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LIE SERIES SOLUTION OF INVARIANT IMBEDDING

NEUTRON TRANSPORT EQUATIONS

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ABSTRACT

The invariant imbedding neutron transport equations were studied by the iterative Lie series perturbation method, taking the first three terms of a Taylor series expansion as the solutions to the perturbed system, which has the same initial condition as the given equations, and using it as a starting solution in the iteration method. Three terms of the Lie series expansion were used in each process as a correction.

The principal result from this investigation is that the iterative Lie series perturbation method is effective in solving the invariant imbedding transport equations. Numerical results obtained for the reflection function of a semi-infinite homogeneous slab were in excellent agreement with those by the Runge-Kutta method given by Bellman. The local truncation errors in the calculations were below 10^{-12}. Furthermore, because of fast convergence, the higher orders of the method are not necessary to yield accurate results. Also, with the variable step size controls, decreases in rounding errors are expected in comparison with the fixed step size calculations.

From indirect comparison, the iterative Lie series perturbation method is faster than the Runge-Kutta method.

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INDEX TO NOTATIONS

c	particle speed
$k(\Omega \leftarrow \Omega_0)$	number of secondary particles produced per collision in direction Ω
mfp	mean free path
$\bar{n}$	outward normal
$p_1(\Delta)$	the probability of suffering no collision for a particle in traversing a distance Δ
$p_2(\Delta)$	the probability for a particle having a collision in traversing a distance Δ
R_a	reflection flux due to process a
R	reflection function (particle density)
$\hat{R}(x,\mu,\mu_0)$	$= c\mu R(x,\mu,\mu_0)$
$r(x,\Omega,\Omega_0)$	$= 4\pi\mu_0\hat{R}(x,\Omega,\Omega_0)$
γ	the total expected number of secondaries per collision
Δ	thickness of a layer
μ	cosine of polar angle θ
σ	the probability of collision per unit path length
Ω	direction of motion of a particle (solid angle)
θ	polar angle
ϕ	azimuthal angle

LIE SERIES SOLUTION OF INVARIANT IMBEDDING
NEUTRON TRANSPORT EQUATIONS

CHAPTER I

INTRODUCTION

Neutron transport theory is concerned with the neutron flux distribution as a function of position, velocity, and time in the medium. The engineers' interest in neutron transport theory is in the study and design of nuclear reactors, such as their radiation shielding and reactor criticality calculations. The methods developed in neutron transport theory can be applied to other transport problems in engineering and physics, such as the study of wave propagation, electron motion in plasmas, electron discharge, penetration of gamma rays through scattering media, radiative transfer in stellar atmosphere[1], etc.

The invariant imbedding formulation of the equations in engineering and the physical sciences, including neutron transport has been developed during the past fifteen years with a goal of achieving a formulation more suitable for application of a digital computer. The reason for this is that the invariant imbedding formulation of the equation is a non-linear initial value problem, which can be solved relatively simply. But in the classical approach, the result is a boundary value problem, which requires at some stage the solution of a large system of linear equations[2].

There is another advantage of invariant imbedding approach over the classical Boltzmann approach. For the invariant imbedding approach, an extension to a slab of large thickness is easy; the transmission function for a thickness 2X can be obtained readily from the function for a thickness X by utilizing the functional relationship [3].

The essence of the invariant imbedding concept is to imbed a particular problem in a family of problems. The family provides a means of advancing gradually from the solution for one member to the solution of the total problem.

The mathematical tool we employed here to solve the invariant imbedding transport equations is a Lie series method. The Lie series is a special power series containing differential operators. It is named after Sophus Lie, a Norwegian mathematician, and was introduced by Wolfgang Gröbner [4], Innsbruck University, Austria, in the early 60's to solve special problems in algebraic geometry.

The possibility of solving differential equations by power series expansions has been known since B. Taylor (1685-1731) first undertook this work. As a practical computational scheme this approach has been deemed infeasible because of the task of deriving expressions for higher derivatives. On the other hand, series methods can give results of arbitrarily high accuracy, provided that the series converges, and that a sufficient amount of time is expended on their calculation. Good numerical results obtained by Taylor series expansions have been reported for certain differential equations [5-6].

Due to the impact of modern high speed digital computers, the calculation of higher derivatives in the series expansion is no longer a problem. Thus, the feasibility of studying the differential equations by the series method is coming to the fore.

CHAPTER II

RESEARCH OBJECTIVES

The primary objective of this work is to show that the Lie series method is suitable for solving invariant imbedding transport equations. It is also to be shown that accuracy is maintained and that savings in computer time are realized.

There are four variable parameters encountered in applying the iterative Lie series perturbation method to a problem: one can choose from several approximations (for example, the number of terms in the Taylor series expansion) to the solution of the perturbed system, its initial conditions are the same as the given equations; number of terms used in the Lie series expansion, in each iterative process, as correction; local error bound used in the step size control; and the number of nodes in the Gaussian quadrature formula used to approximate the integrations.

A secondary objective is, then, to study the effects of these parameters on the precision of the results, and on the computing time required.

CHAPTER III

LITERATURE SURVEY

When a neutron balance equation is written for any region of a reactor core, we get the classical starting point of neutron transport theory, the Boltzmann transport equation. The most comprehensive work in this area is probably the book of Davison and Sykes(7); other notable publications are by Kourganoff(8), and by Case and Zweifel(9). However, by reviewing some relatively new theoretical efforts as well as the new development of the old ideas which arose from other fields, the basis of neutron transport theory has been broadened. The following is the summary of the recent (from mid 50's to date) developments in neutron transport theory, with emphasis on the invariant imbedding concept.

1. Quantum mechanical approach. The most fundamental of this type approach has been the works of Osborn and Yip(10), and of Jauho and Riska (11). They gave conditions for the validity of the transport equation: the side of a cube of the smallest region of validity is $\Lambda \gg 10^{-4}$ cm., and the energies of the neutron must be $\gg 10^{-3}$ mev. To date the quantum mechanical transport theory has not been used to obtain any results which are not more simply found from classical theory.

2. Stochastic transport theory. Neutron transport theory is ordinarily concerned with the mean or expected value of a neutron density. The actual neutron density in a system may or may not resemble the mean value of the density. Generally speaking, if there are few neutrons in a system, the actual density at certain time within certain region of the system is

unlikely to resemble the mean density of the system. But if there are many neutrons in a system, the actual density for a certain region of the system is likely to be very similar to the expected one.

These two extreme examples typify the two most important types of practical questions which have motivated the development of a stochastic theory of neutron transport.

In formulating the transport equations, we generally treat the function as continuous in space, velocity, and time. But it is more fruitful in the stochastic process to treat neutrons a generation at a time. It appears that there are mathematical advantages in doing so. The neutrons in successive generations are simply related and their probability generating functions are connected by a transformation which can be studied.

There are several people working on this area: notably, Pal[12], who in 1958 published the theory of stochastic processes in nuclear reactors. He framed his theory in terms of the probability that a single neutron would lead to "n" neutrons at a later time. By considering the various possible first collisions, and introducing the generating function, he derived the nonlinear integrodifferential equation satisfied by the generating function. Because of ignorance of the space and energy dependence in his theory, it ends up as a point reactor model.

In 1965, Bell[13] clarified the relationship of the extinction probability to the generating function, and in 1966 Otsuka and Saito[14] elucidated the relationship between generating functions and product densities of all orders.

Abu-Shumays[15] uses the generating functions to treat the one energy group transport problem for slab geometry with axial symmetry and constant cross-section. Numerical results for reflection and transmission functions were carried out based on the X- and Y-functions of Chandrasekhar.

Also in 1966, the Langevin's equation was introduced by Sheff and Akcasu[16]. It borrows from the Langevin theory of Brownian motion. In this form of transport theory it is postulated that the difference between the actual neutron density and the mean one is maintained by a random source which arises from the fluctuations in neutron density itself. The physical basis of the Langevin approach seems less rigorous than the first two.

3. Invariant imbedding approach. The invariant imbedding, or the method of continuity, is a fundamental technique of mathematics [17]. It stems from the principle of invariance, introduced by Ambarzumian[18] in astrophysics in 1943, and was extended to radiative transfer by Chandrasekhar [19] in about 1950. After 1956[20-31], through the efforts of Bellman and his colleagues, it found a wide application in radiative transfer[19][33-41], neutron transport [3][35][42-75], random walk and scattering [3][76-82], wave propagation[83-85], rarefied gas dynamics [86-87], Hamilton's equation of motion[88], nonlinear filtering theory[89], quantum mechanics[90], and solid state physics[91]. In short, the invariant imbedding concept is applicable to the treatment of boundary value problems [92-106] and eigenvalue problems[107-108].

The invariant imbedding equations can be obtained from an analysis of the original physical process (in transport theory it is called the "particle counting" technique), which we will employ in the next section, or from the classical equations representing the process [21][26][29][44] [93-104]. Wing[73] illustrated that the reflection equations derived from the classical method and those from invariant imbedding are equivalent.

The "particle counting" technique is fairly prone to error for cylindrical and spherical geometries[43]. However, it does provide a clear picture of the processes.

Although the bulk of papers deal with invariant imbedding formulation of the transport equation, and little work has been done actually on the methods of solution, the problem of deriving invariant imbedding equations for internal flux[53-55][65][70], anisotropic scattering[78a], etc. are still under investigation.

Thus, there are two ways we can approach the study of transport theory: one is from the analytical aspect, the other is from the numerical aspect. From the analytical aspect, one deals with the formulation of the equations and proof for the existence and uniqueness of the solutions. From the numerical aspect, our goal is to find an efficient technique for obtaining numerical solutions on digital computers.

There are four numerical results which related to the invariant imbedding approach. In 1963, Bellman and his colleagues[2] solved the reflection equations for a one-dimensional slab; in 1966, Mathews[61] applied the invariant imbedding method to a shielding problem; in about the same time, Shimizu[3] applied the invariant imbedding method to the reflection and transmission problem of gamma rays. In 1970, Zink[75] extended the method to the two-dimensional slab problem for both reflection and transmission functions.

In the four above cases the Gaussian quadrature formula were used in performing the integrations with respect to the angular variable. Bellman, Mathews, and Zink used tha Runge-Kutta method in the numerical calculations. Mathews also tried the Adams method, which was faster than the Runge-Kutta method and gave a relatively easy estimate of truncation error per step, but was discarded because of problems with numerical instability and because of the large amount of computer memory storage required. The Runge-Kutta method was easy to apply, but it was very time consuming, and it also lacked an easy method of estimating the local truncation error. Shimizu used the iteration method, starting from the first order first degree solutions by neglecting the second degree terms in the equations. Solutions of adequate accuracy was obtained upon two to ten iterations.

Numerical results for the one-dimensional monoenergetic reflection function were found within one to two percent of those that obtained from Chandrasekhar's X- and Y-function[(19)(109)]. For the two dimensional case it was found that the results were smaller than those of a one-dimensional slab because of the leakage from the sides. The theoretical results of Shimizu's calculation of the energy and angular distribution of transmitted photons were compared with experiments and found to be in good agreement.

CHAPTER IV

FORMULATION OF THE THEORY

4.1 Hypotheses

The most important assumptions made in derivation of the transport equations are:

1. The neutron is a point particle, moving at speed c, characterized only by its position and direction of motion.

2. The moving neutrons may collide only with the fixed nuclei of the medium; interaction between the moving neutrons will not be considered.

3. A collision always results in the disappearance of the original neutron and the appearance of "γ" new neutrons at the point of collision. These are emitted isotropically and intantaneously.

4. No more than a single collision is assumed.

5. No internal source.

The assumptions made in solving the transport equations are:

1. The transport functions are continuous, and satisfy a Lipschitz condition, so that the solutions exist and are unique.

2. The transport functions are sufficiently differentiable, so that all the terms in Taylor series and Lie series exist.

4.2 Formulation of the Reflection Equations

In the following derivation of the transport equations, a new graphical representation of collision process was introduced. The one-dimensional invariant imbedding reflection equation was first formulated by Bellman[2], and explained partly due to Wing[73]. The two-dimensional equation was introduced by Zink[75]. In this Chapter clarifications were made on both one- and two-dimensional equations.

4.2 One dimensional reflection function

Consider an input of one particle per second per unit area of a slab surface, and moving in the direction Ω_0.

The probability of one of the injected particles having a collision in passing from $x+\Delta$ to x (Fig. 1, top) is

$$p_2^0 = \sigma(x)\frac{\Delta}{|\mu_0|}$$

where $\sigma(x)$ is the probability of collision per unit path length, and μ_0 is the cosine of angle θ_0. The expected flux thus produced in direction Ω is

$$R_a = p_2^0 k(x,\Omega,\Omega_0),$$

where k is the expected number of secondaries per cm^2 per second going in direction Ω as a result of a collision at x, due to a unit incident flux in direction Ω_0. This process is symbolically represented by R_a in Fig. 1.

Now, consider an injected particle having a collision in $(x,x+\Delta)$ and giving rise to particle going in direction Ω'', where $\frac{1}{2}\pi \leq \theta'' \leq \pi$, and θ'' is the polar-angle component of Ω''. The number of such particles impinging per second on a unit area perpendicular to the outward normal $\bar{n}$ at x and moving in direction Ω'' is

$$p_2^0 k(x,\Omega'',\Omega_0).$$

This amount of flux at x produces a neutron density at $x+\Delta$, moving in direction Ω, due to reflection, of

$$p_2^0 k(x,\Omega'',\Omega_0)R(x,\Omega,\Omega'')(1-p_2),$$

and thus contributes to a flux (moving in direction Ω) normal to $\bar{n}$ of

$$R_b = p_2^0 kc\mu\int_0^{2\pi}\int_{-1}^{0} R(x,\Omega,\Omega'')d\mu''d\phi'' + o(\Delta),$$

where c is the speed of neutron. $R(x,\Omega,\Omega'')$ is called the reflection function, and it has unit of density [#/cm^3]. This result is shown by R_b in Fig. 1, where $p_1^{\prime} = 1 - p_2^0$ is the probability of suffering no collision in traversing a distance Δ/μ_0. The term involving $p_2^0 p_2$ is small in comparsion with p_2^0 and thus is represented here by $o(\Delta)$.

Here we also assume the collision process is isotropic, so that k is independent of Ω and x. Thus,

$$\int_{\Omega} k d\Omega = \gamma,$$

or $\quad k = \gamma/4\pi,$

where γ is the total expected number of secondaries per collision.

Continuing this kind of analysis we will obtain the expressions for R_c, R_d, and R_e respectively.

All other processes involve two or more collisions in $(x,x+\Delta)$ and only contribute a term $o(\Delta)$ to those already included.

By combining five processes in Fig. 1, we get the flux at $x+\Delta$ which is normal to the slab and moving in direction Ω,

$$c\mu R(x+\Delta,\mu,\mu_0) = R_a + R_b + R_c + R_d + R_e.$$

Let $c\mu R(x,\mu,\mu_0) = \hat{R}(x,\mu,\mu_0)$.

Then, $\quad \lim_{\Delta\to 0} \dfrac{\hat{R}(x+\Delta,\mu,\mu_0)}{\Delta} = \dfrac{\partial\hat{R}(x,\mu,\mu_0)}{\partial x}$

$$= \sigma\gamma/4\pi|\mu_0| + \sigma\gamma/4\pi|\mu_0| \cdot \int_0^{2\pi} d\phi'' \int_0^1 \hat{R}(x,\mu,\phi;\mu'',\phi'')d\mu''$$

$$- \hat{R}(x,\mu,\phi;\mu_0,\phi_0)\left[\frac{1}{|\mu_0|} + \frac{1}{\mu}\right]$$

$$+ \frac{\sigma\gamma}{4\pi}\int_0^{2\pi} d\phi' \int_0^1 \hat{R}(x,\mu',\phi';\mu_0,\phi_0)\frac{d\mu'}{\mu'} \tag{4-1}$$

$$+ \frac{\sigma\gamma}{4\pi}\int_0^{2\pi} d\phi \int_0^1 \hat{R}(x,\mu',\phi';\mu_0,\phi_0)\frac{d\mu'}{\mu'}\int_0^{2\pi} d\phi''\int_0^1 \hat{R}(x,\mu,\phi;\mu'',\phi'')d\mu'',$$

with the initial condition that

$$\hat{R}(0,\mu,\phi;\mu_0,\phi_0) = 0. \tag{4-2}$$

The purpose of introducing $\hat{R}$ is to change the function R, which is odd in its argument, μ, to an even function. Thus we can confine the domain of angular variables μ to (0,1), and simplify the transport equation.

By substituting

$$r(x,\mu,\phi;\mu_0,\phi_0) = 4\pi\mu_0\hat{R}(x,\mu,\phi;\mu_0,\phi_0)$$

into Eq.(4-1) and letting $\sigma = 1$ (unitary mean free path), Eq.(4-1) becomes

$$\frac{\partial r}{\partial x} + \left(\frac{1}{\mu_0} + \frac{1}{\mu}\right)r = \frac{1}{\gamma}\left[1 + \frac{1}{2}\int_0^1 r(x,\mu,\mu'')\frac{d\mu''}{\mu''}\right]$$

$$\times \left[1 + \frac{1}{2}\int_0^1 r(x,\mu',\mu_0)\frac{d\mu'}{\mu'}\right] \tag{4-3}$$

with $\quad r(0,\mu,\mu_0) = 0.$

We see that (4-3) is symmetric in its angular arguments.

4.3 Two-dimensional reflection functions

The physical model of the collision processes is shown in Fig. 2. From this one can determine the flux at angle Ω from the normal to the slab at position $(x+\Delta,y)$ due to the neutrons incident at $(x+\Delta,y_0)$ in direction Ω_0 at the rate of 1 per cm^2 per second. This can be written as:

$$c\mu R(x+\Delta,y,\Omega,\Omega_0) = R_a + R_b + R_c + R_d + R_e.$$

Let $\quad c\mu R(x,y,\Omega,\Omega_0) = \hat{R}(x,y,\Omega,\Omega_0),$

then,
$$\lim_{\Delta\to 0}\frac{\hat{R}(x+\Delta,y,\Omega,\Omega_0)-\hat{R}(x,y,\Omega,\Omega_0)}{\Delta}=\frac{\partial\hat{R}}{\partial x}$$

$$=\frac{\gamma}{4\pi\mu_0}-\left(\frac{1}{\mu_0}+\frac{1}{\mu}\right)\hat{R}(x,y,\Omega,\Omega_0)$$

$$+\frac{\gamma}{4\pi\mu_0}\int_{\Omega''}\hat{R}(x,y,\Omega,\Omega'')d\Omega''$$

$$+\frac{\gamma}{4\pi}\int_{y'}\int_{\Omega'}\frac{\hat{R}(x,y',\Omega',\Omega_0)}{\mu'}d\Omega'dy'$$

$$+\frac{\gamma}{4\pi}\int_{\Omega''}\int_{y'}\int_{\Omega'}\frac{\hat{R}(x,y',\Omega',\Omega_0)}{\mu'}d\Omega'dy'\hat{R}(x,y,\Omega,\Omega'')d\Omega'', \qquad (4\text{-}5)$$

with the initial condition that

$$\hat{R}(0,y,\Omega,\Omega_0)=0. \qquad (4\text{-}6)$$

Although we assume that the collision process is isotropic, it does not mean that the reflected flux is azimuthally-symmetric. To simplify the equation, introduce a new function[75]

$$\bar{R}(x,y,\mu,\phi;\mu_0)=2\pi\int_0^{2\pi}\hat{R}(x,y,\Omega,\Omega_0)d\phi_0. \qquad (4\text{-}7)$$

Operating on (4-5) by $\int_0^{2\pi}d\phi_0\int_0^{2\pi}d\phi''$ and applying (4-7), we have

$$\frac{\partial}{\partial x}\bar{R}(x,y,\mu,\phi;\mu_0)=\frac{\gamma\pi}{\mu_0}-\left(\frac{1}{\mu_0}+\frac{1}{\mu}\right)\bar{R}(x,y,\mu,\phi;\mu_0)$$

$$+\frac{\gamma}{2\mu_0}\int_0^1\bar{R}(x,y,\mu,\phi;\mu'')d\mu''+\frac{\gamma}{4\pi}\int_0^1dy'\int_0^{2\pi}\int_0^1\frac{\bar{R}(x,y,\mu',\phi';\mu_0)}{\mu'}d\mu'd\phi'$$

$$+\frac{\gamma}{8\pi^2}\int_0^{2\pi}\int_0^1\int_0^1\int_0^1\frac{\bar{R}(x,y',\mu',\phi';\mu_0)\bar{R}(x,y,\mu,\phi;\mu'')}{\mu'}dy'd\mu''d\mu'd\phi', \qquad (4\text{-}8)$$

with the initial condition that

$$\bar{R}(0,y,\mu,\phi;\mu_0)=0. \qquad (4\text{-}9)$$

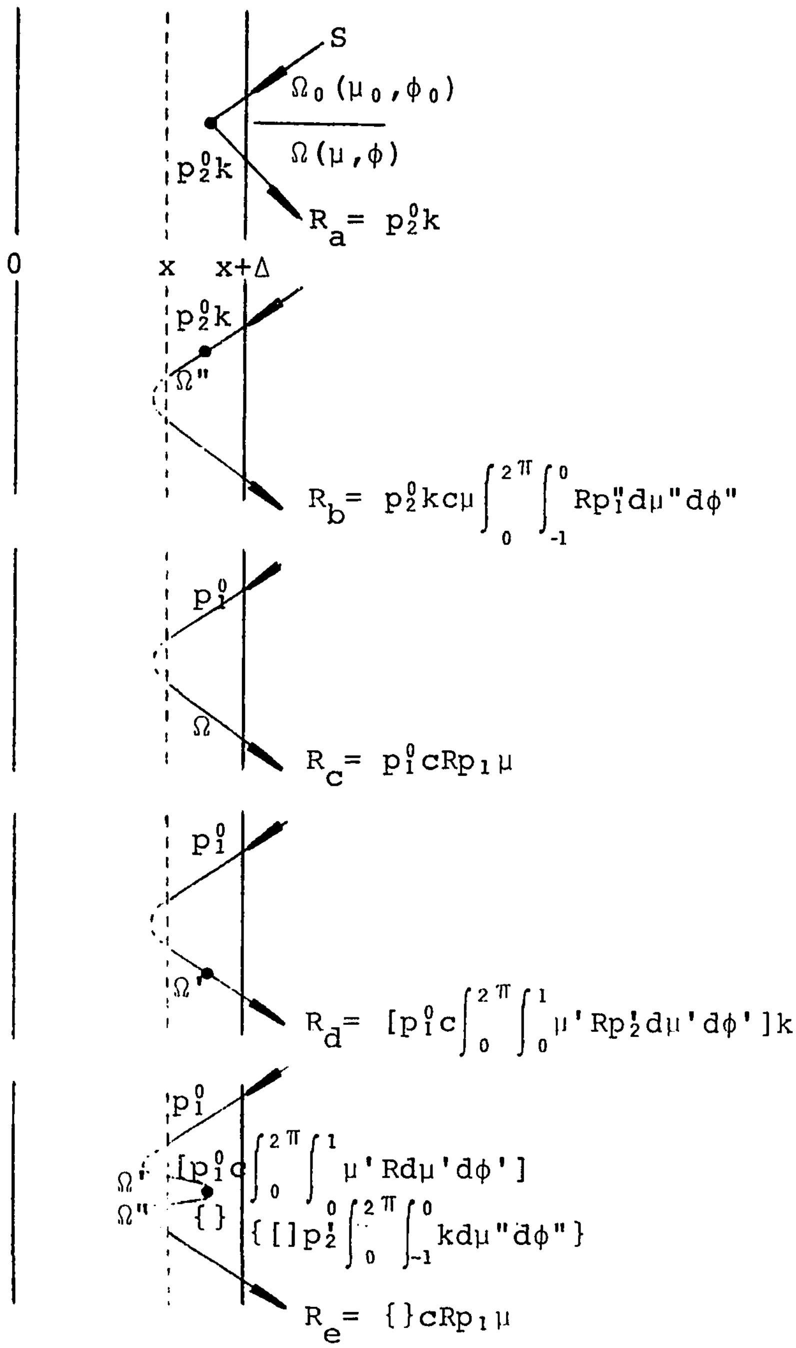
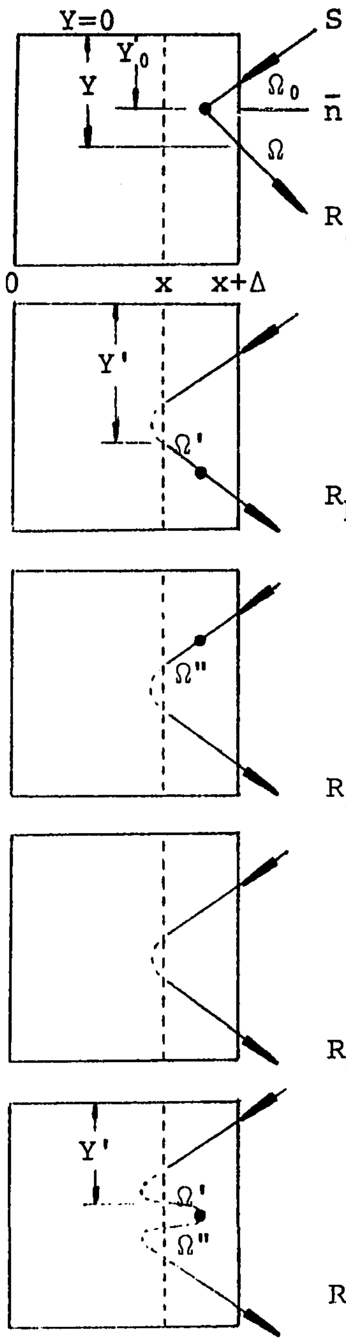


Fig. 1. The physical model of the collision processes.

A dot indicates that a collision has taken place in $(x, x+\Delta)$. The normal components are represented by R_a, R_b, ..., R_e.

The paths are dashed in $(0,x)$ to emphasize the fact that the details of the particle behavior there are of no interest to us.



$$R_a = \frac{\sigma\Delta}{|\mu_0|}\,\frac{\gamma}{4\pi},$$

$$R_b = \int_{Y'}\int_{\Omega'}\left[1-\frac{\sigma\Delta}{|\mu_0|}\right]c\mu' R(x,Y',\Omega',\Omega_0)\,d\Omega'\,dY'\,\frac{\sigma\Delta\gamma}{\mu' 4\pi}$$

$$= \frac{\sigma\Delta\gamma}{4\pi}\int_{Y'}\int_{\Omega'} cR(x,Y',\Omega',\Omega_0)\,d\Omega'\,dY' + o(\Delta),$$

$$R_c = \frac{\sigma\Delta}{|\mu_0|}\,\frac{\gamma}{4\pi}\int_{\Omega''} c\mu R(x,Y,\Omega,\Omega'')\,d\Omega''\left[1-\frac{\sigma\Delta}{\mu}\right]$$

$$= \frac{\sigma\Delta}{|\mu_0|}\,\frac{\gamma}{4\pi}\int_{\Omega''} c\mu R(x,Y,\Omega,\Omega'')\,d\Omega'' + o(\Delta),$$

$$R_d = \left[1-\frac{\sigma\Delta}{|\mu_0|}\right]c\mu R(x,Y,\Omega,\Omega_0)\left[1-\frac{\sigma\Delta}{\mu}\right]$$

$$= c\mu R(x,Y,\Omega,\Omega_0) - \sigma\Delta\left(\frac{1}{|\mu_0|}+\frac{1}{\mu}\right)c\mu R(x,Y,\Omega,\Omega_0)$$

$$+ o(\Delta),$$

R_e

$$R_e = \int_{\Omega''}\int_{Y'}\int_{\Omega'}\left[1-\frac{\sigma\Delta}{|\mu_0|}\right]c\mu' R(x,Y',\Omega',\Omega_0)\,\frac{\sigma\Delta\gamma}{4\pi\mu'}\,c\mu R(x,Y,\Omega,\Omega'')\,d\Omega'\,dY'\,d\Omega''$$

$$\times\left[1-\frac{\sigma\Delta}{\mu}\right].$$

Fig. 2. The physical model of the collision processes for a two-dimensional slab. o(Δ) represents higher-order terms, or terms involving powers of Δ higher than first.

CHAPTER V

LIE SERIES PERTURBATION METHOD

The development of the Lie series method are due to Gröbner, Knapp, Wanner et al.[110], of Innsbruck University, Austria. In this section we try to clarify some of their ideas, consolidate scattered materials into one piece, and to use them to solve the invariant imbedding transport equations.

5.1 Definition of the Lie series

Let F(x,y) be an analytic or a sufficiently differentiable function of the variables x, $y_1, \ldots\ldots, y_n$.

Where

$$y = \begin{bmatrix} y_1 \\ \cdot \\ \cdot \\ y_n \end{bmatrix} \tag{5-1}$$

are solutions of a system of first order ordinary differential equations.

$$y_i' = f_i(x, y_1, \ldots, y_n). \qquad (i = 1, \ldots, n) \tag{5-2}$$

Inserting solutions $y_1(x), \ldots, y_n(x)$ in the place of $y_1, \ldots, y_n$ we find a function that depends on x only. By the chain rule, its derivative with respect to this variable is

$$\frac{d}{dx} F(x,y(x)) = \left[F_x + F_{y_1}\frac{dy_1}{dx} + \ldots + F_{y_n}\frac{dy_n}{dx}\right]_{x,y(x)} . \tag{5-3}$$

Since the functions y(x), which we have inserted in Eq.(5-3), are supposed to be the solutions of Eq.(5-2), we have

$$\frac{dy_i}{dx} = f_i(x,y(x))$$
$$= \Big[f_i(x,y)\Big]_{x,y(x)},$$

and

$$\frac{d}{dx} F(x,y(x)) = \left[F_x + f_1(x,y)\frac{\partial F}{\partial y_1} + \ldots. + f_n(x,y)\frac{\partial F}{\partial y_n}\right]_{x,y(x)}$$
$$= \Big[DF\Big]_{x,y(x)}. \tag{5-4}$$

We have defined the linear differential operator

$$D \equiv \frac{\partial}{\partial x} + f_1(x,y)\frac{\partial}{\partial y_1} + \ldots + f_n\frac{\partial}{\partial y_n}, \tag{5-5}$$

as the Lie series operator.

By iteration of Eq.(5-4), we find for the higher derivatives,

$$\frac{d^s}{dx^s} F(x,y(x)) = \Big[D^s F\Big]_{x,y(x)} \qquad (s = 1,\ 2,\ 3,\ \ldots). \tag{5-6}$$

A series of the following kind:

$$\sum_{s=0}^{\infty} \frac{x^s}{s!} D^s F = F(x,y(x)) + xDF(x,y(x)) + \frac{x^2}{2!}D^2F(x,y(x)) + \ldots \tag{5-7}$$

satisfies these relationships and is called a Lie series, frequently denoted by the symbol $e^{xD}F$.

5.2 The properties of e^{xD}

(1) The symbol e^{xD} fulfills the following relations:

$$e^{xD}\Big[f(x,y) + g(x,y)\Big] = \Big[e^{xD}f(x,y)\Big] + \Big[e^{xD}g(x,y)\Big]. \tag{5-8}$$

and

$$e^{xD}\Big[f(x,y)g(x,y)\Big] = \Big[e^{xD}f(x,y)\Big]\Big[e^{xD}g(x,y)\Big]. \tag{5-9}$$

(2) Commutation theorem: If $F(y)$ denotes any function which is analytic in a neighborhood of $(y_1, y_2, \ldots, y_n)$, and $Y_i = \phi_i(x,y) = e^{xD}y_i$, $\left(Y_i\right)_{x=0} = \phi_i(0,y) = y_i$, $i = 1,2,\ldots, n$, then the corresponding power series expansion $F(Y)$ still converges at the point $(Y_1, Y_2, \ldots, Y_n)$. That is

$$F(Y) = \sum_{s=0}^{\infty} \frac{x^s}{s!} D^s F(y),$$

or
$$F(e^{xD}y) = e^{xD}F(y). \tag{5-10}$$

Thus, the function sign F of an analytic function may be commuted with the symbol e^{xD} for the Lie series.

(3) Multiple Lie series:

Let m linear differential operators be given by

$$D_k = f_{k1}(y) \frac{\partial}{\partial y_1} + \ldots + f_{kn}(y) \frac{\partial}{\partial y_n}, \qquad (k = 1, \ldots, m) \tag{5-11}$$

all of which be holomorphic in a certain domain C. We introduce m new independent variables $x_1, x_2, \ldots, x_m$, using them for the multiple Lie series:

$$e^{x_1D_1 + \ldots + x_mD_m} f(y) = \sum_{s=0}^{\infty} \frac{1}{s!}\left[x_1D_1 + \ldots + x_mD_m\right]^s f(y). \tag{5-12}$$

Positive constants T_k can be found such that the series (5-12) is absolutely convergent for $(y) \varepsilon C$ at $|x| \leq T_k$.

The commutation theorem is valid also for these series:

$$Y_j = e^{x_1D_1 + \ldots + x_mD_m} y_j, \quad (j = 1, 2, \ldots, n). \tag{5-13}$$

We may change the function sign F of each function analytic in C with the symbol of the Lie series:

$$F(Y) = F(e^{x_1 D_1 + \dots + x_m D_m} y)$$

$$= e^{x_1 D_1 + \dots + x_m D_m} F(y). \tag{5-14}$$

The operators D_k in general cannot be exchanged.

5.3 Lie series perturbation method

(1) The problem and the main theroem.

Consider a system of ordinary differential equations

$$y_i'(x) = f_i(x, y_1(x), \dots, y_n(x)), \ (i = 1, \dots, n) \tag{5-15}$$

with the initial conditions

$$y_i(x_0) = y_{i_0}. \tag{5-16}$$

If the functions f_i satisfy a Lipschitz condition, than the solutions

$$y_1(x), \dots, y_n(x) \tag{5-17}$$

are unique. But only in a few simple cases can the solutions be written in terms of well-known elementry functions. In most cases, approximation methods must be used.

The method we introduce here is a perturbation method. Hence, there should be given another system of differential equations

$$\hat{y}_i'(x) = \hat{f}_i(x, \hat{y}_1(x), \dots, \hat{y}_n(x)), \ (i = 1, \dots, n) \tag{5-18}$$

whose solutions:

$$\hat{y}_1(x), \dots, \hat{y}_n(x) \tag{5-19}$$

are known, which satisfy the same initial conditions

$$\hat{y}_i(x_0) = y_{i_0}. \tag{5-20}$$

Next, we define the differential operators

$$D \equiv \frac{\partial}{\partial x} + \sum_{k=1}^{n} f_k(x, y_1, \dots, y_n) \frac{\partial}{\partial y_k} \tag{5-20}$$

belonging to the given system (5-15), and

$$D_1 \equiv \frac{\partial}{\partial x} + \sum_{k=1}^{n} \hat{f}_k(x,\hat{y}_1,\ldots,\hat{y}_n) \frac{\partial}{\partial y_k} \qquad (5\text{-}22)$$

belong to the chosen system (5-18). Then

$$D_2 = \sum_{k=1}^{n} \left[f_k(x,y_1,\ldots,y_n) - \hat{f}_k(x,\hat{y}_1,\ldots,\hat{y}_n) \right] \frac{\partial}{\partial y_k}$$

$$= D - D_1 \qquad (5\text{-}23)$$

taking into account the discrepancy between the original and the chosen system.

Now, let $f_k(x,y_1,\ldots,y_n)$ and $\hat{f}_k(x,\hat{y}_1,\ldots,\hat{y}_n)$ $(k = 1,\ldots,n)$

be continuous in the compact domain

$$C = \{(x,y_1,\ldots,y_n)\ x \in [x_0,\ x_0+a]\ ,\ |y_i - y_{i_0}| \leqslant b\ (i=1,\ldots,n)\} \qquad (5\text{-}24)$$

of the $(x,y_1,\ldots,y_n)$-space. Then they are bounded in C. The conditions

$$|f_k(x,y_1,\ldots,y_n) \leqslant A_0, \quad |\hat{f}_k(x,\hat{y}_1,\ldots,\hat{y}_n)| \leqslant B_0, \qquad (5\text{-}25)$$

and the Peano-theorem [111] guarantee the existence of corresponding solutions $y_k(x)$ of (5-15), and $\hat{y}_k(x)$ of (5-18) at least for

$$x \in [x_0, x_0 + h],$$

where

$$h = \min(a, \frac{b}{A_0}, \frac{b}{B_0}). \qquad (5\text{-}26)$$

<u>The main theorem</u>

If $f_1(x,y_1,\ldots,y_n),\ldots,f_n(x,y_1,\ldots,y_n)$ and $\hat{f}_1(x,\hat{y}_1,\ldots,\hat{y}_n),\ldots,$ $\hat{f}_n(x,\hat{y}_1,\ldots,\hat{y}_n)$ are s-times continuously differentiable $(s \geqslant 0)$ in C then the following formula holds, at least for $x \in [x_0, x_0+h]$:

$$y_k(x) = \hat{y}_k(x) + \sum_{\alpha=0}^{s} \int_{x_0}^{x} \frac{(x-\xi)^\alpha}{\alpha!} [D_2 D^\alpha y_k]_{\xi,\hat{y}(\xi)} d\xi + R_{ks}(x)$$

$$= \bar{y}_k(x) + R_{ks}(x) \qquad (k = 1,\ldots, n) \qquad (5\text{-}27)$$

where

$$R_{ks}(x) = \int_{x_0}^{x} \frac{(x-\xi)^s}{s!} \{[D^{s+1}y_k]_{\xi,y(\xi)} - [D^{s+1}y_k]_{\xi,\hat{y}(\xi)}\} d\xi \quad (5\text{-}28)$$

and

$$\bar{y}_k(x) = \hat{y}_k(x) + \sum_{\alpha=0}^{s} \int_{x_0}^{x} \frac{(x-\xi)^\alpha}{\alpha!} [D_2 D^\alpha y_k]_{\xi,\hat{y}(\xi)} d\xi. \qquad (5\text{-}29)$$

Proof:

First, we introduce three auxiliary formulas. We recognize that:

(a) If $g(\xi)$ and $h(\xi)$ are continuously differentiable functions with $g(\xi_0) = h(\xi_0)$, then

$$g(\xi) - h(\xi) = \int_{\xi_0}^{\xi} \left(g'(\eta) - h'(\eta)\right) d\eta. \qquad (5\text{-}30)$$

(b) Let $y_1(x),\ldots,y_n(x)$ be the solutions of the original problem and $\hat{y}_1(x),\ldots,\hat{y}_n(x)$ be the solutions of the chosen equations. If now $F(x,y_1,\ldots,y_n)$ is sufficiently differentiable, then for the derivative of $F(x,y)$ and $F(x,\hat{y})$, we have, by the chain rule:

$$\frac{d}{dx} F(x,y) = \left[\frac{\partial F(x,y)}{\partial x} + \frac{\partial F(x,y)}{\partial y_1} f_1(x,y) + \ldots + \frac{\partial F(x,y)}{\partial y_n} f_n(x,y)\right]_{x,y(x)}$$

$$= \left[DF(x,y)\right]_{x,y(x)}, \qquad (5\text{-}31)$$

and
$$\frac{d}{dx} F(x,\hat{y}) = \left[\frac{\partial F(x,\hat{y})}{\partial x} + \frac{\partial F(x,\hat{y})}{\partial y_1} f_1(x,\hat{y}) + \ldots + \frac{\partial F(x,\hat{y})}{\partial y_n} f_n(x,\hat{y})\right]_{x,\hat{y}(x)}$$

$$= \left[DF(x,\hat{y})\right]_{x,\hat{y}(x)}, \qquad (5\text{-}31)$$

respectively. Finally,

$$(c)\int_{x_0}^{x} \frac{(x-\xi)^{\alpha}}{\alpha!}\int_{x_0}^{\xi} k(\eta)d\eta d\xi = \int_{x_0}^{x}\frac{(x-\xi)^{\alpha+1}}{(\alpha+1)!}k(\xi)d\xi. \tag{5-33}$$

This can be proven easily by integration by parts.

The left hand side of (5-33) gives:

$$\left[-\frac{(x-\xi)^{\alpha+1}}{(\alpha+1)!}\int_{x_0}^{\xi} k(\eta)d\eta\right]\Bigg|_{\xi=x_0}^{\xi=x} + \int_{x_0}^{x}\frac{(x-\xi)^{\alpha+1}}{(\alpha+1)!}k(\xi)d\xi$$

where the first term vanishes by inserting the integral bounds.

