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MONETARY GROWTH THEORY: CRITICISMS AND MODELS

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CHAPTER I

INTRODUCTION

Purpose

The purpose of the present study is twofold: first to critically evaluate the present state of monetary growth theory and second to remove some of the shortcomings of this theory by suggesting alternative models.

Monetary growth theory is a new area of economic research. The last decade witnessed an ever increasing interest in the field of monetary growth theory and many theoretical economists attempted to investigate problems related to this field. It is not hard to find reasons for this trend. The pathbreaking articles of Solow (1956), Swan (1956), Tobin (1955) and the heavy research activity that these articles stimulated, initiated investigation of the fundamental theoretical issues associated with the problem of economic growth over time. Economists soon came to realize that real growth models, if they were to be theoretically sound and useful in economic policy decision-making, had to consider the very meaningful economic variable, money. The original stimulus for the birth of monetary growth theory having been provided by both theoretical and applied considerations was reinforced by the appropriate responses of very talented economists who found that sphere an appealing field for theoretical and empirical research. Today, monetary growth theory has evolved to become an independent area of research for the economic specialist.

It is not the purpose of this dissertation to survey the area of monetary growth theory, though such a survey might be valuable and contributing. Instead, the author proposes a slightly more ambitious undertaking, as indicated above, i.e., to provide a critique of the present trends in monetary growth theory and then proceed in building alternative models. For this purpose, a convenient point of departure is the Stein article (1970) which presents the main issues and results of the research activity in the area of monetary growth theory during the last decade.

Scope

Monetary growth theory covers a rather wide range of questions of both theoretical and practical significance. There are however some very basic questions that formulate the core of monetary growth theory. These basic questions and the tentative answers to these questions provided in the economic literature have been skillfully surveyed in Stein's article (1970). It is the basic questions and methods of approach of monetary growth theory as interpreted by Stein (1970) that define the scope of this dissertation. Though such a scope is not precisely defined, it is sufficiently defined for the purpose of this dissertation. Part I of this dissertation consists of a critique of monetary growth theory.

The scope of this critique includes both criticisms of Stein's article (1970) already available in the economic literature as well as additional ones suggested by the author. With respect to the models of monetary growth theory presented in Part II of this dissertation, their scope is limited to static and dynamic models of the introduction of money in a barter economy. Economists have come to realize that stochastic models describe the real world more faithfully than the deterministic ones, but the construction of stochastic models presents many conceptual difficulties which are due to the fact that the study of uncertainty is at its infancy in economics. Hence, the scope of this dissertation is not extended to include the construction of dynamic stochastic models of the introduction of money in a barter economy.

Methodology

It is particularly interesting to discuss the method of approach followed in this dissertation. In Part I, the critique of monetary growth theory follows the methodology of positive economics. The theoretical constructions of monetary growth theory are critically evaluated by the utilitarian approach[1] to economic theory, where emphasis is given to the ability of these constructions to adequately

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[1]For some interesting comments of the utilitarian approach to economics, see for example Georgescu-Roegen (1966), pp. 3 - 16.

represent the real world. The same approach motivated the development of the models of Part II. In particular, in all four models, synthesis, solvability and realism compose the very essence of the methodology followed. All models are general equilibrium models, with consumer and producer behavior derived from utility and profit maximization. In addition, all models are not only presented and analyzed but also a set of solutions is obtained to allow for comparisons with the economic reality. Finally, both varieties of static as well as dynamic models appear to represent the two most often used methods of approach in economics.

The Importance of Additional Work in this Area

The modern approach to science is undoubtedly the utilitarian approach. In accordance with this spirit, the importance of the present research in the field of monetary growth theory stems from the fact that there are many unsolved issues in this area. These issues demand solutions and unless additional research is done, solutions cannot be obtained. Furthermore, additional research in the area of monetary growth theory is of particular significance to the economic theorist who is interested in expanding human knowledge, regardless whether such knowledge is motivated by practical considerations or not. In this respect, the importance of further research is due to the fact that a theoretical integration is needed between real growth theory and monetary theory.

PART I

MONETARY GROWTH THEORY: CRITICISMS

CHAPTER II

A CRITIQUE OF MONETARY GROWTH THEORY

One of the most intellectually exciting problems that has attracted the attention of economic theorists the last decade or so is the introduction of money in real growth models. This problem arose naturally because economic growth theorists recognized that the exclusion of money from growth models was a temporary simplification rather than a permanent omission.

From a less abstract and theoretically formal standpoint, the question of introducing money in real growth models may be reformulated to read: what is the role of money in a growing economy? This reformulation is obviously dictated by the utilitarian approach to science, which, if it is interpreted in the present case, will mean that monetary growth models are built for the purpose of shedding some light on matters of monetary policy in modern growing economies. Thus monetary growth theory has a largely overlapping dual role: first, to construct a theoretical framework which would allow for a meaningful integration of monetary theory and real growth theory, and second, to provide meaningful suggestions on questions of monetary policy.

It could safely be argued that monetary growth theory started with the significant article of Tobin (1955). The decade 1955-64, with the exception of Hahn's article (1961), was rather sterile in this area of economic research. However, in the last five years, several powerful articles have appeared in the economic literature.[1] Professor Stein (1970) presented an excellent analysis of the techniques used in the literature in attacking questions of monetary growth theory.[2] His paper may be considered to be a methodological essay on the techniques of monetary growth models[3] and his purpose is, presumably, to intensify "the rivalry of scholars" which will "increase wisdom." Professor Stein's (1970) "main conclusion is that equally plausible models yield fundamentally different results," (p. 85), and he ends by asking: "the crucial question is: which is the correct monetary growth model?" (p. 105). It is the opinion of the present author that the techniques used so far in monetary

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[1]The interested reader is referred to the Journal of Money, Credit and Banking, Volume I, No. 2, May 1969, which is entirely devoted to the Proceedings of the Conference on Money and Economic Growth, June 17-19, 1968. The literature cited at the end of the articles published in the journal includes almost all major articles in monetary growth theory.

[2]A less rigorous presentation of Professor Stein's ideas on the same topic could be found in Stein (1969).

[3]For an interesting discussion of the concepts "model" and "theory", see Nagel (1961), Spector (1965) and DeShon (1970).

growth theory are not fully appropriate and that the crucial question is: what alternative techniques might yield more "interesting and useful" results?

Before such an alternative technique is proposed one must, to maintain continuity, discuss briefly the techniques employed so far in monetary growth theory and criticisms that have been made of them. Once this is done, the author will present some additional criticisms common to the techniques used so far, which have not been widely recognized by monetary growth theorists. On the grounds of the criticisms by the author in this chapter, a new approach will be outlined. The main purpose of this chapter, however, is not the detailed construction of a new technique,[4] but the presentation of some critical comments about the nature of the techniques of monetary growth theory as illustrated in Stein's (1970) article.

Professor Stein (1969, 1970) distinguished between the neoclassical approach and the Keynes-Wicksell approach to monetary growth theory.[5] With respect to the neoclassical approach, Professor Stein derives in a very elegant way equation (11) which describes the time rate of change of the capital intensity.[6] Certain modifications of this equation

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[4]This is the topic of Chapters III and IV.

[5]Actually, Professor Stein states three different approaches, but he goes on to critically evaluate only the two approaches mentioned above. See Stein (1970), p. 85.

[6]This equation reads $Dk = y(k) - nk - C[k + \theta L(k, \mu - n)]$.

follow suggested mainly by the work of Johnson (1967) and the work of Levhari and Patinkin (1968). The latter half of Stein's article (1970) is devoted to the discussion of the Keynes-Wicksell approach. The fundamental equation of this section is equation (30).[7] For both approaches Professor Stein succeeds in demonstrating very logically the essence of these models, although his basic tool is the single differential equation -- as opposed to the two differential equations -- system.[8]

Both approaches have been criticized in the monetary growth theory literature and Professor Stein is most certainly aware of these criticisms. More specifically, Marty (1968, 1969) has raised several valuable criticisms on the neoclassical approach to monetary growth theory. In particular, the neoclassical growth model assumptions of: a 'well behaved' two input -- capital and labor -- production function; savings being a constant proportion of disposable income; the rate of capital formation being identically equal to planned savings, i.e., no independent investment function; the markets being always in equilibrium regardless of price changes; nonexistence of a tradeoff between inflation

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[7]This equation reads $Dp = k\left[(M(t)/N(t)) - p(t)L(k, \mu_0 - n)\right]$ For the detailed derivation of both equations, the reader is referred to the Stein (1970) article.

[8]The two differential equations system approach for the neoclassical growth model may be found for instance in Nagatani (1970) and for the Keynes-Wicksell growth model in Stein (1966).

and unemployment; all these among other assumptions are indeed very restrictive.[9]

With respect to the Keynes-Wicksell approach, Professor Hahn (1969) has criticized it on the grounds that "one has in the present state of knowledge, great latitude in the construction of disequilibrium models; that is one of the reasons why they are so unattractive, and so a great variety of results can be produced" (p. 186). Also, Burmeister and Dobell (1970) seem to agree with Hahn.[10] One has, however, to also state that Hahn (1969) believes that ". . . the picture (in the case of disequilibrium models) does not look as bleak as it does when continuous equilibrium is a requirement" (p. 186).[11]

Critics have gone a little further in criticizing monetary growth theory. Not only each approach -- i.e., neoclassical and Keynes-Wicksell -- seems to be defective for reasons of the nature stated above, but also both of them seem to have some common methodological errors. If it is accepted that the theoretical motivation of monetary growth theory is the construction of a model of the intro-

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[9]Marty (1969) p. 252.

[10]Burmeister and Dobell (1970) p. 166.

[11]This statement with Hahn's overall criticisms of the neoclassical model in Hahn (1969) do not necessarily imply Stein's (1970) assertion on footnote 20 stating that Hahn "also seems to lean in a Keynes-Wicksell approach."

duction of money into a barter economy,[12] then both approaches are intellectually inappropriate for the nature of such a task. The reason is rather simple. Both approaches implicitly assume a given, constant financial structure of the economy. As Professor Marty (1969) has argued ". . . no attempt is made to answer fundamental questions concerning the way a money economy develops, the optimal degree of financial intermediation, and its impact on the savings-investment process, the nature and classification of financial innovations -- a subject which in comparison with the state of analysis of technical change is a terra incognita of economics" (p. 253). Thus, monetary growth models do not actually show the introduction of money in a barter economy, and do not show the relationship between a barter economy and a monetary economy. According to Marty (1968) in monetary growth models "a barter economy is one in which the stock of real cash balances has been reduced to

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[12]A distinction is made between "introduction of money in real growth models" and "construction of a model of the introduction of money in a barter economy." For the former there is need only for a consistent money growth model; for the latter there is need for a model of the economy as a barter and as a money economy. In constructing a model of the introduction of money in a barter economy one does not intend to actually describe the process, institutional and otherwise, through which a barter economy is transformed into a monetary one. What is really meant here by the introduction of money in a barter economy is a study in comparative dynamics; an analysis of the barter economy at a given period of time and an analogous analysis of the same economy after it has been monetized by a process which is not the object of study.

negligibility by a sufficiently rapid inflation and not an economy in which money has not yet been introduced." (p. 868).

All the above criticisms are very valuable. However, monetary growth theory as presented in Stein's (1970) article could further be criticized along the following methodological lines. What both monetary growth theorists and monetary growth critics have failed to recognize so far is that a meaningful integration of real growth theory and monetary theory could only be achieved if the integrated theory is constructed under a set of assumptions consistent with both constituent theories.[13] The nonrigorous state of monetary theory has influenced monetary growth theorists to attempt an integration by leaning very heavily on the assumptions of real growth theory. There is very little doubt that monetary growth theory is primarily an extension of real growth theory along its methodological lines, rather than a meaningful integration of both theories under common assumptions. However, an integration of real growth theory and monetary theory along the assumptions and the methodology substantially of the former is neither practically interesting

[13]In his research, the author has failed in finding any published work in the area that could broadly speaking be called "integration of theories," or "integration of systems." The discipline of mathematics offers a number of such "integrations," for instance, analytical geometry, algebraic topology, algebraic probability, etc. In general, the problem may be formulated as follows: given the axioms, definitions and theorems of a theory or system, say A, and the axioms, definitions and theorems of theory B, how would the scientist construct a theory C by combining or integrating theories A and B?

nor theoretically sound. From the practical standpoint, monetary growth theory is not interesting because it can offer very few monetary policy recommendations. The neutrality and non-neutrality theorems of monetary growth theory do not lead to monetary policy formulations. This is to be expected due to the theoretical weaknesses of monetary growth theory as presented in Stein's (1970) article.

The weakness of monetary growth theory is due to the methodological error of attempting to integrate it with real growth theory without careful consideration of the assumptions of these two theories. More specifically, any monetary theory is constructed on at least one of two basic assumptions: (a) there exists an economy with at least three goods, or (b) there exists uncertainty in the economy. Unfortunately, monetary growth tneory ignores these two necessary assumptions; thus Stein's (1970) article describing the introduction of money in a real economy that does not satisfy these two assumptions has no solid theoretical foundations. Since monetary growth theory has not been criticized along these methodological lines in the available literature, it might be helpful to discuss in some detail the nature of the inconsistent foundations of monetary growth theory.

If monetary growth theory is to study the introduction of money in a barter economy, it is necessary to start with a barter economy. Unfortunately, both the neoclassical and the Keynes-Wicksell models are one-good models.[14] Obviously, such models could not describe a barter economy. It seems rather unnecessary to indicate that for a barter economy to exist more than one good needs to be produced, among other things of course. As a matter of fact, two goods are not sufficient to define a barter economy which would be constructed for the purpose of introducing money later on. The reason is rather simple: in a two good economy the double coincidence of wants occurs very naturally and there seems to be no necessity for a medium of exchange, i.e., money. Thus, Stein's (1970) argument for introducing money in one good models on the grounds that "if there were no medium of exchange, then the inefficiencies of a barter economy would result" (p. 90) is both logically and economically incorrect.[15] On logical grounds it is incorrect because it is inconsistent with the foundations of any monetary theory; economically,

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[14] Pigou (1935) has long ago suggested that there exists no need for money in a one good economy. For possible interpretations of this view see Kuenne (1963).

[15] Stein (1970) in footnote 7 presents a rather unconvincing argument on the way a one-sector model could remove the complications of a multi-sector model. If his argument were correct, there would be no need in studying multi-sector growth models. For the significance of multi-sector analysis see Morishima (1969). This is not however to be interpreted that one-sector models are unimportant for real growth theory; they simply are inappropriate for monetary growth theory, for reasons presented in this paper. In particular, in order to act as if the economy produced a single good we would have to assume that the relative quantities as well as the relative prices of a vector of goods are fixed. In this case the problems of barter would be far less than under less restrictive assumptions.

it could not be argued that a one-good economy suffers from inefficiency. Generally speaking, it might be conjectured that monetary growth theorists have paid very little attention to the economics of a barter economy and that they tried very hurriedly to introduce money in real growth models, justifying such a step by the axiom that barter of any sort is inefficient.[16] Once money has been introduced into a real, i.e., one-good economy rather than a barter economy on the basis of the above axiom, then because of the fact that a barter economy was not constructed, theorists had to rely extensively on their intuition of a monetary economy and asked questions that their models were not suitable for answering.

Furthermore, monetary growth theory has paid very little attention to uncertainty.[17] If the three traditional functions of money are accepted, i.e., money being a (i) measure of value, (ii) medium of exchange, and (iii) store of value, then it could safely be argued that an economy with perfect certainty is inconsistent with the third function of money. Money fulfils its third function as a store of value if and only if there is uncertainty about the

[16] It is interesting to note that social scientists have seriously questioned the validity of the statement that money was introduced historically because of the inefficiencies of a barter economy. See for instance Polanyi (1957).

