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DELLVA, WILFRED LEO

# EVALUATION OF RISKY, INTERTEMPORAL, IMPERFECTLY, CORRELATED CASH FLOWS

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# EVALUATION OF RISKY, INTERTEMPORAL, IMPERFECTLY, CORRELATED CASH FLOWS

A DISSERTATION

SUBMITTED TO THE GRADUATE FACULTY

in partial fulfillment of the requirements for the

degree of

DOCTOR OF PHILOSOPHY

by

WILFRED LEO DELLVA

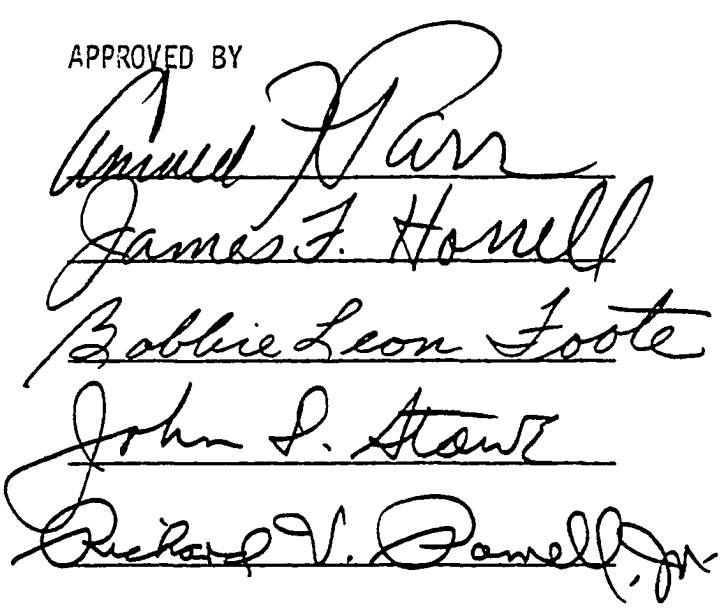
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1980

EVALUATION OF RISKY, INTERTEMPORAL,

IMPERFECTLY, CORRELATED CASH FLOWS

APPROVED BY



DISSERTATION COMMITTEE

TABLE OF CONTENTS

page

LIST OF ILLUSTRATIONS . . . . . vii

Chapter

I. INTRODUCTION. . . . . 1

Problem Identification. . . . . 1

Overview of Solution Methodology. . . . . 2

Limiting Assumptions. . . . . 4

II. REVIEW OF RELATED LITERATURE. . . . . 5

Introduction. . . . . 5

Probabilistic Models. . . . . 7

Popular Risk Adjustment Techniques. . . . . 10

Risk Adjusted Discount Rates . . . . . 10

Certainty Equivalence. . . . . 14

Coefficient of Variation . . . . . 16

Portfolio Theory and Capital Asset Pricing Model. . . . . 18

Portfolio Theory . . . . . 18

Capital Market Theory. . . . . 23

Capital Asset Pricing Model and Capital Budgeting. . . 32

Multi-Period Capital Asset Pricing Models and Capital Budgeting . . . . . 40

Mean-Variance Synthesis. . . . . 42

Alternative Market Models . . . . . 45

Mean/Semi-Variance Model . . . . . 46

Stochastic Dominance Models. . . . . 54

Time-State Preference Models . . . . . 61

Comparison of Market Models . . . . . 69

Practice versus Theory. . . . . 78

III. MULTIVARIATE CAPITAL BUDGETING MODEL. . . . . 82

Introduction. . . . . 82

Asset-by-Asset Analysis versus Portfolio Approach . . . . 84

Independence versus Correlated Cash Flows . . . . . 85

Multivariate Normal Distribution as an Underlying Structure. . . . . 90

Assumption of Multivariate Normality . . . . . 92

Multivariate Normal Distribution . . . . . 95

Measurement/Evaluation of Return. . . . . 100

Certain Cash Flows . . . . . 101

Uncertain Cash Flows . . . . . 103

Measurement/Evaluation of Risk. . . . . 108

Variance-Covariance Matrix . . . . . 110

Estimates of the population variance-covariance matrix. . . . . 114

Some geometric relationships of concentration ellipsoids and the variance-covariance matrix . . . . . 115

Need for scalar risk measures . . . . . 121

Discounted Variance-Covariance Matrix. . . . . 123

Geometric interpretation of the discounted variance-covariance matrix . . . . . 126

Variance of the net present value . . . . . 130

Boundary cases-independence and perfect correlation . . . . . 131

Imperfect correlation. . . . . 136

Generalized Variance . . . . . 138

Geometric interpretation of det $\Sigma$ . . . . . 139

Determinant as volume. . . . . 140
Generalized variance and the isovariance ellipsoid . . . . . 145
Statistical Properties of det Σ . . . . . 149
Eigenvalues and det Σ . . . . . 151
IV. EVALUATION OF SCALAR MULTIVARIATE RISK MEASURES . . . . . 157
Introduction. . . . . 157
Interpretation of Scalar Risk Measures. . . . . 168
Time Adjustment . . . . . 162
Scale Adjustment. . . . . 167
Transformation of Cash Flows . . . . . 167
Multivariate Analogues of the Coefficient of Variation . . . . . 168
Decomposition Analysis. . . . . 170
Limitations of Scalar Multivariate Risk Measures. . . . . 171
V. SUMMARY AND SUGGESTIONS FOR ADDITIONAL RESEARCH . . . . . 177
Summary . . . . . 177
Conclusions . . . . . 181
Limitations of the Study. . . . . 182
Suggestions for Additional Research . . . . . 183
BIBLIOGRAPHY. . . . . 185

# LIST OF ILLUSTRATIONS


page

Figure

1. Mean-Standard Deviation Efficient Frontier . . . . . 25
2. Capital Market Line. . . . . 25
3. Security Market Line in Terms of Beta. . . . . 29
4. Security Market Line in Terms of Covariance. . . . . 31
5. Comparison of WACC and SML Criteria. . . . . 43
6. E-S Efficient Frontier, Capital Market Line. . . . . 50
7. First Order Stochastic Dominance . . . . . 56
8. Second Order Stochastic Dominance. . . . . 57
9. Third Order Stochastic Dominance . . . . . 59
10. Isodensity Contours as $\rho$ varies. . . . . 98
11. Ninety Percent Ellipses for $\sigma_1=\sigma_2$ and $\rho_{12}=.6$ and $\rho_{12}=0.0$ . . 116
12. Ninety Percent Ellipses Before and After Discounting . . . . 128
13. Two-dimensional Parallelogram. . . . . 142
14. Three-dimensional Parallelotope. . . . . 143

# EVALUATION OF RISKY, INTERTEMPORAL IMPERFECTLY CORRELATED CASH FLOWS

## CHAPTER I

## INTRODUCTION

Capital-budgeting is a subset of financial management that is concerned with the evaluation of long-term investment choices. In general, capital-budgeting analysis is concerned with the development of decision rules to select, evaluate, or rank real asset opportunities on the basis of expected return and risk. These decisions receive considerable managerial attention because the investment opportunities require a large commitment of resources, the effects of the decision extend well into the future, and the decisions are difficult to reverse.

### Problem Identification

The objective of this study is the development of a capital-budgeting model that explicitly considers the impact of intertemporally correlated cash flows. Traditional capital-budgeting risk analysis often assumes intertemporal independence or utilizes a single-period market model. An important assumption of this study is that intertemporal correlations contribute to the total risk of a capital-budgeting

project. Therefore, risk analysis techniques that assume intertemporal independence may understate the total risk of a project. Failure to consider all aspects of risk may lead to incorrect selection and ranking decisions that adversely affect the value of the firm.

## Overview of Solution Methodology

Chapter II, Review of Related Literature, provides a comprehensive review of existing methods of risk analysis. Two broad classes of solution techniques are presented: variations of present value analysis and portfolio or market models. Within these classes there exist many different approaches to risk analysis. Dissatisfaction with the existing methods of risk analysis, when the cash flows are intertemporally correlated, motivated this research to identify a capital-budgeting model that explicitly considers these interrelationships.

Chapters III and IV develop and evaluate a multivariate capital-budgeting model. The multivariate approach was selected to accommodate autocorrelated cash flows. Multivariate statistical methods have been developed to simultaneously analyze dependence structures. Capital-budgeting analysis requires a modification to traditional multivariate procedures to accommodate differences in the timing of the cash flows. The element of time is accommodated by transforming the cash flow distribution to present time.

The assumption of normality is crucial to the development of this multivariate capital-budgeting model. With multivariate normally distributed cash flows, analysis of capital-budgeting return and risk reduces to the evaluation of a mean vector and a variance-covariance matrix.

The expected net present value is a widely accepted measure of return. The expected mean cash flows are discounted at the risk-free rate to accommodate differences in timing.

Risk is defined as the variability of the future cash flow stream. With such a general definition, there is considerable controversy as to the most appropriate risk measure. In the case of the univariate normal distribution, the variance and the standard deviation can be shown to have desirable statistical properties. With the assumption of a multivariate normal distribution, the variance-covariance matrix is the multivariate extension of the univariate variance. The variance-covariance matrix identifies the total variation about the centroid of the multivariate normal distribution.

Multivariate distributions may be evaluated geometrically using isodensity ellipsoids. This geometric representation lends considerable insight to the interrelationships depicted by the variance-covariance matrix. Capital-budgeting problems often require ranking of alternative investment opportunities in terms of return and risk. The variance-covariance matrix depicts total variability; however, there is considerable difficulty comparing matrices. Because of these difficulties, scalar representations of the variance-covariance structure are needed.

The remainder of this study will develop and evaluate three scalar multivariate risk measures: 1. variance of the net present value distribution; 2. volume of the isovariance ellipsoid; and 3. generalized variance which is the determinant of the variance-covariance matrix. Each measure will be developed algebraically and geometrically. Statistical

properties, capabilities, and limitations of each measure will be evaluated. Finally, the scalar measures will be compared to identify the preferred scalar multivariate risk measure.

## Limiting Assumptions

Throughout the study many assumptions will be made to facilitate the development of the multivariate capital-budgeting model. The most important assumptions are

1. Asset-by-asset selection methods are appropriate for selecting and ranking capital-budgeting projects.
2. Intertemporal correlation contributes significantly to total risk.
3. Timing of the cash flow stream must be accommodated.
4. Decision makers value real assets in terms of two-parameters, return and risk.
5. Multivariate normal distribution is an appropriate model of capital-budgeting cash flows.
6. Risk-free rate(s) for future periods is (are) known.
7. Project useful life is known.
8. Decision makers are capable of estimating either the probability distribution of future cash flows or the relevant moments of the probability distribution.
9. Intertemporal correlations vary from independence to perfect positive correlation.
10. Alternative capital-budgeting projects are significantly different and can be meaningfully ordered.

# CHAPTER II

# REVIEW OF RELATED LITERATURE

## Introduction

The theory of asset selection under conditions of perfect certainty has been well developed and readily accepted by the finance community.[1] The essential assumptions of the certainty models are: 1. perfect capital markets; 2. complete and certain knowledge about investment outcomes; 3. independence of alternatives; and 4. indivisible investment projects. The above assumptions describe an ideal situation; useful for academic study, but usually quite different from the "real world".

An approach to studying more difficult or more complicated situations is to start with the most simple case, then relax the simplifying assumptions to develop a more realistic model. Under conditions of perfect certainty, the theoretically acceptable approach to capital budgeting analysis is the use of discounted cash flow methods (DCF). Given the future cash flows associated with acceptance of a project, the present value of these cash flows are computed and the present value of the

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[1]Haley, C. W. and Schall, L. D. (1973) *The Theory of Financial Decisions*. New York: McGraw-Hill, Inc. Chapters 2 and 3, pp. 15-72. In the second edition (1979) a discussion of asset selection with perfect certainty will also be found in Chapters 2 and 3.

benefits are compared to the present value of the cost. The appropriate decision rule then is to accept all projects that have a positive net present value or accept if the present value of the benefits is greater than or equal to the present value of the cost. See Robichek and Myers[2] and Haley and Schall[3] for a detailed discussion of financial decision making with perfect markets and certainty.

The objective of discounting is to account for the differences in the timing of the cash flows. The appropriate discount rate is the opportunity cost of funds (a premium for waiting, or Time Value of Money). The DCF analysis transforms the future cash flows into their present time equivalent. As long as the assumptions of perfect certainty are met, decisions using the DCF rule will result in either maintaining or increasing the value of the firm; with the amount of increase in value equal to the net present value of the project.

Relaxing the strong assumptions of perfect certainty results in the need for more sophisticated capital budgeting approaches. The most commonly analyzed departure from perfect certainty is the uncertainty associated with the future benefits and costs. Therefore, these future cash flows must be estimated and evaluated.

A broad class of analytical techniques have been developed to study the capital budgeting decision. In general, these solution methods focus on two aspects of the decision - an analysis of the return to

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[2]Robichek, A. A. and Myers, S. C. (1965) *Optimal Financing Decisions*. Englewood Cliffs, New Jersey: Prentice-Hall. pp. 9-16.

[3]Haley and Schall (1973) Chapters 2 and 3.

the owners and the risk or uncertainty in that return. Here risk is defined as the variability of the future cash flows. The remainder of Chapter II will systematically review models that incorporate risk and return in financial decision making.

## Probabilistic Models

Academics and business practitioners have long recognized that the risk associated with capital expenditures is a significant dimension of financial management. The finance literature for the past twenty years has been dominanted by models and/or procedures that explicitly study the impact of risk under varying assumptions. The goals of these studies have been twofold: 1. to rigorously study the impact of risk on decision making, and 2. to develop techniques that can easily be used in the field.

Hillier[4] developed a probabilistic approach to the capital budgeting problem. Recognizing the random nature of future cash flows, he developed the Probability Distribution of Net Present Value. More specifically, he assumed that the periodic cash flows are normally distributed. Relying on the Central Limit Theorem, he argued that the cash flows, as sums of random variables would be distributed normal or nearly normal.

Given the assumptions about the cash flows, the expected present worth may be defined as:

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[4]Hillier, F. S. (1963) "The Derivation of Probabilistic Information for the Evaluation of Risky Investments," *Management Sciences*. Vol. 9: pp. 443-457.

$$E(PV) = \mu_p = \sum_{j=0}^{n} \left[ \frac{\mu_j}{(1+i)^j} \right] \tag{1}$$

where $\mu_j$ is the mean cash flow during the jth year and i is the rate of interest which properly reflects the decision makers time value of money. Using a decision rule based on return alone, if $\mu_p > 0$, the investment should be made since this choice would increase the expected total wealth of the firm given that i represents the opportunity rate of return.

Extending his work to explicitly consider risk, Hillier studied three cases: 1. intertemporally independent cash flows, 2. perfectly correlated cash flows, and 3. a combination of 1. and 2. with some cash flows independent and others perfectly correlated. These special cases result in different values for the standard deviation of the cash flows. The standard deviation for the independent case is

$$\sigma_p = \left[ \sum_{j=0}^{n} \frac{\sigma_j^2}{(1+i)^{2j}} \right]^{\frac{1}{2}}, \tag{2}$$

while the standard deviation of the perfectly correlated case is

$$\sigma_p = \left[ \sum_{j=0}^{n} \frac{\sigma_j}{(1+i)^j} \right]. \tag{3}$$

Given fixed values of $\sigma_j$, $\sigma_p$ is smallest in the case of independent cash flows and largest for perfect positive correlation. The third

case, the combination case, results in a measure of risk somewhere between independence and perfect correlation.

Wagle[5] extended Hillier's analysis by including cases where the cash flows are less than perfectly positive correlated. Wagle then discounted these variances and covariances to arrive at a measure of risk,

$$\sigma_p^2 = \sum_{t=0}^{n} \frac{\sigma_t^2}{(1+i)^{2t}} + 2 \sum_{t \neq t'}^{n} \frac{\sigma_{t,t'}}{(1+i)^{t+t'}} \quad . \tag{4}$$

For a three period project, the variance of the net present value can be written explicitly as

$$\sigma_p^2 = \underbrace{\frac{\sigma_1^2}{(1+i)^2} + \frac{\sigma_2^2}{(1+i)^4} + \frac{\sigma_3^2}{(1+i)^6}}_{\text{(Variance terms)}} + \underbrace{2\left(\frac{\sigma_{12}}{(1+i)^3} + \frac{\sigma_{13}}{(1+i)^4} + \frac{\sigma_{23}}{(1+i)^5}\right)}_{\text{(Covariance terms)}} \tag{5}$$

To derive the discounted risk measure, the variance terms are discounted by a factor of 2t and the covariance terms are discounted by the sum of the exponents that reflect the time periods.

With autocorrelated cash flows, the above discounting procedure assumes that variance and covariance terms can be combined in an additive manner and result in a meaningful measure of total risk.

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[5]Wagle, B. (1967) "A Statistical Analysis of Risk in Capital Investment Projects," *Operational Research Quarterly*. Vol. 18: pp. 13-33.

Hillier, in a later paper[6] and a monograph[7], extended his earlier analysis to include interrelated projects. Here, he analyzed both the riskiness of individual projects and the effects on risk of relationships between projects (portfolios of real assets). The significance of this approach is the continued expansion of the probabilistic analysis that was developed in his earlier work. Hillier's[8] monograph treats the full range of the real asset selection problem - from the cash flow estimates to combining assets into firms. Much of this work is an extension of Weingartner's[9] classic capital budgeting work that employed linear and integer programming, dynamic programming and discrete optimization models to address the problem of interrelationships between projects.

## Popular Risk Adjustment Techniques

### Risk Adjusted Discount Rates

Using present value models with uncertain cash flows, one needs to account for both the time value of money and for risk. Two of the most popular approaches that incorporate both time and risk are the Risk Adjusted Discount Rate (RAD) and Certainty Equivalence (CE). Both

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[6]Hillier, F. S. (1971) "A Basic Model for Capital Budgeting of Risky Interrelated Projects," *Engineering Economist*. Vol. 17: pp. 1-30.

[7]Hillier, F. S. (1969) *The Evaluation of Risky Interrelated Investments*. Amsterdam: North-Holland Pub. Co.

[8]Ibid.

[9]Weingartner, H. M. (1963) *Mathematical Programming and the Analysis of Capital Budgeting Problems*. Englewood Cliffs, New Jersey: Prentice-Hall.

methods are conceptually simple and find significant practitioner support.[10] The foundations of the RAD approach to risk analysis can be traced to the pioneer stock valuation models of Williams[11], Gordon[12], and Solomon[13].

The valuation models of Gordon, for example, capitalize future earnings and/or dividends to arrive at an equilibrium price for the stock. In the simplest case, current price is

$$P_0 = \frac{(1-b)Y_0}{k-rb}, \quad [14] \qquad (6)$$

where

$Y_t$ = income per share during period t;

$P_t$ = price of a share at end of period t;

b = fraction of income retained in every future period;

k = stockholders' required rate of return;

r = return on investment the firm es expected to earn in every future period.

Using the Gordon model, one can determine a risk-adjusted discount rate for use in capital budgeting by solving for k in equation 6 above, where

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[10]Klammer, T. (1972) "Empirical Evidence of the Adoption of Sophisticated Capital Budgeting Techniques," *Journal of Business*. Vol. 45: pp. 387-397.

[11]Williams, J. B. (1938) *The Theory of Investment Value*. Cambridge, Mass.: Harvard University Press.

[12]Gordon, M. J. (1962) *The Investment, Financing, and Valuation of the Corporation*. Homewood, Ill.: Richard D. Erwin.

[13]Solomon, E. (1963) *The Theory of Financial Management*. New York: Columbia University Press.

[14]Gordon, M. J. (1962) pp. 44-45.

$$k = \frac{(1-b)Y_0}{P_0} + rb. \tag{7}$$

The value k is the often discussed cost of capital for the firm. Then the present value of a stream of uncertain returns can be expressed as

$$PV = \sum_{t=1}^{\infty} \frac{\bar{R}_t}{(1+k)^t} \tag{8}$$

where $\bar{R}_t$ = the expected value of the return to be received at time t, and k = required rate of return appropriate for the risky stream $\bar{R}_1, \bar{R}_2, \ldots$.

Essentially, k is a rate of return that reflects both the time value of money and an adjustment for risk. Using the RAD approach, the risk adjustment factor and the time value of money factor are combined to form the discount rate used in the present value computation. RAD approach adjusts for risk by varying the discount rate.

As a simple technique to consider risk in capital budgeting analysis, the risk adjusted rate approach is intuitively appealing. A very logical approach is to apply higher discount rates to the more risky projects and lower discount rates to less risky projects. In addition, the theoretical background of stock valuation models provides a plausible linkage between the investment or capital budgeting decision and the goal of increasing stockholder welfare.

Several authors[15] have criticized this approach on both theoretical and practical grounds. First, as time and risk are essentially

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[15]Robichek and Myers (1965) pp. 79-86 and Osteryoung, J. (1979) *Capital Budgeting: Long Term Asset Selection,* 2nd ed. Columbus: Grid, Inc. pp. 102-105.

separate variables, combining their effects into one number, a risk adjusted rate or cost of capital, assumes that uncertainty increases through time or that uncertainty is expected to be resolved at a constant rate over time.[16] When this assumption is not satisfied, the RAD approach incorrectly evaluates future opportunities. One can overcome this objection by using different rates for each period and recognizing the fact that there is some single rate that is equivalent to the series of rates,

$$k = \prod_{t=1}^{n} (1+k_t).$$

This remedy, however, leads to the second significant objection. Viewing equation 7, the risk adjusted rate, k, is equal to the sum of the dividend yield and a constant growth rate. Using this valuation model does not give any clues to the appropriate $k_t$ to be evaluated. This model says nothing about how a practitioner might determine the correct risk adjustment factor. The third objection is that the RAD adjusts the wrong element in the present value computation. The RAD adjusts the discount rate and does not adjust for the variability of the cash flow. One normally associates risk with the variability of the future cash flows not the uncertainty of the discount rate. Uncertain risk-free rates add another dimension to the risk analysis problem. Finally, RAD is not an "efficient" estimator of risk because it does not use all the available information from the probability distribution of the projects' cash flows.

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[16]Robichek and Myers (1965) p. 84.

## Certainty Equivalence

Certainty Equivalence (CE) is an approach to risk analysis that overcomes some of the objections to the risk adjusted rate. The CE approach separates time and risk in the present value framework. Robichek and Myers[17] argue that whatever the risk of an expected cash flow, $\overline{CF}_t$, there is some certain equivalent, $\alpha_t$, where $0 \leq \alpha_t \leq 1$, such that the risk-averse decision maker is indifferent between $CF_t$ and a cash flow $CF_t^* = \alpha_t \overline{CF}_t$ which is certain to be paid. If the decision makers utility function in year t is known, the certainty equivalence can easily be determined.

More explicitly, one can define $CE_t = \alpha_t$ as the ratio $CF_t^*/\overline{CF}_t$. The present value of an uncertain stream of cash flows using RAD approach is

$$PV = \sum_{t=1}^{\infty} \frac{\overline{CF}_t}{(1+k)^t} , \tag{9}$$

where $\overline{CF}_t$ = expected value of the cash flow to be received at time t;

k = risk adjusted discount rate.

Using a certainty equivalent approach, the present value of a stream of future cash flows is

$$PV = \sum_{t=1}^{\infty} \frac{\alpha_t \overline{CF}_t}{(1+i)^t} = \sum_{t=1}^{\infty} \frac{CF_t^*}{(1+i)^t} \tag{10}$$

where $\alpha_t$ = certainty equivalent, $CF_t^*/\overline{CF}_t$

i = riskless rate of interest that reflects only the time value of money.

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[17]Robichek and Myers (1965) p. 84.

For both equations (9) and (10) to assign identical values to each period's cash flows, then

$$\frac{CF_t^*}{(1+i)^t} = \frac{\overline{CF}_t}{(1+k)^t} \quad . \tag{11}$$

If $\alpha_o = 1$, then the certainty equivalent for any period t can be expressed as:

$$\alpha_t = \frac{(1+i)^t}{(1+k)^t} \quad . \tag{12}$$

Using equation (12), and assuming $i_t$ is constant for all t, the impact of the RAD approach is obvious when one analyzes expected future cash flows $\overline{CF}_t$ and $\overline{CF}_{t+1}$, which are equal and are considered equally risky; i.e., $\alpha_t = \alpha_{t+1}$. For period t+1,

$$\alpha_{t+1} = \frac{(1+i)^{t+1}}{(1+k)^{t+1}} = \frac{(1+i)^t(1+i)}{(1+k)^t(1+k)} \tag{13}$$

Since $\alpha_t = \alpha_{t+1}$, the only way Equations (12) and (13) will be equal is if $(1+k) = (1+i)$. This result is contradictory because k is a risk adjusted rate, therefore k is supposed to be greater than i to compensate for the additional riskiness of $\overline{CF}_t$ and $\overline{CF}_{t+1}$. The only way the RAD approach will be consistent with CE approach is to define separate risk adjusted rates $k_t$ for each future period. But this means that $k_t$ cannot be identical even if the expected cash flows to which they apply are considered equally risky. Also, if $k_1 = k_2 = k_{t+1}$, then the expected cash flows in period t cannot be equally risky.

Although the CE approach satisfies the objection of not combining time and risk elements in one number, this approach still has some major deficiencies. CE models require the assumption of intertermporally independent cash flows.[18] This restriction is quite severe when the actual cash flows are intertemporally, imperfectly correlated. Also, the utility functions used to develop the $\alpha_t$ values must be independent of the cash flows in prior periods. Once again, the CE like the RAD only uses part of the information available from the probability distribution of a project's cash flow.[19]

## Coefficient of Variation

The coefficient of variation (CV), defined as the ratio of the standard deviation ($\sigma$) to the expected value (EV), $\sigma/EV$.[20] CV is an intuitively appealing statistic that brings together both risk and return into a single measure. In capital-budgeting applications, the CV may be interpreted as the risk per dollar of return. As such, the CV is a pure number independent of project size; and is useful as a measure of relative risk.

The CV is particularly useful in comparing capital budgeting projects that have considerably different absolute magnitudes of expected-value and standard deviation.[21] In the above case, CV is preferred

---

[18]Haley and Schall (1973) p. 185.

[19]Osteryoung (1979) pp. 103-105.

[20]Gibbons, J. D., Olken, I., and Sobel, M. (1977) *Selecting and Ordering Populations: A New Statistical Methodology*. New York: John Wiley and Sons. p. 89.

[21]Gordon, M. J. (1962) pp. 68-69 and Van Horne, J. C. (1977) *Financial Management and Policy*, 4th ed. Englewood Cliffs, New Jersey: Prentice-Hall. p. 118.

to the use of $\sigma$ as a risk measure because $\sigma$ is highly sensitive to the scale of the probability distribution of the cash flows, whereas CV is a pure number that measures relative risk.

A deficiency of the CV approach to risk analysis is that there is no clear-cut decision rule for project acceptance in terms of the value of the firm. Osteryoung, Scott and Roberts[22] in a recent article propose a modified coefficient of variation (MCVAR) that incorporates the market price of risk concept from the capital asset pricing model (CAPM). Assuming that the CAPM is relevant for real asset selection, they define MCVAR as the ratio of the standard deviation of return of the project to the expected risk premium for the asset as determined by the security market line

$$\text{MCVAR} = \frac{1}{\lambda \rho_{i,m}} = \frac{\sigma_i}{E(RP_i)} \quad .$$

Using MCVAR, they established decision rules for project acceptance overcoming the lack of a decision rule criticism of CV.

The use and acceptance of MCVAR is dependent upon the acceptance of the capital asset pricing model in capital budgeting problems. Portfolio theory and the CAPM will be evaluated in the context of capital budgeting in a later section of this paper.

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[22]Osteryoung, J. S., Scott, E., and Roberts, G. S. (1977) "Selecting Capital Projects with the Coefficient of Variation," *Financial Management*. Vol. 6: pp. 65-70.

## Portfolio Theory and Capital Asset Pricing Model

In an attempt to develop a normative theory of asset selection under uncertainty, much of the theoretical study in finance has focused on the behavior of security prices and the capital markets. The portfolio or "market model" approach to valuation has dominated the literature for nearly twenty-five years. This approach has also been extended to the valuation of real assets. A brief review of modern portfolio theory and its implications for capital budgeting will follow.

### Portfolio Theory

The origins of modern portfolio theory can be traced to the classic works of Markowitz[23] and Tobin.[24] From these seeds, the Capital Asset Pricing Model was developed by Sharpe,[25] Lintner,[26] and Mossin.[27] Empirical testing, additional development, and refinements of market models has been at the forefront of research in finance. Because of the vast literature, after a brief review of these early works cited

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[23]Markowitz, H. M. (1952) "Portfolio Selection," *Journal of Finance*. Vol. 7: pp. 77-91.

[24]Tobin, J. (1958) "Liquidity Preference as Behavior Towards Risk," *Review of Economic Studies*. Vol. 26: pp. 65-86.

[25]Sharpe, W. F. (1963) "A Simplified Model for Portfolio Analysis," *Management Science*. Vol. 9: pp. 272-293; and (1964) "Capital Asset Prices: A Theory of Market Equilibrium," *Journal of Finance*. Vol. 19: pp. 425-442.

[26]Lintner, J. (1965) "The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets," *Review of Economics and Statistics*. Vol. 47: pp. 13-37; and (1965) "Security Prices, Risk and Maximal Gains from Diversification," *Journal of Finance*. Vol. 20: pp. 587-615.

[27]Mossin, J. (1966) "Equilibrium in a Capital Assets Market," *Econometrica*. Vol. 34: pp. 768-783.

above, the remainder of this section will focus on the implication for capital budgeting.

Observing that investors recognize the importance of diversification, Markowitz[28] developed a model of portfolio selection based on the rule that investors consider increased expected returns desirable and increased variability of returns as undesirable. His famous mean-variance model is based on four basic assumptions:

1. All investors maximize one-period expected utility and exhibit diminishing marginal utility of wealth.
2. Investors' risk estimates are proportional to the variability of the expected returns.
3. Investors are willing to base their decisions solely in terms of expected return and risk. That is, utility (U) is a function of variability of return ($\sigma$) and expected return [E(r)]. Symbolically, $U = f[\sigma, E(r)]$.
4. For any given level of risk, investors prefer higher returns to lower returns. Symbolically, $\partial U/\partial E(r) > 0$. Conversely, for any given level of return, investors prefer less risk to more risk. Symbolically, $\partial U/\partial \sigma < 0$.[29]

Given the above assumptions, the mean-variance portfolio problem can be formulated as

$$E = \sum_{i=1}^{n} X_i \mu_i \tag{14}$$

$$V = \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} X_i X_j \tag{15}$$

where: $E$ = expected return of the portfolio

$X_i$ = percentage of the investor's assets which are allocated to

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[28]Markowitz, H. M. (1952) pp. 77-91.

[29]For a lucid description of the assumptions underlying modern portfolio theory, see Francis, J. C. and Archer, S. H. (1979) *Portfolio Analysis*. 2nd ed. Englewood Cliffs: Prentice Hall.

the ith security

$\mu_i$ = expected return of the ith security

$V$ = variance of the return of the portfolio

$\sigma_{ij}$ = covariance between the returns of security i and security j (thus $\sigma_{ii}$ is the variance of return of security i).

Two additional conditions are imposed on the model. One that all funds are invested; symbolically,

$$\sum_{i=1}^{n} X_i = 1.$$

The second which disallows short sales; symbolically, $X_i \geq 0$.

Markowitz[30] illustrated geometrically the solution of the portfolio problem for the three and four asset cases. Using isomean lines and isovariance curves, he derived the set of efficient portfolios in E,V space. In a later monograph, Markowitz[31] described the derivation of the efficient set using classical optimization and mathematical programming techniques.[32]

Responsing to classical criticisms of the Keynsian liquidity preference schedule, an inverse relationship between the demand for cash balances and the rate of interest, Tobin[33] developed a rational explanation of liquidity preference based on the ideas of uncertainty, risk

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[30]Markowitz, H. M. (1952) pp. 77-91.

[31]Markowitz, H. M. (1959) *Portfolio Selection: Efficient Diversification of Investments*. New York: John Wiley & Sons.

[32]For a detailed mathematical solution of the Markowitz variance-covariance portfolio selection problem, see Francis and Archer (1979) Chapters 5 and 6.

[33]Tobin, J. (1958) pp. 65-86.

aversion, and investor diversification. Given idealized uncertainty and a two-asset world, cash and consols, he also developed the mean-variance portfolio model. Although more concerned with the implications for economic theory, Tobin's contribution to modern portfolio theory is his original development of the portfolio separation theorem. Assuming a riskless asset, cash, with an expected return of zero and a single risky asset (or portfolio), consols, then an opportunity locus is defined which describes feasible linear combinations of riskless and risky assets. This opportunity locus is independent or separate from the individual's attitudes toward risk. For the investment portfolio problem, the separation theorem is critical to the development of an equilibrium capital market theory.[34]

Sharpe[35] extended Markowitz's work in determining the efficient set of portfolios by introducing a simplified or diagonal model that greatly reduces computational difficulty.[36] Sharpe's diagonal model is described by

> ... the assumption that the returns of various securities are related only through common relationships with some basic underlying factor. The return from any security is determined solely by random factors and this single outside element; more explicitly:

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[34]For additional discussion of the importance of Tobin's separation theorem to the development of equilibrium capital market theory, see Haley & Schall (1973), pp. 125-127 and Francis and Archer (1979), pp. 152-153.

[35]Sharpe, W. F. (1963) pp. 277-293.

[36]Francis and Archer (1979) p. 127 - When using the full covariance portfolio analysis technique, $(N^2-N)/2=4950$, covariances must be estimated if 100 securities are considered. Using Sharpe's model only 100 regression coefficients must be estimated.

$$R_i = A_i + B_i I + C_i \tag{16}$$

where: $A_i$ and $B_i$ are regression parameters for the $i^{th}$ firm
$R_i$ is the return for the $i^{th}$ security
I is the return on some market index
C is a random error term.[37,38]

Using the convenient notation that the index is treated as the n+1 security where $A_{n+1}$ is the expected value of I and $Q_{n+1}$ is the variance about the expected value, the Markowitz model can be specified using the Sharpe formulation.

Sharpe showed that the mean-variance index model can be formulated as:

$$E = \sum_{i=1}^{n+1} X_i A_i \tag{17}$$

$$V = \sum_{i=1}^{n+1} X_i^2 Q_i \tag{18}$$

The major significance of this formulation and the reason for its name as the diagonal model relates to the specification of the variance-covariance matrix. Whereas the Markowitz formulation results in a n x n matrix with all non-zero elements, the Sharpe formulation results in a n+1 x n+1 diagonal matrix with all off-diagonal elements equal to zero. The diagonal matrix is much easier to invert, matrix inversion being the principal computational bottleneck in the various solution techniques.

