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Abstract

The dissertation talks about the study the higher Nash blow-ups introduced by T. Yasuda, an approach towards resolution of singularities. The important step towards the resolution problem is the higher Nobile's theorem. We investigate this theorem for the graded case and the 2nd order Nash blow-ups. We see a possible idea that connects the graded case to the general case. Finally, we show a special case of the (higher) Zariski-Lipman conjecture and how it relates to the (higher) Nobile's theorem.

Chapter 1

Introduction

Algebraic geometry is the branch of mathematics devoted to the study of solution sets of systems of polynomial equations. These solution sets, called algebraic varieties, are the central objects of the subject: they are at once geometric objects one can visualize and draw, and algebraic objects one can compute with. A fundamental example is a plane curve, defined by a single polynomial equation $f(x, y) = 0$ in the plane. Even in this simplest case one quickly encounters a rich variety of geometric phenomena such as smooth arcs, cusps, nodes, crossings that reflect deep algebraic properties of the polynomial f.

A central and recurring theme in algebraic geometry is the study of singularities: points at which a variety fails to be smooth. Informally, a point x on a variety X is singular if the variety has some form of self-intersection, a cusp, or an abnormal tangent behavior at x, and it is smooth (or non-singular) if the variety looks locally like a piece of flat space. Algebraically, smoothness at a point x is detected by the local ring $\mathcal{O}_{X,x}$: the point is smooth if and only if $\mathcal{O}_{X,x}$ is a regular local ring, a condition that can be tested through the module of Kähler differentials. Singularities encode deep geometric and topological information about the variety but they also obstruct many of the most powerful tools of the subject, which are only available in the smooth setting.

The resolution of singularities is a classical problem which asks, given a possibly singular variety X, whether there exists a smooth variety $\tilde{X}$ together with a proper birational morphism $\phi : \tilde{X} \longrightarrow X$ that is an isomorphism over the smooth locus of X. Such a map resolves the singularities by replacing them with smooth geometry, while leaving the smooth part of X untouched.

The history of the problem stretches back to Newton's study of plane curves and runs through two centuries of gradually deepening understanding. Resolution of singularities of curves (dimension one) was understood classically: every algebraic curve has a unique nonsingular model, obtained by normalization. For surfaces (dimension two), rigorous proofs were more difficult; the first algebraic proof was given by R. J. Walker [Wal35] in 1935, and independently by Oscar Zariski [Zar39] in 1939.

The landmark breakthrough came in 1964, when Heisuke Hironaka, a student of Zariski published his two-part paper: Resolution of singularities of an algebraic variety over a field of characteristic zero. Hironaka [Hir64] proved that every algebraic variety over a field of characteristic zero admits a resolution of singularities by a finite sequence of blow-ups along smooth centers, and moreover that this resolution can be made to satisfy strong functoriality properties. The proof, spanning over 200 pages and proceeding by a subtle induction on dimension, was recognized as one of the most difficult in twentieth-century mathematics.

Despite this complete resolution in characteristic zero, the situation in positive characteristic remains far more mysterious and continues to be one of the deepest open problems in algebraic geometry. Resolution is known for curves and surfaces in all characteristics, and for threefolds in characteristic greater than five (following work of Abhyankar [Abh66] and, more recently, Cossart–Piltant [CP09]), but the general case is wide open. The fundamental difficulty is that Hironaka's argument makes essential use of characteristic zero features that fail in positive characteristic. This makes alternative approaches to resolution of singularities, including Nash and higher Nash blow-ups.

Nash blow-ups were first studied by John F. Nash [Nas95], where one should replace each singular point of a variety by the set of all limit positions of tangent spaces approaching from the smooth locus, and take the closure of the resulting graph. More precisely, let X be an algebraic variety of pure dimension d embedded in affine space $\mathbb{A}^n$, and let X_{sm} denote its smooth locus. At each smooth point x, the tangent space $T_{X,x}$ is a well-defined element of the Grassmannian $\mathrm{Gr}(d, n)$ of d-dimensional subspaces of n-dimensional space. This gives a map $\sigma : X_{\text{sm}} \longrightarrow \mathrm{Gr}(d, n)$, $x \mapsto T_xX$. The Nash blow-up of X, denoted $\mathrm{Nash}(X)$, is defined to be the Zariski closure of the graph of σ inside $X \times \mathrm{Gr}(d, n)$, equipped with the natural projection $\pi : \mathrm{Nash}(X) \longrightarrow X$. Equivalently, and more algebraically, $\mathrm{Nash}(X)$ is the blow-up of X along the sheaf of Kähler differentials $\Omega_{X/k}$. The module of Kähler differentials $\Omega_{X/k}$ is the algebraic counterpart of the cotangent bundle. For a k-algebra R it is defined via the universal derivation $d : R \longrightarrow \Omega_{R/k}$. The failure of $\Omega_{X/k}$ to be locally free is a nature of singular behavior.

The term Nash blow-up itself was first used by Nobile in his 1975 paper [Nob75b]. The question of whether finite iteration of Nash blow-ups resolves singularities has been studied for a long time. For curves in characteristic zero, Nobile proved in 1975 that a finite number of normalized Nash blow-ups suffices for resolution. For surfaces in characteristic zero, Hironaka showed in 1983 [Hir83] that after a finite sequence of normalized Nash blow-ups, one obtains a surface with only sandwiched singularities; and in his celebrated 1990 paper, Mark Spivakovsky [Spi90] completed the argument by proving that sandwiched singularities are always resolved by further normalized Nash blow-ups, thereby establishing the conjecture for surfaces. In positive characteristic, Nobile showed that there are singular curves whose Nash blow-up is an isomorphism.

In dimension four and higher, the conjecture has been proven false. In a result published in 2026, Castillo, Duarte, Leyton-Álvarez, and Liendo [CDLÁL26] proved that for every $d \geq 4$ and every algebraically closed field k, there exists a normal singular variety of dimension d such that iterating Nash blow-ups or normalized Nash blow-ups does not resolve its singularities. Their counterexamples are normal affine toric varieties, and the proof uses a combinatorial description of Nash blow-ups of toric varieties due to González-Sprinberg [GS77]. This development does not diminish the importance of Nash blow-ups, the theory remains rich and valuable.

An essential prerequisite for resolution of singularities based on Nash blow-ups is to know that the Nash blow-up actually modifies a singular variety, that is, the Nash blow-up of a singular variety is never isomorphic to the variety itself. Without this, a Nash blow-up could be a trivial step, and no finite iteration could hope to resolve anything. It is given by the following foundational theorem.

Theorem. ([Nob75b]) Let X be an equidimensional algebraic variety over $\mathbb{C}$. Then the Nash blow-up $\mathrm{Nash}(X)$ is isomorphic to X if and only if X is non-singular.

Nobile's original proof used the hypothesis $\mathrm{char(k)} = 0$ and used some technical tools that may fail in positive characteristic. Indeed, Nobile himself noted that the theorem fails in characteristic $p > 0$ without extra hypotheses: there are explicit examples of singular curves in positive characteristic whose Nash blow-up is an isomorphism. The correct statement in positive characteristic, with a normality hypothesis, was proved by D. Duarte and L. Núñez-Betancourt [DNB22] in 2022.

Nobile's theorem is not merely a technical lemma; it is a foundational result that underlies the entire program of resolving singularities by Nash blow-ups. It guarantees that the process is non-trivial at every singular step and thus that iterating it has a chance of eventually reaching a smooth variety at least in the cases where the conjecture holds.

Before introducing higher Nash blow-ups and higher Nobile's theorem, it is useful to describe the algebraic framework that governs them. The key object is the module of principal parts of order n, a generalization of the module of Kähler differentials introduced by Grothendieck in his foundational Éléments de Géométrie Algébrique [Gro67].

For a k-algebra R, the module of principal parts of order n is $P^n_{R/k,+} = I_R/I_R^{n+1}$ where I_R is the kernel of the multiplication map $\varphi : R \otimes_k R \longrightarrow R$. It is also equipped with the canonical derivation of order n, $d^n_R : R \longrightarrow P^n_{R/k,+}$. When $n = 1$, $P^1_{R/k,+}$ is canonically isomorphic to the module of Kähler differentials $\Omega_{R/k}$. Each P^n_R captures the local geometry of R to order n, with $n = 1$ corresponding to the tangent space level information encoded by Kähler differentials.

Geometrically, the sheaf $\mathcal{P}^n_{X/k}$ encodes the n-th order infinitesimal neighborhood structure of the diagonal in $X \times_k X$. Over the smooth locus of a variety X, $\mathcal{P}^n_{X/k,+}$ is locally free, and its sections are the principal parts of local functions. The failure of $\mathcal{P}^n_{X/k,+}$ to be locally free is a nature of singular behavior.

The higher Nash blow-up was introduced by Takehiko Yasuda in his 2007 paper [Yas07]. Yasuda's idea was to replace the first-order data of a tangent space with the higher-order data of an infinitesimal neighborhood. For a d-dimensional variety X and a smooth point $x \in X$, the n-th infinitesimal neighborhood of x is the scheme $x^{(n)} = \operatorname{Spec} \mathcal{O}_{X,x}/m_x^{n+1}$. When x is smooth, then $[x^{(n)}]$ defines a point in the Hilbert scheme of points $\mathrm{Hilb}_{\binom{n+d}{d}}(X)$. This gives a map $\sigma : X_{\mathrm{sm}} \longrightarrow \mathrm{Hilb}_{\binom{n+d}{d}}(X)$, $x \mapsto [x^{(n)}]$. The Nash blow-up of order n of X, denoted $\mathrm{Nash}_n(X)$, is defined to be the Zariski closure of the graph of σ inside $X \times \mathrm{Hilb}_{\binom{n+d}{d}}(X)$, equipped with the natural projection $\pi_n : \mathrm{Nash}_n(X) \longrightarrow X$. Equivalently, $\mathrm{Nash}_n(X)$ is the blow-up of X along the sheaf of principal parts $\mathcal{P}^n_{X/k,+}$. When n = 1, this recovers the classical Nash blow-up: the

Hilbert scheme $\mathrm{Hilb}_{\binom{n+d}{d}}(X)$ maps to the Grassmannian $\mathrm{Gr}(d,n)$ via a natural closed embedding, and $\mathrm{Nash}(X)$ is the classical construction.

Equivalently, and more algebraically, $\mathrm{Nash}_n(X)$ is related to the module of principal parts by the formula $\mathrm{Nash}_n(X) \cong \mathrm{Nash}(X, \mathcal{P}^n_{X/k})$. Moreover, $\mathrm{Nash}_n(X) \cong X$ if and only if the torsion-free quotient $P^n_{R/k,+}/\operatorname{tors}(P^n_{R/k,+})$ is locally free. This translation from geometry to algebra is the foundation of our approach in this thesis.

Yasuda proved that for curves in characteristic zero, a single higher Nash blow-up $\mathrm{Nash}_n(X)$ resolves singularities for n sufficiently large, a one-step resolution result with no analogue for the classical Nash blow-up. The next question we can raise is whether an analogue of Nobile's theorem holds for higher Nash blow-ups. This is the higher Nobile conjecture:

Conjecture. (Higher Nobile). Let X be an algebraic variety and $n \in \mathbb{N}$. Then $\mathrm{Nash}_n(X) \cong X$ if and only if X is non-singular.

Prior to the results of this thesis, the conjecture had been verified for curves [Yas07], for normal hypersurfaces in characteristic zero [Dua17], and for normal hypersurfaces in positive characteristic [DNB22]. These cases all exploited special structural features of hypersurfaces. The conjecture remained open for general varieties.

The central contribution of this thesis is the proof of the higher Nobile conjecture in several new and significant cases. Our strategy is conceptually clean and has a unifying structure: in all cases, we reduce the higher Nobile conjecture to the classical Nobile theorem of Nobile [1975] (in characteristic zero) or of Duarte–Núñez-Betancourt [2022] (in positive characteristic). The reduction proceeds through an injectivity criterion, the HN_n-property which, when verified, allows one to transfer the freeness of $P^n_{R/k,+}/\operatorname{tors}(P^n_{R/k,+})$ back to the freeness of $\Omega_{R/k}/\operatorname{tors}(\Omega_{R/k})$, and then to $\mathrm{Nash}(X) \cong X$. We investigate the higher version of Nobiles's theorem for the graded case.

Theorem. Let X be a graded algebraic variety over an algebraically closed field k. Suppose $\mathrm{Nash}_n(X) \cong X$ for some $n \in \mathbb{N}$, then

(1) if $\mathrm{char}(k) = 0$, then X is a non-singular variety.

(2) if X is normal and $\mathrm{char}(k) > 0$, then X is a non-singular variety.

We also investigate the higher version of Nobiles's theorem for the case when $n = 2$, i.e. the 2nd order Nash blow-ups.

Theorem. Let X be an algebraic variety over an algebraically closed field k with $\mathrm{char}(k) = 0$. Suppose $\mathrm{Nash}_2(X) \cong X$, then X is a non-singular variety.

We show that both of these theorems are achieved as corollaries to the following theorem.

Theorem. Assume the hypothesis of both these theorems, then $\mathrm{Nash}(X) \cong X$.

As a byproduct of the methods used in the proof of the above theorems, we obtain the following corollary, a result that was previously proved by Brenner, Jeffries, and Núñez-Betancourt in [BJNB19] by different techniques, for which our approach gives a transparent, conceptually motivated proof using the filtration of the associated graded ring.

Corollary. Let X be an irreducible algebraic variety of dimension d. Let $R = \mathcal{O}_{X,x}$ be a local ring at a closed point $x \in X$. Suppose P^n_R is a free R-module, then R is regular.

Closely related to the themes of this thesis is a long-standing open conjecture on the relationship between freeness of module of derivations and regularity of rings. The conjecture arose from the work of Oscar Zariski in the 1960s and was formalized by Joseph Lipman in his 1965 paper [Lip65]: Free derivation modules on algebraic varieties.

The dual module of Kähler differentials, $\mathrm{Der}_k(R) = \mathrm{Hom}_R(\Omega_{R/k}, R)$, is the module of k-derivations of R. Zariski and Lipman conjectured:

Conjecture. (Zariski-Lipman) Let X be an algebraic variety over a field of characteristic zero, and $R = \mathcal{O}_{X,x}$ the local ring at a closed point $x \in X$. If the module of derivations $\mathrm{Der}_k(R)$ is a free R-module, then R is regular.

Lipman proved in 1965 [Lip65] that the hypotheses of the conjecture imply R is a normal domain. The full conjecture was subsequently verified in several important special cases: for rings of Krull dimension one (also Lipman 1965), for hypersurface rings (Scheja–Storch [SS72], 1972), for the graded case (Hochster [Hoc77], 1977), and for locally complete intersections (Källström [Käl11], 2006). The general conjecture remains open and is considered a major problem in commutative algebra and algebraic geometry. The difficulty of the conjecture stems from the fact that freeness of the dual module is a priori weaker than freeness of $\Omega_{R/k}$ itself.

In this thesis, we investigate a higher version of the Zariski-Lipman conjecture, replacing the module of Kähler differentials by the module of principal parts of order n. We prove this conjecture for homogeneous hypersurfaces.

Theorem. Let $X = V(f)$ be a hypersurface over a field of characteristic zero such that f is a homogeneous polynomial. Let $R = \mathcal{O}_{X,x}$ be a local ring at a closed point $x \in X$. If $\mathcal{P}^{n^*}_{R/k,+} := \mathrm{Hom}(\mathcal{P}^n_{R/k,+}, R)$ is a free R-module, then R is regular.

Conventions

In this thesis, we will follow notations from the book Algebraic Geometry by Robin Hartshorne.