[17] The role of uncertainty is very lucidly presented in Shackle (1967).

future in an economy.[18] If there is perfect certainty in the economy, then money is simply a measure of value and a medium of exchange.[19] These two functions alone may not yield interesting results, in the sense that classical economists viewed money as simply being a "veil". Consider for instance an economy and assume perfect certainty. On what economic justification would individuals hold cash balances? Professor Rosenstein-Rodan (1936) has long ago provided an answer by stating that "in an economy without 'frictions' where everybody foresaw with perfect certainty his tastes, income, future prices and the dates as well as the size of his purchases, nobody would keep a cash balance" (p. 271).[20] Thus he reaches the conclusion that "a state of general foresight and the existence of money . . . are mutually incompatible" (p. 272). Of course, Professor Rosenstein-Rodan (1936) attempted to build the foundations for a meaningful integration of the theories of price and money, and one may argue that economic theory has greatly

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[18]Definitions of the three functions of money may be found in Rosenstein-Roden (1936), p. 259.

[19]This follows from the definition suggested by Rosenstein-Rodan (1936), p. 259.

[20]Rosenstein-Rodan's views are very similar to Hicks' views. They both argued that money is meaningless in a static general equilibrium model. Marget and Kuenne on the other hand hold an opposite view. For an interesting discussion of these and other economists' views on whether money and static general equilibrium theory are consistent see Kuenne (1963), Chapter 5.

progressed since then.[21] Monetary growth theorists recently are attempting to integrate real growth theory and monetary theory and they should raise the methodological question whether the introduction of money in real growth models is compatible with stationary states. There seems to be very little doubt that the framework of real growth models, although dynamic, does not emphasize uncertainty, a condition which may be sina qua non for any meaningful monetary theory. It is true that monetary growth theory has incorporated a simple price expectations[22] function. The motivation, however behind the introduction of this function is not the detailed study of the implications of uncertainty. The price expectations function has been chosen according to Stein (1970) in such a way that it "is both a stabilizing influence and is consistent with the steady state solution" (p. 89).

Having presented some general methodological criticisms of modern monetary growth theory, a brief answer will be

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[21]There is very little doubt that economic theory has progressed since 1936. It might be interesting, however, to note that the integration of static price theory and monetary theory on the one hand, and the integration of real growth theory and monetary theory on the other hand, seem to raise some of the same fundamental methodological issues. It is hoped that economists have learned something from the famous "Patinkin Controversy" and that the recent efforts of integrating real growth theory and monetary theory will avoid another analogous controversy.

[22]The nature of expectation and uncertainty is very carefully discussed in Georgescu-Roegen's (1958) classic article. See also, Hicks (1946).

provided in the remainder of this chapter to the question already asked above, i.e., what alternative techniques might yield "interesting" results?[23, 24]

It is the opinion of the author that a stochastic, dynamic, general equilibrium approach[25] to monetary growth theory might yield more interesting results in the sense of taking into consideration the various criticisms raised above. Such a task is by no means easy. A great synthesis of various branches of economic theory will be required.[26] Such a synthesis will have to start with a static general equilibrium approach to a barter economy with more than two

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[23]The insufficiency of criticisms in attacking well established paradigms is presented in Kuhn (1970). For an overall view of the structure of scientific revolutions in economics see Coats (1969).

[24]Professor Stein's (1970) paper has actually a misleading title. What the reader is lead to understand is that monetary growth theory has been developed only along the lines of the neoclassical growth model and the Keynes-Wicksell model. This is not however true. There are at least two additional papers in the area of monetary growth theory that have used entirely different techniques. See for instance, Drandakis (1966) and Shapley (1964). As a matter of fact, the approach suggested by the author could be considered as an extension of that followed by Drandakis (1966). Furthermore, Friedman (1969) has also contributed in the area of monetary growth theory.

[25]Professor Hahn (1969) has suggested an alternative technique to monetary growth theory. He called it a "Monetary Debreu." Since Professor Hahn did not explain what he understands by "Monetary Debreu," no comparisons could be made between his suggested technique and the one suggested in this paper. It could safely be argued however, that there are probably more similarities than dissimilarities between Hahn's approach and the author's approach. See also Hahn (1965).

[26]The role of synthesis, among other concepts, has been illustrated in Caws (1969).

commodities. In such an economy, the introduction of money can theoretically be explained by viewing it as a producer and consumer good. The static monetary general equilibrium construction will have to be followed by a dynamic model[27] to account for the use of money as a store of value. Finally, uncertainty will need to be built in the dynamic general equilibrium approach to the barter economy,[28] generalizing the dynamic model into a stochastic dynamic monetary general equilibrium model.

It is hoped[29] that once a barter economy with at least three goods and uncertainty is constructed and then money is introduced in it, a meaningful integration of real growth theory and monetary theory will be achieved and the role of money in a growing monetary economy will be revealed. Having such a great task ahead of us, one has to agree with Stein (1969) who states "research in this exciting field has just begun" (p. 136).

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[27] See, for instance, Morishima (1969).

[28] For a valuable contribution in the area of general equilibrium with uncertainty, the reader is referred to Borch (1968) and the references cited at the end of his paper. See also Radner (1970).

[29] Nothing more than a hope can be expressed at this stage. There is always the fear that a synthesis of the nature suggested above may prove to be fruitless. See Kuhn (1970), in particular pp. 77-91.

PART II

MONETARY GROWTH THEORY: MODELS

CHAPTER III

MODELS I AND II

In this chapter two basic models are presented: model I and model II. Model I is a static, barter, general equilibrium model[1] and model II is a static, monetary, general equilibrium model. The two models are closely related since model I has been constructed to accommodate the introduction of money in a barter economy. The results of introducing money in a barter economy are analyzed in model II and the similarities and dissimilarities of a barter and monetary, static economy are revealed by inspecting the solutions of these two models. In what follows, model I will first be developed and a set of solutions will be obtained; the discussion and presentation of a set of solutions of model II will follow; finally some interpreting remarks will be made.

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[1]Model I is an extension of a model presented in Brems (1968, Chapter 25). Model I differs from Brems' model however in one crucial way: it has three outputs instead of only two. The need for extending Brems' model to include a third output is motivated by the intention of using model I to introduce money in it.

Model I

The notation used in this model is the following:

Variables

C_{ij} = i^{th} output consumed by j^{th} person; $i = 1,2,3$, $j = 1,2$

P_i = price of i^{th} output

P_K, P_L = price of capital and price of labor, respectively

U_j = utility to the j^{th} person

X_i = i^{th} output

K_i = capital absorbed by i^{th} output

L_i = labor absorbed by i^{th} output

Y_j = income of j^{th} person

Parameters

A = elasticity of utility with respect to X_1

B = elasticity of utility with respect to X_2

C = elasticity of utility with respect to X_3

u_i = elasticity of i^{th} output with respect to capital

λ_i = elasticity of i^{th} output with respect to labor

K = total available quantity of capital

L = total available quantity of labor

The production functions of the model are of the Cobb-Douglas form described in equations (1) - (3).

(1) $X_1 = K_1^{u_1} L_1^{\lambda_1}$, $X_1 > 0$, $K_1 > 0$, $L_1 > 0$, $0 < u_1, \lambda_1 < 1$, $u_1 + \lambda_1 = 1$

(2) $X_2 = K_2^{u_2} L_2^{\lambda_2}$, $X_2 > 0$, $K_2 > 0$, $L_2 > 0$, $0 < u_2, \lambda_2 < 1$, $u_2 + \lambda_2 = 1$

(3) $X_3 = K_3^{u_3} L_3^{\lambda_3}$, $X_3 > 0$, $K_3 > 0$, $L_3 > 0$, $0 < u_3, \lambda_3 < 1$, $u_3 + \lambda_3 = 1$

It is interesting to note that constant returns to scale are assumed. The restriction that all variables should be strictly greater than zero is required by the fact that divisions by zero are not allowed.

Assume that firms seek to maximize their profits under conditions of perfect competition. Equations (4) - (9) represent the first order conditions of an unconstrained perfect competitive maximization of profits.

$$(4)\quad p_K = P_1 \frac{u_1}{K_1} X_1$$

$$(5)\quad p_L = P_1 \frac{\lambda_1}{L_1} X_1$$

$$(6)\quad p_K = P_2 \frac{u_2}{K_2} X_2$$

$$(7)\quad p_L = P_2 \frac{\lambda_2}{L_2} X_2$$

$$(8)\quad p_K = P_3 \frac{u_3}{K_3} X_3$$

$$(9)\quad p_L = P_3 \frac{\lambda_3}{L_3} X_3$$

It is useful at this point to attempt to apply Euler's Theorem to the distributive shares. Such an application is both economically useful and algebraically convenient. Equations (4) - (9) are used to derive the distributive shares equations. Rearrange (4) and (5) and add them to obtain equation (4a). Repeat the process for equations (6) and (7) to obtain equation (6a) and for equations (8) and (9) to obtain equation (8a).

(4a) $P_1 X_1 = p_k K_1 + p_L L_1$

(6a) $P_2 X_2 = p_k K_2 + p_L L_2$

(8a) $P_3 X_3 = p_k K_3 + p_L L_3$

All the above analysis refers to the production aspects of the model. The consumption aspects are considered next.

Assume that there are two consumers. The number of consumers could be increased to more than two with only notational complications. No loss of generality is suffered from the present treatment.[2] Let both consumers have the same utility function which for reasons of the solvability of the model is assumed to be of the Cobb-Douglas form.

$$U_1 = C_{11}^{A} \, C_{21}^{B} \, C_{31}^{C} \qquad 0 < A, B, C < 1$$

$$U_2 = C_{12}^{A} \, C_{22}^{B} \, C_{32}^{C} \qquad 0 < A, B, C < 1$$

In the present model it is assumed that all income is spent on consumption. This assumption is not very restrictive for a static model like the present one. In models III and IV of Chapter IV, the "no savings" assumption is dropped. The budget constraints for the two persons are:

$$Y_1 = P_1 C_{11} + P_2 C_{21} + P_3 C_{31}$$

$$Y_2 = P_1 C_{12} + P_2 C_{22} + P_3 C_{32}$$

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[2]For an illustration of the case where more than two consumers are considered see Stephens' article (1970). Although Professor Stephens analyzes questions of inflation and unemployment that are not directly related to the questions of this chapter, the consumption aspects of his paper (1970) are similar to the ones discussed above.

Constrained utility maximization yields the following demand functions:

(10) $C_{11} = \frac{Y_1}{P_1} \cdot \frac{A}{A+B+C}$

(11) $C_{21} = \frac{Y_1}{P_2} \cdot \frac{B}{A+B+C}$

(12) $C_{31} = \frac{Y_1}{P_3} \cdot \frac{C}{A+B+C}$

(13) $C_{12} = \frac{Y_2}{P_1} \cdot \frac{A}{A+B+C}$

(14) $C_{22} = \frac{Y_2}{P_2} \cdot \frac{B}{A+B+C}$

(15) $C_{32} = \frac{Y_2}{P_3} \cdot \frac{C}{A+B+C}$

Three final aspects of the model need to be considered: personal incomes, equilibrium in input markets and equilibrium in output markets.

To simplify the solvability of the model, assume that one individual is the capitalist and the other the worker. Personal incomes then are described by equations (16) and (17).

(16) $Y_1 = p_k K$

(17) $Y_2 = p_L L$

Equations (18) and (19) refer to the equilibrium condition in the input markets.

(18) $K = K_1 + K_2 + K_3$

(19) $L = L_1 + L_2 + L_3$

Finally, equilibrium in the output markets is described as follows:

(20) $X_1 = C_{11} + C_{12}$

(21) $X_2 = C_{21} + C_{22}$

(21a) $X_3 = C_{31} + C_{32}$

Note that equation (21a) implies that equilibria in X_1 and X_2 force equilibrium in X_3.[3] Thus (21a) is not necessary as an independent condition.

The model as described consists of 21 equations and 22 unknowns. The unknowns are: L_1, L_2, L_3, K_1, K_2, K_3, X_1, X_2, X_3, P_1, P_2, P_3, p_K, p_L, C_{11}, C_{21}, C_{31}, C_{12}, C_{22}, C_{32}, Y_1 and Y_2. Let p_L be the Walrasian numeraire. In what follows, a set of solutions is presented and discussed.

To solve for L_1, L_2, L_3, K_1, K_2 and K_3 use equations (4a), (6a), (8a), (10) - (15) and (20) - (21a) to obtain:

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[3]To demonstrate that Walras' law holds true, combine equations (4a), (6a), (8a), (16), (17), (18) and (19) and obtain

$$P_1X_1 + P_2X_2 + P_3X_3 = Y_1 + Y_2$$

Furthermore, combine equations (10) - (15), use equations (20) and (21) to obtain

$$P_1X_1 + P_2X_2 + P_3(C_{31} + C_{32}) = Y_1 + Y_2$$

which directly implies $X_3 = C_{31} + C_{32}$, i.e., equation (21a).

$$AB(p_K K_3 + p_L L_3) = AC(p_K K_2 + p_L L_2) = BC(p_K K_1 + p_L L_1)$$

From profit maximization conditions, i.e., equations (4) - (9) obtain:

$$\frac{p_K}{p_L} = \frac{u_1}{\lambda_1} \cdot \frac{L_1}{K_1} = \frac{u_2}{\lambda_2} \cdot \frac{L_2}{K_2} = \frac{u_3}{\lambda_3} \cdot \frac{L_3}{K_3}$$

Use the above with equations (18) and (19) to find

(22) $$L_1 = \frac{A\lambda_1}{A\lambda_1 + B\lambda_2 + C\lambda_3} \cdot L$$

(23) $$L_2 = \frac{B\lambda_2}{A\lambda_1 + B\lambda_2 + C\lambda_3} \cdot L$$

(24) $$L_3 = \frac{C\lambda_3}{A\lambda_1 + B\lambda_2 + C\lambda_3} \cdot L$$

(25) $$K_1 = \frac{Au_1}{Au_1 + Bu_2 + Cu_3} \cdot K$$

(26) $$K_2 = \frac{Bu_2}{Au_1 + Bu_2 + Cu_3} \cdot K$$

(27) $$K_3 = \frac{Cu_3}{Au_1 + Bu_2 + Cu_3} \cdot K$$

Direct insertion of input solutions, i.e., equations (22) - (27) into the production functions, i.e., equations (1) - (3) gives the solutions for outputs:

(28) $$X_1 = \left[\frac{AKu_1}{Au_1 + Bu_2 + Cu_3}\right]^{u_1} \left[\frac{AL\lambda_1}{A\lambda_1 + B\lambda_2 + C\lambda_3}\right]^{\lambda_1}$$

(29) $$X_2 = \left[\frac{BKu_2}{Au_1 + Bu_2 + Cu_3}\right]^{u_2} \left[\frac{BL\lambda_2}{A\lambda_1 + B\lambda_2 + C\lambda_3}\right]^{\lambda_2}$$

(30) $$X_3 = \left[\frac{CKu_3}{Au_1 + Bu_2 + Cu_3}\right]^{u_3} \left[\frac{CL\lambda_3}{A\lambda_1 + B\lambda_2 + C\lambda_3}\right]^{\lambda_3}$$

