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R.S. PRABHU

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BUCKLING OF CYLINDRICAL SHELLS

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APPROVED BY

Thomas M. Murray

Fulton K. Fears

Michael G. Szmuchin

Lloyd A. Iverson

DISSERTATION COMMITTEE

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LIST OF SYMBOLS

A_0, A_1, A_{mn} B_0, B_1, B_{mn} C_{mn}	= arbitrary constants of the assumed prebuckling and buckling displacement functions.
a_0	= half length of the imperfection in the axial direction
a_1	= sum of half length and the decay length in the axial direction
b_0	= half width of the imperfection in circumferential direction
b_1	= sum of half width and the decay length in the circumferential direction
C_1, C_2	= stretching rigidity $(ET_1/(1-\nu^2))$
C_a, C_{2a}	= general symbol for (Cos(a)/a, etc.,)
D_1, D_2	= bending rigidity of the shell and the imperfection $[ET_1^3/(12*(1-\nu^2))]$
D_{2a}, D_{2b}	= as defined in equation 3.19
d_a, d_c	= location of the imperfection in the axial and in the circumferential direction
E	= stiffness of the material
G	= elements of the prebuckling matrix defined in 3.18
i, j, k	= running integers in the series functions
L	= half length of the cylindrical shell

L_a, L_c = decay lengths in the axial and circumferential direction

M, N = derived imperfection parameters

N_x = axial load in X direction

P = potential energy (general)

$P(\varepsilon)$ = potential energy for stretching

$P(\chi)$ = potential energy for bending

$P(u)$ = potential energy in terms of the displacement functions

p = pressurized parameter

R = principal radius of the shell

r = ratio of decay length to half length of rectangular imperfection in the axial direction

S_a, S_b, S_{2a}, S_{2b} = general symbol for (Sin(a)/a) etc.,

s = ratio of decay length to half width of rectangular imperfection in the circumferential direction

T_1 = thickness of the shell

T_2 = thickness of the imperfection

T_a, T_b, T_{2a}, T_{2b} = as defined in equation 3.19 of prebuckling

u^0, v^0, w^0 = assumed displacement function in the axial, circumferential and normal direction for unstressed state

u, v, w = combined axial, circumferential and normal direction displacement function for the stressed state

$u_{,x}, v_{,x}, w_{,x}$ = partial differentiation of the displacement functions with respect to x once.

$u_{,xx}, v_{,xx}, w_{,xx}$ = partial differentiation of the displacement functions with respect to x twice.

$u_{,y}, v_{,y}, w_{,y}$ = partial differentiation of the displacement functions with respect to y once

$u_{,yy}, v_{,yy}, w_{,yy}$ = partial differentiation of the displacement functions with respect to y twice

X,Y = global axis of coordinates

Z = cylinder geometry parameter

$$(1-\nu^2)^{1/2} * L^2 / RT_1$$

α = as defined in Equation 3.2a

β = as defined in Equation 3.2a

ε_{a_0} = ratio of the area of imperfection to the surface area of the shell.

ε_{a_1} = ratio of the distorted area of the imperfection to the surface area of the shell

ε_b = bending stiffness parameter

ε_s = membrane stiffness parameter

ε^a = resultant stretching strain due to the imperfection

ε^0 = strains in the unstressed state

ε^T = strains in the stressed state

ζ = as defined in Equation 3.11b

η	=	as defined in Equation 3.1b
θ	=	as defined in Equation 3.1b
κ	=	as defined in Appendix C-2
λ	=	axial compressive load parameter
μ	=	rectangular imperfection parameter
ν	=	Poisson's ratio
ξ	=	as defined in Equation 3.11b
ρ	=	pressurization parameter
τ	=	as defined in Appendix C-2
ϕ	=	as defined in Appendix C-2
Φ	=	as defined in Appendix C-2
χ^a	=	resultant curvature strain due to imperfection
χ^0	=	curvature strains in the unstressed state
χ^T	=	curvature strains in the stressed state

ABSTRACT

The purpose of this study is to illustrate how the Koiter's concept of initial-postbuckling behavior can be applied to thin shell structures which exhibit rectangular imperfections. Throughout the study, shallow shell strain displacement is assumed in deriving the final potential energy expression for the buckling analysis.

Decay length in the circumferential direction is varied to obtain the minimum potential energy since the shell buckling has a weak dependence on the shell curvature. Linear prebuckling is assumed for the displacements in the axial direction and for mid-surface displacements, but from the definition of prebuckling the effect of nonlinearity with repect to the axial load is considered.

A computer program to calculate the ratio of bifurcation point load of the imperfect shell to the classical axial load of the perfect shell is developed.

CHAPTER 1

INTRODUCTION

No thin cylindrical shell structure is free of imperfection. The purpose of this study is to illustrate how the Koiter's concept of initial-postbuckling behavior can be applied to cylindrical shell structures which have rectangular finite imperfections. A rectangular imperfection is defined as a rectangular area where the thickness of the imperfection is greater or smaller than the thickness of the remainder of the shell. A rectangular cutout is considered to be a finite imperfection of the shell.

Review of Previous Research

Many thin cylindrical shell structures require circular or rectangular openings and therefore, considerable research has been conducted on the subject. Previous research has been concerned with the experiments of shells having a single cutout or shells having diametrically opposite cutouts.

Starnes [1] has conducted experiments on the effect of a single circular or rectangular cutout in cylindrical shells and has found that the prebuckling is dependent on the non-dimensional geometric parameter $\bar{r} = r/(R*T_1)^{1/2}$ where r is characteristic hole dimension, R is the principal radius of the shell, and T_1 is the shell thickness. An approximate theoretical analysis was also done by Starnes [1] using the Rayleigh-Ritz type of approach. In another paper [2] based on the work of Lekkerkerker [3], Starnes has shown that the parameter

$$\mu = (1/2)*(12*(1-\nu^2))^{1/4}*(r^2/R*T)^{1/2}$$

where ν is Poissons ratio, governs the solution of the prebuckling stress distribution and displacements for a circular hole in its side. Starnes showed that the increase in the prebuckling stress field due to cutouts is very much restricted to the local area of the cutout region and contains both membrane and bending stress increments. Such stress increments are found to be maximum at the cutouts and decay rapidly away from the cutout.

Kraus [4] has provided a formula

$$L_c = \pi(R^2*T^2/3*(1-\nu^2))^{1/4}$$

for axial or circumferential decay lengths in cylindrical shells.

A theoretical and experimental study of the behavior of cylindrical shells with rectangular diametrically opposite cutouts and subjected to axial compressive load was presented by Almroth and Holmes [5,6,7,8]. The analytical studies were made using the computer program STAGS. The method predicted the experimentally observed shell behavior within a reasonable accuracy.

Almroth and Holmes studied the influence of reinforcements around the cutouts and they showed that the critical load was increased by use of the reinforcement, but because of the effect of random geometrical imperfection in the reinforcement, the agreement between the test and theory was not as good as for the cylinders with unreinforced cutouts. In addition, Almroth [8] has used a modified version of the computer program STAGS for geometric nonlinear analysis of shells of general shape with general imperfections. The modified STAGS program uses a bifurcation type analysis and Almroth found this approach does not represent a true solution to the physical problem. From his geometrical nonlinear analysis it was concluded that the shell structure allows considerable redistribution of stresses so the bifurcation buckling load may be only a small fraction of the collapse load. The reason for this is considerable distortion in geometry in the vicinity of the cutout. This fact was verified by comparison with the test results reported in Reference 8.

Gorman et al [9] conducted an experiment on the buckling of cylindrical shells with general imperfections and studied the effect of large prebuckling deformation by photoelastic techniques. These techniques provided a means to observe and record the rapid changes that take place during the transition from prebuckling to the buckling mode. Gorman *et al* used Donnel type equilibrium equations and the Galerkin method to obtain theoretical stresses. They did not consider the initial-postbuckling effects in their analytical studies. The effect of nonuniform prebuckling deformations caused by edge supports which reduces the buckling loads of cylindrical shell was not taken into account in the theoretical analysis but realized in the test. Hence the assumptions used in theory did not agree with the experimental conditions.

Almroth [10] defined prebuckling as a rotation of the structural elements before buckling, thus introducing the concept of nonlinearity with respect to the load but not with respect to the geometry. According to Reference 10 the ultimate load of shell with a cutout when subjected to axial compressive load is governed by collapse after considerable distortion of geometry by bifurcation buckling.

A survey article [11] on postbuckling theory shows that numerous papers describing theories for imperfections are available. Yet there is no general method to find the realistic bifurcation load for a thin shallow cylindrical shell of finite length having a rectangular imperfection subjected to axial compressive load. The survey article further suggests calculations to explore the possibilities of initial-postbuckling theory are necessary to predict the realistic bifurcation point load for thin imperfect cylindrical shells.

Most of the above cited literatures places emphasis on the experimentally determined collapse load. The analytical studies reported in these literatures considered only analysis based on bifurcation type buckling. The predicted bifurcation buckling loads were only a fraction of the experimentally determined collapse load. Studies [5,6,7,8] of general shells with cutouts, some of which were for cylindrical shells, have shown that the analysis of shells subjected to moderate axial loads must include the effect of geometrical nonlinearity. Therefore, an initial-postbuckling analysis must be made if a realistic buckling is to be obtained.

Initial-postbuckling theory as developed by Koiter [12], is suited for the development of the differential equation for potential energy of an imperfect thin cylindrical shell subjected to axial compressive load. Koiter used the principle of minimum potential energy to derive an asymptotic technique that gives an approximation for initial-postbuckling behavior and imperfection sensitivity of structures. Koiter's technique, and parallel methods that have been developed independently by other researchers, have been used to investigate the behavior of a number of thin shell structures [13,14,15,16]. A necessary and absolute condition to apply Koiter's method is the determination of the primary equilibrium path of the perfect structure U, as an explicit function of the applied load parameter λ. The main feature of Koiter's method of analysis is that, after the primary equilibrium path has been determined, the remaining nonlinear problem is converted into a series of linear problems. Because the interest is pointed towards the region of the bifurcation load the nonlinear analysis is reduced to an equivalent of one or two linear analyses. This is only a slight departure from the linearized stability analysis but is a lot easier than the direct nonlinear stability analysis.

The stability of thin cylindrical shells is more complex in nature since they are very sensitive to any sort of imperfection that is inherent in the shell structures. For an imperfect shell according to the Koiter's theory there are two stages of buckling, one is the prebuckled stage (primary equilibrium path) and the other the buckled stage (nonlinear behavior). Hence it is realistic to consider the linear prebuckling behavior because during the initial stages of loading the elements of the structure are considered to be rotation free,that is, the geometry of the shell structure is substantially unaltered prior to buckling [16], and also if this primary equilibrium path is not linear the analytical application of Koiter's method becomes a difficult task.

The concept of initial-postbuckling theory is developed by Koiter [12] appealed to the author as the most efficient and realistic analytical approach that can be fitted to the present study.

Method of Analysis

In this presentation,the prebuckling stages are analyzed by writing the potential energy expression for both stretching and bending. A suitable displacement function in conformity with the nature of the imperfection is assumed and integrated over the area of the imperfection and the shell. The important factors that have to be considered for the assumption of the displacement function is the possible mode of buckling in the vicinity of the imperfection and the effect of discontinuity due to the presence of the rectangular imperfection. The integrated potential energy expression for the prebuckling stage, including the imperfect displacement function, is then partially differentiated with respect to the constants in the assumed displacement function. The result is a symmetric matrix on the left hand side and the load matrix on the right hand side. The order of the matrix will be equal to the number of constants present in the assumed displacement function. The prebuckling equations are then solved to determine the constants of the assumed displacement function for a unit applied load parameter λ. The constants are then substituted into the buckling differential equation.

The buckling differential equation is derived by using Koiter's stability concept, which is as follows: Let the displacement of the equilibrium stage be denoted by U_i and those of the neighboring state by $(U_i + u_i)$. Then the energy criterion of stability requires that the potential energy of the structure in the basic equilibrium state is,

$$P(U_i + u_i) \geq P(U_i) \quad \ldots\ldots \quad (1.1)$$

where the potential energy P is the integral over the entire structure. The integrands of $P(U_i + u_i)$ may be expanded using Taylor expansion. For a homogeneous function of M^{th} order in U_i, the sum of the integrals for the stability criterion is

$$P(U_i + u_i) - P(U_i) = P_1(u) + P_2(u) + P_3(u) + P_4(u) + \ldots\ldots \geq 0 \qquad (1.2)$$

[the subscripts for u are left out for simplicity]

where $P_1(u)$ are the first order terms of the integrand, $P_2(u)$ are the second order terms, etc. For an equilibrium state $P_1(u) = 0$, $P_2(u) \geq 0$ establishes the stability criterion, if $P_2(u)$ vanishes the criterion for stability must be reexamined. The necessary and absolute condition is then

$$P_3(u) = 0 \quad \text{and} \quad P_4(u) \geq 0 \qquad (1.3)$$

and so forth. It is significant that Koiter's initial-postbuckling technique is applicable only to structure which exhibit a bifurcation point type of buckling. Taking the second order terms for the stability criterion, a suitable buckling displacement series solution is assumed. The assumed displacement function as a solution of the stability expression must be continuously differentiable, should have finite integrals and should exhibit uniform convergence. The Fourier double series so assumed represents various buckling modes both in the axial and in the circumferential direction that the shell structure can assume before it bifurcates to the buckling configuration. The derivatives of the displacement functions are then included in the buckling differential equation, which is to be integrated over the area of imperfection and the shell for both stretching and bending. The integrated stability equation is then partially differentiated with respect to the constants of the assumed buckling displacement function to form a buckling determinant. Since the linear prebuckling is assumed, the analysis of stability is relatively simple. All the prebuckling displacement components can be expressed as a load factor λ and their values for a unit load. There is no need in that case to evaluate the determinant of the coefficient matrix for a series of increasing values of the load parameter. The critical value of λ can be directly determined as the eigenvalue λ_{cr} defined by an equation of the form

$$\mathrm{Det}[\, A + \lambda B \,] = 0 \qquad .....(1.4)$$

for the buckling analysis.

Organization of the Present Study

The first part of Chapter 1 describes experiments on the imperfect cylindrical shell structure subjected to axial compressive load. However, it seemed desirable to give a brief summary on the initial-postbuckling theory developed by Koiter which is best suited for imperfect cylindrical shells subjected to axial compressive load.

In Chapter 2 the general potential energy expression is derived for the buckling analysis considering the initial imperfections as the coefficients of derivatives of the buckling displacement functions. Nonlinear strain displacement relations are considered while deriving the resultant strains due to the initial and total strains. These resultant strains are substituted in the potential energy expression and the concept of Koiter is fitted (second order terms for stability) to obtain the final potential energy expression for the buckling analysis.

In Chapter 3, the prebuckling analysis is carried out by considering the linear strain displacement relationship. The effect of circumferential decay length is included in assuming the prebuckling displacement functions. The effect of prebuckling on the stability of a cylindrical shell in the presence of rectangular imperfection is considered and the corresponding parameters derived. The constants assumed in the prebuckling displacement functions are obtained by partially differentiating the prebuckling energy with respect to the constants, and assuming the load parameter λ as unity.

Chapter 4 is concerned with the buckling analysis of the shell with rectangular imperfection. The displacement function as a double Fourier series is assumed. The series displacement functions are assumed because of the fact that the buckling modes in circumferential direction is dependent upon the curvature strains in the presence of a rectangular imperfection. These curvature strains predominate in making the shell structure to bifurcate earlier than the so called classical buckling. The derivatives and their constituents of the assumed displacement function for the buckling analysis are formed before substituting them in the final potential energy expression. The integration of the final potential energy expression over the area of rectangular imperfection and the area of the cylindrical shell is separated in such a manner so as to give the effect of discontinuity of energy in the presence of a rectangular imperfection. The separated potential energy expressions are energy due to stretching and energy due to bending. The final potential energy contains only the second order derivatives of the assumed displacement functions and the first order derivatives of the imperfection displacement functions which act as the coefficients of the final potential energy equation. The functions, all of which are under the summation sign, are partially differentiated with respect to the constants to form homogeneous equations in terms of the constants of the assumed buckling displacement functions. Since there are three constants in the assumed buckling displacement functions, there will be a three by three determinant. The elements of the determinant are symmetric with respect to the main diagonal. The least and consistant positive eigenvalue will give raise to bifurcation point load parameter, which is the ratio of critical load and the classical buckling load.

Chapter 5 describes the development of computer program to obtain the bifurcation point load parameter numerically. Most of the elements of the entire analysis (prebuckling and buckling) are converted into non-dimensional form, which facilitates the obtaining of the main parameters of the thin cylindrical shell. The variation of these parameters along with the number of terms in the series expansion decides the bifurcation point load parameter. The last of this chapter is the discussion of the results from the computer program.

The entire study on the effect of rectangular imperfection on the axial compressive load carrying capacity of thin cylindrical shell is summarized in Chapter 6.

CHAPTER 2

POTENTIAL ENERGY FORMULATION

2.1 Final Potential Energy

In this section the final potential energy expression for the general case for the buckling analysis is derived by considering the Lagrangian and the nonlinear strain displacement relations. Final potential energy is defined here, as the terms containing the second derivatives of the buckling displacement function and the coefficients as the first derivatives of the imperfection displacement functions. Here general case means the imperfect strain displacement relationship (unstressed state) is included in formulating the final potential energy expression for the bifurcation point load analysis. The resultant strains of the imperfect strain displacement functions are explicitly derived. It is noted that the middle surface strains and nonlinear terms are neglected in Reference 16 in arriving at the resultant strains due to imperfection, and the same has been done here for the total potential energy formulation. The resultant strains so obtained are then substituted in the total potential energy expression.

Total potential energy is the sum of the stressed state and the unstressed potential energies. The energies in the stressed and the unstressed state consist of both the stretching and bending strain displacement derivatives. The total potential energy consists of terms up to fourth order in terms of partial derivatives of the strain displacement functions. By using the initial-postbuckling concept as developed by Koiter, the final potential energy is formulated,which is the second order terms in the total potential energy. These second order terms in the final potential energy expression will satisfy the stability criterion and are used for the calculation of bifurcation point buckling. This method of analysis is asymptotically exact at the bifurcation point load [11]. In other words it can be called as initial-postbuckling theory.

The second order terms of the final potential energy formulation consist of the second derivatives of the buckling displacement functions and the coefficients as the first derivatives of the prebuckling displacement functions. This will form a final stability expression for an imperfect thin shallow cylindrical shell for bifurcation point load analysis.

Following is the potential energy expressions for the general case considering the Lagrangian.

The potential energy for the unstressed is :

$$P(u^0,w^0) = \iint (C/2(\varepsilon_x^{0^2} + \varepsilon_y^{0^2} + 2\nu\varepsilon_x^0\varepsilon_y^0 + 2(1-\nu)\varepsilon_{xy}^{0^2})$$

$$+(D/2(\chi_x^{0^2} + \chi_y^{0^2} + 2\nu\chi_x^0\chi_y^0 + 2(1-\nu)\chi_{xy}^{0^2})$$

$$+\lambda N^0\varepsilon_x^0 + pw^0)\,dx.dy \qquad (2.1)$$

Similarly, potential energy for the stressed state is given by:

$$P(U,V,W) = \iint (C/2(\varepsilon_x^{T^2} + \varepsilon_y^{T^2} + 2\nu\varepsilon_x^T\varepsilon_y^T + 2(1-\nu)\varepsilon_{xy}^{T^2})$$

$$+D/2(\chi_x^{T^2} + \chi_y^{T^2} + 2\nu\chi_x^T\chi_y^T + 2(1-\nu)\chi_{xy}^{T^2})$$

$$+\lambda N^0\varepsilon_x^T + pW)\,dx.dy \qquad (2.2)$$

where $U = u + u^0,\ V = v,\ W = w + w^0$ (2.3)

and C,D are the stretching and bending rigidities.

By Koiter's concept, the total potential energy is

$$\bar{P} = P(U,V,W) - P(u_0,w_0)$$

$$= P_1(U,V,W) + P_2(U,V,W) + P_3(U,V,W) + P_4(U,V,W) + \ldots\ldots \geq 0 \qquad (2.4)$$

2.2 Nonlinear Strain Displacement Relations

The following equations are the stretching nonlinear strain displacement relations for both the stressed ε^T and the unstressed ε^0 states.

$$\varepsilon_x^T = u_{,x} + 1/2(u_{,x}^2 + v_{,x}^2 + w_{,x}^2) \tag{2.5a}$$

$$\varepsilon_y^T = v_{,y} + 1/2(u_{,y}^2 + v_{,y}^2 + w_{,y}^2) + w/R \tag{2.5b}$$

$$\varepsilon_{xy}^T = 1/2(u_{,y} + v_{,x} + u_{,x}u_{,y} + v_{,x}v_{,y} + w_{,x}w_{,y}) \tag{2.5c}$$

$$\varepsilon_x^0 = u_{,x}^0 + 1/2(u_{,x}^{0^2} + w_{,x}^{0^2}) \tag{2.5d}$$

$$\varepsilon_y^0 = w^0/R + 1/2(u_{,y}^0 + w_{,y}^0) \tag{2.5e}$$

$$\varepsilon_{xy}^0 = 1/2(u_{,y}^0 + u_{,x}^0 u_{,y}^0 + w_{,x}^0 w_{,y}^0) \tag{2.5f}$$

w is considered positive if the deformation is bulging outside.

The resultant strains ε^a due to imperfection are in general

$$\varepsilon^a = \varepsilon^T - \varepsilon^0 \qquad \text{thus} \tag{2.6}$$

$$\varepsilon_x^a = \varepsilon_x^T - \varepsilon_x^0 \tag{2.7}$$

$$= u_{,x} + u_{,x}u_{,x}^0 + w_{,x}w_{,x}^0 + 1/2(u_{,x}^2 + v_{,x}^2 + w_{,x}^2) \tag{2.8}$$

Similarly,

$$\varepsilon^a_y = v_{,y} + w/R + u_{,y}u^0_{,y} + w_{,y}w^0_{,y} + 1/2(u_{,y}^2 + v_{,y}^2 + w_{,y}^2) \quad (2.9)$$

$$\varepsilon^a_{xy} = 1/2(u_{,y} + v_{,x} + u_{,x}u_{,y} + v_{,x}v_{,y} + w_{,x}w_{,y} + u_{,x}u^0_{,y} + u_{,y}u^0_{,x} + w_{,x}w^0_{,y} + w_{,y}w^0_{,x}) \quad (2.10)$$

The curvature strains for both the stressed χ^T and unstressed χ^0 states are given by

$$\left.\begin{array}{ll} \chi^T_x = w^T_{,xx} & \chi^T_y = w^T_{,yy} - v^T_{,y}/R \\ \multicolumn{2}{c}{\chi^T_{xy} = w^T_{,xy} + u^T_{,y}/4R - 3/4R\, v^T_{,x}} \end{array}\right\} \quad (2.11)$$

$$\left.\begin{array}{ll} \chi^0_x = w^0_{,xx} & \chi^0_y = w^0_{,yy} \\ \multicolumn{2}{c}{\chi^0_{xy} = w^0_{,xy} + u^0_{,y}/4R} \end{array}\right\} \quad (2.12)$$

The resultant curvature strains are in general given by

$$\chi^a = \chi^T - \chi^0 \qquad \text{thus} \quad (2.13)$$

$$\chi^a_x = \chi^T_x - \chi^0_x \quad (2.14)$$

$$\chi_x^a = w_{,xx} \qquad \chi_y^a = w_{,yy} - v_{,y}/R$$

$$\left.\chi_{xy}^a = w_{,xy} + u_{,y}/4R - 3/4R\, v_{,x}\right\} \qquad (2.15)$$

where R is the radius of the cylinder.

Substituting Equations 2.6 and 2.13 in to Equation 2.4, simplifying and collecting similar terms, the total potential energy is given by

$$\bar{P} = \iint (C((\varepsilon_x^0 + \varepsilon_y^0)(\varepsilon_x^a + \varepsilon_y^a) + 2(1-\nu)(\varepsilon_{xy}^0 \varepsilon_{xy}^a -$$

$$1/2(\varepsilon_x^a \varepsilon_y^0 + \varepsilon_y^a \varepsilon_x^0))) + D(\chi_x^0 + \chi_y^0)(\chi_x^a + \chi_y^a) +$$

$$2(1-\nu)(\chi_{xy}^0 \chi_{xy}^a - 1/2(\chi_x^a \chi_y^0 + \chi_y^a \chi_x^0)) + C/2((\varepsilon_x^a + \varepsilon_y^a)^2$$

$$+2(1-\nu)(\varepsilon_{xy}^{a2} - \varepsilon_x^a \varepsilon_y^a)) + D/2((\chi_x^a + \chi_y^a)^2 + 2(1-\nu)(\chi_{xy}^{a2}$$

$$- \chi_x^a \chi_y^a)) + \lambda N_0 \varepsilon_x^a + pw)dx.dy \qquad (2.16)$$

Substituting Equations(2.8-2.12)in Equation 2.16 and considering second order terms only, the final potential energy is given by

$$P_2 = \iint \bar{P}_2 dx.dy \tag{2.17}$$

where

$$2\bar{P}_2/C = u_{,x}^2(1+3u_{,x}^0+\nu w^0/R) + u_{,x}u_{,y}(2u_{,y}^0) + u_{,y}^2(u_{,x}^0+$$

$$w^0/R + (1-\nu)/2 + (1-\nu)/96(T_1/R)^2) + u_{,x}v_{,x}(($$

$$1-\nu)u_{,y}^0) + u_{,y}v_{,x}((1-\nu)(1+u_{,x}^0)) + v_{,x}^2(u_{,x}^0 +$$

$$w^0/R + 3(1-\nu)/32(T_1/R)^2 + (1-\nu)/2) + u_{,x}v_{,y}*$$

$$(2\nu(1+u_{,x}^0)) + u_{,y}v_{,y}(2u_{,y}^0) + v_{,x}v_{,y}(1-\nu)u_{,y}^0+$$

$$v_{,y}^2(1+\nu u_{,x}^0 + w^0/R +1/12(T_1/R)^2) + w/R*u_{,x}($$

$$2\nu(1+u_{,x}^0)) +w/R*u_{,y}(2u_{,y}^0) + w/R(0)v_{,x} +$$

$$w/R(2)v_{,y} +(w/R)^2 +u_{,x}w_{,x}(2w_{,x}^0) + u_{,y}w_{,x}(1-$$

$$)w_{,y}^0+ v_{,x}w_{,x}(1-\nu)w_{,y}^0 + v_{,y}w_{,x}(2\nu w_{,x}^0) + w/R*$$

$$w_{,x}(2\nu w_{,x}^0) + w_{,x}^2(u_{,x}^0+\nu w^0/R) + u_{,x}w_{,y}(2\nu w_{,y}^0)+$$

$$u_{,y}w_{,y}((1-\nu)w_{,x}^0) + v_{,x}w_{,y}((1-\nu)w_{,x}^0) + v_{,y}w_{,y}($$

$$2w^0_{,y} + w/R^* w_{,y}(2w^0_{,y}) + w_{,x}w_{,y}(1-\nu)u^0_{,y} \quad +$$

$$w^2_{,y}(\ u^0_{,x}+w^0/R) + u_{,y}w_{,xy}\ (1-\nu)/R(T_1^2/12) \quad +$$

$$v_{,x}w_{,xy}(-(1-\nu)/R(T_1^2/4)) + w_{,xx}v_{,y}(1-\nu)/R$$

$$*(T_1^2/6) + v_{,y}(w_{,xx}+ w_{,yy}\ )(-1/R(T_1^2/6)) \quad +$$

$$(w_{,xx} + w_{,yy}\ )^2(T_1^2/12) +(\ w^2_{,xy} - w_{,xx}w_{,yy}\ )$$

$$*(1-\nu)\ (T_1^2/6) - u_{,y}v_{,x}((1-\nu)(T_1/R)^2 \qquad (2.18)$$

Equation 2.18 is the final potential energy expression for the buckling analysis. Only the second order terms are considered for both membrane and curvature displacement function. This expression will be used in chapter 4 to develop the buckling determinant.

CHAPTER 3

PREBUCKLING STRESSES

3.1 Assumption for Prebuckling Displacement Functions.

Prebuckling was defined as the rotation of the structural elements before buckling, thus introducing the nonlinearity with respect to the load. Decay length both in the axial and circumferential direction in the immediate vicinity of the rectangular imperfection is assumed to account for the possibility of local buckling before the shell structure bifurcates into a buckling mode.

In this section potential energy for an unstressed state in both stretching and bending is calculated by considering the linear strain displacement equation and integrating over the area of the rectangular imperfection and the area of the shell. It is found after performing the integration that the imperfection parameters RT_1/α^2 and RT_1/β^2 are obtained during the process of arriving at the potential energy for an unstressed state. These parameters show that the decay length in the circumferential direction accelerates bifurcation of the imperfect shell into a buckling mode configuration. The above mentioned imperfection parameters appear in the potential energy expression showing the weak dependence on the shell curvature.

The prebuckling matrix is obtained by taking the first variation of the integrated final potential energy with respect to the constants of the assumed displacement functions.

3.2 Effect of Discontinuity.

Discontinuity in the membrane displacements occurs when there is a change in the wall thickness. The discontinuity forces must be computed along hoop circles [19], however, the discontinuity stresses decay rapidly. According to Reference 4 the decay length L_c and L_a must be greater than or equal to $\Pi/(3*(1-\nu^2)/R^2*T_1^2)^{1/4}$.