- The base field, denoted k, will be algebraically closed.
- We will consider varieties to be integral separated schemes of finite type over k.
- A homogeneous variety is an affine variety whose defining polynomials are homogeneous.

Chapter 2

Principal Parts

In this chapter, we introduce the notion of principal parts using two different perspectives, geometric and algebraic. We can think of the module of principal parts as a generalized version of the module of Kähler differentials. We now start with the geometric version, called the sheaf of principal parts.

2.1 Sheaf of Principal Parts

Definition 2.1.1. [Gro67] Let X be a Noetherian k-scheme. Consider the diagonal morphism $\Delta : X \longrightarrow X \times_k X$ and $\mathcal{I}_\Delta$ be the ideal sheaf defining the diagonal $\Delta(X) \subset X \times_k X$. We define the sheaf of principal parts of order n, denoted $\mathcal{P}^n_{X/k}$, as

$$\mathcal{P}^n_{X/k} := \mathcal{O}_{X \times_k X}/\mathcal{I}_\Delta^{n+1}.$$

Similarly, define $\mathcal{P}^n_{X/k,+} := \mathcal{I}_\Delta/\mathcal{I}_\Delta^{n+1}$.

It is easy to see that $\mathcal{P}^1_{X/k,+} \cong \mathcal{I}_\Delta/\mathcal{I}_\Delta^2 \cong \Omega_{X/k}$. It is known as the sheaf of Kähler differentials.

Remark 2.1.2. We have the following isomorphism

$$\mathcal{P}^n_{X/k} \cong \mathcal{P}^n_{X/k,+} \oplus \mathcal{O}_X$$

We now define the algebraic version called the module of principal parts.

Definition 2.1.3. Let $X = \operatorname{Spec} R$ for a k-algebra R. Consider the multiplication map

$$\varphi : R \otimes_k R \longrightarrow R, \quad a \otimes b \longmapsto ab.$$

Then $\mathcal{P}^n_{X/k}$ in the previous definition corresponds to

$$P^n_{R/k} := R \otimes_k R/I_R^{n+1}.$$

where $I_R := \ker(\varphi)$. Similarly, define the R-module $P^n_{R/k,+} := I_R/I_R^{n+1}$. Notice that when $n = 1$, $P^1_{R/k,+} \cong I_R/I_R^2 \cong \Omega_{R/k}$. It is known as the module of Kähler differentials.

Definition 2.1.4. Let R be a k-algebra and M be a R-module, A k-linear map $D_n : R \longrightarrow M$ is said to be a derivation of order n if the following condition holds for any $x_0, \cdots, x_n \in R$

$$D_n(x_0 \cdots x_n) = \sum_{l=1}^{n} (-1)^{l-1} \sum_{i_1 < \ldots < i_l} x_{i_1} \cdots x_{i_l} D_n(x_0 \cdots \hat{x}_{i_1} \cdots \hat{x}_{i_l} \cdots x_{i_n})$$

where $\hat{x}_{i_j}$ means that this element is eliminated from the product.

Remark 2.1.5. Let R be a k-algebra. Then we have the following isomorphism

$$P^n_{R/k} \cong R \oplus P^n_{R/k,+}.$$

We now have the following canonical derivation of order n.

$$d^n_R : R \longrightarrow P^n_{R/k,+}, \quad x \longmapsto (1 \otimes x - x \otimes 1) + I_R^{n+1}.$$

The R-module $P^n_{R/k,+}$ is generated by the image of the map d^n_R.

To lighten the notation, we will write $\mathcal{P}^n_X = \mathcal{P}^n_{X/k}$, and $P^n_{R,+} = P^n_{R/k,+}$ when the base field is understood from the context. Similarly, we will write $d^n = d^n_R$ when R is clear from the context. We have the following universal property of the derivation of order n.

Remark 2.1.6. Let R be a k-algebra and M be a R-module. Let $D_n : R \longrightarrow M$ be a derivation of order n. Then there exists a unique R-linear map $g : P^n_{R,+} \longrightarrow M$ such that $D_n = g \circ d^n_R$.

We now discuss a fact about canonical derivation of order n.

Remark 2.1.7. [BD20] Let $x, y \in R$, then a simple computation shows that

$$d^n(xy) = d^n(x)d^n(y) + yd^n(x) + xd^n(y).$$

Note that $d^n(xy) = 1 \otimes xy - xy \otimes 1 + I^{n+1}_R$. Similarly, $d^n(x) = 1 \otimes x - x \otimes 1 + I^{n+1}_R$ and $d^n(y) = 1 \otimes y - y \otimes 1 + I^{n+1}_R$. The remark follows form the following computation:

$$1 \otimes xy - xy \otimes 1 = (1 \otimes x - x \otimes 1)(1 \otimes y - y \otimes 1) + x(1 \otimes y - y \otimes 1) + b(1 \otimes x - x \otimes 1)$$

We now see the functoriality of the module of principal parts.

Remark 2.1.8. Let R and S be k-algebras and $f : R \longrightarrow S$ be a k-algebra homomorphism. Seeing S as a R-module, via $r \cdot s = f(r)s$ where $r \in R$ and $s \in S$, we have the following R-module map

$$\phi : R \otimes_k R \longrightarrow S \otimes S$$
$$a \otimes b \mapsto f(a) \otimes f(b).$$

And thus, the commutative diagram

$$\begin{array}{ccc} R \otimes_k R & \longrightarrow & S \otimes_k S \\ \downarrow & & \downarrow \\ R & \longrightarrow & S. \end{array}$$

where the vertical maps are the multiplication maps. It is then easy to see that $\phi(I_R) \subseteq I_S$. Hence, we have the following R-module homomorphisms

$$P^n_R \longrightarrow P^n_S,$$

and

$$P^n_{R,+} \longrightarrow P^n_{S,+}.$$

Similarly, we have the following S-module homomorphisms

$$P^n_R \otimes_R S \longrightarrow P^n_S,$$

and

$$P^n_{R,+} \otimes_R S \longrightarrow P^n_{S,+}.$$

2.2 Higher order Jacobian matrix

In this section, we define the higher order Jacobian matrix, also called Jacobian matrix of order n. We use the following notation for the rest of this subsection:

- Let k be an algebraically closed base field of characteristic zero.
- Let $S = k[x_1, \ldots, x_s]$.
- $X = V(f) \subset \mathbb{A}^s$ be a hypersurface on k, where $f \in S$ is an irreducible homogeneous polynomial.
- $B = S/(f)$.
- Let $R := B_x$ denote a local ring at a closed point $x \in X$.
- Let $M = \binom{n+s-1}{s}$ and $N = \binom{n+s}{s}$.

Remark 2.2.1. Let $R = k[x_1, \ldots, x_s]$ is a polynomial ring, then $P^n_{R,+}$ is generated by

$$S := \{d^n_R(x)^\alpha \mid 1 \leq |\alpha| = \alpha_1 + \cdots + \alpha_s \leq n\}$$

where $d^n_R(x)^\alpha = d^n_R(x_1)^{\alpha_1} \cdots d^n_R(x_s)^{\alpha_s}$. Moreover, S is a free basis of $P^n_{R,+}$. Let $f \in R$, then we have

$$d^n_R(f) = \sum_{1 \leq |\alpha| \leq n} \frac{1}{\alpha!} \frac{\partial^\alpha(f)}{\partial x^\alpha} d^n_R(x)^\alpha$$

Let where $\alpha, \beta \in \mathbb{N}^s$. We use the following multi-index notation.

- $|\alpha| = \alpha_1 + \cdots + \alpha_s$
- $\alpha! = \alpha_1! \alpha_2! \cdots \alpha_s!$
- $\binom{\alpha}{\beta} = \binom{\alpha_1}{\beta_1}\binom{\alpha_2}{\beta_2} \cdots \binom{\alpha_s}{\beta_s}$
- $\frac{\partial^\alpha}{\partial x^\alpha} = \frac{\partial^{\alpha_1}}{\partial {x_1}^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial {x_2}^{\alpha_2}} \cdots \frac{\partial^{\alpha_s}}{\partial {x_s}^{\alpha_s}}$
- $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_s^{\alpha_s}$

Remark 2.2.2. Let R be a Noetherian ring of Krull dimension d, then

$$\operatorname{rank}(P^n_{R/k,+}) = \binom{n+d}{d} - 1.$$

We now define the higher order Jacobian matrix.

Definition 2.2.3. [Dua17] Let f be a polynomial in $S = k[x_1, \ldots, x_s]$. Then the rows of the Jacobian matrix of order n, denoted $\operatorname{Jac}_n(f)$ are the following:

$$r_\beta := \beta! \cdot \left(\binom{\alpha}{\beta} \frac{1}{\alpha!} \frac{\partial^{\alpha-\beta} f}{\partial x^{\alpha-\beta}} \middle| \; 1 \leq |\alpha| \leq n \right)$$

where $\alpha, \beta \in \mathbb{N}^s$, and we arrange r_β using graded lexicographical order on α.

We also note that $\operatorname{Jac}_n(f)$ is a $(M \times N - 1)$ matrix.

Example 2.2.4. Let $f(x,y) = y^2 - x^3 \in \mathbb{C}[x,y]$ and $X = V(f)$. Then the usual Jacobian matrix of f is given by $\mathrm{Jac}(f) = (-3x^2 \quad 2y)$. We now compute the Jacobian matrix of order 2 of f, denoted $\mathrm{Jac}_2(f)$. Here, by the above definition $1 \leq |\alpha| \leq 2$ and $0 \leq |\beta| \leq 1$. Thus, α can be $\{(1,0),(0,1),(2,0),(1,1),(0,2)\}$, and similarly β can be $\{(0,0),(1,0),(0,1)\}$. Using graded lexicographic order, the rows of $\mathrm{Jac}_2(f)$ are the following

$$r_{(0,0)} = \left(\frac{\partial^\alpha f}{\alpha!} \mid 1 \leq |\alpha| \leq n\right) = \begin{pmatrix} -3x^2 & 2y & 3x & 0 & 1 \end{pmatrix}.$$

Similarly,

$$r_{(1,0)} = \begin{pmatrix} f & 0 & -3x^2 & 2y & 0 \end{pmatrix}.$$

And,

$$r_{(0,1)} = \begin{pmatrix} 0 & f & 0 & -3x^2 & 2y \end{pmatrix}.$$

Thus, the Jacobian matrix of order 2 of f is given by

$$\mathrm{Jac}_2(f) = \begin{pmatrix} -3x^2 & 2y & 3x & 0 & 1 \\ f & 0 & -3x^2 & 2y & 0 \\ 0 & f & 0 & -3x^2 & 2y \end{pmatrix}.$$

If we evaluate $\mathrm{Jac}_2(f)$ at $p = (x,y) \in X$, then we get

$$\mathrm{Jac}_2(f)|_p = \begin{pmatrix} -3x^2 & 2y & 3x & 0 & 1 \\ 0 & 0 & -3x^2 & 2y & 0 \\ 0 & 0 & 0 & -3x^2 & 2y \end{pmatrix}.$$

2.3 Formal completion of the module of principal parts

In this subsection, we give some facts about the completion of the module of principal parts which we need in the chapters. These are given by T. Yasuda in [Yas07]. We work with completion because when R is a complete local Noetherian ring, then $\widehat{P}_R^n$ is finitely generated as R-module. But on the other hand, P_R^n is generally not finitely generated.

Let $S = k[x_1, \cdots, x_s]$ be a polynomial ring and $R = S/J$ its quotient ring. We know

$$P_R^n = R \otimes_k R / I_R^{n+1}$$

where I_R is the ideal $(x_i \otimes 1 - 1 \otimes x_i \mid i = 1, \ldots, s)(R \otimes_k R)$. It is easy to see that P_R^n is generated by the image of the map

$$R \longrightarrow P_R^n, \quad x \longmapsto (x \otimes 1) + I_R^{n+1}.$$

We now move to the completion and give similar definitions and facts.

Let $\hat{S} = k[[x_1, \cdots, x_s]]$ be the formal power series ring and $\hat{R} = \hat{S}/\hat{J}$ its quotient ring where $\hat{J} = J\hat{S}$. Let $m_R = (x_1, \cdots, x_s)R$ and $m_{\hat{R}} = (x_1, \cdots, x_s)\hat{R}$ be maximal ideals in R and $\hat{R}$ respectively such that $J \subseteq m_R$ and $\hat{J} \subseteq m_{\hat{R}}$. We can view $\hat{R}$ as the completion with respect to m_R-adic filtration. We then have

$$\hat{P}_{\hat{R}}^n = \hat{R} \hat{\otimes}_k \hat{R} / \hat{I}_{\hat{R}}^{n+1}$$

where $\hat{R} \hat{\otimes}_k \hat{R} = (\widehat{\hat{R} \otimes_k \hat{R}})$ is the completion with respect to $(m_{\hat{R}} \otimes \hat{R}) + (\hat{R} \otimes m_{\hat{R}})$-adic filtration, and $\hat{I}_{\hat{R}} = I_R(\hat{R} \hat{\otimes}_k \hat{R})$.

Lemma 2.3.1. *[Yas07] We have the following isomorphism*

$$\hat{P}_{\hat{R}}^n \cong P_R^n \otimes_R \hat{R}.$$

Proof. Consider the map

$$\varphi : k[x_1, \ldots, x_s] \otimes_k k[x_1, \ldots, x_s] \longrightarrow k[x_1, \ldots, x_s, y_1, \cdots, y_s]$$
$$1 \otimes x_i \mapsto x_i - y_i.$$

Clearly, φ is an isomorphism of $k[x_1, \ldots, x_s]$-algebras. Similarly, we have the following isomorphism of $k[[x_1, \ldots, x_s]]$-algebras

$$\hat{\varphi} : k[[x_1, \ldots, x_s]]\hat{\otimes}_k k[[x_1, \ldots, x_s]] \longrightarrow k[[x_1, \ldots, x_s, y_1, \cdots, y_s]].$$

Let $k[\bar{x}] := k[x_1, \cdots, x_s]$, $k[\bar{y}] := k[y_1, \cdots, y_s]$, and $k[\bar{x}] := k[[x_1, \cdots, x_s]]$.

We can then see that

$$R \otimes_k R \cong \frac{k[\bar{x}, \bar{y}]}{\varphi(J \otimes_k k[\bar{x}] + k[\bar{x}] \otimes_k J)}.$$

And

$$P_R^n \cong \frac{k[\bar{x}, \bar{y}]}{(\varphi(J \otimes_k k[\bar{x}] + k[\bar{x}] \otimes_k J) + (y_1, \cdots y_s)^{n+1})}.$$

Similarly, we have that

$$\hat{P}_{\hat{R}}^n \cong \frac{k[[\bar{x}, \bar{y}]]}{(\hat{\varphi}(\hat{J}\hat{\otimes}_k k[[\bar{x}]] + k[[\bar{x}]]\hat{\otimes}_k \hat{J}) + (y_1, \cdots, y_s)^{n+1})}.$$

where $\hat{J} = Jk[[x_1, \ldots, x_s]]$. We can now write

$$\begin{aligned} P_R^n \otimes_R \hat{R} &\cong \frac{k[[\bar{x}]][\bar{y}]}{(\varphi(J \otimes_k k[\bar{x}] + k[\bar{x}] \otimes_k J) + (y_1, \cdots y_s)^{n+1})} \\ &\cong \frac{k[[\bar{x}, \bar{y}]]}{\hat{\varphi}(\hat{J}\hat{\otimes}_k k[[\bar{x}]] + k[[\bar{x}]]\hat{\otimes}_k \hat{J}) + (y_1, \cdots, y_s)^{n+1})} \\ &\cong \hat{P}_{\hat{R}}^n \end{aligned}$$

□

Remark 2.3.2. From the above lemma, we have

$$\hat{P}^n_{\hat{R}} \cong P^n_R \otimes_R \hat{R}.$$

We can write

$$\begin{aligned} P^n_R \otimes_R \hat{R} &\cong (R \oplus P^n_{R,+}) \otimes_R \hat{R} \\ &\cong (R \otimes_R \hat{R}) \oplus (P^n_{R,+} \otimes_R \hat{R}) \\ &\cong \hat{R} \oplus (P^n_{R,+} \otimes_R \hat{R}) \end{aligned}$$

We also have

$$\begin{aligned} \hat{P}^n_{\hat{R}} &\cong (\widehat{\hat{R} \oplus P^n_{\hat{R},+}}) \\ &\cong \hat{R} \oplus \hat{P}^n_{\hat{R},+} \end{aligned}$$

Thus, we get

$$\hat{P}^n_{\hat{R},+} \cong P^n_{R,+} \otimes_R \hat{R}.$$

2.4 Presentation of the module of principal parts

In this section, we construct an explicit presentation of the module of differentials of order n for a finitely generated k-algebra. We use the following notations for the rest of this subsection.