The solutions of relative prices follow. As it was indicated above, the price of labor, P_L , has been chosen to be the numeraire. Use equations (4) - (9) and equations (22) - (30) to obtain:

$$(31) \quad \frac{P_K}{P_L} = \frac{L\,(Au_1 + Bu_2 + Cu_3)}{K\,(A\lambda_1 + B\lambda_2 + C\lambda_3)}$$

$$(32) \quad \frac{P_1}{P_L} = \frac{1}{u_1^{u_1} \cdot \lambda_1^{\lambda_1}} \cdot \left[\frac{(Au_1 + Bu_2 + Cu_3)}{(A\lambda_1 + B\lambda_2 + C\lambda_3)} \cdot \frac{L}{K} \right]^{u_1}$$

$$(33) \quad \frac{P_2}{P_L} = \frac{1}{u_2^{u_2} \cdot \lambda_2^{\lambda_2}} \cdot \left[\frac{(Au_1 + Bu_2 + Cu_3)}{(A\lambda_1 + B\lambda_2 + C\lambda_3)} \cdot \frac{L}{K} \right]^{u_2}$$

$$(34) \quad \frac{P_3}{P_L} = \frac{1}{u_3^{u_3} \cdot \lambda_3^{\lambda_3}} \cdot \left[\frac{(Au_1 + Bu_2 + Cu_3)}{(A\lambda_1 + B\lambda_2 + C\lambda_3)} \cdot \frac{L}{K} \right]^{u_3}$$

The solutions of consumption are very simple to obtain. Direct insertion of equations (16) and (17) and equations (31) - (34) into equations (10) - (15) yields the following results:

$$(35) \quad C_{11} = \frac{AK}{A+B+C}\, u_1^{u_1} \lambda_1^{\lambda_1} \left[\left(\frac{Au_1 + Bu_2 + Cu_3}{A\lambda_1 + B\lambda_2 + C\lambda_3} \right) \cdot \frac{L}{K} \right]^{\lambda_1}$$

$$(36) \quad C_{21} = \frac{BK}{A+B+C}\, u_2^{u_2} \lambda_2^{\lambda_2} \cdot \left[\left(\frac{Au_1 + Bu_2 + Cu_3}{A\lambda_1 + B\lambda_2 + C\lambda_3} \right) \cdot \frac{L}{K} \right]^{\lambda_2}$$

$$(37) \quad C_{31} = \frac{CK}{A+B+C}\, u_3^{u_3} \lambda_3^{\lambda_3} \cdot \left[\left(\frac{Au_1 + Bu_2 + Cu_3}{A\lambda_1 + B\lambda_2 + C\lambda_3} \right) \cdot \frac{L}{K} \right]^{\lambda_3}$$

$$(38) \quad C_{12} = \frac{AL}{A+B+C}\, u_1^{u_1} \lambda_1^{\lambda_1} \cdot \left[\left(\frac{A\lambda_1 + B\lambda_2 + C\lambda_3}{Au_1 + Bu_2 + Cu_3} \right) \cdot \frac{K}{L} \right]^{u_1}$$

$$(39) \quad C_{22} = \frac{BL}{A+B+C}\, u_2^{u_2} \lambda_2^{\lambda_2} \cdot \left[\left(\frac{A\lambda_1 + B\lambda_2 + C\lambda_3}{Au_1 + Bu_2 + Cu_3} \right) \cdot \frac{K}{L} \right]^{u_2}$$

$$(40)\quad C_{32} = \frac{CL}{A+B+C}\, u_3^{u_3} \cdot \lambda_3^{\lambda_3} \cdot \left[\left(\frac{A\lambda_1 + B\lambda_2 + C\lambda_3}{Au_1 + Bu_2 + Cu_3}\right) \frac{K}{L}\right]^{u_3}$$

For the model to be completed the solutions of Y_1 and Y_2 are needed. These solutions are presented below in equations (41) and (42). Combine equations (16) and (17) where Y_1 and Y_2 are defined with equation (31) to obtain:

$$(41)\quad \frac{Y_1}{P_L} = \left(\frac{Au_1 + Bu_2 + Cu_3}{A\lambda_1 + B\lambda_2 + C\lambda_3}\right) \cdot L$$

$$(42)\quad \frac{Y_2}{P_L} = L$$

Model II

The presentation of model II requires some additional notation. Notation that has already been introduced in model I will not be repeated again. Only new notation is presented below.

Variables

M_i = money considered as an input and absorbed by the i^{th} output; $i = 1,2,3$

$\mathcal{M}_j$ = money considered as an output and demanded by the j^{th} person; $j = 1,2$

P_4 = the price of money viewed as a consumer's good[4]

P_M = the price of money viewed as a producer's good[5]

[4,5]The reader may interpret the price of money to be what is commonly called an "interest rate", rather than the reciprocal of a price index. Money in the present model has a more meaningful function than simply being a medium of exchange (in which case its price would be the reciprocal of a price index).

Parameters

M_T = total supply of money available

D = elasticity of utility with respect to money viewed as a consumer's good

a = proportion of the total money supply held by person 1

μ_i = elasticity of i^{th} output with respect to money as a producer's good

The production functions of model II are similar to those of model I with one basic difference. In model II, money[6] is introduced in the production function to express the function of money as a producer good. The theoretical foundations of this function of money will be discussed in the last section of this chapter. Mathematically, to express the function of money as a producer good is sufficient to introduce it in the production functions of the model. This is done presently by considering real cash balances as an input in the production functions below.

$$(43)\quad X_1 = K_1^{u_1} L_1^{\lambda_1} \left(\frac{M_1}{P_L}\right)^{\mu_1}, \quad X_1>0,\ K_1>0,\ L_1>0,\ \frac{M_1}{P_L}>0,\ 0<u_1,\lambda_1,\mu_1<1,\ u_1+\lambda_1+\mu_1=1$$

$$(44)\quad X_2 = K_2^{u_2} L_2^{\lambda_2} \left(\frac{M_2}{P_L}\right)^{\mu_2}, \quad X_2>0,\ K_2>0,\ L_2>0,\ \frac{M_2}{P_L}>0,\ 0<u_2,\lambda_2,\mu_2<1,\ u_2+\lambda_2+\mu_2=1$$

$$(45)\quad X_3 = K_3^{u_3} L_3^{\lambda_3} \left(\frac{M_3}{P_L}\right)^{\mu_3}, \quad X_3>0,\ K_3>0,\ L_3>0,\ \frac{M_3}{P_L}>0,\ 0<u_3,\lambda_3,\mu_3<1,\ u_3+\lambda_3+\mu_3=1$$

As in model I, also in model II, constant returns to scale

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[6] In what follows "money" and "real cash balances" are used interchangeably.

are assumed. Furthermore, assume that firms seek to maximize profits under conditions of perfect competition. The first order conditions of an unconstrained perfect competitive maximization of profits are presented in the following equations:

(46) $p_K = P_1 \dfrac{u_1}{K_1} \cdot X_1$

(47) $p_L = P_1 \cdot \dfrac{\lambda_1}{L} \cdot X_1$

(48) $p_M = P_1 \dfrac{\mu_1}{\left(\frac{M_1}{p_L}\right)} \cdot X_1$

(49) $p_K = P_2 \dfrac{u_2}{K_2} \cdot X_2$

(50) $p_L = P_2 \dfrac{\lambda_2}{L_2} \cdot X_2$

(51) $p_M = P_2 \dfrac{\mu_2}{\left(\frac{M_2}{p_L}\right)} \cdot X_2$

(52) $p_K = P_3 \dfrac{u_3}{K_3} \cdot X_3$

(53) $p_L = P_3 \dfrac{\lambda_3}{L_3} \cdot X_3$

(54) $p_M = P_3 \dfrac{\mu_3}{\left(\frac{M_3}{p_L}\right)} \cdot X_3$

In a similar way as in model I, Euler's equations are obtained for convenience in deriving the algebraic solutions of model II.

(46a) $P_1 X_1 = p_K K_1 + p_L L_1 + p_M \left(\dfrac{M_1}{p_L}\right)$

(49a) $P_2 X_2 = p_K K_2 + p_L L_2 + p_M \left(\dfrac{M_2}{p_L}\right)$

(52a) $P_3 X_3 = p_K K_3 + p_L L_3 + p_M \left(\frac{M_3}{P_L}\right)$

Having presented the production aspects of model II, emphasis is given next to the consumption side. Model II is based on assumptions similar to those of model I; more specifically it is assumed that there are two consumers having the same utility function which is of the Cobb-Douglas form. An important difference, however, needs to be pointed out. Money is introduced in the utility function to account for the hypothesis that real cash balances function as a consumer good. The utility functions with the budget constraints are presented below.

$$U_1 = C_{11}^A \, C_{21}^B \, C_{31}^C \left(\frac{M_1}{P_L}\right)^D$$

$$U_2 = C_{12}^A \, C_{22}^B \, C_{32}^C \left(\frac{M_2}{P_L}\right)^D$$

$$Y_1 = P_1 C_{11} + P_2 C_{21} + P_3 C_{31} + P_4 \left(\frac{M_1}{P_L}\right)$$

$$Y_2 = P_1 C_{12} + P_2 C_{22} + P_3 C_{32} + P_4 \left(\frac{M_2}{P_L}\right)$$

To obtain the demand function of model II, maximize utility under the above constraints.

(55) $C_{11} = \frac{Y_1}{P_1} \quad \frac{A}{A+B+C+D}$

(56) $C_{21} = \frac{Y_1}{P_2} \quad \frac{B}{A+B+C+D}$

(57) $C_{31} = \frac{Y_1}{P_3} \quad \frac{C}{A+B+C+D}$

(58) $$\frac{M_1}{P_L} = \frac{Y_1}{P_4} \cdot \frac{D}{A+B+C+D}$$

(59) $$C_{12} = \frac{Y_2}{P_1} \cdot \frac{A}{A+B+C+D}$$

(60) $$C_{22} = \frac{Y_2}{P_2} \cdot \frac{B}{A+B+C+D}$$

(61) $$C_{32} = \frac{Y_2}{P_3} \cdot \frac{C}{A+B+C+D}$$

(62) $$\frac{M_2}{P_L} = \frac{Y_2}{P_4} \cdot \frac{D}{A+B+C+D}$$

For the model to be completed, a definition of personal incomes is necessary and the equilibrium conditions need to be stated. First consider the personal incomes.

(63) $$Y_1 = p_K K + a\, p_M \left(\frac{M_T}{P_L}\right)$$

(64) $$Y_2 = p_L L + (1-a)\, p_M \left(\frac{M_T}{P_L}\right)$$

As in model I, equations (63) and (64) show that the first person is assumed to be the capitalist, while the second is assumed to be the worker. An important difference, however, should be noted. Equations (63) and (64) differ from equations (16) and (17) because they contain an additional term. This term reveals that the initial stock of real cash balances has been distributed between the two persons according to some fixed proportion a .

The equilibrium conditions in the input markets are stated in equations (65) - (67).

(65) $$K = K_1 + K_2 + K_3$$

(66) $L = L_1 + L_2 + L_3$

(67) $\frac{M}{P_L} = \frac{M_1}{P_L} + \frac{M_2}{P_L} + \frac{M_3}{P_L}$

Furthermore, equations (68) - (70) describe equilibrium conditions in output markets.

(68) $X_1 = C_{11} + C_{12}$

(69) $X_2 = C_{21} + C_{22}$

(69a) $X_3 = C_{31} + C_{32}$

(70) $\frac{\mathcal{M}}{P_L} = \frac{\mathcal{M}_1}{P_L} + \frac{\mathcal{M}_2}{P_L}$

Equation (69a) simply indicates that equilibria in the output markets X_1 and X_2 would automatically force equilibrium on output market X_3. Thus equation (69a) does not add any necessary information for the solution of the model.

Finally, some emphasis is given to the money market. Equations (71) and (72) describe the equilibrium conditions in the money market. They indicate that total supply of real cash balances is equal to the demand of real cash balances by the production sector where money is viewed as an input, plus the demand of real cash balances demanded by the consumer sector, where money is viewed as a consumption good.

(71) $\frac{M_T}{P_L} = \frac{M}{P_L} + \frac{\mathcal{M}}{P_L}$

(72) $P_4 = P_m$

Note that equation (72) implies that the price of real cash balances in a state of equilibrium is equated to the rate of return of real cash balances.[7]

The model presented in equations (43) - (72) consists of 30 equations and 31 unknowns. The unknowns are: L_1, L_2, L_3, K_1, K_2, K_3, M_1, M_2, M_3, X_1, X_2, X_3, P_1, P_2, P_3, P_4, p_K, p_L, p_M, Y_1, Y_2, C_{11}, C_{12}, C_{22}, C_{21}, C_{31}, C_{32}, $\mathcal{M}_1$, $\mathcal{M}_2$, M, $\mathcal{M}$. Let p_L be the Walrasian numeraire.[8] In what follows, a set of solutions of the model is presented with an indication of the algebraic manipulations involved.

To obtain the solutions of L_1, L_2, L_3, K_1, K_2 and K_3 perform the same algebraic operations as in the real model I. The introduction of money in model II leaves these solutions unaffected. More specifically they are:

$$(73)\quad L_1 = \frac{A\lambda_1}{A\lambda_1 + B\lambda_2 + C\lambda_3} \cdot L$$

[7]Note that the interest rate that producers pay for using money need not be the same as the interest rate that consumers pay for using money in cases where the economic system is off equilibrium. Equation (72) however forces the equality of the two interest rates as an equilibrium condition.

[8]The use of the price of labor, i.e., wage rate, as a numeraire for model II is consistent with both the Keynesian tradition and the derivation of meaningful results. For suppose that the wage rate was allowed to vary; then the units of the economy motivated by utility maximization considerations would attempt to increase the real quantities of economic goods by reducing the wage rate. This would imply that utility maximization would eventually drive the wage rate to zero; such an effect is obviously unobservable in economic reality.

(74) $L_2 = \dfrac{B\lambda_2}{A\lambda_1 + B\lambda_2 + C\lambda_3} \cdot L$

(75) $L_3 = \dfrac{C\lambda_3}{A\lambda_1 + B\lambda_2 + C\lambda_3} \cdot L$

(76) $K_1 = \dfrac{Au_1}{Au_1 + Bu_2 + Cu_3} \cdot K$

(77) $K_2 = \dfrac{Bu_2}{Au_1 + Bu_2 + Cu_3} \cdot K$

(78) $K_3 = \dfrac{Cu_3}{Au_1 + Bu_2 + Cu_3} \cdot K$

For the solutions of money as an input, some preliminary algebraic steps are necessary.