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[37]Sharpe, W. F. (1963) p. 281.

[38]For a formal statement of the linear regression model with an unspecified distribution of error terms, see Neter, J. and Wasserman, W. (1974) *Applied Linear Statistical Models*. Homewood, Ill: Irwin. pp. 30-31.

Sharpe compared the index model with the full Markowitz model obtaining the following results. First, Sharpe's model is extremely economical in terms of computer time and storage space requirements. Second, he found that the efficient portfolios, while somewhat different, are very much alike. Subsequent tests have confirmed Sharpe's results.

## Capital Market Theory

With the development of the two-parameter, mean-variance portfolio models, research interest shifted to the stock market implications of actual use of such models. This study of market behavior has acquired the familiar name - Capital Market Theory (CMT). The foundations of the Capital Market Theory can be traced to the capital asset pricing models of Sharpe,[39] Lintner,[40] and Mossin.[41] Essentially, each author believed that he had developed the more general formulation of capital market equilibrium using mean-variance analysis. Fama,[42] however, showed the similarity of the models and demonstrated that one model can be derived directly from another. Many other authors have contributed to the continued development and empirical testing of what is conveniently called the capital asset pricing model. Because the subject of this study is capital-budgeting, not Capital Market Theory, only a brief

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[39]Sharpe, W. F. (1964) pp. 425-442.

[40]Lintner, J. (1965) pp. 587-615.

[41]Mossin, J. (1966) pp. 768-783.

[42]Fama, E. F. (1968) "Risk, Return, and Equilibrium: Some Clarifying Comments," *Journal of Finance*. Vol. 23: pp. 29-40.

overview of the theory will be presented with greater emphasis on the implications of CMT for capital budgeting.

A convenient starting point in the discussion of Capital Market Theory is a review of the assumptions of the model. Because CMT is a logical extension of portfolio theory, CMT assumes that all investors are "Markowitz-efficient investors."[43] Additional assumptions are:

1. Money can be borrowed and lent at a risk-free rate of interest. The return on short-term U.S. government securities is a suitable proxy for this rate.
2. All investors visualize identical probability distributions for future rates of return, idealized uncertainty or homogeneous expectations.
3. All investors have the same one-period investment horizon.
4. All investments are infinitely divisible.
5. There are no taxes or transactions costs.
6. All changes in the level of interest rates are fully anticipated.
7. The capital markets are in equilibrium.[44]

Much of the discussion as to the validity of the CAPM centers around the implications of these assumptions and the effects of relaxing the assumptions on investor decisions and capital budgeting applications.

Given the assumptions or Markowitz-efficient investors with homogeneous expectations, all investors will envision identical opportunity sets that can be illustrated as in Figure 1. Figure 1 represents the mean-standard deviation "Efficient Frontier." Points on the

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[43]Markowitz, H. M. (1952) pp. 77-91 and Francis & Archer (1979) p. 148.

[44]The above assumptions used to develop the Mean-Variance Capital Market Theory are conveniently organized in Francis & Archer (1979) pp. 148-149. In the original papers of Sharpe, Lintner et al., there is considerable discussion of the role of simplifying assumptions in economic theory in general and more specifically the importance of these assumptions in the development of what is called Capital Market Theory.

efficient frontier (EF) dominate all interior combinations of risk and return; while points above and to the left are not feasible.

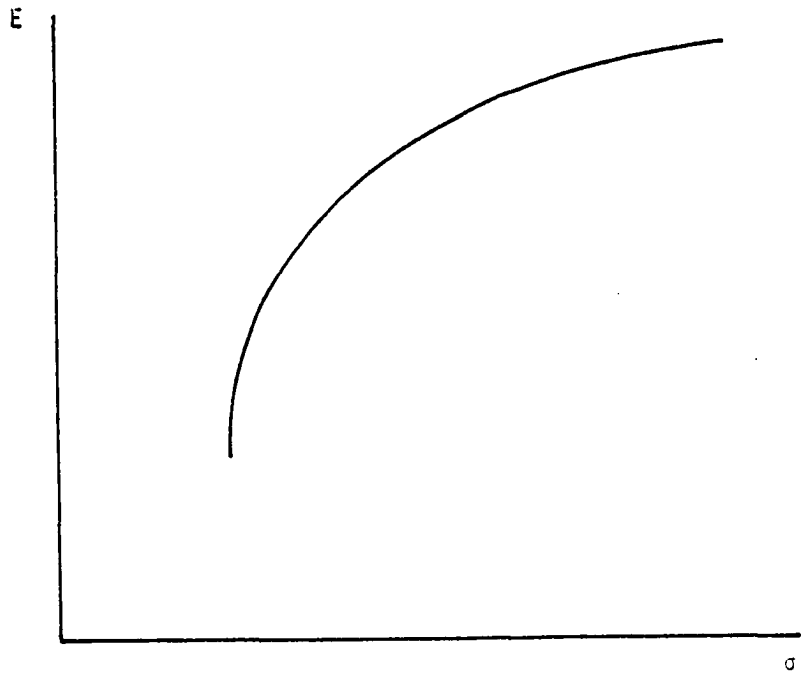



Figure 1: Mean standard deviation
Efficient frontier

Applying assumption 1, borrowing and lending at a fisk-free rate, the Capital Market Line, CML, follows. See Figure 2 below.

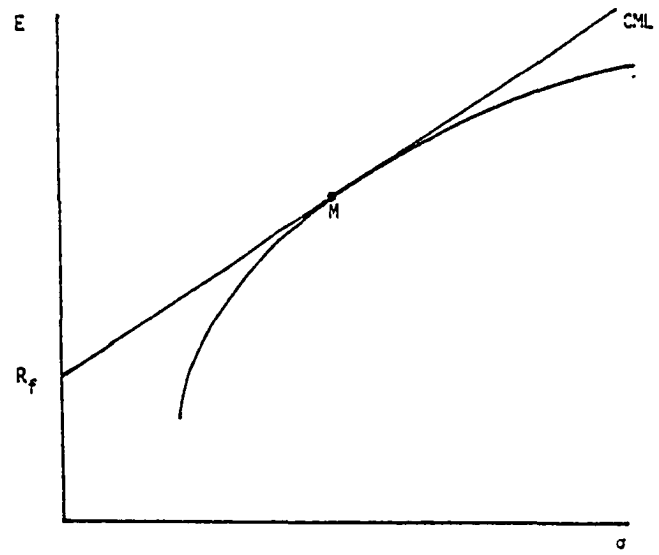



Figure 2: Capital Market Line

The point of tangency of the CML with the efficient frontier is the market portfolio, M. Assuming the market is in equilibrium (assumption 7), all securities must be held. Analyzing their feasible opportunities, investors recognize that choices along the CML (Combinations of Rf and M) dominate all portfolios along the efficient frontier with the exception of portfolio M. Therefore, all investors will want to invest in M; and it follows that M must, in equilibrium, be a portfolio containing all securities, i.e., the market.

The CML is linear in E, $\sigma$ space and can be readily derived algebraically. Let x represent the percentage of resources invested in the market portfolio, M; then (1-x) is the amount invested in the risk-free asset. Let $E(R_m)$, $E(R_p)$ respectively represent the expected returns on the market and the expected return of the investor's portfolio. Similarly, let $\sigma_m$ be the standard deviation of the return on the market portfolio. The expected return on the investor's portfolio is given by

$$E(R_p) = (1-x)R_f + XE(R_m) \tag{19}$$

$$= R_f + X[E(R_m)-R_f]. \tag{20}$$

With the assumption of a risk-free security, the standard deviation of the investor's portfolio is given by

$$\sigma(R_p) = x\sigma_m \tag{21}$$

Solving Equations (20) and (21), the CML can be written as

$$E(R_p) = R_f + \left[\frac{E(R_m) - R_f}{\sigma_m}\right]\sigma(R_p), \tag{22}$$

which is linear in E,$\sigma$ space. The CML can be viewed as the indifference line between risk and return. With this interpretation, the risk-free asset, $R_f$, is the certainty equivalent of all risky assets that lie along the CML. The slope coefficient,

$\left[\frac{E(R_m) - R_f}{\sigma_m}\right]$, is then viewed as the market price of risk.

Tobin's[45] Separation Theorem can readily be interpreted in terms of the capital market line. Since all investors will seek to be on the CML, they will all purchase portfolio M. Based on personal risk-return preferences, investors will either borrow at $R_f$ to move along the CML above M, or lend at $R_f$ to achieve some combination between $R_f$ and M. These choices essentially separate the investment decision into two parts. First, the identification of portfolio M; and second, an appropriate financing decision to achieve the desired risk-return preference.

Analysis of investment decisions using the CML only relates to portfolios. Individual securities will be located within the efficient set, not on the frontier; therefore, in equilibrium only portfolios can be located on the CML.

To analyze individual securities, Sharpe[46] used his single index model to develop equilibrium conditions for individual assets. The single-index model readily shows the decomposition of total risk into risks associated with the market (called systematic risk) and risk that

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[45]Tobin, J. (1958) pp. 65-86.

[46]Sharpe, W. F. (1963) pp. 272-293.

is unique to the firm (called unsystematic or residual risk). Sharpe's diagonal model is

$$R_i = A_i+B_iI+C \tag{23}$$

where $A_i$ and $B_i$ are regression parameters for the $i^{th}$ firm;

$R_i$ is the return on the $i^{th}$ security;

I is the return on some market index often represented as $R_m$;

C is random error term.[47]

Using least-squares regression, the error term C has an expected value of zero, constant variance, and is independent of other error terms, i.e., COV $(C_t, C_{t+\tau}) = 0$. The variance of $R_i$ = variance of $(A_i+B_iI+C)$ can be shown to result from two sources.

$$Var(R_i) = VAR(B_iI)+Var(C) \text{ or} \tag{24}$$

Total Risk = Systematic Risk + Unsystematic Risk

In the regression model, the sign and magnitude of the slope coefficient depends on the correlation between the returns of the firm and the market index. If the returns are perfectly positively correlated, i.e., a one-to-one positive relationship, then $B_i = 1$ and $C = 0$. This case describes the condition where total risk = systematic risk. If there is no relationship between returns of the firm and the market, then $B_i = 0$ and C will be large. This case describes the condition where total risk = unsystematic risk. Since most firms' returns are affected by the general movement of the economy, the returns for firms will generally show a positive relationship to the market, with $B_i$ and $C > 0$. The

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[47]Sharpe, W. F. (1963) pp. 272-293.

$B_i$ coefficient is an index of systematic risk or a measure of the volatility of the firm's return as compared to the market. When there is less than perfect correlation, C measures the vertical deviation of the returns from the regression line. The sum of the squared error terms (deviations) is a measure of unsystematic risk and is that portion of total risk that can be reduced or eliminated through diversification.

In equilibrium, only portfolios will lie on the CML. Since the CML measures the risk-return trade-off in terms of systematic risk only, and individual assets contain unsystematic risk, individual assets must be more risky than points on the CML.

The previous discussion explained the CML, the linear trade-off between risk and return of portfolios. Sharpe[48] derived the relationship for individual securities, the security market line (SML), in terms of the slope coefficient of his single-index model. The security market line in terms of beta is shown in figure 3,

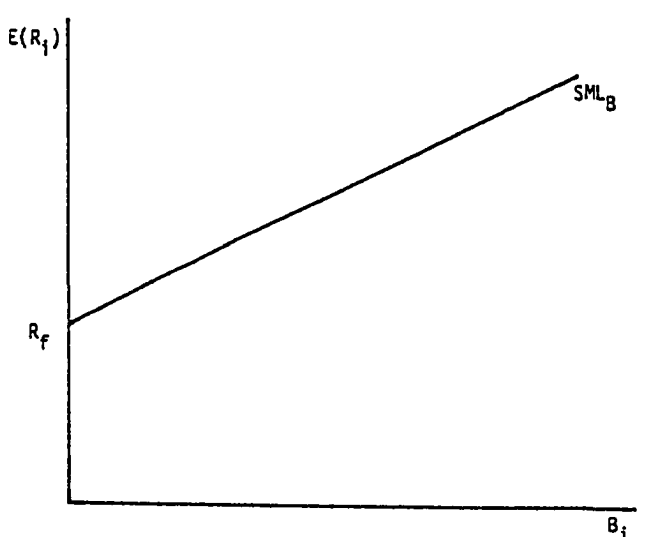



Figure 3: Security Market Line (B)

[48]Sharpe, W. F. (1964) pp. 425-442.

and the equation for the SML is

$$E(R_i) = R_f + \beta_i [E(R_m) - R_f] \quad [49] \tag{25}$$

The SML shows that the expected return on the $i^{th}$ security is equal to the risk-free rate plus the product of $\beta_i$ times the risk premium of the market. Where $\beta_i$ measures the correlation of returns of asset i and the market, or is an index of the degree of systematic risk. Note that in this model, only systematic risk taking is rewarded with increased expected returns.

Lintner[50] derived the SML using a different approach by maximizing angle $\theta$ which is the slope of a line from the risk-free security to an asset in risk-return space. The asset with the larger angle $\theta$ is more desirable. In terms of portfolios of assets, $\theta$ will vary depending on the weights of the individual assets, their expected returns, and the variances and covariances of the assets. Once the maximum $\theta$ is identified, the system of equations can be solved to define the relationship

$$E(R_i) - R_f = \frac{E(R_p) - R_f}{\sigma_p^2} \left( \sum_{i=1}^{n} X_i \sigma_{ij} \right) \tag{26}$$

or

$$E(R_i) - R_f = \lambda \mathrm{cov}(R_i, R_m) \tag{27}$$

if the portfolio in question contains all assets in the market. Lintner's deviation of the SML is equivalent to Sharpe's development in

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[49]Sharpe, W. F. (1964) Footnote 22, p. 438.

[50]Lintner, J. (1965) pp. 13-37.

terms of Beta, since $\beta_i = \frac{cov(R_i,R_m)}{\sigma^2_m}$ when Lintner's portfolio represents the market portfolio. The SML in terms of covariance is shown as Figure 4.

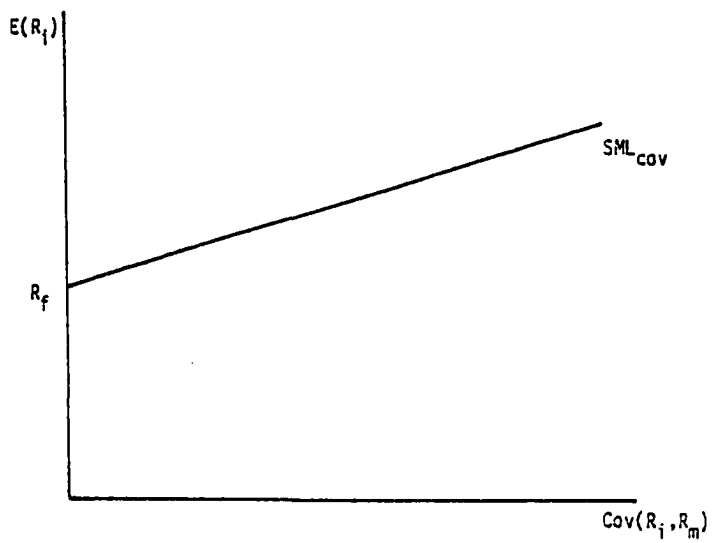



Figure 4: Security Market Line (Cov)

Fama[51] also derived the SML in terms of covariance of asset i and the market. In developing the third derivation, Fama showed the equivalence of the various forms of the SML as developed by Sharpe and Lintner.

Additional work on the CAPM focused on empirical testing and relaxation of various assumptions. A brief current summary of this work can be found in Francis and Archer.[52]

The importance of capital market theory for capital budgeting will now be addressed. One of the significant problems of simple capital budgeting rules, such as the risk-adjusted rate or certainty equivalence is the absence of a theoretical basis for the RADR or CE factors.

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[51]Fama, E. F. (1968) pp. 29-40.

[52]Francis & Archer (1979) Chapter 10, pp. 211-244.

The CAPM can be extended to develop market determined decision rules for project selections. The decision rules, however, must be evaluated in the context of the strenuous assumption of the model.

## Capital Asset Pricing Model and Capital Budgeting

The Capital Asset Pricing Model has dominated finance research in the last two decades. Relaxing assumptions and empirical testing has led to the study of corporate finance problems in a market context. One of the initial and logical extensions of the CAPM was in the area of real asset selection or capital budgeting.

Lintner[53], in a companion piece to his derivation of the security market line, first addressed the implications for capital budgeting. In addition to the assumptions of the CAPM, additional assumptions relating to firms are made to extend Capital Market Theory to the capital budgeting decision. The assumptions are:

1. Corporate management assigns probability zero to default on its debt, and all investors treat corporate debt as a riskless asset -- thus the riskless investment (or borrowing) alternative is extended from individuals to corporations.
2. Investment opportunities available in any time period are independent of the size and composition of the capital budget in any other time period.
3. No limited liability to corporate stock, no institutional or legal restrictions on investors.
4. The riskless rate R* is expected by everyone to remain constant over time.[54]

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[53]Lintner, J. (1965) pp. 13-37.

[54]Ibid., p. 39.

Given these assumptions, the famous Modigliani and Miller[55] Propositions I and II, that for any size and composition of assets, investors are indifferent to the financing decisions of the firm, can be developed from the capital asset pricing model. In this context, the present values of the cash flows to any company from its real and financial assets are equal to the total market value of investors claims to those assets. The change in market value of the equity, $V_o$ as a result of a capital budgeting decision is equal to

$$\Delta V_{oi} = \frac{\Delta(\overline{H}_i - W_i)}{(1+R^*)} \tag{28}$$

where

$\Delta\overline{H}_i$ = the net change in expected present value at the end of the first period as a result of the acquisition of the asset;

$W_i = \gamma_i \Sigma_j R_{ij}$ = market price of risk times the covariance of the aggregate dollar returns of the i and j stocks.

Lintner's results of the capital market approach to capital budgeting can be summarized as

1. Even with highly idealized uncertainty (model assumptions), the minimum expected return required to justify the allocation of funds to a given risky project is an increasing function of each of the following factors: the risk-free rate of return; the market price of risk; the variance of the projects' rate of return; the covariance between the project and existing assets;

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[55]Modigliani, F. and Miller, H. M. (1958) "The Cost of Capital, Corporation Finance and the Theory of Investment," *American Economic Review*. Vol. 49: pp. 261-297.

and the covariance between the project and other projects in the capital budget.

2. Investments that significantly reduce risk rationally belong in capital budgets even at the expense of lowering expected present value returns.
3. Due to the various components of risk, in practice it will be extremely difficult, if not impossible to classify projects into to homogeneous risk classes.
4. Following the requirements of market equilibrium, all means and (co)variances have been calculated using the riskless rate. Recognizing the non-linear effect of varying the discount rate, it necessarily follows that there can be no single risk discount rate to use in computing present values for different projects for accept/reject decisions even if all projects have the same degree of risk.
5. The "cost of capital" (as defined for uncertainty anywhere in the literature) is not the appropriate discount rate to use in capital budgeting decisions.
6. The CAPM does, however, define a "required rate of return as a positively sloped linear function of the ratio of the project's aggregate incremental present-value-variance-covariance to its cost. The slope coefficient is the market price of risk with the risk-free rate as the intercept.[56]

Lintner's conclusions follow logically, but are dependent upon the validity of the strong set of assumptions detailed earlier. Lintner himself recognized that his results are not directly applicable to practical decisions and that additional research was needed to develop practical capital budgeting rules consistent with capital market theory.

Tuttle and Litzenberger[57] addressed the problem of capital budgeting in a capital market framework by looking at the capability to diversify by investors. In the first case, they assumed perfect capital markets made up of small, risk-averse investors with non-diversified portfolios. In addition, they assumed that all prospective

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[56]Lintner, J. (1965) pp. 44-47.

[57]Tuttle, D. L. and Litzenberger, R. H. (1968) "Leverage, Diversifications and Capital Market Effects on a Risk-Adjusted Capital Budgeting Framework," *Journal of Finance*. Vo. 23: pp. 427-443.

investments expected returns are perfectly correlated with the expected return on existing assets. In this environment, the firm can make the returns from capital budgeting projects risk-equivalent to the firm's cost of equity capital by employing more or less financial leverage. This process will then eliminate the need to subjectively determine the desirability of one risk-return combination versus another. Defining risk as the estimated standard error of returns and distinguishing between the risk and return of the project itself and the comparable risk and return to equity from the project which reflects the amount of borrowing and lending; Tuttle and Litzenberger show that risk and return of equity is a simple linear combination of the risk and return of the project. They then showed how this approach could be used to develop a hurdle rate when the firm has budget constraints or faces mutually exclusive projects.

When faced with diversifying investors, Tuttle and Litzenberger essentially reiterated the CAPM solution of Lintner's by hypothesizing a required rate of return determined by the risk-free rate, the market price of risk, and the covariance of return on the project with the market. The significance of this paper lies in the discussion of leverage by firm in comparison to leverage of investors in a capital market framework. Brennan[58] criticized this approach in that it "takes as given the effects on a firm's market risk of adopting a particular project. The paper, therefore, leaves open the questions both of the

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[58]Brennan, M. J. (1973) "An Approach to the Valuation of Uncertain Income Streams," *Journal of Finance*. Vol. 28. pp. 661-674.

determinants of this risk effect, and of the way in which the risk of projects is to be assessed."

Hamada[59] integrated the CAPM into the mainstream of corporate finance by deriving the Modigliani and Miller (M & M)[60] propositions in the context of capital market equilibrium. He was able to show that M and M's risk-class assumption was unnecessary in the market model. In the no tax case, the capital budgeting criteria must ensure that "the change in equity value, as a result of project selection, will at least be larger than any new equity required to finance this project"[61] to be consistent with the CAPM. An approximate cost of capital, or required rate of return[62] can be expressed as

$$C = R_f + \frac{\lambda}{S_T} \frac{d\Sigma_k \mathrm{cov}(X_A, X_k)}{dI} \quad .[63] \tag{29}$$

This formulation differs from the implied CAPM formulation

$$E(R_A) = R_f + \lambda \mathrm{cov}(R_A, R_M) \ , \tag{30}$$

in that Hamada incorporates an adjustment for the firm's change in risk per dollar of invested capital caused by the investment. For pure scale

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[59]Hamada, R. S. (1969) "Portfolio Analysis, Market Equilibrium and Corporation Finance," *Journal of Finance*. Vol. 24: pp. 13-31.

[60]Modigliani and Miller (1958) pp. 261-297.

[61]Hamada, R. S. (1969) p. 21.

[62]With the development of the CAPM, the term "the cost of capital" has generally been replaced by the terminology of required rates of return. This change has been made to call attention to the deep philosophical differences in the derivation in each term; in the context of capital budgeting, both terms attempt to identify a discount rate for future, risky cash flows.

[63]Hamada develops EQ 29 and attributes this form of the CAPM to be essentially the Modigliani and Miller position in the framework of capital market equilibrium.

or non-diversifying investments that do not change the risk of the firm, Hamada shows that Eq. (30) above defines the appropriate required rate.

Hamada also addresses the impact of corporate income taxes on the cost of capital or required rate. He once again derived the M & M position showing the impact of financial leverage via the subsidy given to shareholders through tax-deductible interest payments. The after-tax cost of capital is given by

$$C_\tau = R_f(1-\tau\frac{dD}{dI}) + \frac{\lambda}{S}(\frac{k\, d\Sigma cov(_\tau X_A, _\tau X_k)}{dI}). \qquad (31)$$

Hamada[64] then compares the results and implications of Eq. (29) to Lintner's decision criteria, Eq. (28) this paper. After describing Lintner's various omissions and preoccupation with minor points, Hamada then recalls the limiting nature of both models based on the stringent assumptions.

Litzenberger and Budd[65] continued the search to develop appropriate capital budgeting rules in a capital market framework. The importance of market evaluations lies in the fact that the "firm needs to know how a change in its riskiness will affect its value."[66] Based on the CAPM, Litzenberger and Budd show that the capital budgeting models

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[64]Hamada, R. S. (1969) p. 25.

[65]Litzenberger, R. H. and Budd, A. P. (1970) "Corporate Investment Criteria and the Valuation of Risk Assets," *Journal of Financial and Quantitative Analysis*. Vol. 5: pp. 395-419.

[66]Ibid., p. 397.

of Lintner,[67] Tuttle and Litzenberger,[68] and Hamada[69] develop essentially equivalent decision rules. They conclude that in the absence of taxes, the cost of capital is not firm dependent; however, when taxes are included, the cost of capital depends on the firm undertaking the project.

To attempt to explain the linkage between the financial riskiness of an equity share and the earnings riskiness of the underlying real assets, Litzenberger and Budd reviewed the empirical tests of the CAPM and proposed additional empirical work. Results of these studies have shown a marked difference of the importance of earnings variability on share variability. The major significant problem of market tests is the use of historical information as a surrogate for future expectations. The inherent "static" nature of the CAPM poses severe problems particularly for the evaluation of real assets. Under a dynamic framework the relationship between real and financial sectors is more complex. "Changes in interest rates and/or the general structure of risk asset prices affects the return to investors independently of the firm's earnings. This market risk is independent of the firm's portfolio of productive assets."[70]

Stapleton[71] developed capital budgeting decision rules based on the CAPM. The focus of his study was toward the use of certainty

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[67]Lintner, J. (1965) pp. 13-37.

[68]Tuttle, D. S. and Litzenberger, R. H. (1968) pp. 427-443.

[69]Hamada, R. S. (1969) pp. 13-31.

[70]Litzenberger, R. H. and Budd, A. P. (1970) p. 407.

[71]Stapleton, R. C. (1971) "Portfolio Analysis, Stock Valuation and Capital Budgeting Decision Rules for Risky Projects," *Journal of Finance,* Vol. 26: pp. 95-117.

equivalents based on non-diversifyable risk to the shareholders. The diversification effect of investments within the firm can be ignored and individual projects are appraised separately. Essentially, his technique is to derive the certainty equivalent factor based on the correlation between the investment cash flows and the market index. This approach to evaluating multiperiod real assets was criticized by Brennan[72]. Brennan states that Stapleton's purported multiperiod model

> Actually employs a single-period certainty-equivalent approach in which it assumed that the joint probability distribution of all the future dividend payments on a share can be reduced to a probability distribution on the present values at the end of one period of all possible future dividend streams. . . . This procedure would be legitimate only if, at the end of one period, the investor were to know with certainty the precise pattern of dividend payments over the remainder of the horizon...."[73]

Myers and Turnbull[74] summarized the development of CAPM approaches to the solution of the capital-budgeting problems. Assuming the validity of the CAPM, they conclude that relatively simple and general valuation formulae can be developed. Unfortunately, they find that the real determinants of beta are more complicated than is generally suspected. They address the problems of measurements of beta, but the more fundamental problem of how to handle growth opportunities is also addressed. If the firm has growth opportunities, the observed beta will generally lead to biased capital-budgeting hurdle rates. The results of this bias are to create serious questions about the validity of the CAPM to evaluate real assets.

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[72]Brennan, M. J. (1973) pp. 661-674.

[73]Ibid., p. 662.

[74]Myers, S. C. and Turnbull, S. M. (1977) "Capital Budgeting and the Capital Asset Pricing Model: Good News and Bad News," *Journal of Finance,* Vol. 32: pp. 321-336.

## Multi-Period Capital Asset Pricing Models and Capital Budgeting

To remedy the obvious (single-period nature) shortcoming of the CAPM in the valuation of real assets, Brennan[75] develops a model that describes multi-period capital market equilibrium. Brennan essentially extends the work of Merton[76] and Black-Scholes[77] to develop a differential equation approach to the evaluation of a cash flow stream that captures the dynamic nature of information arrival. Solution of this differential equation yields the present value of a claim to a risky future cash flow. By aggregating the net present value of claims to risky cash flows, the net present values of the assets that generate the claims can be computed.

Although the model is founded upon the portfolio equilibrium of the individual investor, projects are risk independent and do not require the firm to use a portfolio selection approach to project selection. Also, the net present value of a project is a given number in this model, not a random variable that must be studied prior to the acceptance of the project.

In addition to the above features, this model defines the certainty equivalent of a particular cash flow as an objective, market

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[75]Brennan, M. J. (1973) pp. 661-674.

[76]Merton, R. C. (1969) "Lifetime Portfolio Selection Under Uncertainty: The Continuous Time Case," *Review of Economics and Statistics.* Vol. 51: pp. 247-257.

[77]Black, R. and Scholes, M. (1973) "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy.* Vol. 81: pp. 637-654.

determined measure and not derived from the decison makers' utility function.

Bogue and Roll[78] also formulated a multi-period approach to the CAPM solution to capital budgeting problems. While reviewing basic capital market theory and the CAPM, they stressed the importance of the homogeneous expectations assumption. Without common investor beliefs about returns, there is no unique price for a risky security that would prevail in the marketplace. Furthermore, the firm would be uncertain as to which particular beliefs should be used in reaching a decision.

As a point of departure, Bogue and Roll studied the single-period capital budgeting problem. The CAPM provides a theoretically acceptable solution to this problem. In addition, under certain circumstances, the single period analysis will provide an appropriate solution to the multi-period model. When "perfect" secondary markets exist for the asset,

> ... the firm only needs to compare the current cost of the project with the value of forecasted cash flows during the first period and with forecasted end-of-period secondary market price. The decision next period as to whether the project should be used in subsequent periods is completely unaffected by whether it is owned at the end of the present period.[79]

With observed imperfect secondary markets for real assets, one can still use one-period forecasts. In this case, the project is clearly acceptable if the one-period cash flow and its net salvage value at the end of period one is greater than the cost. Obviously, using this decision rule many worthwhile projects would be rejected because

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[78]Bogue, M. C. and Roll, R. (1974) "Capital Budgeting of Risky Projects with 'Imperfect' Markets for Physical Capital," *Journal of Finance*. Vol. 29: pp. 601-613.

[79]Ibid., p. 604.

their salvage values do not reflect the future earning power of the asset. In this context, additional periods cash flows must be considered and describe a need for multi-period valuation models.

Bogue and Roll propose an n-period, infinite state, dynamic programming solution to the multi-period capital budgeting problem. At each step, one applies the one-period valuation model with the parameters of the model depending on the state of the world at the beginning of that period.

## Mean-Variance Synthesis

Rubinstein[80] elegantly showed that the major problems of corporate finance: security valuation, asset expansion and capital structure, can be studied in terms of capital market equilibrium via the capital asset pricing model. Given the strict assumptions of the model, he derived the alternative forms of the CAPM and then showed the logical application to capital budgeting decisions.

The implication of the use of the CAPM versus traditional weighted average cost of capital measures can be readily identified using Rubinstein's graphical analysis. In Figure 5, the SML identified the boundary of acceptable choices in return-risk space. The firm should accept a project if the project's return-risk ordered pair plots on or above the market line. Accepting projects A or B will result in an upward revision of share prices as investors observe the disequilibrium condition of greater return for the risk, or less risk for the return

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[80]Rubinstein, M. (1973) "A Mean-Variance Synthesis of Corporate Financial Theory," *Journal of Finance*. Vol. 28: pp. 167-180.

than warranted by the market risk-return trade-off, the SML. The constant slope of the SML, $\lambda$, is the market price of risk and is appropriate to all firms and all projects. All firms in the economy may use $\lambda$ as the cut-off value for all projects.

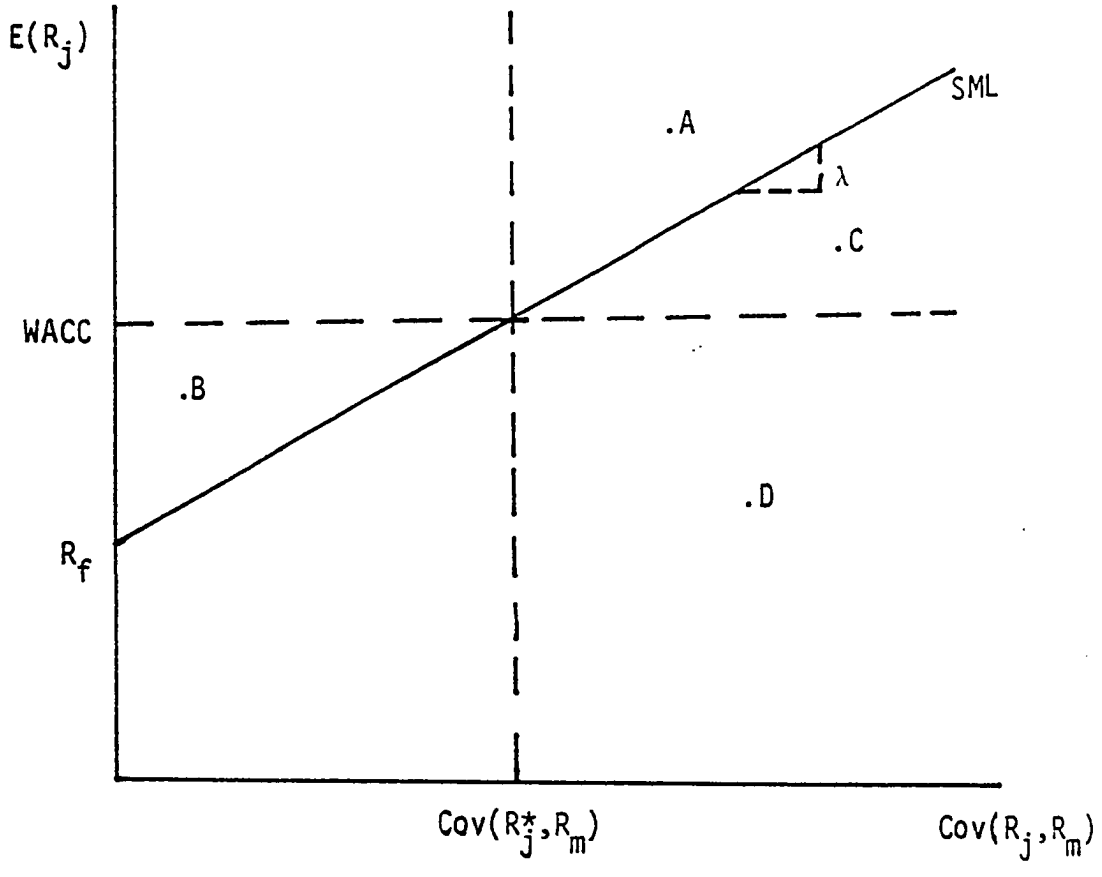



Figure 5: Comparison of WACC and CML Criteria

Rubinstein's conclusions differ from the Lintner,[81] Tuttle and Litzenberger[82] and Hamada[83] formulation, in that in the absence of synergy, each project is evaluated on its own merits without consideration of the firm's existing assets. Therefore, within the firm, diversification can be ignored in capital budgeting decisions. Two arguments can be made for this non-diversifying behavior. First, in computing the CAPM decision rule (Eq. 32), the relevant return

$$\frac{E(R_j)-R_f}{cov(R_j,R_M)} > \lambda \qquad (32)$$

and covariance relates to the asset in question, not the firm. Therefore, the contribution of the project to the firm's variance of equity rate of return does not affect the decision given by CAPM criterion. Second, investors by their own diversification can eliminate any diversifiable risk so the firm need not diversify for investors.