Let us consider a thin cylindrical shell as shown in Fig 3.1 Let the thickness of the shell be T_1 and that of the rectangular imperfection be T_2. Figure 3.2 shows how the prebuckling can be affected in the presence of two different thickness both in the circumferential and in the axial direction. According to Starnes [1], the deformations decay rapidly from one thickness to the other, and require a quadratic displacement function. At the interface of

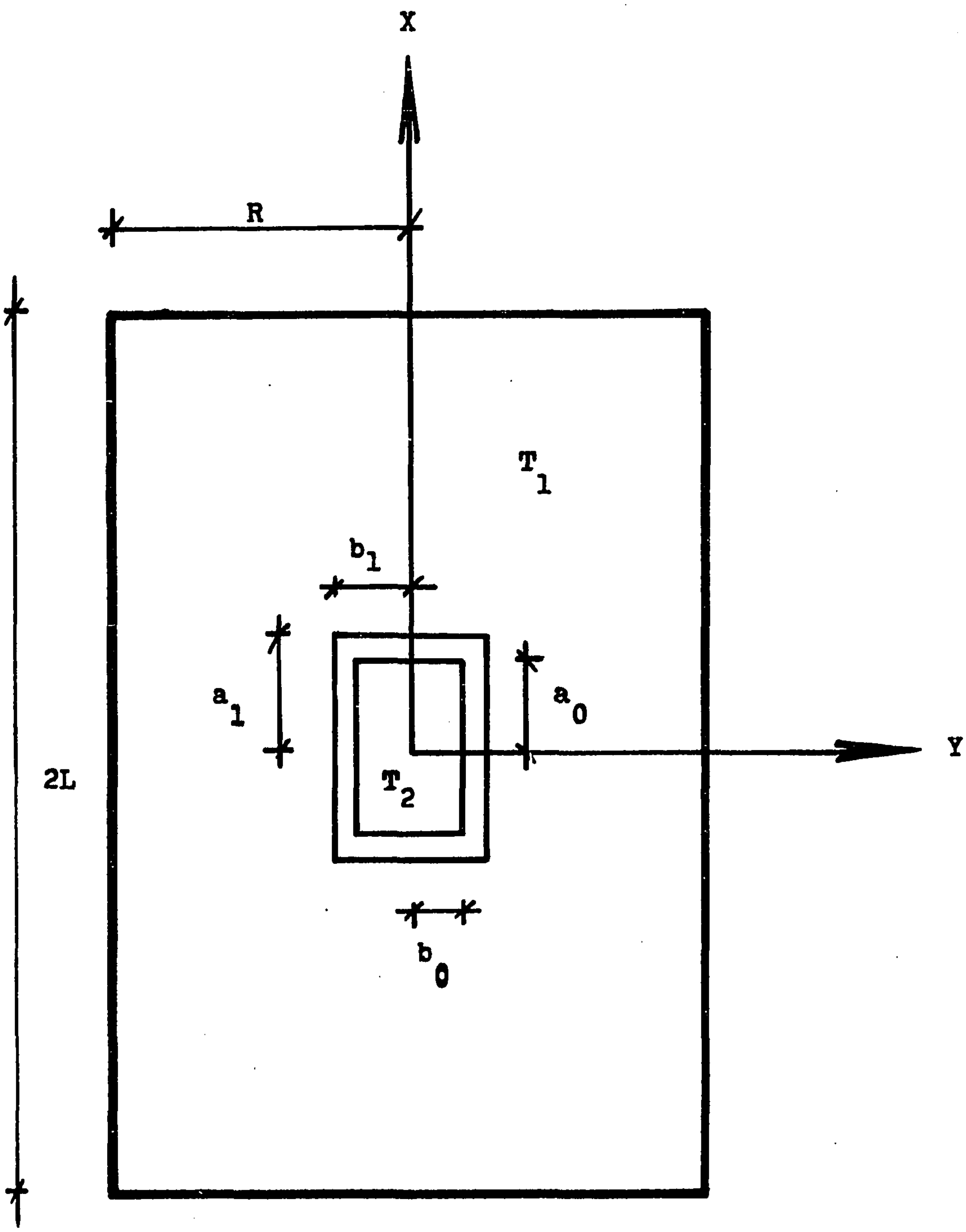

Figure 3.1 Initial geometry of the shell.

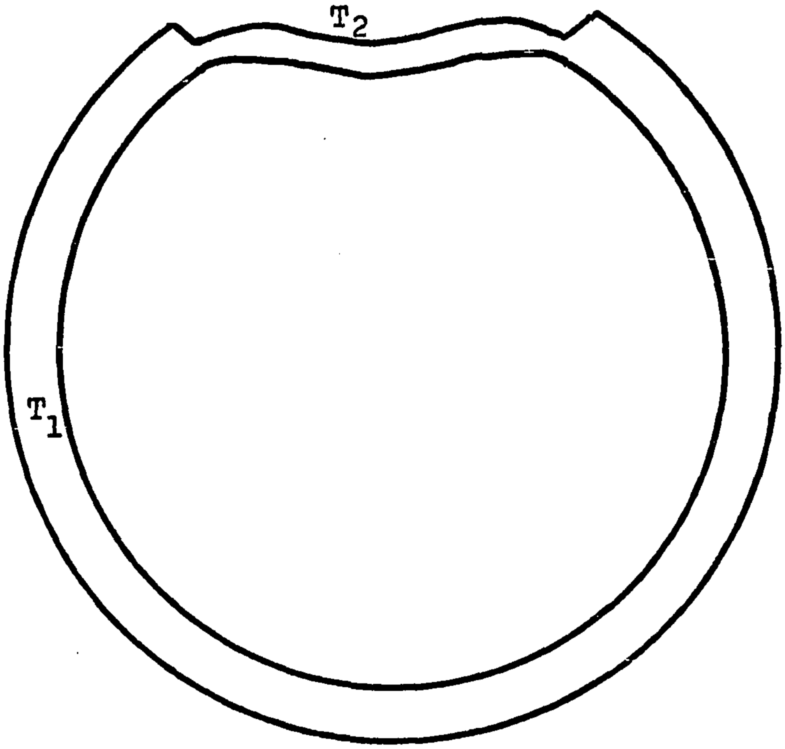

Figure 3.2a Shell geometry in circumferential direction.

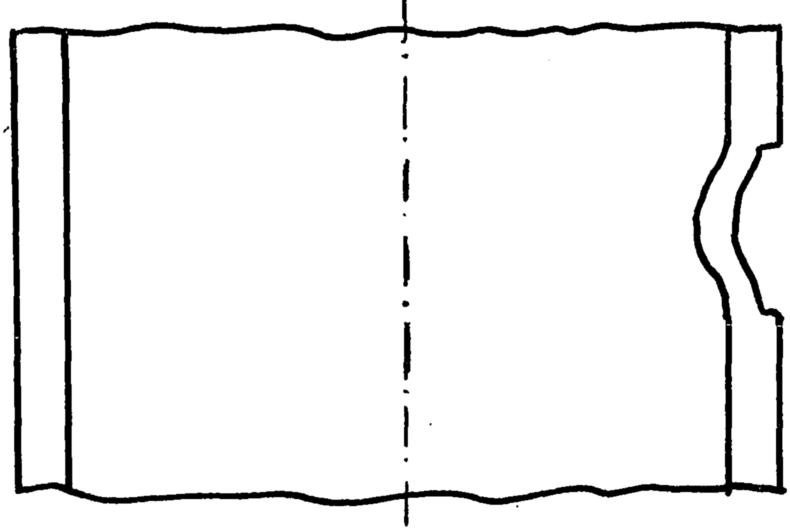

Figure 3.2b Shell geometry in axial direction.

the shell and the imperfection there exists both stretching and bending strains. The magnitude of these strains will be higher in the area of imperfection than at the junction of the imperfection and that of the shell. In conformity with the above, the displacement function is written as follows.

$$\left.\begin{aligned} u^0 &= B_0X + B_1X\theta\cos^2(\pi Y/2(b_0+L_c)) \\ w^0 &= A_0 + A_1\theta\cos^2(\pi X/2(a_0+L_a))*\cos^2(\pi Y/2(b_0+L_c)) \\ v_0 &= 0 \end{aligned}\right\} \quad (3.1_a)$$

Where A_0, A_1, B_0, B_1 are the arbitrary constants which can be obtained from minimization of the potential energy and θ is defined as follows:

$$\theta = \begin{cases} 1 & |X| \leq (a_0+L_a) \quad \text{and} \quad |Y| \leq (b_0+L_c) \\ 0 & |X| > (a_0+L_a) \quad \text{and} \quad |Y| > (b_0+L_c) \end{cases} \qquad (3.1_b)$$

Using these displacement functions the prebuckling strains are now determined.

3.3 Prebuckling Calculations.

Neglecting the higher order strains for the unstressed state, the stretching strains are given by:

$$\varepsilon_x^0 = u^0_{,x} = B_0 + B_1\theta \mathrm{Cos}^2(Y/2\beta)$$

$$\varepsilon_y^0 = -w^c/R = -1/R(A_0 + A_1 \ \mathrm{Cos}^2(X/2\alpha) * \mathrm{Cos}^2(Y/2\beta)) \qquad (3.2_a)$$

$$\varepsilon_{xy}^0 = 1/2(u^0_{,y}) = -B_1X\theta/2\beta \ \mathrm{Sin}(Y/\beta)$$

where $\alpha = (a_0 + L_a)/\Pi$ and $\beta = (b_0 + L_c)/\Pi$

Similarly, the bending strains are

$$\chi_x^0 = w^0_{,xx} = -A_1\theta/2\alpha^2 \ \mathrm{Cos}(X/\alpha) * (\mathrm{Cos}^2(Y/2\beta)$$

$$\chi_y^0 = w^0_{,yy} = -A_1\theta/2\beta^2 \ \mathrm{Cos}^2(X/2\alpha) * (\mathrm{Cos}(Y/\beta) \qquad (3.2_b)$$

$$\chi_{xy}^0 = w^0_{,xy} + 1/4R \ u^0_{,y}$$

$$= -A_1\theta/4\alpha\beta \mathrm{Sin}(X/\alpha) * (\mathrm{Sin}(Y/\beta) - B_1X\theta/8\beta R(\mathrm{Sin}(Y/\beta))$$

Assuming that the area of imperfection is at midheight of the shell, then only one quarter of the shell need be considered because of symmetry. Thus potential energy for one quarter of the shell in the unstressed state is given by.

$$P_0 = C/2\iint\{ \varepsilon_x^{0^2} + 2\nu\varepsilon_x^0\varepsilon_y^0 + \varepsilon_y^{0^2} + 2(1-\nu)\varepsilon_{xy}^{0^2} \}dx \ dy +$$

$$D/2\iint\{ \chi_x^{0^2} + 2\nu\chi_x^0\chi_y^0 + \chi_y^{0^2} + 2(1-\nu)\chi_{xy}^{0^2} \}dx \ dy +$$

$$\iint N_x^a \varepsilon_x^0 \, dx \, dy + \iint p w^0 \, dx \, dy \tag{3.3}$$

Where N_x^a is axial compressive force and p is the internal pressure. From (3.1) if $|X| > a_1$ or $|Y| > b_1$, the only non-zero terms are $\varepsilon_x^0 = B_0$, $\varepsilon_y^0 = -A_0/R$. But for the terms containing axial compression, to give the effect of non-linearity N_x^a is multiplied by the higher order strains. Hence (3.3) can be written as

$$P_0 = C_1/2\int_{a_1}^{L}\int_{b_1}^{\Pi R}(B_0^2 + A_0^2/R^2 - 2\nu A_0B_0/R) \, dx \, dy +$$

$$C_1/2\int_{0}^{a_1}\int_{b_1}^{R}(B_0^2 + A_0^2/R^2 - 2\nu A_0B_0/R) \, dx \, dy +$$

$$C_1/2\int_{a_1}^{L}\int_{0}^{b_1}(B_0^2 + A_0^2/R^2 - 2\nu A_0B_0/R) \, dx \, dy +$$

$$\int_{a_1}^{L}\int_{0}^{\Pi R} N_x^a\varepsilon_x^0 dx \, dy + \int_{0}^{a_1}\int_{b_1}^{\Pi R} N_x^a\varepsilon_x^0 \, dx \, dy \int_{0}^{a_1}\int_{0}^{b_1} N_x^a\varepsilon_x^0 \, dx \, dy +$$

$$\int_{0}^{\Pi R}\int_{0}^{L} p w^0 \, dx \, dy + P_A \tag{3.4}$$

where P_A is the potential energy in the region of imperfection where both stretching and bending are significant. The nonlinear strain ε_x^0 due to the axial load is included to give the beam column effect and to satisfy the condition of prebuckling. Thus

$$\varepsilon_x^0 = u^0_{,x} + 1/2(u^{0^2}_{,x} + w^{0^2}_{,x}) \tag{3.5a}$$

Substituting for the derivatives of the imperfect displacement function from Equation 3.2 in 3.4 we get

$$\varepsilon_x^0 = B_0 + B_0^2/2 + \theta\ [B_1/2(1+\mathrm{Cos}(Y/\beta))(1+B_0)]$$

$$+\theta\ [1/16B_1^2(3+4\mathrm{Cos}(Y/\beta)+\mathrm{Cos}(2Y/\beta)$$

$$+1/128(A_1/\alpha)^2(1-\mathrm{Cos}(2X/\alpha)(3+4\mathrm{Cos}(Y/\beta)$$

$$\mathrm{Cos}(2Y/\beta)) \tag{3.5b}$$

Incorporating Equations 3.5_b in 3.4, the potential energy in the unstressed state becomes

$$P_0 = P_A + C_1/2(B_0^2 + A_0^2/R^2 - 2\nu A_0 B_0/R)[(L-a_1)(\Pi R-b_1)$$

$$+ a_1(\Pi R-b_1) + (L-a_1)b_1] + P_{Nx} + P_p \tag{3.6}$$

Where

$$P_{Nx} = \int_{a_1}^{L}\int_{0}^{\Pi R} N_x^a(B_0 + B_0^2/2)\ dx\ dy + \int_{0}^{a_1}\int_{b_1}^{\Pi R} N_x^a(B_0 + B_0^2/2)\ dx\ dy$$

$$\int_{0}^{a_1}\int_{0}^{b_1} N_x^a(B_0 + B_0^2/2)\ dx\ dy + \int_{0}^{a_1}\int_{0}^{b_1} N_x^a[\ B_0/2(1+B_0)^*$$

$$(1+\mathrm{Cos}(Y/\beta)\ ) + (1/16)B_1^2(3+4\mathrm{Cos}(Y/\beta)\ +\mathrm{Cos}(2Y/\beta))$$

$$+\ 1/128(A\ /\alpha)^2(1-\mathrm{Cos}(2X/\alpha)(3+4\mathrm{Cos}(Y/\beta)\ +\mathrm{Cos}(2Y/\beta))]dx\ dy$$

$$P_{Nx} = N_x^a \pi RL[B_0 + B_0^2/2 + a_1 b_1/2\pi RL(B_1 + 3/8(B_1^2) + B_1 B_0$$

$$+3/64(A_1/\alpha)^2)] \qquad (3.7)$$

$$P_p = pA_0\pi RL + p\int_0^{a_1}\int_0^{b_1} A_1\theta \mathrm{Cos}^2(X/2\alpha) * \mathrm{Cos}^2(Y/2\beta)\ dx\ dy$$

$$= pA_0\pi RL + pA_1 a_1 b_1/4 \qquad (3.8)$$

Now considering P_A, Let $P_A = P_1(\varepsilon) + P_2(\chi)$ (3.9)

where $P_1(\varepsilon)$ = potential energy for stretching

$P_2(\chi)$ = potential energy for bending

$$= C_2/2 \int_0^{a_0}\int_0^{b_0}\{\ \varepsilon_x^{0^2} + 2\nu\varepsilon_x^0\varepsilon_y^0 + \varepsilon_y^{0^2} + 2(1-\nu)\varepsilon_{xy}^{0^2}\}dx\ dy\ +$$

$$C_1/2 \int_0^{a_1}\int_{b_0}^{b_1}\{\ \varepsilon_x^{0^2} + 2\nu\varepsilon_x^0\varepsilon_y^0 + \varepsilon_y^{0^2} + 2(1-\nu)\varepsilon_{xy}^{0^2}\}dx\ dy\ +$$

$$C_1/2 \int_{a_0}^{a_1}\int_0^{b_0}\{\ \varepsilon_x^{0^2} + 2\nu\varepsilon_x^0\varepsilon_y^0 + \varepsilon_y^{0^2} + 2(1-\nu)\varepsilon_{xy}^{0^2}\}dx\ dy\ +$$

$$D_2/2 \int_0^{a_0}\int_0^{b_0}\{\ \chi_x^{0^2} + 2\nu\chi_x^0\chi_y^0 + \chi_y^{0^2} + 2(1-\nu)\chi_{xy}^{0^2}\}dx\ dy\ +$$

$$D_1/2 \int_{a_0}^{a_1} \int_0^{b_1} \{ \chi_x^{0^2} + 2\nu\chi_x^0\chi_y^0 + \chi_y^{0^2} + 2(1-\nu)\chi_{xy}^{0^2} \} dx\, dy +$$

$$D_1/2 \int_0^{a_1} \int_{b_0}^{b_1} \{ \chi_x^{0^2} + 2\nu\chi_x^0\chi_y^0 + \chi_y^{0^2} + 2(1-\nu)\chi_{xy}^{0^2} \} dx\, dy \quad (3.10)$$

Consider $P_A/C_1\pi RL = P_1(\varepsilon)/C_1\pi RL + P_2(\chi)/C_1\pi RL(12/T_1^2)$ (3.11_a)

For stretching

$$P_1(\varepsilon)/C_1\pi RL = \varepsilon_{a_1}/2\ [\ B_0^2(1+\zeta+3/8\zeta^2) + (A_0/R)^2(1+\zeta/2+9/64\zeta^2)$$

$$-2\nu A_0B_0/R(L+\zeta/2+\xi/4 + 3/16\zeta\xi\ ) + B_0^2/12(1-\nu$$

$$)\zeta^2(a_0/\beta)^2(1+r)^2] + \varepsilon_{a_0}\varepsilon_s/2\ [B_0^2(1+\xi(1+S_b)$$

$$+\zeta^2/8(3+4S_b+S_{2b})) + (A_0/R)^2(1+\xi/2(1+S_a)*$$

$$(1+S_b)+\xi^2/64(3+4S_a+S_{2a})(3+4S_b+S_{2b}))$$

$$-2\nu A_0B_0/R(1+\zeta/2(1+S_b)+\xi/4(1+S_a)(1+S_b)+$$

$$\zeta\xi/16(1+S_a)(3+4S_b+S_{2b})) + B_0^2/12(1-\nu)*$$

$$\zeta^2(a_0/\beta)^2(1-S_{2b})] \quad (3.11_b)$$

For bending

$$P_2(\chi)/C_1\pi RL = \varepsilon_{a_1}/1536[A_1^2(3m^2+3n^2+2mn)/R^2+2(1-\nu)/6*$$

$$B_1^2(n/m)(\pi T_1/R)^2 + 4(1-\nu)A_1B_1nT_1/R^2] \ +$$

$$\varepsilon_{a_0}\ \varepsilon_b/1536[A_1/R\ (m\ (1+S_{2a})(3+4S_b+S_{2b}) +$$

$$n^2(1+S_{2b})(3+4S_a+S_{2a})+2\nu mn(1+2S_a+S_{2a})($$

$$1+2S_b+S_{2b})\ +2(1-\nu)mn(1-S_{2a})(1-S_{2b})] +$$

$$2(1-\nu)/6B_1^2(nT_1/R)(a_0/R)^2(1-S_{2b}) +$$

$$4(1-\nu)A_1B_1/R(nT_1/R)(1-S_{2b})(S_a-C_a)) \qquad (3.11_c)$$

Hence

$$P_0/C_1\Pi RL = P_A/C_1\Pi RL + P_p + P_{Nx} + 1/2(B_0^2+(A_0/R)^2$$

$$-2\nu A_0B_0/R)(1-a_1b_1/\Pi RL) \qquad (3.12)$$

To determine the minimum of potential energy Equation (3.12) is differentiated with respect to the constants and equated to zero. Thus

$$\partial P_0/\partial A_0 = 0\ ,\quad \partial P_0/\partial B_0 = 0\quad ,\partial P_0/\partial A_1 = 0\quad ,\partial P_0/\partial B_1 = 0$$

Following are the equations to obtain the (3.13) constants of the assumed prebuckling displacement functions:

$$A_0/R^2(1+\varepsilon_{a_0}\varepsilon_s) - \nu B_0/R(1+\varepsilon_{a_0}\varepsilon_s) + A_1/R^2(\varepsilon_{a_1}/4 + \varepsilon_a\ \varepsilon_s T_a T_b)$$

$$-\nu B_1/R(\varepsilon_{a_1}/2 + \varepsilon_{a_0}\varepsilon_s T_b) + p/C_1 = 0 \quad (3.14)$$

$$-\nu A_0/R(1+\varepsilon_{a_0}\varepsilon_s) + B_0(1+\varepsilon_{a_0}\varepsilon_s + N_x/C_1) - \nu A_1/R($$

$$\varepsilon_{a_1}/4 + \varepsilon_{a_0}\varepsilon_s T_a T_b) + B_1(\varepsilon_{a_1}N_x/2C_1 + \varepsilon_{a_1}/2 + \varepsilon_a \varepsilon_s T_b)$$

$$+ N_x/C_1 = 0 \quad (3.15)$$

$$A_0/R^2(\varepsilon_{a_1}/4 + \varepsilon_{a_0}\varepsilon_s T_a T_b) - \nu B_0/R(\varepsilon_{a_1}/4 + \varepsilon_{a_0}\varepsilon_s T_a T_b)$$

$$+ A_1/R^2(3/64\varepsilon_{a_1}R^2N_x/C_1{}^2 + 9/64\varepsilon_{a_1} + \varepsilon_{a_0}\varepsilon_s(T_a - T_{2a})*$$

$$(T_b - T_{2b}) + \varepsilon_{a_1}/768(3m^2+3n^2+2mn) + \varepsilon_{a_0}\varepsilon_b/12[m^2D_{2a}($$

$$T_b - T_{2b}) + n^2D_{2b}(T_a - T_{2a}) + \nu mn/2(T_b - 2T_{2b})*$$

$$(T_a - 2T_{2a})] - B_1/R(3/16\nu\varepsilon_{a_1} + \varepsilon_{a_0}\varepsilon_s T_a(T_b - T_{2b}) -$$

$$(1-\nu)/384)nT_1/R(\varepsilon a_1 + 16\varepsilon_{a_0}\varepsilon_b T_{2b}(T_a - C_{pa}))) +$$

$$p/C_1\varepsilon_{a_1}/4 = 0 \quad (3.16)$$

$$-\nu A_0/R(\varepsilon_{a_1}/2 + \varepsilon_{a_0}\varepsilon_s T_b) + B_0(N_x/2C_1\varepsilon_{a_1} + \varepsilon_{a_1}/2 +$$

$$\varepsilon_{a_0}\varepsilon_s T_b) - A_1R(3\varepsilon_{a_1}\nu/16 + \nu\varepsilon_{a_0}\varepsilon_s T_a(T_b - T_{2b}) -$$

$$(1-\nu)/384(nT_1/R)(\varepsilon_{a_1} + 16T_{2a}(T \;\; -C_{pa})\varepsilon_{a_0}\varepsilon_b)) +$$

$$B_1(3/8\varepsilon_{a_1}N_x/C_1 + 3/8\varepsilon_{a_1} + \varepsilon_{a_1}(1-\nu)a_1^2/12\beta^2 +$$

$$\varepsilon_{a_0}\varepsilon_s(T_b \;-T_{2b}) + 2/3\varepsilon_{a_0}\varepsilon_s(1-\nu)(a_0^2/\beta^2)T_{2b} \;\; +$$

$$(1-\nu)/384(\varepsilon_{a_1}/6(n/m)(\Pi T_1/R)^2 + \varepsilon_{a_0}\varepsilon_b/6(nT_1/R)*$$

$$(a_1/R)^2)) + N_x^a/C_1\varepsilon_{a_1}/2 \qquad = 0 \qquad (3.17)$$

and writing in matrix form the following expression is obtained:

$$\begin{bmatrix} G_1 & -\nu G_1 & G_2 & -\nu G_3 \\ -\nu G_1 & G_1 + N_1 & -\nu G_2 & G_3 + N_2 \\ G_2 & -\nu G_2 & G_5 + N_3 & -G_6 \\ -\nu G_3 & G_3 + N_2 & -G_6 & G_7 + N_4 \end{bmatrix} \begin{vmatrix} A_0/R \\ B_0 \\ A_1/R \\ B_1 \end{vmatrix} = \begin{vmatrix} -\rho_1 \\ -N_1 \\ -\rho_2 \\ -N_2 \end{vmatrix} \qquad (3.18)$$

where G_1 etc.,are given by

$$G_1 = 1 + \varepsilon_{a_0}\varepsilon_s \qquad G_2 = \varepsilon_{a_1}/4 + \varepsilon_{a_0}\varepsilon_s T_a T_b$$

$G_3 = \varepsilon_{a_1}/2 + \varepsilon_{a_0}\varepsilon_s T_b$ $\quad N_1 = N_x^a/C_1$ $\quad N_2 = N_1\varepsilon_{a_1}/2$

$G_4 = G_1 + N_1$ $\quad G_5 = [9/64\varepsilon_{a_1} + \varepsilon_{a_0}\varepsilon_s(T_a -$

$T_{2a})(T_b - T_{2b}) + m^2/768(3\varepsilon_{a_1} + 64\varepsilon_{a_0}\varepsilon_b D_{2a}(T_b - T_{2b})) +$

$n^2/768(3\varepsilon_{a_1} + 64\varepsilon_{a_0}\varepsilon_b D_{2b}(T_a - T_{2a})) + mn/384(\varepsilon_{a_1} +$

$192\nu(T_b - 2T_{2b})(T_a - 2T_{2a}))]$

$N_3 = 3/32\ N_2\ R^2/\alpha^2$

$G_6 = 3/16\nu\varepsilon_{a_1} + \nu\varepsilon_{a_0}\varepsilon_s T_a(T_b - T_{2b}) - (1-\nu)/384(nT_1/R)($

$\varepsilon_{a_1} + 16\varepsilon_{a_0}\varepsilon_b T_{2b}(T_a - C_{pa}))$

$G_7 = 3/8\varepsilon_{a_1} + \varepsilon_{a_1} a_1^2(1-\nu)/12\beta^2 + \varepsilon_{a_0}\varepsilon_s(T_b - T_{2b}) +$

$2/3\varepsilon_{a_0}\varepsilon_s(1-\nu)(a_0/\beta)^2 T_{2b} + (1-\nu)/384(nT_1/R)($

$\varepsilon_{a_1}/6(\Pi T_1/mR) + \varepsilon_{a_0}\varepsilon_b(a_1/R)^2)$

$N_4 = 3/4N_2$ $\quad \rho_1 = pR/C_1$ $\quad \rho_2 = \varepsilon_{a_1}/4\rho_1$

where

$$(T_b - T_{2b}) = (1/8)[4(1+S_b) - (1-S_{2b})] \qquad (3.19a)$$

$$(T_a - T_{2a}) = (1/8)[4(1+S_a) - (1-S_{2a})] \qquad (3.19b)$$

$$T_b = (1+S_b)/2 \qquad T_{2b} = (1-S_{2b})/8 \qquad (3.19c)$$

$$T_a = (1+S_a)/2 \qquad T_{2a} = (1-S_{2a})/8 \qquad (3.19d)$$

$$D_{2a} = (1+S_{2a})/8 \qquad D_{2b} = (1+S_{2b})/8 \qquad (3.19e)$$

$$C_{pa} = (1+C_a)/2 \qquad (3.19f)$$

By giving a value for the axial compressive load N_1, pressure $\rho_1 = 0$ and knowing the geometry of the shell structure, the constants A_0/R, B_0, A_1/R, B_1 are determined by solving equation 3.18. These constants are then substituted into the equations of the buckling analysis, which are developed in the following chapter.

CHAPTER 4

BUCKLING ANALYSIS

4.1 Priliminary Remarks

The final potential energy equation, as derived in Chapter 2, which contains the second order terms of the buckling displacement functions and the corresponding coefficients from the prebuckling analysis, is integrated over the area of the rectangular imperfection and the area of the shell to arrive at a set of equations which can be used for the buckling analysis. The integration of the final potential energy expression for the buckling analysis is separated in such a manner so as to give the effect of discontinuity of energy in the presence of a rectangular imperfection. The separated potential energy expressions are: (a) energy due to stretching, and (b) energy due to bending. (The bending terms can be identified as the terms which contain second order derivatives of the buckling displacement functions and, hence, the square of the thickness of the shell) The final buckling determinant is formed by taking the first variation of the integrated final potential energy expression with respect to the constants of the assumed buckling displacement function.

The buckling displacement functions are assumed by giving due consideration to the effect of decay lengths in the axial and in the circumferential directions. To do so, a Fourier series solution as a displacement function is choosen for the buckling analysis. This assumed displacement function, as a solution of the stability expression, is continuously differentiable, and has finite integrals and exhibits uniform convergence. The Fourier double series so assumed permits various buckling modes both in the axial and in the circumferential direction.

Figure 4.1 shows a cylindrical shell of radius R and length 2L and d_a and d_c define the location of the center of the imperfection. The displacement functions from the prebuckling analysis are:

$$u^0 = B_0\xi + B_1\xi\theta \mathrm{Cos}^2(\pi\eta/2(b_0+L_c))$$

$$w^0 = A_0 + A_1\theta(\mathrm{Cos}^2(\pi\xi/2(a_0+L_a)))(\mathrm{Cos}^2(\pi\eta/2(b_0+L_c))) \quad (3.1_a)$$

$$\theta = \left\{\begin{array}{lll} 1 & |\xi| < a_0+L_a \text{ and } & |\eta| < b_0+L_c \\ 0 & |\xi| > a_0+L_a \text{ and } & |\eta| > b_0+L_c \end{array}\right\} \quad (3.1_b)$$

The assumed displacement components for buckling are the following double Fourier series expressions:

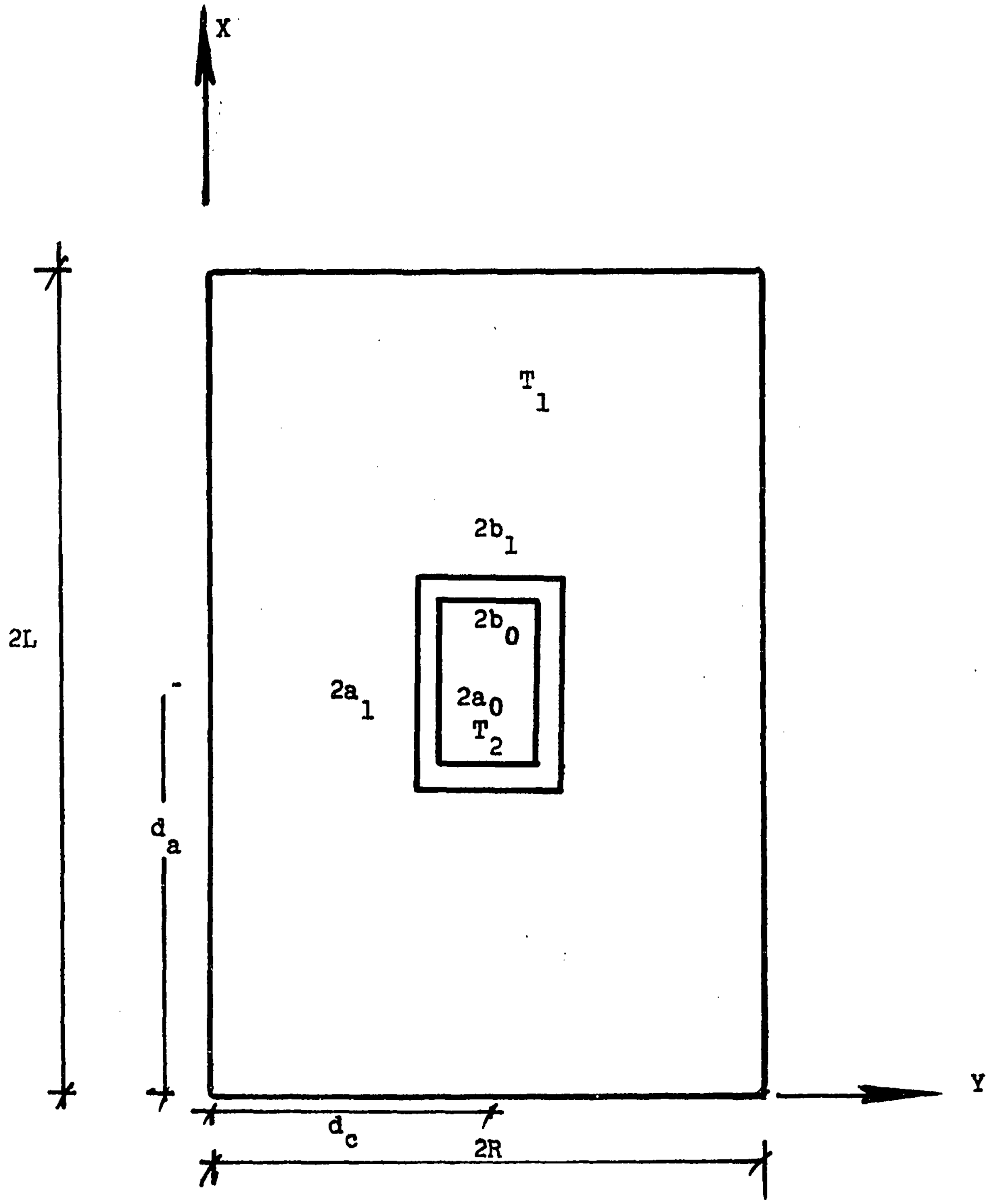

Figure 4.1 Initial geometry of the shell.

$$
\left.
\begin{aligned}
u &= \sum_{i=1}^{n}\sum_{1}^{n} A_{ji}\,\mathrm{Cos}(j\pi X/2L)\ \mathrm{Cos}(iY/R) \\
v &= \sum_{1}^{n}\sum_{1}^{n} B_{ji}\,\mathrm{Sin}(j\pi X/2L)\ \mathrm{Sin}(iY/R) \\
w &= \sum_{1}^{n}\sum_{1}^{n} C_{ji}\,\mathrm{Sin}(j\pi X/2L)\ \mathrm{Cos}(iY/R)
\end{aligned}
\right\} \qquad (4.1)
$$

4.2 Buckling Analysis.

Having assumed the displacement functions, it is now necessary to evaluate the different terms in the final potential energy expression, Equation(1.16). This will be accomplished by first determining the various derivatives of the displacement functions and then forming the numerous products of the various derivatives.

