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INELASTIC STABILITY OF TAPERED, SINGLY-SYMMETRIC BEAM-COLUMNS

A DISSERTATION

SUBMITTED TO THE GRADUATE FACULTY

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degree of

DOCTOR OF PHILOSOPHY

By

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Norman, Oklahoma

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INELASTIC STABILITY OF TAPERED,

SINGLY-SYMMETRIC BEAM-COLUMNS

A DISSERTATION

APPROVED FOR THE DEPARTMENT OF CIVIL ENGINEERING

AND ENVIRONMENTAL SCIENCE

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TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

LIST OF SYMBOLS

A = Total area of the cross-section

A_1 = Area of the lower flange

A_2 = Area of the upper flange

A_3 = Area of the web

b_1 = Width of the lower flange

b_2 = Width of the upper flange

C = Distance from neutral axis to the center of the upper flange

d = Depth of the section

E = Modulus of elasticity

G = Shear modulus of elasticity

h = Length of each segment of the member

J = Torsional constant

I_x = Moment of inertia with respect to x-axis

I_y = Moment of inertia with respect to y-axis

I_t, I_c = Moment of inertia of the tension and compression flange with respect to y axis, respectively

L = Length of the member

M_x = Bending moment

M_1, M_2 = End moments acting at the larger and smaller ends, respectively

N = Number of segments for a member

P = Axial load

R.S. = Residual Stress

r_x, r_y = Radius of gyration with respect to x and y axis, respectively

S = Distance from any point on a cross-section to the shear center

S_T = Section modulus with respect to top flange

t_1 = Thickness of the lower flange

t_2 = Thickness of the upper flange

t_3 = Thickness of the web

u = Horizontal displacement of the shear center during buckling

v = Vertical displacement of the shear center during buckling

x,y = Coordinates of the cross-section

y_0 = Distance between shear center and centroid

y_s = Distance from shear center to the centerline of the lower flange

y_f = Distance from the centroid of the elastic (unyielded) section to the bottom flange

Z = The longitudinal coordinate of the member

σ = Normal stress

σ_y = Yield stress in analysis

F_y = Yield stress in design equation

σ_{rt} = Maximum tensile residual stress

σ_{rc} = Maximum compressive residual stress in top flange

σ_{rb} = Maximum compressive residual stress in bottom flange

τ = Shear stress

θ = Curvature of the section

ϕ = Rotation of the shear center during buckling

$\psi, \nu, k,$
f, r_b = Yield parameters

Subscripts

o = Smaller end

L = Larger end

ABSTRACT

The purpose of this study is to determine the inelastic lateral-torsional buckling load of a linearly web-tapered member having unequal flanges and subjected to unequal end moments and axial load. Triangular residual stress patterns in flanges and uniform tensile stress across the web are assumed to exist. The total potential energy functional for such a member is developed and the inelastic effects are included. Using the standard techniques of calculus of variations, the three differential equations describing the behavior of the member are derived and the natural and force boundary conditions are determined. Two of these equations, which are coupled, describe the out-of-plane behavior of the member. Central difference expressions are used to transform these two equations into difference equations. In order to determine the yield pattern at each station along the member, a new method based on the solution of the equations of equilibrium is proposed. By solving these two simultaneous, nonlinear equations, the yield parameters are computed. The section properties of the partially yielded cross-sections are computed using the yield parameters. The section properties are used to satisfy the difference equations at different nodes along the member. The value of the determinant of these equations is tested for different trial values of the buckling load until a zero value is found which insures buckling. The buckling mode shape is then obtained by assuming one of the displacements equal to unity, cancelling one equation and solving the rest of the equations using Gauss-Jordan elimination method.

The results of the analysis are compared with solutions available in the literature and also with those of a limited experimental study. Good agreement is shown to exist. The effect of the curvature of the neutral axis and the contribution of the yielded portions in lateral-torsional buckling are explained. The results of the study are used to develop a design criterion for singly-symmetric, tapered members and an interaction-type equation is developed in Load Resistance Factor Design format.

CHAPTER I

INTRODUCTION

1.1 Background

The use of tapered or non-prismatic structural elements has been increasingly popular since the 1960's. Their use was suggested for the first time in 1952 by Amirikian[1], and since then a significant amount of research has been devoted to investigating the strength and stability of both singly and doubly-symmetrical web tapered members. In such members, the web dimensions vary along the length to permit a more efficient use of the material and provide the required strength at different locations. It is not feasible to fabricate the structural elements to match the required strength over the entire member length because of labor cost and manufacturing difficulties. However, the use of non-prismatic members may result in significant savings of material.

It was not until 1969 that the design provisions for tapered members appeared in North American specifications. Appendix D of the 1969 AISC "Specification for the Design, Fabrication and Erection of Structural Steel for Buildings"[2] provides criteria for the design of doubly-symmetrical, tapered beams, columns and beam-columns. In order to satisfy these requirements, it is necessary to maintain double symmetry, which results in inefficient use of material especially in the tension flange of members subjected to axial load and bending moment. Another shortcoming of the given specifications is that the interaction equation proposed for the design of beam-columns (tapered or prismatic), is based on the in-plane behavior of such members, while for common dimensions of the cross-sections and lengths, the laterally unsupported members are more likely to fail by the lateral-torsional buckling (LTB).

In order to achieve greatest economy, high strength steels are used to fabricate tapered members, which along with the need for material optimization, result in members with high flange stresses. This causes the potential for lateral-torsional buckling to increase for laterally unsupported beams, columns and beam-columns. For comparatively long members, with no residual stresses, lateral-torsional buckling will occur in the elastic region. Shorter members are capable of being loaded into the inelastic region and this becomes more probable when residual stresses exist. The presence of residual stresses gives rise to complicated problems in the analysis and design of steel structures. Due to the non-uniform distribution of such stresses, at high stress levels some fibers reach the yield point and the effective section properties vary from their values for an elastic section. Because of the complicated nature of the inelastic stability problems, the design specifications have not been modified to take into account the inelastic effects for tapered and singly-symmetrical members.

The purpose of this study is to develop a procedure for the design of tapered, singly-symmetrical beam-columns assuming that the lateral-torsional buckling is the governing failure mode and that the yield stress is reached at some points on the cross-section due to the presence of residual stresses combined with the stresses caused by external loading.

In this study, a review of the literature in the related areas is first presented. Texts and papers giving historical background to the stability problems are reviewed. Then the governing differential equations of the problem are obtained by minimizing the total potential energy functional and the boundary conditions are specified. The finite difference technique is used to solve these equations. The results of theoretical study are used to develop a design methodology and comparisons are done with those of experimental investigations and with other analytical techniques used for different cases.

1.2 Review of Previous Work

1.2.1 Elastic Stability of Prismatic Beams and Columns

The elastic stability of compression members was studied first by Euler[3] in the 18th century. He developed the theoretical solution to the problem of a concentrically loaded column and his results were experimentally verified by Van Musschenbrock[4] who performed tests on the columns with solid rectangular cross-sections. The increasing use of steel during the 19th century resulted in the use of laterally unsupported beams with high flange stresses. Prandtl[5] was the first to consider the elastic lateral-torsional buckling of beams having deep and narrow rectangular cross-sections. He used St. Venant torsional properties and neglected warping torsion. An extensive theoretical study by Timoshenko[6] was performed on the elastic buckling behavior of the doubly-symmetric I-beams and he included the warping effects for such open sections.

DeVries[7], while attempting to simplify the theoretical formulations, derived simple rules for computing the critical stress in the elastic range of buckling. He showed that such a stress is a function of the flange dimensions, depth of the section and the length of the beam. Definition of the shear center by Eggenschwyler and Maillart[8] made it possible to analyze the torsional behavior of singly-symmetrical I-sections. Based on this, Winter[9] analyzed the lateral instability of non-symmetrical sections in 1943 and Ostenfeld[10] noted that at buckling the center of twist does not coincide with the shear center. Wagner[11], in 1929, investigated the torsional buckling of compression members and explained that at buckling, a distrubing torque is exerted on the section due to the twist of the member. This has been referred to as the Wagner effect.

The developments of prismatic beam-column theory in the first half of the 20th century have been reviewed by Bleich[12]. In 1937, Shanley[13], developed the interaction formulas considering the in-plane behavior of beam-columns. Massonet[14], proposed the "equivalent uniform moment" for columns subjected to unequal end moments. Wright[15] was the first one to consider the lateral deformation of a column loaded by a single concentrated force at the mid-span. Ketter[16] has also studied this case and has developed interaction curves to predict the strength of wide-flange beam-columns. A comprehensive treatment of the theory of such members is provided by Atsuta and Chen[17,18], which considers both their in-plane and out-of-plane behavior. The differential equations describing the behavior of non-symmetrical, thin walled beam-columns have been derived by Vlasov[19], and Bleich[12] has derived similar equations by minimizing the total potential energy functional. Similar attempts have been made by Timoshenko[21] and Galambos[20], to study the behavior of structural members subjected to axial load and bending moments.

"Guide to Stability Design Criteria for Metal Structures" by SSRC[22], covers a complete account of buckling of structural members. An extensive list of references on the lateral stability of beams has been assembled by Lee[23].

1.2.2 Elastic Stability of Tapered Members

Shortly after the use of tapered members was suggested by Amirikian[1] in 1952, research started to consider the lateral stability of such members. Lee[24], in 1956, studied the torsional behavior of tapered, doubly-symmetrical I-beams, ignoring the effect of web deformation. He demonstrated the fact that the contribution of web to out-of-plane bending is negligible and also noted that the torsional term derived by Timoshenko for prismatic beams remains unchanged in tapered beams.

Culver and Preg[25] have used Vlasov's approach to derive the differential equations describing the in-plane and out-of-plane behavior of doubly-symmetrical tapered members. They employed the finite difference approach to find the critical moments for beams having different taper ratios, support conditions and end moments. Their results have been experimentally verified by Butler and Anderson[26] and Krefeld et al.[27].

The minimization of the total potential energy has frequently been used to develop the governing differential equations and to obtain approximate solutions for members with tapered web or flanges. Vankayya[28] has investigated the stability of simply supported I-beams in which the depths increased parabolically from the center and were loaded by central concentrated loads or uniformly distributed loads. The potential energy approach was also used by Lee, Morrel and Ketter[29] who investigated the stability of doubly-symmetric tapered members under varying end moments and used the Rayleigh-Ritz method to find the solution. Chi[30] extended this approach to singly-symmetrical members and obtained the solution using polynomial displacement functions. E. Heidarzadeh[31] has also used polynomials to study the elastic stability of tapered members with discrete lateral braces. Sheu[32] has added the effect of lateral loads to Chi's problem and he has obtained the solution in a similar way.

Gere and Carter[33] have developed formulas and graphs for the determination of elastic buckling loads of uniformly tapered and doubly-symmetric columns, having different boundary conditions. The same effort was made by Girijavallahhan [34] who used the finite difference technique to obtain the buckling loads of nonuniform columns. Lee and Hsu[35] used the finite element method to determine the buckling loads for tapered columns having unequal flanges.

Trahair and his research team[36,37,38] have provided several studies on the stability of singly and doubly-symmetric, stepped and uniformly tapered beams. They noted that for singly symmetrical, tapered members the centroidal axis and the shear center axis are inclined toward each other. They also neglected the fact that the centroidal axis in such members is not a straight line, as will be shown in Section 2, Chapter II. Using the method of equilibrium, they have derived differential equations describing the lateral-torsional behavior of tapered beams and solved them by the finite integral method. The same approach was used by Nelson[39] who studied the wedge beams under moment gradient.

1.2.3 Inelastic Stability of Prismatic Members

Experimental study of the inelastic stability started as early as 1913 by Moore[40], who tested standard I-beams and studied their lateral buckling. Later, Ketchum and Draffin[41] made similar experiments. The results obtained showed much lower critical stresses than what was predicted by the theory of elastic stability. In 1939, Chwalla[42] published a comprehensive discussion of the problem, considering the lateral buckling of symmetrical I-sections.

In order to predict the buckling load of columns, two concepts were stated by Engesser[77]. One is called the tangent modulus approach, which is based on the assumption that unloading follows the same path as loading. The second approach is the reduced modulus, which assumes different moduli of elasticity in loading and unloading. This approach leads to a larger value for the critical buckling load. It was not until 1947, when Shanley[43] explained the differences and showed that the inelastic buckling load predicted by the reduced modulus can be reached only for infinite displacements. Bleich[44], has indicated that the inelastic buckling stress can be obtained by substituting the tangent modulus in the elastic buckling formula. Test results reported by Hechtman, Hattrap, Styer and Tiedmann[45] and studies by Austin[46], show that this approach is somewhat unconservative.

One of the first who made experimental and analytical studies on the inelastic lateral-torsional buckling (LTB), was Neal[47]. Using the finite difference approach, he obtained the theoretical buckling loads based on the reduced modulus theory and also performed tests on beams having rectangular cross-sections. Several tests were made to show that, both in beams in pure bending and in bending combined with shear, the initial torsional rigidity remains unaltered at its elastic value when partial yielding of the cross-section has occurred.

Wittrick[48] extended Neal's work to rectangular beams of materials in which strain hardening occurs and followed the tangent modulus approach. White[49] considered the inelastic LTB of a member subjected to unequal end moments by assuming that the cross-sections of the beam are either entirely elastic or completely inelastic. This resulted in a member with abrupt changes in stiffness.

Galambos[50,51,52] studied the LTB of partially yielded, wide flange shapes subjected to eccentric axial loading and included the effect of residual stresses to obtain the buckling loads and to develop the interaction curves. Ketter[53] developed ultimate strength interaction curves for an axially loaded column with a concentrated transverse load at mid-depth, assuming in-plane bending. The theoretical solution to the inelastic stability of a wide-flange beam-column subjected to unequal end moments was given by Fukumoto[54]. Galambos and Fukomoto[55] introduced the idea of Column Deflection Curves (CDC) and Column Curvature Curves (CCC) in solving the buckling problems.

In the 1970's, the finite element method became one of the most important techniques used in solving the stability problems. Murray and Rajasekaran[56,57] have used this method to analyze a thin walled, inelastic member subjected to biaxial bending and axial compression. Mallet and Marcal[58] have proposed a finite element analysis approach for non-linear structures. Considering the out-of-plane behavior of beam-columns, Hsu and Lee[59] used the finite element analysis to analyze such members considering lateral-torsional end restraints and including the effect of residual stresses. In this study only the elastic core has been used in computing the section properties of partially yielded sections. As will be discussed in Chapter III, Section 1 this can result in some error.

The finite difference technique has also been used by several authors to solve the inelastic stability problems. Miranda and Ojalvo[60] used it to solve the governing differential equations of a doubly-symmetric beam-column. Hartman[61] has derived the total potential energy expression for inelastic, singly-symmetrical beam-columns with discrete bracing points and ahs also used the finite difference technique to study the instability of such members. Nowacki[62] has also used the minimization of the total potential energy for inelastic, wide-flange beam-columns and has obtained the critical load by using Holzer's method.

Since the existence of residual stresses is the primary cause for the inelastic behavior of steel members, their study has always been of great importance to those conducting research on the steel structures. For wide-flange shapes, a linear variation of stress along the width of flange is assumed[52,55,62]. The ends of the flanges are assumed to be in compression and the tensile stress along the web is taken as constant. The distribution of residual stresses in singly-symmetrical sections has been studied by Lee and Nwokedi[63].

1.2.4 <u>Inelastic Stability of Non-Prismatic Members</u>

Investigation of the inelastic instability of tapered members has mostly been concentrated on columns. Newton[64], Appl and Sheets[65], Young[66], Klein[67], Goldberg[68], Appl and Smith[69], have all studied the buckling strength of inelastic, tapered columns, using various methods including those discussed here. Emphasis has been placed on doubly-symmetrical members and due to the complexity of the problem, little attention has been devoted to the effects of the residual stresses on the behavior of singly-symmetrical columns.

Only a recent paper by Lee and Hsu[35] was found, which has analyzed the problem of inelastic stability of columns ina more general manner. A singly-symmetrical, tapered beam-column is analyzed subjected to one end moment and the residual stresses are assumed to exist in the flanges only. The finite element method is applied and the average values of the cross-sectional properties at the two nodes are used for each element. The results are used to modify the interaction equation used by AISC.

1.3 Current Methods Used in Stability Analysis of Beam-Columns

The failure analysis of laterally unsupported beam-columns usually involves one of the following methods:

(1) Timoshenko[21] derived the differential equations describing the behavior of thin walled members by considering the equilibrium of a finite portion of the beam-column and obtained three simultaneous equations. Two of these equations are second order and one is a third order equation.

(2) Vlasov[19] considered the equilibrium of an infinitely small element of the buckled beam and obtained three fourth order differential equations.

(3) Bleich[12] minimized the total potential energy functional and derived the governing differential equations. He defined the shear

center axis as the axis of rotation of the member.

In all of the above cases, symmetry of the cross-section causes the differential equations to become uncoupled, i.e., a doubly symmetric member has three uncoupled equations and for a singly symmetric section, two equations are coupled and one is uncoupled. It is obvious that for inelastic cases, a doubly-symmetric section is, in effect, a singly-symmetric since the shear center and the centroid are separated after yielding.

A closed form solution of the governing differential equations is available only for special cases of loading and symmetry of the cross-section. In order to solve stability problems in a more general form, it is necessary to use approximate methods. In some methods, such as Rayleigh-Ritz and the Galerkin method, the buckled configuration of the member is approximated by a function satisfying the appropriate boundary conditions. Finding the buckling load by such an approach is thus reduced to solving an eigenvalue problem.

In the finite element method, the member is modeled as an assemblage of inter-connected elements. For an element of varying cross-section, the section properties are assumed to be equal to the average of their nodal values. A polynomial represents the deformed configuration and the total potential energy functional is used to develop the stiffness matrix of each element. The structure stiffness matrix is assembled considering compatibility of displacements and equilibrium of forces at connecting nodes. By setting the determinant of this matrix equal to zero, the critical buckling load can be obtained.

The finite difference technique gives a pointwise approximation to the governing equations. The values of the derivatives of all terms in the equations are substituted by forward, central or backward difference expressions. The governing equations are satisfied at several points along the member and the determinant of the resulting equations is set equal to zero in order to find the buckling load.

The above methods can be used to analyze inelastic members as well as the elastic ones. It has been conventional practice to find the yield pattern and modify the equations by using the section properties of the elastic core at different cross-sections. It will be shown in Chapter III that neglecting the yielded portions in lateral-torsional buckling problems is unconservative and the results are more realistic if the elastic core is used and the effects of the yielded parts included.

1.4 Current Methods Used in the Design of Beam-Columns

The stability interaction equation used by AISC (Formula 1.6-1a of Reference 2) has generally been recognized as a convenient design tool to design members subjected to axial and bending stresses. This equation, which is derived based on the in-plane behavior of beam-columns, does not have a rational basis when used for laterally unsupported elements. However, as indicated by Massonet[76], with only slight error, the same equation can be used to predict the failure by lateral-torsional buckling. Based on this, the same equation has been recommended by AISC to be used for laterally unsupported beam-columns.

For tapered members used as beam-columns, a similar interaction equation is proposed by AISC (Formula D.1-1a of Reference 2). In this equation, an effective length factor has been used to identify a prismatic member with the equivalent length of a tapered member. The equivalent length approach has also been used in the design of tapered beams and columns with similar factors for both types of members. Another approach to the design of the tapered members is suggested by Salvadori(70). Again, modifying factors to doubly symmetrical solutions are used. Specific coefficients have been incorporated into the AISC specifications to account for moment are a function of gradient.

A new design philosophy, recently introduced into practice in the U.S. by Galambos and his research team(74) is termed Load Resistance Factor Design (LRFD). This approach interprets the safety of a structure in terms of its "limit states" unlike the customary allowable stress design. It was first developed and used in Europe in the early 60's for the design of reinforced concrete structures. Probabilistic approaches have been employed to compute different load and resistance factors with the following general formula as the LRFD design criterion:

$$\phi R_n \geq \Sigma \gamma_k Q_m \qquad \text{(Equation 1 in Reference 74)}$$

where the left side represents the resistance (strength) of the structure and the right side the factored loads.

On the resistance side, ϕ = the resistance factor ≤ 1. This factor reflects the uncertainties included in computing R_n, the nominal resistance, which is computed based on the nominal values of the cross-sectional and material properties. On the right side, Q_m is the "mean load effect" and γ_k is the corresponding load factor which reflects the potential of the overloads. Q_m is a generalized force or action (axial force, bending moment or shearing force), obtained by using the mean loads for which the structure is designed.

The resistance of a structure is always defined with respect to a certain "Limit State". Two sets of limit states have been defined: the ultimate limit states and the serviceability limit states. The ultimate limit states include the maximum strength, different types of instability of structures and members, progressive collapse, etc. Serviceability limit states are observed in excessive deflection and vibration.

The LRFD is a broad subject and the study of its details is beyond the scope of this work. Since this study is limited to the lateral-torsional buckling of beam-columns, only the design formulae proposed for the limit state will be discussed. In the proposed LRFD format[74], the interaction equation currently used by AISC in the allowable stress design format has been translated into the LRFD format using ϕ factors. The ϕ factors proposed in Reference 75 have been used.

1.5 The Scope of the Present Study

This study is concerned with the lateral stability of tapered members subjected to axial loads and unequal end moments. Only singly-symmetrical members, linearly tapered in depth are considered. The effects of the residual stresses are included and the results of the analysis have been used to develop a design formula based on the LRFD method. The objectives can be listed as follows:

1. To develop the governing differential equations of the problem in the most general case to include the effects of the residual stresses and tapering. Inconsistencies in the formulation of the problem by different authors are to be discussed.

2. To determine the critical combination of axial load and end moments when failure is by lateral-torsional buckling.

3. To compare the theoretical critical load with the current design requirements and to develop a design equation based on the LRFD approach.

In Chapter II, the procedure for the theoretical analysis is explained. Assumptions made regarding the values of the residual stresses, limitations on the sizes and those used in the stability analysis of a beam-column are explained. The effect of the curvature of a singly-symmetric, tapered member is investigated and within the limitations of this study, this effect has been neglected. The total potential energy functional for tapered beam-columns is developed and the inelastic effects are included. Using the standard techniques of the calculus of variations, the three differential equations describing the behavior of such elements are derived. The central difference expressions are used to transform these differential equations into difference equations. Two of these three equations are coupled and simultaneously describe the out-of-plane behavior of the beam-column. These two equations need to be satisfied at internal stations along the element, and at each station the coefficients of the difference equations must be computed using the section properties which in turn depend on the yield patterns at the proper points.

In this study a method to find the yield pattern and to determine the yield parameters is proposed which is based on the solution of the equations of equilibrium. The idea is similar to the ultimate strength design of reinforced concrete. It can be shown that the two equations of equilibrium and also the yield parameters for different yield patterns can all be defined in terms of two unknowns: the curvature of the section and the location of the neutral axis (distance to the top flange). The two equations of equilibrium are nonlinear and simultaneous in terms of the above unknowns. Solution of these equations results in the needed yield parameters.

At each node the above analysis is performed and the cross-sectional properties are computed using the yield parameters. The difference equations are then satisfied and the determinant of the resulting equations is assembled. The effect of the boundary conditions (fixed or pinned) is included. Effects of one intermediate brace are included in the formulation.

The value of the determinant is tested for different trial values of the buckling load (using the method of bisection) until the one that vanishes the determinant is found. The details of the computer program written to do the above analysis are explained in Chapter II.

The results obtained are compared with the results given by different sources in Chapter III. Results are obtained for inelastic, prismatic beam-columns, elastic tapered beams and columns, inelastic tapered beam-columns, etc. The effect of an intermediate bracing is shown. The contribution of the yielded portions in LTB is discussed and the error resulting from using the elastic core is noted.

In Chapter IV a design equation is proposed in the LRFD format. Regression analysis is performed to determine the parameters in the interaction equation used for tapered beam-columns and design aids are performed which include the effects of the tapering, singly-symmetric and residual stresses.

CHAPTER II

THEORETICAL ANALYSIS

2.1 General

In this chapter the method of stability analysis for a tapered, singly-symmetrical member subjected to unequal end moments and axial load is explained. The effect of the residual stresses is included. A computer program is developed to find the buckling load and attention is focussed on the out-of-plane behavior of the beam-column.

2.2 Assumptions

The assumptions used in this study are as follows:

1. The material is elastic-perfectly plastic and no strain reversal occurs (Figure 2.1).
2. The member is initially straight and free of imperfections.
3. The moment of inertia about the x-axis is significantly larger than that about the y-axis (Figure 2.2).
4. The external moments are applied about the x-axis of the member.
5. The taper angle is less than 10^{o}.
6. Plane cross-sections do not change in geometric shape during buckling.
7. The effect of the shear strains on the deformations may be neglected.
8. During buckling, the internal forces and the external loads remain in a vertical plane and no unloading occurs.
9. Warping stresses and strains are developed only in the elastic core during buckling.

10. The curvature of the neutral axis is neglected.

11. The Bernaulli-Navier hypothesis (bending strain is proportional to the distance from the neutral axis) applies to inelastic as well as elastic deformations.

12. The material in the flanges of the beam-column is concentrated at the centerline of the flanges. The distance from the top flange to the bottom flange and the height of the web is then equal to the center-to-center distance of the flanges of the real beam (Figure 2.2).

13. The area of the compression flange is not smaller than the area of the tension flange.

Assumption (3) was stated by Salvadori[(70)] and allows vertical deflections to be neglected for lateral-torsional buckling. Boley[(71)] has shown that the standard flexural formulas can be used for less than 10^{o} tapered beams and the resulting error would be less than 2%. Assumption (5) is made regarding his findings. As will be discussed later, for singly-symmetric, tapered members, the neutral axis is not a straight line and the deviation of this curve from a straight line increases as the taper ratio becomes larger. Assumption (5) also justifies assumption (10) because of limiting the deviation of the neutral axis as stated above. Assumptions (6) and (7) are made according to Vlasov[(19)], who used these to separate the combined bending and torsion effect in a member into two independent actions. Assumption (12) results in adding an area equal to the thickness of the web times the average thickness of the two flanges to the area of the cross-section. This is justified by the fact that the tabulated area of a cross-section is more than the sum of the areas of the rectangular components. The rest of the assumptions are common in mechanics of materials.

In addition to the above, the following size limitations have also been assumed to exist (Reference 39):

$$1 \leq \frac{\text{minimum depth}}{\text{flange width}} \leq 6$$

$$\frac{\text{length}}{\text{minimum depth}} \leq 24$$

$$15 \leq \frac{\text{flange width}}{\text{flange thickness}} \leq 50$$

6 inch $\leq$ depth of small end $\leq$ 24 inch

3 inch $\leq$ flange width $\leq$ 12 inch

3/16 inch $\leq$ flange thickness $\leq$ 3/4 inch

2.3 The Curvature of the Neutral Axis

As mentioned earlier, the neutral axis of a singly-symmetric, tapered member (the line that connects the centroids of different cross-sections), is not a straight line. This effect, which is commonly neglected in formulation found in the literature, can be observed by locating the centroid of a cross-section at a distance Z from the smaller end (Figure 2.3).

$$\bar{y} = \frac{A_2 D_z + \tfrac{1}{2} A_3 D_z}{A_1 + A_2 + A_3} = \frac{A_2 D_o (1+\gamma\frac{z}{L}) + \tfrac{1}{2} A_{03} D_o (1+\gamma\frac{z}{L})^2}{A_1 + A_2 + A_{03}(1+\gamma\frac{z}{L})} \qquad (2.3.1)$$

It can be seen that this function does not represent a linear variation in terms of z, so it is concluded that such members can be taken as curved elements in their plane of loading. It is also observed that when $A_1 = A_2$, i.e., for doubly-symmetrical sections,

$$\bar{y} = \frac{D_o}{2} (1+\gamma\frac{z}{L}) = \frac{D}{2}$$

which represents a straight line. In order to determine the amount of the

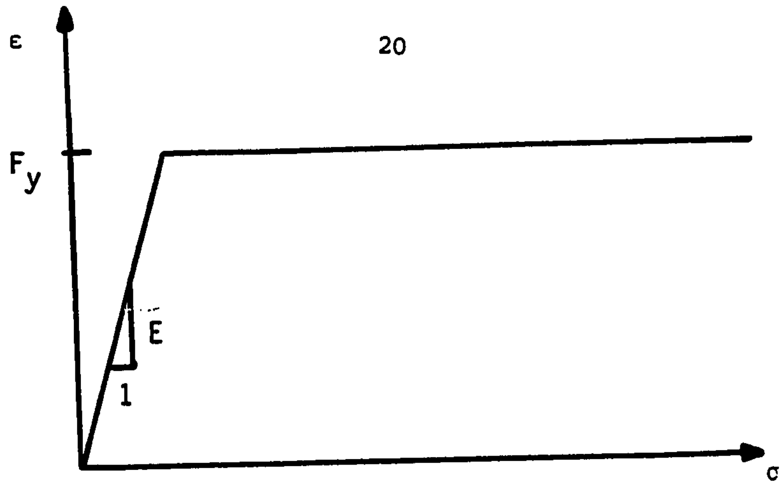


Figure 2.1 Idealized Stress Strain Diagram

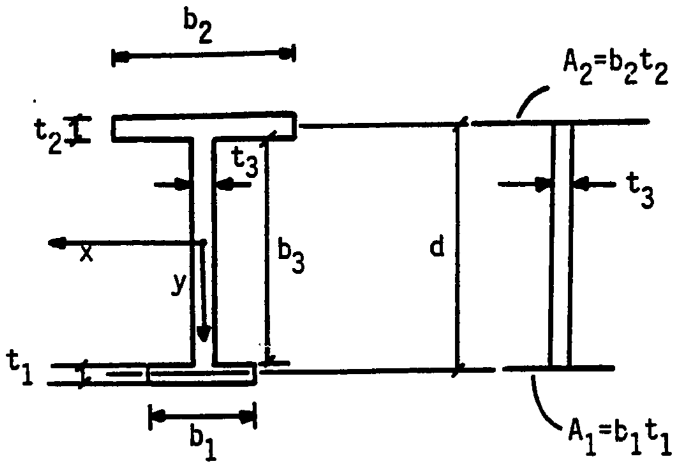


Figure 2.2 The Real and Idealized Cross-Section

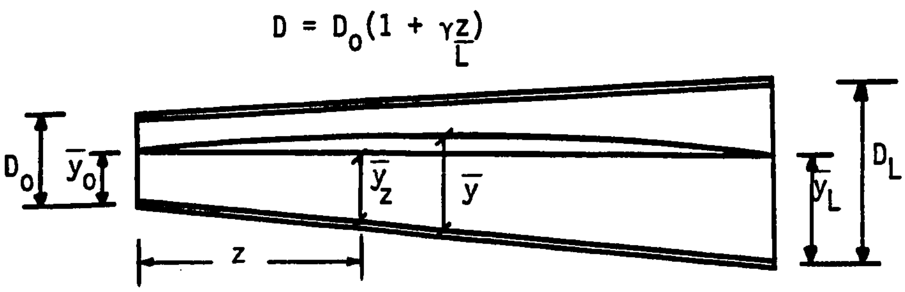


Figure 2.3 The Curvature of the Neutral Axis (Not Drawn to Scale)

deviation from a straight line, the equation of the straight line connecting the centroids of the two ends is written and the difference between the two ordinates shows the amount of this deviation (Figure 2.3):

$$\bar{y}_1 = \frac{A_z D_o + \frac{1}{2} A_3 D_o}{A_1 + A_2 + A_{o3}} \tag{2.3.2}$$

$$\bar{y}_2 = \frac{A_2 D_L + \frac{1}{2} A_3 D_L}{A_1 + A_2 + A_{L3}} \tag{2.3.3}$$

$$\bar{y}_z = \bar{y}_1 + (\bar{y}_2 - \bar{y}_1) \frac{z}{L} \tag{2.3.4}$$

$$\text{Dev.} = \bar{y} - \bar{y}_z \tag{2.3.5}$$

For all positive values of z, this deviation is positive, i.e., the neutral axis is curved, concave down and deflected toward the larger flange. Another important conclusion is that when singly-symmetric, tapered members are subjected to axial compression and bending and the smaller flange is in tension, this curvature is reverse to the deflected shape of the member due to loading. Therefore, it is concluded that, in such beam-columns, the $P - \Delta$ moments are reduced due to the curvature of the neutral axis.

The values of the centroidal deviations in the mid-span are tabulated in Table 2.1 for two different cross-sectional dimensions at the smaller end and a taper angle of 10^o. It is noted that the amount of this deviation is a function of the section properties at the smaller end, taper ratio, length of the member and z, the distance from the smaller end. However, it is observed that for dimensions used in practice, such deviation does not exceed 0.2 inch. Therefore, in this study, the effect of the curvature of the neutral axis is neglected.

Table 2.1

The Values of the Centroidal Deviations for $\alpha = 10^{o}$ at Mid-Span

	L=60"	L=120"	L=180"	L=240"
D_o=17.375, A_1=2.5, A_2=3.75, A_{03}=3.26	0.018	0.06	0.099	0.145
D_o=9.5, A_1=1.125, A_2=1.5, A_{03}=1.78	0.016	0.044	0.068	0.09

2.4 Residual Stresses

The behavior of hot rolled and flame cut members is usually affected by the existence of residual stresses. These stresses are the primary cause of the inelastic behavior of structural members and therefore their consideration is essential in determining the stability of tapered beam-columns. Several experimental investigations to measure the magnitude of such stresses have been reported and it has been shown that the residual stresses can reach the yield stress in some instances.

The stress pattern that has generally been used to model the existing residual stresses is known as the triangular pattern (Figure 2.4). In this model, used for wide flange sections by several authors[55,61,62], the variation of stress across the flanges is taken to be linear, the tips being in compression with tension near the web. The tensile stress across the web is assumed to be constant. (The stress in the web has been disregarded by some authors[59].) Usually, the maximum compressive residual stress at the tips of the flanges is assumed equal to $0.3\sigma_y$[55,62], or $0.5\sigma_y$[59], and the equilibrium of the forces will determine the magnitude of the tensile stresses across the web (Figure 2.4).

However, the use of the above model with the same values of stresses for singly-symmetric sections will result in the inequality of the flange forces and therefore the section will not be in equilibrium. Lee and Hsu[35] have neglected the tensile stress in the web and have assumed equal values for the tensile and compressive stresses in the flanges (Figure 2.6). The forces in this model are in equilibrium but it lacks the compatibility of strains since at the intersection of the web and flanges there is some strain in the flange and none in the web.

In this study another model to represent the distribution of residual stresses for the singly-symmetric sections is proposed. It is assumed that the maximum values of the compressive stresses in the flanges are different while the tensile stress in the web is uniform (Figure 2.5). By specifying the maximum value of the compressive stress in the upper flange, the compressive stress in the lower flange and the tensile stress in the web can be obtained using the equations of equilibrium (See Figure 2.5):

$$\Sigma M \text{ (bottom flange)} = 0$$

$$A_3\sigma_{rt}\left(\frac{D}{2}\right) - \tfrac{1}{2}A_2 D\,(\sigma_{rc} - \sigma_{rt}) = 0 \qquad (2.4.1)$$

$$\Sigma F_z = 0$$

$$A_3\sigma_{rt} - \tfrac{1}{2}A_2\,(\sigma_{rc} - \sigma_{rt}) - \tfrac{1}{2}A_1\,(\sigma_{rb} - \sigma_{rt}) = 0 \qquad (2.4.2)$$

The first equation gives

$$\sigma_{rt} = \frac{A_2}{A_2 + A_3}\,\sigma_{rc}$$

and substituting in Equation (2.4.2),

$$\sigma_{rb} = \frac{1 + A_3/A_1}{1 + A_3/A_2}\,\sigma_{rc}$$

It can be shown that within the practical limitations of the dimensions, the values of σ_{rb} and σ_{rc} are essentially equal. For example, if $A_1 = 2.5''^2$, $A_2 = 3.75''^2$ and $A_3 = 3.26''^2$ (from Table 2.1), the above relationship gives:

$$\sigma_{rb} = 1.23\,\sigma_{rc}$$

This model has the advantage of being in equilibrium and in the special case (doubly-symmetrical section), is the same as the pattern used in the literature by several authors.