Use Euler's equations (46a), (49a) and (52a) and perform the necessary algebraic computations to obtain:

$$BC\left(p_K K_1 + p_L L_1 + p_M \frac{M_1}{p_L}\right) = AC\left(p_K K_2 + p_L L_2 + p_M \frac{M_2}{p_L}\right)$$
$$= AB\left(p_K K_3 + p_L L_3 + p_M \frac{M_3}{p_L}\right)$$

Also from equations (46) - (54) solve to get the following results:

$$\frac{p_K}{p_M} = \frac{u_1}{\mu_1} \cdot \frac{\left(\frac{M_1}{p_L}\right)}{K_1} = \frac{u_2}{\mu_2} \cdot \frac{\left(\frac{M_2}{p_L}\right)}{K_2} = \frac{u_3}{\mu_3} \cdot \frac{\left(\frac{M_3}{p_L}\right)}{K_3}$$

$$\frac{p_L}{p_M} = \frac{\lambda_1}{\mu_1} \cdot \frac{\left(\frac{M_1}{p_L}\right)}{L_1} = \frac{\lambda_2}{\mu_2} \cdot \frac{\left(\frac{M_2}{p_L}\right)}{L_2} = \frac{\lambda_3}{\mu_3} \cdot \frac{\left(\frac{M_3}{p_L}\right)}{L_3}$$

Combine the above equations with equations (65), (66) and (67) to solve for real cash balances as inputs. The solutions obtained are:

(79) $\dfrac{M_1}{p_L} = \dfrac{A\mu_1}{A\mu_1 + B\mu_2 + C\mu_3} \cdot \left(\dfrac{M}{p_L}\right)$

(80) $$\frac{M_2}{P_L} = \frac{B\mu_2}{A\mu_1 + B\mu_2 + C\mu_3} \cdot \left(\frac{M}{P_L}\right)$$

(81) $$\frac{M_3}{P_L} = \frac{C\mu_3}{A\mu_1 + B\mu_2 + C\mu_3} \left(\frac{M}{P_L}\right)$$

Direct insertion of input solutions presented in equations (73) - (81) into the production functions of the model yields the following output solutions:

(82) $$X_1 = \left[\frac{AKu_1}{Au_1 + Bu_2 + Cu_3}\right]^{u_1} \left[\frac{AL\lambda_1}{A\lambda_1 + B\lambda_2 + C\lambda_3}\right]^{\lambda_1} \left[\frac{A\left(\frac{M}{P_L}\right)\mu_1}{A\mu_1 + B\mu_2 + C\mu_3}\right]^{\mu_1}$$

(83) $$X_2 = \left[\frac{BKu_2}{Au_1 + Bu_2 + Cu_3}\right]^{u_2} \left[\frac{BL\lambda_2}{A\lambda_1 + B\lambda_2 + C\lambda_3}\right]^{\lambda_2} \left[\frac{B\left(\frac{M}{P_L}\right)\mu_2}{A\mu_1 + B\mu_2 + C\mu_3}\right]^{\mu_2}$$

(84) $$X_3 = \left[\frac{CKu_3}{Au_1 + Bu_2 + Cu_3}\right]^{u_3} \left[\frac{CK\lambda_3}{A\lambda_1 + B\lambda_2 + C\lambda_3}\right]^{\lambda_3} \left[\frac{C\left(\frac{M}{P_L}\right)\mu_3}{A\mu_1 + B\mu_2 + C\mu_3}\right]^{\mu_3}$$

The solutions for relative prices are obtained with the help of equations (46) - (54) and the input and output solutions presented above. In what follows, the input and output solutions have not been inserted into the relative prices solutions to eliminate unduly lengthy expressions.

(85) $$\frac{P_1}{P_L} = \frac{L_1}{\lambda_1} \cdot \frac{1}{X_1}$$

(86) $$\frac{P_2}{P_L} = \frac{L_2}{\lambda_2} \cdot \frac{1}{X_2}$$

(87) $$\frac{P_3}{P_L} = \frac{L_3}{\lambda_3} \cdot \frac{1}{X_3}$$

(88) $$\frac{P_4}{P_L} = \frac{P_M}{P_L} = \frac{\mu_1}{\lambda_1} \cdot \frac{L_1}{\left(\frac{M_1}{P_L}\right)}$$

(89) $\frac{P_K}{P_L} = \frac{u_1}{\lambda_1} \cdot \frac{L_1}{K_1}$

The solutions of real incomes are in order. Use equations (63) and (64) that define absolute incomes. Divide equations (63) and (64) by P_L and use the solutions for relative prices presented above. Note again that direct insertions of solutions already discussed above have not been made for the sake of simplicity.

(90) $\frac{Y_1}{P_L} = \frac{P_K}{P_L} \cdot K + a \frac{P_M}{P_L} \cdot \left(\frac{M_T}{P_L}\right)$

(91) $\frac{Y_2}{P_L} = \frac{P_L}{P_L} \cdot L + (1-a) \frac{P_M}{P_L}\left(\frac{M_T}{P_L}\right)$

In order to obtain the solutions for consumption, use equations (55) - (62) and several of the solutions already presented above. As before, note that not all necessary insertions are made.

(92) $C_{11} = \frac{Au_1 + Bu_2 + Cu_3}{A+B+C+D} \cdot X_1 + a \cdot \frac{A\mu_1 + B\mu_2 + C\mu_3}{A+B+C+D} \cdot X_1$

(93) $C_{21} = \frac{Au_1 + Bu_2 + Cu_3}{A+B+C+D} \cdot X_2 + a \cdot \frac{A\mu_1 + B\mu_2 + C\mu_3}{A+B+C+D} \cdot X_2$

(94) $C_{31} = \frac{Au_1 + Bu_2 + Cu_3}{A+B+C+D} \cdot \lambda_3 + a \cdot \frac{A\mu_1 + B\mu_2 + C\mu_3}{A+B+C+D} \cdot \lambda_3$

(95) $\frac{M_1}{P_L} = \frac{D}{A+B+C+D} \left[\frac{\mu_1 (Au_1 + Bu_2 + Cu_3)}{u_1 (A\mu_1 + B\mu_2 + C\mu_3)} \cdot \left(\frac{M}{P_L}\right) + a\left(\frac{M_T}{P_L}\right)\right]$

(96) $C_{12} = \frac{A\lambda_1 + B\lambda_2 + C\lambda_3}{A+B+C+D} \cdot X_1 + (1-a) \frac{A\mu_1 + B\mu_2 + C\mu_3}{A+B+C+D} \cdot X_1$

$$(97)\quad C_{22} = \frac{A\lambda_1 + B\lambda_2 + C\lambda_3}{A+B+C+D} \cdot X_2 + (1-a)\frac{A\mu_1 + B\mu_2 + C\mu_3}{A+B+C+D} \cdot X_2$$

$$(98)\quad C_{32} = \frac{A\lambda_1 + B\lambda_2 + C\lambda_3}{A+B+C+D} \cdot X_3 + (1-a)\frac{A\mu_1 + B\mu_2 + C\mu_3}{A+B+C+D} \cdot X_3$$

$$(99)\quad \frac{\mathcal{M}_2}{P_L} = \frac{D}{A+B+C+D}\left[\frac{\mu_1(A\lambda_1 + B\lambda_2 + C\lambda_3)}{\lambda_1(A\mu_1 + B\mu_2 + C\mu_3)}\left(\frac{M}{P_L}\right) + (1-a)\frac{M_T}{P_L}\right]$$

The last two solutions below refer to the distribution of real cash balances between their uses as consumer good and producer good. To obtain such solutions, use equations (95), (99) and (71).

$$(100)\quad \frac{M}{P_L} = \frac{(A+B+C)\lambda_1 u_1 (A\mu_1 + B\mu_2 + C\mu_3)}{(A+B+C+D)\lambda_1 u_1 (A\mu_1 + B\mu_2 + C\mu_3) + \lambda_1 \mu_1 D (Au_1 + Bu_2 + Cu_3) + u_1 \mu_1 D (A\lambda_1 + B\lambda_2 + C\lambda_3)}$$

Finally, use equations (71) and equation (100) to get:

$$(101)\quad \frac{\mathcal{M}}{P_L} = \frac{M_T}{P_L} - \frac{M}{P_L}$$

Thus model II is presented and a set of solutions is obtained. The remainder of this chapter is devoted to a comparison of the two models and interpreting comments.

Comparisons and Interpretations

There are some basic similarities as well as dissimilarities between model I and model II. The basic similarities include that both are static, general equilibrium models of the Cobb-Douglas form, each model having three commodities and two classes of persons, i.e., the capitalists and the workers. Similarities between the two models have been encouraged so that dissimilarities could be studied more effectively. An important characteristic of both models is that a set of solutions can be obtained so that the effect of basic dissimilarities in the construction of the models could be traced in the solutions of the models.

Model I has been constructed with the basic purpose to be extended in order to accommodate money. Money in the form of real cash balances is introduced in model II which can be viewed as an extension of model I. In this respect, model I and model II have a basic dissimilarity. Model II is a monetary model while model I is a barter model. A question naturally arises. How has money in the form of real cash balances been introduced in model II? A simple examination of the way model II has been constructed reveals that real cash balances are included in the production functions of the model as well as in the utility functions of the consumers.

The theoretical motivation for such a construction is the discussion in the current literature, such as Patinkin (1965) and Levhari and Patinkin (1968), where money is viewed as a producer good or a consumer good. Perhaps a brief discussion of Levhari and Patinkin's ideas may prove to be of interest. It has been argued that the existence of cash balances in an economy facilitates production. Resources of the economy are more efficiently utilized in a monetary economy than a barter economy. Resources that would have to be consumed in an attempt to overcome the difficulties of a barter economy are efficiently utilized in a monetary economy. More precisely, as Levhari and Patinkin (1968) put it, "money is held only because it enables the economic unit in question to acquire or produce a larger quantity of commodities in the usual sense of the term" (p. 737).

The question that arises is this: how can money mathematically express the function attributed to it by monetary theory? To account for this function of cash balances, money is introduced as an input argument in the production functions of model II. This is not the most theoretically desirable technique. The optimum-inventory approach suggested by Baumol (1952) has theoretical appeal because when money is viewed as a producer good, it is indeed viewed as an inventory. However, the optimum-inventory approach is analytically more complex compared to the mathematically simpler approach of considering money as an argument in the production functions.

In addition to considering money as a producer good, this chapter presents an additional novelty, i.e., viewing money as a consumer good. Classical economists viewed money as a "veil" which had to be removed before the real variables of the economy could be studied. Money was simply a unit of account and a means of exchange. Because of these two functions of money, classical economists believed that money had no direct utility itself. The only utility money had was the indirect utility obtained from goods and services purchased with the help of money. It was viewed as a double counting to compute utility derived from real goods and services and utility derived from money. As Hansen (1970) argues, "this does not follow logically however" (p. 56). If utility is to be attributed to money, it will be imputed because of money's function as a store of value. Real cash balances yield utility because of the liquidity they provide to their holders. As Hansen (1970) puts it, ". . . money in real terms is useful not only because it can be exchanged for useful commodities, but because it facilitates the process of transactions, bridges gaps between payments and receipts, makes unexpected purchases possible, and so forth" (pp. 56-7). On the basis of the above theoretical grounds, money is introduced in the utility functions of the model.

At this point, the reader must be reminded that the separate discussion of money as a producer good and money as a consumer good should not be interpreted to imply that money can not be viewed as fulfilling both these roles simultaneously. The mathematical complexities of viewing real cash balances as a producer good and consumer good simultaneously have prevented economic theorists such as Levhari and Patinkin (1968) from studying the general case. It is interesting however to note Levhari and Patinkin's (1968) conclusion: "hence, a truly general model would analyze the total demand for money from both these viewpoints" (p. 752). It is in accordance with this spirit that models I and II have been built, i.e., to study the general case.

In addition to the theoretical arguments already presented above for the introduction of money in the production and utility functions, one may give some additional evidence from economic history. There seems to be very little doubt about money's contribution to rapid economic growth through extensive specialization, through saving time and facilitating the production and exchange of goods and services, and finally through adding to convenience and security of consumers as well as producers.

The simultaneous introduction of money as a consumer and as a producer good in model II allows the monetary theorists to study the allocation of the stock of real cash balances between producers and consumers. Less general cases could have been studied, i.e., models could have been developed where money is considered to be only a consumer good or only a producer good, but not both. The treatment of the general case, i.e., model II, has been preferred over alternative special cases.

With respect to the incomes of the two classes of persons, a less arbitrary assumption is made concerning the distribution of the stock of real cash balances. While capitalists still own all capital and workers all labor resources, cash balances are distributed between these two classes according to some proportion α. Such a definition of incomes in model II causes dissimilarities in the consumption solutions of models I and II.

An additional difference that may be pointed out between model I and model II is the nature of the production functions of the two models. In model I, the production functions do not include real cash balances as an argument. In a barter economy, money is not used. Extending the barter economy into a monetary one, the production functions of model II do include real cash balances as an argument. As a matter of fact, in a monetary economy, production functions could not but include real cash balances since production is not feasible without the use of money. So, once an economy is monetized, its reversal to a barter economy is impossible. Another basic difference between model I and model II is that model II has a money market while model I does not. Such a money market is necessary if solutions are to be obtained in model II. Finally, an important characteristic of model II only is the separation of ownership and use of money. The use of money by both the consumers and producers is not restricted by ownership of money. This quality that the model possesses is, of course, in accordance with economic reality.

Thus, the dissimilarities between model I and model II include (a) money as a producer good, (b) money as a consumer good, (c) incomes contain a portion of the stock of money and finally (d) money market.

Model I and model II as constructed and solved above contribute in the removal of a major criticism raised in the literature and more carefully elaborated in Chapter II of the present study. Model I describes a barter economy quite realistically and the existence of three commodities naturally creates barter difficulties since "the double coincidence" is not likely to occur with the same ease as in a two commodity barter economy or even more trivially in a one good economy. It is these barter difficulties that the introduction of money in model II attempts to remove. The presence of three commodities contributes to a sound theoretical foundation for the introduction of money. However, not all basic criticisms of modern monetary growth theory have been removed by the construction and solution of models I and II. In order to contribute to the removal of some of the additional criticisms, models III and IV are presented in the next chapter.

CHAPTER IV

MODELS III AND IV

In this chapter, models III and IV will be developed and solved. Model III is a dynamic, barter, general equilibrium model. It is developed with the basic purpose of providing a theoretically sound framework for the introduction of money. Such an introduction of money in a dynamic economy is achieved in model IV. As in Chapter III, the two models will be presented and solved and then critically compared and evaluated. In relating the two models of the previous chapter to the ones of the present chapter, it should be noted that models III and IV are extensions of the earlier ones. Instead of having three commodities, models III and IV, because of their dynamic nature, include two consumption goods and an investment good. Some fundamental differences are obvious between the models of the previous chapter and the present one. They are due to the methodological differences that are present always between static and dynamic models. Apart from this methodological dissimilarity, the models of the last chapter and the models of the present chapter continue to demonstrate the same basic idea: the fundamentals of a model of the introduction of money in a barter economy. Aspects of the models related to their stability are not discussed.

Model III

Some additional notation is necessary for the present model. Only notation not presented before is introduced below.

Variables

I = investment good

S = savings

P_3 = price of the investment good

Parameters

n = rate of growth of labor force

b = proportion of capitalist's income saved

Note that all variables are subscripted by t, where t = time. t = 0 indicates the values of the variables at the beginning of the period, i.e., the initial conditions of the model.

The production functions of model III are described in equations (1) - (3) below:

(1) $X_1(t) = [K_1(t)]^{u_1}[L_1(t)]^{\lambda_1}, \quad K_1(t)>0, \ L_1(t)>0, \ 0<u_1,\lambda_1<1, \ u_1+\lambda_1=1$

(2) $X_2(t) = [K_2(t)]^{u_2}[L_2(t)]^{\lambda_2}, \quad K_2(t)>0, \ L_2(t)>0, \ 0<u_2,\lambda_2<1, \ u_2+\lambda_2=1$

(3) $I(t) = [K_3(t)]^{u_3}[L_3(t)]^{\lambda_3}, \quad K_3(t)>0, \ L_3(t)>0, \ 0<u_3,\lambda_3<1, \ u_3+\lambda_3=1$

Observe that Cobb-Douglas production functions are postulated with constant returns to scale. This is in accordance with the preceding models.

Assume profit maximization under conditions of perfect competition. The first order conditions of such a profit maximization yield the input demand equations (4) - (9).