The derivatives of the buckling displacement function are as follows:

$$u,_x = -\sum_{i=1}^{n} A_{ji}(j\pi/2L)\mathrm{Sin}(j\pi X/2L)\mathrm{Cos}(iY/R) \qquad (4.2_a)$$

$$u,_y = -\sum_{i=1}^{n} A_{ji}(i\ /\ R)\mathrm{Cos}(j\pi X/2L)\mathrm{Sin}(iY/R) \qquad (4.2b)$$

$$v,_x = +\sum_{i=1}^{n} B_{ji}(j\pi/2L)\mathrm{Cos}(j\pi X/2L)\mathrm{Sin}(iY/R) \qquad (4.2c)$$

$$v,_y = +\sum_{i=1}^{n} B_{ji}(i\ /\ R)\mathrm{Sin}(j\pi X/2L)\mathrm{Cos}(iY/R) \qquad (4.2d)$$

$$w/R = +\sum_{i=1}^{n} C_{ji}(1 / R)\mathrm{Sin}(j\pi X/2L)\mathrm{Cos}(iY/R) \qquad (4.2_e)$$

$$w_{,x} = +\sum_{i=1}^{n} C_{ji}(j\pi/2L)\mathrm{Cos}(j\pi X/2L)\mathrm{Cos}(iY/R) \qquad (4.2f)$$

$$w_{,y} = -\sum_{i=1}^{n} C_{ji}(i / R)\mathrm{Sin}(j\pi X/2L)\mathrm{Sin}(iY/R) \qquad (4.2g)$$

$$w_{,xx} = -\sum_{i=1}^{n} C_{ji}(j\pi/2L)^2\mathrm{Sin}(j\pi X/2L)\mathrm{Cos}(iY/R) \qquad (4.2h)$$

$$w_{,xy} = -\sum_{i=1}^{n} C_{ji}(j\pi/2L)(i/R)\mathrm{Cos}(j\pi X/2L)\mathrm{Sin}(iY/R) \qquad (4.2i)$$

$$w_{,yy} = -\sum_{i=1}^{n} C_{ji}(i/R)^2\ \mathrm{Sin}(j\pi X/2L)\mathrm{Cos}(iY/R) \qquad (4.2j)$$

The derivatives of the imperfection displacement function are as follows:

$$w^0_{,x} = -\ A_1\theta/2\alpha\ \mathrm{Sin}(\xi/\alpha)\mathrm{Cos}^2(\eta/2\beta) \qquad (4.3_a)$$

$$w^0_{,y} = -\ A_1\theta/2\beta\ \mathrm{Cos}^2(\xi/2\alpha)\mathrm{Sin}(\eta/\beta) \qquad (4.3b)$$

$$u^0_{,x} = B_0 + B_1\theta\mathrm{Cos}^2(\eta/2\beta) \qquad (4.3c)$$

$$u^0_{,y} = -\ B_1\xi\theta\ /2\beta\mathrm{Sin}(\eta/\beta) \qquad (4.3d)$$

$$v^0 = 0 \qquad (4.3e)$$

The various products of the derivatives are as follows:

$$(u_{,x})^2 = 1/2(j\pi/2L)^2 * (1-\mathrm{Cos}(j\pi X/L))(\sum_{i=1}^{n} A_{ji}\mathrm{Cos}(iY/R))^2$$

$$1 + 3u^0_{,x} + \nu w^0/R = (1 + 3B_0 + \nu A_0/R) + \theta[3B_1/2(1 +$$

$$\mathrm{Cos}(\eta/\beta) + \nu A_1/4R(1 + \mathrm{Cos}(\xi/\alpha)*$$

$$(1 + \mathrm{Cos}(\eta/\beta) \qquad (4.4)$$

$$(u_{,x}u_{,y}) = (j\pi/2L)\mathrm{Sin}(j\pi X/2L)\mathrm{Cos}(j\pi X/2L)*$$

$$\sum_{i=1}^{n} A_{ji}\mathrm{Cos}(jY/R)\sum_{i=1}^{n} A_{ji}\mathrm{Sin}(iY/R)\ (i/R)$$

$$2u^0_{,y} = - B_1\xi\theta/\beta\mathrm{Sin}(\eta/\beta) \qquad \text{Exists only for } \theta=1 \qquad (4.5)$$

$$(u_{,y})^2 = (\mathrm{Cos}\ (j\pi X/2L))^2(\sum_{i=1}^{n} (i/R)A_{ji}\mathrm{Sin}(iY/R))^2$$

$$(1-\nu)/2[1+(T_1/R)^2/48]+w^0/R + u^0_{,x} = (1-\nu)/2[1+(T_1/R)^2/$$

$$48]+A_0/R+B_0+ \theta[A_1/4R(1+\mathrm{Cos}(\xi/\alpha)(1+\mathrm{Cos}(\eta/\beta)\ )$$

$$B_1/2(1+\mathrm{Cos}(\eta/\beta)] \qquad (4.6)$$

$$(u_{,x}v_{,x}) = -(j\pi/2L)^2\mathrm{Sin}(j\pi X/2L)\mathrm{Cos}(j\pi X/2L)\sum_{i=1}^{n} A_{ji}$$

$$\mathrm{Cos}(iY/R) \sum_{i=1}^{n} B_{ji}\mathrm{Sin}(iY/R)$$

$$(1-\nu)u^0_{,y} = -(1-\nu)B_1\xi\theta/2\beta\ \mathrm{Sin}(\eta/\beta) \quad \text{Exists only for } \theta = 1 \tag{4.7}$$

$$u_{,y}v_{,x} = -j\pi/4L(1+\mathrm{Cos}(j\pi X/L))\sum_{i=1}^{n} A_{ji}(i/R)\mathrm{Sin}(iY/R)*$$

$$\sum_{i=1}^{n} B_{ji}\mathrm{Sin}(iY/R)$$

$$(1-\nu)[\quad] = (1-\nu)[(1-1/16(T_1/R)^2+ B_0)+\theta B_1/2(1+$$

$$\mathrm{Cos}(\eta/\beta)] \tag{4.8}$$

$$(v_{,x})^2 = 1/2(j\pi/2L)^2(1+\mathrm{Cos}(j\pi X/L)(\sum_{i=1}^{n} B_{ji}\mathrm{Sin}(iY/R))^2$$

$$(1-\nu)/2[\quad] + u^0_{,x}+ w^0/R = (1-\nu)/2[1+3/16(T_1/R)^2] +$$

$$+\nu A_0/R +\theta(B_1/2(1+\mathrm{Cos}(\eta/\beta))+\nu A_1/4R(1+$$

$$\mathrm{Cos}(\xi/\alpha))(1+\mathrm{Cos}(\eta/\beta))) \tag{4.9}$$

$$u_{,x}v_{,y} = -(j\pi/2L)/2(1-\mathrm{Cos}(j\pi X/L))\sum_{i=1}^{n} A_{ji}\mathrm{Cos}(iY/R)$$

$$\sum_{i=1}^{n}(i/R)B_{ji}\mathrm{Cos}(iY/R)$$

$$2\ (1+u^0_{,x}) = 2\nu(1+B_0)+\theta B_1\nu(1+\mathrm{Cos}(\eta/\beta)) \tag{4.10}$$

$$(u_{,y}v_{,y}) = -(\mathrm{Sin}(j\pi X/2L))/2\sum_{i=1}^{n} A_{ji}(i/R)\mathrm{Sin}(iY/R)*$$

$$\sum_{i=1}^{n} B_{ji}(i/R)\mathrm{Cos}(iY/R)$$

$$2u^0_{,y} = -B_1\theta\xi/\beta\,\mathrm{Sin}(\xi/\beta) \quad \text{Exists for } \theta = 1 \tag{4.11}$$

$$(v_{,x}v_{,y}) = (j\pi/4L)\mathrm{Sin}(j\pi X/L)\sum_{i=1}^{n} B_{ji}\mathrm{Sin}(iY/R)$$

$$\sum_{i=1}^{n}(i/R)B_{ji}\mathrm{Cos}(iY/R)$$

$$(1-\nu)u^0_{,y} = -(1-\nu)\theta B_1\xi/2\beta\,\mathrm{Sin}(\eta/\beta) \quad \text{Exists for } \theta = 1 \tag{4.12}$$

$$(v_{,y})^2 = 1/2(1-\mathrm{Cos}(j\pi X/L))(\sum_{i=1}^{n} B_{ji}i/R\,\mathrm{Cos}(iY/R))^2$$

$$1+u^0_{,x}+w^0/R+(T_1/R)^2/12 = 1+(T_1/R)^2/12+\nu B_0+A_0/R+$$

$$[\theta B_1/2(1+\mathrm{Cos}(\eta/\beta))+A_1/4R(1+\mathrm{Cos}(\xi/\alpha)($$

$$1+\mathrm{Cos}(\eta/\beta)) \tag{4.13}$$

$$(w/R)u_{,x} = -(j\pi/4RL)(1-\mathrm{Cos}(j\pi X/L))(\sum_{i=1}^{n} C_{ji}\mathrm{Cos}(iY/R)$$

$$(\sum_{i=1}^{n} A_{ji}\mathrm{Cos}(iY/R)$$

$$2\nu(1+u^0_{,x}) = 2\nu(1+B_0)+B_1\theta\nu(1+\mathrm{Cos}(\eta/\beta) \tag{4.14}$$

$$(w/R)u_{,y} = -(\mathrm{Sin}(j\pi X/L))/R^2\sum_{1}^{n} C_{ji}\mathrm{Cos}(iY/R)\sum_{i=1}^{n} A_{ji}\, i\,\mathrm{Sin}(iY/R)$$

$$2u^0_{,y} = -B_1\theta\xi/\beta\,\mathrm{Sin}(\eta/\beta) \quad \text{Exists for } \theta = 1 \qquad (4.15)$$

$$2w/Rv_{,y} = 1/R^2(1-\mathrm{Cos}(j\pi X/L))\sum_{i=1}^{n} C_{ji}\mathrm{Cos}(iY/R)*$$

$$\sum_{i=1}^{n} B_{ji}\, i\,\mathrm{Cos}(iY/R) \qquad (4.16)$$

$$(w/R)^2 = 1/2R^2(1-\mathrm{Cos}(j\pi X/L)(\sum_{i=1}^{n} C_{ji}\mathrm{Cos}(iY/R))^2 \qquad (4.17)$$

$$(u_{,x}w_{,x}) = -1/2(j\pi/2L)^2\mathrm{Sin}(j\pi X/L)\sum_{i=1}^{n} A_{ji}\mathrm{Cos}(iY/R)*$$

$$\sum_{i=1}^{n} C_{ji}\mathrm{Cos}(iY/R)$$

$$2w^0_{,x} = -A_1\theta/2\alpha\,\mathrm{Sin}(\xi/\alpha)(1+\mathrm{Cos}(\eta/\beta) \quad \begin{matrix}\text{Exist for}\\ \theta = 1\end{matrix} \qquad (4.18)$$

$$(u_{,y}w_{,x}) = -(j\pi/4L)(1+\mathrm{Cos}(j\pi X/L)\sum_{i=1}^{n} A_{ji}\, i/R\,\mathrm{Sin}(iY/R)$$

$$\sum_{i=1}^{n} C_{ji}\mathrm{Cos}(iY/R)$$

$$(1-\nu)w^0_{,y} = -(1-\nu)A_1/4\beta(1+\mathrm{Cos}(\xi/\alpha))\mathrm{Sin}(\eta/\beta) \qquad (4.19)$$

$$(v_{,x}w_{,x}) = 1/2(j\pi/2L)^2(1+\mathrm{Cos}(j\pi X/L))*$$

$$\sum_{i=1}^{n} B_{ji}\mathrm{Sin}(iY/R)\sum_{i=1}^{n} C_{ji}\mathrm{Cos}(iY/R)$$

$$(1-\nu)w^{0}_{,y} = -\theta(1-\nu)/4\beta A_1(1+\mathrm{Cos}(\xi/\alpha))\mathrm{Sin}(\eta/\beta) \qquad (4.20)$$

$$(v_{,y}w_{,x}) = (j\Pi/4L)\mathrm{Sin}(j\Pi X/L)\sum_{i=1}^{n}(i/R)B_{ji}\mathrm{Cos}(iy/R)*$$

$$\sum_{i=1}^{n} C_{ji}\mathrm{Cos}(iY/R)$$

$$2\nu w^{0}_{,x} = -\nu\theta A_1/2\alpha\mathrm{Sin}(\xi/\alpha)(1+\mathrm{Cos}(\eta/\beta) \quad \text{Exist for } \Theta=1 \qquad (4.21)$$

$$(w/R)w_{,x} = (j\Pi/4RL)\mathrm{Sin}(j\Pi X/L)\Sigma C_{ji}\mathrm{Cos}(iY/R)$$

$$\sum_{i=1}^{n} C_{ji}\mathrm{Cos}(iY/R)$$

$$2\nu w^{0}_{,x} = -\nu A_1\theta/2\alpha\mathrm{Sin}(\xi/\alpha)(1+\mathrm{Cos}(\eta/\beta) \quad \text{Exist for } \Theta=1 \qquad (4.22)$$

$$(w_{,x})^2 = 1/2(j\Pi/2L)^2(1+\mathrm{Cos}(j\Pi X/L)*$$

$$(\sum_{i=1}^{n} C_{ji}\mathrm{Cos}(iY/R))^2$$

$$u^{0}_{,x}+\nu w^{0}/R = (B_0+\nu A_0/R)+\theta[B_1/2(1+\mathrm{Cos}(\eta/\beta) +$$

$$\nu A_1/4R(1+\mathrm{Cos}(\xi/\alpha))(1+\mathrm{Cos}(\eta/\beta))] \qquad (4.23)$$

$$(u_{,x}w_{,y}) = (j\Pi/4RL)(1-\mathrm{Cos}(j\Pi X/L))*$$

$$\sum_{i=1}^{n}(A_{ji}\mathrm{Cos}(iY/R))(\sum_{i=1}^{n} C_{ji}i\mathrm{Sin}(iY/R))$$

$2\nu w^0_{,y} \quad = -\nu\theta A_1/2\beta(1+\cos(\xi/\alpha)\sin(\eta/\beta)$ Exist for $\theta = 1$ (4.24)

$(u_{,y}w_{,y}) \quad = 1/2R^2\sin(j\pi X/L)\sum_{i=1}^{n} A_{ji}\, i\sin(iY/R)$

$\sum_{i=1}^{n} C_{ji}\, i\sin(iY/R)$

$(1-\nu)w^0_{,x} \quad = -\theta(1-\nu)A_1/4\alpha\sin(\xi/\alpha)(1+\cos(\eta/\beta)$ for $\theta = 1$ (4.25)

$(v_{,x}w_{,y}) \quad = -(j\pi/4RL)\sin(j\pi X/L)\sum_{i=1}^{n} B_{ji}\sin(iY/R)*$

$\sum_{i=1}^{n} C_{ji}\, i\sin(iy/R)$

$(1-\nu)w^0_{,x} \quad = -(1-\nu)\theta A_1/4\alpha\sin(\xi/\alpha)(1+\cos(\eta/\beta)$ (4.26)

$(v_{,y}w_{,y}) \quad = -1/2R^2(1-\cos(j\pi X/L))\sum_{i=1}^{n} B_{ji}\, i\cos(iY/R)*$

$\sum_{i=1}^{n} C_{ji}\, i\sin(iY/R)$

$2w^0_{,y} \quad = -\theta A_1/2\beta(1+\cos(\xi/\alpha))\sin(\eta/\beta)$ (4.27)

$(w/R)w_{,y} \quad = -1/2R^2(1-\cos(j\pi X/L)\sum_{i=1}^{n} C_{ji}\cos(iY/R)$

$\sum_{i=1}^{n} C_{ji}\, i\sin(iY/R)$

$2w^0_{,y} \quad = -\theta A_1/2\beta(1+\cos(\xi/\alpha))\sin(\eta/\beta)$ (4.28)

$$(w_{,x}w_{,y}) = -(j\pi/4RL)\mathrm{Sin}(j\pi X/L)\sum_{i=1}^{n} C_{ji}\mathrm{Cos}(iY/R)$$

$$\sum_{i=1}^{n} C_{ji}\, i\mathrm{Cos}(iY/R)$$

$$(1-\nu)u^0_{,y} = -(1-\nu)B_1\xi\Theta/2\beta\mathrm{Sin}(\eta/\beta) \tag{4.29}$$

$$(w_{,y})^2 = 1/2R^2(1-\mathrm{Cos}(j\pi X/L)(\sum_{i=1}^{n} C_{ji}\, i\mathrm{Sin}(iY/R))^2$$

$$(\nu u^0_{,x}+w_0/R) = (\nu B_0+A_0/R)+\theta[\nu B_1/2(1+\mathrm{Cos}(\eta/\beta) +$$

$$A_1/4R(1+\mathrm{Cos}(\xi/\alpha)(1+\mathrm{Cos}(\eta/\beta))] \tag{4.30}$$

$$(u_{,y}w_{,xy}) = (j\pi/4R^2L)(1+\mathrm{Cos}(j\pi X/L)*$$

$$\sum_{i=1}^{n} A_{ji}\, i\mathrm{Sin}(iY/R)\sum_{i=1}^{n} C_{ji}\, i\mathrm{Sin}(iY/R)$$

$$(1-\nu)D_1/C_1R = (1-\nu)/R(T_1^2/12) \tag{4.31}$$

$$(v_{,x}w_{,xy}) = -1/2R(j\pi/2L)^2(1+\mathrm{Cos}(j\pi X/L))*$$

$$\sum_{i=1}^{n} B_{ji}\mathrm{Sin}(iY/R)\sum_{i=1}^{n} C_{ji}\, i\mathrm{Sin}(iY/R)$$

$$3(1-\nu)/4R^2(D_1/C_1) = -(1-\nu)/R(T_1^2/4) \tag{4.32}$$

$$(v_{,y}w_{,xx}) = -1/2R(j\pi/2L)^2(1-\mathrm{Cos}(j\pi X/L))*$$

$$\sum_{i=1}^{n} C_{ji} \text{Cos}(iY/R) \sum_{i=1}^{n} B_{ji}\, i\, \text{Cos}(iY/R)$$

$$2(1-\nu)/R(D_1/C_1) = (1-\nu)/R\ (T_1^2/6) \qquad (4.33)$$

$$v,_y(w,_{xx}+w,_{yy}) = -1/2R(1-\text{Cos}(j\pi X/L))*$$

$$\sum_{i=1}^{n} B_{ji}\, i\, \text{Cos}(iY/R) \sum_{i=1}^{n} C_{ji}[(j\pi/2L)^2+(i/R)^2]*$$

$$\text{Cos}(iY/R)$$

$$-2D_1/C_1R \quad = -\ T_1^2/6R \qquad (4.34)$$

$$(w,_{xx}+w,_{yy})^2 = [\sum_{i=1}^{n} (C_{ji} \text{Sin}(j\ X/2L)\text{Cos}(iY/R)*$$

$$((j\pi/2L)^2 + (i/R)^2)]^2(T_1^2/12) \qquad (4.35)$$

$$2(1-\nu)T_1^2/12[(w,_{xy})^2 - w,_{xx}w,_{yy}] = (1-\nu)T_1^2/6[$$

$$(j\pi/2L)^2(1/2R^2)((1+\text{Cos}(j\pi X/L))(\sum_{i=1}^{n} C_{ji}\, i\, \text{Sin}(iY/R))^2$$

$$-(1-\text{Cos}(j\pi X/L))\sum_{i=1}^{n} C_{ji} \text{Cos}(iY/R) \sum_{i=1}^{n} C_{ji}\, i^2 \text{Cos}(iY/R)] \qquad (4.36)$$

Having these terms the final potential energy expression can be formulated and integrated.

4.3 Formulation of Equations for Buckling analysis.

From Chapter 2 the potential energy expression is

$$P_2 = \iint \bar{P}_2 \, dx \, dy$$

where $\bar{P}_2$ is already defined.

Letting $2\bar{P}_2/C_1(1/4\pi RL) = I_1 + I_2 + I_3$ (4.37a)

where I_1 , I_2 and I_3 are the integrals

I_1 is then separated into $I_1^0 + I_1^1$, where

I_1^0 is the integral for terms when $\theta = 0$

I_1^1 is the integral for terms when $\theta = 1$

From the prebuckling analysis

$$D_2 = D_1(1 + \varepsilon_b) = (1 + \varepsilon_b)(ET_1^3/12(1-\nu^2)$$

$$D_2/C_1 = (1 + \varepsilon_b)(T_1^2/12) \qquad \varepsilon_s = (C_2/C_1 - 1)$$

where ε_b is the bending stiffness

and ε_s is the stretching stiffness,

d_a is the distance in the axial direction to the centre of of imperfection from Y axia,

d_c is the distance in the circumferential direction from X axis.

These coordinates are so choosen such that the imperfection may be located anywhere in the shell.

substituting

$$I_1 = (1/4\pi RL)\left[\int_0^{2L}\int_0^{2\pi R} \bar{P}_2^0 \, dx\, dy + \int_{d_a-a_1}^{d_a+a_1}\int_{d_c-b_1}^{d_c+b_1} \bar{P}_2^1 \, dx\, dy\right]$$

$$[\,I_1^0\,] \qquad + \qquad [\,I_1^1\,]$$

$$(4.37b) \qquad\qquad (4.37c)$$

The limits of integration over the imperfection are given by

$$\xi = X - d_a \qquad \text{when } \xi = +a_1 \qquad X = d_a + a_1$$

$$X = \xi + d_a \qquad \text{when } \xi = -a_1 \qquad X = d_a - a_1 \tag{4.37d}$$

$$\eta = Y - d_c \qquad \text{when } \eta = +b_1 \qquad Y = d_c + b_1$$

$$Y = \eta + d_c \qquad \text{when } \eta = -b_1 \qquad Y = d_c - b_1$$

$$I_2 = (\varepsilon_s/4\pi RL)\left[\int_{d_a-a_0}^{d_a+a_0}\int_{d_c-b_0}^{d_c+b_0} \bar{P}_2^s \, dx\, dy\right] \tag{4.37e}$$

Where $\bar{P}_2^s$ is the stretching potential energy in the area of the rectangular imperfection.

$$I_3 = (\varepsilon_b/4\pi RL)(T_1^2/12)\left[\int_{d_a-a_0}^{d_a+a_0}\int_{d_c-b_0}^{d_c+b_0} \bar{P}_2^b \, dx\, dy\right] \tag{4.37f}$$

Where $(T_1^2/12)\bar{P}_2^b$ is the bending potential energy in the area of the rectangular imperfection.

In all of the above integrals the symbol $\bar{P}$ represents the potential energy, and the subscript represents the degree of the derivatives in the potential energy expression. The superscripts correspond to the potential caused by stretching (s) or bending (b), and (0) or (1) means that the potential energy is within the area of rectangular imperfection or outside of the imperfection, respectively.

The integration of the above integrals are carried in appendix B-2 and results in:

$$I_1^0 = \sum_{i=1}^{n} A_{ji}^2[(j\pi/2L)^2(1+3B_0+\nu A_0/R) + i^2/R^2((1-\nu)/2)*$$

$$(1+1/48(T_1/R)^2) + A_0/R + B_0)]\pi RL -$$

$$\sum_{i=1}^{n} A_{ji}B_{ji}[(j\pi/2L)(i/R)(1-\nu)(1-(T_1/R)^2/16 + B_0)+$$

$$(j\pi/2L)(i/R)2\nu(1+B_0)\ ]\pi RL +$$

$$\sum_{i=1}^{n} B_{ji}^2[(j\pi/2L)^2((1-\nu)/2(1+(T_1/R)^2 3/16+B_0+\nu A_0/R)+$$

$$(i/R)^2(1+(T_1/R)^2/12+\nu B_0+A_0/R)]\pi RL +$$

$$\sum_{i=1}^{n} B_{ji}C_{ji}[2i/R^2+ (j\pi/2L)^2/12(1-\nu)/12(T_1/R)^2 i +$$

$$i/6(T_1/R)^2((j\pi/2L)^2+(i/R)^2)]\pi RL +$$

$$\sum_{i=1}^{n} C_{ji}^2[1/R^2 + (j\pi/2L)^2(B_0+\nu A_0/R)+(i/R)^2(\nu B_0+A_0/R)+$$

$$((j\pi/2L)^2+(i/R)^2)^2(T_1^2/12)]\pi RL -$$

$$\sum_{i=1}^{n} C\quad A\quad [(j\pi/2L)(2\nu/R)(1+B_0)-(j\pi/2L)(i^2/R)($$

$$(1-\nu)/12\ (T_1/R)^2]\pi RL \qquad (4.38a)$$

$$I_1^1 = a_1 b_1[\sum_{k=1}^{n}\sum_{i=1}^{n} A_{ji}A_{kj}((j\pi/2L)^2((3B_1/2(g_{-1}) +\nu A_1/4R($$

$$S_{a_1}+g_{-1}-(f_{+1}+f_{-2})/2))(t_{+1}+(T_{+2}+S'_{-2})/2))-$$

$$(j\pi/2L)/2R(B_1 d_a/\beta)(iq_{-1}(T'_{+2}+S_{-2})) +$$

$$2ik/R^2(t_{-1}+(T_{-2}+S'_{+2})/2)(A_1/4R(S_{a_1}+g_{+1}+$$

$$(f_{+1}+f_{-2})/2)+B_1/2(g_{+1}))) +\sum_{k=1}^{n}\sum_{i=1}^{n} B_{ji}B_{kj}((1/R^2)($$

$$ki(\nu B_1/2(g_{-1}) +A_1/4R(S_{a_1}+g_{-1}-(f_{+1}+f_{-2})/2))*$$

$$(t_{+1}+(T_{+2}+S'_{-2})/2)) + (t_{-1}+(T_{-2}+S'_{+2})/2)($$

$$B_1 g_{+1}/2 +\nu A_1/4R(S_{a_1}+g_{+1}+(f_{+1}+f_{-2})/2))(j\pi/2L)^2$$

$$-(1-\nu)/2R(j\pi/4L)d_a B_1/\beta k q_{-1}(T'_{+2}+S_{-2}))$$

$$\sum_{k=1}^{n}\sum_{i=1}^{n} A_{ki}\ B_{kj}(B_1 d_a/4\beta(j\pi/2L)^2 q_{-1}(T'_{+2}+S_{-2})(1-\nu)-$$

$$(ji/4L)/R(B_1 k)(t_{-1}+(T_{-2}+S'_{+2})/2)g_{+1}(1-\nu)-$$

$$\nu/R(j\pi/2L)g_{-1}B_1 i(t_{+1}+(T_{+2}+S'_{-2})/2)\ +B_1/\beta($$

$$d_a/2R^2)(kiq_{-1}(T'_{+2}+S_{-2})))+\sum_{k=1}^{n}\sum_{i=1}^{n} A_{ki}C_{kj}\ ($$

$$-A_1/4\alpha(j\pi/2L)^2(f_{-1}+f_{+2})(t_{+1}+(T_{+2}+S'_{-2})/2)-$$

$$(j\pi/4RL)(\nu A_1/2\beta)(S_{a_1}+g_{-1}-(f_{+1}+f_{-2}))*$$

$$(i/2)(T'_{+2}+S_{-2})\ +\ (1-\nu/4\alpha)(A_1/2R^2)(f_{-1}+$$

$$f_{+2})ki(t_{-1}+(T_{-2}+S'_{+2})/2))\ +\sum_{k=1}^{n}\sum_{i=1}^{n}-B_{ki}C_{kj}\ ($$

$$(j\pi/4L)/R(\nu A_1/2\alpha)(f_{-1}+f_{+2})k(t_{+1}+(T_{+2}+$$

$$S'_{-2})/2)\ +(1-\nu)/4R(j\pi/L)(A_1/4\alpha)(f_{-1}+f_{+2})*$$

$$i(t_{-1}+(T_{-2}+S'_{+2})/2)\ +(A_1/R)/8\beta R(S_{a_1}+g_{-1}-$$

$$(f_{+1}+f_{-2})/2)ik(T'_{+2}+S_{-2})\ +\sum_{k=1}^{n}\sum_{i=1}^{n} C_{ki}C_{kj}(-\nu A_1/4\alpha R($$

$$\pi j/2L)(f_{-1}+f_{+2})(t_{+1}+(T_{+2}+S'_{-2})/2)\ +(j\pi/2L)^2($$

$$t_{+1}+(T_{+2}+S'_{-2}))(B_1/2(g_{+1})+\nu A_1/4R(S_{a_1}+$$

$$g_{+1}+(f_{+1}+f_{-2})/2))\ +(1/4\beta)(i/R)(A_1/R)($$

$$S_{a_1}+g_{-1}-(f_{+1}+f_{-2})/2)(T'_{+2}+S_{-2})+(1-\nu)/4\beta($$

$$B_1/R)d_a(j\pi/2\ )iq_{-1}(G^S_{-2}-G^T_{+2})\ +\ (i/R)(k/R)($$

$$(t_{-1}+(T_{-2}+S'_{+2})/2)(\nu B_1/2(g_{-1})+A_1/4R(S_{a_1}+$$

$$g_{-1}-(f_{+1}+f_{+2})/2))\ +\sum_{k=1}^{n}\sum_{i=1}^{n}C_{ki}A_{kj}\ (-\nu j\pi B_1/2RL($$

$$t_{+1}+(T_{+2}+S'_{-2})/2)g_{-1}\ +\ id_aB_1/R^2\beta(q_{-1})($$

$$T'_{+2}+S_{-2}))\ +\sum_{k=1}^{n}\sum_{i=1}^{n}B_{ki}C_{kj}(-(1-\nu)/8(A_1/\beta)(j\pi/2L)^2($$

$$S_{a_1}+g_{+1}+(f_{+1}+f_{-2})/2)(T'_{+2}+S_{-2}))\sum_{k=1}^{n}\sum_{i=1}^{n}A_{ki}C_{kj}\ ($$

$$(1-\nu)/4\beta(A_1/2R)(j\pi/2L)(S_{a_1}+g_{+1})+(f_{+1}+f_{-2})/2)*$$

$$i(T'_{+2}+S_{-2}))] \tag{4.38b}$$

The integration of the I_2 integral is identical except for the integration limits.

and for bending are :

$$I_3/\varepsilon_b = a_0 b_0 [\sum_{k=1}^{n}\sum_{i=1}^{n} A_{ji}A_{jk}(1-\nu)/R^2(T_1/R)^2/96(ki)g^0_{+1}t^0_{-1} +$$

$$\sum_{k=1}^{n}\sum_{i=1}^{n} A_{ji}B_{jk}(1-\nu)(T_1/R)^2/32(j\pi/4RL)g^0_{+1}kt^0_{-1} +$$

$$\sum_{k=1}^{n}\sum_{i=1}^{n} B_{ji}B_{jk}(T_1/R)^2/12R^2(ki)g^0_{-1}t^0_{+1} +$$

$$(1-\nu)/2(T_1/R)^2(j\pi/2L)^2 g^0_{+1}t^0_{-1} +$$

$$\sum_{k=1}^{n}\sum_{i=1}^{n} C_{ji}C_{jk}((T_1^2/12)g^0_{-1}((j\pi/2L)^2 + (k/R)^2)($$

$$(j\pi/2L)^2+(i/R)^2)t^0_{+1}+(1-\nu)/6(T_1/R)^2($$

$$j\pi/2L)^2(g^0_{+1}kit^0_{-1}-g^0_{-1}i^2t^0_{+1}) -$$

$$\sum_{k=1}^{n}\sum_{i=1}^{n} C_{ji}B_{jk}((1-\nu)/6R(J\pi/2L)^2T_1^2g^0_{-1}t^0_{+1})$$

$$\sum_{k=1}^{n}\sum_{i=1}^{n} A_{ji}C_{jk}((1-\nu)/R^3(j\pi/2L)(T_1^2/12)g^0_{+1}kit^0_{-1})$$

$$\sum_{k=1}^{n}\sum_{i=1}^{n} B_{ji}C_{jk}((1-\nu)/4R^2(j\pi/2L)^2T_1^2\ g^0_{+1}kt^0_{-1} +$$

$$T_1^2/6R^2g^0_{-1}k((j\pi/2L)^2+(i/R)^2)t^0_{+1}) \qquad (4.38d)$$

substitution of the integrals into Equation (4.37) is the final potential energy for the system.

