UNIVERSITY OF OKLAHOMA

GRADUATE COLLEGE

PARAMETER ESTIMATION FOR DAMPED SINE-GORDON EQUATION WITH NEUMANN BOUNDARY CONDITION

A DISSERTATION

SUBMITTED TO THE GRADUATE FACULTY

in partial fulfillment of the requirements for the

Degree of

DOCTOR OF PHILOSOPHY

By

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Norman, Oklahoma
2010

PARAMETER ESTIMATION FOR DAMPED SINE-GORDON EQUATION WITH NEUMANN BOUNDARY CONDITION

A DISSERTATION APPROVED FOR THE DEPARTMENT OF MATHEMATICS

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Acknowledgments

First, I wish to express my gratitude to my research advisor, Professor Semion Gutman, for teaching me a great deal of mathematics, for encouraging to explore and work in mathematics, for his guidance and unconditional support during all these years of graduate school, and for being a excellent mentor. I am grateful for having had the opportunity to work with him.

I also wish to sincerely thank to Professor Luther White, Professor Kevin Grasse, Professor Shihshu Walter Wei, and Professor Alfred G. Striz for their unconditional support, for their advice and help in various ways.

I would like to give a special thanks to my loving wife, Sarala Thapa-Shrestha, who has always been there for me, helping and supporting me in every step of my life. In addition, I would like to give a special thanks to my respectful parents and parents-in-law who have given me their encouragement and their love and care that has helped me become who I am today. Also I would like to thank to my wonderful children, Namisha and Nishan, who have shown me the beauty of life through their precious smiles and special thanks to all my other relatives for their continuous faith in me.

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Abstract

In this thesis we study an identification problem for physical parameters associated with damped sine-Gordon equation with Neumann boundary conditions. The existence, uniqueness, and continuous dependence of weak solution of sine-Gordon equations are established. The method of transposition is used to prove the Gâteaux differentiability of the solution map. The Gâteax differential of the solution map is characterized. The optimal parameters are established. Fŕechet differentiability of the cost functional J is established. Computational algorithm and numerical results are presented.

Chapter 1

Introduction

Sine-Gordon equation models the dynamics of a series of small-area Josephson junctions driven by a current source by taking into the account a damping effect. It is numerically verified in Bishop *et al* [1] that the solution of the sine-Gordon equation with periodic boundary conditions shows a chaotic behavior. However, there are no proofs of existence, uniqueness, and chaotic behavior of solutions in [1]. The chaotic behavior suggests that the problem of controlling the solutions of sine-Gordon equations by forcing and initial functions is very delicate and important. In recent years, some attentions has also been paid to models which possess soliton-like structures in higher dimensions [13], in particular, the Josephson junction model [14] which consists of two layers of superconducting material separated by an isolating barrier. This model can be described by sine-Gordon equations. In addition, sine-Gordon equations possess soliton-like solutions [15]. Solitons have been shown to play a central role in the theory of nonlinear differential equations.

Let Ω be an open bounded set of $\mathbb{R}^n$ with C^1 boundary. Let us consider the following sine-Gordon equation

$$\begin{aligned}
&u_{tt}(t,x) + \alpha u_t(t,x) - \beta \triangle u(t,x) + \delta \sin u(x,t) = f(x,t);\ (t,x) \in Q \\
&\frac{\partial u}{\partial n}(t,x)|_{x\in\Gamma} = 0,\ t \in\ (0,T) \\
&u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in \Omega \qquad (1.1)
\end{aligned}$$

where $T > 0,\ Q = (0,T)\times\Omega,\ f \in L^2(Q),\ u_0 \in V = H^1(\Omega)$ and $u_1 \in H = L^2(\Omega)$.

Solutions of (1.1) furnish a description of the dynamic behavior of the Josephson junction tunnel. The Josephson junction tunnel consists of two super conducting strips separated by a thin dielectric film. The dependent variable $u(x,t)$ is related to the current passing through dielectric. The boundary condition (1.1) implies that the current at the end of the junction vanishes.

Many scientists have had great interests in damping effects as appeared in (1.1). For instance, Nakajima and Onodera [2], studied parameters by numerical simulations based on the finite difference method. Levi [3], verified numerically that for special choices of parameters and forcing functions (1.1) leads to chaotic behaviors. Temam [4], has extensively studied the stability of (1.1). In Gutman [5], Fréchet differentiability of solution of the (1.1) is shown for Dirichlet boundary condition settings. The main goal of this thesis consists in finding the parameters α, β, and δ such that the solution of (1.1) exhibits the desired behavior.

More precisely, let

$$\mathcal{P} = \{q = (\alpha, \beta, \delta) \in [\alpha_{min}, \alpha_{max}] \times [\beta_{min}, \beta_{max}] \times [\delta_{min}, \delta_{max}]\}, \tag{1.2}$$

where $\beta_{min} > 0$. Define the cost functional $J(q)$ by

$$J(q) = k_1 |u(q;T) - z_d^1|^2 + k_2 \|u(q;t) - z_d^2\|^2_{L^2(0,T;H)} \tag{1.3}$$

where $z_d^1 \in H$, $z_d^2 \in L^2(0,T;H)$ and $k_i \geq 0$ for $i = 1, 2$ with $k_1 + k_2 > 0$. The data z_d^1 and z_d^2 can be thought of as the targeted behavior of (1.1). The parameter identification problem for (1.1) with the objective function (1.3) is to find $q^* = (\alpha^*, \beta^*, \delta^*) \in \mathcal{P}_{ad}$ satisfying

$$J(q^*) = \inf_{q \in \mathcal{P}_{ad}} J(q). \tag{1.4}$$

For solving the above identification problem, we utilize the method which is used by Lions [6] for solving the optimal control problems. We show the Gâteaux differentiability of the solution map u. Since the second order evolution equation (1.1) has the forcing term containing the diffusion operator, it is not easy or impossible to solve the equation by the standard variational manner as in [7]. In order to overcome this difficulty, we use the method of transposition studied in Lions and Magenes [8]. In our identification problem we use the method of transposition to prove the Gâteaux differentiability of the solution map, and to characterize the Gâteaux differential of the solution map.

The thesis is organized as follows. In Chapter 2 we introduce appropriate function spaces with their respective inner products and norms. In addition, we show the existence of eigenvalues and eigenfunctions of the operator $-\beta\Delta + I$. In general, equation (1.1) does not have a classical solution. To overcome such a problem, we define weak solution of (1.1) in Chapter 3. In Chapter 4 we prove the uniqueness of weak solutions of (1.1). The existence of weak solutions of (1.1) is proved by using approximate solutions. Continuity of the weak solution of (1.1) with respect to the parameters is proved in Chapter 5. In Chapter 6 we show that the weak solution of (1.1), as a function of q, is weakly Gâteaux differentiable by using the method of transposition by Lions and Magenes [8]. In Chapter 7 we show that the cost functional (1.3) is Gâteaux differentiable on $\mathcal{P}$. We derive the optimal parameters and finally we show that the cost functional (1.3) is differentiable. In Chapter 8 we develop a computational algorithm. In Chapter 9 we present numerical results. We present the conclusion of the thesis in Chapter 10.

Chapter 2

Problem Setup

Let $H = L^2(\Omega)$ be a Hilbert space with following inner product and norm

$$(\phi, \psi) = \int_\Omega \phi(x)\psi(x)dx, \qquad |\phi| = (\phi, \phi)^{\frac{1}{2}} \tag{2.1}$$

for all ϕ , $\psi \in L^2(\Omega)$. Let $V = H^1(\Omega)$ be a Hilbert space with following inner product and norm

$$((\phi, \psi)) = (\phi, \psi) + (\nabla\phi, \nabla\psi), \qquad \|\phi\| = ((\phi, \phi))^{\frac{1}{2}} \tag{2.2}$$

for all ϕ , $\psi \in H^1(\Omega)$. The dual H' is identified with H leading to $V \subset H \subset V'$ with compact, continuous, and dense injections [9]. Hence there exists a constant $K_1 = K_1(\Omega)$ such that

$$|w| \leq K_1\|w\| \quad \text{for any} \quad w \in V. \tag{2.3}$$

Let $< u, v >_{V, V'}$ denote the duality pairing between V and V'. To use the variational formulation let us define the following bilinear form on $V \times V$

$$a_\beta(u, v) = \int_\Omega u\ v dx + \beta \int_\Omega \nabla u \nabla v dx \tag{2.4}$$

for any $u, v \in H^1(\Omega)$ and diffusion coefficient β.

Lemma 2.1. *Let $\beta > 0$, then $a_\beta(u, v)$ is bounded and coercive in V.*

Proof. Using Cauchy-Schwartz inequality in (2.4) we have,
$|a_\beta(u, v)| = |\int_\Omega uvdx + \beta \int_\Omega \nabla u \nabla v dx| \leq C(|u||v| + |\nabla u||\nabla v|) \leq C\|u\|\|v\|$.
Similarly,
$a_\beta(u, u) = \int_\Omega u^2 dx + \beta \int_\Omega \nabla u \nabla u dx \geq min\{1, \beta\}(\int_\Omega u^2 dx + \int_\Omega \nabla u \nabla u dx) \geq c\|u\|^2$.
where c is some positive constant. □

Define a linear operator $A_\beta : D(A_\beta) = \{u : u \in V, A_\beta u \in H\}$ into H by $a_\beta(u, v) = (A_\beta u, v)$ for all $u \in D(A_\beta)$ and for all $v \in V$. Let the norm on $D(A_\beta)$ be $\|u\|_\beta^2 = \int_\Omega |u|^2 dx + \beta \int_\Omega |\nabla u|^2 dx$

Lemma 2.2. *A_β is an isomorphism between $D(A_\beta)$ and H.*

Proof. I) A_β is linear:
Let $u_1, u_2 \in D(A_\beta)$ then $(A_\beta(u_1 + u_2), v) = a_\beta(u_1 + u_2, v)$
$= \int_\Omega (u_1 + u_2) v dx + \beta \int_\Omega \nabla(u_1 + u_2) \nabla v dx$
$= \int_\Omega u_1 v dx + \int_\Omega u_2 v dx + \beta \int_\Omega \nabla u_1 \nabla v dx + \beta \int_\Omega \nabla u_2 \nabla v dx$
$= (A_\beta u_1, v) + (A_\beta u_2, v)$.
Similarly,
$(A_\beta \alpha u, v) = a_\beta(\alpha u, v) = \int_\Omega \alpha uv dx + \beta \int_\Omega \nabla(\alpha u) \nabla v dx = \alpha[\int_\Omega uv dx + \beta \int_\Omega \nabla u \nabla v dx]$
$= \alpha(A_\beta u, v)$

II) A_β is one to one:

Let $u_1, u_2 \in D(A_\beta)$ with $A_\beta u_1 = A_\beta u_2$, then for any $v \in V$ $(A_\beta u_1, v) = (A_\beta u_2, v)$ which implies $(A_\beta(u_1 - u_2), v) = 0$ for any $v \in V$. If $A_\beta(u_1 - u_2) \in V$, we can choose $v = A_\beta(u_1 - u_2)$. which implies $u_1 = u_2$. But if $A_\beta(u_1 - u_2)$ does not belong to V, being V dense in H there exist a sequence $v_n \in V$ such that $\{v_n\}$ converges to $A_\beta(u_1 - u_2)$ in V but V is complete so $A_\beta(u_1 - u_2) \in V$ hence $u_1 = u_2$.

III) A_β is onto:

For any $f \in H$ we can define $L(v) = \int_\Omega f v dx = a_\beta(u, v)$ so L is bounded linear functional on H hence by Riesz Representation Theorem there exist unique $u \in D(A_\beta)$ such that $A_\beta u = f$. Hence $R(A_\beta) = H$.