(4) $p_K(t) = P_1(t) \dfrac{u_1}{K_1(t)} \cdot X_1(t)$

(5) $p_L(t) = P_1(t) \dfrac{\lambda_1}{L_1(t)} \cdot X_1(t)$

(6) $p_K(t) = P_2(t) \dfrac{u_2}{K_2(t)} \cdot X_2(t)$

(7) $p_L(t) = P_2(t) \dfrac{\lambda_2}{L_2(t)} \cdot X_2(t)$

(8) $p_K(t) = P_3(t) \dfrac{u_3}{K_3(t)} \cdot I(t)$

(9) $p_L(t) = P_3(t) \dfrac{\lambda_3}{L_3(t)} \; I(t)$

Euler's theorem can easily be derived from the above equations. Simple algebraic manipulations yield:

(4a) $P_1(t) X_1(t) = p_K(t) K_1(t) + p_L(t) L_1(t)$

(6a) $P_2(t) X_2(t) = p_K(t) K_2(t) + p_L(t) L_2(t)$

(8a) $P_3(t) I(t) = p_K(t) K_3(t) + p_L(t) L_3(t)$

Having presented the elements of the production side of the model, the utility functions and demand functions follow. Assume, as in Chapter II, that there are two persons, or two groups of individuals, having similar utility functions of the Cobb-Douglas type. Assume furthermore that only the capitalist saves while the worker consumes his entire income. More formally, these assumptions are expressed in the following equations:

$$U_1(t) = C_{11}^A(t)\, C_{21}^B(t)$$
$$U_2(t) = C_{12}^A(t)\, C_{22}^B(t)$$

The budget constraints under the assumptions discussed are:

$$Y_1(t) - S(t) = P_1(t)C_{11}(t) + P_2(t)C_{21}(t)$$
$$Y_2(t) = P_1(t)C_{12}(t) + P_2(t)C_{22}(t)$$

Utility maximization under the budget constraint yields the demand functions of model III. They are presented in equations (10) - (13).

(10) $C_{11} = \dfrac{Y_1(t)(1-b)}{P_1(t)} \cdot \dfrac{A}{A+B}$

(11) $C_{21} = \dfrac{Y_1(t)(1-b)}{P_2(t)} \cdot \dfrac{B}{A+B}$

(12) $C_{12} = \dfrac{Y_2(t)}{P_1(t)} \cdot \dfrac{A}{A+B}$

(13) $C_{22} = \dfrac{Y_2(t)}{P_2(t)} \cdot \dfrac{B}{A+B}$

The definitions of personal incomes are similar to the ones of model I. More specifically they are:

(14) $Y_1(t) = p_K(t) \cdot K(t)$

(15) $Y_2(t) = p_L(t) \cdot L(t)$

An important feature of this model is the existence of savings. It is postulated that savings is a proportion of the income of the capitalist. This is one out of several different savings assumptions to be found in the literature of economic growth.[1] In general, there are two broad categories of savings assumptions: first, the neoclassical assumption where savings is proportional to total income and second, the Cambridge assumption where capitalists save all their income and consumers spend all their income. The assumption introduced above is believed to be a more realistic version of the Cambridge savings assumption. It is more realistic because one could not expect capitalists to live without consuming. In general, a model postulating savings and consumption for both capitalists and workers would obviously come closer to the real world. This, however, would cause unnecessary mathematical complications. Furthermore, equilibrium between savings and investment is stated below to demonstrate the usual assumption that desired savings and desired investment are equal. This assumption is very common in the literature of neoclassical growth models. In such models, once a certain proportion of output is saved, the level of investment required to maintain the economy at full employment is automatically determined. Unlike the neoclassical models, the savings equals investment

[1]See for example Stiglitz and Uzawa (1969), pp. 309 - 313.

assumption of model III needs to be interpreted differently because this model postulates an investment function described by equation (3). All equation (17) demonstrates is an equilibrium condition for income. Observe that savings is deflated by the price of the investment good since $I(t)$ is expressed in real terms and $S(t)$ is expressed in monetary terms.

(16) $S(t) = bY_1(t)$

(17) $I(t) = \dfrac{S(t)}{P_3(t)}$

For the model to be completed, the equilibrium conditions in the input and output markets need to be stated. With respect to the output markets, equilibrium in the one market automatically guarantees equilibrium in the other. To indicate this only one equation is numbered.

(18) $X_1(t) = C_{11}(t) + C_{12}(t)$

(18a) $X_2(t) = C_{21}(t) + C_{22}(t)$

It is important to demonstrate that Walras' law holds once again, and that in particular the introduction of an independent investment function described by equation (3) is consistent with this law. Combine equations (4a), (6a), (8a), (14), (15) and equations (19) and (20) below to obtain:

$$P_1(t)X_1(t) + P_2(t)X_2(t) + P_3(t)I_3(t) = Y_1(t) + Y_2(t)$$

On the other hand, combine equations (10) - (13) and obtain:

$$P_1(t)X_1(t) + P_2(t)\left[C_{21}(t) + C_{22}(t)\right] = Y_1(t)(1-b) + Y_2(t)$$

Making use of equations (16) and (17), this last equation becomes:

$$P_1(t)X_1(t) + P_2(t)\left[C_{21}(t) + C_{22}(t)\right] + P_3(t)I(t) = Y_1(t) + Y_2(t)$$

which immediately implies equation (18a). The dynamic nature of the model indicates that Walras' law holds true for all time periods.

In the input markets, note the definition of capital, i.e., $K(t)$. It is the sum of capital at the beginning of the period, plus capital accumulated from $t = 0$ to present time t . For simplicity, capital consumption allowance is assumed to be zero. Furthermore, capital is assumed homogeneous and although technical progress could be introduced, since the model is built to shed some light in the introduction of money rather than other aspects, it is assumed that there is no technical progress.

(19) $K(t) = K_1(t) + K_2(t) + K_3(t)$

(20) $L(t) = L_1(t) + L_2(t) + L_3(t)$

(21) $K(t) = K(0) + \int_0^t I(t)\,dt$

(21a) $L(t) = L(0)e^{nt}$

Note the assumption described in equation (21a). It is assumed that labor force grows exogeneously at a rate n .

This assumption is rather usual in growth theory.

The model as described above consists of 22 unknowns and 21 equations. Note that $L(t)$, i.e., labor force at period t, although a variable, is not included among the other 22 variables since equation (21a) provides the solution in the form of an assumption. The unknowns of the model are: L_1, L_2, L_3, K_1, K_2, K_3, K, X_1, X_2, I, P_1, P_2, P_3, p_K, p_L, Y_1, Y_2, S, C_{11}, C_{21}, C_{12}, C_{22}. Let p_L, the price of labor, be the numeraire. Before a set of solutions is presented, a remark is in order. To keep notation as simple as possible, the time subscript is dropped from the variables. Since the present model is a dynamic one, it is understood that the values of the 22 variables stated above depend upon time. Furthermore, assume that the initial values of all the variables are given. With the above introductory comments, we embark on presenting a set of solutions of the model.

First, to solve for the values of the inputs rearrange equations (4a), (6a), (8a), (10) - (13), (14) and (15) to obtain the subsystem:

$$X_1 P_1 = P_1 C_{11} + P_1 C_{12} = Y_1 (1-b) \frac{A}{A+B} + Y_2 \frac{A}{A+B} = p_K K_1 + p_L L_1$$

$$X_2 P_2 = P_1 C_{21} + P_2 C_{22} = Y_1 (1-b) \frac{B}{A+B} + Y_2 \frac{B}{A+B} = p_K K_2 + p_L L_2$$

$$I P_3 = b Y_1 = p_K K_3 + p_L L_3$$

Perform the necessary algebraic operations to solve for K_1, K_2, K_3. These solutions are not as complicated as the solutions of L_1, L_2, L_3. In equations (22) - (27), the solutions are explicitly presented.

$$(22) \quad K_1 = \frac{Au_1(1-bu_3)}{Au_1 + Bu_2} \cdot K$$

$$(23) \quad K_2 = \frac{Bu_2(1-bu_3)}{Au_1 + Bu_2} \cdot K$$

$$(24) \quad K_3 = bu_3 K$$

$$(25) \quad L_1 = \frac{A\lambda_1[1-b(1-\lambda_3)]}{A(\lambda_1 + b\lambda_3 - b\lambda_1) + B(\lambda_2 + b\lambda_3 - b\lambda_2)} \cdot L$$

$$(26) \quad L_2 = \frac{B\lambda_2[1-b(1-\lambda_3)]}{A(\lambda_1 + b\lambda_3 - b\lambda_1) + B(\lambda_2 + b\lambda_3 - b\lambda_2)} \cdot L$$

$$(27) \quad L_3 = \frac{Ab\lambda_3(1-\lambda_1) + Bb\lambda_3(1-\lambda_2)}{A(\lambda_1 + b\lambda_3 - b\lambda_1) + B(\lambda_2 + b\lambda_3 - b\lambda_2)} \cdot L$$

Although some critical comments concerning the comparison of the solutions of model I and the present one will be made in the last section of this chapter, it is interesting to note the great significance of the propensity to save. From a mathematical standpoint, the significance of the propensity to save is witnessed by the presence of the coefficient b, i.e., the proportion of the capitalist's income saved, in all solutions demonstrated by equations (22) - (27). More interestingly, from an economic standpoint, the introduction of an investment output and savings (among other things) in model III reveals the powerful significance of the propensity to save, not observed in previous models. Such an economic significance of the propensity to save may be interpreted as follows. The propensity to save directly affects the allocation of the two inputs, i.e., capital and labor. In particular, the propensity to save causes a decrease in the proportion of inputs allocated in the production of consumption outputs as indicated by equations (22), (23), (25) and (26), in favor of an increase in the proportion of inputs allocated in the production of the investment good as shown in equations (24) and (27). In general, the higher the propensity to save is, the smaller the proportion of inputs allocated in the production of consumption outputs will be (and consequently, the higher the proportion of inputs allocated in the production of the investment good). Such an observation is clearly in accordance with the principles of economic theory.

Having solved for the inputs, direct insertions of equations (22) - (27) in equations (1) - (3) yield the solutions X_1, X_2 and I.

$$(28)\quad X_1 = \left[\frac{AKu_1(1-bu_3)}{Au_1+Bu_2}\right]^{u_1}\left[\frac{AL\lambda_1[1-b(1-\lambda_3)]}{A(\lambda_1+b\lambda_3-b\lambda_1)+B(\lambda_2+b\lambda_3-b\lambda_2)}\right]^{\lambda_1}$$

$$(29)\quad X_2 = \left[\frac{BKu_2(1-bu_3)}{Au_1+Bu_2}\right]^{u_2}\left[\frac{BL\lambda_2[1-b(1-\lambda_3)]}{A(\lambda_1+b\lambda_3-b\lambda_1)+B(\lambda_2+b\lambda_3-b\lambda_2)}\right]^{\lambda_2}$$

$$(30)\quad I = \left[bu_3K\right]^{u_3}\left[\frac{ALb\lambda_3(1-\lambda_1)+BLb\lambda_3(1-\lambda_2)}{A(\lambda_1+b\lambda_3-b\lambda_1)+B(\lambda_2+b\lambda_3-b\lambda_2)}\right]^{\lambda_3}$$

The solutions of relative prices are presented next. They are very simple to obtain. Use equations (4) - (9) and solve directly for relative prices. The solutions are given below in equations (31) - (34). Note that insertions of solutions already obtained have not been made to keep the expressions in their simplest forms.

$$(31) \quad \frac{P_1}{P_L} = \frac{L_1}{\lambda_1} \cdot \frac{1}{X_1}$$

$$(32) \quad \frac{P_2}{P_L} = \frac{L_2}{\lambda_2} \cdot \frac{1}{X_2}$$

$$(33) \quad \frac{P_3}{P_L} = \frac{L_3}{\lambda_3} \cdot \frac{1}{I}$$

$$(34) \quad \frac{P_K}{P_L} = \frac{u_1}{\lambda_1} \cdot \frac{L_1}{K_1}$$

Following the solutions of relative prices are the solutions of relative incomes. They are necessary for the solutions of consumption.

$$(35) \quad \frac{Y_1}{P_L} = \frac{u_1}{\lambda_1} \cdot \frac{L_1}{K_1} \cdot K$$

$$(36) \quad \frac{Y_2}{P_L} = L$$

Since the usual assumption of "savings equals investment" is introduced in equation (17), the solution of savings is directly obtained from equations (16) and (35).

$$(37) \quad \frac{S}{P_L} = b \cdot \frac{u_1}{\lambda_1} \cdot \frac{L_1}{K_1} \cdot K$$

At this point, it may be interesting to check whether the solutions of $I(t)$ and $S(t)$ given above satisfy equation (17) of the model. Such a check would reveal both internal consistency of the model as well as correct solutions. From equation (17) obtain:

$$P_3 I = S$$

From equation (37), solve for $S(t)$ and obtain:

$$S = P_L \cdot b \cdot \frac{u_1}{\lambda_1} \cdot \frac{L_1}{K_1} \cdot K$$

Thus we have:

$$P_3 I = P_L \cdot b \cdot \frac{u_1}{\lambda_1} \cdot \frac{L_1}{K_1} K$$

Divide both sides by P_L , use equations (33), (25), (27) and (22). Simple algebraic operations show that equality holds.

Now, to obtain the solutions of consumption, use equations (10) - (13) and solutions already obtained above to get:

(38) $C_{11} = u_1(1-b)\,\frac{K}{K_1} \cdot X_1 \cdot \frac{A}{A+B}$

(39) $C_{21} = u_1(1-b)\,\frac{K}{K_1} \cdot X_2 \,\frac{B}{A+B}$

(40) $C_{12} = \lambda_1 \cdot \frac{L}{L_1} \cdot X_1 \cdot \frac{A}{A+B}$

(41) $C_{22} = \lambda_2 \cdot \frac{L}{L_2} \cdot X_2 \cdot \frac{B}{A+B}$

Finally, in order to complete the solutions of this model, it is necessary to indicate the time paths of labor growth and capital accumulation. Direct substitutions of such solutions in all the solutions presented above will allow one to perfectly determine the value of any variable at any time t . The path of labor growth has already been assumed to be that of an exponential growth. For convenience, it is reproduced below:

(42) $L(t) = L(0)e^{nt}$

The path of capital accumulation needs to be solved for. Since it is fairly complicated to find, the complete set of derivations is presented.

From equation (21) it is obtained that:

$$K(t) = K(0) + \int_0^t I(\xi)d\xi \quad , \qquad \xi = \text{variable of integration}$$

Substitute the solution of $I(t)$ obtained in equation (30); furthermore, substitute $L(t)$ by its equivalent in equation (42).

$$K(t) = K(0) + \int_0^t [bu_3 K(\xi)]^{u_3} \left[\frac{Ab\lambda_3(1-\lambda_1) + Bb\lambda_3(1-\lambda_2)}{A(\lambda_1+b\lambda_3-b\lambda_1) + B(\lambda_2+b\lambda_3-b\lambda_2)} L(0)e^{n\xi}\right]^{\lambda_3} d\xi$$

$$K(t) = K(0) + (bu_3)^{u_3}\left[\frac{Ab\lambda_3(1-\lambda_1) + Bb\lambda_3(1-\lambda_2)}{A(\lambda_1+b\lambda_3-b\lambda_1) + B(\lambda_2+b\lambda_3-b\lambda_2)} L(0)\right]^{\lambda_3} \cdot \int_0^t [K(\xi)]^{u_3} e^{n\lambda_3\xi} d\xi$$

Let

$$G = (bu_3)^{u_3}\left[\frac{Ab\lambda_3(1-\lambda_1) + Bb\lambda_3(1-\lambda_2)}{A(\lambda_1+b\lambda_3-b\lambda_1) + B(\lambda_2+b\lambda_3-b\lambda_2)} L(0)\right]^{\lambda_3}$$

Hence the last equation becomes:

$$K(t) = K(0) + G\int_0^t [K(\xi)]^{u_3} e^{n\lambda_3\xi} d\xi$$

Differentiate both sides with respect to t .

$$K'(t) = G[K(\xi)]^{u_3} e^{n\lambda_3\xi}$$

Divide by $[K(\xi)]^{u_3}$ and integrate, $[K(\xi)]^{u_3} \neq 0$.

$$\int_0^t \frac{K'(\xi)}{[K(\xi)]^{u_3}} d\xi = G\int_0^t e^{n\lambda_3\xi} d\xi$$

$$\left.\frac{[K(\xi)]^{1-u_3}}{1-u_3}\right|_0^t = \left.\frac{G}{n\lambda_3} e^{n\lambda_3\xi}\right|_0^t$$

$$\frac{[K(t)]^{1-u_3}}{1-u_3} - \frac{[K(0)]^{1-u_3}}{1-u_3} = \frac{G}{n\lambda_3} e^{n\lambda_3 t} - \frac{G}{n\lambda_3}$$

$$[K(t)]^{1-u_3} = (1-u_3)\left\{\frac{[K(0)]^{1-u_3}}{1-u_3} - \frac{G}{n\lambda_3}\left[e^{n\lambda_3 t} - 1\right]\right\}$$

Thus:

$$(43) \quad K(t) = \left\{[K(0)]^{1-u_3} + \frac{G(1-u_3)}{n\lambda_3}\left[e^{n\lambda_3 t} - 1\right]\right\}^{\frac{1}{1-u_3}}$$

and equivalently,

$$(43a) \quad K(t) = \left[K(0)^{\lambda_3} + \frac{G}{n}\left(e^{n\lambda_3 t} - 1\right)\right]^{\frac{1}{\lambda_3}}$$

Some economic interpretations are in order for a better understanding of model III. This model is a dynamic, general equilibrium model with two consumption goods and an investment good. Unlike neoclassical growth models, model III explicitly postulates an investment function. It has been shown that Walras' law holds true for all time periods and that total output is in equilibrium. A question naturally arises. Is such an output equilibrium a full employment equilibrium? The answer is provided by the solutions of the model. Observe that all solutions refer to some period of time t . Obviously, the solutions (25), (26) and (27) indicate that the proportions of labor force allocated to the production of the three outputs add to give 1. Thus, model

III is a full employment model.