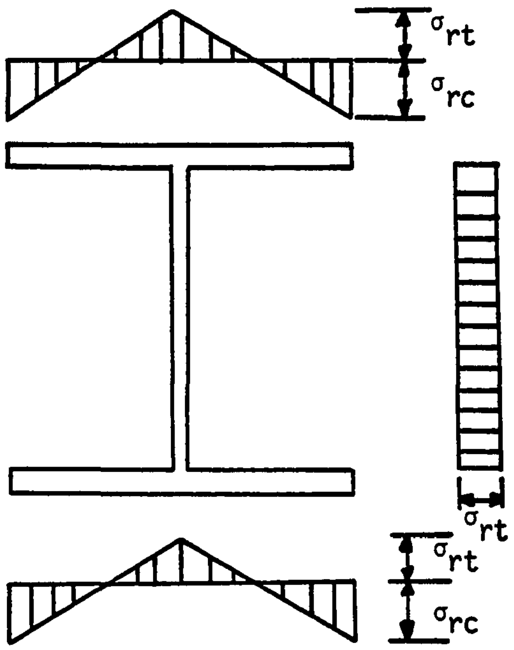


Figure 2.4 Residual Stresses for a Steel Wide Flange Section (Ref. 20)

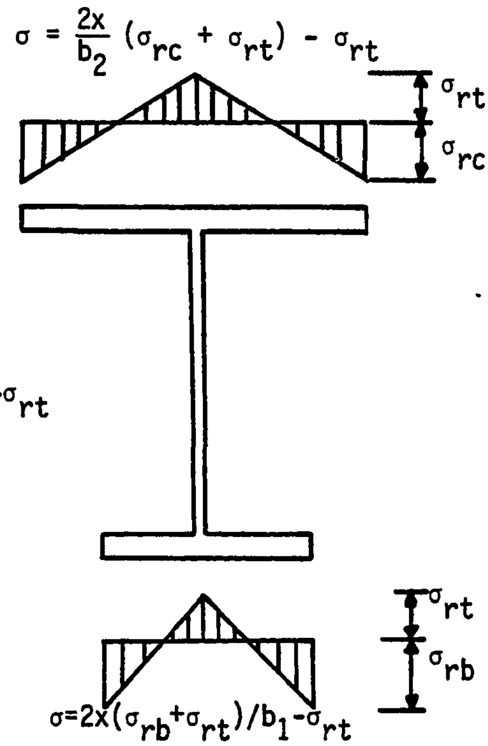


Figure 2.5 The Residual Stress Pattern Used in this Study for Singly Symmetric Sections

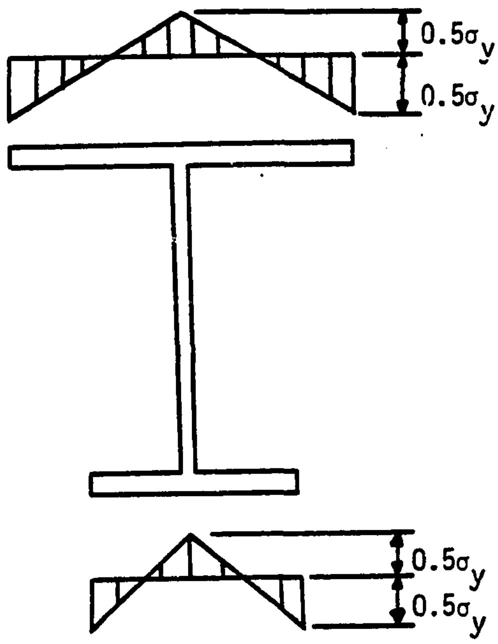


Figure 2.6 The Residual Stress Pattern Used by Lee & Hsu[35]

2.5 Theoretical Analysis and Solution Method

The total potential energy expression for an inelastic, tapered, singly-symmetric beam-column is derived in Appendix A and is repeated here:

$$\pi = \tfrac{1}{2}\int_0^L \{EI_c(\frac{d^2}{dz^2}(u+(d-y_s)\phi))^2 + EI_t(\frac{d^2}{dz^2}(u-y_s\phi))^2 + GJ\phi'^2$$

$$+ \frac{P^2}{EA} + EI_x v''^2 + 2M_x v'' - 2M_x u'\phi' + M_x\beta_x\phi'^2 - p(u'^2 + v'^2 + r^2\phi'^2$$

$$+ 2u'\phi' y_o) + \int_A (R.S. + \sigma_y)(2yv'' + u'^2 + v'^2 + \phi'^2 x^2 + \phi'^2(y_o - y)^2$$

$$+ 2u'\phi'(y_o - y))dA\} \, dz \qquad (2.5.1)$$

The three governing differential equations which describe the behavior of the beam-column, can be derived by minimizing the above functional. It is sufficient to set the first variation of the total potential energy expression equal to zero or use Euler's equation, given as:

$$\frac{d^2}{dz^2}\left(\frac{\partial\pi}{\partial x''_i}\right) - \frac{d}{dz}\left(\frac{\partial\pi}{\partial x'_i}\right) + \frac{\partial\pi}{\partial x_i} = 0 \qquad (2.5.2)$$

and the associated boundary conditions are:

and $$\left[\left(\frac{d}{dz}\frac{\partial\pi}{\partial x''_i} - \frac{\partial\pi}{\partial x'_i}\right)\delta x_i\right]_o^L = 0 \qquad (2.5.3)$$

$$\left[\frac{\partial\pi}{\partial x''_i}\,\delta x'_i\right]_o^L = 0 \qquad (2.5.4)$$

where x_i denotes each of the deformations v, u and ϕ.

Substituting Equation (2.5.1) into Equation (2.5.2) gives three equations by taking first variations in v, u and ϕ, respectively:

$$\frac{d^2}{dz^2}\left(EI_x v'' + M_x + \int_A (R.S. + \sigma_y)\, y dA\right) - \frac{d}{dz}\left(-pv' + v'\int_A (R.S. + \sigma_y) dA\right) = 0$$

(variation in v) (2.5.5)

$$\frac{d^2}{dz^2}\left(EI_c(d - y_s)(u - y_s\phi + d\phi)'' - EI_t\, y_s(u - y_s\phi)''\right) - \frac{d}{dz}\left(2EI_c\right.$$

$$(d' - y'_s)(u - y_s\phi + d\phi)'' - 2EI_t\, y'_s\phi'(u - y_s\phi)'' + GJ\phi' - M_x u' + M_x\beta_x\phi' - pr^2\phi'$$

$$\left. - pu'y_o + \phi'\int_A (R.S. + \sigma_y)S^2\, dA + u'\int_A (R.S. + \sigma_y)(y_o - y) dA\right) - Ey''_s$$

$$\left(I_c(u - y_s\phi + d\phi)'' + I_t(u - \phi y_s)''\right) = 0 \quad \text{(variation in } \phi\text{)} \qquad (2.5.6)$$

$$\frac{d^2}{dz^2}\left(EI_c(u - y_s\phi + d\phi)'' + EI_t(u - y_s\phi)''\right) - \frac{d}{dz}\left(-M_x\phi' - pu' - p\phi' y_o\right.$$

$$\left. + u'\int_A (R.S. + \sigma_y)\, dA + \phi'\int_A (R.S. + \sigma_y)(y_o - y) dA\right) = 0$$

(variation in u) (2.5.7)

The above equations describe the behavior of inelastic, tapered beam-columns. Equation (2.5.5) governs the in-plane behavior and since this study is concerned with lateral-torsional buckling, it will not be used and its boundary conditions are not of interest. The two other equations, which are coupled, simultaneously describe the out-of-plane behavior of such members. It is observed that for simple or fixed boundary conditions, δu and $\delta\phi$ at the ends are zero and the boundary condition (2.5.3) is satisfied. Also for fixed ends,

$\delta u'$ and $\delta\phi'$ are equal to zero which satisfy (2.5.4). The natural boundary conditions for simply supported ends are:

$$\frac{\partial\pi}{\partial u''} = EI_c(u+d\phi-y_s\phi)'' + EI_t(u-y_s\phi)'' = EI_y u''-2EI_y y_s'\phi' - Ey_s\phi'' I_y +$$

$$2EI_c d'\phi' - Ey_s'' I_y\phi \qquad (2.5.8)$$

$$\frac{\partial\pi}{\partial\phi''} = EI_c(d-y_s)(d\phi)'' - (y_s\phi)'' E(I_c d-I_c y_s-I_t y_s) = EI_c(d-y_s)(d\phi''+2d'\phi')$$

$$(2.5.9)$$

The boundary condition (2.5.9) has been previously mentioned by Lee[24]. Assumption 3 of Section 2.2 implies that the lateral-torsional buckling will be the governing failure mode, so the buckling load will be obtained by solving Equations (2.5.6) and (2.5.7), simultaneously. The details of simplification of these equations are given in Appendix B. It is to be noted that the section properties in these equations vary along the beam-column, so a numerical solution method is required which takes such variations into account.

In this study, the finite difference technique is used to solve the above equations and determine the buckling load. The procedure of transforming the two coupled differential equations into difference equations is also shown in Appendix B. The central difference expressions have been used to approximate all derivatives since the round-off error is smaller than what caused by using backward or forward differences. The resulting difference equations are linear combinations of u and ϕ values at different stations. The member is divided into n equal segments and the difference equations must be satisfied at different stations (nodes) along the member. The boundary conditions are $\phi_o = u_o = \phi_n = u_n = 0$, Figure 2.7(a). Also, for simple ends (lateral-torsional), $u_{-1} = -u_1$ and $\phi_{-1} = -\phi_1$, Figure 2.7(b). Fixed ends (lateral-torsional) require: $u_{-1} = u_1$ and $\phi_{-1} = \phi_1$, $u_{n+1} = u_{n-1}$ and $\phi_{n+1} = \phi_{n-1}$, Figure 2.7(c).

Since the related difference equations must be satisfied at different nodes along the beam-column and the coefficients in these equations are functions of the section properties which vary with the different yield patterns, it is necessary to determine the yield pattern at each node. For prismatic members, development of the moment-curvature-thrust relationship about the strong axis of the cross-section, has been proposed by Galambos[55]. For tapered members, due to the change of the depth, the above procedure would be quite difficult to apply, since at each node a new set of curves would need to be developed.

Another approach, suggested by Morrel[72] and Movaghar[73] can be used to find the yield pattern through an iterative procedure based on incremental loading. The cross-section is divided into small segments, the load is increased and the yielded portions are removed until the equilibrium is satisfied. This method is applicable for tapered members but the amount of computer time required is comparatively high and the accuracy of the result will depend on the number of segments at each cross-section.

In this study, another approach, based on the solution of the equations of equilibrium is proposed. The idea is similar to that of the ultimate strength design of reinforced concrete. This approach is based on the fact that the two equations of equilibrium at a cross-section and also the yield parameters (which specify the depth of the yield penetration in the web

(a) Node Numbers and Segmental Divisions

(b) Lateral Deformed Shape of a Simple Supported Member

(c) Lateral Deformed Shape of a Fixed End Member

Figure 2.7 Node Numbering and Boundary Conditions or flanges), can all be written in terms of two unknowns: The value of the curvature due to loading and the location of the neutral axis (point of zero strain due to loading). The two equations of equilibrium are nonlinear and simultaneous in terms of the above unknowns and they are also functions of the yield parameters (which in turn depend on the same unknowns). Since it is not possible to determine in advance which yield pattern will prevail at a cross-section, the procedure to determine the yield pattern (among different patterns) is a trial and error procedure. This method, enjoys the advantage of being non-iterative and so the amount of book-keeping effort and computational time required is considerably less than what is needed for other methods discussed.

Since the aim of the whole procedure is to determine the buckling load, and a value for buckling load is needed to find the yield patterns, an iterative procedure must be followed. In this study, two values for the buckling load (upper and lower limits) are estimated and investigation is performed to locate the value of the buckling load between the two limits. The method of Bisection (or halfing) is used to determine the critical buckling load.

The method used to find the inelastic buckling load of a tapered beam-column, based on the above discussion, can be summarized as follows:

1. The two limits of the buckling load (axial load or bending moment) are estimated and the critical load is investigated between these two limits.
2. Assuming a trial value of the buckling load, the yield pattern is sought at each node along the member. In order to determine the yield pattern, the following steps are needed: first based on the elastic behavior, the extreme values of the stresses on the section

are checked (one out of five patterns will be chosen) and if no acceptable result is obtained by solving the equations of equilibrium, another yield pattern is tried. (Figure 2.8).

3. By solving the equations of equilibrium and finding the value of the curvature and the location of the neutral axis, the yield parameters are obtained.
4. The section properties are calculated using the yield parameters.
5. The difference equations are satisfied at each unrestrained node. By dividing the member into n segments, there will be 2(n-1) difference equations to be satisfied, which can be written in matrix form as:

   $$(D)\ \left\{ {u \atop \phi} \right\} = 0$$

6. Setting the determinant of matrix (D) equal to zero, provides the condition of lateral-torsional instability. The value of this determinant is tested for different trial values of the buckling load till the load that vanishes the determinant is found.

In order to insure that this load is the first buckling load, the buckling mode shape is found. After determination of the buckling load, one of the degrees-of-freedom, for example, u_1, is taken equal to 1. This value is substituted into all equations, the corresponding row and column to u_1 are cancelled and the remaining 2n-3 linear equations are solved to find the relative values of the displacements and rotations.

2.6 Computer Program

A computer program for this study was written using Fortran - IV language for an IBM 370/158 computer. The steps used to find the buckling load of a tapered beam-column are as follows:

Step I - Input Data. The input data includes the dimensions of the web and the flanges at the smaller end, the taper ratio, the yield stress of the material, the length of the member and the type of the boundary conditions (fixed or pinned). The value of the maximum compressive residual stress in the upper flange as a percentage of the yield stress must also be given. The location of the intermediate brace (distance from the smaller end) must be specified and if such a brace does not exist, this value will be equal to zero. An upper bound estimate of the buckling load must be given. This load should not be higher than the plastic capacity of the section and some preliminary calculation is required to estimate its value. The lower bound is specified as 40% of the upper bound. The program can work in two modes: Beam type, in which the value of the axial load is specified and the larger end moment is taken as buckling load, and column type, for which the end moments are specified and the axial load is taken as the buckling load. In both modes, the ratio of the end moments is an input data. The values of E, the modulus of elasticity, G, the shear modulus of elasticity and the required accuracy for the buckling load must be given.

Step II - Preliminary Computations. The beam-column is divided into n equal segments (n+1 nodes) and at each node, the following quantities are computed: external bending moment, the values of the maximum compressive residual stresses in the flanges and the tensile stress in the web, the torsional constant J (which is assumed to remain unaltered for all yield patterns), the location of the centroid and the moment of inertia about the strong axis. Assuming that the cross-section remains elastic, the extreme values of stresses on the cross-section are computed as sum of the residual stresses and the stresses due to bending moment and axial load.

Step III - Determining the Yield Pattern. Using the values of the stresses computed above, the yield pattern which is deemed most probable to govern is found. An attempt is then made to solve the two equations of equilibrium using a subroutine written to solve the simultaneous nonlinear equations. The two unknowns of the equations are: the curvature θ times E and the distance from the top flange to the point of zero strain (due to loading). If the convergence criterion is met and acceptable values are obtained for the yield parameters, it is assumed the proper yield pattern and the yield parameters obtained are used to compute the section properties. Otherwise, the next yield pattern is tried according to Figure 2.8.

Step IV - Assembling the Determinant of the Equations. The section properties at each node are computed using the yield parameters and they are used to satisfy the difference equations at that node. The procedure is repeated for all (n+1) nodes and the determinant of these difference equations are assembled for (n-1) unrestrained nodes. The effect of the boundary conditions is included and if there is an intermediate braced point, the corresponding row and column of that node are cancelled.

Step V - Obtaining the Buckling Load and Mode Shape. The value of the determinant assembled is then tested and different buckling loads are tried until a zero value for the determinant is found. The IMSL library routine LiNV3F is used to evaluate this determinant[78]. The convergence criterion depends on the required accuracy which is specified in the input data. The buckling mode is then specified by computing the relative displacements. This is done by assuming $u_1 = 1$ and cancelling the first row and column of the determinant and solving the remaining 2n-3 linear equations by Gauss-Jordan elimination method[79].

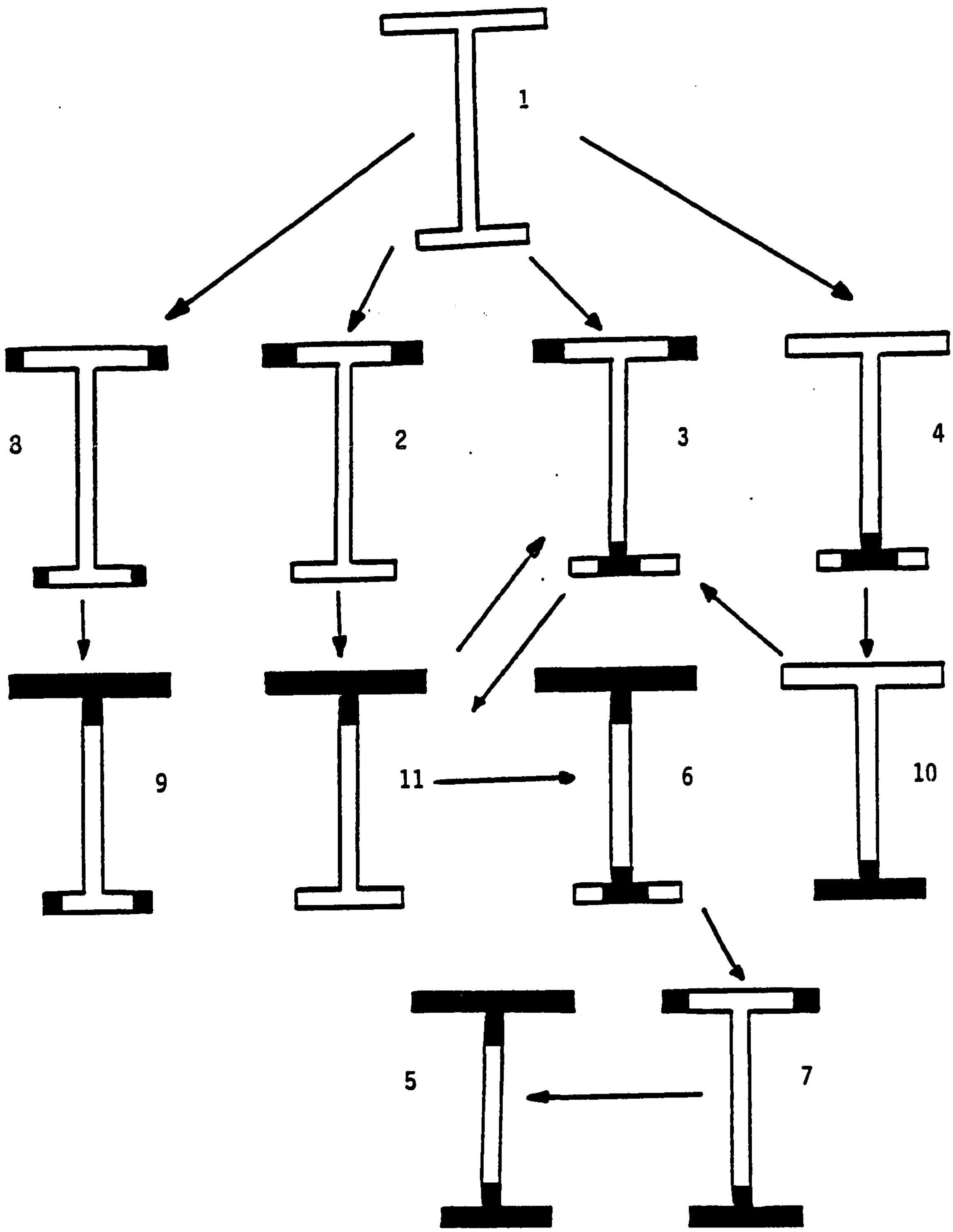


Figure 2.8 Possible Yield Patterns and Order of Assumption in Analysis

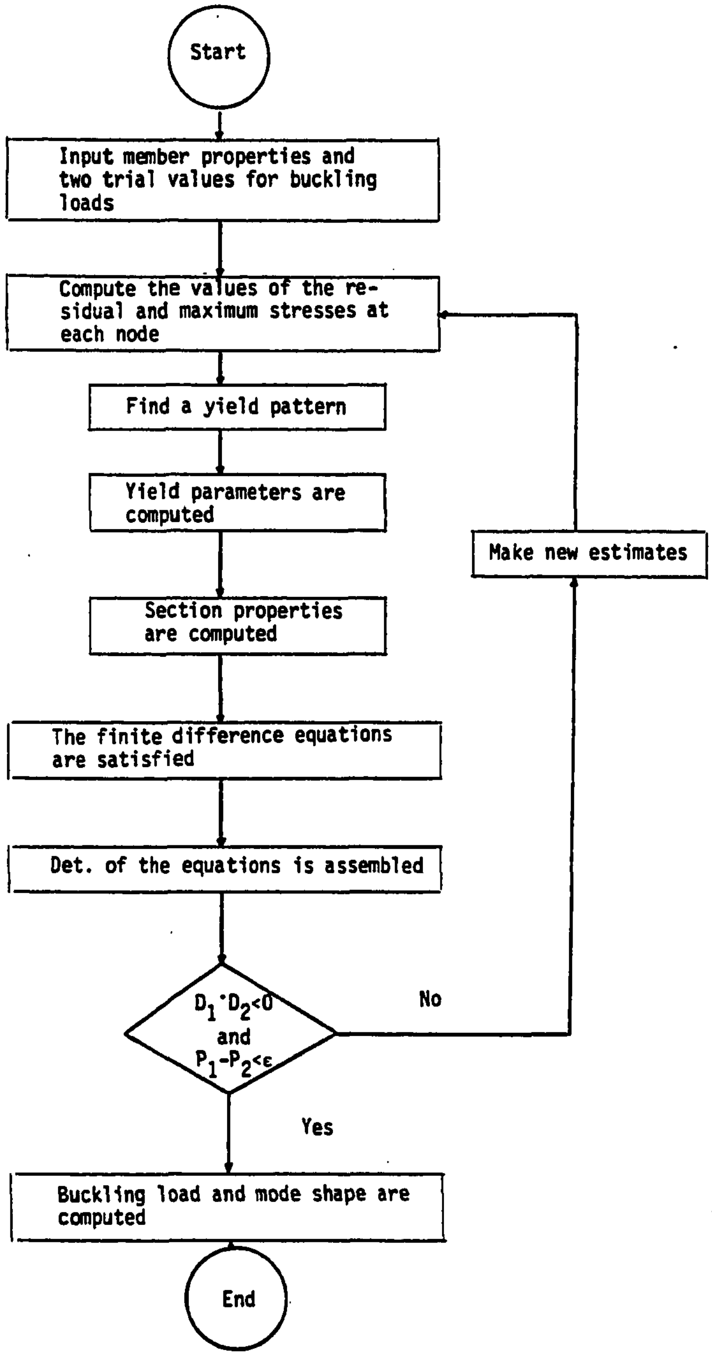


Figure 2.9 Macro Computer Program Flow Chart

2.7 Computer Program Capability

The program can analyze a linearly web tapered, singly-symmetrical member subjected to axial load and unequal end moment. Options of including the residual stresses (inelastic analysis) and neglecting them (elastic analysis), are available. The triangular residual stress pattern is used and the value of the maximum stress is specified as a percentage of the yield stress. For an elastic analysis this percentage is taken equal to zero. The boundary conditions may be fixed or pinned at both ends. One braced point, along the beam-column may be specified. The number of segments must be chosen in a way that the braced point coincides with a node.

Small modifications by the user are required to handle the following cases:

1. Tapered flanges or tapered web and flanges.
2. Equally spaced multiple braces.
3. Combination of different types of boundary conditions.

It is noted, however, that considerable modification is required for extensions to the double curvature case because of the numerous combinations of possible yield patterns.

CHAPTER III

NUMERICAL AND EXPERIMENTAL STUDIES

3.1 General

In the previous chapter, the method of determining the buckling load of a beam-column with variable cross-section was explained. Results of the numerical studies are presented in this chapter. These results are compared to results found in the literature by different authors and also with those observed in a limited experimental investigation.

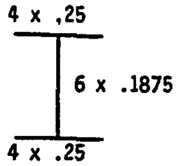
3.2 Numerical Studies

3.2.1 Elastic Stability of Tapered Beams

The elastic stability of tapered, singly-symmetrical beams under moment gradient has been investigated by Nelson[39]. In this reference, the critical buckling moment is found using trigonometric functions, finite element analysis and polynomial functions (reported from Reference 30). These results and those obtained using the present program are shown in Tables 3.1 and 3.2 for two different cross-sections. Comparisons are made for prismatic and tapered beams, doubly and singly-symmetric sections. A stress ratio, r, is defined as the ratio of the maximum compressive stresses at the ends of the beam. This ratio is positive for a beam in single curvature and negative for beams having double curvature. In all cases, it is assumed that the larger moment is acting at the larger end.

The results obtained in this study are in good agreement with

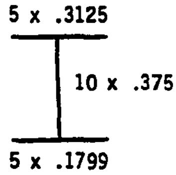
Table 3.1
Results of Critical Moment Calculations
Section #1



Taper Radians	Length Inches	Stress Ratio r	Polynomial Functions (Chi) (K-ft)	Trigonometric Functions (Nelson) (K-ft)	Finite Element (BASP 15) (K-ft)	Present Study (K-ft)
0.00	60.0	+1.0*	62.79	62.79	63.08	62.45
		+0.5	82.79	82.92	83.09	79.92
		0.0	114.84	116.19	116.40	112.73
		-0.5	155.76	163.40	164.83	166.00
0.01	180.0	+1.0	10.67	10.63	10.82	10.54
		+0.5	14.63	14.77	15.08	19.42
		0.0	20.69	22.35	22.85	22.0
		-0.5	25.15	33.21	33.71	28.03

*exact solution = 62.79 K-ft.

Table 3.2
Results of Critical Moment Calculations
Section #2



Taper α Radians	Length L Inches	Stress Ratio r	Polynomial Functions (Chi) (K-ft)	Trigonometric Functions (Nelson) (K-ft)	Finite Element BASP 15 (K-ft)	Present Study (K-ft)
0.00	120.0	+1.0*	71.91	71.91	74.01	71.34
		+0.5	94.44	94.90	97.67	91.74
		0.0	127.62	132.29	136.11	128.0
		-0.5	156.82	181.38	186.24	183.07
0.05	120.0	+1.0	62.69	61.04	62.87	59.46
		+0.5	89.31	89.37	91.94	133.86
		0.0	138.58	151.26	154.87	161.21
		-0.5	167.31	204.15	201.99	196.12

*exact solution = 71.91 K-ft.

those obtained by other methods. Comparison of results with closed form solutions indicates that the finite difference method yield buckling loads which are slightly (less than 1 percent) on the conservative side. The same conclusion is reached through different numerical studies presented in other sections in this chapter.

3.2.2 Elastic Stability of Tapered Beam-Columns

The elastic stability of a tapered beam-column has been studied by Morrel[72], Figure 3.1. For an eccentricity of e = 6", length L = 144" and taper ratio $\gamma = 2$, the critical moment is found by Morrel to be 160 k-in., using both the finite element method and the Rayleigh-Ritz method.

The same problem has been solved using the present computer program. A constant axial load equal to 160 k/6 = 26.67 k was specified and the analysis performed by dividing the member into 10 segments to find M_{cr} = 160.4 k-in., which is in excellent agreement with the result given in the above study.

3.2.3 Inelastic Stability of Wide Flange Column

The inelastic stability of wide flange columns has been investigated by Lee and Hsu[59]. The finite element method was used with the member divided into 10 elements. The same residual stress pattern as shown in Figure 2.4 and two different cross-sections were studied by Lee and Hsu. The results were compared with the curves proposed by AISC (Figure 3.2). In this study, only the resistance provided by the elastic core is taken into account.

The present computer program was used to develop results for the above problems. Again, the member was divided into 10 segments. Excellent agreement was found in both the elastic and inelastic region,

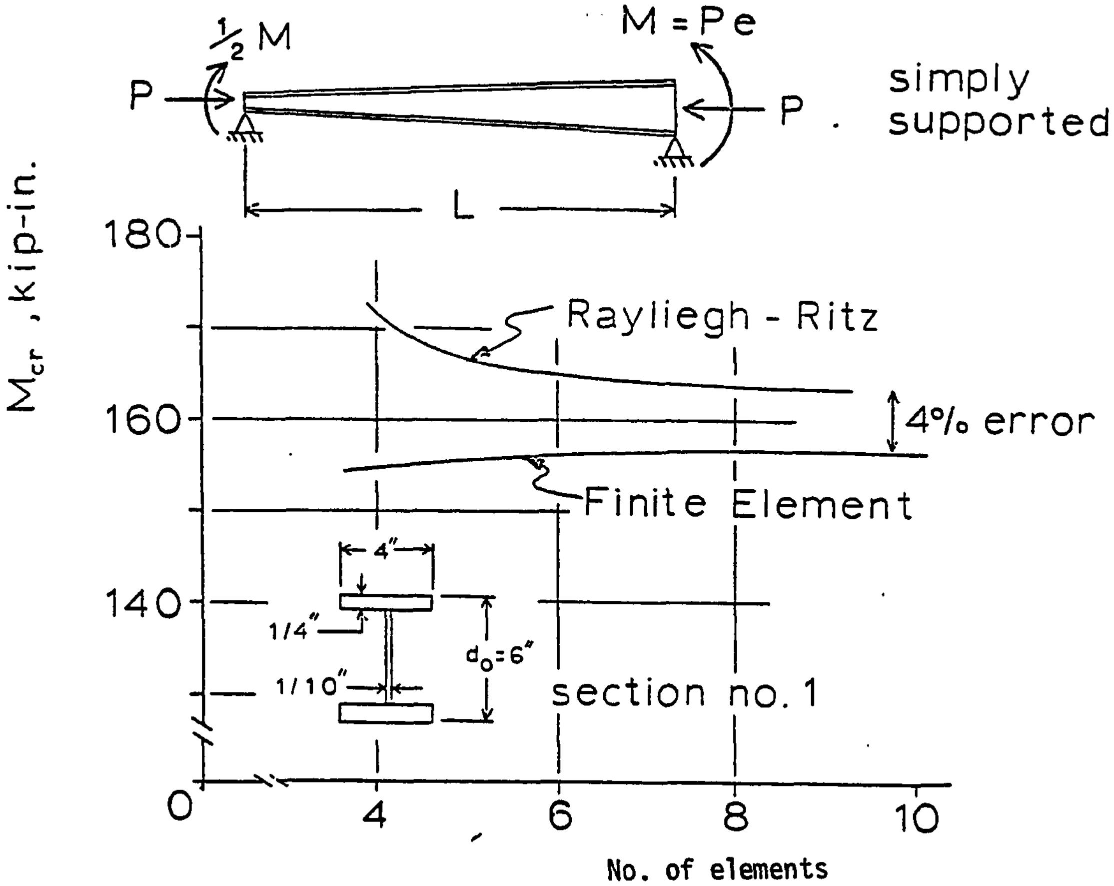


Figure 3.1 Elastic Stability of a Tapered Beam-Column
Reference 72

Figure 3.2. It is observed that due to the symmetry of the cross-section, buckling modes are separated and the weak axis buckling governs. In such cases, neglecting the contribution of the yielded portions does not affect the critical stress in the inelastic region.

3.2.4 Inelastic Stability of Wide Flange Beam-Columns

The results of the present study have been compared with those presented in Reference 55 for a W8x31 beam-column. For three different levels of axial load, 0.2 P_y, 0.4 P_y and 0.6 P_y, the critical moments were determined. Two different curves are presented: one for equal end moments and one for one end moment only (Figure 3.3). The same residual stress pattern (Figure 2.4) was used in both studies. The procedure used in Reference 55 included the effect of P-Δ moments, which are neglected in this study.

For the case of equal end moments, no differences are observed between the results obtained by the two methods. For one end moment, a maximum difference of approximately 5 percent was found. This difference might be attributed to the difference in number of segments, 15 in the referenced study and 10 in this study, or to the fact that some terms in the governing equations have been neglected in the referenced study. It was also found that neglecting P-Δ moments does not result in unconservative estimates of buckling loads, therefore such effect can be safely disregarded for the range of problems studied here.