- $S = k[x_1, \cdots, x_s]$
- $J = (f_1, \cdots, f_c)$
- $B = S/J$

Remark 2.4.1. We know when $n = 1$, $P^1_{B,+} = \Omega_B$, and is given by the following presentation

$$B^c \xrightarrow{\mathrm{Jac}^T} B^s \longrightarrow \Omega_B \longrightarrow 0.$$

where Jac^T is the transpose of the Jacobian matrix.

Consider the k-linear map

$$\varphi : J/J^{n+1} \longrightarrow P^n_{S,+} \otimes_S B$$
$$\bar{x} \mapsto d^n(x) \otimes 1$$

Let $\langle \mathrm{Im}(\varphi) \rangle_B$ be the B-submodule of $P^n_{S,+} \otimes_S B$ generated by the image of φ.

Theorem 2.4.2. *[BD20] We have the following short exact sequence of B-modules*

$$0 \longrightarrow \langle \mathrm{Im}(\varphi) \rangle_B \longrightarrow P^n_{S,+} \otimes_S B \xrightarrow{g} P^n_{B,+} \longrightarrow 0.$$

where $g(d^n(x) \otimes b) = bd^n(\bar{x})$.

Because of the short exact sequence, we have

$$P^n_{B,+} \cong (P^n_{S,+} \otimes_S B)/\langle \mathrm{Im}(\varphi) \rangle_B$$

So, to give the presentation of $P^n_{B,+}$, we have to explicitly describe $\langle \mathrm{Im}(\varphi) \rangle_B$.

Lemma 2.4.3. *[BD20]*

$$\mathrm{Im}(\varphi) = \Big\{ \sum_{i=1}^{c} (d^n(g_i)d^n(f_i) \otimes 1 + g_i(d^n(f_i) \otimes 1)) \mid g_1, \cdots, g_c \in S \Big\}$$

Proof. Let $\varphi(\bar{h}) \in \mathrm{Im}(\varphi)$ and $h = \sum_{i=1}^{c} f_i g_i \in I$. Then we have

$$\varphi(\bar{h}) = \sum_{i=1}^{c} \varphi(\overline{f_i g_i}) = \sum_{i=1}^{c} d^n(f_i g_i) \otimes 1$$

By the Remark 2.1.7, we get

$$\sum_{i=1}^{c} d^n(f_i g_i) \otimes 1 = \sum_{i=1}^{c} d^n(f_i)d^n(g_i) \otimes 1 + g_i(d^n(f_i) \otimes 1) + f_i(d^n(g_i) \otimes 1)$$

It is enough to show that $f_i(d^n(g_i) \otimes 1) = 0$ for all $1 \le i \le c$. We observe that all the tensor products are over S. Then we have

$$f_i(d^n(g_i) \otimes 1) = d^n(g_i) \otimes \bar{f}_i$$

where $\bar{f}_i \in B$ for all $1 \leq i \leq c$. But $f_i \in I$, and so $\bar{f}_i = 0 \in B$ for all $1 \leq i \leq c$. Then we get

$$f_i(d^n(g_i) \otimes 1) = 0$$

for all $1 \leq i \leq c$ and thus,

$$\operatorname{Im}(\varphi) \subseteq \Big\{ \sum_{i=1}^{c} (d^n(g_i) d^n(f_i) \otimes 1 + g_i(d^n(f_i) \otimes 1)) \mid g_1, \cdots, g_c \in S \Big\}$$

The reverse inequality follows by the similar argument as all the statements and calulations we did are equivalent. $\square$

We now introduce new notation F^i_β for $\beta \in \mathbb{N}^s$ such that $0 \leq |\beta| \leq n-1$ given by

$$F^i_\beta := d^n(x)^\beta d^n(f_i)$$

where $d^n(x)^\beta := d^n(x_1)^{\beta_1} d^n(x_2)^{\beta_2} \cdots d^n(x_s)^{\beta_s}$. Further, we know

$$F^i_\beta = d^n(x)^\beta d^n(f_i) = \sum_{1 \leq |\gamma| \leq n} \frac{1}{\gamma!} \frac{\partial^\gamma (f_i)}{\partial x^\gamma} d^n(x)^{\gamma+\beta}$$

Let $\alpha = \gamma + \beta$, then we get

$$F^i_\beta = d^n(x)^\beta d^n(f_i) = \sum_{1+|\beta| \leq |\alpha| \leq n+|\beta|} \frac{1}{(\alpha-\beta)!} \frac{\partial^{\alpha-\beta}(f_i)}{\partial x^{\alpha-\beta}} d^n(x)^\alpha$$

We also observe that $\beta \leq \alpha$ which means $\beta_i \leq \alpha_i$ for all $1 \leq i \leq c$, and $\beta \neq 0$ because $\gamma \neq 0$. Since $P^n_{S,+} = I_S / I_S^{n+1}$, then $|\alpha| \leq n$. This implies

$$F^i_\beta = \sum_{1+|\beta| \leq |\alpha| \leq n} \frac{1}{(\alpha-\beta)!} \frac{\partial^{\alpha-\beta}(f_i)}{\partial x^{\alpha-\beta}} d^n(x)^\alpha$$

We now define

$$\frac{1}{(\alpha-\beta)!}\frac{\partial^{\alpha-\beta}(f_i)}{\partial x^{\alpha-\beta}}d^n(x)^\alpha = 0 \quad \text{when} \quad \begin{cases} |\alpha| < 1+|\beta|, \\ \alpha_i < \beta_i \text{ for some i}, \\ \alpha = \beta \end{cases}$$

Then it follows that

$$F^i_\beta = \sum_{1\le|\alpha|\le n} \frac{1}{(\alpha-\beta)!}\frac{\partial^{\alpha-\beta}(f_i)}{\partial x^{\alpha-\beta}}d^n(x)^\alpha.$$

Lemma 2.4.4. *[BD20] Let $\mathcal{N}$ be the B-submodule of $P^n_{S,+} \otimes_S B$ generated by $\{F^i_\beta \otimes 1 \mid 0 \le |\beta| \le n-1, 1 \le i \le c\}$. Then $\mathcal{N} = \langle \mathrm{Im}(\varphi)\rangle_B$.*

Proof. We start by showing $\mathcal{N} \subseteq \langle \mathrm{Im}(\varphi)\rangle_B$. We first observe that

$$F^i_0 \otimes 1 = d^n(f_i) \otimes 1 = \varphi(\bar{f}_i) \in \mathrm{Im}(\varphi)$$

We now assume $1 \le \beta \le n-1$. By [Nak70, Proposition II-2], $\{d^n(x^\alpha) \mid 1 \le |\alpha| \le n\}$ is also a basis for $P^n_{S,+}$. Then we have

$$d^n(x)^\beta = \sum_\alpha g_\alpha d^n(x^\alpha)$$

where $g_\alpha \in S$. Further, by Lemma 2.4.3, we have

$$d^n(x^\alpha)d^n(f_i) \otimes 1 + x^\alpha d^n(f_i) \otimes 1 \in \mathrm{Im}(\varphi)$$

Thus,

$$g_\alpha(d^n(x^\alpha)d^n(f_i) \otimes 1 + x^\alpha d^n(f_i) \otimes 1) \in \langle \mathrm{Im}(\varphi)\rangle_B.$$

But we know $d^n(f_i) \otimes 1 \in \mathrm{Im}(\varphi)$. It follows that $g_\alpha x^\alpha(d^n(f_i) \otimes 1) \in \langle \mathrm{Im}(\varphi)\rangle_B$. Thus,

$$g_\alpha d^n(x^\alpha)d^n(f_i) \otimes 1 \in \langle \mathrm{Im}(\varphi)\rangle_B.$$

And, so $F^i_\beta \otimes 1 \in \langle \mathrm{Im}(\varphi)\rangle_B$.

We now prove the reverse inequality $\langle \mathrm{Im}(\varphi)\rangle_B \subseteq \mathcal{N}$. Let $g \in S$, then by Remark 2.1.7

$$\begin{aligned} d^n(g)d^n(f_i) &= \left(\sum_{1\le|\alpha|\le n} \frac{1}{\alpha!}\frac{\partial^\alpha(g)}{\partial x^\alpha} d^n(x)^\alpha\right) d^n(f_i) \\ &= \left(\sum_{1\le|\alpha|\le n-1} \frac{1}{\alpha!}\frac{\partial^\alpha(g)}{\partial x^\alpha} d^n(x)^\alpha\right) d^n(f_i) \\ &= \left(\sum_{1\le|\alpha|\le n-1} \frac{1}{\alpha!}\frac{\partial^\alpha(g)}{\partial x^\alpha} d^n(x)^\alpha\right) F^i_\alpha. \end{aligned}$$

Then

$$d^n(g)d^n(f_i) \in \mathcal{N}. \tag{2.1}$$

We also observe that

$$g(d^n(f_i) \otimes 1) = g(F^i_0 \otimes 1) \in \mathcal{N}. \tag{2.2}$$

□

Thus, by (2.1), (2.2) and Lemma 2.4.3, we have $\mathrm{Im}(\varphi) \in \mathcal{N}$.

Proposition 2.4.5. *[BD20]*

$$P^n_{B,+} \cong \bigoplus_{1\le|\alpha|\le n} Bd^n(x)^\alpha / \sum_{\substack{0\le|\beta|\le n-1\\ 1\le i\le c}} BF^i_\beta.$$

Proof. We know from Theorem 2.4.2 that

$$P^n_{B,+} \cong (P^n_{S,+} \otimes_S B)/\langle \mathrm{Im}(\varphi)\rangle_B.$$

Since S is a polynomial ring, we also know that

$$P^n_{S,+} \cong \bigoplus_{1\le|\alpha|\le n} S(d^n(x))^\alpha.$$

Thus,

$$P^n_{S,+} \otimes_S B \cong B(d^n(x))^\alpha.$$

and the result follows from Lemma 2.4.4 $\square$

We now define the higher Jacobian matrix which will be useful to give the presentation of the module of principal parts.

Definition 2.4.6. [BD20] Let $M = \binom{s+n-1}{s}$ and $N = \binom{s+n}{s}$. We define the Jacobian matrix of order n, denoted $\mathrm{Jac}_n(f_1, \cdots, f_c)$ as the matrix whose rows are the vectors $C^i_\beta \in B^{N-1}$, where C^i_β are the vectors whose entries are the coefficients of F^i_β.

Notice that $\mathrm{Jac}_n(f_1, \cdots, f_c)$ is a $(cM \times N - 1)$-matrix.

Theorem 2.4.7. *[BD20] The B- module $P^n_{B,+}$ has the following presentation*

$$B^{cM} \xrightarrow{\mathrm{Jac}_n^T} B^{N-1} \longrightarrow P^n_{B,+} \longrightarrow 0,$$

where Jac_n^T *is the transpose of the Jacobian matrix of order* n.

Proof. By the definition of the Jacobian matrix of order n and Proposition 2.4.5, the result follows. $\square$

Example 2.4.8. Let $X = V(f)$ where $f = y^2 - x^3 \in k[x, y]$. Let $n = 2$ and $B = k[x, y]/(f)$. Here, $c = 1$, $M = 3$ and $N = 6$. We know $P^1_{B,+} = I_B/I^2_B$, and using Remark 2.1.7 we have

$$\begin{aligned} d^2 f &= d^2(y^2 - x^3) \\ &= d^2(y^2) - d^2(x^3) \\ &= d^2(y)^2 + 2yd^2(y) - 3x^2 d^2(x) - 3xd^2(x)^2 \end{aligned}$$

Then we get

$$\begin{aligned} F_{(0,0)} &= d^2(x)^{(0,0)} d^2(f) \\ &= d^2(y)^2 + 2yd^2(y) - 3x^2 d^2(x) - 3xd^2(x)^2 \end{aligned}$$

And,

$$\begin{aligned} F_{(1,0)} &= d^2(x)^{(1,0)}d^2(f) \\ &= 2yd^2(x)d^2(y) - 3x^2d^2(x)^2 \end{aligned}$$

And,

$$\begin{aligned} F_{(0,1)} &= d^2(x)^{(0,1)}d^2(f) \\ &= 2yd^2(y)^2 - 3x^2d^2(x)d^2(y) \end{aligned}$$

Further, by the definition of the Jacobian matrix of order n, we get

$$\mathrm{Jac}_2(f) = \begin{pmatrix} -3x^2 & 2y & 3x & 0 & 1 \\ 0 & 0 & -3x^2 & 2y & 0 \\ 0 & 0 & 0 & -3x^2 & 2y \end{pmatrix}.$$

Let $\mathrm{Jac}_2(f)^T$ be the transpose of $\mathrm{Jac}_2(f)$. Then by the Theorem 2.4.7, we have the following presentation of $P^2_{B,+}$

$$0 \longrightarrow B^3 \xrightarrow{\mathrm{Jac}_2^T} B^5 \longrightarrow P^2_{B,+} \longrightarrow 0.$$

2.5 Filtration and associated graded ring of P^n_R

We start by constructing a filtration of P^n_R which gives us the associated graded module of P^n_R. One can then compare it with the module of principal parts of the associated graded ring. For this section, we use the Setup 4.2.1 and the following notations.