Norms $\|u\|^2 = \int_\Omega |u|^2 dx + \int_\Omega |\nabla u|^2 dx$ and $\|u\|_\beta^2 = \int_\Omega |u|^2 dx + \beta \int_\Omega |\nabla u|^2 dx$ are equivalent. From (2.1) $\alpha_1 \|u\|^2 \leq a_\beta(u, u) = \|u\|_\beta^2 = \int_\Omega |u|^2 dx + \beta \int_\Omega |\nabla u|^2 dx \leq \alpha_2 \|u\|^2$. Since $|A_\beta u|^2 = (A_\beta u, A_\beta u) = a_\beta(u, A_\beta u) \leq C\|u\|_\beta |A_\beta u|$ which implies $|A_\beta u| \leq C\|u\|_\beta$ for all $u \in D(A_\beta)$, hence A_β is bounded. Since A_β from $D(A_\beta) \subseteq V$ to H is bounded bijective linear operator so its inverse exist. $\|A_\beta^{-1}\| = sup\{\frac{\|A_\beta^{-1} v\|}{\|v\|} : \|v\| \neq 0\}$ for any $v \in H$. Since A_β is surjective, for $v \in H$ there exist $w \in D(A_\beta)$ such that $A_\beta w = v$. Hence

$$\|A_\beta^{-1}\| = sup\{\frac{\|A_\beta^{-1} A_\beta w\|}{\|A_\beta w\|} : \|A_\beta w\|` \neq 0\} \leq \frac{\|w\|}{\nu \|w\|} < \frac{1}{\nu} < \infty$$

for some $\nu = \beta_{min} > 0$. □

Lemma 2.3. *The operator $A_\beta :\ D(A_\beta) \subset H$ into H is a self-adjoint.*

Proof. Enough to show that A_β is symmetric and $R(A_\beta) = H$. For any $u, v \in D(A_\beta)$, we have $(A_\beta u, v) = a_\beta(u, v)$ and $(u, A_\beta v) = a_\beta(v, u)$ so $(A_\beta u, v) = (u, A_\beta v)$. Hence A_β is symmetric bounded linear operator. From Lemma (2.2) $\mathrm{R}(\mathrm{A}_\beta) = H$. Therefore A_β is self adjoint operator. □

Since A_β is bounded self-adjoint operator with A_β^{-1} as an inverse, A_β^{-1} is self-adjoint. Now it remains to show that A_β^{-1} is compact. Let B be any bounded set in H. A_β^{-1} is bounded thus for any $h \in H$, $\|A_\beta^{-1}h\| \leq \|A_\beta^{-1}\||h|$. Hence the set $A_\beta^{-1}(B)$ is bounded in V. A_β^{-1} is compact [9]. So there exist λ_k for $k = 1, 2, ...$ such that $(\beta\nabla w_k, \nabla v) + (w_k, v) = \lambda_k(w_k, v)$ for all $v \in V$. which shows that λ_k and w_k respectively are the nonzero eigenvalues and eigenfunctions for the operator A_β defined in V such that $\{w_K\}_{k=1}^{\infty}$ form an orthonormal basis in H.

Lemma 2.4. *Functions $\{\frac{w_k}{\sqrt{\mu}_k}\}_{K=1}^{\infty}$ form an orthonormal basis in V.*

Proof. Since λ_k are nonzero eigenvalues of A_β, we have $(w_k, w) + (\beta\nabla w_k, \nabla w) = \lambda_k(w_k, w)$ for any $w \in V$. Since $\{w_k\}$ forms an orthonormal basis in H, $\{\frac{w_k}{\sqrt{\mu}_k}\}$ forms an orthonormal set in V. It remains to show that orthonormal set $\{\frac{w_k}{\sqrt{\lambda_k}}\}_{k=1}^{\infty}$ in V is complete. Assume $(w_k, h) + (\beta\nabla w_k, \nabla h) = 0$ for $h \in H$. We have $(w_k, h) + (\beta\nabla w_k, \nabla h) = \lambda_k(w_k, h) = 0$. Since $\lambda_k \neq 0$, (w_k, h) has to be 0 for all $h \in H$. Hence $h = 0$ a.e. in H. Thus $\{\frac{w_k}{\sqrt{\lambda_k}}\}_{k=1}^{\infty}$ is a complete orthonormal set in V and thus forms a basis for V. □

Remark 2.5.

The computations in Chapter 8 is done with $\Omega = (0{,}1)$. Thus the computations of the eigenvalues and eigenfunctions for the $-\triangle$ with Neumann Boundary conditions is explicit in this case. These eigenvalues and eigenfunctions can be used to compute the eigenvalues and eigenfunctions of the operator $A_\beta = -\beta\triangle + I$. Thus we relate the eigenvalues and the eigenfunctions of the operator A_β to the eigenfunctions and the eigenvalues of the operator $-\triangle$ with Neumann boundary conditions.

Let μ_k and y_k be the eigenvalues and the eigenfunctions of the operator $-\triangle$ respectively. Thus we have

$$-\triangle y_k = \mu_k y_k \quad \text{for} \quad k = 0, 1, 2.... \tag{2.5}$$

Similarly, let λ_n and w_n be the eigenvalues and the eigenfunctions of the operator $A_\beta = -\beta\triangle + I$ respectively. Thus we have

$$-\triangle w_n = \frac{1}{\beta}(\lambda_n - 1)w_n, \quad \text{for} \quad n = 1, 2, 3, ..., \tag{2.6}$$

Comparing (2.5) and (2.6) we have $y_k = w_n$ and $\mu_k = \frac{1}{\beta}(\lambda_n - 1)$. Let $k = n - 1$. then we have $\mu_{n-1} = [\pi(n-1)]^2$ for $n = 1, 2, 3, ...,$ and

$$y_{n-1} = \begin{cases} \sqrt{2}\cos(\pi(n-1)x), & n = 2, 3, 4, ..., \\ 1, & n = 1. \end{cases} \tag{2.7}$$

Hence, $\lambda_n = \beta[\pi(n-1)]^2 + 1$ and

$$w_n = \begin{cases} \sqrt{2}\cos(\pi(n-1)x), & n = 2, 3, 4, ..., \\ 1, & n = 1. \end{cases} \tag{2.8}$$

Chapter 3

Weak formulation of the sine-Gordon equation

From now on the dependency on x is suppressed, and $'$ and $''$ stand for the time derivatives. Let

$$W(0,T) = \{u \ : \ u \in L^2(0,T;V), u' \in L^2(0,T;H), u'' \in L^2(0,T;V')\}. \tag{3.1}$$

u' and u'' are the derivatives in the distributional sense. That is, $u' \in L^2(0,T;H)$ is derivative of $u \in L^2(0,T;V)$ in the distributional sense if for any $\phi \in C_0^\infty(0,T)$ and $v \in V$

$$\int_0^T (u'(t),v)\phi(t)dt = -\int_0^T (u(t),v)\phi'(t)dt \tag{3.2}$$

similarly, $u'' \in L^2(0,T;V')$ is second derivative of $u \in L^2(0,T;V)$ in the distributional sense if for any $\phi \in C_0^\infty(0,T)$ and $v \in V$

$$\int_0^T (u''(t),v)\phi(t)dt = \int_0^T (u(t),v)\phi''(t)dt. \tag{3.3}$$

For more details see [10].

Definition 3.1. Let $\{w_j\}_{j=1}^{\infty}$ be the eigenfunctions of the operator A_β as introduced in (2.4). The weak solution of (1.1) is a function $u \in W(0,T)$ satisfying

$$\langle u'', w_j\rangle + \alpha(u', w_j) + a_\beta(u, w_j) + \delta(\sin(u), w_j) = (f, w_j) + (u, w_j),\ \forall j \in \mathbb{N},$$
$$u(0) = u_0 \in V, \quad u'(0) = u_1 \in H, \tag{3.4}$$

where the equations in t are satisfied in the distributional sense. Since the span $\{w_1, w_2, w_3, ...\}$ is dense in V, (3.4) is satisfied for any $v \in V$

$$\langle u''+\alpha u'+A_\beta u+\delta \sin u, v\rangle = \langle f+u, v\rangle, \quad u(0) = u_0 \in V, \quad u'(0) = u_1 \in H. \tag{3.5}$$

Thus

$$u'' + \alpha u' + A_\beta u + \delta \sin u = f + u, \quad u(0) = u_0 \in V, \quad u'(0) = u_1 \in H \tag{3.6}$$

which is understood in the sense of distributions on $(0,T)$ with the values in V'. For more details see [4].

Remark : The Neumann boundary condition does not explicitly appear in the weak formulation (3.4) but it is implicitly contained in it.

Suppose that the solution $u \in C^2(\overline{\Omega} \times [0,T])$. Let $v \in \mathcal{D}(\overline{\Omega}) = \{v|_\Omega : v \in \mathcal{D}(\mathcal{R}^{\mathcal{N}})\} \subseteq H^1(\Omega)$. Then by Green's Theorem

$$\int_\Omega (u^{''} + \alpha u^{'} - \beta \Delta u + \delta \sin\ u - f) v dx + \int_{\partial\Omega} v \frac{\partial u}{\partial n} ds = 0. \tag{3.7}$$

Suppose $v \in \mathcal{D}(\Omega)$. Since $v = 0 \in \partial\Omega$, then in (3.8) $\int_{\partial\Omega} v \frac{\partial u}{\partial n} ds = 0$. Therefore for all $v \in \mathcal{D}(\Omega)$

$$\int_{\Omega} (u^{''} + \alpha u^{'} - \beta \Delta u + \delta \sin\ u - f) v \, dx = 0. \tag{3.8}$$

Since $\mathcal{D}(\Omega)$ is dense in $L^2(\Omega)$, we conclude that (3.8) is true for any $v \in L^2(\Omega)$. Let us choose $v = u^{''} + \alpha u^{'} - \beta \Delta u + \delta \sin\ u - f$. Then (3.8) can be written as

$$\int_{\Omega} |u^{''} + \alpha u^{'} - \beta \Delta u + \delta \sin\ u - f|^2 \, dx = 0, \tag{3.9}$$

which implies that $u^{''} + \alpha u^{'} - \beta \Delta u + \delta \sin\ u - f = 0$ a.e. on Ω.

Suppose $v \in C^1(\overline{\Omega})$. Then (3.8) can be written as

$$\int_{\partial\Omega} v \frac{\partial u}{\partial n} ds = 0 \tag{3.10}$$

for any $v \in C^1(\overline{\Omega})$. Since Ω is bounded and $\partial\Omega$ is C^1, then there exist a bounded linear operator $T : V \to H(\partial\Omega)$ such that $Tv = v|_{\partial\Omega}$ for all $v \in V(\Omega) \cap C(\overline{\Omega})$, [11]. Thus

$$\int_{\partial\Omega} v \frac{\partial u}{\partial n} ds = 0 \tag{3.11}$$

is true for any $v \in L^2(\partial\Omega)$. Take $v = \frac{\partial u}{\partial n}$ in (3.12) to get

$$\int_{\partial\Omega} \left| \frac{\partial u}{\partial n} \right|^2 ds = 0 \tag{3.12}$$

which implies that $\frac{\partial u}{\partial n} = 0$ a.e. on $\partial\Omega$. Since we assume u, v, and f are continuous up to the boundary, then $\frac{\partial u}{\partial n}$ in fact, equals to zero at each point on the boundary $\partial\Omega$.

Chapter 4

Existence and Uniqueness of Weak Solutions

Now we first show the uniqueness of the solutions of equation (3.6) which we later use to show the existence of a solution of the equation (3.6). The following two Lemmas are of critical importance for the existence and uniqueness of weak solutions.

Lemma 4.1. *Let $w \in L^2(0,T;V)$, $w' \in L^2(0,T;H)$ and $w'' + A_\beta w \in L^2(0,T;H)$. Then, after a modification on the set of measure zero, $w \in C([0,T];V)$, $w' \in C([0,T];H)$ and, in the sense of distributions on $(0,T)$ one has*

$$(w'' + A_\beta w, w') = \frac{1}{2}\frac{d}{dt}\{|w'|^2 + a_\beta(w,w)\}. \tag{4.1}$$

For proof see [4].

Lemma 4.2. *(Gronwall's Lemma) Let $\xi(t)$ be a nonnegative, summable function on [0,T] which satisfies the integral inequality*

$$\xi(t) \leq C_1 \int_0^t \xi(s)ds + C_2 \ \ for\ constants\ C_1\ , C_2 \geq 0 \tag{4.2}$$

almost everywhere $t \in [0,T]$. Then

$$\xi(t) \leq C_2(1 + C_1 t e^{C_1 t})\ a.e.\ on\ 0 \leq t \leq T. \tag{4.3}$$

In particular, if

$$\xi(t) \leq C_1 \int_0^t \xi(s)ds\ \ a.e.\ on\ 0 \leq t \leq T,\ then\ \ \xi(t)\ =\ 0\ \ a.e.\ on\ [0,\ T] \tag{4.4}$$

For proof see [11].