3.2.5 The Contribution of the Yielded Portions in LTB

The behavior of structural steel is usually idealized as elastic-perfectly plastic (Figure 2.1). For such material, when the stress at some fibers reaches the yield stress no further resistance is provided

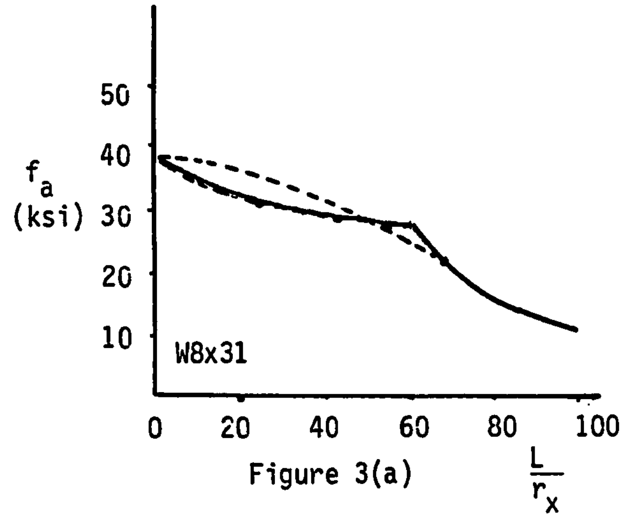


Figure 3(a)

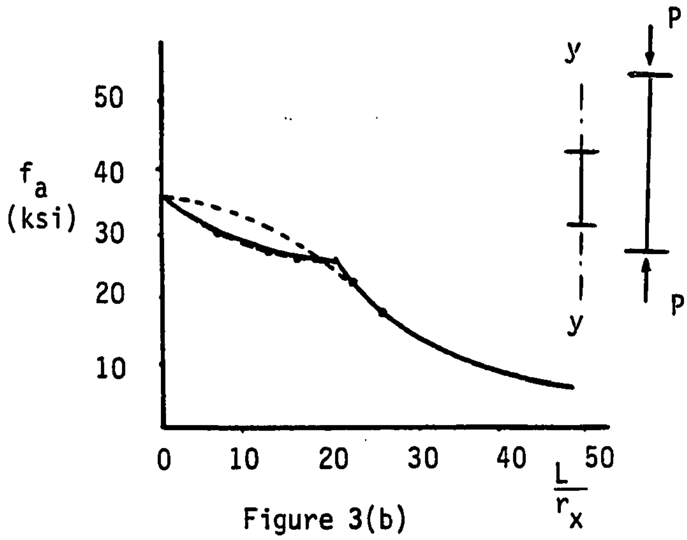
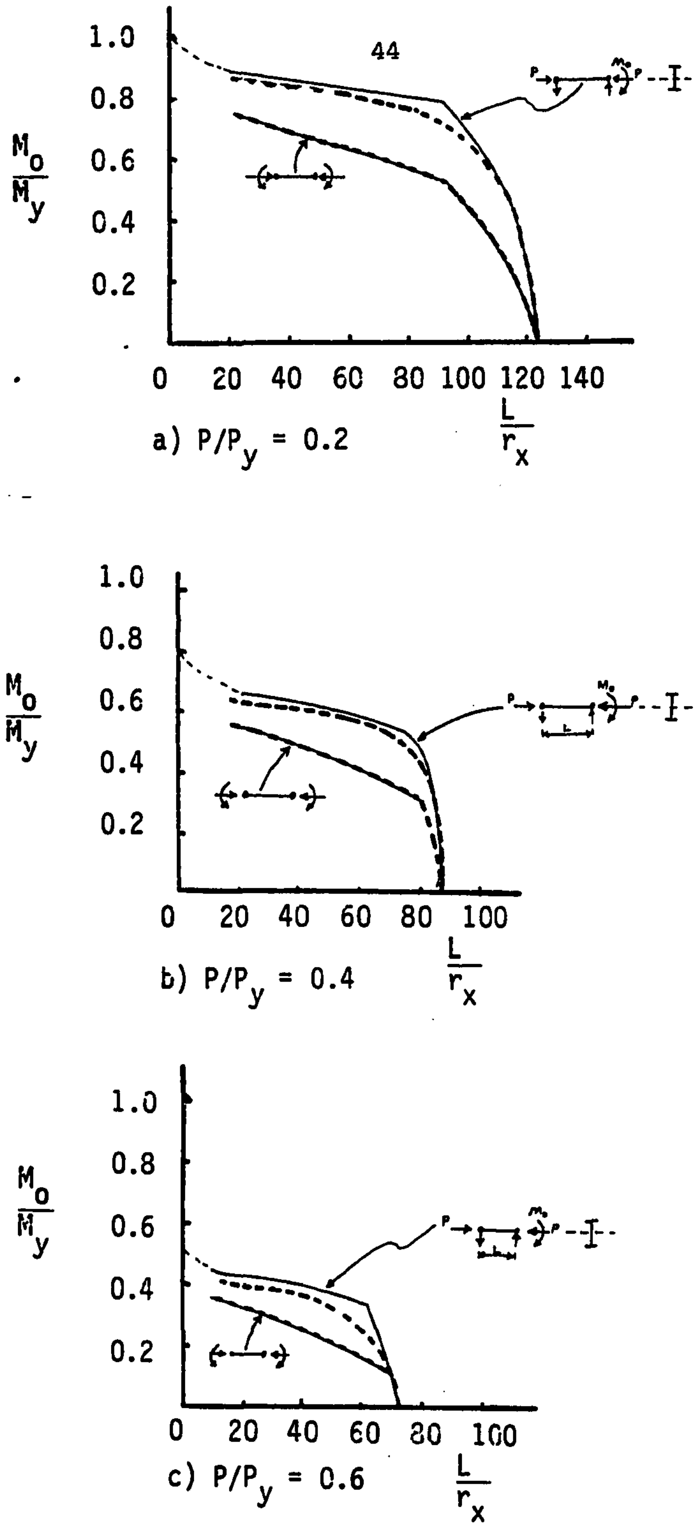


Figure 3(b)

Figure 3.2 Weak Axis Buckling Stress f_a

——— Lee and Hsu
-·-·-·- Present Study
- - - - AISC



- - - - Present Study

Figure 3.3 Lateral-Torsional Buckling Curves for 8WF31
Galambos and Fukumoto (Ref. 55)

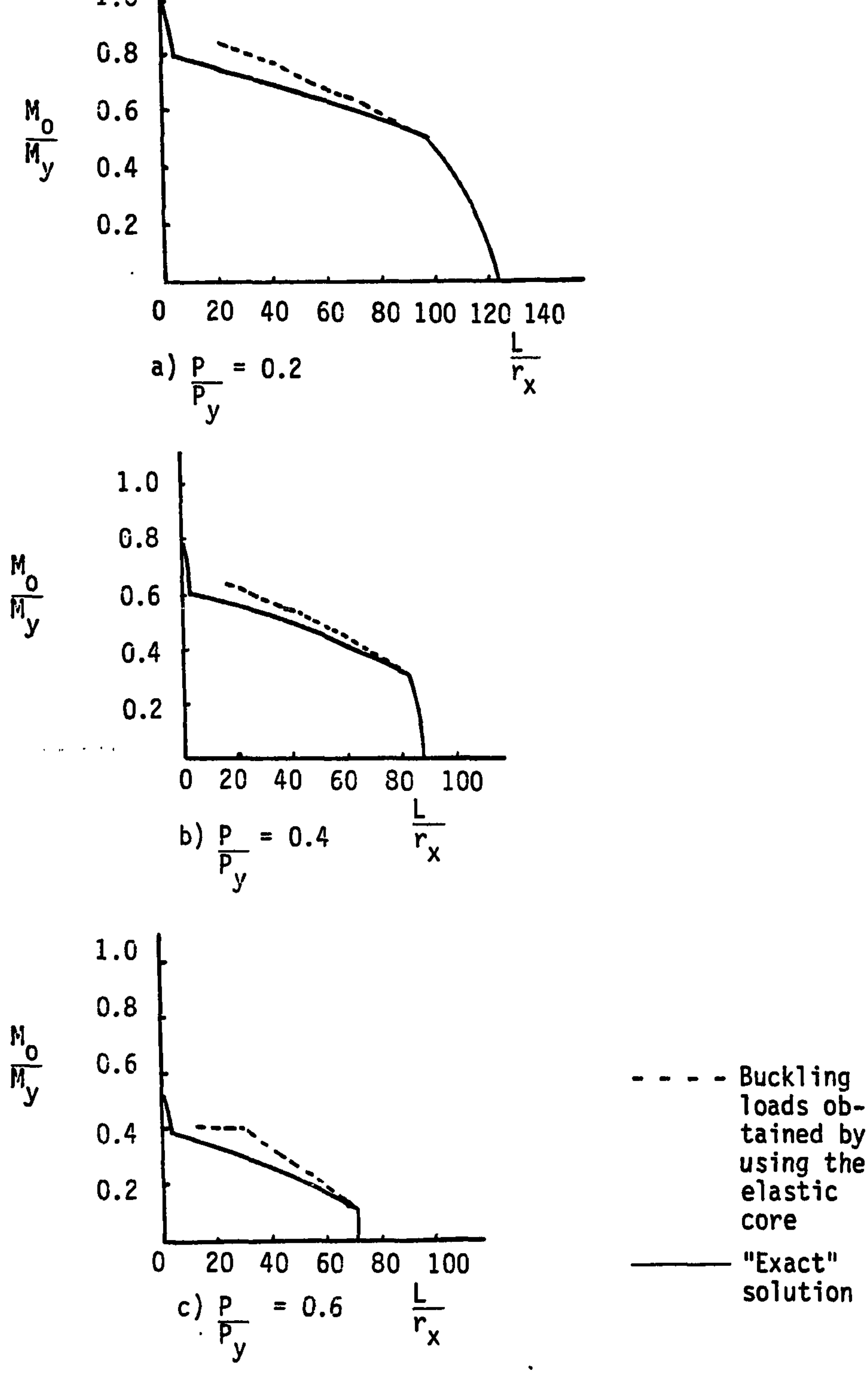


Figure 3.4 Effect of the Yielded Portions in LTB

by the yielded portions. It has been common practice in analysis of inelastic members to remove the yielded portions and use the section properties of the elastic core in the governing equations. This approach, which is applicable to laterally supported beams, is unconservative when applied to LTB problems.

For LTB of laterally unsupported beams or columns, all normal stresses contribute to twist of the cross-section and neglecting the yielded portions can result in a significant error in computing the buckling load. This effect has been observed by Galambos and Fukumoto[55] while using the method of equilibrium to derive the governing differential equations. Still, several sources[35,59,61] who have minimized the potential energy functional to derive these equations, have used the section properties of the elastic core. The error resulting from this approach will be explained in this section.

The derivation of the total potential energy functional is given in Appendix A and it is shown that the yield stresses contribute to the external work. Physically, this work can be interpreted as the displacement of the yield stresses due to the elongation of the yielded fibers during LTB. Neglecting the yielded portions in such problems means ignoring a disturbing torque imposed by the yield stresses about the shear center which will result in obtaining unconservative values for the buckling loads.

In order to show the error caused by using the elastic core in inelastic LTB problems, the curves in Figure 3.4 are presented. The study is done for the same cross-section as in the previous section, subjected to equal end moments. At the same levels of axial load, the analysis is performed by using the elastic core only and the "exact" solution has also been presented. It is shown that the error resulting from neglecting the yielded portions for this cross-section can be as high as 10%. It is also observed that the error is larger for shorter members.

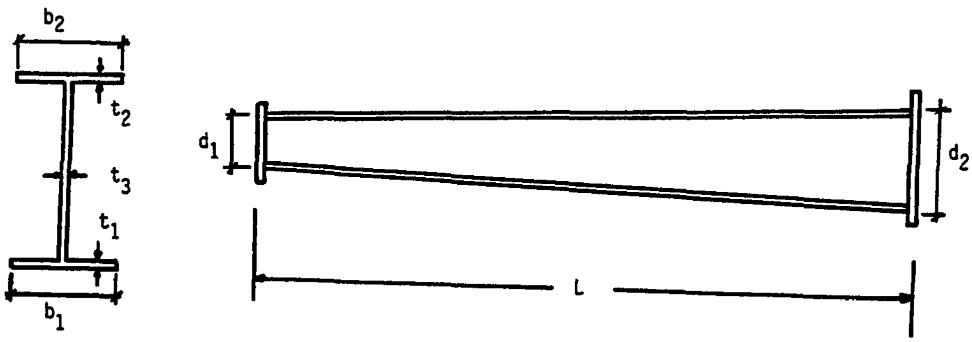
3.3 Experimental Investigation

The results of the present study have been experimentally verified in a limited study at the Fears Structural Engineering Laboratory, University of Oklahoma. Three different tapered beam-columns have been tested and the buckling loads are estimated using the load vs. lateral deflection curves. The specimens were fabricated from A572 Gr50 steel. Details and dimensions of the three specimens are given in Figure 3.5

3.3.1 Test Details

The test set-up consisted of the test specimens with two columns bolted to the end-plates. A test frame supported the beams on the reaction floor of the laboratory. A hydraulic ram was used to draw the two columns together in order to produce axial load and end moments in the specimens. A steel strap and a load cell were connected to the hydraulic ram as shown in Figure 3.6.

To stabilize the supporting columns, lateral brace mechanisms were used which are shown in Figure 3.6. These mechanisms also provide lateral support to the ends of the specimens. In addition, lateral support mechanisms were attached to the flanges of the beams to simulate intermediate brace locations. Intermediate brace locations at 1/4, 1/2 and 3/4 of span were investigated.



	L	d_1	d_2	b_1	t_1	b_2	t_2	t_3
Beam #1	240	10	14	4.932	0.256	4.978	0.259	0.125
Beam #2	300	10	20	4.984	0.18	5.016	0.241	0.18
Beam #3	292	15.97	20	4.91	0.246	4.91	0.246	0.126*

*Average web thickness

Figure 3.5. Details and Dimensions of Three Specimens.

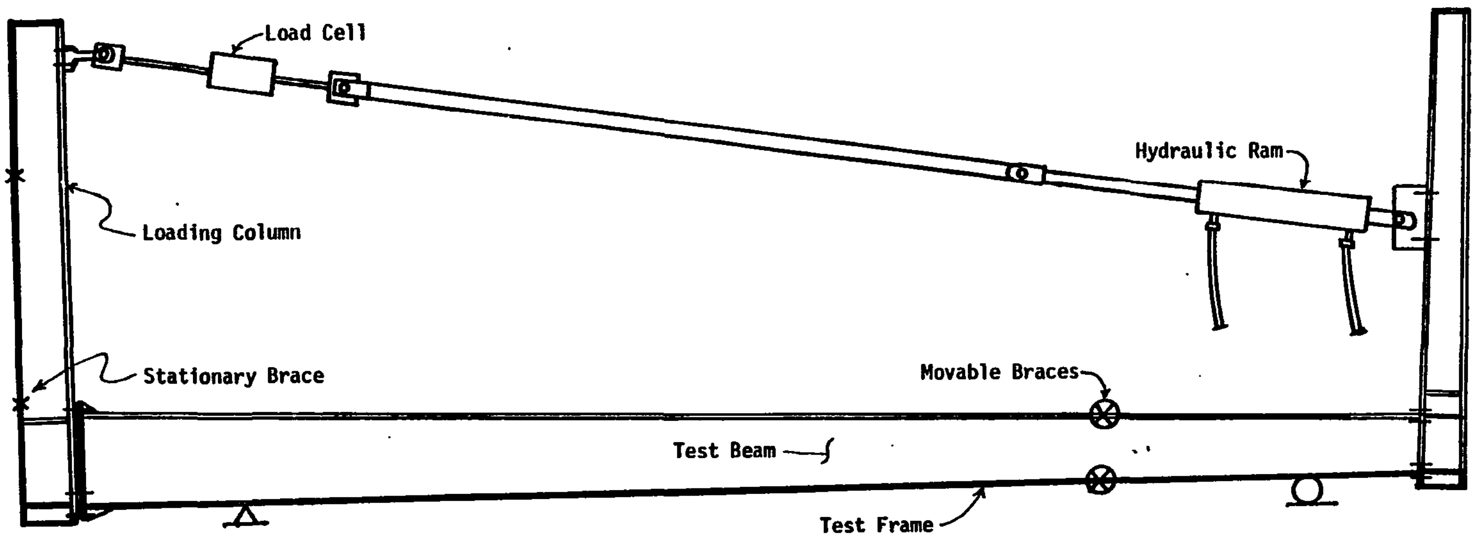
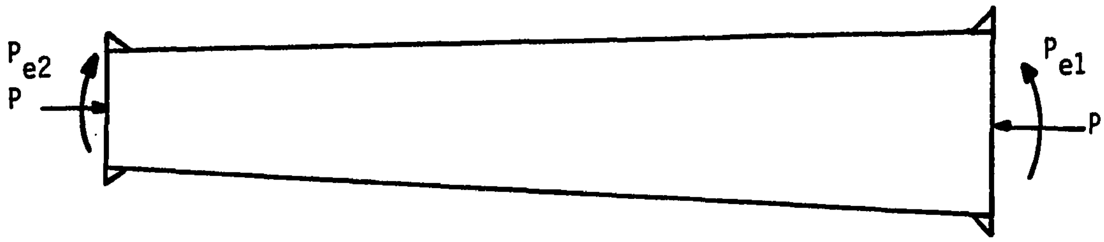
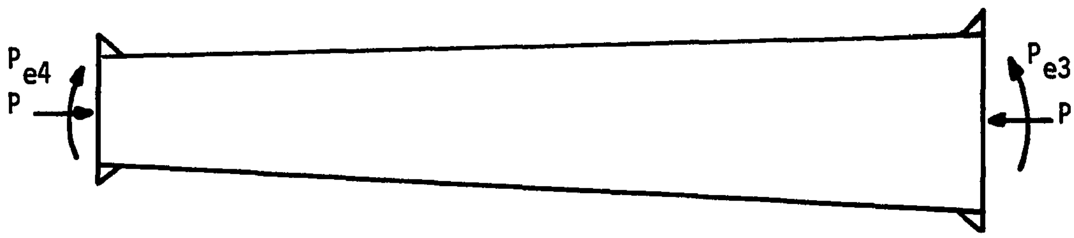


Figure 3.6 Side View of Test Set-Up

Loading Cases



a) Loading Case I - Beam-Type Loading

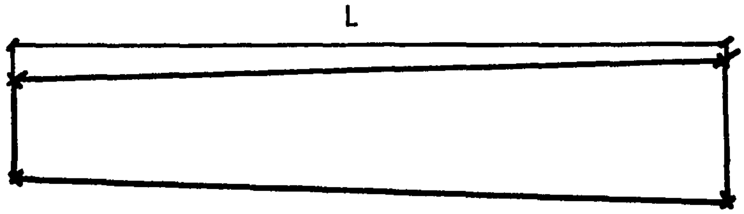


b) Loading Case II - Column-Type Loading

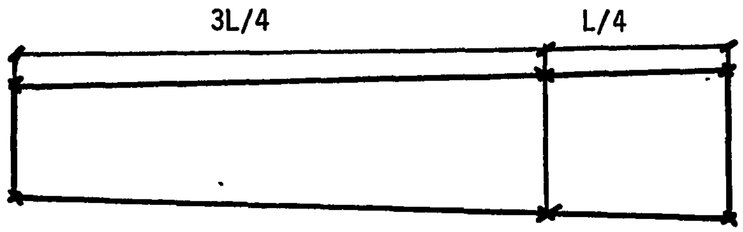
	e_1	e_2	e_3	e_4
Beam #1	77	54	77	15
Beam #2	74	30	74	15
Beam #3	73.75	57.25	73.75	14

Figure 3.7 Different Loading Conditions of the Beams

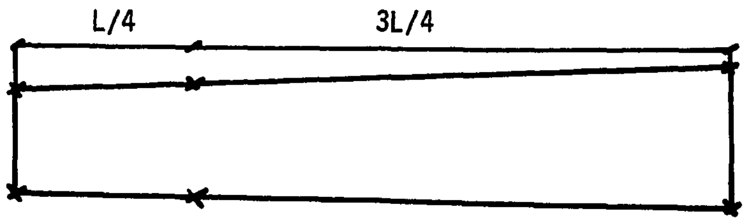
Brace Cases



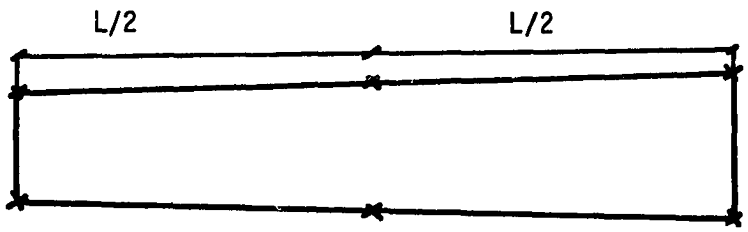
Case I - Unbraced Beam



Case II - Brace at 3/4L from the Smaller End



Case III - Brace at L/4 from the Smaller End



Case IV - Brace at the Middle

Figure 3.8 Different Brace Cases

3.3.2 Instrumentation

Instrumentation consisted of a calibrated load cell, two extensometers and several lateral deflection scales. These scales were graduated to 0.1 inch and were attached horizontally to both the top and bottom flanges of the beam section. A digital strain indicator, along with the load cell were used to monitor the applied load. Compressive stress in the top flange was measured with the extensometers, a switch and balance unit and a digital strain indicator. A transit was set in a fixed position to measure the lateral deflection of the beams with its telescope free to move only in a vertical plane.

3.3.3 Testing Procedure

Prior to testing, all testing apparatus and instrumentation were checked and calibrated and zero readings were recorded. The load was increased incrementally until near the buckling load. At each increment, the lateral deflection and the compressive stresses in the upper flange were recorded. When lateral deflections increased considerably, the maximum load was recorded as the failure load and the load was then removed. In all tests, the beams remained elastic and no indication of yielding was observed at any point along the members.

A total of 18 tests were performed on three beams. The various loading conditions and bracing cases are shown in Figures 3.7 and 3.8. Results of the experimental investigation were compared to the buckling load predicted by the BASP 15 computer program using the finite element method[80] and also with the buckling loads obtained using the present program. In using the present program, since both the axial load and the bending moment are variables, a trial and error procedure was followed.

Beam #1

Loading Case	Bracing Case	BASP 15	Present Program	Failure Load
1	1	2.90	2.72	3.50
	2	6.92	7.40	7.75
	3	6.60	6.38	6.75
	4	9.81	9.48	9.75
2	1	4.02	3.65	4.75
	2			
	3	7.70	7.57	7.75
	4			

Beam #2

Loading Case	Bracing Case	BASP 15	Present Program	Failure Load
1	1	2.57	1.86	4.0
	2			
	3			
	4			
2	1	3.03	2.1	4.25
	2			
	3	6.30	5.07	6.78
	4			

Beam #3

Loading Case	Bracing Case	BASP 15	Present Program	Failure Load
1	1	2.76	2.03	4.2
	2	6.57	6.08	7.5
	3	6.11	5.45	6.9
	4	9.04	7.79	9.7
2	1	4.12	2.81	4.75
	2	10.14	10.39	9.75
	3	7.30	6.55	8.0
	4	11.74	10.35	13.5

Table 3.3 The Results of the Experimental and Analytical Investigations

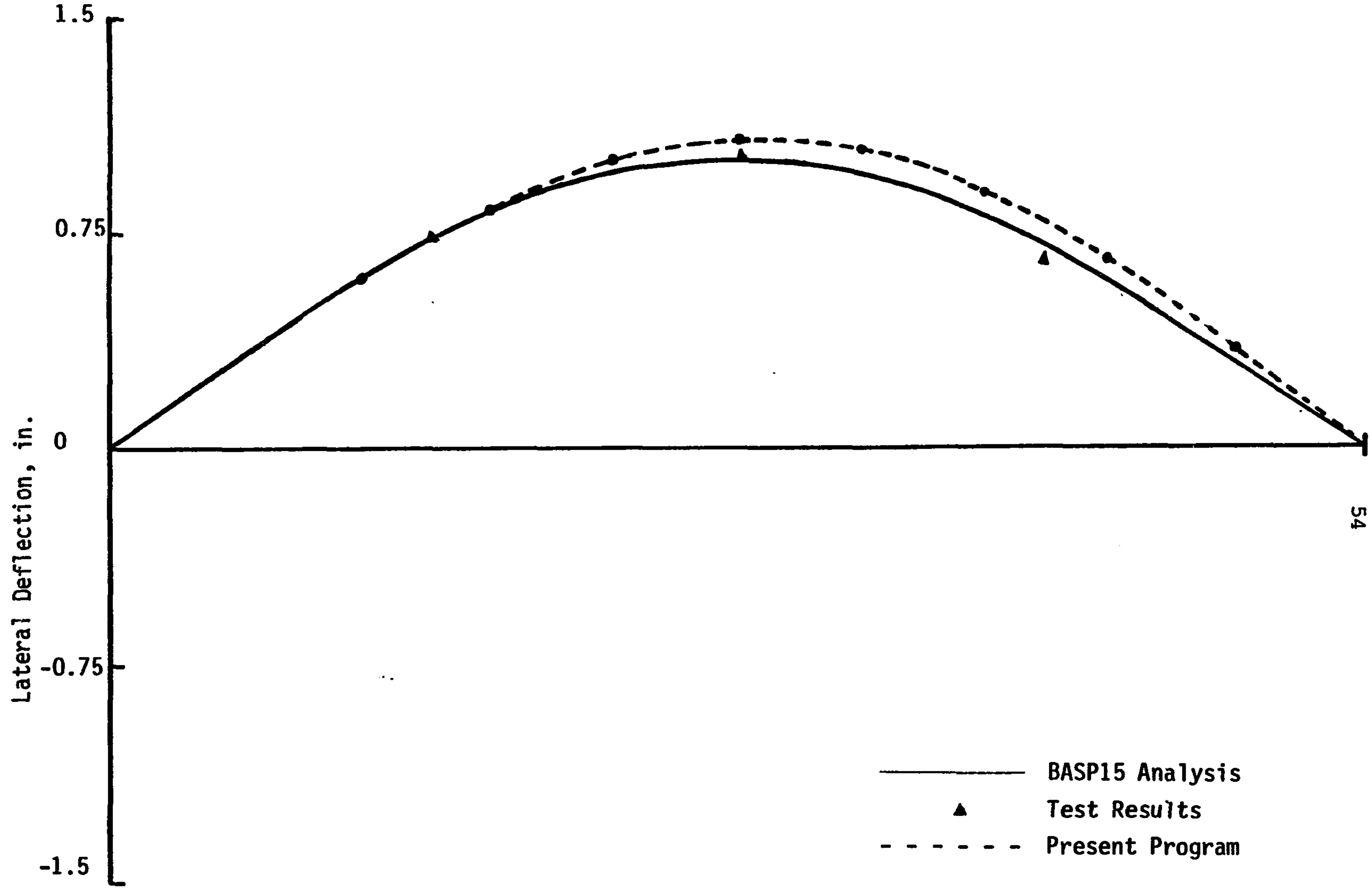


Figure 3.9 Test 1-I: Buckled Shape From BASP15 Analysis

Only two or three trial values of the axial load were needed to obtain a bending moment with the desired eccentricity. The values of the failure loads obtained from the tests and also those found by using the finite element procedure and the present program are presented in Table 3.3. In the case of members with intermediate bracings, each member is analyzed using 8 segments so the braced points coincide with the nodes in the present program. The lateral deflection of the centerline of the Beam 1-Loading 1 - Brace Case 1 from all three methods is shown in Figure 3.9.

3.3.4 Discussion of the Test Results

The critical buckling loads obtained through tests are considerably higher than what was predicted by the two different methods of analysis. This difference arises from the boundary conditions of the beams. Bolting the ends of the beams on top and bottom to the end plates has virtually caused torsional fixity at the ends, while analysis was performed assuming torsionally simple ends which means that the ends must be free to warp. The results obtained by the finite element method (BASP 15 program) and the finite difference method (present study) are in close agreement and also the deflected shape which is found using all three methods, shows good agreement between the results.

3.4 General Remarks and Conclusions

The following conclusions have been reached through the studies reported in this chapter:

1. The present program is comparatively efficient with respect to CPU time and storage requirements.

2. The results obtained using the present procedure are in good agreement with previous studies. It is shown that the simplification of the governing equations, as is commonly done, tends to achieve results which are slightly on the unconservative side.
3. Comparison with closed form solutions, showed that the finite difference method gives buckling loads which are 1 to 2 percent on the conservative side. To obtain more accurate loads, it is possible to change the number of segments and extrapolate the results.
4. There is no indication that the P-Δ moments affect the critical LTB loads of the beam-columns studied. Little difference was found in results obtained neglecting such moments as compared to methods that include the effects.
5. Using the elastic core of partially yielded sections in the governing equations can yield unconservative results. The contribution of the yielded portions in reducing the buckling load can become significant, especially for short members.
6. Since the end conditions were not properly provided, the results of the experimental investigation give higher buckling loads than what was predicted using the present program. However, the results obtained by the finite element method verify the accuracy of the results obtained through this study.

CHAPTER IV

DEVELOPMENT OF A DESIGN METHODOLOGY

4.1 General

In this chapter, the results of the analysis performed by the present computer program are used to develop a design guideline for tapered members having unequal flanges and subjected to axial load and end moments. A design example is given to illustrate the design procedure.

4.2 Design Procedure

For the development of this design methodology, the results of an extensive study by Lee and Hsu[35] are used and the same parameters are chosen. In the above reference, tapered members subjected to axial load and one end moment acting at the larger end were studied. In the present study, an attempt is made to include the effect of a second end moment acting at the smaller end and the relative variation of the magnitude of these moments. Since the number of parameters affecting the solution of the problem is comparatively large, common values for some parameters were assumed based on statistical studies. The effect of the other parameters is then investigated by specifying values. The total number of constant and variable parameters is large enough to define the geometry of the member.

Based on the studies performed by Lee and Hsu, the following section properties and parameters were used in this study:

Web thickness = 0.1 in.

Dimensions of the tension flange = 6 in. x 0.25 in.

Total depth of the section at the smaller end = 12 in.

$E = 29000$ ksi $\qquad G = 11200$ ksi $\qquad F_y = 45$ ksi

$$\gamma = \text{Taper ratio} = \frac{d_L - d_o}{d_o} = 0, 2, 4$$

The ratio of the area of the compression flange to that of the tension flange =

$$\frac{A_2}{A_1} = \frac{b_2 t_2}{b_1 t_1} = 1, 2$$

$$\rho = \frac{b_2/t_2}{b_1/t_1} = 1, 2$$

$$\frac{L}{r_{yo}} = 30, 60, 90, 120$$

The four cross-sections (at the smaller end) shown in Figure 4.1 were studied. Three taper ratios and four values for length of the members were used. It was assumed that the larger moment acts at the larger end of the member and the larger flange is in compression at that end. It was also assumed that lateral-torsional simple boundary conditions exist at the ends. The member was divided into 10 segments for the finite difference analysis and a value of $\sigma_{rc} = 0.5\ F_y$ was used for the maximum compressive residual stresses at the tips of the compression flange.

In order to include the effect of the moment acting at the smaller end, the practical conditions of using such members must be considered.

Tapered beam-columns are generally used in frames where the smaller end of the member resists either a small moment or zero moment. Based on this, the range of the variation of moment at the smaller end is assumed not to exceed the following limits: A minimum of zero and a maximum moment that produces a stress equal to the stress produced at the larger end at the centerline of the compression flange (Basic Case). Results for these two loading conditions will be presented in graphical form. Interpolation of the values is suggested for other types of the moment diagrams.

The same interaction equation proposed by Lee and Hsu is used in this study, that is, of the form:

$$\frac{P}{P_{cr}} + \delta \frac{M}{M_{cr}} = 1 \qquad (4.1)$$

In which, P and M are the applied axial load and bending moment, respectively. P_{cr} is the critical axial load in the absence of bending moment and M_{cr} is the critical moment at the larger end in the absence of axial load. The term, δ, is the moment modification factor, values of which are computed for different cross-sections and loading conditions in the following.

To transform Equation (4.1) into LRFD format, resistance factors must be used and the values of P and M must be obtained from factored loads. Using the resistance factors, Equation (4.1) is written as:

$$\frac{P}{\phi P_{cr}} + \delta \frac{M}{\phi M_{cr}} = 1 \qquad (4.2)$$

As suggested by Galambos and his research team[75], $\phi = 0.86$ is appropriate in lateral-torsional buckling.

To determine the values of P_{cr}, Figure 4.2 was developed using the

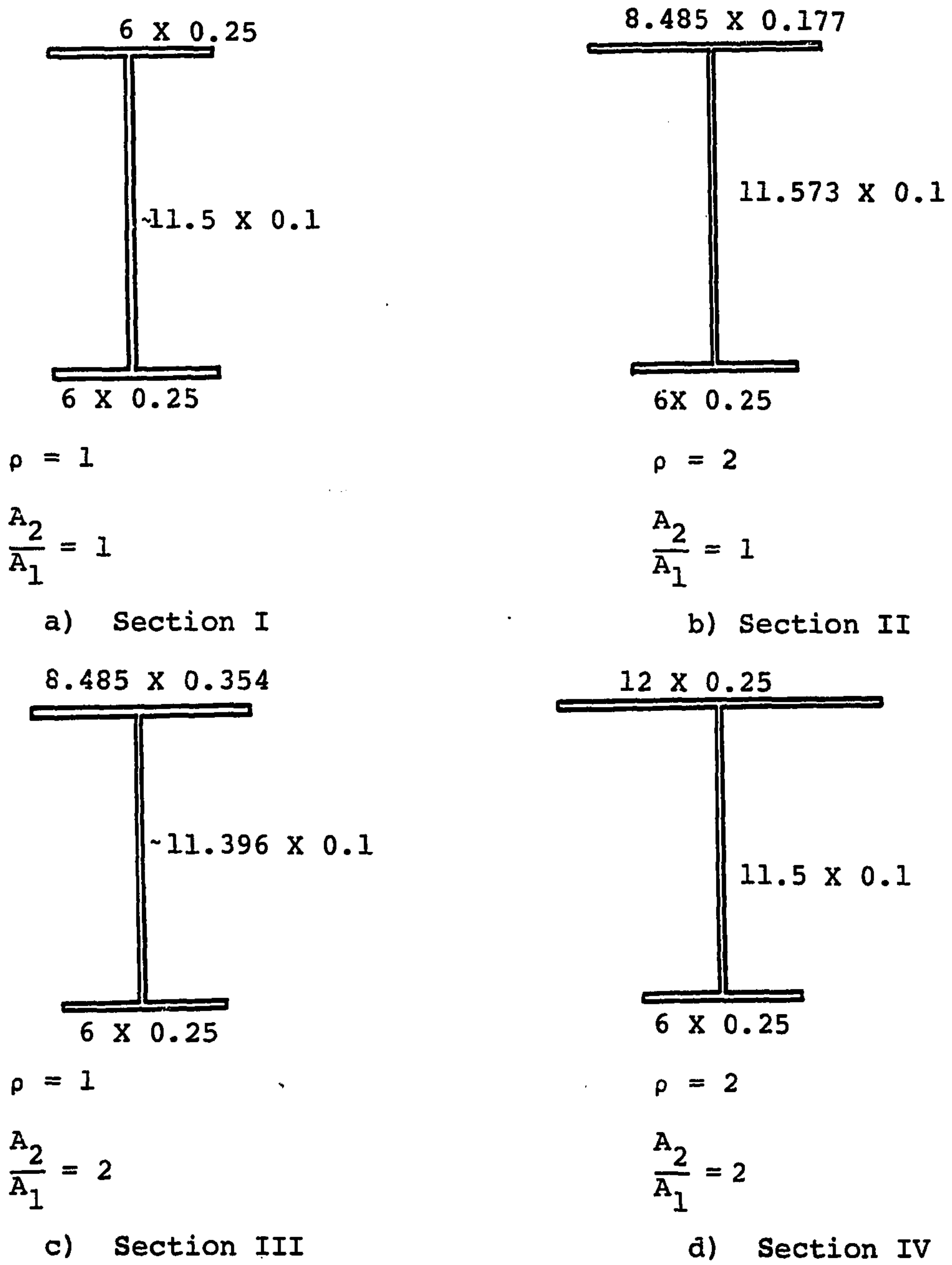


Figure 4.1. Four cross sections used to develop the design criteria.

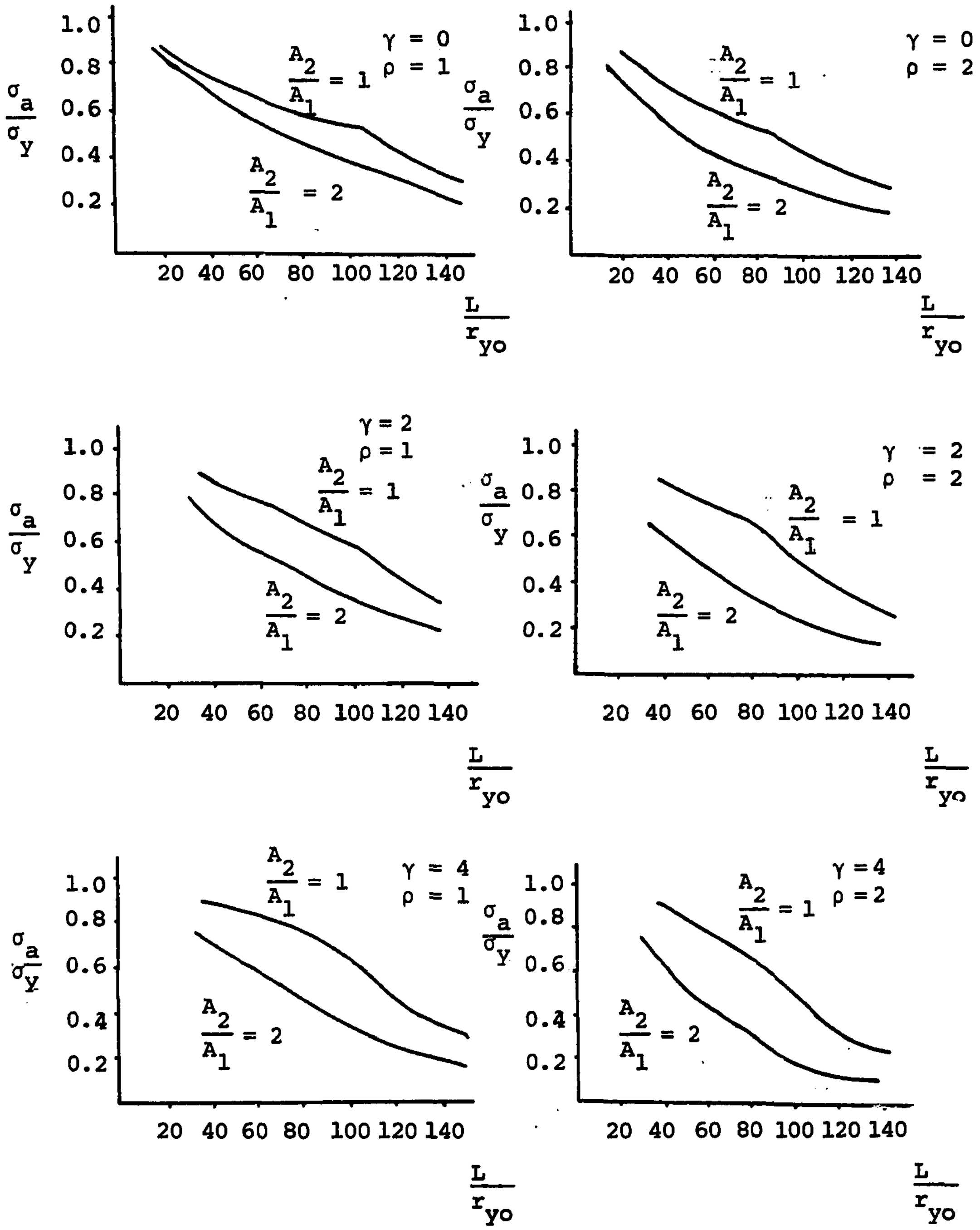


Figure 4.2. Graphs for determining the critical stress for axially loaded monosymetric tapered columns.

