $X_c \cap D = \phi$. For each $c \in C$, let c' denote

the first member of C such that $c < c'$. Then, for each $c \in C$, $D \cap [\bigcup_{c<\alpha<c'} X_\alpha]$ is a countable discrete set. By 4.3.5, Y is a regular T_2 space, and so there exists a pairwise disjoint open collection $V(c,c')$ in Y which separates the points in $D \cap [\bigcup_{c<\alpha<c'} X_\alpha]$. Therefore, $\bigcup_{c\in C} V(c,c')$ is a pairwise disjoint open collection in Y which separates the points in D.

4.3.8 <u>Comment</u>: If points of the form (f_α,α) were included in Y, then Y would not be CWH. For then, a pressing down argument would show that some uncountable subset of $\{(f_\lambda,\lambda)$: λ is a limit ordinal in $\omega_1\}$ could be separated in Y by basic "product" neighborhoods. This in turn would show that the associated f_λ's could be separated in X, which is impossible.

These first seven properties will hold in all of the attempts to follow. Note that Fleissner's space also has these properties. Before proving that Y is not CWN, a preliminary result is needed.

4.3.9 <u>Lemma</u>: If U_α is open in Y with $D_\alpha \subset U_\alpha$ and $\beta \in (\alpha,\omega_1)$, then there exists a basic open set $B(\beta)$ in X_β with $(f_\alpha,\beta) \in B(\beta)$ such that $B(\beta) \times (\alpha,\beta] \subset U_\alpha$.

<u>Proof</u>: If β is not a limit ordinal, select basic open sets $U(\beta)$ and $U(\beta - 1)$ in X_β and $X_{\beta-1}$, respectively, such that $(f_\alpha,\beta) \in U(\beta) \subset U_\alpha$ and $(f_\alpha,\beta - 1) \in U(\beta - 1) \subset U_\alpha$. Then $U(\beta) \cap U(\beta - 1)$ is a basic open set in both X_β and $X_{\beta-1}$ which contains (f_α,β) and $(f_\alpha,\beta - 1)$ and is a subset of U_α. If β is a limit ordinal, then there exists a basic open set $V \times (\rho,\beta]$

which contains (f_α,β) and is a subset of U_α. Therefore, for each $\gamma \in [\rho + 1,\beta]$, V is a basic open set in X_γ with $(f_\alpha,\beta) \in V \subset U_\alpha$. Thus, in either case, there exists an ordinal β' such that $\beta' < \beta$ and there exists a basic open set $B(\beta)$ in X_β such that $(f_\alpha,\beta) \in B(\beta)$ and $B(\beta) \times [\beta',\beta] \subset U_\alpha$. Since one can back up in the ordinals only finitely many steps, the result follows.

4.3.10 <u>Y is not CWN</u>.

<u>Proof</u>: Suppose $\{U_\alpha: \alpha < \omega_1\}$ is a collection of open sets in Y such that, for each $\alpha < \omega_1$, $D_\alpha \subset U_\alpha$. Fix $\alpha < \omega_1$. Define the function $t: (\alpha,\omega_1) \to N$ by $t(\beta) = \min \{\deg B(\alpha,\beta): B(\alpha,\beta)$ is basic open in X_β and $(f_\alpha,\beta) \in B(\alpha,\beta) \times (\alpha,\beta] \subset U_\alpha\}$. This is well-defined due to the previous lemma. There exists $n_\alpha \in N$ such that $t(\beta) = n_\alpha$ for cofinally many $\beta \in (\alpha,\omega_1)$. It follows that $t(\beta) \leq n_\alpha$ for each $\beta \in (\alpha,\omega_1)$. Letting α vary, define the function $\psi: [0,\omega_1) \to N$ by $\psi(\alpha) = n_\alpha$. For some $n \in N$, $\psi^{-1}(n)$ is uncountable. Select a sequence $\{\alpha_i\}_1^\infty$ of distinct members of $\psi^{-1}(n)$, and let $\beta = (\sup \alpha_i) + 1$. Then $\{B(\alpha_i,\beta)\}_1^\infty$ is an infinite collection of basic open sets in X_β, each having degree n. By 4.2(4), $\{B(\alpha_i,\beta)\}_1^\infty$ is not pairwise disjoint. Therefore $\{U_i\}_1^\infty$, and hence $\{U_\alpha: \alpha < \omega_1\}$, is not a pairwise disjoint collection.

4.3.11 <u>Y is not normal</u>.