Lemma 4.3. *The solution of equation (3.6) is unique.*

Proof. Let z_1 and z_2 be two solutions of (3.6). Then we have the following equations

$$z_1'' + \alpha z_1' + A_\beta z_1 + \delta \sin z_1 = f + z_1, \quad z_1(0) = z_0 \in V, \quad z_1'(0) = z_1 \in H. \tag{4.5}$$

$$z_2'' + \alpha z_2' + A_\beta z_2 + \delta \sin z_2 = f + z_2, \quad z_2(0) = z_0 \in V, \quad z_2'(0) = z_1 \in H. \tag{4.6}$$

Subtracting (4.6) from (4.5) one has

$$w'' + \alpha w' + A_\beta w + \delta(\sin z_2 - \sin z_1) = w, \quad w(0) = 0 \in V, \quad w'(0) = 0 \in H, \tag{4.7}$$

where $w = (z_2 - z_1)$. Using lemma (4.1) one can obtain

$$\frac{1}{2}\frac{d}{dt}\{|w'|^2 + a_\beta(w,w)\} = -\alpha|w^{'}|^2 - \delta(\sin(z_2) - \sin(z_1), w^{'}) + (w, w') \tag{4.8}$$

Integrating (4.8) over $0 \le t \le T$, we get

$$\int_0^t \frac{1}{2}\frac{d}{dt}\{|w'|^2 + a_\beta(w,w)\}ds = \int_0^t [-\alpha|w^{'}|^2 - \delta(\sin(z_2) - \sin(z_1), w^{'}) + (w, w')]ds$$

$$|w'|^2 + a_\beta(w,w) = 2\int_0^t [-\alpha|w^{'}|^2 - \delta(\sin(z_2) - \sin(z_1), w^{'}) + (w, w')\ ]ds$$

$$\le 2|\alpha|\int_0^t |w^{'}|^2 ds + 2|\delta|\int_0^t |(\sin(z_2) - \sin(z_1), w^{'})|ds + 2\int_0^t |(w, w')|ds$$

Let $\epsilon > 0$. Using Cauchy Schwartz inequality and the fact that $V \subset\subset H$, we have

$$\begin{aligned}
|w'(t)|^2 + \|w(t)\|^2 &\le 2|\alpha|\int_0^t |w'(s)|^2 ds + 2|\delta|\int_0^t |w(s)|.|w'(s)|ds \\
&\quad + 2\int_0^t |w(s)|.|w'(s)|ds \\
&\le 2|\alpha|\int_0^t |w'(s)|^2 ds + |\delta|\int_0^t (\frac{1}{\epsilon}|w(s)|^2 + \epsilon|w'(s)|^2)ds \\
&\quad + \int_0^t (\frac{1}{\epsilon}|w(s)|^2 + \epsilon|w'(s)|^2)ds \\
&\le c(\int_0^t |w'(s)|^2 ds + \int_0^t \|w(s)\|^2 ds)
\end{aligned} \tag{4.9}$$

where $c = \max\ \{2|\alpha| + \epsilon|\delta| + \epsilon, \frac{1+K^2|\delta|}{\epsilon}\}$.

By lemma (4.2) $|w'(t)|^2 + \|w(t)\|^2 = 0$. Therefore $w = 0$ a.e. in $W(0,T)$ Hence $z_1 = z_2$ a.e. in $W(0,T)$. □

Fix $m \in \mathbb{N}$ and let $V_m = span\{w_1, w_2, ..., w_m\}$. Let $P_m : H \to V_m$ be the projection operator defined by $P_m v = \sum_{k=1}^m (v, w_k)w_k$ for any $v \in H$.

The approximate solution of (3.4) is a function $u_m(t) \in W(0,T)$ that satisfies

$$u_m'' + \alpha u_m' + A_\beta u_m + \delta P_m \sin(u_m) = P_m f + u_m$$
$$u_m(0) = P_m u_0 \quad u_m'(0) = P_m u_1. \tag{4.10}$$

Lemma 4.4. *The solution of equation* (4.10) *is unique.*

Proof. Assume z_1 and z_2 be two solutions of (4.10). Then their difference $w = z_1 - z_2$ satisfies

$$w'' + A_\beta(w) = w - \alpha w' - \delta P_m((\sin z_2) - (\sin z_1)) \in L^2(0,T;H) \tag{4.11}$$

with zero initial conditions. The fact $|P_m u| \leq |u|$ for any $u \in H$ and lemma (4.3) provides the result. □

Let

$$z_m(t) = \sum_{j=1}^{m} g_{jm}(t) w_j(x) \tag{4.12}$$

satisfy

$$\frac{d^2}{dt^2}(z_m, w_j) + \alpha \frac{d}{dt}(z_m, w_j) + a_\beta(z_m, w_j) + \delta(P_m \sin z_m, w_j)$$
$$= (P_m f, w_j) + (z_m, w_j)$$
$$z_m(0) = P_m z_0 \quad \text{and} \quad \frac{d}{dt} z_m(0) = P_m z_1 \quad \text{for} \quad \text{any} \quad j \in \mathbb{N} \tag{4.13}$$

Theorem 4.5. *For each integer* $m = 1, 2, ...,$ *there exist a unique function* $z_m(t) = \sum_{j=1}^{m} g_{jm}(t) w_j(x)$ *satisfying (4.13).*

Proof. Let $P_m : H \to V_m$ be the projection operator defined by $P_m v = \sum_{k=1}^{m} (v, w_k) w_k$ for any $v \in H$. We can write equation (4.13) as the vector differential equation

$$\frac{d^2}{dt^2}\vec{g}_m(t) + \alpha\frac{d}{dt}\vec{g}_m(t) + \beta\Lambda\vec{g}_m(t) = \vec{F}(t, \vec{z}_m) \tag{4.14}$$

with the initial values

$$\vec{g}_m(0) = \begin{bmatrix} (P_m z_0, w_1) \\ (P_m z_0, w_2) \\ . \\ . \\ . \\ (P_m z_0, w_m) \end{bmatrix},$$

and

$$\frac{d}{dt}\vec{g}_m(0) = \begin{bmatrix} (P_m z_1, w_1) \\ (P_m z_1, w_2) \\ . \\ . \\ . \\ (P_m z_1, w_m) \end{bmatrix}.$$

Here

$$\vec{g}_m(t) = \begin{bmatrix} g_{1m}(t) \\ g_{2m}(t) \\ . \\ . \\ . \\ g_{1m}(t) \end{bmatrix}.$$

Similarly

$$\vec{F}(t, z_m) = \begin{bmatrix} (P_m f(t), w_1) + (z_m, w_1) - \delta(\sin(z_m), w_1) \\ (P_m f(t), w_2) + (z_m, w_2) - \delta(\sin(z_m), w_2) \\ . \\ . \\ . \\ (P_m f(t), w_m) + (z_m, w_m) - \delta(\sin(z_m), w_m) \end{bmatrix}$$

and

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 & . & . & . & 0 \\ 0 & \lambda_2 & 0 & . & . & . & 0 \\ 0 & 0 & \lambda_3 & & & & \\ . & . & . & & & & \\ 0 & 0 & 0 & . & . & . & \lambda_m \end{bmatrix}.$$

Lemma 4.6. *Function $\vec{F}(t, \vec{z}_m)$ is Lipschitz continuous.*

Proof. Let $z_m(t) = \sum_{j=1}^m g_{jm}(t) w_j$ and $v_m(t) = \sum_{j=1}^m h_{jm}(t) w_j$. For any ϕ, $\psi \in H$. We have the following inequality

$$\int_\Omega |\sin\phi(x) - \sin\psi(x)|^2 dx \leq \int_\Omega |\phi(x) - \psi(x)|^2 dx. \tag{4.15}$$

Using (4.15) and Schwartz inequality we have

$$|\vec{F}(t, z_m(t)) - \vec{F}(t, v_m(t))|^2 = \delta^2 \sum_{i=1}^{m} |(\sin(\sum_{j=1}^{m} g_{jm}(t)w_j) - \sin(\sum_{j=1}^{m} h_{jm}(t)w_j), w_i)|^2$$
$$+|(\sum_{j=1}^{m} g_{jm}(t)w_j - \sum_{j=1}^{m} h_{jm}(t)w_j, w_i)|^2$$

$\leq \delta^2 m|(\sin(\sum_{j=1}^{m} g_{jm}(t)w_j) - \sin(\sum_{j=1}^{m} h_{jm}(t)w_j)|^2 + m|\sum_{j=1}^{m}(g_{jm}(t) - h_{jm}(t))|^2$
$\leq \delta^2 m^2 \sum_{j=1}^{m} |g_{jm}(t) - h_{jm}(t)|^2 + m^2|g_{jm}(t) - h_{jm}(t)|^2 \leq M \sum_{j=1}^{m} |g_{jm}(t) - h_{jm}(t)|^2$
$\leq M|\vec{g}_m - \vec{h}_m|^2$. Hence $\vec{F}(t, z_m)$ is Lipschitz continuous. □

Definition 4.7. Carathéodory Condition: $\vec{f}(x, \vec{y})$ is continuous as a function of $\vec{y}$ for fixed x and measurable as a function of x for each fixed $\vec{y}$.

Theorem 4.8. *Let $J = [\xi, \xi + a]$, $S = J \times \mathbb{R}^n$, and assume that the function $\vec{f}$: $S \to \mathbb{R}^n$ satisfies the Carathéodory condition in S. Let $\vec{f}$ satisfy $\vec{f}(x, \vec{y}) \in L(J)$, the class of functions that are integrable and measurable over J for each fixed $\vec{y}$, and satisfying the generalized Lipschitz condition*

$$|\vec{f}(x, \vec{y}) - \vec{f}(x, \vec{y}_1)| \leq l(x)|\vec{y} - \vec{y}_1| \text{ in } S \tag{4.16}$$

where $l(x) \in L(J)$. Then there exists a unique solution of $\vec{y}' = \vec{f}(x, \vec{y})$, $\vec{y}(\xi) = \vec{\eta}$ in J. For details see [16].

Hence the system of m second order vector differential equations admits a unique solution $\vec{g}_m(t)$ on $[0, T]$. This is shown by reducing it into a system of first order vector differential equations and by applying Carathéodory type extension Theorem 4.8.

Lemma 4.9. *Function* $z_m(t) = \sum_{j=1}^m g_{jm}(t)w_j(x)$ *satisfies*

$$\frac{d^2}{dt^2}(z_m, w_j) + \alpha\frac{d}{dt}(z_m, w_j) + a_\beta(z_m, w_j) + \delta(P_m \sin z_m, w_j)$$
$$= (P_m f, w_j) + (z_m, w_j),$$
$$z_m(0) = P_m z_0 \quad \text{and} \quad \frac{d}{dt}z_m(0) = P_m z_1 \tag{4.17}$$

for $j > m$.

Proof. It sufficies to show that $(A_\beta z_m, w_j) = a_\beta(z_m, w_j)$ is zero for $j > m$. Since $\{w_j\}_{j=1}^\infty$ are the eigenfunctions of the operator A_β, we have $(z_m, w_j) + \beta(\nabla z_m, \nabla w_j) = \lambda_j(z_m, w_j)$. This implies $\beta(\nabla z_m, \nabla w_j) = \lambda_j(z_m, w_j) - (z_m, w_j) = (\lambda_j - 1)(z_m, w_j)$. For $j > m$, $\beta(\nabla z_m, \nabla w_j) = 0$. Hence, $(A_\beta z_m, w_j) = 0$ for $j > m$.

□

Hence z_m is a weak solution of the sine-Gordon equation. Furthermore, z_m also satisfies (4.10). By Lemma 4.4 the approximate solution u_m is in fact a weak solution of the sine-Gordon equation (1.1). □

Theorem 4.10. *Let* $q = (\alpha, \beta, \delta) \in \mathcal{P}$, $u_0 \in V$, $u_1 \in H$ *and* $f \in L^2(0, T; H)$. *Then*

(i). There exists a unique weak solution $u(t; q)$ *of (1.1). This solution satisfies* $u \in C([0, T]; V) \cap W(0, T)$, $u' \in C([0, T]; H)$, *and*

$$\max_{0\le t\le T}(\|u(t)\|^2 + |u'(t)|^2) + \|u''(t)\|^2_{L^2(0,T;V')} \le C\left[\|u_0\|^2 + |u_1|^2 + \|f\|^2_{L^2(0,T;H)}\right], \tag{4.18}$$

where C *is a constant independent of* $q \in \mathcal{P}$. *The approximate solutions* $u_m(t; q)$ *also satisfy the energy estimate (4.18) with the same constant* C.