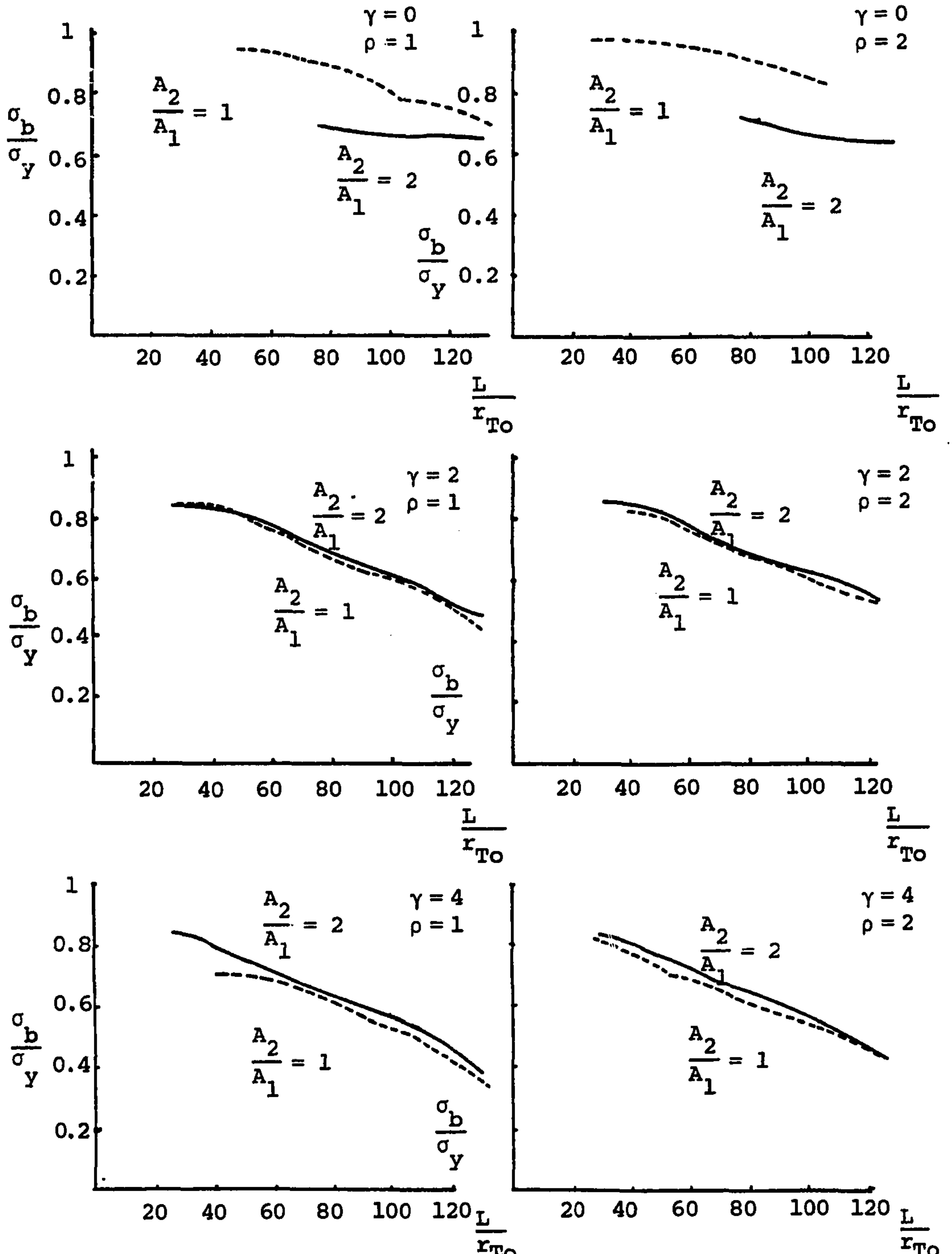


Figure 4.3. Graphs for Determining the Critical Stress in the Centerline of the Compression Flange of the Beams Subjected to one end moment.

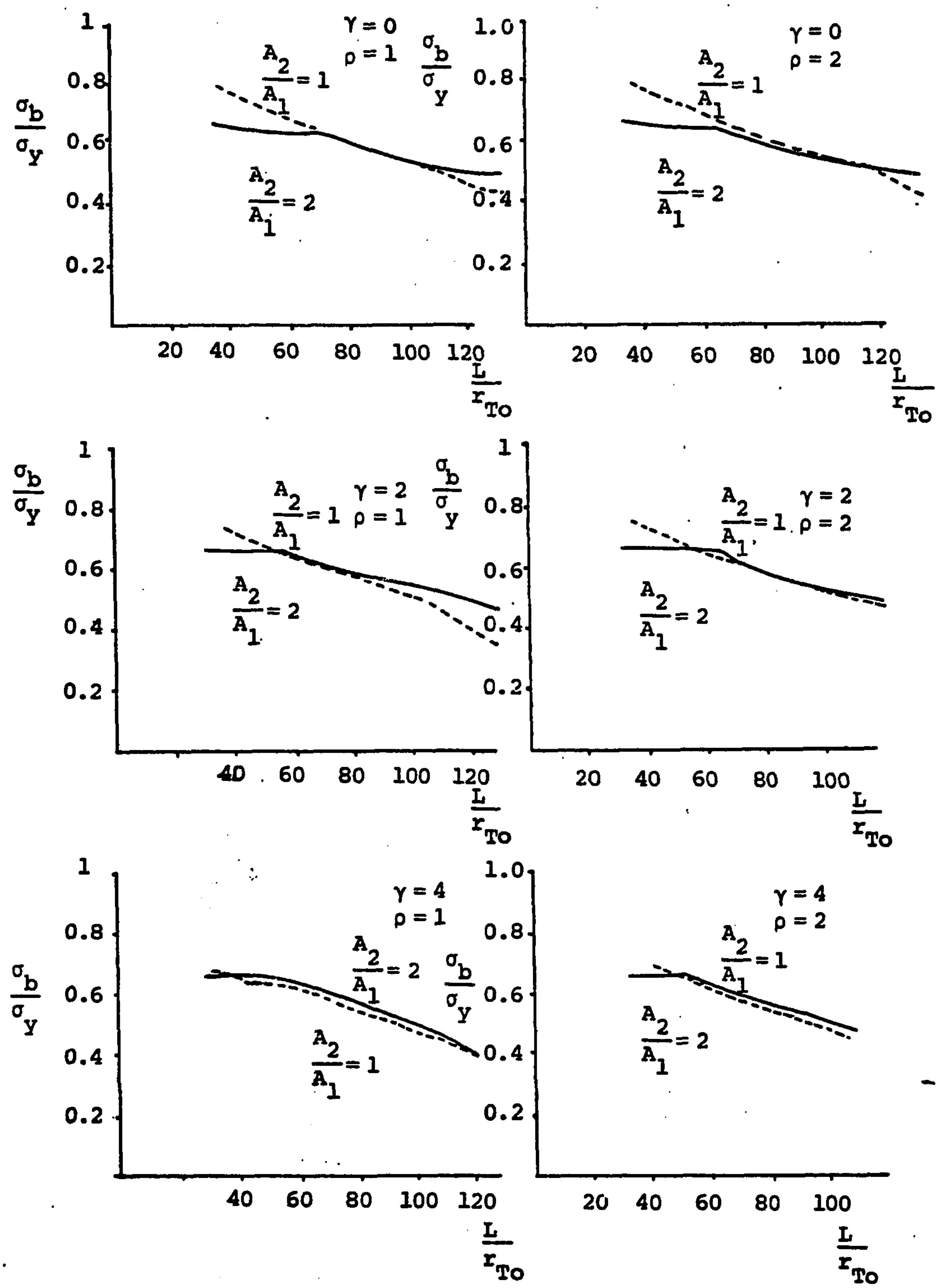


Figure 4.4. Graphs for Determining the Critical Stress in the Centerline of the Compression Flange of the Beams Subjected to Two end Moments (Basic Case)

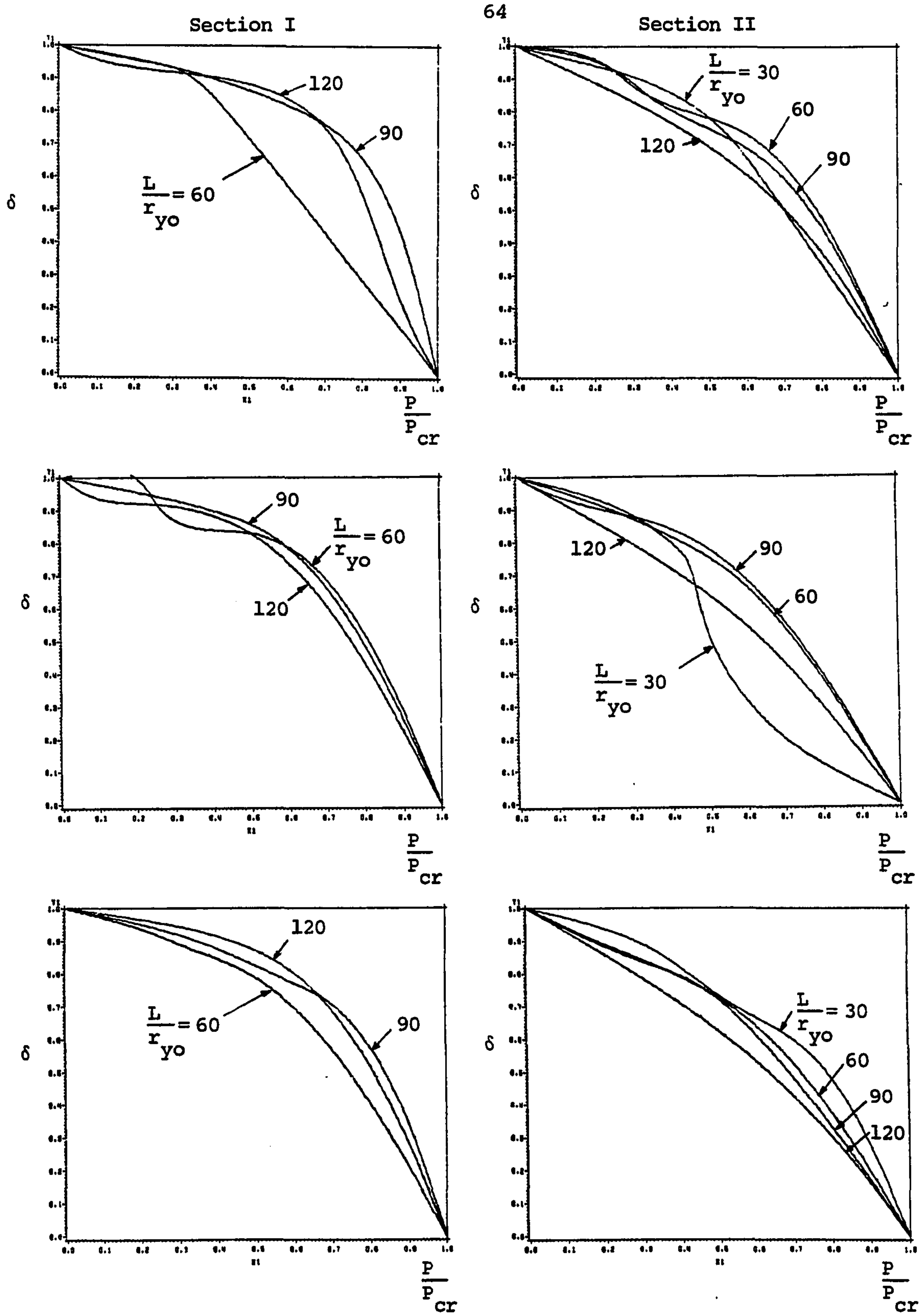


Figure 4.5. Graphs for Determining the Moment Modification Factor (One End Moment)

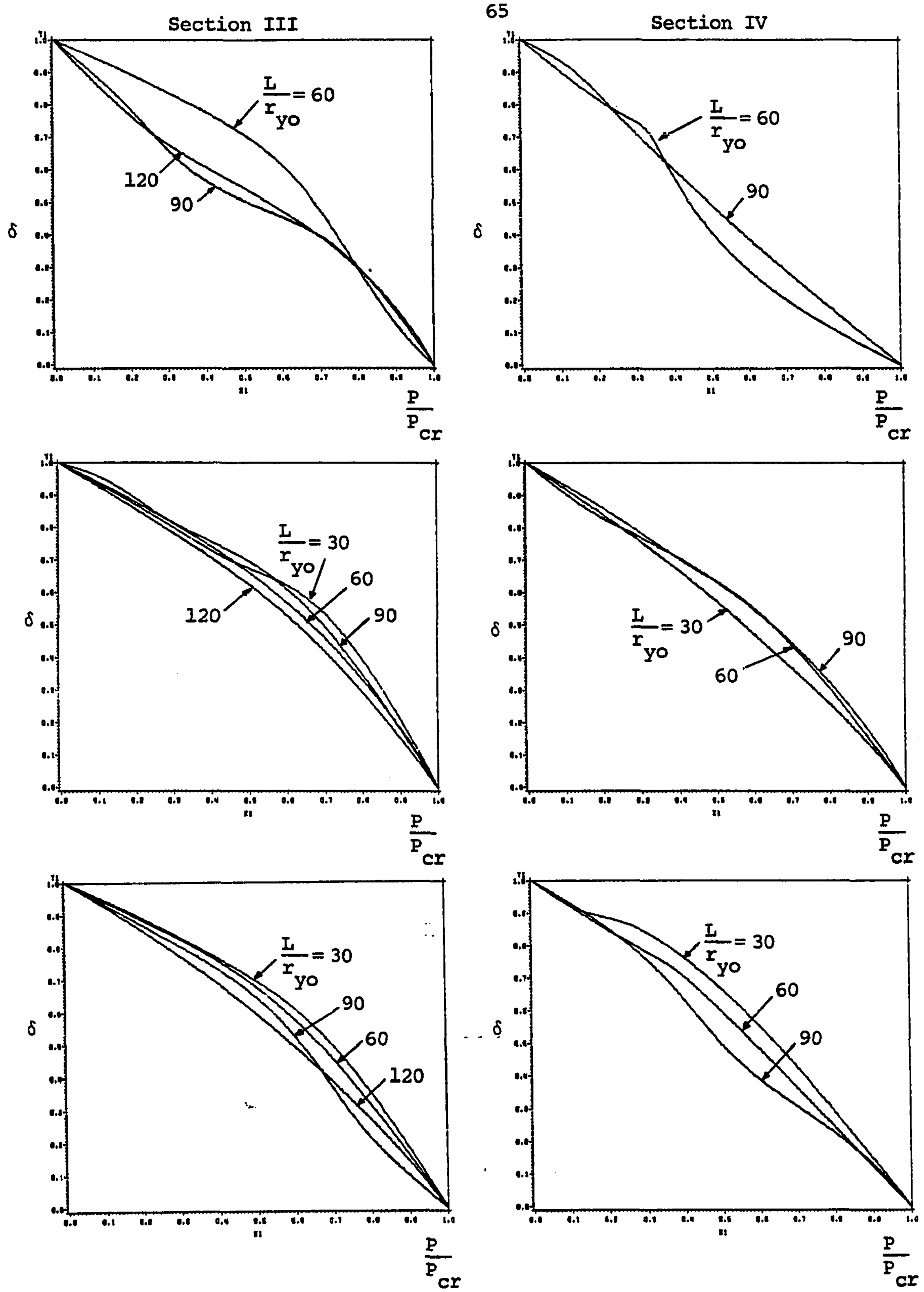


Figure 4.5 Continued

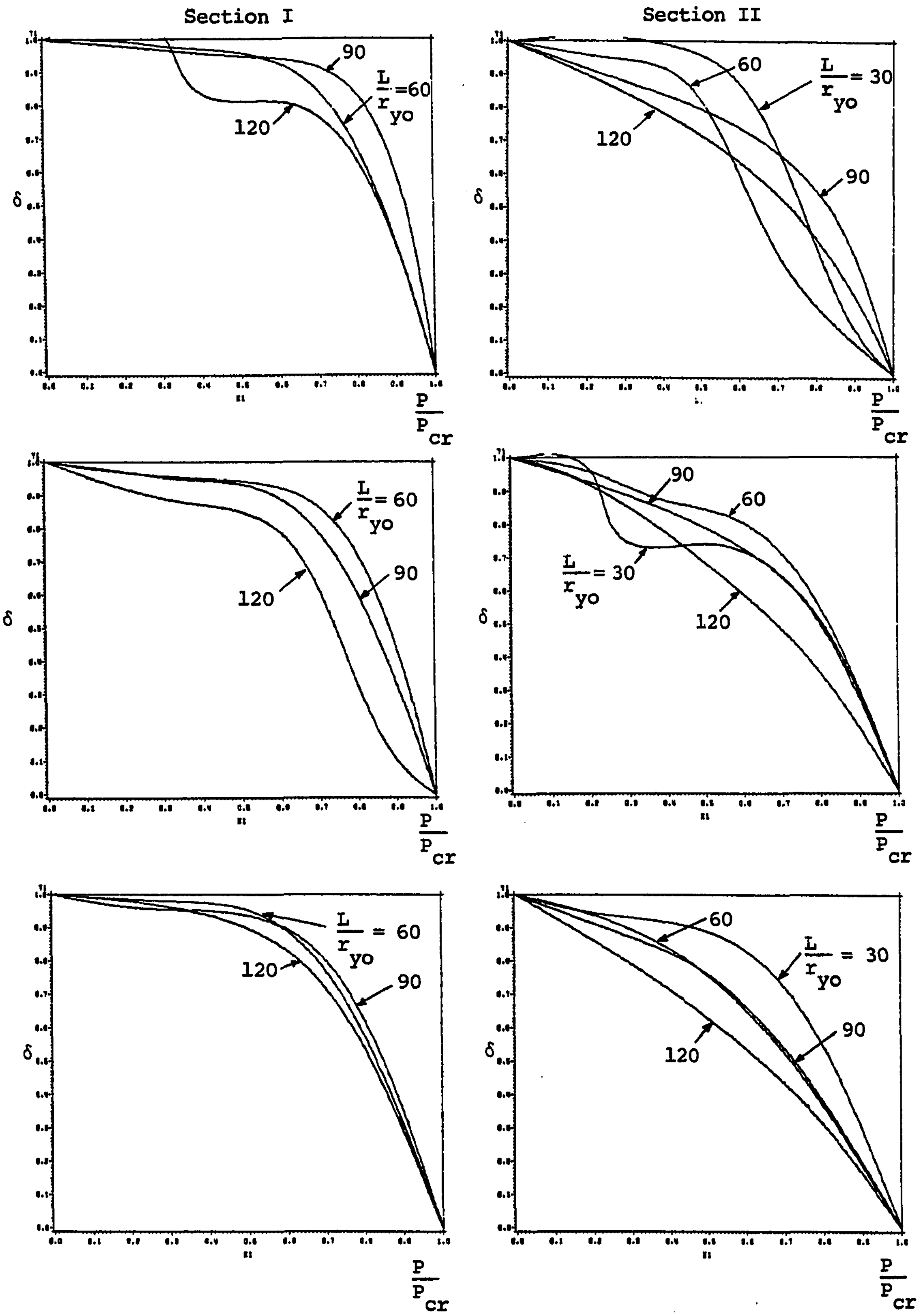


Figure 4.6. Graphs for Determining the Moment Modification Factor (Basic Case)

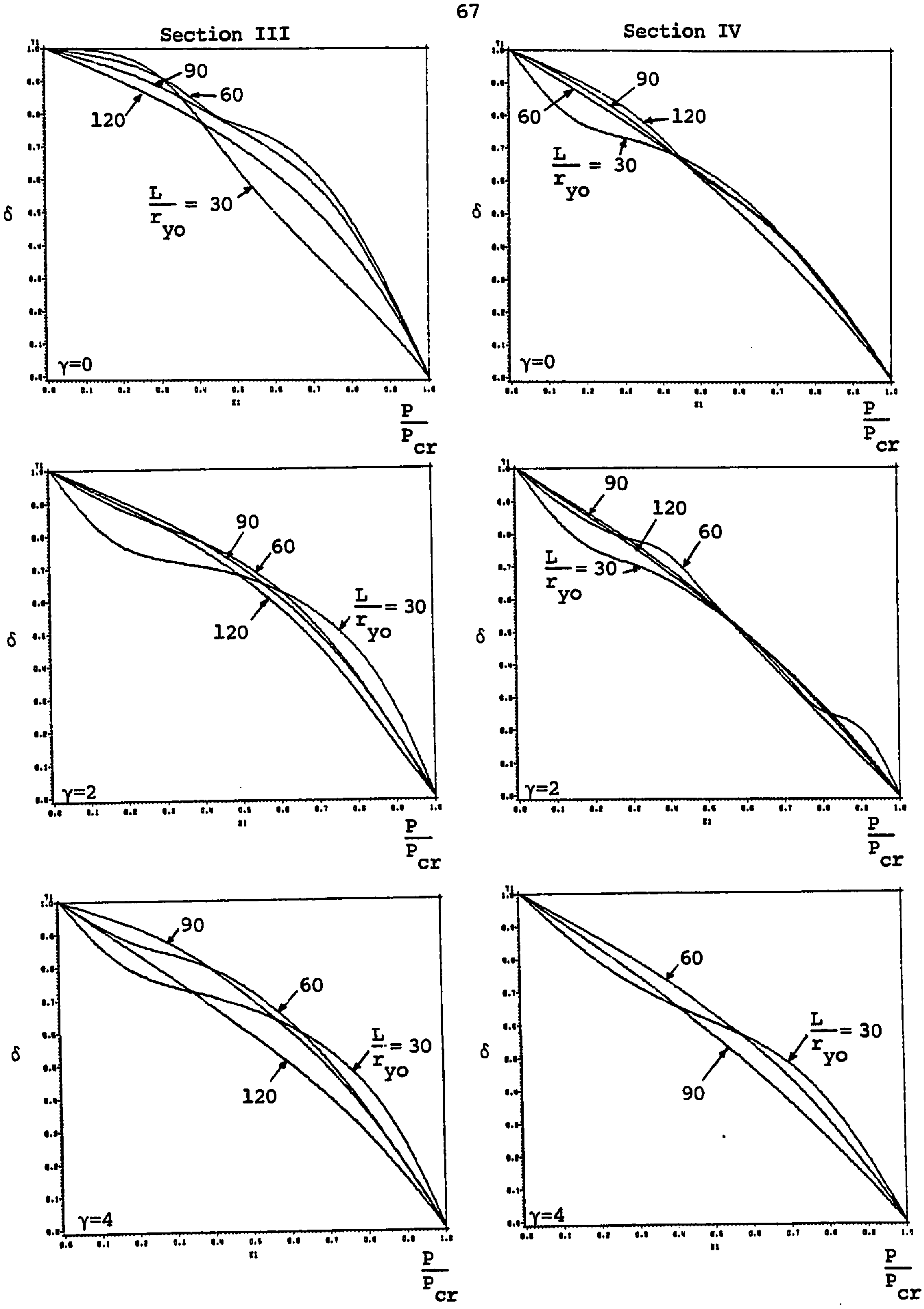


Figure 4.6. Continued

computer program developed through this study. For a concentrically loaded column, the critical axial load was determined and the critical stress at the smaller end was computed as a percentage of the yield stress. The critical stress is plotted versus ℓ/r_{yo} length divided by the radius of gyration about the weak axis at the smaller end. Curves are given for four cross-sections, three taper ratios, two ratios of the areas of the flanges and two different values of $\rho = (b_2/t_2)/(b_1/t_1)$. The present computer program was also used to determine the critical moment of the beams for the following cases: one end moment acting at the larger end (Figure 4.3), and end moments producing equal stresses in the compression flange at both ends, e.g. the basic case (Figure 4.4). The compressive stress at the centerline of the compression flange at the larger end is computed as a percentage of the yield stress and plotted versus ℓ/r_{To} which is the parameter used in the AISC specifications. The term r_{To} is the radius of gyration of the compression flange plus one third of the web, with respect to y axis, at the smaller end.

The values of the moment modification factor are determined from Equation (4.1). Rearranging terms in this equation, it is written as:

$$\delta = \frac{1-P/P_{cr}}{M/M_{cr}} \tag{4.3}$$

The curves previously discussed were used to find the values of P_{cr} and M_{cr}, and for different values of P, the corresponding values of M were determined using the program. Curves are plotted for four values of ℓ/r_{yo} as mentioned before. These curves specify the values of δ versus the applied load as a percentage of the critical load (Figures 4.5 and 4.6). It is shown that the values of the moment modification factor vary between unity for a beam and zero for a column.

Summarizing the discussion made above, the following limitations must be considered using the design criteria proposed in this study:

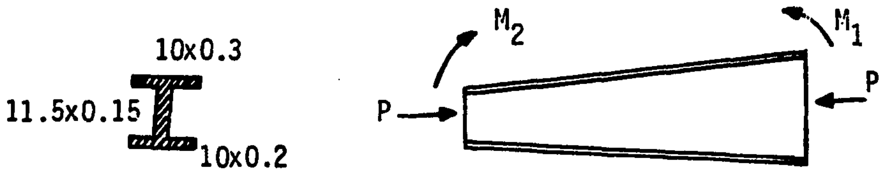
1. The larger moment is acting at the larger end of the tapered member and the larger flange is in compression.

2. The compressive stress due to bending at the smaller end does not exceed that at the larger end. This assumption implies that the moment at the smaller end remains reasonably small.

The data used to develop the design criteria is given in Appendix E. Linear interpolation and extrapolation methods are suggested to be used for problems involving parameters other than those used in the present criteria. The following example illustrates the application of the proposed design method.

4.3 Design Example

The beam-column shown in Figure 4.7 is to be checked for the given loading conditions. It is assumed that the specified loads are factored and the LRFD criteria is used. Interpolation between the curves shown in Figures 4.2 through 4.6 is necessary to obtain results.



F_y = 60 ksi

γ = 1.5

L = 200 in.

M_1 = 2000 in.-k

M_2 = 150 in.-k

$\frac{A_2}{A_1}$ = 1.5

A_o = 6.725 in^2

ρ = 0.67

I_y = 41.67 in^4

D_o = 11.75 in

r_{yo} = 2.49 in.

$\frac{L}{r_{yo}}$ = 80.3

$\bar{y}_o$ = 6.74 in.

I_{xo} = 187.75 in^4

S_{To} = 37.475 in^3

D_L = 29.375 in

$\bar{y}_L$ = 16.25 in

I_{xL} = 1372.5 in^4

S_{TL} = 104.57 in^3

r_{To} = 2.64 in

$\frac{L}{r_{To}}$ = 75.76

Figure 4.7 Design Example Data

Solution:

Step I. Critical Axial Load

Using the parameters of the problem and Figure 4.2, Table 4.1 is developed. The critical axial stress is obtained for a value of L/r_{yo} = 80.3 at α = 0, 2; p = 1, 2 and A_2/A_1 = 1, 2, 1.5 from Figure 4.2. Results are given in the first 4 rows of Table 4.1. Interpolating between rows 1 and 2, and rows 3 and 4, results in rows 5 and 6, respectively. Interpolating between rows 5 and 6 results in row 7. Finally, the values on row 7 are interpolated to result in row 8. From this procedure, a critical axial load equal to 232^k is obtained. An exact analysis of the same member was done using the present computer program. The critical axial load was found to be 214.65^k, a difference of 8%.

Step II. Critical Bending Moment

Tables 4.2 and 4.3 were developed to determine the critical stress due to moment only. Figures 4.3 and 4.4 were used and the same order of interpolation as was done for critical axial load was followed. For the example problem, having a moment ratio of 0.075, interpolation is done between 0.715 (moment ratio of zero), and 0.6 (moment ratio of S_{To}/S_{TL} = 0.36). The result is:

$$\sigma_{cr} = 0.69\ F_y$$

and

$$M_{cr} = 0.69\ S_{TL}F_y = 0.69 \times 104.57 \times 60 = 4329 \text{ in-k}$$

The exact value of the critical moment, obtained from the present computer program is 4110.8 in-k, a 5.3% difference.

Table 4.1

Critical Stresses in Axial Loading

Row No.	γ	ρ	A_2/A_1	σ_a/F_y	γ	ρ	A_2/A_1	σ_a/F_y
1	0	2	1	0.54	2	2	2	0.32
2	0	2	2	0.34	2	2	1	0.63
3	0	1	1	0.59	2	1	2	0.44
4	0	1	2	0.46	2	1	1	0.67
5	0	2	1.5	0.44	2	2	1.5	0.475
6	0	1	1.5	0.525	2	1	1.5	0.555
7	0	0.67	1.5	0.553	2	0.67	1.5	0.582
8	1.5	0.67	1.5	0.575				

$$P_{cr} = A_o\sigma_a = 6.725 \times 0.575 \times 60 = 232 \text{ k}$$

Table 4.2
Critical Bending Stresses (one end moment)

Row No.	γ	ρ	A_2/A_1	σ_b/F_y	γ	ρ	A_2/A_1	σ_b/F_y
1	0	1	1	0.89	2	1	1	0.68
2	0	1	2	0.68	2	1	2	0.71
3	0	2	1	0.91	2	2	1	0.69
4	0	2	2	0.72	2	2	2	0.70
5	0	1	1.5	0.785	2	1	1.5	0.695
6	0	2	1.5	0.815	2	2	1.5	0.695
7	0	0.67	1.5	0.775	2	0.67	1.5	0.695
8	1.5	0.67	1.5	0.715				

Table 4.3
Critical Bending Stresses (Basic Case)

Row No.	γ	ρ	A_2/A_1	σ_b/F_y	γ	ρ	A_2/A_1	σ_b/F_y
1	0	1	1	0.63	2	1	1	0.59
2	0	1	2	0.62	2	1	2	0.6
3	0	2	1	0.63	2	2	1	0.6
4	0	2	2	0.62	2	2	2	0.6
5	0	1	1.5	0.625	2	1	1.5	0.595
6	0	2	1.5	0.625	2	2	1.5	0.6
7	0	0.67	1.5	0.625	2	0.67	1.5	0.593
8	1.5	0.67	1.5	0.6				

Table 4.4

Moment Modification Factor (one end moment)

Row No.	γ	ρ	A_2/A_1	δ	γ	ρ	A_2/A_1	δ
1	0	1	1	0.71	2	1	1	0.92
2	0	1	2	0.95	2	1	2	0.79
3	0	2	1	0.86	2	2	1	0.82
4	0	2	2	0.71	2	2	2	0.75
5	0	1	1.5	0.83	2	1	1.5	0.855
6	0	2	1.5	0.785	2	2	1.5	0.785
7	0	0.67	1.5	0.845	2	0.67	1.5	0.878
8	1.5	0.67	1.5	0.87				

Table 4.5

Moment Modification Factor (Basic Case)

Row No.	γ	ρ	A_2/A_1	δ	γ	ρ	A_2/A_1	δ
1	0	1	1	0.97	2	1	1	0.96
2	0	1	2	0.82	2	1	2	0.84
3	0	2	1	0.92	2	2	1	0.88
4	0	2	2	0.78	2	2	2	0.76
5	0	1	1.5	0.895	2	1	1.5	0.9
6	0	2	1.5	0.85	2	2	1.5	0.82
7	0	0.67	1.5	0.91	2	0.67	1.5	0.927
8	1.5	0.67	1.5	0.923				

Step III. Moment Modification Factor

The values of the moment modification factor from Figures 4.5 and 4.6 are tabulated in Tables 4.4 and 4.5. A value of

$$\frac{P}{P_{cr}} = \frac{60}{6.725 \times 0.47 \times 60} = 0.315$$

is used and the same order of interpolation and extrapolation as previously defined, is followed. Interpolation is done between the values of δ specified by Tables 4.4 and 4.5, and for the desired moment diagram $\delta = 0.881$ is obtained. To find an exact value, an axial load equal to 60^k is specified and analysis is performed to find the critical moment using the present program. M = 3301 in-k is obtained and Equation 4.3 is used to find δ:

$$\delta = \frac{1-P/P_{cr}}{M/M_{cr}} = \frac{1-60/214.65}{3301/4110.8} = 0.897$$

The difference between the two values obtained for δ is less than 2%.

Equation 4.2 is to be used to check the adequacy of the member, as:

$$\frac{P}{\phi P_{cr}} + \delta \frac{M}{\phi M_{cr}} \leq 1$$

$$\frac{60}{0.86 \times 232} + 0.881 \times \frac{2000}{0.86 \times 4329} = 0.295+0.437 = 0.77 < 1$$

The design is acceptable in accordance with LRFD criteria.

CHAPTER V

SUMMARY AND CONCLUSIONS

The purpose of this study was to investigate the elastic and inelastic stability of tapered beam-columns having unequal flanges. It was assumed that the lateral-torsional buckling was the governing failure mode. A detailed literature survey was conducted to determine the scope of the investigation done by different authors on the stability of structural members. It was observed that little effort had been devoted to investigate the inelastic stability of singly-symmetric, tapered members subjected to unequal end moments and axial load. It was also noticed that approximate methods with insufficient accuracy were used to determine the yield patterns and a general formulation of the governing equations had also not been made. Present design specifications need to be modified to allow the design of tapered members having unequal flanges.

The method of solution proposed in this study is based on the minimization of the total potential energy functional and then to solve the governing equations using the finite difference technique. The total potential energy expression is derived and the effect of the residual stresses are included. The governing differential equations are derived and the boundary conditions are specified using calculus of variations. These equations are transformed into difference equations using the central difference expressions. The difference equations are satisfied at each station along the member. To find the depth of the yield penetration at each station, a new method based on the solution of the equations of equilibrium is presented. The yield pattern at each station is determined, the section properties are computed and the difference equations are satisfied. The determinant of the resulting equations is evaluated and an iterative procedure, using the method of Bisection is followed until a zero value for the determinant is obtained which specifies the critical combination of the bending moment and axial load. The buckling mode shape is then specified by assuming one of the displacements equal to one, cancelling an equation and solving the rest of the simultaneous equations using Gauss-Jordan elimination method. A computer program is developed using this procedure and it is used to analyze several different types of members having different loadings.

The results obtained using this program are in good agreement with the studies found in literature for different kinds of members and loading conditions. It is noticed that for tapered, singly-symmetric members, the neutral axis is not a straight line. It is also mentioned that in partially yielded cross-sections, the yielded portions contribute in twisting the section during lateral-torsional buckling and using the section properties of the elastic core in the governing equations can yield unconservative results. The results of a limited experimental investigation on three different members are also presented and comparisons are made with the buckling loads obtained by using the present program and those obtained through a finite element analysis.

In order to establish a design criteria for singly-symmetric, tapered beam-columns, the present program is used and an interaction equation was developed. Four different cross-sections with three different taper ratios are studied and curves are presented to determine the critical buckling stresses on the compression flange of the beams and on the cross-section of the columns. A modifying factor is introduced and for two types of moment diagrams (one end moment only and two moments introducing equal stresses at the ends), the values of this factor are given for different types of cross-sections and taper ratios. The resistance factors are used to transform the interaction equation into the limit states design format. A design example is worked using the above equation.

From the results of this study it is concluded that: 1) The yielded portions have a significant effect in LTB and they should not be disregarded. 2) The effect of the P-Δ moments is negligible for practical dimensions of the tapered members. 3) The solution methods provided in the literature yield buckling loads which are slightly in the unconservative side in the inelastic region.