Now, we may begin the proof of the main theorem. We write

$$y_k(x) = \hat{y}_k(x) + \left[y_k(x) - \hat{y}_k(x)\right] \qquad (k = 1,\ldots, n)$$

by (a) and (b) we have

$$\begin{aligned}
y_k(x) - \hat{y}_k(x) &= \int_{x_0}^{x}\left[y_k'(\eta) - \hat{y}_k'(\eta)\right]d\eta \\
&= \int_{x_0}^{x}\{\left[Dy_k\right]_{\eta,y(\eta)} - \left[D_1 y_k\right]_{\eta,\hat{y}(\eta)}\}d\eta \\
&= \int_{x_0}^{x}\{\left[Dy_k\right]_{\eta,\hat{y}(\eta)} - \left[D_1 y_k\right]_{\eta,\hat{y}(\eta)} + \left[Dy_k\right]_{\eta,y(\eta)} \\
&\qquad - \left[Dy_k\right]_{\eta,\hat{y}(\eta)}\}d\eta \\
&= \int_{x_0}^{x}\left[D_2 y_k\right]_{\eta,\hat{y}(\eta)}d\eta + R_{k_0}(x),
\end{aligned} \tag{5-34}$$

where

$$R_{k0}(x) = \int_{x_0}^{x}\{\left[Dy_k\right]_{\eta,y(\eta)} - \left[Dy_k\right]_{\eta,\hat{y}(\eta)}\}d\eta.$$

This proves our main theorem for $s = 0$.

Now, for any positive integer s, and $\alpha<s$, then by (a), (b), and (c):

$$\begin{aligned}
R_{k\alpha}(x) &= \int_{x_0}^{x}\frac{(x-\xi)^{\alpha}}{\alpha!}\{\left[D^{\alpha+1}y_k\right]_{\xi,y(\xi)} - \left[D^{\alpha+1}y_k\right]_{\xi,\hat{y}(\xi)}\}d\xi \\
&= \int_{x_0}^{x}\frac{(x-\xi)^{\alpha}}{\alpha!}\int_{x_0}^{\xi}\{(\left[D^{\alpha+1}y_k\right]_{\eta,y(\eta)})' - (\left[D^{\alpha+1}y_k\right]_{\eta,\hat{y}(\eta)})'\}d\eta d\xi
\end{aligned}$$

$$= \int_{x_0}^{x} \frac{(x-\xi)^{\alpha}}{\alpha!} \int_{x_0}^{\xi} \{ \left[DD^{\alpha+1}y_k\right]_{\eta,y(\eta)} - \left[D_1 D^{\alpha+1} y_k\right]_{\eta,\hat{y}(\eta)} \} d\eta d\xi$$

$$= \int_{x_0}^{x} \frac{(x-\xi)^{\alpha+1}}{(\alpha+1)!} \{ \left[DD^{\alpha+1}y_k\right]_{\xi,\hat{y}(\xi)} - \left[D_1 D^{\alpha+1} y_k\right]_{\xi,\hat{y}(\xi)}$$

$$+ \left[DD^{\alpha+1}y_k\right]_{\xi,y(\xi)} - \left[DD^{\alpha+1}y_k\right]_{\xi,\hat{y}(\xi)} \} d\xi$$

$$= \int_{x_0}^{x} \frac{(x-\xi)^{\alpha+1}}{(\alpha+1)!} \left[D_2 D^{\alpha+1} y_k\right]_{\xi,\hat{y}(\xi)} d\xi + R_{k,\alpha+1}(x), \tag{5-35}$$

where

$$R_{k,\ \alpha+1}(x) = \int_{x_0}^{x} \frac{(x-\xi)^{\alpha+1}}{(\alpha+1)!} \{ \left[D^{\alpha+2}y_k\right]_{\xi,y(\xi)} - \left[D^{\alpha+2}y_k\right]_{\xi,\hat{y}(\xi)} \} d\xi. \tag{5-36}$$

Thus, the main theorem is valid if $\alpha<s$. From Eq.(5-29), we notice that our method is an expansion of the solutions in the neighborhood of chosen functions, whereas the Taylor series is an expansion in the neighborhood of a point.

(2) Order of the remainder term.

If, in addition to the assumptions in the main theorem,

$$| D^s f_i(x,y_1,\ldots,y_n)_{x,y^*} - D^s f_i(x,y_1,\ldots,y_n)_{x,y^{**}} |$$

$$\leq K_s \sum_{k=1}^{n} | y_k^* - y_k^{**} | \tag{5-37}$$

for $(x,y_1^*,\ldots,y_n^*)\in C$ and $(x,y_1^{**},\ldots y_n^{**})\in C$ (that is the Lipschitz condition for the functions $D^s f_1,\ldots,D^s f_n$ with a constant K_s in C),

and $$|y_i(x) - \hat{y}_i(x)| \leq M \frac{|x_0 - x|^{m+1}}{(m+1)!} \quad \text{for } x\in[x_0, x_0 + h], \tag{5-38}$$

where M is a constant and m is an integer, then

$$|R_{is}(x)| \leq M \frac{K_s n|x - x_0|^{m+s+2}}{(m+s+2)!} \quad \text{for} \quad x\varepsilon[x_0, x_0 + h]. \tag{5-39}$$

Proof: Since $D^{s+1}y_i = D^s(Dy_i) = D^s f_i$, and because of (5-37) and (5-38), we can estimate R_{is}

$$|R_{is}(x)| = \left|\int_{x_0}^{x} \frac{(x-\xi)^s}{s!}\{\left[D^{s+1}y_i\right]_{\xi,y(\xi)} - \left[D^{s+1}y_i\right]_{\xi,\hat{y}(\xi)}\}d\xi\right|$$

$$\leq \left|\int_{x_0}^{x} \frac{|x-\xi|^s}{s!} K_s \sum_{k=1}^{n} |y_k(\xi) - \hat{y}_k(\xi)|d\xi\right|$$

$$\leq \left|\int_{x_0}^{x} \frac{|x-\xi|^s}{s!} K_s nM \frac{|\xi - x_0|^{m+1}}{(m+1)!} d\xi\right|.$$

Introducing

$$\int_{x_0}^{x} \frac{(x-\xi)^{\alpha}}{\alpha!}\frac{(\xi - x_0)^{\beta}}{\beta!} d\xi = \frac{(x - x_0)^{\alpha+\beta+1}}{(\alpha+\beta+1)!}, \tag{5-40}$$

where α and β are positive integers, we get (5-39).

(3) Iterative method interpretation

It is indicated in section V - 3 that the solutions to the perturbed system must be known. In our case, for example, we use the first m terms of a Taylor series. We get the first approximate solutions from Eq.(5-27) by substituting the perturbed solutions for $\hat{y}_k$ and neglecting the remainders. In this way, we have an iteration process which yields the desired solutions. To put it another way, let us introduce a left subscript ν ($\nu = 0, 1, \ldots$) to denote the index of steps of the iteration. Then,

$${}_0y_i(x) = \hat{y}_i(x), \qquad (i = 1,\ldots,n) \tag{5-41}$$

and by Eq.(5-27), we have

$$ {}_1y_i(x) = \hat{y}_i(x) + \sum_{\alpha=0}^{s} \int_{x_0}^{x} \frac{(x-\xi)^\alpha}{\alpha!} \left[{}_0D_2 D^\alpha y_i\right]_{\xi,y(\xi)} d\xi. $$

In general, we calculate ${}_{\nu+1}y_i(x)$ by

$$ {}_{\nu+1}\, y_i(x) = {}_\nu y_i(x) + \sum_{\alpha=0}^{s} \int_{x_0}^{x} \frac{(x-\xi)^\alpha}{\alpha!} \left[{}_\nu D_2 D^\alpha y_i\right]_{\xi,{}_\nu y(\xi)} d\xi \qquad (5\text{-}42) $$

where

$$ {}_\nu D_2 \equiv \sum_{k=1}^{n} \{ f_k(x,y_1,\ldots,y_n) - {}_\nu f_k(x) \} \frac{\partial}{\partial y_k} . \qquad (5\text{-}43) $$

(4) Convergence proof

If the functions $D^s f_1(x,y_1,\ldots,y_n),\ldots,\ D^s f_n(x,y_1,\ldots,y_n)$ are continuous and satisfy a Lipschitz condition (5-37) in C, given by Eq.(5-24), then

$$ \lim_{\nu\to\infty} {}_\nu y_i(x) = y_i(x). $$

<u>Proof</u>:

Recall that,

(a) Under these assumptions the existence and uniqueness of the solution are guaranteed, at least for $x\in[x_0,\ x_0+h]$;

(b) If the functions ${}_\nu y_i(x)$ are (s+1)-times differentiable and $(x,\ {}_\nu y_i(x))\in C$ for $x\in[x_0,x_0+h_\nu]$, then ${}_{\nu+1}y_i(x)$ are also (s+1)-times differentiable for $x\in[x_0,x_0+h_\nu]$. The functions ${}_\nu f_i(x)$ are, therefore, s-times differentiable.

Note that

$$\frac{d^\alpha}{dx^\alpha}\int_{x_0}^{x}\frac{(x-\xi)^{\alpha-1}}{(\alpha-1)!}\left[{}_\nu D_2 D^\alpha y_i\right]_{\xi,\,{}_\nu y(\xi)}d\xi = \left[{}_\nu D_2 D^\alpha y_i\right]_{x,\,{}_\nu y(x)}$$

is still $(s-\alpha)$-times differentiable; therefore each term of

$${}_{\nu+1}f_i(x) = f_i(x,{}_\nu y_1(x),\ldots,{}_\nu y_n(x)) + \sum_{\alpha=1}^{s}\int_{x_0}^{x}\frac{(x-\xi)^{\alpha-1}}{(\alpha-1)!}\left[{}_\nu D_2 D^\alpha y_i\right]_{\xi,\,{}_\nu y(\xi)}d\xi,$$

and the sum itself are s-times differentiable. This together with

$${}_\nu y_i(x) = y_{i_0} + \int_{x_0}^{x}{}_\nu f_i(\xi)d\xi$$

proves the assertion.

Thus the iterated functions ${}_{\nu+1}y_1(x),\ldots,{}_{\nu+1}y_n(x)$ remain $(s+1)$-times differentiable in $\bigcap_{\mu=0}^{\nu}[x_0,x_0+h_\mu]$ if the starting functions ${}_0y_i(x)$ have this property. Hence they may be inserted into (5-42).

Now, we first have to show that there exists an interval $[x_0,x_0+h]$ with $h \neq 0$ such that $(x,{}_\nu y_i(x))\varepsilon C$ for $x\varepsilon[x_0,x_0+h]$. (Or in other words, we have to show that there is an $h \neq 0$ such that $[x_0,x_0+h]\subset\bigcap_{\mu=0}^{\infty}[x_0,x_0+h_\mu]$.)

For this we do the following:

Functions which are continuous in the compact domain C are uniformly continuous. The assumptions stated lead to numbers A_0, and $C_0,\ldots,C_s$ such that

$$\left.\begin{aligned} &|f_i(x,y_1,\ldots,y_n)|\leq A_0 && (i=1,\ldots,n)\\ &\left|\left[\frac{\partial}{\partial y_k}D^\alpha y_i\right]_{x,y}\right|\leq C_\alpha && (k=1,\ldots,n)\\ & && (\alpha=0,\ldots,s)\end{aligned}\right\}\qquad (5\text{-}44)$$

for $(x,y_1,\ldots,y_n)\varepsilon C$.

Now let

$$|{}_0f_i(x)| \leqslant B_0 \text{ for } x \varepsilon [x_0, x_0+h]. \tag{5-45}$$

Restricting the interval to $[x_0, x_0+\bar{h}]$, with

$$\bar{h} = \min[a, \frac{b}{B_0}] \qquad \text{(first restriction)}, \tag{5-46}$$

we have the estimate that

$$\left|{}_0y_k(x) - y_{k_0}\right| = \left|\int_{x_0}^{x} {}_0f_k(\xi)d\xi\right| \leqslant B_0\left|x - x_0\right| \leqslant b. \tag{5-47}$$

Thus we obtain the rough bounds

$$|\ {}_0D_2D^{\alpha}y_i \quad {}_{x,y}| \leqslant (A_0 + B_0)nC_{\alpha} = A_{\alpha} \tag{5-48}$$

and $|{}_1f_i(x)| \leqslant |f_i(x, {}_0y_1(x), \ldots, {}_0y_n(x))|$

$$+\left|\sum_{\alpha=1}^{s}\int_{x_0}^{x} \frac{(x - \xi)^{\alpha-1}}{(\alpha - 1)!}[{}_0D_2D^{\alpha}y_i]_{\xi, {}_0y(\xi)}d\xi\right|$$

$$\leqslant \sum_{\alpha=0}^{s} \frac{|x - x_0|^{\alpha}}{\alpha!} A_{\alpha} = B_1. \tag{5-49}$$

Under the assumption $A_0 < B_0$, we restrict the x-interval to $[x_0,\ x_0+h]$, with

$$h = \min[a, \frac{b}{B_0}, \bar{a}] \qquad \text{(second restriction)},$$

where $\bar{a}$ is the positive root of the polynomial

$$p(t) = B_0 - \sum_{\alpha=0}^{s} \frac{t^{\alpha}}{\alpha!}A_{\alpha}.$$

Then $B_1 \leqslant B_0$, from which follows

$$|{}_{\nu+1}\ f_i(x)| \leqslant |f_i(x, {}_{\nu}y_1(x), \ldots, {}_{\nu}y_n(x))|$$

$$+\left|\sum_{\alpha=1}^{s}\int_{x_0}^{x} \frac{(x - \xi)^{\alpha-1}}{(\alpha - 1)!}[{}_{\nu}D_2D^{\alpha}y_i]_{\xi, {}_{\nu}y(\xi)}d\xi\right|$$

$$\leqslant \sum_{\alpha=0}^{s} \frac{|x - x_0|^\alpha}{\alpha!} A_\alpha \leqslant B_0,$$

and

$$\left|{}_{\nu+1}\, y_k(x) - y_{k_0}\right| = \left|\int_{x_0}^{x} {}_{\nu+1}f_k(\xi)d\xi\right| \leqslant b, \tag{5-50}$$

for all $\nu = 0,1,\ldots$. Since, for the estimation of ${}_{\nu+1}f_i(x)$, we have only to repeat the above techniques beginning with (5-45) by considering ${}_\nu f_k$, ${}_\nu y_k$, and ${}_\nu D_2$ instead of ${}_0f_k$, ${}_0y_k$, and ${}_0D_2$ $(k = 1,\ldots,n)$, no further restrictions on the x-interval are necessary to ensure that $|{}_{\nu+1}f_i(x)| \leqslant B_0$.

Now, we show that the sequences of functions ${}_\nu y_i(x)$ converge toward $y_i(x)$. This is easily done, because in $[x_0,x_0+h]$ the main theorem holds. Written with ${}_\nu y_1(x),\ldots, {}_\nu y_n(x)$ as approximate solutions, we have:

$$\begin{aligned} y_i(x) &= {}_\nu y_i(x) + \sum_{\alpha=0}^{s}\int_{x_0}^{x} \frac{(x-\xi)^\alpha}{\alpha!}\left[{}_\nu D_2 D^\alpha y_i\right]_{\xi,{}_\nu y(\xi)} d\xi + {}_\nu R_{is}(x) \\ &= {}_{\nu+1}y_i(x) + {}_\nu R_{is}(x). \end{aligned} \tag{5-51}$$

If in the interval $[x_0,x_0+h]$

$$|y_i(x) - {}_0y_i(x)| \leqslant 2\bar{b}, \qquad (\bar{b}<b)$$

then we have (using the Lipschitz condition (5-37))

$$|y_i(x) - {}_1y_i(x)| = |{}_0R_{is}(x)| \leqslant 2\bar{b}\,\frac{K_s n|x - x_0|^{s+1}}{(s + 1)!}\,.$$

Induction over ν then leads to the following result:

If $\quad |y_i(x) - {}_0y_i(x)| \leqslant 2\bar{b} \quad$ for $\quad x\varepsilon[x_0,x_0+h]$ $(\bar{b}\leqslant b)$

where $\quad b = \operatorname*{Mix}_i|\, y_i - y_{i_0}|,$

then

$$|y_i(x) - {}_\nu y_i(x)| < 2\bar{b}\frac{\left(K_s n|x - x_0|^{s+1}\right)^\nu}{[(s + 1)\nu]!} \qquad \begin{matrix}(i = 1,2,\dots,n)\\(\nu = 0,1,\dots\dots)\end{matrix} \quad (5\text{-}52)$$

To prove this we use (5-51) for the corresponding index ν, and the Lipschitz condition (5-37), and integrate with the help of (5-40)

According to (5-38) the errors decrease with increasing ν.

Hence,

$$\lim_{\nu \to \infty} {}_\nu y_i(x) = y_i(x)$$

for $x\epsilon[x_0,x_0+h]$. This completes the proof of the convergence. Incidentally, if

$$|y_i(x) - {}_0y_i(x)| \leqslant M\frac{|x-x_0|^{m+1}}{(m+1)!}$$

then (5-52) becomes

$$|y_i(x) - {}_0y_i(x)| \leqslant M|x - x_0|^{m+1}\frac{[K_s n|x-x_0|^{s+1}]^\nu}{[m+1+\nu(s+1)]!}. \qquad (5\text{-}52')$$

Next, we try to find the error bounds without knowledge about the solution. We have

$${}_{\nu+1}y_i(x) - {}_\nu y_i(x) = y_i(x) - {}_\nu y_i(x) + {}_{\nu+1}y(x) - y_i(x)$$

$$= {}_{\nu-1}R_{is}(x) - {}_\nu R_{is}(x) \qquad (5\text{-}53)$$

$$= \int_{x_0}^{x}\frac{(x - \xi)^s}{s!}\{[D^{s+1}y_i]_{\xi,{}_\nu y(\xi)} - [D^{s+1}y_i]_{\xi,{}_{\nu-1}y(\xi)}\}d\xi.$$

If for the first iterated functions ${}_1y_1(x),\dots,{}_1y_n(x)$,

then by induction and applying the Lipschitz condition (5-37),(5-38) and (5-40), we have

$$|{}_{\nu+1}y_i(x) - y_i(x)| \leqslant 2b'\frac{(K_s n|x - x_0|^{s+1})^\nu}{[(s + 1)\nu]!}, \qquad (5\text{-}55)$$

where $x\epsilon[x_0,x_0+h]$, $(i = 1,\dots, n)$, $(\nu = 0,1,\dots\dots)$.

Specifically, if

$$|{}_1y_i(x) - {}_0y_i(x)| \leq M\frac{|x-x_0|^{m+1}}{(m+1)!}$$

then Eq.(5-55) becomes

$$|{}_{\nu+1}y_i(x) - {}_\nu y_i(x)| \leq M|x-x_0|^{m+1}\frac{[K_s n|x-x_0|^{s+1}]^\nu}{[m+1+\nu(s+1)]!} \qquad (5\text{-}55')$$

From

$$_\mu y_i(x) - {}_\nu y_i(x) = \sum_{\rho=\nu}^{\mu-1}\{{}_{\rho+1}y_i(x) - {}_\rho y_i(x)\}, \qquad (\mu>\nu)$$

we can write

$$y_i(x) - {}_\nu y_i(x) = \sum_{\rho=\nu}^{\infty}\{{}_{\rho+1}y_i(x) - {}_\rho y_i(x)\}, \qquad (5\text{-}56)$$

and by (5-55') we have

$$|y_i(x) - {}_\nu y_i(x)| \leq M|x-x_0|^{m+1}\sum_{\rho=\nu}^{\infty}\frac{(K_s n|x-x_0|^{s+1})^\rho}{(m+1)![(s+1)\rho]!}. \qquad (5\text{-}57)$$

Similarly, we can iterate the main formula Eq.(5-27). Recall that, ${}_0y_i(x) = \hat{y}_i(x)$, and ${}_1y_i(x) = \bar{y}_i(x)$. Thus, from (5-27) we have

$$\begin{aligned} y_i(x) - \bar{y}_i(x) &= R_{is}(x) \\ &= \int_{x_0}^{x}\frac{(x-\xi)^s}{s!}\{[D^{s+1}y_i]_{\xi,y(\xi)} - [D^{s+1}y_i]_{\xi,\hat{y}(\xi)}\}d\xi \\ &= \int_{x_0}^{x}\frac{(x-\xi)^s}{s!}\{[D^{s+1}y_i]_{\xi,\bar{y}(\xi)} - [D^{s+1}y_i]_{\xi,\hat{y}(\xi)}\}d\xi \\ &\quad + \int_{x_0}^{x}\frac{(x-\xi)^s}{s!}\{[D^{s+1}y_i]_{\xi,y(\xi)} - [D^{s+1}y_i]_{\xi,\bar{y}(\xi)}\}d\xi \\ &= R^*_{is}(x) + R^{**}_{is}(x). \end{aligned} \qquad (5\text{-}58)$$

We see that $R^{**}_{is}(x)$ is of higher order than $R^*_{is}(x)$, Hence, in actual computation we can use $R^*_{is}(x)$ instead of $R_{is}(x)$.

5.4 Numerical Consideration

Recurrence Relationships

Recurrence Relationships of Simple Derivatives. In a system of differential equations, like

$$\frac{dy_i}{dx} = f_i(x,y_1,\ldots, y_n), \quad (i = 1,\ldots, n),$$

one can display each f_i as the result of the final operation in a finite sequence of arithmetic operations, each of which defines a partial result

$$R_s(x,y_1,\ldots,y_n) = P_s(x,y_1,\ldots,y_n)*Q_s(x,y_1,\ldots,y_n),$$

where * represents one of the arithmetic operations +, −, ·, /, etc. and $s = 1, 2, \ldots, S_i$ is a finite sequence.

We want to apply the appropriate rule for the j^{th} derivative of a sum, difference, product, or quotient etc., corresponding to each partial result $R_s(x,y_1,\ldots,y_n)$ so that we obtain a sequence of subroutines for each $d^jR_s(x,y_1,\ldots,y_n)/dx^j$, and hence for each $f_i^{(j)}$, since $d^{j+1}y_i/dx^{j+1} = f_i^{(j)}$.

Let us abbreviate $\frac{1}{j!}\frac{d^jR}{dx^j}$ of a function $R(x,y_1,\ldots,y_n)$ as $(R)_j$. Then for two functions $P(x,y_1,\ldots,y_n)$, and $Q(x,y_1,\ldots,y_n)$ we have

$$(P\pm Q)_j = (P)_j \pm (Q)_j, \quad (j = 1,2,\ldots, J)$$

$$(PQ)_j = \sum_{r=0}^{j} (P)_r(Q)_{j-r},$$

and

$$\left(\frac{P}{Q}\right)_j = \frac{1}{Q}\left[(P)_j - \sum_{r=0}^{j}(P)_r(Q)_{j-r}\right].$$

The second relationship follows directly from the expression

$$\frac{d^j}{dx^j}(PQ) = \sum_{r=0}^{j} \binom{j}{r} P^{(r)} Q^{(j-r)}.$$

Thus,
$$\frac{1}{j!}\frac{d^j}{dx^j}(PQ) = \sum_{r=0}^{j} \frac{1}{r!} P^{(r)} \frac{1}{(j-r)!} Q^{(j-r)}$$

$$= \sum_{r=0}^{j} (P)_r (Q)_{j-r},$$

and so
$$(PQ)_j = \sum_{r=0}^{j} (P)_r (Q)_{j-r}.$$

The third relationship is the consequence of direct application of the second relationship.

Each of the quantities P and Q in the above expressions can be: (1) a constant, (2) one of the variables y_i, or (3) one of the intermdiate results R.

These relationships can be found in the literature of the mid-50's[112]. More relationships were established in the 60's, mostly due to Gibbons[113] or Moore[114], and Leawitt[115].

Since for any function F, we have $((F)_1)_j = (j+1)(F)_{j+1}$, then for a system of differential equations

$$\frac{dy_i}{dx} = f_i(x, y_1, \ldots, y_n),$$

we add
$$(y_i)_{j+1} = \frac{1}{j+1}(f_i)_j$$

with which the procedure can be repeated, leading to a recurrent calculation of the expression $(y_i)_{j+1}$.

<u>Recurrence Relationships for Expressions with Two Differential Operators</u>. As was shown in Eq.(5-27), combining the perturbation method with the Lie series method gives rise to a D_2D^{α} term in the expression. This requires additional recurrence relationships to carry out the calculation of the differentiation involving the operations D_2D^{α}.

Let the second differential operator be represented by $\bar{D}$. Operating with $\bar{D}$ on

$$(R)_j = \frac{1}{j!} D^j R(x, y_1, \ldots, y_n),$$

we get

$$\bar{R}_j = \bar{D}(R)_j .$$

Applying the operator $\bar{D}$ to the formulas obtained for D^j, and using the notation we just introduced, we have:

for a sum or difference $r = p \pm q$,

$$\bar{R}_j = \bar{P}_j \pm \bar{Q}_j ;$$

for a product $r = p \cdot q$,

$$R_j = \sum_{r=0}^{j} \left[P_r (Q)_{j-r} + (P)_r \bar{Q}_{j-r} \right] ;$$

and for a quotient $r = p/q$,

$$\bar{R}_j = \{ \bar{P}_j - \sum_{r=0}^{j} \left[\bar{P}_r (Q)_{j-r} + (P)_r \bar{Q}_{j-r} \right] - (P)_j Q_0 \} / (Q)_0 .$$

In deriving this expression, we first multiplied the simple recurrence relationship by $(Q)_0$, then operated $\bar{D}$ on it.

Finally,

$$\bar{y}_{i,\mu+1} = \frac{1}{\mu+1} \bar{f}_{i,\mu} .$$

The operator $\bar{D}$ here can be either (1) D_2, or (2) $\frac{\partial}{\partial y_k}$, which appears in a simple Lie series expansion. In the first case, we have the expression

$$\bar{y}_{i,\alpha} = \frac{D_2 D^{\alpha} y_i}{\alpha!},$$

and it has to be started with the initial values

$$\bar{y}_{i,0} = D_2 y_i = f_i(x,y_1,\ldots,y_n) - \hat{f}_i(x,y_1,\ldots,y_n).$$

In the second case we have

$$\bar{y}_{i,\alpha} = \frac{1}{\alpha!}\frac{\partial}{\partial y_k} D^{\alpha} y_i.$$

Here, it starts with

$$\bar{y}_{i,0} = \begin{cases} 1, & \text{for } i = k, \\ 0, & \text{for } i \neq k. \end{cases}$$

In both events, if we encounter the independent variable x, or a constant, the results vanish. That is,

$$\bar{X}_0 = \bar{X}_1 = \ldots = 0,$$

and

$$\bar{C}_0 = \bar{C}_1 = \ldots = 0.$$

(2) The concept of transfer matrix.

Let $y_i(x)$ be solutions of the given differential equations for the initial values y_{k_0}. The matrix

$$H(x) = \left[H_{ik}(x)\right] = \left[\frac{\partial y_i(x)}{\partial y_{k_0}}\right] \qquad (5\text{-}62)$$

which consists of the derivatives of the solutions $y_i(x)$ with respect to the k^{th} initial value y_{k_0}, is then called the transfer matrix pertaining to y(x).

In other words, the transfer matrix describes, in first approximation, the variation of the solutions y_i at the point x if the initial values y_{k_0} are changed. When we change the initial values y_{i_0}, by ε_{i_0}, the solutions y_i at the point x will in the first approximation change by

$$\varepsilon_i(x) = \frac{\partial y_i(x)}{\partial y_{1_0}}\varepsilon_{1_0} + \dots + \frac{\partial y_i(x)}{\partial y_{n_0}}\varepsilon_{n_0}.$$

Thus,

$$\begin{bmatrix} \varepsilon_1(x) \\ \cdot \\ \cdot \\ \varepsilon_n(x) \end{bmatrix} = \begin{bmatrix} \dfrac{\partial y_i(x)}{\partial y_{1_0}} & \dots\dots\dots & \dfrac{\partial y_1(x)}{\partial y_{n_0}} \\ \cdot & \dots\dots\dots & \cdot \\ \dfrac{y_n(x)}{\partial y_{1_0}} & \dots\dots\dots & \dfrac{y_n(x)}{\partial y_{n_0}} \end{bmatrix} \begin{bmatrix} \varepsilon_{1_0} \\ \cdot \\ \cdot \\ \varepsilon_{n_0} \end{bmatrix}$$

or, in vector form,

$$\varepsilon(x) = H(x)\varepsilon_0. \tag{5-63}$$

From (5-63) we see that errors committed in the numerical integration can be mapped to any fixed point by means of the transfer matrices. These also describe how an error committed at a certain place influences the final result.

(3) Step size control.

There are three possible schemes for step size control. These are: (a) constant step size, (b) step size control using the local truncation errors, and (c) optimal step size control.

The idea of step size control using the local truncation errors is to choose a step size which keeps the local truncation errors below the preset bounds. Let $R_{is}(x_0+h)$ be the remainders of the solution $y_i(x_0+h)$, and γ_i the desired sizes of the local truncation errors. Then we define the error ratio as

$$\eta = \operatorname*{Max}_{i} \frac{|R_{is}(x_0+h)|}{\gamma_i} .$$

For a control of the step size h, one tries to keep $\eta \approx 1$.

From Eq.(5-39), it shows that $R_{is}(x_0+h)$ is on the order of m+s+2; it follows that η is proportional to h^{m+s+2}. That is, if we calculate one step with step size h, then

$$\eta \propto h^{m+s+2}.$$

Suppose $\bar{h}$ is the second step size, and $\bar{\eta}$ the error ratio for the second step, and $\bar{\eta} \propto \bar{h}^{\,m+s+2}$,

then
$$\frac{\eta}{\bar{\eta}} = \left(\frac{h}{\bar{h}}\right)^{m+s+2}.$$

Since we want to keep $\bar{\eta} \approx 1$, then

$$\bar{h} = h \sqrt[m+s+2]{\frac{1}{\eta}} ,$$

and we see that we have the equation of a parabola. One can plot $\bar{h}$ against η as shown in the following figure.

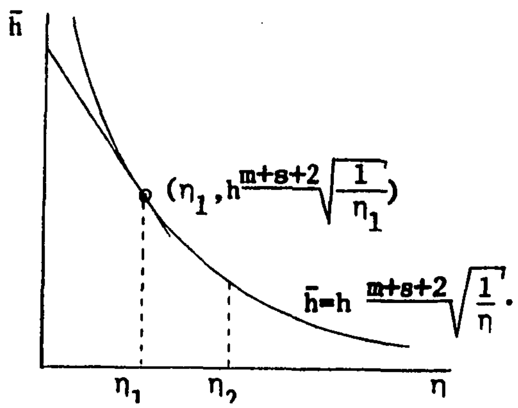


Fig. 3. Step Size Control Using the Local Truncation Error.

If $\eta > 1$, it is evident from the definition of η that the remainder is greater than the desired size of the local truncation error we set. In this case the step size, $\bar{h}$, is recalculated. Otherwise, the original $\bar{h}$ is taken as the step size for the calculation of the solution.

If $\eta << 1$, as is indicated by the sketch, a too-rapid increases of the step size occurs and causes oscillation in the solution. Thus, from a practical point of view, we choose upper and lower limits of η, for example, $\eta_1 = 0.1$ or 0.01, and $\eta_2 = 1.1$ or 1.2.

For $\eta_1 < \eta < \eta_2$ we use the formula

$$\bar{h} = h \sqrt[m+s+2]{\frac{1}{\eta}}$$

to calculate the step size; and for $\eta < \eta_1$, we replace the parabola by its tangent at $\eta = \eta_1$, that is,

$$\bar{h} = h \frac{m+s+3}{m+s+2} \sqrt[m+s+2]{\frac{1}{\eta_1}} \{1 - \frac{\eta}{(m+s+3)\eta_1}\}.$$

Optimal step size control: From the concept of the transfer matrix if e_j is an error committed at the point x_j, then its propagation at x_N is given by

$$e^{(N)} = H(x_N)H^{-1}(x_j)e_j.$$

We want to know how the step sizes $h_1, h_2, \ldots, h_N$ are to be chosen to give minimal final error.

Neglecting rounding errors, we may write that

$$e_j = \psi_j h_j^{p+1}$$

where p is the order of the method, that is m+s+1. Then, we have the relationship that

$$e^{(N)} = H(x_N)H^{-1}(x_j)\psi_j h_j^{p+1} = \chi_j h_j^{p+1}.$$

We try to minimize

$$\sum_{j=1}^{N} e_j^{(N)} = \sum_{j=1}^{N} \chi_j h_j^{p+1}$$

under the condition that

$$\sum_{j=1}^{N} h_j = x_N - x_0$$

$$= \text{constant}.$$

The method of Lagrange-multipliers yields the result

$$h_j = \frac{C_1}{\sqrt[p]{\chi_j}}.$$

Thus, for local error we can write

$$e_j = \psi_j h_j^{p+1}$$

$$= \frac{\psi_j C_2}{\chi_j \sqrt[p]{\chi_j}}$$

$$= \frac{C_3 h_j}{H(x_N) H^{-1}(x_j)},$$

that is, the h_j are optimal if

$$\frac{H(x_N) H^{-1}(x_j) e_j}{h_j} \simeq C_3.$$

In other words, if the contribution of each step to the final result is proportional to its step size, the total truncation error will be the minimum. In the above expression, $H(x_N)$ is a constant and need not be known at the moment, but it can be calculated when the solution at x_N

is obtained, since $H(x_N) = \partial y(x_N)/\partial y_0$.

Thus, for optimal step size, one has to keep

$$\frac{H^{-1}(x_j)e_j}{h_j} \simeq k,$$

a constant.

From this we also find that the propagated error of e_j at x_N is

$$h_j H(x_N)k,$$

and the total error at x_N due to the propagation of local error is estimated at

$$E \simeq \sum_{j=1}^{N} h_j H(x_N)k$$

$$= (x_N - x_0)H(x_N)k.$$

5.5 Application of the Lie Series Method to the Reflection Equation

In order to perform an integration in Eq.(4-3), we employed the Gauss-Legendre quadrature formula

$$\int_0^1 f(x)dx \simeq \sum_{k=1}^{N} c_k f(x_k),$$

where c_k are the Christoffel numbers, defined by

$$c_k = \int_0^1 \frac{P_N(x)dx}{(x-x_k)P_N'(x_k)}, \quad k = 1,2,\ldots, N,$$

and x_k are roots of Legendre polynomial of degree N, Eq.(4-3) is then replaced by the approximate equation

$$\frac{dr}{dx} \simeq \frac{1}{\gamma}\Big[1+ \frac{1}{2}\sum_{k=1}^{N} \frac{c_k}{\mu_k''} r(x,\mu,\mu_k'')\Big]\Big[1+ \frac{1}{2}\sum_{k=1}^{N} \frac{c_k}{\mu_k'} r(x,\mu_k',\mu_0)\Big] - \left(\frac{1}{\mu_0} + \frac{1}{\mu}\right)r \quad (5\text{-}64)$$

with the same initial condition,

$$r(0,\mu,\mu_0) = 0.$$

The sets $\{\mu_k''\}$ and $\{\mu_k'\}$ are the same. In Eq.(5-64) discretize μ and μ_0 by letting μ assume only the values $\mu_1', \mu_2', \ldots, \mu_N'$ and μ_0 only the values $\mu_1'', \mu_1'', \ldots, \mu_N''$ and define

$$r_{ij}(x) = r(x,\mu_i',\mu_j'').$$

Eq.(5-64) then is replaced by the following set of N^2 nonlinear ordinary differential equations:

$$\frac{dr_{ij}}{dx} = \frac{1}{\gamma}\left[1 + \frac{1}{2}\sum_{k=1}^{N} \frac{c_k}{\mu_k''} r_{ik}\right]\left[1 + \frac{1}{2} \sum_{k=1}^{N} \frac{c_k}{\mu_k'} r_{kj}\right] - \left(\frac{1}{\mu_i'} + \frac{1}{\mu_j''}\right) r_{ij}\,, \qquad (5\text{-}65)$$

with the initial conditions

$$r_{ij}(0) = 0.$$

By the symmetry condition,

$$r_{ij}(x) = r_{ji}(x), \quad i,j = 1,2,\ldots, N.$$

This allows one to reduce the dimension of (5-65) from N^2 to $N(N+1)/2$.

In applying the Lie series perturbation method to Eq.(5-65), first three terms of a Taylor series expansion of the functions were taken as the solutions to the perturbed system, which has the same initial conditions as the given equations, and make use of it as a starting solutions in the iteration method, in which three terms of the Lie series expansion were used in each process as correction.

In order to compare the results of the Lie series method with that of the Runge-Kutta method given by Bellman, seven nodes were used in the Gauss-Legendre quadrature formula.

CHAPTER VI

RESULTS AND CONCLUSIONS

In this investigation calculations were made for $\gamma = 0.1$, 0.6, and 1.0. The results* for $\gamma = 0.1$ are in exact agreement with those from Bellman's Runge-Kutta method, and the results from $\gamma = 0.6$ and 1.0 deviate slightly or not at all (the maximum percentage deviation from Bellman's results is found to be less than - 0.02%).

No direct comparison of the speed of calculation for both methods has been made. But from indirect estimation, the iterative Lie series perturbation method is many times faster than the Runge-Kutta fourth-order method. The comparison is based on the following facts and assumptiom:

1. The speed ratio of UNIVAC 1108 to IBM 360/50 computers in solving a restricted-three-body-problem by the Lie series method is about 3.

2. The time for solving a two-dimensional reflection function, with five y-divisions, five polar angles, five azimuthal angles (125 equations), and 25 x-steps by the Runge-Kutta method is about one hour on UNIVAC 1107 computer[75]. It is about 1.663 sec./Eq.-step on UNIVAC 1107.

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* The results from the computer output are the solutions of the symmetric functions $r(x,\mu_i,\mu_j)$. In order to compare with the results given by Bellman, we have to divided these figures by four times the cosine of angle of reflection.

3. The speed of UNIVAC 1107 and 1108 computers are assumed as the same.

4. The rate for solving a one-dimensional reflection function with seven polar angles, or 28 equations by the Lie series perturbation method is about 0.938 sec./Eq.-step on IBM 360/50 computer.

The iterative Lie series perturbation method can be operated with some flexibility. In addition to the calculation of the reflection function with three γ values at the same order of the method, the effect of other parameters on the accuracy of the results and on the computing time were also compared. These comparsions were (1) double precision vs. single precision, and (2) varying the order of the method.

Based on the three point approximate reflection function, the first three terms in Taylor series expansion were taken as first approximate solutions. Three terms in the Lie series expansion are used in each iteration as a correction, with $\gamma = 0.1$, with the thickness of the slab 0.6(0.6)3.0 mean free paths (mfp), and with the lower bound of local truncation errors of 10^{-8}. The results from single precision calculation tend to be smaller than that those from double precision calculation by the order of about 10^{-6}, as shown in Table 1. The computing time in double precision calculation is about 1.66 times that of single precision.

Thus, for a slab thickness of 3 mfp or less, a double precision calculation seems to be unnecessary.

The purpose of using double precision calculations is to reduce the rounding errors. Thus for a thick slab, say 20 mfp, and $\gamma = 1.0$, the differences in the results of double precision and single precision calculations still need to be investigated.

Table 2 shows the comparison of the results by varying the order of the method. These results are also from three-point approximate reflection functions, with $\gamma = 0.1$, the preset bound of local truncation errors are 10^{-8}. We see that oscillation occurs at almost every solution. In this case, the proportion of the computing time for the order of the method of 7, 10, 15, and 20 is 1 : 1.21 : 1.26 : 1.56.

The oscillation occurring in the solution can be reduced by tightening the error bound. Table 3 shows how the situation was improved by choosing a proper error bound for each method. It is because when we use the order of the method of p, and if the step size is on the order of 10^{-1}, then the truncation error involved in the solutions is on the order of $10^{-(p+1)}$. If the error bound we selected is greater than $10^{-(p+1)}$, than a bigger step size will be taken in the next step of calculation. This leads to a less accurate result, or causes the oscillation in the solutions.

In the actual computations, the step size often varies from 10^{-1} to 10^{-2}. Thus, for a given order of the method p, the truncation errors can be varied from the order of $10^{-(p+1)}$ to $10^{-2(p+1)}$. Because the error bound remains constant throughout the computation, small oscillations in the solutions are inevitable in the Lie series method.

The proportion of the computation time in the later case, for the order of the method of 7, 10, 15, and 20 is 1 : 1.38 : 1.69 : 2.11.