(ii). The solution $u(t;q)$ and its approximations $u_m(t;q)$ satisfy the following convergence estimate

$$
\begin{aligned}
&|u'(t) - u_m'(t)|^2 + \|u(t) - u_m(t)\|^2 \le C_2(|u_1 - P_m u_1|^2 + \|u_0 - P_m u_0\|^2 \\
&+\|f - P_m f\|^2_{L^2(0,T;H)} + \int_0^t |\sin u(s;q) - P_m \sin u(s;q)|^2 ds) \qquad (4.19)
\end{aligned}
$$

where C_2 is a constant independent of $q \in \mathcal{P}$.

(iii). Furthermore, $u_m \to u$ in $C([0,T];V)$ and $u_m' \to u'$ in $C([0,T];H)$ as $m \to \infty$.

Proof. Part I. A priori estimates. Multiply (4.17) by $g_{jm}^{'}(t)$ on both sides and sum from $j = 1$ to m to get

$$
\begin{aligned}
\sum_{j=1}^{m} \frac{d^2}{dt^2}(u_m(t), w_j) g_{jm}^{'}(t) + \alpha \sum_{j=1}^{m} \frac{d}{dt}(u_m(t), w_j) g_{jm}^{'}(t) \sum_{j=1}^{m} a_\beta(u_m(t), w_j) g_{jm}^{'}(t) \\
= \sum_{j=1}^{m}(f(t), w_j) g_{jm}^{'}(t) + \sum_{j=1}^{m}(u_m(t), w_j) g_{jm}^{'}(t) \\
- \sum_{j=1}^{m} \delta(\sin\ u_m(t), w_j) g_{jm}^{'}(t).
\end{aligned}
$$

We claim that

$$\sum_{j=1}^{m} \frac{d^2}{dt^2}(u_m(t), w_j) g_{jm}^{'}(t) = \frac{1}{2}\frac{d}{dt}|u_m^{'}|^2, \qquad (4.20)$$

$$\alpha \sum_{j=1}^{m} \frac{d}{dt}(u_m(t), w_j) g_{jm}^{'}(t) = \alpha |u_m^{'}|^2, \qquad (4.21)$$

$$\sum_{j=1}^{m} a_\beta(u_m, w_j) g_{jm}^{'}(t) = \frac{1}{2}\frac{d}{dt} a_\beta(u_m, u_m), \qquad (4.22)$$

$$\sum_{j=1}^{m}(f,w_j)g_{jm}^{'}(t)=(f,u_m^{'}), \tag{4.23}$$

and

$$\sum_{j=1}^{m}(u_m(t),w_j)g_{jm}^{'}(t)=(u_m,u_m^{'}). \tag{4.24}$$

Verification of (4.20)

$$\sum_{j=1}^{m}\frac{d^2}{dt^2}(u_m(t),w_j)g_{jm}^{'}(t)=\sum_{j=1}^{m}(u_m^{''},w_j)\ g_{jm}^{'}=\sum_{j=1}^{m}\int_\Omega u_m^{''}w_j\ g_{jm}^{'}dx$$
$$=\int_\Omega u_m^{''}\sum_{j=1}^{m}g_{jm}^{'}w_jdx=(u_m^{''},u_m^{'})=\frac{1}{2}[(u_m^{''},u_m^{'})+(u_m^{'},u_m^{''})]=\frac{1}{2}\frac{d}{dt}|u_m^{'}|^2$$

Verification of (4.21)

$$\alpha\sum_{j=1}^{m}\frac{d}{dt}(u_m(t),w_j)g_{jm}^{'}(t)=\alpha\sum_{j=1}^{m}(u_m^{'},w_j)g_{jm}^{'}=\alpha(u_m^{'},\sum_{j=1}^{m}g_{jm}^{'}w_j)$$
$$=\alpha(u_m^{'},u_m^{'})=\alpha|u_m^{'}|^2.$$

Verification of (4.22)

$$\sum_{j=1}^{m}a_\beta(u_m(t),w_j(x))g_{jm}^{'}(t)=\sum_{j=1}^{m}\int_\Omega u_m(t)w_j(x)g_{jm}^{'}(t)dx+$$
$$\sum_{j=1}^{m}\int_\Omega\beta\nabla u_m\nabla w_j(x)g_{jm}^{'}(t)dx=\int_\Omega u_m(t)\sum_{j=1}^{m}g_{jm}^{'}(t)w_j(x)$$
$$+\int_\Omega\beta\nabla u_m\sum_{j=1}^{m}g_{jm}^{'}(t)\nabla w_j(x)g_{jm}^{'}(t)dx=\int_\Omega u_mu_m^{'}dx+$$
$$\int_\Omega\beta\nabla u_m\nabla u_m^{'}dx=a_\beta(u_m,u_m^{'}).$$

Verification of (4.23)

$$\sum_{j=1}^{m}(f(t), w_j)g_{jm}^{'}(t) = \int_{\Omega} f(t)\sum_{j=1}^{m} g_{jm}^{'}(t)w_j(x)dx = \int_{\Omega} f(t)u_m^{'}dx = (f, u_m^{'}).$$

Verification of (4.24)

$$\sum_{j=1}^{m}(u_m(t), w_j)g_{jm}^{'}(t) = \int_{\Omega} u_m \sum_{j=1}^{m} g_{jm}^{'}(t)w_j(x) = (u_m, u_m^{'})$$

Using (4.20), (4.21), (4.22), (4.23), and (4.24) in (4.20) we get

$$\frac{1}{2}\frac{d}{dt}\left[|u_m^{'}|^2 + a_\beta(u_m, u_m)\right] = (f(t), u_m^{'}) + (u_m, u_m^{'}) - \alpha(u_m^{'}, u_m^{'}) - \delta(\sin(u_m), u_m^{'}). \tag{4.25}$$

Integrate (4.25) from 0 to t and use Cauchy Schwartz Inequality to get

$$\begin{aligned}
&\left[|u_m^{'}|^2 + a_\beta(u_m, u_m)\right] \leq 2\int_0^t |(f, u_m^{'})|ds + 2\int_0^t |(u_m, u_m^{'})|ds \\
&+2|\alpha|\int_0^t |(u_m^{'}, u_m^{'})|ds + 2|\delta|\int_0^t |(\sin(u_m), u_m^{'})|ds \\
&\leq |P_m u_1|^2 + \|P_m u_0\|^2 + 2\int_0^t |f(s)||u_m^{'}(s)|ds \\
&+2\int_0^t |u_m(s)||u_m^{'}(s)|ds + 2|\alpha|\int_0^t |u_m^{'}(s)|^2 ds + 2|\delta|\int_0^t |u_m(s)||u_m^{'}(s)|ds.
\end{aligned}$$

Using the coerciveness estimate $a_\beta(u, u) \geq \nu\|u\|^2$ for some constant $\nu > 0$ we have

$$\begin{aligned}
&|u_m^{'}|^2 + \nu\|u_m\|^2 \leq |u_m^{'}|^2 + a_\beta(u, u) \leq |P_m u_1|^2 + \|P_m u_0\|^2 \\
&+2\int_0^t |f(s)||u_m^{'}(s)|ds + 2\int_0^t |u_m(s)||u_m^{'}(s)|ds + 2|\alpha|\int_0^t |u_m^{'}(s)|^2 ds \\
&+2|\delta|\int_0^t |u_m(s)||u_m^{'}(s)|ds.
\end{aligned}$$

Therefore

$$|u_m^{'}|^2 + \nu\|u_m\|^2 \geq min\{1, \nu\} \left[|u_m^{'}|^2 + \|u_m\|^2 \right] = c \left[|u_m^{'}|^2 + \|u_m\|^2 \right]$$

where $c = min\{1, \nu\}$. Thus

$$|u_m^{'}|^2 + \|u_m\|^2 \leq c_1 \left[|u_m^{'}|^2 + \nu\|u_m\|^2 \right] \leq c_1(|P_m u_1|^2 + \|P_m u_0\|^2$$
$$+2\int_0^t |f(s)||u_m^{'}(s)|ds + 2\int_0^t |u_m(s)||u_m^{'}(s)|ds + 2|\alpha|\int_0^t |u_m^{'}(s)|^2 ds$$
$$+2|\delta|\int_0^t |u_m(s)||u_m^{'}(s)|ds).$$

Using $|ab| \leq \frac{a^2+b^2}{2}$ we get

$$|u_m^{'}|^2 + \|u_m\|^2 \leq c_1(|P_m u_1|^2 + \|P_m u_0\|^2 + \|f\|^2_{L^2(0,T;H)}$$
$$+(1 + |\alpha| + |\delta|)\int_0^t |u_m^{'}|^2 ds) + (1 + |\delta|)\int_0^t |u_m|^2 ds)$$
$$\leq max\{(1 + |\delta|), (2 + |\alpha| + |\delta|)\}(|P_m u_1|^2 + \|P_m u_0\|^2$$
$$+\|f\|^2_{L^2(0,T;H)} + \int_0^t |u_m^{'}|^2 ds) + \int_0^t |u_m|^2 ds)$$
$$= c_2\ (|P_m u_1|^2 + \|P_m u_0\|^2 + \|f\|^2_{L^2(0,T;H)} + \int_0^t |u_m^{'}|^2 ds) + \int_0^t |u_m|^2 ds)$$

where $c_2 = \max\ \{(1 + |\delta|), (2 + |\alpha| + |\delta|)\}$. Using Poincare inequality for the last integral we get

$$|u_m^{'}|^2 + \|u_m\|^2 \leq c_2(|P_m u_1|^2 + \|P_m u_0\|^2 + \|f\|^2_{L^2(0,T;H)}$$
$$+c_3\int_0^t (|u_m^{'}|^2 + \|u_m\|^2 ds)$$

where $c_3 = \max\{1, K_1^2\}$. Hence we have

$$|u_m^{'}|^2 + \|u_m\|^2 \leq C(|u_1|^2 + \|u_0\|^2 + \|f\|^2_{L^2(0,T;H)} + \int_0^t (|u_m^{'}|^2 + \|u_m\|^2) ds), \tag{4.26}$$

where $C = \max\{c_2, c_3\}$. The Gronwall's Lemma gives

$$|u_m^{'}|^2 + \|u_m\|^2 \leq C\left[|u_1|^2 + \|u_0\|^2 + \|f\|^2_{L^2(0,T;H)}\right], t \in [0, T]. \tag{4.27}$$

Since u_m is an approximate solution of (1.1) and for any $v \in V$ with $\|v\| \leq 1$, we have

$$|\langle u_m^{''}, v\rangle| \leq c(|f| + |u_m^{'}| + |u_m| + \|u_m\|) \tag{4.28}$$

where $c = max\{1, (1 + |\delta|), |\alpha|\}$. Using $|u_m| \leq K_1\|u_m\|$ and integrating from 0 to T we get

$$\|u_m''\|^2_{L^2(0,T;V')} \leq c(|f|^2_{L^2(0,T;H)} + |u_m'|^2_{L^2(0,T;H)} + \|u_m\|^2_{L^2(0,T;V)}). \tag{4.29}$$

From (4.27) and (4.29) we conclude that

$$\max_{0\leq t\leq T}(\|u_m(t)\|^2 + |u_m'(t)|^2) + \|u_m''(t)\|^2_{L^2(0,T;V')} \leq C\left[\|u_0\|^2 + |u_1|^2 + \|f\|^2_{L^2(0,T;H)}\right], \tag{4.30}$$

where C is a constant independent of $q \in \mathcal{P} = \{q = (\alpha, \beta, \delta) \in [\alpha_{min}, \alpha_{max}] \times [\beta_{min}, \beta_{max}] \times [\delta_{min}, \delta_{max}]\}$.

Part II. Existence and convergence.