Returning to the WACC criterion where a firm would accept a project only if $E(R_j)>WACC_j$, one can see in Figure 5 the WACC and CAPM lead to contradictory decisions. Project B is acceptable using CAPM, but unacceptable using WACC; while project C is acceptable using WACC, but unacceptable using CAPM. The WACC is invalid because in fails to consider both risk and return in the decision. WACC sets a rate of return and ignores risk resulting in a bias against low-risk projects and a bias for high risk projects. At the point of intersection

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[81]Lintner, J. (1965) pp. 13-37.

[82]Tuttle, D. L. and Litzenberger, R. H. (1968) pp. 427-443.

[83]Hamada, R. S. (1969) pp. 13-31.

of the SML and WACC lines, the covariance between the project and the market and the covariance between the firm and the market are equal, identifying project of the same risk as the firm.

Rubinstein then derived the Modigliani and Miller propositions in the framework of the CAPM. The significance of this paper is that the author addresses and integrates the major problems of corporate finance in a capital market framework at a relatively introductory level.

## Alternative Market Models

The Capital Market Theory based on the Markowitz mean-variance portfolio model has received wide attention in the finance literature. Empirical testing has generally confirmed the approach, but throughout the literature there has been continued concern with the strict assumptions required to validate the theory. One of the assumptions that has strongly been attacked is the adequacy of variance as measure of risk. Joyce and Vogel[84] reach the conclusion that variance is ambiguous and yields conflicting results when calculated from information generally available to decision makers. In addition to questions relating to variance, the comparability of the analysis of financial versus real assets has directed research into more general models. The remainder of this review chapter will look to alternate theories of asset selection: Semi-variance and Stochastic Dominance models that allow for different risk measures; Time-State-Preference Theory which provides a

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[84]Joyce, J. M. and Vogel, R. C. (1970) "The Uncertainty in Risk: Is Variance Ambiguous?," *Journal of Finance*. Vol. 25: pp. 127-134.

more general approach to the theory of capital market equilibrium; a review of simulation studies that evaluate all of these models in a comparative context; and finally, a comparison of theoretical models and the techniques used by practitioners in the field. This last section will essentially describe the state-of-the-art and will lead to the development of capital budgeting risk measures that focus on intertemporal, imperfectly correlated cash flows.

## Mean/Semi-Variance Model

The adequacy of the two-parameter, mean-variance portfolio model depends on two critical factors: 1) the shape of the probability distribution of the cash flows, and, 2) the form of the decision maker's utility function. If the cash flows can be described by a symmetric distribution then the variance is an adequate measure of dispersion or risk. However, if the distribution is skewed, then decisions based on a mean-variance criteria will be insensitive to the assymetric nature of the distribution. In terms of the utility function, mean-variance analysis is consistent with maximization of expected utility for quadratic or any other two parameter distribution. Once again, if investor's utility is described by higher moments, then the mean-variance criteria is inadequate.

Markowitz[85] recognized the limitations of the variance as a measure of portfolio risk and proposed the semi-variance as an alternate measure. The semi-variance measures the down-side, squared deviations from a reference point. More specifically, let R represent a random

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[85]Markowitz, H. M. (1959).

variable with a known probability distribution; h is the critical value against which the actual values of R are compared[86] and let $(R-h)^-$ represent (R-h) for $(R-h) \leq 0$, then semi-variance can be defined as

$$S_h = E[(R-h)^-]^2. \quad ^{87} \tag{33}$$

In comparing variance and semi-variance as a risk measure in portfolio analysis, Markowitz[88] selected the variance based on the following considerations: variance is a familiar statistical measure that is much easier to use with computer analysis; for portfolio problems based on variance inputs needed are the means, variance and, covariances while the portfolio problem based on semi-variance requires the estimation of the entire joint probability distribution of returns. Intuitively, the semi-variance is a better measure of investor risk. Analysis based on semi-variance concentrates on the elimination of down-side variation or losses, while variance considers extremely high and low returns as equally undesirable. With symmetric distributions of returns, portfolio models will select the same efficient set using either variance or the semi-variance. Given the assumption of normal or nearly normal returns, Markowitz focused his attention to the more economical, more familiar variance as a measure of risk.

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[86]Common critical values reported are the mean and zero. Using the mean, semi-variance measures the deviations below the mean while using zero for the reference point, semi-variance measures deviations resulting in a loss.

[87]Mao, J. C. T. (1970) "Survey of Capital Budgeting: Theory and Practice," *Journal of Finance*. Vol. 25: pp. 349-360.

[88]Markowitz, H. M. (1959).

Mao,[89] and Mao and Brewster[90] argue that the semi-variance is a better risk measure than the variance particularly for capital budgeting analysis. These papers revived Markowitz's arguments for the semi-variance and described the utility implications for a decision rule based on semi-variance. Using Tobin's[91] development of the quadratic utility function for the mean-variance case, Mao[92] developed a hybrid utility function that would imply a mean/semi-variance criterion. This utility function is quadratic for returns less than the critical value and linear for returns greater than the critical value. This hybrid function depicts a decision maker who is risk averse at low return levels, but is neutral toward risk at high returns.[93] Mao states "It does not seem unreasonable for an investor who is conservative at low returns to become aggressive a high returns." [94]

Mao[95] also addressed the computational problems of portfolio analysis using semi-variance. Recognizing that there is no easy way to derive the semi-variance for a portfolio from the semi-variances of the individual projects, Mao shows that to compute portfolio semivariance

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[89]Mao, J. C. T. (1970) "Models of Capital Budgeting, E-V vs. E-S," *Journal of Financial and Quantitative Analysis*. Vol. 4: pp. 657-675.

[90]Mao, J. C. T. and Brewster, J. F. (1970) "An E-Sh Model of Capital Budgeting," *Engineering Economist*. Vol. 15: pp. 103-121.

[91]Tobin, J. (1958) pp. 65-86.

[92]Mao, J. C. T. (1970) pp. 658-662.

[93]Here risk and variability are used interchangeably.

[94]Mao, J. C. T. (1970) p. 661.

[95]Ibid, pp. 668-674.

one must specify the complete joint probability distribution of portfolio returns. For example, with three individual projects, with three possible outcomes each, twenty-seven possible portfolios must be evaluated to determine the joint probability distribution. Mao[96] recognizes the operational difficulties associated with semi-variance analysis, but lays the ground work for further research in this area.

Hogan and Warren[97] added to the intuitive appeal of semi-variance by developing an equilibrium capital-market model based on semivariance. They define a riskless rate in E-$S_T$ space, where $S_T$ represents the square root of the semi-variance, as an investment for which $S_T = 0$. Following closely with a procedure that parallels Sharpe[98], Hogan and Warren shown that the CML can be drawn in mean/semi-standard deviation (E-S) space analogously to the E-V model. Defining semivariance as

$$E(\min[0,R_X-T]^2) \tag{34}$$

where

$R_X$ = return on portfolio X

T = fixed reference point;

then the riskless rate is the largest value for T for which there exists a portfolio such that $S_T = 0$. Using this riskless asset, they then show the existence of a CML by demonstrating that the combination of the

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[96]Mao, J. C. T. (1970) p. 661.

[97]Hogan, W. W. and Warren, J. M. (1974) "Toward the Development of an Equilibrium Capital-Market Model Based on Semivariance," *Journal of Financial and Quantitative Aanlysis*. Vol. 9: pp. 1-11.

[98]Sharpe, W. F. (1964) pp. 425-442.

riskless asset and any risky portfolio produces a new portfolio for which $S_T$ is proportional to the risk measure of the original portfolio. Let Y represent the riskless asset and X is any other portfolio, then for portfolio $Z(\alpha) = \alpha X+(1-\alpha)Y$, $\alpha \geq 0$; the semi-standard-deviation of the return of portfolio $Z(\alpha)$ is

$$S_T[R \cdot Z(\alpha)] = [E(\min(0,\alpha(R_X-T) + (1-\alpha)(R_Y-T))^2)]^{\frac{1}{2}} \qquad (35)$$

$$= \alpha S_T(R_X).$$

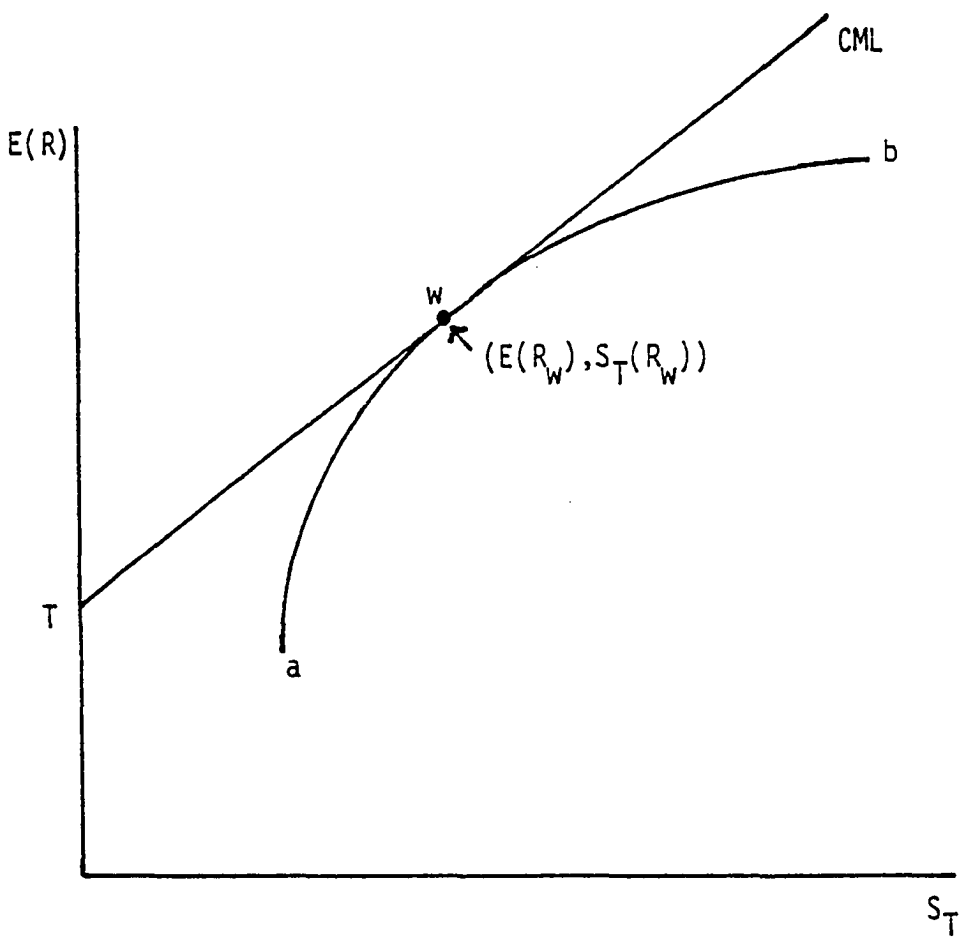



Figure 6: E-S Efficient Frontier, CML

Figure 6 above depicts the capital market line in E-S space; as before, the point of tangency of the efficient frontier and the CML identifies the market portfolio, W. The equation of this CML can be written as

$$E(R_p) = T + \left[\frac{E(R_w)-T}{S_w}\right] S_p.^{99} \qquad (36)$$

To develop a relationship for individual assets similar to the security market line, Hogan and Warren define the cosemivariance as the proper measure of realtive security risk. At equilibrium, the expected return on a security is a linear function of the securities' cosemivariance with the tangency or market portfolio. Following Sharpe[100] and exploiting the fact that the slope of the curve a b in Figure 6, in equilibrium, must equal the slope of the CML, one can solve for this tangency relation and derive the expression for the expected return for any asset, i.e., the SML.

$$E(R_i) = T + \left[\frac{E(R_w)-T}{S_w}\right] S_{wi} \qquad (37)$$

where

$$S_w = E[\min(0,R_w-T)^2]$$

$$S_{wi} = E([\min(0,R_w-T)][R_i-T]).$$

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[99]The critical value used to compute the semi-variance is the risk-free rate. This is a reasonable, but not mandatory, choice. Hogan and Warren defend this choice as the minimum rate that all investors would strive to exceed.

[100]Sharpe, W. F. (1964) pp. 425-442.

Porter, Bey, and Lewis[101] extended Hogan and Warren's theoretical model to develop capital budgeting rules based on semi-variance. They followed Lintner's[102] lead to develop an expression for the market value of the equity which then serves as a basis for capital budgeting decisions. Assuming that individual firms are small relative to the market so that capital budgeting decisions do not affect the market price of risk, they developed a valuation equation in semi-variance space analagous to Lintner's[103] valuation in variance space.[104]

Haskins[105] developed a simplified capital budgeting rule based on the extension of Sharpe's single index model to incorporate the semi-variance as the risk measure. Using Hogan and Warren's[106] E-S capital market model, Haskin's decision rule for diversified investors is to accept a project if

$$E(R_j) > T+[E(R_w)-T]\gamma_j \qquad (38)$$

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[101]Porter, R. B., Bey, R. P., and Lewis, D. C. (1975) "The Development of a Mean-Semivariance Approach to Capital Budgeting," *Journal of Financial and Quantitative Analysis*. Vol. 10: pp. 639-649.

[102]Lintner, J. (1965) pp. 13-37.

[103]Refer to Equation (28), p. 33 this paper.

[104]The derivation of the valuation equation in semi-variance space is quite tedious because of the assymetric nature of the cosemi-variance terms. See Porter, Bey, and Lewis, p. 644, for the derivation of the valuation equation.

[105]Haskins, C. G. (1978) "Capital Budgeting Decision Rules for Risky Projects Derived from a Capital Market Model Based on Semi-variance," *Engineering Economist*. Vol. 23: pp. 211-222.

[106]Hogan, W. W. and Warren, J. M. (1974) pp. 1-11, see also Equation (37), p. 51 this paper.

where

$E(R_j)$ = expected return of project j

$T$ = semi-variance risk-free rate

$E(R_w)$ = expected return of the market

$\gamma_j$ = semi-variance analogue to the mean-variance $\beta$, i.e., the cosemivariance of the asset and the market divided by the semi-variance of the market.

The volatility measure, $\gamma_j$, is only affected by project deviations associated with events for which $R_w < T$, whereas $\beta_j$ is affected by project return deviations whenever $R_m \neq E(R_m)$. According to the E-S model, a diversified investor is not concerned with adverse returns from a project when $R_w > T$.

Semi-variance has received considerable recent attention in the literature. Intuitively a risk-measure that accounts for down-side-risk can be quite valuable. If, however, normal or other symmetric distributions adequately characterize capital budgeting cash flows, then the additional computational difficulty of the semi-variance does not seem warranted. Also, the model, as developed, suffers from the single period perspective as does the capital asset pricing model. Since the primary concern of this paper is to identify risk evaluation techniques that incorporate the intertemporal nature of the cash flows in capital budgeting, more general approaches to risk evaluation need to be explored.

## Stochastic Dominance Models

"Decision-making under uncertainty may be viewed as choices between alternative probability distributions of returns, and the individual chooses between them in accordance to a consistent set of preferences."[107] Decision rules based on the moments of the probability distribution (i.e., mean-variance or mean semi-variance) require a particular form of utility function or a specific distribution of returns. For example, the mean-variance decision rule requires either a two parameter distribution of returns (more specifically as assumption of normality) or a two parameter utility function. Dissatisfaction with the strenuous assumptions of the mean-variance capital asset pricing model has lead researchers to look for more general models that allow for less restrictive assumptions for the utility function and the underlying probability distribution of returns. There is an obvious tradeoff when one looks to more general formulations to solve financial decision problems.

> The more restrictive the class of utility functions, the smaller will be the admissible set and thus more useful will it be in practical situations. However, more restrictions on the utility functions imply that the admissible set is relevant for a smaller group of individuals and may involve a severe loss of generality. Thus, one is interested in determining the admissible set of alternatives for the most restrictive class of utility functions that is consistent with observed economic phenomena.[108]

The concept of dominance has been used in decision making to eliminate inferior alternatives from further consideration. For example,

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[107]Bawa, V. S. (1975) "Optimal Rules for Ordering Uncertain Prospects," *Journal of Financial Economics*. Vol. 2: pp. 95-121. (p. 95).

[108]Ibid., p. 96.

using the mean-variance decision rule, dominated choices can be readily identified. If a project has a larger expected return and a variance at least as small, or a smaller variance and an expected return at least as large, one can identify mean-variance dominance. Unfortunately, these two polar cases do not eliminate many alternatives from further study.

Analyzing the entire probability distribution of returns rather than just the first two moments, Stochastic Dominance decision rules have been developed. These decision rules have been identified as First Order Stochastic Dominance (FSD), Second Order Stochastic Dominance (SSD), and Third Order Stochastic Dominance (TSD). The orderings from first through third degree are based on more restrictive assumptions on the decision maker's utility function.

Quirk and Saposnik[109] developed First Order Stochastic Dominance as the optimal decision rule for the entire class of increasing utility functions (i.e., the utility function is an increasing function of returns). The probability functions are taken over the interval $I = [0,1]$ or $I = [0,\infty]$; also these dominance rules apply to either discrete or continuous functions. The FSD rule is that probability distribution F dominates probability distribution G if and only if F never lies above and somewhere lies below G; more specifically,

$$F >_1 G \text{ if and only if } G(X) \geq F(X) \text{ for all } X \in I.^{110} \qquad (39)$$

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[109]Quirk, J. P. and Saposnik, R. (1962) "Admissibility and Measurable Utility Functions," *Review of Economic Studies*. Vol. 29: pp. 140-146.

[110]The stochastically larger random variable has the smaller distribution function. Whitmore, G. A. and Findlay, M. C. (1978) *Stochastic Dominance*. Lexington, Mass.: Lexington Books, p. 66.

Graphically, FSD is depected in Figure 7 below; in this case (F(X) = G(X) in all cases except for the 3, 4 interval where G(X)>F(X). Any decision maker observing these two alternatives will select alternative F as F(X) clearly dominates G(X) by FSD. Unfortunately, among the class of all distribution functions, the subclass that can be ordered by the FSD rule is quite small.

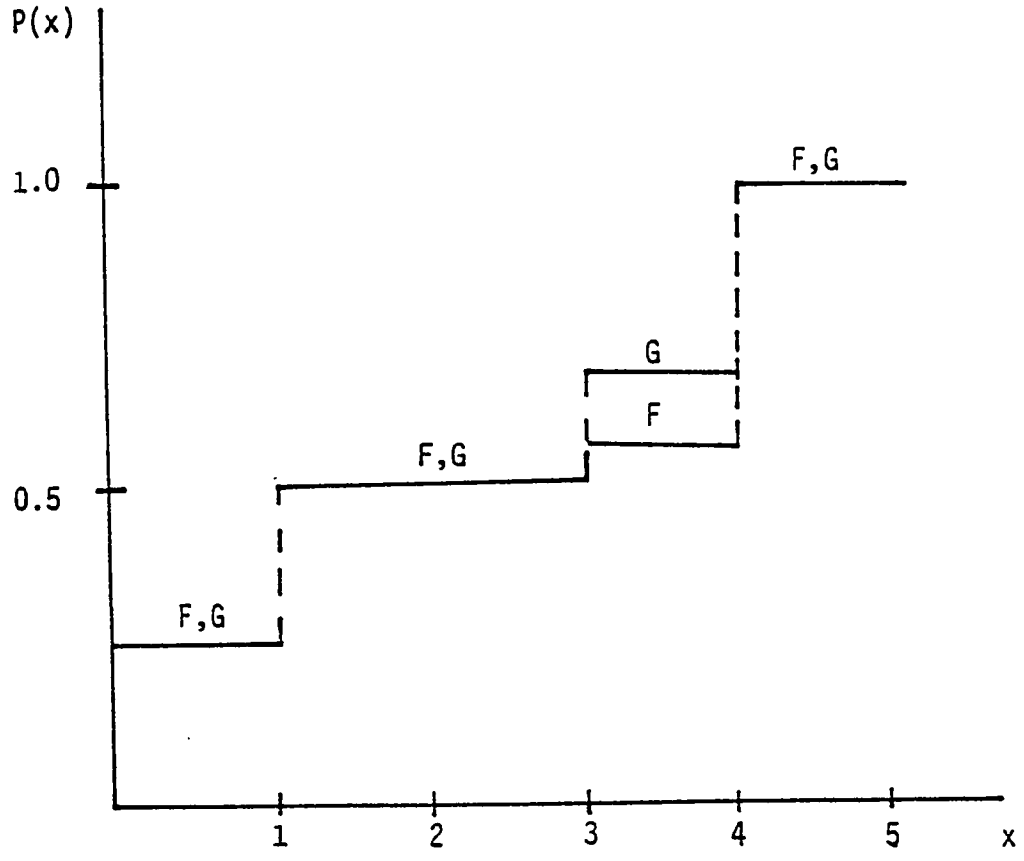



Figure 7: First Order Stochastic Dominance, $F >_1 G$

Recognizing the limitations of the FSD decision rule, Hadar and Russell[111] developed a weaker dominance criteria called Second Order Stochastic Dominance, (SSD). SSD considers the restricted class of increasing utility functions that depict risk aversion (i.e., the utility functions are increasing with decreasing marginal utility everywhere). The SSD rule is that probability distribution F dominates a probability distribution G if and only if the integral of F never lies above and somewhere lies below the integral of G; more specifically,

$$F >_2 G \text{ if and only if } G^2(X) \geq F^2(X) \text{ for all } X \in I. \qquad (40)$$

Graphically, SSD is depicted in Figure 8 below; in this case neither $F(X) >_1 G(X)$ nor $G(X) >_1 F(X)$, however $F^2$ clearly dominates $G^2$.

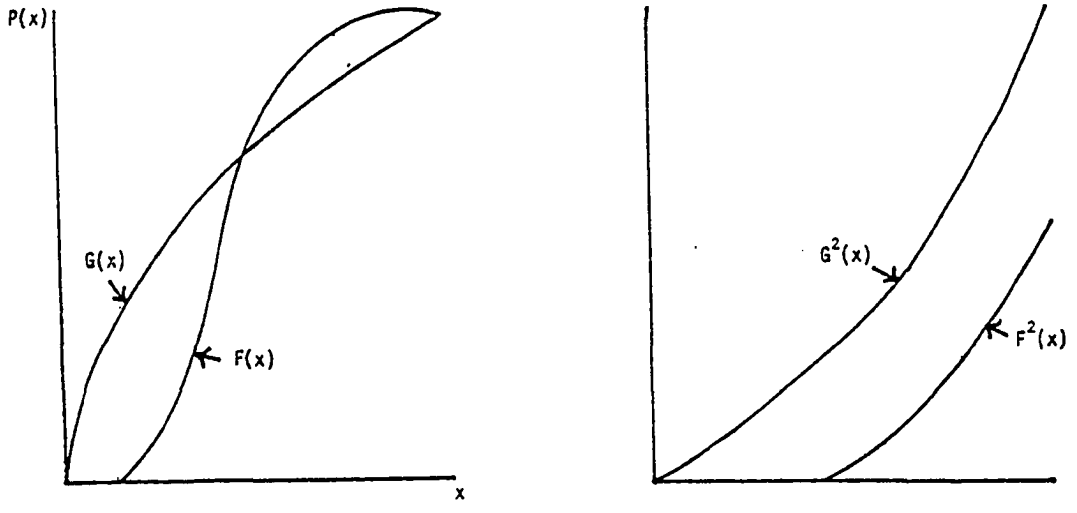



Figure 8: Second Order Stochastic Dominance

$F^2 >_2 G^2$

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[111]Hadar, J. and Russel, W. R. (1969) "Rules for Ordering Uncertain Prospects," *American Economic Review*. Vol. 49: pp. 25-34.

Whitmore[112] further restricted the class of utility functions to develop Third Order Stochastic Dominance, (TSD). TSD considers the class of increasing and risk averse utility functions with the additional restriction that the third derivative of the utility be positive. The economic rational for considering positive third derivatives is that this condition implies decreasing absolute risk aversion.[113] The TSD rule is that probability distribution F dominates a probability distribution G if and only if the mean of F is greater than the mean of G and the integral of the integral of F never lies above and somewhere lies below the integral of the integral of G; more specifically,

$$F >_3 G \text{ if and only if: a) } \mu_F \geq \mu_G, \text{ and b) } G^3(X) \geq F^3(X) \text{ for all } X \in I. \quad (41)$$

Graphically, TSD is depicted in Figure 9 below; in this case neither $F(X) >_X G(X)$ nor $G(X) >_2 F(X)$, however, with $\mu_F = \mu_G$, $F^3$ dominates $G^3$.

The Stochastic Dominance Rules (FSD, SSD, and TSD) have tremendous potential due to their generality. However, significant shortcomings of the procedure are the need for complete knowledge of the entire distribution and the requirement to make pairwise comparisons to identify dominating choices. Much empirical work has been done to develop algorithms that allow for the efficient computations of the

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[112]Whitmore, G. A. (1970) "Third Order Stochastic Dominance," *American Economic Review*. Vol. 59: 457-459.

[113]Pratt, J. W. (1964) "Risk Aversion in the Small and in the Large," *Econometrica*. Jan-Apr: pp. 122-136. Pratt, in this classic paper differentiates between local and global measures of risk aversion. In addition, he discussed the questions of constant risk aversion versus increasing and decreasing risk aversion and constant proportional risk aversion versus increasing and decreasing proportional risk aversion.

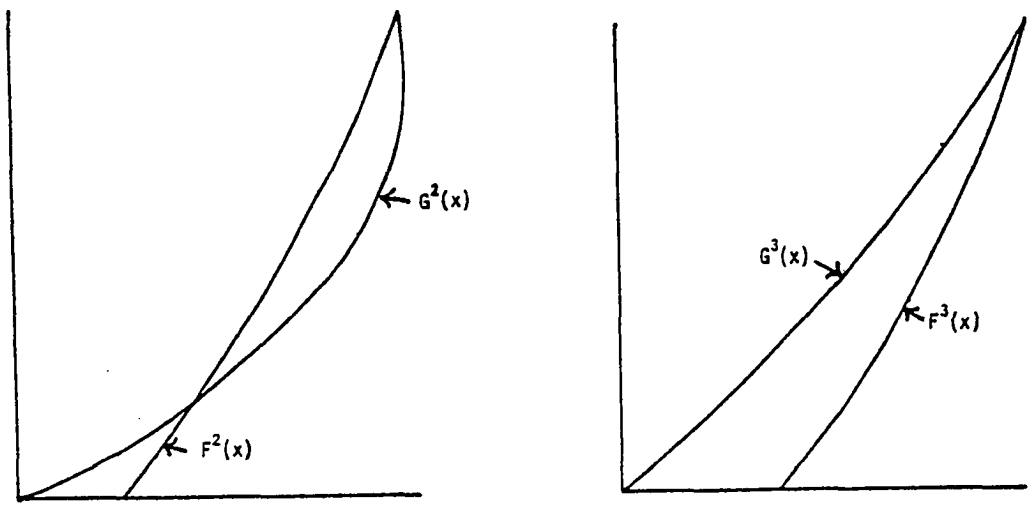



Figure 9: Third Order Stochastic Dominance $F^3 >_3 G^3$

Stochastic Dominance Rules. Whitmore and Findlay[114] in a recent monograph have brought together the theory and empirical work on SD. In addition, their extensive bibliography will direct the student to the pertinent original work. As a summary to this section, Levy and Sarnat[115] have constructed a table comparing SD requirements with other selected decision rules. Table 1 below is a modified version of the Levy and Sarnat work.

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[114]Whitmore, G. A. and Findley, M. C. (1978).

[115]Levy, H. and Sarnat, M. (1972) *Investment and Portfolio Analysis*. New York: John Wiley and Sons.

| Criterion (1) | Conditions for Dominance (2) | Restrictions on Utility Functions (3) | | Restrictions on Distributions of Returns (4) | Information Required (5) | Optimality for Class given in (3) and (4) (6) | Relation to Other Criteria (7) |
|---|---|---|---|---|---|---|---|
| FSD | $F(X) \leq G(X)$ for all X, $F(X_0) < G(X_0)$ for some $X_0$ | Non decreasing utility functions $(U'(X) \geq 0)$. | | None | Entire Probability Distribution of Returns | Optimal | FSD efficient set includes SSD and TSD efficient set |
| SSD | $F^2(X) \leq G^2(X)$ for all X, $F^2(X_0) < G^2(X_0)$ for some $X_0$ | Non decreasing concave utility functions. $(U'(X) \geq 0,\ U''(X) \leq 0)$ | | None | Entire Probability Distribution of Returns | Optimal | SSD efficient set is a subset of FSD efficient set |
| TSD | $F^3(X) \leq G^3(X)$ for all X, $F^3(X_0) < G^3(X_0)$ for some X, and $\mu_F \geq \mu_G$ | Non decreasing concave utility functions with decreasing absolute risk aversion. $(U'(X) \geq 0, U''(X) \leq 0, U'''(X) \geq 0)$ | | None | Entire Probability Distribution of Returns | Optimal | TSD efficient set is a subset of SSD and FSD efficient set |
| Mean Variance Criterion MVC | $E(X) \geq E(Y)$ $(X) \leq (Y)$ with at least one strong inequality | This criterion is relevant for the following two different cases, a) and b) | a) Non decreasing concave utility functions | $a_1$) All prospects belong to the same family of distributions with two independent parameters | $a_1$) First two moments of Distribution of Returns | $a_1$) Not optimal under 3a and $4a_1$ | $a_1$) MVA efficient set includes the TSD efficient set. |
| | | | | $a_2$) Normal distribution of returns | $a_2$) First two moments of Distribution of Returns | $a_2$) Optimal under 3a and $4a_2$ | $a_2$) MVA efficient set is identical to TSD efficient set. |
| | | | b) Quadratic utility functions $U'(X) \geq 0$ $U''(X) \leq 0$. | b) None | b) First two moments of Distribution of Returns | b) Not optimal under 3b and 4b | b) Not related to SD sets for other than quadratic utility functions |

TABLE I

COMPARISON OF STOCHASTIC DOMINANCE AND MEAN VARIANCE EFFICIENT SETS

Source: Levy & Sarnat [115]

## Time-State Preference Models

The Time-State Preference Model (TSP) presents the most general formulation and solution to the problem of decision making under uncertainty. Arrow[116], Luce and Raffia,[117] Debreu,[118] and Borch[119] made early contributions to the TSP approach to decision making. The work of Arrow resulted in the original formulation of the state-preference approach in an atemporal context. Debreu extended Arrow's work to include choices over time, while Borch addressed the problem of incomplete markets for the time-state claims.

In two companion articles, Hirshleifer[120] showed that the state-preference formulation was the logical extension of Fisher's[121] model of certain intertemporal choice. Uncertainty in the TSP model takes the form of uncertainty as to what state of the world will occur in what time period. Once a state is observed, the outcome is known and can be evaluated as in the case of certainty.

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[116]Arrow, K. J. (1951) "Alternative Approaches to the Theory of Choice in Risk-Taking Situations," *Econometrica*. Vol. 19: pp. 404-437.

[117] Luce, R. D. and Raffia, H. (1957) *Games and Decisions*. New York: John Wiley and Sons.

[118] Debreu, G. (1959) *Theory of Value*. New York: John Wiley and Sons.

[119] Borch, K. H. (1968) *The Economics of Uncertainty*. Princeton, New Jersey: Princeton University Press.

[120] Hirshleifer, J. (1965) "Investment Decision under Uncertainty: Choice-Theoretic Approaches," *Quarterly Journal of Economics*. Vol. 79: pp. 509-536 and (1966) "Investment Decision under Uncertainty: Application of the State-Preference Approach," *Quarterly Journal of Economics*. Vol. 80: pp. 252-277.

[121] Fisher, I. (1930) *The Theory of Interest*. New York: MacMillan.

Myers applied the TSP approach to the analysis of security valuation[122] and to capital-budgeting.[123] Myers' formulation will be reviewed in greater detail since he explicitly addresses the subject matter of this paper.

While addressing the problem of security valuation, Myers[124] developed the basic time-state preference valuation model that relates the present value of a security to the present value of the contingent returns the security may pay to its owner. Essential to the development of the model is the concept of states or states of nature. Defining a set of states is a way of describing the risk of the security, since the security can be thought of as a contract to pay an amount depending on which state actually occurs. The basic assumptions of Myers' TSP model are:

1. States - A state of nature which may occur at time $\tau$ is defined as a particular sequence of events during the time span from t=1 to t=$\tau$. State s is the set of all possible outcomes from the present to time t.
2. Investor expectations - Investors agree on the definition of the relevant states {(s,t)}. Conditions at t=0 are known with certainty.
3. Outcomes - The set {(s,t)} is sufficiently detailed that, if state s occurs at time t, then returns on every security are uniquely specified for period t and all previous periods. Therefore, a security's contingent returns {R(s,t)} are *not* random

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[122]Myers, S. C. (1968) "A Time-State Preference Model of Security Valuation," *Journal of Financial and Quantitative Analysis*. Vol. 3: pp. 1-33.

[123]Myers, S. C. (1968) "Procedures for Capital Budgeting Under Uncertainty," *Industrial Management Review*. Vol. 9: pp. 1-19.

[124]Myers, S. C. (1968) pp. 1-33.

variables; the return R(s,t) is certain to be paid in period t if state s occurs.