4.4 Minimization of Potential energy.

Taking the partial derivatives of the potential energy with respect to the arbitrary constants of the assumed buckling displacement function, and for homogenuity leaving the indices of the constants the buckling determinant is formed.

$$\begin{aligned}
\partial I_1^0/\partial A_{ji} &= \Sigma 2A_{ji}C_{11}^0 - \Sigma B_{ji}C_{12}^0 - \Sigma C_{ji}C_{13}^0 \\
\partial I_1^0/\partial B_{ji} &= -\Sigma A_{ji}C_{21}^0 + \Sigma 2B_{ji}C_{22}^0 + \Sigma C_{ji}C_{23}^0 \qquad (4.39) \\
\partial I_1^0/\partial C_{ji} &= -\Sigma A_{ji}C_{31}^0 + \Sigma B_{ji}C_{32}^0 + \Sigma 2C_{ji}C_{33}^0
\end{aligned}$$

The final buckling determinant in general is given by

$$C(K,L) = C(K,L)^0 + \varepsilon_{a_1} C(K,L)^1 + \varepsilon_s \, \varepsilon_{a_0} \, C(K,L)^S + \varepsilon_b \, \varepsilon_{a_0} C(K,L)^B \qquad (4.40)$$

where K varies from 1 to 3

and L varies from 1 to 3.

C(K,L) is the general representation of the elements of the buckling determinant. These elements require a great deal of numerical computaion before applying the Jacobi's method for finding the eigenvalue for a real symmetric matrix. The elements were converted into non-dimensionalized form to facilitate easier computation.

The superscripts 0 or 1 in the equation of the elements of the determinant represent whether the term is for the imperfection or for the cylindrical shell respectively, and S or B is for the stretching or bending of the imperfection respectively. As can be seen from equation 4.40, the second term of the element consists of ε_{a_1} which is the distortion parameter, the third term consists of the product of stretching stiffness and the undistorted parameter whereas, the fourth term consists of the product of the bending stiffness and the undistorted parameter.

CHAPTER 5

ANALYTICAL STUDIES

5.1 Method of Analysis.

Based on the theoretical developments of Chapter 3 Prebuckling and Chapter 4 Buckling, a FORTRAN 1V computer program was developed to perform the calculations. A complete listing is reproduced in Appendix A.

Only one data card which includes the aspect ratio, length to radius ratio, width of rectangular imperfection to the radius ratio, and the location of the rectangular imperfection is required to define the shell and its imperfection. Permutations and combinations of the above can be made to obtain various values of the buckling load parameter, which is the least positive eigenvalue. Number of terms in the various series can be varied in the program.

A macro-flowchart of the program is shown in Figure A.1. First, the constants of the assumed prebuckling displacement functions are calculated. Then, after the summation of the required trigonometric function pertaining to the buckling analysis is completed, the elements of the buckling determinant are found and the symmetric buckling matrix is developed. The resulting homogeneous buckling equations are then solved using Jacobi method for finding the eigenvalue of a real symmetric matrix.

The computer program was used to study the effect of rectangular imperfection on cylindrical shell buckling strength.

5.2 Selection of Shells for Study.

Two shells having the same radius but different thicknesses were selected for studying the effect of rectangular imperfection on the bifurcation point load. The following sections describe the various parameters for the prebuckling and buckling analysis that were made using the computer program of Appendix A.

5.2.1 Prebuckling:

(a) The imperfection parameter μ (rectangular imperfection) was derived in the prebuckling analysis. This parameter depends on the ratio of the principal radius of the cylindrical shell to the thickness of the shell and the aspect ratio of the rectangular imperfection. This parameter is obtained for the corresponding ratio of the decay length to the shell radius in the circumferential direction and governs the prebuckling stress distribution and the displacements for a circular cylindrical shell with a rectangular imperfection in its side. In the prebuckling analysis, the prebuckling displacements are localized around the periphery of the rectangular imperfection and account for both the stretching and bending stresses. For a constant aspect ratio of the rectangular imperfection and the ratio of imperfection width to radius of the cylindrical shell, the ratio of decay length in the circumferential direction to the principal radius is varied. This strategy is required since the circumferential decay length has a weak dependence on the shell curvature.

(b) The only load parameter considered was the ratio of the critical load to the classical load in the axial direction. The pressurization parameter was taken as zero for this study. The axial load parameter was set at 0.5 and 1.0 to obtain the least and consistant positive eigenvalues from the buckling analysis.

5.2.2 Buckling:

The main parameters for the bifurcation point load analysis are: the ratio of principal radius of the cylindrical shell to the thickness of the shell, ratio of length of the cylindrical shell to the radius of the shell, aspect ratio of the rectangular imperfection, the ratio of imperfection width along the circumferential direction to the principal radius of the cylindrical shell, the ratio of imperfection length to the length of the cylinder, and the location parameters of the rectangular imperfection both in the axial and circumferential direction, and the ratio of the thickness of the rectangular imperfection to the thickness of the shell. The location parameter in the circumferential direction was taken as zero, since it was assumed to pass through the axis of symmetry of the cylindrical shell.

(b) The membrane stiffness parameter and the bending stiffness parameter derived in the prebuckling analysis act as multiplying factors for the corresponding terms of the elements in the buckling matrix. The membrane stiffness parameter varies linearly with respect the ratio of thickness of the imperfection to that of the shell. The membrane and bending stiffness parameters are so formed to be equal to zero when there is no variation in the thickness of the shell and the imperfection. These parameters are taken as positive or negative depending upon whether the thickness of the rectangular imperfection is more or less than the thickness of the cylindrical shell respectvely. It was found that the bending stiffness parameter predominates in making the imperfect shell to bifurcate prematurely. Each element of the buckling matrix consists of the membrane stiffness parameter and the bending stiffness parameter, together with the combination of imperfection stiffness parameters.

(c) The imperfection stiffness parameters depend on the decay length in the circumferential direction. The imperfection stiffness parameter in the circumferential direction is approximately three times less than the imperfection stiffness parameter in the axial direction. From this fact, the cylindrical shell with rectangular imperfection in its side, under the action of axial compressive load, bifurcates with considerable distortion of geometry at the point of discontinuity.

Based on the above, shells with radius to thickness ratio of

100 and 200 were selected for numerical study. The other parameters, the aspect ratio of the rectangular imperfection, length to radius ratio, width of the rectangular imperfection to the radius, thickness of the imperfection to the thickness of the shell, which were used to study the subject shells are shown in Tables 5.1 and 5.2. In these tables the last two columns are from the computer results.

mm	a_0/b_0	2L/R	$2b_0/R$	$T_2/T_1 = 0$ Load in lbs	$T_2/T_1 = .5$ Load in lbs	n T_2/T_1 0.0	n T_2/T_1 0.5	Comp. Result	
1	.25	.4	.1570	1811.08	1356.92	14	15	.7481	.5605
2	.25	.8	.1570	1799.95	1177.36	16	17	.7435	.4863
3	.25	1.2	.1570	1632.66	1224.74	17	18	.6744	.5059
4	.25	1.6	.1570	1645.25	978.78	18	16	.6796	.4043
5	.50	.4	.1570	1555.68	1453.99	10	13	.6426	.6006
6	.50	.8	.1570	1394.20	1228.13	11	15	.5759	.5073
7	.50	1.2	.1570	1478.45	1137.59	12	16	.6107	.4699
8	.50	1.6	.1570	1433.47	1169.78	12	17	.5921	.4832
9	.75	.4	.1570	1581.82	1400.25	10	12	.6534	.5784
10	.75	.8	.1570	1630.24	1267.59	10	14	.6734	.5236
11	.75	1.2	.1570	1568.26	1224.74	14	15	.6478	.5050
12	.75	1.6	.1570	1423.01	1300.27	11	16	.5878	.5371
13	1.00	.4	.1570	1538.97	1336.58	13	12	.6357	.5521
14	1.00	.8	.1570	1555.44	1108.05	10	10	.6425	.4577
15	1.00	1.2	.1570	1465.38	1115.80	10	14	.6053	.4609
16	1.00	1.6	.1570	1394.44	1224.50	11	15	.5760	.5058
17	.25	.4	.6280	1758.55	1145.33	10	10	.7264	.4731
18	.25	.8	.6280	1520.57	1146.54	11	12	.6281	.4736
19	.25	1.2	.6280	1462.71	1148.96	11	13	.6042	.4746
20	.25	1.6	.6280	1173.90	841.02	12	13	.4849	.3474
21	.50	.4	.6280	1496.12	997.42	9	9	.6180	.4112
22	.50	.8	.6280	1395.65	854.34	10	10	.5765	.3529
23	.50	1.2	.6280	1238.05	967.88	10	11	.5114	.3998
24	.50	1.6	.6280	1181.40	901.79	10	11	.4880	.3725
25	.75	.4	.6280	1130.08	1295.91	9	10	.4668	.5353
26	.75	.8	.6280	1109.26	1117.00	9	10	.4582	.4614
27	.75	1.2	.6280	1106.11	853.13	9	10	.4569	.3524
28	.75	1.6	.6280	1062.78	872.74	9	10	.4390	.3605
29	1.00	.4	.6280	1094.74	967.40	8	9	.4522	.3996
30	1.00	.8	.6280	1080.21	730.39	7	10	.4462	.3017
31	1.00	1.2	.6280	1026.47	1045.11	8	10	.4240	.4317
32	1.00	1.6	.6280	1038.57	852.89	9	10	.4290	.3523

$E_{al} = 10^7$ psi, $R/T_1 = 100.00$, $T_1 = 0.04$ ins , $\nu = 0.3$, $N_{xa} = (\lambda) * 2420.91$

Table 5.1 Shell imperfection parameters.

mm	a_0/b_0	2L/R	$2b_0/R$	$T_2/T_1=0$ Load in Lbs	$T_2/T_1=1.5$ Load in Lbs	n T_2/T_1 0.0	n T_2/T_1 0.5	Comp. Result	
1	.25	.4	.1570	670.00	901.91	17	39	.5535	.7451
2	.25	.8	.1570	600.27	823.35	19	28	.4959	.6802
3	.25	1.2	.1570	519.04	702.43	20	29	.4288	.5803
4	.25	1.6	.1570	498.95	537.68	21	29	.4122	.4442
5	.50	.4	.1570	653.16	802.17	15	22	.5396	.6627
6	.50	.8	.1570	661.39	710.78	17	25	.5464	.5872
7	.50	1.2	.1570	599.66	634.16	18	26	.4954	.5239
8	.50	1.6	.1570	599.66	486.36	18	26	.4954	.4018
9	.75	.4	.1570	643.11	777.84	14	20	.5313	.6426
10	.75	.8	.1570	600.14	696.62	15	28	.4958	.5755
11	.75	1.2	.1570	565.52	630.89	16	25	.4672	.5212
12	.75	1.6	.1570	555.96	535.02	17	25	.4593	.4420
13	1.00	.4	.1570	676.16	795.27	14	18	.5586	.6570
14	1.00	.8	.1570	652.92	723.00	15	23	.5394	.5973
15	1.00	1.2	.1570	630.43	675.43	16	24	.5206	.5580
16	1.00	1.6	.1570	565.52	522.92	16	24	.4672	.4320
17	.25	.4	.6280	505.85	1027.93	13	19	.3693	.7889
18	.25	.8	.6280	447.02	954.93	13	19	.3693	.7889
19	.25	1.2	.6280	440.48	686.45	14	19	.3639	.5671
20	.25	1.6	.6280	469.54	666.96	15	20	.3879	.5510
21	.50	.4	.6280	505.49	939.31	11	17	.4176	.7760
22	.50	.8	.6280	492.53	843.93	19	17	.4069	.6972
23	.50	1.2	.6280	459.97	736.56	12	21	.3800	.6085
24	.50	1.6	.6280	446.90	698.67	13	18	.3692	.5772
25	.75	.4	.6280	485.99	808.22	11	13	.4015	.6677
26	.75	.8	.6280	476.19	786.19	15	17	.3934	.6495
27	.75	1.2	.6280	487.08	690.20	19	17	.4024	.5702
28	.75	1.6	.6280	463.73	613.94	12	19	.3831	.5072
29	1.00	.4	.6280	486.72	797.57	10	14	.4021	.6589
30	1.00	.8	.6280	504.88	755.21	11	16	.4171	.6239
31	1.00	1.2	.6280	473.17	742.49	16	15	.3909	.6134
32	1.00	1.6	.6280	484.55	595.67	19	16	.4003	.4921

$E_{a1}= 10^7$ psi, $R/T_1=200.00$. $T_1= 0.04$ ins , $\nu = 0.3$, $N_{xa}=(\lambda)*1210.46$

Table 5.2 Shell imperfection parameters.

5.3 Discussion of the Results of Computer Analysis.

Figures 5.1 to 5.5 show the variation of the load as a function of value of the determinant for the parameters stated in the previous section. Each figure is for two shells having radius to thickness ratio of 100 and 200.

Figures 5.1 and 5.2 are for a rectangular imperfection having an aspect ratio of 1.5. Figure 5.1 is for a cutout and Figure 5.2 is for a general imperfection of the same thickness as the shell.

Figures 5.3 and 5.4 are for a rectangular imperfection with an aspect ratio of 0.5 and varying imperfection thicknesses.

The analyses represented by Figure 5.1 to 5.5 show that, since the buckling load parameter λ is the ratio of the buckling load of imperfect to the classical buckling load of the perfect shell, the buckling load parameter approaches 1.0 as the thickness of the imperfection approaches the thickness of the shell.

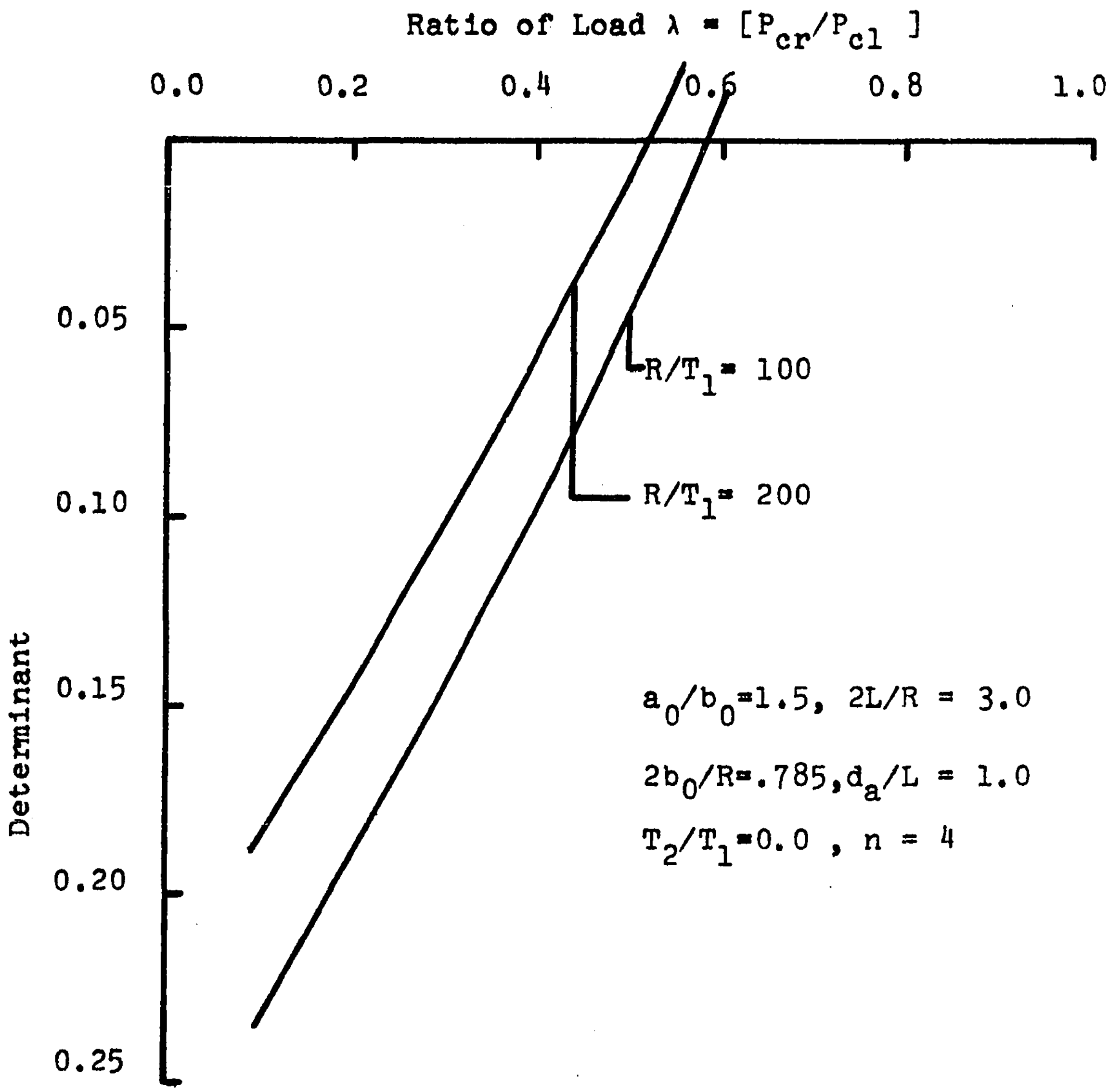

Figure 5.1 Variation of buckling load with determinant.

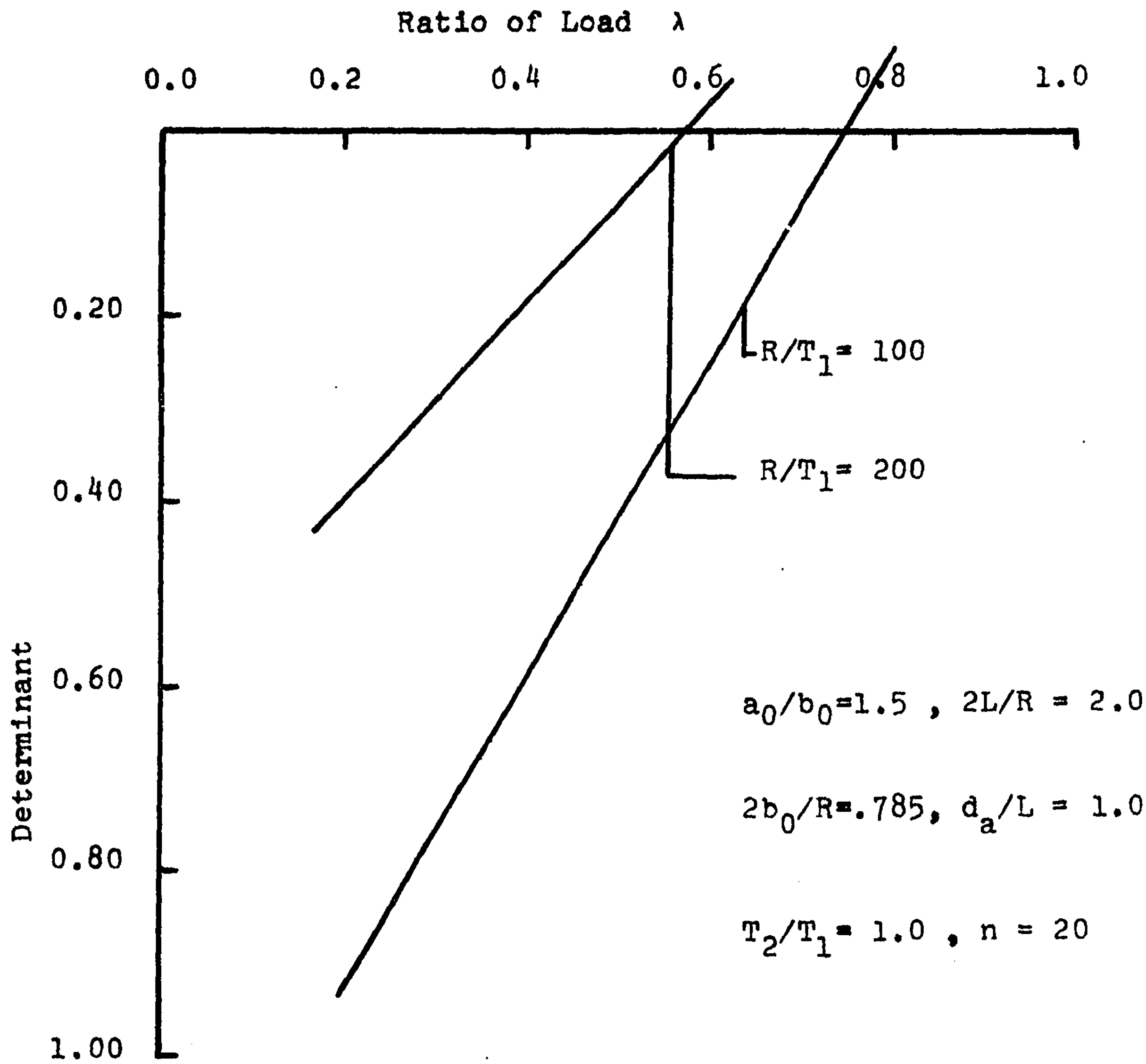

Figure 5.2 Variation of buckling load with determinant.

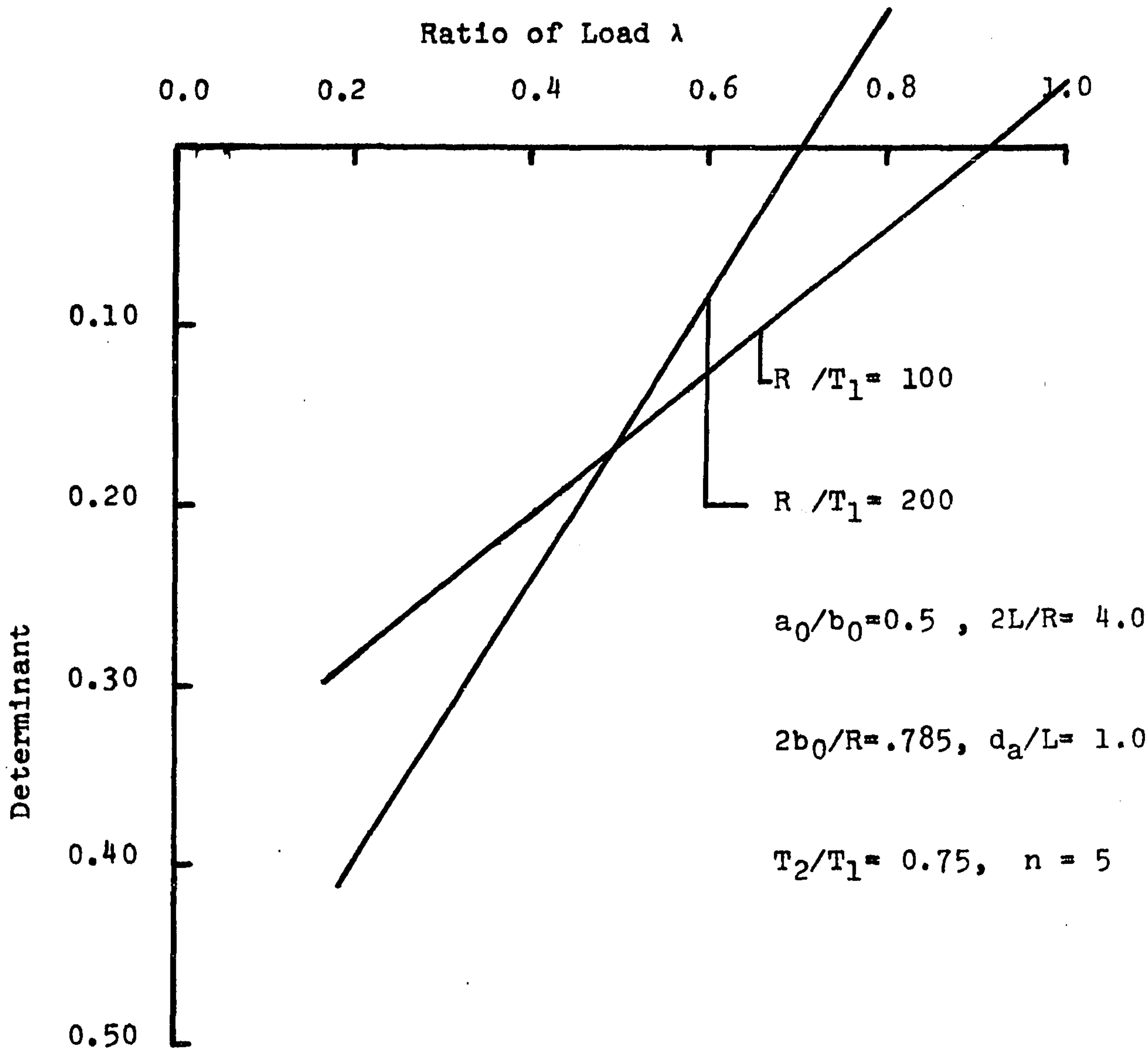

Figure 5.3 Variation of buckling load with determinant.

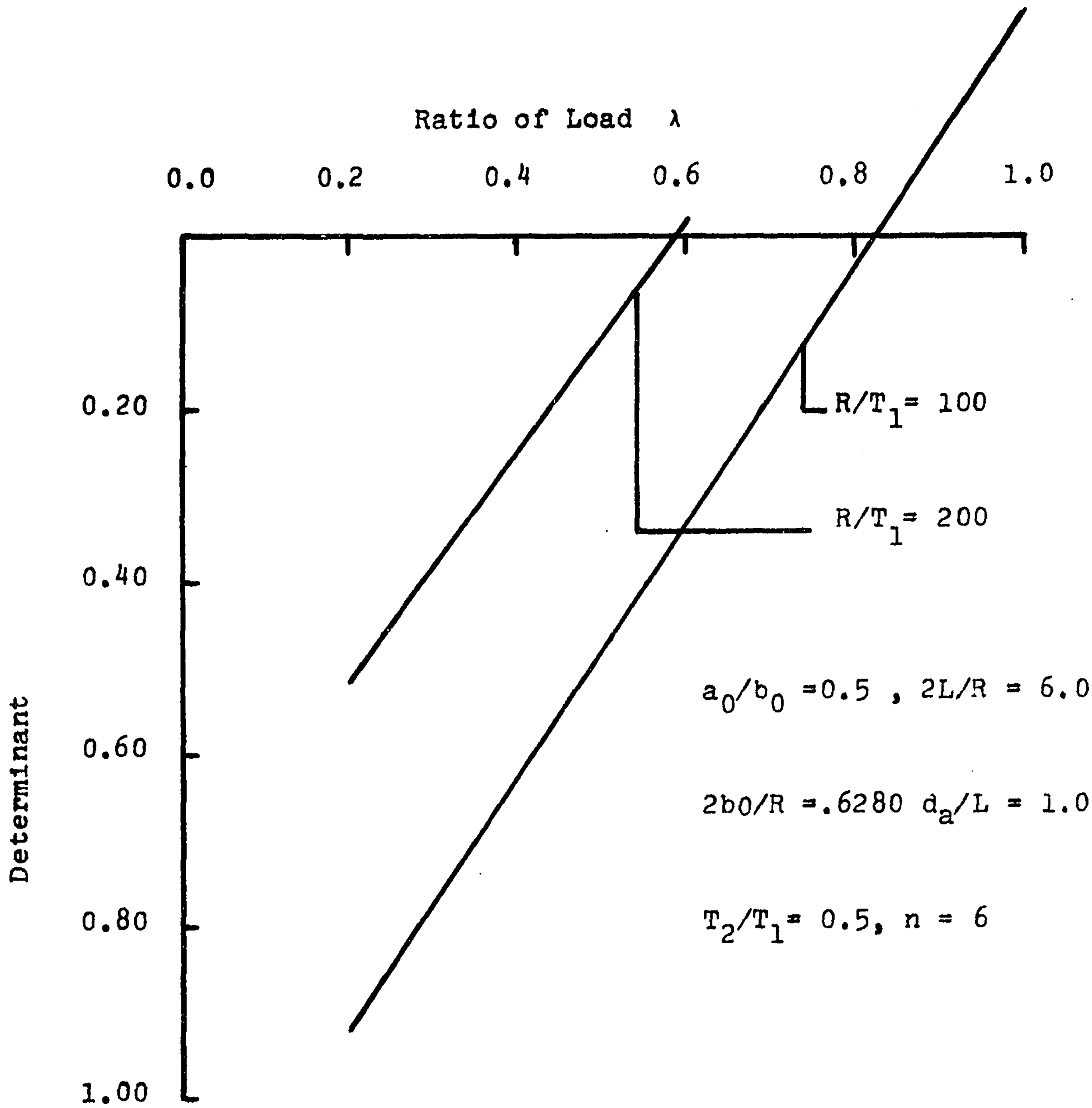

Figure 5.4 Variation of buckling load with determinant.