Estimate (4.30) shows that for any $q \in \mathcal{P}$ and $m \in \mathbb{N}$ the approximate solutions $u_m(q)$ belong to same bounded convex ball $\|w\|_W \leq C$ of $W(0,T)$ for the same $C > 0$. Fix a $q \in \mathcal{P}$. Since $W(0,T)$ is a reflexive space, there exists a subsequence u_{m_k} of u_m that converges weakly to a function $z \in W(0,T)$. According to the energy estimate (4.30) we see that the sequence $\{u_m\}_{m=1}^{\infty}$ is bounded in $L^2(0,T;V)$, $\{u_m^{'}\}_{m=1}^{\infty}$ is bounded in $L^2(0,T;H)$, and $\{u_m^{''}\}_{m=1}^{\infty}$ is bounded in $L^2(0,T;V^{'})$, where $V^{'}$ is the dual space of V. Since $L^2(0,T;V)$, $L^2(0,T;H)$, and $L^2(0,T;V^{'})$ are reflexive spaces, there exist a subsequence $\{u_{m_k}\}_{k=1}^{\infty} \subset \{u_m\}_{k=1}^{\infty}$ and $z \in L^2(0,T;V)$, $d^1 \in L^2(0,T;H)$, $d^2 \in L^2(0,T;V^{'})$ such that

$$\begin{aligned} u_{m_k} &\rightharpoonup z, \quad \text{in} \quad L^2(0,T;V), \\ u_{m_k}^{'} &\rightharpoonup d^1, \quad \text{in} \quad L^2(0,T;H), \\ u_{m_k}^{''} &\rightharpoonup d^2, \quad \text{in} \quad L^2(0,T;V^{'}), \end{aligned} \tag{4.31}$$

where $\rightharpoonup$ indicates the weak convergence. Since the convergence in $W(0,T)$ is the distributional convergence, we have

$$\begin{aligned} u_{m_k}^{'} &\rightharpoonup z^{'}, \quad \text{in} \quad L^2(0,T;H), \\ u_{m_k}^{''} &\rightharpoonup z^{''} \quad \text{in} \quad L^2(0,T;V^{'}) \quad \text{as} \quad k \to \infty. \end{aligned} \tag{4.32}$$

But the weak limit is unique when it exists. So $d^1 = z^{'}$ and $d^2 = z^{''}$. Energy estimate (4.30) also implies that $\{u_m\}_{m=1}^{\infty}$ is bounded in $L^{\infty}(0,T;V)$ and the sequence $\{u_m^{'}\}_{m=1}^{\infty}$ is bounded in $L^{\infty}(0,T;H)$. By the Alaoglu Theorem, [15] we can find subsequences $\{u_{m_k}\}_{m=1}^{\infty}$ and $\{u_{m_k}^{'}\}_{m=1}^{\infty}$ of $\{u_m\}_{m=1}^{\infty}$ and $\{u_m^{'}\}_{m=1}^{\infty}$

respectively such that

$$
\begin{aligned}
u_{m_k} &\rightharpoonup z \quad \text{weak star in} \quad L^\infty(0,T;V),\\
u'_{m_k} &\rightharpoonup z' \quad \text{weak star in} \quad L^\infty(0,T;H).
\end{aligned} \tag{4.33}
$$

Now we show that z is a weak solution. Since V is compactly imbedded in H, then by the classical compactness theorem [4] $u_{m_k} \to z$ in $L^2(0,T;H)$. Using Cauchy Schwartz inequality, $|(\sin(u_{m_k}) - \sin(z), w_k)_{L^2(0,T;H)}| \leq \|\sin(u_{m_k}) - \sin(z)\|_{L^2(0,T;H)} \ \|w_k\|_{L^2(0,T;H)}$. Since $\{w_k\}_{k=1}^\infty$ is orthonormal in H the sequence $\{w_k\}_{k=1}^\infty$ is bounded in $L^2(0,T;H)$.
Thus $|(\sin(u_{m_k}) - \sin(z), w_k)_{L^2(0,T;H)}| \leq \|\sin(u_{m_k}) - \sin(z)\|_{L^2(0,T;H)} \to 0$ as $k \to \infty$ by (4.15). Hence $\sin(u_{m_k}) \to \sin(z)$ in $L^2(0,T;H)$. Rewrite (4.17) as

$$
\begin{aligned}
&\langle u''_m, w_j\rangle + \alpha(u'_m, w_j) + a_\beta(u_m, w_j) + \delta(P_m \sin(u_m), w_j)\\
&= (P_m f, w_j) + (u_m, w_j),\\
&u_m(0) = P_m u_0, \quad u'_m(0) = P_m u_1 \quad \text{for} \quad j = 1,2,...,m.
\end{aligned} \tag{4.34}
$$

We pass to the limit in (4.34) to obtain

$$
\begin{aligned}
&\langle z'', w_j\rangle + \alpha(z', w_j) + a_\beta(z, w_j) + \delta(\sin(z), w_j) = (f, w_j) + (z, w_j)\\
&z(0) = u_0, \quad z'(0) = u_1 \quad \text{for} \quad j = 1,2,...,m.
\end{aligned} \tag{4.35}
$$

Thus z is a weak solution of (1.1). It satisfies the energy estimate

$$
\max_{0\leq t\leq T} [\|z(t)\|^2 + |z(t)'|^2] + \|z(t)''\|^2_{L^2(0,T;V')} \leq C_1[\|u_0\|^2 + |u_1|^2 + \|f\|_{L^2(0,T;H)}],
$$

where C_1 is a constant independent of $q \in \mathcal{P} = \{q = (\alpha, \beta, \delta) \in [\alpha_{min}, \alpha_{max}] \times [\beta_{min}, \beta_{max}] \times [\delta_{min}, \delta_{max}]$. By Lemma (4.3) the solution z is unique. Therefore $u_m \to z$ as $m \to \infty$ in $L^2(0,T;H)$ for the entire sequence. Hence (3.6) can be rewritten as $z^{''} + A_\beta z = f + z - \alpha z^{'} - \delta \sin\ z$. Hence $z^{''} + A_\beta z \in L^2(0,T;H)$. Similarly (4.17) can be rewritten as $u_m^{''} + A_\beta u_m = P_m f + u_m - \alpha u_m^{'} - \delta P_m \sin\ u_m$. Therefore $u_m^{''} + A_\beta u_m \in L^2(0,T;H)$. Subtract (4.34) from (4.35) to get

$$(z - u_m)^{''} + A_\beta(z - u_m) = f - P_m f - \alpha(z - u_m)^{'} \tag{4.36}$$
$$-\delta(\sin(z) - P_m \sin(u_m)) + (z - u_m) \in L^2(0,T;H).$$

Therefore by Lemma (4.1) we have

$$\frac{1}{2}\frac{d}{dt}\{|z' - u_m'|^2 + a_\beta(z - u_m, z - u_m)\} = ((z - u_m)'' + A_\beta(z - u_m), z' - u_m'))$$
$$= (f - P_m f - \alpha(z' - u_m') - \delta(\sin(z) - P_m \sin(u_m)) + z - u_m, z' - u_m')$$
$$= (f - P_m f, z' - u_m') - \alpha|z' - u_m'|^2 - \delta(\sin(z) - P_m \sin(u_m), z' - u_m')$$
$$+(z - u_m, z' - u_m').$$

Integrating both sides over $[0,t]$ we get

$$|z'(t) - u_m'(t)|^2 + a_\beta(z(t) - u_m(t), z(t) - u_m(t)) \leq |u_1 - P_m u_1|^2$$
$$+(u_0 - P_m u_0, u_0 - P_m u_0) + 2\int_0^t |(f - P_m f)(z' - u_m')|ds$$
$$+2|\alpha|\int_0^t |(z' - u_m')|^2\ ds + 2|\delta|\int_0^t |(\sin(z) - P_m \sin(u_m))(z' - z_m')|ds$$
$$+\int_0^t |(z - u_m)(z' - u_m')|ds.$$

Use $|ab| \leq \frac{a^2+b^2}{2}$ to get

$$|z'(t) - u'_m(t)|^2 + \|z(t) - u_m(t)\|^2 \leq |u_1 - P_m u_1|^2 + \|u_0 - P_m u_0\|^2$$
$$+\|f - P_m f\|^2_{L^2(0,T;H)} + (2 + |\alpha| + |\delta|) \int_0^t |z' - u'_m|^2(s)ds$$
$$+ \int_0^t |z - u_m|^2(s)ds + \int_0^t |\sin(z) - P_m \sin(u_m)|^2(s)ds. \tag{4.37}$$

Since V is compactly embedded in H, (4.37) can be rewritten as

$$|z'(t) - u'_m(t)|^2 + \|z(t) - u_m(t)\|^2 \leq C[|u_1 - P_m u_1|^2 + \|u_0 - P_m u_0\|^2$$
$$+\|f - P_m f\|^2_{L^2(0,T;H)} + \int_0^t |\sin(z) - P_m \sin(u_m)|^2(s)ds$$
$$+ \int_0^t |z' - u'_m|^2(s)ds + \int_0^t \|z - u_m\|^2(s)ds] \tag{4.38}$$

where $C = max\{1, (2 + |\alpha| + |\delta|), 4K_1^2\}$.

Using Gronwall's lemma we get

$$|z'(t) - u'_m(t)|^2 + \|z(t) - u_m(t)\|^2 \leq C[|u_1 - P_m u_1|^2 + \|u_0 - P_m u_0\|^2$$
$$+\|f - P_m f\|^2_{L^2(0,T;H)} + \int_0^t |\sin(z) - P_m \sin(u_m)|^2(s)ds]. \tag{4.39}$$

Therefore $|z'(t) - u'_m(t)|^2 + \|z(t) - u_m(t)\|^2 \rightarrow 0$ as $m \rightarrow \infty$. This implies $u_m \rightarrow z$ in $L^\infty(0,T;V)$ and $u'_m \rightarrow z'$ in $L^\infty(0,T;H)$. But u_m, $u'_m \in C([0,T];V)$, being the solutions of the systems of ODEs. This implies $z \in C([0,T];V)$ and $z' \in C([0,T];H)$ after a modification on a set of measure zero on $[0,T]$. □

Chapter 5

Continuity of the Solution Map

Lemma 5.1. *Let $v \in V$. Then the mapping $\beta \to A_\beta v$ from $[\beta_{min}, \beta_{max}]$ into V' is continuous.*

Proof. Suppose that $\beta_n \to \beta$ in $\mathbb{R}$ as $n \to \infty$. We denote $A = A_\beta$ and $A_n = A_{\beta_n}$. We claim that $\|(A_n - A)v\|_{V'} \to 0$ as $n \to \infty$. Let $w \in V$ with $\|w\| \leq 1$. Then

$$|\langle (A_n - A)v, w\rangle|^2 \leq \left(\int_\Omega |\beta_n - \beta| |\nabla v(x)| |\nabla w(x)| dx \right)^2$$
$$\leq |\beta_n - \beta|^2 \int_\Omega |\nabla v(x)|^2 dx \to 0 \quad \text{as} \quad n \to \infty.$$

□

Lemma 5.2. *Suppose that $\beta_n \to \beta$ in $\mathbb{R}$, and $v_n \rightharpoonup v$ weakly in V, as $n \to \infty$. Then $A_n v_n \rightharpoonup Av$ weakly in V'.*

Proof. Let $w \in V$, then

$$|\langle A_n v_n, w\rangle - \langle Av, w\rangle| = |\langle A_n w, v_n\rangle - \langle Aw, v\rangle|$$
$$\leq |\langle (A_n - A)w, v_n\rangle| + |\langle Aw, v_n - v\rangle|. \tag{5.1}$$

Since a weakly convergent sequence is bounded, we have

$$|\langle (A_n - A)w, v_n\rangle| \le \|A_n w - Aw\|_{V'}\|v_n\| \le c\|A_n w - Aw\|_{V'} \to 0$$

as $n \to \infty$ by Lemma 5.1. The second term $|\langle Aw, v_n - v\rangle| \to 0$ since $v_n \rightharpoonup v$. □

Lemma 5.3. *Let $q \in \mathcal{P}$. Then the solution map $q \to u(q)$ from $\mathcal{P}$ into $C([0,T];H)$ is continuous.*