4. Markets - assume perfect markets.
5. Utility functions - Investors act to maximize the expected utility of future returns of their portfolios. The total expected utility associated with any portfolio is a linear function of the utility functions defined for each state. Specifically, if $\pi(s,t)$ represents the probability of the contingency (s,t) and U(s,t) is the associated utility, then the overall utility can be represented as

$$\Psi = \underset{s,t}{\Sigma}\pi(s,t)U(s,t) \tag{42}$$

where $\underset{s,t}{\Sigma}$ is the summation over all states in the set {(s,t)}. The utility of returns in state (s,t) is independent of the utility of returns in all other states.[125]

In addition to the above stated assumptions, Myers goes into considerable detail about the nature of the economy, the available securities, and the reinvestment assumptions. The portfolio selection problem in a TSP framework is to select a vector of investments, $(h_0,h_1,\ldots h_n)$, where $h_k$ is the number of shares of the kth security to maximize expected utility $\Psi$, where

$$\Psi = \underset{s,t}{\Sigma}\pi(s,t)U(s,t)+U(o)$$

with

$$U(s,t) = f\left[\sum_{k=1}^{N} h_k R_k(s,t)\right] \quad \text{and} \tag{43}$$

$$U(o) = f(h_o).$$

The investor is constrained by his wealth endowment, W, which is

$$\phi = \sum_{k=0}^{N} h_k P_k - W = 0. \tag{44}$$

If no short selling or borrowing is allowed, then $h_k \geq 0$ for all k. With

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[125]Myers, S. C. (1968) pp. 2-5.

these restrictions, maximizing Eq. (43) subject to the wealth constraint is a non-linear programming problem.

The basic valuation formula in a TSP framework is detailed in Eq. (45) below.

$$P_k \geq \sum_{s,t} q(s,t)R_k(s,t) \tag{45}$$

where $P_k$ = price of security k and $q(s,t) = \pi(s,t)\frac{U'(s,t)}{U'(0)}$

In words, Eq. (45) states

> ... when an investor maximizes the expected utility of his portfolio, the price of each security is at least equal to the expectation of the marginal utility associated with a small increment in his holdings of that security, when the utility of money in future contingencies is measured in terms of the utility of money used for present consumption.... The terms q(s,t) thus indicate the present value to this investor of an incremental dollar of portfolio return to be received at time t is state (s,t) occurs.[126]

Relaxing the assumption prohibiting short selling in a perfect market, a necessary condition for market equilibrium is that Eq. (45) holds as an equality for each investor and for each security. Given the market equilibrium conditions, Myers relates TSP valuation to conventional risk-adjusted rate and certainty equivalent valuation formulas. Recall,

$$P_k = \sum_{t=1}^{T} \frac{\overline{R}_k(t)}{(1+r)^t} = \sum_{t=1}^{T} \frac{C_k(t)}{(1+i)^t} \tag{46}$$

where

$\overline{R}_k(t)$ = expected return in t;
r = required rate of return;
$C_k(t)$ = certainty equivalent of $\overline{R}_k(t)$; and
i = risk-free rate.

In a TSP framework the size of r or $C_k(t)$ will depend on 1) the pattern across states of stock k's contingent dividends, 2) investor's

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[126]Myers, S. C. (1968) p. 8.

valuation of those dividends, and (3) the investor's probability assessments. More specifically,

$$C_k(t) = Z_k(t)Q'(t)D_k(t) \tag{47}$$

where

$$Z_k(t) = \frac{1}{D_k(t)}[R_k(1,t)\ldots R_k(m(t),t)]$$

$$Q(t) = \frac{1}{\sum_{s=1}^{m(t)} q(s,t)} [q(1,t)\ldots q(m(t),t)] \text{ and}$$

$$D_k(t) = \sum_{s=1}^{m(t)} R_k(s,t)$$

Then the relationship between an investor's expected return, $\overline{R}_k(t)$ and its certainty equivalent, $C_k(t)$ is

$$C_k(t) = \alpha(t)\overline{R}_k(t) = \frac{Z_k(t)Q'(t)}{Z_k(t)\pi'(t)}\overline{R}_k(t) \tag{48}$$

Relating the TSP model to risk averse investors, Myers attacks the perception that the market as a whole is risk averse. Market risk aversion implies that the certainty equivalent of an uncertain return should always be less than or equal to expected return, i.e., $\alpha(t)\leq 1$ or $r\geq i$. The generality of the TSP approach is shown by the reasonable explanation of cases where $\alpha(t)>1$ or $r<i$. Referring to Eq. (48), the numerator and denominator are simply weighted averages of relative prices and probabilities. The relative price in a particular state and time can be more or less than the probability assessment which leads to the conclusion that "a bundle of contingent returns will be relatively

more valuable if it pays higher returns in states in which contingent returns have a high value."[127]

In a companion article, Myers[128] adresses the application of the TSP model to capital budgeting. The central theme of the paper is that portfolio selection models are entirely *inappropriate* for making investment decisions. The basis for this disagrement lies with the basic assumption of the portfolio approaches that projects are *risk-interdependent*, which results in the value of a capital budgeting project being dependent on the covariance of the returns of the project and the firm. Myers demonstrates that *risk-independence* is a necessary condition for equilibrium in the security markets. Independence then allows for the use of asset-by-asset selection techniques.[129]

Risk-independence requires that the increase in share price as a result of the acceptance of project A is not affected if some other project B is also accepted. A necessary condition for independence is that the projects are physically independent. Here we are concerned with economic dependencies such as complementary or prerequisite projects. These projects can and should be combined and evaluated as if they were one project. Assuming physically independent projects, the proof of risk-independence is related to a theory of security valuation. The TSP security valuation model, Eq. (45) is the basis for the

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[127]Myers, S. C. (1968) p. 25.

[128]Myers, S. C. (1968) pp. 1-19.

[129]Myers' result that asset-by-asset capital budgeting selection techniques are consistent with capital market equilibrium relates to the central theme of this paper. In Chapter 3 scalar multivariate risk measures will be presented to evaluate individual capital budgeting projects.

analysis. Referring to Myers' statement of market equilibrium, the value of any security depends on both the scale and risk of its stream of returns. To adjust for scale, divide both sides of Eq. (45) by $S_k = \sum_{s,t} R_k(s,t)$. Equation (45) can now be rewritten as

$$\mu_k = \frac{P_k}{S_k} \geq \sum_{s,t} q_i(s,t) \frac{\Sigma R_k(s,t)}{S_k} \tag{49}$$

To simplify notation define row vectors $Q_i$ and $X_k$ to represent sets $\{q_i(s,t)\}$ and $\{R_k(s,t)/S_k\}$ which results in

$$\mu_k = \frac{P_k}{S_k} \geq Q_i X'_k \tag{50}$$

Let $P_o$ imply acceptance of neither project A nor B; $P_a$ and $P_b$ imply acdeptance of projects A or B; and $P_{ab}$ imply acceptance of both A and B. Then the requirement for risk-independence can be shown as

$$\mu_A S_A - \mu_o S_o = \mu_{AB} S_{AB} - \mu_B S_B \tag{51}$$

Using Eqs. (49, 50, and 51), Myers proves risk-independence from the point of view of the individual investor and from a more general condition of market equilibrium.[130]

Myers addresses the question of when this partial equilibrium approach would not be appropriate for evaluation of capital budgets. He agrees that the TSP approach may break down when

1) Projects are large compared to the value of the firm's existing assets,

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[130]Refering to Myers pp. 6-11 for detailed proof of risk-independence, the implication of mergers on firm value, and an additional proof of Modigliani and Miller's Proposition I.

2) the scale and/or risk characteristics of the projects' incremental cash flows are unanticipated,
3) the firm's shareholders do not hold diversified portfolios, and
4) the shareholders are "locked in" to their holding.[131]

The conclusions of Myers' paper are quite relevant to the study of intertemporal, imperfectly correlated cash flows. For a firm attempting to maximize market value to the shareholders, risk-independence implies that each investment project can be evaluated independently of the firm's other activities; diversification per se is not an appropriate objective for the firm. Finally, the concept of risk-independence is consistent with the idea that value of a project depends on the covariance of the returns of the project with other opportunities. But here the relevant relationships are those facing the investor, not the firm. Risk-independence will break down only when the risk characteristics of the firm's opportunities are not equivalent to those provided by securities in the market.

As with any general model, applying the TSP approach to decision making poses very real practical problems. The requirement to specify the state-contingent claims presents formidable problems. As with all the models previously discussed, the models that provide simple decision criteria tend to be based on highly restrictive assumptions; while the more general approaches are difficult to implement. The primary contribution of the TSP approach is that the model is a useful way of viewing the array of possible outcomes in the future. In addition, the TSP model directly attacks the concept of risk-interdependence for capital budgeting.

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[131]Myers, S. C. (1968) p. 12.

## Comparison of Market Models

Early attempts to incorporate risk in capital budgeting utilized simulation models to evaluate uncertain cash flows. Hertz[132] and Hespos and Strassmann[133] propose that the computer can be used to enhance traditional "sensitivity analysis." This approach centers on the specification of the probability distributions of the components of the cash flows, i.e., sales, investment, useful life, costs, etc. Once these distributions are developed, then computer simulation routines are used to develop either the probability distribution of present values or rate of return. The most significant advantage of simulation according to Hertz is that

> the program allows management to ascertain the sensitivity of the results to each or all of the input factors. Simply by running the program with changes in the distribution of an input factor, it is possible to determine the effect of added or changed information.[134]

Lewellen and Long[135] challenged the use of simulation techniques by comparing simulation results with the results from traditional single-point estimates. For example, they show that when the future cash flows are symmetrically distributed, the mean of the distribution of the IRR obtained by simulating over the cash flows is lower than the

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[132]Hertz, D. B. (1964) "Risk Analysis in Capital Investment," *Harvard Business Review*. Vol. 42: pp. 95-106.

[133]Hespos, R. F. and Strassman, P. A. (1965) "Stochastic Decision Trees for the Analysis of Investment Decisions," *Management Science*. Vol. 11: pp. 244-259.

[134]Hertz, D. B. (1964) p. 103.

[135]Lewellen, W. G. and Long, M. S. (1972) "Simulation Versus Single-Value Estimates in Capital Expenditure Analysis," *Decision Sciences*. Vol. 3: pp. 19-33.

IRR produced by discounting the means of the respective cash flow distributions. This bias occurs because upward variations in the cash flows result in smaller proportionate changes in IRR than do identical downward variations. This assymetry occurs because the cash flow, IRR plot is concave to the origin. A simulation in terms of NPV, however, would eliminate this bias since the present value of any future cash flow is a linear function of the size of the flow.[136] In addition to the comparability problem of IRR vs. NPV, Lewellen and Long noted significant biases when simulations consider variable project life, asymetrical outcomes, and interdependencies among flows. This last problem is of particular interest to this paper.

Interdependent projects where the cash flows depend on multiplicative combination of stochastic variables are particularly vulnerable to simulation induced bias. The significance of the bias can be easily observed by viewing the formula for the covariance:

$$\text{cov}(x,y) = \sigma_{xy} = E(x\cdot y)-E(x)\cdot E(y), \tag{52}$$

therefore,

$$E(x\cdot y) = E(x)\cdot E(y)+\sigma_{xy}, \tag{53}$$

and

$$\sigma_{xy} = \rho\sigma_x\sigma_y \tag{54}$$

where $\rho$ is the coefficient of correlation between x and y. Simulations disassemble these multiplicative relationships creating biases depending on the direction of the relationship. When the variables are directly related, $\rho>0$; a disassembled single point estimate will understate expected returns, i.e.,

$$E(x\cdot y) > E(x)\cdot E(y). \tag{55}$$

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[136]Lewellen, W. G. and Long, M. S. (1972) p. 21.

While the bias for cases where inverse relationships exist, $\rho<0$; a disassembled estimate will overstate the actual mean; i.e.,

$$E(x \cdot y) < E(x) \cdot E(y). \tag{56}$$

If the cash flow represents quotients of stochastic variables, the biases would be reversed. Additive combinations create no difficulties since $E(x+y) = E(x)+E(y)$ regardless of the nature of the x,y relationship.[137]

In addition to the bias problems noted above, effective simulation studies require the specification of the entire probability distribution of the cash flow elements and may be quite costly. If a decision maker can specify the underlying distribution, more powerful decision making tools such as Stochastic Dominance should be used to evaluate capital budgeting projects.

This paper has addressed many alternative ways to evaluate capital budgeting problems. The degree of sophistication has widely varied from simple estimates of expected present value with sensitivity analysis to highly technical mathematical models that require strong assumptions and relate the capital budgeting problem to the study of equilibrium in security markets. The last two sections of this review chapter will review tests of many of these models in simulated environments and will look at surveys of what capital budgeting techniques are actually being used.

Sundem[138] reports the results of a simulation study that compared the performance of various capital budgeting models. Sundem

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[137]Lewellen, W. G. and Long, M. S. (1972) p. 28.

[138]Sundem, G. L. (1975) "Evaluting Capital Budgeting Models in Simulated Environments," *Journal of Finance*. Vol. 30: pp. 977-992.

hypothesized that the Time-State Preference model (TSP) provided the correct market values for each project. Given a large set of projects (Weingartner's data set[139]) and various decision environments (states of nature) he compared the performance of simple capital budgeting models to the TSP metric. The capital budgeting models evaluated were: 1) mean variance portfolio model, 2) mean-variance diagonal model, 3) variability of returns model, 4) chance-constrained programming model, 5) net present value model, and 6) payback model. In a companion paper, two additional simple models are evaluated: 1) net present value with a payback constraint, and 2) net present value based on risk classes.[140]

The simulation study was structured to allow each model to perform as well as possible in each environment. Project parameters were assumed to be predicted perfectly and the environmental parameters were chosen ex post to maximize the value of each model. Each model selected a set of projects in each decision environment. The value of a model is the increase in the value of the firm associated with acceptance of the projects. The metric value is the sum of the values of all projects with positive TSP values; then the amount by which the value of a given model falls short of the TSP value can be viewed as the amount paid for the simplification of the model.

The results of the study are summarized in the table "Model Value as Percent of TSP Model Value."[141] To summarize briefly: The

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[139]Weingartner, H. M. (1963) *Mathematical Programming and the Analysis of Capital Budgeting Problems.* Englewood Cliffs, New Jersey: Prentice Hall.

[140]Sundem, G. L. (1974) "Evaluating Simplified Capital Budgeting Models Using a Time-State Preference Metric," *Accounting Review.* pp. 306-320.

[141]Sundem, G. L. (1974) p. 988.

mean-variance (MV) and the mean-variance diagonal (MVD) models yielded results quite similar to the TSP model. In highly uncertain environments the MVD outperformed the MV model, however, the difference is slight and may be due to chance. 2) The variability of returns model (VR) despite theoretical shortcomings performed very well. VR model achieved over 90% of the maximum possible value in all environments except one. The VR may be a very cost effective model because it can evaluate projects singly rather than depend on portfolio of projects. The VR model clearly outperforms the chance-constrained programming and net present value models, especially in highly uncertain environments. 3) There is a large drop in performance with the chance-constrained programming model, CCP. As the CCP model is very complicated and with such poor performance, the CCP model seems to offer little potential when compared with the VR model. 4) The net present value model (NPV) performance deteriorated very rapidly with increases in uncertainty. In risky environments the NPV approach rejected many alternatives that were found acceptable using the metric. 5) Finally the payback model was outperformed by all other models in relatively certain environments. However, under conditions of greater uncertainty, payback performance increased. This behavior may suggest that the payback may be used efficiently in highly uncertain environments or as an initial screening device or as a constraint to eliminate very risky projects.

In a companion paper, Sundem[142] used the payback as a constraint to the net present value model and compared the results of this

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[142]Sundem, G. L. (1974) pp. 306-320.

model (NPV-PB) and of a risk class net present value model (NPV-RC) to the same TSP metric. Using the same methodology as was previously reported (Sundem, 1975), the results of these two additional models are: 1) The NPV-PB outperforms both the NPV and the PB models alone. However, there does not appear to be any synergistic effects from combining the two models. The NPV-PB simply selects the highest values of the NPV or PB applied separately. 2) The NPV-RC uses the capital asset pricing model to identify a cost of capital that is the discount rate that equates a project's net present value to its TSP metric value. Therefore, ranking projects by this cost of capital is a ranking in terms of riskiness. The effectiveness of the NPV-RC model depends on the number of risk classes identified. At one extreme, assign N projects to N risk classes which results in a set of projects identical to the TSP set; while at the other extreme, one risk class, the set is identical to the NPV set. Sundem found a significant increase in proficiency when he used two or three risk classes while there was only a slight increase for five or six classes. He concluded that a two or three risk class model is likely to be cost/benefit efficient while additional refinements may not be worth their cost.

Bey and Porter[143] also used a simulation study to evaluate portfolio approaches to capital budgeting. This study differs from the Sundem work in that Bey and Porter used a Second Degree Stochastic Dominance model (SSD) as their metric as opposed to Sundem's TSP

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[143]Bey, R. P. and Porter, R. B. (1977) "An Evaluation of Capital Budgeting Portfolio Models Using Simulated Data," *Engineering Economist*. Vol. 23: pp. 41-65.

framework. In addition, Bey and Porter varied the probability distribution of the cash flows to allow for skewness while Sundem limited his study to normal distributions.

Taking a subset of the projects found in Weingartner's (1963) study, Bey and Porter use the annual cash flows from this study to represent the annual expected cash flows in an uncertain environment. The standard deviation of the cash flows for each project i in period t was projected by the formula

$$\sigma(CF_{it}) = Z_i \overline{CF}_{it} e^{G_i t} \tag{57}$$

where $Z_i$ = degree of uncertainty associated with project i,

$\overline{CF}_{it}$ = expected cash flow of project i in period t,

$G_i$ = an annual growth factor for project i representing the situation where risk increases over time.[144]

$Z_i$ was assumed to be constant over time for a project, but varied among projects. Low values of $Z_i$ represent low degrees of uncertainty such as replacement or scale expansion of the firm while high values of $Z_i$ represent highly uncertain new projects. The study addressed the problem of non-normal distributions by analyzing five general groups:

1. The cash flow distributions are all normal with positive inter-period correlations;
2. The underlying distribution is not normal, but the general shape is the same for all projects and is constant over time. In

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[144]Bey, R. P. and Porter, R. B. (1977) p. 46.

addition to positive correlation, this environment created both positive and negative skewness;

3. The shape of the distribution varied among projects but was constant over time;
4. The distribution varied among projects and over time;
5. Variable distributions with negative correlations over time.[145]

Using these inputs, the joint probability functions of the net present values was simulated for ten economic states. The result of the simulation was the development of cash flows associated with each state, environment and project. Then these cash flows were used to evaluate the following capital budgeting models:

1. NPV: Risk adjusted discount rate model, where a single adjustment value was used for all cases.[146]
2. EV-I: A mean-variance portfolio model adapted to the capital budgeting problem.
3. EV-II: A multiperiod mean-variance certainty equivalent model which is an extension of Lintner's single period portfolio model.
4. $ES_h$: A mean-semivariance portfolio model.
5. CCP: A chance-constrained-programming model with a NPV constraint.

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[145]Bey, R. P. and Porter, R. B. (1977) p. 46-47.

[146]The use of a single risk-adjusted rate across environments of widely varying risk severely handicaps this model in the Bey and Porter study.

The basis of comparison, or metric, was the Second Degree Stochastic Dominance Model (SSD). As stated, the SSD model was assumed to evaluate all projects and combinations of projects correctly and was used as the basis for comparison. The results of the study are stated in terms of the consistency between the efficient sets selected by the decision models tested and the SSD efficient set. The SSD efficient set contained more combinations than the efficient set associated with any other model. This performance results from the less restrictive utility function associated with the SSD model. As greater restrictions are placed on the utility function the efficient set becomes smaller, resulting in a more effective decision rule to order or rank alternatives.

Ranking the models on the basis of mis-classifications, the Mean-Semi-Variance model clearly outperformed all other models. This performance can be explained since the $ES_h$ efficient set is a proper subset of the SSD efficient set. Therefore, using an $ES_h$ model assures the decision maker a SSD efficient set - a choice which is consistent with both risk aversion and nondecreasing utility of wealth. The next best performance was accomplished by the chance-constrained model. Both of the mean-variance models did not select as well as the $ES_h$ or CCP models; since the project means and variances were not affected by changes in the decision environments, the efficient sets selected by the EV models were stable as environments changed. The NPV model's performance was viewed to be quite poor in comparison to the portfolio models. Since the NPV model selects individual projects on an accept/reject basis while the portfolio models select efficient sets, a

direct comparison of the NPV and the portfolio models is not possible.

In analyzing the effects of correlated cash flows, Bey and Porter found the mean-variance models to be insensitive to changes in the characteristics of project cash flow distributions. This shortcoming can be particularly severe if the distributions are significantly skewed. On the other hand, the SSD, $ES_h$, and CCP capital budgeting models selected different efficient sets as the distributions changed shape. If the actual underlying distribution of the cash flows is assymetric, use of the variance alone becomes suspect as an appropriate measure of risk.

This study is significant in that it addresses both the problems of skewness and of correlated cash flows. Bey and Porter show that when the cash flow distributions depart from standard assumptions of independent normal distributions, capital budgeting models need to incorporate these departures to adequately measure risk.

## Practice versus Theory

The previous portion of Chapter II systematically reviewed the capital budgeting literature to identify the various ways in which risk has been incorporated into the decision making process. A wide range of techniques are available to the practitioner, from relatively simple risk-adjusted rate analysis to highly sophisticated market models such as time state preference theory. In this section of the paper, a comparison of capital budgeting theory and actual techniques used in the field will be reviewed.

There has been continual interest in the past decade to evaluate the actual capital budgeting technqiues used in the field. Mao[147] interviewed the management of eight medium to large companies to compare current theory with actual practice. Mao found that there was a wide difference between the sophisticated theoretical capital budgeting models found in the literature and actual practice. His findings are briefly summarized as: 1. Management either explicitly or implicitly stated that maximizing the value of the firm was a primary goal; 2. top executives view the value of the firm independently of the effects of diversification by investors; 3. executives tend to view risk in terms of downside variation which indicates that the semi-variance may be a better description of risk than the variance of the cash flows; 4. the primary method of incorporating risk is through the use of risk-adjusted discount rates rather than the probabilistic approaches advocated in the literature; 5. finally, most firms use the less-sophisticated payback period as the primary risk measure rather than the theoretically superior net present value or internal rate of return analysis. Mao's findings show that there is a considerable lag between finance theory in the literature and practice in the field.

Klammer[148] in a large survey confirmed Mao's small sample observations. Klammer found that nearly all firms have some method of dealing with risk but less than 40% of the respondents admitted to using a specific formal method of risk analysis. Comparing capital budgeting

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[147]Mao, J. C. T. (1970) "Survey of Capital Budgeting: Theory and Practice," *Journal of Finance*. Vol. 25: pp. 349-360.

[148]Klammer, T. (1972) pp. 387-397.

techniques over a ten-year period, Klammer found an increase in the use of discounted cash flow methods while reliance on the sole use of payback methods declined.

In a recent article, Schall, Sundem, and Geijsbeek[149] reported the results of a sample survey of a large firm's capital budgeting methods. Their main conclusion was that practitioners are becoming increasingly more sophisticated in the analysis of risk in capital budgeting. Although payback is still the most popular capital budgeting technique used in the field, only two percent of the respondents use payback as the only capital budgeting tool. Most companies use payback along with a discounted cash flow technique (either net present value or internal rate of return). Looking at procedures for assessing risk, more than thirty-six percent of the firms use some quantitative methodology, only four percent do not assess risk, while the remaining sixty percent assess risk subjectively. More firms look at risk in terms of a measure of total risk than use portfolio covariance analysis.

The trend to greater sophistication in capital budgeting has been documented by the studies reported in this section. There still exists the need to develop techniques of risk analysis that are theoretically acceptable and yet simple enough to understand and use by practitioners. Current theory seems to be going the way of greater sophistication rather than increased simplicity. In Chapter III a theoretically sound, yet simple, approach to risk analysis in capital

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[149]Schall, L. D., Sundem, G. L., and Geijsbeek, W. R. (1978) "Survey and Analysis of Capital Budgeting Methods," *Journal of Finance*. Vol. 33: pp. 281-287.

budgeting will be presented. This approach will develop simple scalar measures of return and risk that can be easily used in the field. As with any new approach, this initial work requires some simplifying assumptions that can be relaxed as the approach gains acceptance.

# CHAPTER III

# MULTIVARIATE CAPITAL BUDGETING MODEL

## Introduction

The capital budgeting problem is to develop decision rules to select, evaluate or rank real asset opportunities on the basis of expected returns, the timing of those returns and the risk associated with the returns. The use of return and risk as decision criteria is well-documented in the finance literature. While general agreement exists as to the measurement of returns, controversy attends to an appropriate measure of risk. The after-tax cash flows related to the project, either measured absolutely or relatively, have been accepted as an adequate measure of return. Consideration of the timing of the cash flows requires explicit consideration of the multiperiod nature of the problem. Risk has been described in the literature in many ways: as the total variation of returns, as the sum of systematic and residual variation in a market context, or as the downside variation from some standard goal. In addition to the risk measures described above, total risk can be viewed as a function of two components; the variation of the periodic cash flows, and the intertemporal correlations among cash flows. This partition of total risk allows the analyst the opportunity to study explicitly the impact of correlated cash flows on the total risk.

Recognition of the random nature of the capital-budgeting problem leads the decision maker to the study of various possible probability distributions of the cash flows. The univariate normal distribution has been used extensively in the finance literature with assumed intertemporal independence or with single-period market models. When the cash flows are assumed independent and symmetric, the location and scale parameters of the univariate normal distribution provide simple measures of return and risk.

Dissatisfaction with the strict assumptions of the mean-variance decision rule has resulted in more general capital budgeting models. Models based on semivariance incorporate skewness into the decision process. Stochastic dominance decision rules relate to the entire probability distribution as opposed to the first two moments. Time-state preference models specify outcomes in the form of state variables with risk measured as the likelihood of state occurrence. The intertemporal nature of the capital-budgeting problem has either been assumed away or treated quite selectively.

An approach to the solution of the capital budgeting problem that explicitly considers the intertemporal nature of the risk assessment problem is the use of multivariate statistical techniques. More specifically, the periodic cash flows are assumed to be a realization from an underlying multivariate distribution. With the added assumption of multivariate normality, the analysis of risk is equivalent to the analysis of the variance-covariance matrix. The remainder of this chapter will address the need for a multivariate approach and the appropriateness of the multivariate normal distribution as an underlying

structure. Given the rationale for the multivariate approach, the variance-covariance matrix is offered as a multivariate analogue to the univariate variance. Two scalar risk measures are presented and analyzed: the variance of the net present value and the generalized variance (generalized variance - determinant of variance-covariance matrix).

## Asset-by-Asset Analysis versus Portfolio Approach

Many of the capital-budgeting models found in the literature are extensions of stock market valuation models. Arguments for investor diversification leading to stock portfolio models have been discussed in detail in Chapter II. However, when these models are extended to the evaluation of real assets the need for complicated portfolio models has been questioned.

Rubinstein[1] states that, in a setting of perfect capital markets and in the absence of synergy, diversification effects within the firm can be ignored in capital-budgeting decisions. The primary argument against firm diversification is that an investor can more efficiently diversify his portfolio of assets than can the firm diversify its portfolio of assets. Also, since the life of real assets normally is long, portfolio revision techniques which can be readily applied to stock portfolios may be uneconomical in the case of real assets.

Myers[2] also argues for asset-by-asset selection using the equilibrium conditions of the security markets. Myers states that a

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[1] Rubinstein, M. (1973) "A Mean-Variance Synthesis of Corporate Financial Theory," *Journal of Finance*. Vol. 28: pp. 167-180.

[2] Myers, S. C. (1968) "Procedures for Capital Budgeting Under Uncertainty," *Industrial Management Review*. Vol. 9: pp. 1-19.

necessary condition for equilibrium in the capital market is that projects are risk-independent of the firm. There are cases however, when portfolio effects must be considered. These effects are cited in detail in Chapter II, pp. 67-86. Assuming projects that do not significantly change the scale and/or risk characteristics of the firm and diversified investors, Myers proves that asset-by-asset capital budgeting approaches are on sound theoretical footing.

Using the above arguments as a basis, the capital budgeting model developed in this paper will concentrate on the selection and ranking of single projects. The implications of the model for portfolios will be addressed in Chapter V.

## Independence versus Correlated Cash Flows

Ideally, a normative capital budgeting model would have sufficient flexibility to incorporate all the dimensions of a project's return and risk. An underlying hypothesis of this study is that the assumption of intertemporal independence, while simplifying the capital-budgeting analysis, may lead to incorrect decisions that adversely affect the value of the firm.

Very little research has been reported as to the impact of the effects of correlation on capital-budgeting decisions. Initial work in this area was reported in Bey and Singleton's[3] paper on the effects of autocorrelation in a portfolio context. Using a simulated environment based on a Markov process, Bey and Singleton compared the efficient sets

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[3]Bey, R. P. and Singleton, J. C. (1978) "Autocorrelated Cash Flows and the Selection of a Portfolio of Capital Assets," *Decision Sciences*. Vol. 9: pp. 640-657.

generated while incorrectly assuming independence with the efficient sets that incorporated the known autocorrelations. Using mean-variance (E-V), second degree stochastic dominance (SSD), and mean-semivariance (E-S) market models, the size of the efficient sets varied positively with the degree of correlation. Certain models were found to be more sensitive to changes in correlation, but Bey and Singleton were unable to identify the exact level at which autocorrelation becomes critical. In this portfolio context, ignoring the effects of autocorrelation could lead to the selection of a nonoptimal portfolio.

Although Bey and Singleton's work is the first attempt at identification of the effects of correlated cash flows, a number of prominant researchers have incorporated autocorrelation into their capital-budgeting models. Hillier[4] developed the probability distribution of net present values for three cases: independent cash flows, perfect positive correlation, and the case where some cash flows were independent and others perfectly correlated. Wagle[5] followed Hillier's work closely and considered the case of imperfect correlation. Levy and Sarnat[6] extended this work to include both autocorrelated and cross-correlated cash flows.

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[4]Hillier, F. S. (1963) "The Derivation of Probabilistic Information for the Evaluation of Risky Investments," *Management Sciences*. Vol. 9: pp. 443-457.

[5]Wagle, B. (1967) "A Statistical Analysis of Risk in Capital Investment Projects," *Operational Research Quarterly*. Vol. 18: pp. 13-33.

[6]Levy, H. and Sarnat, M. (1970) "The Portfolio Analysis of Multiperiod Capital Investment Under Conditions of Risk," *Engineering Economist*. Vol. 16: pp. 1-19.

In all these cases, the risk measure developed is the variance of the net present value distribution. The appropriateness of this discounted risk measure will be discussed in detail in a later part of this chapter.

In addition to incorporating autocorrelated cash flows into capital-budgeting models, significant work has been reported on estimation of correlation parameters. Hillier[7] used regression analysis to estimate correlation coefficients. Initially, $W_1$ and $W_2$ are assumed to be random variables with variances, $\sigma_1^2$ and $\sigma_2^2$ and with a correlation coefficient $\rho$. An additional assumption is that $W_1$ and $W_2$ follow a bivariate normal distribution. The expected value of $W_2$, given the value of $W_1$, is merely $\rho$ times the value of $W_1$. Symbolically,

$$E(W_2|W_1 = w) = E(W_2) + \rho(\sigma_2/\sigma_1)(w-E(W_1)) \tag{1}$$

which can be rewritten as

$$\frac{E(W_2|W_1 = w) - E(W_2)}{\sigma_2} = \frac{\rho(w-E(W_1))}{\sigma_1} \tag{2}$$

The above regression procedure is general in that even if the assumption of normality does not hold, the expression still provides the best linear estimate of $E(W_2|W_1=w)$ according to the principle of least squares.

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[7]Hillier, F. S. (1963) pp. 443-457.

Bussey and Stevens[8] used simulated cash flow patterns and Box and Jenkins[9] time series analysis to estimate autocorrelation patterns. Hypothesizing a first-order stationary autoregressive model of the form

$$Y_t = \phi_1 Y_{t-1} + \varepsilon_t \tag{3}$$

where $Y_t$ = the t-th realization of the cash flow time series at time

$t = 0,1,\ldots;$

$\phi_1$ = the one-period lag coefficient, $|\phi_1| < 1;$

$\varepsilon_t$ = random error term with $\mu = 0$ and $\sigma^2 = \sigma_\varepsilon^2$ .

Inputs to the model ($Y_t$) are simulated cash flow streams using the estimator's values of optimistic, pessimistic, and most likely values as parameters of the simulation model to generate pseudo random cash flows. Using these inputs in a least squares regression results in the estimate of one period lag coefficient, $\phi_1$. This estimate then is used to compute the autocorrelation matrix using

$$\rho_{\tau\theta} = \phi_1^{|\theta-\tau|} \tag{4}$$

where $\tau,\theta = 1,2,3,\ldots n$ are the periods of interest.

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[8]Bussey, L. E. and Stevens, G. T. (1972) "Formulating Correlated Cash Flow Streams," *Engineering Economist*. Vol. 18: pp. 1-30; and (1975) "Reply to 'Comment on Formulating Correlated Cash Flow Streams'," *Engineering Economist*. Vol. 20: pp. 215-221.

[9]Box, G. E. P., and Jenkins, G. M. (1970) *Time Series Analysis, Forecasting and Control*. San Francisco: Holden-Day, Inc.

Gnanadesikan[10] reviewed in detail the development of robust estimates of location and dispersion parameters in a multivariate setting. Estimates of covariance and correlation were emphasized, using an elemental concept for estimating the covariance between two variables $Y_1$ and $Y_2$:

$$COV(Y_1,Y_2) = \tfrac{1}{4}\{var(Y_1+Y_2) - var(Y_1-Y_2)\}. \tag{5}$$

Given the covariance estimate in Eq. 5, a corresponding robust estimator of the correlation coefficient between $Y_1$ and $Y_2$ is

$$r_{12} = \frac{S_{12}}{(S_{11}S_{22})^{\frac{1}{2}}} \tag{6}$$

where $S_{jj}$ is a robust estimator of the jth response.

The estimate of the correlation coefficient in Eq. 6 may not necessarily lie in the admissable range [-1, +1]. A modification to ensure the estimate is in the valid range is to standardize the $Y_j$'s. Let $Z_j = Y_j/S_{jj}^{(\frac{1}{2})}$, then

$$COV(Z_1,Z_2) = \tfrac{1}{4}\{var(Z_1+Z_2) - var(Z_1-Z_2)\} . \tag{7}$$

Using the transformed variances and covariances, an estimate of the correlation coefficient in the valid range can be obtained.

The works referenced above have shown the need for accommodating intertemporal correlation in capital-budgeting analysis. In

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[10]Gnanadesikan, R. (1977) *Methods for Statistical Data Analysis of Multi-Variate Observations*. New York: John Wiley and Sons. Chapter 5.

addition, relatively simple statistical techniques have been presented to estimate the correlation matrix which along with estimates of the periodic variances allows the decision maker to develop a variance-covariance matrix to evaluate the risk of a project in an intertemporal, multivariate perspective.