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APPENDIX A

DEVELOPMENT OF THE TOTAL POTENTIAL ENERGY EXPRESSION

The strain energy for a unit volume of an elastic body is defined as[20]:

$$U_o = \frac{1}{2}(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{xz}\gamma_{xz} + \tau_{yz}\gamma_{yz}) \quad (A.1)$$

where σ_x, σ_y and σ_z are the normal stresses and τ_{xy}, τ_{xz} and τ_{yz} are the shearing stresses; ε_x, ε_y, ε_z, γ_{zy}, γ_{xz} and γ_{yz} are the corresponding normal and shearing strains. For a member subjected to uniaxial bending and axial load, normal stresses exist in the z direction only, and shear stresses exist in the xy plane. Thus, for a member of length L, the above relationship can be transformed into the following, using Hooke's Law ($\varepsilon = \frac{\sigma}{E}$ and $\gamma = \frac{\tau}{G}$):

$$U = \frac{1}{2} \int_0^L \int_0^e \left(\frac{\sigma^2}{E} + \frac{\tau^2}{G}\right) t \, ds \, dz \quad (A.2)$$

where t is the thickness, E is the modulus of elasticity and G is the shear modulus of elasticity and s is the variable defining the distance on the cross-section and e is the end point on the cross-section. The components of the normal and shearing stresses and the corresponding strain energy expressions are:

1. Normal stress due to axial load:

$$\sigma_1 = \frac{P}{A} \text{ and } U_1 = \frac{1}{2} \int_0^L \frac{P^2}{EA} \, dz \quad (A.3)$$

P is the axial load and A is the cross-sectional area which for a tapered member is a function of Z.

2. Normal stresses due to strong axis bending:

$$\sigma_2 = \frac{M_x y}{I_x} = -EyV'' \quad \text{and} \quad U_2 = \frac{1}{2} \int_0^L EI_x \, V''^2 \, dz \quad (A.4)$$

I_x is the moment of inertia w.r.t. x axis and v is the deformation in the vertical plane.

3. Normal stresses due to weak axis bending and warping:

As proved by Vankayya[28] and mentioned by Lee, Morrel and Ketter[29], the resistance to weak axis bending and warping comes from flange bending. Refering to Figure A.1, the following equations are obtained for a singly-symmetrical section:

$$\sigma_3 = Ex\ (u - \phi y_s)'' \tag{A.5}$$

$$\sigma_3 = Ex\ [u + (d - y_s)\phi] \tag{A.6}$$

$$U_3 = \frac{1}{2} \int_0^L \{EI_c\ [u + (d - y_s)\phi]''^2 + EI_t\ (u - \phi y_s)''^2\}\ dz \tag{A.7}$$

Equations (A.5) and (A.6) give the values of stresses in the tension flange and the compression flange, respectively. U is the horizontal deformation of the shear center and ϕ is the angle of rotation (Figure A.1). I_c and I_t are the moments of inertia of the compression flange and the tension flange with respect to y axis and y_s is the distance from the shear center to the tension flange.

4. Shear stresses due to torsion.

The shearing stress, resulting from torsion, can be written as:

$$\tau_{st} = 2G\phi'\tilde{\chi} \tag{A.8}$$

where $\tilde{\chi}$ is the distance from any point on the cross-section to the mid-line. The corresponding value of U is:

$$U_4 = \frac{1}{2} \int_0^L GJ\ \phi'^2\ dz \tag{A.9}$$

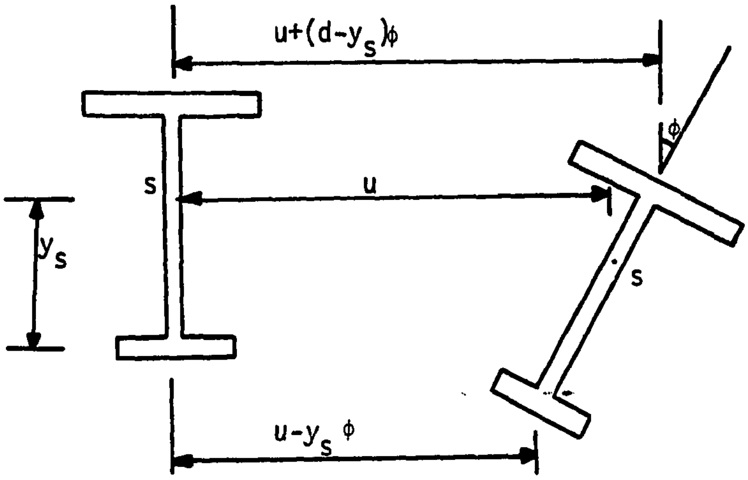

Figure A.1 The Cross-Section Before and After Buckling where J is the torsional constant. The total strain energy expression is obtained by adding together U_1 through U_4 which yields:

$$U = \frac{1}{2}\int_0^L \{\frac{P^2}{EA} + EI_x v''^2 + EI_c [U+(d-y_s)\phi]''^2 + EI_t (U-\phi y_s)''^2 + GJ\phi'^2\}dz \qquad (A.10)$$

The existing stress on the cross-section is of the following form:

$$\sigma = \frac{P}{A} + \frac{M_x y}{I_x} + R.S. \qquad (A.11) \qquad \text{and } \sigma = \sigma_y \qquad (A.12)$$

(A.11) gives the stress on the elastic core and (A.12) is the value of stress on the yielded portions which is equal to the yield stress. R.S. indicates residual stresses. The potential of the external forces can be written as[20]:

$$V_p = -\int_0^e \Delta L\sigma t ds \qquad (A.13)$$

where L, the elongation of a fiber is given as:

$$\Delta L = \int_0^L \{- y v'' - \chi U'' + \omega_n\phi'' - \frac{1}{2}[u'+\phi'(y_0-y)]^2 - \frac{1}{2}[v'-\phi(\chi_0-\chi)]^2\}dz \qquad (A.14)$$

in which ω_n is the warping function, χ_0 and y_0 are the coordinates of the shear center. Setting $\chi_0 = 0$ in Eq. (A.14) for a singly-symmetrical section and noting the following:

$$\int y\, dA = \int \chi\, dA = \int \chi y\, dA = \int y\omega_n\, dA = \int R.S.\ \omega_n\, dA = \int \omega_n\, dA = 0$$

$$\int \chi\sigma dA = \int \chi\ R.S. dA = \int \sigma_y\ \omega_n\, dA = 0 \qquad \int y^2 dA = I_\chi$$

$$\frac{\int y(\chi^2 + y^2)dA}{I_\chi} - 2y_0 = \beta_\chi \qquad \frac{I_\chi + I_y}{A} + y_0^2 = r^2 \qquad (A.15)$$

The following expression is obtained by substituting (A.14) into (A.13) and including the results of (A.15):

$$V_p = \int_0^L M_\chi \left(V''-u'\,\phi' + \frac{1}{2}\,\beta_\chi \phi'^2\right)dz - \frac{P}{2}\int_0^L \left(u'^2+v'^2+r^2\phi'^2 + 2u'\phi' y_o\right)dz$$

$$+\int_0^L \left\{\int_A (R.S.+\sigma_y)\left[y_V''+\frac{1}{2}\,u'^2+\frac{1}{2}\,v'^2+\frac{1}{2}\,\phi'^2\chi^2+\frac{1}{2}\,\phi'^2(y_o-y)^2+u'\phi'(y_o-y)\right]dA\right\} dz \qquad \text{(A.16)}$$

Everywhere in this study any integral including the term $(R.S.+\sigma_y)$ means integral of residual stresses on the elastic core and the yield stresses on the yielded portions. M_χ is the external bending moment. Since the total potential energy, π, is the sum of the strain energy and the potential of the external forces, then it can be obtained by adding equations (A.10) and (A.16), as:

$$\pi= \frac{1}{2}\int_0^L \{EI_c\,[u+(d-y_s)\phi]''^2+EI_t\,(u-y_s\phi)''^2+GJ\phi'^2+\frac{P^2}{EA} + EI_\chi\,v''^2+M_\chi v''$$

$$-M_\chi u'\phi' +M_\chi\beta_\chi\phi'^2 -P(u'^2+v'^2+r^2\phi'^2+2u'v'y_o)+\int_A (R.S.+\sigma_y)\,[\,2yv'' +$$

$$u'^2 + v'^2 +\phi'^2\chi' +\phi'^2(y_o-y)^2 + 2u'\phi'\,(y_o-y)\,]\,dA\,\}\,dz \qquad \text{(A.17)}$$

In this expression, all primes indicate differentiation with respect to z.

APPENDIX B

DEVELOPMENT OF THE DIFFERENCE EQUATIONS

Equations (2.5.6) and (2.5.7) which govern the out-of-plane behavior of a beam-column, are developed in section 2.2 and they are repeated here as:

$$\frac{d^2}{dz^2}[EI_c(d - y_s)(u - y_s\phi + d\phi)'' - EI_t\, y_s\, (u - y_s\phi)''] - \frac{d}{dz}[2(d'-y'_s)EI_c$$

$$(u - y_s\, \phi + d\, \phi)'' - 2EI_t\, y'_s\, \phi'\, (u-y_s\phi)'' + GJ\phi' - M_x u' + M_x \beta_x \phi' - Pr^2 \phi' + Pu'y_o$$

$$+ \phi' \int_A (R.S.+\sigma_y)S^2 dA + u' \int_A (R.S.+\sigma_y)(y_o-y)dA\,\} - Ey''_s[I_c\, (u-y_s\phi+d\phi)''$$

$$+ I_t\, (u - \phi y_s)''] = 0 \qquad (B.1)$$

$$\frac{d^2}{dz^2}[EI_c\, (u-y_s\phi+d\phi)'' + EI_t\, (u-y_s\phi)''] - \frac{d}{dz}[-M_x\phi' - Pu' - P\phi' y_o + u' \int_A$$

$$(R.S. + \sigma_y)\, dA + \phi' \int_A (R.S. + \sigma_y)(y_o-y)\, dA\,] = 0 \qquad (B.2)$$

Rearranging terms and noting that $I_t = I_y - I_c$ and $y_s = \frac{I_c}{I_y} d$, the above equations are simplified as:

$$\frac{d^2}{dz^2}[EI_c\, (d-y_s)\, (d\phi)''] - \frac{d}{dz}[2E(I_c d' - I_y y'_s)(u-y_s\phi)'' + 2EI_c(d'-y'_s)$$

$$(d\phi)'' + GJ\phi' - M_x u' + M_x\beta_x\phi' - Pr^2\, \phi' - Pu'y_o + \phi' \int_A (R.S. + \sigma_y)S^2 dA +$$

$$u' \int_A (R.S. + \sigma_y)(y_o-y)\, dA\,] - Ey''_s\, [I_y(u-y_s\phi)'' + I_c\, (d\phi)''] = 0 \quad (B.1.a)$$

$$\frac{d^2}{dz^2}[EI_y\, (u-y_s\phi)'' + EI_c\, (d\phi)''] + (M_x\phi'' + M'_x\phi') + Pu'' + P\,(\phi'' y_o + \phi' y'_o)$$

$$+ u'' \int_A (R.S. + \sigma_y)\, dA + u' \frac{d}{dz} \int_A (R.S. + \sigma_y)\, dA + \phi'' \int_A (R.S.+\sigma_y)(y_o-y)\, dA +$$

$$\phi' \frac{d}{dz} \int_A (R.S. + \phi_y)\ (y_o - y)\ dA = 0 \qquad \text{(B.2.a)}$$

Performing differentiations with respect to z in Eqs. (B.1.a) and (B.1.b) the following equations are obtained:

$$E\ [I_c\ (d - y_s)]''\ (d\phi)'' + 2E[I_c\ (d - y_s)]'\ (d\phi)''' + EI_c\ (d - y_s(d\phi)^{IV} -$$

$$\{2E(I_c d' - I_y y'_s)(u - \phi y_s)''' + 2E(I_c d' - I_y y'_s)'(u-\phi y_s)''+2E[I_c(d'-y'_s)]'$$

$$(d\phi)'' + 2E(I_c d' - I_c y'_s)(d\phi)''' + G(J'\phi'+J\phi'') - (M_x u''+M'_x u') + (M'_x$$

$$\beta_x \phi' + M_x \beta'_x \phi' + M_x \beta_x \phi'')-P(r^2\phi''+2rr'\phi') - P(u''y_o+ u'y'_o) +\phi'' \int_A$$

$$(R.S.+\phi_y)\ S^2 dA +\phi' \frac{d}{dz} \int_A (R.S.+\phi_y)S^2\ dA +u'' \int_A (R.S.+\phi_y)(y_o-y)\ dA$$

$$+u'\frac{d}{dz} \int_A (R.S.+\sigma_y)(y_o-y)\ dA\ \} - Ey''_x [I_y(u-y_s\phi)'' + I_c\ (d\phi)''] = 0 \quad \text{(B.1.b)}$$

$$E[I''_y\ (u''-y_s\phi-2y'_s\phi'-y_s'') + 2I'_y(u'''-y_s\phi'''-3y'_s\phi''-3y''_s\phi'-y'''_s\phi)$$

$$+ I_y(u^{IV}-y_s\phi^{IV}-4y'_s\phi'''-6y''_s\phi''-4y'''_s\ \phi'\ -y^{IV}_s\phi)+I''_c(d\phi''+2d'\phi')+2I'_c$$

$$(d\phi'''+3d'\phi'')+I_c(d\phi^{IV}+4d'\phi''')] +M_x\phi''+M'_x\phi'+Pu''+P(\phi''y_o+\phi'y'_o)$$

$$+u'' \int_A (R.S.+\sigma_y)dA+u'\frac{d}{dz} \int_A (\sigma_y+R.S.)dA+\phi'' \int_A (R.S.+\sigma_y)(y_o-y)dA+$$

$$\phi' \frac{d}{dz} \int_A (R.S.+\sigma_y)\ (y_o-y)dA = 0 \qquad \text{(B.2.b)}$$

In the above equations, for all the terms that include integrals, a sign change has taken place. This is because the derivation of the equation is done based on taking the tensile stress as positive, while the integrals for different patterns are computed by assuming compressive stress being positive. It must be noted that d" and all the higher derivatives of d are zero, because of the linear variation of d. Rearranging the terms in Eq. (B.2.a) and simplifying, yields:

$$-2E(u'''-\phi''' y_s-3\phi'' y'_s-3\phi' y''_s-\phi y'''_s)(I_c d'-I_y y'_s) - E(u''-\phi'' y_s-2\phi' y'_s$$

$$-\phi y''_s)(2I'_c d'-I_y y''_s-2I_y y'_s)+EI_c(d-y_s)(d\phi^{IV}+4d'\phi''')+2EI'_c$$

$$(d-y_s)(d\phi'''+3d'\phi'')+EI''_c(d-y_s)(d\phi''+2d'\phi')-G(J'\phi'+J\phi'')+$$

$$M_x u''+M'_x u'-(M'_x\beta_x\phi'+M_x\beta'_x\phi'+M_x\beta_x\phi'')+P(r^2\phi''+2rr'\phi')+$$

$$P(u''y_o+u'y'_o)+\phi''\int_A(R.S.+\sigma_y)S^2dA+\phi'\frac{d}{dz}\int_A(R.S.+\sigma_y)S^2dA+u''$$

$$\int_A(R.S.+\sigma_y)(y_o-y)dA+u'\frac{d}{dz}\int_A(R.S.+\sigma_y)(y_o-y)dA = 0 \qquad \text{(B.1.c)}$$

All derivatives of u and ϕ are substituted by central difference expressions, which are[79]:

$$u' = \frac{u_{j+1} - u_{j-1}}{2h} \qquad \text{(B.3.a)}$$

$$u'' = \frac{u_{j+1} - 2u_j + u_{j-1}}{h^2} \qquad \text{(B.3.b)}$$

$$u''' = \frac{-u_{j-2}+2u_{j-1}-2u_{j+1}+u_{j+2}}{2h^3} \qquad \text{(B.3.c)}$$

$$u = \frac{u_{j-2}-4u_{j-1}+6u_j-4u_{j+1}+u_{j+2}}{h^4} \qquad \text{(B.3.d)}$$

$$\phi' = \frac{\phi_{j+1} - \phi_{j-1}}{2h} \qquad \text{(B.3.e)}$$

$$\phi'' = \frac{\phi_{j+1} - 2\phi_j + \phi_{j-1}}{h^2} \qquad \text{(B.3.f)}$$

$$\phi''' = \frac{-\phi_{j-2} + 2\phi_{j-1} - 2\phi_{j+1} + \phi_{j+2}}{2h^3} \qquad \text{(B.3.g)}$$

$$\phi^{IV} = \frac{\phi_{j-2} - 4\phi_{j-1} + 6\phi_j - 4\phi_{j+2} + \phi_{j+2}}{h^4} \qquad \text{(B.3.h)}$$

Substituting (B.3.a) through (B.3.h) into equations (B.2.b) and (B.1.c) yields two equations of the following forms:

$$A_1 u_{j-2}+A_2\phi_{j-2}+A_3 u_{j-1}+A_4 \phi_{j-1}+A_5 u_j+A_6 \phi_j+A_7 u_{j+1}+A_8 \phi_{j+1}+A_9 u_{j+2}$$

$$+ A_{10} \phi_{j+2} = 0 \qquad \text{(B.4)}$$

$$B_1 u_{j-2}+B_2\phi_{j-2}+B_3 u_{j-1}+B_4 \phi_{j-1}+B_5 u_j+B_6\phi_j+B_7 u_{j+1}+B_8\phi_{j+1}+B_9 u_{j+2}$$

$$+B_{10} \phi_{j+2} = 0 \qquad \text{(B.4.2)}$$

where the values of A_1 through A_{10} and B_1 through B_{10} are:

$$A_1 = - E(I'_c d - I'_y y_s)/h^3 \qquad \text{(B.5.1)}$$

$$A_2 = Ey_s(I'_c d-I'_y y_s)/h^3 + EI_c d(d-y_s)/h^4 - 2EI_c d'(d-y_s)/h^3 - EI'_c$$

$$d(d-y_s)/h^3 \qquad \text{(B.5.2)}$$

$$A_3 = 2E(I'_c d - I'_y y_s)/h^3 + E(I''_c d - I''_y y_s)/h^2 + (M_x + P y_o + \int_A (R.S.+\sigma_y)(y_o - y)\,dA$$
$$)/h^2 - (M'_x + P y'_o + \frac{d}{dz}\int_A (R.S.+\sigma_y)(y_o - y)\,dA)/2h \tag{B.5.3}$$

$$A_4 = -2E y_s (I'_c d - I'_y y_s)/h^3 - 6E y'_s (I'_c d - I'_y y_s)/h^2 + 3E y''_s (I'_c d - I'_y y_s)$$
$$/h - E y_s (I''_c d - I''_y y_s)/h^2 + E y'_s (I''_c d - I''_y y_s)/h - 4E I_c d(d - y_s)/h^4$$
$$+ 4E I_c d'(d - y_s)/h^3 + 2E I'_c d(d - y_s)/h^3 + 6E I'_c d'(d - y_s)/h^2 +$$
$$E I''_c d(d - y_s)/h^2 - E I''_c d'(d - y_s)/h + GJ'/2h - GJ/h^2 + (M'_x \beta_x + M_x \beta'_x)/2h - M_x \beta_x/h^2$$
$$+ (Pr^2 + \int_A (R.S.+\sigma_y) S^2 dA)/h^2 - Prr'/h - (\frac{d}{dz}\int_A (R.S.+\sigma_y) S^2 dA)/2h \tag{B.5.4}$$

$$A_5 = -2E(I''_c d - I''_y y_s)/h^2 - 2M_x/h^2 - 2P y_o/h^2 - 2\int_A (R.S.+\sigma_y)(y_o - y)\,dA/h^2 \tag{B.5.5}$$

$$A_6 = 12E y'_s (I'_c d - I'_y y_s)/h^2 - 2E y_s''' (I'_c d - I'_y y_s) + 2E y_s (I''_c d - I''_y y_s)/h^2$$
$$- E y''_s (I''_c d - I''_y y_s) + 6E I_c d(d - y_s)/h^4 - 12E I'_c d'(d - y_s)/h^2 - 2E I''_c d(d - y_s)/h^2$$
$$+ 2GJ/h^2 + 2M_x \beta_x/h^2 - 2Pr^2/h^2 - 2\int_A (R.S.+\sigma_y) S^2 dA/h^2 \tag{B.5.6}$$

$$A_7 = -2E(I'_c d - I'_y y_s)/h^3 + E(I''_c d - I''_y y_s)/h^2 + (M_x + P y_o + \int_A (R.S.+\sigma_y)(y_o - y)\,dA$$
$$)/h^2 + (M'_x + P y'_o + \frac{d}{dz}\int_A (R.S.+\sigma_y)(y_o - y)\,dA)/2h \tag{B.5.7}$$

$$A_8 = 2Ey_s(I'_c d-I'_y y_s)/h^2 - 6Ey'_s(I'_c d-I'_y y_s)/h^2 - 3Ey''_s(I'_c d-I'_y y_s)$$

$$/h - Ey_s(I''_c d-I''_y y_s) - Ey'_s(I''_c d-I''_y y_s)/h - 4EI_c d(d-y_s)/h^4 -$$

$$4EI_c d'(d-y_s)/h^3 - 2EI'_c d(d-y_s)/h^3 + 6EI'_c d'(d-y_s)/h^2 + EI''_c d(d-y_s)/$$

$$h^2 + EI''_c d'(d-Y_s)/h - GJ'/2h - GJ/h^2 - (M'_\chi \beta_\chi + M_\chi \beta'_\chi)/2h - M_\chi \beta_\chi/h^2$$

$$+(Pr^2 + \int_A (R.S.+\sigma_y)S^2 dA)/h^2 + Prr'/h + (\frac{d}{dz}\int_A (R.S.+\sigma_y)S^2 dA)/2h \qquad \text{(B.5.8)}$$

$$A_9 = E(I'_c d - I'_y y_s)/h^3 \qquad \text{(B.5.9)}$$

$$A_{10} = -Ey_s(I'_c d-I'_y y_s)/h^3 + EI_c d(d-y_s)/h^4 + 2EI_c d'(d-y_s)/h^3 + EI'_c d(d-y_s)/h^3$$

(B.5.10)

$$B_1 = EI_y/h^4 - EI'_y/h^3 \qquad \text{(B.5.11)}$$

$$B_2 = EI'_y y_s/h^3 + 2EI_y y'_s/h^3 - EI'_c d/h^3 - 2EI_c d'/h^3 \qquad \text{(B.5.12)}$$

$$B_3 = EI''_y/h^2 + 2EI'_y/h^3 - 4EI_y/h^4 + (P + \int_A (R.S.+\sigma_y)dA)/h^2 - \frac{d}{dz}\int_A (R.S.+\sigma_y)dA/2h$$

B.5.13)

$$B_4 = EI''_y y'_s/h - EI''_y y_s/h^2 - 2EI'_y y_s/h^3 - 6EI'_y y'_s/h^2 + 3EI'_y y''_s/h$$

$$-4EI_y y'_s/h^3 - 6EI_y y''_s/h^2 + 2EI_y y'''_s/h + EI''_c d/h^2 - EI''_c d'/h$$

$$+2EI'_c d/h^3 + 6EI'_c d'/h^2 + 4EI_c d'/h^3 + M_\chi/h^2 - M'_\chi/2h + Py_o/h^2 - Py'_o/2h$$

$$+\int_A (R.S.+\sigma_y)(y_o-y)\,dA/h^2-\frac{d}{dz}\int_A (R.S.+\sigma_y)\,dA/2h \qquad (B.5.14)$$

$$B_5=-2EI''_y/h^2+6EI_y/h^4-2P/h^2-2\int_A (R.S.+\sigma_y)\,dA/h^2 \qquad (B.5.15)$$

$$B_6=-EI''_y y''_s+2EI''_y y_s/h^2+12EI'_y y'_s/h^2-2EI'_y y'''_s+12EI_y y''_s/h^2-EI_y y^{IV}_s$$

$$-2EI''_c d/h^2-12EI'_c d'/h^2-2M_\chi/h^2-2Py_o/h^2-2\int_A (R.S.+\sigma_y)(y_o-y)\,dA/h^2 \qquad (B.5.16)$$

$$B_7=EI''_y/h^2-2EI'_y/h^2-4EI_y/h^4+P/h^2+\int_A (R.S.+\sigma_y)\,dA/h^2+$$

$$\left(\frac{d}{dz}\int_A (R.S.+\sigma_y)\,dA\right)/2h \qquad (B.5.17)$$

$$B_8=-EI''_y y'_s/h-EI''_y y_s/h^2+2EI'_y y_s/h^3-6EI'_y y'_s/h^2-3EI'_y y''_s/h+4E$$

$$I_y y'_s/h^3-6EI_y y''_s/h^2-2EI_y y'''_s/h+EI''_c d/h^2+EI''_c d'/h-2EI'_c d/h^3+6E$$

$$I'_c d'/h^2-4EI_c d'/h^3+M_\chi/h^2+M'_\chi/2h+Py_o/h^2+Py'_o/2h+\int_A (R.S.+\sigma_y)(y_o-y)\,dA/h^2$$

$$+\left(\frac{d}{dz}\int_A (R.S.+\sigma_y)(y_o-y)\,dA/2h \qquad (B.5.18)$$

$$B_9 = EI''_y/h^3+EI_y/h^4 \qquad (B.5.19)$$

$$B_{10}=-EI'_y y_s/h^3-2EI_y y'_s/h^3+EI'_c d/h^3+2EI_c d'/h^3 \qquad (B.5.20)$$

In the above equations $h=\frac{L}{n}$ and $S^2=\chi^2+(y-y_o)^2$. The rest of the parameters are defined in Appendix A.

APPENDIX C

THE YIELD PATTERNS

In this Appendix, the different yield patterns along with their yield parameters and the equations of equilibrium are presented. Ten different yield patterns are recognized which, along with the elastic case, (Case 1), are considered to be all the possible patterns. (Figure 2.8).

For a doubly symmetrical section subjected to bending and axial compression, yielding will always start at the extreme ends of the compression flange (Case 2). Additional loading will cause either complete yielding of the compression flange plus part of the web (Case 11), or partial yielding of the tension flange (Case 3). Depending on the dimensions of the cross-section, increasing the load will cause one of the two cases: Complete yielding of the compression flange plus part of the web and partial yielding of the tension flange (Case 6), or complete yielding of the tension flange plus part of the web and partial yielding of the compression flange (Case 7). The ultimate pattern consists of both flanges completely yielded plus parts of the web on both sides (Case 5).

Partial yielding of the tension flange and web, (Case 4), may occur when the compression flange is larger than the tension flange and the amount of axial force is low. When the difference between the areas of the two flanges is large, the complete yielding of the lower flange (Case 10), is a possibility. In any event, these two cases rarely happen within the practical size limitations and they are encountered only in the case of low axial load and singly symmetrical sections.

High values of axial load and small bending moment can cause the partial yielding of both flanges in compression (Case 8). Increasing the moment in this case will cause the complete yield of the compression flange and part of the web (Case 9).

For each yield pattern, the yield parameters are computed in terms of the curvature of the section, the location of the neutral axis and the values of the residual stresses. Referring to Figure (5), the equation representing the variation of the residual stresses across the upper flange can be written as:

$$\sigma = \frac{2x}{b_2} (\sigma_{rc} + \sigma_{rt}) - \sigma_{rt} \tag{C.1}$$

and similarly for the lower flange:

$$\sigma = \frac{2x}{b_1} (\sigma_{rb} + \sigma_{rt}) - \sigma_{rt} \tag{C.2}$$

where x is measured from the centerline of the web to both sides and compressive stress is positive.

In order to find the yield parameters, at the point where yielding starts, sum of the stresses due to loading and the residual stresses is set equal to the yield stress. For example, for the yield pattern of Case (2), in order to find k, the yield parameter, the following procedure is followed: The compressive stress due to loading in the upper flange would be equal to $E\theta C$ and the residual stress where yielding starts is:

$$\frac{2}{b_2} \left(\frac{k}{2} b_2\right)(\sigma_{rc} + \sigma_{rt}) - \sigma_{rt} = k(\sigma_{rc} + \sigma_{rt}) - \sigma_{rt} \tag{C.3}$$

Sum of these two stresses must equal σ_y, so the following relationship is obtained:

$$E\theta C + k(\sigma_{rc} - \sigma_{rt}) = \sigma_y \tag{C.4}$$

from which k is found.

The two equations of equilibrium are derived on the same basis. The forces are summed in a direction perpendicular to the cross-section. These forces consist of the following components: axial force, the yield stresses, the residual stresses and the stresses caused by loading. The moments are summed about the neutral axis (point of zero stress due to loading). In both equations, sum of the internal forces and moments are set equal to external axial force and bending moment. For the elastic section (Case 1), since the section properties do not depend on the curvature, the equations of equilibrium have not been written.

Case 1

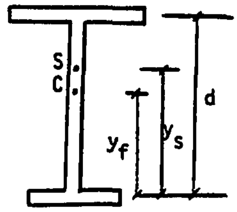

Figure C.1 Case 1 yield pattern

Section properties

$$I_c = \frac{t_2 b_2^3}{12} \qquad I_y = \frac{t_2 b_2^3}{12} + \frac{t_1 b_1^3}{12}$$

$$Y_s = \frac{I_c d}{I_y} \qquad y_f = d(A2 + \tfrac{1}{2} A_3)/(A_1 + A_2 + A_3) \qquad Y_o = Y_f - Y_s$$

$$I_x = A_1 y_f^2 + A2(d-y_f)^2 + \frac{d^3 t3}{12} + A3\ (y_f - \frac{d}{2})^2$$

$$J = \frac{1}{3}\ (B_1 t_1^3 + b_2 t_2^3 + b_3 t_3^3) \qquad \text{(unchanged for different patterns)}$$

$$\beta_x = -[t_2 b_2^3 (D-y_f)/_{12} - t_1 b_1^3 y_f/_{12} + A2(D-y_f)^3 - A_1 y_f^3 + (\frac{t_3}{4})((D-y_f)^4$$

$$-y_f^4)]/_{I_x} - 2y_o$$

$$\int(R.S. + \sigma_y) S^2 dA = t_1 b_1^3 (\sigma_{rb} - \frac{\sigma_{rt}}{3})/_{16} + t_2 b_2^3 (\sigma_{rc} - \frac{\sigma_{rt}}{3})/_{16} - (\frac{t_3}{3})$$

$$\sigma_{rt}[(D-y_f)^3 + y_f^3] + A_1 (D-y_f)^2 (\sigma_{rb} - \sigma_{rt})/_2 + A_2 y_f^2 (\sigma_{rc} - \sigma_{rt})/_2$$

$$\int(R.S. + \sigma_y)\ dA = 0$$

$$\int(R.S. + \sigma_y)\ (y_o - y)\ dA = 0$$

Case 2

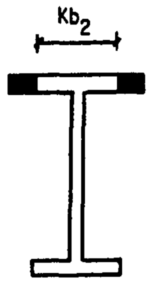

Figure C.2 Case 2 yield pattern

1. Section properties

$$I_c = \frac{t_2(Kb_2)^3}{12} \qquad I_y = \frac{t_2(Kb_2)^3}{12} + \frac{t_1 b_1^3}{12}$$

$$y_s = \frac{I_c d}{I_y} \qquad y_b = d(KA2 + \tfrac{1}{2}A3)/(A1+KA2+A3)$$

$$I_\chi = KA_2(d-y_b)^2 + A_1 y_b^2 + t_3 b_3^3/_{12} + A_3(y_b - d/2)^2$$

$$\beta_\chi = [-t_2(Kb_2)^3(d-y_b)/_{12} + y_b t_1 b_1^3/_{12} - KA_2(d-y_b)^3 + y_b^3 A_1 + (t_{3/4})(y_b^4$$
$$-(d-y_b)^4)]/_{I_\chi} - 2y_o$$

$$\int(R.S.+\sigma_y)S^2 dA = A_1[b_1^2(\sigma_{rb}+\sigma_{rt})/_{16} - b_1^2\ rt/_{12} + y_s^2(\sigma_{rb}+\sigma_{rt})/_2 - \sigma_{rt} y_s^2]$$
$$-\sigma_{rt} t_3[y_s^3 + (d-y_s)^3]/_3 + A_2[b_1^2 k^4(\sigma_{rc}+\sigma_{rt})/_{16} - b_1^2 k^3 \sigma_{rt}/_{12} + (d-y_s)^2$$
$$k^2(\sigma_{rc}+\sigma_{rt})/_2 - \sigma_{rt} k(d-y_s)^2] + \sigma_y A_2[b_2^2(1-k^3)/_{12} + (d-y_s)^2(1-k)]$$

$$\int(R.S.+\sigma_y)dA = A_2\sigma_y(1-k) - \sigma_{rt}A_3 + 0.5A_1(\sigma_{rb}-\sigma_{rt}) + A_2[0.5k^2(\sigma_{rc}+\sigma_{rt}) - k\sigma_{rt}]$$

$$\int(R.S.+\sigma_y)(y_o - y)dA = \sigma_y A_2(1-k)(d-y_s) + (d-y_s)A_2[0.5k^2(\sigma_{rc}+\sigma_{rt}) - k\sigma_{rt}]$$
$$-0.5A_1 y_s(\sigma_{rb}-\sigma_{rt}) + 0.5t_3\sigma_{rt}[y_s^2 - (d-y_s)^2]$$

2. The yield parameter: $k = (\sigma_y + \sigma_{rt} - E\theta C)/(\sigma_{rc}+\sigma_{rt})$

3. Equations of equilibrium:

$$P = A\sigma_y(1-k)-A_3\sigma_{rt}+E\theta CkA_2-A_1E\theta(d-c)+0.5t_3E\theta C^2-0.5t_3E\theta$$

$$(d-c)^2+0.5A_1(\sigma_{rb}-\sigma_{rt})+A_2[0.5k^2(\sigma_{rc}+\sigma_{rt})-k\sigma_{rt}]$$

$$M = -P(y_f+C-d)-0.5A_1(d-c)(\sigma_{rb}-\sigma_{rt})+CA_2[0.5k^2(\sigma_{rc}+\sigma_{rt})-k\sigma_{rt}]$$

$$+A_3\sigma_{rt}(d/_2-c)+E\theta C^2kA_2+A_1E\theta(d-c)^2+(t_3/_3)E\theta C^3+(r_3/_3)E\theta$$

$$(d-c)^3+A_2\sigma_yC(1-k)$$

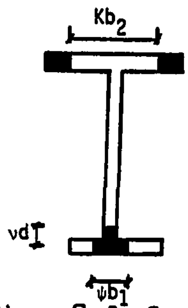

Figure C.3. Case 3 yield pattern

Case 3

1. Section properties

$$I_c=\frac{t_2(kb_2)^3}{12} \qquad I_y=\frac{t_2(kb_2)^3}{12}+\frac{t_1b_1^3}{12}(1-\psi^3)$$

$$Y_s=\frac{I_c d}{I_y} \qquad y_b=[kA_2d+\tfrac{1}{2}dA_3(1-\nu^2)]/[kA_2+A_3(1-\nu)+A_1(1-\psi)]$$

$$I_x= kA_2(d-y_b)^2+y_b^2A_1(1-\psi)+(\frac{t_3}{12})(d-\nu d)^3+t_3(d-\nu d)[y_b-\nu d-0.5(d-\nu d)]^2$$

$$\beta_x=[(\frac{t_3}{4})((y_b-\nu d)^4-(d-y_b)^4)+y_b^3A_1(1-\psi)-kA_2(d-y_b)^3-k^3(d-y_b)\frac{t_2b_2^3}{12}$$

$$+y_b(1-\psi^3)\frac{t_1b_1^3}{12}]/I_x-2y_o$$

$$\int(R.S.+\sigma_y)S^2dA=\sigma_yA_2[b_2^2(1-k^3)/_{12}+(1-k)(d-y_s)^2]+A2[b_2^2k^4(\sigma_{rc}-\sigma_{rt})/_{16}$$

$$+(d-y_s)^2(\sigma_{rc}+\sigma_{rt})\frac{k^2}{2}-\sigma_{rt}b_2^2\,k^3/_{12}-\sigma_{rt}(d-y_s)^2k]-\sigma_{rt}t_3[(y_s-\nu d)^3+(d-y_s)^3]/_3$$

$$-\sigma_yt_3[y_s^3-(y_s-\nu d)^3]/_3-\sigma_yA_1(\frac{b_1^2\psi^3}{12}+\psi y_s^2)+A_1\,b_1^2(\sigma_{rb}+\sigma_{rt})(1-\psi^4)/_{16}$$

$$-\sigma_{rt}b_1^2(1-\psi^3)/_{12}]+A_1[\frac{y_s^2}{2}(1-\psi^2)(\sigma_{rb}+\sigma_{rt})-\sigma_{rt}y_s^2\,(1-\psi)]$$

$$\int(R.S.+\sigma_y)dA=A_2[0.5k^2(\sigma_{rc}+\sigma_{rt})-\sigma_{rt}k]+A_1[0.5\,(\sigma_{rb}+\sigma_{rt})(1-\psi^2)$$

$$-\sigma_{rt}(1-\psi)]-A_3\sigma_{rt}(1-\nu)-\nu A_3\sigma_y-\psi A_1\sigma_y+A_2\sigma_y(1-k)$$

$$\int(R.A.+\sigma_y)(y_o-y)dA=A_2[0.5k^2(\sigma_{rc}+\sigma_{rt})-k\sigma_{rt}](d-y_s)-A_1y_s[0.5(\sigma_{rb}+\sigma_{rt})$$