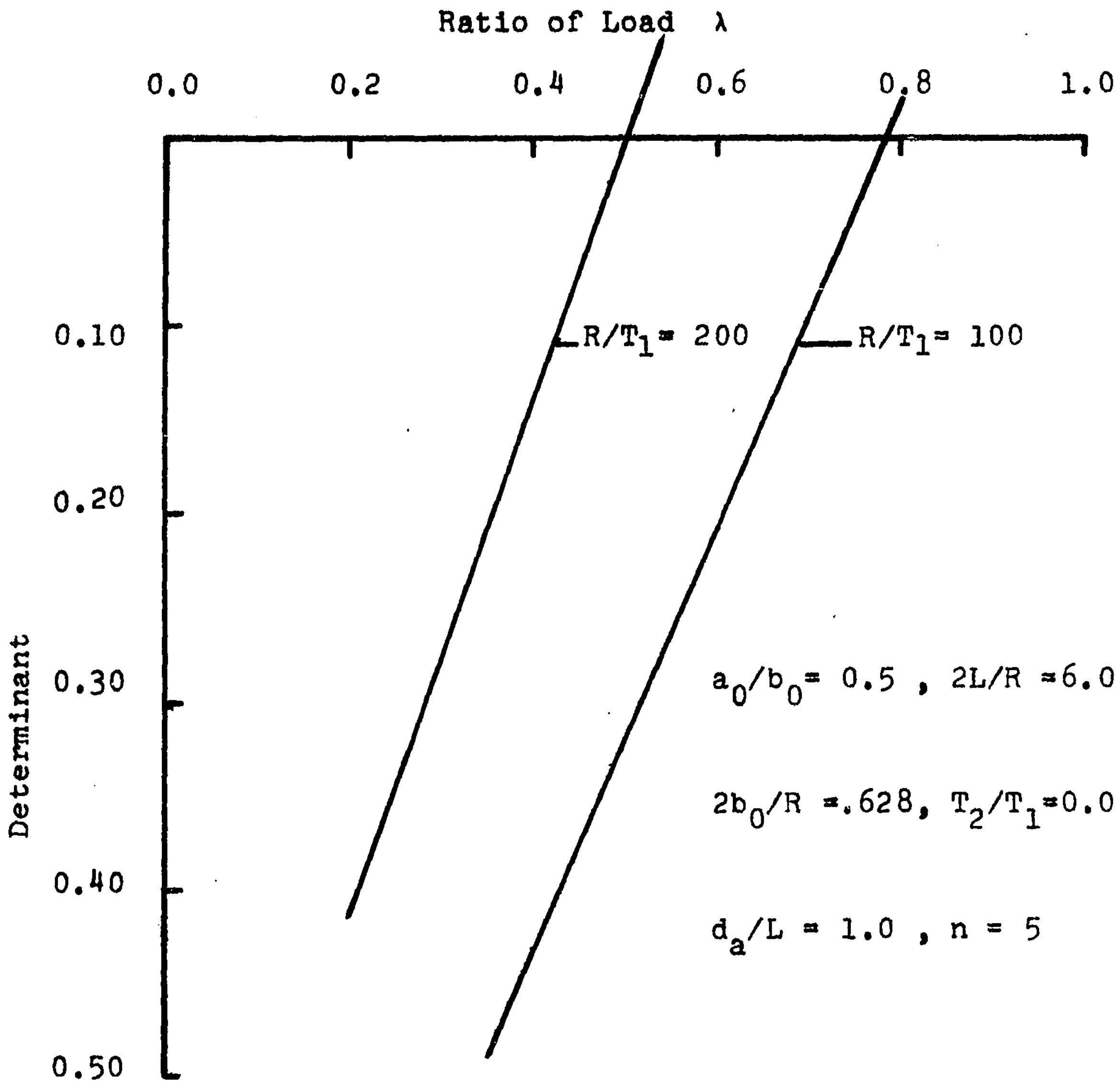

Figure 5.5 Variation of buckling load with determinant.

Figures 5.6 to 5.9 show the variation of the load versus the aspect ratio of the imperfection for different shell length to radius ratios.

Figure 5.6 is for width of imperfection to radius ratio of 0.1570 and Figure 5.7 is for 0.6280. Both figures are for a shell having a radius to thickness ratio of 200.0.

The behavior of the shell for different length to radius ratios is almost unchanged as shown in Figure 5.6, whereas, there is a drastic change shown in Figure 5.7.

Figures 5.8 and 5.9 are for different aspect ratios of the rectangular imperfection, but otherwise the same as the shells for Figures 5.6 and 5.7. The analyses represented by Figures 5.8 and 5.9 show that the buckling load decreases as the aspect ratio of the rectangular imperfection increases.

The rectangular imperfection parameter given by

$$\mu = \bar{C}*[(2b_0/R)^2*(R/T_1)^{1/2}*(1+a_0/b_0)]$$

where $$\bar{C} = 1/8 *(12(1-\nu))^{1/4}$$

was used to develop the curves shown in Figures 5.10 and 5.11. Figure 5.10 is for an imperfection width to radius ratio of 0.1570 and Firgure 5.11 is for 0.6280. The

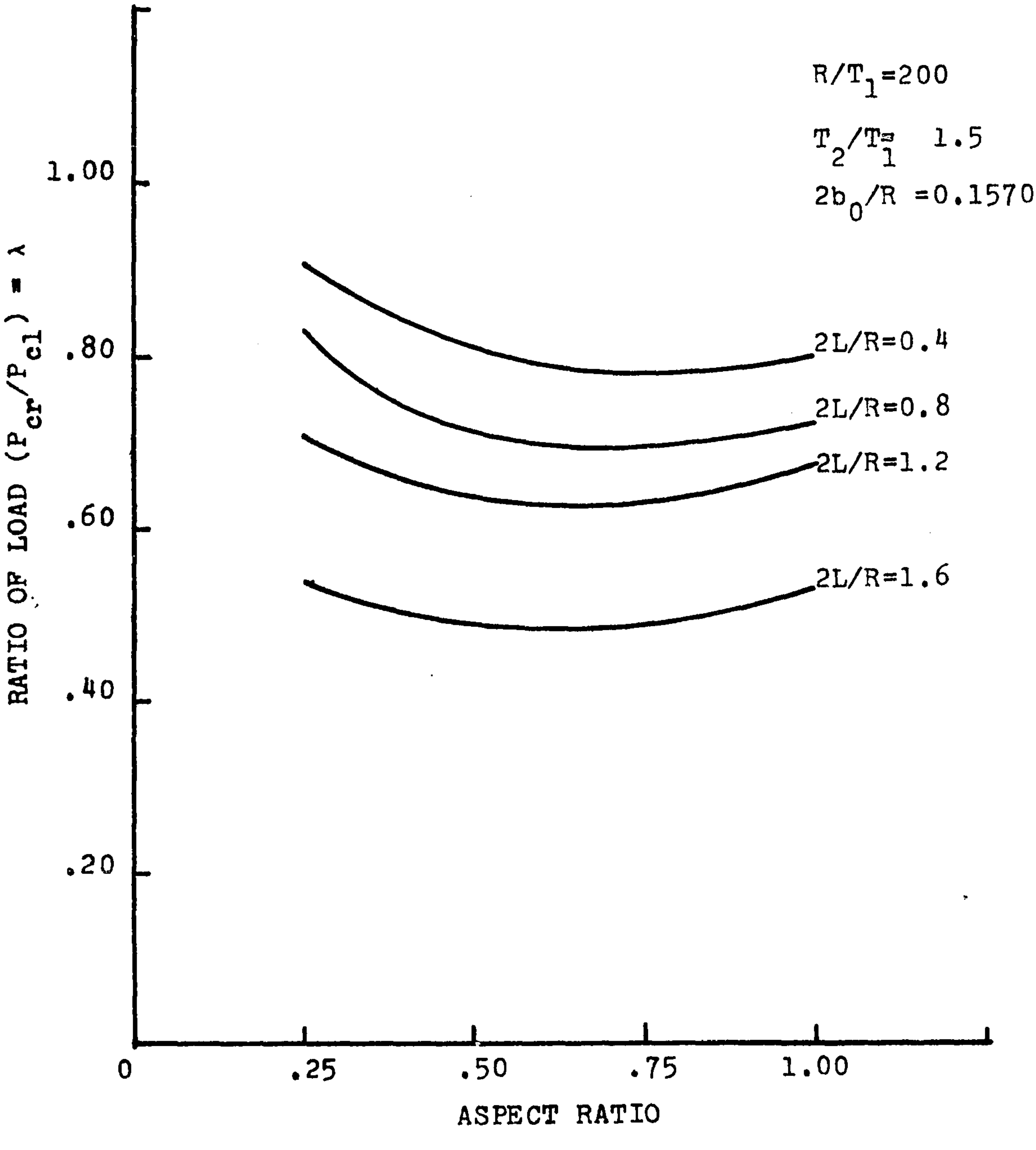

Figure 5.6 Variation of buckling load with aspect ratio of rectangular imperfection.

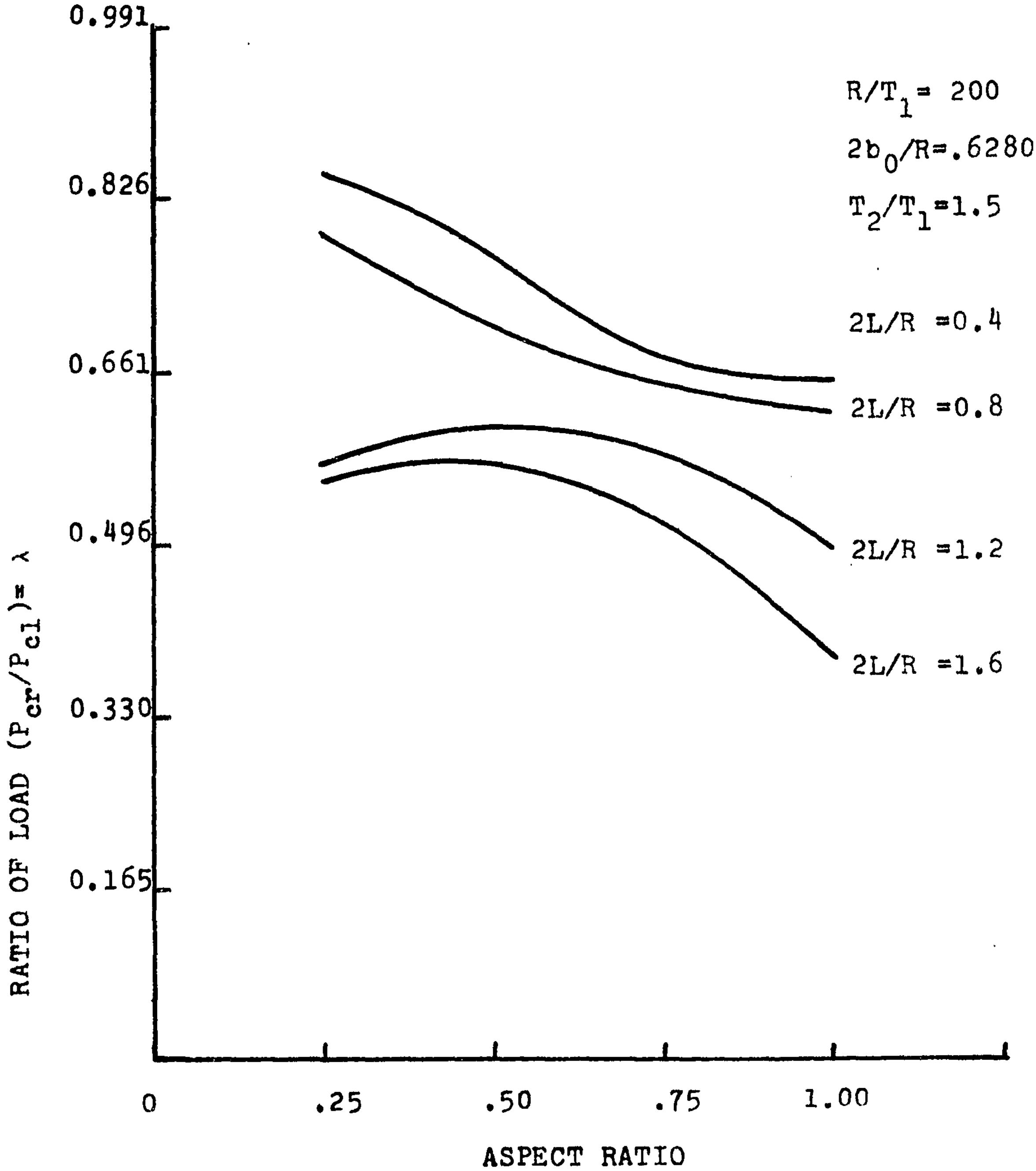

Figure 5.7 Variation of buckling load with aspect ratio of rectangular imperfection.

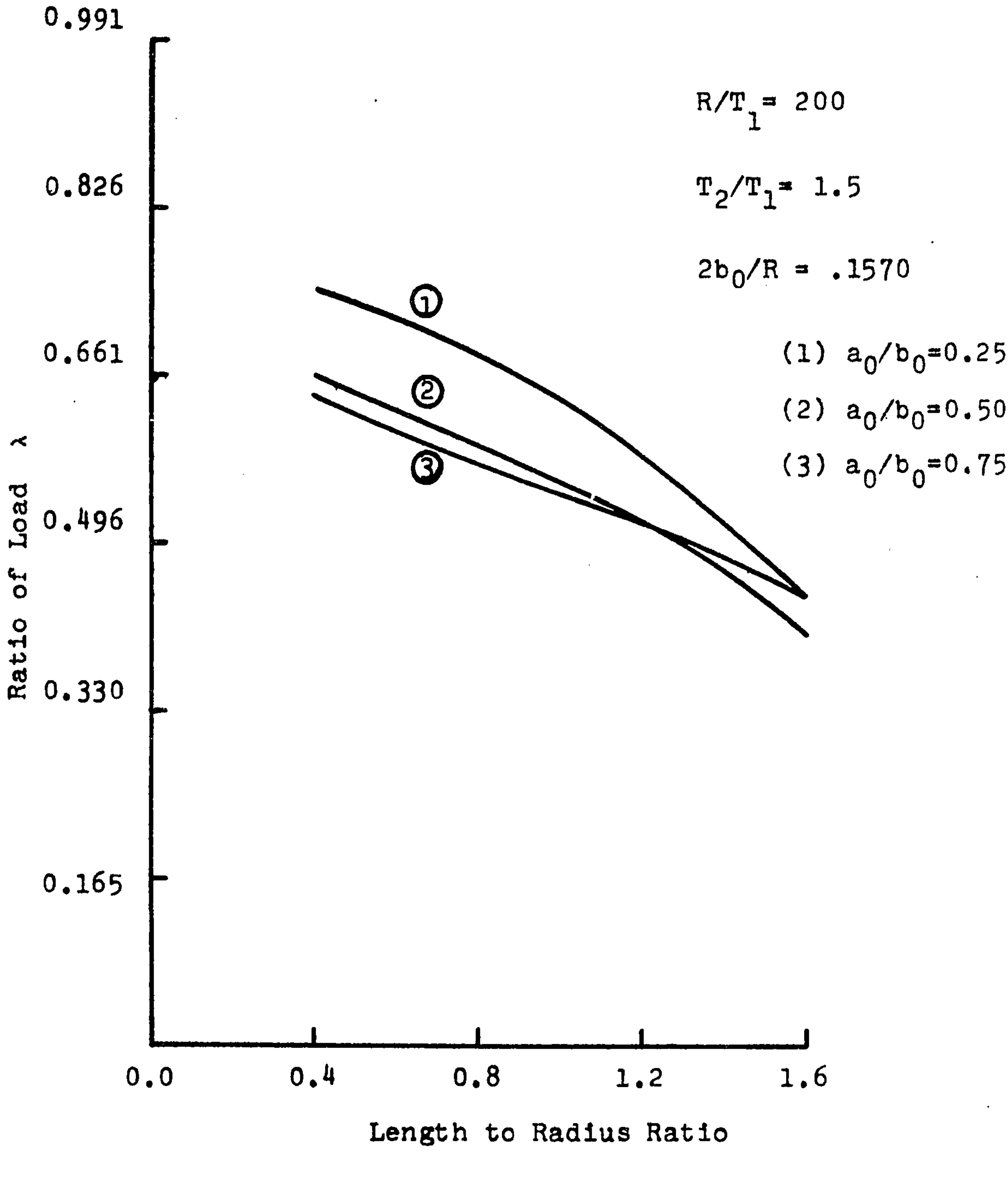

Figure 5.8 Variation of buckling load with length to radius ratio of the shell.

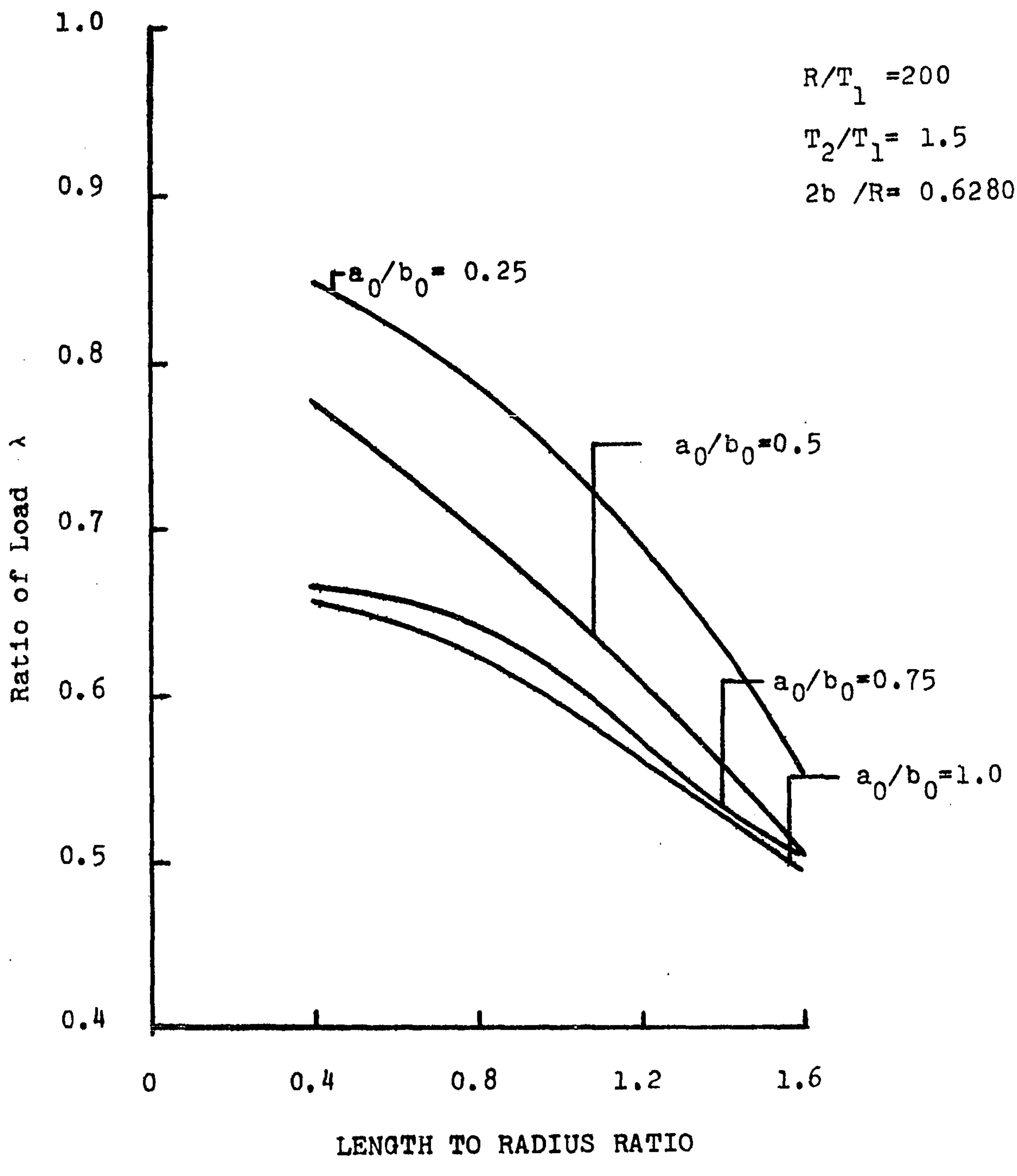

Figure 5.9 Variation of buckling load with length to radius of the shell.

ratio of the thickness of the imperfection to that of the shell is 1.5. These figures show the variation of the load as a function of the imperfection parameter for the same shells. The buckling load parameter decreases with increase of the imperfection parameter. The shape of curves shows general agreement with curves developed by Koiter for shells with general imperfections. The steepness of the curves shows that the shell structure is very much sensitive to a rectangular imperfection.

Figure 5.12 is for a rectangular imperfection having a width to radius ratio 0.1570 and Figure 5.13 is for 0.6280. Both of these curves are for a ratio of thickness of the imperfection to that of the shell of 1.5. Figures 5.12 and 5.13 show the variation of the load as a function of the curvature parameter. These two figures reveal the reduction in load as the geometric parameter increases.

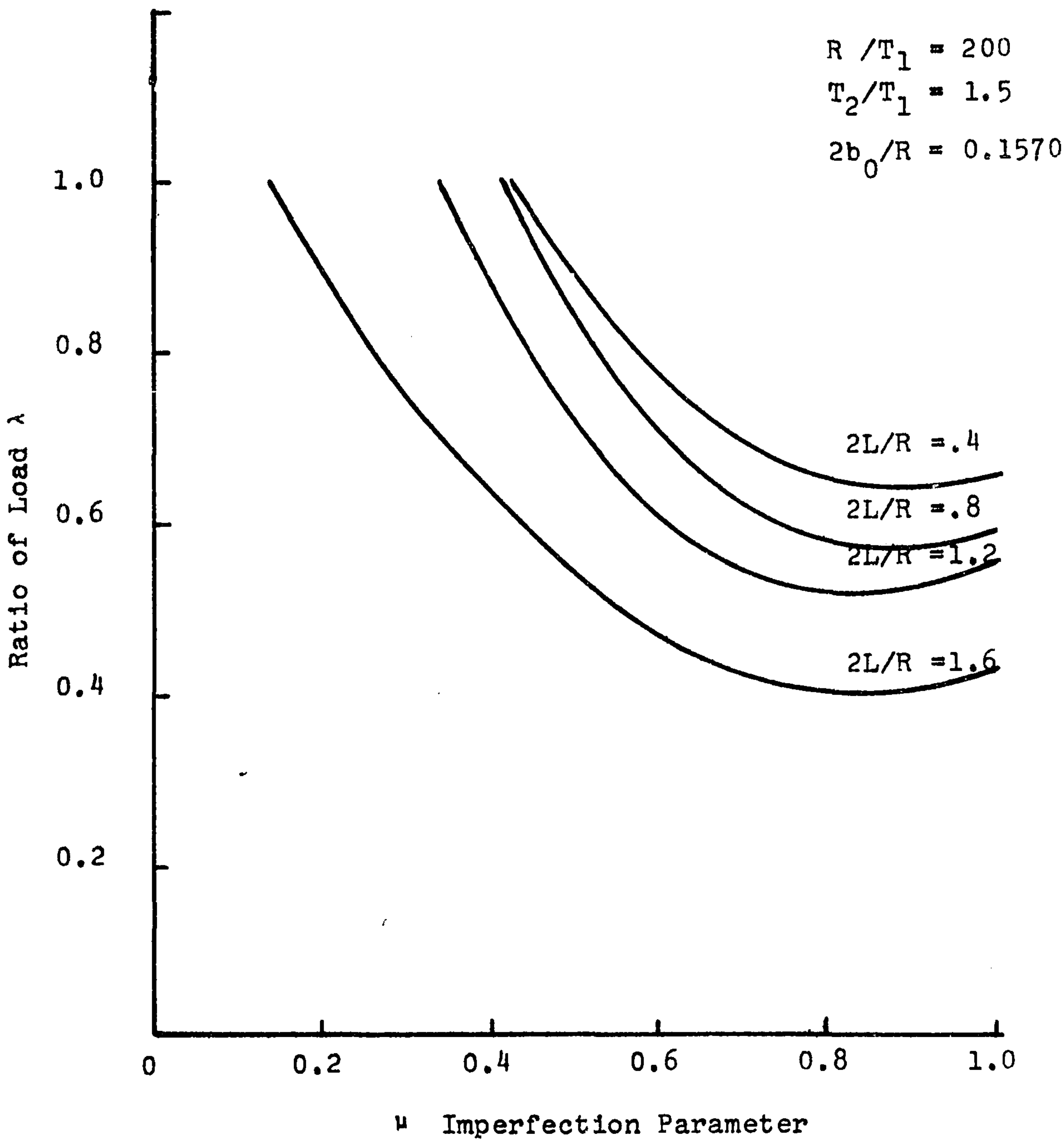

Figure 5.10 Variation of buckling load with imperfection parameter.

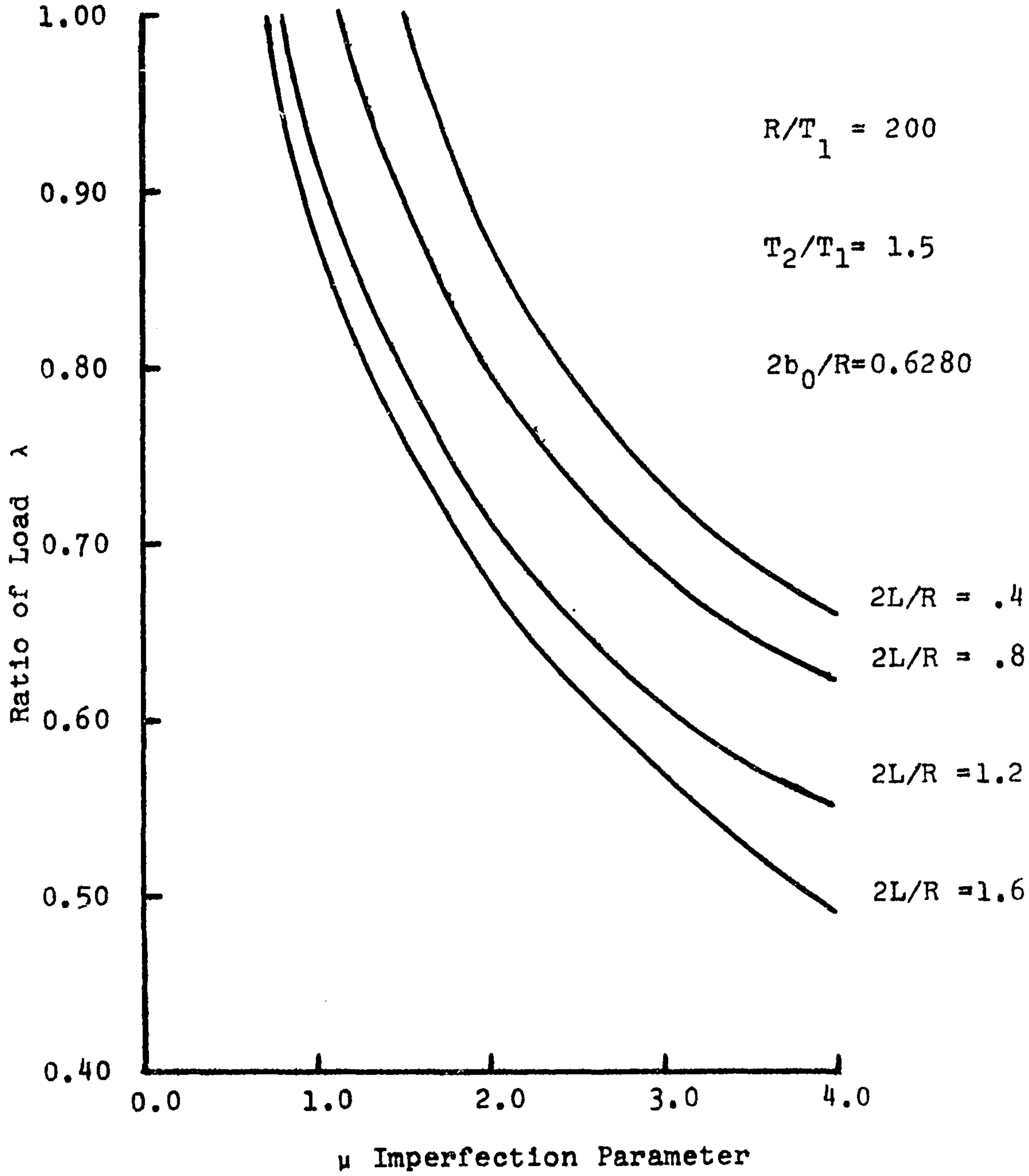

Figure 5.11 Variation of buckling load with imperfection parameter.

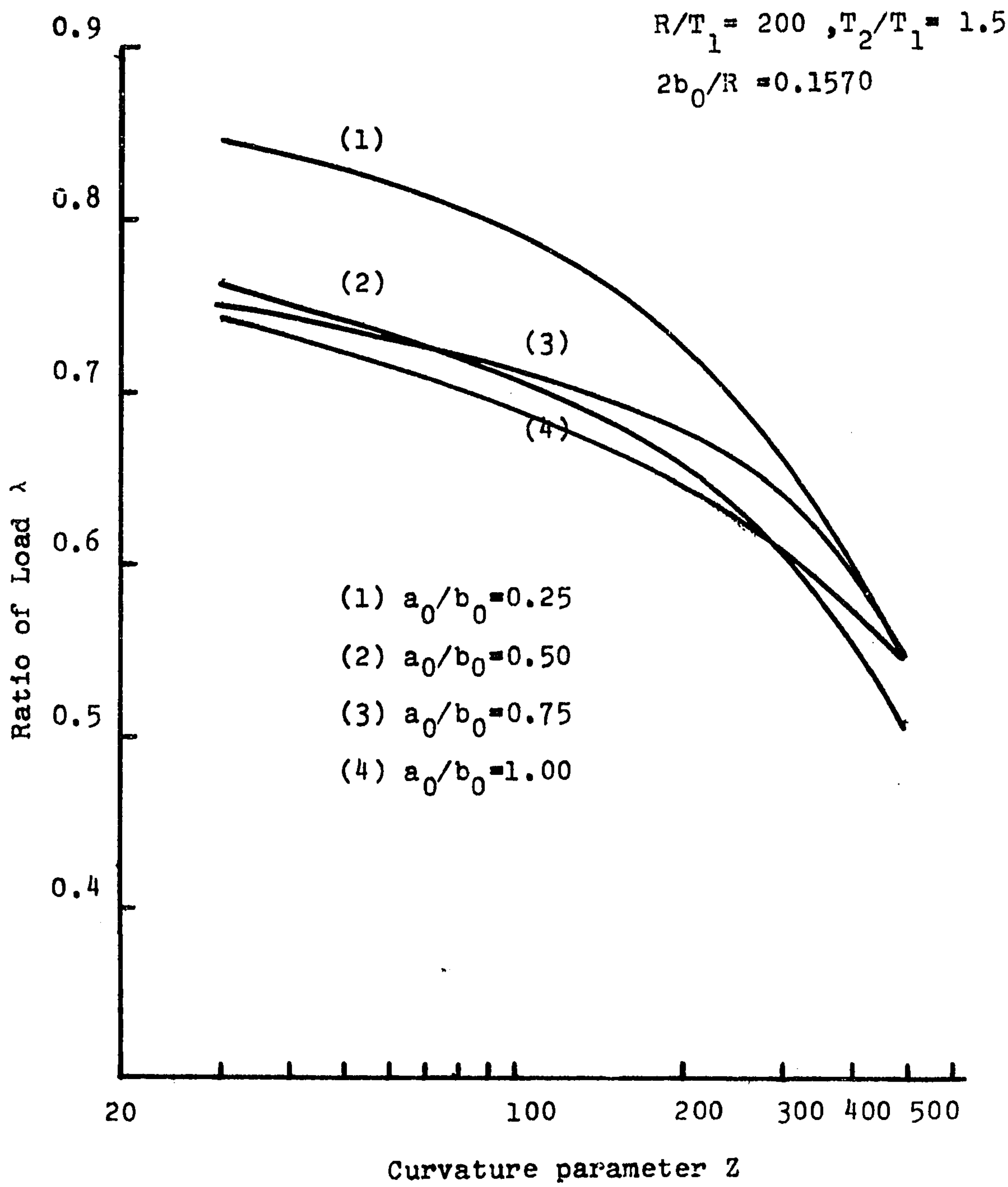

Figure 5.12 Variation of buckling load with curvature parameter.