The computing time can be varied appreciably in the same order of the method. The order of the method is defined as $p \equiv m + s + 1$, here m is the order of the Taylor series expansion, and s is the order of the Lie series correction. Generally speaking, for a given order of the method, the more terms taken from the Taylor series, the better the starting solution we get, and the fewer terms needed in the Lie

Table 1

Comparison of the Results
from
Double Precision and Single Precision Calculations

	Double Precision	Single Precision	Relative Error	I,J*
r1(0.6)	.5813125D-02	.5813103D-02	-3.96E-06	1,1
r2(0.6)	.9566489D-02	.9566460E-02	-3.03E-06	1,2
r3(0.6)	.2859589D-01	.2859582E-01	-2.45E-06	1,3
r4(0.6)	.2381252D-01	.2381246E-01	-2.52E-06	2,2
r5(0.6)	.2884260D-01	.2884253E-01	-2.43E-06	2,3
r6(0.6)	.3553846D-01	.3553836E-01	-2.81E-06	3,3
r1(3.0)	.5826996D-02	.5826946E-02	-8.58E-06	1,1
r2(3.0)	.9624760D-02	.9624705E-02	-5.71E-06	1,2
r3(3.0)	.3467144D-01	.3466594E-01	-1.58E-04	1,3
r4(3.0)	.2647722D-01	.2647680E-01	-1.58E-05	2,2
r5(3.0)	.3508860D-01	.3508293E-01	-1.61E-04	2,3
r6(3.0)	.5041230D-01	.5034984E-01	-1.23E-03	3,3

* I,J = 1,2, and 3 represent the cosine of the polar angles:
1 is .1127017, 2 is .5000000, and 3 is .8872983.

Table 2

Comparison of the Results by Varying the Order of the Method

(with the error bound of 10^{-8})

	Order of 20	Relative Errors Order of 15	Order of 10	Order of 7	I,J*
r1(0.6)	0.5813107E-02	-0.69E-06	0.00	-0.69E-06	1,1
r2(0.6)	0.9566456E-02	0.00	+0.42E-06	+0.42E-06	1,2
r3(0.6)	0.2859852E-01	0.00	0.00	0.00	1,3
r4(0.6)	0.2381246E-01	+0.42E-06	-0.42E-06	0.00	2,2
r5(0.6)	0.2884254E-01	0.00	-0.35E-06	-0.35E-06	2,3
r6(0.6)	0.3553836E-01	0.00	-0.28E-06	0.00	3,3
r1(3.0)	0.5826958E-02	+0.51E-06	-0.69E-06	-0.21E-05	1,1
r2(3.0)	0.9624705E-02	-0.42E-06	0.00	0.00	1,2
r3(3.0)	0.3466954E-01	0.00	+0.29E-06	0.00	1,3
r4(3.0)	0.2647679E-01	+0.38E-06	0.00	+0.38E-06	2,2
r5(3.0)	0.3508290E-01	+0.86E-06	+0.86E-06	+0.86E-06	2,3
r6(3.0)	0.5034985E-01	0.00	0.00	-0.20E-06	3,3

* I,J = 1, 2, and 3 represent the cosine of the polar angles:
1 is .1127017, 2 is .5000000, and 3 is .8872983.

Table 3

Comparison of the Results by Varying the Order of the Method

(with the error bound of $10^{-(p+1)}$)

		Relative Errors			
	Order of 20	Order of 15	Order of 10	Order of 7	I,J
r1(0.6)	0.5813096E-02	+0.52E-06	+1.89E-06	+1.20E-06	1,1
r2(0.6)	0.9566449E-02	+0.31E-06	+1.15E-06	+1.15E-06	1,2
r3(0.3)	0.2859805E-01	+0.35E-06	+0.70E-06	+0.70E-06	1,3
r4(0.6)	0.2381242E-01	+0.84E-06	+1.68E-06	+1.68E-06	2,2
r5(0.6)	0.2884251E-01	+0.35E-06	+0.69E-06	+0.69E-06	2,3
r6(0.6)	0.3553833E-01	+0.56E-06	+0.84E-06	+0.84E-06	3,3
r1(3.0)	0.5826954E-02	+0.69E-06	+1.20E-06	-1.37E-06	1,1
r2(3.0)	0.9624701E-02	0.00	+0.42E-06	+0.42E-06	1,2
r3(3.0)	0.3466590E-01	0.00	+1.15E-06	+1.15E-06	1,3
r4(3.0)	0.2647677E-01	0.00	+0.76E-06	+1.13E-06	2,2
r5(3.0)	0.3508287E-01	0.00	+1.41E-06	+1.71E-06	2,3
r6(3.0)	0.5034978E-01	0.00	+1.19E-06	+1.19E-06	3,3

*I,J = 1, 2, and 3 represent the cosine of the polar angles:
1 is .1127017, 2 is .5000000, and 3 is .8872983.

Table 4

Comparison of the Result from the Order of the Method of 15
by Taking the Different Starting Solutions*

	Case I	Case II
r1(0.6)	.5813103E-02	.5813103E-02
r2(0.3)	.9566460E-02	.9566456E-02
r3(0.3)	.2859853E-01	.2859852E-01
r4(0.3)	.2381246E-01	.2381247E-01
r5(0.3)	.2884253E-01	.2884254E-01
r6(0.3)	.3553836E-01	.3553836E-01
r1(3.0)	.5826961E-02	.5826961E-02
r2(3.0)	.9624705E-02	.9624701E-02
r3(3.0)	.3466594E-01	.3466594E-01
r4(3.0)	.2647697E-01	.2647680E-01
r5(3.0)	.3508291E-01	.3508293E-01
r6(3.0)	.5034984E-01	.5-34985E-01

* In case I, first 10 terms of the Taylor series were taken as the starting solutions, and first 4 terms of the Lie series were taken as the correction. In case II, first 11 terms of the Taylor series were taken as the starting solutions, and first 3 terms of the Lie series were taken as the correction. The computing time in the Case I and Case II are 0.5702 and 0.4375 second per step per equation respectively.

series correction process. Since it is easier to calculate the Taylor series than the Lie series, in the actual computation, it is favorable to take as many terms in the Taylor series as possible. For example, an order of the method of 15 (partial results are listed in Table 4), in one case, we use the first 11 terms in the Taylor series as a starting solution, and the first 3 terms in the Lie series as correction. The computing time is 0.4375 sec./step-Eq.; in another case, we use the first 10 terms in the Taylor series as starting solution, and the first 4 terms in the Lie series as correction. The computing time is 0.5702 sec./step-Eq. The ratio of the computing time for the two cases is 1.3. The differences in the computing time will widen if a higher order of quadrature formula is used, and if more terms of the Lie series are calculated, as one might expect.

The symmetrical property of the reflection function $r_{ij}(x)$ with respect to its angular arguments is best explained by the reciprocity property of transport function. In terms of the principle of reciprocity[12], the symmetrical reflection function can be explained as follows:

(1) The interchange of the direction of incidence and the direction of reflection of particles effects the interchange of the magnitude of source strength and the magnitude of the reflection function.

(2) For a given thickness of the slab, the component of the reflected flux normal to the slab surface in direction Ω due to a given strength of incident beam in direction Ω_0 is the same as the normal component of the flux of the reflected beam in the direction $-\Omega_0$ due to the same strength of the incident beam in the direction $-\Omega$.

Figs. 4 through 7 show the normalized reflection curves in the polar coordinate. Only two limiting cases for a given value of γ are plotted. These results are obtained from symmetrical reflection functions divided by four times the cosine of reflection angle, then normalized. An interesting characteristic common to these is that no matter what direction the incident particle may come from, the maximum reflection flux always occurs at the maximum reflection angle.

Figs. 8 and 9 show the typical normalized reflection curves, obtained by normalizing the symmetrical reflection functions. We see that for γ of 0.1 all the curves level off near the slab thickness of 3.0 mfp, and for γ of 0.6, the curves level off near the slab thickness of 4.0 mfp.

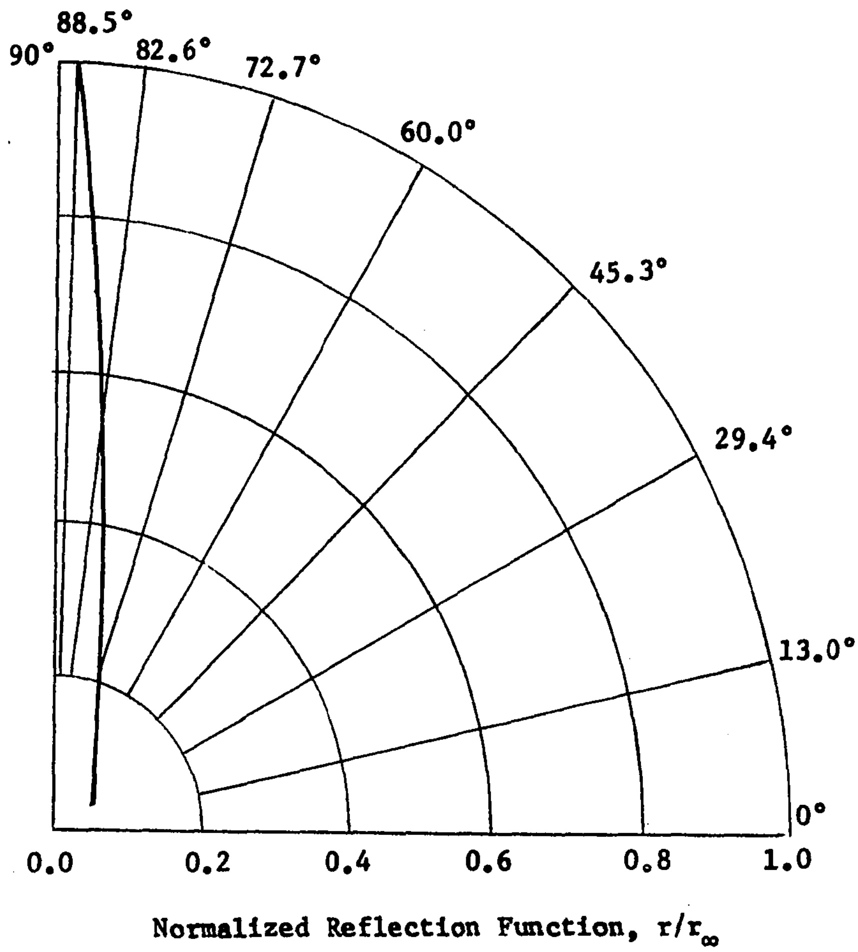

Fig. 4. Normalized Reflection Curve with Gamma = 0.1, the Angle of Incidence of 88.5°, and the Slab Thickness of 0.1 mfp.

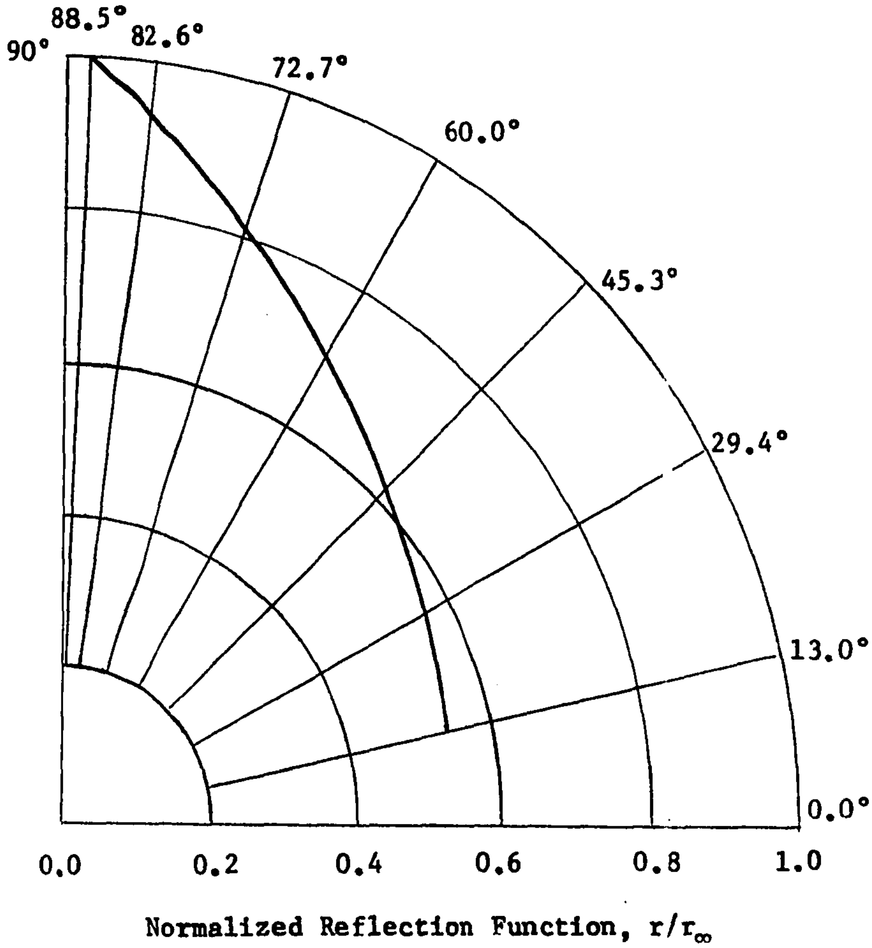

Fig. 5. Normalized Reflection Curve with Gamma = 0.1, the Angle of Incidence of 13.0°, and the Slab Thickness of 3.0 mfp.

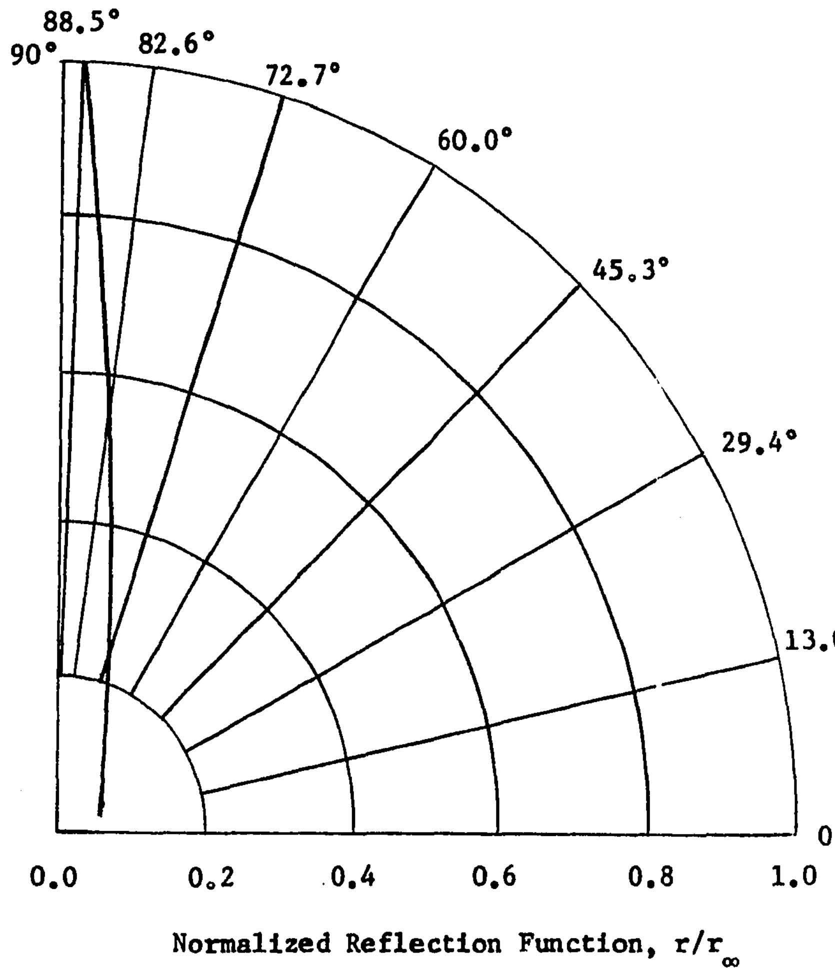

Fig. 6. Normalized Reflection Curve with Gamma = 0.6, the Angle of Incidence of 88.5°, and the Slab Thickness of 0.1 mfp.

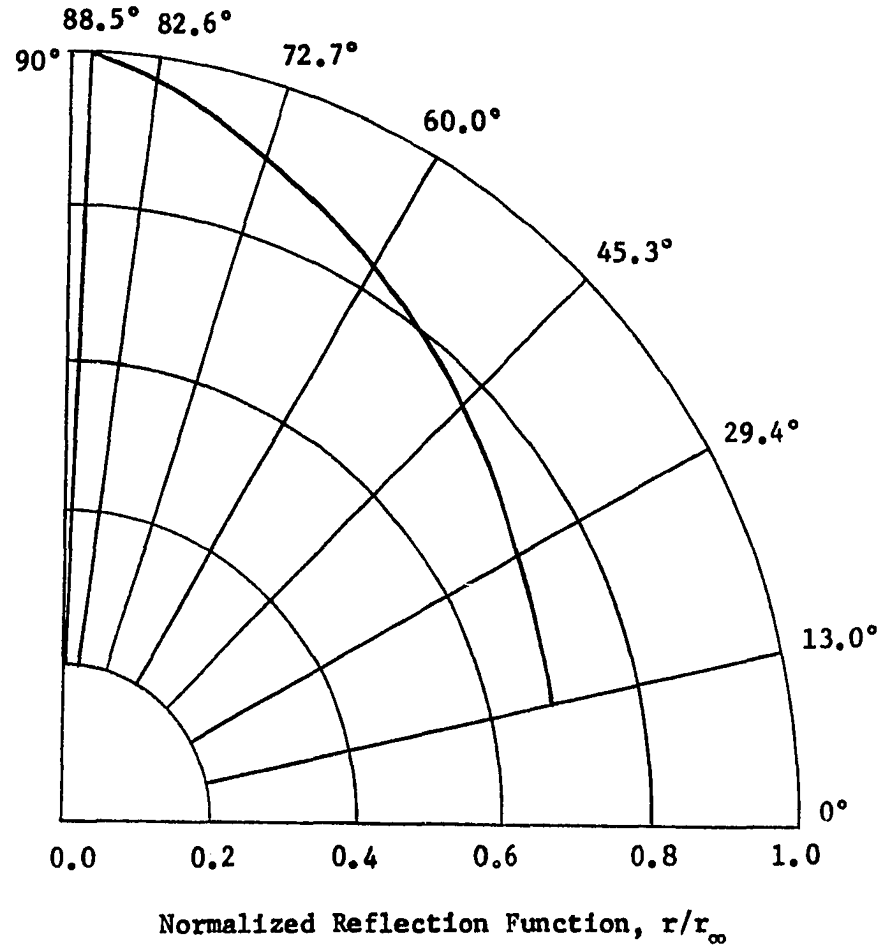

Fig. 7. Normalized Reflection Curve with Gamma = 0.6, the Angle of Incidence of 13.0°, and the Slab Thickness of 3.0 mfp.

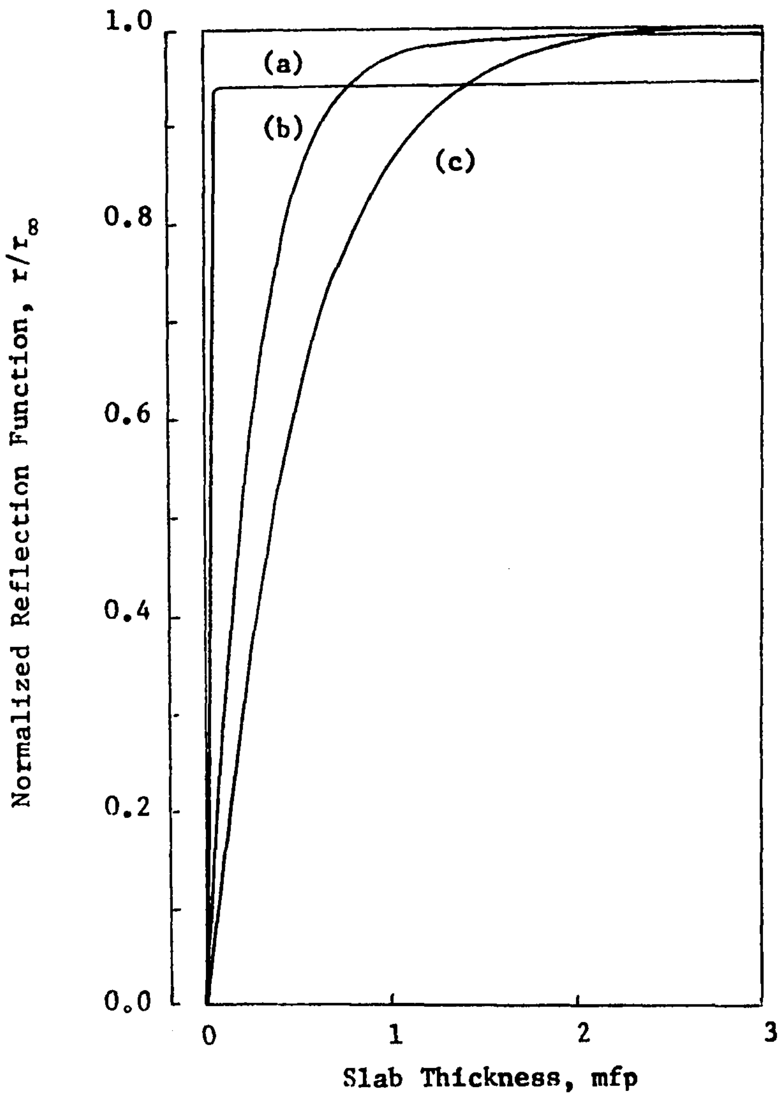

Fig. 8. Normalized Reflection Curves with Gamma = 0.1, and the Angle of Incidence/Angle of Reflection are: (a) 13.0°/13.0°, (b) 60.0°/60.0°, and (c) 88.5°/88.5°.

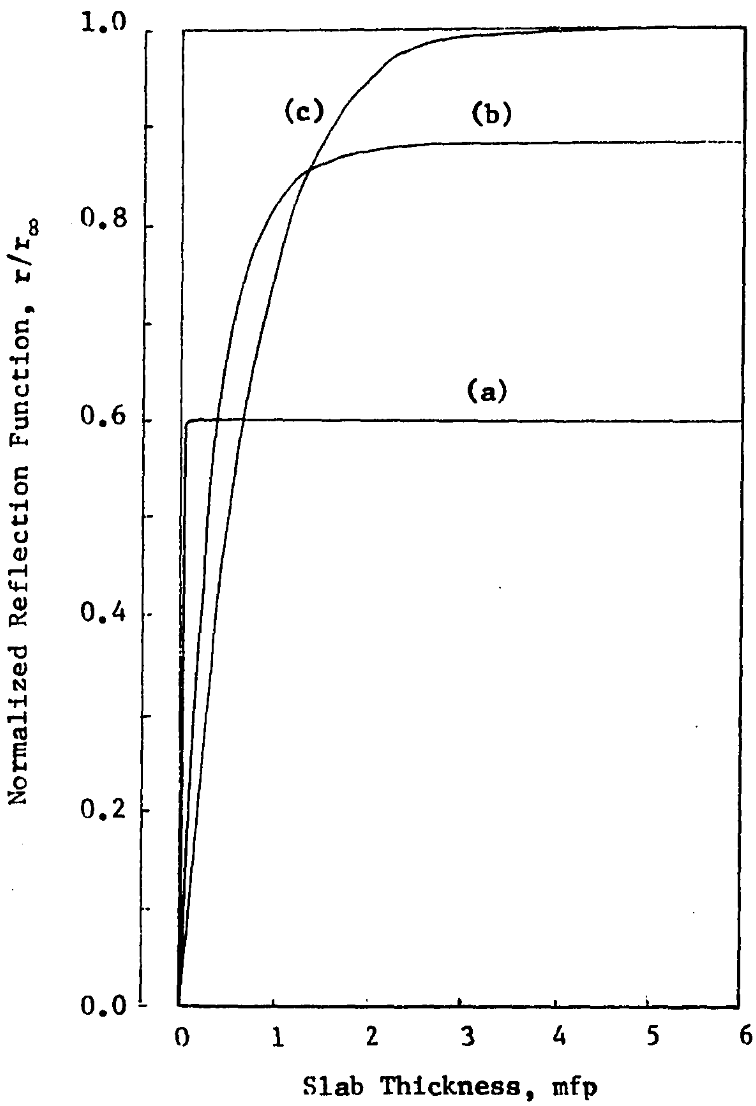

Fig. 9. Normalized Reflection Curves with Gamma = 0.6, and the Angle of Incidence/Angle of Reflection are: (a) 13.0°/13.0°, (b) 60.0°/60.0°, and (c) 88.5°/88.5°.

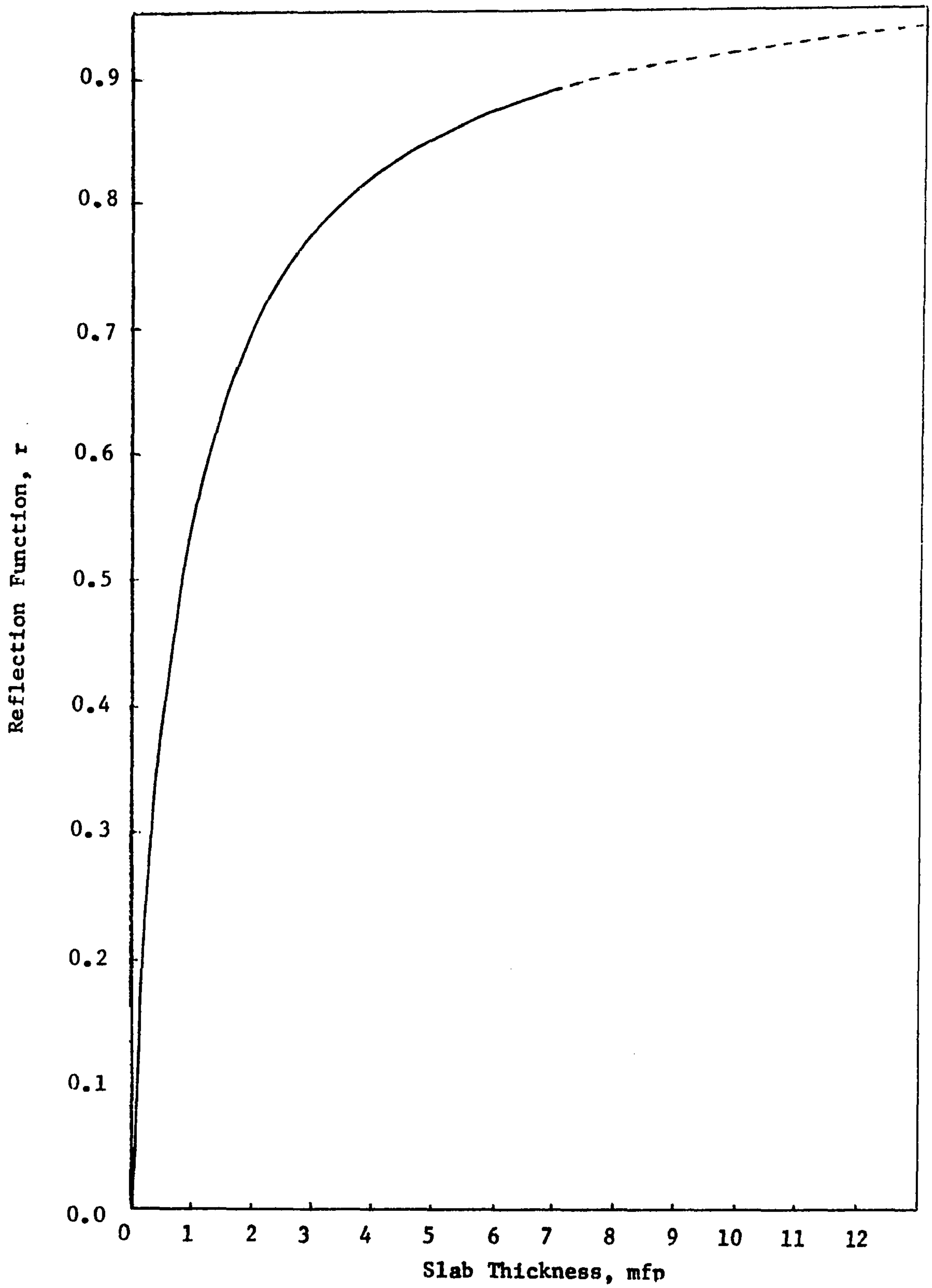

Fig. 10. Reflection Curve with Gamma = 1.0, and the Angle of Incidence = Angle of Reflection = 60.0°.

From these results the following conclusions are reached:

1. The iterative Lie series perturbation method is effective in solving the invariant imbedding transport equations. It can give results of arbitrarily high accuracy by simply increasing the order of the method, provided that a sufficient amount of time is expended on their calculation. Also, because of the rapid convergence of this method, the same accuracy can be obtained as with the Runge-Kutta method, at much faster speed. But in the Lie series method, a bigger computer memory is required(248K vs. 32K).

2. We cannot estimate the rounding errors involved in the calculation by the iterative Lie series method, but decreases in rounding errors are expected with the variable step size controls in comparison with the fixed step size calculations.

3. The invariant imbedding reflection functions satisfy a reciprocity relation.

4. The reflection function increases with increasing angle between the direction of normal to the slab and the direction of reflection while the other paramters are fixed.

5. For γ less than or equal to 0.6, a slab thickness of 4.0 mfp is essentialy equivalent to infinite thickness.

CHAPTER VII

RECOMMENDATIONS

1. The effectiveness of the Lie series perturbation method in solving the one-dimensional invariant imbedding reflection function implies that it can be applied equally well to similar problem of greater complexity. Thus, to gain more knowledge of the usefulness of the Lie series method in invariant imbedding transport calculations, it is worthwhile to recalculate the two-dimensional problem and the problem of gamma ray penetration.

2. The invariant imbedding formulation of the transport equation for cylindrical geometry with finite height has not been done yet. Since it is of practical importance, any efforts to derive the transport equation for this geometry should be encouraged.

3. One advantage of iterative Lie series perturbation method, as indicated by the transfer matrix, is its capability to calculate the total truncation error. There is no existing numerical method to estimate the total truncation error easily for the problem of this complexity. Thus if one is interested in knowing the total truncation error in his solution he may start from the Lie series perturbation method.

4. Although the Lie series perturbation method allows easy estimation of the truncation error, it is even better if one can confine the error to a certain interval. Such a method has been proposed by Moore(114), and a

computer program has been written by Ladner and Yohe[117]. In order to find a better numerical method to solve the invariant imbedding transport equation, the method of interval analysis should be investigated.

5. According to Bellman's result, for γ equal to 1.0, in a 20 mfp slab, the reflection function decreases with increasing the angle of reflection. The reason for the inversion of this property is not clear, a further study seems to be needed.

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Appendix A

Description and Listing of the Computer Programs

The Lie series computer programs were originally written by Mathematics Research Center, The University of Wisconsin[116], in Fortran V, for UNIVAC 1108 computer. Because of extreme machine dependence of the subroutine NXTCHR, it was completely rewritten to fit IBM 360/50 computer. Others were also modified to meet our purpose.

Subroutines MMULT and INVERT in the original programs were replaced by subroutines GMPRD and MINV of IBM system/360 Scientific Subroutine Package Version III, respectively.

The stacking of the deck is as follows:

1. Main program.
2. Sub-programs.
3. Fixed input data: Quadrature coefficients A and C.
4. System of differential equations.
5. Variable parameters:

a. Order of the Taylor series expansion (MM), order of the Lie series correction (LL), number of nodes in the Gauss quadrature formula (KK), number of equations (NN).

b. Switch for the table control (ITAB), number of abscissas (NNTAB), switch for step size control (ISTV), switch for intermediate step values outputs (NZWI), switch for the calculation of the transfer matrix (IJAC), and the order for the calculation of the transfer matrix (MMJ).

c. Initial step size (STH), initial value of independent variable (XX0), and the initial values of dependent variables (YY0(NN)).

d. Bounds on local truncation errors (GAM(NN)).

e. The lower tolerance bound (ETA1) and the upper tolerance bound (ETA2) for the error ratio.

f. The abscissas at which the results should be printed out (TAB(NN)).

or f'. The table-difference (DTAB), in case a constant step size is choosen.

and g. A signal of the end of the program (any negative MM).

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```
C       MAIN PROGRAM
C       THE MAIN PROGRAM MANAGES THE INPUT, THE OUTPUT, THE STEP SIZE CONTROL
C       AND THE TABLE LOOP.
C
C       LIST OF PRINCIPAL VARIABLES
C
C MM    THE ORDER OF THE APPROX. SOLUTION OR THE NUMBER OF TERMS IN TAYLOR SERIES
C       (0≤MM≤5).
C LL    THE ORDER OF THE LIE SERIES CORRECTION (0≤LL≤10).
C KK    THE NUMBER OF NODES IN THE GAUSS-QUADRATURE FORMULA (1≤KK≤6).
C NN    THE NUMBER OF DEPENDENT VARIABLES IN THE SYSTEM OF DIFFERENTIAL EQS.
C ITAB SWITCH FOR THE WANTED TABLE
C       ITAB¬=0: THE SOLUTIONS SHOULD BE PRINTED OUT AT DIFFERENT ABSCISSAS TAB(1)
C       .......TAB(NNTAB).
C       ITAB=0: THE SOLUTIONS SHOULD BE PRINTED OUT AT EQUIDISTANT ABSCISSAS.
C       AND ONLY THE TABLE-DIFFERENCE DTAB WILL BE READ IN LATER.
C NNTAB NUMBER OF ABSCISSAS TAB(I)
C TAB(I)=X(I) THE ABSCISSAS AT WHICH THE RESULTS SHOULD BE PRINTED OUT.
C ISTV SWITCH FOR THE STEP SIZE CONTROL
C       ISTV=0: CONSTANT STEP SIZE. THE PROGRAM WILL USE THE STEP SIZE STH WHICH
C          IS READ IN LATER.
C       ISTV=1: STEP SIZE CONTROL IS USED WHICH KEEPS THE LOCAL TRUNCATION ERRORS
C          BELOW BOUNDS  WHICH ARE ALSO READ IN LATER.
C       ISTV=2: OPTIMAL STEP SIZE CONTROL USING THE CONNECTION MATRIX (IN THIS
C          CASE, IJAC=1).
C NZWI SWITCH FOR INTERMEDIATE STEP VALUES OUTPUTS (IN ADDITION TO THOSE OF THE
C          TABLE ABSCISSAS)
C       NZWI=0: NO  INTERMEDIATE RESULTS ARE PRINTED OUT.
C       NZWI=J>0: THE RESULTS AFTER EACH J-TH STEP ARE PRINTED OUT.
C IJAC SWITCH FOR THE CALCULATION OF THE CONNECTION MATRIX
C       IJAC=0: THE MATRIX IS NOT CALCULATED.
C       IJAC=1: THE MATRIX IS CALCULATED AND PRINTED OUT TOGETHER WITH THE
C          OTHER RESULTS.
C MMJ THE ORDER FOR THE CALCULATION OF THE CONNECTION MATRIX (0≤MMJ≤MM).
C STH THE CURRENT STEP SIZE H. IF ISTV¬=0, STH IS USED AS THE INITIAL APPROX.
C       FOR THE ITERATIVE CALCULATION OF A SUITABLE STEP SIZE.
```

```
C STHC THE NEW STEP SIZE, (=STH*(1/ETA)**(1/(M+L+2))
C                              IF ETA≤ETA2,UPPER TOLERANCE BOUND.
C                               =STH*(ZETA-ZETA1*ETA)
C                              IF ETA<ETA1, THE LOWER TOLERANCE BOUND).
C XXO THE INITIAL POINT OF THE INDEPENDENT VARIABLE X.
C YYO(N) THE INITIAL VALUES FOR THE SOLUTIONS YI.
C R(I) THE LOCAL TRUNCATION ERRORS OR THE LOCAL ERROR ESTIMATES.
C GAM(N) UPPER BOUND ON THE LOCAL TRUNCATION ERRORS (REMINDERS).
C ETA  THE ERROR RATIO(=|R(I)|/GAM(I)).
C ETA1  THE LOWER TOLERANCE BOUND. (0<ETA1<1).
C ETA2 THE UPPER TOLERANCE BOUND. (FROM 1.1 TO 2.0).
C DTAB THE TABLE-DIFFERENCE OR CONSTANT STEP SIZE H.
C HJ   THE CONNECTION MATRIX.
C IZ   THE NUMBER OF THE STEPS, WHICH ARE USED FOR THE CONTINUATION OF THE
C      SOLUTION.
C IZI  THE NUMBER OF THE REPEATED STEPS, WHICH WERE NOT USED FOR THE CONTINUATION
C      BECAUSE THE CORRESPONDING TRUNCATION ERRORS WERE ABOVE THE ALLOWED BOUNDS.
C ISW  THE SWITCH. IF ISW=1, THE SUBROUTINES CALCULATE THE MU-TH ORDER TAYLOR
C      COEFFICIENT.
C      IF ISW=2, THE COEFFICIENTS OF GROEBNER'S FORMULA ARE CALCULATED.
C YHAT(I) THE TAYLOR SERIES APPROXIMATION.
C YQUER(I) THE LIE SERIES SOLUTION.
C CJ(I,K) THE LOCAL CONNECTION MATRIX WITH ORDER MMJ.
C
                  IMPLICIT REAL*4(A-H,O-Z)
                  COMMON/ L/LL,LL1/MMM/MM,MM1/K/KK/NNN/NN/STHH/STH/MJ/MMJ,MMJ1
                  COMMON/ JAC/IJAC
                  COMMON/XY/X0,Y0(28)
                  COMMON/OUT/YHAT(28),YQUER(28)/STORE/STOER(30,11)/QQ/Q(28)
                  COMMON/CJJ/CJ(28,28)/RR/R(28)
                  COMMON/AA/A( 6,6,6)/CC/C( 6,6,6)/COEF/CM(6,11)
                  DIMENSION YYO(28),GAM(28),TAB(60),HJ(28,28),YJ(28,28)
                  DIMENSION YOST(28),ZJ(28,28),ROST(28),TEMP(28,28)
                  READ (5,104)(((A(I,J,K0),I=1, 6),J=1,6),K0=1,6)
                  READ (5,104)(((C(I,J,K0),I=1, 6),J=1,6),K0=1,6)
                  CALL GETEQ(NN)
              75  READ (5,1000) MM,LL,KK,NN
```

```
 1000 FORMAT (15I5)
      IF (MM .LT. 0) GO TO 20
      READ (5,1000) ITAB,NNTAB,ISTV,NZWI,IJAC,MMJ
      READ (5,1001) STH, XX0
      READ (5,1001) (YY0(N), N=1,NN)
      MMJ1=MMJ+1
 104  FORMAT (E24.7,2E17.7,E13.7)
 1001 FORMAT    (E23.7 ,E30.7 )
      KONTR=NZWI
      IZ=0
      WRITE(6,2000) MM,LL,KK,NN
 2000 FORMAT (6H0MM = I3, 7H  LL = I3, 7H  KK = I3,7H  NN = I3)
      WRITE (6,2012)ITAB,NNTAB,ISTV,NZWI,IJAC,MMJ
 2012 FORMAT (8H0ITAB = I3,10H  NNTAB = I3,9H  ISTV = I3,9H  NZWI = I3,
     1  9H  IJAC = I3,8H  MMJ = I3 )
      WRITE (6,2001) XX0,STH,IZ
 2001 FORMAT (1H0  / 7H0XX0 = E31.7 ,7H STH = E31.7 ,8H   IZ = I6 )
      READ (5,1001) (GAM(N), N=1,NN)
      READ (5,1001)ETA1,ETA2
      WRITE (6,2011) ETA1, ETA2
 2011 FORMAT( 8H0ETA1 = E31.7 ,9H  ETA2 = E31.7  )
      WRITE (6,2006)
 2006 FORMAT (7X,4HY = 30X,6HGAM = )
      WRITE (6,2005) (N,YY0(N),GAM(N), N=1,NN)
 2005 FORMAT (1X,I3,3X,E31.7 ,3X,E31.7 )
      IF (ITAB .EQ. 0) GO TO 2
      READ (5,1001) (TAB(N), N=1,NNTAB)
      GO TO 3
 2    READ (5,1001) DTAB
      DO 4 N=1,NNTAB
      SSS=N
 4    TAB(N)=XX0+SSS*DTAB
 3    IF (NZWI .EQ. 0) NZWI=13421
      NZWIL=NZWI
      IF (IJAC .EQ. 0) GO TO 8
      WRITE (6,2003)
 2003 FORMAT (6H0HJ = )
```

```
      DO 10  I=1,NN
      DO 9  KO = 1,NN
9     HJ(I,KO)= 0.0E0
      HJ(I,I) = 1.0E0
10    WRITE (6,2004) I, (HJ(I,KO), KO = 1,NN)
2004  FORMAT (I5,5X,3E35.7 ,/ (10X,3E35.7 ))
8     II = -5
      IZ = 0
      IZ1 = 0
      NTAB = 1
      IT = 1
      ITT = -1
      MM1 = MM + 1
      LL1 = LL + 1
      SM2 = MM1 + LL1
      IF (ISTV .EQ. 2) SM2 = SM2 - 1.0E0
      ZETA = (SM2+1.0E0)/(SM2*ETA1**(1.0E0/SM2))
      ZETA1 = ZETA/((SM2+1.0E0)*ETA1)
      DO 5 KO = 1,KK
      CM(KO,1) = C(MM1,KK,KO)
      DO 5 IALPH = 2, LL1
5     CM(KO,IALPH) = CM(KO,IALPH - 1)*(1.0E0 - A(MM1,KK,KO))
      XO = XXO
      DO 11 N = 1,NN
11    YO(N) = YYO(N)
      CALL JACOB
12    CONTINUE
      CALL GREEN
      IF (IT .NE. 1) GO TO 14
      IF (ISTV .EQ. 0) GO TO 16
      IF (ISTV .EQ. 1) GO TO 63
      CALL GMPRD(CJ,HJ,TEMP,NN,NN,NN)
      CALL   MINV(TEMP,NN,D,LO,M)
      CALL GMPRD(TEMP,R,Q,NN,NN,1)
      DO 61 I = 1,NN
61    Q(I) = ABS(Q(I))/STH
```