$(1-\psi^2)-\sigma_{rt}(1-\psi)]+\sigma_y\psi A_1 y_s+A_2(1-k)(d-y_s)\sigma_y+\sigma_y t_3[y_s{}^2-(y_s-\nu d)^2]/_2$

$-\sigma_{rt}t_3[(y_s-\nu d)^2-(d-y_s)^2]/_2$

2. <u>Yield parameters</u>:

$k = (\sigma_y+\sigma_{rt}-E\theta C)/(\sigma_{rc}+\sigma_{rt})$

$\nu = [\sigma_{rt}-\sigma_y+E\theta(d-c)]/_{E\theta d}$

$\psi = [E\theta(d-c)+\sigma_{rt}-\sigma_y]/_{(\sigma_{rb}+\sigma_{rt})}$

3. <u>Equations of equilibrium</u>:

$P=\frac{1}{2}A_1[(\sigma_{rt}+\sigma_{rb})(1-\psi^2)-2\sigma_{rt}(1-\psi)]+A_2[0.5k^2(\sigma_{rc}+\sigma_{rt})-k\sigma_{rt}]$

$+E\theta CkA_2-E\theta(d-c)A_1(1-\psi)+\frac{1}{2}\ t_3E\theta C^2-\frac{1}{2}t_3E\theta(d-c-\nu d)^2$

$-\sigma_{rt}A_3(1-\nu)+A_2\sigma_y(1-k)-\nu A_3\sigma_y-\psi A_1\sigma_y$

$M=-P(y_f+c-d)-A_1(d-c)[0.5(\sigma_{rt}+\sigma_{rb})(1-\psi^2)-\sigma_{rt}(1-\psi)]$

$+A_2C[0.5k^2(\sigma_{rc}+\sigma_{rt})-k\sigma_{rt}]+E\theta C^2kA_2+E\theta A_1(d-c)\ \ (1-\psi)$

$+({}^{t}3/_3)E\theta C^3+({}^{t}3/_3)E\theta(d-c-\nu d)^3-\sigma_{rt}(1-\nu)[C-0.5d(1-\nu)]A_3$

$+A_2(1-k)\sigma_yC+\nu A_3\sigma_y(d-c-0.5\nu d)+\psi A_1\sigma_y(d-c)$

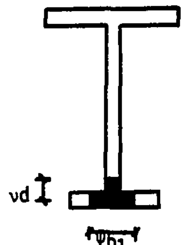

Figure C.4 Case 4 yield pattern

Case 4

1. Section properties

$$I_c = t_2 b_2^{\,3}/12 \qquad I_y = I_c + \frac{t_1 b_1^{\,3}}{12}(1-\psi^3)$$

$$y_s = \frac{dI_c}{I_y} \qquad y_b = [A_2 d + 0.5A_3 d(1-\nu^2)]/[A_2 + A_3(1-\nu) + A_1(1-\psi)]$$

$$I_x = A_2(d-y_b)^2 + y_b^{\,2}A_1(1-\psi) + (t_3/12)(d-\nu d)^3 + t_3(d-\nu d)(y_b - 0.5(d-\nu d))^2$$

$$\beta_x = \{(t_3/4)[(y_b-\nu d)^4 - (d-y_b)^4] + y_b^{\,3}A_1(1-\psi) - A_2(d-y_b)^3 - (d-y_b)t_2 b_2^{\,3}/12$$

$$+ y_b(1-\psi^3)t_1 b_1^{\,3}/12\}/I_x - 2y_o$$

$$\int(R.S.+\sigma_y)S^2 dA = A_2[b_2^{\,2}(\sigma_{rc}+\sigma_{rt})/16 + 0.5(d-y_s)^2(\sigma_{rc}+\sigma_{rt}) - \sigma_{rt}\,b_2^{\,2}/12$$

$$-\sigma_{rt}(d-y_s)^2] - \sigma_{rt}t_3[(y_s-\nu d)^3 + (d-y_s)^3]/3 - \sigma_y[y_s^{\,3} - (y_s^{\,3} - (y_s-\nu d)^3]t_3/3$$

$$-\sigma_y A_1(\psi^3 b_1^{\,2}/12 + y_s^{\,2}\psi) + A_1[(\sigma_{rb}+\sigma_{rt})b_1^{\,2}(1-\psi^4)/16 - \sigma_{rt}b_1^{\,2}(1-\psi^3)/12 +$$

$$(\sigma_{rt}+\sigma_{rb})y_s^{\,2}(1-\psi^2)/2 - \sigma_{rt}y_s^{\,2}(1-\psi)]$$

$$\int(R.S.+\sigma_y)dA = 0.5A_2(\sigma_{rc}-\sigma_{rt}) + A_1[0.5(\sigma_{rb}+\sigma_{rt})(1-\psi^2) - \sigma_{rt}(1-\psi)] - A_3\sigma_{rt}(1-\nu)$$

$$-\nu A_3\sigma_y - \psi A_1\sigma_y$$

$$\int(R.S.+\sigma_y)(y_o-y)dA = A_2(\sigma_{rc}-\sigma_{rt})(d-y_s)/2 + A_1\sigma_y y_s\psi - A_1[(\sigma_{rb}+\sigma_{rt})(1-\psi^2)/2$$

$$-\sigma_{rt}(1-\psi)] + \sigma_y t_3[(\nu d-y_s)-^2 y_s^{\,2}]/2 + \sigma_{rt}t_3[(d-y_s)^2 + (\nu d-y_s^{\,2})]$$

2. <u>The yield parameters</u>:

$$\psi=[E\theta(d-c)+\sigma_{rt}-\sigma_y]/(\sigma_{rt}+\sigma_{rb})$$

$$\nu=[\sigma_{rt}-\sigma_y+E\theta(d-c)]/E\theta d$$

3. <u>Equation of equilibrium</u>

$$P=A_1[0.5(\sigma_{rt}+\sigma_{rb})(1-\psi^2)-\sigma_{rt}(1-\psi)]+A_2(\tfrac{1}{2})(\sigma_{rc}-\sigma_{rt})+E\theta cA_2$$

$$-E\theta A_1(d-c)(1-\psi)+0.5C^2E\theta t_3-0.5t_3E\theta(d-c-\nu d)^2-A_3\sigma_{rt}(1-\nu)$$

$$-\nu A_3\sigma_y-\psi A_1\sigma_y$$

$$M=-P(y_f+c-d)-A_1(d-c)[0.5(\sigma_{rb}+\sigma_{rt})(1-\psi^2)-\sigma_{rt}(1-\psi)]+0.5A_2C(\sigma_{rc}-\sigma_{rt})$$

$$+E\theta C^2A_2+E\theta(d-c)^2A_1(1-\psi)+(t_{3/_3})E\theta C^3+(t_{3/_3})E\theta(d-c-\nu d)^3-d(1-\nu)$$

$$\sigma_{rt}[c-0.5d(1-\nu)]t_3+\nu A_3\sigma_y(d-c-0.5\nu d)+\psi A_1\sigma_y(d-c)$$

Case 5

1. **Section properties:**

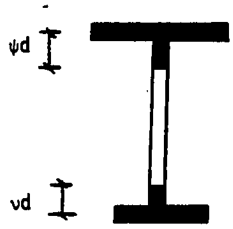

Figure C.5 Case 5 yield pattern

$I_c = 0 \qquad I_y = 0$

$y_s = 0.5d(1-\psi+\nu) \qquad y_b = y_s$

$I_x = [d(1-\psi-\nu)]^3 t_3/_{12} \qquad \beta_x = 0$

$$\int(R.S.+\sigma_y)S^2dA = A_2\sigma_y(d-y_s)^2 - A_1\sigma_y y_s^2 + \sigma_y(^{t}3/_3)(\frac{d^3}{8})[-1-\nu-\psi)^3 + (1-\nu+\psi)^3] - \sigma_y(\frac{t_3}{24})d^3[(1+\nu-\psi)^3 - (1-\nu-\psi)^3] + \sigma_{rt}(\frac{t_3}{12})d^3(1-\nu-\psi)^3 + \sigma_y A_2 b_2^2/_{12} - \sigma_y A_1 b_1^2/_{12}$$

$$\int(R.S.+\sigma_y)dA = A_2\sigma_y - A_1\sigma_y + \psi A_3\sigma_y - \nu A_3\sigma_y - \sigma_{rt}A_3(1-\psi-\nu)$$

$$\int(R.S.+\sigma_y)(y_o-y)dA = A_2\sigma_y(d-y_s) + A_1\sigma_y y_s - 0.125\sigma_y t_3 d^2[(1-\nu-\psi)^2 - (1-\nu+\psi)^2] + 0.125 t_3\sigma_y d^2[(1+\nu-\psi)^2 - (1-\nu-\psi)^2]$$

2. **The yield parameters:**

$\psi = (C - \frac{\sigma_y+\sigma_{rt}}{E\theta})/d \qquad \nu = (d - C - \frac{\sigma_y-\sigma_{rt}}{E\theta})/d$

3. **Equations of equilibrium:**

$$P = A_2\sigma_y - A_1\sigma_y - \nu A_3\sigma_y + \psi A_3\sigma_y - \sigma_{rt}A_3(1-\nu-\psi) + 0.5E\theta t_3(c-\psi d)^2 - 0.5E\theta t_3(d-c-\nu d)^2$$

$$M=A_1\sigma_y(d-c)+A_2\sigma_y C+\psi A_3\sigma_y(C-0.5\psi d)+\nu A_3\sigma_y(d-c-0.5\nu d)$$

$$+({}^{t}3/_3)E\theta(c-\psi d)^3+({}^{t}3/_3)E\theta(d-c-\nu d)^3+P(d-c-y_f).$$

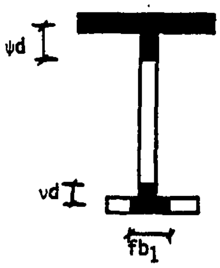

Figure C.6 Case 6 yield pattern

Case 6

1-section properties

$I_c=0 \qquad I_y=\frac{t_1b_1^3}{12}(1-f^3)$

$y_s=0 \qquad y_b=0.5d\ t_3(1-\psi-\nu)(1-\psi+\nu)/[A_3(1-\psi-\nu)+A_1(1-f)]$

$I_x= t_3d^3(1-\psi-\nu)^3/12+A_1(1-f)y_b^2+A_3(1-\nu-\psi)[y_b-\frac{d}{2}(1-\psi-\nu)]^2$

$\beta_x=\{(t_3/4)[(y_b-\nu d)^4-(y_b-d+d\psi)^4]+A_1y_b[b_1^2(1-f^3)/12+y_b^2(1-f)]\}/I_x-2y_o$

$$\int(R.S.+\sigma_y)S^2dA=A_2f_yd^2+\sigma_y(t_3/3)[(\psi d-d)^3+d^3]-(t_3/3)\sigma_{rt}[-(\nu d)^3$$
$$-(\psi d-d)^3]-(t_3/3)\sigma_y(\nu d)^3-(\sigma_y/12)t_1(fb_1)^3+A_1b_1$$
$$[(1-f^4)(\sigma_{rb}+\sigma_{rt})/16-\sigma_{rt}(1-f^3)/12]+A_2\sigma_yb_2^2/12$$

$$\int(R.S.+\sigma_y)dA=A_2\sigma_y+\psi A_3\sigma_y-\nu A_3\sigma_y-fA_1\sigma_y+A_1[0.5(1-f\ )(\sigma_{rb}+\sigma_{rt})$$
$$-\sigma_{rt}(1-f)]-A_3\sigma_{rt}(1-\nu-\psi)$$

$$\int(R.S.+\sigma_y)(y_o-y)dA=A_2\sigma_yd+0.5t_3\sigma_yd^2[1-(1-\psi)^2]-0.5d^2t_3\sigma_{rt}[(1-\psi)^2-\nu^2]$$
$$+0.5t_3\sigma_y\nu^2d^2-A_1y_s[0.5(\sigma_{rb}+\sigma_{rt})(1-f^2)-\sigma_{rt}(1-f)]$$

2. The yield parameters:

$\nu = (d-c-\frac{\sigma_y-\sigma_{rt}}{E\theta})/d \qquad \psi = (C-\frac{\sigma_y+\sigma_{rt}}{E\theta})/d$

$f = [E\theta(d-c)+\sigma_{rt}-\sigma_y]/(\sigma_{rt}+\sigma_{rb})$

3. Equations of equilibrium:

$$P=A_2\sigma_y+\psi A_3\sigma_y-\nu A_3\sigma_y-fA_1\sigma_y+0.5E\theta(C-\psi d)^2t_3-0.5E\theta t_3(d-c-\nu d)^2$$

$$-E\theta A_1(1-f)(d-c)+A_1[0.5(\sigma_{rb}+\sigma_{rt})(1-f^2)-\sigma_{rt}(1-f)]-\sigma_{rt}A_3(1-\nu-\psi)$$

$$M=A_2\sigma_yC+\psi A_3\sigma_y(C-0.5\psi d)+\nu A_3\sigma_y(d-c-0.5\nu d)+fA_1\sigma_y(d-c)+$$

$$({}^{t}3/_3)E\theta(C-\psi d)^3+({}^{t}3/_3)E\theta(d-c-\nu d)^3+E\theta A_1(1-f)(d-c)^2-P(y_f+c-d)$$

$$-A_1(d-c)[0.5(\sigma_{rb}+\sigma_{rt})(1-f^2)-\sigma_{rt}(1-f)]+\sigma_{rt}A_3(1-\nu-\psi)(d-2c-\nu d+\psi d)/_2$$

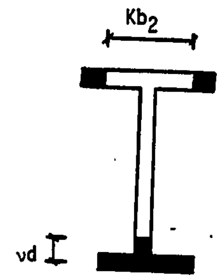

Figure C.7 Case 7 yield pattern

Case 7

1. Section properties

$$I_c=\frac{(kb_2)^3 t_2}{12} \qquad I_y=I_c$$

$$y_s=d \qquad y_b=[kA_2d+A_3(1-\nu^2)d/2]/[kA_2+(1-\nu)A_3]$$

$$I_x= kA_2(d-y_b)^2+[d(1-\nu)]^3 t_3/_{12}+A_3(1-\nu)[d(1-\nu)/_2-d+y_b]^2$$

$$\beta_2=\{-A_2(d-y_b)[k^3b_2^2/_{12}+k(d-y_b)^2]+(^{t}3/_4)[(y_b-\nu d)^4-(d-y_b)^4]\}/_{I_x}-2y_o$$

$$\int(R.S.+\sigma_y)S^2dA=-A_1\sigma_y(d^2+\frac{b_1^2}{12})-(\frac{t_3}{3})\sigma_y d^3(1-\nu^3)-(\frac{t_3}{3})\sigma_{rt}(\nu d)^3+$$

$$A_2b_2^2[k^4(\sigma_{rc}+\sigma_{rt})/_{16}-\sigma_{rt}k^3/_{12}]+\sigma_y A_2b_2^2(1-k^3)/_{12}$$

$$\int(R.C.+\sigma_y)dA=-A_1\sigma_y-\nu A_3\sigma_y+(1-k)A_2\sigma_y-\sigma_{rt}A_3(1-\nu)+A_2[0.5k^2(\sigma_{rc}+$$

$$\sigma_{rt})-k\sigma_{rt}]$$

$$\int(R.S.+\sigma_y)(y_o-y)dA=-A_1\sigma_y d+0.5\sigma_{rt}t_3d^2(1-\nu)^2-0.5\sigma_y t_3[d^2-d^2(1-\nu)^2]$$

2. The yield parameters:

$$\nu=[\sigma_{rt}-\sigma_y+E\theta(d-c)]/E\theta d \qquad k=(\sigma_y+\sigma_{rt}-E\theta C)/(\sigma_{rc}+\sigma_{rt})$$

3. Equations of equilibrium:

$$P=-A_1\sigma_y+A_2(1-k)\sigma_y-\nu A_3\sigma_y+0.5E\theta C^2t_3+kA_2E\theta C-0.5t_3E\theta(d-c-\nu d)^2$$

$$-\sigma_{rt}A_3(1-\nu)+A_2[0.5k^2(\sigma_{rc}+\sigma_{rt})-k\sigma_{rt}]$$

$$M=A_1\sigma_y(d-c)+A_2\sigma_y(1-k)C+\nu A_3\sigma_y(d-c-0.5\nu d)+({}^{t}3/{}_{3})E\theta(d-c-\nu d)^3$$

$$+({}^{t}3/{}_{3})E\theta C^3+CA_2[0.5k^2(\sigma_{rc}+\sigma_{rt})-k\sigma_{rt}]+kA_2E\theta C^2+\sigma_{rt}A_3(1-\nu)[d-c$$

$$-0.5d(1+\nu)]-P(y_f+c-d)$$

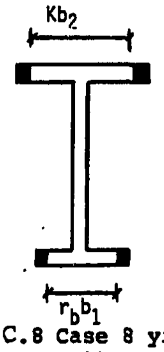

Figure C.8 Case 8 yield pattern

Case 8

1-Section properties

$$I_c = \frac{t_2 (kb_2)^3}{12} \qquad I_y = \frac{t_2 (kb_2)^3}{12} + \frac{t_1 (r_b b_1)^3}{12}$$

$$y_s = \frac{I_c d}{I_y} \qquad y_b = d(kA_2 + 0.5A_3)/(A_3 + kA_2 + r_b A_1)$$

$$I_x = r_b A_1 y_b^2 + kA_2 (d-y_b)^2 + d^3 t_3/12 + A_3 (y_b - 0.5d)^2$$

$$\beta_x = \{-A_2 (d-y_b) [k^3 b_2^2/12 + k(d-y_b)^2] + A_1 y_b (r_b^3 b_1^2/12 + r_b y_b^2) + (\frac{t_3}{4}) [y_b^4 -$$

$$(d-y_b)^4]\}/I_x - 2y_o$$

$$\int (R.S. + \sigma_y) S^2 dA = A_1 \sigma_y [b_1^2/12 - r_b^3 b_1^2/12 + y_s^2 (1-r_b)] + A_2 \sigma_y [b_2^2/12 (1-k^3)$$

$$+ (d-y_s)^2 (1-k)] - (\frac{\sigma_{rt}}{3}) [y_s^3 + (d-y_s)^3] + A_2 [b_2^2 k^4 (\sigma_{rc} + \sigma_{rt})/16 - \sigma_{rt} K^3 b_2^2/12$$

$$+ 0.5k^2 (d-y_s)^2 (\sigma_{rc} + \sigma_{rt}) - \sigma_{rt} k (d-y_s)^2] + A_1 [b_1^2 r_b^4 (\sigma_{rb} + \sigma_{rt})/16 - \sigma_{rt}$$

$$r_b^3 b_1^2/12 + 0.5 r_b^2 y_s^2 (\sigma_{rb} + \sigma_{rt}) - \sigma_{rt} r_b y_s^2]$$

$$\int (R.S. + \sigma_y) dA = A_1 \sigma_y (1-r_b) + A_2 \sigma_y (1-k) - \sigma_{rt} A_3 + A_2 [0.5k^2 (\sigma_{rc} + \sigma_{rt}) - k\sigma_{rt}]$$

$$+ A_1 [0.5 r_b^2 (\sigma_{rb} + \sigma_{rg}) - r_b \sigma_{rt}]$$

$$\int (R.S. + \sigma_y)(y_o - y) dA = +A_2 (d-y_s) [0.5k^2 (\sigma_{rc} + \sigma_{rt}) - k\sigma_{rc}] - y_s A_1 [0.5 r_b^2$$

$$(\sigma_{rb} + \sigma_{rt}) - \sigma_{rt} r_b] + A_2 \sigma_y (1-k)(d-y_s) - A_1 \sigma_y y_s (1-r_b) + \frac{1}{2} t_3 \sigma_{rt} [y_s^2 - (y_s - d)^2]$$

2. <u>The yield parameters</u> :

$k=(\sigma_y+\sigma_{rt}-E\theta C)/(\sigma_{rc}+\sigma_{rt})$ $\qquad r_b=[\sigma_y+\sigma_{rt}-E\theta(c-d)]/(\sigma_{rb}+\sigma_{rt})$

3. <u>Equations of equilibrium</u> :

$$P=A_2[0.5k\ (\sigma_{rc}+\sigma_{rt})-k\sigma_{rt}]+A_1[0.5r_b{}^2(\sigma_{rb}+\sigma_{rt})-r_b\sigma_{rt}]-\sigma_{rt}A_3+E\theta$$

$$A_1r_b(c-d)+E\theta CA_2k+A_2\sigma_y(1-k)+A_1\sigma_y(1-r_b)+A_3E\theta(C-d/2)$$

$$M=CA_2[0.5k^2(\sigma_{rc}+\sigma_{rt})-\sigma_{rt}k]+A_1(c-d)[0.5r_b{}^2(\sigma_{rb}+\sigma_{rt})-\sigma_{rt}r_b]$$

$$-P(c-d+y_f)+A_1r_bE\theta(c-d)^2+kA_2E\theta C^2+(1-k)\sigma_yA_2C+A_1\sigma_y(1-r_b)$$

$$(c-d)+E\theta A_3(c-d)(c-{}^{d}/2)+E\theta dA_3(C-{}^{d}/3)/_2$$

Case 9

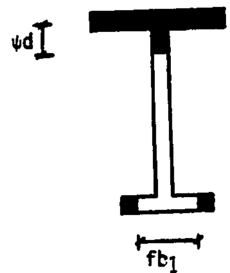

Figure C.9. Case 9 yield pattern

1-Section properties

$I_c=0 \qquad I_y=\dfrac{(fb_1)^3 t_1}{12}$

$y_s=0 \qquad y_b=0.5(1-\psi)^2 d^2/(fA_1+A_3-\psi A_3)$

$I_x=(1-\psi)^3 d^3 t_3/12+(1-\psi)A_3[y_b-0.5d(1-\psi)]^2+fA_1 y_b^2$

$\beta_x=\{(\frac{t_3}{4})[y_b^4-(y_b+d\psi-d)^4]+A_1 y_b f(y_b^2+\frac{f^3 b_1^2}{12})\}/I_x-2y_o$

$\int(\text{R.S.}+\sigma_y)S^2 dA=\sigma_y A_2(d^2+b_2^2/12)+(t_3/3)\sigma_y[d^3-(d-\psi d)^3]-(t_3/3)\sigma_{rt}(d-\psi d)^3$

$\qquad +A_1 b_1^2[(\sigma_{rb}+\sigma_{rt})f^4/16-\sigma_{rt}f^3/12]+\sigma_y A_1 b_1^2(1-f^3)/12$

$\int(\text{R.S.}+\sigma_y)dA=A_2\sigma_y+\psi A_3\sigma_y-\sigma_{rt}A_3(1-\psi)+(1-f)A_1\sigma_y+A_1[0.5f^2(\sigma_{rb}+\sigma_{rt})-f\sigma_{rt}]$

$\int(\text{R.S.}+\sigma_y)(y_o-y)dA=A_2\sigma_y d+\frac{t_3}{2}\sigma_y(d^2-\psi^2 d^2)-\frac{t_3}{3}\sigma_{rt}d^2\psi^2$

2. The yield parameters

$\psi=\frac{1}{d}(c-\frac{\sigma_y+\sigma_{rt}}{E\theta}) \qquad f=[\sigma_y+\sigma_{rt}-E\theta(c-d)]/(\sigma_{rb}+\sigma_{rc})$

3. Equations of equilibrium

$P=A_2\sigma_y+\psi A_3\sigma_y+(1-f)A_1\sigma_y-\sigma_{rt}A_3(1-\psi)+A_1[0.5f^2(\sigma_{rb}+\sigma_{rt})-f\sigma_{rt}]$

$\qquad +E\theta(c-d)A_1 f+E\theta A_3(1-\psi)[C-0.5d(1+\psi)]$

$$M=A_2\sigma_y C+\psi A_3\sigma_y\left(C-\frac{\psi}{2}d\right)+(1-f)A_1\sigma_y(c-d)-\sigma_{rt}A_3(1-\psi)\left(C-\frac{d}{2}-\frac{\psi d}{2}\right)$$

$$+A_1(c-d)[0.5f^2(\sigma_{rb}+\sigma_{rt})-f\sigma_{rt}]+E\theta(c-d)^2A_1f-P(C+y_f-d)$$

$$+E\theta A_3(c-d)(1-\psi)\left(C-\frac{d}{2}-\psi\frac{d}{2}\right)+\frac{1}{2}E\theta d^2t_3(1-\psi)^2\left[C-\psi d-\frac{1}{3}d(1-\psi)\right]$$

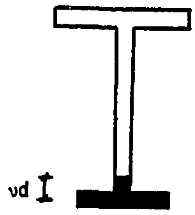

Figure C.10 Case 10 yield pattern

Case 10

1-Section properties

$I_c=\frac{b_2^3 t_2}{12}$ $\qquad I_y=I_c$

$y_s=d \qquad y_b=[A_2 d+A_3(1-\nu^2)d/2]/[A_3(1-\nu)+A_2]$

$I_x= A_2(d-y_b)^2+d^3(1-\nu)^3 t_{3/12}+A_3(1-\nu)[d(1-\nu)/_2-d+yb]$

$\beta_x=\{-A_2(d-y_b)[b_2^2/_{12}+(d-y_b)^2]+(\frac{t_3}{4})[(y_b-\nu d)^4-(d-y_b)^4]\}/_{I_x}-2y_o$

$\int(R.S.+\sigma_y)S^2dA=-A_1\sigma_y(d^2+\frac{b_1^2}{12})-\frac{t_3}{3}\sigma_y d^3(1-\nu^3)-\frac{t3}{3}\sigma_{rt}\nu^3 d^3+A_2 b_2^2(\frac{\sigma_{rc}+\sigma_{rt}}{16}-\frac{\sigma_{rt}}{12})$

$\int(R.S.+\sigma_y)dA=-A_1\sigma_y-\nu A_3\sigma_y-\sigma_{rt}A_3(1-\nu)+0.5A_2(\sigma_{rc}-\sigma_{rt})$

$\int(R.S.+\sigma_y)(y_o-y)dA=-A_1\sigma_y d+0.5\sigma_{rt}t_3 d^2(1-\nu)^2-0.5\sigma_y t_3[d^2-d^2(1-\nu)^2]$

2. The yield parameter

$\nu=[\sigma_{rt}-\sigma_y+E\theta(d-c)]/E\theta d$

3. Equations of equilibrium

$P=-A_1\sigma_y-\mu A_3\sigma_y+0.5E\theta C^2 t_3+A_2E\theta C-0.5t_3E\theta(d-c-\nu d)^2-\sigma_{rt}A_3(1-\nu)$

$\quad +0.5A_2(\sigma_{rc}-\sigma_{rt})$

$M=A_1\sigma_y(d-c)+\nu A_3\sigma_y(d-c-0.5\nu d)+(\frac{t_3}{3})E\theta(d-c-\nu d)^3+(\frac{t_3}{3})E\theta C^3+0.5C$

$\quad A_2(\sigma_{rc}-\sigma_{rt})+A_2E\theta C^2+\sigma_{rt}A_3(1-\nu)[d-c-0.5d(1+\nu)]-P(y_f+C-d)$

Case 11

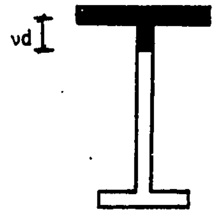

Figure C.11 Case 11 yield pattern

1. Section properties

$$I_c=0 \qquad I_y=\frac{t_1 b_1^{\,3}}{12}$$

$$y_s=0 \qquad y_b=0.5A_3(1-\nu)^2/[A_1+(1-\nu)A_3]$$

$$I_x=[d(1-\nu)]^3 t_3/_{12}+A_1 y_b^{\,2}+d(1-\nu)t_3[y_b-0.5d(1-\nu)]^2$$

$$\beta_x=(t_3/_4)[y_b^{\,4}-(d(1-\nu)-y_b)^4]+y_b A_1(b_1^{\,2}/_{12}+y_b^{\,2})$$

$$\int(R.S.+\sigma_y)S^2dA=\sigma_y A_2(\frac{b_2^{\,2}}{12}+d^2)+A_1 b_1^{\,2}[(\sigma_{rt}+\sigma_{rb})/_{16}-\sigma_{rt}/_{12}]$$

$$+(\frac{A_3}{3})\sigma_y d^2[1-(1-\nu)^3]-(\frac{A_3}{3})\sigma_{rt}d^2(1-\nu)^3$$

$$\int(R.S.+\sigma_y)dA=\frac{1}{2}A_1(\sigma_{rb}-\sigma_{rt})+A_2\sigma_y+\nu A_3\sigma_y-\sigma_{rt}A_3(1-\nu)$$

$$\int(R.S.+\sigma_y)(y_o-y)dA=-A_2\sigma_y d-0.5A_3\sigma_y d(1-\nu^2)+0.5A_3 d\sigma_{rt}\nu^2$$

2. The yield parameter : $\nu = (C-\frac{\sigma_y+\sigma_{rt}}{E\theta})/d$

3. Equations of equilibrium :

$$P=A_2\sigma_y+\nu A_3\sigma_y-\sigma_{rt}A_3(1-\nu)-E\theta A_1(d-c)+0.5A_1(\sigma_{rb}-\sigma_{rt})+$$

$$0.5t_3E\theta(c-\nu d)^2-0.5t_3E\theta(d-c)^2$$

$$M=CA_2\sigma_y+\nu A_3\sigma_y(C-0.5\nu d)+E\theta A_1(d-c)^2-0.5A_1(\sigma_{rb}-\sigma_{rt})(d-c)$$

$$-P(c+y_f-d)+A_3(1-\nu)[d-C-0.5d(1-\nu)]\sigma_{rt}+({}^{t}3/3)E_\theta(C-\nu d)^3$$

$$+({}^{t}3/3)E\theta(d-c)^3$$

APPENDIX D

COMPUTER PROGRAM

COMPUTER PROGRAM

D.1 Description of the Computer Program

The computer program used in this study computes the buckling load of a linearly web-tapered member having unequal flanges and subjected to axial load and end moments. The program consists of a main program and four subroutines as follows:

MAIN--Reads and prints the input data, computes the section properties, the bending moment and the values of the residual stresses at each station. Sum of the axial, bending and residual stresses are computed at extreme points of the cross-section, based on which the type of the most probable yield pattern is determined. For each yield pattern, computes the section properties. Computes the terms of the difference equations at each station and assembles the determinant of the difference equations. Determines the value of this determinant and follows the iterative procedure until a zero value is obtained. Then uses Gauss-Jordan elimination to determine the buckling mode shape.

NONLIN--This subroutine, written by Dr. K. Brown at the University of Minnesota, solves a system of N simultaneous nonlinear equations. The method does not require the user to furnish any derivatives. In this study, the subroutine is used to solve the two equations of equilibrium. The two unknowns are: The curvature times E, and the distance from the neutral axis to the top flange.

BACK--This subroutine back - solves the first N rows of a triangularized linear system.

COLUMN--This subroutine computes the value of stress and the yield parameters in the absence of bending moment since in this case the location of the neutral axis cannot be determined.

AUXFCN--Lists the equations of equilibrium for different yield patterns. The yield parameters are defined before each equation.

D.2 Input Data Format

CONTROL CARD

Cols. 1-2 JG - Number of members to be analyzed

MEMBER DATA CARD (Two cards per member)

Card 1

Cols. 1-2	N	- Number of segments which a member is divided into
3	NB	- Code number for boundary conditions: 1 = simple lateral and torsional supports; 2 = fixed conditions
4	NH	= Analysis option: 1 = beam analysis (the larger end moment is taken as buckling load); 2 = axial load analysis
5-10	B1	- Width of the lower flange
11-16	T1	- Thickness of the lower flange
17-22	B2	- Width of the upper flange
23-28	T2	- Thickness of the upper flange
29-34	BO3	- Center to center distance of the flanges at the smaller end
35-40	T3	- Thickness of the web
41-46	DEL	- Taper ratio

47-52 CC - The ratio of the moment at the smaller end to that at the larger end. Positive for single curvature and negative otherwise.

53-57 xxx - The specified load: axial load for NH = 1; larger end moment for NH = 2 (Moment producing tension in the lower flange is positive).

58-62 FY - Yield stress of steel

63-67 COF - Ratio of the maximum compressive residual stress in the upper flange to the yield stress.

68-72 AJ - Distance from the braced point to the smaller end. In the absence of intermediate bracing, AJ = 0

73-77 AL - Length of the member

Card 2

Cols. 1-10 E - Modulus of elasticity

11-20 G - Shear modulus of elasticity

21-30 ES - Acceptable error in computing the buckling load, 0.1 to 0.5 percent of the estimated buckling load is recommended.

31-40 xp(2) - Upper bound estimate of the buckling load; preferably not to exceed 50 percent of the estimated value.