<u>Proof</u>: Let $H = \{(f_\alpha,\alpha + 1): \alpha < \omega_1\}$ and $K = T - H$. Since both $H \cap T$ and $K \cap T$ are closed in D_α for all α, both H and K are closed in Y. Suppose U and V are open sets with

$H \subset U$ and $K \subset V$. For each $\alpha < \omega_1$, let Q_α be a basic open set in $X_{\alpha+1}$ such that $(f_\alpha, \alpha + 1) \varepsilon Q_\alpha \subset U$. Then $Q_\alpha \cup V$ is open in Y and $D_\alpha \subset Q_\alpha \cup V$. By Lemma 4.3.9, for each (f_α, β) there exists a basic open set $B(\alpha,\beta)$ in X_β such that $B(\alpha,\beta) \times (\alpha,\beta] \subset Q_\alpha \cup V$ and $(f_\alpha,\beta) \varepsilon B(\alpha,\beta)$. Let $Z_\alpha = \bigcup_{\alpha<\beta} B(\alpha,\beta) \times (\alpha,\beta]$, an open set in Y with $D_\alpha \subset Z_\alpha \subset Q_\alpha \cup V$. Since Y is not CWN, there are ordinals α_1 and α_2 such that $\alpha_1 < \alpha_2$ and $Z_{\alpha_1} \cap Z_{\alpha_2} \neq \phi$. Therefore, $Q_{\alpha_2} \cap Z_{\alpha_1} \neq \phi$. But $Q_{\alpha_2} \cap Q_{\alpha_1} = \phi$, so $Q_{\alpha_2} \cap Z_{\alpha_1} \subset U \cap V$. Thus, $U \cap V \neq \phi$ and Y is not normal.

From the method of proof it is evident that any topology on Y which is not CWN because $\{D_\alpha : \alpha < \omega_1\}$ is a closed discrete collection of closed sets that cannot be separated and in which Lemma 4.3.9 is valid will not be normal. In the next attempt, normality will be achieved by adding more open sets in such a way that Lemma 4.3.9 is not valid.

4.4 <u>Second attempt</u>: In order to describe this second topology on Y, a preliminary result is needed. Note that this is simply a special case of Lemma 4.3.6.

4.4.1 <u>Lemma</u>: If H and K are disjoint closed subsets of T in the old topology, and $H \cap D_\alpha$ is bounded for each α, then there is a closed unbounded set C in ω_1 such that $0 \varepsilon C$ and if $c \varepsilon C$, then $X_c \cap H = \phi$.

4.4.2 <u>The new (second) topology</u>: Let H, K, and C be as in the previous lemma. For each $\gamma \in C$, let γ' be the first element in C such that $\gamma < \gamma'$. Let $U(H,\gamma,\gamma')$ be an open set in Y such that

$$H \cap (\bigcup_{\gamma<\beta<\gamma'} X_\beta) \subset U(H,\gamma,\gamma') \subset \bigcup_{\alpha<\beta<\gamma'} X_\beta - K.$$

Note that $\bigcup_{\gamma\in C} U(H,\gamma,\gamma')$ is open in the old topology, but is not necessarily closed. The second topology on Y is obtained by declaring such sets closed. That is, a subbasis for the second topology consists of the old subbasic sets along with these new sets.

4.4.3 <u>Y is normal</u>.

<u>Proof</u>: Suppose H and K are disjoint closed sets in Y. Without loss of generality, assume that $H \cup K = T$. For each $\alpha < \omega_1$, either $H \cap D_\alpha$ or $K \cap D_\alpha$ is bounded. Let H_B, K_B, H_U, K_U denote the bounded and unbounded parts of H and K, respectively. Let C be clopen in Y such that $H_B \cup K_B \subset C \subset Y - (H_U \cup K_U)$. Let W and W' be disjoint open sets such that $H_U \subset W$ and $K_U \subset W'$. Let Q and Q' be disjoint open sets such that $H_B \subset Q$ and $K_B \subset Q'$. Let

$$U = (Q \cap C) \cup [W \cap (Y - C)] \quad\text{and}\quad V = (Q' \cap C) \cup [W' \cap (Y - C)].$$

Clearly, U and V are disjoint open sets with $H \subset V$ and $K \subset V$. Therefore, Y is normal.