## Multivariate Normal Distribution as an Underlying Structure

Capital-budgeting decisions refer to the commitment of current resources in the hope of realizing future benefits over time. Significant characteristics of this problem are the uncertainty associated with the expenditure of current resources and the uncertainty associated with the future benefits. The need to explicitly estimate and evaluate the future benefits and their associated uncertainty has lead to the development of many capital-budgeting models. Chapter II of this paper has systematically reviewed the capital-budgeting literature.

Existing capital budgeting models fall into two broad categories: 1. Simple models with highly restrictive assumptions; 2. General models that are difficult to implement. The model presented here will attempt to reconcile and adapt these models to the end that the model developed here is general while still resulting in simple decision criteria. Because of the random nature of the future benefits, most capital budgeting models are probabilistic in nature; i.e., the decision maker hypothesizes a probability distributuion of future cash flows and then develops analytical techniques to evaluate the probability distribution.

The objective of this study is to report on a general probabilistic approach to real-asset selection that results in relatively

simple measures to evaluate uncertain future benefits. A significant difference between the model to be developed and the existing models is a multi-dimensional perspective. This multi-dimensional view leads to the study of Multivariate Statistical methods for an appropriate solution.

Multivariate Statistical techniques are characterized by the analysis of data that consist of sets of measurements, or multiple dimensions of a particular problem. The capital budgeting problem can be viewed in this perspective with each period's cash flow as a dimension. For example, let the firm be faced with a current outlay of Y dollars for a project that will provide X dollars for p periods in the future. The cash flow in each of p periods is a random variable to be estimated and evaluated. A natural way to view this problem is that the cash flow of each period can be represented by a probability distribution that can be evaluated using well-developed statistical techniques. Because of dependencies or autocorrelation among the periodic cash flow estimates, multivariate statistical models provide an effective framework for return-risk analysis. Consideration of these dependence structures requires an analysis of all the dimensions of the problem simultaneously as opposed to a period-by-period analysis. The effect of ignoring possible dependencies is that the decision maker may understate the risk of the project.

The need for the analysis of dependent structures had lead to the development of multivariate statistics. Initial developments in the field were to study problems in genetics, biology, and anthropology. The desire to develop hybrid plants, to classify populations into

groups, and to evaluate test scores has lead to well-developed multivariate statistical tools. The problem of capital-budgeting is quite similar to the traditional multivariate applications with one significant difference: the element of time. The problem of selecting assets whose outcomes will not be known until some time in future requires the multivariate measures to accommodate this additional dimension. Fortunately, multivariate distributions can be easily transformed to incorporate this time element.

## Assumption of Multivariate Normality

A crucial assumption to the development of this multivariate capital-budgeting model is the assumption of normality.

> The univariate normal distribution arises frequently because the effect studied is the sum of many independent random effects. Similarly, the multivariate normal distribution often occurs because the multiple measurements are sums of small independent effects. Just as the central limit theorem leads to the univariate normal distribution for single variables, so does the general central limit theorem for several variables lead to the multivariate normal distribution.[11]

Consider the capital budgeting problem in which an investment Y results in cash flows during the next p years. Let $X_i$ be a random variable which takes on the value of the net cash flow during the $i^{th}$ year, where i = 1,2,3,...p. The realized or observed cash flow in any particular period will depend on many factors internal and external to the firm. Some of these factors such as production and marketing decisions are controlled or influenced by the firm. However, many other factors relating to government policies or the state of the economy will

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[11]Anderson, T. W. (1958) *An Introduction to Multivariate Statistical Analysis.* New York: John Wiley and Sons. p. 2.

force the firm to react rather than act. Since there are many possible combinations of factors that influence the cash flow stream, the assumption of a normal distribution of the cash flows via the central limit theorem appears reasonable. In addition, the consideration of the time element results in measures of merit (net present value and risk) that are linear combinations of the cash flow variables, more specifically, sums of random variables.

Thus, the central limit theorem for means, the Lindeberg-Levy Theorem, is a major justification for the use of both the univariate and multivariate normal distribution assumptions. If the estimated cash flows come from *any* population with a finite variance, $\sigma^2$, then the limiting distribution of the statistic

$$Z = \frac{\overline{X} - \mu}{\sigma/n^{(\frac{1}{2})}}$$

is standard normal. Consequently, the sample mean is asymptotically normally distributed with mean, $\mu$, and variance $\sigma^2/n$. The multivariate extension of the theorem is

> Let $x_1,\ldots,x_n$ be an independent random sample from a k dimensional population with finite mean vector $\mu$ and finite covariance matrix A; then the vector $n^{-\frac{1}{2}}\Sigma_i(X_i-\mu_i)$ has a multinormal limiting distribution with zero mean vector and covariance matrix A. Equivalently, the vector $\overline{X} = (1/n)\Sigma X_i$ is asymptotically normally distributed with mean vector $\mu$ and covariance matrix $(1/n)A$.[12]

Further arguments for assuming multivariate normality can be found in the finance literature. The assumption of multivariate normal returns is important to the development of the two-parameter portfolio

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[12]Theil, H. (1971) *Principles of Econometrics*. New York: John Wiley and Sons. pp. 367-370.

model. Fama[13] summarized the development of the "market model" focusing on the joint probability distribution of security returns. Assuming that the returns on individual securities are distributed normally, then the return on a portfolio, which is simply a linear combination of securities, is also normal.

> This result implies that the joint distribution of the returns on any two different portfolios is bivariate normal... the result also implies that the return on any security i and the return on any portfolio p is bivariate normal.... Bivariate normality of security and portfolio returns is the foundation of our theoretical and empirical work on the so-called 'market model' relationships between the returns on securities and the return on a portfolio of securities taken to be representative of the market.[14]

Looking to empirical work on security prices, Fama states "our conclusions that the distributions of monthly portfolio returns and security returns are approximately normal is consistent with the assumption that the joint distribution of returns is multivariate normal."[15]

The mean-variance security model has been adapted to the capital-budgeting problem. This extension was reported in detail in Chapter II. The purpose of this brief review here is to support the multivariate approach by reporting previous multivariate applications in the finance literature.

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[13]Fama, E. F. (1976) *Foundations of Finance*. New York: Basic Books.

[14]Ibid., pp. 65-66.

[15]Ibid., p. 65.

## Multivariate Normal Distribution[16]

The identification of multivariate measures of return and risk follow quite logically from the multivariate normal distribution function, the properties of the distribution, and the moments of the distribution. The multivariate normal distribution is a generalization of the univariate normal distribution to p dimensions.

The univariate normal density function can be written

$$f(x) = \frac{1}{(2\pi)^{\frac{1}{2}}\sigma} e^{-\frac{1}{2}\sigma^2(x-\mu)^2} = ke^{-\frac{1}{2}(x-\mu)'1/\sigma^2(x-\mu)}. \tag{8}$$

In matrix notation, the similarity between the univariate and multivariate normal distributions is clear. The scalar variable x is replaced by a vector $\underline{x} = [x_1 \ldots x_p]'$; the scalar constant $\mu$ is replaced by a vector of constants $\underline{\mu} = [\mu_1 \ldots \mu_p]'$; and the positive constant $1/\sigma^2$ is replaced by the inverse of the positive definite variance-covariance matrix

$$\underline{\Sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \cdots \sigma_{1p} \\ \sigma_{21} & \sigma_{22} \cdots \sigma_{2p} \\ \sigma_{p1} & \sigma_{p2} \cdots \sigma_{pp} \end{bmatrix} . \tag{9}$$

The squared term in the exponent $(x-\mu)'1/\sigma^2(x-\mu)$ is replaced by the quadratic form

$$(\underline{x}-\underline{\mu})'\underline{\Sigma}^{-1}(\underline{x}-\underline{\mu}) = \sum_{i,j=1}^{p} \frac{1}{\sigma_{ij}} (x_i-\mu_i)(x_j-\mu_j). \tag{10}$$

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[16]This presentation of the multivariate normal distribution function, its properties and its moments follows closely that of Anderson (1958), Chapter 2.

Therefore, the density of a p-variate normal distribution is

$$f(x_1,\ldots,x_p) = Ke^{-\frac{1}{2}(\underline{x}-\underline{\mu})'\underline{\Sigma}^{-1}(\underline{x}-\underline{\mu})} \tag{11}$$

where $K > 0$ is chosen so that the integral over the entire p-dimensional Euclidean space of $x_1,\ldots,x_p = 1.0$. To determine K, evaluate the multiple integral of (Eq. 11) such that the integral over the p-dimensional space is one. Evaluating

$$K = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\frac{1}{2}(\underline{x}-\underline{\mu})'\underline{\Sigma}^{-1}(\underline{x}-\underline{\mu})} dx_1 \ldots dx_n \quad , \tag{12}$$

the result of the integration is that $K = \sqrt{|\underline{\Sigma}|}/(2\pi)^{n/2}$ , which results in the multivariate normal density

$$f(x_1,\ldots,x_p) = (2\pi)^{-p/2}|\underline{\Sigma}|^{-\frac{1}{2}} e^{-\frac{1}{2}(\underline{x}-\underline{\mu})'\Sigma^{-1}(\underline{x}-\underline{\mu})} \quad .^{17} \tag{13}$$

As a special case of the multivariate normal distribution, the bivariate normal distribution is often studied because of the capability to geometrically evaluate the two-variable space. The bivariate normal density can be written explicitly as

$$f(x_1,x_2) = \frac{1}{2\pi\sigma_1\sigma_2(1-\rho^2)^{\frac{1}{2}}} \left| e^{\frac{-1}{2(1-\rho^2)}\left\{\frac{(x_1-\mu_1)^2}{\sigma_1^2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2} - \right.} \right.$$

$$\left. \left. \frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2}\right\} \right| \tag{14}$$

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[17]For a detailed derivation of the multivariate normal density see Anderson (1958) pp. 11-19. Also to differentiate between scalars and vectors, the vector will be underlined.

where $\mu_i$ and $\sigma_i^2$ are the mean and variance of $x_i$, $i = 1,2$, and $\rho$ is the correlation coefficient. The surface depicted by this equation is a bell shaped mound; the shape of the mound is dependent on the value of $\rho$ and the ratio $\sigma_1/\sigma_2$. The customary way to depict the bivariate normal surface is through the use of isodensity contours which are a cross section of the surface made by a plane parallel to the $(x_1,x_2)$ plane. The portion of the exponent of the bivariate density (Eq. 14) in braces, Eq. 15 below,

$$\frac{(x_1-\mu_1)^2}{\sigma_1^2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2} - \frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} \tag{15}$$

when set equal to some positive constant, c identifies the isodensity contour. Equation 15 represents an ellipse with the center at the point $(\mu_1,\mu_2)$ which is called the centroid of the bivariate population. The major and minor axes of the ellipse pass through the centroid and make the following angle with the positive $x_1$-axis:

$$\theta = \begin{cases} 45^\circ & \text{when } \sigma_1 = \sigma_2 \\ 1/2 \operatorname{Arctan} \dfrac{2\rho\sigma_1\sigma_2}{\sigma_1^2 - \sigma_2^2} & \text{when } \sigma_1 > \sigma_2 \end{cases} . \tag{16}$$

If $\rho > 0$ the line contains the major axis, if $\rho < 0$ the minor axis, and if $\rho = 0$ and $\sigma_1 = \sigma_2$ the ellipse reduces to a circle with all axes equal. Angle $\theta$ is a function of $\sigma_1$, $\sigma_2$, and $\rho$; therefore, by varying c one can

generate a family of concentric ellipses, all with the same orientation.[18]

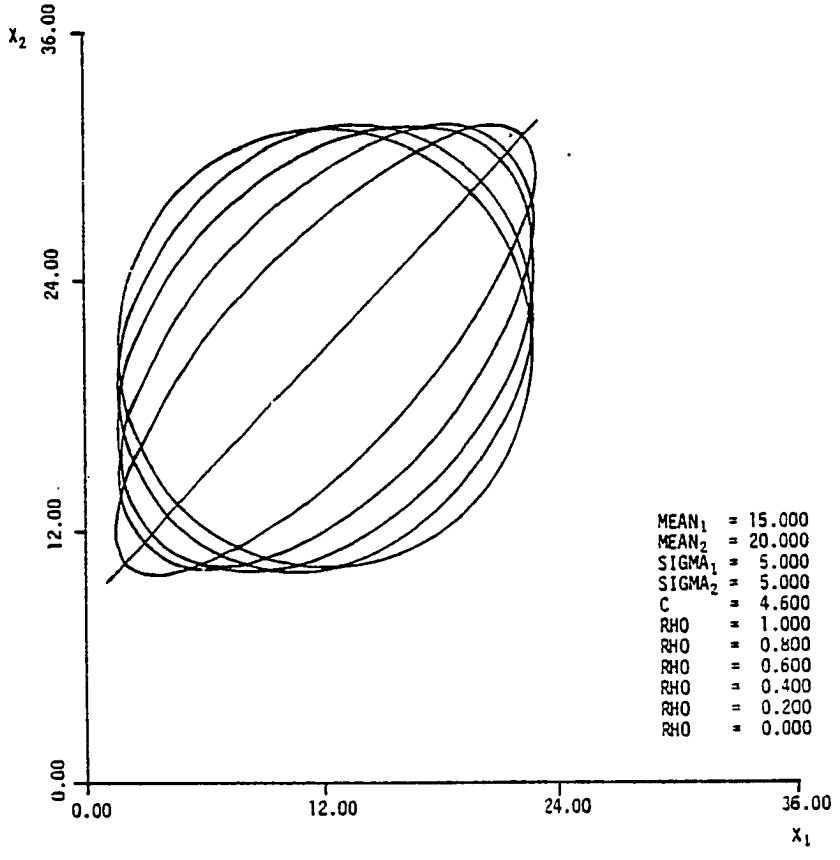



Figure 10: Isodensity Contours as $\rho$ varies.

The above bivariate description is readily generalized to three or more dimensions (although only the three dimension ellipsoid can be depicted geometrically.) Refer to the multivariate density, (Eq. 13), the exponent

$$(\underline{x}-\underline{\mu})'\underline{\Sigma}^{-1}(\underline{x}-\underline{\mu}) = c \qquad (17)$$

is constant on ellipsoids for every positive value of c in a p-dimensional Euclidean space. The centroid is determined by the mean vector, $\underline{\mu}$. The shape and orientation of the ellipsoid are determined by $\underline{\Sigma}$, and the size (given $\underline{\Sigma}$) is determined by c.

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[18]Refer to Figure 10 which depicts various isodensity contours as $\rho$ varies.

The isovariance ellipse in the bivariate case and the isovariance ellipsoid for the multivariate case are a way in which one can evaluate and compare variance-covariance matrices. For a given c level, where the c level corresponds to a chi-square variate with p degrees of freedom, one can directly compare two or more populations using the area/volume of a particular isovariance ellipse/ellipsoid as a surrogate measure of the variance-covariance structure. This technique will be evaluated in a later part of this paper under the heading of evaluation of risk in capital budgeting.

The importance of the multivariate normality assumption to the capital budgeting problem relates to the need to include the time element in the decision. The process of discounting transforms the multivariate cash flow stream to present time. Discounting is necessary to make the various alternative projects comparable. A particularly attractive property of the multivariate normal distribution is that linear combinations of normal variates are also normally distributed. Anderson states and proves the following theorem: "Let $\underline{X}$ (with p components) be distributed according to $N(\underline{\mu},\underline{\Sigma})$, then

$$\underline{Y} = \underline{CX} \tag{18}$$

is distributed according to $N(\underline{C\mu},\underline{C\Sigma C'})$ for C nonsingular."[19]
This theorem is particularly relevant if the C matrix represents a diagonal matrix of discount factors which results in the probability distribution of present values behaving according to the above cited theorem. In addition, Anderson proves that "it is *only* the multivariate

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[19]Anderson (1958) pp. 19-22.

normal distribution that has the property that every linear combination of variates is normally distributed."[20]

The moments of multivariate normal distributions are also a generalization of the univariate normal. Just as the distribution law of the univariate normal is $N(\mu,\sigma^2)$, the multivariate distribution law is $N(\underline{\mu},\underline{\Sigma})$, where $\underline{\mu}$ is a p dimensional vector and $\Sigma$ is a p x p symmetric positive semi-definite variance-covariance matrix. The mean vector and variance-covariance matrix fully specifies the multivariate normal distribution. The multivariate capital budgeting model to be offered will use the sum of the discounted mean cash flows as the measure of return; while alternative representations of the variance-covariance structure will be interpreted as surrogate risk measures.

## Measurement/Evaluation of Return

The capital-budgeting problem is to select real assets such that the value of the firm is maintained or increased. Any proposed capital-budgeting criterion then must consider the impact of the decision on the value of the firm. The relationship between the value of the firm and the capital-budgeting decision has been reviewed and documented in Chapter II. To identify appropriate measures of return, the capital-budgeting decision will be studied first from the standpoint of perfect certainty and perfect capital markets.[21] Once reasonable

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[20]Anderson (1958) pp. 19-22.

[21]Haley, C. W. and Schall, L. D. (1973) *The Theory of Financial Decisions*. New York: McGraw-Hill, Inc. Chapters 2 and 3 for a detailed discussion of the implications of perfect certainty and perfect capital markets on the capital-budgeting decisions.

decision rules have been reported for the certain case, the assumption of perfect certainty will be relaxed by considering the impact of random cash flows.

## Certain Cash Flows

Under conditions of perfect certainty and perfect capital markets, discounted-cash-flow models have been accepted as the appropriate method to evaluate capital budgeting alternatives. The need to consider the timing as well as the magnitude of the cash flows affords a clear advantage to discounted-cash-flow models as opposed to ad hoc methods such as the payback period and the accounting rate of return.

The discounted-cash-flow methods most often cited are the Net Present Value (NPV) and the Internal Rate of Return (IRR). Although both models result in consistent decisions for conventional cash flow patterns,[22] the NPV model is often preferred to the IRR model for capital budgeting.

Reasons for the preference of NPV over IRR are: 1. As a ratio measure, IRR does not reflect the scale of the project; 2. In the case of unconventional cash flow patterns there can be more than one IRR or cases where there is no unique real rate of return.[23]

Given the very real problem of computing meaningful rates of return for many capital-budgeting projects, the remainder of this study

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[22]Conventional cash flow patterns can be described as an outflow/inflow or a series of outflows/inflows followed by a series of inflows/outflows. The significant feature of a conventional pattern is that there is at most one sign change.

[23]Bierman, H., Jr. and Smidt, S. (1975) *The Capital Budgeting Decision*. New York: Macmillan. p. 51.

will use extensions of the NPV model only.

The net present value of an investment project is

$$NPV = X_o + \frac{X_1}{(1+i_1)} + \frac{X_2}{(1+i_1)(1+i_2)} + \ldots +$$

$$\frac{X_p}{(1+i_1)(1+i_2)\ldots(1+i_p)} = \sum_{t=0}^{p} \frac{X_t}{\prod_{j=0}^{t} (1+i_j)} \qquad (19)$$

where NPV = the net present value;

$X_t$ = the cash flow at the end of period t;

$i_j$ = the interest rate in period t;

p = the useful life of the project.

This general form allows for different interest rates in each period, but in this certain environment the appropriate rate for each future period is known at the time of the decision, t=o. Conventional capital-budgeting practices often use a single discount rate rather than the product in the denominator of Eq. 19. The rationale for the use of a single rate is either that the periodic rates are equal or that the single rate represents an equivalent formulation of the varying rates.

In a certain world, little controversy attends the choice of the discount rate or with the following present value decision rules.

PV Rule 1: Accept projects where the net present value, NPV > 0; Indifferent where NPV = 0; and reject if NPV < 0.

PV Rule 2: If projects or combinations of projects are mutually exclusive, accept the project with the largest net present value.

Use of these decision rules in the case of perfect certainty and perfect capital markets ensures that the capital budgeting decision criteria are consistent with the goal of the firm to maximize shareholder wealth.

## Uncertain Cash Flows

Relaxing the assumption of perfect certainty requires the explicit consideration of the nature of the uncertainty. The uncertain elements associated with the capital budgeting decision are the uncertainty as to:

1.) the future outcomes or periodic cash flows;

2.) the appropriate discount rate; and

3.) the useful life of the project.

As the main thrust of this study is the development of scalar multivariate risk measures that explicitly consider autocorrelated cash flows, the appropriate discount rates and the project lives are assumed to be known. This idealized uncertainty is structured to direct attention toward the primary objective of this paper. The issues of the choice of discount rates, the uncertainty as to future rates, and the uncertainty as to project life are best left for future research.

Uncertainty as to the future cash flow streams leads to a probabilistic approach to capital budgeting, where the future cash flows are viewed as random variables following some known or assumed probability distribution. The idea of parametric decision criteria based on return and risk leads to the analysis of the parameters/moments of the probability distribution. The arguments leading to the assumption of multivariate normally distributed cash flows have been discussed in detail in

an earlier part of this study. Therefore, the remainder of this chapter will relate to the presentation of measures of return and risk when the cash flow stream is modeled using the multivariate normal distribution.

Critical to the application of *all* capital budgeting models is the assumption that the analyst can estimate the future cash flows, or the probability distribution of future cash flows, or the relevant moments of the probability distribution. The difficulties of providing inputs to capital budgeting models are significant. The lack of historical data to use as the basis for forecasts of future cash flows has been recognized and discussed in the literature. Remedies suggested have included the use of check lists and flow charts, simulation studies, and the development of trained subjective estimators. In the case where the multivariate normal distribution is assumed, the analyst must be able to estimate the expected value of estimated mean cash flow for each period, the estimated variance of the net cash flow for each period, and the estimated autocorrelation among the periodic cash flows. Obviously, the performance of any model is dependent upon the quality of the estimates. In Chapter IV, the performance of alternative risk measures is evaluated using simple examples. Even with this testing, the practical use of any model can only be evaluated on real capital budgeting problems using real data. This shortcoming is not unique to this study.

As the purpose of this study is the theoretical analysis of scalar multivariate risk measures, an important assumption is that the analyst can make realistic estimates of the parameters of the multivariate normally distributed cash flows. In Chapter V, under "Suggestions

for Further Research", the problems of data estimation and sampling issues are addressed.

In the multivariate normal approach to capital-budgeting, the necessary parameter estimates are the estimated mean vector, $\underline{\hat{\mu}}$ and the estimated variance-covariance matrix, $\underline{\hat{\Sigma}}$. To account for the timing of the cash flows, the distribution characterized by $N(\underline{\hat{\mu}},\underline{\hat{\Sigma}})$ is transformed to present time using a diagonal matrix of present value factors, $\underline{C}$ which results in the time transformed cash flow distribution, $N(\underline{C\hat{\mu}},\underline{C\hat{\Sigma}C})$. This time-adjusted distribution is then evaluated in terms of return and risk.

The proposed measure of return is simply the expected net present value, E(NPV). The E(NPV) is a linear combination of the present value factors and the estimated cash flow mean vector. More specifically,

$$E(PV) = \underline{C}\,\underline{\hat{\mu}}, \tag{20}$$

where $\underline{C}$ is a p x p diagonal matrix of discount rates and

$\underline{\hat{\mu}}$ is a p x 1 vector of estimated expected cash flows,

explicitly

$$\underline{C} = \begin{bmatrix} 1/(1+i) & & & \\ & 1/(1+i)^2 & & \\ & & \ddots & \\ & & & 1/(1+i)^p \end{bmatrix} \quad \text{and} \quad \underline{\hat{\mu}} = \begin{bmatrix} \hat{\mu}_1 \\ \hat{\mu}_2 \\ \vdots \\ \hat{\mu}_p \end{bmatrix}, \tag{21}$$

with

$$E(\underline{PV}) = \begin{bmatrix} \mu_1/(1+i) \\ \mu_2/(1+i)^2 \\ \vdots \\ \mu_p/(1+i)^p \end{bmatrix} . \quad (22)$$

Then the expected net present value, E(NPV) is equal to the sum of the elements in the E(PV) vector plus or minus the expected cash flow at time zero. That is,

$$E(NPV) = \hat{\mu}_o + \hat{\mu}_1/(1+i) + \hat{\mu}_2/(1+i)^2 + \ldots + \hat{\mu}_p/(1+i)^p . \quad (23)$$

The most controversial aspect of the above measure of return (Eq. 23) is the selection of the appropriate $\underline{C}$ matrix, i.e., the appropriate present value factors to be used. The discounting mechanism above is designed to account only for the difference in the timing of the cash flows. The literature relating to capital budgeting under uncertainty offers three alternative choices for the discount rate: 1.) cost of capital, 2.) required rate of return, and 3.) risk-free rate measuring the time value of money.

The weighted average cost of capital (WACC) is often used as the discount rate. The WACC approach has been criticized due to the required assumptions that the risk of the project is exactly equal to that of the firm acquiring the project and that the capital structure and dividend policy remain constant. In addition, the components of the WACC, the cost of debt and the cost of equity, include risk premiums

for the risk assessed by lenders and stockholders. The use of a risk-adjusted rate to adjust for timing results in a double-counting of risk.[24]

The required rate of return, as defined by the "market model", is defined as that rate of return a project must earn to maintain shareholder wealth. Once again, this required rate includes both the time value of money and a market determined risk premium. Due to the strict assumptions of the Capital Asset Pricing model and the single-period nature of the model, the required rate of return has questionable validity because of market imperfections associated with real asset selection.

The use of the risk-free rate as the time value of money discount rate has wide support in the finance literature.[25] The choice of the risk-free rate is supported by capital market theory. If the assumption of perfect or nearly perfect capital markets is valid, then the risk-free rate identifies the reasonable opportunity cost of funds expended in the capital budget. Formulation of Eqs. 20-23 does not require that the risk-free rate be constant over the life of the project. In fact, the analyst often must estimate relevant risk-free rates for

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[24]See the discussion in Chapter II for a detailed anlaysis of the pitfalls of using risk-adjusted discount rates. Also, see Rubinstein, M. (1973), pp. 167-180 for additional discussion of the biases associated with the cost of capital approach to capital budgeting.

[25]For example, Lintner, J. (1965) "The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets," *Review of Economics and Statistics*. Vol. 49: pp. 13-37. Robichek, A. A. and Myers, S. C. (1965) *Optimal Financing Decisions*. Englewood Cliffs, New Jersey: Prentice-Hall. Van Horne, J. (1966) "Capital-Budgeting Decisions Involving Combinations of Risky Investments," *Management Science*. Vol. 13: pp. 84-92.

future time periods. The uncertainty as to the future risk-free rates, of course, adds an additional dimension to the problem of capital budgeting under uncertainty. Given the previous discussion, the elements of the C matrix used in Eq. 23 can be viewed as the expected time value of money discount rates for period t.

## Measurement/Evaluation of Risk

The use of the Expected Net Present Value, E(NPV) has been readily accepted as an adequate measure of the return on an investment or capital-budgeting project. The appropriate measure of risk finds little agreement in the literature. In fact, considerable controversy exists in terms of both the definition of risk and to the subsequent measurement of risk consistent with a definition. Chapter II of this study systematically reviewed the significant alternate approaches to the evaluation of risky assets. Although many different risk-evaluation models have been presented in the literature, most if not all current techniques follow either the portfolio approach or assume intertemporal independent cash flows. Those models that do not follow either of the above patterns have been found to be too general to be useful in practical capital-budgeting situations.[26]

To limit the scope of a complex problem, two assumptions have been made: 1.) That asset-by-asset selection techniques are appropriate

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[26]The Time-State-Prefence (TSP) model of Myers, S. C. (1968) pp. 1-19, is an example of a general model that does not use either the portfolio approach or assume independent intertemporal cash flows. The TSP approach is general in that the model will accommodate any possible distribution of cash flows; unfortunately, the requirement to specify the state-contingent claims for all possible future states of nature presents application problems to the analyst.

for the analysis of real assets, and 2.) That the future cash flow stream can be modeled using the multivariate normal distribution. These assumptions, then, define risk as the uncertainty of the future cash flows. Risk is viewed as the variability of future outcomes. In this sense, the total variability of the returns is the measure to be studied. In a probabilistic sense, risk can then be expressed as the probability of not achieving the expected return. Using this probabilistic approach, the appropriate risk measure depends on the probability distribution of the cash flows. For example, if the univariate normal distribution is assumed, the variability of return can be measured by either the variance ($\sigma^2$) or the standard deviation ($\sigma$). Moreover, the coefficient of variation ($CV = \sigma/E(v)$) can be used to measure the relative risk/return properties of a project. The CV measures the risk per dollar return and is often used to compare two or more investment projects.

For the assumption of multivariate normality, similar test and ranking procedures can be used. The multivariate extension of the univariate variance is the p x p variance-covariance matrix, where p x p refers to the number of parameters or dimensions. In this capital budgeting application, p refers to the number of periodic cash flows. The remainder of this paper studies the variance-covariance matrix and related statistical measures to incorporate multivariate risk analysis into the capital-budgeting decision.

## Variance-Covariance Matrix

The variance-covariance matrix is a multivariate analogue to the univariate variance. As such, the variance-covariance matrix measures the dispersion about the centroid of the multivariate normal distribution. The amount of dispersion depends on two factors: 1.) the variance of each dimension, and 2.) the correlation between dimensions. These factors become more obvious when the variance-covariance matrix ($\underline{\Sigma}$) is decomposed into the matrix product of the standard deviation diagonal matrix and the symmetric correlation matrix. Let $\underline{B}$ represent the diagonal standard deviation matrix, and let $\underline{R}$ represent the positive definite correlation matrix. Then

$$\underline{\Sigma} = \underline{BRB} \, . \tag{24}$$

More specifically,

$$\underline{\Sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} & \cdots & \sigma_{1p} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} & & \vdots \\ \sigma_{31} & \sigma_{32} & \sigma_{33} & & \vdots \\ & & & \ddots & \vdots \\ \sigma_{p1} & \cdots & \cdots & \cdots & \sigma_{pp} \end{bmatrix} \text{ or } \begin{bmatrix} \sigma_1^2 & \sigma_1\sigma_2\rho_{12} & \sigma_1\sigma_3\rho_{13} & \cdots & \sigma_1\sigma_p\rho_{1p} \\ \sigma_2\sigma_1\rho_{21} & \sigma_2^2 & \sigma_2\sigma_3\rho_{23} & & \vdots \\ \sigma_3\sigma_1\rho_{31} & \sigma_3\sigma_2\rho_{32} & \sigma_3^2 & & \vdots \\ \vdots & & & \ddots & \vdots \\ \sigma_p\sigma_1\rho_{p1} & & & & \sigma_p^2 \end{bmatrix} \tag{25}$$

and the decomposition is

$$\underline{\Sigma}=\underline{BRB}= \begin{bmatrix} \sigma_1 & & & & \\ & \sigma_2 & & & \\ & & \sigma_3 & & \\ & & & \ddots & \\ & & & & \sigma_p \end{bmatrix} \begin{bmatrix} 1 & \rho_{12} & \rho_{13} & \cdots & \rho_{1p} \\ \rho_{21} & 1 & \rho_{23} & & \vdots \\ \rho_{31} & \rho_{32} & 1 & & \vdots \\ \vdots & & & \ddots & \vdots \\ \rho_{p1} & & \cdots & & 1 \end{bmatrix} \begin{bmatrix} \sigma_1 & & & & \\ & \sigma_2 & & & \\ & & \sigma_3 & & \\ & & & \ddots & \\ & & & & \sigma_p \end{bmatrix} . \tag{26}$$

The above decomposition permits the decision maker the opportunity to analyze the components of the project's variability in terms of the standard deviations and correlation.

The total variation is affected by the timing and the magnitude of the periodic standard deviations. The effect of timing is discussed in detail in Chapter IV. The total variation or dispersion increase as the $\sigma_j$ elements increase. A risk-averse decision maker prefers, ceteris paribus, smaller standard deviations.

The total variation is also dependent upon the correlation structure. The interperiod correlations range from -1 to +1. The major limitation on the individual correlation coefficients is that the variance-covariance matrix must be positive semi-definite; i.e., the determinant of the variance-covariance matrix must be greater than or equal to zero. Computing the determinant quickly shows whether the estimated correlation structure results in a feasible variance-covariance matrix.[27] When analyzing a time series such as a cash flow stream, a positive definite variance-covariance also insures that if the underlying distribution is multivariate normal then the stochastic process is stationary. Stationarity provides a simple stochastic structure in that a stationary time series can be located in the neighborhood of its mean value. Although the series may vary significantly, it will return repeatedly to the mean value. Many economic time series display no affinity for a mean value; but fortunately, the differences in many

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[27]From the definition of the non-degenerate multivariate normal distribution, the variance-covariance matrix must be positive definite for the existence of the multivariate normal density. See Anderson, T. W. (1958) p. 17. By including the semi-definite case where det $\Sigma=0$, the degenerate case of a singular $\Sigma$ matrix will be addressed in this study.

nonstationary time series are stationary. The property of stationarity is important to the development of simple models in that a stationary time series can be meaningfully modeled using means, variances, and the autocorrelation function.[28]

For capital-budgeting decisions, a reasonable assumption is that the interperiod correlations will vary from independence ($\rho=0$)[29] to perfect positive correlation ($\rho = +1$). The rationale for the assumption of this non-symmetric correlation of the periodic cash flows in the capital budgeting problem relates to the sources of the interperiod correlations. Most project cash flows will be related to general movements in the economy. While many project returns vary positively with the economy, some firms and projects move contra-cyclically. The assumption of positive interperiod correlations does not preclude contra-cyclical behavior. What this assumption does reject is the violent plus to minus, minus to plus swings in the cash flow stream that are implied by negative autocorrelations.[30] Negative interperiod correlations mean that high/low cash flows in a particular period would be followed by low/high cash flows in subsequent periods. Though cash flows for many projects vary significantly, the case of negative

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[28]Box, G. E. P. and Jenkins, G. M. (1976) pp. 23-30.

[29]For normal distributions, zero correlation implies independence.