Proof. Let $q_n \to q$ in $\mathcal{P}$ as $n \to \infty$. Since $u(t;q)$ is the weak solution of (1.1) for any $q \in \mathcal{P}$, we have the following estimate

$$\begin{aligned} &\max_{0\le t\le T}(\|u(t;q_n)\|^2 + |u'(t;q_n)|^2) + \|u''(t;q_n)\|^2_{L^2(0,T;V')} \\ &\le C\left[\|u_0\|^2 + |u_1|^2 + \|f\|^2_{L^2(0,T;H)}\right], \end{aligned} \tag{5.2}$$

where C is a constant independent of $q \in \mathcal{P}$. Estimate (5.2) shows that $u(t;q_n)$ is bounded in $W(0,T)$. Since $W(0,T)$ is reflexive, we can choose a subsequence $u(t;q_{n_k})$ weakly convergent to a function z in $W(0,T)$. The fact that $u(t;q_n)$ is bounded in $W(0,T)$ implies that $u(t;q_n)$ is bounded in $L^2(0,T;V)$, so $u(t;q_{n_k})$ weakly convergent to a function z in $L^2(0,T;V)$. Since V is compactly imbedded in H, then by the classical compactness theorem [4] $u(t;q_n) \to z$ in $L^2(0,T;H)$. Using Cauchy Schwartz inequality, $|(\sin(u_{m_k})-\sin(z), w_k)_{L^2(0,T;H)}| \le \|\sin(u_{m_k})-\sin(z)\|_{L^2(0,T;H)}\ \|w_k\|_{L^2(0,T;H)}$. Since $\{w_k\}_{k=1}^{\infty}$ is orthonormal in H the sequence $\{w_k\}_{k=1}^{\infty}$ is bounded in $L^2(0,T;H)$. Thus $|(\sin(u_{m_k}) - \sin(z), w_k)_{L^2(0,T;H)}| \le \|\sin(u_{m_k}) - \sin(z)\|_{L^2(0,T;H)} \to 0$ as $k \to \infty$ by (4.15) By (4.18) the derivatives $u'(t;q_{n_k})$ and z' are uniformly bounded in $L^\infty(0,T;H)$. Therefore functions $\{u(t;q_{n_k}), z\}_{k=1}^{\infty}$ are equicontinuous in $C([0,T];H)$. Thus $u(t;q_{n_k}) \to z$ in $C([0,T];H)$. In particular, $u(t;q_{n_k}) \to z(t)$ in H and $u(t;q_{n_k}) \rightharpoonup z(t)$ weakly in V

for any $t \in [0,T]$. By Lemma 5.2, $A_{n_k} u(t; q_{n_k}) \rightharpoonup Az(t)$ weakly in V'. Now we see that z satisfies equation (3.4), i.e. it is the weak solution $u(q)$. The uniqueness of the weak solutions implies that $u(q_n) \to u(q)$ as $n \to \infty$ in $C([0,T]; H)$ for the entire sequence $u(q_n)$ and not just for its subsequence. Thus $u(t; q_n) \to u(q)$ in $C([0,T]; H)$ as $q_n \to q$ in P as claimed. □

Theorem 5.4. *Let $q \in \mathcal{P}$. Then the solution maps $q \to u(q)$ from $\mathcal{P}$ into $C([0,T]; V)$ and $q \to u'(q)$ from $\mathcal{P}$ into $C([0,T]; H)$ are continuous.*

Proof. Part I. First, we establish the continuity of the approximate solution maps $q \to u_m(q)$ from $\mathcal{P}$ into $C([0,T]; V)$, and $q \to u_m'(q)$ from $\mathcal{P}$ into $C([0,T]; H)$.

Fix $m \in \mathbb{N}$. Suppose that $q_n \to q$ in $\mathbb{R}^3$ as $n \to \infty$. That is $\alpha_n \to \alpha$, $\beta_n \to \beta$, and $\delta_n \to \delta$ in $\mathbb{R}$. The approximate solutions $u_m(q_n)$ and $u_m(q)$ satisfy

$$u_m''(q_n) + A_n u_m(q_n) = P_m f + u_m(q_n) - \alpha_n u_m'(q_n) - \delta_n P_m \sin(u_m(q_n)),$$
$$u_m''(q) + A u_m(q) = P_m f + u_m(q) - \alpha u_m'(q) - \delta P_m \sin(u_m(q)), \quad (5.3)$$

where we write $A = A_\beta$ and $A_n = A_{\beta_n}$ to simplify the notation. In each case the initial conditions are the same for q and q_n: $u(0,q) = P_m u_0$ and $u'(0;q) = P_m u_1$.

Let $w = u_m(q_n) - u_m(q)$. Subtracting the equations in (5.3) gives

$$w'' + A_n(w) = (A - A_n)u_m(q) + w - \alpha_n w' + (\alpha - \alpha_n)u_m'(q)$$
$$-\delta_n P_m(\sin(u_m(q_n)) - \sin(u_m(q))) + (\delta - \delta_n)P_m \sin(u_m(q)). \quad (5.4)$$

Take the H inner product of each side with w' to get

$$(w'' + A_n(w), w') = ((A - A_n)u_m(q), w') + (w, w') - \alpha_n|w'|^2$$
$$+(\alpha - \alpha_n)(u'_m(q), w') - \delta_n(P_m(\sin(u_m(q_n)) - \sin(u_m(q))), w')$$
$$+(\delta - \delta_n)(P_m \sin(u_m(q)), w'). \tag{5.5}$$

Since $w(t) \in L^2(0, T; V)$, $w'(t) \in L^2(0, T; H)$ and $w'' + A_n(w) \in L^2(0, T; H)$, then by Lemma 4.1 we have

$$\frac{1}{2}\frac{d}{dt}\{|w'|^2 + a_n(w, w)\} = ((A - A_n)u_m(q), w') + (w, w') - \alpha_n|w'|^2$$
$$+(\alpha - \alpha_n)(u'_m(q), w') - \delta_n(P_m(\sin(u_m(q_n)) - \sin(u_m(q))), w')$$
$$+(\delta - \delta_n)(P_m \sin(u_m(q)), w'). \tag{5.6}$$

Integrate both sides from 0 to t and use Cauchy-Schwartz Inequality to get

$$|w'(t)|^2 + \|w(t)\|^2 \leq 2\int_0^t |(A - A_n)u_m(q)||w'(s)|ds$$
$$+2|\alpha - \alpha_n|\int_0^t |u'_m(s; q)||w'(s)|ds + 2|\delta - \delta_n|\int_0^t |u_m(s; q)||w'(s)|ds$$
$$+2|\alpha_n|\int_0^t |w'(s)|^2 ds + 2|\delta_n|\int_0^t |w(s)||w'(s)|ds. \tag{5.7}$$

Use $|ab| \leq \frac{a^2+b^2}{2}$ and use the fact that V is compactly embedded in H to get

$$|w'(t)|^2 + \|w(t)\|^2 \leq \int_0^t \|(A - A_n)u_m(q)\|^2_{V'} ds + \int_0^t |w'(s)|^2 ds$$
$$+|\alpha - \alpha_n|\int_0^t |u'_m(s; q)|^2 ds + |\alpha - \alpha_n|\int_0^t |w'(s)|^2 ds$$
$$+|\delta - \delta_n|\int_0^t \|u_m(s; q)\|^2 ds + |\alpha_n|\int_0^t |w'(s)|^2 ds + |\delta_n|\int_0^t \|w(s)\|^2 ds$$
$$+|\delta_n|\int_0^t |w'(s)|^2 ds. \tag{5.8}$$

In a finite dimensional normed space all norms are equivalent. Hence there exists a constant $C(m)$ such that $\|w'(s)\| \le C(m)|w'(s)|$ for any $s \in [0,T]$.

Now the Gronwall's inequality and the energy estimate (4.18) give

$$
\begin{aligned}
&|u'_m(t;q_n) - u'_m(t;q)|^2 + \|u_m(t;q_n) - u_m(t;q)\|^2 \\
&\le c(m)\left(\int_0^T \|(A - A_n)u_m(s;q)\|_{V'}^2 ds + |\alpha - \alpha_n| + |\delta - \delta_n|\right). \qquad (5.9)
\end{aligned}
$$

By the assumption $q_n \to q$ in $\mathcal{P}$, that is $\alpha_n \to \alpha,\ \delta_n \to \delta$ and $\beta_n \to \beta$ in $\mathbb{R}$ as $n \to \infty$. The integral term in the right hand side of (5.9) approaches zero by Lemma 5.1 and the Lebesgue Dominated Convergence Theorem. Hence the required convergence $u_m(q_n) \to u_m(q)$ in $C([0,T];V)$ and $u'_m(q_n) \to u'_m(q)$ in $C([0,T];H)$ as $n \to \infty$ follows.

Part II. Next we prove that $u_m(q) \to u(q),\ m \to \infty$ in $C([0,T];V)$ uniformly on $\mathcal{P}$.

Estimate (4.39) shows that it is enough to establish the uniform convergence of

$$\int_0^T |\sin(u(s;q)) - P_m \sin(u(s;q))|^2 ds \to 0, \quad m \to \infty \qquad (5.10)$$

for $q \in \mathcal{P}$. Note that the mapping $[0,T] \times \mathcal{P} \to H$ defined by $(s,q) \to u(s;q)$ is continuous, since $q \to u(q) \in C([0,T];H)$ is continuous by Lemma 5.3. Therefore the mapping $[0,T] \times \mathcal{P} \to H$ defined by $(s,q) \to \sin(u(s;q))$ is continuous. Thus it takes the compact set $[0,T] \times \mathcal{P}$ into a compact set in H, and the uniform convergence of the integrals in (5.10) follows from the Dini's Theorem.

Finally, let $q_n \to q$ in $\mathcal{P}$. By Part I the map $q \to u_m(q)$ is continuous on $\mathcal{P}$ for every $m \in \mathbb{N}$. By Part II the convergence $u_m(q) \to u(q)$ is uniform on $\mathcal{P}$. Therefore $u(q_n) \to u(q),\ m \to \infty$ in $C([0,T];V)$ as claimed. This argument applied to the estimate (4.19) also shows the convergence of the derivatives $u'(q_n) \to u'(q)$ in $C([0,T];H)$. $\square$

Chapter 6

Weak Gâteaux Differentiability of the Solution Map

Let

$$\mathcal{H} = \left\{ G = \begin{pmatrix} \xi \\ g \end{pmatrix} : \xi \in H \quad \text{and} \quad g \in L^2(0,T;H) \right\}. \tag{6.1}$$

Then H is a Hilbert space with the following inner product and the norm

$$(G_1, G_2)_{\mathcal{H}} = (\xi_1, \xi_2)_H + (g_1, g_2)_{L^2(0,T;H)}, \quad \|G\|_{\mathcal{H}} = (G,G)_{\mathcal{H}}^{\frac{1}{2}}, \tag{6.2}$$

where $G_1 = \begin{pmatrix} \xi_1 \\ g_1 \end{pmatrix} \in \mathcal{H}$ and $G_2 = \begin{pmatrix} \xi_2 \\ g_2 \end{pmatrix} \in \mathcal{H}$.

To show the Gâteaux differentiability of $J(q)$ at $q^* \in \mathcal{P}$ we have to estimate the quotient

$$z_\lambda = \frac{u(q_\lambda) - u(q^*)}{\lambda}, \tag{6.3}$$

where $q_\lambda = q^* + \lambda(q - q^*), \quad \lambda \in (0,1]$. Generally it is desirable to estimate z_λ in the solution space $W(0,T)$. Since the second order evolution equations for z_λ in (6.24) have the forcing term containing a diffusion operator, it is not easy or impossible to solve the equation (6.24) by standard variational manner as in [7]. Hence we will restrict ourselves to an estimate of $\begin{pmatrix} z_\lambda(T) \\ z_\lambda(t) \end{pmatrix} \in H \times L^2(0,T;H)$ as $\lambda \to 0$ based on the method of transposition presented in [8].

Now we show the Gâteaux differentiability of the solution map $q \to \begin{pmatrix} u(q;T) \\ u(q;t) \end{pmatrix}$ of $\mathcal{P}$ into $H \times L^2(0,T;H)$ via the method of transposition and characterize its Gâteaux derivative.