```
   63 ETA = 0.0E0
      DO 51 N = 1, NN
      SSS = G(N)/GAM(N)
      IF (ETA .LT. SSS) ETA =SSS
   51 CONTINUE
      IF (KCNTR .EQ. 0) GO TO 69
      WRITE (6,2009) STH, (G(I),I=1,NN)
 2009 FORMAT (7H0STH = E11.4,5X, 7E12.4, / (23X, 7E12.4))
   69 IF (ETA .GE. ETA1) GO TO 52
      STHG = (ZETA-ZETA1*ETA)*STH
      IF (II .EQ. 1) GO TO 23
      GO TO 22
   52 EPS = 1.0E0/ETA
      STHG = STH*EPS**(1.0E0/SM2)
      IF (ETA .GE. ETA2) GO TO 24
      II = 1
   23 IZ = IZ + 1
      GO TO 25
   22 II = II + 1
   24 STH = STHG
      IZ1 = IZ1 + 1
      GO TO 12
   16 STHG = STH
      GO TO 25
   14 WRITE (6,2008) XTAB,STHST,IZ,IZ1
 2008 FORMAT (1H0 / 5H0X = E31.7 ,6H  H = E31.7 ,7H  IZ = I6,8H  IZ1 =
     1 I6)
      WRITE (6,2002)
 2002 FORMAT (7X,4HR = 30X,4HW = )
      DO 80 N =1, NN
   80 R(N) = ROST(N)
      WRITE (6,2005)(N,YQUER(N),R(N),N=1,NN)
      IF (IJAC .EQ. 0) GO TO 27
      CALL GMPRD(CJ,HJ,YJ,NN,NN,NN)
      WRITE (6,2003)
      DO 57 I=1,NN
   57 WRITE (6,2004) I,(YJ(I,KO), KO=1,NN)
```

```
   27 IF (NTAB .EQ. NNTAB) GO TO 75
      NTAB = NTAB + 1
      STH = STHST
   25 SSS = XO + STH - TAB(NTAB)
      SSS = SSS*STH
      IF (SSS .GE. 0.0E0) GO TO 31
      ITT = -1
      IF (IT .NE. 1) GO TO 33
      XO = XO + STH
      DO  53 N = 1, NN
   53 YO(N) = YCUER(N)
      IF (IJAC .NE. 0) CALL GMPRD(CJ,HJ,HJ,NN,NN,NN)
      GO TO 35
   33 XO = XOST
      DO 54  N = 1, NN
   54 YO(N) = YOST(N)
      IT = 1
      IF (IJAC .EQ. 0) GO TO 35
      DO 55 I = 1,NN
      DO 55 KO=1,NN
      HJ(I,KO) = ZJ(I,KO)
   55 CONTINUE
   35 NZWIL = NZWIL - 1
      IF (NZWIL .NE. 0) GO TO 42
      NZWIL = NZWI
      WRITE (6,2008) XO,STH,IZ,IZ1
      WRITE (6,2002)
      WRITE (6,2005) (N,YO(N),R(N),N=1,NN)
      IF (IJAC .EQ. 0) GO TO 42
      WRITE (6,2003)
      DO 58 I=1,NN
   58 WRITE (6,2004) I,(HJ(I,KO), KO= 1,NN)
   42 STH=STHQ
      GO TO 12
   31 IT = 0
      ITT = ITT + 1
      IF (ITT .NE. 0) GO TO 39
```

```
      XOST = XO + STH
      DO 56 N =1, NN
      ROST(N) = R(N)
   56 YOST(N) = YCUER(N)
      STHST = STH
      IF (IJAC .NE. 0) CALL GMPRD(CJ,HJ,ZJ,NN,NN,NN)
   39 STH = TAB(NTAB) - XO
      XTAB = TAB(NTAB)
      GO TO 12
   20 STOP
      END
```

```
0001   SUBRCUTINE JACCE
C      SOUBRCUTINE JACCE CALCULATES THE JACOBI-MATRIX OF THE SYSTEM CF FUNCTICNS.
C      WHICH IS DETERMINED BY THE SUBROUTINE EQLATN. IT MUST BE GIVEN THE INTEGER
C      NN(# CF EGS) AND THE COMMON VARIABLES X0, Y0(1),..........,Y0(NN)
C      (THE PCINT AT WHICH THE JACOBIAN IS WANTED). IT CALCULATES THE VALUES OF
C      THE FUNCTIONS FI(X,Y1,...YN) AT THE GIVEN POINT.  THESE RESULTS APPEAR
C      IN THE LCCATICNS T(1000+I,1,1) = FI, THE JACOBI-MATRIX AT THIS POINT GOES
C      INTC THE CJ(I,K).
C
0002               IMPLICIT REAL*4(A-H,O-Z)
0003               CCMMCN/XY/X0,Y0(28)/M0/M/ISWH/ISW
0004               COMMON//T(4000,2,4)
0005               COMMCN/CJJ/CJ(28,28)/NNN/NN
0006               CO 2  J=1, 4
0007               T(2000,1,J) = 0.0E0
0008        2      T(2000,2,J) = 0.0E0
0009               T(2000,1,2) = 1.0E0
0010               T(2000,1,1) = X0
0011               DO 7  J = 1,NN
0012        7      T(J,1,1) = Y0(J)
0013               ISW = 1
0014               M = 1
0015               CALL EGUATN
0016               CO 60  I=1,NN
0017               CO 61  K=1,NN
0018        61     T(K,2,1) = 0.0E0
0019               T(I,2,1) = 1.0E0
0020               ISW = 2
0021               CALL EGUAIN
0022               CO 60  K=1,NN
0023        60     CJ(K,I) = T(1000+K,2,1)
0024               RETURN
0025               END
```

```
      SUBROUTINE GREEN
C     SUBROUTINE GREEN CALCULATES FOR GIVEN
C        NUMBER OF EQS. NN,
C        PARAMETERS MM, LL, KK  (AND MMJ IF IJAC=1),
C        INITIAL VALUES X0, Y0(1),....,Y0(N()
C        STEP SIZE STH
C        QUADRATURE COEFFICIENTS A(*,*,*) AND CM(*,*), AND
C        IJAC, THE SWITCH, WHICH EFFECTS THE CALCULATION OF THE CONNECTION
C        MATRIX.
C     THE RESULTS
C        YHAT(I), THE TAYLOR SERIES APPROXIMATION.
C        YQUER(I), THE LIE-SERIES SOLUTION.
C        R(I), ESTIMATES FOR THE LOCAL TRUNCATION ERRORS.
C     AND IF IJAC=1, IT ALSO CALCULATES
C        CJ(I,K), THE LOCAL CONNECTION MATRIX WITH ORDER MMJ.
C     THE SUBROUTINE TAYLOR CALCULATES, FOR THE SAME GIVEN CONDTIONS, THE
C     RESULTS
C        T(I,1,M), THE (M-1)ST TAYLOR COEFFIEIENT OF YI(X) FOR M=1,MM1.
C        YHAT, THE TAYLOR APPROXIMATION OF ORDER MM FOR YI(X0+STH).
C        CJ(I,K), THE CONNECTION MATRIX.
C
            IMPLICIT REAL*4(A-H,O-Z)
            COMMON/AA/A( 6,6,6)/COEF/CM(6,11)/STHH/STH/M0/M/MJ/MMJ,MMJ1
            COMMON/XY/X0,Y0(28)/K/KK/L/LL,LL1/MMM/MM,MM1/NNN/NN/ISWH/ISW
            COMMON/OUT/YHAT(28),YQUER(28)/STORE/STOER(30,11)/ QQ/Q(28)
            COMMON/CJJ/CJ(28,28)/RR/R(28)
            COMMON/ JAC/IJAC
            COMMON//T(4000,2,4)
            DIMENSION PHI(28),YHATA(28, 11),FHATA(28,11),YHATAK(28)
            KORET = 1
            GO TO 42
            ENTRY TAYLOR
            KORET = 0
```

```
42    MMM1 = MM - 1
      LLP2 = LL + 2
      DO 2 J = 1, 4
      T(2000,2,J) = 0.0E0
2     T(2000,1,J) = 0.0E0
      T(2000,1,2) = 1.0E0
      ISW = 1
      CALL RECUR(X0,Y0,MM)
      DO 7 N = 1,NN
      SSS = T(N,1,MM1)
      IF (MM .EQ. 0) GO TO 31
      DO 5 J = 1,MM
5     SSS = SSS*STH + T(N,1,MM1 - J)
31    YHAT(N)  = SSS
      DO 7 L0= 1,LL1
7     STDER(N,L0) = 0.0E0
      IF (IJAC .EQ. 0) GO TO 4
      ISW = 2
      DO 60 I = 1, NN
      DO 61 K0= 1, NN
61    PHI(K0)= 0.0E0
      PHI(I) = 0.0E0
      CALL RECUR(X0,PHI,MMJ)
      DO 60 K0=1,NN
      SSS =T(K0,2,MMJ1)
      IF (MMJ .EQ. 0)  GO TO 64
      DO 65  J = 1, MMJ
65    SSS = SSS*STH +T(K0,2,MMJ1 -J )
64    CJ(K0,I) = SSS
60    CONTINUE
4     IF (KORET .EQ. 0) GO TO 58
      ISW = 1
      DO 11 K0= 1,KK
      AKH = A(MM1,KK,K0)*STH
      DO 11 N =1,NN
      SSS = T(N,1,MM1)
```

```
      IF (MM .EQ. 0) GO TO 36
      DO 6 J =1,MM
 6    SSS = SSS*AKH + T(N,1,MM1 - J)
 36   YHATA(N,KO)= SSS
      SSS = 0.0E0
      IF (MM .EQ. 0) GO TO 41
      SSS = T(1000+N,1,MM)
      IF (MM .EQ. 1) GO TO 41
      DO 8  J =1,MMM1
 8    SSS = SSS*AKH + T(1000+N,1,MM-J)
 41   FHATA(N,KO)= SSS
 11   CONTINUE
      DO 19  KO= 1,KK
      ISW = 1
      DO 13 N=1,NN
 13   YHATAK(N) = YHATA(N,KO)
      AKH=A(MM1,KK,KO)*STH
      CALL RECUR (XO+AKH,YHATAK,LL)
      DO 15  N=1,NN
 15   PHI(N) =T(1000+N,1,1) - FHATA(N,KO)
      ISW = 2
      CALL RECUR(XO+AKH,PHI,LL)
      DO 19 M=1,LL1
      DO 19 N=1,NN
      STOER(N,M)=STOER(N,M)+CM(KO,M)*T(N,2,M)
 19   CONTINUE
      DO 20  N=1, NN
      SSS=0.0E0
      DO 21 LO=1,LL1
 21   SSS=SSS*STH+STOER(N,LLP2-LO)
      YQUER(N)= SSS*STH + YHAT(N)
 20   CONTINUE
      ISW=1
      CALL RECUR (XO+STH,YQUER, LL1)
      DO 53 N=1,NN
 53   G(N) =T(1000 + N,1,LL1)
```

```
      CALL RECUR (X0+STH, YHAT , LL1)
      MM2=MM+2
      REALM=MM2
      SSS=1.0E0/REALM
      IF (LL .EQ. 0) GO TO 59
      DO 56  J=1,LL
      REALJ =J
      REALM = MM2 +J
      SSS=SSS*REALJ/REALM
56    CONTINUE
59    SSS=SSS*(STH**LL1)
      DO 57 N=1,NN
      R(N) =  (C(N) - T(1000+ N,1,LL1))*SSS
57    C(N)= ABS(R(N))
58    RETURN
      END
```

```
0001   SUBROUTINE RECUR (XTO,YTO,JJ)
C      SUBROUTINE  RECUR HAS THE FOLLOWING  FUNCTIONS
C      1) IF THE STATEMENTS ARE
C         ISW=1
C         CALL RECUR(X0,Y0,MM)
C      WITH GIVEN INITIAL VALUES
C         X0, Y0(U),....,Y0(NN).
C      IT CALCULATES THE TAYLOR COEFFICIENTS T(I,1,M)
C         I=1,NN  M=MU+1, MU=0,MM.
C      2) IF THE STATEMENTS ARE
C         ISW=2
C         CALL RECUR(X0,PHI,MMJ)
C      WITH  THE INITIAL VALUES PHI(1)=0 ,PHI(K)=1,..... PHI(NN)=0
C      IT CALCULATES THE TAYLOR COEFFICIENTS T(I,2,M)
C      OF THE K-TH COLUMN OF THE LOCAL CONNECTION MATRIX.
C      3) IF THE STATEMENTS ARE
C         ISW=1
C         CALL RECUR(X0+AKH,YHATAK,LL)
C      WITH THE STARTING VALUES
C         X0+AKH, YHATAK(I)     (I=1,NN)
C      OR ISW=2
C         CALL RECUR(X0+AKH,PHI,LL)
C      WITH THE INITIAL VALUES
C
C               ^   ^
C      PHI(I)=FI-YI       (I=1,NN)
C
C                                                                         J
C      IT CALCULATES THE TAYLOR COEDFICIENTS T(I,2,M) WITH TWO CPERATORS DD
C
0002               IMPLICIT REAL*4(A-H,O-Z)
0003               COMMON//T(4000,2,4)
0004               COMMON/ISWH/ISW/MO/M/NNN/NN
0005               DIMENSION YTO(6)
0006               JJJ=JJ
0007               T(2000,1,1) = XTO
0008               DO 7  J=1,NN
0009           7   T(J,ISW,1) = YTO(J)
```

```
      IF (JJJ .EQ. 0) JJJ=1
      DO 6 M=1,JJJ
      RM = M
      CALL EQUATN
      DO 2  N=1, NN
    2 T(N,ISW,M+1) = T(1000+N,ISW,M)/RM
    6 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE EQUATN
C     SUBROUTINE EQUATN INTERPRETS THE INTERNAL CODE SET UP BY THE SUBROUTINE
C     GETEQ  AND CALLS THE APPROPRIATE RECURSION SUBROUTINES FOR INTEGRATING THE
C     SYSTEM BEING STUDIED.
C
      COMMON /IRPQ/IRR,IPP,IQQ
      COMMON/CKK/CK
      COMMON/COD /NXTCOD,CODE(5000),NXTLIT,LITERL(100)
      INTEGER NXTCOD,CODE,NXTLIT
      REAL LITERL
      I=1
      GO TO 9999
 9993 I = I + 3
      GO TO 9999
 9994 I=I+4
 9999 IFUNC=CODE(I)
      IRR=CODE(I+1)
      IPP=CODE(I+2)
      IQQ=CODE(I+3)
      GO TO (1, 2,3,4,5,6,7, 8,14,14,14,12,14),IFUNC
    1 GO TO 14
    2 CK = LITERL(IPP)
      CALL KNST
      GO TO 9993
    3 CK=LITERL(IQQ)
      CALL SUMK
      GO TO 9994
    4 CK=LITERL(IQQ)
      CALL PRDK
      GO TO 9994
    5 CALL SUM
      GO TO 9994
    6 CALL DIFF
      GO TO 9994
    7 CALL PRCD
```

```
      GO TO 9994
    8 CALL CUCT
      GO TO 9994
   12 CK=LITERL(IGQ)
      CALL CCPO
      GO TO 9994
   14 RETURN
      END
```

```
C      THE FOLLOWING RECURSION SUBROUTINES ARE PROGRAMMED ACCORDING TO  THE
C      FOLLOWING  RELATIONS
C                      CONSTANT  CK=T(IR,*,*)
C          SUM WITH A CONSTANT  T(IP,*,*) + CK = T(IR,*,*)
C      PRODUCT WITH A CONSTANT  T(IP,*,*)*CK = T(IR,*,*)
C                          SUM  T(IP,*,*)+T(IQ,*,*)=T(IR,*,*)
C                   DIFFERENCE  T(IP,*,*)-T(IQ,*,*)=T(IR,*,*)
C                      PRODUCT  T(IP,*,*)*T(IQ,*,*)=T(IR,*,*)
C      WHERE IP, IQ, OR IR IS THE STORE ADDRESS ASSIGNED TO THE EACH OPERAND.
C      ARGUMENTS ARE OBTAINED FROM THE COMMON BLOCKS IRPQ, ISW AND M.
C
      SUBROUTINE FOCK
      IMPLICIT REAL*4(A-H,O-Z)
      COMMON//T(4000,2,4)/IRPQ/IRR,IPP,IQQ/CKK/CK
      COMMON/ISWH/ISW/MO/M
      T(IRR,ISW,M) = CK*T(IPP,ISW,M)
      RETURN
      END
```



```
      SUBROUTINE KNST
      IMPLICIT REAL*4(A-H,O-Z)
      COMMON//T(4000,2,4)/IRPQ/IRR,IPP,IQQ/CKK/CK
      COMMON/ISWH/ISW/MO/M
      GO TO (1,2), ISW
    1 IF (M .NE. 1) GO TO 2
      T(IRR,1,I) = CK
      RETURN
    2 T(IRR,ISW,M) = 0.0E0
      RETURN
      END
```

```
      SUBROUTINE CUCT
      IMPLICIT REAL*4(A-H,C-Z)
      COMMON//T(4000,2,4)/IRPQ/IRR,IPP,IQQ/CKK/CK
      COMMON/ISWH/ISW/MO/M
      ML1 = M - 1
      M1 = M + 1
      GO TO (1,2) , ISW
    1 SSS = T(IPP,1,M)
      IF (ML1 .EQ. 0) GO TO 4
      DO 5  J = 1, ML1
    5 SSS =SSS - T(IRR,1,J)*T(IQQ,1,M1-J)
    4 T(IRR,1,M) = SSS/T(IQQ,1,1)
      RETURN
    2 SSS = T(IPP,2,M)
      IF (ML1 .EQ. 0) GO TO 8
      DO 6 J=1,ML1
      M1J=M1-J
    6 SSS=SSS-T(IRR,2,J)*T(IQQ,1,M1J)-T(IRR,1,J)*T(IQQ,2,M1J)
    8 SSS=SSS-T(IRR,1,M)*T(IQQ,2,1)
      T(IRR,2,M)=SSS/T(IQQ,1,1)
      RETURN
      END
```

```
      SUBROUTINE SUMK
      IMPLICIT REAL*4(A-H,O-Z)
      COMMON//T(4000,2,4)/IRPQ/IRR,IPP,IQQ/CKK/CK
      COMMON/ISWH/ISW/MO/M
      GO TO (1,2), ISW
    1 IF (M .NE. 1)  GO TO 2
      T(IRR,1,M) = T(IPP,1,M) + CK
      GO TO 3
    2 T(IRR,ISW,M) = T(IPP,ISW,M)
    3 RETURN
      END
```

```
      SUBROUTINE SUM
      IMPLICIT REAL*4(A-H,O-Z)
      COMMON//T(4000,2,4)/IRPQ/IRR,IPP,IQQ/CKK/CK
      COMMON/ISWH/ISW/MO/M
      T(IRR,ISW,M) = T(IPP,ISW,M) + T(IQQ,ISW,M)
      GO TO 3
      ENTRY DIFF
      T(IRR,ISW,M)=T(IPP,ISW,M)-T(IQQ,ISW,M)
    3 RETURN
      END

      SUBROUTINE PRCD
      IMPLICIT REAL*4(A-H,O-Z)
      COMMON//T(4000,2,4)/IRPQ/IRR,IPP,IQQ/CKK/CK
      COMMON/ISWH/ISW/MO/M
      SSS=0.0E0
      M1 = M + 1
      GO TO (1,2), ISW
    1 DO 5  J =1,M
    5 SSS = SSS + T(IPP,1,J)*T(IQQ,1,M1 -J)
      T(IRR,1,M) = SSS
      GO TO 3
    2 DO 6  J = 1 , M
      M1J = M1 - J
    6 SSS=SSS+T(IPP,2,J)*T(IQQ,1,M1J)+T(IPP,1,J)*T(IQQ,2,M1J)
      T(IRR,2,M) = SSS
    3 RETURN
      END
```

```
      SUBROUTINE COPC
      IMPLICIT PEAL*4(A-H.O-Z)
      COMMON//T(4000.2.4)/IRPQ/IRR,IPP,IQQ/CKK/CK
      COMMON/ISWH/ISW/MO/M
      IF (M .NE. 1) GC TC 2
      GO TC (4.8).ISW
    2 ML1=M-1
      RM=ML1
      M1=M+1
      GO TO (3.7).ISW
    3 SSS=0.0E0
      DO 5 J=1.ML1
      RMJ=CK*RM-(CK+1.0E0)*(J-1)
    5 SSS=SSS+RMJ*T(IRR,1,J)*T(IPP.1.M1-J)
      T(IRR.1.M)=SSS/(RM*T(IPP.1.1))
      GO TO 9
    4 T(IRR,1.M)=T(IPP.1.1)**CK
      RETURN
    7 SSS=0.0E0
      DO 6 J=1.ML1
      RMJ=CK*RM-(CK+1.0E0)*(J-1)
      MJ=M1-J
    6 SSS=SSS+RMJ*(T(IRR.2.J)*T(IPP.1.MJ)+T(IRR.1.J)*T(IPP.2.MJ))
      T(IRR.2.M)=(SSS/RM-T(IPP.2.1)*T(IRR.1.M))/T(IPP.1.1)
      GO TO 9
    8 T(IRR.2.1)=CK*T(IRR.1.1)*T(IPP.2.1)/T(IPP.1.1)
    9 RETURN
      END
```

```
0001   SUBROUTINE GETEQ(NN)
C      SUBROUTINE  GETEQ IS THE MAIN ROUTINE RESPONSIBLE FOR READING EQS. AND
C      COMPILING CODE.  GETEQ FIRST INITIALIZED ALL VARIABLES USED IN THE
C      COMPILATION PROCESS. THEN  GETEQ, USING THE ROUTINE NXTTOK, CONVERTS THE
C      EXPRESSIONS OCCURRING IN THE INPUT STREAM TO POSTFIX NOTATION AND CALLS
C      THE ROUTINE GENRAT TO  PRODUCE THE APPROPRIATE CODE WHICH WILL BE
C      INTERPRETED AT EXECUTION TIME. THE CONVERSION IS DONE IN THE CONVENTIONAL
C      MANNER.
C      AFTER COMPILATION IS COMPLETE,  GETEQ PRINTS OUT THE GENERATED CODE.
C
0002              COMMON/ALOCAT/TAVAIL,TUSE(4000),GARBAG,TMPADR(1000),
                 1 LAVAIL, LUSE(1000)
0003              INTEGER TAVAIL,TUSE,GARBAG,TMPADR,LAVAIL,LUSE
0004              COMMON/COD /NXTCOD,CODE(5000),NXTLIT,LITERL(100)
0005              INTEGER NXTCOD,CODE,NXTLIT
0006              REAL LITERL
0007              COMMON /SCAN/NCLASS,NSYMB,LCLASS,COLUMN,CARD(80),NUMERR,MAXY
0008              INTEGER NCLASS,NSYMB,LCLASS,COLUMN,CARD,NUMERR,MAXY
0009              COMMON/STACKS/ ATOP,BTOP,ASTACK(100),BSTACK(100)
0010              INTEGER ATOP,BTOP,ASTACK,BSTACK
0011              COMMON//T(4000,2,4)
0012              LOGICAL GENRAT
0013              INTEGER KIND,ADDRES,TOPOP,OPNAME
0014              DIMENSION OPNAME(13)
0015              DATA OPNAME /3HEND,4HKNST,4HSUMK,4HPODK,3HSUM,4HDIFF,4HPROD,
                 1 4HQUOT,4HEXPO,4HLOGA,4HSINK,4HCOPO,4HWURZ/
           C      ************************************************************
           C      INITIALZE ALL VARIABLES
           C      INITIALIZE VARIABLES IN COMMON BLOCK ALOCAT
0016              DO 1 I = 1,4000
0017        1     TUSE(I)= I - 1
0018              TUSE(29) = TUSE(1)
0019              TUSE(1029) = TUSE(1001)
0020              TUSE(2001) = TUSE(2000)
0021              TAVAIL = 4000
0022              DO 2 I = 1, 1000
```

```
      2     TMPADR(1)=0
            GARBAG=0
            DO 3 I = 3, 1000
      3     LUSE(I)=I+1
            LUSE(1000) = 0
            LAVAIL=3
C           ******************************************************************
C           INITIALIZE VARIABLES IN COMMON BLOCK CODE
            NXTCOD=1
            LITERL(1)=0.0E0
            LITERL(2)=1.0E0
C           ******************************************************************
C           INITIALIZE VARIABLES IN COMMON BLOCK SCAN
            MAXY=0
            NUMERR=0
            LCLASS=6
            COLUMN=81
            CALL NXTCHR
C           INITIALIZE STACKS TO PREPARE FOR NEXT STATEMENT
     80     ATOP=0
            BTOP=1
            BSTACK(BTOP)=-2
     99     CALL NXTTOK(KIND,ADDRES)
            GO TO (100,200,300,400,500,600,700,800),KIND
C           OPERAND, PUT IT ON THE ASTACK
    100     IF (ATOP .GE. 100) GO TO 101
            ATOP = ATOP + 1
            ASTACK(ATOP)=ADDRES
            GO TO 99
C           OPERAND STACK OVERFLOW
    101     CALL ERROR(2)
            WRITE (6,102)
            GO TO 800
    102     FORMAT (32H0OPERAND STACK CAPACITY EXCEEDED    )
C           OPERATOR
    200     TOPOP=BSTACK(BTOP)
```

```
      IF (ADDRES .GT. TOPOP) GO TO 201
C     CURRENT OPERATOR HAS LOWER PRECEDENCE, SO EMIT SOME CODE
      BTOP=BTOP-1
      IF(GENRAT(TOPOP)) GO TO 800
      GO TO 200
C     CURRENT OPERATOR HAS HIGHER PRECEDENCE, THEREFORE STACK IT
  201 IF (BTOP .GE. 100) GO TO 202
      BTOP = BTOP + 1
      BSTACK(BTOP)=ADDRES
      GO TO 99
C     OPERATOR STACK OVERFLOW
  202 CALL ERROR(2)
      WRITE (6,203)
      GO TO 800
  203 FORMAT (33HOOPERATOR STACK CAPACITY EXCEEDED)
C     FUNCTION, TREAT IT AS A UNARY OPERATOR
  300 GO TO 200
C     LEFT PARENTHESIS, STACK A ZERO ON THE BSTACK
  400 ADDRES=0
      GO TO 201
C     RIGHT PARENTHESIS
  500 TOPOP= BSTACK(BTOP)
      BTOP=BTOP-1
      IF (TOPOP) 501,99,502
C     EMIT ALL STACKED OPERATORS UNTIL YOU GET TO A LEFT PARENTHESIS
  502 IF (GENRAT(TOPOP)) GO TO 800
      GO TO 500
C     THERE WAS NO LEFT PRRENTHESIS IN THE STACK
  501 CALL ERROR(1)
      WRITE (6,503)
      GO TO 800
  503 FORMAT (24HOEXTRA RIGHT PARENTHESIS)
C     DOLLAR SIGN
  600 TOPOP=BSTACK(BTOP)
      BTOP=BTOP-1
      IF (TOPOP) 601,602,603
```

```
C                 OPERATOR STACK IS NOW EMPTY
           601    IF (ATOP .EQ. 1) GO TO 80
                  CALL ERROR(2)
                  WRITE (6,610)
                  GO TO 80
           610    FORMAT(15H0INTERNAL ERROR)
C                 OPERATOR STACK CONTAINS AN UNBALANCED LEFT PARENTHESIS
           602    CALL ERROR(2)
                  WRITE (6,620)
                  GO TO 80
           620    FORMAT(55H0ABOVE STATEMENT COMTAINS AN UNMATCHED LEFT PARENTHESIS)
C                 EMIT CODE FOR ALL OPERATORS STILL IN THE OPERATOR STACK
           603    IF (GENRAI(TOPOP)) GO TO 80
                  GO TO 600
C                 END OF INPUT,EMIT THE END OPERATOR
           700    CALL EMIT(1)
C                 LIST THE CELLS ASSIGNED TO TEMPORARY VARIABLES
                  IFLAG=0
                  DO 741 I = 1, 1000
                  J=TMPADR(I)
                  IF (J .EQ. 0) GO TO 741
                  IF (IFLAG .NE. 0) GO TO 743
                  WRITE (6,740)
                  IFLAG=1
           743    WRITE (6,742)I,J
           741    CONTINUE
           740    FORMAT(55H0TEMPORARY CELLS WERE ASSIGNED TO THE TARRAY AS FOLLOWS)
           742    FORMAT (1H0,30X,1HT,I3,21H WAS ASSIGNED TO CELL,I5   )
C                 LIST THE LITERALS ASSIGNED DIRECTLY TO THE T-ARRAY
                  IFLAG=0
                  DO 751 I =1, 4000
                  IF (TUSE(I) .NE. 9999) GO TO 751
                  IF (IFLAG .NE. 0) GO TO 753
                  WRITE (6,750)
                  IFLAG = 1
           753    WRITE (6,752) T(I,1,1),I
```

```
  751 CONTINUE
  750 FORMAT (48H0LITERALS WERE ASSIGNED TO THE TARRAY AS FOLLOWS)
  752 FORMAT(1H0,30X,E34.7 ,21H WAS ASSIGNED TO CELL,I5)
C     EDIT AND PRINT THE GENERATED CODE
      WRITE (6,709)
  709 FORMAT (52H0THE FOLLOWING CODE WAS GENERATED FROM THE EQUATIONS  )
      I=1
  710 J=CODE(I)
      GO TO  (730,711,712,712,713,713,713,713,714,714,713,712,714),J
  711 LIT=CODE(I+2)
      WRITE(6,721) OPNAME(J),CODE(I+1),LITERL(LIT)
      I= I+3
      GO TO 710
  712 LIT =CODE(I+3)
      WRITE (6,722) OPNAME(J),CODE(I+1),CODE(I+2),LITERL(LIT)
      I=I+4
      GO TO 710
  713 WRITE(6,723)OPNAME(J),CODE(I+1),CODE(I+2),CODE(I+3)
      I=I+4
      GO TO 710
  714 WRITE(6,723) OPNAME(J),CODE(I+1),CODE(I+2)
      I=I+3
      GO TO 710
  721 FORMAT(1H0,30X,A6,I10,E34.7 )
  722 FORMAT(1H0,30X,A6,2I10,E34.7 )
  723 FORMAT(1H0,30X,A6,3I10)
  729 FORMAT (1H0,30X,I4,38H LOCATIONS WERE USED IN THE CODE ARRAY  )
  730 WRITE (6,723) OPNAME(1)
      WRITE (6,729)I
      DO 760 I=1,MAXY
      IF (TUSE(I+1000).LT. 0) GO TO 760
      CALL ERROR (2)
      WRITE (6,761)I
  760 CONTINUE
  761 FORMAT(32H0NO VALUE HAS BEEN ASSIGNED TO Y,I2,6H PRIME)
      NN=MAXY
```

```
      WRITE(6,799) NUMERR
  799 FORMAT (1H0,I4,21H ERRORS WERE DETECTED )
      IF (NUMERR .EQ. 0) GO TO 4
      WRITE (6,7999)
 7999 FORMAT (42H0**** EXECUTION WILL NOT BE ATTEMPTED ****     )
C     ERROR RUNOUT LOOP
  800 CALL NXTTOK(KIND,ADDRES)
      IF (KIND .NE. 6) GO TO 800
      GO TO 80
    4 RETURN
      END
```

```
      LOGICAL FUNCTION GENRAT(OP)
C     LOGICAL FUNCTION GENRAT IS INVOKED WHENEVER IT IS NECESSARY TC PRCDUCE
C     THE CCDE PERTAINING TO ANY ARITHMETIC OPERATOR. THE PARAMETER OP INDICATES
C     WHICH CPERATOR IS BEING CONSIDERED. GENRAT AUTOMATICALLY POPS THE PRCPER
C     NUMBER CF OFERANCS OFF THE ASTACK, CALLS DEFCHK TC  TEST IF THE OPERANDS
C     ARE CURRENTLY DEFINED AND THEN GOES TC THE GENERATOR APPRCPRIATE TO  OP.
C     THE VARICUS GENERATORS WILL USUALLY CALL CONCHK IN AN ATTEMPT TC PERFCRM
C     SOME LCCAL CPTIMIZATION BEFORE ACTUALLY PRODUCING ENTRIES IN THE CODE
C     ARRAY.  THE VALUE RETURNED BY GENRAT IS TRUE  IF AND CNLY IF ERRORS WERE
C     DETECTED DURING THE ABOVE PROCESSING.
C
      COMMON/ALLCAT/TAVAIL,TUSE(4000),GARBAG,TMPADR(1000),
     1 LAVAIL, LUSE(1000)
      INTEGER TAVAIL,TUSE,GARBAG,TMPADR,LAVAIL,LUSE
      COMMON/COD /NXTCOD,CODE(5000),NXTLIT,LITERL(100)
      INTEGER NXTCOD,CODE,NXTLIT
      REAL LITERL
      COMMON /STACKS/ ATOP,BTOP,ASTACK(100),BSTACK(100)
      INTEGER ATOP,BTOP,ASTACK,BSTACK
      INTEGER OP,CP1,CP2,OPCODE,GETCEL,TRUE,FALSE
C     *****************************************************************
      GENRAT=.FALSE.
      FALSE=0
      TRUE=1
C     DETERMINE NUMBER OF OPERANDS AND POP THEM OFF OF THE STACK
      IF (OP .GT. 6) GO TO 9991
      CP2 = ASTACK(ATOP)
      ATOP=ATOP-1
      CALL DEFCHK(OP2)
 9991 CP1=ASTACK(ATOP)
      IF (ATOP .LE. 0) GO TO 9992
      IF (OP .NE. 1) CALL DEFCHK(OP1)
      GO TO (1,2,3,4,5,6,7,8,14,14,14,14,14),OP
 9992 GENRAT=.TRUE.
      CALL ERROR(2)
      WRITE (6,9993)
```

```
      GO TO  14
 9993 FORMAT(17H0MISSING OPERAND )
C     ********************************************************************
C     ASSIGNMENT OPERATOR =
 1    IF ((OP1 .LT. 1) .OR. (OP1 .GT. 5000))GO TO 101
      IF (TUSE(OP1) .LT. 0) GO TO 105
      TUSE(OP1)=-OP1
      IF (OP2 .LT. 0) GO TO 102
      IF (OP2 .GT. 5000) GO TO 103
      CALL EMIT(3)
      CALL EMIT(OP1)
      CALL EMIT(OP2)
      CALL EMIT(1)
      GO TO  14
 103  CALL EMIT(2)
      CALL EMIT(OP1)
      CALL EMIT(OP2-5000)
      GO TO  14
 102  CODE(-OP2)=OP1
      GO TO  14
 101  CALL ERROR(2)
      WRITE (6,104)
      GENRAT=.TRUE.
      GO TO  14
 104  FORMAT (64H0A CONSTANT OR EXPRESSION MAY NOT OCCUR TO THE LEFT OF
     1AN = SIGN )
 105  CALL ERROR(2)
      WRITE(6,106)
      GENRAT=.TRUE.
      GO TO  14
 106  FORMAT (85H0ATTEMPT TO ASSIGN A VALUE TO A VARIABLE WHICH ALREADY
     1HAS A VALUE ASSIGNED TO IT           )
C     ********************************************************************
C     +
 2    CALL CONCHK(NUMCON,OP1,OP2,TRUE)
      GO TO (200,201,202),NUMCON
 200  OPCODE=5
```

```
      GO TO 9999
  201 OPCODE=3
      GO TO 9999
  202 LITERL(OP1)=LITERL(OP1)+LITERL(OP2)
      GO TO 9801
C     ****************************************************************
C     MINUS
    3 CALL CONCHK(NUMCON,OP1,OP2,FALSE)
      GO TO (300,301,302),NUMCON
  300 OPCODE=6
      GO TO 9999
  301 LITERL(OP2)=-LITERL(OP2)
      OPCODE=3
      GO TO 9999
  302 LITERL(OP1)=LITERL(OP1)-LITERL(OP2)
      GO TO 9801
C     ****************************************************************
C     *
    4 CALL CONCHK(NUMCON,OP1,OP2,TRUE)
      GO TO (400,401,402),NUMCON
  400 OPCODE=7
      GO TO 9999
  401 OPCODE=4
      GO TO 9999
  402 LITERL(OP1)=LITERL(OP1)*LITERL(OP2)
      GO TO 9801
C     ****************************************************************
C     /
    5 CALL CONCHK(NUMCON,OP1,OP2,FALSE)
      GO TO (500,501,502),NUMCON
  500 OPCODE=8
      GO TO 9999
  501 LITERL(OP2)=1.0E0/LITERL(OP2)
      OPCODE=4
      GO TO 9999
  502 LITERL(OP1)=LITERL(OP1)/LITERL(OP2)
```

```
      GO TO 9801
C     ******************************************************************
C     **
6     CALL CCNCHK(NUMCCN,OP1,OP2,FALSE)
      GO TO (600,601,602),NUMCON
600   CALL ERROR(2)
      WRITE(6,609)
      GENRAT=.TRUE.
      GO TO  14
609   FORMAT (43H0EXPCNENTIATION MUST BE TO A CONSTANT POWER)
601   CPCCDE=12
      GO TO 9999
602   LITERL(OP1)=LITERL(OP1)**LITERL(OP2)
      GO TO 9801
C     ******************************************************************
C     UNARY +
7     GO TO 14
C     ******************************************************************
C     UNARY -
8     IF (OP1 .GT. 5000) GO TO 801
      CP2=2
      CPCODE=4
      GO TO 9999
801   LITERL(OP1 - 5000) = -LITERL(OP1 - 5000)
      GO TO  14
C     ******************************************************************
C     RETURN A LITERAL CELL TO AVAILABLE STCRAGE FOR FLTURE USE
9801  LUSE(OP2)=LAVAIL
      LAVAIL=OP2
      GO TO  14
C     ******************************************************************
C     EMIT THE CODE SET UP BY ABOVE GENERATCRS
9999  CALL EMIT(OPCCDE)
      ASTACK(ATCP)=-NXTCOD
      CALL EMIT(0)
      CALL EMIT(OP1)
      IF (OP2 .NE. 0) CALL EMIT(OP2)
14    RETURN
      END
```

```
      SUBROUTINE CONCHK (NUMCON,LEFT,RIGHT,COMMUT)
C        CONCHK IS USED TO OBTAIN A CERTAIN AMOUNT OF LOCAL OPTIMIZATION IN THE
C        CODE BEING PRODUCED. IT IS CALLED BEFORE GENERATING CODE FOR ANY BINARY
C        OPERATOR.
C     NUMCON=3 IF BOTH OPERANDS ARE CONSTANTS. THE OPERATION IS PERFORMED AT
C        COMPILE TIME BY THE APPROPRIATE GENERATOR.
C     NUMCON=1 IF NEITHER OPERAND IS A CONSTANT.
C     NUMCON=2 IF THE RIGHT OPERAND IS A CONSTANT. OR  IF THE LEFT OPERAND IS A
C        CONSTANT AND THE OPERATOR IS COMMUTATIVE. IN BOTH CASE A MORE EFFICIENT
C        OPERATION CODE CAN THEN BE EMITTED. E.G. A PODK   INSTEAD OF A PROD.
C     IF LEFT OPERAND IS A CONSTANT AND THE OPERATOR IS NOT  COMMUTATIVE. THEN THE
C        CONSTANT IS ASSIGNED PERMANENTLY TO A CELL IN THE TARRAY AND TREATED AS A
C        VARIABLE.
C
            COMMON/ALLCAT/TAVAIL,TUSE(4000),GARBAG,TMPADR(1000),
           1 LAVAIL, LUSE(1000)
            INTEGER TAVAIL,TUSE,GARBAG,TMPADR,LAVAIL,LUSE
            COMMON//T(4000,2,4)
            COMMON/COD /NXTCOD,CODE(5000),NXTLIT,LITERL(100)
             INTEGER NXTCOD,CODE,NXTLIT
            REAL LITERL
            INTEGER NUMCON,LEFT,RIGHT,COMMUT,GETCEL
            NUMCON=1
            IF (RIGHT .GT. 5000) GO TO 3
            IF (LEFT .LE. 5000) GO TO 4
C           ******************************************************************
C           LEFT OPERAND IS A CONSTANT, RIGHT OPERAND IS NOT
            IF (COMMUT .EQ. 1) GO TO 2
C           ******************************************************************
C           OPERATOR IS NOT COMMUTATIVE, ASSIGN CONSTANT TO T-ARRAY
            K=GETCEL(DUMMY)
            TUSE(K)=9999
            DO 1 I=1,4
            T(K,1,I)=0.0E0
    1       T(K,2,I)=0.0E0
            LIT = LEFT - 5000
```