Furthermore, one may ask about the nature of the growth paths of the variables of the model. A detailed presentation of such growth paths for the 22 variables is not shown, not only because of the complexity of the expressions, but also because it lies beyond the scope of this thesis. Recall that emphasis is given to a meaningful integration of real and monetary growth models rather than to a full presentation of a real growth theory. All the details, however, for a discussion of growth paths are available. The basic exogenous force of growth is provided by the growth of labor force postulated by equation (21a). Assuming full employment, the growth of labor determines the growth path of the worker's income since the wage rate has been postulated to be maintained fixed, playing the role of the economy's numeraire. Also, labor force causes partially the growth of the three outputs according to the proportion of its allocation. However, the most interesting growth effect of the labor force is the effect labor force has on capital accumulation. The growth path of capital is determined directly by the influence which the labor force's growth has on the investment function. Naturally, the determination of capital's growth path is subject to two basic constraints: Walras' law and the "savings equals investment" assumption. So, once the "right" rate of growth of capital is determined, capital growth (endogeneously determined) and labor growth (exogenously determined) help determine the growth paths of the remaining variables. Note that the simplicity of growth paths observed in simple neoclassical growth models where all variables grow at the same rates, i.e., stationary states, is not present in model III. Growth rates vary and in general are determined by the rate of growth of labor, the rate of growth of capital, savings coefficient and all the remaining parameters of the model. Equipped with such a powerful tool, the analysis proceeds to model IV.

Model IV

The analysis of this chapter continues with the presentation and solution of the dynamic general equilibrium model with money. This model is considered to be the most complete model of the dissertation; and former models, although contributing in themselves, were built as preliminary steps to the construction of model IV.

The production sector of model IV consists of two consumption goods and an investment good. This is similar to model III. The difference between model III and the present model is the introduction of real cash balances in the production functions and utility functions. The theoretical issues of money viewed as a producer good and consumer good have been touched upon in Chapter III. More formally, the production functions of the model are:

$$(44)\quad X_1(t) = \left[K_1(t)\right]^{u_1}\left[L_1(t)\right]^{\lambda_1}\left[\frac{M_1(t)}{P_L(t)}\right]^{\mu_1}$$

$$(45)\quad X_2(t) = [K_2(t)]^{u_2}[L_2(t)]^{\lambda_2}\left[\frac{M_2(t)}{P_L(t)}\right]^{\mu_2}$$

$$(46)\quad I(t) = [K_3(t)]^{u_3}[L_3(t)]^{\lambda_3}\left[\frac{M_3(t)}{P_L(t)}\right]^{\mu_3}$$

The usual assumptions of positive inputs, positive elasticities of outputs with respect to inputs and constant returns to scale are retained. Profit maximization under perfectly competitive market structure yields the following first order conditions:

$$(47)\quad P_K(t) = P_1(t) \cdot \frac{u_1}{K_1(t)} \cdot X_1(t)$$

$$(48)\quad P_L(t) = P_1(t) \frac{\lambda_1}{L_1(t)} \cdot X_1(t)$$

$$(49)\quad P_M(t) = P_1(t) \frac{\mu_1}{\left[\frac{M_1(t)}{P_L(t)}\right]} \cdot X_1(t)$$

$$(50)\quad P_K(t) = P_2(t) \frac{u_2}{K_2(t)} \cdot X_2(t)$$

$$(51)\quad P_L(t) = P_2(t) \frac{\lambda_2}{L_2(t)} \cdot X_2(t)$$

$$(52)\quad P_M(t) = P_2(t) \frac{\mu_2}{\left[\frac{M_2(t)}{P_L(t)}\right]} X_2(t)$$

$$(53)\quad P_K(t) = P_3(t) \frac{u_3}{K_3(t)} \cdot I(t)$$

$$(54)\quad P_L(t) = P_3(t) \frac{\lambda_3}{L_3(t)} \cdot I(t)$$

$$(55)\quad P_M(t) = P_3(t) \frac{\mu_3}{\left[\frac{M_3(t)}{P_L(t)}\right]} \cdot I(t)$$

Following the pattern established in previous models, Euler's theorem when applied in this model yields:

$$(47a)\quad P_1(t)\,X_1(t) = p_K(t)\,K_1(t) + p_L(t)\,L_1(t) + p_M(t)\,\frac{M_1(t)}{p_L(t)}$$

$$(50a)\quad P_2(t)\,X_2(t) = p_K(t)\,K_2(t) + p_L(t)\,L_2(t) + p_M(t)\,\frac{M_2(t)}{p_L(t)}$$

$$(53a)\quad P_3(t)\,I(t) = p_K(t)\,K_3(t) + p_L(t)\,L_3(t) + p_M(t)\,\frac{M_3(t)}{p_L(t)}$$

Having presented the basic elements of the production side of the model, a discussion of the consumption side of the model follows. Note that the utility function includes real cash balances as one of the arguments. The introduction of money in the utility function has been discussed in Chapter III and similar arguments hold true in the present model. Note also that only one utility function is postulated instead of two as was the case in the previous models. The idea behind this change is purely computational. The introduction of money in a rather large and complex model such as model III presents difficult computations. Theoretically, not very important information is lost by considering the special case of a classless society, in particular since the purpose of model IV is to provide a meaningful dynamic integration rather than to study the relative shares of the society's different classes. The new notation needed is the following:

C_i = aggregate consumption of i^{th} output; i = 1,2

U = aggragate utility

Y = total income

$\frac{M}{P_L}$ = total cash balances viewed as a consumption good

On the basis of the above, the utility function postulated in model IV is of the form:

$$U = \left[C_1(t) \right]^A \left[C_2(t) \right]^B \left[\frac{M(t)}{P_L(t)} \right]^D$$

To derive the demand functions, the above utility function needs to be maximized under the budget constraint stated below.

$$Y(t) - S(t) = P_1(t)C_1(t) + P_2(t)C_2(t) + P_4(t)\left[\frac{M(t)}{P_L(t)}\right]$$

where:

(56) $S(t) = bY(t)$

Two remarks are in order at this point. First, observe that not all total income is spent on consumption. A proportion of total income is saved and the savings function is explicitly postulated in equation (56). Second, the process of maximizing aggregate utility under the budget constraint to obtain demand functions is not entirely necessary. It contributes to the completeness of the model but also might be criticized along the conceptual lines of the existence of such a social utility function.

If the reader has serious objections against the process of maximizing social utility under the given constraint in order to obtain the demand functions described in equations (57) - (59), then he may want to consider equations (57) - (59) as being postulated rather than derived. This is a very usual approach in the growth theory literature.

The demand functions are:

$$(57)\quad C_1(t) = \frac{(1-b)Y(t)}{P_1(t)} \cdot \frac{A}{A+B+D}$$

$$(58)\quad C_2(t) = \frac{(1-b)Y(t)}{P_2(t)} \cdot \frac{B}{A+B+D}$$

$$(59)\quad \frac{M(t)}{P_L(t)} = \frac{(1-b)Y(t)}{P_4(t)} \cdot \frac{D}{A+B+D}$$

Total income is defined next. According to this definition, total money income is the sum of the yields received from the services of the capital stock, the services of the labor force and the services of total real cash balances. It is assumed that all inputs are homogeneous so that their total quantities could be measured and their monetary values would be meaningful.

$$(60)\quad Y(t) = p_K(t)\,K(t) + p_L(t)\,L(t) + p_M(t)\,\frac{M_T(t)}{P_L(t)}$$

Following the definition of income is the usual equilibrium condition stating that desired investment is equal to desired savings.

$$(61)\quad I(t) = \frac{S(t)}{P_3(t)}$$

The presentation of model IV is near completion. Equilibrium conditions need to be stated to fully complete the model. Previous discussions should facilitate the meaning of the following three types of equilibrium conditions.

First, equilibrium on Output Markets;

(62) $X_1(t) = C_1(t)$

(62a) $X_2(t) = C_2(t)$

Second, equilibrium in Input Markets;

(63) $K(t) = K_1(t) + K_2(t) + K_3(t)$

(64) $L(t) = L_1(t) + L_2(t) + L_3(t)$

(65) $\frac{M(t)}{p_L(t)} = \frac{M_1(t)}{p_L(t)} + \frac{M_2(t)}{p_L(t)} + \frac{M_3(t)}{p_L(t)}$

Third, equilibrium conditions in the Money Market;

(66) $\frac{M_T(t)}{p_L(t)} = \frac{M(t)}{p_L(t)} + \frac{\mathcal{M}(t)}{p_L(t)}$

(67) $P_4(t) = p_M(t)$

It seems worthwhile to establish Walras' law once again. This is both mathematically and economically desirable. Mathematically, the demonstration of Walras' law, is shown to hold, establishes the truth of equation (62a). Economically, Walras' law establishes the consistency of the equilibrium conditions of the model. Such a property if particularly desirable in the present model which is very complex.

To show that Walras' law holds true, use equations (47a), (50a), (53a), (63) - (65) and (60) to obtain (note that the subscript t has been dropped):

$$P_1X_1 + P_2X_2 + P_3I = p_K K + p_L L + p_M\left(\frac{M}{p_L}\right)$$

Add to both sides of the above expression $P_4\left(\frac{\mathcal{M}}{p_L}\right)$; make use of equation (67) and obtain:

(*) $P_1X_1 + P_2X_2 + P_3I + P_4\left(\frac{\mathcal{M}}{p_L}\right) = p_K K + p_L L + p_M\left[\frac{M}{p_L} + \frac{\mathcal{M}}{p_L}\right] = Y$

On the other hand, combine equations (57) - (59) to get the following expression:

$$P_1 C_1 + P_2 C_2 + P_4 \left(\frac{M}{P_L} \right) = (1-b)Y$$

Use equation (62) to rewrite this last expression as

$$P_1 X_1 + P_2 C_2 + P_4 \left(\frac{M}{P_L} \right) = (1-b)Y$$

Finally, make use of equations (56) and (61) to get:

$$(**) \quad P_1 X_1 + P_2 C_2 + P_4 \left(\frac{M}{P_L} \right) + P_3 I = Y$$

Obviously, equations (*) and (**) imply equation (62a). The above verifies that Walras' law holds true.

The dynamic nature of the model requires some assumptions concerning the growth of inputs. Presently, it is assumed that labor grows exponentially at a rate of n per period of time and that also the money supply grows exponentially at a rate of μ per period of time. Assumptions of this nature are not too restrictive and have been widely used in the current literature. No assumption has been made concerning the growth of capital. Capital accumulation will be determined within the context of solutions of the model. Algebraically, the assumptions can be stated as follows:

$$(67) \quad L(t) = L(0)\, e^{nt}$$

$$(68) \quad \frac{M_T(t)}{P_L(t)} = \frac{M_T(0)}{P_L(0)}\, e^{\mu t}$$

$$(69) \quad K(t) = K(0) + \int_0^t I(t)\,dt$$

Assume that all initial values are known. The model consists of 27 equations and 28 unknowns. Use $p_L(t)$ as a numeraire. The unknowns are (note that the subscript t is dropped):

L_1, L_2, L_3, L, K_1, K_2, K_3, K, M_1, M_2, M_3, M, X_1, X_2, I, Y, S, P_1, P_2, P_3, P_4, p_K, p_L, p_M, C_1, C_2, $\mathcal{M}$, M_T.

The solutions of model IV are a little complicated. The dynamic nature of the model is not the only reason. The complications of model II, caused mainly by the introduction of an independent investment function in an already large general equilibrium model, are multiplied in model IV by the introduction of money.

To proceed with the discussion and presentation of solutions, a convenient point of departure is the solutions of input allocation in the production of two consumption goods and the investment good.

Use Euler's equations (47a), (50a), (53a) and equations (47) - (62a) to obtain the following basic expressions:

$$\frac{BbK_1}{u_1} = \frac{AbK_2}{u_2} = \frac{AB}{A+B+D} \cdot \frac{K_3}{u_3}$$

$$\frac{BbL_1}{\lambda_1} = \frac{AbL_2}{\lambda_2} = \frac{AB}{A+B+D} \cdot \frac{L_3}{\lambda_3}$$

$$\frac{Bb\left(\frac{M_1}{p_L}\right)}{\mu_1} = \frac{Ab\left(\frac{M_2}{p_L}\right)}{\mu_2} = \frac{AB}{A+B+D} \cdot \frac{\left(\frac{M_3}{p_L}\right)}{\mu_3}$$

Use equations (63) - (65) and perform the necessary algebraic manipulations to obtain the following input solutions. Note that throughout the presentation of solutions, the subscript t has been dropped to simplify the expressions.

$$(70) \quad K_1 = \frac{Au_1}{Au_1 + Bu_2 + bu_3(A+B+D)} \cdot K$$

$$(71) \quad K_2 = \frac{Bu_2}{Au_1 + Bu_2 + bu_3(A+B+D)} \cdot K$$

$$(72) \quad K_3 = \frac{bu_3(A+B+D)}{Au_1 + Bu_2 + bu_3(A+B+D)} \cdot K$$

$$(73) \quad L_1 = \frac{A\lambda_1}{A\lambda_1 + B\lambda_2 + b\lambda_3(A+B+D)} \cdot L$$

$$(74) \quad L_2 = \frac{B\lambda_2}{A\lambda_1 + B\lambda_2 + b\lambda_3(A+B+D)} \cdot L$$

$$(75) \quad L_3 = \frac{b\lambda_3(A+B+D)}{A\lambda_1 + B\lambda_2 + b\lambda_3(A+B+D)} \cdot L$$

$$(76) \quad \frac{M_1}{P_L} = \frac{A\mu_1}{A\mu_1 + B\mu_2 + b\mu_3(A+B+D)} \cdot \frac{M}{P_L}$$

$$(77) \quad \frac{M_2}{P_L} = \frac{B\mu_2}{A\mu_1 + B\mu_2 + b\mu_3(A+B+D)} \cdot \frac{M}{P_L}$$

$$(78) \quad \frac{M_3}{P_L} = \frac{b\mu_3(A+B+D)}{A\mu_1 + B\mu_2 + b\mu_3(A+B+D)} \cdot \frac{M}{P_L}$$

Two remarks are appropriate at this point. They refer to the input solutions. First, observe the significance of the propensity to save b. Mathematically, it sufficies to indicate the presence of the savings coefficient b in all solutions. Economically, savings reduces total income spent for consumption which directly reduces the demand for the two consumption goods and indirectly it reduces the demand for inputs used in the production of these two consumption goods. Furthermore, the assumption of an independent investment function and the equality of investment with savings once again economically justifies the significant role of the savings coefficient.