4.4.4 Y is CWN.

Proof: It is sufficient to show that there exists a pairwise disjoint open collection $\{U_\alpha : \alpha < \omega_1\}$ such that $D_\alpha \subset U_\alpha$ for each α. If $\alpha < \beta < \omega_1$, let $V(\alpha,\beta)$ denote a clopen set in X_β such that $(f_\alpha,\beta) \in V(\alpha,\beta)$. Also, suppose that, for fixed β, the collection $\{V(\alpha,\beta): \alpha < \beta\}$ is pairwise disjoint. If $\alpha < \lambda < \omega_1$ and λ is a limit ordinal, let $\rho_{\alpha\lambda}$ denote an ordinal such that $\alpha < \rho_{\alpha\lambda} < \lambda$. Fix $\alpha < \omega_1$. Suppose λ is a limit ordinal and $\alpha < \lambda < \omega_1$. Let H be the closed set which satisfies the following:

(1) $H \cap D_\alpha = (f_\alpha, \alpha + 1)$

(2) $H \cap D_\beta = f_\beta \times (\beta,\lambda]$ if $\beta < \lambda$ and $\beta \neq \alpha$.

(3) $H \cap D_\gamma = (f_\gamma, \gamma + 2)$ if $\gamma \geq \lambda$.

Let $K = T - H$. Then H and K are disjoint closed sets in Y and $H \cap D_\tau$ is bounded for each $\tau < \omega_1$. The set $C = \{0, \lambda + 1, \lambda' : \lambda'$ is a limit ordinal $> \lambda\}$ is a closed unbounded set in ω_1 such that $c \in C$ implies that $X_c \cap H = \phi$. The following set Q is one of the new clopen sets:

$$Q = V(\alpha,\alpha + 1) \cup \left(\bigcup_{\alpha \geq \lambda} V(\gamma,\gamma + 2)\right) \cup \left(\cup \{V(\beta,\delta + 1): \beta < \lambda, \delta < \lambda, \beta \neq \alpha\}\right)$$

$$\cup \left(\cup \{V(\beta,\lambda') \times (\rho_{\beta\lambda'}, \lambda']: \beta < \lambda' \leq \lambda, \beta \neq \alpha, \lambda' \text{ is a limit ordinal}\}\right).$$

Let $V'(\alpha,\lambda) = V(\alpha,\lambda) \times (\rho_{\alpha\lambda},\lambda] - Q$. Then $V'(\alpha,\lambda)$ is open in Y. Finally, the set

$$U_\alpha = [\bigcup_{\beta<\omega_1} V(\alpha,\beta + 1)] \cup [\cup \{V'(\alpha,\lambda)\colon \lambda \text{ is a limit ordinal}\}]$$

is open and $\{U_\alpha\}_{\alpha<\omega_1}$ is a pairwise disjoint open collection with $D_\alpha \subset U_\alpha$ for each α. Therefore, Y is CWN.

The basic reason that this second topology on Y is CWN is that it is too close to the discrete topology, which is obviously CWN. However, since the first topology was not even normal, any topology which is to contain the old topology must contain quite a few additional open sets in order to make the resulting space normal. Part of the difficulty with the second topology seems to be that the new sets introduced were tied too closely to special types of disjoint closed sets in Y. What is needed, and will be achieved in the next attempt, is a new type of clopen set which is more global in nature. However, the primary motivation for the new clopen sets is still to insure normality.

4.5 Third attempt: Let $n \in N$. Suppose U is a subset of Y which satisfies the following:

(1) $U \cap D_\alpha$ is clopen in D_α for each $\alpha < \omega_1$.

(2) If $(f_\alpha,\beta) \in U$, then there is a basic open set $F(\alpha,\beta)$ of degree $\leq n$ in X_β such that $(f_\alpha,\beta) \in F(\alpha,\beta) \subset U$.

(3) If $(f_\gamma,\delta) \notin U$, then there exists a basic open set $F(\gamma,\delta)$ of degree $\leq n$ in X_δ such that $(f_\gamma,\delta) \in F(\gamma,\delta) \subset X_\delta - U$.

The third topology on Y is obtained by declaring such a set U to be clopen in Y. Note the differences here between these new clopen sets and the ones added in the second attempt: these new clopen sets were not necessarily either open or closed in the old topology, and these new clopen sets are symmetric; i.e., if U is a new clopen set, then $Y - U$ is also a new clopen set (depending on the same $n \in N$ as U). Note also that these new clopen sets utilize the degree concept for the first time.

In particular, a subbasis for this third topology on Y is given by the following types of sets:

Type 1: Old subbasic sets.