```
      T(K,1,1)=LITERL(LIT)
      LUSE(LIT)=LAVAIL
      LAVAIL=LIT
      LEFT=K
      GO TO 4
C     ******************************************************************
C     CPERATCR IS CCMMUTATIVE. SO WE CAN SWAP THE OPERANDS
    2 K=RIGHT
      RIGHT = LEFT - 5000
      LEFT=K
      NUMCON=2
      GO TO 4
C     ******************************************************************
C     RIGHT CPERAND IS A CONSTANT
    3 RIGHT = RIGHT - 5000
      NUMCON=2
      IF (LEFT .LE. 5000) GO TO 4
      LEFT = LEFT - 5000
      NUMCON =3
    4 RETURN
      END
```

```
      SUBROUTINE DEFCHK(OPRAND)
C     SUBROUTINE DEFCHK IS USED TO CHECK THAT AN OPERAND HAS A VALUE ASSIGNED TO
C     IT BEFORE IT IS USED IN AN EXPRESSION. THE ONLY OPERANDS WHICH ALWAYS HAVE
C     A VALUE ASSIGNED TO THEM ARE X AND ALL Y  VALUES. VARIABLIES OF THE FORM T
C     AND YNNP CANNOT BE REFERENCED UNTIL A  VALUE HAS BEEN ASSIGNED TO THEM.
C
      COMMON/ALLOCAT/TAVAIL,TUSE(4000),GARBAG,TMPADR(1000),
     1 LAVAIL, LUSE(1000)
      INTEGER TAVAIL,TUSE,GARBAG,TMPADR,LAVAIL,LUSE
      DATA I1HT/'T'/, I1HY/'Y'/, I6HP/'   P'/, I6H/'     '/
      INTEGER OPRAND
      IF (OPRAND .LT. 0) GO TO 4
      IF (OPRAND .GT. 5000)  GO TO 4
      IF (TUSE(OPRAND) .LT. 0) GO TO 4
      IF ((OPRAND .GE.1001) .AND. (OPRAND .LE.1028)) GO TO 2
      NAME1=I1HT
      NAME2=TUSE(OPRAND)
      NAME3=I6H
      GO TO 3
    2 NAME1=I1HY
      NAME2 = OPRAND - 1000
      NAME3=I6HP
    3 CALL ERROR(2)
      WRITE(6,1) NAME1,NAME2,NAME3
    1 FORMAT(75H0A VARIABLE CANNOT BE REFERENCED BEFORE A VALUE HAS BEEN
     1 ASSIGNED TO IT        ,A1,I3,A6)
    4 RETURN
      END
```

```
      SUBROUTINE NXTCHR
C     NXTCHR OBTAINS THE NEXT NON-BLANK CHARACTER FROM THE INPUT STREAM AND SETS
C     THE COMMON VARIABLES NCLASS AND NSYMB ACCORDING TO THE FOLLOWING TABLE
C
C                   SCANNED SYMBOL       NCLASS     NSYMB
C
C                         A                 1          A
C                         B                 1          B
C                         .                 .          .
C                         .                 .          .
C                         Z                 1          Z
C                         0                 2          0
C                         1                 2          1
C                         .                 .          .
C                         .                 .          .
C                         9                 2          9
C                         =                 3          1
C                         +                 3          2
C                         -                 3          3
C                         *                 3          4
C                         /                 3          5
C                         (                 3          6
C                         )                 3          7
C                         $                 3          8
C                         .                 3          9
C
      COMMON/SCAN/NCLASS,NSYMB,LCLASS,COLUMN,CARD(80),NUMERR,MAXY
      INTEGER NCLASS,NSYMB,LCLASS,COLUMN,CARD,NUMERR,MAXY
      INTEGER   ISYMB(5), INTG(10), ISS(9),I1H,INTGR(10)
      DATA INTGR/0,1,2,3,4,5,6,7,8,9/
      DATA I1H/' '/
      DATA INTG/'0','1','2','3','4','5','6','7','8','9'/
      DATA ISYMB /'X','Y','T','E','P'/
      DATA ISS/'=','+','-','*','/','(',')','$','.'/
  100 IF (COLUMN .GT. 80) GO TO 2
   11 LSYMB = CARD(COLUMN)
      COLUMN = COLUMN + 1
```

```
      IF (LSYMB .EQ. I1H) GO TO 100
      DO 30 J=1,5
      IF (LSYMB .EQ.ISYMB(J)) GO TO 10
30    CONTINUE
      DO 40 I =1,10
      NCLASS = 2
      NSYMB = INTGR(I)
      IF (LSYMB .EQ. INTG(I)) GO TO 6
40    CONTINUE
      DO 50 J = 1,9
      NCLASS = 3
      NSYMB = INTGR(J+1)
      IF (LSYMB .EQ. ISS(J)) GO TO 6
50    CONTINUE
      GO TO 5
10    NCLASS = 1
      NSYMB = LSYMB
      GO TO 6
C     *******************************************************************
C     READ A NEW CARD AND SCAN CHARACTER IN COLUMN 1
2     READ(5,3,END=6)     (CARD(I),I=1,80)
      WRITE (6,4) (CARD(I), I = 1, 80)
3     FORMAT (80A1)
4     FORMAT (10X, 80A1)
      COLUMN = 1
      GO TO 11
C     *******************************************************************
C     ILLEGAL CHARACTER, PRINT MASSAGE AND DISCARD THE CHARACTER
5     CALL ERROR(1)
      WRITE (6,1) LSYMB
1     FORMAT (19H0ILLEGAL CHARACTER , A1, 12H WAS IGNORED   )
C    ********************************************************************
6     RETURN
      END
```

```
C       SUBROUTINE NXTTOK
C       NXTTOK OBTAINS THE NEXT TOKEN FROM THE INPUT STREAM. IT USES NXTCHR TO
C       ACTUALLY READ CHARACTERS FROM THE INPUT STREAM. THE PARAMETERS CLOUT AND
C       INDOUT ARE SET ACCORIDNG TO THE FOLLOWING TABLE
C
C     TOKEN        CLOUT                    INDOUT
C
C   OPERAND          1      ADDRESS OF THE OPERAND IN THE TARRAY
C   OPERATOR         2      PRECEDENCE OF THE OPERATOR
C   FUNCTION         3      INDEX OF THE FUNCTION (I.E. SIN=1, COS=2, ETC.)
C       (            4      MEANINGLESS
C       )            5           "
C       $            6           "
C END OF INPUT       7           "
C ILLEGAL TOKEN      8           "
C
C      NXTTOK ALSO CONSULTS A PRECEDENCE TABLE TO MAKE SURE THAT THE CURRENT
C      TOKEN MAY LEGALLY FOLLOW THE PRECEDING  TOKEN. THE PRECEDENCE MATRIX IS
C      GIVEN IN THE FOLLOWING TABLE
C
C      CURRENT TOKEN CLASS >   1   2   3   4   5   6
C      LAST   TOKEN  CLASS  V
C                           1  E   L   E   E   L   L
C                           2  L   C   L   L   E   E
C                           3  E   E   E   L   E   E
C                           4  L   C   L   L   E   E
C                           5  E   L   E   E   L   L
C                           6  L   E   E   E   E   L
C
C      ACTION E = ERROR
C             L = ILLEGAL TOKEN PAIR
C             C = CONDITIONALLY VALID (I.E. VALID IF THE CURRENT TOKEN IS A
C                 UNARY OPERATOR)
C      INTEGER FUNCTION SCNINT IS USED TO SCAN AN OPTIONALLY SIGNED INTEGER FROM
C      THE INPUT STREAM.
C
```

```
      SUBROUTINE NXTTCK(CLOUT,INDOUT)
      COMMON/ALLCAT/TAVAIL,TUSE(4000),GARBAG,TMPADR(1000),
     1 LAVAIL, LUSE(1000)
      INTEGER TAVAIL,TUSE,GARBAG,TMPADR,LAVAIL,LUSE
      COMMON/COD /NXTCOD,CODE(5000),NXTLIT,LITERL(100)
      INTEGER NXTCOD,CODE,NXTLIT
      REAL LITERL
      COMMON /SCAN/NCLASS,NSYMB,LCLASS,COLUMN,CARD(80),NUMERR,MAXY
      INTEGER NCLASS,NSYMB,LCLASS,COLUMN,CARD,NUMERR,MAXY
      INTEGER CLASS,INDEX,GETCEL,SCNINT,EXPCN,CLOUT,INDOUT,TABLE
      DATA I1HE/'E'/
      DATA I1HX/'X'/, I1HY/'Y'/, I1HT/'T'/, I1HP/'P'/
      GO TO (100,200,300), NCLASS
C     ****************************************************************
C     FIRST CHARACTER IS ALPHABETIC
  100 IF (NSYMB .EQ. I1HX) GO TO 110
      IF (NSYMB .EQ. I1HY) GO TO 120
      IF (NSYMB .EQ. I1HT) GO TO 130
      GO TO 999
C     ****************************************************************
C     INDEPENDENT VARIABLE (IE.X)
  110 CLASS=1
      INDEX = 2000
      TUSE(2000) = -2000
      GO TO 998
C     ****************************************************************
C     DEPENDENT VARIABLE OR ITS DERIVATIVE (IE.Y28 OR Y28P)
  120 INDEX=SCNINT(DUMMY)
      IF ((INDEX .LT. 1) .OR. (INDEX .GT.28)) GO TO 121
      CLASS=1
      IF(INDEX .GT. MAXY) MAXY=INDEX
      TUSE(INDEX)=-INDEX
C     CHECK IF NEXT CHARACTER IS P, THUS INDICATING A DERIVATIVE
      IF (NSYMB .NE. I1HP) GO TO 999
      INDEX = INDEX + 1000
      GO TO 998
```

```
  121 CALL ERROR(1)
      WRITE (6,122)
      GO TO 4
  122 FORMAT (46HODEPENCENT VARIABLES MUST BE NAMED Y1 THRU Y30 )
C    ******************************************************************
C     TEMPORARY VARIABLE (IE. T20)
  130 INDEX=SCNINT(DUMMY)
      IF ((INDEX .LT. 1) .OR. (INDEX .GT.1000))GO TO 132
      CLASS=1
      IF (TMPADR(INDEX) .NE. 0) GO TO 131
      I=GETCEL(DUMMY)
      TUSE(I)=INDEX
      TMPADR(INDEX)=I
  131 INDEX=TMPADR(INDEX)
      GO TO 999
  132 CALL ERROR(1)
      WRITE (6,133)
      GO TO 4
  133 FORMAT (48HOTEMPORARY VARIABLES MUST BE NAMED T1 THRU T2000)
C     *****************************************************************
C     FIRST CHARACTER IS NUMERIC
  200 CONS=NSYMB
C     PROCESS THE INTEGER PORTION
  210 CALL NXTCHR
      GO TO (211,212,213),NCLASS
C     NEXT CHARACTER IS ALPHABETIC, CHECK FOR E INDICATING EXPCNENT
  211 IF (NSYMB .EQ. I1HE) GO TO 230
      GO TO 290
      ACCRUE NUMBER AND CONTINUE INTEGER LOOP
  212 CONS=CCNS*10  +NSYMB
      GO TO 210
C     NEXT CHARACTER IS SPECIAL. CHECK FOR DECIMAL POINT.
  213 IF (NSYMB .EQ. 9) GO TO 220
      GO TO 290
C     *****************************************************************
C     PROCESS THE FRACTION PORTION
```

```
      220 FRACT=0.1E0
      229 CALL NXTCHR
          GO TO (211,222,290),NCLASS
C         ACCRUE NUMBER AND CONTINUE THE FRACTION LOOP
      222 CONS=CONS+NSYMB*FRACT
          FRACT=FRACT*0.1E0
          GO TO 229
C         ********************************************************
C         PROCESS THE EXPONENT PORTION
      230 EXPON=SCNINT(DUMMY)
          CONS=CONS*(10.0E0**EXPON)
C         ********************************************************
C         ASSIGN A PLACE IN THE LITERAL ARRAY FOR THE CONSTANT
      290 INDEX=LAVAIL
          CLASS=1
          IF(INDEX .EQ. 0) GO TO 291
          LAVAIL=LUSE(INDEX)
          LITERL(INDEX)=CONS
          INDEX = INDEX + 5000
          GO TO 999
C         ********************************************************
C         NO MORE SPACE AVAILABLE FOR LITERALS
      291 CALL ERROR(2)
          WRITE(6,292)
          GO TO 4
      292 FORMAT(39H0NO MORE SPACE AVAILABLE FOR CONSTANTS )
C         ********************************************************
C         FIRST CHARACTER IS A SPECIAL CHARACTER
      300 GO TO (399,399,399,304,399,388,388,388,309),NSYMB
C                 =   +   -   *   /   (   )   $   .
C         ********************************************************
C         DECIMAL POINT .
      309 CONS=0.0
          GO TO 220
C         *******************************************************
C         =  +  -  /
      399 CLASS=2
```

```
0078              INDEX=NSYMB
0079              GO TO 998
C                 ********************************************************************
C                 STAR, CHECK IF TIMES OR EXPONENTIATION IS DESIRED
0080        304   CLASS=2
0081              INDEX=4
0082              CALL NXTCHR
0083              IF ((NCLASS  .NE. 3) .OR. (NSYMB .NE. 4)) GO TO 999
0084              INDEX=6
0085              GO TO 998
C                 ********************************************************************
C                   (   )   $
0086        388   CLASS = NSYMB - 2
0087              IF (LCLASS .EQ. 6) GO TO 999
0088              GO TO 998
C                 ********************************************************************
C                 ADVANCE THE SCANNER FOR NEXT ENTRANCE
0089        998   CALL NXTCHR
C                 TEST FOR LEGAL ADJACENT TOKENS IN INPUT STRING
0090        999   GO TO (10,20,30,40,50,60,60,80),LCLASS
0091        10    GO TO (2,1,2,2,1,1),CLASS
0092        20    GO TO (1,29,1,1,2,2), CLASS
0093        30    GO TO  (2,2,2,1,2,2),CLASS
0094        40    GO TO (1,29,1,1,2,2),CLASS
0095        50    GO TO (2,1,2,2,1,1),CLASS
0096        60    GO TO (1,2,2,2,2,69), CLASS
0097        80    GO TO (4,4,4,4,4,1) , CLASS
C                 ********************************************************************
C                 CHECK FOR UNARY OPERATORS + AND -
0098        29    IF (INDEX .EQ. 2) GO TO 28
0099              IF (INDEX .EQ. 3) GO TO 28
0100              GO TO 2
0101        28    INDEX=INDEX+5
0102              GO TO 1
C                 ********************************************************************
C                 KLUDGE FOR END-OF-INPUT DETECTION
0103        69    CLASS=7
```

```
      GO TO 1
C     ****************************************************************
    2 CALL ERROR(1)
      WRITE (6,2222)
 2222 FORMAT (23H0ILLEGAL CHARACTER PAIR)
    4 CLASS=8
    1 CLOUT=CLASS
      LCLASS=CLASS
      INDOUT=INDEX
      RETURN
      END
```

```
      INTEGER FUNCTION SCNINT(DUMMY)
C     INTEGER FUNCTION SCNINT IS USED TO SCAN AN OPTIONALLY SIGNED INTEGER FROM
C     THE INPUT STREAM.
C
                COMMON/SCAN/NCLASS,NSYMB,LCLASS,COLUMN,CARD(80),NUMERR,MAXY
                INTEGER  NCLASS,NSYMB,LCLASS,COLUMN,CARD,NUMERR,MAXY
                ISIGN=1
                CALL NXTCHR
                GO TO (999,100,300),NCLASS
C               ****************************************************************
C               SPECIAL CHARACTER, SHOULD BE + OR -
          300   IF (NSYMB .EQ. 3) GO TO 301
                IF (NSYMB .EQ. 2) GO TO 302
                GO TO 999
          301   ISIGN=-1
          302   CALL NXTCHR
                GO TO (999,100,999),NCLASS
C               ****************************************************************
C               ACCRUE NUMBER
          100   SCNINT=NSYMB
          101   CALL NXTCHR
                GO TO (103,102,103),NCLASS
          102   SCNINT=SCNINT*10+NSYMB
                GO TO 101
          103   SCNINT=SCNINT*ISIGN
                GO TO 14
C               ****************************************************************
C               ERROR, NO INTEGER FOUND WHERE ONE WAS EXPECTED
          999   CALL ERROR(1)
                WRITE(6,9999)
                SCNINT=0
                GO TO 14
          9999  FORMAT (31H0INTEGER EXPECTED BUT NOT FOUND  )
          14    RETURN
                END
```

```
      SUBROUTINE EMIT(ARG)
C     SUBROUTINE EMIT IS USED TO PACK GENERATED CODE INTO THE CODE ARRAY. A
C     POSITIVE ARGUMENT INDICATES THAT ARG SHOULD BE PUT INTO THE NEXT
C     SEQUENTIAL LOCATION IN THE CODE ARRAY. A NEGATIVE VALUE FOR ARG MEANS
C     THAT A REFERENCE IS BEING MADE TO THE RESULT WHICH WAS COMPUTED AT
C     LOCATION ABS(ARG) IN THE CODE ARRAY. A TARRAY CELL IS ALLOCATED FOR THIS
C     RESULT, AND A REFERENCE TO IT IS GENERATED IN THE NEXT SEQUENTIAL LOCATION
C     OF THE CODE ARRAY.
C
            COMMON/COD /NXTCOD,CODE(5000),NXTLIT,LITERL(100)
            INTEGER NXTCOD,CODE,NXTLIT
            REAL LITERL
            INTEGER ARG,TEMP,GETCEL
C           ****************************************************************
C           A NEGATIVE ARGUMENT INDICATES A REFERENCE TO A PREVIOUS RESULT
            IF (ARG .GE. 0) GO TO 1
C           ALLOCATE A TEMPORARY STORAGE CELL AND BACKPATCH THE REFERENCE
            TEMP=GETCEL(DUMMY)
            CODE(-ARG)=TEMP
            ARG=TEMP
C           ****************************************************************
C           STORE THE SYLLABLE IN THE CODE ARRAY
      1     IF (NXTCOD .GT. 5000) GO TO 2
            CODE(NXTCOD)=ARG
            NXTCOD=NXTCOD+1
            RETURN
C           ****************************************************************
C           CODE ARRAY OVERFLOW. FATAL ERROR
      2     CALL ERROR(2)
            WRITE(6,3)
      3     FORMAT(55H0INSUFFICIENT SPACE FOR STORING EQUATIONS IN CODE ARRAY)
            RETURN
            END
```

```
      SUBROUTINE ERROR(IFLAG)
C     SUBROUTINE ERROR IS USED TO INDICATE THAT AN ERROR HAS BEEN DETECTED AND
C     TO INDICATE THE APPROXIMATE PLACE IN THE INPUT CARD STREAM WHERE
C     THE ERROR OCCURRED.
C
      COMMON/SCAN/NCLASS,NSYMB,LCLASS,COLUMN,CARD(80),NUMERR,MAXY
      INTEGER NCLASS,NSYMB,LCLASS,COLUMN,CARD,NUMERR,MAXY
      DIMENSION ICARD(81)
      DATA I1H/' '/, J1/'*'/
      DO 10 I=1,81
   10 ICARD(I)=I1H
      IF (IFLAG .EQ.1) ICARD(COLUMN)=J1
      WRITE(6,11) ICARD
   11 FORMAT (14H0*** ERROR *** ,27X,81A1    )
      NUMERR=NUMERR+1
      RETURN
      END
```

```
      INTEGER FUNCTION GETCEL(DUMMY)
C     GETCEL IS USED TO OBTAIN AN AVAILABLE CELL IN THE TARRAY. IT SIMPLY POPS
C     THE TOP ENTRY OFF THE TUSE STACK.
C
      COMMON/ALLOCAT/TAVAIL,TUSE(4000),GARBAG,TMPADR(1000),
     1 LAVAIL, LUSE(1000)
      INTEGER TAVAIL,TUSE,GARBAG,TMPADR,LAVAIL,LUSE
      IF (TAVAIL .EQ. 0) GO TO 1
      GETCEL=TAVAIL
      TAVAIL=TUSE(TAVAIL)
      RETURN
C     ******************************************************************
C     NO MORE AVAILABLE CELLS IN THE T-ARRAY
    1 CALL ERROR(2)
      WRITE(6,2)
      GETCEL=1
    2 FORMAT (43H0INSUFFICIENT SPACE FOR COMPILING EQUATIONS )
      RETURN
      END
```

```
      SUBROUTINE GMPRD(A,B,R,N,M,L)
      DIMENSION A(1),B(1),R(1)
      IR = 0
      IK = -M
      DO 10 K=1,L
      IK = IK + M
      DO 10 J = 1,N
      IR = IR + 1
      JI = J - N
      IB = IK
      R(IR) = 0
      DO 10 I = 1,M
      JI = JI + N
      IB = IB + 1
   10 R(IR) = R(IR) + A(JI)*B(IB)
      RETURN
      END
```

```
      SUBROUTINE MINV(A,N,D,L,M)

      DIMENSION A(1),L(1),M(1)
      D = 1.0
      NK = -N
      DO 80 K = 1,N
      NK = NK + N
      L(K) = K
      M(K) = K
      KK = NK + K
      BIGA = A(KK)
      DO 20 J = K,N
      IZ = N*(J-1)
      DO 20 I=K,N
      IJ = IZ + I
   10 IF (ABS(BAGA)-ABS(A(IJ))) 15,20,20
   15 BIGA = A(IJ)
      L(K) = I
      M(K) = J
   20 CONTINUE
C
C     INTERCHANGE ROWS
C
      J = L(K)
      IF (J-K) 35,35,25
   25 KI = K-N
      DO 30 I = 1, N
      KI = KI + N
      HOLD = -A(KI)
      JI = KI - K + J
      A(KK) = 1.0/BIGA
   30 A(JI) = HOLD
C
C     INTERCHANGE COLUMNS
C
```

```
0029          35  I = M(K)
0030              IF (I-K) 45,45,38
0031          38  JP = N*(I-1)
0032              DO 40 J = 1,N
0033              JK = NK + J
0034              JI = JP + J
0035              HOLD = -A(JK)
0036              A(JK) = A(JI)
0037          40  A(JI) = HOLD
C
C                 DIVIDE COLUMN BY MINUS PIVOT (VALUE OF PIVOT ELEMENT IS
C                 CONTAINED IN BIGA)
C
0038          45  IF (BIGA) 48,46,48
0039          46  D = 0.0
0040              RETURN
0041          48  DO 55 I = 1,N
0042              IF (I-K) 50,55,50
0043          50  IK = NK + I
0044              A(IK) = A(IK)/(-BIGA)
0045          55  CONTINUE
C
C                 REDUCE MARTIX
C
0046              DO 65 I=1,N
0047              IK = NK + I
0048              HOLD = A(IK)
0049              IJ = I - N
0050              DO 65 J=1,N
0051              IJ = IJ + N
0052              IF (I-K) 60,65,60
0053          60  IF (J-K) 62,65,62
0054          62  KJ = IJ - I + K
0055              A(IJ) = HOLD*A(KJ) + A(IJ)
0056          65  CONTINUE
```

```
C
C          DIVIDE ROW BY PIVOT
C
                KJ = K - N
                DO 75 J = 1,N
                KJ = KJ + N
                IF (J-K) 70,75,70
          70    A(KJ) = A(KJ)/BIGA
          75    CONTINUE
C
C          PRODUCT OF PIVOTS
C
                D = D*BIGA
C
C          REPLACE PIVOT BY RECIPROCAL
C
          80    CONTINUE
C
C          FINAL ROW AND COLUMN INTERCHANGE
C
                K = N
         100    K = (K-1)
                IF (K) 150,150,105
         105    I = L(K)
                IF (I-K) 120,120,108
         108    JQ = N*(K-1)
                JR = N*(I-1)
                DO 110 J = 1,N
                JK = JQ + J
                HOLD = A(JK)
                JI = JR + J
                A(JK) = -A(JI)
         110    A(JI) = HOLD
         120    J = M(K)
                IF (J-K) 100,100,125
         125    KI = K -N
```