[30]Box, G. E. P. and Jenkins, G. M. (1976) p. 57 and p. 62 for an example of the oscillating effect with negative autocorrelations.

interperiod correlations will not be discussed further in this chapter.[31] In the independent case, total variation is reflected by the variance diagonal matrix alone, since the correlation matrix is the identity matrix. Using Eq. 27, the matrix $\Sigma$ is the product of the standard deviation diagonal matrices which results in a diagonal matrix of variances. With independence between periodic cash flows, the multivariate normal distribution collapses into the sum of independent univariate normal distributions which can be evaluated using well-known univariate statistical methods. In the other boundary case to be considered, perfect positive correlation, the total variation is represented by the sum of squares and cross-products matrix, SSCP. Perfect positive correlation results in a degenerate multivariate distribution such that the correlation matrix is a matrix of all ones and is singular. Degenerate cases also exist if any rows or columns of the correlation matrix are identical or are linear combinations. These degeneracy conditions may very well be found in capital-budgeting problems. The degenerate cases require additional special analysis and are discussed in detail in a later part of this study.

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[31] When analyzing groups of assets or portfolios, the case of negative correlation between assets becomes more important. The optimal choice in portfolio selection problems is to find and acquire negatively correlated projects to maximize the effects of diversification. Since the purpose of this paper is to study individual project selection techniques, portfolio effects among projects will not be addressed. In the case of asset-by-asset selection the intertemporal or interperiod correlation is of primary interest. Chapter V will address the extension of the model to consider the standard single-period portfolio problem. In this application, both positive and negative correlations between projects will be discussed.

Recapping, the total variation is a function of both the standard deviations and the intertemporal correlation: with total variation increasing as the periodic variance increases and also increasing as the correlation increases positively from independence to perfect positive correlation.

Estimates of the population variance-covariance matrix

In capital-budgeting problems, the population variance-covariance matrix may not be known. In fact, since only a single realization from the distribution will be observed, the ex-post distribution of the cash flows will be difficult to evaluate. This shortage of data is a common problem with economic time series applications. To evaluate risk using a multivariate perspective, an estimated variance-covariance matrix must be determined.

The estimated variance-covariance matrix, $\hat{\underline{\Sigma}}$, can be obtained either by using conditional probability distributions of the cash flow stream or by developing a computer simulation model. The estimate $\hat{\underline{\Sigma}}$ differs from the classical sample variance-covariance matrix $\underline{S}$, in that $\hat{\underline{\Sigma}}$ is not a summary measure of random sample data, but represents a sample of size one. The importance of this distinction between $\hat{\underline{\Sigma}}$ and $\underline{S}$ is that classical significance tests and confidence intervals using $\hat{\underline{\Sigma}}$ are not appropriate. The dependencies of the cash flow estimates used in either the conditional probability or the simulation approach result in the analysis "of such data by classical multivariate procedures will lead to spuriously short confidence intervals and overly significant tests of hypothesis."[32]

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[32]Morrison, D. F. (1967) *Multivariate Statistical Methods,* 2nd ed. New York: McGraw-Hill Book Co. p. 98.

Some geometric relationships of concentration ellipsoids and the variance-covariance matrix

The bivariate normal distribution is often studied to gain insight into the higher dimensional multivariate normal distributions. In the bivariate case, the behavior of the variance-covariance structure can be depicted geometrically. Although graphical representation is not possible, the results can be extended to the more general p-variate cases.

The bivariate normal distribution was discussed in detail in an earlier part of this study (see pp. 97 - 99); the bivariate density function (Eq. 14) was reported to be constant on ellipsoids. Or,

$$(\underline{X}-\underline{\mu})'\underline{\Sigma}^{-1}(\underline{X}-\underline{\mu}) = C \qquad (27)$$

for every positive value of C in a p-dimensional Euclidean space. The shape and orientation of the ellipsoids are determined by the variance-covariance structure and the size of the ellipsoids (for a given $\underline{\Sigma}$) are determined by C.

The variance-covariance matrix may be evaluated in terms of volumes of the isovariance ellipsoid. Given a cash flow stream $X_1, X_2, \ldots X_p$, the probability that this vector observation will fall inside the ellipsoid $\underline{X}'\underline{\Sigma}^{-1}\underline{X}=\chi^2_p$ is $1-\alpha$, where $\alpha$ is the probability level of interest. For example, in the bivariate case, let $\alpha = 0.10$, then the 90 percent ellipse (that ellipse inside or on which 90 percent of the population lies) is specified by evaluating

$$\underline{X}'\underline{\Sigma}^{-1}\underline{X} = 4.605 \qquad (28)$$

where 4.605 is the value of a chi-square variate with 2 degrees of freedom.[33]

Since the shape and orientation of the isovariance ellipse is determined by the $\Sigma$ matrix, different variance-covariance structures may be compared to the volumes of the ellipses or ellipsoids. Suppose the analyst has two, two-period projects to evaluate in terms of return and risk. Let the NPV be a proper measure of return and let the two projects' variance-covariance matrices be

$$\Sigma_1 = \begin{bmatrix} 25 & 15 \\ 15 & 25 \end{bmatrix} \text{ and } \Sigma_2 \begin{bmatrix} 25 & 0 \\ 0 & 25 \end{bmatrix} \tag{29}$$

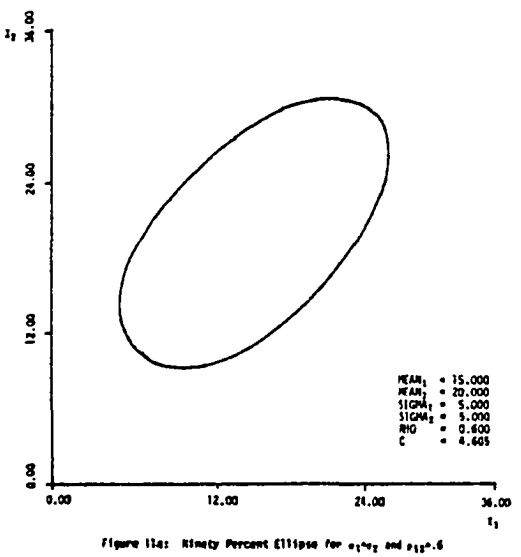



Figure 11a: Ninety Percent Ellipse for $\sigma_1=\sigma_2$ and $\rho_{12}=.6$

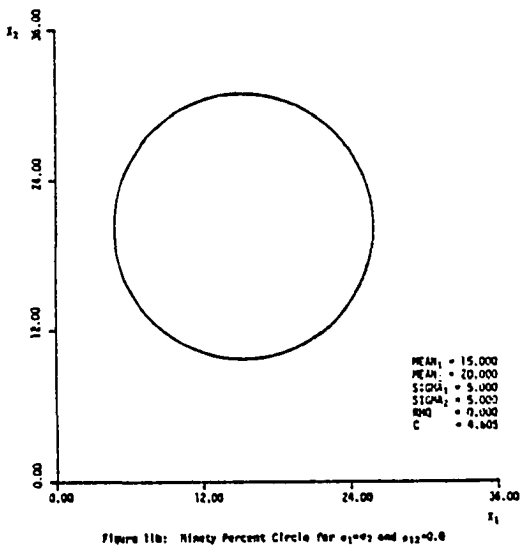



Figure 11b: Ninety Percent Circle for $\sigma_1=\sigma_2$ and $\rho_{12}=0.0$

[33]Tatsuoka, M. M. (1971) pp. 65-73.

Project 1 has $\sigma_1 = \sigma_2 = 5$ and $\rho_{12} = 0.6$, while project 2 has $\sigma_1 = \sigma_2 = 5$ with independence between periods. The 90 percent isovariance ellipses for each structure are shown above in Figure 11. Note that the major axis for $\Sigma_1$ lies along the 45° angle with the horizontal axis while $\Sigma_2$ plots as a circle since $\rho_{12} = 0$. One way to directly compare these two structures is to compute and evaluate the area of the ellipse/circle. (In three or more dimensions rather than two, volumes or hypervolumes are compared). From Anderson[34] the volume of the isovariance ellipsoid generated from the exponent of the multivariate normal distribution is

$$\text{Volume} = C(p)|\Sigma|^{\frac{1}{2}}[\chi^2_p(\alpha)]^{p/2}/p \tag{30}$$

where $C(p) = \frac{2\pi^{p/2}}{\Gamma(p/2)}$ = the surface area of a sphere of unit radius in p dimensions.

In the above example with p = 2, the area is easily computed as

$$\text{Area/Volume} = \frac{2\pi}{\Gamma(1)}|\Sigma|^{\frac{1}{2}}[\chi^2_2(\alpha)]/2 \tag{31}$$

Evaluating Eq. 31 for the 90 percent isovariance ellipse and noting that $\Gamma(1)$ is equal to one,[35] then

$$\text{Area/Volume} = 2\pi|\Sigma|^{\frac{1}{2}}4.605/2 = \pi|\Sigma|^{\frac{1}{2}}4.601. \tag{32}$$

Then for the example projects, $V_1 = 289.0894$ and $V_2 = 361.3617$. Relating area/volume to risk measurement, project 2 would be viewed

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[34]Anderson, T. W. (1958) p. 170.

[35]Widder, D. V. (1961) *Advanced Calculus*, Englewood Cliffs, New Jersey: Prentice Hall. p. 36.

intuitively as less risky than project 1 because variances are equal for both and project 2 exhibits the smaller intertemporal correlation. Then for a given probability level, the interpretation of the volume of the isovariance ellipse is that the larger the area/volume, the smaller the risk. To consistently apply this concept, differences in timing and scale must be accommodated. In the example given, timing is not a significant factor nor is the scale, however, in most capital budgeting problems, differences in the timing and scale of the project cash flows will affect the use of the volume of the isovariance ellipsoid as an effective multivariate risk measure. The differences in the timing of the cash flows are accommodated by transforming the probability distribution to present time. To account for differences in scale, multivariate analogues to the coefficient of variation (coefficient of variation = $\sigma/\mu$) are developed. The transformation to present time and multivariate coefficients of variation are discussed in detail in Chapter IV.

The volume of the isovariance ellipsoid has been shown to give a geometric interpretation to the variance-covariance matrix. Volume of the ellipsoid is directly related to another multivariate risk measure, the Generalized Variance, which is offered as a scalar multivariate analogue to the univariate variance. The Generalized Variance is also a measure of distance or volume and is discussed in detail in a later part of this chapter.

Analysis of eigenvalues and eigenvectors provide additional insights to the relationships between the concentration ellipsoids and the variance-covariance matrix. The eigenvalues of a p x p matrix $\underline{\Sigma}$ are the

solution to the determinantal equation

$$|\underline{\Sigma}-\lambda\underline{I}| = 0. \tag{33}$$

The determinant is a $\rho$th-degree polynomial in $\lambda$, and thus $\underline{\Sigma}$ has just p eigenvalues. Associated with every eigenvalue $\lambda_i$ of the matrix $\underline{\Sigma}$ is an eigenvector $\underline{x}_i$ whose elements satisfy the homogeneous system of equations

$$[\underline{\Sigma}-\lambda_i\underline{I}]\underline{x}_i = 0.$$

By the definition of the eigenvalue, the determinant of the system vanishes, and a nontrivial solution $\underline{x}_i$ always exists.[36]

Eigenvalues and eigenvectors are used in many multivariate statistical procedures. The purpose of this brief introduction is to gain additional insights to the relationships between the concentration ellipsoids of the multivariate normal distribution and the variance-covariance matrix.

If the eigenvalues of $\underline{\Sigma}$ are all different, then the corresponding eigenvectors are all orthogonal. In this case, the positions of the axes of the isodensity ellipsoid are uniquely specified by p mutually perpendicular axes. The largest eigenvalue identifies the first principal axis of a particular ellipsoid. The coordinates specifying the principal axis are proportional to the elements of the eigenvector of $\underline{\Sigma}$ that correspond to the largest eigenvalue. The squared length of the principal axis for a given C is equal to $4\lambda_1 C$, where $\lambda_1$ is the largest eigenvalue of $\underline{\Sigma}$. The next longest axis of the ellipsoid has an orientation given by the elements of the vector of the second largest

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[36]Morrison, D. F. (1967) pp. 64-65.

eigenvalue. This process continues until the equations of the p new axes have been determined. If two successive eigenvalues are equal, the ellipsoid is circular through the plane generated by the corresponding eigenvectors and the position of the axes in the circle are not unique.[37]

The singular - or degenerate - multivariate normal distribution may be interpreted geometrically. The degenerate multivariate normal distribution is identified when one or more of the eigenvalues of $\underline{\Sigma}$ are equal to zero. In addition, if any eigenvalues are negative, the variance-covariance matrix is not positive semi-definite and the matrix $\underline{\Sigma}$ is not a feasible variance-covariance matrix. Anderson[38] describes the degenerate multivariate normal distribution as

> A singular distribution is a distribution in p-space which is concentrated on a lower dimensional set; that is, the probability associated with any set not intersecting the given set is zero. In the case of the singular normal distribution the mass is concentrated on a given linear set that is, the intersection as a number of (p-1)-dimensional hyperplanes.

Viewing the isovariance ellipsoid for the trivariate case, the variance-covariance structure is depicted by the bivariate ellipse or by the univariate density depending on the degree of singularity. Degenerate multivariate normal distributions occur in capital-budgeting problems when the future cash flows are perfectly correlated. Identification of these degenerate cases are important because of their effect on scalar multivariate risk measures.

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[37]Morrison, D. F. (1967) pp. 87-88, Giri, N. C. (1977) pp. 59-61.

[38]Anderson, T. W. (1958) p. 25.

Need for scalar risk measures

Financial decision-makers are often faced with alternative investment opportunities that must be evaluated. In many situations, the analyst will have to choose between mutually exclusive alternatives. In situations where capital is rationed, the analyst is faced with competing uses for the capital investment dollars. In either case, project selection requires the ranking of alternative investment choices in terms of some measure of "attractiveness." Traditional finance theory, which assumes risk-aversion, equates "more attractive" with more return and less risk. The objective of this study is to present and evaluate alternative risk measures that can be used in the ranking process. Ideally, there is a need to develop ranking and selection criteria that are in some sense "optimal". Optimality implies that the criterion chosen:

1. Is reasonable and has intuitive appeal to the analyst;
2. Uses all available information;
3. Can be applied across a wide range of problems; and,
4. Is theoretically supportable.

Basic assumptions in this study were: 1) Return and risk are appropriate decision criteria; 2) Capital-budgeting cash flow streams can be modeled with the multivariate normal distribution; and, 3) Timing of cash flows is an improtant consideration in the decision. Given these assumptions, appropriate ordering and selection techniques use the estimated mean vector ($\hat{\mu}$) of the cash flows, the estimated variance-covariance matrix ($\hat{\Sigma}$), and the diagonal matrix of discount factors ($\underline{C}$) as the bases for ordering the unknown populations in terms of return and risk.

Because of the difficulty in directly comparing vectors and matrices, the mean vector and variance-covariance matrix are represented by scalar variables that maintain the information content of the respective vector and matrix. In the case of the mean vector, the sum of the discounted means

$$\left( \sum_{i=1}^{N} C_i \hat{\mu}_i \right),$$

the Net Present Value, has been previously discussed in detail and is readily accepted in the finance literature as an appropriate measure of project return. The remaining part of this chapter presents alternative ways of representing the variance-covariance matrix (risk) as a scalar. In a previous section of this study, the isovariance ellipse/ellipsoid has been shown to be the geometric representation of the variance-covariance matrix of a multivariate normal distribution. The area/volume of this ellipse/ellipsoid for a particular probability level has been shown to be a scalar measure of the dispersion structure. Two additional scalar risk measures are presented and evaluated: the familiar variance of the net present value and the determinant of the variance-covariance matrix (Wilk's Generalized Variance). The Generalized Variance (GV) is shown to be a natural multivariate generalization of the univariate variance. Also GV is shown to be a measure that is consistent with and proportional to the volume of the isovariance ellipse that portrays the variance-covariance structure while being a more simple measure to compute, interpret, and evaluate.

## Discounted Variance-Covariance Matrix

Throughout this study, probabilistic approaches to project evaluation have been emphasized. The review of finance literature in Chapter II identifies many alternative methods of accommodating risk analysis in the capital-budgeting decision. One such technique, the probability distribution of net present values, is discussed in detail because this method explicitly considers the timing and the risk of the cash flow stream.

Assuming normally distributed cash flows, Hillier[39] developed discounted risk measures, the variance and the standard deviation of the net present value. His initial work concentrated on three special cases: 1) the cash flows are intertemporally independent; 2) the cash flows are perfectly correlated; and 3) an intermediate case where some flows are independent while other flows are perfectly correlated. Extending this initial work, Wagle[40] and Hillier[41], in a later monograph, developed the more general case where the cash flow streams are less than perfectly correlated. Here, the variance of the net present value V(NPV) is

$$V(NPV) = \sum_{t=0}^{N} \frac{\sigma_t^2}{(1+i)^{2t}} + 2 \sum_{t \neq t'}^{N} \frac{\sigma_{t,t'}}{(1+i)^{t+t'}} \qquad (34)$$

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[39]Hillier, F. S. (1963) pp. 443-457.

[40]Wagle, B. (1967) pp. 13-33.

[41]Hillier, F. S. (1969) *The Evaluation of Risky Interrelated Investments*. Amsterdam: North-Holland Pub. Co.

where

$\sigma_t^2$ = variance of the cash flow in period t,

$\sigma_{t,t'}$ = covariance between the cash flows in periods t and t',

i = risk-free rate of interest.

The variance of the net present value may be presented more compactly and is easier to interpret when matrix notation is used. Given a p x p estimated variance-covariance matrix ($\hat{\underline{\Sigma}}$) and a p x 1 vector of present value factors ($\underline{C}$) then the variance of the net present value may be expressed compactly as

$$V(NPV) = \underline{C}'\hat{\underline{\Sigma}}\underline{C}. \tag{35}$$

Using vector/matrix multiplication, the p x p variance-covariance matrix is pre-multiplied by a 1 x p row vector and post-multiplied by a p x 1 column vector to arrive at the scalar variance of the net present value. For example, a project of p cash flows

$$V(NPV) = \left[1/(1+i)^1/(1+i)^2 \ldots 1/(1+i)^p\right] \begin{bmatrix} \sigma_{11} & \sigma_{12} \ldots & \sigma_{1p} \\ \sigma_{21} & \sigma_{22} & \vdots \\ \vdots & & \vdots \\ \sigma_{p1} & \ldots & \sigma_{pp} \end{bmatrix} \begin{bmatrix} 1/(1+i) \\ 1/(1+i)^2 \\ \vdots \\ 1/(1+i)^p \end{bmatrix} . \tag{36}$$

Discounting the variance-covariance matrix results in a scalar risk measure that reflects the variability, the interperiod correlation, and the timing of the cash flows. An alternate matrix formulation of the variance of the net present value lends considerable insight into the interpretation and evaluation of the V(NPV) as an appropriate scalar multivariate risk measure.

The alternate formulation relates to the vector of discount rates used in Eqs. 35 and 36. Rather than presenting the discount rates as a p x 1 vector, discount factors may be organized into a p x p diagonal matrix,

$$\underline{C} = \begin{bmatrix} 1/(1+i) & & & \\ & 1/(1+i)^2 & & \\ & & \ddots & \\ & & & 1/(1+i)^p \end{bmatrix} . \qquad (37)$$

Then, the transformation of the variance-covariance matrix to present time becomes the discounted variance-covariance matrix (DVC) where

$$\underline{DVC} = \underline{C}\,\underline{\hat{\Sigma}}\,\underline{C} \qquad (38)$$

More specifically,

$$DVC = \begin{bmatrix} 1/(1+i) & & & \\ & 1/(1+i)^2 & & \\ & & \ddots & \\ & & & 1/(1+i)^p \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1p} \\ \sigma_{21} & \sigma_{22} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{p1} & \cdots & \cdots & \sigma_{pp} \end{bmatrix} \begin{bmatrix} 1/(1+i) & & & \\ & 1/(1+i)^2 & & \\ & & \ddots & \\ & & & 1/(1+i)^p \end{bmatrix} . \qquad (39)$$

After the matrix multiplication,

$$DVC = \begin{bmatrix} \sigma_{11}/(1+i)^2 & \sigma_{12}/(1+i)^3 & \cdots & \sigma_{1p}/(1+i)^{p+1} \\ \sigma_{21}/(1+i)^3 & \sigma_{22}/(1+i)^4 & & \vdots \\ \vdots & & \ddots & \vdots \\ \sigma_{p1}/(1+i)^{p+1} & \cdots & \cdots & \sigma_{pp}/(1+i)^{2p} \end{bmatrix} . \qquad (40)$$

Referring to Eq. 40, the V(NPV) is determined by summing all elements of the discounted variance-covariance matrix. Algebraically, let $a_{ij}$ represent the element of the ith row and jth column, then

$$V(NPV) = \sum_{i,j=1}^{p} a_{ij} \tag{41}$$

Using the DVC formulation, Eq. 40, all variance elements are discounted at the power of 2t while the exponent of the discount rate for covariance elements is raised to the $t+t'$ power; where t and t' represent the subscripts of a particular covariance element of the variance-covariance matrix. For example, the discount factor for the covariance between the cash flows in periods 3 and 5 is $1/(1+i)^8$.

With independent cash flows, the V(NPV) is the sum of the diagonal elements of the discounted variance-covariance matrix. In this special case, the V(NPV) is equal to the trace of the DVC matrix. Appealing properties of the trace are that: 1.) the trace is a scalar representation of a matrix; and 2.) the trace is invariant when the axes of the distribution are transformed.

Geometric interpretation of the discounted variance-covariance matrix

The discounted variance-covariance matrix results from the linear combination of the original variance-covariance structure and the matrix of discount factors. Assuming multivariate normality and recognizing the invariance property of linear combinations of normal variates, this transformation of the cash flow stream to present time preserves the relative return and risk structure to allow comparisons between alternative capital budgeting projects. The impact of this

transformation to present time can be readily interpreted geometrically using the isovariance ellipse in the bivariate case. When the original cash flows are distributed $N(\hat{\underline{\mu}},\hat{\underline{\Sigma}})$ then the present time transformed flows are distributed $N(C\hat{\underline{\mu}}, C\hat{\underline{\Sigma}}C)$. The effect of this transformation is to adjust the centroid of the isovariance ellipse to present values and to compress the size of the ellipse while rotating the axes. Figure 12a below shows the 90 percent ellipse for a sample project with $\hat{\mu}_1=15$, $\hat{\mu}_2=20$, $\hat{\sigma}_1=\hat{\sigma}_2=5$, and $\rho_{12}=.6$. Figure 12b shows the effects of transforming the bivariate normal cash flows to present time using a 10 percent risk-free rate.

Discounting reduces the size of the isovariance ellipse and also rotates the angle that the major axis makes with the $X_1$-axis. This axis rotation occurs because the slope of the major-axis is determined by the variance-covariance structure. Viewing the original pre-transformed standard deviations $\hat{\sigma}_1=\hat{\sigma}_2=5$, the major-axis makes a $45^\circ$ angle with the $X_1$-axis. After discounting, the transformed variance-covariance matrix is

$$DVC = \begin{bmatrix} 20.66 & 11.27 \\ 11.27 & 17.08 \end{bmatrix} \tag{42}$$

Transforming the cash flow stream to present time results in $\hat{\sigma}_1 \neq \hat{\sigma}_2$. With $\sigma_1 > \sigma_2$, the angle between the major axis of the ellipse and the $X_1$-axis is

$$\text{Angle } \theta = \tfrac{1}{2} \text{ Arctan } \frac{2\rho\hat{\sigma}_1\hat{\sigma}_2}{\hat{\sigma}_1^2 - \hat{\sigma}_2^2} \quad . \tag{43}$$

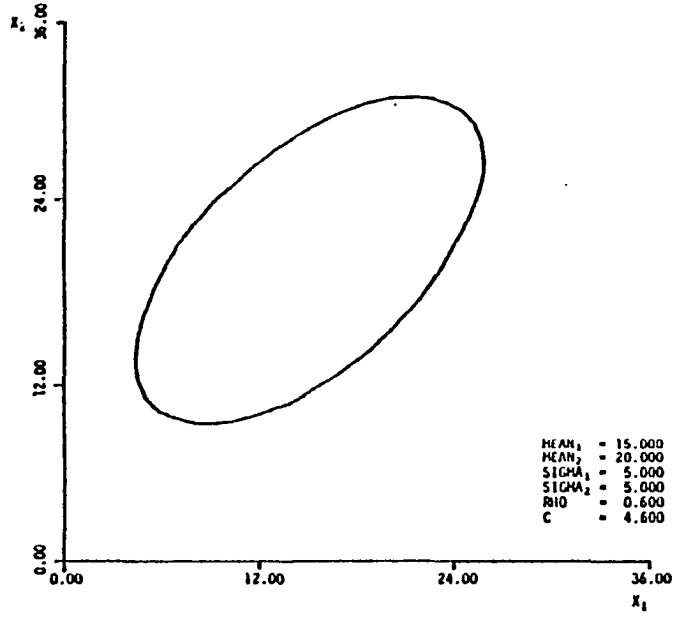



Figure 12a: Ninety Percent Ellipse Before Discounting

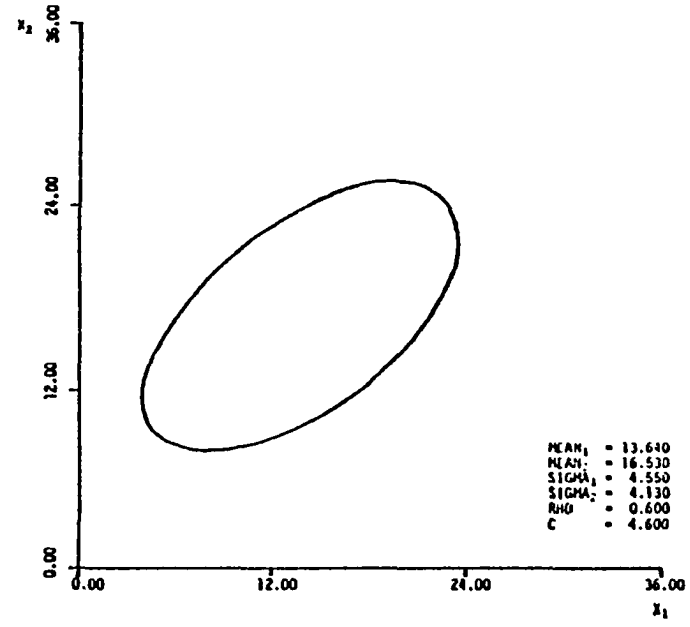



Figure 12b: Ninety Percent Ellipse After Discounting

For the example problem, discounting at a 10 percent risk-free rate causes the major-axis of the isovariance ellipse to rotate clockwise to the point where angle $\theta$ is approximately equal to $40.5^{\circ}$. When the variances are not equal, the slope of the major axis of the isovariance ellipse is determined by three factors: 1) the difference between the variances, 2) the size of the discount rates, and 3) the degree of autocorrelation. To better understand the geometric implications of discounting the variance-covariance matrix and the impact of autocorrelated cash flows, the effects of each factor may best be identified by allowing one factor to vary and holding the other factors constant. Holding the degree of autocorrelation constant at $\rho_{12}=.6$ and keeping $\hat{\sigma}_1=\hat{\sigma}_2=5$, then the effect of increasing the discount rate is to cause the major-axis of the isovariance ellipse to rotate clockwise from $45^{\circ}$ when the discount rate is zero to approximately $34.7^{\circ}$ when the discount rate is 25 percent. Higher discount rates will result in greater rotation. In terms of the bivariate normal hill mass, transforming the cash flows to present time twists the hill mass. Higher discount rates result in a larger twist.

The slope of the major axis of the isovariance ellipse is also affected by different autocorrelation patterns. Holding the discount rate constant at 10 percent and varying the correlation from zero to plus one, the slope increases; i.e., angle $\theta$ increases as the slope rotates counter-clockwise in response to higher correlation. For independent cash flows, the major-axis is not unique (independent cash flows result in an isovariance circle in the bivariate case), therefore, angle $\theta$ for $\rho_{12}=0$ is not defined. However, increasing the

autocorrelation from +.1 to +1.0 results in angle $\theta$ increasing from approximately $23.2^{\circ}$ for $\rho$=.1 to approximately $42.8^{\circ}$ for $\rho$=1.0.

In addition to the axis rotation in response to discounting, the impact of the time transformation may be observed by comparing the area of the isovariance ellipses. Equation 31, p. 118, computes the area/volume of the isovariance ellipse/ellipsoid. Using the same example cash flow structure, the area of the 90 percent non-discounted isovariance ellipse is 289.09; while the area of the time transformed ellipse for a 10 percent discount rate is 217.23. Comparing the area of the ellipses shows that discounting the variance-covariance matrix compresses the isovariance ellipse. The higher the discount rate, the smaller is the area of the ellipse. In terms of the bivariate normal density surface, discounting has been shown to twist and compress the bivariate surface. The higher the discount rate the greater is the distortion.

Variance of the net present value

The variance of the net present value, V(NPV), has wide acceptance in the finance literature as a scalar risk measure that incorporates both the time value of money and autocorrelated cash flows. Traditional development of this risk measure emphasized the application of the mathematical expectation operator as the primary justification for its use. Effects of the discounting process when the cash flows exhibit interperiod correlations was little discussed.[42] In a previous section of this study, discounting the variance-covariance matrix has

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[42]Haley, C. W. and Schall, L. D. (1973) *The Theory of Financial Decisions*. New York: McGraw-Hill, Inc. pp. 81-87.

been given a geometrical meaning. Viewing the future cash flows from a multivariate normal perspective, discounting transforms the iso-density ellipse/ellipsoid by compressing the area/volume and by rotating the axes. This geometric view allows for the explicit analysis of the effects of different discount rates on the probability distribution of the future cash flows. The purpose of this section of the study is to evaluate an extension of the discounting process, i.e., the scalar risk measure derived from the discounted variance-covariance matrix, the variance of the net present value, V(NPV).

Using the discounted variance-covariance matrix (DVC), Eq. 40, p.126, as a point of departure; V(NPV) has been shown to be the sum of all the elements of the DVC matrix (Eq. 41, p.127). Continuing the geometric approach to capital budgeting decision making, two critical questions need to be answered: What is the rationale for the use of the V(NPV) as a scalar multivariate risk measure? What is the geometric interpretation of the addition of the variance-covariance elements that result in the V(NPV)?

Boundary cases-independence and perfect correlation. As a starting point for further analysis, the boundary cases of intertemporal independent cash flows and cash flows with correlation +1 have been often studied. Due to their mathematical tractability, these two cases have been widely reviewed in the literature.

Bussey[43] defines these boundary cases as

> Cash flow increments from a project are said to be *completely independent* if there is no causative or consequential relationship

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[43]Bussey, L. E. (1978) *The Economic Analysis of Industrial Projects*. Englewood Cliffs, New Jersey: Prentice Hall. p. 382.

> between any two cash flow increments in the cash flow stream. That is, if any given cash flow increment can be determined solely from causative events occurring within that period and if these events bear no relationship to or dependency on events in prior or succeeding periods, then the cash flow increments are said to be independent among periods. . . . *Complete dependence* between cash flow increments exists if there is a one-to-one relationship among events in succeeding periods.

A close look at these definitions of the boundary capital-budgeting cases leads to the realization that analysis based on these cases is really an oversimplification of the capital-budgeting problem. Real world problems do not fit neatly into either case. Project returns are related, but not on a strict one-to-one basis. If imperfect correlation is the norm, then a substantial part of the rationale for studying these boundary cases is to develop analytical techniques that may be applied to the more interesting intermediate imperfectly correlated cases.

The basis for the use of the variance of the net present value, V(NPV), as a measure of risk relates to the behavior of this scalar risk measure when the cash flow stream is intertemporally independent. Cash flows are statistically independent if and only if the joint distribution function of the cash flows $X_1, X_2, \ldots X_p$ is capable of being factored into the product of the individual densities. More specifically,

$$F(x_1, x_2, \ldots x_p) = F_1(x_1) \cdot F_2(x_2) \cdot \cdots F_p(x_p). \tag{44}$$

For independent normal cash flows, the estimated dispersion matrix, $\hat{\underline{\Sigma}}$, is a diagonal matrix of variances with all off diagonal elements equal to zero,

$$\hat{\underline{\Sigma}} = \begin{bmatrix} \hat{\sigma}_{11} & & & \\ & \hat{\sigma}_{22} & & \\ & & \ddots & \\ & & & \hat{\sigma}_{pp} \end{bmatrix} . \tag{45}$$

For the two-parameter case, when the variables are independent normal, the quadratic form in the exponent of the bivariate normal density, $\underline{x}'\hat{\underline{\Sigma}}^{-1}\underline{x} = C$, identifies isovariance circles rather than elipses that are found when the variables are correlated. The area/volume of the ellipse/ellipsoid are scalar measures of the dispersion matrix. The area is at a maximum when the variables are independent. The area then gets smaller as the degree of correlation increases to the point where the area is undefined.[44]

Transforming the independent variances to present time results in a discounted variance matrix,

$$\underline{DV} = \begin{bmatrix} \hat{\sigma}_{11}/(1+i)^2 & & & \\ & \hat{\sigma}_{22}/(1+i)^4 & & \\ & & \ddots & \\ & & & \hat{\sigma}_{pp}/(1+i)^{2p} \end{bmatrix} . \tag{46}$$

Summing the diagonal elements above results in the trace of the discounted variance matrix which is the scalar risk measure, the variance of the net present value. The rationale for summing variances for the

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[44]Area is no longer a meaningful measurement for the perfectly correlated case because the isovariance ellipse collapses into a straight line for this degenerate bivariate distribution.

independent case comes from elementary statistical theory. Morrison[45] shows that the variance of the sum of *independent* random variables is merely the sum of the individual variances,

$$VAR(X_1+X_2+\ldots+X_p) = VAR(X_1)+VAR(X_2)+\ldots+VAR(X_p). \qquad (47)$$

This additivity principle of independent normal moments provides the theoretical basis for the scalar risk measure, the variance of the net present values.

The other boundary case, where the cash flow stream is perfectly positively correlated, has also received wide attention in finance literature. Hillier[46], Wagle[47], Bussey et al.[48] have shown that the variance of the net present value, V(NPV) is at a maximum when the interperiod correlations are equal to plus one. Because the correlation matrix is a matrix of all ones, the variance-covariance matrix is

$$\hat{\underline{\Sigma}} = \begin{bmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \sigma_p \end{bmatrix} \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & & \vdots \\ \vdots & & \ddots & \vdots \\ 1 & \cdots & \cdots & 1 \end{bmatrix} \begin{bmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \sigma_p \end{bmatrix} \qquad (48)$$

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[45]Morrison, D. F. (1967) p. 7.

[46]Hillier, F. S. (1963) pp. 443-457.

[47]Wagle, B. (1967) pp. 13-33.