Type 2: U (as above).

A basis for this third topology is easily described because the intersection of two of the new clopen sets is also a new clopen set. Thus, a basis consists of the following types of sets:

Type I: Old basic open sets.

Type II: $U \cap B$, where U is a Type 2 set and B is a Type I set.

Note that this actually is a clopen basis. In order to verify that this topology is normal and not CWN, a few lemmas are needed.

4.5.1 Lemma: If U_α is open in Y and $D_\alpha \subset U_\alpha$, then for every $\beta \in (\alpha,\omega_1)$ there exists an ordinal $\beta' \in (\alpha,\beta)$ and a positive integer $N(\beta',\beta)$ such that if $\beta' \leq \gamma \leq \beta$ then there is a basic open set $B(\gamma)$ in X_γ such that $(f_\alpha,\gamma) \in B(\gamma) \subset U_\alpha$ and $\deg B(\gamma) \leq N(\beta',\beta)$.

Proof: Select $\beta \in (\alpha,\omega_1)$. Suppose β is not a limit ordinal. Then there is a basic open set $B(\beta)$ of degree $N(\beta)$ in X_β such that $(f_\alpha,\beta) \in B(\beta) \subset U_\alpha$. Also there is a basic open set $B(\beta - 1)$ of degree $N(\beta - 1)$ in $X_{\beta-1}$ such that $(f_\alpha,\beta -1) \in B(\beta - 1) \subset U_\alpha$. The lemma follows in this case by setting $\beta' = \beta - 1$ and $N(\beta',\beta) = N(\beta) + N(\beta - 1)$ because $B(\beta) \cap B(\beta - 1)$ is a basic open set of degree $\leq N(\beta) + N(\beta - 1)$. Now suppose β is a limit ordinal. If there is a Type I basic open set B with $(f_\alpha,\beta) \in B \subset U_\alpha$, then the conclusion of the lemma is an obvious consequence of the nature of Type I sets. Suppose there is no Type I set B with $(f_\alpha,\beta) \in B \subset U_\alpha$. Then, there must be a Type II set, $U \cap B$, such that $(f_\alpha,\beta) \in U \cap B \subset U_\alpha$. Let N_U be the positive integer associated with U. Suppose $B = V \times (\rho,\beta]$, where V is a basic open set in X of degree N_B. The conclusion of the lemma is obtained by setting $\beta' = \rho + 1$ and $N(\beta',\beta) = N_U + N_B$.

4.5.2 Comment: It is possible to sharpen the conclusion of the last lemma so that $\deg B(\gamma) = N(\beta',\beta)$. This can be done by intersecting with more subbasic sets if necessary.

4.5.3 Lemma: If U_α is open in Y and $D_\alpha \subset U_\alpha$, then for every $\beta \varepsilon (\alpha,\omega_1)$ there is a positive integer N_β such that if $\gamma \varepsilon (\alpha,\beta]$ there exists a basic open set $B(\gamma)$ of degree N_β in X_γ such that $(f_\alpha,\gamma) \varepsilon B(\gamma) \subset U_\alpha$.

Proof: The result follows by repeated application of the previous lemma and the fact that one can back up in the ordinals only finitely many steps.

4.5.4 Lemma: If U_α is open in Y and $D_\alpha \subset U_\alpha$, then there is a positive integer N_α such that if $\beta \varepsilon (\alpha,\omega_1)$ there is a basic open set $B(\beta)$ in X_β of degree N_α such that $(f_\alpha,\beta) \varepsilon B(\beta) \subset U_\alpha$.

Proof: For each $\beta \varepsilon (\alpha,\omega_1)$ obtain N_β as in the previous lemma. There exists a cofinal subset Γ of (α,ω_1) such that each $\gamma \varepsilon \Gamma$ is associated with the same natural number, say N_α. Thus, if $\beta \varepsilon (\alpha,\omega_1)$, there exists $\gamma \varepsilon \Gamma$ such that $\beta < \gamma < \omega_1$. Hence, there is a basic open set $B(\beta)$ in X_β of degree N_α such that $(f_\alpha,\beta) \varepsilon B(\beta) \subset U_\alpha$. Since β is arbitrary, this completes the proof.

4.5.5 Y is not CWN.

Proof: Suppose, for each $\alpha < \omega_1$, U_α is open in Y and $D_\alpha \subset U_\alpha$. For each α, choose N_α as in the previous lemma.