```
      DO 130 I=1,N
      KI = KI + N
      HOLD = A(KI)
      JI = KI - K + J
      A(KI) = -A(JI)
  130 A(JI) = HOLD
      GO TO 100
  150 RETURN
      END
```

COEFFICIENTS A(6,6,6)

0.5000000E 00	0.6666666E 00	0.7500000E 00	0.8000000E 00	0.8333333E 00
0.8571427E 00	0.7886751E 00	0.8449489E 00	0.8774852E 00	0.8987135E 00
0.9136634E 00	0.9247639E 00	0.8872983E 00	0.9114119E 00	0.9270059E 00
0.9379241E 00	0.9459975E 00	0.9522110E 00	0.9305681E 00	0.9428958E 00
0.9514993E 00	0.9578471E 00	0.9627240E 00	0.9665886E 00	0.9530899E 00
0.9601901E 00	0.9654210E 00	0.9694357E 00	0.9726143E 00	0.9751938E 00
0.9662347E 00	0.9706835E 00	0.9740953E 00	0.9767953E 00	0.9789852E 00
0.9807972E 00	0.0	0.0	0.0	0.0
0.0	0.0	0.2113248E 00	0.3550510E 00	0.4558481E 00
0.5298579E 00	0.5863366E 00	0.6307915E 00	0.5000000E 00	0.5905331E 00
0.6529962E 00	0.6988112E 00	0.7338893E 00	0.7616239E 00	0.6699905E 00
0.7231569E 00	0.7613991E 00	0.7902831E 00	0.8128915E 00	0.8310789E 00
0.7692346E 00	0.8019866E 00	0.8265197E 00	0.8456051E 00	0.8608863E 00
0.8734027E 00	0.8306047E 00	0.8519214E 00	0.8684360E 00	0.8816167E 00
0.8923851E 00	0.9013507E 00	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.1127017E 00
0.2123405E 00	0.2949978E 00	0.3632646E 00	0.4201130E 00	0.4679832E 00

0.3300095E 00	0.4164096E 00	0.4829527E 00	0.5348464E 00	0.5789157E 00
0.6146694E 00	0.5000000E 00	0.5620252E 00	0.6101136E 00	0.6485699E 00
0.6800592E 00	0.7063338E 00	0.6193095E 00	0.6630152E 00	0.6975631E 00
0.7256027E 00	0.7488365E 00	0.7684135E 00	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.1397598E 00	0.2041485E 00	0.2614778E 00
0.3121355E 00	0.3568937E 00	0.2307653E 00	0.3045357E 00	0.3656664E 00
0.4171002E 00	0.4609336E 00	0.4987098E 00	0.3806904E 00	0.4413285E 00
0.4909636E 00	0.5324115E 00	0.5675724E 00	0.5977890E 00	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.4691007E-01	0.9853500E-01
0.1489457E 00	0.1962120F 00	0.2397920E 00	0.2796931E 00	0.1693953E 00
0.2307661E 00	0.2843189E 00	0.3313004E 00	0.3727511E 00	0.4095333E 00

0.C	0.0	0.0	0.0	0.0
0.C	0.0	0.0	C.0	0.0
0.0	0.0	0.0	C.0	0.0
0.0	0.0	0.0	C.0	0.0
0.C	0.0	0.0	0.0	0.0
0.0	0.0	0.0	C.0	0.0
0.3376524E-01	0.7305425E-01	0.1131943E 00	0.1522731E 00	0.1894696E 00
0.2244689E 00				

COEFFICIENTS C(6,6,6)

0.1000000E 01	0.7500000E 00	0.5925925E 00	0.4882813E 00	0.4147200E 00
0.3602323E 00	0.5000000E 00	0.3764030E 00	0.3020174E 00	0.2522389E 00
0.2165682E 00	0.1897451E 00	0.2777777E 00	0.2204621E 00	0.1828570E 00
0.1562502E 00	0.1364201E 00	0.1210651E 00	0.1739274E 00	0.1437135E 00
0.1224810E 00	0.1067317E 00	0.9457964E-01	0.8491641E-01	0.1184634E 00
0.1007941E 00	0.8772683E-01	0.7766646E-01	0.6968027E-01	0.6318581E-01
0.8566225E-01	0.7449412E-01	0.6590915E-01	0.5910210E-01	0.5357164E-01
0.4898897E-01	0.0	0.0	0.0	0.0
0.0	0.0	0.5000000E 00	0.5124857E 00	0.4850196E 00
0.4497610E 00	0.4152777E 00	0.3838795E 00	0.4444444E 00	0.3881934E 00
0.3429757E 00	0.3065149E 00	0.2767369E 00	0.2520615E 00	0.1994007E 00
0.2813560E 00	0.2474584E 00	0.2208490E 00	0.1994007E 00	0.1817443E 00
0.2393143E 00	0.2084506E 00	0.1847351E 00	0.1659088E 00	0.1505875E 00
0.1378693E 00	0.1803808E 00	0.1591021E 00	0.1423749E 00	0.1288624E 00
0.1177105E 00	0.1083460E 00	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.2777777E 00
0.3288442E 00	0.3441672E 00	0.3437661E 00	0.3360796E 00	0.3250821E 00

0.3260726E 00	0.3118265E 00	0.2942587E 00	0.2764993E 00	0.2596964E 00
0.2442204E 00	0.2844444E 00	0.2604634E 00	0.2396317E 00	0.2215590E 00
0.2058293E 00	0.1920667E 00	0.2339569E 00	0.2123519E 00	0.1943656E 00
0.1791571E 00	0.1661325E 00	0.1548567E 00	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.1739274E 00	0.2231039E 00	0.2483943E 00	0.2605729E 00
0.2650933E 00	0.2650126E 00	0.2393143E 00	0.2426935E 00	0.2397361E 00
0.2336878E 00	0.2262012E 00	0.2181501E 00	0.2339569E 00	0.2235548E 00
0.2127798E 00	0.2022694E 00	0.1923066E 00	0.1829978E 00	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.1184634E 00	0.1598203E 00
0.1854340E 00	0.2013093E 00	0.2108042E 00	0.2160023E 00	0.1803808E 00
0.1904749E 00	0.1944685E 00	0.1948467E 00	0.1930448E 00	0.1899267E 00

0.0	0.0	0.0	C.0	0.0
0.C	0.0	0.0	0.0	0.0
0.C	0.0	0.0	C.0	0.0
0.0	C.0	0.0	C.0	0.0
0.C	0.0	0.0	C.0	0.0
0.0	0.0	0.0	C.0	0.0
0.8566225E-01	0.1196136E 00	0.1429C80E 00	0.1590899E 00	0.1702574E 00
0.1778027E 0C				

Appendix B. Computer Output

SYMMETRICAL REFLECTION FUNCTION R(X,I,J) FOR GAMMA = 0.1

I,J	X = 0.1	X = 0.2	X = 0.3	X = 0.4	X = 0.5	NN
1,1	0.1283550E-02	0.1284326E-02	0.1284352E-02	0.1284366E-02	0.1284373E-02	1
1,2	0.2136011E-02	0.2162914E-02	0.2165527E-02	0.2166411E-02	0.2166754E-02	2
1,3	0.2345844E-02	0.2392465E-02	0.2399802E-02	0.2403559E-02	0.2405690E-02	3
1,4	0.2419388E-02	0.2474848E-02	0.2485089E-02	0.2491062E-02	0.2494923E-02	4
1,5	0.2451749E-02	0.2511533E-02	0.2523249E-02	0.2530539E-02	0.2535498E-02	5
1,6	0.2467396E-02	0.2529328E-02	0.2541849E-02	0.2549850E-02	0.2555462E-02	6
1,7	0.2474433E-02	0.2537348E-02	0.2550247E-02	0.2558607E-02	0.2564526E-02	7
2,2	0.5168006E-02	0.6312598E-02	0.6570127E-02	0.6629929E-02	0.6644562E-02	8
2,3	0.6137464E-02	0.8228969E-02	0.8946564E-02	0.9196855E-02	0.9286590E-02	9
2,4	0.6495927E-02	0.9029824E-02	0.1002383E-01	0.1041974E-01	0.1058161E-01	10
2,5	0.6656613E-02	0.9405930E-02	0.1054749E-01	0.1102861E-01	0.1123669E-01	11
2,6	0.6734945E-02	0.9593189E-02	0.1081247E-01	0.1134040E-01	0.1157495E-01	12
2,7	0.6770305E-02	0.9678584E-02	0.1093426E-01	0.1148457E-01	0.1173200E-01	13
3,3	0.7397372E-02	0.1127417E-01	0.1330094E-01	0.1436145E-01	0.1491748E-01	14
3,4	0.7866688E-02	0.1258527E-01	0.1540525E-01	0.1708979E-01	0.1809661E-01	15
3,5	0.8077595E-02	0.1320744E-01	0.1645217E-01	0.1850295E-01	0.1979941E-01	16
3,6	0.8180529E-02	0.1351863E-01	0.1698745E-01	0.1923961E-01	0.2070201E-01	17
3,7	0.8227028E-02	0.1366083E-01	0.1723463E-01	0.1958302E-01	0.2112623E-01	18
4,4	0.8378465E-02	0.1413016E-01	0.1805945E-01	0.2073904E-01	0.2256469E-01	19
4,5	0.8608624E-02	0.1486557E-01	0.1938953E-01	0.2265369E-01	0.2500595E-01	20
4,6	0.8720983E-02	0.1523390E-01	0.2007173E-01	0.2365744E-01	0.2631153E-01	21
4,7	0.8771747E-02	0.1540232E-01	0.2038723E-01	0.2412660E-01	0.1692771E-01	22
5,5	0.8847527E-02	0.1565591E-01	0.2086592E-01	0.2484322E-01	0.1787479E-01	23
5,6	0.8964162E-02	0.1605198E-01	0.2162419E-01	0.2599383E-01	0.2941480E-01	24
5,7	0.9016857E-02	0.1623315E-01	0.2197506E-01	0.2653220E-01	0.3014288E-01	25
6,6	0.9082898E-02	0.1646209E-01	0.2242199E-01	0.2722296E-01	0.3108356E-01	26
6,7	0.9136543E-02	0.1664969E-01	0.2279129E-01	0.2794321E-01	0.3198970E-01	27
7,7	0.9190626E-02	0.1684024E-01	0.2316915E-01	0.2839130E-01	0.3269221E-01	28

I,J = 1,2,....,7, WHERE 1 IS 88.5 DEG FROM $\bar{N}$, 2 IS 82.6 DEG FROM $\bar{N}$, 3 IS 72.7 DEG FROM $\bar{N}$, 4 IS 60.0 DEG FROM $\bar{N}$, 5 IS 45.3 DEG FROM $\bar{N}$, 6 IS 29.4 DEG FROM $\bar{N}$, 7 IS 13.0 DEG FROM $\bar{N}$.

SYMMETRICAL REFLECTION FUNCTION R(X,I,J) FOR GAMMA = 0.1

I,J	X = 0.6	X = 0.7	X = 0.8	X = 0.9	X = 1.0	NN
1,1	0.1284379E-02	0.1284382E-02	0.1284385E-02	0.1284386E-02	0.1284386E-02	1
1,2	0.2166897E-02	0.2166960E-02	0.2166992E-02	0.2167010E-02	0.2167018E-02	2
1,3	0.2406935E-02	0.2407675E-02	0.2408124E-02	0.2408397E-02	0.2408567E-02	3
1,4	0.2497491E-02	0.2499227E-02	0.2500418E-02	0.2501241E-02	0.2501814E-02	4
1,5	0.2538996E-02	0.2541496E-02	0.2543309E-02	0.2044632E-02	0.2545607E-02	5
1,6	0.2559513E-02	0.2562486E-02	0.2564699E-02	0.2566360E-02	0.2567614E-02	6
1,7	0.2568851E-02	0.2572063E-02	0.2574481E-02	0.2576319E-02	0.2577722E-02	7
2,2	0.6648447E-02	0.6649617E-02	0.6650019E-02	0.6650198E-02	0.6650273E-02	8
2,3	0.9320177E-02	0.9333547E-02	0.9339321E-02	0.9342052E-02	0.9343464E-02	9
2,4	0.1065059E-01	0.1068182E-01	0.1069711E-01	0.1070527E-01	0.1071002E-01	10
2,5	0.1133049E-01	0.1137539E-01	0.1139862E-01	0.1141174E-01	0.1141979E-01	11
2,6	0.1168353E-01	0.1173691E-01	0.1176528E-01	0.1178173E-01	0.1179210E-01	12
2,7	0.1184791E-01	0.1190558E-01	0.1193659E-01	0.1195479E-01	0.1196640E-01	13
3,3	0.1520986E-01	0.1536419E-01	0.1544604E-01	0.1548972E-01	0.1551317E-01	14
3,4	0.1869912E-01	0.1906034E-01	0.1927743E-01	0.1940830E-01	0.1948747E-01	15
3,5	0.2061979E-01	0.2113972E-01	0.2146993E-01	0.2168021E-01	0.2181455E-01	16
3,6	0.2165240E-01	0.2227096E-01	0.2267436E-01	0.2293813E-01	0.2311116E-01	17
3,7	0.2214118E-01	0.2280965E-01	0.2325082E-01	0.2354271E-01	0.2373647E-01	18
4,4	0.2380797E-01	0.2465454E-01	0.2523101E-01	0.2562366E-01	0.2589120E-01	19
4,5	0.2669980E-01	0.2791901E-01	0.2879634E-01	0.2942761E-01	0.2988184E-01	20
4,6	0.2827439E-01	0.2972528E-01	0.3079740E-01	0.3158949E-01	0.3217465E-01	21
4,7	0.2092416E-01	0.3059234E-01	0.3176496E-01	0.3264164E-01	0.3329700E-01	22
5,5	0.3018311E-01	0.3193942E-01	0.3327497E-01	0.3429016E-01	0.3506156E-01	23
5,6	0.3209003E-01	0.3418035E-01	0.3581262E-01	0.3708658E-01	0.3808047E-01	24
5,7	0.3300033E-01	0.3525975E-01	0.3704514E-01	0.3845522E-01	0.3956841E-01	25
6,6	0.3418421E-01	0.3667222E-01	0.3866724E-01	0.4026603E-01	0.4154668E-01	26
6,7	0.3527886E-01	0.3794989E-01	0.4011737E-01	0.4187518E-01	0.4330003E-01	27
7,7	0.3622985E-01	0.3913688E-01	0.4152391E-01	0.4348274E-01	0.4508938E-01	28

I,J = 1,2,....,7. WHERE 1 IS 88.5 DEG FROM $\bar{N}$. 2 IS 82.6 DEG FROM $\bar{N}$. 3 IS 72.7 DEG FROM $\bar{N}$. 4 IS 60.0 DEG FROM $\bar{N}$. 5 IS 45.3 DEG FROM $\bar{N}$. 6 IS 29.4 DEG FROM $\bar{N}$. 7 IS 13.0 DEG FROM $\bar{N}$.

SYMMETRICAL REFLECTION FUNCTION R(X,I,J) FOR GAMMA = 0.1

I,J	X = 1.1	X = 1.2	X = 1.3	X = 1.4	X = 1.5	NN
1,1	0.1284389E-02	0.1284390E-02	0.1284389E-02	0.1284390E-02	0.1284392E-02	1
1,2	0.2167027E-02	0.2167031E-02	0.2167034E-02	0.2167035E-02	0.2167038E-02	2
1,3	0.2408675E-02	0.2408741E-02	0.2408784E-02	0.2408810E-02	0.2408830E-02	3
1,4	0.2502219E-02	0.2502502E-02	0.2502702E-02	0.2502845E-02	0.2502949E-02	4
1,5	0.2546330E-02	0.2546866E-02	0.2547267E-02	0.2547567E-02	0.2547795E-02	5
1,6	0.2568570E-02	0.2569298E-02	0.2569856E-02	0.2570286E-02	0.2570620E-02	6
1,7	0.2578804E-02	0.2579638E-02	0.2580285E-02	0.2580788E-02	0.2581184E-02	7
2,2	0.6650325E-02	0.6650355E-02	0.6650377E-02	0.6650388E-02	0.6650403E-02	8
2,3	0.9344239E-02	0.9344690E-02	0.9344976E-02	0.9345144E-02	0.9345252E-02	9
2,4	0.1071298E-01	0.1071490E-01	0.1071621E-01	0.1071711E-01	0.1071775E-01	10
2,5	0.1142508E-01	0.1142873E-01	0.1143134E-01	0.1143324E-01	0.1143464E-01	11
2,6	0.1179909E-01	0.1180404E-01	0.1180768E-01	0.1181040E-01	0.1181246E-01	12
2,7	0.1197431E-01	0.1197999E-01	0.1198420E-01	0.1198739E-01	0.1198984E-01	13
3,3	0.1552587E-01	0.1553281E-01	0.1553665E-01	0.1553879E-01	0.1554000E-01	14
3,4	0.1953557E-01	0.1956493E-01	0.1958297E-01	0.1959411E-01	0.1960104E-01	15
3,5	0.2190072E-01	0.2195624E-01	0.2199220E-01	0.2201563E-01	0.2203101E-01	16
3,6	0.2322509E-01	0.2330045E-01	0.2335055E-01	0.2338406E-01	0.2340663E-01	17
3,7	0.2386556E-01	0.2395196E-01	0.2401009E-01	0.2404942E-01	0.2407622E-01	18
4,4	0.2607356E-01	0.2619794E-01	0.2628282E-01	0.2634079E-01	0.2638042E-01	19
4,5	0.3020871E-01	0.3044397E-01	0.3061333E-01	0.3073529E-01	0.3082315E-01	20
4,6	0.3260697E-01	0.3292638E-01	0.3316243E-01	0.3333690E-01	0.3346590E-01	21
4,7	0.3378692E-01	0.3415319E-01	0.3442707E-01	0.3463191E-01	0.3478514E-01	22
5,5	0.3564756E-01	0.3609264E-01	0.3643063E-01	0.3668726E-01	0.3688209E-01	23
5,6	0.3885562E-01	0.3946000E-01	0.3993112E-01	0.4029828E-01	0.4058438E-01	24
5,7	0.4044692E-01	0.4114000E-01	0.4168668E-01	0.4211777E-01	0.4245765E-01	25
6,6	0.4257209E-01	0.4339283E-01	0.4404957E-01	0.4457493E-01	0.4499510E-01	26
6,7	0.4445451E-01	0.4538958E-01	0.4614670E-01	0.4675955E-01	0.4725550E-01	27
7,7	0.4640656E-01	0.4748600E-01	0.4837034E-01	0.4909462E-01	0.4968765E-01	28

I,J = 1,2,....,7, WHERE 1 IS 88.5 DEG FROM $\bar{N}$, 2 IS 82.6 DEG FROM $\bar{N}$, 3 IS 72.7 DEG FROM $\bar{N}$, 4 IS 60.0 DEG FROM $\bar{N}$, 5 IS 45.3 DEG FROM $\bar{N}$, 6 IS 29.4 DEG FROM $\bar{N}$, 7 IS 13.0 DEG FROM $\bar{N}$.

SYMMETRICAL REFLECTION FUNCTION R(X,I,J) FOR GAMMA = 0.1

I,J	X = 1.6	X = 1.7	X = 1.8	X = 1.9	X = 2.0	NN
1,1	0.1284392E-02	0.1284392E-02	0.1284392E-02	0.1284392E-02	0.1284392E-02	1
1,2	0.2167041E-02	0.2167041E-02	0.2167041E-02	0.2167043E-02	0.2167044E-02	2
1,3	0.2408843E-02	0.2408850E-02	0.2408855E-02	0.2408859E-02	0.2408862E-02	3
1,4	0.2503022E-02	0.2503075E-02	0.2503111E-02	0.2503139E-02	0.2503158E-02	4
1,5	0.2547966E-02	0.2548095E-02	0.2548191E-02	0.2548266E-02	0.2548322E-02	5
1,6	0.2570876E-02	0.2571074E-02	0.2571228E-02	0.2571349E-02	0.2571443E-02	6
1,7	0.2581492E-02	0.2581733E-02	0.2581924E-02	0.2582073E-02	0.2582191E-02	7
2,2	0.6650414E-02	0.6650422E-02	0.6650422E-02	0.6650425E-02	0.6650429E-02	8
2,3	0.9345327E-02	0.9345379E-02	0.9345412E-02	0.9345435E-02	0.9345453E-02	9
2,4	0.1071820E-01	0.1071852E-01	0.1071875E-01	0.1071892E-01	0.1071903E-01	10
2,5	0.1143570E-01	0.1143648E-01	0.1143707E-01	0.1143752E-01	0.1143786E-01	11
2,6	0.1181404E-01	0.1181526E-01	0.1181620E-01	0.1181692E-01	0.1181749E-01	12
2,7	0.1199174E-01	0.1199321E-01	0.1199437E-01	0.1199527E-01	0.1199598E-01	13
3,3	0.1554070E-01	0.1554111E-01	0.1554136E-01	0.1554151E-01	0.1554160E-01	14
3,4	0.1960540E-01	0.1960816E-01	0.1960992E-01	0.1961106E-01	0.1961180E-01	15
3,5	0.2204117E-01	0.2204795E-01	0.2205251E-01	0.2205561E-01	0.2205773E-01	16
3,6	0.2342194E-01	0.2343242E-01	0.2343965E-01	0.2344468E-01	0.2344823E-01	17
3,7	0.2409462E-01	0.2410736E-01	0.2411625E-01	0.2412252E-01	0.2412697E-01	18
4,4	0.2640752E-01	0.2642609E-01	0.2643882E-01	0.2644755E-01	0.2645355E-01	19
4,5	0.3088646E-01	0.3093212E-01	0.3096505E-01	0.3098882E-01	0.3100598E-01	20
4,6	0.3356130E-01	0.3363189E-01	0.3368415E-01	0.3372285E-01	0.3375152E-01	21
4,7	0.3489981E-01	0.3498566E-01	0.3504995E-01	0.3509812E-01	0.3513424E-01	22
5,5	0.3703000E-01	0.3714228E-01	0.3722752E-01	0.3729222E-01	0.3734134E-01	23
5,6	0.4080729E-01	0.4098095E-01	0.4111622E-01	0.4122158E-01	0.4130364E-01	24
5,7	0.4272559E-01	0.4293678E-01	0.4310323E-01	0.4323440E-01	0.4333777E-01	25
6,6	0.4533107E-01	0.4559967E-01	0.4581435E-01	0.4598543E-01	0.4612304E-01	26
6,7	0.4765676E-01	0.4798134E-01	0.4824386E-01	0.4845614E-01	0.4862777E-01	27
7,7	0.5017310E-01	0.5057041E-01	0.5089552E-01	0.5116149E-01	0.5137906E-01	28

I,J = 1,2,...,7, WHERE 1 IS 88.5 DEG FROM $\bar{N}$, 2 IS 82.6 DEG FROM $\bar{N}$, 3 IS 72.7 DEG FROM $\bar{N}$, 4 IS 60.0 DEG FROM $\bar{N}$, 5 IS 45.3 DEG FROM $\bar{N}$, 6 IS 29.4 DEG FROM $\bar{N}$, 7 IS 13.0 DEG FROM $\bar{N}$.

SYMMETRICAL REFLECTION FUNCTION R(X,I,J) FOR GAMMA = 0.1

I,J	X = 2.1	X = 2.2	X = 2.3	X = 2.4	X = 2.5	NN
1,1	0.1284392E-02	0.1284392E-02	0.1284392E-02	0.1284392E-02	0.1284392E-02	1
1,2	0.2167044E-02	0.2167044E-02	0.2167044E-02	0.2167044E-02	0.2167044E-02	2
1,3	0.2408862E-02	0.2408863E-02	0.2408864E-02	0.2408865E-02	0.2408866E-02	3
1,4	0.2503172E-02	0.2503183E-02	0.2503190E-02	0.2503197E-02	0.2503200E-02	4
1,5	0.2548365E-02	0.2548397E-02	0.2548422E-02	0.2548441E-02	0.2548455E-02	5
1,6	0.2571516E-02	0.2571573E-02	0.2571618E-02	0.2571654E-02	0.2571681E-02	6
1,7	0.2582284E-02	0.2582358E-02	0.2582417E-02	0.2582463E-02	0.2582500E-02	7
2,2	0.6650433E-02	0.6650433E-02	0.6650433E-02	0.6650433E-02	0.6650433E-02	8
2,3	0.9345461E-02	0.9345464E-02	0.9345472E-02	0.9345472E-02	0.9345476E-02	9
2,4	0.1071912E-01	0.1071918E-01	0.1017922E-01	0.1071926E-01	0.1071928E-01	10
2,5	0.1143812E-01	0.1143832E-01	0.1143847E-01	0.1143858E-01	0.1143866E-01	11
2,6	0.1181793E-01	0.1181827E-01	0.1181855E-01	0.1181875E-01	0.1181891E-01	12
2,7	0.1199654E-01	0.1199699E-01	0.1199734E-01	0.1199726E-01	0.1199784E-01	13
3,3	0.1554165E-01	0.1554169E-01	0.1554171E-01	0.1554173E-01	0.1554175E-01	14
3,4	0.1961229E-01	0.1961261E-01	0.1961284E-01	0.1961299E-01	0.1961310E-01	15
3,5	0.2205919E-01	0.2206022E-01	0.2206095E-01	0.2206147E-01	0.2206185E-01	16
3,6	0.2345075E-01	0.2345255E-01	0.2345387E-01	0.2345483E-01	0.2345555E-01	17
3,7	0.2413018E-01	0.2413251E-01	0.2413423E-01	0.2413549E-01	0.2413645E-01	18
4,4	0.2645768E-01	0.2646053E-01	0.2646250E-01	0.2646386E-01	0.2646480E-01	19
4,5	0.3101839E-01	0.3102735E-01	0.3103385E-01	0.3103856E-01	0.3104198E-01	20
4,6	0.3377379E-01	0.3378856E-01	0.3380027E-01	0.3380898E-01	0.3381545E-01	21
4,7	0.3516133E-01	0.3518167E-01	0.3519695E-01	0.3520843E-01	0.3521708E-01	22
5,5	0.3737862E-01	0.3740692E-01	0.3742842E-01	0.3744473E-01	0.3745711E-01	23
5,6	0.4136755E-01	0.4141732E-01	0.4145608E-01	0.4141627E-01	0.4150977E-01	24
5,7	0.4341921E-01	0.4348338E-01	0.4353394E-01	0.4357377E-01	0.4360516E-01	25
6,6	0.4623260E-01	0.4632011E-01	0.4639003E-01	0.4644587E-01	0.4649048E-01	26
6,7	0.4876652E-01	0.4887866E-01	0.4896930E-01	0.4904255E-01	0.4910174E-01	27
7,7	0.5155700E-01	0.5170251E-01	0.5182150E-01	0.5191877E-01	0.5199829E-01	28

I,J = 1,2,....,7, WHERE 1 IS 88.5 DEG FROM $\bar{N}$, 2 IS 82.6 DEG FROM $\bar{N}$, 3 IS 72.7 DEG FROM $\bar{N}$, 4 IS 60.0 DEG FROM $\bar{N}$, 5 IS 45.3 DEG FROM $\bar{N}$, 6 IS 29.4 DEG FROM $\bar{N}$, 7 IS 13.0 DEG FROM $\bar{N}$.

SYMMETRICAL REFLECTION FUNCTION R(X,I,J) FOR GAMMA = 0.1

I,J	X = 2.6	X = 2.7	X = 2.8	X = 2.9	X = 3.0	NN
1,1	0.1234392E-02	0.1284392E-02	0.1284392E-02	0.1284392E-02	0.1284392E-02	1
1,2	0.2167044E-02	0.2167044E-02	0.2167044E-02	0.2167044E-02	0.2167044E-02	2
1,3	0.2408867E-02	0.2408867E-02	0.2408867E-02	0.2408867E-02	0.2408867E-02	3
1,4	0.2503202E-02	0.2503205E-02	0.2503208E-02	0.2503208E-02	0.2503209E-02	4
1,5	0.2548467E-02	0.2548475E-02	0.2548481E-02	0.2548486E-02	0.2548491E-02	5
1,6	0.2571720E-02	0.2571719E-02	0.2571733E-02	0.2571744E-02	0.2571751E-02	6
1,7	0.2582529E-02	0.2582552E-02	0.2582570E-02	0.2582586E-02	0.2582597E-02	7
2,2	0.6650433E-02	0.6650433E-02	0.6650433E-02	0.6650433E-02	0.6650433E-02	8
2,3	0.9345479E-02	0.9345483E-02	0.9345483E-02	0.9345483E-02	0.9345483E-02	9
2,4	0.1071930E-01	0.1071931E-01	0.1071932E-01	0.1071932E-01	0.1071933E-01	10
2,5	0.1143873E-01	0.1143878E-01	0.1143882E-01	0.1143885E-01	0.1143887E-01	11
2,6	0.1181905E-01	0.1181914E-01	0.1181922E-01	0.1181929E-01	0.1181934E-01	12
2,7	0.1199801E-01	0.1199815E-01	0.1199826E-01	0.1199834E-01	0.1199841E-01	13
3,3	0.1554177E-01	0.1554177E-01	0.1554178E-01	0.1554178E-01	0.1554178E-01	14
3,4	0.1961317E-01	0.1961322E-01	0.1961326E-01	0.1961329E-01	0.1961330E-01	15
3,5	0.2206213E-01	0.2206233E-01	0.2206247E-01	0.2206258E-01	0.2206266E-01	16
3,6	0.2345607E-01	0.2345647E-01	0.2345677E-01	0.2345700E-01	0.2345717E-01	17
3,7	0.2413715E-01	0.2413770E-01	0.2413812E-01	0.2413843E-01	0.2413868E-01	18
4,4	0.2646544E-01	0.2646590E-01	0.1646621E-01	0.2646643E-01	0.2646658E-01	19
4,5	0.3104445E-01	0.3104626E-01	0.3104756E-01	0.3104851E-01	0.3104921E-01	20
4,6	0.3382027E-01	0.3382387E-01	0.3382654E-01	0.3382853E-01	0.3383002E-01	21
4,7	0.3522358E-01	0.3522848E-01	0.3523218E-01	0.3523497E-01	0.3523707E-01	22
5,5	0.3746653E-01	0.3747367E-01	0.3747909E-01	0.3748321E-01	0.3748634E-01	23
5,6	0.4152808E-01	0.4154233E-01	0.4155343E-01	0.4156207E-01	0.4156880E-01	24
5,7	0.4362988E-01	0.4364936E-01	0.4366469E-01	0.4367679E-01	0.4368631E-01	25
6,6	0.4652610E-01	0.4655454E-01	0.4657726E-01	0.4659539E-01	0.4660987E-01	26
6,7	0.4914956E-01	0.4918820E-01	0.4921940E-01	0.4924462E-01	0.4926498E-01	27
7,7	0.5206328E-01	0.5211641E-01	0.5215981E-01	0.5219530E-01	0.5222428E-01	28

I,J = 1,2,....,7, WHERE 1 IS 88.5 DEG FROM $\bar{N}$, 2 IS 82.6 DEG FROM $\bar{N}$, 3 IS 72.7 DEG FROM $\bar{N}$, 4 IS 60.0 DEG FROM $\bar{N}$, 5 IS 45.3 DEG FROM $\bar{N}$, 6 IS 29.4 DEG FROM $\bar{N}$, 7 IS 13.0 DEG FROM $\bar{N}$.

SYMMETRICAL REFLECTION FUNCTION R(X,I,J) FOR GAMMA = 0.6

I,J	X = 0.1	X = 0.2	X = 0.3	X = 0.4	X = 0.5	NN
1,1	0.8086778E-02	0.8112945E-02	0.8120809E-02	0.8125819E-02	0.8129302E-02	1
1,2	0.1378705E-01	0.1427338E-01	0.1442047E-01	0.1448975E-01	0.1452686E-01	2
1,3	0.1523420E-01	0.1608516E-01	0.1645788E-01	0.1668994E-01	0.1684150E-01	3
1,4	0.1574508E-01	0.1676418E-01	0.1727105E-01	0.1762395E-01	0.1787947E-01	4
1,5	0.1497046E-01	0.1707108E-01	0.1764938E-01	0.1807248E-01	0.1839369E-01	5
1,6	0.1607956E-01	0.1722132E-01	0.1783722E-01	0.1829875E-01	0.1865734E-01	6
1,7	0.1612856E-01	0.1728930E-01	0.1792280E-01	0.1840265E-01	0.1877938E-01	7
2,2	0.3370528E-01	0.4294711E-01	0.4572248E-01	0.4668551E-01	0.4708253E-01	8
2,3	0.4010014E-01	0.5648902E-01	0.6346840E-01	0.6667614E-01	0.6829286E-01	9
2,4	0.4246643E-01	0.6217918E-01	0.7164347E-01	0.7650942E-01	0.7923019E-01	10
2,5	0.4352741E-01	0.6485641E-01	0.7564121E-01	0.8146840E-01	0.8488560E-01	11
2,6	0.4404466E-01	0.6619048E-01	0.7766962E-01	0.8402252E-01	0.8783585E-01	12
2,7	0.4427822E-01	0.6679910E-01	0.7860309E-01	0.8520651E-01	0.8921212E-01	13
3,3	0.4837373E-01	0.7776242E-01	0.9545290E-01	0.1061870E 00	0.1127835E 00	14
3,4	0.5145622E-01	0.8693767E-01	0.1109972E 00	0.1273136E 00	0.1384284E 00	15
3,5	0.5284154E-01	0.9129417E-01	0.1187457E 00	0.1383033E 00	0.1522718E 00	16
3,6	0.5351762E-01	0.9347379E-01	0.1227107E 00	0.1440427E 00	0.1596346E 00	17
3,7	0.5382301E-01	0.9446985E-01	0.1245424E 00	0.1467201E 00	0.1631004E 00	18
4,4	0.5381258E-01	0.9770370E-01	0.1304611E 00	0.1552856E 00	0.1740292E 00	19
4,5	0.5632206E-01	0.1028297E 00	0.1402234E 00	0.1699923E 00	0.1935568E 00	20
4,6	0.5705896E-01	0.1053973E 00	0.1452323E 00	0.1777088E 00	0.2040170E 00	21
4,7	0.5739187E-01	0.1065715E 00	0.1475943E 00	0.1813167E 00	0.2089573E 00	22
5,5	0.5788783E-01	0.1083285E 00	0.1510236E 00	0.1867287E 00	0.2163597E 00	23
5,6	0.5864231E-01	0.1110843E 00	0.1565714E 00	0.1955276E 00	0.2286116E 00	24
5,7	0.5899771E-01	0.1123447E 00	0.1591388E 00	0.1996451E 00	0.2344059E 00	25
6,6	0.5943030E-01	0.1139351E 00	0.1623991E 00	0.2049043E 00	0.2418453E 00	26
6,7	0.5978183E-01	0.1152392E 00	0.1650969E 00	0.2092943E 00	0.2481082E 00	27
7,7	0.6013612E-01	0.1165633E 00	0.1678552E 00	0.2138127E 00	0.2545950E 00	28

I,J = 1,2,....,7, WHERE 1 IS 88.5 DEG FROM $\bar{N}$, 2 IS 82.6 DEG FROM $\bar{N}$, 3 IS 72.7 DEG FROM $\bar{N}$, 4 IS 60.0 DEG FROM $\bar{N}$, 5 IS 45.3 DEG FROM $\bar{N}$, 6 IS 29.4 DEG FROM $\bar{N}$, 7 IS 13.0 DEG FROM $\bar{N}$.

SYMMETRICAL REFLECTION FUNCTION R(X,I,J) FOR GAMMA = 0.6

I,J	X = 0.6	X = 0.7	X = 0.8	X = 0.9	X = 1.0	NN
1,1	0.8131832E-02	0.8133713E-02	0.8135144E-02	0.8136250E-02	0.8137111E-02	1
1,2	0.1454903E-01	0.1456352E-01	0.1457364E-01	0.1458106E-01	0.1458667E-01	2
1,3	0.1694344E-01	0.1701364E-01	0.1706294E-01	0.1709818E-01	0.1712378E-01	3
1,4	0.1806834E 01	0.1820998E-01	0.1831729E-01	0.1839926E-01	0.1846227E-01	4
1,5	0.1864214E-01	0.1883674E-01	0.1899043E-01	0.1911258E-01	0.1921009E-01	5
1,6	0.1894102E-01	0.1916809E-01	0.1935131E-01	0.1949997E-01	0.1962106E-01	6
1,7	0.1908047E-01	0.1932389E-01	0.1952223E-01	0.1968470E-01	0.1981829E-01	7
2,2	0.4727761E-01	0.4738924E-01	0.4746093E-01	0.4751078E-01	0.4754729E-01	8
2,3	0.6918895E-01	0.6972975E-01	0.7007873E-01	0.7031566E-01	0.7048237E-01	9
2,4	0.8088851E-01	0.8197886E-01	0.8273834E-01	0.8328915E-01	0.8369941E-01	10
2,5	0.8706599E-01	0.8856344E-01	0.8965081E-01	0.9047145E-01	0.9110636E-01	11
2,6	0.9032458E-01	0.9207165E-01	0.9336787E-01	0.9436655E-01	0.9515488E-01	12
2,7	0.9185320E-01	0.9372586E-01	0.9512895E-01	0.9622037E-01	0.9709018E-01	13
3,3	0.1169023E 00	0.1195217E 00	0.1212209E 00	0.1223462E 00	0.1231073E 00	14
3,4	0.1460557E 00	0.1513380E 00	0.1550348E 00	0.1576514E 00	0.1595246E 00	15
3,5	0.1623023E 00	0.1695578E 00	0.1748511E 00	0.1787490E 00	0.1816473E 00	16
3,6	0.1710853E 00	0.1795511E 00	0.1858600E 00	0.1906031E 00	0.1942018E 00	17
3,7	0.1752530E 00	0.1843278E 00	0.1911570E 00	0.1963408E 00	0.2003105E 00	18
4,4	0.1881625E 00	0.1988209E 00	0.2068673E 00	0.2129523E 00	0.2175640E 00	19
4,5	0.2121496E 00	0.2267933E 00	0.2383171E 00	0.2473844E 00	0.2545209E 00	20
4,6	0.2252456E 00	0.2423344E 00	0.2560714E 00	0.2671059E 00	0.2759677E 00	21
4,7	0.2314888E 00	0.2498082E 00	0.2646787E 00	0.2767387E 00	0.2865149E 00	22
5,5	0.2408227E 00	0.2609448E 00	0.2774515E 00	0.2909644E 00	0.3020089E 00	23
5,6	0.2565451E 00	0.2800288E 00	0.2997067E 00	0.3161525E 00	0.3298678E 00	24
5,7	0.2640553E 00	0.2892310E 00	0.3105333E 00	0.3285079E 00	0.3436393E 00	25
6,6	0.2737429E 00	0.3011509E 00	0.3246102E 00	0.3446261E 00	0.3616588E 00	26
6,7	0.2819658E 00	0.3113490E 00	0.3367453E 00	0.3586222E 00	0.3774138E 00	27
7,7	0.2905333E 00	0.3220342E 00	0.3495277E 00	0.3734388E 00	0.3941711E 00	28

I,J = 1,2,....,7. WHERE 1 IS 88.5 DEG FROM $\bar{N}$, 2 IS 82.6 DEG FROM $\bar{N}$, 3 IS 72.7 DEG FROM $\bar{N}$.
4 IS 60.0 DEG FROM $\bar{N}$, 5 IS 45.3 DEG FROM $\bar{N}$, 6 IS 29.4 DEG FROM $\bar{N}$, 7 IS 13.0 DEG FROM $\bar{N}$.

SYMMETRICAL REFLECTION FUNCTION R(X,I,J) FOR GAMMA = 0.6

I,J	X = 1.1	X = 1.2	X = 1.3	X = 1.4	X = 1.5	NN
1,1	0.8137785E-02	0.8138321E-02	0.8138750E-02	0.8139085E-02	0.8139860E-02	1
1,2	0.1459100E-01	0.1459438E-01	0.1459704E-01	0.1459917E-01	0.1460087E-01	2
1,3	0.1714263E-01	0.1715671E-01	0.1716735E-01	0.1717547E-01	0.1718175E-01	3
1,4	0.1851097E-01	0.1854879E-01	0.1857828E-01	0.1860137E-01	0.1861950E-01	4
1,5	0.1928822E-01	0.1935098E-01	0.1940152E-01	0.1944232E-01	0.1947530E-01	5
1,6	0.1972000E-01	0.1980101E-01	0.1986749E-01	0.1992212E-01	0.1996707E-01	6
1,7	0.1992846E-01	0.2001949E-01	0.2009485E-01	0.2015733E-01	0.2020919E-01	7
2,2	0.4757491E-01	0.4759625E-01	0.4761296E-01	0.4762618E-01	0.4763670E-01	8
2,3	0.7060277E-01	0.7069165E-01	0.7075828E-01	0.7080883E-01	0.7084775E-01	9
2,4	0.8401054E-01	0.8424920E-01	0.8443385E-01	0.8457756E-01	0.8468997E-01	10
2,5	0.9160537E-01	0.9200144E-01	0.9231782E-01	0.9257179E-01	0.9277630E-01	11
2,6	0.9578675E-01	0.9629798E-01	0.9671420E-01	0.9705430E-01	0.9733313E-01	12
2,7	0.9779370E-01	0.9836811E-01	0.9883994E-01	0.9922892E-01	0.9955055E-01	13
3,3	0.1236327E 00	0.1240027E 00	0.1242682E 00	0.1244620E 00	0.1246058E 00	14
3,4	0.1608813E 00	0.1618749E 00	0.1626108E 00	0.1631611E 00	0.1635767E 00	15
3,5	0.1838232E 00	0.1854722E 00	0.1867332E 00	0.1877056E 00	0.1884613E 00	16
3,6	0.1969572E 00	0.1990856E 00	0.2007438E 00	0.2020461E 00	0.2030762E 00	17
3,7	0.2033783E 00	0.2057698E 00	0.2076497E 00	0.2091390E 00	0.2103273E 00	18
4,4	0.2210678E 00	0.2237370E 00	0.2257765E 00	0.2273394E 00	0.2285408E 00	19
4,5	0.2601410E 00	0.2645709E 00	0.2680663E 00	0.2708278E 00	0.2730123E 00	20
4,6	0.2830850E 00	0.2888030E 00	0.2934001E 00	0.2970893E 00	0.3000762E 00	21
4,7	0.2944396E 00	0.3008644E 00	0.3060756E 00	0.3103051E 00	0.3137401E 00	22
5,5	0.3110249E 00	0.3183774E 00	0.3243688E 00	0.3292481E 00	0.3332202E 00	23
5,6	0.3412861E 00	0.3507774E 00	0.3586575E 00	0.3651925E 00	0.3706074E 00	24
5,7	0.3563533E 00	0.3670178E 00	0.3759511E 00	0.3834248E 00	0.3896710E 00	25
6,6	0.3761199E 00	0.3883727E 00	0.3987368E 00	0.4074897E 00	0.4148718E 00	26
6,7	0.3935161E 00	0.4072836E 00	0.4190331E 00	0.4290433E 00	0.4375588E 00	27
7,7	0.4121010E 00	0.4275705E 00	0.4408907E 00	0.4523389E 00	0.4621621E 00	28

I,J = 1,2,....,7, WHERE 1 IS 88.5 DEG FROM $\bar{N}$, 2 IS 82.6 DEG FROM $\bar{N}$, 3 IS 72.7 DEG FROM $\bar{N}$, 4 IS 60.0 DEG FROM $\bar{N}$, 5 IS 45.3 DEG FROM $\bar{N}$, 6 IS 29.4 DEG FROM $\bar{N}$, 7 IS 13.0 DEG FROM $\bar{N}$.

SYMMETRICAL REFLECTION FUNCTION R(X,I,J) FOR GAMMA = 0.6

I,J	X = 1.6	X = 1.7	X = 1.8	X = 1.9	X = 2.0	NN
1,1	0.8139588E-02	0.8139763E-02	0.8139908E-02	0.8140024E-02	0.8140117E-02	1
1,2	0.1460224E-01	0.1460333E-01	0.1460421E-01	0.1460493E-01	0.1460552E-01	2
1,3	0.1718664E-01	0.1719046E-01	0.1719350E-01	0.1719590E-01	0.1719783E-01	3
1,4	0.1863379E-01	0.1864509E-01	0.1865406E-01	0.1866118E-01	0.1866686E-01	4
1,5	0.1950200E-01	0.1952364E-01	0.1954122E-01	0.1955549E-01	0.1956710E-01	5
1,6	0.2000411E-01	0.2003463E-01	0.2005983E-01	0.2008063E-01	0.2009781E-01	6
1,7	0.2025227E-01	0.2028807E-01	0.2031787E-01	0.2034268E-01	0.2036333E-01	7
2,2	0.4764514E-01	0.4765191E-01	0.4765738E-01	0.4766180E-01	0.4766539E-01	8
2,3	0.7087803E-01	0.7090163E-01	0.7092023E-01	0.7093501E-01	0.7094687E-01	9
2,4	0.8477843E-01	0.8484811E-01	0.8490324E-01	0.8494693E-01	0.8498180E-01	10
2,5	0.9294134E-01	0.9307486E-01	0.9318298E-01	0.9327072E-01	0.9334213E-01	11
2,6	0.9756202E-01	0.9775031E-01	0.9790546E-01	0.9803325E-01	0.9813869E 00	12
2,7	0.9981692E-01	0.1000378E 00	0.1002212E 00	0.1003736E 00	0.1005003E 00	13
3,3	0.1247141E 00	0.1247965E 00	0.1248600E 00	0.1249095E 00	0.1249484E 00	14
3,4	0.1638935E 00	0.1641366E 00	0.1643247E 00	0.1644711E 00	0.1645858E 00	15