Fix $q = (\alpha, \beta, \delta) \in \mathcal{P}$ and $h \in L^2(0,T;H)$. Let $G = \begin{pmatrix} \xi \\ g \end{pmatrix} \in \mathcal{H}$. Let us consider the following linear terminal value problem

$$\begin{aligned} &\phi'' - \alpha\phi' + A_\beta \phi + (\delta h - 1)\phi = g \quad \text{in} \quad (0,T) \\ &\phi(T) = 0, \quad \phi'(T) = \xi. \end{aligned} \tag{6.4}$$

Let $\phi(T-s,x) = w(s,x)$ for any $x \in (0,1)$, then we have $\phi_t(T-s,x) = -w_s(s,x)$ and $\phi_{tt}(T-s,x) = w_{ss}(s,x)$, then (6.4) can be written as

$$\begin{aligned} &w'' + \alpha w' + A_\beta w + (\delta h - 1)w = g \quad \text{in} \quad (0,T) \\ &w(0) = 0, \quad w'(0) = -\xi. \end{aligned} \tag{6.5}$$

Arguing as in Chapter 4, we can conclude that (6.5) has a unique weak solution. Hence (6.4) has a unique weak solution $\phi = \phi(\xi, g) \in W(0,T)$ that satisfies the energy estimate

$$|\phi'(t)|^2 + \|\phi(t)\|^2 \leq c(|\xi|^2 + \|g\|^2_{L^2(0,T;H)}), \quad t \in [0,T]. \tag{6.6}$$

Definition 6.1. Solution map: Given $G \in \mathcal{H}$ define the solution map from $\mathcal{H}$ into $W(0,T)$ by $\tau(G) = \phi$, where ϕ is the weak solution of (6.4).

Definition 6.2. Fix $q = (\alpha, \beta, \delta) \in \mathcal{P}$ and $h \in L^2(0,T;H)$. Let the solution space $\mathcal{X}(q;h) = \tau(\mathcal{H})$ be defined by

$$\mathcal{X}(q,h) = \{\phi : \phi \text{ is solution of (6.4) for each G} \in \mathcal{H}\}.$$

Let the linear operator $\mathcal{L}(q;h)$ from $\mathcal{X}(q;h)$ into $\mathcal{H}$ be defined by

$$\mathcal{L}(q;h)\phi = \begin{pmatrix} \phi'(T) \\ \phi'' - \alpha\phi' + A_\beta\phi + (\delta h - 1)\phi. \end{pmatrix} = \begin{pmatrix} \phi'(T) \\ g \end{pmatrix}. \tag{6.7}$$

Let the inner product $(.\ ,\ .)$ in $\mathcal{X}(q;h)$ be defined by

$$(\phi, \psi)_{\mathcal{X}(q;h)} = (\mathcal{L}(q;h)\phi, \mathcal{L}(q;h)\psi)_{\mathcal{H}}. \tag{6.8}$$

In terms of the operator $\mathcal{L}(q;h)$ the energy estimate (6.6) can be written as

$$|\phi'(t)|^2 + \|\phi(t)\|^2 \leq c(\|\mathcal{L}(q;h)\phi\|^2_{\mathcal{H}}) = c\|\phi\|^2_{\mathcal{X}(q,h)}. \tag{6.9}$$

Definition 6.3. Given $q \in \mathcal{P}$, $h \in L^2(0,T;H)$, and $f \in L^2(0,T;V')$, the element $\bar{z} = \begin{pmatrix} z_1 \\ z \end{pmatrix} \in \mathcal{H}$, $z_1 \in H$, $z \in L^2(0,T;H)$ is called a weakened solution of the problem

$$z''(t) + \alpha z'(t) + A_\beta z(t) + (\delta h(t) - 1)z(t) = f(t)$$
$$z(0) = 0\ ,\ z'(0) = 0, \quad t \in (0, T), \tag{6.10}$$

if

$$(\bar{z}, \mathcal{L}(q; h)\phi)_{\mathcal{H}} = \int_0^T \langle f(t), \phi(t) \rangle dt \tag{6.11}$$

for any $\phi \in \mathcal{X}(q; h)$. That is,

$$(z_1, \xi)_H + \int_0^T (z(t), g(t)) dt = \int_0^T \langle f(t), \phi(t) \rangle dt \tag{6.12}$$

for all $\phi \in \mathcal{X}(q, h)$.

Remark 6.4. If $f \in L^2(0, T; H)$ and $z(t)$ is the weak solution (in the sense of Chapter 4) of the problem (6.10), then the integration by parts shows that $\bar{z} = \begin{pmatrix} z'(T) \\ z(t) \end{pmatrix}$ also is its weakened solution.

Lemma 6.5. *If $f \in L^2(0, T; V')$, then there exists a unique weakened solution of the problem* (6.10).

Proof. By the method of transposition of Lions, if F is a bounded linear functional on $\mathcal{X}(q; h)$, then there exists a unique $\bar{\xi} \in \mathcal{H}$ such that

$$F(\phi) = (\bar{\xi}(t), \mathcal{L}(q; h)(\phi)(t))_{\mathcal{H}} \quad \text{for} \quad \text{any} \quad \phi \in \mathcal{X}(q; h). \tag{6.13}$$

Let

$$F(\phi) = \int_0^T \langle f(t), \phi(t) \rangle dt, \ \ \phi \in \mathcal{X}(q, h).$$

Using the energy estimate (6.9) we get

$$\begin{aligned}
&|F(\phi)| \le \|f\|_{L^2(0,T:V')}\|\phi\|_{L^2(0,T;V)} = \|f\|_{L^2(0,T:V')}\sqrt{\int_0^T \|\phi(t)\|_V^2 dt} \\
&\le \|f\|_{L^2(0,T:V')}\ \sqrt{c}\int_0^T \|\phi(t)\|^2_{\mathcal{X}(q,h)} dt \\
&\le \sqrt{cT}\|f\|_{L^2(0,T:V')}\ \|\phi(t)\|_{\mathcal{X}(q,h)} \qquad (6.14)
\end{aligned}$$

and the result follows. □

Let $\hat{u}$ and $\hat{v}$ be two measurable functions on Ω. Define the function $B(\hat{u},\hat{v})(x)$ for $x \in \Omega$ by

$$B(\hat{u},\hat{v})(x) = \begin{cases} \frac{\sin(\hat{u}(x)) - \sin(\hat{v}(x))}{\hat{u}(x)-\hat{v}(x)}, & \hat{u}(x) \neq \hat{v}(x), \\ \cos(\hat{v}(x)), & \hat{u}(x) = \hat{v}(x), \end{cases} \qquad (6.15)$$

Then B is an integrable function on Ω with $|B(\hat{u},\hat{v})(x)| \le 1$ for any $x \in \Omega$. If $\hat{u}_1 = \hat{u}$ a.e. on Ω , and $\hat{v}_1 = \hat{v}$ a.e. on Ω, then $B(\hat{u}_1,\hat{v}_1) = B(\hat{u},\hat{v})$ a.e. on Ω. Thus $B(u,v) : H \times H \to H$ is well defined by (6.15).

Furthermore, the inequality

$$|\cos(b) - \frac{\sin(a)-\sin(b)}{a-b}| \le |a-b| \qquad (6.16)$$

for $a,b \in \mathbb{R}$, $a \neq b$ implies that

$$|\cos(b) - B(u,v)|_H \le |u-v|_H \qquad (6.17)$$

for any $u,v \in H$.

Definition 6.6. Let $q,q^* \in \mathcal{P}$. Let $q_\lambda = q^* + \lambda(q-q^*)$ for $\lambda \in (0,1]$. The solution map $q \to \bar{u}(q) = \begin{pmatrix} u'(T;q) \\ u(t;q) \end{pmatrix}$ of $\mathcal{P}$ into $\mathcal{H}$ is said to be weakly Gateaux differentiable at q^* in the direction $q - q^*$ if there exist $\bar{z} \in \mathcal{H}$ such that

$$\lim_{\lambda \to 0+} \frac{1}{\lambda}(\bar{u}(q_\lambda) - \bar{u}(q^*), \bar{v})_{\mathcal{H}} = (\bar{z}, \bar{v})_{\mathcal{H}} \tag{6.18}$$

for any $\bar{v} \in \mathcal{H}$.

Theorem 6.7. *Let $q = (\alpha, \beta, \delta), q^* = (\alpha^*, \beta^*, \delta^*) \in \mathcal{P}$. Then the weak Gâteaux derivative $\bar{z} \in \mathcal{H}$ at $q^* \in \mathcal{P}$ in the direction $q - q^*$ is the unique weakened solution of the problem*

$$\begin{aligned} &z''(t) + \alpha^* z'(t) + A_{\beta^*} z(t) + (\delta^* \cos\, u(t;q^*) - 1)z(t) = f_0(t), \\ &z(0) = 0,\ z'(0) = 0,\ t \in (0,T), \end{aligned} \tag{6.19}$$

where $f_0(t) = (\alpha^* - \alpha)u'(t;q^*) + (A_{\beta^*} - A_\beta)u(t;q^*) + (\delta^* - \delta)\sin(u(t;q^*))$.

Remark 6.8. For $\mathcal{X}$ and $\mathcal{L}$ defined by (6.8) and (6.7) respectively with q^* and $h = \cos(u(q^*))$ the solution $\bar{z} = \begin{pmatrix} z(T) \\ z(t) \end{pmatrix}$ satisfies

$$(\bar{z}(t), \mathcal{L}(q^*; \cos u(t;q^*)\phi(t))_{\mathcal{H}} = \int_0^T \langle f_0(t), \phi(t)\rangle dt \tag{6.20}$$

for any $\phi \in \mathcal{X}(q^*; \cos(u(q^*)))$.

Proof. Let $q_\lambda = q^* + \lambda(q - q^*) = (\alpha_\lambda, \beta_\lambda, \delta_\lambda)$ and denote $A_\lambda = A_{\beta_\lambda}$. Then $A_0 = A_{\beta^*}$. By (3.6) functions $u(q_\lambda)$ and $u(q^*)$ are the weak solutions of the equations

$$u''(q_\lambda) + \alpha_\lambda u'(q_\lambda) + A_\lambda u(q_\lambda) + \delta_\lambda \sin(u(q_\lambda)) = f + u(q_\lambda)$$
$$u_\lambda(0, q) = u_0, \ u'_\lambda(0; q) = u_1 \tag{6.21}$$

and

$$u''(q^*) + \alpha^* u'(q^*) + A_{\beta^*} u(q^*) + \delta^* \sin(u(q^*)) = f + u(q^*)$$
$$u(0, q^*) = u_0, \ u'(0; q^*) = u_1 \tag{6.22}$$

correspondingly.

Then the quotient $z_\lambda = (u(q_\lambda) - u(q^*))/\lambda$ satisfies

$$z''_\lambda + \alpha^* z'_\lambda + A_{\beta^*} z_\lambda + \delta^* \frac{\sin(u(q_\lambda)) - \sin(u(q^*))}{\lambda} - z_\lambda$$
$$= (\alpha^* - \alpha) u'(q_\lambda) + (A_{\beta^*} - A_\beta) u(q_\lambda) + (\delta^* - \delta) \sin(u(q_\lambda)),$$
$$z_\lambda(0) = 0, \quad z'_\lambda(0) = 0. \tag{6.23}$$

Let

$$f_\lambda(t) = (\alpha^* - \alpha) u'(t; q_\lambda) + (A_{\beta^*} - A_\beta) u(t; q_\lambda) + (\delta^* - \delta) \sin(u(t; q_\lambda)).$$

Using the notation (6.15) we let $B_\lambda(t) = B(u(t; q_\lambda), u(t; q^*)) \in H$ for $0 \leq t \leq T$.

Then

$$z''_\lambda + \alpha^* z'_\lambda + A_{\beta^*} z_\lambda + (\delta^* B_\lambda(t) - 1) z_\lambda = f_\lambda,$$
$$z_\lambda(0) = 0, \quad z'_\lambda(0) = 0. \tag{6.24}$$

Since H is continuously imbedded in V' there exists a constant $K_2 = K_2(\Omega)$ such that $\|v\|_{V'} \leq K_2|v|$ for any $v \in H$. Therefore one can estimate

$$\|f_\lambda(t)\|_{V'} \leq K_2(|\alpha^* - \alpha||u'(t;q_\lambda)| + 2\mu K_1\|u(t;q_\lambda)\| + K_1|\delta^* - \delta|\|u(t;q_\lambda)\|). \quad (6.25)$$

Now the energy estimate (4.18) shows that there exists $C_2 > 0$ independent of $q \in \mathcal{P}$ such that

$$\|f_\lambda\|_{L^2(0,T;V')} \leq C_2 \quad (6.26)$$

for all $\lambda \in (0, 1]$.