- (R, m_R) be a local ring as before (not necessary homogeneous)
- $G := \mathrm{gr}_{m_R}(R)$ - associated graded ring of R (with respect to the m_R-adic filtration)
- $m_G := \mathrm{gr}^+_{m_R}(R)$ - homogeneous maximal ideal

We define the filtration of P^n_R as follows:

Let $S' := k[x_1, \ldots, x_s, \xi_1, \ldots, \xi_s]$ and $m_{S'} := (x_1, \ldots, x_s, \xi_1, \ldots, \xi_s)$ be its maximal ideal. Define

$$S_l = \{f \in S' \mid \mathrm{order}(f) \geq l\}$$

We can also write it as

$$S_l = (x_1, \ldots, x_s, \xi_1, \ldots, \xi_s)^l$$

It is easy to see that $S_{l_1} \cdot S_{l_2} \subseteq S_{l_1+l_2}$ This defines a grading on S', and we get

$$\mathrm{gr}(S') = \bigoplus_{l \geq 0} S_l/S_{l+1}.$$

Let $R' = S'/I'$ where $I' = (f_1, \ldots, f_c, g_1, \ldots, g_c) + (x_i - \xi_i, \ldots, x_s - \xi_s)^{n+1}$ is an ideal in S'. Here, $(f_1, \ldots, f_c)$ is the ideal in $k[x_1, \ldots, x_s]$ defining our variety X, and $(g_1, \ldots, g_c)$ is the ideal defining X but with different coordinates. It means that f_i and g_i are exactly the same polynomials for all $1 \leq i \leq c$ which we get by replacing x_i by ξ_i and vice versa. We define the filtration on R' similar to S' by letting

$$R_l := (S_l + I')/I' \cong S_l/(I' \cap S_l).$$

Then it follows that $R_{l_1} \cdot R_{l_2} \subseteq R_{l_1+l_2}$ and we get

$$\mathrm{gr}(R') = \bigoplus_{l \geq 0} R_l/R_{l+1}.$$

We know that $P^n_R = (R \otimes R)/I^{n+1}_R$ where $R := (k[x_1, \ldots, x_s]/(f_1, \ldots, f_c))_{(x_1, \ldots, x_s)}$.Then, P^n_R can be identified with $R'_{(x_1, \ldots, x_s, \xi_1, \ldots, \xi_s)}$, i.e,

$$P^n_R = \frac{k[x_1, \ldots, x_s, \xi_1, \ldots, \xi_s]_{(x_1, \ldots, x_s, \xi_1, \ldots, \xi_s)}}{(f_1, \ldots, f_c, g_1, \ldots, g_c) + (x_1 - \xi_1, \ldots, x_s - \xi_s)^{n+1}}. \tag{2.3}$$

We also have the initial map associated to the above filtration on S, denoted,

$$\mathrm{In} : S' \longrightarrow \mathrm{gr}(S'). \tag{2.4}$$

This map gives the homogeneous part of lowest degree of a polynomial. The above defined filtration is m_R-compatible. It tells us that the associated graded ring of a localization of a ring is the same as the associated graded ring of the ring, and so we get

$$\mathrm{gr}(P^n_R) = \frac{k[x_1, \ldots, x_s, \xi_1, \ldots, \xi_s]}{\mathrm{In}((f_1, \ldots, f_c, g_1, \ldots, g_c) + (x_1 - \xi_1, \ldots, x_s - \xi_s)^{n+1})}.$$

Definition 2.5.1. Let $I = (h_1, \ldots, h_l)$ be an ideal in S'. We say $\{h_1, \ldots, h_l\}$ is a Gröbner basis with respect to In if $\mathrm{In}(I) = (\mathrm{In}(h_1), \ldots, \mathrm{In}(h_l))$.

From now on, we always assume that $\{f_1, \ldots, f_c\}$ and $\{g_1, \ldots, g_c\}$ are Gröbner bases with respect to In. Then $G := k[x_1, \ldots, x_s]/(\mathrm{In}(f_1), \ldots, \mathrm{In}(f_c))$ and it follows that

$$P^n_G = \frac{k[x_1, \ldots, x_s, \xi_1, \ldots, \xi_s]}{(\mathrm{In}(f_1), \ldots, \mathrm{In}(f_c), \mathrm{In}(g_1), \ldots, \mathrm{In}(g_c)) + (x_1 - \xi_1, \ldots, x_s - \xi_s)^{n+1}}.$$

Let $J_1 := (\mathrm{In}(f_1), \ldots, \mathrm{In}(f_c), \mathrm{In}(g_1), \ldots, \mathrm{In}(g_c)) + (x_1 - \xi_1, \ldots, x_s - \xi_s)^{n+1}$ and $J_2 := \mathrm{In}((f_1, \ldots, f_c, g_1, \ldots, g_c) + (x_1 - \xi_1, \ldots, x_s - \xi_s)^{n+1})$. Clearly, $J_1 \subseteq J_2$. We then have the following surjective G-module homomorphism

$$\psi : P^n_G \longrightarrow \mathrm{gr}(P^n_R)$$

Let $m \in J_2 \setminus J_1$. Then we write

$$m = \mathrm{In}\bigg(\sum_i a_i f_i + \sum_j b_j g_j + \sum_{|\alpha|=n+1} r_\alpha (x_1 - \xi_1)^{\alpha_1} \cdots (x_s - \xi_s)^{\alpha_s} \bigg).$$

for some $a_i, b_j, r_\alpha \in k[x_1, \dots, x_s, \xi_1, \dots, \xi_s]$, and assume that

$$\sum_i \mathrm{In}(a_i)\mathrm{In}(f_i) + \sum_j \mathrm{In}(b_j)\mathrm{In}(g_j) + \sum_{|\alpha|=n+1} \mathrm{In}(r_\alpha)(x_1-\xi_1)^{\alpha_1}\cdots(x_s-\xi_s)^{\alpha_s} = 0.$$

Otherwise $m \in J_1$. By the construction of the initial map (2.4), m is a homogeneous polynomial and $\deg(m) \geq n+1$. Then the image of m in $\mathrm{gr}(P_R^n) \otimes_G G/m_G$ lies in $(\xi_1, \dots, \xi_s)^{n+1}$. It is easy to see that

$$P_G^n \otimes_G G/m_G = \frac{k[\xi_1, \dots, \xi_s]}{(\mathrm{In}(g_1), \dots, \mathrm{In}(g_c)) + (\xi_1, \dots, \xi_s)^{n+1}},$$

and

$$\mathrm{gr}(P_R^n) \otimes_G G/m_G = \frac{k[\xi_1, \dots, \xi_s]}{\mathrm{In}((g_1, \dots, g_c) + (\xi_1, \dots, \xi_s)^{n+1})}.$$

Since $\{g_1, \dots, g_c\}$ is a Gröbner basis and with the similar argument as above, notice that $(\mathrm{In}(g_1), \dots, \mathrm{In}(g_c)) + (\xi_1, \dots, \xi_s)^{n+1}) = \mathrm{In}(g_1, \dots, g_c + (\xi_1, \dots, \xi_s)^{n+1})$. Therefore,

$$P_G^n \otimes_G G/m_G \cong \mathrm{gr}(P_R^n) \otimes_G G/m_G \tag{2.5}$$

Chapter 3

Nash and Higher Nash Blow-ups

In this chapter, we introduce Nash blow-up and later higher Nash blow-ups. We also give important properties of both of these blow-ups, especially how they are related to the sheaf of principal parts.

3.1 Nash Blow-up

In this section, we define Nash blow-up and some of its properties.

Definition 3.1.1. [Nob75a] Let $X = V(f_1, \cdots, f_c)$ be a variety of dimension d over algebraically closed field k. Let $T_{X,x} \in \mathrm{Gr}(d, n)$ be the tangent space of X at x, where $\mathrm{Gr}(d, n)$ is the grassmanian of d-dimensional spaces in n-dimensional space. Let $S(X)$ be the set of singular points of X, and X_{sm} its complement called smooth locus in X. Consider the morphism

$$\sigma : X_{\mathrm{sm}} \longrightarrow \mathrm{Gr}(d, n), \quad x \longmapsto T_{X,x}.$$

The Nash blow-up of X, denoted $\mathrm{Nash}(X)$, is defined to be the closure of the graph Γ_σ

inside $X \times_k \mathrm{Gr}(d,n)$. It comes equipped with the natural projection

$$\pi : \mathrm{Nash}(X) \longrightarrow X.$$

The above definition is local. We will give a more general definition in the next subsection.The next theorem says that the Nash blow-up is the usual blowup, and is due to A. Nobile.

Proposition 3.1.2. *Let $X = V(f_1, \cdots, f_c) \subseteq \mathbb{A}^s$ be a variety of dimension d over an algebraically closed field k of characteristic zero. A Nash blowing up is the usual blow-up along an ideal I. Moreover, there exists $(s-d) \times s$ submatrix B of the Jacobian matrix $[\partial f_i / \partial x_j]$, $1 \leq i \leq c$ and $1 \leq j \leq s$ such that I is generated by the maximal minors of B.*

We give the classical Nobile's theorem which is only true over a field of characteristic zero in the next chapter. We also see a counter-example where it fails in positive characteristic.

3.2 Higher Nash Blow-up

In this section, We define the notion of higher Nash blow-up which is a higher version of the Nash blow-up of a variety. We define it using the following definition of Hilbert scheme of points.

Definition 3.2.1. (Hilbert scheme of points) Let X be a k-scheme and $l \in \mathbb{Z}_{\geq 0}$. Let $F_{X,l} : \mathcal{S}ch_k^{\mathrm{op}} \longrightarrow \mathrm{Set}$ be a contravariant functor given by

$$F_{X,l}(Z) = \{Y \subset X \times_k Z\}$$

such that $\pi : Y \to Z$ is flat, and $\pi^{-1}(Z)$ is a 0-dimensional closed subscheme of X length

l. The Hilbert scheme of points on X, denoted $\mathrm{Hilb}_l(X)$, is the scheme representing the functor $F_{X,l}$.

Definition 3.2.2. [Yas07] Let X be a variety of dimension d over k. For a point $x \in X$, the n-th infinitesimal neighborhood of x is $x^{(n)} := \operatorname{Spec} \mathcal{O}_{X,x}/m_x^{n+1}$. If x is a smooth point, then $[x^{(n)}] \in \mathrm{Hilb}_{\binom{n+d}{d}}(X)$. This gives us a morphism

$$\sigma_n : X_{\mathrm{sm}} \longrightarrow \mathrm{Hilb}_{\binom{n+d}{d}}(X), \quad x \longmapsto [x^{(n)}].$$

The Nash blow-up of order n of X, denoted $\mathrm{Nash}_n(X)$, is defined to be the closure of the graph Γ_{σ_n} inside $X \times_k \mathrm{Hilb}_{\binom{n+d}{d}}(X)$. It comes equipped with the natural projection

$$\pi_n : \mathrm{Nash}_n(X) \longrightarrow X.$$

Definition 3.2.3 (Torsion submodule)**.** Let R be a Noetherian ring and M be a R-module. We define torsion submodule of M, denoted $\mathrm{tors}_R(M)$, as

$$\mathrm{tors}_R(M) = \{m \in M \mid \exists \text{ a non-zero divisor } r \in R \text{ such that } rm = 0\}.$$

Remark 3.2.4. From now on, we will denote

$$\widetilde{M} = M/\mathrm{tors}_R(M).$$

This definition extend naturally to the global case. For a quasi-coherent module $\mathcal{M}$ on X, we write

$$\widetilde{\mathcal{M}} = \mathcal{M}/\mathrm{tors}(\mathcal{M}).$$

We now give the universal property of higher Nash blow-up.

Proposition 3.2.5. *[Yas07, OZ91] Let X be a variety of dimension d over k. Let $n \in \mathbb{N}$, and we have the natural projection map*

$$\pi_n : \mathrm{Nash}_n(X) \longrightarrow X.$$

Suppose we also have a morphism $f : Y \longrightarrow X$ such that $\widetilde{f^*(\mathcal{P}^n_{X,+})}$ is locally free, then there exists a unique morphism $g : Y \longrightarrow \mathrm{Nash}_n(X)$ such that $\pi_n \circ g = f$.

Higher Nash blow-ups and modules of principal parts are related via the following classical construction.

Definition 3.2.6. [OZ91] Let X be reduced Noetherian scheme and $\mathcal{M}$ a coherent $\mathcal{O}_X$-module locally free of constant rank r on an open dense subscheme $U \subseteq X$. The Nash tranasformation of X associated to $\mathcal{M}$, denoted $\mathrm{Nash}(X, \mathcal{M})$, is defined to be the closure of the graph of the natural map $U \longrightarrow \mathrm{Grass}_r(\mathcal{M})$.

We have the natural projective morphism $\pi_{\mathcal{M}} : \mathrm{Nash}(X, \mathcal{M}) \longrightarrow X$, and by definition $\widetilde{\pi^*_{\mathcal{M}}(\mathcal{M})}$ is locally free.

We now give the universal property of $\mathrm{Nash}(X, \mathcal{M})$

Remark 3.2.7. Let X be a reduced Noetherian scheme, $\mathcal{M}$ a locally free coherent $\mathcal{O}_X$-module and $\pi_{\mathcal{M}} : \mathrm{Nash}(X, \mathcal{M}) \longrightarrow X$ the natural projective morphism. If $f : Y \longrightarrow X$ is a birational morphism such that $\widetilde{f^*(\mathcal{M})}$ is locally free, then there exists a unique morphism $g : Y \longrightarrow \mathcal{M}$ such that $\pi_{\mathcal{M}} \circ g = f$.

We can also construct higher Nash blow-ups using the following definition of the relative Hilbert scheme. The next result relates the above two definitions.

Proposition 3.2.8. *[Yas07] Let X be a variety of dimension d, then for every $n \in \mathbb{N}$, we have*

$$\mathrm{Nash}_n(X) \cong \mathrm{Nash}(X, P^n_{X,+}) \cong \mathrm{Nash}(X, P^n_X)$$

Proof. We can view $\mathrm{Nash}(X, P^n_{X,+})$ and $\mathrm{Nash}(X, P^n_X)$ as $\mathcal{O}_X$-modules. We know that $\mathcal{P}^n_X \cong \mathcal{P}^n_{X,+} \oplus \mathcal{O}_X$. Since $\mathcal{O}_X$ is locally free, we have that $\widetilde{\pi^*_{\mathcal{P}^n_{X,+}}(\mathcal{P}^n_X)}$ is locally free. It is also easy to see that $\widetilde{\pi^*_{\mathcal{P}^n_X}(\mathcal{P}^n_{X,+})}$ is locally free because $\widetilde{\pi^*_{\mathcal{P}^n_X}(\mathcal{P}^n_X)}$ is locally free, and

$\widetilde{\pi^*_{\mathcal{P}^n_X}(\mathcal{P}^n_{X,+})} \cong \widetilde{\pi^*_{\mathcal{P}^n_X}(\mathcal{P}^n_X)} \oplus \mathcal{O}_X$. Then by the universal property, we get that $\mathrm{Nash}_n(X) \cong \mathrm{Nash}(X, P^n_{X,+}) \cong \mathrm{Nash}(X, P^n_X)$.

The first isomorphism is due to T. Yasuda [Yas07, Proposition 1.8]. $\square$

Chapter 4

Higher Nobile's Theorem

In this chapter, we prove our main theorems. We prove higher Nobile's theorem in the graded case and the case when $n = 2$, i.e. for the Nash blow-ups of order 2. We also construct an approach which could be helpful to prove the higher Noible's theorem in the general setting. We first start by stating the Nobile's theorem.

4.1 Nobile's Theorem

In this section, we give the classical Nobile's theorem over a field of characteristic zero. We also give a version of this theorem over a field of arbitrary characteristic.

Theorem 4.1.1. *[Nob75a] (Classical Nobile's Theorem) Let X be an algebraic variety over an algebraically closed field k with $\mathrm{char}(k) = 0$. Then $\mathrm{Nash}(X) \cong X$ if and only if X is a non-singular variety.*

The above theorem fails in positive characteristic. The simplest example is the cusp.

Remark 4.1.2. Let $X = V(f = y^2 - x^3) \subset \mathbb{A}^2$ be the curve over a field k such that $\mathrm{char}(k) = 2$. The Jacobian matrix of f is given by $\mathrm{Jac(f)} = [-3x^2 \quad 2y]$. By the

Proposition 3.1.2, $\mathrm{Nash}(X)$ is the blow-up along the ideal $I = (3x^2, 2y) = (x^2)$. Since I is the principal ideal, $\mathrm{Bl}_I(X) \cong X$. Thus, $\mathrm{Nash}(X) \cong X$, but X is a singular variety.

We now show the well known result about the resolution of singularities for curves using Nash blow-ups.

Corollary 4.1.3. *Let X be an algebraic curve over an algebraically closed field of characteristic zero. Then a finite sequence of Nash blow-ups desingularizes the curve.*

Proof. Consider the following sequence of Nash blow-ups

$$\cdots \longrightarrow X_2 \xrightarrow{\sigma_2} X_1 \xrightarrow{\sigma_1} X_0 = X,$$

where $X_i = \mathrm{Nash}(X_{i-1})$ for all $i \geq 0$.