Second, observe that the introduction of money in the form of real cash balances in model IV affected the input solutions. Mathematically, the elasticity of utility of money viewed as a consumption good is present in all input solutions. More interestingly, from an economic standpoint, the demand for money viewed as a consumption good directly affects the demand for the other two consumption goods and indirectly the input allocation solutions are also affected. Obviously, then, model IV demonstrates a meaningful integration of real growth theory and monetary theory, where solutions are definitely affected by the presence of real cash balances.

Having presented and discussed the input solutions, the remaining solutions follow relatively easily.

With respect to output solutions, direct insertions of equations (70) - (78) into equations (44) - (46) yield equations (79) - (81).

$$(79)\; X_1 = \left[\frac{Au_1K}{Au_1+Bu_2+bu_3(A+B+D)}\right]^{u_1}\left[\frac{A\lambda_1L}{A\lambda_1+B\lambda_2+b\lambda_3(A+B+D)}\right]^{\lambda_1}\left[\frac{A\mu_1\left(\frac{M}{P_L}\right)}{A\mu_1+B\mu_2+b\mu_3(A+B+D)}\right]^{\mu_1}$$

$$(80)\quad X_2 = \left[\frac{Bu_2K}{Au_1+Bu_2+bu_3(A+B+D)}\right]^{u_2}\left[\frac{B\lambda_2L}{A\lambda_1+B\lambda_2+b\lambda_3(A+B+D)}\right]^{\lambda_2}\left[\frac{B\mu_2\left(\frac{M}{P_L}\right)}{A\mu_1+B\mu_2+b\mu_3(A+B+D)}\right]^{\mu_2}$$

$$(81)\quad I = \left[\frac{bu_3(A+B+D)K}{Au_1+Bu_2+bu_3(A+B+D)}\right]^{u_3}\left[\frac{b\lambda_3(A+B+D)L}{A\lambda_1+B\lambda_2+b\lambda_3(A+B+D)}\right]^{\lambda_3}\left[\frac{b\mu_3(A+B+D)\left(\frac{M}{P_L}\right)}{A\mu_1+B\mu_2+b\mu_3(A+B+D)}\right]^{\mu_3}$$

The solutions for relative prices are rather simple. Make use of equations (47) - (55) to obtain the following expressions. Note that these expressions are not in final form since direct insertions of solutions already obtained above have not been made to eliminate unnecessary long and messy formulae.

$$(82)\quad \frac{P_K}{P_L} = \frac{u_1}{\lambda_1} \cdot \frac{L_1}{K_1}$$

$$(83)\quad \frac{P_M}{P_L} = \frac{\mu_1}{\lambda_1} \quad \frac{L_1}{\left(\frac{M_1}{P_L}\right)}$$

$$(84)\quad \frac{P_1}{P_L} = \frac{L_1}{\lambda_1} \cdot \frac{1}{X_1}$$

$$(85)\quad \frac{P_2}{P_L} = \frac{L_2}{\lambda_2} \cdot \frac{1}{X_2}$$

$$(86)\quad \frac{P_3}{P_L} = \frac{L_3}{\lambda_3} \quad \frac{1}{I}$$

Furthermore, by equations (67) and (83), the relative price of real cash balances viewed as a consumer good is

$$(87)\quad \frac{P_4}{P_L} = \frac{\mu_1}{\lambda_1} \cdot \frac{L_1}{\left(\frac{M_1}{P_L}\right)}$$

The solutions already obtained above immediately determine the total income solution:

$$(88)\quad \frac{Y}{P_L} = \frac{u_1}{\lambda_1}\cdot\frac{L_1}{K_1}\cdot K + L + \frac{\mu_1}{\lambda_1}\cdot\frac{L}{\left(\frac{M_1}{P_L}\right)}\cdot\frac{M_T}{P_L}$$

Having obtained the total income solution, the savings solution immediately follows.

$$(89)\quad \frac{S}{P_L} = b\left[\frac{u_1}{\lambda_1}\cdot\frac{L_1}{K_1}\cdot K + L + \frac{\mu_1}{\lambda_1}\cdot\frac{L}{\left(\frac{M_1}{P_L}\right)}\cdot\frac{M_T}{P_L}\right]$$

At this point, it may be interesting to check whether the solutions of $I(t)$ and $S(t)$ given above satisfy equation (61). Such a check would reveal both internal consistency of the model as well as correct solutions. From equation (61) obtain

$$P_3 I = S$$

From equation (89), solve for S and obtain

$$S = P_L b\left[\frac{u_1}{\lambda_1}\cdot\frac{L_1}{K_1} K + L + \frac{\mu_1}{\lambda_1}\cdot\frac{L_1}{\left(\frac{M_1}{P_L}\right)}\cdot\frac{M_T}{P_L}\right]$$

Thus, we have

$$P_3 I = P_L b\left[\frac{u_1}{\lambda_1}\cdot\frac{L_1}{K_1} K + L + \frac{\mu_1}{\lambda_1}\ \frac{L_1}{\left(\frac{M_1}{P_L}\right)}\cdot\frac{M_T}{P_L}\right]$$

Divide both sides by P_L, use equations (86), (73), (75), (71) and (76). Use needs also to be made of equation (93) presented below. Rather complicated algebraic manipulations show that equality holds.

Now to obtain the solutions for consumption, use equations (57) - (59) and solutions already obtained above to get:

$$(90)\quad C_1 = \frac{(1-b)AX_1}{A+B+D}\left[\frac{u_1}{K_1}K + \frac{\lambda_1}{L_1}L + \frac{\mu_1}{\left(\frac{M_1}{P_L}\right)}\ \frac{M_T}{P_L}\right]$$

$$(91)\quad C_2 = \frac{(1-b)BX_2}{A+B+D}\left[\frac{u_2}{K_2}K + \frac{\lambda_2}{L_2}L + \frac{\mu_2}{\left(\frac{M_2}{P_L}\right)}\frac{M_T}{P_L}\right]$$

$$(92)\quad \frac{\mathcal{M}}{P_L} = \frac{(1-b)D}{A+B+D}\left[\frac{u_1}{\lambda_1}\cdot\frac{L_1}{K_1}\cdot K + L + \frac{\mu_1}{\lambda_1}\cdot\frac{L_1}{\left(\frac{M_1}{P_L}\right)}\frac{M_T}{P_L}\right]$$

Few steps remain for the model to be fully solved. The solution of the proportion of the total money supply that is allocated to the use of money as an input is given below. To obtain this solution, use equations (65) and (66) and solutions expressed in equations (70), (73) and (76).

$$(93)\quad \frac{M}{P_L} = \frac{[A+B+D-(1-b)D][A\mu_1+B\mu_2+b\mu_3(A+B+D)]}{A+B+D+(1-b)D[Au_1+Bu_2+bu_3(A+B+D)+A\lambda_1+B\lambda_2+b\lambda_3(A+B+D)]}\cdot\frac{M_T}{P_L}$$

or equivalently,

$$(93a)\quad \frac{M}{P_L} = F\,\frac{M_T}{P_L}$$

where

$$F = \frac{[A+B+D-(1-b)D][A\mu_1+B\mu_2+b\mu_3(A+B+D)]}{A+B+D+(1-b)D[Au_1+Bu_2+bu_3(A+B+D)+A\lambda_1+B\lambda_2+b\lambda_3(A+B+D)]}$$

Naturally, the solution of the proportion of money used as a consumer good is

$$(94)\quad \frac{\mathcal{M}}{P_L} = \frac{M_T}{P_L} - \frac{M}{P_L}$$

The solution of capital accumulation will complete the set of solutions. Since it is rather involved, a detailed presentation of this solution follows.

From equation (69) it is obtained that:

$$K(t) = K(0) + \int_0^t I(\xi)d\xi$$

Substitute the solution of $I(t)$ obtained in equation (83); furthermore, substitute $L(t)$ and $\frac{M_T(t)}{P_L(t)}$ by its equivalent in equations (68) and (69) after making use of equation (93a).

$$K(t) = K(0) + \int_0^t \left[\frac{bu_3(A+B+D)\cdot K(\xi)}{Au_1+Bu_2+bu_3(A+B+D)}\right]^{u_3}\left[\frac{b\lambda_3(A+B+D)L(0)e^{n\xi}}{A\lambda_1+B\lambda_2+b\lambda_3(A+B+D)}\right]^{\lambda_3}\left[\frac{b\mu_3(A+B+D)F\frac{M_T(0)}{P_L(0)}e^{\mu\xi}}{A\mu_1+B\mu_2+b\mu_3(A+B+D)}\right]^{\mu_3}d\xi$$

Let

$$H = \left[\frac{bu_3(A+B+D)}{Au_1+Bu_2+bu_3(A+B+D)}\right]^{u_3}\left[\frac{b\lambda_3(A+B+D)L(0)}{A\lambda_1+B\lambda_2+b\lambda_3(A+B+D)}\right]^{\lambda_3}\left[\frac{b\mu_3(A+B+D)F}{A\mu_1+B\mu_2+b\mu_3(A+B+D}\cdot\frac{M_T(0)}{P_L(0)}\right]^{\mu_3}$$

Hence, the last equation becomes

$$K(t) = K(0) + H\int_0^t [K(\xi)]^{u_3} e^{(n\lambda_3+\mu\mu_3)\xi}d\xi$$

Differentiate both sides with respect to t

$$K'(t) = H[K(\xi)]^{u_3} e^{(n\lambda_3+\mu\mu_3)\xi}$$

Divide by $[K(\xi)]^{u_3}$ assuming $[K(\xi)]^{u_3} \neq 0$ and integrate

$$\int_0^t \frac{K'(\xi)}{[K(\xi)]^{u_3}}d\xi = H\int_0^t e^{(n\lambda_3+\mu\mu_3)\xi}d\xi$$

$$\left.\frac{[K(\xi)]^{1-u_3}}{1-u_3}\right|_0^t = \frac{H}{n\lambda_3+\mu\mu_3}\left. e^{(n\lambda_3+\mu\mu_3)\xi}\right|_0^t$$

$$\frac{[K(t)]^{1-u_3}}{1-u_3} - \frac{[K(0)]^{1-u_3}}{1-u_3} = \frac{H}{n\lambda_3+\mu\mu_3}e^{(n\lambda_3+\mu\mu_3)t} - \frac{H}{n\lambda_3+\mu\mu_3}$$

$$[K(t)]^{1-u_3} = (1-u_3)\left\{\frac{[K(0)]^{1-u_3}}{1-u_3} + \frac{H}{n\lambda_3+\mu\mu_3}\left(e^{(n\lambda_3+\mu\mu_3)t} - 1\right)\right\}$$

Thus:

$$(95)\quad K(t) = \left\{ \left[K(0)\right]^{1-u_3} + \frac{H}{(1-u_3)(n\lambda_3+\mu\mu_3)}\left[e^{(n\lambda_3+\mu\mu_3)t} - 1\right]\right\}^{\frac{1}{1-u_3}}$$

Having obtained equation (95), some interpreting comments are in order.

First, observe that by letting enough time elapse, the rate of growth of capital denoted by g_K becomes:

$$(96)\quad g_K = \frac{n\lambda_3+\mu\mu_3}{1-u_3} = \frac{n\lambda_3+\mu\mu_3}{\lambda_3+\mu_3}$$

This rate of growth is very characteristic. As one might have expected, the rate of growth of capital is determined by the rate of growth of the labor force and the rate of growth of the money supply, as well as the elasticities of investment with respect to labor and money viewed as an input. Naturally, capital accumulation is determined by the independent investment function where the two elasticities mentioned first appeared. The dependence of the rate of growth of capital on the rate of monetary expansion, among other things, is of particular interest. It establishes once again the importance of money in a dynamic economy. Furthermore, equation (97) clearly indicates that increases in the rate of monetary expansion cause increases in the rate of capital accumulation.

$$(97)\quad \frac{dg_K}{d\mu} = \frac{\mu_3}{\lambda_3+\mu_3} > 0$$

Second, the reader should note that a well known result of neoclassical growth theory is present in model IV. This result refers to the fact that the savings coefficient b does not appear in equation (96). This implies that the proportion of income saved does not affect the rate of capital accumulation. This is consistent with the findings of neoclassical growth theory.[2]

Third, it is of interest to study the rate of growth of investment. Denote such a rate by g_I. Once again, if enough time is allowed to elapse, the rate of growth of investment would become

$$(98)\quad g_I = u_3 g_{K_3} + \lambda_3 n + \mu_3 \mu$$

where g_{K_3} is the rate of growth of capital allocated in the investment good industry. But since the allocation of capital stock in the investment good industry is a fixed proportion of total capital stock, g_{K_3} is given by equation (99).

$$(99)\quad g_{K_3} = \frac{n\lambda_3 + \mu\mu_3}{1-u_3}$$

Inserting equation (99) into equation (98), the latter becomes:

$$(100)\quad g_I = u_3\left[\frac{n\lambda_3 + \mu\mu_3}{1-u_3}\right] + n\lambda_3 + \mu\mu_3$$

Very simple algebraic manipulations yield

$$(101)\quad g_I = \frac{n\lambda_3 + \mu\mu_3}{1-u_3}$$

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[2] A useful reference may be Brems (1968; chapter 46).

which means that the rate of growth of investment is equal to the rate of growth of capital. This result is not only aesthetically beautiful but also economically meaningful since it establishes the long run stability of the model.

Finally, one may study the rate of growth of the two consumption goods. This is particularly interesting in view of the fact that economies grow in order to guarantee to their members, in the long run, high levels of consumption. Since all output produced by both the two consumption goods is consumed, the rate of growth of C_1 denoted by g_{C_1} and the rate of growth of C_2 denoted by g_{C_2} are given by:

(102) $g_{C_1} = g_{X_1}$

(103) $g_{C_2} = g_{X_2}$

where g_{X_1} and g_{X_2} denote the rate of growth of X_1 and X_2 respectively. Allowing for sufficiently long time to elapse, g_{X_1} and g_{X_2} are given by

(104) $g_{X_1} = u_1 g_{K_1} + \lambda_1 n + \mu_1 \mu$

(105) $g_{X_2} = u_2 g_{K_2} + \lambda_2 n + \mu_2 \mu$

where

(106) $g_{K_1} = g_{K_2} = g_{K_3} = \dfrac{n\lambda_3 + \mu\mu_3}{1 - u_3}$

Observe that once again the rate of growth of the money supply plays an important role in determining the growth rates of the two consumption goods.

Comparisons and Interpretations

Models III and IV have both similarities and dissimilarities. In general, the basic similarity is that they both are dynamic, general equilibrium models. In particular, both models consist of three productive sectors, two of which produce two different consumer goods and a third sector producing an investment good. These two models are complete in the sense that output demand functions are derived from constrained utility maximization instead of simply being postulated bo be of a certain form. Also, input demand functions are derived from unconstrained profit maximization under conditions of perfect competition. A savings function is postulated in both models which in conjunction with the investment function provides the stimulus for economic growth.

The basic dissimilarity stems from introducing money. Model III is a real dynamic general equilibrium model while model IV is a monetary dynamic general equilibrium one. The introduction of money in model III is achieved by considering real cash balances to be a consumer as well as a producer good. So, real cash balances enter both the utility functions as well as the production functions. Hence, a basic dissimilarity between model III and model IV is that real cash balances are assumed to be an argument of the utility and production functions of model IV while such an assumption is not made in model III. Such an assumption is crucial because the introduction of real cash balances in the utility and production functions further generates demand functions for real cash balances viewed as an input and viewed as an output. Clearly then, model IV is larger and closer to interpreting economic reality where undoubtedly money is demanded for the services it renders to both producer and consumer.