Note:

Kips and inches are assumed for all units.

```
$JOB TIME=(0,30),PAGES=70
        DIMENSION BET(20),YI(20),YS(20),X(30),RN(20),CI(20),BM(20)
        DIMENSION UY(20),WKAREA(18),U(20),W(20),XP(2),FNC(2),URV(20)
        DIMENSION T(20),YN(20),A(18,18),B(20),DT(18,18),UR(20),URR(20)
       DIMENSION DU(20,20),DV(20),DC(18,18),DG(20,20)
        COMMON JS,A1,A2,T3,SIBR,SICR,SITR,FY,BK,SAI,UN,D,P,XM,YF,A3
        COMMON BR,M1,PAI
       READ(5,483) JG
483     FORMAT(I2)
         DO 401 NX=1,JG
C
C      N=NO. OF ELEMENTS WHICH A MEMBER IS DIVIDED INTO
C      AL=LENGTH OF A MEMBER
C      B1,T1=THE DIMENSIONS OF THE LOWER FLANGE
C      B2,T2=THE DIMENSIONS OF THE UPPER FLANGE
C      B3,T3=THE DIMENSIONS OF THE WEB
C      BO3=THE WIDTH OF THE WEB AT THE SMALLER END
C      DEL=TAPER RATIO
C      OM1,OM2=THE TWO END MOMENTS
C      FY=THE YIELD STRESS
C      AJ=DISTANCE FROM THE SMALLER END TO THE BRACED POINT
C       XXX=SPECIFIED LOAD.AXIAL LOAD FOR NH=1 AND BENDING MOMENT FOR NH=2
C       CC=RATIO OF THE MOMENT AT THE SMALLER END TO THAT AT THE LARGER END
C  POSITIVE FOR SINGLE CURVATURE AND NEGATIVE OTHERWISE
C       COF=RATIO OF THE MAXIMUM COMPRESSIVE RESIDUAL STRESS TO THE YIELD STR
C       E,G=MODULUS AND SHEAR MODULUS OF ELASTICITY
C      JG= NUMBER OF MEMBERS TO BE ANALYZED
C      H=THE LENGTH OF EACH SEGMENT
C
       READ(5,31) N,NB,NH,B1,T1,B2,T2,BO3,T3,DEL,CC,XXX,FY,COF,AJ,AL
31     FORMAT(I2,2I1,8F6.3,5F5.1)
        READ(5,681) E,G,ES,XP(2)
681     FORMAT(4F10.1)
        WRITE(6,491) NX
491    FORMAT(//1H1,1X,'INPUT DATA',4X,'MEMBER #',I2)
        PRINT  492
492     FORMAT(' **********')
       PRINT 129
129    FORMAT(//' INCH AND KIPS ARE TO BE USED FOR ALL UNITS')
       PRINT 459
459    FORMAT(//1X,'WIDTH-L.FLG.',1X,'THICK.L.FLG.',1X,'WIDTH-UPP.FLG.',
      C 1X,'THICK.UPP.FLG.',1X,'DEPTH',1X,'WEB-THICK.',1X,'TAPER RATIO')
        WRITE(6,7)  B1,T1,B2,T2,BO3,T3,DEL
7       FORMAT(//,2X,F10.4,1X,F10.4,4X,F10.4,3X,F10.3,4X,3F10.4)
        PRINT 433
433    FORMAT(//1X,'NUM.SEG.',1X,'B.C.',1X,'E',1X,'ANAL.TYPE.',1X,'G',1X,
      C'MOM.RATIO',1X,'YIELD STR.',1X,'LOC.BRACE',1X,'R.S./FY',1X,'SPEC.L
      COAD',1X,'LENGTH')
        WRITE(6,495) N,NB,E,NH,G,CC,FY,AJ,COF,XXX,AL
495    FORMAT(//I6,I5,2X,F7.1,I3,2X,F7.1,F6.2,1X,F6.2,3X,2F9.3,2X,F9.3,
      C 1X,F6.2)
        PRINT  122
122      FORMAT(//' ACCEPTABLE ERROR',2X,'ESTIMATED BUCKLING LOAD')
        WRITE(6,123) ES,XP(2)
123     FORMAT(//2F15.4)
C
C       JX=NO. OF POINTS (STATIONS) ALONG THE MEMBER
C       ES=ACCEPTABLE ERROR IN COMPUTING THE BUCKLING LOAD
C       XP(1),XP(2) ARE TWO ESTIMATES OF THE BUCKLING LOAD
C       METHOD OF BISECTION IS USED TO FIND THE "ZERO" OF THE DETERMINANT
```

```
C       OF THE LINEAR EQUATIONS.THE TWO ESTIMATES ABOVE WILL BE USED AND
C       THE VALUE OF THE BUCKLING LOAD IS EXPECTED TO BE BETWEEN THE TWO
C
        OM2=XXX
        XP(1)=0.4*XP(2)
         H=AL/N
         NZ=AJ/H+1
         JX=N+1
         XP(1)=(XP(1)+XP(2))/2
   14    DO 93 IN=1,2
       IF(NH.EQ.1) GO TO 377
         P=XP(IN)
        GO TO 378
  377   OM2=XP(IN)
          P=XXX
  378   CONTINUE
        OM1=CC*OM2
         DO 8 I=1,JX
         Z=H*(I-1)
C
C      XM=BENDING MOMENT AT EACH POINT
C      XMP=THE DERIVATIVE OF THE BENDING MOMENT
         XM=OM1+(OM2-OM1)*Z/AL
         XMP=(OM2-OM1)/AL
        BM(I)=XM
          D=BO3*(1+DEL*Z/AL)
C
C      A1,A2,A3 ARE THE AREAS OF THE LOWER FLANGE,UPPER FLANGE AND WEB
C       SICR=COMPRESSIVE STRESS AT THE SIDES OF THE UPPER FLANGE
C       SITR=TENSILE STRESS ALONG THE  WEB
C       SIBR=COMPRESSIVE STRESS AT THE SIDES OF THE LOWER FLANGE
C       YP=DISTANCE FROM THE CENTROID TO THE BOTTOM FLANGE
C
         A1=B1*T1
         A2=B2*T2
        A3=D*T3
         AT=A1+A2+A3
        SICR=COF*FY
         SITR=(A2/(A2+A3))*SICR
         SIBR=(A2*(A1+A3)/(A1*(A2+A3)))*SICR
C
C      THIS PART COMPUTES THE PROPERTIES OF THE ELASTIC SECTION
C
          YP=(A2*D+0.5*A3*D)/AT
C
C       YS=DISTANCE FROM THE SHEAR CENTER TO LOWER FLANGE
C       YB=DISTANCE FROM THE CENTROID TO LOWER FLANGE
C         T=THE TORSIONAL CONSTANT
C       XI=MOMENT OF INERTIA W.R.T. X-AXIS
C      X(1) IS AN ESTIMATE OF E TIMES CURVATURE
C      X(2) IS AN ESTIMATE OF C (DISTANCE FROM NEUTRAL AXIS TO UPPER FLANGE)
C
          YT=D-YP
          T(I)=(B2*T2**3+B1*T1**3+ D*T3**3)/3
          XI=A1*YP**2+A2*YT**2+D**3*T3/12+  A3*(YT-D/2)**2
          IF(XM.EQ.0)GO TO 358
          X(1)=XM/XI
          X(2)=D-YP+P*XI/(AT*XM)
  358     CONTINUE
C      THE EXTREME VALUES OF STRESSES ARE CHECKED TO DECIDE THE PROPER YIELD
```

```
C       PATTERN
         SCOM=P/AT-XM*YF/XI+SIBR-FY
         SBOT=-P/AT+XM*(YF)/XI+SITR-FY
         STOP=P/AT+XM*(D-YF)/XI+SICR-FY
         IF(SCOM)91,91,92
   91    IF(SBOT)80,80,81
   81    IF(STOP)82,82,83
   80    IF(STOP)84,84,85
   84    CONTINUE
C       CASE # 1
        JS=0.0
C
C        YI=MOMENT OF INERTIA W.R.T. Y-AXIS
C        CI=MOMENT OF INERTIA OF THE UPPER FLANGE W.R.T.Y-AXIS
C       INTEGRAL OF(STRESSES TIMES SQUARE OF THEIR DISTANCE TO THE SHEAR CEN
C        UR=INTEGRAL OF STRESSES
C        UY=INTEGRAL OF STRESSES TIMES THEIR DISTANCE TO THE SHEAR CENTER
C
         YI(I)=(T2*B2**3+T1*B1**3)/12
         YS(I)=D*B2**3*T2/(12*YI(I))
         YR=YS(I)
        YN(I)=YF-YS(I)
         CI(I)=T2*B2**3/12
        BET(I)=-(T2*B2**3*(D-YF)/12-YF*T1*B1**3/12+A2*(D-YF)**3-A1*YF**3
      &  +(T3/4)*((D-YF)**4-YF**4))/XI-2*YN(I)
       URR(I)=T1*B1**3*(SIBR-SITR/3)/16+T2*B2**3*(SICR-SITR/3)/16-(T3/3)
      &  *SITR*((D-YR)**3+YR**3)+A1*(D-YR)**2*(SIBR-SITR)*0.5+A2*YR**2
      &  *0.5*(SICR-SITR)
         UR(I)=0.
         UY(I)=0.
         RN(I)=SQRT((XI+YI(I))/AT+YN(I)**2)
         GO TO 8
C
C       THE CROSS SECTION WITH THE YIELDED PORTIONS
C       IN THE COMPRESSION FLANGE ONLY
C       CASE # 2
C
   85    JS=1
         IZ=20
         CALL NONLIN(2,5,IZ,2,X,0.01)
        IF(BK.GT.1) GO TO 83
        IF(M1.GT.9) GO TO 83
        IF(BK.LT.0) GO TO 83
        AT=A1+A3+BK*A2
         YI(I)=(T2*BK**3*B2**3+T1*B1**3)/12
         YS(I)=D*T2*(BK*B2)**3/(12*YI(I))
         YR=YS(I)
         YB=D*(BK*A2+0.5*A3)/(A1+A3+BK*A2)
         XI=BK*A2*(D-YB)**2+A1*YB**2+T3* D**3/12+A3*(YB-D/2)**2
         CI(I)=T2*BK**3*B2**3/12
         YN(I)=YB-YS(I)
       BET(I)=((T3/4)*(YB**4-(D-YB)**4)+YB**3*A1-BK*A2*(D-YB)**3-BK**3*(D
      C-YB)*T2*B2**3/12+YB*T1*B1**3/12)/XI-2*YN(I)
       UR(I)=A2*FY*(1-BK)-SITR*A3+0.5*A1*(SIBR-SITR)+A2*(0.5*BK**2*(SICR
      C +SITR)-BK*SITR)
       URR(I)=A1*(B1**2*(SIBR+SITR)/16-B1**2*SITR/12+YR**2*(SIBR+SITR)/2
      C-SITR*YR**2)-SITR*T3*(YR**3+(D-YR)**3)/3+A2*(B1**2*BK**4*(SICR+
      CSITR)/16-B1**2*BK**3*SITR/12+(D-YR)**2*BK**2*(SICR+SITR)/2-SITR*
      CBK*(D-YR)**2)+FY*A2*(B2**2*(1-BK**3)/12+(D-YR)**2*(1-BK))
       UY(I)=FY*A2*(D-YR)*(1-BK)+(D-YR)*A2*(0.5*BK**2*(SICR+SITR)-BK*SITR
```

```
     C)-0.5*A1*YR*(SIBR-SITR)+0.5*T3*SITR*(YR**2-(D-YR)**2)
      RN(I)=SQRT((XI+YI(I))/AT+YN(I)**2)
      GO TO 8
C
C    PARTIAL COMPRESSION YIELD IN THE UPPER FLANGE AND TENSION YIELD
C    IN THE LOWER FLANGE AND WEB
C    CASE # 3
C
83    JS=2
     IF(COF.EQ.0) GO TO 28
      IZ=20
      CALL NONLIN(2,5,IZ,2,X,0.01)
     IF(M1.GT.9) GO TO 56
     IF(BK.LT.0) GO TO 56
     IF(UN.LT.0) GO TO 56
     IF(BK.GT.1) GO TO 56
     IF(UN.GT.1) GO TO 56
      IF(SAI.GT.1) GO TO 56
      IF(SAI.LT.0) GO TO 56
      CI(I)=T2*(BK*B2)**3/12
      YI(I)=CI(I)+T1*B1**3*(1-SAI**3)/12
      YS(I)=D*T2*(BK*B2)**3/(12*YI(I))
      YR=YS(I)
     YB=(BK*A2*D+D*A3*0.5*(1-UN**2))/(BK*A2+A3*(1-UN)+A1*(1-SAI))
      XI=BK*A2*(D-YB)**2+YB**2*A1*(1-SAI)+(T3/12)*(D-UN*D)**3+T3*(D-
     & UN*D)*(YB-UN*D-0.5*(D-UN*D))**2
      YN(I)=YB-YS(I)
     AT=A1*(1-SAI)+A2*BK+A3*(1-UN)
      BET(I)=((T3/4)*((YB-UN*D)**4-(D-YB)**4)+YB**3*A1*(1-SAI)-BK*A2
     & *(D-YB)**3-BK**3*(D-YB)*T2*B2**3/12+YB*(1-SAI**3)*T1*B1**3/12)/
     & XI-2*YN(I)
      UR(I)=A2*(0.5*BK**2*(SICR+SITR)-SITR*BK)+A1*(0.5*(SIBR+SITR)*
     & (1-SAI**2)-SITR*(1-SAI))- (1-UN)*SITR*A3-UN* A3*FY-SAI*A1*FY+
     & A2*FY*(1-BK)
      URR(I)=FY*A2*(B2**2*(1-BK**3)/12+(1-BK)*(D-YR)**2)+A2*(B2**2*BK
     &**4*(SICR+SITR)/16+(D-YR)**2*(SICR+SITR)*BK**2*0.5-SITR*B2**2*BK
     & **3/12-SITR*(D-YR)**2*BK)-SITR*T3*((YR-UN*D)**3+(D-YR)**3)/3-FY*
     & T3*(YR**3-(YR-UN*D)**3)/3-FY*A1*(B1**2*SAI**3/12+YR**2*SAI)+A1*(
     & B1**2*(SIBR+SITR)*(1-SAI**4)/16-SITR*B1**2*(1-SAI**3)/12)
      URV(I)=A1*(YR**2*0.5*(1-SAI**2)*(SIBR+SITR)-SITR*YR**2*(1-SAI))
      URR(I)=URR(I)+URV(I)
     UY(I)=+A2*(0.5*BK**2*(SICR+SITR)-SITR*BK)*(D-YR)-A1*(0.5*(SIBR+
     CSITR)*(1-SAI**2)-SITR*(1-SAI))*YR+SAI*A1*FY*YR+A2*(1-BK)*(D-YR)*FY
     C+FY*T3*(YR**2-(YR-UN*D)**2)/2+SITR*T3*((YR-UN*D)**2-(D-YR)**2)/2
      RN(I)=SQRT((XI+YI(I))/AT+YN(I)**2)
      GO TO 8
C
C    COMPLETE COMPRESSION YIELD OF THE UPPER FLANGE PLUS PART OF THE
C    WEB
C     CASE # 11
C
56    JS=3
      X(1)=XM/XI
      X(2)=D-YF+P*XI/(AT*XM)
      IZ=20
      CALL NONLIN(2,5,IZ,2,X,0.01)
     IF(M1.GT.9) GO TO 29
     IF(UN.LT.0) GO TO 29
     IF(UN.GT.1) GO TO 29
     AT=A1+A3*(1-UN)
```

```
          YB=D*A3*0.5*(1-UN)**2/(A1+(1-UN)*A3)
            XI=(D*(1-UN))**3*T3/12+(YB-0.5*D*(1-UN))**2*A3*(1-UN)+A1*YB**2
            CI(I)=0.
            YI(I)=B1**3*T1/12
            YS(I)=0.
            YN(I)=YB-YS(I)
            UR(I)=0.5*A1*(SIBR-SITR)+A2*FY+UN*A3*FY-SITR*A3*(1-UN)
            URR(I)=FY*A2*(B2**2/12+D**2)+A1*B1**2*((SITR+SIBR)/16-SITR/12)
     &      +(A3/3)*FY*D**2*(1-(1-UN)**3)-(A3/3)*SITR*D**2*(1-UN)**3
            UY(I)=A2*FY*D+A3*D*FY*0.5*(1-(1-UN))**2-A3*D*0.5*(1-UN)**2*SITR
            BET(I)=(T3/4)*(YB**4-(D*(1-UN)-YB)**4)+YB*A1*(B1**2/12+YB**2)
            RN(I)=SQRT((XI+YI(I))/AT+YN(I)**2)
           GO TO 8
C
C          PARTIAL YIELDING OF THE UPPER AND LOWER
C          CASE # 8
C          FLANGES IN COMPRESSION
C
92          JS=4
           IF(COP.EQ.0) GO TO 28
            IZ=20
            IF(XS.NE.0) GO TO 355
            CALL COLUMN(X,ROOT)
            GO TO 356
355         CONTINUE
            CALL NONLIN(2,5,IZ,2,X,0.01)
           IF(N1.GT.7) GO TO 78
            IF(BK.GT.1) GO TO 78
            IF(BK.LT.0) GO TO 78
            IF(BR.GT.1) GO TO 78
            IF(BR.LT.0) GO TO 78
356         CONTINUE
            CI(I)=T2*(BK*B2)**3/12
            YI(I)=CI(I)+T1*(BR*B1)**3/12
            YS(I)=D*T2*(BK*B2)**3/(12*YI(I))
            YB=(BK*A2*D+A3*0.5*D)/(BK*A2+A3+BR*A1)
            YN(I)=YB-YS(I)
            XI=BR*A1*YB**2+BK*A2*(D-YB)**2+D**3*T3/12+A3*(YB-0.5*D)**2
            BET(I)=(-A2*(D-YB)*(BK**3*B2**2/12+(D-YB)**2*BK)+A1*YB*(BR**3*
     &      B1**2/12+YB**2*BR)+(T3/4)*(YB**4-(D-YB)**4))/XI-2*YN(I)
            YR=YS(I)
            UR(I)=A1*FY*(1-BR)+A2*FY*(1-BK)-SITR*A3+A2*(0.5*BK**2*(SICR+
     &      SITR)-SITR*BK)+A1*(0.5*BR**2*(SIBR+SITR)-SITR*BR)
            UY(I)=+A2*(0.5*BK**2*(SICR+SITR)-BK*SITR)*(D-YR)-YR*A1*(0.5*BR
     &      **2*(SIBR+SITR)-BR*SITR)+A2*(1-BK)*FY*(D-YR)-A1*(1-BR)*FY*YR+
     &      T3*SITR*(YR**2-(YR-D)**2)*0.5
            URR(I)=A1*FY*((B1**2/12)-(BR**3*B1**2/12)+YR**2*(1-BR))+A2*FY*
     &      ((B2**2/12)*(1-BK**3)+(D-YR)**2*(1-BK))-(SITR/3)*T3*(YR**3+
     &      (D-YR)**3)+A2*((B2**2/16)*BK**4*(SICR+SITR)-(SITR/12)*BK**3*B2
     &      **2+(SICR+SITR)*(D-YR)**2*0.5*BK**2-SITR*BK*(D-YR)**2)+A1*((B1**
     &      2/16)*BR**4*(SIBR+SITR)-SITR*BR**3*B1**2/12+(SIBR+SITR)*YR**2*0.
     &      5*BR**2-SITR*BR*YR**2)
           AT=BK*A2+BR*A1+A3
            RN(I)=SQRT((XI+YI(I))/AT+YN(I)**2)
           GO TO 8
C
C        COMPLETE YIELDING OF BOTH FLANGES
C          CASE # 5
C
28          JS=6
```

```
      X(1)=XM/XI
      X(2)=D-YF+P*XI/(AT*XM)
     IZ=20
      CALL NONLIN(2,5,IZ,2,X,0.01)
     IF(UN.LT.0)  GO TO 422
     IF(M1.GT.7)  GO TO 422
     IF(SAI.LT.0)  GO TO 422
     CI(I)=0.0
      YI(I)=0.0
     XI=(D*(1-SAI-UN))**3*T3/12
      YS(I)=0.5*D*(1-SAI+UN)
     YR=YS(I)
     BET(I)=0.0
     UR(I)=A2*FY-A1*FY+SAI*A3*FY-UN*A3*FY-SITR*A3*(1-UN-SAI)
      YN(I)=0.0
     URR(I)=A2*FY*(D-YR)**2-A1*FY*YR**2+FY*(T3/3)*(D**3/8)*(-(1-UN-SAI)
    C**3+(1-UN+SAI)**3)-FY*(T3/24)*D**3*((1+UN-SAI)**3-(1-UN-SAI)**3)+
    CSITR*(T3/12)*D**3*(1-UN-SAI)**3+FY*A2*B2**2/12-FY*A1*B1**2/12
     UY(I)=+A2*FY*(D-YR)+A1*FY*YR-FY*T3*0.125*D**2*((1-UN-SAI)**2-(1-UN
    C +SAI)**2)+FY*T3*0.125*D**2*((1+UN-SAI)**2-(1-UN-SAI)**2)
      AT=A3*(1-UN-SAI)
      RN(I)=SQRT((XI+YI(I))/AT+YN(I)**2)
      GO TO 8
422   PRINT 423
423   FORMAT(/' A PROPER YIELD PATTERN NOT FOUND.TRY ANOTHER ESTIMATE')
       GO TO 401
C
C      PRTIAL YIELDING OF THE UPPER FLANGE AND COMPLETE YIELDING OF THE LOWE
C  FLANGE PLOS PART OF THE WEB
C      CASE # 7
C
   27   JS=7
      X(1)=XM/XI
      X(2)=D-YF+P*XI/(AT*XM)
     IZ=20
      CALL NONLIN(2,5,IZ,2,X,0.01)
     IF(M1.GT.7)  GO TO 28
     IF(UN.GT.1)  GO TO 28
     IF(BK.GT.1)  GO TO 28
     IF  (BK.LT.0)  GO TO 28
     IF  (UN.LT.0)  GO TO 28
     CI(I)=(BK*B2)**3*T2/12
     YI(I)=CI(I)
      YS(I)=D
     YB=(BK*A2*D+D*(1-UN)*T3*0.5*D*(1+UN))/(BK*A2+A3*(1-UN))
      YN(I)=YB-YS(I)
     BET(I)=(-A2*(D-YB)*(BK**3*B2**2/12+BK*(D-YB)**2)+(T3/4)*((YB-UN*D)
    C**4-(D-YB)**4))/XI-2*YN(I)
     XI=BK*A2*(D-YB)**2+(D*(1-UN))**3*T3/12+A3*(1-UN)*(0.5*D*(1+UN)-YB)
      UY(I)=A1*FY*D-0.5*SITR*T3*D**2*(1-UN)**2+FY*0.5*T3*(D**2-D*(1-UN)
    C  **2)
      URR(I)=-A1*FY*(D**2+B1**2/12)-FY*(T3/3)*D**3*(1-UN**3)-SITR*(T3/
    C3)*(UN*D)**3+A2*B2**2*(BK**4*(SICR+SITR)/16-SITR*BK**3/12)+FY*
    C A2*B2**2*(1-BK**3)/12
      UR(I)=-A1*FY-UN*A3*FY+(1-BK)*A2*FY-SITR*A3*(1-UN)+A2*(0.5*BK**2
    C  *(SICR+SITR)-BK*SITR)
      AT=BK*A2+A3*(1-UN)
      RN(I)=SQRT((XI+YI(I))/AT+YN(I)**2)
      GO TO 8
C
```

```
C      PARTIAL YIELDING OF THE LOWER FLANGE AND CO-PLETE YIELDING OF THE
C      UPPER FLANGE PLUS PART OF THE WEB
C       CASE # 6
C
 29     JS=8
          X(1)=XN/XI
          X(2)=D-YF+P*XI/(AT*XN)
        IZ=20
          CALL NONLIN(2,5,IZ,2,X,0.01)
         IF(N1.GT.9) GO TO 27
         IF(UN.LT.0) GO TO 27
         IF(SAI.LT.0) GO TO 27
         IF(FAI.LT.0) GO TO 27
         IF(UN.GT.1)GO TO 27
          IF(SAI.GT.1) GO TO 27
          IF(FAI.GT.1) GO TO 27
         YS(I)=0.0
         CI(I)=0.0
         YI(I)=B1**3*T1/12-(FAI*B1)**3*T1/12
         YB=D*(1-SAI-UN)*T3*0.5*D*(1-SAI+UN)/(D*(1-SAI-UN)*T3+(1-FAI)*A1)
          YN(I)=YB-YS(I)
         XI=T3*(D*(1-SAI-UN))**3/12+A1*(1-FAI)*YB**2+T3*D*(1-SAI-UN)*(YB
       C  -0.5*D*(1-SAI+UN))**2
         UR(I)=A2*FY+SAI*A3*FY-UN*A3*FY-FAI*A1*FY+A1*(0.5*(1-FAI**2)*(SIBR
       C +SITR)-(1-FAI)*SITR)-A3*SITR*(1-UN-SAI)
            AT=A3*(1-SAI-UN)+A1*(1-FAI)
        UY(I)=-A2*FY*D-FY*T3*0.5*D**2*(1-(1-SAI)**2)+0.5*D**2*T3*SITR*((
       C 1-SAI)**2-UN**2)-0.5*T3*FY*(UN*D)**2+A1*YR*(0.5*(SIBR+SITR)*
       C (1-FAI**2)-SITR*(1-FAI))
         URR(I)=A2*FY*D**2+FY*(T3/3)*((SAI*D-D)**3+D**3)-SITR*(T3/3)*
       C  (-(UN*D)**3-(SAI*D-D)**3)-FY*(T3/3)*(UN*D)**3-(FY/12)    *T1*(FAI
       C  *B1)**3+A1*B1**2*((1-FAI**4)*(SIBR+SITR)/16-(SITR/12)*(1-FAI**3
       C  )) +A2*FY*B2**2/12
        BET(I)=((T3/4)*((YB-UN*D)**4-(YB-D*(1-SAI))**4)+A1*YB*(B1**2*(1-
       CFAI**3)/12+YB**2*(1-FAI)))/XI-2*YN(I)
         RN(I)=SQRT((XI+YI(I))/AT+YN(I)**2)
            GO TO 8
C
C       COMPLETE YIELDING OF THE UPPER FLANGE AND PARTIAL YIELDING OF THE
C       LOWER FLANGE,BOTH IN COMPRESSION
C       CASE # 9
C
 78     JS=9
          IZ=20
          CALL NONLIN(2,5,IZ,2,X,0.01)
         IF(FAI.LT.0) GO TO 422
         IF(SAI.LT.0) GO TO 422
         IF(SAI.GT.1) GO TO 422
         IF(FAI.GT.1) GO TO 422
         IF(N1.GT.9) GO TO 422
          YS(I)=0.0
          YR=YS(I)
          YN(I)=YB-YS(I)
          CI(I)=0.0
         YI(I)=(FAI*B1)**3*T1/12
         YB=0.5*(1-SAI)**2*D**2/(FAI*A1+A3-SAI*A3)
         XI=(1-SAI)**3*D**3*T3/12+(1-SAI)*A3*(YB-0.5*D*(1-SAI))**2+FAI*A1*
       C YB**2
        BET(I)=((T3/4)*(YB**4-(YB+SAI*D-D)**4)+A1*YB*FAI*(YB**2+FAI**3*B1
       C **2/12))/XI-2*YN(I)
```

```
            URR(I)=FY*A2*(D**2+B2**2/12)+FY*(T3/3)*(D**3-(D-SAI*D)**3)-(T3/3)
          C *SITR*(D-SAI*D)**3+A1*B1**2*(FAI**4*(SIBR+SITR)/16-SITR*FAI**3/12
          C)+FY*A1*B1**2*(1-FAI**3)/12
           UR(I)=A2*FY+SAI*A3*FY-SITR*A3*(1-SAI)+FY*A1*(1-FAI)+A1*(0.5*FAI**2
          C *(SIBR+SITR)-FAI*SITR)
           UY(I)=A2*FY*D+(T3/2)*FY*(D**2-SAI**2*D**2)-(T3/3)*SITR*D**2*SAI**2
            RN(I)=SQRT((XI+YI(I))/AT+YN(I)**2)
             GO TO 8
C
C        COMPLETE YIELDING OF THE LOWER FLANGE IN TENSION
C          CASE # 10
C
 79          JS=10
             IZ=20
             CALL NONLIN(2,6,IZ,2,X,0.001)
             YS(I)=D
            YB=YS(I)
            CI(I)=B2**3*T2/12
             YI(I)=CI(I)
            YB=(A2*D+A3*(1-UN)*0.5*D*(1+UN))/(A2+A3*(1-UN))
             YN(I)=YB-YS(I)
            XI=A2*(D-YB)**2+T3*(D*(1-UN))**3/12+A3*(1-UN)*(YB-0.5*D*(1+UN))**
          C2
           BET(I)=(-A2*(D-YB)*(B2**2/12+(D-YB)**2)+(T3/4)*((YB-UN*D)**4-(D-YB
          C)**4))/XI-2*YN(I)
           UY(I)=A1*FY*D-0.5*SITR*T3*D**2*(1-UN)**2+FY*0.5*T3*(D**2-(D*(1-UN)
          C)**2)
            URR(I)=-A1*FY*(D**2+B1**2/12)-FY*(T3/3)*D**3*(1-UN**3)-SITR*(T3/
          C3)*(UN*D)**3+A2*B2**2*((SICR+SITR)/16-SITR/12)
            AT=A2+A3*(1-UN)
           UR(I)=-A1*FY-UN*A3*FY-SITR*A3*(1-UN)+0.5*(SICR-SITR)*A2
            RN(I)=SQRT((XI+YI(I))/AT+YN(I)**2)
             GO TO 8
C
C          PARTIAL YIELDING OF THE LOWER FLANGE IN TENSION PLUS
C          PART OF THE WEB
C          CASE # 4
C
 82          JS=5
             IF(COF.EQ.0) GO TO 79
             IZ=20
             X(1)=XM/XI
             X(2)=D-YF+P*XI/(AT*XM)
             CALL NONLIN(2,5,IZ,2,X,0.01)
            IF(N1.GT.7) GO TO 83
             IF(UN.LT.0) GO TO 83
             IF(UN.GT.1) GO TO 83
            IF(SAI.LT.0) GO TO 83
            IF(SAI.GT.1) GO TO 83
             CI(I)=T2*B2**3/12
             YI(I)=CI(I)+T1*B1**3*(1-SAI**3)/12
             YS(I)=D*T2*B2**3/(12*YI(I))
             YR=YS(I)
          YB=(A2*D+D*A3*0.5*(1-UN**2))/(A2+A3*(1-UN)+A1*(1-SAI))
             XI=A2*(D-YB)**2+YB**2*A1*(1-SAI)+(T3/12)*(D-UN*D)**3+T3*(D-UN*D
          &  )*(YB-0.5*(D+UN*D))**2
             YN(I)=YB-YS(I)
             BET(I)=((T3/4)*((YB-UN*D)**4-(D-YB)**4)+YB**3*A1*(1-SAI)-A2*(D-
          &  YB)**3-(D-YB)*T2*B2**3/12+YB*(1-SAI**3)*T1*B1**3/12)/XI-2*YN(I)
            AT=A2+A1*(1-SAI)+A3*(1-UN)
```

```
          UR(I)=A2*0.5*(SICR-SITR)+A1*(0.5*(SIBR+SITR)*(1-SAI**2)-SITR*(1
     &   -SAI))-D*(1-UN)*SITR*T3-UN*D*T3*FY-SAI*A1*FY
         UY(I)=0.5*A2*(SICR-SITR)*(D-YR)+A1*FY*YR*SAI-A1*(0.5*(SIBR+SITR)
     C*(1-SAI**2)-SITR*(1-SAI))+0.5*FY*T3*((UN*D-YR)**2-YR**2)+0.5*SITR
     C*T3*((D-YR)**2-(UN*D-YR)**2)
          RN(I)=SQRT((XI+YI(I))/AT+YN(I)**2)
          URR(I)=A2*(B2**2*(SICR+SITR)/16+(D-YR)**2*(SICR+SITR)*0.5-SITR*
     &    B2**2/12-SITR*(D-YR)**2)-SITR*T3*((YR-UN*D)**3+(D-YR)**3)/3-FY*
     &   T3*(YR**3-(YR-UN*D)**3)/3-FY*A1*(B1**2*SAI**3/12+YR**2*SAI)+A1*
     &    (B1**2*(SIBR+SICR)*(1-SAI**4)/16-SITR*B1**2*(1-SAI**3)/12+YR**2
     &    *0.5*(1-SAI**2)*(SIBR+SITR)-SITR*     YR**2*(1-SAI))
    8     CONTINUE
          JL=2*(N-1)
          DO 180 JF=1,JL
          DO 180 JU=1,JL
  180     DT(JF,JU)=0.
C
C       THIS PART COMPUTES THE
C         DERIVATIVES OF THE SECTION PROPERTIES
C
          DO 191 I=2,N
         Z=(I-1)*H
          D=BO3*(1+DEL*Z/AL)
          IPP=I-2
          IP=I-1
          IA=I+1
          IAA=I+2
          IF(I.EQ.2) GO TO 89
          IF(I.EQ.N) GO TO 88
          YSFO=(YS(IPP)-4*YS(IP)+6*YS(I)-4*YS(IA)+YS(IAA))/H**4
          YSTH=(-YS(IPP)+2*YS(IP)-2*YS(IA)+YS(IAA))/(2*H**3)
           GO TO 555
   89    YSFO=(YS(IAA)-4*YS(IA)+7*YS(I)-4*YS(IP))/H**4
         YSTH=(-YS(I)+2*YS(IP)-2*YS(IA)+YS(IAA))/(2*H**3)
           GO TO 87
   88     YSFO=(YS(IPP)-4*YS(IA)+7*YS(I)-4*YS(IP))/H**4
         YSTH=(-YS(IPP)+2*YS(IP)-2*YS(IA)+YS(I))/(2*H**3)
  555      CONTINUE
   87    CONTINUE
          YSPP=(YS(IA)-2*YS(I)+YS(IP))/H**2
          YSP=(YS(IA)-YS(IP))/(2*H)
          YR=YS(I)
          YIN=YI(I)
          YIP=(YI(IA)-YI(IP))/(2*H)
          YIPP=(YI(IA)-2*YI(I)+YI(IP))/H**2
          CIN=CI(I)
          CIP=(CI(IA)-CI(IP))/(2*H)
          CIPP=(CI(IA)-2*CI(I)+CI(IP))/H**2
          RP=(RN(IA)-RN(IP))/(2*H)
          BETAP=(BET(IA)-BET(IP))/(2*H)
          XN=BN(I)
          VP=CIP*D-YIP*YR
          VPP=CIPP*D-YIPP*YR
          YO=YN(I)
         DP=BO3*DEL/AL
          R=RN(I)
          BETA=BET(I)
          TJ=T(I)
          RS=UR(I)
          RAS=URR(I)
```

```
      RSP=(UR(I+1)-UR(I-1))/(2*H)
      RASP=(URR(I+1)-URR(I-1))/(2*H)
      YTN=UY(I)
      YTP=(UY(I+1)-UY(I-1))/(2*H)
      TJP=(T(I+1)-T(I-1))/(2*H)
      YOP=(YN(I+1)-YN(I-1))/(2*H)
C
C     THIS PART DEFINES THE DIFFERENT TERMS OF THE FINITE DIFFERENCE
C     EQUATIONS.
C
      U(1)=        -E*VP/H**3
      U(2)=(E/H**4)*(CIN*D**2-2*CIN*D*YR+YIN*YR**2)-(E/H**3)*(2*CIN*D
     & *DP-2*CIN*DP*YR-2*YR*CIP*D+YR**2*YIP-2*YSP*CIN*D+2*YSP*YR*YIN+
     & CIP*D**2)
     U(3)=2*E*VP/H**3+(E*VPP+P*YO+XM+YTN)/H**2+(-P*YOP-XMP-YTP)/(2*H)
     U(4)=-4*E*CIN*D*(D-YR)/H**4+E*(-2*VP*YR+(D-YR)*(4*CIN*DP+2*CIP*D))
    C/H**3+(-6*E*VP*YSP-E*VPP*YR+E*(D-YR)*(6*CIP*DP+CIPP*D)-G*TJ+P*R**2
    C-XM*BETA+RAS)/H**2-(-6*E*VP*YSPP-2*VPP*E*YSP+2*E*CIPP*DP*(D-YR)-G*
    CTJP+2*P*R*RP-XMP*BETA-XM*BETAP+RASP)/(2*H)
      U(5)=         -2*(E*VPP+P*YO+XM+YTN)/H**2
     U(6)=6*E*CIN*D*(D-YR)/H**4-2*(-6*E*VP*YSP-E*VPP*YR+E*(D-YR)*(6*CIP
    C*DP+D*CIPP)-G*TJ+P*R**2-XM*BETA+RAS)/H**2-(2*VP*YSTH+VPP*YSPP)*E
     U(7)=-2*E*VP/H**3+(E*VPP+P*YO+XM+YTN)/H**2+(XMP+P*YOP+YTP)/(2*H)
     U(8)=-4*E*CIN*D*(D-YR)/H**4-E*(-2*VP*YR+(D-YR)*(4*CIN*DP+2*CIP*D))
    C/H**3+(-6*E*VP*YSP-E*VPP*YR+E*(D-YR)*(6*CIP*DP+CIPP*D)-G*TJ+P*R**2
    C-XM*BETA+RAS)/H**2+(-6*E*VP*YSPP-2*VPP*E*YSP+2*E*CIPP*DP*(D-YR)-G*
    CTJP+2*P*R*RP-XMP*BETA-XM*BETAP+RASP)/(2*H)
      U(9)=        E*VP/H**3
      U(10)=(E/H**4)*(CIN*D**2-2*CIN*D*YR+YIN*YR**2)+(E/H**3)*(2*CIN*D
     & *DP-2*CIN*DP*YR-2*YR*CIP*D+YR**2*YIP-2*YSP*CIN*D+2*YSP*YR*YIN+
     & CIP*D**2)
      W(1)=E*(YIN/H**4-YIP/H**3)
      W(2)=        -E*(VP+2*CIN*DP-2*YIN*YSP)/H**3
      W(3)=-4*E*YIN/H**4+2*E*YIP/H**3+(E*YIPP+P+RS)/H**2-RSP/(2*H)
      W(4)=          2*E*(VP+2*CIN*DP-2*YIN*YSP)/H**3+(E*(VPP-6*(YSP
     & *YIP+YIN*YSPP-CIP*DP))+P*YO+XM+YTN)/H**2-(-2*E*YIPP*YSP-6*E*YIP*
     & YSPP-4*E*YIN*YSTH+2*E*CIPP*DP-P*YOP+XMP+YTP)/(2*H)
      W(5)=6*E*YIN/H**4-(P+E*YIPP+RS)*2/H**2
      W(6)=         -2*(E*(VPP-6*YIP*YSP-6*YIN*YSPP+6*DP*CIP)+P*YO+XM
     & +YTN)/H**2-E*(YIPP*YSPP+2*YIP*YSTH+YIN*YSFO)
      W(7)=-4*E*YIN/H**4-2*E*YIP/H**3+(P+YIPP*E+RS)/H**2+RSP/(2*H)
      W(8)=           -2*E*(VP+2*CIN*DP-2*YIN*YSP)/H**3+(E*(VPP-6*(YSP
     & *YIP+YIN*YSPP-CIP*DP))+P*YO+XM+YTN)/H**2+(-2*E*YIPP*YSP-6*E*YIP*
     & YSPP-4*E*YIN*YSTH+2*E*CIPP*DP-P*YOP+XMP+YTP)/(2*H)
      W(9)=E*YIN/H**4+E*YIP/H**3
      W(10)=        E*(VP+2*CIN*DP-2*YIN*YSP)/H**3
C
C     THIS PART ASSEMBLES THE DETERMINANT OF THE FINITE DIFFERENCE
C     EQUATIONS.
C
      IF(I.EQ.2) GO TO 71
      IF(I.EQ.N) GO TO 72
      K=2*I-2
      DO 19 J=1,JL
      IF(J-2*I.GT.2) GO TO 19
      IF(8+J-2*I.LT.1) GO TO 19
      DT(K,J)=W(8-2*I+J)
   19 CONTINUE
      DO 18 J=1,JL
      IF(J-2*I.GT.2) GO TO 18
```

```
             IF(8+J-2*I.LT.1) GO TO 18
             DT(K-1,J)=U(8-2*I+J)
      18     CONTINUE
             GO TO 75
      71     DT(1,1)=U(5)-U(1)
             DT(1,2)=U(6)-U(2)
             DT(2,1)=W(5)-W(1)
             DT(2,2)=W(6)-W(2)
             IF(NB.EQ.1)GO TO 222
             DT(1,1)=U(5)+U(1)
             DT(1,2)=U(6)+U(2)
             DT(2,1)=W(5)+W(1)
             DT(2,2)=W(6)+W(2)
      222    DO 96 J=7,10
             DT(1,J-4)=U(J)
      96     DT(2,J-4)=W(J)
             GO TO 75
      72     DT(2*N-3,2*N-3)=U(5)-U(9)
             DT(2*N-2,2*N-3)=W(5)-W(9)
             DT(2*N-3,2*N-2)=U(6)-U(10)
             DT(2*N-2,2*N-2)=W(6)-W(10)
             IF(NB.EQ.1)GO TO 223
             DT(2*N-3,2*N-3)=U(5)+U(9)
             DT(2*N-2,2*N-3)=W(5)+W(9)
             DT(2*N-3,2*N-2)=U(6)+U(10)
             DT(2*N-2,2*N-2)=W(6)+W(10)
      223    DO 97 J=1,4
             DT(2*N-3,2*N+J-8)=U(J)
      97     DT(2*N-2,2*N+J-8)=W(J)
      75     CONTINUE
      191    CONTINUE
      C
      C      HERE A SUBROUTINE IS CALLED TO COMPUTE THE
      C      VALUE OF THE ABOVE DETERMINANT.
      C
             IJOB=4
             D1=1.
            IF(AJ.NE.0) GO TO 817
            DO 669  IV=1,JL
            DO 669  JV=1,JL
      669   DG(IV,JV)=DT(IV,JV)
            IA=JL
            NN=JL
             CALL LINV3F(DT,B,IJOB,NN,IA,D1,D2,WKAREA,IER)
             FNC(IN)=D1*2**D2
            GO TO 93
      C
      C    THE PROPER ROW AND COLUMN ARE CANCELLED IN CASE OF INTERMEDIATE BRACIN
      C
      817    IA=JL
             JA=JL-2
              DO 840 NN=1,JA
             DO 840 MM=1,JA
             IF(NN.GT.2*NZ-4) GO TO 343
             IF(MM.GT.2*NZ-4) GO TO 341
             DC(MM,NN)=DT(MM,NN)
              GO TO 840
      341    IF(NN.GT.2*NZ-4) GO TO 344
             DC(MM,NN)=DT(MM+2,NN)
              GO TO 840
```

```
343     IF(MM.GT.2*NZ-4) GO TO 344
        DC(MN,NN)=DT(MM,NN+2)
        GO TO 840
344     DC(MM,NN)=DT(MM+2,NN+2)
840     CONTINUE
        DO 657 IV=1,JA
        DO 657 JV=1,JA
657      DG(IV,JV)=DC(IV,JV)
        NN=JL-2
         CALL LINV3F(DC,B,IJOB,NN,IA,D1,D2,WKAREA,IER)
         FNC(IN)=D1*2**D2
93       CONTINUE
        IF(FNC(1).EQ.0) GO TO 276
         DE=FNC(1)/FNC(2)
         IF(DE) 20,20,40
20       IF(XP(2)-XP(1)-2*ES) 30 ,30,70
276      ROOT=XP(1)
         GO TO 277
30       ROOT=(XP(1)+XP(2))/2
277      CONTINUE
        PRINT 461
461    FORMAT(//,3X,'OUTPUT')
       PRINT 462
462    FORMAT(   3X,'******')
        PRINT  463
463     FORMAT(///,2X,'BUCKLING  LOAD')
         WRITE(6,61) ROOT
61      FORMAT(F12.3)
        IF(AJ.NE.0) GO TO 617
         JA=JL-1
         GO TO 618
617      JA=JL-3
          JL=JL-2
618      CONTINUE
        DO 801 NS=1,JA
        DO 801 ND=1,JA
 801    DU(NS,ND)=DG(NS+1,ND+1)
        DO 802 NS=1,JA
802     DU(NS,JA+1)=-DG(NS+1,1)
        DO 888 K=1,JA
        TEMP=DU(K,K)
        DO 999 J=1,JL
999    DU(K,J)=DU(K,J)/TEMP
        DO 777 I=1,JA
         AAAAAA=DU(I,K)
         DO 777 J=1,JL
        IF(I.EQ.K) GO TO 777
        DU(I,J)=DU(I,J)-AAAAAA*DU(K,J)
777     CONTINUE
  888   CONTINUE
        PRINT 467
467      FORMAT(//2X,'MODE SHAPE')
        PRINT 466
466     FORMAT(//2X,'FI1',2X,'U2',2X,'FI2',3X,'U3',3X,'FI3',3X,'U4',2X,
      C'FI4',2X,'U5',2X,'FI5',3X,'U6',2X,'FI6',3X,'U7',3X,'FI7',2X,'U8',
      C2X,'FI8',2X,'U9',3X,'FI9')
         WRITE(6,803) (DU(NS,JL),NS=1,JA)
803     FORMAT(17F5.2)
         GO TO 9
70       XP(1)=(XP(2)+XP(1))/2
```

```
         GO TO 14
   40    TEN=XP(2)-XP(1)
         IF(TEN.LT.0.01*ES) GO TO 419
         XP(2)=XP(1)
         XP(1)=XP(2)-(TEN/2)
         GO TO 14
  419   PRINT 418
  418   FORMAT(/' THE SOLUTION IS MISSING.TRY ANOTHER ESTIMATE')
   9     CONTINUE
  401   CONTINUE
         STOP
        END
C
C       THIS SUBROUTINE ,WRITTEN BY DR K.BROWN AT THE UNIVERSITY OF
C       MINNESOTA,SOLVES A SYSTEM OF N SIMULTANEOUS NONLINEAR EQUATIONS
C       THE METHOD REQUIRES ONLY (N**2/2+3*N/2) FUNCTION EVALUATION PER
C       ITERATIVE STEP AS COMPARED WITH (N**2+N) EVALUATIONS FOR NEWTON'S
C       METHOD.THE METHOD DOES NOT REQUIRE THE USER TO FURNISH ANY
C       DERIVATIVES.HERE THE SUBROUTINE IS USED TO SOLVE THE EQUATIONS
C       OF EQUILIBRIUM AT ANY CROSS SECTION AND TO FIND THE LOCATION OF
C       THE NEUTRAL AND THE VALUE OF THE CURVATURE AT THE SAME POINT.
C

         SUBROUTINE NONLIN(N,NUMSIG,MAXIT,IPRINT,X,EPS)
         REAL     X(30),PART(30),TEMP(30),COE(30,31),RELCON,F
     &  ,FACTOR,HOLD,H,FPLUS,DERMAX,TEST
         DIMENSION ISUB(30),LOOKUP(30,30)
         COMMON JS,A1,A2,T3,SIBR,SICR,SITR,FY,BK,SAI,UN,D,P,XN,YF,A3
         COMMON BR,M1,FAI
         IFLAG=0.
         DELTA=1.E-7
         RELCON=10.E+0**(-NUMSIG)
         JTEST=1
         IF(IPRINT.EQ.1)PRINT 48
   48    FORMAT(1H1)
         DO 700 M=1,MAXIT
         IQUIT=0
         FMAX=0
         M1=M-1
         IF(IPRINT.NE.1)GO  TO 9
         PRINT  49,M1,(X(I),I=1,N)
   49    FORMAT(I5,3E18.8/(E23.8,2E18.8))
    9    DO 10 J=1,N
   10    LOOKUP(1,J)=J
         DO 500 K=1,N
         IF(K-1) 134,134,131
  131    KMIN=K-1
         CALL BACK(KMIN,N,X,ISUB,COE,LOOKUP)
  134    CALL AUXFCN(X,F,K)
         FMAX=AMAX1(FMAX,ABS(F))
         IF(ABS(F).GE.EPS)  GO TO 1345
         IQUIT=IQUIT+1
         IF(IQUIT.NE.N)GO TO 1345
         GO TO 725
 1345    FACTOR=0.001E+00
  135    ITALLY=0
         DO 200 I=K,N
         ITEMP=LOOKUP(K,I)
         HOLD=X(ITEMP)
         PREC=5.E-4
```

```
      ETA=FACTOR*ABS(HOLD)
      H=AMIN1(FMAX,ETA)
      IF(H.LT.PREC) H=PREC
      X(ITEMP)=HOLD+H
      IF(K-1) 161,161,151
151   CALL BACK (KMIN,N,X,ISUB,COE,LOOKUP)
161   CALL AUXFCN(X,FPLUS,K)
      PART(ITEMP)=(FPLUS-F)/H
      X(ITEMP)=HOLD
      IF(ABS(PART(ITEMP)).LT.DELTA)  GO TO 190
      IF(ABS(F/PART(ITEMP)).LE.1.E+15) GO TO 200
190   ITALLY=ITALLY+1
200   CONTINUE
      IF(ITALLY.LE.N-K)  GO TO 202
      FACTOR=FACTOR*10.0E+00
      IF (FACTOR.GT.11.)  GO TO 775
      GO TO 135
202   IF(K.LT.N)  GO TO 203
      IF(ABS(PART(ITEMP)).LT.DELTA)  GO TO 775
      COE(K,N+1)=0.0E+00
      KMAX=ITEMP
      GO TO 500
203   KMAX=LOOKUP(K,K)
      DERMAX=ABS(PART(KMAX))
      KPLUS=K+1
      DO 210 I=KPLUS,N
      JSUB=LOOKUP(K,I)
      TEST=ABS(PART(JSUB))
      IF(TEST.LT.DERMAX)  GO TO 209
      DERMAX=TEST
      LOOKUP(KPLUS,I)=KMAX
      KMAX=JSUB
      GO TO 210
209   LOOKUP(KPLUS,I)=JSUB
210   CONTINUE
      IF(ABS(PART(KMAX)).EQ.0.0)GO TO 775
      ISUB(K)=KMAX
      COE(K,N+1)=0.0E+00
      DO 220 J=KPLUS,N
      JSUB=LOOKUP(KPLUS,J)
      COE(K,JSUB)=-PART(JSUB)/PART(KMAX)
      COE(K,N+1)=COE(K,N+1)+PART(JSUB)*X(JSUB)
220   CONTINUE
500   COE(K,N+1)=(COE(K,N+1)-F)/PART(KMAX)+X(KMAX)
      X(KMAX)=COE(N,N+1)
      IF(N.EQ.1)  GO TO 610
      CALL BACK(N-1,N,X,ISUB,COE,LOOKUP)
610   IF(N-1)  650,650,625
625   DO 630 I=1,N
      IF(ABS(TEMP(I)-X(I)).GT.ABS(X(I))*RELCON)  GO TO 649
630   CONTINUE
      JTEST=JTEST+1
      IF(JTEST-3)650,725,725
649   JTEST=1
650   DO 660 I=1,N
660   TEMP(I)=X(I)
700   CONTINUE
      PRINT 1753
1753  FORMAT(/'                                        ')
      IF(IPRINT.NE.1)GO TO 800
```

```
              PRINT 1763
      1763    FORMAT('FUNCTION VALUES AT THE LAST APPROXIMATION FOLLOW:'/)
              IFLAG=1
              GO TO 7777
      725     IF(IPRINT.NE.1) GO TO 800
      7777    DO 750 K=1,N
              CALL AUXFCN(X,PART(K),K)
      750     CONTINUE
              IF(IFLAG.NE.1)GO TO 8777
              PRINT 7788,(PART(K),K=1,N)
      7788    FORMAT(3E20.8)
              GO TO 800
      8777    PRINT 751
      751     FORMAT(//'CONVERGENCE HAS BEEN ACHIEVED.THE FUNCTION VALUES')
              PRINT 7515,(PART(K),K=1,N)
      7515    FORMAT('AT THE FINAL APPROXIMATION FOLLOW:'//(3E20.8))
              GO TO 800
      775     PRINT 752
      752     FORMAT(//'MODIFIED JACOBIAN IS SINGULAR.TRY A DIFFERENT')
              PRINT 7525
      7525    FORMAT('INITIAL APPROXIMATION.')
      800     MAXIT=M1+1
              RETURN
              END
      C     THIS SUBROUTINE BACK-SOLVES THE FIRST KMIN ROWS OF A TRIANGULARI-
      C    -ZED LINEAR SYSTEM FOR IMPROVED X VALUES IN TERMS PREVIOUS ONES.

              SUBROUTINE BACK(KMIN,N,X,ISUB,COE,LOOKUP)
              DIMENSION  X(30),COE(30,31),ISUB(30),LOOKUP(30,30)
              DO 200 KK=1,KMIN
              KM=KMIN-KK+2
              KMAX=ISUB(KM-1)
              X(KMAX)=0.0E+00
              DO 100 J=KM,N
              JSUB=LOOKUP(KM,J)
              X(KMAX)=X(KMAX)+COE(KM-1,JSUB)*X(JSUB)
      100     CONTINUE
              X(KMAX)=X(KMAX)+COE(KM-1,N+1)
      200     CONTINUE
              RETURN
              END
      C    THIS SUBROUTINE COMPUTES THE BUCKLING LOAD OF A COLUMN

              SUBROUTINE COLUMN(X,ROOT)
              DIMENSION X(2),FNC(2)
              COMMON JS,A1,A2,T3,SIBR,SICR,SITR,FY,BK,SAI,UN,D,P,XM,YF,A3
              COMMON BR,M1,PAI
              EPS=1.0
             X(1)=0.1*FY
              X(2)=FY+SITR
              X(1)=(X(1)+X(2))/2
      314     DO 393 I=1,2
              BR=(FY+SITR-X(I))/(SIBR+SITR)
              BK=(FY+SITR-X(I))/(SICR+SITR)
              IF(BK.GT.1) GO TO 692
              GO TO 393
      692    BK=1
      393     FNC(I)=A2*(1-BK)*FY+A1*(1-BR)*FY+X(I)*BK*A2+X(I)*BR*A1+X(I)*A3
           &  -SITR*A3+A1*(0.5*BR**2*(SIBR+SITR)-SITR*BR)+A2*(0.5*BK**2*(SICR
           &  +SITR)-SITR*BK)-P
```

```
              DE=FNC(1)*FNC(2)
              IF(DE)302,302,304
      302     IF(X(2)-X(1)-2*EPS)303,303,307
      303     ROOT=(X(2)+X(1))/2
              GO TO 309
      307     X(1)=(X(1)+X(2))/2
              GO TO 314
      304     TEM=X(2)-X(1)
              X(2)=X(1)
              X(1)=X(2)-(TEM/2)
              GO TO 314
      309     CONTINUE
             RETURN
              END
      C
      C       THIS SUBROUTINE LISTS THE EQUATIONS OF EQUILIBRIUM FOR DIFFERENT
      C       YIELD PATTERNS.THE YIELD PARAMETERS ARE DEFINED BEFORE EACH
      C       EQUATION.
      C

              SUBROUTINE AUXFCN(X,Y,K)
              DIMENSION X(2)
              COMMON JS,A1,A2,T3,SIBR,SICR,SITR,FY,BK,SAI,UN,D,P,XM,YP,A3
              COMMON BR,M1,PAI
              IF(JS.EQ.1)GO TO 51
              IF(JS.EQ.2)GO TO 52
              IF(JS.EQ.3)  GO TO 57
              IF(JS.EQ.4)GO TO 36
              IF(JS.EQ.5)GO TO 37
             IF(JS.EQ.6)  GO TO 90
              IF(JS.EQ.7)  GO TO 98
             IF(JS.EQ.8)  GO TO 91
             IF(JS.EQ.9)  GO TO 250
              IF(JS.EQ.10)  GO TO 950
      91     GO TO (23,24),K
      23      CONTINUE
             UN=(D-X(2)-(FY-SITR)/X(1))/D
             SAI=(X(2)-(FY+SITR)/X(1))/D
              PAI=(X(1)*(D-X(2))+SITR-FY)/(SITR+SIBR)
             Y=A2*FY+SAI*A3*FY-UN*A3*FY-PAI*A1*FY+0.5*X(1)*(X(2)-SAI*D)**2*T3
            C -0.5*X(1)*(D-X(2)-UN*D)**2*T3-X(1)*A1*(1-PAI)*(D-X(2))+A1*(0.5
            C *(SIBR+SITR)*(1-PAI**2)-SITR*(1-PAI))-P-SITR*A3*(1-UN-SAI)
             RETURN
      24      CONTINUE
             UN=(D-X(2)-(FY-SITR)/X(1))/D
             SAI=(X(2)-(FY+SITR)/X(1))/D
              FAI=(X(1)*(D-X(2))+SITR-FY)/(SITR+SIBR)
             Y=A2*FY*X(2)+SAI*A3*FY*(X(2)-0.5*SAI*D)+UN*A3*FY*(D-X(2)-0.5*UN*D
            C)+PAI*A1*FY*(D-X(2))+(T3/3)*X(1)*(X(2)-SAI*D)**3+(T3/3)*X(1)*(D-
            CX(2)-UN*D)**3+X(1)*A1*(1-PAI)*(D-X(2))**2-P*(YP+X(2)-D)-A1*(D-X(2)
            C)*(0.5*(SIBR+SITR)*(1-PAI**2)-SITR*(1-PAI))-XM+SITR*A3*(1-UN-SAI)*
            C 0.5*(D-2*X(2)-UN*D+SAI*D)
              RETURN
      51      GO TO (1,2),K
      1       CONTINUE
              BK=(FY+SITR-X(1)*X(2))/(SICR+SITR)
              Y=A2*FY*(1-BK)-A3*SITR+X(1)*X(2)*BK*A2-A1*X(1)*(D-X(2))+0.5*X(2
            & )**2*T3*X(1)-0.5*(D-X(2))**2*X(1)*T3+0.5*A1*(SIBR-SITR)+A2*(BK
            & **2*0.5*(SICR+SITR)-BK*SITR)-P
              RETURN
```

```
2       CONTINUE
        BK=(FY+SITR-X(1)*X(2))/(SICR+SITR)
        Y=-P*(YF-D+X(2))-0.5*A1*(D-X(2))*(SIBR-SITR)+X(2)*A2*(0.5*BK**2
     &  *(SICR+SITR)-BK*SITR)+A3*SITR*(D/2-X(2))+X(1)*X(2)**2*BK*A2+A1
     &  *X(1)*(D-X(2))**2+(1./3)*T3*X(1)*X(2)**3+(1./3)*(D-X(2))**3*T3*
     &  X(1)-XM+A2*FY*(1-BK)*X(2)
        RETURN
52      GO TO (3,4),K
3       CONTINUE
        BK=(FY+SITR-X(1)*X(2))/(SICR+SITR)
        SAI=(X(1)*(D-X(2))+SITR-FY)/(SITR+SIBR)
        UN=(SITR-FY+X(1)*(D-X(2)))/(X(1)*D)
        Y=0.5*A1*((SITR+SIBR)*(1-SAI**2)-2*SITR*(1-SAI))+A2*(0.5*BK**2*
     &  (SICR+SITR)-BK*SITR)+X(1)*X(2)*BK*A2-X(1)*(D-X(2))*A1*(1-SAI)+
     &  0.5*X(2)**2*X(1)*T3-0.5*X(1)*T3*(D-X(2)-UN*D)**2-   (1-UN)*SITR*
     &  A3+A2*(1-BK)*FY-UN*  A3*FY-SAI*A1*FY-P
        RETURN
4       CONTINUE
        SAI=(X(1)*(D-X(2))+SITR-FY)/(SITR+SIBR)
        BK=(FY+SITR-X(1)*X(2))/(SICR+SITR)
        UN=(SITR-FY+X(1)*(D-X(2)))/(X(1)*D)
        Y=-P*(YF+X(2)-D)-A1*(D-X(2))*(0.5*(SITR+SIBR)*(1-SAI**2)-SITR*(1
     &  -SAI))+A2*X(2)*(0.5*BK**2*(SICR+SITR)-SITR*BK)+X(1)*X(2)**2*BK*
     &  A2+X(1)*(D-X(2))**2*A1*(1-SAI)+(T3/3)*X(1)*X(2)**3+(X(1)/3)
     &  *T3*(D-X(2)-UN*D)**3-   (1-UN)*SITR*(X(2)-0.5*D*(1-UN))*A3+A2*(1
     &  -BK)*FY*X(2)+UN*  A3*FY*(D-X(2)-0.5*UN*D)+SAI*A1*FY*(D-X(2))-XM
        RETURN
57      GO TO (58,59),K
58      CONTINUE
        UN=(X(2)-(FY+SITR)/X(1))/D
        Y=A2*FY+UN*A3*FY-SITR*(1-UN)*A3-X(1)*(D-X(2))*A1+0.5*A1*(SIBR-
     &  SITR)+X(1)*(X(2)-UN*D)**2*0.5*T3-X(1)*(D-X(2))**2*0.5*T3-P
        RETURN
59      CONTINUE
        UN=(X(2)-(FY+SITR)/X(1))/D
        Y=A2*FY*X(2)+UN*A3*FY*(X(2)-0.5*UN*D)+X(1)*(D-X(2))**2*A1-0.5*A1
     &  *(SIBR-SITR)*(D-X(2))-P*(X(2)+YF-D)-XM+A3*(1-UN)*(D-X(2)-0.5*D*
     &  (1-UN))*SITR+(T3/3)*X(1)*(X(2)-UN*D)**3+(T3/3)*X(1)*(D-X(2))**3
        RETURN
36      GO TO (32,33),K
32      CONTINUE
        BK=(FY+SITR-X(1)*X(2))/(SICR+SITR)
        BR=(FY+SITR-X(1)*(X(2)-D))/(SIBR+SITR)
        Y=A2*(0.5*BK**2*(SITR+SICR)-SITR*BK)+A1*(0.5*BR**2*(SIBR+SITR)
     &  -SITR*BR)-SITR*A3+X(1)*A1*BR*(X(2)-D)+X(1)*X(2)*A2*BK+(1-BK)*A2
     &  *FY+(1-BR)*A1*FY+A3*X(1)*0.5*(2*X(2)-D)-P
       RETURN
33      CONTINUE
        BK=(FY+SITR-X(1)*X(2))/(SICR+SITR)
        BR=(FY+SITR-X(1)*(X(2)-D))/(SIBR+SITR)
        Y=X(2)*A2*(0.5*BK**2*(SITR+SICR)-SITR*BK)+A1*(X(2)-D)*(0.5*BR
     &  **2*(SIBR+SITR)-SITR*BR)-P*(X(2)-D+YF)+A1*BR*X(1)*(X(2)-D)**2
     &  +A2*BK*X(1)*X(2)**2+(1-BK)*A2*FY*X(2)+(1-BR)*A1*FY*(X(2)-D)+X(1
     &  )*A3*(X(2)-D)*(X(2)-D/2)+X(1)*D*A3*0.5*(X(2)-D/3)-XM
        RETURN
37      GO TO (38,39),K
38      CONTINUE
        SAI=(X(1)*(D-X(2))+SITR-FY)/(SITR+SIBR)
        UN=(SITR-FY+X(1)*(D-X(2)))/(X(1)*D)
        Y=A1*(0.5*(SITR+SIBR)*(1-SAI**2)-SITR*(1-SAI))+A2*0.5*(SICR-SITR
```

```
      &   )+X(1)*X(2)*A2-X(1)*(D-X(2))*A1*(1-SAI)+0.5*X(2)**2*X(1)*T3-0.5
      &   *X(1)*T3*(D-X(2)-UN*D)**2-D*(1-UN)*SITR*T3-UN*D*T3*FY-SAI*A1*FY
      &   -P
          RETURN
 39       CONTINUE
          SAI=(X(1)*(D-X(2))+SITR-FY)/(SITR+SIBR)
          UN=(SITR-FY+X(1)*(D-X(2)))/(X(1)*D)
          Y=-P*(YF+X(2)-D)-A1*(D-X(2))*(0.5*(SITR+SIBR)*(1-SAI**2)-SITR*
      &   (1-SAI))+A2*X(2)*0.5*(SICR-SITR)+X(1)*X(2)**2*A2+X(1)*(D-X(2))**
      &   2*A1*(1-SAI)+(T3/3)*X(1)*X(2)**3+(X(1)/3)*T3*(D-X(2)-UN*D)**3-D
      &   *(1-UN)*SITR*(X(2)-0.5*D*(1-UN))*T3+UN*D*T3*FY*(D-X(2)-0.5*UN*D
      &   )+SAI*A1*FY*(D-X(2))-XM
          RETURN
 98       GO TO (25,26),K
 25       CONTINUE
          UN=(SITR-FY+X(1)*(D-X(2)))/(X(1)*D)
        BK=(FY+SITR-X(1)*X(2))/(SICR+SITR)
         Y=-A1*FY+A2*(1-BK)*FY-UN*A3*FY+0.5*X(1)*X(2)**2*T3+BK*A2*X(1)*X(
      C 2)-0.5*X(1)*(D-X(2)-UN*D)**2*T3-SITR*D*(1-UN)*T3+A2*(0.5*BK**2
      C*(SICR+SITR)-BK*SITR)-P
         RETURN
 26       CONTINUE
          UN=(SITR-FY+X(1)*(D-X(2)))/(X(1)*D)
        BK=(FY+SITR-X(1)*X(2))/(SICR+SITR)
        Y=A1*FY*(D-X(2))+A2*(1-BK)*FY*X(2)+UN*D*T3*FY*(D-X(2)-0.5*D*UN)+
      C(T3/3)*X(1)*(D-X(2)-UN*D)**3+(T3/3)*X(1)*X(2)**3+X(2)*A2*(0.5*BK**
      C2*(SICR+SITR)-BK*SITR)+BK*A2*X(1)*X(2)**2+SITR*A3*(1-UN)*(D-X(2)-
      C0.5*D*(1+UN))-XM-P*(YF+X(2)-D)
         RETURN
 950     GO TO (951,952),K
 951      CONTINUE
          UN=(SITR-FY+X(1)*(D-X(2)))/(X(1)*D)
          Y=-A1*FY-UN*A3*FY+0.5*X(1)*X(2)**2*T3+A2*X(1)*X(2)-0.5*X(1)*(D-
      C X(2)-UN*D)**2*T3-SITR*A3*(1-UN)+0.5*A2*(SICR-SITR)-P
         RETURN
 952     CONTINUE
          UN=(SITR-FY+X(1)*(D-X(2)))/(X(1)*D)
         Y=A1*FY*(D-X(2))+UN*A3*FY*(D-X(2)-0.5*UN*D)+(T3/3)*X(1)*(D-X(2)-
      C UN*D)**3+(T3/3)*X(1)*X(2)**3+0.5*X(2)*A2*(SICR-SITR)+A2*X(1)*X(2)
      C **2+SITR*A3*(1-UN)*(D-X(2)-0.5*D*(1+UN))-XM-P*(YF+X(2)-D)
         RETURN
 90      GO TO (21,22),K
 21       CONTINUE
         SAI=(X(2)-(FY+SITR)/X(1))/D
        UN=(D-X(2)-(FY-SITR)/X(1))/D
        Y=A2*FY-A1*FY+SAI*A3*FY-UN*A3*FY-SITR*A3*(1-UN-SAI)+0.5*X(1)*
      C (X(2)-SAI*D)**2*T3-0.5*X(1)*(D-X(2)-UN*D)**2*T3-P
         RETURN
 22       CONTINUE
         SAI=(X(2)-(FY+SITR)/X(1))/D
        UN=(D-X(2)-(FY-SITR)/X(1))/D
        Y=A1*FY*(D-X(2))+A2*FY*X(2)+SAI*A3*FY*(X(2)-0.5*SAI*D)+UN*A3*FY*
      C(D-X(2)-0.5*UN*D)+(T3/3)*X(1)*(X(2)-SAI*D)**3+(T3/3)*X(1)*(D-X(2
      C )-UN*D)**3+P*(D-X(2)-YF)-XM+SITR*A3*(1-UN-SAI)*(D-2*X(2)-UN*D+SAI
      C *D)*0.5
         RETURN
 250     GO TO (251,252), K
 251     CONTINUE
        SAI=(X(2)-(FY+SITR)/X(1))/D
         FAI=(FY+SITR-X(1)*(X(2)-D))/(SIBR+SITR)
```

```
      Y=A2*FY+SAI*A3*FY+(1-FAI)*A1*FY-SITR*A3*(1-SAI)+A1*(0.5*FAI**2*(
     C SIBR+SITR)-FAI*SITR)+X(1)*(X(2)-D)*A1*FAI+X(1)*A3*(1-SAI)*(X(2)-
     C 0.5*D*(1+SAI))-P
      RETURN
  252 CONTINUE
      FAI=(FY+SITR-X(1)*(X(2)-D))/(SIBR+SITR)
      SAI=(X(2)-(FY+SITR)/X(1))/D
      Y=A2*FY*X(2)+SAI*A3*FY*(X(2)-0.5*SAI*D)+(1-FAI)*A1*FY*(X(2)-D)-
     CSITR*A3*(1-SAI)*(X(2)-0.5*D-0.5*SAI*D)+A1*(X(2)-D)*(0.5*FAI**2*(
     C SIBR+SITR)-FAI*SITR)+X(1)*(X(2)-D)**2*A1*FAI-P*(X(2)+YP-D)+X(1)*
     C A3*(X(2)-D)*(1-SAI)*(X(2)-0.5*D-0.5*SAI*D)+0.5*X(1)*D**2*T3*(1-
     C SAI)**2*(X(2)-SAI*D-D*(1-SAI)/3)
      RETURN
      END

$EXEC
```

```
 INPUT DATA    MEMBER # 1
**********

INCH AND KIPS ARE TO BE USED FOR ALL UNITS

WIDTH-L.FLG. THICK.L.FLG. WIDTH-UPP.FLG. THICK.UPP.FLG. DEPTH WEB-THICK. TAPER RATIO

     5.0000      0.1800         5.0000         0.312        10.0000     0.3750     0.6000

NUM.SEG. B.C. E ANAL.TYPE. G MOM.RATIO YIELD STR. LOC.BRACE R.S./FY SPEC.LOAD LENGTH

   10    1  29000.0  1  11200.0 -0.51 990.00       0.000    0.000      0.000 120.00

ACCEPTABLE ERROR  ESTIMATED BUCKLING LOAD

       1.0000       2900.0000
*** TERMINAL ERROR            (IER = 129) FROM IMSL ROUTINE LUDATF
*** TERMINAL ERROR            (IER = 130) FROM IMSL ROUTINE LINV3F

  OUTPUT
  ******

 BUCKLING  LOAD
   2853.267

 MODE SHAPE

 FI1  U2  FI2  U3   FI3  U4  FI4  U5  FI5  U6  FI6  U7   FI7  U8  FI8  U9   FI9
0.05 2.01 0.12 2.97 0.21 3.81 0.29 4.40 0.36 4.62 0.39 4.32 0.37 3.41 0.30 1.91 0.17

STATEMENTS EXECUTED=  283529

CORE USAGE       OBJECT CODE=   91832 BYTES,ARRAY AREA=   16360 BYTES,TOTAL AREA AVAI

DIAGNOSTICS        NUMBER OF ERRORS=       0, NUMBER OF WARNINGS=       0, NUMBER OF

COMPILE TIME=     0.53 SEC,EXECUTION TIME=     2.12 SEC,     21.30.29    WEDNESDAY

        C$BTCHEND
```