Let $\bar{X}$ be the normalization of X. Then $\nu : \bar{X} \longrightarrow X$ is finite. Since X is a curve, $\bar{X}$ is non-singular. Therefore, $\widetilde{\nu^*(\Omega_X)}$ is locally free. Then by the universal property of the Nash blow-up, there exists a unique morphism $\bar{\nu} : \bar{X} \longrightarrow X_1$ such that $\sigma_1 \circ \bar{\nu} = \nu$. We can observe that $\bar{\nu}$ is finite because ν is finite and σ_1 is isomorphism over smooth locus. Further, let $\bar{X}_1$ be the normalization of X_1. We know $\bar{X}$ is non-singular, and $\bar{\nu}$ is finite, birational, then by the universal property of normalization $\bar{X} \cong \bar{X}_1$.

Using the above argument for all $X_i, i \geq 0$, we have the following sequence of Nash blow-ups dominated by the normalization

$$\bar{X} \cdots \longrightarrow X_2 \xrightarrow{\sigma_2} X_1 \xrightarrow{\sigma_1} X_0 = X.$$

We know that ν is finite. Then it follows that after a finite number of Nash blow-ups, the sequence stabilizes and reaches the normalization, i.e, there exists $n \in \mathbb{N}$ such that $X_n = \bar{X}$.

□

The classical Nobile's theorem is true in positive characteristic if we add some hypothesis.

Theorem 4.1.4. *[DNB22] (Nobile's theorem in positive characteristic) Let X be a normal algebraic variety over an algebraically closed field of arbitrary characteristic. Suppose* $\mathrm{Nash}(X) \cong X$*, then X is a non-singular variety.*

We now move towards proving higher Nobile's theorem for few classes of varieties. For this, we need some construction and tools.

4.2 HN_n Property

We will use the following setup for the rest of this paper.

Setup 4.2.1. *We use the following notations.*

- *We fix an affine algebraic variety $X \subset \mathbb{A}^s$ of dimension d over the base field k.*
- *We write $\mathbb{A}^s = \operatorname{Spec} S$ and $X = \operatorname{Spec} B$ where $B = S/I$.*
- *We fix a closed point $x \in X$, and set $R := B_{m_x} = \mathcal{O}_{X,x}$ where m_x is the maximal ideal corresponding to x.*
- *We Let K be the field of fractions of R.*

To prove the generalized Nobile's theorem we need a key property. We define this property as follows:

Definition 4.2.1 (HN_n)**.** Let R be a k-algebra, $\mathfrak{m}_R$ be a maximal ideal and $\dim R = d$. Given $n \in \mathbb{N}$, we say that R satisfies the HN_n property if there exists algebraically independent elements $x_1, \ldots, x_d \in \mathfrak{m}_R$, corresponding to a ring homomorphism $A := k[x_1, \ldots, x_d] \longrightarrow R$, such that the induced map

$$P^n_{A,+} \otimes_A R/\mathfrak{m}_R \longrightarrow \widetilde{P}^n_{R,+} \otimes_R R/\mathfrak{m}_R$$

is injective. Notice that this is equivalent to saying that

$$P^n_A \otimes_A R/\mathfrak{m}_R \longrightarrow \widetilde{P}^n_R \otimes_R R/\mathfrak{m}_R$$

is injective.

4.3 Reducing higher Nobile's theorem to classical Nobile's theorem

We now see how higher Nobile's theorem reduces down to the classical Nobile's theorem.

Theorem 4.3.1. *Let X be an affine algebraic variety, and R be as in Setup 4.2.1. Suppose R satisfies the HN_n property, and $\mathrm{Nash}_n(X) \cong X$ for some $n \in \mathbb{N}$, then $\mathrm{Nash}(X) \cong X$.*

Proof. We assume $\mathrm{Nash}_n(X) \cong X$. Then by the definition of n-th Nash blow-up we have that $\pi_n^*\widetilde{(\mathcal{P}^n_{X,+})}$ is locally free of rank $D-1 := \binom{n+d}{d} - 1$. But π_n is an isomorphism, so $\widetilde{\mathcal{P}}^n_{X,+}$ is locally free of rank $D-1$ which implies $\widetilde{P}^n_{\mathcal{O}_{X,x},+}$ is free of rank $D-1$ for all $x \in X$. Let $R := \mathcal{O}_{X,x}$. By hypothesis, we have $A = k[x_1, \ldots, x_d]$ such that

$$P^n_A \otimes_A R/\mathfrak{m}_R \longrightarrow \widetilde{P}^n_R \otimes_R R/\mathfrak{m}_R,$$

is injective. This is equivalent to saying that

$$P^n_{A,+} \otimes_A R/\mathfrak{m}_R \longrightarrow \widetilde{P}^n_{R,+} \otimes_R R/\mathfrak{m}_R,$$

is injective. But $\dim(P^n_{A,+} \otimes_A R/\mathfrak{m}_R) = \dim(\widetilde{P}^n_{R,+} \otimes_R R/\mathfrak{m}_R) = D-1$, and so the above map is an isomorphism. Thus, $\widetilde{P}^n_{R,+} \otimes_R R/\mathfrak{m}_R$ is generated by $\{[\overline{d^n(x_1)}]^{\alpha_1} \cdots [\overline{d^n(x_d)}]^{\alpha_d} \mid$

$1 \leq |\alpha| \leq n\}$ as an $R/\mathfrak{m}_R$-vector space. Consider the surjective map

$$\widetilde{P}_R^n \longrightarrow \widetilde{P}_R^n \otimes_R R/\mathfrak{m}_R \cong \frac{\widetilde{P}_R^n}{\mathfrak{m}_R \widetilde{P}_R^n}.$$

Further, it follows from Nakayama's lemma that the R-module $\widetilde{P}_{R,+}^n$ is freely generated by $\left\{\overline{d^n(x_1)}^{\alpha_1} \cdots \overline{d^n(x_d)}^{\alpha_d} \mid 1 \leq |\alpha| \leq n\right\}$. Furthermore, we have a surjective R-module homomorphism

$$P_{R,+}^n = I_R/I_R^{n+1} \longrightarrow \Omega_R = I_R/I_R^2.$$

This induces the following surjective map

$$\widetilde{P}_{R,+}^n \longrightarrow \widetilde{\Omega}_R.$$

It follows that the R-module $\widetilde{\Omega}_R$ is generated by $\left\{\overline{d(x_1)}, \cdots, \overline{d(x_d)}\right\}$. But $\operatorname{rank}(\widetilde{\Omega}_R) = \dim R = d$. Thus, $\widetilde{\Omega}_R = \Omega_{\mathcal{O}_{X,x}}/\operatorname{tors}_{\mathcal{O}_{X,x}}(\Omega_{\mathcal{O}_{X,x}})$ is free of rank d. This is equivalent to $\widetilde{\Omega}_X$ being locally free of rank d. So, by the universal property 3.2.5 of Nash blow-up, there exists a unique map $\varphi : X \longrightarrow \operatorname{Nash}(X)$. Thus, $\pi : \operatorname{Nash}(X) \longrightarrow X$ is an isomorphism. □

4.4 Arc Lemma

In this section, we construct an arc and observe some of its important properties.

Remark 4.4.1. We can assume $x \in X$ is the origin of $\mathbb{A}^s$. Now we consider linear projections of the form $\mathbb{A}^s \longrightarrow Y$, where Y is an affine subspace of $\mathbb{A}^s$ of dimension d containing 0. For each such projection we can choose affine coordinates $x_1, \ldots, x_s$ in $\mathbb{A}^s$ centered at 0 such that Y is defined by the equations $x_{d+1} = \ldots = x_s = 0$ and the projection is given by $(x_1, \ldots, x_s) \in \mathbb{A}^s \mapsto (x_1, \ldots, x_d) \in Y$ (notice that $(x_1, \ldots, x_d)$ give coordinates in Y). Let Z be the subspace of $\mathbb{A}^s$ given by $x_1, \ldots, x_d = 0$. This Z gives the direction of projection. The fibers of the projection are translations of Z, that is, the affine subspaces of the form $y+Z$, where $y \in Y$ and $+$ is the addition for the vector space structure of $\mathbb{A}^s$ induced by the coordinates $x_1, \ldots, x_s$. Each such projection induces a map $X \longrightarrow Y$, and let $\mathcal{H}$ be the set of all such induced projections. When this induced map is finite we get a Noether normalization which means that $k[x_1, \ldots, x_d] \longrightarrow B$ is a finite integral extension. The proof of the Noether normalization lemma shows that the subset $\mathcal{U} \subseteq \mathcal{H}$ of the projections that give a Noether normalization is an open dense subset of $\mathcal{H}$.

Firstly, we give the graded arc lemma.

Lemma 4.4.2 (Graded arc lemma). *Let X be a graded algebraic variety, and R be as in Setup 4.2.1. Then for a generic choice of coordinates $x_1, \ldots, x_s$ for $\mathbb{A}^s$, we get that $\{x_1, \ldots, x_d\}$ is a transcendental basis of K over k. Further, there also exists a field L, and an arc $\beta : R \longrightarrow L[[t]]$ such that β is an inclusion and $\beta(x_i) = u_i t$, where $u_i \in L$ for all $1 \leq i \leq s$, and $\{u_1, \ldots, u_d\}$ is a transcendental basis of L over k.*

Proof. From Setup 4.2.1, we know B is a finitely generated k-algebra. By the above Remark 4.4.1, we can pick a projection and coordinates $x_1, \ldots, x_s$ for $\mathbb{A}^s$. We can assume that $\{x_1, \ldots, x_d\}$ is a transcendental basis of K over k such that $A \subseteq B$, where $A = k[x_1, \ldots, x_d]$ is a Noether normalization. Since B is a domain, we have $B \subseteq R$, and so $A \subseteq R$. Let $u_1, \ldots, u_d$ be new transcendental elements, and let L be the algebraic closure of the field of fractions of $k[u_1, \ldots, u_d]$.

Consider the generic point of X, and since R is integral over A, we have a map $\varphi : R \longrightarrow L$ such that $\varphi(x_i) = u_i$ for all $1 \leq i \leq d$. For $d+1 \leq i \leq s$, set $u_i = \varphi(x_i) \in L$. We can now scale the generic point. More precisely, since X is graded, we get an arc $\widetilde{\beta} : S \longrightarrow L[[t]]$ via $\widetilde{\beta}(x_i) = u_i t$, for all $1 \leq i \leq s$. Then, since the ideal I is homogeneous, $\widetilde{\beta}$ factors via a ring map $\beta : R \longrightarrow L[[t]]$ still satisfying $\beta(x_i) = u_i t$ for all $1 \leq i \leq s$. $\square$

For general rings, we have the following version.

Lemma 4.4.3 (Arc lemma). *Let X and R be as in Setup 4.2.1, and assume char$(k) = 0$. Then for a generic choice of coordinates $x_1, \ldots, x_s$ for $\mathbb{A}^s$, we get that $\{x_1, \ldots, x_d\}$ is a transcendental basis of K over k. Further, there also exists a field L, and an arc $\beta : R \longrightarrow L[[t]]$ such that β is an inclusion, and*

(a) $\beta(x_i) = u_i t^a$, where $u_i \in L$ for all $1 \leq i \leq d$, and $\{u_1, \ldots, u_d\}$ is a transcendental basis of L over k.

(b) $\beta(x_j) = u_j t^{b_j}$, where $u_j \in L[[t]]$ and $b_j \geq a$ for all $d+1 \leq j \leq s$.

Proof. We start by proving statement (a) of the theorem. Again by the Remark 4.4.1, we can pick a projection and coordinates $x_1, \ldots, x_s$ for $\mathbb{A}^{\mathrm{s}}$. We can assume that $\{x_1, \ldots, x_d\}$ is a transcendental basis of K over k such that $A \subseteq B$, where $A = k[x_1, \ldots, x_d]$ is a Noether normalization. Since B is a domain, we have $B \subseteq R$, and so $A \subseteq R$. Let $u_1, \ldots, u_d$ be new transcendental elements, and let L be the algebraic closure of the field of fractions of $k[u_1, \ldots, u_d]$. Then consider the arc $\beta_0 : A \longrightarrow L[[s]]$ such that $\beta_0(x_i) = u_i s$. Notice that β_0 is clearly an inclusion. Further, let K and K_0 be the fields of fractions of R and A respectively. Then K is an algebraic extension of K_0. We denote by K_0^{ac} the algebraic closure of K_0, so we have $K_0 \subseteq K \subseteq K_0^{\mathrm{ac}}$. Next we denote by $L((s))$ the field of fractions of $L[[s]]$, and we let $L((s))^{\mathrm{ac}}$ be the algebraic closure of $L((s))$. Let $\delta_0 : K_0 \longrightarrow L((s))$ be the map induced by β_0. Then notice that the morphism $\delta^{\mathrm{ac}} : K_0^{\mathrm{ac}} \longrightarrow L((s))^{\mathrm{ac}}$ exists by the definition of algebraic closure. Since L is an algebraically closed field of characteristic zero and by the Newton-Puiseux theorem, we have $L((s))^{\mathrm{ac}} = \bigcup_{a \geq 1} L((s^{1/a}))$. Furthermore, we know that K is a finite field extension of K_0 since R is a finite ring extension of A. By looking at the images via δ^{ac} of each of the finitely many generators of K over K_0, we see that $\delta^{\mathrm{ac}}(K)$ lies inside of $L((s^{1/a}))$ for some non-negative integer a. This gives the map $\beta : K \longrightarrow L((s^{1/a}))$

and the following commutative diagram:

$$\begin{array}{ccc} K_0 & \xrightarrow{\delta_0} & L((s)) \\ \downarrow & & \downarrow \\ K & \xrightarrow{\delta} & L((s^{1/a})) \\ \downarrow & & \downarrow \\ K_0^{\mathrm{ac}} & \xrightarrow{\delta^{\mathrm{ac}}} & L((s))^{\mathrm{ac}}. \end{array}$$

Next notice that $L[[s^{1/a}]]$ is the integral closure of $L[[s]]$ inside of $L((s^{1/a}))$. Since R is integral over A, we see that $\delta(R) \subseteq L[[s^{1/a}]]$ giving the map $\beta : R \longrightarrow L[[s^{1/a}]]$ and the following commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{\beta_0} & L[[s]] \\ \downarrow & & \downarrow \\ R & \xrightarrow{\beta} & L[[s^{1/a}]]. \end{array}$$

Also, by the construction, β is an inclusion. Let $t = s^{1/a}$ and then we have constructed the arc $\beta : R \longrightarrow L[[t]]$ such that $\beta(x_i) = u_i t^a$ where $u_i \in L$ for all $1 \leq i \leq d$.

The next step is to prove the part (b) of the theorem. Consider the maximal ideal $m_x \subset \mathcal{O}_X$ and let $b \geq 1$ be order of contact of β with m_x which means $\beta^{-1}(m_x) \cdot L[[t]] = (t^b)L[[t]]$. Here, β, $L[[t]]$ and a are from part (a). Notice that m_x is generated by $x_1, \dots, x_s$. Therefore, for each $\{1, \dots, s\}$ we have $\beta(x_i) = t^{b_i}.u_i(t)$ with $u_i(t) \in L[[t]] \setminus (t)$, $b_i \in \mathbb{Z}_{\geq 1} \cup \{\infty\}$ and $b = \min\{b_1, \dots, b_s\} \in \mathbb{Z}_{\geq 1}$. We see that $a = b_1 = \dots = b_d \geq b$. Here, we use the convention $t^\infty = 0$.