3,5	0.1890527E 00	0.1895183E 00	0.1898871E 00	0.1901805E 00	0.1904151E 00	16
3,6	0.2038965E 00	0.2045533E 00	0.2050819E 00	0.2055094E 00	0.2058563E 00	17
3,7	0.2112814E 00	0.2120518E 00	0.2126771E 00	0.2131868E 00	0.2136039E 00	18
4,4	0.2294675E 00	0.2301844E 00	0.2307410E 00	0.2311743E 00	0.2315128E 00	19
4,5	0.2747428E 00	0.2761160E 00	0.2772073E 00	0.2780757E 00	0.2787682E 00	20
4,6	0.3024764E 00	0.3044130E 00	0.3059775E 00	0.3072426E 00	0.3082669E 00	21
4,7	0.3165325E 00	0.3188044E 00	0.3206547E 00	0.3221632E 00	0.3233944E 00	22
5,5	0.3364525E 00	0.3390825E 00	0.3412221E 00	0.3429626E 00	0.3443785E 00	23
5,6	0.3750906E 00	0.3788000E 00	0.3818675E 00	0.3844030E 00	0.3864978E 00	24
5,7	0.3948866E 00	0.3992382E 00	0.4028664E 00	0.4058898E 00	0.4084079E 00	25
6,6	0.4210900E 00	0.4263220E 00	0.4307200E 00	0.4344136E 00	0.4375130E 00	26
6,7	0.4447928E 00	0.4509309E 00	0.4561330E 00	0.4605373E 00	0.4642630E 00	27
7,7	0.4705782E 00	0.4777790E 00	0.4839321E 00	0.4891840E 00	0.4936622E 00	28

I,J = 1,2,....,7. WHERE 1 IS 88.5 DEG FROM $\bar{N}$, 2 IS 82.6 DEG FROM $\bar{N}$, 3 IS 72.7 DEG FROM $\bar{N}$, 4 IS 60.0 DEG FROM $\bar{N}$, 5 IS 45.3 DEG FROM $\bar{N}$, 6 IS 29.4 DEG FROM $\bar{N}$, 7 IS 13.0 DEG FROM $\bar{N}$.

SYMMETRICAL REFLECTION FUNCTION R(X,I,J) FOR GAMMA = 0.6

I,J	X = 2.1	X = 2.2	X = 2.3	X = 2.4	X = 2.5	NN
1,1	0.8140195E-02	0.8140262E-02	0.8140314E-02	0.8140352E-02	0.8140393E-02	1
1,2	0.1460600E-01	0.1460639E-01	0.1460670E-01	0.1460696E-01	0.1460718E-01	2
1,3	0.1719937E-01	0.1720062E-01	0.1720162E-01	0.1720243E-01	0.1720310E-01	3
1,4	0.1867140E-01	0.1867504E-01	0.1867796E-01	0.1868031E-01	0.1868220E-01	4
1,5	0.1957657E-01	0.1958427E-01	0.1959056E-01	0.1959569E-01	0.1959988E-01	5
1,6	0.2011202E-01	0.2012377E-01	0.2013349E-01	0.2014153E-01	0.2014820E-01	6
1,7	0.2038055E-01	0.2039489E-01	0.2040685E-01	0.2041682E-01	0.2042514E-01	7
2,2	0.4766830E-01	0.4767068E-01	0.4767261E-01	0.4767420E-01	0.4767550E-01	8
2,3	0.7095629E-01	0.7096392E-01	0.7097000E-01	0.7097501E-01	0.7097906E-01	9
2,4	0.8500957E-01	0.8503193E-01	0.8504975E-01	0.8506411E-01	0.8507568E-01	10
2,5	0.9340000E-01	0.9344721E-01	0.9348565E-01	0.9351701E-01	0.9354258E-01	11
2,6	0.9822577E-01	0.9829772E-01	0.9835714E-01	0.9840631E-01	0.9844702E-01	12
2,7	0.1006057E 00	0.1006936E 00	0.1007667E 00	0.1008276E 00	0.1008785E 00	13
3,3	0.1249790E 00	0.1250035E 00	0.1250229E 00	0.1250385E 00	0.1250513E 00	14
3,4	0.1646760E 00	0.1647474E 00	0.1648040E 00	0.1648493E 00	0.1648855E 00	15
3,5	0.1906032E 00	0.1907547E 00	0.1908767E 00	0.1909754E 00	0.1910554E 00	16
3,6	0.2061389E 00	0.2063696E 00	0.2065583E 00	0.2067131E 00	0.2068403E 00	17
3,7	0.2139460E 00	0.2142276E 00	0.2144598E 00	0.2146516E 00	0.2148105E 00	18
4,4	0.2317781E 00	0.2319868E 00	0.2321512E 00	0.2322813E 00	0.2323844E 00	19
4,5	0.2793209E 00	0.2797631E 00	0.2801173E 00	0.2804016E 00	0.2806299E 00	20
4,6	0.3090973E 00	0.3097714E 00	0.3103190E 00	0.3107646E 00	0.3111275E 00	21
4,7	0.3244004E 00	0.3252233E 00	0.3258970E 00	0.3264493E 00	0.3269025E 00	22
5,5	0.3455306E 00	0.3464681E 00	0.3472310E 00	0.3478521E 00	0.3483577E 00	23
5,6	0.3882281E 00	0.3896568E 00	0.3908364E 00	0.3918100E 00	0.3926136E 00	24
5,7	0.4105040E 00	0.4122485E 00	0.4136996E 00	0.4149066E 00	0.4159101E 00	25
6,6	0.4401120E 00	0.4422897E 00	0.4441132E 00	0.4456395E 00	0.4469162E 00	26
6,7	0.4674115E 00	0.4700704E 00	0.4723139E 00	0.4742059E 00	0.4758004E 00	27
7,7	0.4974766E 00	0.5007229E 00	0.5034832E 00	0.5058285E 00	0.5078197E 00	28

I,J = 1,2,....,7, WHERE 1 IS 88.5 DEG FROM $\bar{N}$, 2 IS 82.6 DEG FROM $\bar{N}$, 3 IS 72.7 DEG FROM $\bar{N}$, 4 IS 60.0 DEG FROM $\bar{N}$, 5 IS 45.3 DEG FROM $\bar{N}$, 6 IS 29.4 DEG FROM $\bar{N}$, 7 IS 13.0 DEG FROM $\bar{N}$.

SYMMETRICAL REFLECTION FUNCTION R(X,I,J) FOR GAMMA = 0.6

I,J	X = 2.6	X = 2.7	X = 2.8	X = 2.9	X = 3.0	NN
1,1	0.8140419E-02	0.8140437E-02	0.8140456E-02	0.8140478E-02	0.8140482E-02	1
1,2	0.1460735E-01	0.1460749E-01	0.1460760E-01	0.1460770E-01	0.1460778E-01	2
1,3	0.1720365E-01	0.1720408E-01	0.1720444E-01	0.1720474E-01	0.1720498E-01	3
1,4	0.1868373E-01	0.1868496E-01	0.1868597E-01	0.1868678E-01	0.1868744E-01	4
1,5	0.1960332E-01	0.1960611E-01	0.1960840E-01	0.1961028E-01	0.1961182E-01	5
1,6	0.2015371E-01	0.2015825E-01	0.2016206E-01	0.2016520E-01	0.2016780E-01	6
1,7	0.2043208E-01	0.2043787E-01	0.2044270E-01	0.2044673E-01	0.2045010E-01	7
2,2	0.4767656E-01	0.4767743E-01	0.4767814E-01	0.4767874E-01	0.4767921E-01	8
2,3	0.7098240E-01	0.7098508E-01	0.7098728E-01	0.7098913E-01	0.7099056E-01	9
2,4	0.8508497E-01	0.8509260E-01	0.8509868E-01	0.8510369E-01	0.8510762E-01	10
2,5	0.9356350E-01	0.9358060E-01	0.9359455E-01	0.9360600E-01	0.9361541E-01	11
2,6	0.9848058E-01	0.9850854E-01	0.9853154E-01	0.9855062E-01	0.9856641E-01	12
2,7	0.1009208E 00	0.1009561E 00	0.1009856E 00	0.1010101E 00	0.1010306E 00	13
3,3	0.1250616E 00	0.1250700E 00	0.1250768E 00	0.1250824E 00	0.1250868E 00	14
3,4	0.1649144E 00	0.1649379E 00	0.1649566E 00	0.1649719E 00	0.1649842E 00	15
3,5	0.1911204E 00	0.1911733E 00	0.1912162E 00	0.1912514E 00	0.1912801E 00	16
3,6	0.2069449E 00	0.2070312E 00	0.2071022E 00	0.2071608E 00	0.2072092E 00	17
3,7	0.2149422E 00	0.2150514E 00	0.2151420E 00	0.2152174E 00	0.2152801E 00	18
4,4	0.2324665E 00	0.2325320E 00	0.2325842E 00	0.2326261E 00	0.2326598E 00	19
4,5	0.2808139E 00	0.2809620E 00	0.2810815E 00	0.2811782E 00	0.2812565E 00	20
4,6	0.3114234E 00	0.3116649E 00	0.3118622E 00	0.3120237E 00	0.3121559E 00	21
4,7	0.3272747E 00	0.3275808E 00	0.3278325E 00	0.3280401E 00	0.3282112E 00	22
5,5	0.3487696E 00	0.3491052E 00	0.3493788E 00	0.3496017E 00	0.3497836E 00	23
5,6	0.3932766E 00	0.3938237E 00	0.3942751E 00	0.3946477E 00	0.3949552E 00	24
5,7	0.4167443E 00	0.4174376E 00	0.4180138E 00	0.4184927E 00	0.4188905E 00	25
6,6	0.4479836E 00	0.4488757E 00	0.4496207E 00	0.4502431E 00	0.4507626E 00	26
6,7	0.4771433E 00	0.4782737E 00	0.4792247E 00	0.4800246E 00	0.4806969E 00	27
7,7	0.5095090E 00	0.5109415E 00	0.5121552E 00	0.5131832E 00	0.5140532E 00	28

I,J = 1,2,....,7, WHERE 1 IS 88.5 DEG FROM $\bar{N}$, 2 IS 82.6 DEG FROM $\bar{N}$, 3 IS 72.7 DEG FROM $\bar{N}$, 4 IS 60.0 DEG FROM $\bar{N}$, 5 IS 45.3 DEG FROM $\bar{N}$, 6 IS 29.4 DEG FROM $\bar{N}$, 7 IS 13.0 DEG FROM $\bar{N}$.

SYMMETRICAL REFLECTION FUNCTION R(X,I,J) FOR GAMMA = 0.6

I,J	X = 3.1	X = 3.2	X = 3.3	X = 3.4	X = 3.5	NN
1,1	0.8140504E-02	0.8140512E-02	0.8140516E-02	0.8140527E-02	0.8140527E-02	1
1,2	0.1460785E-01	0.1460790E-01	0.1460794E-01	0.1460799E-01	0.1460802E-01	2
1,3	0.1720518E-01	0.1720535E-01	0.1720547E-01	0.1720559E-01	0.1720569E-01	3
1,4	0.1868798E-01	0.1868833E-01	0.1868878E-01	0.1868908E-01	0.1868932E-01	4
1,5	0.1961308E-01	0.1961412E-01	0.1961497E-01	0.1961567E-01	0.1961624E-01	5
1,6	0.2016996E-01	0.2017175E-01	0.2017323E-01	0.2017446E-01	0.2017549E-01	6
1,7	0.2045291E-01	0.2045526E-01	0.2045721E-01	0.2045885E-01	0.2046022E-01	7
2,2	0.4767961E-01	0.4767993E-01	0.4768020E-01	0.4768043E-01	0.4768062E-01	8
2,3	0.7099175E-01	0.7099271E-01	0.7099360E-01	0.7099432E-01	0.7099491E-01	9
2,4	0.8511096E-01	0.8511364E-01	0.5811585E-01	0.8511764E-01	0.8511913E-01	10
2,5	0.9362310E-01	0.9362936E-01	0.9363461E-01	0.9363878E-01	0.9364223E-01	11
2,6	0.9857959E-01	0.9859043E-01	0.9859943E-01	0.9860700E-01	0.9861314E-01	12
2,7	0.1010477E 00	0.1010619E 00	0.1010379E 00	0.1010839E 00	0.1010922E 00	13
3,3	0.1250904E 00	0.1250933E 00	0.1250959E 00	0.1250981E 00	0.1250997E 00	14
3,4	0.1649942E 00	0.1650023E 00	0.1650090E 00	0.1650144E 00	0.1650189E 00	15
3,5	0.1913036E 00	0.1913226E 00	0.1913383E 00	0.1913512E 00	0.1913617E 00	16
3,6	0.2072491E 00	0.2072821E 00	0.2073097E 00	0.2073323E 00	0.2073511E 00	17
3,7	0.2153322E 00	0.2153755E 00	0.2154118E 00	0.2154419E 00	0.2154670E 00	18
4,4	0.2326869E 00	0.2327088E 00	0.2327265E 00	0.2327408E 00	0.2327523E 00	19
4,5	0.2813199E 00	0.2813713E 00	0.2814132E 00	0.2814472E 00	0.2814748E 00	20
4,6	0.3122643E 00	0.3123531E 00	0.3124261E 00	0.3124862E 00	0.3125355E 00	21
4,7	0.3283526E 00	0.3284690E 00	0.3285655E 00	0.3286453E 00	0.3287113E 00	22
5,5	0.3499321E 00	0.3500534E 00	0.3501524E 00	0.3502331E 00	0.3502992E 00	23
5,6	0.3952090E 00	0.3954821E 00	0.3955910E 00	0.3957336E 00	0.3958513E 00	24
5,7	0.4192211E 00	0.4194957E 00	0.4197239E 00	0.4199132E 00	0.4200706E 00	25
6,6	0.4511963E 00	0.4515578E 00	0.4518596E 00	0.4521112E 00	0.4523209E 00	26
6,7	0.4812618E 00	0.4817362E 00	0.4821345E 00	0.4824689E 00	0.4827493E 00	27
7,7	0.5147893E 00	0.5154116E 00	0.5159377E 00	0.5163821E 00	0.5167574E 00	28

I,J = 1,2,....,7. WHERE 1 IS 88.5 DEG FROM $\bar{N}$, 2 IS 82.6 DEG FROM $\bar{N}$, 3 IS 72.7 DEG FROM $\bar{N}$, 4 IS 60.0 DEG FROM $\bar{N}$, 5 IS 45.3 DEG FROM $\bar{N}$, 6 IS 29.4 DEG FROM $\bar{N}$, 7 IS 13.0 DEG FROM $\bar{N}$.

SYMMETRICAL REFLECTION FUNCTION R(X,I,J) FOR GAMMA = 0.6

I,J	X = 3.6	X = 3.7	X = 3.8	X = 3.9	X = 4.0	NN
1,1	0.8140527E-02	0.8140530E-02	0.8140542E-02	0.8140545E-02	0.8140545E-02	1
1,2	0.1460803E-01	0.1460805E-01	0.1460808E-01	0.1460810E-01	0.1460810E-01	2
1,3	0.1720576E-01	0.1720582E-01	0.1720588E-01	0.1720592E-01	0.1720595E-01	3
1,4	0.1868951E-01	0.1868967E-01	0.1868981E-01	0.1868992E-01	0.1869001E-01	4
1,5	0.1967671E-01	0.1961709E-01	0.1961741E-01	0.1961768E-01	0.1961790E-01	5
1,6	0.2017632E-01	0.2017702E-01	0.2017762E-01	0.2017810E-01	0.2017850E-01	6
1,7	0.2046135E-01	0.2046230E-01	0.2046309E-01	0.2046376E-01	0.2046432E-01	7
2,2	0.4768077E-01	0.4768089E-01	0.4768099E-01	0.4768108E-01	0.4768114E-01	8
2,3	0.7099527E-01	0.7099563E-01	0.7099593E-01	0.7099622E-01	0.7099652E-01	9
2,4	0.8512032E-01	0.8512127E-01	0.8512199E-01	0.8512270E-01	0.8512330E-01	10
2,5	0.9364510E-01	0.9364748E-01	0.9364945E-01	0.9365100E-01	0.9365237E-01	11
2,6	0.9861833E-01	0.9862256E-01	0.9862608E-01	0.9862900E-01	0.9863150E-01	12
2,7	0.1010990E 00	0.1011047E 00	0.1011096E 00	0.1011136E 00	0.1011171E 00	13
3,3	0.1251010E 00	0.1251022E 00	0.1251032E 00	0.1251040E 00	0.1251047E 00	14
3,4	0.1650224E 00	0.1650254E 00	0.1650277E 00	0.1650298E 00	0.1650315E 00	15
3,5	0.1913704E 00	0.1913775E 00	0.1913833E 00	0.1913881E 00	0.1913922E 00	16
3,6	0.2073666E 00	0.2073796E 00	0.2073902E 00	0.2073991E 00	0.2074065E 00	17
3,7	0.2154880E 00	0.2155054E 00	0.2155200E 00	0.2155321E 00	0.2155424E 00	18
4,4	0.2327617E 00	0.2327694E 00	0.2327757E 00	0.2327809E 00	0.2327852E 00	19
4,5	0.2814974E 00	0.2815158E 00	0.2815308E 00	0.2815431E 00	0.2815533E 00	20
4,6	0.3125760E 00	0.3126094E 00	0.3126368E 00	0.3126595E 00	0.3126783E 00	21
4,7	0.3287658E 00	0.3288110E 00	0.3288486E 00	0.3288798E 00	0.3289058E 00	22
5,5	0.3503532E 00	0.3503975E 00	0.3504336E 00	0.3504632E 00	0.3504876E 00	23
5,6	0.3959484E 00	0.3960285E 00	0.3960947E 00	0.3961495E 00	0.3961949E 00	24
5,7	0.4202015E 00	0.4203103E 00	0.4204004E 00	0.4204755E 00	0.4205379E 00	25
6,6	0.4524955E 00	0.4526411E 00	0.4527622E 00	0.4528633E 00	0.4529476E 00	26
6,7	0.4829844E 00	0.4831816E 00	0.4833468E 00	0.4834855E 00	0.4836017E 00	27
7,7	0.5170740E 00	0.5173414E 00	0.5175667E 00	0.5177569E 00	0.5179172E 00	28

I,J = 1,2,....,7. WHERE 1 IS 88.5 DEG FROM $\bar{N}$, 2 IS 82.6 DEG FROM $\bar{N}$, 3 IS 72.7 DEG FROM $\bar{N}$, 4 IS 60.0 DEG FROM $\bar{N}$, 5 IS 45.3 DEG FROM $\bar{N}$, 6 IS 29.4 DEG FROM $\bar{N}$, 7 IS 13.0 DEG FROM $\bar{N}$.

SYMMETRICAL REFLECTION FUNCTION R(X,I,J) FOR GAMMA = 0.6

I,J	X = 4.1	X = 4.2	X = 4.3	X = 4.4	X = 4.5	NN
1,1	0.8140545E-02	0.8140545E-02	0.8140545E-02	0.8140545E-02	0.8140545E-02	1
1,2	0.1460811E-01	0.1460812E-01	0.1460813E-01	0.1460813E-01	0.1460814E-01	2
1,3	0.1720598E-01	0.1720600E-01	0.1720602E-01	0.1720604E-01	0.1720605E-01	3
1,4	0.1869008E-01	0.1869014E-01	0.1869019E-01	0.1869023E-01	0.1869026E-01	4
1,5	0.1961808E-01	0.1961822E-01	0.1961834E-01	0.1961844E-01	0.1961852E-01	5
1,6	0.2017883E-01	0.2017911E-01	0.2017934E-01	0.2017953E-01	0.2017968E-01	6
1,7	0.2046478E-01	0.2046516E-01	0.2046549E-01	0.2046575E-01	0.2046598E-01	7
2,2	0.4768122E-01	0.4768125E-01	0.4768131E-01	0.4768132E-01	0.4768138E-01	8
2,3	0.7099670E-01	0.7099688E-01	0.7099700E-01	0.7099706E-01	0.7099718E-01	9
2,4	0.8512378E-01	0.8512408E-01	0.8512443E-01	0.5812467E-01	0.8512491E-01	10
2,5	0.9365344E-01	0.9365433E-01	0.9365511E-01	0.9365571E-01	0.9365618E-01	11
2,6	0.9863353E-01	0.9863526E-01	0.9863663E-01	0.9863782E-01	0.9863877E-01	12
2,7	0.1011199E 00	0.1011223E 00	0.1011242E 00	0.1011258E 00	0.1011272E 00	13
3,3	0.1251052E 00	0.1251057E 00	0.1251061E 00	0.1251063E 00	0.1251066E 00	14
3,4	0.1650328E 00	0.1650339E 00	0.1650349E 00	0.1650356E 00	0.1650363E 00	15
3,5	0.1913954E 00	0.1913981E 00	0.1914003E 00	0.1914022E 00	0.1914037E 00	16
3,6	0.2074126E 00	0.2074177E 00	0.2074220E 00	0.2074255E 00	0.2074283E 00	17
3,7	0.2155508E 00	0.2155579E 00	0.2155639E 00	0.2155688E 00	0.2155729E 00	18
4,4	0.2327885E 00	0.2327913E 00	0.2327937E 00	0.2327955E 00	0.2327970E 00	19
4,5	0.2815615E 00	0.2815683E 00	0.2815739E 00	0.2815784E 00	0.2815821E 00	20
4,6	0.3126938E 00	0.3127066E 00	0.3127172E 00	0.3127260E 00	0.3127332E 00	21
4,7	0.3289271E 00	0.3289449E 00	0.3289598E 00	0.3289721E 00	0.3289824E 00	22
5,5	0.3505075E 00	0.3505238E 00	0.3505372E 00	0.3505482E 00	0.3505573E 00	23
5,6	0.3962322E 00	0.3962631E 00	0.3962888E 00	0.3963098E 00	0.3963273E 00	24
5,7	0.4205898E 00	0.4206330E 00	0.4206687E 00	0.4206984E 00	0.4207233E 00	25
6,6	0.4530177E 00	0.4530761E 00	0.4531247E 00	0.4531652E 00	0.4531989E 00	26
6,7	0.4836991E 00	0.4837806E 00	0.4838490E 00	0.4839061E 00	0.4839540E 00	27
7,7	0.5180523E 00	0.5181661E 00	0.5182620E 00	0.5183428E 00	0.5184107E 00	28

I,J = 1,2,....,7. WHERE 1 IS 88.5 DEG FROM $\bar{N}$, 2 IS 82.6 DEG FROM $\bar{N}$, 3 IS 72.7 DEG FROM $\bar{N}$, 4 IS 60.0 DEG FROM $\bar{N}$, 5 IS 45.3 DEG FROM $\bar{N}$, 6 IS 29.4 DEG FROM $\bar{N}$, 7 IS 13.0 DEG FROM $\bar{N}$.

SYMMETRICAL REFLECTION FUNCTION R(X,I,J) FOR GAMMA = 0.6

I,J	X = 4.6	X = 4.7	X = 4.8	X = 4.9	X = 5.0	NN
1,1	0.8140545E-02	0.8140545E-02	0.8140545E-02	0.8140545E-02	0.8140545E-02	1
1,2	0.1460814E-01	0.1460814E-01	0.1460814E-01	0.1460814E-01	0.1460815E-01	2
1,3	0.1720606E-01	0.1720607E-01	0.1720608E-01	0.1720608E-01	0.1720608E-01	3
1,4	0.1869029E-01	0.1869031E-01	0.1869033E-01	0.1869035E-01	0.1869036E-01	4
1,5	0.1961859E-01	0.1961864E-01	0.1961868E-01	0.1961872E-01	0.1961876E-01	5
1,6	0.2017982E-01	0.2017993E-01	0.2018002E-01	0.2018009E-01	0.2018015E-01	6
1,7	0.2046617E-01	0.2046632E-01	0.2046646E-01	0.2046656E-01	0.2046665E-01	7
2,2	0.4768140E-01	0.4768140E-01	0.4768140E-01	0.4768140E-01	0.4768145E-01	8
2,3	0.7099724E-01	0.7099730E-01	0.7099730E-01	0.7099736E-01	0.7099742E-01	9
2,4	0.8512509E-01	0.8512521E-01	0.8512533E-01	0.8512539E-01	0.8512551E-01	10
2,5	0.9365660E-01	0.9365696E-01	0.9365720E-01	0.9365743E-01	0.9365761E-01	11
2,6	0.9863955E-01	0.9864026E-01	0.9864080E-01	0.9864122E-01	0.9864157E-01	12
2,7	0.1011284E 00	0.1011293E 00	0.1011301E 00	0.1011307E 00	0.1011313E 00	13
3,3	0.1251068E 00	0.1251070E 00	0.1251072E 00	0.1251074E 00	0.1251075E 00	14
3,4	0.1650369E 00	0.1650372E 00	0.1650375E 00	0.1650379E 00	0.1650380E 00	15
3,5	0.1914049E 00	0.1914060E 00	0.1914068E 00	0.1914074E 00	0.1914080E 00	16
3,6	0.2074307E 00	0.2074327E 00	0.2074344E 00	0.2074357E 00	0.2074369E 00	17
3,7	0.2155763E 00	0.2155792E 00	0.2155815E 00	0.2155836E 00	0.2155852E 00	18
4,4	0.2327983E 00	0.2327994E 00	0.2328002E 00	0.2328008E 00	0.2328014E 00	19
4,5	0.2815853E 00	0.2815877E 00	0.2815899E 00	0.2815915E 00	0.2815930E 00	20
4,6	0.3127393E 00	0.3127441E 00	0.3127483E 00	0.3127517E 00	0.3127545E 00	21
4,7	0.3289909E 00	0.3289981E 00	0.3290039E 00	0.3290088E 00	0.3290129E 00	22
5,5	0.3505647E 00	0.3505708E 00	0.3505759E 00	0.3505799E 00	0.3505833E 00	23
5,6	0.3963417E 00	0.3963535E 00	0.3963634E 00	0.3963715E 00	0.3963783E 00	24
5,7	0.4207439E 00	0.4207609E 00	0.4207752E 00	0.4207870E 00	0.4207968E 00	25
6,6	0.4532269E 00	0.4532503E 00	0.4532697E 00	0.4532859E 00	0.4532992E 00	26
6,7	0.4839939E 00	0.4840276E 00	0.4840556E 00	0.4840789E 00	0.4840986E 00	27
7,7	0.5184679E 00	0.5185159E 00	0.5185564E 00	0.5185903E 00	0.5186190E 00	28

I,J = 1,2,....,7, WHERE 1 IS 88.5 DEG FROM $\bar{N}$, 2 IS 82.6 DEG FROM $\bar{N}$, 3 IS 72.7 DEG FROM $\bar{N}$, 4 IS 60.0 DEG FROM $\bar{N}$, 5 IS 45.3 DEG FROM $\bar{N}$, 6 IS 29.4 DEG FROM $\bar{N}$, 7 IS 13.0 DEG FROM $\bar{N}$.

SYMMETRICAL REFLECTION FUNCTION R(X,I,J) FOR GAMMA = 0.6

I,J	X = 5.1	X = 5.2	X = 5.3	X = 5.4	X = 5.5	NN
1,1	0.8140545E-02	0.8140545E-02	0.8140545E-02	0.8140545E-02	0.8140545E-02	1
1,2	0.1460815E-01	0.1460815E-01	0.1460815E-01	0.1460815E-01	0.1460815E-01	2
1,3	0.1720610E-01	0.1720610E-01	0.1720610E-01	0.1720610E-01	0.1720610E-01	3
1,4	0.1869037E-01	0.1869037E-01	0.1869039E-01	0.1869039E-01	0.1869039E-01	4
1,5	0.1961878E-01	0.1961880E-01	0.1961881E-01	0.1961883E-01	0.1961885E-01	5
1,6	0.2018021E-01	0.2018025E-01	0.2018028E-01	0.2018031E-01	0.2018034E-01	6
1,7	0.2046673E-01	0.2046679E-01	0.2046684E-01	0.2046689E-01	0.2046693E-01	7
2,2	0.4768148E-01	0.4768148E-01	0.4768149E-01	0.4768149E-01	0.4768149E-01	8
2,3	0.7099742E-01	0.7099748E-01	0.7099748E-01	0.7099748E-01	0.7099748E-01	9
2,4	0.8512557E-01	0.8512563E-01	0.8512568E-01	0.8512586E-01	0.8512568E-01	10
2,5	0.9365785E-01	0.9365803E-01	0.9365809E-01	0.9365815E-01	0.9365821E-01	11
2,6	0.9864193E-01	0.9864223E-01	0.9864247E-01	0.9864265E-01	0.9864283E-01	12
2,7	0.1011318E 00	0.1011322E 00	0.1011325E 00	0.1011328E 00	0.1011329E 00	13
3,3	0.1251075E 00	0.1251076E 00	0.1251076E 00	0.1251076E 00	0.1251036E 00	14
3,4	0.1650383E 00	0.1650385E 00	0.1650386E 00	0.1650388E 00	0.1650388E 00	15
3,5	0.1914085E 00	0.1914089E 00	0.1914092E 00	0.1914095E 00	0.1914097E 00	16
3,6	0.2074378E 00	0.2074386E 00	0.2074392E 00	0.2074398E 00	0.2074401E 00	17
3,7	0.2155866E 00	0.2155877E 00	0.2155887E 00	0.2155895E 00	0.2155902E 00	18
4,4	0.2328019E 00	0.2328024E 00	0.2328028E 00	0.2328032E 00	0.2328033E 00	19
4,5	0.2815940E 00	0.2815949E 00	0.2815957E 00	0.2815936E 00	0.2815968E 00	20
4,6	0.3127567E 00	0.3127586E 00	0.3127602E 00	0.3127615E 00	0.3127626E 00	21
4,7	0.3290163E 00	0.3290191E 00	0.3290215E 00	0.3290235E 00	0.3290250E 00	22
5,5	0.3505860E 00	0.3505881E 00	0.3505901E 00	0.3505915E 00	0.3505929E 00	23
5,6	0.3963838E 00	0.3963885E 00	0.3963922E 00	0.3963954E 00	0.3963981E 00	24
5,7	0.4208049E 00	0.4208117E 00	0.4208173E 00	0.4208220E 00	0.4208260E 00	25
6,6	0.4533104E 00	0.4533197E 00	0.4533273E 00	0.4533337E 00	0.4533390E 00	26
6,7	0.4841149E 00	0.4841287E 00	0.4841400E 00	0.4841496E 00	0.4841577E 00	27
7,7	0.5186430E 00	0.5186631E 00	0.5186801E 00	0.5186944E 00	0.5187064E 00	28

I,J = 1,2,....,7. WHERE 1 IS 88.5 DEG FROM $\bar{N}$. 2 IS 82.6 DEG FROM $\bar{N}$. 3 IS 72.7 DEG FROM $\bar{N}$. 4 IS 60.0 DEG FROM $\bar{N}$. 5 IS 45.3 DEG FROM $\bar{N}$. 6 IS 29.4 DEG FROM $\bar{N}$. 7 IS 13.0 DEG FROM $\bar{N}$.

SYMMETRICAL REFLECTION FUNCTION R(X,I,J) FOR GAMMA = 0.6

I,J	X = 5.6	X = 5.7	X = 5.8	X = 5.9	X = 6.0	NN
1,1	0.8140545E-02	0.8140545E-02	0.8140545E-02	0.8140545E-02	0.8140545E-02	1
1,2	0.1460815E-01	0.1460815E-01	0.1460815E-01	0.1460815E-01	0.1460815E-01	2
1,3	0.1720610E-01	0.1720610E-01	0.1720610E-01	0.1720611E-01	0.1720612E-01	3
1,4	0.1869040E-01	0.1869040E-01	0.1869041E-01	0.1869041E-01	0.1869041E-01	4
1,5	0.1961885E-01	0.1961886E-01	0.1961887E-01	0.1961888E-01	0.1961888E-01	5
1,6	0.2018036E-01	0.2018037E-01	0.2018039E-01	0.2018040E-01	0.2018041E-01	6
1,7	0.2046696E-01	0.2046698E-01	0.2046700E-01	0.2046702E-01	0.2046704E-01	7
2,2	0.4768149E-01	0.4768149E-01	0.4768149E-01	0.4768149E-01	0.4768149E-01	8
2,3	0.7099748E-01	0.7099748E-01	0.7099748E-01	0.7099748E-01	0.7099754E-01	9
2,4	0.8512568E-01	0.8512574E-01	0.8512574E-01	0.8512574E-01	0.8512574E-01	10
2,5	0.9365821E-01	0.9365827E-01	0.9365833E-01	0.9365839E-01	0.9365839E-01	11
2,6	0.9864289E-01	0.9864300E-01	0.9864306E-01	0.9864312E-01	0.9864318E-01	12
2,7	0.1011331E 00	0.1011333E 00	0.1011335E 00	0.1011336E 00	0.1011336E 00	13
3,3	0.1251076E 00	0.1251076E 00	0.1251076E 00	0.1251077E 00	0.1251078E 00	14
3,4	0.1650388E 00	0.1650388E 00	0.1650389E 00	0.1650389E 00	0.1650390E 00	15
3,5	0.1914099E 00	0.1914101E 00	0.1914101E 00	0.1914102E 00	0.1914104E 00	16
3,6	0.2074405E 00	0.2074409E 00	0.2074411E 00	0.2074413E 00	0.2074416E 00	17
3,7	0.2155908E 00	0.2155911E 00	0.2155915E 00	0.2155918E 00	0.2155922E 00	18
4,4	0.2328035E 00	0.2328037E 00	0.2328038E 00	0.2328039E 00	0.2328039E 00	19
4,5	0.2815973E 00	0.2815976E 00	0.2815980E 00	0.2815983E 00	0.2815986E 00	20
4,6	0.3127635E 00	0.3127642E 00	0.3217649E 00	0.3127654E 00	0.3127658E 00	21
4,7	0.3290262E 00	0.3290274E 00	0.3290284E 00	0.3290293E 00	0.3290299E 00	22
5,5	0.3505939E 00	0.3505948E 00	0.3505956E 00	0.3505962E 00	0.3505968E 00	23
5,6	0.3964001E 00	0.3964019E 00	0.3964035E 00	0.3964048E 00	0.3964059E 00	24
5,7	0.4208292E 00	0.4208319E 00	0.4208341E 00	0.4208360E 00	0.4208373E 00	25
6,6	0.4533435E 00	0.4533473E 00	0.4533503E 00	0.4533572E 00	0.4533548E 00	26
6,7	0.4841643E 00	0.4841699E 00	0.4841746E 00	0.4841784E 00	0.4841817E 00	27
7,7	0.5187165E 00	0.5187249E 00	0.5187319E 00	0.5187379E 00	0.5187429E 00	28

I,J = 1,2,....,7, WHERE 1 IS 88.5 DEG FROM $\bar{N}$, 2 IS 82.6 DEG FROM $\bar{N}$, 3 IS 72.7 DEG FROM $\bar{N}$, 4 IS 60.0 DEG FROM $\bar{N}$, 5 IS 45.3 DEG FROM $\bar{N}$, 6 IS 29.4 DEG FROM $\bar{N}$, 7 IS 13.0 DEG FROM $\bar{N}$.

SYMMETRICAL REFLECTION FUNCTION R(X,I,J) FOR GAMMA = 1.0

I,J	X = 0.1	X = 0.2	X = 0.3	X = 0.4	X = 0.5	NN
1,1	0.1407084E-01	0.1418610E-01	0.1423959E-01	0.1427931E-01	0.1431096E-01	1
1,2	0.2443987E-01	0.2600808E-01	0.2664901E-01	0.2702847E-01	0.2728493E-01	2
1,3	0.2720429E-01	0.2981061E-01	0.3130798E-01	0.3239342E-01	0.3321941E-01	3
1,4	0.2816632E-01	0.3127671E-01	0.3327458E-01	0.3486547E-01	0.3617801E-01	4
1,5	0.2859155E-01	0.3194647E-01	0.3420950E-01	0.3609088E-01	0.3770545E-01	5
1,6	0.2879756E-01	0.3227592E-01	0.3467824E-01	0.3671798E-01	0.3850317E-01	6
1,7	0.2889030E-01	0.3242533E-01	0.3489275E-01	0.3700784E-01	0.3887563E-01	7
2,2	0.6040168E-01	0.8041608E-01	0.8813280E-01	0.9176964E-01	0.9384978E-01	8
2,3	0.7196879E-01	0.1066042E 00	0.1245452E 00	0.1349219E 00	0.1416153E 00	9
2,4	0.7625163E-01	0.1176581E 00	0.1415476E 00	0.1567350E 00	0.1673626E 00	10
2,5	0.7817233E-01	0.1228671E 00	0.1499029E 00	0.1678516E 00	0.1809192E 00	11
2,6	0.7910889E-01	0.1254648E 00	0.1541518E 00	0.1736048E 00	0.1880504E 00	12
2,7	0.7953173E-01	0.1266503E 00	0.1561089E 00	0.1762776E 00	0.1913900E 00	13
3,3	0.8687967E-01	0.1473549E 00	0.1892707E 00	0.2189369E 00	0.2404876E 00	14
3,4	0.9243578E-01	0.1649569E 00	0.2208827E 00	0.2643507E 00	0.2985858E 00	15
3,5	0.9493297E-01	0.1733183E 00	0.2366669E 00	0.2880651E 00	0.3301705E 00	16
3,6	0.9615153E-01	0.1775026E 00	0.2447497E 00	0.3004705E 00	0.3470219E 00	17
3,7	0.9670204E-01	0.1794149E 00	0.2484848E 00	0.3062620E 00	0.3549653E 00	18
4,4	0.9847826E-01	0.1855370E 00	0.2602177E 00	0.3239485E 00	0.3783581E 00	19
4,5	0.1011958E 00	0.1953378E 00	0.2799617E 00	0.3553418E 00	0.4222754E 00	20
4,6	0.1025224E 00	0.2002475E 00	0.2900957E 00	0.3718268E 00	0.4458368E 00	21
4,7	0.1031218E 00	0.2024928E 00	0.2947839E 00	0.3795372E 00	0.4569722E 00	22
5,5	0.1040133E 00	0.2058352E 00	0.3017423E 00	0.3909134E 00	0.4732634E 00	23
5,6	0.1053888E 00	0.2110960E 00	0.3129321E 00	0.4096230E 00	0.5006847E 00	24
5,7	0.1060103E 00	0.2135025E 00	0.3181112E 00	0.4183807E 00	0.5136591E 00	25
6,6	0.1067883E 00	0.2165343E 00	0.3246701E 00	0.4295186E 00	0.5302165E 00	26
6,7	0.1074207E 00	0.2190222E 00	0.3301042E 00	0.4388348E 00	0.5441971E 00	27
7,7	0.1080580E 00	0.2215472E 00	0.3356566E 00	0.4484139E 00	0.5586573E 00	28

I,J = 1,2,....,7, WHERE 1 IS 88.5 DEG FROM $\bar{N}$, 2 IS 82.6 DEG FROM $\bar{N}$, 3 IS 72.7 DEG FROM $\bar{N}$, 4 IS 60.0 DEG FROM $\bar{N}$, 5 IS 45.3 DEG FROM $\bar{N}$, 6 IS 29.4 DEG FROM $\bar{N}$, 7 IS 13.0 DEG FROM $\bar{N}$.

SYMMETRICAL REFLECTION FUNCTION R(X,I,J) FOR GAMMA = 1.0

I,J	X = 0.6	X = 0.7	X = 0.8	X = 0.9	X = 1.0	NN
1,1	0.1433726E-01	0.1435968E-01	0.1437919E-01	0.1439642E-01	0.1441178E-01	1
1,2	0.2747620E-01	0.2762913E-01	0.2775719E-01	0.2786775E-01	0.2796516E-01	2
1,3	0.3387003E-01	0.3439713E-01	0.3483439E-01	0.3520449E-01	0.3552321E-01	3
1,4	0.3728499E-01	0.3823387E-01	0.3905769E-01	0.3978046E-01	0.4042023E-01	4
1,5	0.3911688E-01	0.4036688E-01	0.4148458E-01	0.4249153E-01	0.4340425E-01	5
1,6	0.4009267E-01	0.4152467E-01	0.4282576E-01	0.4401547E-01	0.4510888E-01	6
1,7	0.4055273E-01	0.4207574E-01	0.4346993E-01	0.4475387E-01	0.4594177E-01	7
2,2	0.9524226E-01	0.9628510E-01	0.9712589E-01	0.9783578E-01	0.9845334E-01	8
2,3	0.1463510E 00	0.1499457E 00	0.1528162E 00	0.1551929E 00	0.1572133E 00	9
2,4	0.1754164E 00	0.1818860E 00	0.1872938E 00	0.1919349E 00	0.1959909E 00	10
2,5	0.1911859E 00	0.1997075E 00	0.2070438E 00	0.2135096E 00	0.2192957E 00	11
2,6	0.1996108E 00	0.2093726E 00	0.2179123E 00	0.2255515E 00	0.2324830E 00	12
2,7	0.2035869E 00	0.2139688E 00	0.2231149E 00	0.2313635E 00	0.2388940E 00	13
3,3	0.2565944E 00	0.2689847E 00	0.2787855E 00	0.2867417E 00	0.2933533E 00	14
3,4	0.3259823E 00	0.3482825E 00	0.3667453E 00	0.3822814E 00	0.3955525E 00	15
3,5	0.3650975E 00	0.3944703E 00	0.4195168E 00	0.4411595E 00	0.4600913E 00	16
3,6	0.3863519E 00	0.4199995E 00	0.4491535E 00	0.4747233E 00	0.4974024E 00	17
3,7	0.3964618E 00	0.4322465E 00	0.4634861E 00	0.4910802E 00	0.5157190E 00	18
4,4	0.4249727E 00	0.4651174E 00	0.4999027E 00	0.5302424E 00	0.5568824E 00	19
4,5	0.4817134E 00	0.5345972E 00	0.5817911E 00	0.6240590E 00	0.6620632E 00	20
4,6	0.5127742E 00	0.5733585E 00	0.6282947E 00	0.6782342E 00	0.7237622E 00	21
4,7	0.5275997E 00	0.5920341E 00	0.6509017E 00	0.7047960E 00	0.7542588E 00	22
5,5	0.5490597E 00	0.6187308E 00	0.6827639E 00	0.7416525E 00	0.7958711E 00	23
5,6	0.5860487E 00	0.6658680E 00	0.7404085E 00	0.8099877E 00	0.8749417E 00	24
5,7	0.6037310E 00	0.6886246E 00	0.7685009E 00	0.8435909E 00	0.9141593E 00	25
6,6	0.6263583E 00	0.7178062E 00	0.8045799E 00	0.8867916E 00	0.9646077E 00	26
6,7	0.6456425E 00	0.7429050E 00	0.8358912E 00	0.9246150E 00	0.1009157E 01	27
7,7	0.6656995E 00	0.7691467E 00	0.8687904E 00	0.9645424E 00	0.1056393E 01	28

I,J = 1,2,...,7, WHERE 1 IS 88.5 DEG FROM $\bar{N}$, 2 IS 82.6 DEG FROM $\bar{N}$, 3 IS 72.7 DEG FROM $\bar{N}$, 4 IS 60.0 DEG FROM $\bar{N}$, 5 IS 45.3 DEG FROM $\bar{N}$, 6 IS 29.4 DEG FROM $\bar{N}$, 7 IS 13.0 DEG FROM $\bar{N}$.

SYMMETRICAL REFLECTION FUNCTION R(X,I,J) FOR GAMMA = 1.0

I,J	X = 1.1	X = 1.2	X = 1.3	X = 1.4	X = 1.5	NN
1,1	0.1442561E-01	0.1443816E-01	0.1444961E-01	0.1446012E-01	0.1446980E-01	1
1,2	0.2805218E-01	0.2813074E-01	0.2820220E-01	0.2826763E-01	0.2832786E-01	2
1,3	0.3580175E-01	0.3604830E-01	0.3626888E-01	0.3646806E-01	0.3664931E-01	3
1,4	0.4099097E-01	0.4150366E-01	0.4196706E-01	0.4238831E-01	0.4277319E-01	4
1,5	0.4423584E-01	0.4499697E-01	0.4569634E-01	0.4634136E-01	0.4693818E-01	5
1,6	0.4611802E-01	0.4705272E-01	0.4792114E-01	0.4873028E-01	0.4948608E-01	6
1,7	0.4704509E-01	0.4807309E-01	0.4903359E-01	0.4993324E-01	0.5077776E-01	7
2,2	0.9900075E-01	0.9949255E-01	0.9993857E-01	0.1003461E 00	0.1007206E 00	8
2,3	0.1589657E 00	0.1605093E 00	0.1618859E 00	0.1631262E 00	0.1642532E 00	9
2,4	0.1995814E 00	0.2027912E 00	0.2056834E 00	0.2083065E 00	0.2106996E 00	10
2,5	0.2245272E 00	0.2292923E 00	0.2336570E 00	0.2376737E 00	0.2413845E 00	11
2,6	0.2388314E 00	0.2446831E 00	0.2501029E 00	0.2551416E 00	0.2598407E 00	12
2,7	0.2458348E 00	0.2520707E 00	0.2582650E 00	0.2638674E 00	0.2691184E 00	13
3,3	0.2989626E 00	0.3038072E 00	0.3080563E 00	0.3118320E 00	0.3152241E 00	14
3,4	0.4070425E 00	0.4171194E 00	0.4260461E 00	0.4340314E 00	0.4412340E 00	15
3,5	0.4768361E 00	0.4917912E 00	0.5052634E 00	0.5174903E 00	0.5286593E 00	16
3,6	0.5177214E 00	0.5360867E 00	0.5528149E 00	0.5681528E 00	0.5822966E 00	17
3,7	0.5379338E 00	0.5581325E 00	0.5766340E 00	0.5936876E 00	0.6094917E 00	18
4,4	0.5804304E 00	0.6013795E 00	0.6201330E 00	0.6370210E 00	0.6523147E 00	19
4,5	0.6963733E 00	0.7274729E 00	0.7557762E 00	0.7816353E 00	0.8053507E 00	20
4,6	0.7653968E 00	0.8035886E 00	0.8387327E 00	0.8711716E 00	0.9012035E 00	21
4,7	0.7997782E 00	0.8417825E 00	0.8806522E 00	0.9167200E 00	0.9502778E 00	22
5,5	0.8458636E 00	0.8920332E 00	0.9347500E 00	0.9743466E 00	0.1011121E 01	23