APPENDIX E

DATA REGRESSION ANALYSIS

The data generated to develop the design aid in Chapter IV was statistically analyzed using linear regression analysis technique. The Statistical Analysis System (SAS) program[78] was used to develop linear regression equations for data used to develop the curves shown in Figures 4.2 through 4.6. The parameters used in these equations were previously defined in Chapter IV. Equation B.1 specifies the critical axial stress as a function of ℓ/r_{yo}, A_2/A_1, ρ and γ. Equations B.2 and B.3 specify the critical stress in the compression flange of beams for each of two moment gradients as a function of ℓ/r_{To}, A_2/A_1, ρ and γ. Values of the moment modification factor are given by Equations B.4 and B.5 as a function of P/P_{cr}, ℓ/r_{yo}, A_2/A_1, ρ and γ.

Critical stress for columns:

$$\frac{\sigma_a}{\sigma_y} = 1.456 - 0.0057\frac{\ell}{r_{yo}} - 0.2333\frac{A_2}{A_1} - 0.1091\rho + 0.009486\gamma \qquad (R^2 = 0.947) \tag{B.1}$$

Critical stress for beams subjected to one end moment:

$$\frac{\sigma_b}{\sigma_y} = 1.092 - 0.00368\frac{\ell}{r_{To}} - 0.0236\frac{A_2}{A_1} + 0.0215\rho - 0.04786\gamma \qquad (R^2 = 0.884) \tag{B.2}$$

Critical stress for beams subjected to two end moments (Basic Case):

$$\frac{\sigma_b}{\sigma_y} = 0.8419 - 0.0029\frac{\ell}{r_{To}} - 0.0107\frac{A_2}{A_1} + 0.01085\rho - 0.0165\gamma \qquad (R^2 = 0.914) \tag{B.3}$$

Moment modification factor (one end moment):

$$\delta = 1.32 - 0.991 \frac{P}{P_{cr}} - 0.0849 \frac{A_2}{A_1} - 0.0512\rho + 0.0017\gamma - 0.00048 \frac{\ell}{r_{yo}}$$

$(R^2 = 0.92)$ (B.4)

Moment modification factor (Basic Case):

$$\delta = 1.37 - 0.9817 \frac{P}{P_{cr}} - 0.09776 \frac{A_2}{A_1} - 0.0563\rho - 0.00728\gamma - 0.00024 \frac{\ell}{r_{yo}}$$

$(R^2 = 0.886)$ (B.5)