Let $C \in \mathbb{A}^{\mathrm{s}}$ be the tangent cone to X at x. The ideal of C is $I_C = \mathrm{in}(I) = \{\mathrm{in}(f) : f \in I\}$ where $\mathrm{in}(f)$ denotes the lowest homogeneous part of the polynomial f. Since β is an arc on X, we know that $\beta(f) = 0$ for $f \in I$. For $f \in I$, write $f = f_e + f_{e+1} + \cdots$, where $e \geq 1$ and f_j is homogeneous of degree j. Write $\beta(x_i) = v_i t^b +$ (h.o.t) for $i \in \{1, \dots, s\}$, where (h.o.t) means higher order terms. Let $v = (v_1, \dots, v_s)$. Then we see that $0 = \beta(f) = f_e(v)t^{eb} + (\text{h.o.t}) = \mathrm{in}(f)(v)t^{eb} + (\text{h.o.t})$. Therefore $\mathrm{in}(f)(v) = 0$ and thus $v \in C$. We also know that $v \neq x$, and it follows that $v \in C \setminus \{0\}$.

Recall that Z is given by $x_1 = \cdots = x_d = 0$. Using the notation of Remark 4.4.1, let $\mathcal{V} \subseteq \mathcal{H}$ be the set of all projections for which Z intersects C equally at the origin, i.e. $Z \cap C = \{0\}$. Then $\mathcal{V}$ is a dense open subset of $\mathcal{H}$. Thus, pick any projection in $\mathcal{U} \cap \mathcal{V}$. Notice that $v_i = u_i(0)$ if $b_i = b$ and $v_i = 0$ if $b_i > b$. Then $v \notin Z$ since $v \neq x$ and $v \in C$. Therefore, $a = b$ and $b_j \geq a$ for all $d+1 \leq j \leq s$. □

4.5 Construction

In this section, we are going to construct the following commutative diagram which will be essential in proving the HN_n property.

$$\begin{tikzcd}
P^n_{A,+} \otimes_A R \arrow[r, "d\varphi"] \arrow[d] & \widetilde{P}^n_{R,+} \arrow[r, "d\beta"] \arrow[d] & \widehat{P}^n_{L[[t]]/L,+} \arrow[d] \\
P^n_{A,+} \otimes_A R/\mathfrak{m}_R \arrow[r, "\overline{d\varphi}"] & \widetilde{P}^n_{R,+} \otimes_R R/\mathfrak{m}_R \arrow[r, "\overline{d\beta}"] & \widehat{P}^n_{L[[t]]/L,+}/N_L.
\end{tikzcd} \tag{4.1}$$

Firstly, let $\varphi : A \longrightarrow R$ be the Noether normalization map and $\beta : R \longrightarrow L[[t]]$ be the arc constructed in the graded arc lemma 4.4.2 or the arc lemma 4.4.3. Recall that in the graded case, k has arbitrary characteristic; in the non-graded case, we need to assume $\mathrm{char}(k) = 0$.

We now consider the following composition:

$$A \xrightarrow{\varphi} R \xrightarrow{\beta} L[[t]].$$

This induces the following composition of R-module homomorphisms for all $n \in \mathbb{N}$

$$R \otimes_A P^n_{A,+} \xrightarrow{d\varphi} P^n_{R,+} \xrightarrow{d\beta} P^n_{L[[t]]/L,+}.$$

We also have the ring morphism $P^n_{L[[t]]/L} \longrightarrow \widehat{P}^n_{L[[t]]/L}$ where $\widehat{P}^n_{L[[t]]/L}$ is the (t)-adic completion of the $L[[t]]$-module $P^n_{L[[t]]/L}$. We know $P^n_{L[[t]]/L} \cong P^n_{L[[t]]/L,+} \oplus L[[t]]$, and so we get that $\widehat{P}^n_{L[[t]]/L} \cong \widehat{P}^n_{L[[t]]/L,+} \oplus L[[t]]$. Thus, one gets a morphism $P^n_{L[[t]]/L,+} \longrightarrow \widehat{P}^n_{L[[t]]/L,+}$.

This gives us

$$R \otimes_A P^n_{A,+} \xrightarrow{d\varphi} P^n_{R,+} \xrightarrow{d\beta} \widehat{P}^n_{L[[t]]/L,+}.$$

We work with $\widehat{P}^n_{L[[t]]/L,+}$ because it is free of rank n with basis $\{d^n t, (d^n t)^2, \cdots, (d^n t)^n\}$. On the other hand, $P^n_{L[[t]]/L,+}$ is not finitely generated.

Lemma 4.5.1. *Let* $\mathrm{tors}_R(P^n_{R,+})$ *be the torsion R-submodule of* $P^n_{R,+}$. *Then we have that* $\mathrm{tors}_R(P^n_{R,+}) \subseteq \ker(d\beta)$.

Proof. Let $\omega \in \mathrm{tors}_R(P^n_{R,+})$. Then there exists a non-zero $f \in R$ such that $f\omega = 0$. This implies $d\beta(f\omega) = 0$. But $d\beta(f\omega) = \beta(f)d\beta(\omega)$, and so $\beta(f)d\beta(\omega) = 0$. We know that β is an inclusion. Thus, $\beta(f) \neq 0$. Also, $\widehat{P}^n_{L[[t]]/L,+}$ is free and hence torsion-free. Then we get $d\beta(\omega) = 0$. Therefore, $\mathrm{tors}_R(P^n_{R,+}) \subseteq \ker(d\beta)$. $\square$

Then from the above lemma, $d\beta$ factors through $\widetilde{P}^n_{R,+}$. We then have the following commutative diagram:

$$\begin{tikzcd}
P^n_{A,+} \otimes_A R \arrow[r, "d\varphi"] \arrow[dr] & P^n_{R,+} \arrow[r, "d\beta"] \arrow[d, "\pi"] & \widehat{P}^n_{L[[t]]/L,+} \\
& \widetilde{P}^n_{R,+} \arrow[ur] &
\end{tikzcd}.$$

Consider the quotient map $\pi : P^n_{R,+} \longrightarrow \widetilde{P}^n_{R,+}$. We know that $P^n_{R,+}$ is generated as an R-module by $\{d^n(x_1)^{\alpha_1} \cdots d^n(x_s)^{\alpha_s} | 1 \leq |\alpha| \leq n\}$, where $\alpha = (\alpha_1, \ldots, \alpha_S)$ denotes a multi-index, and $|\alpha| = \alpha_1 + \cdots + \alpha_s$. Then $\widetilde{P}^n_{R,+}$ is generated as an R-module by $\{\pi(d^n(x_1)^{\alpha_1} \cdots d^n(x_s)^{\alpha_s}) \mid 1 \leq |\alpha| \leq n\}$. We will do abuse of notation and say $\widetilde{P}^n_{R,+}$ is generated as an R-module by $\{d^n(x_1)^{\alpha_1} \cdots d^n(x_s)^{\alpha_s} \mid 1 \leq |\alpha| \leq n\}$, and we write

$$P^n_{A,+} \otimes_A R \xrightarrow{d\varphi} \widetilde{P}^n_{R,+} \xrightarrow{d\beta} \widehat{P}^n_{L[[t]]/L,+}.$$

Denote $d\beta_A := d\beta \circ d\varphi : P^n_{A,+} \otimes_A R \longrightarrow \widehat{P}^n_{L[[t]]/L,+}$ which is the induced map from the arc $\beta \circ \varphi : A \longrightarrow L[[t]]$.

Consider the $L[[t]]$-submodule of $\widehat{P}^n_{L[[t]]/L,+}$, denoted N_L, defined as

$$N_L := \operatorname{span}\{(t^a)d\beta(d^n(x_1)^{\alpha_1} \cdots d^n(x_s)^{\alpha_s}) \mid 1 \le |\alpha| \le n\}, \tag{4.2}$$

where a is the non-negative integer from Lemma 4.4.2 or Lemma 4.4.3. Then we have that $d\beta_A(\mathfrak{m}_A(P^n_{A,+} \otimes_A R)) \subseteq d\beta(\mathfrak{m}_R \widetilde{P}^n_{R,+}) \subseteq N_L$ where $\mathfrak{m}_A := (x_1, \dots, x_d)A$ and $\mathfrak{m}_R := (x_1, \dots, x_s)R$ are the maximal ideals of A and R respectively. To see this, let $f \in \mathfrak{m}_R$ and $\omega \in \widetilde{P}^n_{R,+}$, then

$$\begin{aligned} d\beta(f\omega) &= \beta(f)d\beta(\omega) \\ &= wt^b d\beta(\omega), \end{aligned}$$

where $b \ge a$ and w is a unit. Thus, we get the required commutative diagram (4.1).

Remark 4.5.2. Using Remark 2.1.7 with induction, we can see that

$$d\beta_A(d^n(x_1)^{\alpha_1} \cdots d^n(x_d)^{\alpha_d}) = d^n(u_1t^a)^{\alpha_1} \cdots d^n(u_dt^a)^{\alpha_d},$$

where $d^n : L[[t]] \longrightarrow \hat{P}^n_{L[[t]]/L,+}$ is the universal n-th order differential over $L[[t]]$ and $1 \le |\alpha| \le n$. We do it for the basic case and it follows easily in the general case by applying induction.

$$\begin{aligned} d\beta_A(d^n(x_i)d^n(x_j)) &= d\beta_A(d^n(x_ix_j) - x_id^n(x_j) - x_jd^n(x_i)) \\ &= d\beta_A(d^n(x_ix_j)) - d\beta_A(x_id^n(x_j)) - d\beta_A(x_jd^n(x_i)) \\ &= d^n(u_itu_jt) - u_itd^n(u_jt) - u_jtd^n(u_it) \\ &= d^n(u_it)d^n(u_jt). \end{aligned}$$

These formulas will be useful later to prove the HN_n property.

Lemma 4.5.3. *If $\overline{d\beta}_A := \overline{d\beta} \circ \overline{d\varphi} : P^n_{A,+} \otimes_A R/\mathfrak{m}_R \longrightarrow \widehat{P}^n_{L[[t]]/L,+}/N_L$ is injective, then R satisfies the HN_n property.*

Proof. It is clear from the construction of the commutative diagram (4.1). $\square$

4.6 HN_n property for the graded case

Theorem 4.6.1. *Let X be a homogeneous algebraic variety, and R be as in Setup 4.2.1. Then R satisfies the HN_n property.*

Proof. From Lemma 4.5.3, it is enough to show that $\overline{d\beta}_A$ is injective. Consider the R-module homomorphism $d\beta : \widetilde{P}^n_{R,+} \longrightarrow \widehat{P}^n_{L[[t]]/L,+}$. Since R is a graded ring, then by the graded arc lemma 4.4.2, $a = b_{d+1} = \cdots = b_s = 1$. Then from the Remark 4.5.2, we get

$$d\beta_A(d^n(x_1)^{\alpha_1}\cdots d^n(x_d)^{\alpha_d}) = {u_1}^{\alpha_1}\cdots {u_d}^{\alpha_d}(d^n t)^{|\alpha|},$$

where $1 \leq |\alpha| \leq n$. We look at $P^n_{A,+} \otimes_A R/\mathfrak{m}_R$ as a vector space over k. Let $\overline{\omega} \in \ker(\overline{d\beta_A})$, and we write

$$\overline{\omega} = \sum_{1\leq|\alpha|\leq n} p_{\alpha_1\ldots\alpha_d}\overline{d^n(x_1)}^{\alpha_1}\cdots\overline{d^n(x_d)}^{\alpha_d},$$

where $p_{\alpha_1\ldots\alpha_d} \in k$ for all $1 \leq |\alpha| \leq n$. Then $\overline{d\beta_A}(\overline{\omega}) = 0$ and we get

$$\overline{d\beta_A}(\overline{\omega}) = \overline{d\beta_A}\left(\sum_{1\leq|\alpha|\leq n} p_{\alpha_1\ldots\alpha_d}\overline{d^n(x_1)}^{\alpha_1}\cdots\overline{d^n(x_d)}^{\alpha_d}\right)$$

Let $\omega = \sum_{1\leq|\alpha|\leq n} p_{\alpha_1\ldots\alpha_d}d^n(x_1)^{\alpha_1}\cdots d^n(x_d)^{\alpha_d} \in P^n_{A,+}\otimes_A R$ be the natural pre-image of $\overline{\omega}$. Then we get

$$\begin{aligned} d\beta_A(\omega) = &\Bigg(\sum_{|\alpha|=1}(p_{\alpha_1\ldots\alpha_d}{u_1}^{\alpha_1}\cdots {u_d}^{\alpha_d})(d^n t) + \sum_{|\alpha|=2}(p_{\alpha_1\ldots\alpha_d}{u_1}^{\alpha_1}\cdots {u_d}^{\alpha_d})(d^n t)^2 + \\ &+\cdots\cdots+ \\ &\sum_{|\alpha|=n}(p_{\alpha_1\ldots\alpha_d}{u_1}^{\alpha_1}\cdots {u_d}^{\alpha_d})(d^n t)^n\Bigg) \in N_L \end{aligned}$$

Recall from (4.2) that N_L is generated by

$$\left\{(t^a)d^n(u_1t^a)^{\alpha_1}\cdots d^n(u_dt^a)^{\alpha_d}d^n(v_{d+1}t^{b_{d+1}})^{\alpha_{d+1}}\cdots d^n(v_st^{b_s})^{\alpha_s}\,\middle|\, 1\leq|\alpha|\leq n\right\}.$$

Notice that the coefficient of $(d^n t)^i$ in $d\beta_A(\omega)$ does not involve a factor of t. But every element of N_L has a factor of t. Thus,

$$\sum_{|\alpha|=1}(p_{\alpha_1\ldots\alpha_d}{u_1}^{\alpha_1}\cdots{u_d}^{\alpha_d}) = \cdots = \sum_{|\alpha|=n}(p_{\alpha_1\ldots\alpha_d}{u_1}^{\alpha_1}\cdots{u_d}^{\alpha_d}) = 0.$$

But $u_1, \ldots, u_d$ are algebraically independent over k. Thus, $p_{\alpha_1\ldots\alpha_d} = 0$ for all $1 \leq |\alpha| \leq n$. This implies $\overline{\omega} = 0$, and so $\overline{d\beta}_A$ is injective. $\square$

Remark 4.6.2. Let R be a graded ring as above. We know $P^n_A \otimes_A R \cong R \oplus (P^n_{A,+} \otimes_A R)$ which gives us $P^n_A \otimes_A R/\mathfrak{m}_R \cong R/\mathfrak{m}_R \oplus (P^n_{A,+} \otimes_A R/\mathfrak{m}_R)$. We also have $P^n_{R/k} \cong R \oplus P^n_{R,+}$, and it follows that $\widetilde{P}^n_R \cong R \oplus \widetilde{P}^n_{R,+}$. This implies that $\widetilde{P}^n_R \otimes_R R/\mathfrak{m}_R \cong R/\mathfrak{m}_R \oplus (\widetilde{P}^n_{R,+} \otimes_R R/\mathfrak{m}_R)$. Then from the above proposition, it follows that

$$\overline{d\varphi} : P^n_A \otimes_A R/\mathfrak{m}_R \longrightarrow \widetilde{P}^n_R \otimes_R R/\mathfrak{m}_R,$$

is injective.

4.7 HN$_2$ property for R

In this section, we prove the HN$_2$ property for R.