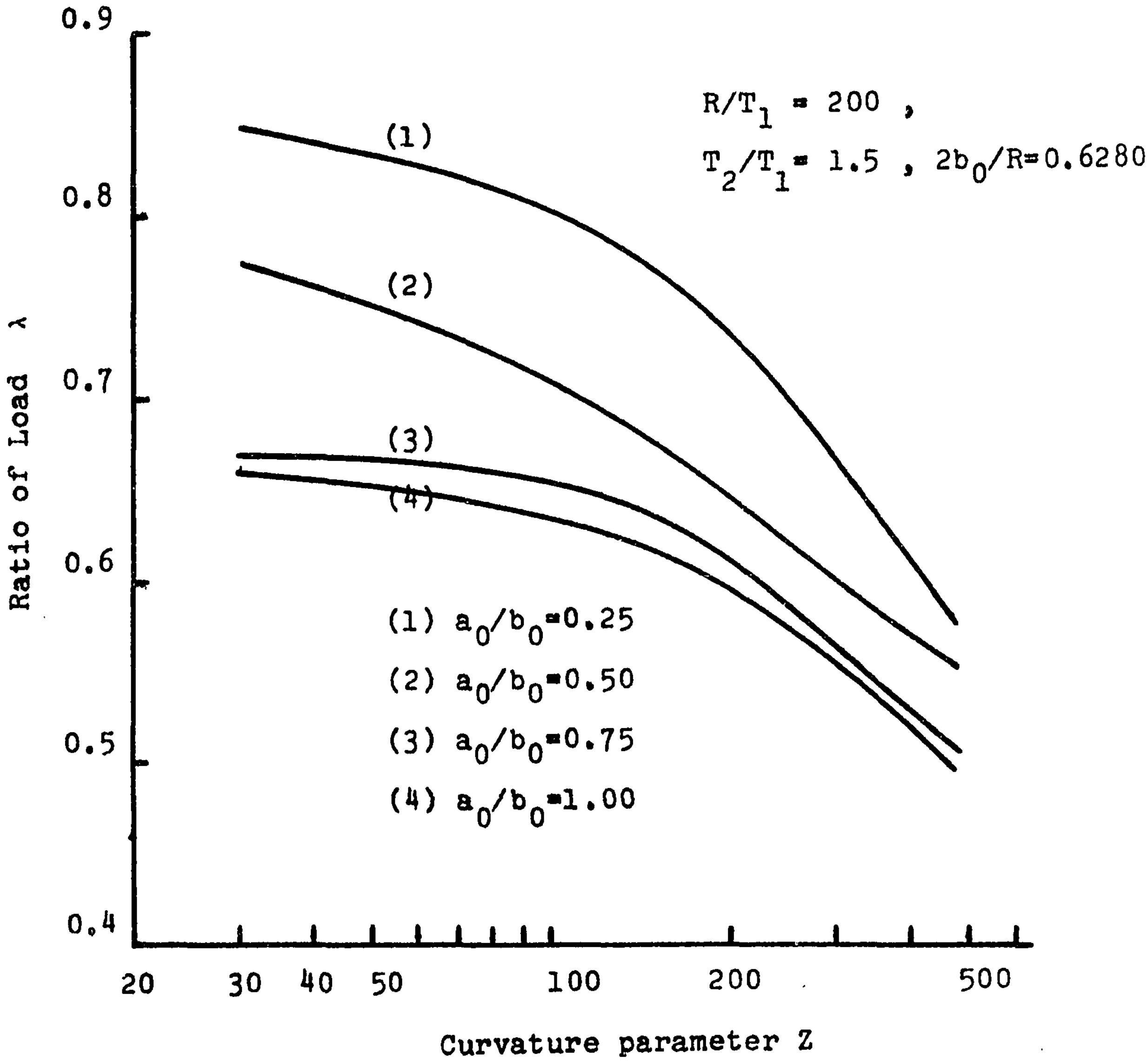

Figure 5.13 Variation of buckling load with curvature parameter

CHAPTER 6

SUMMARY AND CONCLUSION

In this study the bifurcation point load for cylindrical shell having a rectangular imperfection was obtained by adopting the Koiter's concept of initial-postbuckling theory for imperfect shells. Since linear prebuckling behavior is assumed, the bifurcation point load analysis leads to a linear eigenvalue problem. In this case the bifurcation point load analysis is relatively simple, and eminently useful way to determine the stability limit. Although nonlinear strain displacement relationships of shallow shells are assumed for deriving the potential energy expressions for the buckling analysis, when it comes to the question of considering the second order terms of the total potential energy, the equation exhibits a highly linearized form, hence, obtaining the linear eigenvalue of the bifurcation point load is justified.

Variation of decay length in the circumferential direction is considered in obtaining the minimum potential energy for the buckling analysis of imperfect shallow shells. This strategy is required because the shell buckling has a weak dependence on shell curvature. The bifurcation point load for a shell having a rectangular imperfection in its side considerably reduces the axial compressive load carrying capacity of the shell.