5,6	0.9356077E 00	0.9923071E 00	0.1045347E 01	0.1095017E 01	0.1141586E 01	24
5,7	0.9804866E 00	0.1042844E 01	0.1101507E 01	0.1156735E 01	0.1208773E 01	25
6,6	0.1038225E 01	0.1107856E 01	0.1173717E 01	0.1236029E 01	0.1295003E 01	26
6,7	0.1089645E 01	0.1166227E 01	0.1239074E 01	0.1308356E 01	0.1374255E 01	27
7,7	0.1144392E 01	0.1228621E 01	0.1309191E 01	0.1386225E 01	0.1459863E 01	28

I,J = 1,2,....,7. WHERE 1 IS 88.5 DEG FROM $\bar{N}$, 2 IS 82.6 DEG FROM $\bar{N}$, 3 IS 72.7 DEG FROM $\bar{N}$. 4 IS 60.0 DEG FROM $\bar{N}$, 5 IS 45.3 DEG FROM $\bar{N}$, 6 IS 29.4 DEG FROM $\bar{N}$, 7 IS 13.0 DEG FROM $\bar{N}$.

SYMMETRICAL REFLECTION FUNCTION R(X,I,J) FOR GAMMA = 1.0

I,J	X = 1.6	X = 1.7	X = 1.8	X = 1.9	X = 2.0	NN
1,1	0.1447877E-01	C.1448710E-01	0.1449486E-01	0.1450212E-01	0.1450891E-01	1
1,2	0.2838354E-01	C.2843520E-01	0.2848333E-01	0.2852830E-C1	C.2857041E-01	2
1,3	0.3681538E-01	C.3696839E-01	0.3711C07E-01	0.3724183E-01	0.3736480E-01	3
1,4	0.4312648E-01	C.4345218E-01	C.4375362E-01	0.4403359E-01	0.4429446E-01	4
1,5	0.4749207E-01	C.48C0755E-01	C.4848857E-01	0.4893849E-01	0.4936026E-01	5
1,6	0.5019362E-01	C.5085741E-01	C.5148138E-01	0.5206896E-01	0.5262318E-01	6
1,7	0.5157210E-01	C.5232C58E-01	0.53027C4E-01	0.5369489E-01	0.5432714E-01	7
2,2	0.1010663E 00	C.1013869E 0C	0.1C16852E 00	0.1019637E 00	0.1022246E 00	8
2,3	0.1651845E 00	C.1662338E 0C	C.1671122E 0C	0.1679285E C0	0.1686901E 00	9
2,4	0.2128935E 00	C.2149143E 0C	0.2167830E 00	0.2185177E C0	0.2201335E 00	10
2,5	0.2448241E 0C	C.2480224E 0C	0.251C046E 0C	0.2537923E C0	0.2564046E 00	11
2,6	0.2642347E 00	C.2683531E 02	0.2722214E 00	0.2758622E C0	0.2792947E 00	12
2,7	0.2740512E 00	C.2786950E 00	C.2830749E 0C	0.2872130E 00	0.2911286E 00	13
3,3	0.3183000E 00	C.2632111E 02	0.3236971E 0C	0.3260896E C0	0.3283132E 00	14
3,4	0.4477780E 0C	C.4537616E C0	0.4592634E 0C	0.4643469E C0	0.4690647E C0	15
3,5	0.5389190E 00	C.2654839E 02	0.5571689E 00	0.5653384E C0	0.5729652E 00	16
3,6	0.5954024E 00	C.6075973E 00	0.6189858E 0C	0.6296551E 00	0.6396772E 00	17
3,7	0.6242047E 00	C.2663795E 02	0.6508501E 0C	0.6629768E C0	0.6744096E 00	18
4,4	0.6662369E 00	C.6789734E 0C	0.6906785E 0C	0.7014812E 00	0.7114900E 00	19
4,5	0.8271780E 00	C.2684734E 02	0.866C153E 00	0.8833753E 00	0.8995563E C0	20
4,6	0.9290859E 00	C.9550439E 0C	0.9792731E 0C	0.1001945E 01	0.1023206E 01	21
4,7	0.9815798E 00	C.2610107E 03	0.1C38280E 01	0.1064047E 01	0.1088302E 01	22
5,5	0.1045339E 01	C.1077244E 01	C.11C7C50E 01	0.1134948E 01	0.1161107E 01	23
5,6	0.1185299E 01	C.2612263E C3	0.1265047E 01	0.1301482E 01	0.1335857E 01	24
5,7	0.1257849E 01	C.1304176E 01	0.1347954E 01	0.1389366E 01	0.1428579E 01	25
6,6	0.135C844E 01	C.2614038E 03	0.14539C5E 01	0.1501487E 01	0.1546659E 01	26
6,7	0.1436945E 01	C.1496601E 01	0.1553389E 01	0.1607471E 01	0.1659000E 01	27
7,7	0.1530242E 01	C.2615974E 03	0.16618C5E 01	0.1723277E 01	0.1782061E 01	28

I,J = 1,2,....,7, WHERE 1 IS 88.5 DEG FROM $\bar{N}$, 2 IS 82.6 DEG FROM $\bar{N}$, 3 IS 72.7 DEG FROM $\bar{N}$, 4 IS 60.0 DEG FROM $\bar{N}$, 5 IS 45.3 DEG FROM $\bar{N}$, 6 IS 29.4 DEG FROM $\bar{N}$, 7 IS 13.0 DEG FROM $\bar{N}$.

SYMMETRICAL REFLECTION FUNCTION R(X,I,J) FOR GAMMA = 1.0

I,J	X = 2.1	X = 2.2	X = 2.3	X = 2.4	X = 2.5	NN
1,1	0.1451531E-01	0.1452132E-01	0.1452699E-01	0.1453236E-01	0.1453744E-01	1
1,2	0.2860997E-01	0.2864718E-01	0.2868231E-01	0.2871549E-01	0.2874691E-01	2
1,3	0.3747996E-01	0.3758807E-01	0.3768989E-01	0.3778598E-01	0.3787683E-01	3
1,4	0.4453834E-01	0.4475587E-01	0.4498164E-01	0.4518393E-01	0.4537484E-01	4
1,5	0.4975653E-01	0.5012950E-01	0.5048126E-01	0.5081357E-01	0.5112808E-01	5
1,6	0.5314684E-01	0.5364228E-01	0.5411176E-01	0.5455720E-01	0.5498038E-01	6
1,7	0.5492654E-01	0.5549544E-01	0.5603613E-01	0.5655056E-01	0.5704055E-01	7
2,2	0.1024695E 00	0.1026999E 00	0.1029172E 00	0.1031224E 00	0.1033168E 00	8
2,3	0.1694031E 00	0.1700722E 00	0.1707022E 00	0.1712946E 00	0.1718528E 00	9
2,4	0.2216432E 00	0.2230576E 00	0.2243862E 00	0.2256374E 00	0.2268183E 00	10
2,5	0.2588575E 00	0.2611659E 00	0.2633421E 00	0.2653977E 00	0.2673425E 00	11
2,6	0.2825365E 00	0.2856027E 00	0.2885042E 00	0.2912624E 00	0.2938793E 00	12
2,7	0.2948390E 00	0.2983601E 00	0.3017052E 00	0.3048870E 00	0.3079172E 00	13
3,3	0.3303887E 00	0.3323324E 00	0.3341588E 00	0.3358793E 00	0.3375041E 00	14
3,4	0.4734598E 00	0.4775683E 00	0.4814208E 00	0.4850431E 00	0.4884577E 00	15
3,5	0.5801070E 00	0.5868120E 00	0.5931216E 00	0.5990728E 00	0.6046966E 00	16
3,6	0.6491151E 00	0.6580214E 00	0.6664427E 00	0.6744193E 00	0.6819870E 00	17
3,7	0.6852126E 00	0.6954395E 00	0.7051382E 00	0.7143503E 00	0.7231125E 00	18
4,4	0.7207975E 00	0.7294808E 00	0.7376072E 00	0.7452336E 00	0.7524096E 00	19
4,5	0.9146802E 00	0.9288511E 00	0.9421611E 00	0.9546902E 00	0.9665090E 00	20
4,6	0.1043189E 01	0.1062012E 01	0.1079776E 01	0.1096569E 01	0.1112473E 01	21
4,7	0.1111177E 01	0.1132792E 01	0.1153250E 01	0.1172645E 01	0.1191058E 01	22
5,5	0.1185681E 01	0.1208806E 01	0.1230605E 01	0.1251188E 01	0.1270656E 01	23
5,6	0.1368333E 01	0.1399055E 01	0.1428149E 01	0.1455740E 01	0.1481935E 01	24
5,7	0.1465752E 01	0.1501027E 01	0.1534534E 01	0.1566397E 01	0.1596725E 01	25
6,6	0.1589576E 01	0.1630386E 01	0.1669213E 01	0.1706194E 01	0.1741442E 01	26
6,7	0.1708123E 01	0.1754980E 01	0.1799702E 01	0.1842410E 01	0.1883223E 01	27
7,7	0.1838290E 01	0.1892094E 01	0.1943599E 01	0.1992922E 01	0.2040177E 01	28

I,J = 1,2,....,7. WHERE 1 IS 88.5 DEG FROM $\bar{N}$, 2 IS 82.6 DEG FROM $\bar{N}$, 3 IS 72.7 DEG FROM $\bar{N}$, 4 IS 60.0 DEG FROM $\bar{N}$, 5 IS 45.3 DEG FROM $\bar{N}$, 6 IS 29.4 DEG FROM $\bar{N}$, 7 IS 13.0 DEG FROM $\bar{N}$.

SYMMETRICAL REFLECTION FUNCTION R(X,I,J) FOR GAMMA = 1.0

I,J	X = 2.6	X = 2.7	X = 2.8	X = 2.9	X = 3.0	NN
1,1	0.1454226E-01	0.1454683E-01	0.1455120E-01	0.1455534E-01	0.1455930E-01	1
1,2	0.2877671E-01	0.2880501E-01	0.2883193E-01	0.2885755E-01	0.2888200E-01	2
1,3	0.3796292E-01	0.3804460E-01	0.3812226E-01	0.3819617E-01	0.3826663E-01	3
1,4	0.4555546E-01	0.4572658E-01	0.4588902E-01	0.4604342E-01	0.4619045E-01	4
1,5	0.5142614E-01	0.5170908E-01	0.5197802E-01	0.5223400E-01	0.5247794E-01	5
1,6	0.5538291E-01	0.5576620E-01	0.5613162E-01	0.5648035E-01	0.5681352E-01	6
1,7	0.5750773E-01	0.5795362E-01	0.5837961E-01	0.5878692E-01	0.5917674E-01	7
2,2	0.1035010E 00	0.1036759E 00	0.1038423E 00	0.1040006E 00	0.1041518E 00	8
2,3	0.1723905E 00	0.1728956E 00	0.1733755E 00	0.1738322E 00	0.1742677E 00	9
2,4	0.2279348E 00	0.2289926E 00	0.2299967E 00	0.2309511E 00	0.2318597E 00	10
2,5	0.2691854E 00	0.2709346E 00	0.2725970E 00	0.2741791E 00	0.2756867E 00	11
2,6	0.2963679E 00	0.2987376E 00	0.3009965E 00	0.3031518E 00	0.3052107E 00	12
2,7	0.3108056E 00	0.3135623E 00	0.3161955E 00	0.3187129E 00	0.3211219E 00	13
3,3	0.3390418E 00	0.3404995E 00	0.3418844E 00	0.3432018E 00	0.3444569E 00	14
3,4	0.4916837E 00	0.4947376E 00	0.4976342E 00	0.5003864E 00	0.5030056E 00	15
3,5	0.6100211E 00	0.6150703E 00	0.6198664E 00	0.6244287E 00	0.6287743E 00	16
3,6	0.6891777E 00	0.6960177E 00	0.7025344E 00	0.7087497E 00	0.7146842E 00	17
3,7	0.7314578E 00	0.7394154E 00	0.7470120E 00	0.7542714E 00	0.7612156E 00	18
4,4	0.7591776E 00	0.7655747E 00	0.7716340E 00	0.7773836E 00	0.7828492E 00	19
4,5	0.9776796E 00	0.9882567E 00	0.9982892E 00	0.1007818E 01	0.1016884E 01	20
4,6	0.1127556E 01	0.1141885E 01	0.1155517E 01	0.1168502E 01	0.1180884E 01	21
4,7	0.1208567E 01	0.1225235E 01	0.1241125E 01	0.1256290E 01	0.1270780E 01	22
5,5	0.1280903E 01	0.1306580E 01	0.1323192E 01	0.1338990E 01	0.1354035E 01	23
5,6	0.1506832E 01	0.1530522E 01	0.1553093E 01	0.1574615E 01	0.1595162E 01	24
5,7	0.1625622E 01	0.1653183E 01	0.1679494E 01	0.1704634E 01	0.1728676E 01	25
6,6	0.1775064E 01	0.1807161E 01	0.1837828E 01	0.1867149E 01	0.1895208E 01	26
6,7	0.1922247E 01	0.1959585E 01	0.1995334E 01	0.2029583E 01	0.2062414E 01	27
7,7	0.2085471E 01	0.2128906E 01	0.2170579E 01	0.2210582E 01	0.2248998E 01	28

I,J = 1,2,...,7. WHERE 1 IS 88.5 DEG FROM $\bar{N}$. 2 IS 82.6 DEG FROM $\bar{N}$. 3 IS 72.7 DEG FROM $\bar{N}$. 4 IS 60.0 DEG FROM $\bar{N}$. 5 IS 45.3 DEG FROM $\bar{N}$. 6 IS 29.4 DEG FROM $\bar{N}$. 7 IS 13.0 DEG FROM $\bar{N}$.

SYMMETRICAL REFLECTION FUNCTION R(X,I,J) FOR GAMMA = 1.0

I,J	X = 3.1	X = 3.2	X = 3.3	X = 3.4	X = 3.5	NN
1,1	0.1456308E-01	0.1456669E-01	0.1457014E-01	0.1457014E-01	0.1457345E-01	1
1,2	0.2890534E-01	0.2892765E-01	0.2894900E-01	0.2894900E-01	0.2896945E-01	2
1,3	0.3833389E-01	0.3839815E-01	0.3845963E-01	0.3845963E-01	0.3851851E-01	3
1,4	0.4633064E-01	0.4646447E-01	0.4659243E-01	0.4659243E-01	0.4671483E-01	4
1,5	0.5271069E-01	0.5293306E-01	0.5314570E-01	0.5314570E-01	0.5334926E-01	5
1,6	0.5713207E-01	0.5743699E-01	0.5772909E-01	0.5772909E-01	0.5800916E-01	6
1,7	0.5955013E-01	0.5990810E-01	0.6025151E-01	0.6025151E-01	0.6058122E-01	7
2,2	0.1042960E 00	0.1044339E 00	0.1045658E 00	0.1045658E 00	0.1046922E 00	8
2,3	0.1746834E 00	0.1750805E 00	0.1754604E 00	0.1754604E 00	0.1758242E 00	9
2,4	0.2327260E 00	0.2335531E 00	0.2343437E 00	0.2343437E 00	0.2351000E 00	10
2,5	0.2771250E 00	0.2784991E 00	0.2798131E 00	0.2798131E 00	0.2810706E 00	11
2,6	0.3071793E 00	0.3090636E 00	0.3108685E 00	0.3108685E 00	0.3125988E 00	12
2,7	0.3234293E 00	0.3256415E 00	0.3277633E 00	0.3277633E 00	0.3298004E 00	13
3,3	0.3456544E 00	0.3467984E 00	0.3478922E 00	0.3478922E 00	0.3489395E 00	14
3,4	0.5055018E 00	0.5078840E 00	0.5101605E 00	0.5101605E 00	0.5123385E 00	15
3,5	0.6329189E 00	0.6368769E 00	0.6406609E 00	0.6406609E 00	0.6442823E 00	16
3,6	0.7203566E 00	0.7257839E 00	0.7309819E 00	0.7309819E 00	0.7359647E 00	17
3,7	0.7678642E 00	0.7742358E 00	0.7803467E 00	0.7803467E 00	0.7862123E 00	18
4,4	0.7880526E 00	0.7930144E 00	0.7977520E 00	0.7977520E 00	0.8022813E 00	19
4,5	0.1025522E 01	0.1033767E 01	0.1041640E 01	0.1041640E 01	0.1049170E 01	20
4,6	0.1192710E 01	0.1204013E 01	0.1214829E 01	0.1214829E 01	0.1225190E 01	21
4,7	0.1284640E 01	0.1297911E 01	0.1310628E 01	0.1310628E 01	0.1322825E 01	22
5,5	0.1368382E 01	0.1382077E 01	0.1395165E 01	0.1395165E 01	0.1407686E 01	23
5,6	0.1614796E 01	0.1633575E 01	0.1651553E 01	0.1651553E 01	0.1668782E 01	24
5,7	0.1751687E 01	0.1773734E 01	0.1794872E 01	0.1794872E 01	0.1815153E 01	25
6,6	0.1922078E 01	0.1947830E 01	0.1972528E 01	0.1972528E 01	0.1996232E 01	26
6,7	0.2093908E 01	0.2124140E 01	0.2153174E 01	0.2153174E 01	0.2181079E 01	27
7,7	0.2285913E 01	0.2321402E 01	0.2355536E 01	0.2355536E 01	0.2388387E 01	28

I,J = 1,2,....,7. WHERE 1 IS 88.5 DEG FROM $\bar{N}$, 2 IS 82.6 DEG FROM $\bar{N}$, 3 IS 72.7 DEG FROM $\bar{N}$, 4 IS 60.0 DEG FROM $\bar{N}$, 5 IS 45.3 DEG FROM $\bar{N}$, 6 IS 29.4 DEG FROM $\bar{N}$, 7 IS 13.0 DEG FROM $\bar{N}$.

SYMMETRICAL REFLECTION FUNCTION R(X,I,J) FOR GAMMA = 1.0

I,J	X = 3.6	X = 3.7	X = 3.8	X = 3.9	X = 4.0	NN
1,1	0.1457967E-01	0.1458259E-01	0.1458540E-01	0.1458810E-01	0.1459071E-01	1
1,2	0.2900786E-01	0.2902593E-01	0.2904330E-01	0.2906001E-01	0.2907610E-01	2
1,3	0.3862909E-01	0.3868110E-01	0.3873108E-01	0.3877917E-01	0.3882546E-01	3
1,4	0.4694461E-01	0.4705258E-01	0.4715632E-01	0.4725608E-01	0.4735209E-01	4
1,5	0.5373145E-01	0.5391106E-01	0.5408366E-01	0.5424963E-01	0.5440938E-01	5
1,6	0.5853609E-01	0.5878420E-01	0.5902283E-01	0.5925254E-01	0.5947382E-01	6
1,7	0.6120265E-01	0.6149573E-01	0.6177793E-01	0.6204981E-01	0.6231192E-01	7
2,2	0.1049294E 00	0.1050410E 00	0.1051484E 00	0.1052517E 00	0.1053510E 00	8
2,3	0.1765074E 00	0.1768286E 00	0.1771375E 00	0.1774344E 00	0.1777204E 00	9
2,4	0.2365196E 00	0.2371867E 00	0.2378275E 00	0.2384438E 00	0.2390369E 00	10
2,5	0.2834319E 00	0.2845416E 00	0.2856077E 00	0.2866331E 00	0.2876199E 00	11
2,6	0.3158543E 00	0.3173870E 00	0.3188612E 00	0.3202803E 00	0.3216472E 00	12
2,7	0.3336397E 00	0.3354504E 00	0.3371937E 00	0.3388733E 00	0.3404924E 00	13
3,3	0.3509065E 00	0.3518313E 00	0.3527198E 00	0.3535746E 00	0.3543975E 00	14
3,4	0.5164251E 00	0.5183449E 00	0.5201891E 00	0.5219625E 00	0.5236689E 00	15
3,5	0.6510792E 00	0.6542730E 00	0.6573412E 00	0.6602919E 00	0.6631312E 00	16
3,6	0.7453356E 00	0.7497472E 00	0.7539898E 00	0.7580732E 00	0.7620062E 00	17
3,7	0.7972643E 00	0.8024757E 00	0.8074926E 00	0.8123256E 00	0.8169844E 00	18
4,4	0.8107716E 00	0.8147574E 00	0.8185847E 00	0.8222635E 00	0.8258026E 00	19
4,5	0.1063292E 01	0.1069922E 01	0.1076291E 01	0.1082412E 01	0.1088301E 01	20
4,6	0.1244661E 01	0.1253820E 01	0.1262626E 01	0.1271097E 01	0.1279254E 01	21
4,7	0.1345788E 01	0.1356607E 01	0.1367020E 01	0.1377047E 01	0.1386709E 01	22
5,5	0.1431176E 01	0.1442207E 01	0.1452804E 01	0.1462988E 01	0.1472784E 01	23
5,6	0.1701166E 01	0.1716402E 01	0.1731052E 01	0.1745147E 01	0.1758718E 01	24
5,7	0.1853345E 01	0.1871345E 01	0.1888668E 01	0.1905351E 01	0.1921427E 01	25
6,6	0.2040881E 01	0.2061927E 01	0.2082182E 01	0.2101688E 01	0.2120486E 01	26
6,7	0.2233738E 01	0.2258599E 01	0.2282551E 01	0.2305639E 01	0.2327906E 01	27
7,7	0.2450490E 01	0.2479859E 01	0.2508182E 01	0.2535508E 01	0.2561886E 01	28

I,J = 1,2,....,7, WHERE 1 IS 88.5 DEG FROM $\bar{N}$, 2 IS 82.6 DEG FROM $\bar{N}$, 3 IS 72.7 DEG FROM $\bar{N}$, 4 IS 60.0 DEG FROM $\bar{N}$, 5 IS 45.3 DEG FROM $\bar{N}$, 6 IS 29.4 DEG FROM $\bar{N}$, 7 IS 13.0 DEG FROM $\bar{N}$.

SYMMETRICAL REFLECTION FUNCTION R(X,I,J) FOR GAMMA = 1.0

I,J	X = 4.1	X = 4.2	X = 4.3	X = 4.4	X = 4.5	NN
1,1	0.1459322E-01	0.1459565E-01	0.1459797E-01	0.1460023E-01	0.1460240E-01	1
1,2	0.2909160E-01	0.2910655E-01	0.2912097E-01	0.2913489E-01	0.2914834E-01	2
1,3	0.3887007E-01	0.3891306E-01	0.3895454E-01	0.3899460E-01	0.3903330E-01	3
1,4	0.4744456E-01	0.4753369E-01	0.4761966E-01	0.4770263E-01	0.4778280E-01	4
1,5	0.5456324E-01	0.5471155E-01	0.5485459E-01	0.5499265E-01	0.5512600E-01	5
1,6	0.5968709E-01	0.5989281E-01	0.6009131E-01	0.6028301E-01	0.6046826E-01	6
1,7	0.6256467E-01	0.6280875E-01	0.6304437E-01	0.6327212E-01	0.6349236E-01	7
2,2	0.1054468E 00	0.1055391E 00	0.1056281E 00	0.1057141E 00	0.1057973E 00	8
2,3	0.1779960E 00	0.1782617E 00	0.1785179E 00	0.1787652E 00	0.1790043E 00	9
2,4	0.2396080E 00	0.2401586E 00	0.2406896E 00	0.2412022E 00	0.2416973E 00	10
2,5	0.2885705E 00	0.2894865E 00	0.2903700E 00	0.2912228E 00	0.2920465E 00	11
2,6	0.3229647E 00	0.3242354E 00	0.3254615E 00	0.3266457E 00	0.3277900E 00	12
2,7	0.3420543E 00	0.3435618E 00	0.3450175E 00	0.3464243E 00	0.3477843E 00	13
3,3	0.3551901E 00	0.3559543E 00	0.3566915E 00	0.3574033E 00	0.3580908E 00	14
3,4	0.5253124F 00	0.5268964E 00	0.5284243E 00	0.5298988E 00	0.5313232E 00	15
3,5	0.6658658E 00	0.6685014E 00	0.6710433E 00	0.6734966E 00	0.6758659E 00	16
3,6	0.7657965E 00	0.7694523E 00	0.7729802E 00	0.7763870E 00	0.7796785E 00	17
3,7	0.8214782E 00	0.8258153E 00	0.8300035E 00	0.8340504E 00	0.8379627E 00	18
4,4	0.8292100E 00	0.8324934E 00	0.8356595E 00	0.8387148E 00	0.8416652E 00	19
4,5	0.1093969E 01	0.1099432E 01	0.1104699E 01	0.1109783E 01	0.1114693E 01	20
4,6	0.1287112E 01	0.1294689E 01	0.1302000E 01	0.1309058E 01	0.1315877E 01	21
4,7	0.1396026E 01	0.1405017E 01	0.1413695E 01	0.1422080E 01	0.1430185E 01	22
5,5	0.1482218E 01	0.1491309E 01	0.1500072E 01	0.1508530E 01	0.1516695E 01	23
5,6	0.1771794E 01	0.1784402E 01	0.1796566E 01	0.1808309E 01	0.1819654E 01	24
5,7	0.1936929E 01	0.1951888E 01	0.1966328E 01	0.1980279E 01	0.1993763E 01	25
6,6	0.2138612E 01	0.2156100E 01	0.2172982E 01	0.2189290E 01	0.2205050E 01	26
6,7	0.2349394E 01	0.2370142E 01	0.2390182E 01	0.2409554E 01	0.2428287E 01	27
7,7	0.2587359E 01	0.2611974E 01	0.2635767E 01	0.2658779E 01	0.2681045E 01	28

I,J = 1,2,....,7. WHERE 1 IS 88.5 DEG FROM $\bar{N}$, 2 IS 82.6 DEG FROM $\bar{N}$, 3 IS 72.7 DEG FROM $\bar{N}$, 4 IS 60.0 DEG FROM $\bar{N}$, 5 IS 45.3 DEG FROM $\bar{N}$, 6 IS 29.4 DEG FROM $\bar{N}$, 7 IS 13.0 DEG FROM $\bar{N}$.

SYMMETRICAL REFLECTION FUNCTION R(X,I,J) FOR GAMMA = 1.0

I,J	X = 4.6	X = 4.7	X = 4.8	X = 4.9	X = 5.0	NN
1,1	0.1460451E-01	0.1460655E-01	0.1460851E-01	0.1461043E-01	0.1461228E-01	1
1,2	0.2916134E-01	0.2917391E-01	0.2918608E-01	0.2919787E-01	0.2920929E-01	2
1,3	0.3907069E-01	0.3910685E-01	0.3914184E-01	0.3917573E-01	0.3920856E-01	3
1,4	0.4786025E-01	0.4793516E-01	0.4800763E-01	0.4807781E-01	0.4814578E-01	4
1,5	0.5525484E-01	0.5537945E-01	0.5550000E-01	0.5561671E-01	0.5572975E-01	5
1,6	0.6064737E-01	0.6082063E-01	0.6098830E-01	0.6115067E-01	0.6130802E-01	6
1,7	0.6370527E-01	0.6391144E-01	0.6411099E-01	0.6430441E-01	0.6449181E-01	7
2,2	0.1058775E 00	0.1059552E 00	0.1060303E 00	0.1061031E 00	0.1061736E 00	8
2,3	0.1792352E 00	0.1794586E 00	0.1796747E 00	0.1798841E 00	0.1800869E 00	9
2,4	0.2421757E 00	0.2426383E 00	0.2430860E 00	0.2435195E 00	0.2439393E 00	10
2,5	0.2928424E 00	0.2936119E 00	0.2943566E 00	0.2950774E 00	0.2957757E 00	11
2,6	0.3288962E 00	0.3299664E 00	0.3310019E 00	0.3320050E 00	0.3329768E 00	12
2,7	0.3490999E 00	0.3503730E 00	0.3516058E 00	0.3528002E 00	0.3539578E 00	13
3,3	0.3587552E 00	0.3593979E 00	0.3600196E 00	0.3606217E 00	0.3612049E 00	14
3,4	0.5326993E 00	0.5340301E 00	0.5353178E 00	0.5365646E 00	0.5377721E 00	15
3,5	0.6781555E 00	0.6803693E 00	0.6825112E 00	0.6845846E 00	0.6865928E 00	16
3,6	0.7826609E 00	0.7859391E 00	0.7889184E 00	0.7918032E 00	0.7945981E 00	17
3,7	0.8417471E 00	0.8454095E 00	0.8489556E 00	0.8523909E 00	0.8557205E 00	18
4,4	0.8445162E 00	0.8472724E 00	0.8499392E 00	0.8525207E 00	0.8550209E 00	19
4,5	0.1119433E 01	0.1124019E 01	0.1128455E 01	0.1132749E 01	0.1136906E 01	20
4,6	0.1322470E 01	0.1328845E 01	0.1335014E 01	0.1340987E 01	0.1346776E 01	21
4,7	0.1438025E 01	0.1445608E 01	0.1452951E 01	0.1460067E 01	0.1466960E 01	22
5,5	0.1524585E 01	0.1532210E 01	0.1539588E 01	0.1546728E 01	0.1553643E 01	23
5,6	0.1830619E 01	0.1841224E 01	0.1851485E 01	0.1861422E 01	0.1871046E 01	24
5,7	0.2006803E 01	0.2019420E 01	0.2031634E 01	0.2043466E 01	0.2054931E 01	25
6,6	0.2220293E 01	0.2235037E 01	0.2249311E 01	0.2263135E 01	0.2276539E 01	26
6,7	0.2446411E 01	0.2463955E 01	0.2480945E 01	0.2497408E 01	0.2513366E 01	27
7,7	0.2702599E 01	0.2723473E 01	0.2743696E 01	0.2763299E 01	0.2782306E 01	28

I,J = 1,2,....,7, WHERE 1 IS 88.5 DEG FROM $\bar{N}$, 2 IS 82.6 DEG FROM $\bar{N}$, 3 IS 72.7 DEG FROM $\bar{N}$, 4 IS 60.0 DEG FROM $\bar{N}$, 5 IS 45.3 DEG FROM $\bar{N}$, 6 IS 29.4 DEG FROM $\bar{N}$, 7 IS 13.0 DEG FROM $\bar{N}$.

SYMMETRICAL REFLECTION FUNCTION R(X,I,J) FOR GAMMA = 1.0

I,J	X = 5.1	X = 5.2	X = 5.3	X = 5.4	X = 5.5	NN
1,1	0.1461406E-01	0.1461580E-01	0.1461748E-01	0.1461912E-01	0.1462071E-01	1
1,2	0.2922033E-01	0.2923105E-01	0.2924147E-01	0.2925157E-01	0.2926138E-01	2
1,3	0.3924038E-01	0.3927124E-01	0.3930116E-01	0.3933023E-01	0.3935843E-01	3
1,4	0.4821165E-01	0.4827553E-01	0.4833749E-01	0.4839764E-01	0.4845603E-01	4
1,5	0.5583928E-01	0.5594551E-01	0.5604854E-01	0.5614854E-01	0.5624564E-01	5
1,6	0.6146052E-01	0.6160840E-01	0.6175191E-01	0.6189118E-01	0.6202644E-01	6
1,7	0.6467348E-01	0.6484979E-01	0.6502098E-01	0.6518710E-01	0.6534839E-01	7
2,2	0.1062419E 00	0.1063082E 00	0.1063724E 00	0.1064348E 00	0.1064954E 00	8
2,3	0.1802834E 00	0.1804739E 00	0.1806588E 00	0.1808383E 00	0.1810125E 00	9
2,4	0.2443461E 00	0.2447407E 00	0.2451235E 00	0.2454950E 00	0.2458556E 00	10
2,5	0.2964523E 00	0.2971084E 00	0.2977447E 00	0.2983623E 00	0.2989621E 00	11
2,6	0.3339188E 00	0.3348321E 00	0.3357185E 00	0.3365788E 00	0.3374142E 00	12
2,7	0.3550802E 00	0.3561690E 00	0.3572258E 00	0.3582519E 00	0.3592486E 00	13
3,3	0.3617702E 00	0.3623182E 00	0.3628499E 00	0.3633661E 00	0.3638672E 00	14
3,4	0.5389425E 00	0.5400771E 00	0.5411781E 00	0.5422465E 00	0.5432839E 00	15
3,5	0.6885391E 00	0.6904262E 00	0.6922566E 00	0.6940331E 00	0.6957580E 00	16
3,6	0.7973076E 00	0.7999350E 00	0.8024842E 00	0.8049586E 00	0.8073614E 00	17
3,7	0.8589491E 00	0.8620809E 00	0.8651204E 00	0.8680715E 00	0.8709379E 00	18
4,4	0.8574440E 00	0.8597932E 00	0.8620720E 00	0.8642839E 00	0.8664312E 00	19
4,5	0.1140935E 01	0.1144842E 01	0.1148632E 01	0.1152309E 01	0.1155880E 01	20
4,6	0.1352386E 01	0.1357825E 01	0.1363103E 01	0.1368224E 01	0.1373199E 01	21
4,7	0.1473644E 01	0.1480126E 01	0.1486420E 01	0.1492529E 01	0.1498463E 01	22
5,5	0.1560346E 01	0.1566841E 01	0.1573142E 01	0.1579258E 01	0.1585194E 01	23
5,6	0.1880375E 01	0.1889421E 01	0.1898195E 01	0.1906713E 01	0.1914983E 01	24
5,7	0.2066047E 01	0.2076828E 01	0.2087293E 01	0.2097450E 01	0.2107316E 01	25
6,6	0.2289515E 01	0.2302110E 01	0.2314331E 01	0.2326193E 01	0.2337714E 01	26
6,7	0.2528841E 01	0.2543853E 01	0.2558424E 01	0.2572573E 01	0.2586316E 01	27
7,7	0.2800747E 01	0.2818642E 01	0.2836016E 01	0.2852891E 01	0.2869286E 01	28

I,J = 1,2,....,7, WHERE 1 IS 88.5 DEG FROM $\bar{N}$, 2 IS 82.6 DEG FROM $\bar{N}$, 3 IS 72.7 DEG FROM $\bar{N}$, 4 IS 60.0 DEG FROM $\bar{N}$, 5 IS 45.3 DEG FROM $\bar{N}$, 6 IS 29.4 DEG FROM $\bar{N}$, 7 IS 13.0 DEG FROM $\bar{N}$.

SYMMETRICAL REFLECTION FUNCTION R(X,I,J) FOR GAMMA = 1.0

I,J	X = 5.6	X = 5.7	X = 5.8	X = 5.9	X = 6.0	NN
1,1	0.1462224E-01	0.1462375E-01	0.1462521E-01	0.1462663E-01	0.1462800E-01	1
1,2	0.2927091E-01	0.2928017E-01	0.2928917E-01	0.2929794E-01	0.2930646E-01	2
1,3	0.3938583E-01	0.3941246E-01	0.3943837E-01	0.3946357E-01	0.3948806E-01	3
1,4	0.4851275E-01	0.4856789E-01	0.4862148E-01	0.4867363E-01	0.4872433E-01	4
1,5	0.5633996E-01	0.5643162E-01	0.5652072E-01	0.5660740E-01	0.5669171E-01	5
1,6	0.6215784E-01	0.6228556E-01	0.6240974E-01	0.6253046E-01	0.6264800E-01	6
1,7	0.6550515E-01	0.6565762E-01	0.6580585E-01	0.6595010E-01	0.6609040E-01	7
2,2	0.1065543E 00	0.1066114E 00	0.1066671E 00	0.1067212E 00	0.1067738E 00	8
2,3	0.1811818E 00	0.1813462E 00	0.1815062E 00	0.1816618E 00	0.1818132E 00	9
2,4	0.2462060E 00	0.2465464E 00	0.2468774E 00	0.2471995E 00	0.2475128E 00	10
2,5	0.2995448E 00	0.3001106E 00	0.3006611E 00	0.3011964E 00	0.3017171E 00	11
2,6	0.3382258E 00	0.3390145E 00	0.3397815E 00	0.3405276E 00	0.3412535E 00	12
2,7	0.3602170E 00	0.3611584E 00	0.3620740E 00	0.3629646E 00	0.3638314E 00	13
3,3	0.3643540E 00	0.3648271E 00	0.3652871E 00	0.3657346E 00	0.3661700E 00	14
3,4	0.5442916E 00	0.5452708E 00	0.5462229E 00	0.5471491E 00	0.5480499E 00	15
3,5	0.6974335E 00	0.6990615E 00	0.7006444E 00	0.7021841E 00	0.7036817E 00	16
3,6	0.8096957E 00	0.8119642E 00	0.8141702E 00	0.8163159E 00	0.8184037E 00	17
3,7	0.8737234E 00	0.8764310E 00	0.8790643E 00	0.8816261E 00	0.8841189E 00	18
4,4	0.8685172E 00	0.8705441E 00	0.8725147E 00	0.8744314E 00	0.8762962E 00	19
4,5	0.1159348E 01	0.1162718E 01	0.1165995E 01	0.1169181E 01	0.1172279E 01	20
4,6	0.1378030E 01	0.1382726E 01	0.1387292E 01	0.1391733E 01	0.1396055E 01	21
4,7	0.1504230E 01	0.1509832E 01	0.1515283E 01	0.1520586E 01	0.1525745E 01	22
5,5	0.1590960E 01	0.1596562E 01	0.1602009E 01	0.1607306E 01	0.1612460E 01	23
5,6	0.1923017E 01	0.1930823E 01	0.1938414E 01	0.1945797E 01	0.1952978E 01	24
5,7	0.2116901E 01	0.2126219E 01	0.2135279E 01	0.2144094E 01	0.2152670E 01	25
6,6	0.2348907E 01	0.2359786E 01	0.2370362E 01	0.2380651E 01	0.2390663E 01	26
6,7	0.2599671E 01	0.2612656E 01	0.2625282E 01	0.2637567E 01	0.2649521E 01	27
7,7	0.2885223E 01	0.2900717E 01	0.2915790E 01	0.2930456E 01	0.2944730E 01	28

I,J = 1,2,...,7, WHERE 1 IS 88.5 DEG FROM $\bar{N}$, 2 IS 82.6 DEG FROM $\bar{N}$, 3 IS 72.7 DEG FROM $\bar{N}$, 4 IS 60.0 DEG FROM $\bar{N}$, 5 IS 45.3 DEG FROM $\bar{N}$, 6 IS 29.4 DEG FROM $\bar{N}$, 7 IS 13.0 DEG FROM $\bar{N}$.

SYMMETRICAL REFLECTION FUNCTION R(X,I,J) FOR GAMMA = 1.0

I,J	X = 6.1	X = 6.2	X = 6.3	X = 6.4	X = 6.5	NN
1,1	0.1462935E-01	0.1463066E-01	0.1463193E-01	0.1463317E-01	0.1463438E-01	1
1,2	0.2931475E-01	0.2932282E-01	0.2933070E-01	0.2933836E-01	0.2934582E-01	2
1,3	0.3951192E-01	0.3953515E-01	0.3955777E-01	0.3957982E-01	0.3960131E-01	3
1,4	0.4877371E-01	0.4882179E-01	0.4886862E-01	0.4891426E-01	0.4895871E-01	4
1,5	0.5677376E-01	0.5685368E-01	0.5693153E-01	0.5700738E-01	0.5708127E-01	5
1,6	0.6276244E-01	0.6287372E-01	0.6298226E-01	0.6308800E-01	0.6319106E-01	6
1,7	0.6622702E-01	0.6636006E-01	0.6648970E-01	0.6661594E-01	0.6673920E-01	7
2,2	0.1068251E 00	0.1068749E 00	0.1069235E 00	0.1069708E 00	0.1070170E 00	8
2,3	0.1819605E 00	0.1821040E 00	0.1822436E 00	0.1823798E 00	0.1825126E 00	9
2,4	0.2478177E 00	0.2481147E 00	0.2484039E 00	0.2486857E 00	0.2489604E 00	10
2,5	0.3022240E 00	0.3027175E 00	0.3031984E 00	0.3036668E 00	0.3041234E 00	11
2,6	0.3419600E 00	0.3426480E 00	0.3433182E 00	0.3439712E 00	0.3446077E 00	12
2,7	0.3646753E 00	0.3654970E 00	0.3662975E 00	0.3670777E 00	0.3678383E 00	13
3,3	0.3665937E 00	0.3670063E 00	0.3674082E 00	0.3677999E 00	0.3681816E 00	14
3,4	0.5489269E 00	0.5497809E 00	0.5506126E 00	0.5514233E 00	0.5522134E 00	15
3,5	0.7051394E 00	0.7065588E 00	0.7079413E 00	0.7092886E 00	0.7106016E 00	16
3,6	0.8204356E 00	0.8224144E 00	0.8243417E 00	0.8262197E 00	0.8280505E 00	17
3,7	0.8865457E 00	0.8889090E 00	0.8912115E 00	0.8934553E 00	0.8956427E 00	18
4,4	0.8781115E 00	0.8798789E 00	0.8816004E 00	0.8832778E 00	0.8849127E 00	19
4,5	0.1175297E 01	0.1178235E 01	0.1181097E 01	0.1183887E 01	0.1186603E 01	20
4,6	0.1400261E 01	0.1404356E 01	0.1408345E 01	0.1412231E 01	0.1416020E 01	21
4,7	0.1530768E 01	0.1535659E 01	0.1540424E 01	0.1545067E 01	0.1549594E 01	22
5,5	0.1617476E 01	0.1622358E 01	0.1627115E 01	0.1631748E 01	0.1636265E 01	23
5,6	0.1959970E 01	0.1966778E 01	0.1973408E 01	0.1979868E 01	0.1986166E 01	24
5,7	0.2161020E 01	0.2169149E 01	0.2177072E 01	0.2184791E 01	0.2192314E 01	25
6,6	0.2400409E 01	0.2409898E 01	0.2419142E 01	0.2428147E 01	0.2436927E 01	26
6,7	0.2661160E 01	0.2672495E 01	0.2683537E 01	0.2694300E 01	0.2704790E 01	27
7,7	0.2958630E 01	0.2972167E 01	0.2985359E 01	0.2998217E 01	0.3010751E 01	28

I,J = 1,2,....,7. WHERE 1 IS 88.5 DEG FROM $\bar{N}$, 2 IS 82.6 DEG FROM $\bar{N}$, 3 IS 72.7 DEG FROM $\bar{N}$, 4 IS 60.0 DEG FROM $\bar{N}$, 5 IS 45.3 DEG FROM $\bar{N}$, 6 IS 29.4 DEG FROM $\bar{N}$, 7 IS 13.0 DEG FROM $\bar{N}$.

SYMMETRICAL REFLECTION FUNCTION R(X,I,J) FOR GAMMA = 1.0

I,J	X = 6.6	X = 6.7	X = 6.8	X = 6.9	X = 7.0	NN
1,1	0.1463555E-01	0.1463671E-01	0.1463784E-01	0.1463893E-01	0.1464000E-01	1
1,2	0.2935310E-01	0.2936022E-01	0.2936715E-01	0.2937392E-01	0.2938052E-01	2
1,3	0.3962225E-01	0.3964269E-01	0.3966263E-01	0.3968209E-01	0.3970108E-01	3
1,4	0.4900207E-01	0.4904437E-01	0.4908564E-01	0.4912591E-01	0.4916522E-01	4
1,5	0.5715323E-01	0.5722365E-01	0.5729222E-01	0.5735916E-01	0.5742448E-01	5
1,6	0.6329149E-01	0.6338960E-01	0.6348521E-01	0.6357855E-01	0.6366962E-01	6
1,7	0.6685925E-01	0.6697637E-01	0.6709069E-01	0.6720221E-01	0.6731111E-01	7
2,2	0.1070620E 00	0.1071059E 00	0.1071486E 00	0.1071905E 00	0.1072313E 00	8
2,3	0.1826420E 00	0.1827682E 00	0.1828914E 00	0.1830115E 00	0.1831288E 00	9
2,4	0.2492282E 00	0.2494895E 00	0.2497443E 00	0.2499931E 00	0.2502357E 00	10
2,5	0.3045684E 00	0.3050026E 00	0.3054261E 00	0.3058394E 00	0.3062429E 00	11
2,6	0.3452284E 00	0.3458337E 00	0.3464244E 00	0.3470007E 00	0.3475633E 00	12
2,7	0.3685800E 00	0.3693035E 00	0.3700092E 00	0.3706982E 00	0.3713708E 00	13
3,3	0.3685538E 00	0.3689167E 00	0.3692709E 00	0.3696166E 00	0.3699540E 00	14
3,4	0.5529836E 00	0.5537349E 00	0.5544678E 00	0.5551831E 00	0.5558814E 00	15
3,5	0.7118819E 00	0.7131306E 00	0.7143487E 00	0.7155376E 00	0.7166981E 00	16
3,6	0.8298354E 00	0.8315766E 00	0.8332751E 00	0.8349328E 00	0.8365511E 00	17
3,7	0.8977757E 00	0.8998564E 00	0.9018865E 00	0.9038680E 00	0.9058024E 00	18
4,4	0.8865068E 00	0.8880617E 00	0.8895785E 00	0.8910589E 00	0.8925038E 00	19
4,5	0.1189253E 01	0.1191837E 01	0.1194358E 01	0.1196818E 01	0.1199219E 01	20
4,6	0.1419714E 01	0.1423318E 01	0.1426832E 01	0.1430263E 01	0.1433611E 01	21
4,7	0.1554009E 01	0.1558316E 01	0.1562517E 01	0.1566618E 01	0.1570621E 01	22
5,5	0.1640669E 01	0.1644963E 01	0.1649154E 01	0.1653244E 01	0.1657233E 01	23
5,6	0.1992307E 01	0.1998295E 01	0.2004137E 01	0.2009838E 01	0.2015405E 01	24
5,7	0.2199651E 01	0.2206809E 01	0.2213790E 01	0.2220606E 01	0.2227259E 01	25
6,6	0.2445488E 01	0.2453838E 01	0.2461986E 01	0.2469937E 01	0.2477698E 01	26
6,7	0.2715019E 01	0.2724998E 01	0.2734735E 01	0.2744237E 01	0.2753513E 01	27
7,7	0.3022975E 01	0.3034900E 01	0.3046537E 01	0.3057896E 01	0.3068987E 01	28

I,J = 1,2,....,7, WHERE 1 IS 88.5 DEG FROM $\bar{N}$, 2 IS 82.6 DEG FROM $\bar{N}$, 3 IS 72.7 DEG FROM $\bar{N}$, 4 IS 60.0 DEG FROM $\bar{N}$, 5 IS 45.3 DEG FROM $\bar{N}$, 6 IS 29.4 DEG FROM $\bar{N}$, 7 IS 13.0 DEG FROM $\bar{N}$.