Theorem 4.7.1. *Let R be as in Setup 4.2.1 and assume* $\mathrm{char}(k) = 0$. *Then*

$$\overline{d\beta}_A := \overline{d\beta} \circ \overline{d\varphi} : P^2_{A,+} \otimes_A R/\mathfrak{m}_R \longrightarrow \widehat{P}^2_{L[[t]]/L,+}/N_L,$$

in (4.1) is injective. In particular, R satisfies the HN$_n$ *property when $n = 2$.*

Proof. Consider the R-module homomorphism $d\beta_A : \widetilde{P}^2_{R,+} \longrightarrow \widehat{P}^2_{L[[t]]/L,+}$. If $a = 1$, then we are done by the proof of the Theorem 4.6.1. So, we assume $a \geq 2$. Then from Remark 4.5.2, and by Remark 2.1.7 we get

$$d\beta_A(d^2(x_i)) = (au_i t^{a-1})(d^2 t) + \left(\binom{a}{2} u_i t^{a-2}\right)(d^2 t)^2,$$

and

$$d\beta_A((d^2(x_i))(d^2(x_j)) = (a^2u_iu_jt^{2a-2})(d^2t)^2.$$

Let $\overline{\omega} \in \ker(\overline{d\beta_A})$, and we write

$$\overline{\omega} = \sum_{1\leq i\leq d} p_i\overline{d^2(x_i)} + \sum_{1\leq j\leq l\leq d} p_{jl}\overline{d^2(x_j)d^2(x_l)},$$

where $p_i, p_{jl} \in k$ for all $1 \leq i \leq d$ and $1 \leq j \leq l \leq d$. Let $\omega = \sum_{1\leq i\leq d} p_i d^2(x_i) + \sum_{1\leq j\leq l\leq d} p_{jl}(x_j)d^2(x_l) \in P^2_{A,+} \otimes_A R$ be the natural pre-image of $\overline{\omega}$. Then we have $d\beta_A(\omega) \in N_L$ where

$$d\beta_A(\omega) = \sum_{1\leq i\leq d}(ap_iu_it^{a-1})(d^2t) + \left(\sum_{1\leq j\leq l\leq d} p_i\binom{a}{2}u_it^{a-2} + \sum_{1\leq j\leq l\leq d} a^2p_{jl}u_ju_lt^{2a-2}\right)(d^2t)^2.$$

Recall from (4.2) that N_L is generated by

$$\left\{(t^a)d^2(u_1t^a)^{\alpha_1}\cdots d^2(u_dt^a)^{\alpha_d}d^2(v_{d+1}t^{b_{d+1}})^{\alpha_{d+1}}\cdots d^2(v_st^{b_s})^{\alpha_s}\,\middle|\, 1 \leq |\alpha| \leq 2\right\}.$$

Notice that for any element of N_L, the coeffecient of d^2t has a factor of t^{2a-1}. Since $d\beta_A(\omega) \in N_L$, we get

$$\sum_{1\leq i\leq d} ap_iu_i = 0.$$

But $u_1, \ldots, u_d$ are algebraically independent over k. Since $\mathrm{char}(k) = 0$, $p_i = 0$ for all $1 \leq i \leq d$. Further, we reduce down to

$$d\beta_A(\omega) = \left(\sum_{1\leq j\leq l\leq d} a^2p_{jl}u_ju_lt^{2a-2}\right)(d^2t)^2 \in N_L.$$

Then the only possibility is

$$\begin{aligned}\left(\sum_{1\leq j\leq l\leq d} a^2p_{jl}u_ju_lt^{2a-2}\right)(d^2t)^2 &= \sum_{1\leq i\leq d} g_it^a d\beta_A(d^2(x_i)) \\ &= \sum_{1\leq i\leq d}\left((g_iau_it^{2a-1})(d^2t) + \left(g_i\binom{a}{2}u_it^{2a-2}\right)(d^2t)^2\right).\end{aligned}$$

where $g_i \in L[[t]]$. Further, comparing the coefficients of d^2t on both sides, we get

$$\sum_{1\le i\le d} g_i a u_i = a \sum_{1\le i\le d} g_i u_i = 0.$$

And, so

$$\sum_{1\le i\le d} g_i \binom{a}{2} u_i = \binom{a}{2}\Big(\sum_{1\le i\le d} g_i u_i\Big) = 0.$$

It follows that

$$\sum_{1\le j\le l\le d} a^2 p_{jl} u_j u_l = 0.$$

But again we use that $u_1, \ldots, u_d$ are algebraically independent over k. Again since $\mathrm{char}(k) = 0$, $p_{jl} = 0$ for all $1 \le j \le l \le d$. This implies $\overline{\omega} = 0$, and so $\overline{d\beta}_A$ is injective. □

4.8 HN$_n$ property for G

In this section, we prove the HN$_n$ property for G. We know G is a graded ring, but G may not be an integral domain or even may not be reduced. So, we cannot apply Theorem 4.6.1 to G. We now follow a different approach to prove the HN$_n$ property for G.

Remark 4.8.1. Let $G_{\mathrm{red}} := G/\mathrm{rad}(0)$ be the reduction of the associated graded ring of R modulo the radical ideal of (0). Then we have the induced G-module homomorphism

$$P^n_G \longrightarrow P^n_{G_{\mathrm{red}}}.$$

It is also easy to see that if $g \in G$ is a non-zero divisor, then $[g] \in G_{\mathrm{red}}$ is also a non-zero divisor. So, we also get a induced G-module homomorphism

$$dq : \widetilde{P}^n_G \longrightarrow \widetilde{P}^n_{G_{\mathrm{red}}}.$$

Lemma 4.8.2. *G_{red} satisfies the* HN_n *property.*

Proof. Let $B := G_{\mathrm{red}}$ and $\mathfrak{m}_B := (\mathfrak{m}_G)_{\mathrm{red}} = (\mathfrak{m}_G + \mathrm{rad}(0))/\mathrm{rad}(0) = \mathfrak{m}_G/\mathrm{rad}(0)$. We can write $B = k[x_1, \dots, x_s]/J$ where J is a homogeneous ideal with $J = \mathrm{rad}(J)$. It follows that $J = p_1 \cap \cdots \cap p_l$ where $p_1, \dots, p_l$ are homogeneous prime ideals containing J. Let $B_i := B/p_i$ and $Q_i = \mathrm{Frac}(B_i)$ be the fraction field of B_i. Then the total ring of fractions of B, denoted Q is given by $Q := Q_1 \times \cdots \times Q_l$. We have a map $B \longrightarrow Q$ which induces the map $g : P_B^n \longrightarrow P_Q^n = \bigoplus_{i=1}^{l} P_{Q_i}^n$, and $\mathrm{im}(g) = \widetilde{P}_B^n$. We know that there exists $i_0 \in \{1, \dots, l\}$ such that $\dim B_{i_0} = \dim B := d$. Since $B_{i_0} = B/p_{i_0}$, we have a B-module homomorphism $P_B^n \longrightarrow P_{B_{i_o}}^n$. We also have a map $B_{i_0} \longrightarrow Q_{i_0}$ which induces the map $g_{i_0} : P_{B_{i_0}}^n \longrightarrow P_{Q_{i_0}}^n$, and $\mathrm{im}(g_{i_0}) = \widetilde{P}_{B_{i_0}}^n$. Notice that we have a commutative diagram

$$\begin{array}{ccc} P_B^n & \xrightarrow{g} & P_Q^n \\ \downarrow & & \downarrow \\ P_{B_{i_o}}^n & \xrightarrow{g_{i_0}} & P_{Q_{i_o}}^n. \end{array}$$

It follows that we have a B-module homomorphism $dq_0 : \widetilde{P}_B^n \longrightarrow \widetilde{P}_{B_{i_0}}^n$. By the construction in the Section 4.5, we have the following commutative diagram:

$$\begin{array}{ccc} P_A^n \otimes_A B/\mathfrak{m}_B & \xrightarrow{\overline{d\varphi}} & \widetilde{P}_B^n \otimes_B B/\mathfrak{m}_B \\ \downarrow & & \downarrow \\ P_A^n \otimes_A B_{i_0}/\mathfrak{m}_{B_{i_0}} & \xrightarrow{\overline{d\varphi_0}} & \widetilde{P}_{B_{i_0}}^n \otimes_{B_{i_0}} B_{i_0}/\mathfrak{m}_{B_{i_0}}, \end{array}$$

where $A = k[x_1, \dots, x_d]$ is the generic linear projection as defined in the arc lemma 4.4.3. The left vertical map in the above diagram is an isomorphism, and by Remark 4.6.2, $\overline{d\varphi_0}$ is injective. It follows that $\overline{d\varphi}$ is injective. □

Proposition 4.8.3. *G satisfies the* HN_n *property.*

Proof. The Remark 4.8.1 and the construction in the Section 4.5 gives us the following commutative diagram

$$\begin{tikzcd}
P_A^n \otimes_A G/\mathfrak{m}_G \arrow[r, "\overline{d\phi}"] \arrow[d] & \widetilde{P}_G^n \otimes_G G/\mathfrak{m}_G \arrow[d] \\
P_A^n \otimes_A G_{\mathrm{red}}/(\mathfrak{m}_G)_{\mathrm{red}} \arrow[r, "\overline{d\varphi}"] & \widetilde{P}_{G_{\mathrm{red}}}^n \otimes_{G_{\mathrm{red}}} G_{\mathrm{red}}/(\mathfrak{m}_G)_{\mathrm{red}}.
\end{tikzcd}$$

Here, the left vertical map is an isomorphism, and by the above lemma, we know $\overline{d\varphi}$ is injective. It follows that $\overline{d\phi}$ is injective. □

Remark 4.8.4. We know a classical result for the module of Kähler differentials saying that if Ω_R is free, then R is regular. We give a analogous result for the module of principal parts. This was previously obtained by Benner-Jeffries-Nuñez [BJNB19, see Theorem 10.2].

Corollary 4.8.5. *Let R be as in Setup 4.2.1 of Krull dimension d. Suppose P_R^n is free R-module of rank $D := \binom{n+d}{d}$, then R is regular.*

Proof. Consider the following commutative diagram

$$\begin{tikzcd}
P_A^n \otimes_A G/\mathfrak{m}_G \arrow[r, "dp"] \arrow[d] & P_G^n \otimes_G G/\mathfrak{m}_G \arrow[d] \\
P_A^n \otimes_A G/\mathfrak{m}_G \arrow[r, "\overline{dp}"] & \widetilde{P}_G^n \otimes_G G/\mathfrak{m}_G.
\end{tikzcd}$$

Then from Proposition 4.8.3, dp is injective. Recall from (2.3) that

$$P_R^n = \frac{k[x_1, \ldots, x_s, \xi_1, \ldots, \xi_s]_{(x_1, \ldots, x_s, \xi_1, \ldots, \xi_s)}}{(f_1, \ldots, f_c, g_1, \ldots, g_c) + (x_1 - \xi_1, \ldots, x_s - \xi_s)^{n+1}}.$$

And, we get

$$P_R^n \otimes_R R/\mathfrak{m}_R = \frac{k[\xi_1, \ldots, \xi_s]_{(\xi_1, \ldots, \xi_s)}}{(g_1, \ldots, g_c) + (\xi_1, \ldots, \xi_s)^{n+1}}.$$

It is then easy to see that the associated graded ring of $P_R^n \otimes_R R/\mathfrak{m}_R$ with respect to the $(\xi_1, \ldots, \xi_s)$-adic filtration can be written as

$$\operatorname{gr}(P_R^n \otimes_R R/\mathfrak{m}_R) = \frac{k[\xi_1, \ldots, \xi_s]}{\operatorname{In}((g_1, \ldots, g_c) + (\xi_1, \ldots, \xi_s)^{n+1})}.$$

Here, In is the same as in (2.4). It follows from (2.5) that

$$\operatorname{gr}(P_R^n \otimes_R R/\mathfrak{m}_R) \cong \operatorname{gr}(P_R^n) \otimes_G G/\mathfrak{m}_G \cong P_G^n \otimes_G G/\mathfrak{m}_G$$

We then get the following commutative diagram, and notice that the vertical maps are not linear

$$\begin{tikzcd}
P_A^n \otimes_A R/\mathfrak{m}_R \arrow[r, "d\psi"] \arrow[d, "\operatorname{In}"'] & P_R^n \otimes_R R/\mathfrak{m}_R \arrow[d, "\operatorname{In}"] \\
P_{\operatorname{gr}(A)}^n \otimes_{\operatorname{gr}(A)} G/\mathfrak{m}_G \arrow[r, "d\phi"] & P_G^n \otimes_G G/\mathfrak{m}_G.
\end{tikzcd}$$

Suppose $u \in P_A^n \otimes R/\mathfrak{m}_R$, then $\operatorname{In}(u) \neq 0$. We know $d\phi$ is injective, and thus, $d\psi$ is injective. But $\dim(P_{A,+}^n \otimes_A R/\mathfrak{m}_R) = \dim(P_{R,+}^n \otimes_R R/\mathfrak{m}_R) = D - 1$, and so $d\psi$ is an isomorphism. This is equivalent to saying that

$$P_{A,+}^n \otimes_A R/\mathfrak{m}_R \longrightarrow P_{R,+}^n \otimes_R R/\mathfrak{m}_R$$

is an isomorphism. Further, it follows from Nakayama's lemma that the R-module $P_{R,+}^n$ is freely generated by $\left\{d^n(x_1)^{\alpha_1} \cdots d^n(x_d)^{\alpha_d} \mid 1 \leq |\alpha| \leq n\right\}$. Furthermore, we have a surjective R-module homomorphism

$$P_{R,+}^n = I_R/I_R^{n+1} \longrightarrow \Omega_R = I_R/I_R^2.$$

It follows that the R-module Ω_R is generated by $\left\{d(x_1), \cdots, d(x_d)\right\}$. But $\operatorname{rank}(\Omega_R) = d$. Thus, Ω_R is a free R-module, and so R is regular. $\square$

4.9 Equidimensionality of G

Let R be a Noetherian local ring and $\mathfrak{m}_R$ be its maximal ideal. Consider $R[t]$ as a graded ring where t is an indeterminate over R. Then we construct a graded ring $R' \subset R[t]$ as follows

$$R' = \left\{ \sum a_n t^n \mid a_n \in \mathfrak{m}_R^n \right\} = \bigoplus_{n \geq 0} \mathfrak{m}_R^n t^n \subset R[t].$$

For simplicity, set $s = t^{-1}$ define

$$\widetilde{R} = R'[s] = \left\{ \sum a_n t^n \mid a_n \in \mathfrak{m}_R^n \text{ for } n \geq 0 \text{ and } a_n \in R \text{ for } n \leq 0 \right\} \subset R[t, s].$$

Then

$$s\widetilde{R} = \left\{ \sum a_n t^n \mid a_n \in \mathfrak{m}_R^{n+1} \text{ for } n \geq 0 \text{ and } a_n \in R \text{ for } n \leq -1 \right\}.$$

Thus,

$$\widetilde{R}/s\widetilde{R} \cong \bigoplus_{i \geq 0} \frac{\mathfrak{m}_R^i}{\mathfrak{m}_R^{i+1}} = \mathrm{gr}_{\mathfrak{m}_R}(R).$$

Let $\mathfrak{m}_R = (a_1, \dots, a_r)$. Then

$$\widetilde{R} = R[s, a_1 t, \dots, a_r t] = R[s, x_1, \dots, x_r] / (s x_i - a_1 \mid i = 1, \dots, r).$$

It follows that $\widetilde{R}$ is Noetherian.