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APPENDIX A

Computer Program Listing

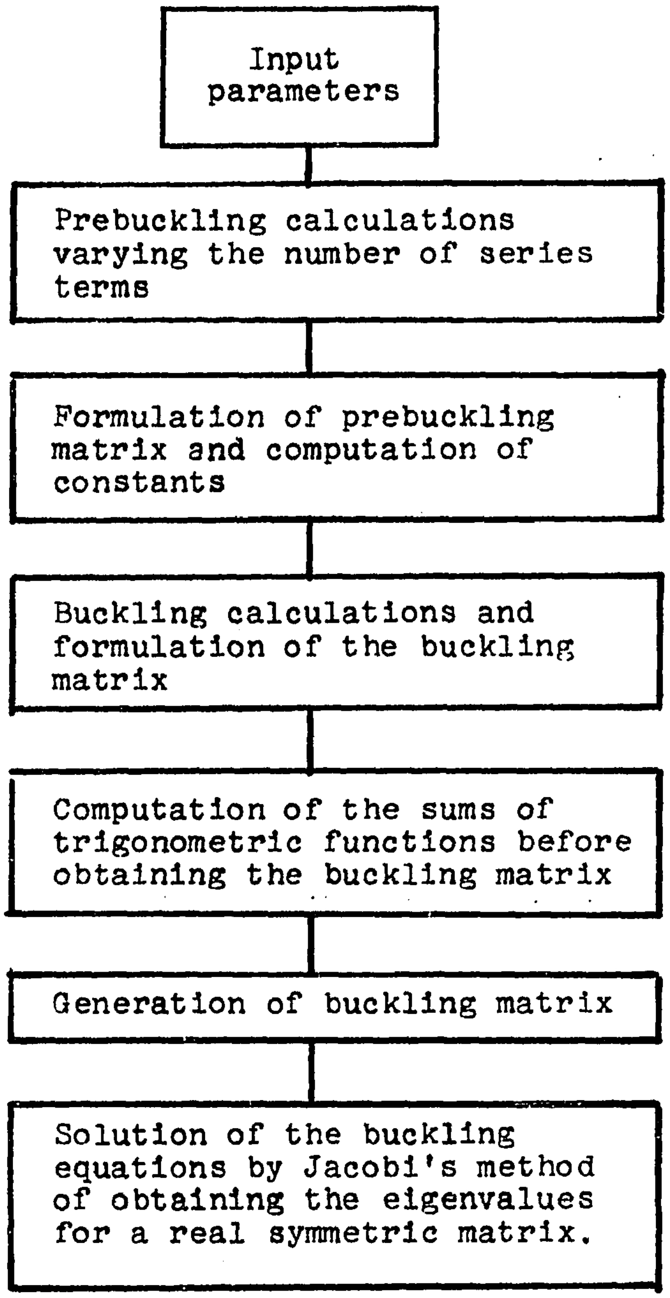

Figure A.1 Macro computer program flow chart

```
C       MAIN PRBUCKLING STRESS
C
C       THIS PROGRAM FIRST CALCULATES THE PREBUCKLING COEFFICI
C       ENTS BY A SUBROUTINE 'GAUSJO' THUS OBTAINED COEFFICIENT
C       S ARE THEN SUBSTITUTED IN THE BUCKLING DETERMINANT.THE
C       BUCKLING DETERMINANT INVOLVES VARIABLE INTEGERS. THE
C       ELEMENTS OF THE PREBUCKLING MATRIX DEPENDS ON THE RATI
C       IO OF THE THICKNESSES OF THE SHELL AND THE IMPERFECTION
C       THE PROGRAM CAN BE EXTENDED TO FORM THE CUTOUT BY
C       GIVING THE VALUE OF T2 AS ZERO AND CAN ALSO BE TREATED
C       WITH OUT ANY IMPERFECTION BY SETTING T2=T1=T WHERE T1
C       IS THE THICKNESS OF THE SHELL. THE VARIABLES ARE AS
C       FOLLOWS.
C       1...A0/B0.....ASPECT RATIO OF THE CUTOUT  OR
C                     IMPERFECTION
C       2...EL/R .....LENGTH OF THE CYLINDER TO RADIUS
C       3... R/T1.....RATIO OF PRINCIPAL RADIUS OF THE CYLINDER
C                     TO THE THICKNESS OF THE CYLINDER
C       4...B0/R .....RATIO OF CUTOUT OR IMPERFECTION BREADTH
C                     TO PRINCIPAL RADIUS OF THE CYLINDER.
C       5...T2/T1.....RATIO OF THICKNESS OF CUTOUT OR IMPERFEC
C                     TION TO THCKNESS OF CYLINDER.
C       6...DA/EL.....LOCATION OF THE CUTOUT OR IMPERFECTION A
C                     ALONG THE LENGTH OF THE CYLINDER.
C       7...DC/R .....LOCATION OF THE CUTOUT OR IMPERFECTION A
C                     LONG THE CIRCUMFERENCE OF THE CYLINDER
C                     THIS CAN BE SET TO ZERO
C       8...POR   .....POISSON RATIO DIFFERS FOR MATERIALS
C       9...SIGA/SIGCL RATIO OF ACTUAL STRESS TO THE CLASSICAL
C                    CAL STRESS FROM WHICH THE LOAD CAR
C                     RYING CAPACITY OF THE CYLINDER WITH
C                     AN IMPERFECTION OR CUTOUT CAN BE AR
C                     RIVED WITH RESPECT TO A PERFECT CYL
C                     INDER
C       10..      .....THE MAIN PARAMETER IS SQRT(RT1/B0**
C                 .....2) AND THE ASPECT RATIO.
C
C       THE SUBROUTINE'GAUSJO' IS USED TO OBTAIN THE BUCK
C       LING DETERMINANT, A PLOT IS MADE OF EACH OF THE D
C       ETERMINANT AGAINST THE RATIO OF 9
C
        IMPLICIT REAL * 8 (A-H, O-Z )
        DIMENSION A(4,5),C(3,3),T(3,3),AIK(3),EIGEN(3)
C       COMMON BLOCKS
        COMMON/BLOCK1/B1TAU,STAUB1,SEPSIY,SCSIY,SOHMB1,SPHIY,
       1SSIY,SPHIOY,SSIOY,SCSIOY,STHBBC,B1TAU1,STAUB0,DCCHM,SESIOY
        COMMON/BLK2/XI,XII,XIJ,XJ,XJJ,XIIII
        COMMON/BLK3/EPSA0,EPSA1
```

```
      COMMON/BLK4/RBALP,ALPBEL,A3
      COMMON/BLK7/C1130,C2570,C3290,C1231,C2481,C3621,C123S,C248S,C362S,
     2           C113B,C215B,C329B,C2610,C3420,C2091,C3981,C209S,C348S,
     3           C206B,C332B,C3530,C3281,C358S,C334B,DELTA
      COMMCN/BLOCK5/Q1,BOBBET,ELBR,AOBBO,BOBR,DABEL,DCBR,T2BT1,
     1ALPBET,CLBR,X0,Y0,X1,Y1,EPS,EPSB,TEMP,AOBEL,
     2FOP1,FOM2,TOP1,TOP2,SOPRM2,QOM1,TOPRP2,SA0,SOM2,GOP1,
     1QM1,TPRP2,SM2,TM1,TM2,GP1,FM1,BETBT1,ALPBT1,GOM1,
     3TOM1,TOM2,BETBR,FOM1,FOP2,GSM2,GTP2,GOSM2,
     4 GOTP2,GM1,FP1,FM2,FP2,TP1,TP2,SPRM2,DABBET,SPRP2
      COMMON/BLK6/PI,B1BR,DCTAU,N
C     CODING
 13   READ 100,MM,AOBBO,ELBR,BOBR,DAEEL,DCBR
      PRINT 1
      PRINT 110
      PRINT 105,MM,AOBBO,ELBR,BOBR,DAEEL,DCBR
      PI       = 3.141592654
      PCR      = 0.3
      P        = 0.0
      Q1       =(3.*(1.-POR*POR))**0.25
      N        = 17
      T2BT1    = 1.5
      PRINT 180,N,T2BT1
      DO 85 KK= 7,12,5
      ZGXBCL   = (KK-2)/10.
      TEMP     = ZGXBCL
      RBT1     = 100
      PRINT 210, RBT1,TEMP
      PRINT 200
 40   ALBEL    = 2.44/(ELBR*DSQRT(RBT1))
      RBO      = BOBR*DSQRT(RBT1)
      AR       = (2.44/AOBBO)/RBO
      AS       = 2.44/RBO
      AOBEL    =(AOBBO)*(BOBR)/(ELBR)
      A3       = AOBEL + ALBEL
      ALPBEL   =A3/PI
      RBALP    =(1./((ELBR)*(ALPBEL)))
      CLBR     = 2.44/DSQRT(RBT1)
      BETAA    = BOBR*BOBR*RBT1
      ALPHA    = (AOBBO)**2*BETAA
      Q2       =1.+AR
      Q3       =1.+AS
      UN       =(PI/Q3)**2/BETAA
      UM       =(PI/Q2)**2/ALPHA
      AOBALP   =PI/Q2
      BOBBET   =PI/Q3
      ALPBET   = AOBBO * ( Q2/Q3 )
      EPSAO    = .85*AOBEL*BOBR/PI
      EPSA1    =EPSAO*Q2*Q3
      SA       =(DSIN(AOBALP))/(AOBALP)
      SB       =(DSIN(BOBBET))/(BOBBET)
      S2A      =(DSIN(2.*AOBALP))/(2.*AOBALP)
```

```
      S2B       =(DSIN(2.*BOBBET))/(2.*BOBBET)
      CA        =(DCOS(AOBALP))
      TA        =(1.+SA)/2.
      TB        =(1. + SB )/2.
      T2A       =(1.-S2A)/8.
      T2B       =(1.-S2B)/8.
      D2A       =(1.+S2A)/8.
      D2B       =(1.+S2B)/8.
      CPA       =(1.+CA )/2.
      EPS       =(T2BT1-1.)
      EPSB      =((T2BT1)**3-1.)
      G1        =1.+EPSAO*EPS
      G2        =(EPSA1/4.)+(EPSAO*EPS*TA*TB)
      G3        =(EPSA1/2.)+(EPSAO*EPS*TB)
      ANXAC1    = TEMP   * (DSQRT((1. - POR*POR)/3.))/(RBT1)
      AN1       = ANXAC1
      G4        =G1+AN1
      G5        =(9./64.)*EPSA1+EPSAO*EPS*(TA-T2A)*(TB-T2B)+(UN*
     1           *2/768.)*(3.*EPSA1+64.*EPSAO*EPSB*D2B*(TA-T2A
     2           ))+(UM*UN/384.)*(EPSA1+192.*POR*(TB-2.*T2B)*(
     3           TA-2.*T2A))
      AN2       =AN1*EPSA1*.5
      AN3       =(3./32.)*AN2*(RBALP)**2
      G6        =( 3./16.)*POR*EPSA1+POR*EPSAO*EPS*TA*(TB-T2B)-
     1           ((1.-POR)/384.)*UN*(EPSA1+16.*EPSAO*EPSB*T2B*
     2           (TA-CPA))/(RBT1)
      A1BBET    =(PI*(AOBBO)*Q2/Q3)
      AOBBET    =(AOBBO)*(BOBBET)
      A1BR      =(AOBBO)*(BOBR)*Q2
      G7        =(3./8.)*EPSA1+((1.-POR)/12.)*EPSA1*(A1BBET)**
     1           2+EPSAO*EPS*(TB-T2B)+.67*EPSAO*EPS*(1.-POR)*T2
     2           B*(AOBBET)**2+((1.-POR)/384.)*UN*(.17*EPSA1*
     3           PI/(UM*(RBT1))+.17*EPSAO*EPSB*(A1BR)**2)/(RBT1)
      AN4       =.75*AN2
      RHO1      =0.0
      RHO2      =EPSA1*RHO1/4.
      DO 12 I = 1,4
      DO 12 J = 1,5
   12 A(I,J)  = 0.0
       A(1,1) =G1
       A(1,2) =-POR*A(1,1)
       A(1,3) =G2
       A(1,4) =-POR*G3
       A(2,1) =A(1,2)
       A(2,2) =G1+AN1
       A(2,3) =-POR*A(1,3)
       A(2,4) =G3+AN2
       A(2,5) =-AN1
       A(3,1) =A(1,3)
       A(3,2  =A(2,3)
       A(3,3) =G5+AN3
       A(3,4) =-G6
```

```
         A(4,1) =A(1,4)
         A(4,2) =A(2,4)
         A(4,3) =A(3,4)
         A(4,4) =G7+AN4
         A(4,5) =-AN2
        CALL GAUSJO ( A,4,1,DETER )
        DO 30 I =1,4
   30   X0       = A( I,5 )
        X0       = A (1,5 )
        Y0       = A ( 2,5 )
        X1       = A ( 3,5 )
        Y1       = A (4 ,5 )
        CALL BUCK (C,RBT1)
C       EIGENVALUES BY THE JACOBI METHOD.
        DO 2 I   = 1,3
        DO 2 J   = 1,3
        T(I,J)   = 0.0
    2   C(I,J)   = 0.0
        NN       = 3
        ITMAX    = 50
        EPS1     = 1.00E-10
        EPS2     = 1.00E-10
        EPS3     = 1.00E-5
        C( 1,1 )=.5*(C1130+EPSA1*C1231+C123S*EPS*EPSA0+C113B*EPSB
       1         *EPSA0 )
        C( 1,2 )=(-C2570+EPSA1*C2481+C248S*EPS*EPSA0+C215B*EPSB
       1         *EPSA0 )*.25
        C( 1,3 )=(-C3290+C3621*EPSA1+C362S*EPS*EPSA0+C329B*EPSB
       1         *EPSA0 )*.25
        C( 2,2 )=.5*(C2610+C2091*EPSA1+C209S*EPS*EPSA0+C206B*EPSB
       1         *EPSA0 )
        C( 2,3 )=(C3420+C3981*EPSA1+C348S*EPS*EPSA0+C332B*EPSB
       1         *EPSA0 )*.25
        C( 3,3 )=.5*(C3530+C3281*EPSA1+C358S*EPS*EPSA0+C334B*EPSB
       1         *EPSA0 )
C
        NM1      = NN-1
C       SET UP INITIAL MATRIX T, COMPUTE SIGMA AND S
        SIGMA1   = 0.0
        OFFDSQ   = 0.0
        DO   5 I = 1,NN
        SIGMA1   = SIGMA1 + C(I,I)**2
        T(I,I)   = 1.0
        IP1      = I + 1
        IF (I.GE.NN ) GO TO 6
        DO   5 J = IP1, NN
    5   OFFDSQ   = OFFDSQ + C(I,J)**2
    6   S        = 2.0*OFFDSQ + SIGMA1
C       BEGIN JACOBI ITERATION
        DO 26 ITER = 1,ITMAX
        DO 20  I = 1,NM1
        IP1      = I + 1
```

```
      DO 20 J = IP1,NN
      Q       = DABS(C(I,I)-C(J,J))
C     COMPUTE SINE AND COSINE OF ROTATION ANGLE
      IF(Q.LE.EPS1) GO TO 9
      IF(DABS(C(I,J)).LE.EPS2) GO TO 20
      P       = 2.0*C(I,J)*Q/(C(I,I)-C(J,J))
      SPQ     = DSQRT(P*P+Q*Q)
      CSA     = DSQRT((1.0+Q/SPQ)/2.)
      SNA     = P/(2.0*CSA*SPQ)
      GO TO 10
 9    CSA     = 1.0/DSQRT(2.0D0)
      SNA     = CSA
 10   CONTINUE
C     UPDATE COLUMNS I AND J OF T EQUIVALENTS TO
C     MULTIPLICATION BY THE ANNHILLATION MATRIX
      DO 11 K = 1,NN
      HOLDKI  = T(K,I)
      T(K,I)  = HOLDKI*CSA + T(K,J)*SNA
 11   T(K,J)  = HOLDKI*SNA - T(K,J)*CSA
C     COMPUTE NEW ELEMENTS OF A IN ROWS I AND J
      DO 16 K = I,NN
      IF (K.GT.J) GO TO 15
      AIK(K)  = C(I,K)
      C(I,K)  = CSA*AIK(K) + SNA*C(K,J)
      IF (K.NE.J ) GO TO 14
      C(J,K)  = SNA*AIK(K) - CSA*C(J,K)
 14   GO TO 16
 15   HOLDIK  = C(I,K)
      C(I,K)  = CSA*HOLDIK + SNA*C(J,K)
      C(J,K)  = SNA*HOLDIK - CSA*C(J,K)
 16   CONTINUE
C     COMPUTE NEW ELEMENTS OF A IN COLUMNS I AND J
      AIK(J)  = SNA*AIK(I) - CSA*AIK(J)
C     WHEN K IS LARGER THAN I
      DO 19 K = 1,J
      IF (K.LE.I) GO TO 18
      C(K,J)  = SNA*AIK(K) - CSA*C(K,J)
      GO TO 19
 18   HOLDKI  = C(K,I)
      C(K,I)  = CSA*HOLDKI + SNA*C(K,J)
      C(K,J)  = SNA*HOLDKI - CSA*C(K,J)
 19   CONTINUE
 20   C(I,J)  = 0.0
C     FIND SIGMA2 FOR TRANSFORMED C AND FOR CONVERGENCE
      SIGMA2  = 0.0
      DO 21 I = 1,NN
      EIGEN(I)= C(I,I)
 21   SIGMA2  = SIGMA2 + EIGEN(I)**2
      IF(1.0 - SIGMA1/SIGMA2.GE.EPS3) GO TO 25
      PRINT 204, ITER,SIGMA2,S,NN
      PRINT 201,(EIGEN(I),I= 1,NN)
      GO TO 84
```

```
 25    PRINT 202, ITER,SIGMA1,SIGMA2
 26    SIGMA1  = SIGMA2
C      IF ITER EXCEEDS ITMAX , NO CONVERGENCE
       PRINT 203,ITER,S,SIGMA1,SIGMA2
 84    RBT1     = RBT1 + 50
       IF ( RBT1 - 100 ) 40,40,85
 85    CONTINUE
       IF (MM) 90,90,13
 90    STOP
C      FORMAT STATEMENTS
 1     FORMAT ( '1' )
 100   FORMAT (I5,5X,5F10.4    )
 105   FORMAT (/,I5,5X,5F10.4 )
 110   FORMAT(3X,'MM',10X,'A0/B0',5X,'L/R',7X,'B0/R'
      1 ,5X,'DA/EL',5X,'DC/R',/)
 180   FORMAT (/,10X,' N     =   ',I2,5X,'T2/T1 = ',F10.2,/ )
 200   FORMAT(5X,' EIGEN VALUES BY JACCBI METHOD',/)
 201   FORMAT (5X,6E18.8)
 202   FORMAT(/,6X,'ITER = ',I3,6X,'SIGMA1 = ',E17.8,6X,'SIGMA2=',E17.8)
 203   FORMAT(/,4X,'NO CONVERGENCE , WITH ',/7X,'ITER = ',I3,5X,
      1         'S = ',E17.8,5X,'SIGMA1 = ',E17.8,5X,'SIGMA2 =
      2         ',E17.8,//)
 204   FORMAT(/,4X,'CONVERGENCE HAS OCCURED , WITH ',/
      1         6X,'ITER = ',I3,5X,'SIGMA2 = ',E17.8,5X,'S = ',
      2         E17.8,5X,'NN = ',I3,/,4X,'EIGENVALUES EIGEN(1 ).
      3         ..EIGEN(NN) ARE ',/)
 210   FORMAT (/,5X,'RBT1   = ',F6.0,5X,' TEMP = ',F6.2,/)
       END
```

```
C     SUBROUTINE BUCKLING
      SUBRCUTINE BUCK (C,RET1)
C     STABILITY OF CYLINDRICAL SHELL WITH RECTANGULAR CUT CU
C     T OR IMPERFECTION.
C.....ASPECT RATIO                                    A0/B0
C.....LENGTH TO RADIUS                                EL/R
C.....RADIUS TO THICKNESS OF CYLINDER                  R/T1
C.....CUTOUT LENGTH TO LENGTH OF CYLINDER             A0/EL
C.....CUTOUT BREADTH TO RADIUS OF CYLINDER            B0/R
C.....RATIO OF THICKNESSES                            T2/T1
C.....T2 IS THE THICKNESS OF IMPERFECTION
C.....T1 IS THE THICKNESS OF CYLINDER
C.....WHEN THERE IS A CUTOUT THICKNESS OF
C.....IMPERFECTION IS TO BE GIVEN AS ZERO
C.....POSITION OF OMPERFECTION IN X-AXIS              DA/EL
C.....POSITION OF OMPERFECTION IN Y-AXIS              DC/R
      IMPLICIT REAL * 8 (A-H, O-Z )
      DIMENSION C ( 3 , 3 )
C     COMMON BLOCKS
      COMMCN/BLOCK1/B1TAU,STAUB1,SEPSIY,SCSIY,SOHMB1,SPHIY,
     1SSIY,SPHIOY,SSIOY,SCSIOY,STHBB0,B1TAU1,STAUB0,DCCHM,SESIOY
      COMMON/BLK2/XI,XII,XIJ,XJ,XJJ,XIIII
      COMMCN/BLK3/EPSA0,EPSA1
      COMMON/BLK4/RBALP,ALPBEL,A3
      COMMCN/BLK7/C1130,C2570,C3290,C1231,C2481,C3621,C123S,C248S,C362S,
     2           C113B,C215B,C329B,C2610,C3420,C2091,C3981,C209S,C348S,
     3           C206B,C332B,C3530,C3281,C358S,C334B,DELTA
      COMMCN/BLOCK5/Q1,B0BEET,ELBR,A0BB0,B0BR,DABEL,DCBR,T2BT1,
     1ALPBET,CLBR,X0,Y0,X1,Y1,EPS,EPSB,TEMP,A0BEL,
     2FCP1,FOM2,TOP1,TOP2,SOPRM2,Q0M1,TOPRP2,SA0,SOM2,GOP1,
     1QM1,TPRP2,SM2,TM1,TM2,GP1,FM1,BETBT1,ALPBT1,GOM1,
     3TOM1,TOM2,BETBR,FOM1,FOP2,GSM2,GTP2,GOSM2,
     4 GOTP2,GM1,FP1,FM2,FP2,TP1,TP2,SPRM2,DABBET,SPRP2
      COMMON/BLK6/PI,B1BR,DCTAU,N
      SA1      =0.0
      POR      = 0.3
      M        = 2
      PI       = 3.141592654
      PHIX     =PI*(M*A3+1. )
      PSIX     =PI*(M*A3-1. )
      SPHIX    =DSIN(PHIX)/PHIX
      SPSIX    =DSIN(PSIX)/PSIX
      IF(M*A3.EQ.1. ) SPSIX = 1.
      DABALP   =(DABEL)/(ALPBEL)
      A5       =( M*PI*A3)
      SPIA1    =DSIN(A5)/A5
      A7       =( M*PI*DABEL)
      CDPIDA   =DCOS(A7)
      CPHIDA   =DCOS(A7+DABALP)
      CPSIDA   =DCOS(A7-DABALP)
      CDALP    =DCOS(DABALP)
      SPHIDA   =DSIN(A7+DABALP)
```

```
      SPSIDA  =DSIN(A7-DABALP)
      GP1     =1.  + CDPIDA*SPIA1
      GM1     =1.  - CDPIDA*SPIA1
      B1BR    =(B0BR)+(CLBR)
C
      CALL SUMS
      ELBT1   = ELBR * RBT1
      DABBET  = DABEL*ELBR*B0BBET/B0BR
      ALPBT1  = ALPBEL * ELBT1
      BETBT1  = B0BR * RBT1/B0BBET
      BETBR   = B0BR / B0BBET
      DCBBET  = DCBR/BETBR
      DCTAU1  = DCBBET
      PHIDC   = DCTAU + DCTAU1
      SIDC    = DCTAU - DCTAU1
      EPSIDC  = DCOHM + DCTAU1
      CPSIDC  = DCOHM - DCTAU1
      CDPHDC  =DCOS(PHIDC)
      CCSICC  =DCOS(CPSIDC)
      CESIDC  =DCOS(EPSIDC)
      SDCSDC  =DSIN(CPSIDC)
      SDSIDC  =DSIN(SIDC)
      SDPHDC  =DSIN(PHIDC)
      SDPSDC  =DSIN(EPSIDC)
      CDT1DC  =DCOS(DCTAU1)
      CDTADC  =DCOS(DCTAU)
      CDOHDC  =DCOS(DCOHM)
      SDT1DC  =DSIN(DCTAU1)
      CDSIDC  =DCOS(SIDC)
      TP1     =(CDTADC*STAUB1+CDOHDC*SOHMB1)
      TM1     =(CDTADC*STAUB1-CDOHDC*SOHMB1)
      STP2    =(CDSIDC*SSIY  +CDPHDC*SPHIY )
      STP3    =(CCSIDC*SCSIY +CESIDC*SEPSIY)
      STM2    =(CDPHDC*SPHIY -CDSIDC*SSIY  )
      STM3    =(CCSIDC*SCSIY -CESIDC*SEPSIY)
      CPIDA   =DCOS(A7)/A7
      SDALP   =DSIN(DABALP)
      CDPIA1  =DCOS(A3)
      SSP2    =(SDPHDC*SPHIY +SDSIDC*SSIY  )
      SSP3    =(SDCSDC*SCSIY +SDPSDC*SEPSIY)
      SSM2    =(SDPHDC*SPHIY -SDSIDC*SSIY  )
      SSM3    =(SDCSDC*SCSIY -SDPSDC*SEPSIY)
      FP1     =(CPHIDA*SPHIX +CPSIDA*SPSIX )*CDALP
      FM1     =(CPHIDA*SPHIX -CPSIDA*SPSIX )*CDALP
      FP2     =(SPHIDA*SPHIX +SPSIDA*SPSIX )*SDALP
      FM2     =(SPHIDA*SPHIX -SPSIDA*SPSIX )*SDALP
      TP2     =CDT1DC*(STP2  +STP3 )
      TM2     =CDT1DC*(STP2  -STP3 )
      TPRP2   =CDT1DC*(STM2  +STM3 )
      TPRM2   =CDT1DC*(STM2  -STM3 )
      SP2     =SDT1DC*(SSP2  +SSP3 )
      SM2     =SDT1DC*(SSP2  -SSP3 )
```

```
      SPRP2   =SDT1DC*(SSM2  +SSM3 )
      SPRM2   =SDT1DC*(SSM2  -SSM3 )
      QP1     =CPIDA *(SPIA1 +CDPIA1)
      QM1     =CPIDA *(SPIA1 -CDPIA1)
      GSP2    =CDT1DC*(SSM2  +SSM3   )
      GSM2    =CDT1DC*(SSM2  -SSM3   )
      GTP2    =SDT1DC*(STP2  +STP3   )
      GTM2    =SDT1DC*(STP2  -STP3   )
      AOBALP  =(AOBEL)/(A3/PI)
      SAO     =DSIN(AOBALP)/(AOBALP)
      SIOX    =(M*PI*(AOBEL)-(AOBALP))
      PHIOX   =(M*PI*(AOBEL)+(AOBALP))
      SPIAO   =DSIN(M*PI*(AOBEL))/(M*PI*(AOBEL))
      SPHIOX  =DSIN(PHIOX)/PHIOX
      SSIOX   =DSIN(SIOX)/SIOX
      STOP2   =(CDSIDC*SSIOY +CDPHDC*SPHIOY)
      STOP3   =(CPSIDC *SCSIOY+CESIDC*SESIOY)
      STOM2   =(CDSIDC*SSIOY -CDPHDC*SPHIOY)
      STOM3   =(CPSIDC *SCSIOY-CESIDC*SESIOY)
      SSOP2   =(SDPHDC*SPHIOY+SDSIDC*SSIOY )
      SSOM2   =(SDPHDC*SPHIOY-SDSIDC*SSIOY )
      SSOP3   =(SDCSDC*SCSIOY+SDPSDC*SESIOY)
      SSOM3   =(SDCSDC*SCSIOY-SDPSDC*SESIOY)
      TOP2    =CDT1DC *(STOP2+STOP3)
      TOM2    =CDT1DC *(STOP2-STOP3)
      TOPRP2  =CDT1DC *(STOM2+STOM3)
      TOPRM2  =CDT1DC *(STOM2-STOM3)
      SOP2    =SDT1DC *(SSOP2+SSOP3)
      SOM2    =SDT1DC *(SSOP2-SSOP3)
      SOPRP2  =SDT1DC *(SSOM2+SSOM3)
      SOPRM2  =SDT1DC *(SSOM2-SSOM3)
      CDPIAO  =DCOS(M *PI*AOBEL)
      SPIAO   =DSIN(M *PI*AOBEL)/(M*PI*AOBEL)
      QOP1    =CPIDA  *(SPIAO+CDPIAO)
      QOM1    =CPIDA  *(SPIAO-CDPIAO)
      GOSP2   =CDT1DC *(SSOM2+SSOM3)
      GOSM2   =CDT1DC *(SSOM2-SSOM3)
      GOTP2   =SDT1DC *(STOP2+STOP3)
      GOTM2   =SDT1DC *(STOP2-STOP3)
      GOP1    =1. + CDPIDA * SPIAO
      GOM1    =1. - CDPIDA * SPIAO
      TOP1    =(CDTADC*STAUBO+CDOHDC*STHBBO)
      TOM1    =(CDTADC*STAUBO-CDOHDC*STHBBO)
      FOP1    =(CPHIDA*SPHIOX+CPSIDA*SSIOX )*CDALP
      FOM1    =(CPHIDA*SPHIOX-CPSIDA*SSIOX )*CDALP
      FOP2    =(SPHIDA*SPHIOX+SPSIDA*SSIOX )*SDALP
      FOM2    =(SPHIDA*SPHIOX-SPSIDA*SSIOX )*SDALP
      A9      =M*PI/2.
C
C
      C1130   = (A9*A9/(ELBT1*ELBT1))*(1.+3.*YO+PCR*XC)+(XII
     1         /(RBT1*RBT1))*(.5*(1.-POR)*(1.+(.021/(RBT1*RBT1
```

```
2          ))) +X0+Y0)
 C2570    = (A9*XI/(ELBT1*RBT1))*((1.+POR)*(1.+Y0)-((1.-
1          POR)/(16.*RBT1*RBT1)))
 C3290    =(A9/(ELBT1*RBT1))*(2.*POR*(1.+Y0)-(XII*(1.-POR)
1          /(12.*RBT1*RBT1)))
 C12311   =(A9*A9/(ELBT1*ELBT1))*(1.5*Y1*GM1+(0.25*POR*X1)
1          *(SA1+GM1-0.5*(FP1+FM2)))*(TP1+0.5*(TP2+SPRM2))
 C12312   =-(A9/(ELBT1*RBT1))*Y1*.5*DABBET*XI*QM1*(TPRP2+SM2)
 C12313   =(2.*XIJ/(RBT1*RBT1))*(TM1+0.5*(TM2+SPRP2))*(0.25
1         *X1*(SA1+GP1+0.5*(FP1+FM2))+0.5*Y1*GP1)
 C1231    = C12311 + C12312 +C12313
 C24811   =(A9*A9/(ELBT1*ELBT1))*.25*Y1*DABBET*QM1*(1.-POR)
1          *(TPRP2+SM2)
 C24822   =-(A9/(ELBT1*RBT1))*.5*(1.-POR)*XJ*(TM1+0.5*(TM2
1          +SPRP2))*GP1
 C24823   =-(A9/(ELBT1*RBT1))*POR*XI*Y1*GM1*(TP1+.5*(TP2+
1          SPRM2))
 C24824   =(Y1*.5*DABBET*XIJ*QM1/(RBT1*RBT1))*(TPRP2+SM2)
 C2481    =C24811 + C24822 + C24823 + C24824
 C36211   =-X1*.25*RBALP*(A9*A9/(ELBT1*ELBT1))*(FM1+FP2)*
1          (TP1+.5*(TP2+SPRM2))
 C36212   =-(X1*POR*.5/BETBT1)*(SA1+GM1-.5*(FP1+FM2))*XI*
1          .5*(TPRP2+SM2)
 C36213   =-(2.*POR*Y1/RBT1)*(TP1+.5*(TP2+SPRM2))*GM1
 C36214   =(X1*.25*(1.-POR)/BETBT1)*(SA1+GP1+.5*(FP1+FM2))
1          *XI*(TPRP2+SM2)
 C36215   =((1.-POR)*.125*X1/(ALPBT1*RBT1))*XIJ*(FM1+FP2)*
1          (TM1+.5*(TM2+SPRP2))
 C36216   =(Y1*XI*DABBET*QM1*(TPRP2+SM2)/(RBT1*RBT1))
 C3621    =C36211+.5*A9*(C36212+C36213+C36214)+C36215+C36216
 C123S1   =(1.+3.*Y0+POR*X0)*GOM1*TOP1
 C123S2   =(1.5*Y1*GOM1+.25*POR*X1*(SA0+GOM1-.5*(FOP1+FOM2
1          )))*(TOP1+.5*(TOP2+SOPRM2))
 C123S3   =-(A9/(ELBT1*RBT1))*.5*Y1*DABBET*XI*QOM1*(TOPRP2
1          +SOM2)
 C123S4   =((.5*(1.-POR)+X0+Y0)*GOP1*TOM1+(TOM1+.5*(TOM2+
1          SOPRP2))*(.25*X1*(SA0+GOP1+.5*(FOP1+FOM2))+.5*
2          Y1*GOP1))*(2.*XIJ/(RBT1*RBT1))
 C123S    =(A9*A9/(ELBT1*ELBT1))*(C123S1+C123S2)+C123S3+C123S4
 C248S1   =(A9*A9/(ELBT1*ELBT1))*.25*Y1*DABBET*QOM1*(TOPRP2+
1          SOM2)*(1. - POR )
 C248S2   =-.5*XJ*(1.-POR)*(2.*(1.+Y0)*GOP1*TOM1+Y1*GOP1*
1          (TOM1+.5*(TOM2+SOPRP2)))
 C248S3   =-XI*POR*(2.*(1.+Y0)*GOM1*TOP1+GOM1*Y1*(TOP1+.5*
1          (TOP2+SOPRM2)))
 C248S4   =(Y1*.5*DABBET/(RBT1*RBT1))*XIJ*QOM1*(TOPRP2+SOM2)
 C248S    =C248S1+(A9/(ELBT1*RBT1))*(C248S2+C248S3)+C248S4
 C362S1   =-(A9*A9/(ELBT1*ELBT1))*X1*.25*RBALP*(FOM1+FOP2)*
1          (TOP1+.5*(TOP2+SOPRM2))
 C362S2   =-(X1*POR*.5/BETBT1)*(SA0+GOM1-.5 *(FOP1+FOM2))*
1          XI*(TOPRP2+SOM2)*.5
 C362S3   =-(2.*POR/RBT1)*(2.*(1.+Y0)*TOP1+Y1*(TOP1+.5*
```

```
      1         (TOP2+SOPRM2)))*GOM1
       C362S4  =(X1*.25*(1.-POR)/BETBT1)*(SA0+GOP1+.5*(FOP1+FOM2
      1         ))*XI*(TOPRP2+SOM2)
       C362S5  =(.125*(1.-POR)/(ALPBT1*RBT1))*X1*(FOM1+FOP2)*
      1         XIJ*(TOM1+.5*(TOM2+SOPRP2))
       C362S6  =(DABBET/(RBT1*RBT1))*Y1*XI*QOM1*(TOPRP2+SOM2)
       C362S   =C362S1+(A9*.5/ELBT1)*(C362S2+C362S3+C362S4)
      1         +C362S5+C362S6
       C113B   =((1.-POR)/(96.*RBT1**4))*GOP1*TOM1*XIJ
       C215B   =((1.-POR)/(16.*RBT1**3*ELBT1))*A9*GOP1*TOM1*XJ
       C329B   =((1.-POR)/(12.*RBT1**3*ELBT1))*A9*GOP1*TOM1*XIJ
       C26101  =(A9*A9/(ELBT1*ELBT1))*(.5*(1.-POR)*(1.+(.1875
      1         /(RBT1*RBT1)))+Y0+POR*X0)
       C26102  =(XII/(RBT1*RBT1))*(1.+(.0833/(RBT1*RBT1))+POR
      1         *Y0+X0)
       C2610   = C26101 + C26102
       C34201  =(2.*XI/(RBT1*RBT1))+((A9*A9/(ELBT1*ELBT1))*XI*
      1         (1.-POR)/(144.*RBT1*RBT1))
       C34202  =(XI*.1667/RBT1**4)*((A9*A9/(ELBR*ELBR))+XII)
       C3420   = C34201 + C34202
       C20911  =(XIJ/(RBT1*RBT1))*(.5*POR*Y1*GM1+.25*X1*(SA1+
      1         GM1-.5*(FP1+FM2)))*(TP1+.5*(TP2+SPRM2))
       C20912  =(A9*A9/(ELBT1*ELBT1))*(TM1+.5*(TM2+SPRP2))*(.5*
      1         Y1*GP1+.25*POR*X1*(SA1+GP1+.5*(FP1+FM2)))
       C20913  =-(1.-POR)*.25*A9*DABBET*Y1*XJ*QM1*((TPRP2+SM2)/
      1         (RBT1*ELBT1))
       C2091   = C20911 + C20912 + C20913
       C39811  =-POR*.5*(FM1+FP2)*XJ*(TP1+.5*(TP2+SPRM2))
       C39812  =.25*(1.-POR)*(FM1+FP2)*XI*(TM1+.5*(TM2+SPRP2))
       C39813  =.125*X1*(SA1+GM1-.5*(FP1+FM2))*XIJ*(TPRP2+SM2)
      1         /(BETBT1*RBT1)
       C39814  =-(.125*X1/BETBR)*(A9*A9/(ELBT1*ELBT1))*(SA1+GP1
      1         +.5*(FP1+FM2))*(TPRP2+SM2)
       C3981   =(.5*A9*X1/(ALPBT1*ELBT1))*(C39811+C39812)
      1         +C39813 + C39814*( 1. - POR )
       C209S1  =(XIJ*GOM1*TOP1*(1.+POR*Y0+X0)/(RBT1*RBT1))
       C209S2  =(XIJ/(RBT1*RBT1))*(TOP1+.5*(TOP2+SOPRM2))*(.5*
      1         POR*Y1*GOM1+.25*X1*(SA0+GOM1-.5*(FOP1+FOM2)))
       C209S3  =(A9*A9/(ELBT1*ELBT1))*GOP1*TOM1*(.5*(1.-POR)+
      1         Y0+POR*X0)
       C209S4  =(A9*A9/(ELBT1*ELBT1))*(TOM1+.5*(TOM2+SOPRP2))
      1         *(.5*Y1*GOM1+.25*POR*X1*(SA0+GOP1+.5*(FOP1+FOM2)))
       C209S5  =-(A9/(ELBT1*RBT1))*.25*Y1*DABBET*XJ*(1.-POR)*
      1         QOM1*(TOPRP2+SOM2)
       C209S   =C209S1+C209S2+C209S3+C209S4+C209S5
       C348S1  =(XI*GOM1*TOP1/(RBT1*RBT1))
       C348S2  =(.5*X1*A9/(ELBT1*ALPBT1))*(-.5*POR*(FOM1+FOP2)*
      1         XJ*(TOP1+.5*(TOP2+SOPRM2))+.25*(1.-POR)*(FOM1+
      2         FOP2)*XI*(TOM1+.5*(TOM2+SOPRP2)))
       C348S3  =(.125*X1/(BETBT1*RBT1))*XIJ*(SA0+GOM1-.5*(FOP1+
      1         FOM2))*(TOPRP2+SOM2)
       C348S4  =-.125*X1*(A9*A9/(ELBT1*ELBT1*BETBR))*(SA0+GOP1+
```

```
      1        .5*(FOP1+FOM2))*(TOPRP2+SOM2)
       C348S   = C348S1+C348S2+C348S3+C348S4
       C206B   =(.0833*XIJ*GOM1*TOP1/(RBT1**4))+1.5*(1.-POR)*
      1         GOP1*TOM1*(A9*A9/(ELBT1*ELBT1*16.*RBT1*RBT1))
       C332B1  =(A9*A9/(ELBT1*ELBT1*RBT1*RBT1))*(1.-POR)*(-.
      1         1667*GOM1*XI*TOP1+.25*GOP1*XJ*TOM1)
       C332B2  =(.1667*GOM1*XJ*TOP1/(RBT1**4))*((A9*A9/(ELBR*EL
      1         BR))+XII)
       C332B   = C332B1 + C332B2
       C35301  =(1./(RBT1*RBT1))+(A9*A9/(ELBT1*ELBT1))*(Y0+POR*X0)
       C35302  =(XII/(RBT1*RBT1))*(POR*Y0+X0)
       C35303  =(((A9**4/(ELBR**4))+XIIII+2*XII*(A9**2/(ELBR**2
      1         )))/(12.*RBT1**4))
       C3530   = C35301 + C35302  + C35303
       C32811  =-(POR*.5*X1/ALPBT1)*(FM1+FP2)*(TP1+.5*(TP2+SPR
      1         M2))
       C32812  =(.5*(1.-POR)*Y1*DABBET*XI*QM1/RBT1)*(GSM2-GTP2)
       C32813  =(A9*A9/ELBT1*ELBT1)*(TP1+.5*(TP2+SPRM2))*(.5*Y1*
      1         GP1+.25*POR*X1*(SA1+GP1+.5*(FP1+FM2)))
       C32814  =(X1*XI*.25/(RBT1*BETBT1))*(SA1+GM1-.5*(FP1+FM2
      1         ))*(TPRP2+SM2)
       C32815  =(XIJ/(RBT1*RBT1))*(TM1+.5*(TM2+SPRP2))*(.5*POR*
      1         Y1*GM1+.25*X1*(SA1+GM1-.5*(FP1+FM2)))
       C3281   =(.5*A9/ELBT1)*(C32811+C32812)+C32813+C32814
      1         +C32815
       C358S1  =(GOM1*TOP1/(RBT1*RBT1))+(.5*A9/ELBT1)*((-.5*POR
      1         *X1/ALPBT1)*(FOM1+FOP2)*(TOP1+.5*(TOP2+SOPRM2))
      2         +.5*(1.-POR)*Y1*(DABBET/RBT1)*QOM1*XI*(GOSM2
      3         -GOTP2))
       C358S2  =(Y0+POR*X0)*GOP1*TOP1+(.5*Y1*GOP1+.25*POR*X1*
      1         (SA0+GOP1+.5*(FOP1+FOM2)))*(TOP1+.5*(TOP2+SOP
      2         RM2))
       C358S3  =(.25*X1/(BETBT1*RBT1))*(SA0+GOM1-.5*(FOP1+FOM2))
      1         *XI*(TOPRP2+SOM2)
       C358S   =C358S1+(A9*A9/(ELBT1*ELBT1))*C358S2+C358S3
       C334B1  =(GOM1/(12.*RBT1**4))*((A9*A9/(ELBR*ELBR))+XJJ)
      1         *((A9*A9/(ELBR*ELBR))+XII)*TOP1
       C334B2  =(1.-POR)*(A9*A9/(6.*ELBR**2*RBT1**4))*
      1         (GOP1*XIJ*TOM1-GOM1*XII*TOP1)
       C334B   = C334B1 + C334B2
       RETURN
       END
```

```
C
      SUBROUTINE GAUSJO (A,N,M,DETER )
C     GAUSS-JORDAN REDUCTION
      IMPLICIT REAL * 8 (A-H, O-Z )
      DIMENSION A(4,5)
      NPM     = N + M
      EPS     =10**(-10)
C     BEGIN ELIMINATION PROCEDURE
      DETER   =1.
      DO 9  K =1,N
C
C     UPDATE THE DETERMINANT VALUE
      DETER   =DETER * A (K,K )
C
C     CHECK FOR PIVOT ELEMENT TOO SMALL
      IF(DABS(A(K,K)).GT.EPS ) GO TO 5
C
C     NORMALIZE THE PIVOT ROW
 5    KP1     =K + 1
      DO 6 J  = KP1, NPM
 6    A(K,J)  =A(K,J)/A(K,K)
      A(K,K)  =1.
C
C     ELIMINATE K(TH) COLUMN ELEMENTS EXCEPT FOR PIVOT
      DO 9 I  =1,N
      IF ( I.EQ.K.OR.A(I,K).EQ.0.) GO TO 9
      DO 8 J  = KP1, NPM
 8    A(I,J)  =A(I,J) - A(I,K)*A(K,J)
      A(I,K)  =0.0
 9    CONTINUE
      RETURN
      END
```

```
      SUBROUTINE SUMS
C     SUBROUTINE SUMS
      IMPLICIT REAL * 8 (A-H, O-Z )
C     SUMS OF SQUARES ETC.
C
C     COMMON BLOCKS
      COMMON/BLOCK1/B1TAU,STAUB1,SEPSIY,SCSIY,SOHMB1,SPHIY,
     1SSIY,SPHIOY,SSIOY,SCSIOY,STHBB0,B1TAU1,STAUB0,DCOHM,SESIOY
      COMMON/BLK2/XI,XII,XIJ,XJ,XJJ,XIIII
      COMMON/BLK3/EPSA0,EPSA1
      COMMON/BLK4/RBALP,ALPBEL,A3
      COMMON/BLK7/C1130,C2570,C3290,C1231,C2481,C3621,C123S,C248S,C362S,
     2            C113B,C215B,C329B,C2610,C3420,C2091,C3981,C209S,C348S,
     3            C206B,C332B,C3530,C3281,C358S,C334B,DELTA
      COMMON/BLOCK5/Q1,BOBEET,ELBR,A0BB0,BOBR,DABEL,DCBR,T2BT1,
     1ALPBET,CLBR,X0,Y0,X1,Y1,EPS,EPSB,TEMP,A0BEL,
     2FOP1,FOM2,TOP1,TOP2,SOPRM2,QOM1,TOPRP2,SA0,SOM2,GOP1,
     1QM1,TPRP2,SM2,TM1,TM2,GP1,FM1,BETBT1,ALPBT1,GOM1,
     3TOM1,TOM2,BETBR,FOM1,FOP2,GSM2,GTP2,GOSM2,
     4 GOTP2,GM1,FP1,FM2,FP2,TP1,TP2,SPRM2,DABBET,SPRP2
      COMMON/BLK6/PI,B1BR,DCTAU,N
      SUMI    = 0.0
      SUMII   = 0.0
      SUMIJ   = 0.0
      SUMJ    = 0.0
      SUMJJ   = 0.0
      SMIIII  = 0.0
      DO 6 J  = 1,N
      SUMJ    = SUMJ  + J
    6 SUMJJ   = SUMJJ + J*J
      DO 4 I  = 1,N
      SUMI    = SUMI + I
      SUMII   = SUMII + I*I
      SMIIII  = SMIIII + I*I*I*I
      DO 5 J  = 1,N
    5 SUMIJ   = SUMIJ + I*J
    4 CONTINUE
      XI      = SUMI
      XII     = SUMII
      XIJ     = SUMIJ
      XJ      = SUMJ
      XJJ     = SUMJJ
      XIIII   = SMIIII
C
      SUM     = 0.0
      DO 50 I = 1,N
      DO 50 J = I,N
      IF(J.EQ.I) GO TO 10
      B1TAU   = (I - J ) * B1BR
      SUM     = SUM + DSIN(B1TAU)/B1TAU
   10 CONTINUE
   50 STAUB1  = 2. * SUM + N
```

```
C
      DO 15 I = 1,N
      DO 15 J = 1,N
      DCTAU   = ( I - J ) * DCBR
 15   CONTINUE
C
      B1TAU1  = PI
      SUMO    = 0.0
      SUMO1   = 0.0
      SUMO2   = 0.0
      DO 40 I = 1, N
      DO 40 J = 1, N
      B1OHM   = ( I + J ) * B1BR
      EPSIY   = B1OHM + B1TAU1
      CSIY    = B1OHM - B1TAU1
      SUMO    = SUMO + DSIN(B1OHM)/B1CHM
      SUMO1   = SUMO1 +DSIN(EPSIY)/EPSIY
 40   SUMO2   = SUMO2 +DSIN(CSIY)/CSIY
      SEPSIY  = SUMO1
      SCSIY   = SUMO2
      SOHMB1  = SUMO
      SUM1    = 0.0
      SUM2    = 0.0
      DO 70 I = 1, N
      DO 70 J = 1, N
      B1TAU   = (I - J ) * B1BR
      PHIY    = B1TAU + B1TAU1
      SIY     = B1TAU - B1TAU1
      SUM1    = SUM1 +DSIN(PHIY)/PHIY
 70   SUM2    = SUM2 +DSIN(SIY)/SIY
      SPHIY   = SUM1
      SSIY    = SUM2
      SUMT1   = 0.0
      SUMT2   = 0.0
      DO 65 I = 1,N
      DO 65 J = 1,N
      TB      = ( I - J ) * BOBR
      PHIOY   = TB + BOBBET
      SIOY    = TB - BOBBET
      SUMT1   = SUMT1 +DSIN(PHIOY)/PHIOY
 65   SUMT2   = SUMT2 +DSIN(SIOY)/SIOY
      SPHIOY  = SUMT1
      SSIOY   = SUMT2
      SUMTH1  = 0.0
      SUMTH2  = 0.0
      SUMTH3  = 0.0
      DO 62 I = 1,N
      DO 62 J = 1,N
      THB     = ( I + J ) * BOBR
      SUMTH3  = SUMTH3 + DSIN(THB)/THB
      ESIOY   = THB + BOBBET
      CSIOY   = THB - BOBBET
```

```
      SUMTH1  = SUMTH1 +DSIN(ESIOY)/ESIOY
   62 SUMTH2  = SUMTH2 +DSIN(CSIOY)/CSIOY
      SESIOY  = SUMTH1
      SCSIOY  = SUMTH2
      STHBBO  = SUMTH3
      SUMT    = 0.0
      DO 60 I = 1,N
      DO 60 J = I,N
      IF ( J.EQ.I ) GO TO 20
      TB      = ( I - J ) * BOBR
      SUMT    = SUMT + DSIN(TB)/TB
   20 CONTINUE
   60 STAUBO  = 2. * SUMT +N
C
C
C
      DO 42 I = 1,N
      DO 42 J = 1,N
   42 DCOHM   = ( I + J ) * DCBR
C
      RETURN
      END
```

APPENDIX B

Intermediate Calculations for Prebuckling and Buckling Analysis.

Following are the intermediate calculations for prebuckling analysis.

Consider Equation 3.10

Letting $P_1(\varepsilon) = I_1(\varepsilon) + I_2(\varepsilon) + I_3(\varepsilon)$

From Appendix C-1

$$2I_1(\varepsilon)/C_2 = B_0^2 a_0 b_0 (1+\eta(1+S_b)+\eta^2/8(3+4S_b+S_{2b}))+$$

$$A_0^2 a_0 b_0/R^2(1+\xi/2(1+S_a)(1+S_b) + \xi^2/64($$

$$3+4S_a+S_{2a}))(3+4S_b+S_{2b})-2\nu A_0 B_0/R(1+$$

$$\eta/2(1+S_b)+\xi/4(1+S_a)(1+S_b) + \eta\xi/16($$

$$1+S_a)(3+4S_b+S_{2b})+B_0^2(1-\nu)/12\eta^2 a_0 b_0($$

$$a_0/\beta)^2(1-S_{2b}) - 2\nu A_0 B_0/R(1+\eta/2(1+S_b)+$$

$$\xi/4(1+S_a)(1+S_b)+\eta\xi/16(1+S_b)(3+4S_b+S_{2b})$$

$$B_0^2(1-\nu)/12\eta^2 a_0 b_0 (a_0/\beta)^2(1-S_{2b}) \qquad \text{(B-1)}$$

$$2I_2(\varepsilon)/C_1 = B_0^2 a_0 b_0 (1+r)[s+\eta(s-S_b)+\eta^2/8(3s-4S_b-S_{2b})]$$

$$+ A_0^2 a_0 b_0/R(1+r)[s+\xi/2(s-S_b)+\xi^2(9s/64-$$

$$3S_b/16 -3S_{2b}/64)]- 2\nu A_0 B_0 a_0 b_0/R(1+r)[$$

$$s+\eta/2(s-S_b)+\xi/4(s-S_b)+\eta\xi/16(3s-4S_b-S_{2b})]$$

$$+B_0^2(1-\nu)\eta^2 a_0 b_0/12(1+r)^3(a_0/\beta)^2(s+S_{2b}) \quad (B-2)$$

$$2I_3(\epsilon)/C_1 a_0 b_0 = B_0 r(1+\eta(1+S_b)+\eta^2/8(3+4S_b+S_{2b}))+A_0^2/R\ [$$

$$r+\xi/2(r-S_a)(1+S_b)+\xi^2/64(3r-4S_a-S_{2a})($$

$$3+4S_b+S_{2b})]-2\nu A_0 B_0/R[r+\eta r/2(1+S_b)+$$

$$\xi/4(r-S_a)(1+S_b)+\eta\xi/16(r-S_a)(3+4S_b+S_{2b})]+$$

$$B_0^2(1-\nu)\eta^2/12(a_0/\beta)^2\ ((r+1)^3-1)(1-S_{2b}) \quad (B-3)$$

Finally

$$2(I_2(\epsilon) + I_3(\epsilon))/C_1 a_0 b_0 =$$

$$B_0^2((1+r)[s+\eta(s-S_b)+\eta^2/8(3s-4S_b-S_{2b})]+r[1+\eta(1+S_b) +$$

$$\eta^2/8(3+4S_b+S_{2b})])+(A_0/R)^2((1+r)[s+\xi/2(s-S_b)+3\xi^2/64($$

$3s-4S_b-S_{2b})]+r+\xi/2(r-S_a)(1+S_b)+\xi^2/64(3r-4S_a-S_{2a})($

$3+4S_b+S_{2b}))-2\nu A_0B_0/R((1+r)[s+\eta/2(s-S_b)+\xi/4(s-S_b)+$

$\eta\xi/16(3s-4S_b-S_{2b})]+r+\eta r/2(1+S_b)+\xi/(r-S_a)(1+S_b)+$

$\eta\xi/16(r-S_a)(3+4S_b+S_{2b}))+B_0^2(1-\nu)\eta^2/12(a_0/\beta)^2[(1+$

$r)^3(1+s)-(1-S_{2b})]$ (B-4)

The potential energy for stretching is given by

$$P_1(\varepsilon) = I_1(\varepsilon) + I_2(\varepsilon) + I_3(\varepsilon)$$

hence

$2P_1(\varepsilon)/C_1a_0b_0 = B_0^2(1+s)(1+r)(1+\eta+3/8\eta^2) + (A_0/R)^2($

$1+s)(1+r)(1+\xi/2+9/64\xi^2) - 2\nu A_0B_0/R($

$1+s)(1+r)(1+\eta/2+\xi/4+3\eta\xi/16)+B_0^2/12($

$1-\nu)\eta^2(a_0/\beta)^2(1+s)(1+r)^3+\varepsilon_s[B_0^2(1+$

$\eta(1+S_b)+\eta^2/8(3+4S_a+S_{2a}))+(A_0/R)^2($

$1+\xi/2(1+S_a)(1+S_b)+\xi^2/64(3+4S_a+S_{2a})($

$3+4S_b+S_{2b}))-2\nu A_0B_0/R(1+\eta/2(1+S_b)+$

$+\xi/4(1+S_a)(1+S_b)+\eta\xi/16(1+S_a)(3+$

$4S_b+S_{2b}))+B_0^2/12(1-\nu)\eta^2(a_0/\beta)^2(1-S_{2b})]$ (B-5)

for bending

Letting $P_1(x) = J_1(x) + J_2(x) + J_3(x)$

$2J_1(x)/D_1a_0b_0 =$

$(1+\epsilon)[(A_1^2/64\alpha^4)(1+S_{2a})(3+4S_b+S_{2b})+(A_1^2/64\beta^4)(1+S_{2b})($

$3+4S_a+S_{2a})+(2\nu A_1^2/64\alpha^2\beta^2)(1+2S_a+S_{2a})(1+2S_b+S_{2b})+$

$(2(1-\nu)A_1^2/64\alpha^2\beta^2)(1-S_{2b})(1-S_{2a}) + (2(1-\nu)B_1^2/384\beta^2)($

$1-S_{2b})(a_0/R)^2+((1-\nu)A_1B_1/16\beta^2R)(1-S_{2b})(S_a-C_a)]$ (B-6)

$(2(J_2(x) + J_3(x)))/D_1a_0b_0 =$

$(A_1^2/64\alpha^4)[3(1+r)(1+s)-(3+4S_b+S_{2b})(1+S_{2a})] +$

$(A_1^2/64\beta^4)[3(1+r)(1+s)-(3+4S_a+S_{2a})(1+S_{2b})] +$

$(2\nu A_1^2/64\alpha^2\beta^2)[\ (1+r)(1+s)-(1+2S_a+S_{2a})(1+2S_b+S_{2b})]+$

$(2(1-\nu)A_1^2/64\alpha^2\beta^2)[(1+r)(1+s)-(1-S_{2a})(1-S_{2b})]\ +$

$(2(1-\nu)B_1^2/384\beta^2\ )[(a_1/R)^2((1+r)(s+S_{2b})+r(1-S_{2b})($

$a_0/R)^2(r^2+(1+r)^3)\ ]\ \ +\ (4(1-\nu)A_1B_1/64\beta^2)\ [(1$

$/R)(1+r)(s+S_{2b})+(1-S_{2b})(1+r+C_a-S_a)\ \ ]$ (B-7)

hence Equation (3.11c)

Define the following terms

$\varepsilon_s=(C_2/C_1-1)\quad \varepsilon_b=(D_2/D_1-1)\qquad D_1/C_1=T_1^2/12$

$\varepsilon_{a_1}=a_1b_1/\pi RL = a_0b_0(1+r)(1+s)\ \pi RL \qquad \varepsilon_{a_0}=a_0b_0/\pi RL$

$\varepsilon_s\ =(T_2/T_1-1)\quad \varepsilon_b = (T_2/T_1)^3-1$

$\varepsilon_b = \ (3(1+\varepsilon_s)+\varepsilon_s^2) \qquad D_2/C_1 = (1+\varepsilon_b)T_1^2/12$

$a_1 = \alpha\pi \qquad\qquad b_1 = \beta\pi$

$m = RT_1/\alpha^2 \qquad n = RT_1/\beta^2$

For Buckling analysis.