[48]Bussey, L. E. (1978)

$$\hat{\underline{\Sigma}} = \begin{bmatrix} \sigma_1^2 & \sigma_1\sigma_2 & \cdots & \sigma_1\sigma_p \\ \sigma_2\sigma_1 & \sigma_2^2 & & \\ \vdots & & \ddots & \\ \sigma_p\sigma_1 & \cdots & & \sigma_p^2 \end{bmatrix} \quad . \tag{49}$$

Discounting this matrix results in the discounted variance-covariance matrix

$$\underline{DVC} = \begin{bmatrix} \sigma_1^2/(1+i)^2 & \sigma_1\sigma_2/(1+i)^3 & \cdots & \sigma_1\sigma_p/(1+i)^{p+1} \\ \sigma_2\sigma_1/(1+i)^3 & \sigma_2^2/(1+i)^4 & & \vdots \\ \vdots & & \ddots & \vdots \\ \sigma_p\sigma_1/(1+i)^{p+1} & \cdots & & \sigma_p^2/(1+i)^{2p} \end{bmatrix} \tag{50}$$

Comparing the DVC matrix when cash flows are perfectly positively correlated, (Eq. 50) with the DVC matrix when the cash flows are independent, (Eq. 46) shows that the variance of the net present value is at a minimum for the independent case while the V(NPV) is at a maximum when cash flows are perfectly positively correlated.

The V(NPV) may easily be computed with perfect positive correlation, however, considerable difficulty attends interpretation of this boundary case. Perfect positive correlation results in a degenerate multivariate distribution because of the singular correlation matrix. Since the multivariate density does not exist, then the geometric analysis of the isodensity ellipse and its associated area/volume is not

meaningful. The fact that the volume of the isovariance ellipse is not defined for this boundary case does not defeat use of volume as a scalar risk measure. In real capital-budgeting problems neither boundary case is reasonable. The likely situation observed in practice is that the cash flows will be intertemporally correlated but not perfectly so. The reason for the emphasis of the boundary cases in the literature has been their simplicity and their mathematical tractability.

Imperfect correlation. The central hypothesis of this study is that capital-budgeting cash flows exhibit positive correlation among time periods. Acceptance of this hypothesis then requires the development of appropriate multivariate scalar risk measures that: 1) are on sound theoretical footing; and 2) discriminate between alternative investment choices to allow the decision maker to rank and select projects based on risk and return criteria.

The expected net present value, E(NPV) and the variance of the net present value, V(NPV) identify the probability distribution of net present values when the capital-budgeting cash flows are normally distributed. In essence, the multivariate capital-budgeting problem has been transformed via the diagonal matrix of present value factors to the analysis of the univariate distribution of present values.

The assumption of normality allows the analyst to evaluate the riskiness of capital-budgeting projects either by comparison of the cumulative probability distributions for dominance or by making probability statements about specific outcomes. For example, suppose a project's distribution of net present values is characterized by E(NPV) = \$100 and V(NPV) = \$250. Using readily available tables for the normal

distribution, the probability that the NPV<$50 is 15.87 percent while the probability that the NPV<$0 is 2.28 percent. Using a series of probability statements, the analyst is able to evaluate the potential variability or risk of the capital-budgeting project. When the choice must be made from two or more projects, the coefficient of variation can be used to evaluate the relative risk per dollar return.

The ability of the analyst to describe the outcomes from a capital-budgeting project using simple probability statements is an attractive feature of the variance of the distribution of the net present values. Unfortunately, the validity of these probability statements is questionable. The suitability of this probabilistic analysis is dependent upon the availability of independent random samples from the underlying probability distribution. The estimated periodic cash flows from which the estimated expected net present values, $E(\hat{NPV})$ and the estimated variance of the net present value $V(\hat{NPV})$ are computed are not independent random samples. In fact, the various estimated cash flow streams form a sample of size one which results in overly significant probability statements.

In addition to the problem of making valid probability statements, the variance of the net present value is difficult to interpret in terms of the isodensity ellipse/ellipsoids of the multivariate normal distribution. The isodensity ellipse/ellipsoid has been shown to be an easily understood means of depicting various variance-covariance structures when the cash flow stream is modeled with the multivariate normal distribution. The area/volume of the ellipse/ellipsoid has also been shown to be a scalar measure of variability or spread in a

multivariate setting analogous to the univariate variance. In addition, area/volume is sensitive to differences in correlation structure and is amenable to the discounting mechanism. If the fact that the isovariance ellipse/ellipsoid properly depicts the variance-covariance structure can be accepted, then the evaluation of any simpler risk measure should be based on the consistency between the rankings of that measure and the rankings in terms of area/volume.

The remainder of this chapter introduces the generalized variance and shows that the generalized variance is a scalar multivariate risk measure that captures the structure of the variance-covariance matrix. Chapter IV then compares the ranking and selection capabilities of both the variance of the net present value and the generalized variance using the area/volume of the isovariance ellipse/ellipsoid as a standard.

## Generalized Variance

The requirement that intertemporally correlated cash flows in capital-budgeting decisions be evaluated leads in turn to evaluation of multivariate probability distributions. In a previous part of this study, the multivariate normal distribution has been offered as a reasonable model of the return and risks associated with capital-budgeting problems. Particular attention has been directed to the similarity between the univariate normal and multivariate normal distributions.[49] The variance-covariance matrix, $\underline{\Sigma}$, is a multivariate analogue of the variance, $\sigma^2$, of the univariate normal distribution. The need to rank

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[49]Refer to the previous discussion of the multivariate normal distribution. pp. 96-101.

and to select alternative capital budgeting projects requires the analyst to inquire beyond the variance-covariance matrix for a suitable scalar risk measure. The area/volume of the isodensity ellipse/ellipsoid has been shown to have attractive properties. Particularly appealing is the fact that the area/volume is a scalar measure that allows ranking in terms of variability. Unfortunately, when the number of periodic cash flows are large, the volume of the ellipsoid is difficult to compute.

The search for a scalar representation of the variance-covariance matrix led S. S. Wilks[50] to propose that the determinant of the variance-covariance matrix could also be viewed as a multivariate analogue of the univariate variance. Wilks called the determinant of the variance-covariance matrix (det $\Sigma$) the Generalized Variance of the multivariate normal distribution. In the following sections, det $\Sigma$ is offered as the preferred alternative scalar multivariate risk measure when the cash flow stream is modeled by the multivariate normal distribution.

Geometric interpretation of det $\Sigma$

The determinant of the variance-covariance matrix is a measure of spread or distance in a multivariate setting. The geometric interpretation of det $\Sigma$ will be developed from two perspectives: 1) in general, the determinant of any matrix will be shown to measure the area/volume of a parallelogram/parallelotope formed by p vectors in Euclidean space; 2) More specifically, the determinant of $\underline{\Sigma}$, the variance-

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[50]Wilks, S. S. (1932) "Certain Generalizations in the Analysis of Variance," *Biometrika*. Vol. 24: pp. 471-494.

covariance matrix, will be shown to be proportional to the volume squared of the isovariance ellipsoid of concentration of the random vector of expected cash flows.

Determinant as volume. The rationale for the interpretation of the determinant as a measure of volume comes from elementary linear algebra.[51] The evaluation of the area of a parallelogram is fundamental to the computation of areas in the plane. If $a_1$, $a_2$ are two linearly independent vectors in a 2-dimensional Euclidean space, $(R_2)$, the totality of all points into which some fixed point is carried by all vectors of the form $\lambda_1 a_1 + \lambda_2 a_2$ where $0 \leq \lambda_1, \lambda_2 \leq 1$, forms a parallelogram. The generalization to the p-dimensional parallelotope follows analogously. If $a_1, a_2, \ldots a_p$ are edge vectors of $R_p$, then the totality of all points into which some fixed point of $R_p$ is carried by any vector of the form $\lambda_1 a_1 + \lambda_2 a_2 + \ldots \lambda_p a_p$ where $0 \leq \lambda_i \leq 1 (i=1,2,\ldots p)$ is called a p-dimensional parallelotope.

To show that the determinant measures volumes, the essential properties of any measure of volume are identified. The measure of volume must be independent of its origin. Area and volume are a function of the edge vectors alone regardless of the location of the parallelotope in space. Therefore, the volume of a parallelotope having edge vectors $a_1, a_2 \ldots a_p$ is defined by $V(a_1, a_2, \ldots a_p)$. The unit vectors are defined in Euclidean space as $e_1, e_2, \ldots e_p$. Unit vectors of length 1 are recognized. Unit volume is then $V(e_1, e_2, \ldots e_p)=1$. Areas and

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[51] Scheier, O. and Sperner, E. (1959) *Modern Algebra and Matrix Theory*. New York: Chelsea Publishing Co. pp. 63-86.

volumes are non-negative; therefore, $V(a_1, a_2,...a_p) \geq 0$. From elementary geometry, parallelograms with equal bases and altitudes have the same area. Therefore, the volume of a p-dimensional parallelotope with edge vectors $a_1, a_3,...a_p$ must be equal to the volume of the parallelotope obtained by the replacement of an $a_i$, $1 \leq i \leq p$, by a vector of the form $a_i + a_k$ where $i \neq k$, $1 \leq k \leq p$. More specifically, $V(a_1, a_2,...a_i...a_p) = V(a_1, a_2,...a_i+a_k...a_p)$. Finally, for a parallelogram with edge vectors $a_1, a_2,...a_p$, if $a_i$ is replaced by $\lambda a_i$ then the area is multiplied by $abs|\lambda|$. Once again, for a p-dimensional parallelotope $V(a_1, a_2,..., \lambda a_i,...a_p) = |\lambda| \cdot V(a_1,a_2,...a_i,...a_p)$. Given the desired properties of volume, Shreier shows that the *only* function that satisfies these properties is the *absolute value of the determinant!* The first property is satisfied in that the determinant is a function of the vectors of the matrix, the second property is satisfied because the determinant of a matrix of ones is one, the third property is satisfied in that the absolute value is non-negative, the fourth property is satisfied by the invariance property of determinants, i.e., $D(a_1, a_2,...a_p)$ remains unchanged if some $a_i$ is replaced by $a_i + a_k$, $(i \neq k)$, the final property is satisfied by the homogeneity property of determinants, i.e., $D(a_1, a_2,...a_p)$ becomes $\lambda D \cdot (a_1, a_2,...a_p)$ if any $a_i$ is replaced by $\lambda a_i$.

From the above analysis, a formal definition of the determinant can be made:

> A p-dimensional parallelotope in Euclidian $R_p$
> with edge vectors $a_1, a_2,...a_p$ has the
>
> $$\text{Volume} = |D(a_1, a_2,...a_p)| \qquad (51)$$

The volume computation for the 2-dimensional parallelogram and the 3-dimensional parallelotope can be easily illustrated both algebraically and geometrically.[52] In two dimensions let $\underline{a}_1 = (a_{11}, a_{12})$ and $\underline{a}_2 = (a_{21}, a_{22})$, then the area of the parallelogram shown in Fig. 13 below

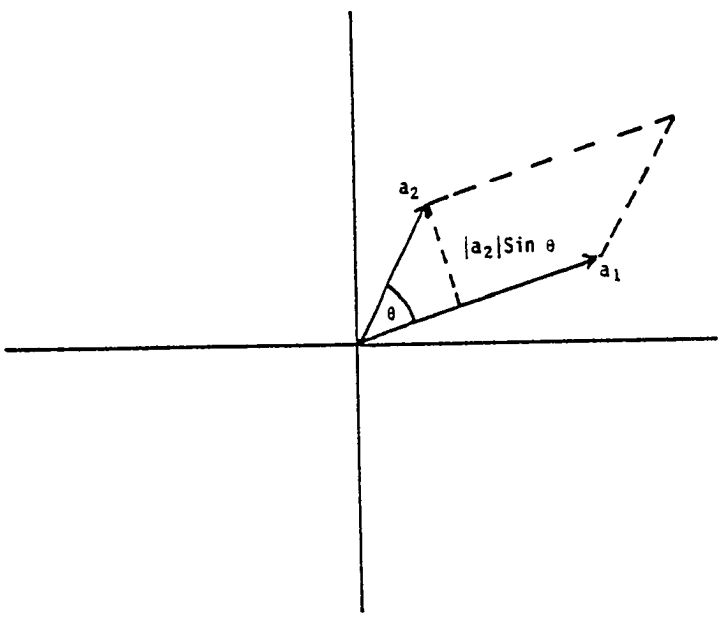



Figure 13: Two-dimensional Parallelogram

may be written as the absolute value of the determinant of the matrix with $a_i$ and $a_2$ as rows or columns. Area equals $|\det A|$ where

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}. \tag{52}$$

To show that the area equals the determinant using the fact that $\det A = \det A^T$, $(\det A)^2$ is computed.

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[52]This presentation of the geometric interpretation of the determinant follows closely that of Williamson, R. E. and Trotter, H. F. (1979) *Multivariate Mathematics*, Englewood Cliffs, New Jersey, Prentice-Hall, pp. 144-152.

$$(\det A)^2 = \det A \cdot \det A^T = \det AA^T$$

$$= \det\left[\begin{pmatrix} a_{11} & a_{22} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}\right] = \det\begin{bmatrix} a_1 \cdot a_1 & a_1 \cdot a_2 \\ a_2 \cdot a_1 & a_2 \cdot a_2 \end{bmatrix}$$

$$= |a_1|^2|a_2|^2 - (a_1 \cdot a_2)^2$$

$$= |a_1|^2|a_2|^2 - |a_1|^2|a_2|^2\text{Cos}^2\theta, \tag{53}$$

where $\theta$ is the angle between the vectors $a_1$ and $a_2$. From basic trigonometry, $1-\text{Cos}^2\theta = \text{Sin}^2\theta$,

therefore

$$(\det A)^2 = |a_1|^2|a_2|^2\text{Sin}^2\theta. \tag{54}$$

Taking the square roots of both sides, the absolute value of the determinant of the matrix A is

$$|\det A| = |a_1||a_2||\text{Sin}\theta| \tag{55}$$

Relating Equation 55 to Figure 13, Eq. 55 is the desired area: base length, $|a_1|$ times the height, $|a_2||\text{Sin}\theta|$ .

The computations illustrated above can be generalized to any number of dimensions. Any k independent vectors in the Euclidean space $R_p$, with $k \leq p$, may be considered to form all linear combinations

$$\lambda_1 a_1 + \lambda_2 a_2 + \ldots \lambda_k a_k \tag{56}$$

where $0 \leq k_i \leq 1$, the resulting set of points is an p-dimensional parallelotope. A 3-dimensional parallelotope is shown in Figure14; the faces are determined by the vector pairs $(\underline{a}_1, \underline{a}_2)$, $(\underline{a}_2, \underline{a}_3)$ and the vectors

($\underline{a}_1$, $\underline{a}_2$) determine 2-dimensional parallelograms in $R_3$.[53]

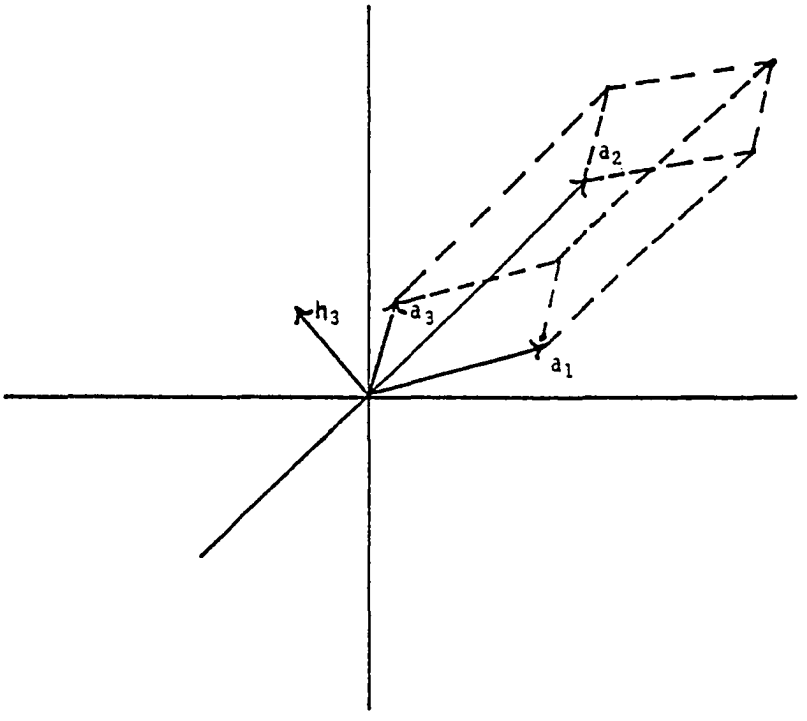



Figure 14: Three-dimensional Parallelotope

The volume of higher dimensional parallelotopes may be easily computed using a recursive relationship. The vector $h_j$ is determined by dropping a perpendicular from vector $\underline{a}_j$ to the (j-1)-dimensional subspace spanned by $a_1$, $a_2$, ... $a_{j-1}$. Then the length $|h_j|$ is the height of the parallelotope determined by $a_1$, $a_2$, ... $a_j$, relative to the base determined by $a_1$, $a_2$, ... $a_{j-1}$. Using the following notation, the volume of k-dimensional parallelotope may be easily defined:

$V_1 = |a_1|$, one-dimensional volume, or length of $a_1$.

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[53]Choosing two vectors in $R_2$ leads to the special case illustrated in Figure 4.

$V_2 = V_1|H_2|$, two-dimensional volume, or area, of the parallelogram determined by $a_1$ and $a_2$.

$V_3 = V_2|h_3|$, three-dimensional volume of the parallelotope determined by $a_1$, $a_2$, and $a_3$.

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$V_k = V_{k-1}|h_k|$, k-dimensional volume of the parallelotope determined by $a_1$, $a_2 \ldots a_k$.

Linking the above formulae, the volume is

$$V_k = |a_1||h_2||h_3| \ldots |h_k| \ . \tag{57}$$

More explicitly, the following definition of Volume is presented.

If $a_1$, $a_2$, ... $a_k$ are vectors in $R_p$, $k \leq p$, then $V_k$, the volume of the k-dimensional parallelotope determined by them, satisfies

$$V_k^2 = \det (a_i \cdot a_j),\ i,\ j,\ = 1 \ldots k \ . \tag{58}$$

For k = p, then,

$$V_p = |\det (a_i \cdot a_j)|^{\frac{1}{2}} = |\det a| \ , \tag{59}$$

where A is the p x p matrix with $a_1$, $a_2$, ... $a_p$ as columns.

Generalized variance and the isovariance ellipsoid. The variance-covariance matrix may be evaluated in terms of the volume of the isovariance ellipsoid. The generalized variance, det $\Sigma$ is also a measure of volume. In a previous section of this study, the volume of the isovariance ellipsoid has been shown to be a sensitive scalar

multivariate risk measure. One shortcoming that has been noted is that the volume of the ellipsoid becomes tedious to compute when the number of cash flow parameters is large. In this section, det $\Sigma$ will be shown to be proportional to the volume of the ellipsoid and easy to compute.

Given multivariate normally distributed cash flows, the volume of the isovariance ellipsoid is

$$\text{Volume} = \frac{2\pi^{p/2}}{\Gamma(p/2)} (\det\Sigma)^{\frac{1}{2}}[\chi_p^2(\alpha)]^{p/2}/p \ .^{54} \tag{60}$$

Once the size of the problem is determined (number of parameters or p in the above equation) and the level of significance is identified ($\alpha$, above) all terms other than det $\Sigma$ are constants. For example, with p=2 and $\alpha$=.1, then the volume of the resulting isovariance ellipse is

$$\text{Volume} = (\det \Sigma)^{\frac{1}{2}}4.61\pi \tag{61}$$

where 4.61 = chi square table value with 2 degrees of freedom and a probability level of ten percent.

The determinant of the variance-covariance matrix measures the area of a parallelotope formed by the edge vectors of the variance-covariance matrix. For the two parameter case, det $\Sigma$ measures the area of the parallelogram formed by using either the row or column vectors of $\Sigma$. Projecting these vectors in Euclidean space, the area of the parallelogram formed by vectors $\underline{a}_1 = (a_{11}, a_{12})$ and $\underline{a}_1 = (a_{21}, a_{22})$ is simply the determinant of the resulting matrix,

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[54]See p. 118 for development of the volume of isovariance ellipsoids.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} .$$

The relationship between the volume of the parallelotope as measured by det $\Sigma$ and the volume of the isovariance ellipsoid, is

$$\text{Ratio of Volumes} = \frac{\det \Sigma}{\text{Volume of Isovariance Ellipsoid}} \tag{62}$$

$$= \frac{\det \Sigma}{\frac{2\pi^{p/2}}{\Gamma(p/2)}(\det \Sigma)^{\frac{1}{2}}[\chi_p^2(\alpha)]^{p/2}/p} . \tag{63}$$

A form of det $\Sigma$ occurs in both the numerator and the denominator of the above ratio. After factoring det $\Sigma$ into the product of the square root of det $\Sigma$, i.e.,

$$\det \Sigma = (\det \Sigma)^{\frac{1}{2}} \cdot (\det \Sigma)^{\frac{1}{2}}, \tag{64}$$

then the ratio of volumes can be expressed generally as

$$\text{Ratio of volumes} = \frac{(\det \Sigma)^{\frac{1}{2}}}{2\pi^{p/2}/\Gamma(p/2) \cdot [\chi_p^2(\alpha)]^{p/2}/p} . \tag{65}$$

Given the number of parameters and the significance level, the ratio of volumes is simply the ratio of square root of the determinant and a constant.

For $p = 2$ and $\alpha = .10$, the ratio of volumes is $(\det \Sigma)^{\frac{1}{2}}/14.48274$. As the number of parameters in the problem increases, the

ratio of the generalized variance to the volume of the isovariance ellipsoid is

$$\text{For } p = (2, 3, 4\ldots) \quad \frac{\text{Ratio of}}{\text{Volumes}} = \frac{(\det \Sigma)^{\frac{1}{2}}}{14.48274}, \frac{(\det \Sigma)^{\frac{1}{2}}}{65.44964}, \frac{(\det \Sigma)^{\frac{1}{2}}}{296.69570} \ldots.$$

As the size of the problem increases (number of periodic cash flows), the ratio of the two volumes tends to move toward zero. Thus, an inverse relationship between the ratio of volumes and the number of parameters results.

Holding the number of parameters constant, one can view the effect of the probability level on the ratio of volumes. As $\alpha$ decreases from 0.1 to .001, the chi-square table value increases from 4.61 to 13.82 for $p = 2$. The effect of raising the level of significance is to increase the value of the denominator of the ratio of volumes, resulting in a lower ratio for a given p parameter. The ratio of volume as the significance level changes is also an inverse relationship. As $\alpha$ increases, the ratio of volumes decreases.

Thus, the volume of the isovariance ellipsoid has been advanced as an appropriate scalar multivariate risk measure and det $\Sigma$ is proportional to the volume of the ellipsoids of concentration of the random vector of cash flows. This functional relationship may be used to support the use of det $\Sigma$ as a measure of total risk when the capital-budgeting cash flows are modeled using the multivariate normal distribution. Det $\Sigma$ is an easy to compute and understand scalar multivariate analogue to the univariate variarce. Furthermore, Chapter IV shows that det $\Sigma$ consistently ranks alternative capital budgeting projects in terms of total risk when compared to the volume of the isovariance ellipsoid.

Statistical Properties of det Σ

The determinant of the variance-covariance matrix plays an important part in multivariate statistical theory. In this section, some of the uses of det Σ will be outlined to show the importance of det Σ as a multivariate analogue to the univariate variance. Wilks,[55] in his initial paper which suggested det Σ, was interested in developing a multivariate generalization of the univariate variance. Following the work of Wishart who developed the distribution of the variance-covariance matrix,[56] Wilks showed that the distribution of the generalized variance for n = 1 results in the well-known distribution of the variance in samples of a single variate.[57] In the same paper, Wilks also recognized that if the nth root of the generalized variance is taken then the resulting measure would be the geometric mean of the variances of the n variates multiplied by the nth root of the correlations among the n variables.[58]

The det Σ plays an important role in the multivariate extension of the F-test used in many multivariate significance tests. Wilks' Λ uses the ratio of the determinants of the within-groups sum of squares

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[55]Wilks, S. S. (1932) pp. 471-494.

[56]Anderson, T. W. (1958) pp. 154-163 for a detailed discussion of the Wishart distribution.

[57]Wilks, S. S. (1932) p. 477.

[58]A preference for the geometric mean versus the arithmetic mean has received considerable attention in the finance literature. The reason for this preference is illustrated in Francis, J. C. and Archer, S. H. (1979) *Portfolio Analysis. 2nd Ed.* Englewood Cliffs, New Jersey: Prentice-Hall; pp. 12-13, where they show cases where the simpler arithmetic mean can result in misleading and often nonsensical results.

and cross-products matrix and the total sample SSCP matrix. In the univariate case, Wilks' $\Lambda$ is inversely proportional to the F-ratio. The effect of this inverse relationship is that F increases as the variation increases; whereas, $\Lambda$ decreases with greater variation. This relationship holds as well for the multivariate case.

Considerable work on the properties of det $\Sigma$ and on its use in practical classification problems has been reported in the statistics literature. Anderson[59] derived the distribution of the sample generalized variance of a multivariate normal distribution to be the same as the distribution of the population generalized variance times the product of p independent factors, the distribution of the ith factor being the $\chi^2$ distribution with (N-i) degrees of freedom. Bagai,[60] Mathai and Rathie,[61] and Mathai[62] developed exact distributions of the sample generalized variance and the ratio of two independent sample generalized variances. Goodman[63] extended the distribution-theory work to include the case where the multivariate random variables are complex, i.e., with real and imaginary parts to the random variables.

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[59]Anderson, T. W. (1958) pp. 170-173.

[60]Bagai, O. P. (1965) "The Distribution of the Generalized Variance," *Annals of Mathematical Statistics*. Vol. 36: pp. 120-130.

[61]Mathai, A. M. and Rathie, P. M. (1967) "The Exact Distribution of Wilks' Generalized Variance in the Non-Central Linear Case," *Sankhya*. Vol. 29: pp. 45-60.

[62]Mathai, A. M. (1972) "The Exact Distribution of Three Multivariate Statistics Associated with Wilks' Concept of Generalized Variance," *Sankhya*. Vol. 34: pp. 161-170.

[63]Goodman, N. R. (1963) "The Distribution of the Determinant of a Complex Wishart Distributed Matrix," *Annals of Mathematical Statistics*. Vol. 34: pp. 178-180.

Sokal[64] advocates the generalized variance for use in systematic work to answer questions such as "is population A more variable than population B" or "what are the overall variabilities of a series of samples?". Goodman[65] used the generalized variance to compare overall variability of different populations of maize and cotton. The reported ability to discriminate successfully between populations while using det $\Sigma$ as the measure of variability shows the usefulness of the generalized variance as a multivariate scalar measure of overall variability. Additional uses of det $\Sigma$ to rank populations have been reported in the statistics literature. These works will be addressed in Chapters IV and V.

Eigenvalues and det $\Sigma$

The generalized variance has been shown to be a multivariate analogue of the univariate variance. Analysis of the relationships between the variance-covariance matrix, the corresponding eigenvalues and eigenvectors, and the generalized variance provides additional insights and/or justification for the use of det $\Sigma$ as a multivariate scalar risk measure.

Analysis of the eigenvalues readily identifies cases where the multivariate normal distribution is degenerate, i.e., the variance-covariance matrix is singular. When one or more eigenvalues of $\Sigma$ is equal to zero, the variance-covariance matrix is singular. Tatsuoka[66] states

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[64]Sokal, R. R. (1965) "Statistical Methods in Systematics," *Biological Review*. Vol. 40: pp. 337-391.

[65]Goodman, M. M. (1968) "A Measure of Overall Variability in Populations," *Biometrics*.Vol. 24: pp. 189-192.

[66]Tatsuoka, M. M. (1971) p. 127.

that "Even though the presence of just one eigenvalue equal to zero is sufficient to assure that (Matrix) A is singular, we somehow feel that it is 'more singular' with a greater number of zero eigenvalues." Evaluating the eigenvalues shows that in the case of a degenerate multivariate normal distribution the aggregate variance of the original p variables can be accounted for by a smaller number of transformed variables.

Analysis of the eigenvalues of a variance-covariance matrix also provides insight into the behavior of the generalized variance or det $\Sigma$. The generalized variance, det $\Sigma$, has been shown to be a scalar multivariate risk measure. The eigenvalues of the variance-covariance matrix provide insights into the analysis of singular or near-singular risk structures. This analysis is derived from the fact that det $\Sigma$ is equal to the product of the eigenvalues. Therefore, if one or more eigenvalues is equal to zero, then det $\Sigma$ must be equal to zero. Tatsuoka[67] outlines the proof that

$$\prod_{i=1}^{p} \lambda_i = \det \Sigma.$$

The essentials of the proof are based on the theory of equations. By definition, the eigenvalues $\lambda_i$ are the roots of the characteristic equation

$$|\underline{\Sigma} - \underline{\lambda I}| = 0, \tag{66}$$

which is a polynomial equation of degree p in $\lambda$. If $\lambda_1, \lambda_2, \ldots \lambda_p$ are the roots of the polynomial

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[67]Tatsuoka, M. M. (1971) p. 126.

$$C_o\lambda^p + C_1\lambda^{p-1} + C_2\lambda^{p-2} + \ldots + C_p = 0, \tag{67}$$

then

$$\prod_{i=1}^{p} \lambda_i = (-1)^p(C_p/C_o), \tag{68}$$

where $C_p$ is the constant term and $C_o$ is the coefficient of $\lambda^p$ in the characteristic equation. By expansion of the characteristic determinant, $C_o = (-1)^p$ and $C_p$ can be found by setting the $\lambda$ vector equal to zero, i.e. $C_p = |\underline{\Sigma} - \underline{0I}| = \det \Sigma$. Therefore, the assertion that the product of the eigenvalues equals det $\Sigma$ follows immediately.

The generalized variance may be easily shown to be invariant to an orthoginal transformation of the original cash-flow vector. From Anderson,[68] let $\underline{V} = \underline{CX}$ represent an orthogonal transformation of a random vector $\underline{X}$. Let $E(X) = 0$ and $E(\underline{XX}') = \Sigma$. Then $E(\underline{V}) = 0$ and $E(\underline{VV}') = \underline{C\Sigma C}'$. Then the generalized variance of V is

$$\begin{aligned} \det(\underline{C\Sigma C}') &= \det \underline{C} \cdot \det \underline{\Sigma} \cdot \det \underline{C}' \\ &= \det \underline{\Sigma} \cdot \det \underline{CC}' = \det \underline{\Sigma}, \end{aligned} \tag{69}$$

which is the generalized variance of X. The ability of the analyst to decompose a complicated matrix into the product of matrices is particularly important for the analysis of risk in capital-budgeting problems. In the context of the problem at hand, the determinant of the discounted variance-covariance matrix may be easily decomposed into the product of the determinants of the discount factor diagonal matrix, the variance diagonal matrix, and the symmetric correlation matrix. This decomposition is discussed in detail in Chapter IV.

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[68]Anderson, T. W. (1958) p. 277.

Eigenvalues and the corresponding eigenvectors also provide insight into the relationship between the generalized variance, det $\Sigma$, and the isovariance ellipsoid, $X'\Sigma^{-1}X = C$, of the multivariate normal density. Det $\Sigma$ measures the volume of the parallelotope formed by the vectors of $\Sigma$. Det $\Sigma$ is invariant to an orthogonal transformation of the original variables. In addition, det $\Sigma$ is equal to the product of the eigenvalues of $\Sigma$. The above relations may be used to evaluate the principal axes of the isovariance ellipsoid. From elementary geometry, the principal axes of ellipsoids are mutually orthogonal.[69]

From Franklin[70] the geometric interpretation of the eigenvalues and eigenvectors is that the principal axes of the isovariance ellipsoid, $\underline{X}'\underline{\Sigma}^{-1}\underline{X} = \underline{C}$ are the eigenvectors of the real symmetric matrix $\Sigma$ and the squared length of the principal axis associated with the eigenvalue $\lambda_i$ is $4\lambda_i C$.

The relationship between det $\Sigma$ and the isovariance ellipse in terms of the eigenvalues of $\Sigma$ may be easily developed for the bivariate case. For ease of computation and explanation, let C = 1.0, i.e., the unit ellipse. The unit ellipse for a sample project with $\hat{\mu}_1 = 15$, $\hat{\mu}_2 = 20$, $\sigma_1 = \sigma_2 = 5$, and $\rho_{12} = .6$ can be constructed. For equal standard diviations, the major-axis makes a $45^\circ$ angle with the $X_1$-axis, while the minor-axis is perpendicular to the major-axis. To determine the length of the axes, the eigenvalues of the variance-covariance matrix are

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[69]For independent cash flow streams the isovariance ellipse/ellipsoid is a circle/sphere and the mutually orthogonal principal axes are not unique.

[70]Franklin, J. N. (1968) *Matrix Theory*. Englewood Cliffs, New Jersey: Prentice-Hall. pp. 94-98.

computed, then axis lengths equal to $\sqrt{4\lambda_i C}$ are computed. For the example project, the sample variance-covariance matrix is

$$\Sigma = \begin{bmatrix} 25 & 15 \\ 15 & 25 \end{bmatrix} . \tag{70}$$

After expansion of the characteristic determinant, the eigenvalues are the roots of the characteristic equation

$$\lambda^2 - 50\lambda + 400 = 0. \tag{71}$$

Using the quadratic formula, the roots are $\lambda_1 = 40$, $\lambda_2 = 10$. Then the squared length of the major-axis is $4(40)(1.0)=160$ and the squared length minor-axis is $4(10)(1.0) = 40$. The value of det $\Sigma$ is 400 which may easily be verified by either computing the determinant of the 2x2 variance-covariance matrix, (Eq. 70 above) or by computing the product of the eigenvalues.

The relationship between the unitary isovariance ellipse and the inscribed parallelogram formed by the principal axes of the ellipse is easily developed. Due to the orthogonal nature of the principal axes, the area of the parallelogram is four times the area of any one of the four similar right triangles. Using one half the major axis as the base and one half the minor axis as the height, the area of the resulting right triangle is:

$$\begin{aligned} A &= \tfrac{1}{2}(2\lambda_1 \cdot 2\lambda_2) \\ &= 2\lambda_1\lambda_2. \end{aligned} \tag{72}$$

The area of each of the right triangles that form the inscribed

parallelogram is equal to two times the generalized variance or the determinant of the variance-covariance matrix. Then, the area of the inscribed parallelogram is equal to eight times det $\Sigma$. Summarizing, det $\Sigma$ is directly proportional to the area of the isovariance ellipse.