Proposition 4.9.1. *Let R be a local Noetherian domain. Then $G := \mathrm{gr}_{\mathfrak{m}_R}(R)$ is equidimensional.*

Proof. From the above construction $G = \widetilde{R}/s\widetilde{R}$. Let $p_1, \dots, p_l$ be the minimal primes associated to $s\widetilde{R}$, i.e, $\mathrm{rad}(s\widetilde{R}) = p_1 \cap p_2 \cap \cdots p_l$. Then we have

$$G_{\mathrm{red}} = \widetilde{R}/\mathrm{rad}(s\widetilde{R}) = \widetilde{R}/(p_1 \cap p_2 \cap \cdots p_l).$$

It follows from the Krull principal ideal theorem that $\mathrm{ht}(p_i) \leq 1$ for all $1 \leq i \leq l$. Suppose there exists $i_0 \in \{1, \ldots, l\}$ such that $\mathrm{ht}(p_{i_0}) = 0$. Then $\widetilde{R}_{p_{i_0}}$ is zero dimensional. It implies that $\widetilde{R}_{p_{i_0}}$ is Artinian, and so some power of the maximal ideal $p_{i_0}\widetilde{R}_{p_{i_0}}$ of $\widetilde{R}_{p_{i_0}}$ vanishes. It means that $\frac{s^k}{1} = 0$ in $\widetilde{R}_{p_{i_0}}$ for some $k \in \mathbb{N}$. Then there exists $r \in \widetilde{R}$ such that $rs^k = 0$ in $\widetilde{R}$. But s is a non-zero divisor of $\widetilde{R}$ as $\widetilde{R} \subset R[t, t^{-1}]$ which results into a contradiction. Thus, $\mathrm{ht}(p_i) = 1$ for all $1 \leq i \leq l$, and so, G is equidimensional. $\square$

Further, we see the behaviour of torsion elements of a module under the initial map. Before proving the result, we introduce some definitions and terminologies.

Remark 4.9.2. Let R be a Noetherian ring (not necessary a domain) and $\mathfrak{D}_R := \{r \in R |$ r is a zero divisor$\} \cup \{0\} = \cup\, p$ where the union is over primes associated to 0_R. Then $\mathfrak{D}_R^c := \{r \in R |$ r is a non-zero divisor$\} = \bigcap (R \setminus p)$. Let M be a R-module with $\mathrm{tors}_R(M) = \{m \in M \mid \exists\, r \in \mathfrak{D}_R^c$ such that $rm = 0\}$, and define $\mathrm{supp}(m) = \{p \in \mathrm{Spec}(R) \mid \frac{m}{1} \neq \frac{0}{1} \in M_p\} = V(\mathrm{ann}_R(m))$ where $\mathrm{ann}_R(m) = \{r \in R |\ rm = 0_M\}$. Then the following statements are equivalent:

(i) $m \in \mathrm{tors}_R(M)$.

(ii) $\mathrm{ann}_R(m) \cap \mathfrak{D}_R^c \neq \emptyset$.

(iii) $\mathrm{ann}_R(m) \nsubseteq \mathfrak{D}_R$.

(iv) $\mathrm{ann}_R(m) \nsubseteq p$ for all primes associated to 0_R.

(v) $p \notin \mathrm{supp}(m)$ for all primes associated to 0_R.

(vi) $\frac{m}{1} = \frac{0}{1} \in M_p$ for all primes associated to 0_R.

We now assume R to be a local Noetherian domain. Let $\mathfrak{m}_R$ be its maximal ideal and $\mathrm{gr}_{\mathfrak{m}_R}(R) = \bigoplus_{i \geq 0} \frac{\mathfrak{m}_R^i}{\mathfrak{m}_R^{i+1}}$. Let M be a R-module and $\mathrm{gr}_{\mathfrak{m}_R}(M) = \bigoplus_{i \geq 0} \frac{\mathfrak{m}_R^i M}{\mathfrak{m}_R^{i+1} M}$. Suppose we have a filtration of M as $M := F_0 \supset F_1 \supset F_2 \supset \cdots$ and $\mathrm{gr}_{F_\bullet}(M) = \bigoplus_{i \geq 0} \frac{F_i}{F_{i+1}}$. If the filtration is $\mathfrak{m}_R$-compatible, i.e. $\mathfrak{m}_R^i F_j \subseteq F_{i+j}$, then $\mathrm{gr}_{F_\bullet}(M)$ is a $\mathrm{gr}_{\mathfrak{m}_R}(R)$-module. We also have the initial maps $\mathrm{in}_R : R \longrightarrow \mathrm{gr}_{\mathfrak{m}_R}(R)$ and $\mathrm{in}_M : M \longrightarrow \mathrm{gr}_{F_\bullet}(M)$. Then we have the following lemma.

Lemma 4.9.3. *If $m \in \mathrm{tors}_R(M)$, then $\mathrm{in}_M(m) \in \mathrm{tors}_{\mathrm{gr}_{\mathfrak{m}_R}(R)}(\mathrm{gr}_{F_\bullet}(M))$.*

Proof. Let R be a domain and $m \in \mathrm{tors}_R(M)$, then by the above remark we get $\frac{m}{1} = \frac{0}{1} \in M_0$. Let $p_1, \ldots, p_l$ be the minimal primes associated to 0 in $\mathrm{gr}_{\mathfrak{m}_R}(R)$. Let $Z = \mathrm{supp}(m)$ and since R is a local domain, $Z \subset X := \mathrm{Spec}(R)$. This means $\mathrm{ann}_R(m) \neq (0)$, and so $\mathrm{in}_R(\mathrm{ann}_R(m)) \neq (0)$. Let $\mathrm{TC}_0(Z)$ and $\mathrm{TC}_0(X)$ be tangent cones of Z and X at 0. Then we have that $\mathrm{TC}_0(Z)$ is contained in $\mathrm{TC}_0(X)$. Let $Y_1, \ldots, Y_l$ be the components of $\mathrm{TC}_0(X)$ corresponding to $p_1, \ldots, p_l$. It follows from Proposition 4.9.1, and Theorem 15.7 [Mat89] that $\dim Y_j = \dim(\mathrm{TC}_0(X)) = \dim X$ for all $1 \leq j \leq l$. Since $\mathrm{in}_R(\mathrm{ann}_R(m)) \neq (0)$, we get $\mathrm{in}_R(\mathrm{ann}_R(m)) \nsubseteq p_j$ for all $1 \leq j \leq l$. But in_R and in_M are multiplicative, and so we have $\mathrm{in}_R(\mathrm{ann}_R(m)) \subseteq \mathrm{ann}_{\mathrm{gr}_{\mathfrak{m}_R}(R)}(\mathrm{in}_M(m))$. Thus, $\mathrm{ann}_{\mathrm{gr}_{\mathfrak{m}_R}(R)}(\mathrm{in}_M(m)) \nsubseteq p_j$ for all $1 \leq j \leq l$ and by the above remark, $\mathrm{in}_M(m) \in \mathrm{tors}_{\mathrm{gr}_{\mathfrak{m}_R}(R)}(\mathrm{gr}_{F_\bullet}(M))$. □

4.10 Proof of Higher Nobile's theorem

We are now ready to prove the higher Nobile's theorem for the graded case, and also for any algebraic variety when $n = 2$.

Theorem 4.10.1. *(Graded case) Let X be a homogeneous algebraic variety, and R be as in Setup 4.2.1. Suppose $\mathrm{Nash}_n(X) \cong X$ for some $n \in \mathbb{N}$, then*

(1) if $\mathrm{char}(k) = 0$, then X is a non-singular variety.

(2) if X is normal and $\mathrm{char}(k) > 0$, then X is a non-singular variety.

Proof. By the Theorem 4.3.1, [Nob75a, Theorem 2] and [DNB22, Theorem 3.10], it concludes the proof of (1) and (2). $\square$

Theorem 4.10.2. *($n = 2$) Let X be an algebraic variety over an algebraically closed field k with* $\mathrm{char}(k) = 0$*. Suppose* $\mathrm{Nash}_2(X) \cong X$*, then X is a non-singular variety.*

Proof. By the Theorem 4.3.1 and [Nob75a, Theorem 2], X is a non-singular variety. $\square$

Chapter 5

Higher Zariski-Lipman Conjecture for Homogeneous Hypersurface

The Zariski-Lipman conjecture is one of the classical problems in algebraic geometry. The conjecture says the following.

Conjecture 5.0.1 (Zariski-Lipman)**.** *Let X be an algebraic variety over a base field of characteristic zero. Let $R := \mathcal{O}_{X,x}$ be the local ring at a closed point $x \in X$. Suppose the dual $\Omega^*_{R/k}$ of the module of Kähler differentials is a free R-module, then R is regular.*

The conjecture has been proved for hypersurfaces [SS72], for the graded case [Hoc75], and for a few other cases. We can ask a similar question for the module of principal parts instead of the module of Kähler differentials. We call it the higher Zariski-Lipman conjecture. It says the following:

Conjecture 5.0.2 (Higher Zariski-Lipman)**.** *Let X be an algebraic variety over a base field of characteristic zero. Let $R := \mathcal{O}_{X,x}$ be the local ring at a closed point $x \in X$. If the dual $(P^n_{R/k,+})^*$ of the module of principal parts of order n is a free R-module, then R is regular.*

5.1 Homogeneous hypersurface case

In this section, we prove the higher version of the Zariski-Lipman conjecture for a homogeneous hypersurface. For the rest of this section, we use the following notation:

- Let k be an algebraically closed base field of characteristic zero.
- Let $S = k[x_1, \ldots, x_s]$.
- $X = V(f) \subset \mathbb{A}^s$ be a hypersurface on k, where $f \in S$ is an irreducible homogeneous polynomial.
- $B = S/(f)$.
- Let $R := B_x$ denote a local ring at a closed point $x \in X$.
- Let $M = \binom{n+s-1}{s}$ and $N = \binom{n+s}{s}$.

Theorem 5.1.1. *Suppose $P^{n^*}_{R/k,+} := \mathrm{Hom}(P^n_{R/k,+}, R)$ is free, then R is regular.*

Proof. We start by assuming that $\deg(f) \geq 2$. Further, we can also assume by k-linear change of variables that

$$f, x_2, \ldots, x_s, \tag{5.1}$$

is a homogeneous system of parameters of S. This is possible because k is an infinite field. By Theorem 2.4.7, we have the following presentation of $P^n_{R/k,+}$

$$R^M \xrightarrow{(\overline{\mathrm{Jac}_n})^T} R^{N-1} \longrightarrow P^n_{R/k,+} \longrightarrow 0.$$

where $(\mathrm{Jac}_n)^T$ is the transpose of the Jacobian matrix of order n, Jac_n. Here, $\bar{g}$ denotes the reduction modulo (f) for $g \in S$. Dualizing the short exact sequence above, we get the following exact sequence

$$0 \longrightarrow \mathcal{P}^{n^*}_{R/k,+} \xrightarrow{A} R^{N-1} \xrightarrow{\overline{J_n}} R^M.$$

By hypothesis, $\mathcal{P}^{n^*}_{R/k,+}$ is a free R-module; it has rank $N-1-M$.

Since f is homogeneous, there exists a positive integer $c \geq 2$ such that

$$cf = \sum_{1 \leq |\alpha| \leq n} \frac{1}{\alpha!} \frac{\partial^\alpha f}{\partial x^\alpha} x^\alpha. \tag{5.2}$$

By (5.2), we can assume that the first column of A is

$$\{\overline{x_1}^{\alpha_1} \cdots \overline{x_s}^{\alpha_s} |\ 1 \leq |\alpha| \leq n\},$$

which is arranged using graded lexicographical order on α. We can rearrange the rows of A and the columns of J_n such that the first column of A becomes $(\bar{x}_1, \bar{x}_1^2, \ldots, \bar{x}_1^n, \cdots)$. Let us define the function $\phi : \mathrm{Mat}(r,t,R) \longrightarrow R$ such that $\phi(B)$ is the determinant of the $(M \times M)$ submatrix in the upper left-hand corner of B. Here, $\mathrm{Mat}(r,t,R)$ is the set of $r \times t$ matrices where the entries are from R. If $M > r$ or $M > t$, then we assume $\phi(B) = 0$. It follows from [BE74, Theorem 1] that $\phi(\overline{J_n})$ is a multiple of the bottom $((N-1-M) \times (N-1-M))$ minor of A. Thus, $\phi(\overline{J_n}) \in (\overline{x_2}, \ldots, \overline{x_s})R$, and hence

$$\phi(J_n) \in (x_2, \ldots, x_s, f)S.$$

Let $\tilde{g}$ denote the reduction modulo $(x_2, \ldots, x_s)S$ for $g \in S$. Then

$$\phi(\tilde{J}_n) \in (\tilde{f})\tilde{S}, \tag{5.3}$$

where $\tilde{S} = k[x_1]$. Then by (5.1), $\tilde{f}$ form a system of parameters for $\tilde{S}$. It follows from [Gul72, Theorem 1] that the image of ϕ generates the socle of the 0-dimensional Gorenstein local ring $\tilde{S}/(\tilde{f})\tilde{S}$ which is a contradiction to (5.3). Thus, $\deg(f) = 1$. This implies R is regular. $\square$

5.2 Zariski-Lipman conjecture and Nobile's theorem

The Zariski-Lipman conjecture is a stronger condition, and we observe how it implies the Nobile's theorem. We assume that the Zariski-Lipman conjecture holds, and show that the Nobile's theorem is true in this case. Recall the Setup 4.2.1 which we will use for the rest of this section.

Lemma 5.2.1. *Let M be a finitely generated R-module where R is from the Setup 4.2.1. Then*

$$M^* \cong (M/\operatorname{tors(M)})^*.$$

Proof. We have the natural quotient map

$$\pi : M \longrightarrow M/\operatorname{tors}(M)$$

This induces the following R-module homomorphism on the dual modules

$$\pi^* : (M/\operatorname{tors(M)})^* \longrightarrow M^*$$

$$\varphi \mapsto \varphi \circ \pi$$

We want to show that π^* is an isomorphism. We start with the injectivity.

Let $\varphi \in \ker(\pi^*)$. This implies $\varphi \circ \pi = 0$. But π is surjective, and it follows that $\varphi = 0$. Thus, π^* is injective.

Let $f \in M^*$. Take $m \in \operatorname{tors(M)}$, then there exists non-zero $r \in R$ such that $rm = 0$. It follows that

$$rf(m) = f(rm) = f(0) = 0$$

But R is a domain and $r \neq 0$. Then $f(m) = 0$, and we have $\operatorname{tor(M)} \subseteq \ker(f)$. It follows that there exists $\bar{f} : M/\operatorname{tors}(M) \longrightarrow R$ such that $f = \bar{f} \circ \pi = \pi^*(\bar{f})$. Thus, π^* is surjective. □

We know from the proof of the Theorem 4.3.1 that $\operatorname{Nash}(X) \cong X$ is equivalent in saying $\Omega_R / \operatorname{tors}(\Omega_R)$ is free. We will then use $\Omega_R / \operatorname{tors}(\Omega_R)$ is free as the hypothesis of the Nobile's theorem.

Theorem 5.2.2. *Let X be a variety of dimension d over algebraically closed base field k. Let R be as in Setup 4.2.1. Suppose the Zariski-Lipman conjecture 5.0.1 holds for X. If $\Omega_R / \operatorname{tors}(\Omega_R)$ is free then R is regular.*

Proof. We assume $\Omega_R / \operatorname{tors}(\Omega_R)$ is free. It is clear that $(\Omega_R / \operatorname{tors}(\Omega_R))^*$ is free. Then it follows from the Lemma 5.2.1 that Ω_R^* is free. Thus, R is regular as the Conjecture 5.0.1 holds. $\square$

Remark 5.2.3. The similar result holds for the higher Zariski-Lipman conjecture and the higher Nobile's theorem. We just replace Ω_R with P_R^n in Theorem 5.2.2.

Corollary 5.2.4. *Let X be a homogeneous hypersurface and R be as in Setup 4.2.1. Suppose $\operatorname{Nash}_n(X) \cong X$ which is equivalent in saying $(P_R^n / \operatorname{tors}(P_R^n))^*$ is free. Then R is regular.*

Proof. The result follows from the Theorem 5.1.1, and the Remark 5.2.3. $\square$

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