Let $\{\alpha_i\}_1^\infty$ be a sequence of distinct ordinals, each associated with the same natural number N. Let $\beta = (\sup \alpha_i) + 1$. For each α_i, there is a basic open set $U(\alpha_i)$ of degree N in X_β such that $(f_{\alpha_i},\beta) \varepsilon U(\alpha_i) \subset U_{\alpha_i}$. Since there are only finitely many pairwise disjoint open sets of degree N in X_β, $\{U(\alpha_i)\}_1^\infty$ is not pairwise disjoint. Since $U(\alpha_i) \subset U_{\alpha_i}$, this implies that $\{U_{\alpha_i}\}_1^\infty$ is not pairwise disjoint. Therefore, Y is not CWN.

4.5.6 <u>Y is normal.</u>

<u>Proof</u>: Suppose H and K are disjoint closed sets in Y. Let $H_T = H \cap T$, $H_0 = H \cap (Y - T)$, $K_T = K \cap T$, $K_0 = K \cap (Y - T)$. Then H_T and K_T are disjoint closed sets in the old topology. Without loss of generality, assume that $H_T \cup K_T = T$. Then, for each $\alpha < \omega_1$, $H \cap D_\alpha$ is clopen in D_α and $(H_T \cap X_\alpha, K_T \cap X_\alpha)$ is a disjoint pair of closed sets in $X_\alpha \cap T$. By 4.2(5), there is a basic clopen set $B(H_T,\alpha)$ in X_α such that $H_T \cap X_\alpha \subset B(H_T,\alpha) \subset Y - (K_T \cap X)$. Thus, $\bigcup_{\alpha<\omega_1} B(H_T,\alpha)$ is a Type 2 subbasic set (using $n = 1$) such that $H_T \subset \bigcup_{\alpha<\omega_1} B(H_T,\alpha) \subset Y - K_T$. Changing notation for convenience, it has been shown that there exist disjoint open sets V_T and W_T with $H_T \subset V_T$ and $K_T \subset W_T$. Now let $V_T' = V_T - \bar{K}_0$ and $W_T' = W_T - \bar{H}_0$. Then $H_T \subset V_T'$, $K_T \subset W_T'$, and $H \subset V = V_T' \cup H_0$, while $K \subset W = W_T' \cup K_0$. Clearly, V and W are disjoint open sets. Therefore, Y is normal.

4.6 <u>Comments</u>: There are certainly many other topologies on Y which could be attempted; for example, Y could be considered as the disjoint union $Y = \bigcup_{\alpha<\omega_1} X'_\alpha$, where X'_α receives the relative topology. However, this obviously produces a CWN space. Other topologies on Y which produce a normal CWH space that is not CWN can be constructed by trivial modifications of the third topology; for example, let $\alpha \in (0,\omega_1)$ and declare X_β discrete if $\beta \leq \alpha$, then do the same construction as before on $\bigcup_{\beta<\gamma<\omega_1} X_\gamma$. This leads to uncountably many distinct topologies on Y, each of which has the desired properties and whose intersection has these same properties because it is the third topology.

4.7 <u>Theorem</u>: There exists a space Z such that (1) Y is open and dense in Z, and (2) X is homeomorphic to Z - Y.

<u>Proof</u>: Let $X_{\omega_1} = X \times \{\omega_1\}$, and let $Z = \bigcup_{\alpha\leq\omega_1} X_\alpha$. Then $Y \subset Z$. Let points in Y have their usual neighborhoods. If $(p,\omega_1) \in X_{\omega_1}$ and $p \notin F$, let an open set about (p,ω_1) be a set of the form $\{p\} \times (\alpha,\omega_1]$, where $\alpha < \omega_1$. If $(f_\alpha,\omega_1) \in X_{\omega_1}$, let an open set about (f_α,ω_1) be a set of the form $A_\alpha \times (\rho,\omega_1]$, where A_α is open in X, $f_\alpha \in A_\alpha$, and $\rho < \omega_1$. Clearly, $X_{\omega_1} = Z - Y$ is homeomorphic to X, Y is open in Z, and $\bar{Y} = Z$, as claimed.

4.8 <u>Theorem</u>: The projection map $p\colon Z \to X_{\omega_1}$ defined by $(x,\alpha) \overset{p}{\to} (x,\omega_1)$ is a continuous open surjection, and p is a retraction of Z onto X_{ω_1}. However, p is not a closed map.