Chapter III of this study has laid the groundwork for a multivariate approach to the analysis of risky capital-budgeting projects. The multivariate approach was shown to be necessary to account for the intertemporal relationships among the future cash flows. The variance-covariance matrix is a natural generalization of the univariate variance; as such, the analysis of project variability requires the analysis of the variance-covariance structure. When the cash flow stream is modeled by the multivariate normal distribution, the ellipse/ellipsoids of concentration (isovariance ellipse/ellipsoid) geometrically depicts the variance-covariance structure. The need to compare alternative capital-budgeting projects or to rank projects requires scalar measures of return and risk. The expected net present value is a well-accepted measure of project return. This study has identified and developed in detail three scalar multivariate risk measures: 1. area/volume of the isovariance ellipse/ellipsoid; 2. variance of the net present value; and, 3. generalized variance. In Chapter IV, these risk measures are evaluated using simple cash flow patterns. In addition, to account for differences in project size, multivariate analogues of the univariate coefficient of variation are presented.

# CHAPTER IV

# EVALUATION OF SCALAR MULTIVARIATE RISK MEASURES

## Introduction

Chapter III of this study has presented a multivariate approach to the analysis of risky intertemporally correlated capital-budgeting projects. Given the assumption of multivariate normally distributed cash flow streams, risk analysis requires the interpretation and comparison of variance-covariance matrices. These matrices, however, are not directly comparable and are not amenable to the ranking and selection problems associated with capital-budgeting decisions. The need to rank alternative projects requires the development of scalar measures that are in some sense "optimal". As previously stated, optimality implies that the scalar measures:

1. are reasonable, and have intuitive appeal;
2. use all available information;
3. can be applied across a wide range of problems; and,
4. are theoretically supportable.

Given these criteria, three scalar multivariate measures of total variability have been presented: 1. area/volume of the isovariance ellipse/ellipsoid; 2. variance of the net present value distribution; and 3. generalized variance.

In this chapter, capabilities of the scalar risk measures to measure project variability are evaluated and compared. Principal areas of interest are: 1. interpretation of each risk measure in terms of a simple idea of risk; 2. treatment of differences in timing and scale of the capital budgeting cash flow streams; 3. decomposition of the risk measure to enhance sensitivity analysis; and 4. analysis of limitations associated with each scalar multivariate risk measure.

## Interpretation of Scalar Risk Measures

Risk has been defined as the variability of the future cash flow stream. With such a general definition, there exists considerable controversy as to the most appropriate risk measure. In the case of the univariate normal distribution, the variance or the standard deviation can be shown to have desirable statistical properties. Depending on the nature of the data and the analysis problem at hand, other measures of variability such as the range or the mean absolute deviation may be appropriate.

When the data are multivariate normal, the variance-covariance matrix $\underline{\Sigma}$ has been shown to be the natural extension of the univariate variance. The need for scalar multivariate measures of variability has resulted in the development of three risk measures (area/volume, variance of net present value, and generalized variance) that depict different aspects of multivariate variability. In this section of the study, these measures are reconciled to reflect different aspects of variability or risk.

In the single-period (univariate) case, the area/volume of the isovariance ellipse/ellipsoid reduces to a measure of Euclidean length or distance. The variance of the net present value and the generalized variance both reduce to the univariate variance. As such, all three scalars are a measure of distance or spread.

The bivariate normal distribution is often studied because this two-dimensional case can be easily described geometrically. In addition, the bivariate results may be immediately generalized to any p-dimensional multivariate normal distribution.

The quadratic form in the exponent of multivariate normal distributions,

$$(\underline{X}-\underline{\mu})'\Sigma^{-1}(\underline{X}-\underline{\mu}) = C \qquad (1)$$

identifies isovariance ellipses in the bivariate case and isovariance ellipsoids for multivariate cases. The area/volume[1] of these ellipses/ellipsoids has been shown to be a scalar multivariate measure that depicts the variance-covariance structure. In the bivariate case the ellipses are cross-sections of the bivariate hill mass formed by planes parallel to the axes. Area is dependent upon the magnitude of the variances, the degree of correlation,[2] and the location on the bivariate hill

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[1]Area/volume $= \dfrac{2\pi^{p/2}}{\Gamma(p/2)}\left|\Sigma\right|^{1/2}\left[\chi^2_p(\alpha)\right]^{p/2}\Big/p$ .

[2]Holding the variances and C constant, area is at a maximum when $\rho_{12} = 0$. Area declines as $\rho_{12}$ increases to the point where $\rho_{12} = +1$; at this point, area is no longer meaningful because the isovariance ellipse has collapsed into a straight line.

mass where the slice is made.[3] Holding C constant, and transforming the distribution such that the centroid of the ellipse is at the origin, area is a function of the variance-covariance matrix. Viewing the special case where $\rho_{12} = 0$, the area of the resulting isovariance circle is a function of the diagonal matrix of variances. Relating area to total risk, projects with the larger area are interpreted to have the smaller variability or total risk.

The variance of the net present value (VNPV) is a weighted sum of the discounted variance-covariance elements.[4] In the special case where $\rho_{12} = 0$, VNPV is equal to the trace of matrix DVC. The trace of a matrix is also equal to the sum of the eigenvalues of a matrix. Relating the VNPV to the isovariance ellipse, the lengths of the axes of the isovariance ellipse are equal to $4\lambda_i C$, where $\lambda_i$ identify the eigenvalues of matrix DVC and C is a positive constant. In the bivariate case, VNPV = $\lambda_1+\lambda_2$. Therefore, VNPV is interpreted as being proportional to the sum of the lengths of the axes of the isovariance ellipse. With this interpretation, project rankings based on VNPV and area/volume of the ellipse/ellipsoid may differ. Although the two risk measures are related (length and direction of the axes determine the size and shape of the ellipse), the sum of the axes and area/volume measure different aspects of the variance-covariance structure. Therefore, they

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[3]The positive constant, C, in equation 1 identifies the chi-square probability level associated with a particular slice. If C = 1.0 and $\rho_{12} = 0.0$, then the resulting isovariance circle is the unit circle.

[4]VNPV = $\underline{C}'\hat{\underline{\Sigma}}\underline{C}$, where C is a vector of discount rates. VNPV is also equal to the sum of the elements of the discounted variance-covariance matrix DVC.

measure different aspects of project variability. A second interpretation of VNPV is in terms of an arithmetic mean of the variance covariance elements. VNPV is computed by adding the weighted variance-covariance elements (the weights being the appropriate discount rates). Arguments of preference for the VNPV must implicitly accept the fact that the individual components of total risk are additive.

The generalized variance (det $\Sigma$) is the determinant of the variance-covariance matrix. Det $\Sigma$ measures the area/volume of the parallelogram/parallelotope formed by either the row or column vectors of the variance-covariance matrix. As such, det $\Sigma$ is the scalar representation of the matrix $\underline{\Sigma}$. Det $\Sigma$ is also equal to the product of the eigenvalues of a matrix. As a measure of area/volume, det $\Sigma$ is directly related to the area/volume of the isovariance ellipse/ellipsoid.[5] Therefore, the area/volume of the parallelogram/parallelotope represented by det $\Sigma$ is directly proportional to the area/volume of the isovariance ellipse/ellipsoid. With this direct relationship, det $\Sigma$ and area/volume measure the same aspects of project variability and will give consistent rankings. A second interpretation of det $\Sigma$ is in terms of a geometric mean of the variance-covariance elements. Det $\Sigma$ can be computed by taking the product of the eigenvalues; taking the pth root

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[5]The relationship between det $\Sigma$ and the isovariance ellipse is that of a ratio. The ratio of volumes is

$$\frac{(\det)^{\frac{1}{2}}}{2\pi^{p/2}\,\Gamma(p/2)\cdot[\chi^2_p(\alpha)]^{p/2}/p}\ .$$

For fixed p and $\alpha$ the denominator is a constant and the ratio is the square-root of det $\Sigma$ to a constant.

of det $\Sigma$ results in the geometric mean of the variance-covariance structure.

Comparing the generalized variance and the variance of the net present value, a central fact emerges: that each measure identifies a *different* dimension of project variability. A significant aspect of this study is that the *exclusive use* of either the variance of the net present value or the generalized variance to describe multivariate variability may cause the analyst to overlook an important dimension of project risk. If multivariate variability is additive, i.e., the addition of discounted variance and covariance elements, then VNPV captures the essence of risk. If, however, multivariate variability interacts in a multiplicative fashion, then the generalized variance may be a more informative measure of risk.

An obvious area for additional research is further study of what is meant by project variability. With a clearer understanding of risk or a more specific definition of multivariate variability, a more informed choice between variance of the net present value and the generalized variance may be made. The remaining sections of this chapter will evaluate the merits of these scalar multivariate risk measures with respect to difficulties associated with capital-budgeting applications.

## Time Adjustment

The need to adjust for the timing of capital-budgeting cash flows has been documented in the finance literature. The time dimension associated with capital-budgeting analysis must be explicitly included to develop meaningful scalar multivariate risk measures.

To accomodate the effects of time, the probability distribution of project cash flows is transformed to a present-time equivalent. The geometric interpretation of the effects of discounting for the bivariate normal assumption was discussed in detail in Chapter III of this study.[6] Transforming the cash flow stream to present-time creates a linear combination of the original cash flow structure. The resulting discounted variance-covariance matrix, DVC may be depicted as the matrix product

$$\underline{DVC} = \underline{C}\hat{\underline{\Sigma}}\underline{C} \tag{2}$$

where $\underline{C}$ is a p x p diagonal matrix of discount factors and

$\hat{\underline{\Sigma}}$ is a p x p positive definite estimated variance-covariance matrix.

The effects of this transformation to present-time on the variance of the net present value and the isovariance ellipse/ellipsoid were discussed in Chapter III.[6] The generalized variance is easily shown to be invariant to a linear transformation. The determinant of DVC matrix may be evaluated in terms of the product of individual matrices. From elementary matrix algebra, the determinant of a matrix product is simply the product of the determinants of the individual matrices. The discounted generalized variance is

$$\begin{aligned} \det \underline{DVC} &= \det \underline{C} \cdot \det \hat{\underline{\Sigma}} \cdot \det C \\ &= (\det \underline{C})^2 \cdot \det \hat{\underline{\Sigma}}. \end{aligned} \tag{3}$$

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[6]Chapter III, pp. 124 to 138. Discounting is shown to reduce the size of the isovariance ellipse and rotate the axes. In terms of the bivariate normal density surface, discounting twists and compresses the bivariate surface.

The determinant of discount factors, det $\underline{C}$ is less than one; therefore, the determinant of the discounted variance-covariance matrix is less than the determinant of the undiscounted variance-covariance matrix. In Chapter III, pp 131, discounting was shown to reduce the volume of the isovariance ellipsoid. Discounting also reduces the generalized variance which has been interpreted as the volume of the parallelotope formed by the row or column vectors of the variance-covariance matrix. The invariance property of linear transformations of determinants may lead to criticism of the generalized variance as a scalar multivariate risk measure. What the invariance property confirms is the fact that there are different dimensions to the analysis of variability. The generalized variance captures the impact of the interaction between variance and covariance elements and depicts a risk relationship that is invariant to linear transformations, i.e., discounting.

The multidimensional nature of the risk assessment problem may to illustrated using simple examples. The examples presented here are designed to show that *no single* scalar multivariate risk measure provides optimal rankings for all cases.

Suppose the analyst is faced with three, two-period mutually exclusive projects. Let the expected net-present values be identical; therefore, the objective is to select the project with the lowest risk. To further simplify the problem, assume that the periodic cash flows are intertemporally independent, i.e., $\rho_{12} = 0$. The estimated variance-covariance matrices are given as

$$\hat{\Sigma}_a = \begin{bmatrix} 12.1 & 0 \\ 0 & 14.641 \end{bmatrix}, \quad \hat{\Sigma}_b = \begin{bmatrix} 6.05 & 0 \\ 0 & 21.961 \end{bmatrix}, \quad \hat{\Sigma}_c = \begin{bmatrix} 18.15 & 0 \\ 0 & 7.321 \end{bmatrix}. \quad (4)$$

Assuming a 10 percent time-value of money, then the discounted variance-covariance matrices, DVC are

$$DVC_a = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}, \quad DVC_b = \begin{bmatrix} 5 & 0 \\ 0 & 15 \end{bmatrix}, \quad DVC_c = \begin{bmatrix} 15 & 0 \\ 0 & 5 \end{bmatrix}. \quad (5)$$

Before computing and evaluating the scalar measures,[7] an interesting exercise is to intuitively determine which project is preferred. Project a has equal variability in each period, project b has low variability in period 1 with high variability in period 2; while project c's variability is just the reverse of project b's. Does the decision maker have a preference as to the point in time when the risk occurs? The answer is yes, maybe! One might argue, using some form of a minimax criterion, that project a with equal periodic variances is preferred because the largest periodic variance of project a is less than the largest periodic variance of either projects b or c. Other plausible arguments relating to lower variability in period 1 may favor selection of project b. The contrived example illustrates the problem of identifying the *best* measure of risk - the ambiguity associated with the idea

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[7]For $\alpha = .10$, Area of Isovariance Ellipse = $\pi(\det \hat{\Sigma})^{\frac{1}{2}}4.601$, Generalized Variance = $\det \Sigma = a_{11} \cdot a_{22} - a_{12} \cdot a_{21}$, and Variance of Net Present Value = $a_{11}+a_{12}+a_{21}+a_{22}$.

of risk. The difficulties associated with intuitive project selection are not entirely resolved by the analysis of the scalar risk measures or by discounting alone.

Computing the Variance of the Net-Present Value results in $VNPV_a = VNPV_b = VNPV_c$.[8] In this example, shifting the variance between periods while keeping a constant trace results in a scalar risk measure that does not discriminate among the three projects. If the decision maker believes that risk is additive, then the three projects are equally risky.

Computing the Area of the Isovariance Ellipse and the Generalized Variance results in a preference for project a with indifference among projects b and c.[9] Project a is selected because of its larger area and larger generalized variance. Both measures are sensitive to shifts in variance. This sensitivity is easily explained geometrically. With equal variances, the variance matrix identifies the isovariance circle. Matrices with unequal variances identify isovariance ellipses. When the corresponding traces are equal, the area of the circle is greater than the area of an ellipse. If the decision maker believes that risk is interactive, i.e., multiplicative, then project a would be selected.

The illustrated example also shows cases where the generalized variance fails to discriminate among projects. Comparing projects b and c, both the area and the generalized variance for each project are

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[8]VNPV = The sum of the discounted variance elements; VNPV = 20.

[9]Generalized Variance = $\det \Sigma$; $GV_a = 100$, $GV_b = GV_c = 75$. Area (for $\alpha = .10$) $= \pi(\det \Sigma)^{\frac{1}{2}}4.601$; $A_a = 144.54$, $A_b = A_c = 125.18$.

equal. Due to the multiplicative nature of both scalar risk measures, reversing the timing does not affect either measure.

This section of the study has confirmed the previous observations that variability or risk in a multivariate setting is multifaceted. Obviously, additional examples can be created to reflect the superiority of one scalar multivariate risk measure or another. An obvious extension of this study is to empirically test these scalar multivariate risk measures using real or simulated data.

## Scale Adjustment

When evaluating competing capital-budgeting alternatives, the analyst is often confronted by projects of differing size or scale. Obviously, the magnitude of the cash flows is an important determinant of the total risk of a capital-budgeting project. However, differences in scale often distort the risk analysis. For example, using the variance of the net present value, a project with less variability may very well exhibit a larger variance of the net present value because of scale differences alone. A similar problem exists when area/volume of the isovariance ellipse/ellipsoid or the generalized variance is used as a scalar risk measure. Both measures are sensitive to the size of the cash flow stream.

### Transformation of Cash Flows

One way to adjust for differences in scale is to transform the probability distribution of cash flows to a zero-one scale. The transformation is analagous to the univariate transformation of $N(\mu,\sigma)$ to the standard normal distribution $N(0,1)$. When the cash flow stream is

characterized by the density

$$f(X_1,\ldots,X_p) = (2\pi)^{-p/2}|\underline{\Sigma}|^{-\frac{1}{2}}e^{-\frac{1}{2}(\underline{x}-\underline{\mu})'\underline{\Sigma}^{-1}(\underline{x}-\underline{\mu})} \quad , \qquad (6)$$

the distribution characterized by $N(\underline{\mu},\Sigma)$ may be standardized, i.e., transformed to $N(\underline{0},I)$ using the transformation

$$\underline{Y} = \underline{\Sigma}^{-\frac{1}{2}}(\underline{x}-\underline{\mu}).^{10} \qquad (7)$$

Then the density of the multivariate unit normal distribution is

$$g(\underline{Y}_1,\ldots,\underline{Y}_p) = (2\pi)^{-p/2}e^{-\frac{1}{2}(\underline{Y}',\underline{Y})}.^{11} \qquad (8)$$

This transformation to the multivariate standard normal distribution allows simple probabilistic comparisons among alternative capital-budgeting projects. The transformation does not, however, solve the scale problem. The transformed vector Y in Equation 8 loses all information about the variance-covariance structure.

## Multivariate Analogues of the Coefficient of Variation

A simple ratio measure can be computed to accommodate problems of differences in scale. The univariate coefficient of variation,

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[10]The transformation shown by Equation 7 is the multivariate analogue of the univariate standard normal transformation,

$$Z = \frac{X-\mu}{\sigma} \quad .$$

[11]Anderson, T. W. (1958) *An Introduction to Multivariate Statistical Analysis*. New York: John Wiley and Sons. pp. 11-16.

the ratio of the standard deviation of the net-present value, $\sigma$, to the expected net present value, E(NPV), $\sigma$/E(NPV) has been widely accepted as a statistic that brings together both risk and return into a single measure. The coefficient of variation is interpreted as the risk per dollar return and is useful as a measure of relative risk.

The univariate coefficient of variation may be logically extended to evaluate multivariate cash flow streams. Each scalar multivariate risk measure may be combined with the expected net present value to form multivariate coefficients of variation. For example, the multivariate VNPV combined with the E(NPV) results in the ratio VNPV/E(NPV) which provides a direct extension of the univariate coefficient of variation. Comparing alternative projects using this ratio measure, the analyst would select the project with the smallest ratio; which results in the selection of the project with the smallest risk per dollar return.

The area/volume of the isovariance ellipse/ellipsoid and the generalized variance must be modified before they may be meaningfully combined with the expected net present value to form multivariate coefficients of variation. Both area and generalized variance measure risk inversely, i.e., larger area and generalized variances identify lower risk. Therefore, the reciprocal of the area or the generalized variance is the more appropriate numerator for these multivariate extensions of the coefficient of variation. The ratios would be

$$\frac{1/\text{Area}}{E(NPV)} \quad \text{or} \quad \frac{1/\det \Sigma}{E(NPV)} \quad .$$

Once again comparing projects, the analyst would select the project with the smallest ratio.

## Decomposition Analysis

Sensitivity analysis is widely used by practitioners to assist in risk evaluation. Recognizing the difficulties in estimating future cash flow streams, simulation studies may be used to assess the sensitivity of measures of risk and return.

The generalized variance is a scalar risk measure that is amenable to sensitivity analysis through decomposition. With the assumption of multivariate normally distributed cash flows, total variation is described by the variance-covariance matrix. In turn, the variance-covariance matrix structure depends on the periodic variances and the interperiod correlation. The variance-covariance matrix may be decomposed into standard deviation diagonal matrices and the correlation matrix. More specifically,

$$\hat{\underline{\Sigma}} = \underline{B}\,\underline{R}\,\underline{B} \tag{9}$$

where B is a diagonal matrix of periodic estimated standard deviations and R is a symmetric positive definite estimated correlation matrix.

When sensitivity analysis is of interest, the generalized variance may be evaluated in terms of products of determinants, i.e.,

$$\begin{aligned} \det \hat{\underline{\Sigma}} &= \det \underline{B} \cdot \det \underline{R} \cdot \det \underline{B} \\ &= (\det \underline{B})^2 \cdot \det \underline{R} \end{aligned} \tag{10}$$

Using this decomposition, the sensitivity of the total variability of the capital-budgeting cash flows may be easily analyzed. One may study the effects of changes in the estimated variances or changes in the correlation structure independently or together. Sensitivity analysis may be extended to include changes in discount rates. This extension provides no additional information if the same discount rates are applied to all projects, because the relative rankings of capital-budgeting projects are not affected by linear transformations.

The decision maker's ability to decompose the generalized variance to allow further analysis is an appealing feature of this scalar multivariate risk measure. Sensitivity analysis may be applied to the other scalar risk measures considered in this study, variance of the net present value and volume of the isovariance ellipsoid. However, these risk measures cannot be decomposed to evaluate the variance and correlation effects independently.

## Limitations of Scalar Multivariate Risk Measures

Three scalar risk measures have been offerred as multivariate analogues to the univariate variance. An important limiting condition to their use relates to model specification. Critical to the acceptance of any of the measures is the assumption of multivariate normally distributed cash flows. Because each of the risk measures are directly derived from the variance-covariance matrix, if the variance-covariance matrix does not accurately represent the risk of the asset, then these risk measures will be inadequate measures of total risk. Arguments for the

validity of the multivariate normal distribution were presented in Chapter III. If the multivariate normal assumption is clearly inappropriate, then these scalar measures should not be used to measure total risk.

A second problem area relates to the appropriate definition of risk associated with capital-budgeting analysis. Each scalar risk measure captures some aspect of total risk. As such, they are only appropriate for asset-by-asset selection. If the portfolio models correctly identify risk, then, in addition to intertemporal correlation, cross correlation between projects must be considered. The variance of the net present value has been extended to accommodate both autocorrelated and cross correlated cash flows.[12] However, neither the volume of the ellipse nor the generalized variance are currently capable of being extended to include the second set of correlations. Tensor analysis is a mathematical area that extends vector and matrix concepts. Possibly an application of tensor analysis will allow for the analysis of both autocorrelated and cross correlated cash flows. Another consideration relating to the definition of risk is the nature of the interaction between variance and covariance elements. If total risk is an appropriate concept for capital-budgeting cash flows and if the multivariate normal distribution is reasonable, the question whether variances and covariance should be combined additively or multiplicatively must be answered. If risk is additive, then the variance of the net present value is preferred to the generalized variance. Conversely, if risk is multiplicative, then

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[12]Levy, H. and Sarnat, M. (1970) "The Portfolio Analysis of Multiperiod Capital Investment Under Conditions of Risk," *Engineering Economist*. Vol. 16: pp. 1-19.

the generalized variance is the preferred scalar multivariate risk measure.

Capital-budgeting analysis differs considerably from the analysis of financial assets. The unavailability of historical data creates problems of data estimation. Application of the multivariate approach requires, at a minimum, estimates of the mean vector and the variance-covariance matrix. The nature of the data precludes the use of classical hypothesis tests and confidence intervals. Classical statistical methods are not appropriate because the expected net present value and variance-covariance matrix estimates are not summary measures of random sample data, but represent a sample of size one. Using traditional capital-budgeting estimation methods results in dependent rather than independent cash flow estimates.

The lack of random sample data constrains the use of scalar risk measures. Hillier[13] initially developed the variance of the net present value to allow the decision maker to make probabilistic statements about the worth of an investment opportunity. Unfortunately, the dependencies in the cash flow estimates result in an overstatement of the particular probabilities. The data problems also affect the use of the generalized variance. Considerable work using the generalized variance has been reported in the statistical literature. The most common use is as an extension to the univariate F-test.[14] As with the

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[13]Hillier, F. S. (1963) "The Derivation of Probabilistic Information for the Evaluation of Risky Investments," *Management Sciences*. Vol. 9: pp. 443-457.

[14]Chapter III, pp. 150-152 reports on many uses of the generalized variance in classical hypotheses testing situations.

variance of the net present value, the lack of random sample data constrains the use of the generalized variance.

Kowal[15] criticized the use of the generalized variance as a measure of multivariate variability. Noting that det $\Sigma$ may be decomposed as

$$\det \Sigma = \rho \mu^p \tag{10}$$

where $\rho$ is the determinant of the correlation matrix and $\mu$ is the geometric mean of the p variances.

An obvious comparison of two projects' variability is to compute and evaluate the ratio of the variability measures. Using generalized variance

$$K = \frac{\det \Sigma_A}{\det \Sigma_B} = \frac{\rho_A \mu_A^p}{\rho_B \mu_B^p} \quad . \tag{11}$$

If the projects have the same correlation structure, $\rho_A = \rho_B$, then

$$K = \frac{\mu_A^p}{\mu_B^p} \tag{12}$$

Equation 12 shows that the ratio of the generalized variances is equivalent to the ratio of the geometric mean of the variances. All information concerning correlation disappears from the comparison.

Kowal's analysis highlights the most severe limitation associated with the use of the generalized variance. When cash flows are perfectly correlated, the multivariate normal distribution is degenerate,

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[15]Kowal, R. R. (1971) "Disadvantages of the Generalized Variance as a Measure of Variability," *Biometrics*. pp. 213-216.

the variance-covariance matrix is singular, and det $\Sigma$ is equal to zero. Considering the case where project A exhibits imperfect correlation while project B's cash flows are perfectly correlated; Ratio K in Equation 11 shows that Project A is infinitely more variable than Project B regardless of the relative sizes of the respective variances. This result is obviously nonsensical. Analysis of the implications of perfect correlation[16] and the economic events that would attend perfectly correlated cash flows, suggests that if the multivariate normal distribution is an appropriate model for real capital-budgeting problems, then perfectly correlated cash flows and singular variance-covariance matrices are extremely unlikely occurrences.

In addition to the problem with singular variance-covariance matrices, projects with different variances and correlation structures may have the same generalized variance. For example, one project may have large periodic variances with low interperiod correlation while another project has smaller variances with larger correlation resulting in the same generalized variance. Given the acceptance of the multiplicative interpretation of total risk, the analyst would accept the fact that the two project reflect the same degree of total risk as represented by equal generalized variances. If the analyst if uncomfortable with the above approach, then the generalized variance may be decomposed into the product of the determinants of the variance diagonal matrix and the correlation matrix. This decomposition allows for additional analysis based on the decision maker's relative preference for variance and

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[16]Chapter III, pp. 135-137.

correlation. This approach accommodates the fact that individual risk preferences may reflect a trade-off between variation and correlation.

This chapter has extended the development of the multivariate approach to capital-budgeting. Primary emphasis was in comparative analysis of three alternative scalar multivariate risk measures: 1. variance of the net present value; 2. area/volume of the isovariance ellipse; and 3. generalized variance. Each measure was given an intuitive interpretation. The effects of time and scale were accomodated. And, finally, limitations associated with each measure were addressed.

# CHAPTER V

# SUMMARY AND SUGGESTIONS FOR ADDITIONAL RESEARCH

## Summary

The primary objective of this study is the development of a capital-budgeting model that explicitly considers the impact of intertemporally correlated cash flows. The need for such a model was documented in Chapter II, Review of Related Literature.

The Review of Related Literature systematically surveyed existing methodology for risk analysis in capital-budgeting. The "state of art" was classified into two broad groups: variations of simple present value techniques and market or portfolio theory adaptations. Present value approaches resulted in the development of the probability distribution of the net present value. Risk analysis then relates to the analysis of variance of the net present value. While this risk measure has been widely reported in the finance literature, there has been significant disagreement as to its merit. Other popular risk adjustment techniques such as risk adjusted rates and certainty equivalents have been offered as simple alternatives. Both of these simple methods of risk analysis have serious shortcomings that have led to more sophisticated capital budgeting models.

The development of portfolio theory, capital asset pricing model and capital market theory resulted in the mean-variance approach to the analysis of financial assets within the context of efficient capital markets. These "market" models were then extended/adapted to the analysis of real assets or capital-budgeting problems. Unfortunately, the strong set of assumptions required for the development of equilibrium conditions in the capital markets do not appear to be satisfied when the portfolio models are adapted to capital-budgeting problems. The violation of assumptions and dissatisfaction with the mean-variance decision criterion led to the development of alternative market models. Mean-semivariance models incorporate skewness into the decision process. Stochastic dominance models moved the analysis away from the study of the moments of the distribution of cash flows to the distribution itself. Finally, time-state preference models, while conceptually elegant and the most general approach reviewed, are generally found to be extremely difficult to implement.

Chapter II closed with comparisons of market models and a report of actual capital-budgeting techniques used in the field. Three surveys were reviewed that show that simple, ad-hoc risk adjustment techniques still dominate in the field. This dominance, however, is declining. Later surveys show a trend of increasing sophistication in capital-budgeting risk analysis. With the increased use of statistical models to analyze risk, there exists the need for the development of capital-budgeting models that are theoretically acceptable, yet simple enough to be used by practitioners. Chapters III and IV of this study

presented a simple, yet theoretically sound, approach to capital-budgeting risk analysis.

Chapters III and IV developed and evaluated a multivariate approach to the solution of capital-budgeting problems. Prior to the development of the multivariate model, arguments in favor of single asset analysis as opposed to portfolio approaches were presented. In addition, the need to explicitly consider the intertemporal correlation in capital-budgeting cash flows was documented. Simple techniques for estimation of the correlation structure were presented.

A multivariate approach to capital-budgeting was selected to accommodate autocorrelation among the periodic cash flows. Consideration of these dependence structures requires the simultaneous analysis of all the dimensions of the capital-budgeting problem. Multivariate statistical methods have been designed to explicitly consider these dependence structures. Capital-budgeting analysis requires a modification to traditional multivariate analysis procedures to accommodate the element of time. Timing differences in the cash flow stream are accommodated by transforming the cash flow distribution to present time; i.e., discounting.

A crucial assumption of the study was that the cash flow stream may be modeled with the multivariate normal distribution. The multivariate normal assumption was justified using arguments from both the statistics and the finance literature. The multivariate normal distribution was shown to be a generalization of the univariate normal distribution. Therefore, the evaluation of return and risk is equivalent to the evaluation of the mean vector and the variance-covariance matrix.

The expected net present value was accepted as an appropriate measure of return. The expected mean cash flows are discounted at the risk-free rate to account for differences in timing and to preclude double counting for risk.

While little controversy attends the use of expected net present value to measure return, there is little agreement to the appropriate measure of risk. With normally distributed cash flows, the variance-covariance matrix is a multivariate analogue of the univariate variance. Therefore, risk analysis is equivalent to the analysis of the variance-covariance structure. This structure may be evaluated geometrically using the ellipsoids of concentration of the multivariate normal distribution. Changes in matrix structure are readily identified through the changes in the size and slope of the ellipsoids.

Analysis of the riskiness of capital-budgeting cash flows often requires comparison of alternative cash flow streams. Project selection requires the ranking of alternative investment choices in terms of some measure of attractiveness. In general, attractiveness relates to more return and less risk. The need to rank and compare alternatives directed this study toward the development of scalar risk measures that capture the information found in the variance-covariance matrix.

Three scalar multivariate risk measures were presented and evaluated. The variance of the net present value, volume of the iso-variance ellipsoid, and the generalized variance were shown to be scalar representations of the variance-covariance matrix. Each measure was described algebraically and geometrically. Statistical

properties and the capabilities of each scalar to measure risk were evaluated. Finally, the measures were compared and contrasted to determine if there exists a preferred scalar multivariate measure of total risk.

## Conclusions

The need to explicitly consider intertemporal, imperfectly correlated cash flows in capital-budgeting analysis provided the movivation for this study. Dissatisfaction with existing methods of risk analysis, directed this research to identify risk measures that:

1. are reasonable, and have intuitive appeal;
2. use all available information;
3. can be applied across a wide range of problems; and,
4. are theoretically supportable.

Given these criteria, a multivariate approach to the evaluation of return and risk was developed.

Traditional capital-budgeting studies have been dominated by time series and portfolio models. Therefore, a significant contribution of this study is the multivariate approach to capital-budgeting. Multivariate statistical methods were developed to analyze dependent structures. The interperiod dependencies of the capital-budgeting cash flows are easily evaluated using multivariate statistical methods.

A second contribution of this study is the application of the scalar multivariate risk measures (Volume of the Isovariance Ellipsoid and Generalized Variance) to capital-budgeting problems. These measures of total risk have not been previously used to evaluate capital-budgeting or any other economic problems. While both measures of variability

have been used in other areas of study (anthropology, horticulture and biology), this application in finance is unique.

Finally, this study has contributed to the study of risk. Risk has been defined as the variability of future outcomes. With such a general definition, there exists considerable controversy as to the most appropriate risk measure. This study has not resolved the controversy, but in fact may have expanded it. Comparing the volume of the isovariance ellipsoid and the generalized variance with the well-established variance of the net present value alerts the analyst that definitions such as "variability of future outcomes" are ambiguous. Each scalar risk measure identifies a *different* aspect of project variability. If multivariate variability is additive, then variance of the net present value captures the essence of risk. If, however, multivariate variability is multiplicative, then volume of the ellipsoid or the generalized variance may be a more informative measure of risk. The significant conclusion is that *exclusive use* of either the variance of the net present value or the volume measures to describe multivariate variability may cause the financial decision maker to overlook an important dimension of project risk.

## Limitations of the Study

The significant limitation of this study is the absence of empirical testing of the multivariate capital budgeting model. The primary emphasis of this study was the expository, theoretical development of the multivariate approach and related scalar risk measures. In addition to this theoretical development, the model needs empirical

testing. This testing may be performed using simulated cash flow streams or better yet, testing by practitioners using real data in the field.

Other limitations relate to simplifying assumptions that were made to develop the model. Assumptions that disregarded negative interperiod correlations, variable and unequal useful lives of competing projects, and uncertainty as to the discount rate need to be relaxed and accommodated in the model.

## Suggestions for Additional Research

This introductory study of a multivariate approach to capital budgeting suggests many areas for additional research. An obvious starting point is empirical testing of the model and relaxation of the simplifying assumptions. Empirical testing of the model may provide insights to the study of risk not apparent from this introductory study. Additional research is needed to refine the definition of risk when the cash flows are modeled by a multivariate distribution. This study has presented scalar risk measures that identify different aspects of project variability. With a more specific definition of project risk, a clear preference for one risk measure or another may be determined. Relating to the definition of risk and risk preferences, the utility theory implications of the multivariate model need to be considered. The properties and characteristics of utility functions in terms of expected return and generalized variance may provide considerable insight to the study of risk. Finally, the complementary nature of the single-period portfolio problem is recognized. The multiple asset, single

period nature of portfolio analysis results in a multivariate structure that is similar to the single asset, multiple period capital-budgeting problem. Therefore, the model developed in this study for capital-budgeting may easily be extended for portfolio analysis.

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