The elements of buckling determinant are multiplied by T_1^2 to make them dimensionless and is as follows.

$$C^0_{11} = (j\pi/2L)^2(1+3B_0+\nu A_0/R)+\sum_1^n (i/R)^2((1-\nu)/2($$

$$1+(1/48R^2/T_1^2))+A_0/R+B_0)$$

$$C^0_{12} = (j\pi/2L)\sum_1^n (i/R)[(1+\nu)(1+B_0)-(1-\nu)/(16(R/T_1)^2)]$$

$$C^0_{13} = (j\pi/2RL)[2\nu(1+B_0)-\sum_1^n i^2(1-\nu)/(12(R/T_1)^2)]$$

$$C^1_{11} = (j\pi/2L)^2[(3B_1/2*g_{-1}+\nu A\ /4R*(S_{a_1}+g_{-1}-(f_{+1}+f_{-2})/2))($$

$$t_{+1}+(T_{+2}+S'_{-2})/2)] - (j\pi/2L)B_1d_a/2\beta R*(\sum_1^n iq_{-1}($$

$$T'_{+2}+S_{-2})) + \sum_1^n\sum_1^n 2ik/R^2*(t_{-1}+(T_{-2}+S'_{+2}))(A_1/4R*($$

$$S_{a_1}+g_{+1}+(f_{+1}+f_{-2})/2) +B_1/2*g_{+1})$$

$$C^1_{12} = (j\pi/2L)(B_1d_a/4\beta)q_{-1}(1-\nu)(T'_{+2}+S_{-2})-(j\pi/2L)($$

$$B_1/2R)(1-\nu)\sum_1^n k(t_{-1}+(T_{-2}+S'_{+2})/2)g_{+1}-(j\pi/2L)($$

$$(\sum_1^n iB_1/R)g_{-1}(t_{+1}+(T_{+2}+S'_{-2}\ )/2) +(B_1d_a\sum_1^n ki/2R^2\beta)*$$

$$q_{-1}(T'_{+2} + S_{-2})$$

$$C^{1}_{13} = -(j\Pi/2L)^2 (A_1/4\alpha)(f_{-1}+f_{+2})(t_{+1}+(T_{+2}+S'_{-2}))+$$

$$[A_1\nu/2\beta R)(S_{a_1}+g_{-1}-(f_{+1}+f_{-2})/2)*\sum_1^n i(T'_{+2} + S_{-2})/2-$$

$$2\nu B_1/R(t_{+1}+(T_{+2}+S'_{-2})/2)g_{-1}+(A_1(1-\nu)/4\beta R)($$

$$S_{a_1}+g_{+1}+(f_{+1}+f_{-2})/2)\sum_1^n i(T'_{+2} + S_{-2})](j\Pi/4L) +$$

$$(A_1(1-\nu)/8\alpha R^2)(f_{-1}+f_{+2})\sum_1^n\sum_1^n ik(t_{-1}+(T_{-2}+S'_{+2})/2)+$$

$$(B_1 d_a/\beta R)\sum_1^n iq_{-1}(T'_{+2} + S_{-2})$$

$$C^{S}_{11} = (j\Pi/2L)^2[(1+3B_0+\nu A_0/R)g^0_{-1}t^0_{+1} + (3B_1/2(g^0_{-1}) +$$

$$(\nu A_1/4R)(S_{a_0}+g^0_{-1}-(f^0_{+1}+f^0_{-2})/2))*(t^0_{+1}+(T^0_{+2}+S'^0_{-2})/2)]$$

$$-(j\Pi/2L)(B_1 d_a/\beta R)\sum_1^n iq^0_{-1}(T'^0_{+2} + S^0_{-2})+(((1-\nu)/2+$$

$$A_0/R+B_0)g^0_{+1}t^0_{-1}+(t_{-1}+(T^0_{-2}+S'^0_{+2})/2)*(A_1/4R(S_{a_0}+$$

$$g^0_{+1}+(f^0_{+1}+f^0_{-2})/2)+B_1 g^0_{+1}/2))2\sum_1^n\sum_1^n ik/R^2$$

$$C^{S}_{12} = (j\Pi/2L)^2(B_1 d_a(1-\nu)/4\beta)q^0_{-1}(T'^0_{+2} + S^0_{-2})-(j\Pi/2L)($$

$$(1-\nu)/2R)\sum_{1}^{n} k(2(1+B_0)g^0_{+1}t^0_{+1}+B_1 g^0_{+1}(t^0_{-1}+(T^0_{-2}+S'^0_{+2})/2))-$$

$$(j\Pi/2L)(\nu\sum_{1}^{n} i/R)(2(1+B_0)g^0_{-1}t^0_{+1}+g^0_{-1}B_1(t^0_{+1}+(T^0_{+2}+S'^0_{-2})/2))$$

$$+(B_1 d_a/2R\beta)\sum_{1}^{n}\sum_{1}^{n} ki q^0_{-1}(T'^0_{+2}+S^0_{-2})$$

$$C^S_{13} = -(j\Pi/2L)\ (A_1/2\alpha)(f^0_{-1}+f^0_{+2})(t^0_{+1}+(T^0_{+2}+S'^0_{-2})/2)+[-$$

$$(A_1\nu/2R\beta)(S_{a_0}+g^0_{-1}-(f^0_{+1}+f^0_{-2})/2)\sum_{1}^{n} i(T'^0_{+2}+S^0_{-2})/2-$$

$$2\nu/R(2(1+B_0)t^0_{+1}+B_1(t^0_{+1}+(T^0_{+2}+S'^0_{-2})/2))g^0_{-1}+$$

$$(A_1(1-\nu)/4\beta R)(S_{a_0}+g^0_{+1}+(f^0_{+1}+f^0_{-2})/2)\sum_{1}^{n} i($$

$$T'^0_{+2}+S^0_{-2})](j\Pi/4L)+(A_1(1-\nu)/8\alpha R^2)(f^0_{-1}+f^0_{+2})\sum_{1}^{n}\sum_{1}^{n} ki($$

$$t^0_{-1}+(T^0_{-2}+S'^0_{+2})/2)+(d_a B_1/\beta R^2)\sum_{1}^{n} iq^0_{-1}(T'^0_{+2}+S^0_{-2})$$

$$C^B_{11} = ((1-\nu)/96)/(R/T_1)^4\ g^0_{+1}t^0_{-1}\sum_{1}^{n}\sum_{1}^{n} ki$$

$$C^B_{12} = ((1-\nu)/32R/(R/T_1)^2)(j\Pi/L)g^0_{+1}t^0_{-1}\sum_{1}^{n} k$$

$$C^B_{13} = ((1-\nu)/12R/(R/T_1)^2)\ (j\Pi/2L)\ g^0_{+1}t^0_{-1}\sum_{1}^{n}\sum_{1}^{n} ki$$

$$C^0_{22} = (j\Pi/2L)^2[((1-\nu)/2*(1+3/(16(R/T_1)^2))+B_0+\nu A_0/R)+$$

$$(\sum_1^n i/R)^2(1+1/(12(R/T_1)^2)+ B_0+A_0/R)$$

$$C_{23}^0 = 2\sum_1^n i/R^2 +(j\pi/2L)\ ((1-\nu)/(144(R/T_1)^2))\sum_1^n i +$$

$$(\sum_1^n i/6)/(R/T_1)^2((j\pi/2L)^2 +(\sum_1^n i/R\ )^2$$

$$C_{22}^1 = \sum_1^n \sum_1^n ik[\ B_1/2(g_{-1})+A_1/4R(S_{a_1}+g_{-1}-(f_{+1}+f_{-2})/2)]*$$

$$(t_{+1}+(T_{+2}+ S'_{-2}\ ))/R^2 +(j\pi/2L)^2(t_{-1}+$$

$$(T_{+2}+S'_{-2}\ )/2)(B_1g_{+1}/2 +(\nu A_1/4R)(S_{a_1}+$$

$$g_{+1}+(f_{+1}+f_{-2})/2))-(j\pi/2L)((1-\nu)d_aB_1/4R\beta)*$$

$$\sum_1^n kq_{-1}(T'_{+2}+S_{-2})$$

$$C_{23}^1 = (j\pi/4L)[-\nu A_1/2\alpha R\ *(f_{-1}+f_{+2})\ \sum_1^n k(t_{+1}+(T_{+2}+S'_{-2})/2)+$$

$$(A_1(1-\nu)/4\alpha R)(f_{-1}+f_{+2})\sum_1^n i(t_{-1}+(T_{-2}+S'_{+2}\ )/2)] +$$

$$(A_1/8_\beta R^2)(S_{a_1}+g_{-1}-(f_{+1}+f_{-2})/2)\sum_1^n\sum_1^n ki(T'_{+2}\ +S_{-2})-$$

$$(A_1(1-\nu)/8\beta)(j\pi/2L)^2(S_{a_1}+g_{+1}+(f_{+1}+f_{-2})/2)*$$

$$(T'_{+2}\ +S_{-2})$$

$$C_{33}^{0} = (1/R^2)+(j\pi/2L)^2(B_0 +\nu A_0/R)+(\sum_1^n i/R)^2(\nu B_0+A_0/R)+$$

$$[(j\pi/2L)^2 + (\sum_1^n i/R)^2]^2(T_1^2/12)$$

$$C_{33}^{1} = (j\pi/4L)(-(\nu A_1/2\ R)(f_{-1}+f_{+2})(t_{+1}+(T_{+2}+S'_{-2})/2)+$$

$$((1-\nu)B_1 d_a/2\beta R)q_{-1}\sum_1^n i(G^S_{-2}-G^T_{+2}))+(j\pi/2L)^2(t_{+1}+$$

$$(T_{+2}+S'_{-2})/2)(B_1 g_{+1}/2 +\nu A_1/4R(S_{a_1}+g_{+1}+(f_{+1}+f_{-2})/2))+$$

$$(A_1/4\beta R^2)(S_{a_1}+g_{-1}-(f_{+1}+f_{-2})/2)\sum_1^n i(T'_{+2} +S_{-2}) +$$

$$(\sum_1^n \sum_1^n ki/R^2)(t_{-1}+(T_{-2}+ S'_{+2})/2)(\nu B_1 g_{-1}/2 +$$

$$(A_1/4R)(S_{a_1}+g_{-1}-(f_{+1}+f_{-2})/2))$$

$$C_{33}^{S} = (g^0_{-1}t^0_{+1}/R^2)+(j\pi/4L)[-\nu A_1/2\alpha R)(f^0_{-1}+f^0_{+2})(t^0_{+1}+$$

$$(T^0_{+2}+ S'^0_{-2})/2)+((1-\nu)B_1 d_a/2R\beta)q^0_{-1} \sum_1^n i(G^{S0}_{-2}-G^{T0}_{+2})]+$$

$$(j\pi/2L)^2((B_0+\nu A_0/R)g^0_{+1}t^0_{+1}+(B_1 g^0_{+1}/2)+(\nu A_1/4R)($$

$$S_{a_0}+g^0_{+1}+(f^0_{+1}+f^0_{-2})/2))(t^0_{+1}+(T^0_{+2}+S'^0_{-2})/2))+$$

$$(A_1/4\beta R^2)(S_{a_0}+g^0_{-1}-(f^0_{+1}+f^0_{-2})/2) \sum_1^n i(T'^0_{+2}+S^0_{-2})$$

$$C^S_{22} = (1+ B_0 + A_0/R)g^0_{-1}\sum_1^n \sum_1^n ikt^0_{+1}/R^2 + (\nu B_1 g^0_{-1}/2 + A_1/4R*($$

$$S_{a_0} + g^0_{-1} - (f^0_{+1} + f^0_{-2})/2))(t^0_{+1} + (T^0_{+2} + S'^0_{-2})/2)\sum_1^n \sum_1^n ik/R^2$$

$$+((1-\nu)/2) + B_0 + \nu A_0/R)(j\Pi/2L)^2 g^0_{+1} t^0_{-1} + (t^0_{-1} + ($$

$$T^0_{-2} + S'^0_{+2})/2)(B_1 g^0_{-1}/2 + \nu A_1/4R(S_{a_0} + g^0_{+1} + (f^0_{+} + f^0_{-2})/2))*$$

$$(j\Pi/2L)^2 - (j\Pi/2L)((1-\nu)/4R)B_1 d_a/ * \sum_1^n kq^0_{-1}(T'^0_{+2} + S^0_{-2})$$

$$C^S_{23} = \sum_1^n ig^0_{-1}t^0_{+1}/R^2 + (j\Pi/4L)[-(\nu A_1/2\alpha R)(f^0_{-1} + f^0_{+2})\sum_1^n k($$

$$t^0_{+1} + (T^0_{+2} + S'^0_{-2})/2) + (A_1(1-\nu)/4\alpha R)(f^0_{-1} + f^0_{+2})\sum_1^n i\ ($$

$$t^0_{-1} + (T^0_{-2} + S'^0_{+2})/2)] + (A_1/4\beta R^2(S_{a_0} + g^0_{-1} - (f^0_{+1} + f^0_{-2})/2)*$$

$$(\sum_1^n \sum_1^n ik/2)(T'^0_{+2} + S^0_{-2}) - (j\Pi/2L)\ (A_1(1-\nu)/8\beta)($$

$$S_{a_o} + g^0_{+1} + (f^0_{+1} + f^0_{-2})/2)(T'^0_{+2} + S^0_{-2})$$

$$C^B_{22} = \sum_1^n \sum_1^n ik(g^0_{-1}t^0_{+1}/12(R/T_1)^2/R^2) + (j\Pi/2L)^2((1-\nu)3/$$

$$32(R/T_1)^2)g^0_{+1}t^0_{-1}$$

$$C^B_{23} = (j\Pi/2L)^2((1-\nu)/(R/T_1)^2)[-g^0_{-1}\sum_1^n it^0_{+1}/6 + g^0_{+1}\sum_1^n kt^0_{-1}]+$$

$$C_{33}^{B} = (g_{-1}^{0}T_{1}^{2}/12)[(j\pi/2L)^2 + (\sum_{1}^{n} k/R)^2][(j\pi/2L)^2 + (\sum_{1}^{n} i/R)^2]$$

$$+((1-\nu)/6(R/T_1)^2)(j\pi/2L)^2(g_{-1}^{0}\sum_{1}^{n}\sum_{1}^{n} kit_{-1}^{0} - g_{-1}^{0}\sum_{1}^{n} i^2 t_{+1}^{0})$$

APPENDIX C

EVALUATION OF INTEGRALS FOR THE

APPENDIX B

C-1 Evaluation of Integrals for Prebuckling.

Consider the Equation (3.10). Each term in the integral is written separately and integrated for the limits specified in the Chapter 3.

$$\varepsilon_x^{0^2} = B_0^2 + B_0B_1 + 3/8B_1^2 + (B_0B_1 + B_1^2/2)\text{Cos}(Y/\beta) + B_1^2/8\text{Cos}(2Y/\beta)$$

$$\iint \varepsilon_x^{0^2}\, dx\, dy = X[(B_0^2 + B_0B_1 + 3/8B_1^2\ )Y + B_1/2(2B_0 + B_1)^* \beta\text{Sin}(Y/\beta) + (B_1^2\beta/16)\text{Sin}(2Y/\beta)]\Big|_{\text{limits}}^{\text{limits}}$$

$$\varepsilon_y^{0^2} = (A_0 + A_1\text{Cos}^2(X/2\alpha)^*\text{Cos}^2(Y/2\beta))^2/R^2$$

$$\iint \varepsilon_y^{0^2}\, dx\, dy = X[A_0^2 + 9A_1^2/64 + A_0A_1/2]Y + ((3A_1^2/16) + (A_0A_1/2)((Y\alpha\text{Sin}(X/\alpha) + X\beta\text{Sin}(Y/\beta)) + (A_1/4 + A_0A_1/2)^* \alpha\beta\text{Sin}(X/\alpha)^*\text{Sin}(Y/\beta) + (3A_1^2/128)(\alpha Y\text{Sin}(2X/\alpha) + (A_1^2\alpha\beta/32)(\text{Sin}(X/\alpha)\text{Sin}(2Y/\beta) + \text{Sin}(2X/\alpha)^* \text{Sin}(Y/\beta) + \text{Sin}(2X/\alpha)\text{Sin}(2Y/\beta)/8)\Big|_{\text{limits}}^{\text{limits}}$$

Letting $A_1/A_0 = \xi$ and $B_1/B_0 = \zeta$

$$2\nu\varepsilon_x^0\varepsilon_y^0 = -2(\nu A\ B\ /R)[1+\zeta(1+\mathrm{Cos}(Y/\beta)/2\ +\xi(1+\mathrm{Cos}(X/\alpha)\ +$$

$$\mathrm{Cos}(Y/\beta)+\mathrm{Cos}(X/\alpha)\mathrm{Cos}(Y/\beta))/4\ +\zeta\xi(1.5+2\mathrm{Cos}(Y/\beta)\ +$$

$$.5\mathrm{Cos}(2Y/\beta)+1.5\mathrm{Cos}(X/\alpha)+2\mathrm{Cos}(X/\alpha)\mathrm{Cos}(Y/\beta)\ +$$

$$.5\mathrm{Cos}(X/\alpha)\mathrm{Cos}(2Y/\beta))/8]$$

$$\int\int 2\nu\varepsilon_x^0\varepsilon_y^0\ dx\ dy = -(2\nu A\ B\ /R)[\ XY+.5\zeta(XY+X\beta\mathrm{Sin}(Y/\beta))\ +$$

$$.25\xi(XY+\alpha Y\mathrm{Sin}(X/\alpha)\ +\beta X\mathrm{Sin}(Y/\beta)\ +\alpha\beta\mathrm{Sin}(X/\alpha)*$$

$$\mathrm{Sin}(Y/\beta))\ +(\zeta\xi/16)(3XY+4X\beta\mathrm{Sin}(Y/\beta)+.5X\beta*$$

$$\mathrm{Sin}(2Y/\beta)\ +3\alpha Y\mathrm{Sin}(X/\alpha)\ +4\alpha\beta\mathrm{Sin}(X/\alpha)\mathrm{Sin}(Y/\beta)+$$

$$.5\alpha\beta\mathrm{Sin}(X/\alpha)\mathrm{Sin}(2Y/\beta)]\ \Bigg|_{\text{limits}}^{\text{limits}}$$

$$2(1-\nu)\varepsilon_{xy}^{0^2} = (B_0^2(1-\nu)\zeta^2/4\beta^2)X^2(1\ -\ \mathrm{Cos}(2Y/\beta))$$

$$\int\int 2(1-\nu)\varepsilon_{xy}^{0^2}\ dx\ dy = (B_0^2(1-\nu)\zeta^2/12\beta^2)X^3(Y-.5\ \mathrm{Sin}(2Y/\beta))\Bigg|_{\text{limits}}^{\text{limits}}$$

Hence $I_1(\varepsilon)$ etc.,

Integrals for the curvature strains are as follows:

$$\chi_x^{0^2} = (A_1^2/64\alpha^4)(1+\mathrm{Cos}(2X/\alpha)(3+4\mathrm{Cos}(Y/\beta)+\mathrm{Cos}(2Y/\beta))$$

$$\int\int \chi_x^{0^2} dx\, dy = (A_1^2/64\alpha^4)[(X+.5\alpha \mathrm{Sin}(2X/\alpha))(3Y+4\beta \mathrm{Sin}(Y/\beta) + .5\beta \mathrm{Sin}(2Y/\beta))] \Bigg|_{\text{limits}}^{\text{limits}}$$

$$\chi_y^{0^2} = (A_1^2/64\beta^4)(1+\mathrm{Cos}(2Y/\beta)(3+4\mathrm{Cos}(X/\alpha)+\mathrm{Cos}(2X/\alpha))$$

$$\int\int \chi_y^{0^2} dx\, dy = (A_1^2/64\beta^4)[(Y+.5\beta \mathrm{Sin}(2Y/\beta))(3X+4\alpha \mathrm{Sin}(X/\alpha) + .5\alpha \mathrm{Sin}(2X/\alpha))] \Bigg|_{\text{limits}}^{\text{limits}}$$

$$2\nu\chi_x^0\chi_y^0 = (A_1^2/32\alpha^2\beta^2)(1+2\mathrm{Cos}(X/\alpha)+\mathrm{Cos}(2X/\alpha))(1+2\mathrm{Cos}(Y/\beta) + \mathrm{Cos}(2Y/\beta))$$

$$\int\int 2\chi_x^0\chi_y^0 \nu\, dx\, dy = (A_1^2/32\alpha^2\beta^2)(X+2\alpha \mathrm{Sin}(X/\alpha)+.5\alpha \mathrm{Sin}(2X/\alpha)* (Y+2\beta \mathrm{Sin}(Y/\beta)+.5\beta \mathrm{Sin}(2Y/\beta) \Bigg|_{\text{limits}}^{\text{limits}}$$

$$2(1-\nu)\chi_{xy}^{0^2} = ((1-\nu)/32\alpha^2\beta^2)(1-\mathrm{Cos}(2Y/\beta))(A_1^2-A_1^2\mathrm{Cos}(2X/\alpha) + (B_1^2\alpha^2X^2/2R^2) + (2\alpha X A_1 B_1/R)\mathrm{Sin}(X/\alpha))$$

$$\int\int 2(1-\nu)\chi_{xy}^{0^2}\, dx\, dy = ((1-\nu)/32\alpha^2\beta^2)(Y-.5\beta \mathrm{Sin}(2Y/\beta))(A_1^2($$

$X-.5\alpha Sin(2X/\alpha))+(B_1^2\alpha^2X^3/6R^2)+(2\alpha A_1B_1/R)($

$$-\alpha XCos(X/\alpha)+\alpha^2Sin(X/\alpha))\Big|_{limits}^{limits}$$

C-2 Evaluation of Integrals for Buckling.

$$\int_{d_a-a_1}^{d_a+a_1}(1\pm Cos(j\Pi X/L))dx = 2a_1[1\pm Cos(j\Pi d_a/L)(Sin(j\Pi a_1/L)/(j\Pi a_1/L))]$$

$$= 2a_1[\ g_{\pm 1}\ ]$$

$$\int_{d_c-b_1}^{d_c+b_1}[\sum_1^n Cos(iY/R)]^2dy = \sum_1^n b_1(1+Cos(2id_c/R)*(Sin(2ib_1/R))/(2ib_1/R))$$

$$\int_{d_a-a_1}^{d_a+a_1}(1\ \pm Cos(i\Pi X/L)(1+Cos((X-d_a)/\alpha))dx =$$

$$2a_1+2\alpha Sin(a_1/\alpha)\ (2/j\Pi/L)(Cos(j\Pi d_a/L))Sin(j\Pi a_1/L)\ \pm$$

$$Cos(d_a/\alpha)[Cos((j\Pi/L)-(1/\alpha))d_a(Sin((j\Pi/L)-(1/\alpha))a_1)/((j\Pi/L)-(1/\alpha))\ +$$

$$(Cos((j\Pi/L+(1/\alpha))d_a)(Sin((j\Pi/L)+(1/\alpha))a_1)/((j\Pi/L)+(1/\alpha))\ ]\ \pm$$

$Sin(d_a/\alpha)[-(Sin((j\Pi/L)-(1/\alpha))d_a)(Sin((j\Pi/L)-(1/\alpha))a_1)/$

$((j\Pi/L)-(1/\alpha)) \quad +$

$Sin((j\Pi/L)+(1/\alpha)d_a(Sin((j\Pi/L)+(1/\alpha))a_1)/((j\Pi/L)+(1/\alpha))]$

Letting

$\phi_x = a_1((j\Pi/L)+(1/\alpha))$ $\qquad$ $\psi_x = a_1((j\Pi/L)-(1/\alpha))$

$S_{a_1} = (Sin(a_1/\alpha))/(a_1/\alpha) = 0$ $\qquad$ $S_{\Pi a_1} = (Sin(j\Pi a_1/L))/(j\Pi a_1/L)$

$S_{\phi_x} = (Sin((j\Pi/L)+(1/\alpha))a_1)/((j\Pi/L)+(1/\alpha))a_1$

$S_{\psi_x} = (Sin((j\Pi/L)-(1/\alpha))a_1)/((j\Pi/L)-(1/\alpha))a_1$

$¢_{\phi_{d_a}} = Cos((j\Pi/L)+(1/\alpha))d_a$ $\qquad$ $\$_{\phi_{d_a}} = Sin((j\Pi/L)+(1/\alpha))d_a$

$¢_{\psi_{d_a}} = Cos((j\Pi/L)-(1/\alpha))d_a$ $\qquad$ $\$_{\psi_{d_a}} = Sin((j\Pi/L)-(1/\alpha))d_a$

$¢_{\alpha} = Cos(d_a/\alpha)$ $\qquad$ $\$_{\alpha} = Sin(d_a/\alpha)$

hence the integral is

$= 2a_1[S_{a_1} + g_{\pm 1} \pm .5(f_{+1} \pm f_{-2})]$

where

$$f_{+1} = ¢_{\alpha}(¢_{\psi_{d_a}} S_{\psi_x} + ¢_{\phi_{d_a}} S_{\phi_x})$$

$$f_{-2} = \$_{\alpha}(\$_{\phi_{d_a}} S_{\phi_x} - \$_{\psi_{d_a}} S_{\psi_x})$$

$$\int_{d_c-b_1}^{d_c+b_1} (\sum_{1}^{n} Cos(iY/R))^2 * (1+Cos((Y-d_c)/\beta)\ dy =$$

$$= b_1 \sum_{1}^{n} \sum_{1}^{n} [t_{+1} + .5¢_{\tau_{1_{d_c}}} (t_{+2} + t_{+3}) + .5S_{\tau_{1_{d_c}}} (s_{-2} - s_{-3})$$

Letting

$$S_{\tau b_1} = (Sin(\tau b_1))/\tau b_1 \qquad ¢_{\tau\ c} = Cos(\tau d_c)$$

$$S_{\kappa b_1} = (Sin(\kappa b_1))/\kappa b_1 \qquad ¢_{\kappa d_c} = Cos(\kappa d_c)$$

$$S_{\psi y} = (Sin(\psi_y))/\psi_y \qquad \$_{\tau_{1_{d_c}}} = Sin(\tau_1 d_c)$$

$$\int_{d_a-a_1}^{d_a+a_1} Sin(j\pi X/2L)Cos(j\pi X/2L) * [X-d_a] =$$

$$= a_1 d_a [\ C_{\Pi_{d_a}} S_{\Pi_{a_1}} - ¢_{\Pi_{a_1}} C_{\Pi_{d_a}}\ ] = a_1 d_a [q_{-1}]$$

$$\int_{d_c-b_1}^{d_c+b_1} \sum_{1}^{n} \sum_{1}^{n} iCos(iY/R)Sin(kY/R)Sin((Y-d_c)/\beta)\ dy =$$

$$= \sum_{1}^{n} \sum_{1}^{n} .5b_1 i[\, ¢_{\tau 1_{d_c}}(t_{-3}+t_{-2}) - \$_{\tau 1_{d_c}}(s_{+3}-s_{+2})]$$

$$\int_{d_a-a_1}^{d_a+a_1} \mathrm{Sin}(j\pi X/L)\mathrm{Sin}((X-d_a)/\alpha)\, dx = -(f_{-1}+f_{+2})a_1$$

$$\int_{d_c-b_1}^{d_c+b_1} \sum_{1}^{n} \sum_{1}^{n} i\mathrm{Cos}(iY/R)\mathrm{Cos}(kY/R)\mathrm{Sin}((Y-d_c)/\beta))\, dy =$$

$$= .5b_1[G^S_{-2} - G^T_{+2}]$$

Following are the symbols for simplicity.

$$¢_{\psi_{d_c}} S_{\psi_y} \pm ¢_{\phi_{d_c}} S_{\phi_y} = t_{\pm 3} \qquad \$_{\phi_{d_c}} S_{\phi_y} \pm \$_{\psi_{d_c}} S_{\psi_y} = s_{\pm 2}$$

$$¢_{\phi_{d_c}} S_{\phi_y} \pm ¢_{\psi_{d_c}} S_{\psi_y} = t_{\pm 2} \qquad \$_{\psi_{d_c}} S_{\psi_y} \pm \$_{\phi_{d_c}} S_{\phi_y} = s_{\pm 3}$$

$$[¢_{\phi_{d_a}} S_{\phi_x} \pm ¢_{\psi_{d_a}} S_{\psi_x}]¢_\alpha = f_{\pm 1} \qquad [\$_{\phi_{d_a}} S_{\phi_x} \pm \$_{\psi_{d_a}} S_{\psi_x}]\$_\alpha = f_{\pm 2}$$

$$¢_{\tau_{d_c}} S_{\tau_{b_1}} \pm ¢_{\kappa_{d_c}} S_{\kappa_{b_1}} = t_{\pm 1} \qquad 1 \pm ¢_{\Pi_{a_1}} S_{\Pi_{a_1}} = g_{\pm 1}$$

$$\text{¢}\tau_{1_{d_c}}(t_{+2} \pm t_{+3}) = T_{\pm 2} \qquad \$\tau_{1_{d_c}}(s_{+2} \pm s_{+3}) = S_{\pm 2}$$

$$\text{¢}\tau_{1_{d_c}}(t_{-2} \pm t_{-3}) = T'_{\pm} \qquad \$\tau_{1_{d_c}}(s_{-2} \pm s_{-3}) = S'_{\pm 2}$$

$$C_{\Pi_{d_a}}(S_{\Pi_{a_1}} \pm C_{\Pi_{a_1}}) = q_{\pm 1}$$

$$\text{¢}\tau_{1_{d_c}}(s_{-2} \pm s_{-3}) = G^{S}_{\pm 2} \qquad \$\tau_{1_{d_c}}(t_{+2} \pm t_{+3}) = G^{T}_{\pm 2}$$

$$(\tau + \tau_1)b_1 = \phi_y \qquad (\tau - \tau_1)b_1 = \psi_y$$

$$(\tau + \tau_1)d_c = \phi_{d_c} \qquad (\tau - \tau_1)d_c = \psi_{d_c}$$

$$(\kappa + \tau_1)b_1 = \Phi_y \qquad (\kappa - \tau_1)b_1 = \Psi_y$$

$$(\kappa + \tau_1)d_c = \Phi_{d_c} \qquad (\kappa - \tau_1)d_c = \Psi_{d_c}$$

$$\tau = (k-1)/R \qquad \tau_1 = (1/\beta) \qquad \kappa = (k+1)/R$$