As a result of introducing real cash balances in model IV, a monetary market is developed to describe the monetary behavior of the model. In particular, it is postulated that the price of real cash balances used as a consumer good and the price of real cash balances used as a producer good must be equal if equilibrium is to be achieved. This is in accordance with orthodox economic theory.

An interesting question arises at this point. With respect to the monetary sector of model IV, how can absolute prices be determined? Recall that solvability of model IV required the reduction of the number of variables by one. The approach followed was suggested by previous models where the wage rate was chosen to play the role of the numeraire. A different approach, however, could have been followed. Orthodox economic theory has long postulated that monetary magnitudes can be determined by making use of the "Quantity Theory of Money". The author has researched this possibility in both model II and model IV. The main shortcoming of introducing in these models a "quantity theory of money" equation is the magnitude of the models. More specifically, recall that a quantity theory of money equation requires the exact enumeration of transactions taking place at a given period of time. Such an enumeration is not simple to obtain, even in an economy with three goods and two persons. The difficulty, however, of introducing a quantity theory of money equation does not reduce the contribution of models II and IV since the determination of absolute prices is not very crucial in models of the introduction of money in barter economies. At this early stage of economic research in this field of study, a theoretically satisfactory integration of barter and monetary theories precedes the derivation of operationally meaningful theorems concerning different aspects of the model.

A final dissimilarity between models III and IV lies in the way incomes are defined. Unlike the definitions of income in model III, which do not consider any income generated by the money stock, the definitions of income in model IV do contain the money stock of the economy as an argument.

Having presented a brief discussion on the similarities and dissimilarities of the models of this chapter, some interpretations are in order. Note that several interpreting comments followed the presentation of the models. It seems rather impossible to exhaust all angles of interpretations, particularly in rich models such as the present two. All that is intended here is to bring the attention of the reader to some crucial points, thus stimulating further valuable insights into these models.

The basic purpose of the two models is to establish an integration of real growth theory and monetary theory at a level higher than that of models I and II. Such a higher level is the dynamic nature of the integration as compared to the static integration of Chapter III. A dynamic integration of real growth theory and monetary theory is of course desirable because it represents economic reality more faithfully than a static integration.

The main force of growth in model III is the growth of labor which, through an investment-savings machanism, causes capital accumulation which in turn, associated with labor growth and the investment-savings mechanism, generates growth to all the remaining variables of the model. All these are under conditions of constrained utility maximization and unconstrained perfectly competitive profit maximization. All solutions of the variables of model III are illustrated and are indicative of the complexities observed in real economies. The solutions as presented illustrate the relationships that exist among the variables and parameters of the model at some given moment of time. No further manipulations of the solutions are made since the main purpose is to establish solvability as the main property of the model rather than any other property. Having achieved the development of a consistent, realistic and solvable model, the analysis proceeded by introducing money in model III.

The introduction of money in model IV along the lines of considering real cash balances as both a consumer and producer good helped establish a meaningful integration of real growth theory and monetary theory. This integration is meaningful because money is a variable which affects all the solutions of the model. Money is not simply a veil but a very important variable. The fact that all monetary variables of model IV are expressed in terms of a numeraire demonstrates that money affects relative magnitudes rather than only absolute levels of prices as classical economists have argued.

Most interestingly, labor and monetary growth now become the great engine transmitting growth. Both labor growth and monetary expansion, with the help of the investment-savings mechanism, determine the rate of capital accumulation, which in turn, associated with labor and monetary expansion, stimulate growth to all the variables of the model. Thus a further step towards a meaningful integration of monetary and real growth theory is achieved in Chapter IV by demonstrating such an integration at a dynamic level.

CHAPTER V

CONCLUSION

The present chapter addresses itself to presenting the conclusions of this dissertation. The reader must be reminded that this study had a twofold purpose: first, to critically evaluate monetary growth theory and, second, to construct models with the introduction of money into barter economies. Both aspects of the dissertation's purpose are quite broad in content. Though the discussion on the research's scope, presented in Chapter I, helped define, as well as narrow down, the range of study, one may still argue that the research territory remains quite extensive. A systematic study of this extensive research territory leads to a rich harvest of theoretical results. It is not difficult to explain the number and nature of results. In essence, this dissertation is an essay in the methodology of monetary growth theory. It is not confined within the existing paradigm. It essentially attempts to demonstrate a different angle for tackling the theoretical puzzles, most of which monetary growth theory inherited from the still unresolved controversy of the integration of value theory and monetary theory. In other words, the richness of results is due to the new way of looking at the old issues and the reexamination of unsolved problems from the methodological standpoint presented in this research. The author believes that the present research has succeeded in providing a meaningful integration of monetary and real growth theory and that many issues can be resolved if use is made of the methodology suggested in this dissertation. A complete list of all results is impossible. The main conclusions, however, are presented below to be followed by suggestions for further research.

Contribution

The detailed critique presented in Chapter II contributes to understanding the main theoretical issues of monetary growth theory. The question "how is money introduced in a real economy" that monetary growth theory attempts to answer is indeed a very difficult question. It calls for a theoretical integration of real growth theory and monetary theory. Such an integration has been attempted in the current economic literature and the critical evaluation of this literature concludes that the results are neither theoretically sound nor empirically meaningful. Again, the methodological critique of monetary growth theory in Chapter II contributes to raising several points. Two points, however, are far superior in importance as far as the contribution of this thesis is concerned. These are the following.

First, it is shown that no theoretically sound integration of monetary theory and real growth theory can be achieved by attempting to introduce money into one good real growth models. A real growth model that consists of only one, or even two, goods does not suffer from any sort of barter inefficiencies. Consequently, the introduction of money in such models is arbitrary and theoretically unjustifiable. Hence, one main contributing point is the result that economic models built for the porpose of being used for introducing money into them should include at least three goods.

Second, it has been argued that the theoretically important function of money is that of being a store of value. The prerequisite for money to perform this function is future uncertainty. In an economy with perfect economic foresight, money is reduced simply to a means of exchange and a unit of account with no meaningful economic role. So, an additional result of Chapter II is the suggestion that uncertainty needs to be introduced in real growth models if they are to be used to study the role of money in a growing economy.

Motivated by a deeper understanding of the theoretical requisites for a meaningful integration of monetary and real growth theory, part II of this study is devoted to the presentation of four models. Again, many interesting results follow from these four models. However, two contributing points deserve most attention. These are the following.

First, the dissertation contributes to a better understanding of what may be called a "static integration of monetary and value theory." The two models of Chapter III illustrate a theoretically sound synthesis of barter and monetary economies. Both models are general equilibrium models. They are static and contain three goods. Model I shows that a three good barter economy, with producers maximizing profits and consumers maximizing utility, reaches a static equilibrium. The existence of three goods justifies the claim that a barter economy is inefficient. Model II removes this inefficiency by introducing money. Such introduction of money is theoretically sound because money is viewed as both a consumer good and a producer good. The introduction of money in both the utility and the production functions of model II affect almost all solutions of this model. Solutions of models I and II, when compared, clearly indicate that money in model II is an essential variable with far-reaching effects. The completeness, simplicity and solvability of models I and II definitely contribute to a static integration of monetary and value theory. Detailed inspection of the construction, analysis and solutions of these two models undoubtedly uncovers many other contributing points. But the static integration of monetary and value theory remains methodologically the foremost contributing aspect of part II.

Second, part II presents a dynamic integration of monetary and real growth theory. Realizing that static integration is not sufficient, models III and IV are developed to study the methodology of a dynamic integration. The models are general equilibrium with output demand functions derived from utility maximization and input demand functions derived from profit maximization. The complexity of such models is not allowed to become an obstacle to penetrating the hidden relationships of a dynamic economy. After a dynamic economy is constructed with two consumption goods and an investment good, a set of solutions is obtained to illustrate the solvability of the models. The existence of three goods justifies the claim of barter inefficiency which motivates the introduction of money. Again, such an introduction of money is achieved by considering money not simply as a means of exchange and a unit of account but, more importantly, as a consumer and producer good. Mathematically, this way of looking at money is demonstrated by including money as an argument in the production and utility functions. This causes direct, as well as indirect, effects on the solutions of the model which demonstrate that money is a very integral variable of the model. Needless to say, a detailed study of the presentation, construction and solutions of models III and IV reveals very interesting results. However, the second most significant aspect of the contributing role of part II is the dynamic integration of monetary and real growth theory.

Some Unresolved Issues

The remainder of this chapter is devoted to a brief discussion of some of the more significant unresolved issues in the area of monetary growth theory. This discussion will enable the reader to develop a better perspective in evaluating the contribution of this dissertation. Furthermore, the reader may be motivated in researching some of the unresolved issues himself.

The critical elaboration of the present day foundations of monetary growth theory, as illustrated in Chapter II, shows that a theoretically sound and empirically meaningful approach seems to be the stochastic, dynamic, general equilibrium method. A stochastic, dynamic, general equilibrium approach to monetary growth theory consists of several building blocks. First, a static, barter, general equilibrium construction with more than two goods has to be developed and then money has to be introduced in it. Second, the static, barter, general equilibrium model has then to be dynamized. This is usually achieved by postulating labor growth and an investment-savings mechanism. In such a dynamic, barter, general equilibrium model, money needs to be introduced and its effects carefully studied. Finally, the dynamic, barter, general equilibrium model should be transformed to become a stochastic rather than a deterministic one. Mathematically, this can be done by assuming stochastic functional dependence with postulated distributions of the error terms. Economically, such a stochastic barter dynamic contruction would illustrate the functioning of a barter economy with imperfect foresight, i.e., uncertainty. In such a barter stochastic dynamic model, money needs to be introduced and once this is achieved, monetary growth theory could claim its first theoretical triumph for integrating monetary theory and real growth theory in the most theoretically satisfactory way. The reader has undoubtedly detected that the first two building blocks, i.e., static and dynamic integration of theories, are laid down in this dissertation. The stochastic dynamic general equilibrium integration remains to be done.

Furthermore, it can be argued that a stochastic dynamic general equilibrium synthesis should not be considered the final goal of monetary growth theory. Actually, it is only the beginning of a new approach. In particular, once such a synthesis is achieved, a great deal of research effort should be devoted to manipulations of the models in order to better understand all the relationships involved. The synthetic and solvable nature of the models would allow a detailed investigation of the preperties of the theoretical synthesis and its ability to explain and solve theory and policy issues. Equipped with a thorough understanding of the methodological foundations and properties of the integration of monetary growth theory, economists will have to attack what seems today to be the ultimate theoretical goal - an abstract construction of a stochastic, dynamic, general equilibrium economy - rather than one that is of the Cobb-Douglas form. Faced with such an extraordinary task, the theoretical economist cannot help but feel highly stimulated and challenged.

BIBLIOGRAPHY

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W. J. Baumol, "The Transactions Demand for Cash: An Inventory Theoretic Approach," Quarterly Journal of Economics, November 1952, 66, 546-56.

K. Borch, "General Equilibrium in the Economics of Uncertainty," in K. Borch and J. Mossin, eds., Risk and Uncertainty, New York 1968.

H. Brems, Quantitative Economic Theory, New York 1968.

E. Burmeister and A. R. Dobell, Mathematical Theories of Economic Growth, New York 1970.

P. Caws, "The Structure of Discovery," Science, 12 December 1969, 166, 1375-1380.

A. W. Coats, "Is There a 'Structure of Scientific Revolutions' in Economics," Kyklos, February 1969, 22, 289-296.

D. S. DeShon, "An Essay on the Nature of Economic Science: Where do we go from here?," paper presented to the Southwestern Social Science Association, Dallas, Texas, March 27, 1970.

R. Dorfman, "General Equilibrium with Public Goods," in J. Margolis and H. Guitton, eds., Public Economics, New York 1969.

E. Drandakis, "On the Competitive Equilibrium in a Monetary Economy," International Economic Review, September 1966, 7, 304-328.

M. Friedman, The Optimum Quantity of Money and Other Essays, Chicago 1969.

N. Georgescu-Roegen, "The Nature of Expectation and Uncertainty," in M. J. Bowman, ed., Expectations, Uncertainty and Business Behavior, New York 1958.

_____, Analytical Economics, Cambridge, Mass. 1966.

F. Hahn, "Money, Dynamic Stability and Growth," Metroeconomica, August, 1961.

_____, "On Some Problems of Proving the Existence of an Equilibrium in a Monetary Economy," in Hahn and Brechling, eds., The Theory of Interest Rates, London 1965.

_____, "On Money and Growth," Journal of Money, Credit and Banking, May 1969, 1, 172-187.

B. Hansen, A Survey of General Equilibrium Systems, New York 1970.

J. R. Hicks, Value and Capital, 2nd ed., London 1946.

H. G. Johnson, Essays in Monetary Economics, Cambridge, Mass. 1967.

R. Kuenne, The Theory of General Economic Equilibrium, New Jersey 1963.

T. Kuhn, The Structure of Scientific Revolutions, 2nd ed., Chicago 1970.

D. Levhari and D. Patinkin, "The Role of Money in a Simple Growth Model," American Economic Review, September 1968, 55, 713-53.

A. Marty, "The Optimal Rate of Growth of Money," Journal of Political Economy, July/August 1968, 76, Part 2, 860-73.

M. Morishima, Theory of Economic Growth, London 1969.

K. Nagatani, A Note on Professor Todin's 'Money and Economic Growth'," Econometrica, January 1970, 38, 171-75.

E. Nagel, The Structure of Science, New York 1961.

D. Patinkin, Money, Interest and Prices, 2nd ed., New York 1965.

A. C. Pigou, The Economics of Stationary States, London 1935.

F. Polanyi, The Great Transformation, paperback, Boston 1957.

R. Radner, "Problems in the Theory of Markets Under Uncertainty," American Economic Review, May 1970, 60, 454-460.

Rosenstein-Rodan, "The Coordination of the General Theories of Money and Price," Economica, August 1936, 3 (new series), 257-280.

G. L. S. Shackle, The Years of High Theory: Invention and Tradition in Economic Thought 1926-1939, Cambridge 1967.

L. S. Shapley, Values of Large Games--VII: A General Exchange Economy with Money, Memorandum RM-4248-PR, December 1964, The Rand Corporation.

R. Solow, "A Contribution to the Theory of Economic Growth," The Quarterly Journal of Economics, February 1956, 70, 65-94.

M. Spector, "Models and Theories," The British Journal for the Philosophy of Science, August 1965, 16, 121-142.

J. L. Stein, "Money and Capacity Growth," Journal of Political Economy, October 1966, 74, 451-465.

_____, "Introduction," Journal of Money, Credit and Banking, May 1969, 1, 131-137.

_____, "Neoclassical and Keynes-Wicksell Monetary Growth Models," Journal of Money, Credit and Banking, May 1969, 1, 153-171.

_____, "Monetary Growth Theory in Perspective," American Economic Review, March 1970, 60, 85-106.

J. K. Stephens, "Two Keynesian Models of Simultaneous Inflation and Unemployment," Nebraska Journal of Economics and Business, Winter 1970, 9, 18-26.

J. E. Stiglitz and H. Uzawa, eds., Readings in the Modern Theory of Economic Growth, Cambridge, Mass. 1969.

T. W. Swan, "Economic Growth and Capital Accumulation," The Economic Record, November 1956, 32, 334-361.

J. Tobin, "Relative Income, Absolute Income, and Saving," in Money, Trade and Economic Growth, New York 1951.

_____, "A Dynamic Aggregative Model," Journal of Political Economy, April 1955, 63, 103-15.