<u>Proof</u>: It is easily seen that p is a continuous open retraction of Z onto X_{ω_1}. Let K be any subset of X such that f_0 is a limit point of K. Then $\bar{K} \times \{0\}$ is closed in X_0, and since X_0 is closed in Z, $\bar{K} \times \{0\}$ is also closed in Z. However, $(f_0,0) \notin \bar{K} \times \{0\}$, and so $p(\bar{K} \times \{0\})$ does not contain $(f_0,\omega_1) \in \overline{p(\bar{K} \times \{0\})}$. This proves that p is not a closed map.

CHAPTER FIVE

MORE ON CIC-SUBSETS

5.1 Definition: A space X satisfies the countable chain condition if every pairwise disjoint open collection in X is at most countable. Such spaces are called CCC-spaces.

5.2 Theorem: Let X be a topological space.

(A) Every countable subset of X is a CIC-subset of X.

(B) If X is CCC, then every subset of X is CIC.

(C) If K is CIC in X and U is an open set in X such that $K \subset U$, then K is a CIC-subset of U.

(D) If K is CIC in X and $A \subset K$, then A is CIC in X.

(E) If X is separable, then X is CCC.

5.3 Theorem: If $X = \bigcup_1^\infty X_n$ and each X_n is a CIC-subset of X, then X is CCC.

5.3.1 Corollary: If $\{X_n\}_1^\infty$ is a sequence of open CIC-subsets of X, then $\bigcup_1^\infty X_n$ is a CIC-subset of X.

5.3.2 Corollary: A countable free union of CCC spaces is CCC.

5.4 Comment: It is clear that if a subspace K of a space

X is a CCC space in the relative topology, then K is a CIC-subset of X. The converse is not true. The subspace X_S of Bing's example G (3.9) is a CIC-subset of G, but it is not CCC in the relative topology -- it is uncountable and discrete. The x-axis in the tangent disc space (3.7) is another example of this.

5.5 Theorem: If X is a Lindelöf space, then X is CCC if and only if every point in X has a CIC-neighborhood.

Proof: If X is CCC, then X is a CIC-neighborhood of each point of X. If every point $x \in X$ has a CIC-neighborhood $N(x)$, then $\{N(x): x \in X\}$ is a cover of X which may be assumed to be open by Theorem 5.2(D). From the Lindelöf property, $\{N(x): x \in X\}$ has a countable subcover, say $\{N(x_i)\}_1^\infty$. Since each $N(x_i)$ is an open CIC-subset of X and $X = \bigcup_1^\infty N(x_i)$, it follows from Theorem 5.3 that X is CCC.

5.6 Comment: The necessity of the Lindelöf property in the theorem is made clear by considering an uncountable discrete space. For such a space, every point x has the CIC-neighborhood $\{x\}$, yet the space is not CCC.

5.7 Theorem: If $\{F_a: a \in A\}$ is a locally countable cover of the Lindelöf space X by CIC-subsets, then X is CCC.

Proof: Let $x \in X$ and let $N(x)$ be an open neighborhood of x meeting at most countably many members of $\{F_a: a \in A\}$, say $\{F_i\}_1^\infty$. Then $N(x) = \bigcup_1^\infty [F_i \cap N(x)]$ is a countable union of CIC-subsets of X. By Theorem 5.3, each $N(x)$ is CCC, hence also CIC. This shows that every point in X has a CIC-neighborhood. Since X is Lindelöf, the result follows from Theorem 5.6.

5.7.1 <u>Corollary</u>: If $\{F_a : a \in A\}$ is a locally finite cover of the Lindelöf space X by CIC-subsets, then X is CCC.

5.8 <u>Theorem</u>: Every screenable CCC space is Lindelöf.

Proof: Let U be an open cover of X, and let $\overset{\infty}{\underset{1}{\cup}} U_n$ be a σ-disjoint open refinement of U. Since X is CCC, each U_n is countable. Therefore, $\overset{\infty}{\underset{1}{\cup}} U_n$ is a countable open refinement of U. Expanding the members of $\overset{\infty}{\underset{1}{\cup}} U_n$, a countable subcover of U is obtained. Therefore X is Lindelöf.

5.8.1 <u>Corollary</u>: In CCC spaces, screenability is equivalent to the Lindelöf property.

5.8.2 <u>Corollary</u>: Every C-space which is CCC is Lindelöf.

5.8.3 <u>Corollary</u>: Every paracompact CCC space is Lindelöf.

5.8.4 <u>Corollary</u>: Every screenable CCC Moore space is metrizable.

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