Since z_λ is a weak solution of (6.24) it is also its weakened solution, i.e.

$$(\bar{z}_\lambda, \mathcal{L}(q^*; B_\lambda)\phi)_{\mathcal{H}} = \int_0^T \langle f_\lambda(t), \phi(t)\rangle dt \quad (6.27)$$

for any $\phi \in \mathcal{X}(q^*; B_\lambda)$.

Since $\bar{z}_\lambda \in \mathcal{H}$ and $\mathcal{L}(q^*; B_\lambda)$ from $\mathcal{X}(q^*; B_\lambda) \to \mathcal{H}$ is surjective, there exists $\phi_\lambda \in \mathcal{X}(q^*; B_\lambda)$ such that $\mathcal{L}(q^*; B_\lambda)\phi_\lambda = \bar{z}_\lambda$.

For such a function ϕ_λ one gets from (6.27)

$$\|\bar{z}_\lambda\|^2_{\mathcal{H}} \leq \|f_\lambda\|_{L^2(0,T;V')}\|\phi_\lambda\|_{L^2(0,T;V)}. \quad (6.28)$$

This inequality and estimates (6.9) and (6.26) give

$$\|\bar{z}_\lambda\|^2_{\mathcal{H}} \leq C_2\|\bar{z}_\lambda\|_{\mathcal{H}}.$$

Thus $\|\bar{z}_\lambda\|_{\mathcal{H}} \leq C_2$ for some constant C_2 independent of $\lambda \in (0, 1]$. Here we used the fact that $|B_\lambda(t)| \leq 1$ for any t, λ and $q, q^* \in \mathcal{P}$. Therefore one can extract a subsequence $\bar{z}_{\lambda_k}$, $\lambda_k \to 0+$, such that $\bar{z}_{\lambda_k} \rightharpoonup \bar{z}$ weakly in $\mathcal{H}$. Now we would like to pass to the limit in (6.27) as $\lambda_k \to 0$ to obtain (6.32). However, the domains of the operators $\mathcal{L}(q^*; B_\lambda)$ depend on λ, so one has to proceed differently. Let

$$f_0(t) = (\alpha^* - \alpha)u'(t; q^*) + (A_{\beta^*} - A_\beta)u(t; q^*) + (\delta^* - \delta)\sin u(t; q^*). \tag{6.29}$$

From Lemma 5.3 we get $u(q_\lambda) \to u(q^*)$ in $L^2(0, T; V)$, and $u'(q_\lambda) \to u'(q^*)$ in $L^2(0, T; H)$. Therefore $f_\lambda \rightharpoonup f_0$ weakly in $L^2(0, T; V')$. In fact, Theorem 5.4 shows that this is a strong convergence. Thus $\|f_0\|_{L^2(0,T;V')} \leq C_2$.

Write $\mathcal{L}_0 = \mathcal{L}(q^*; \cos\, u(q^*))$ and $\mathcal{L}_k = \mathcal{L}(q^*; B_{\lambda_k})$ to simplify the notation. Let $\phi \in \mathcal{X}(q^*; \cos\, u(q^*))$. Then $\mathcal{L}_0\phi \in \mathcal{H}$. Therefore

$$(\bar{z}_{\lambda_k}, \mathcal{L}_0\phi(t))_{\mathcal{H}} \to (\bar{z}(t), \mathcal{L}_0\phi(t))_{\mathcal{H}}, \quad \text{and}$$

$$\int_0^T \langle f_{\lambda_k}(t), \phi(t)\rangle dt \to \int_0^T \langle f_0(t), \phi(t)\rangle dt \tag{6.30}$$

as $\lambda_k \to 0+$.

On the other hand,

$$\begin{aligned}
&(\bar{z}_{\lambda_k}(t), \mathcal{L}_0\phi(t))_{\mathcal{H}} = (z_{1\lambda_k}, \xi)_H + \int_0^T (z''_{\lambda_k}(t) + \alpha^* z'_{\lambda_k}(t) + A_{\beta^*} z_{\lambda_k}(t), \phi(t))dt \\
&+ \int_0^T (\delta^* \cos\, u(t; q^*) - 1) z_{\lambda_k}(t), \phi(t))dt \\
&= \int_0^T (z''_{\lambda_k}(t) + \alpha^* z'_{\lambda_k}(t) + A_{\beta^*} z_{\lambda_k}(t), \phi(t))dt \\
&+ (z_{1\lambda_k}, \xi)_H + \int_0^T ((\delta^* B_{\lambda_k}(t) - 1) z_{\lambda_k}(t), \phi(t))dt \\
&+ \delta^* \int_0^T ((\cos\, u(t; q^*) - B_{\lambda_k}(t))) z_{\lambda_k}(t), \phi(t))dt \\
&= (z_{1\lambda_k}, \xi)_H + \int_0^T \langle f_{\lambda_k}(t), \phi(t)\rangle dt \\
&+ \delta^* \int_0^T ((\cos\, u(t; q^*) - B_{\lambda_k}(t)) z_{\lambda_k}(t), \phi(t))dt.
\end{aligned} \tag{6.31}$$

Using $\|\bar{z}_\lambda\|_{\mathcal{H}} \leq C_2, \phi \in W(0,T)$ and the estimate (6.17), the last term in (6.31) can be estimated by $c\|u(q_{\lambda_k}) - u(q^*)\|_{L^2(0,T;H)}\|\phi\|_{L^\infty(0,T;H)}$. Since the mapping $q \to u(q)$ from $\mathcal{P}$ into $L^2(0,T;H)$ is continuous, then the last term of (6.31) tends to 0 as $\lambda_k \to 0+$.

Now we can pass to the limit as $\lambda_k \to 0+$ in (6.31), and conclude that

$$(\bar{z}, \mathcal{L}(q^*; \cos\ u(t;q^*))\phi)_{\mathcal{H}} = \int_0^T \langle f_0, \phi(t)\rangle dt \tag{6.32}$$

for any $\phi \in \mathcal{X}(q^*; \cos\ u(q^*))$. Since $\|f_0\|_{L^2(0,T;V')} \leq C_2$, Lemma (6.5) shows that that $\bar{z}$ is the unique weakened solution of (6.19). Hence $\bar{z}_\lambda \rightharpoonup \bar{z}$ as $\lambda \to 0+$ weakly in $\mathcal{H}$ by Definition 6.6. This proves that the $\bar{z}$ is the weak Gâteaux derivative of the map $q \to \bar{u}(q)$.

□

Chapter 7

Optimal Parameters

From Theorem 6.7 the map $q \to \bar{u}(q)$ is weakly Gâteaux differentiable at $q = q^* \in \mathcal{P}$ in any direction of $q - q^*$, and its weak Gâteaux derivative $\bar{z}(t,x) = D\bar{u}(q^*; q - q^*)(t,x)$ can be described by (6.20).
Let us consider the functional

$$J(q) = k_1|u(q;T) - z_d^1|^2 + k_2\|u(q;t) - z_d^2\|_{L^2(0,T;H)}^2 \tag{7.1}$$

where $z_d^1 \in H$, $z_d^2 \in L^2(0,T;H)$ and $k_i \geq 0$ for $i = 1,2$ with $k_1 + k_2 > 0$.

Lemma 7.1. *$J(q)$ is Gâteaux differentiable, and its Gâteaux derivative is given by*

$$DJ(q^*; q - q^*) = 2k_1((u(q^*;T) - z_d^1), z_1) + 2k_2 \int_0^T (u(q^*;t) - z_d^2), z)dt \tag{7.2}$$

where $\bar{z}$ is the solution of integral equation (6.20).

Proof. In the previous section we have shown that the weak solution $u(q;t)$ is weakly Gâteaux differentiable in the admissible set of parameters $\mathcal{P}$. Hence the following limits exist

$$\lim_{\lambda \to 0+} \left(\frac{u(q^* + \lambda(q - q^*); T) - u(q^*; T)}{\lambda}, v_1 \right)_H = (z_1, v_1) \tag{7.3}$$

for any $v_1 \in H$ and

$$\lim_{\lambda \to 0+} \left(\frac{u(q^* + \lambda(q - q^*); t) - u(q^*; t)}{\lambda}, v_2 \right)_{L^2(0,T;H)} = (z, v_2)_{L^2(0,T;H)} \tag{7.4}$$

for any $v_2 \in L^2(0, T; H)$.

To show that the cost functional $J(q)$ is Gâteaux differentiable at q^*, it sufficies to show that the following limit exists

$$\lim_{\lambda \to 0+} \left(\frac{J(q^* + \lambda(q - q^*)) - J(q^*)}{\lambda} \right) = DJ(q^*; q - q^*). \tag{7.5}$$

Evaluating the limit in (7.5)

$$\begin{aligned}
&\lim_{\lambda \to 0+} \left(\frac{J(q^* + \lambda(q - q^*)) - J(q^*)}{\lambda} \right) \\
&= k_1 \lim_{\lambda \to 0+} \frac{1}{\lambda} \left([(u(q^* + \lambda(q - q^*); T) - z_d^1, u(q^* + \lambda(q - q^*); T) - z_d^1) \right. \\
&-(u(q^*; T) - z_d^1, u(q^*; T) - z_d^1)]) \\
&+k_2 \lim_{\lambda \to 0+} \frac{1}{\lambda} [(u(q^* + \lambda(q - q^*); t) - z_d^2, u(q^* + \lambda(q - q^*); t) - z_d^2)_{L^2(0,T;H)} \\
&-(u(q^*; t) - z_d^2, u(q^*; t) - z_d^2)_{L^2(0,T;H)}].
\end{aligned} \tag{7.6}$$

Consider the first part of limit from (7.6)

$$\begin{aligned}
&k_1 \lim_{\lambda \to 0+} \frac{1}{\lambda} [(u(q^* + \lambda(q - q^*); T) - z_d^1, u(q^* + \lambda(q - q^*); T) - z_d^1) \\
&-(u(q^* + \lambda(q - q^*); T) - z_d^1, u(q^*; T) - z_d^1) \\
&+(u(q^* + \lambda(q - q^*); T) - z_d^1, u(q^*; T) - z_d^1) - (u(q^*; T) - z_d^1, u(q^*; T) - z_d^1)]
\end{aligned}$$

$$k_1 \lim_{\lambda\to 0+} \frac{1}{\lambda}[(u(q^*+\lambda(q-q^*);T)-z_d^1-u(q^*;T)+z_d^1, u(q^*+\lambda(q-q^*);T)-z_d^1)$$
$$+(u(q^*;T)-z_d^1), u(q^*+\lambda(q-q^*);T)-z_d^1-u(q^*;T)+z_d^1)]$$
$$= 2k_1(u(q^*;T)-z_d^1, z_1). \tag{7.7}$$

Similarly,

$$k_2 \lim_{\lambda\to 0+} \frac{1}{\lambda}[(u(q^*+\lambda(q-q^*);t)-z_d^2, u(q^*+\lambda(q-q^*);t)-z_d^2)_{L^2(0,T;H)}$$
$$-(u(q^*;t)-z_d^2, u(q^*;t)-z_d^2)_{L^2(0,T;H)}]$$
$$= 2k_2(u(q^*;t)-z_d^2, z)_{L^2(0,T;H)}. \tag{7.8}$$

Using (7.7) and (7.8) we get

$$DJ(q^*;q-q^*) = 2k_1((u(q^*;T)-z_d^1), z_1) + 2k_2\int_0^T (u(q^*;t)-z_d^2), z)dt. \tag{7.9}$$

□

Since $\mathcal{P} = \{q = (\alpha,\beta,\delta) \in [\alpha_{min},\alpha_{max}]\times[\beta_{min},\beta_{max}]\times[\delta_{min},\delta_{max}]\}$ is a closed and convex subset of $\mathbb{R}^3$, then we have the following optamility condition

$$2k_1((u(q^*;T)-z_d^1), z_1) + 2k_2\int_0^T (u(q^*;t)-z_d^2), z)dt \ge 0 \quad \text{for} \quad q\in\mathcal{P}, \tag{7.10}$$

where $\begin{pmatrix} z_1 \\ z \end{pmatrix}$ is a solution of the integral equation (6.20).

Let us introduce the adjoint state p defined to be the weak solution of the following adjoint system

$$p'' - \alpha^* p' + A_\beta^* p + (\delta^* \cos(u(q^*) - 1))p = k_2(u(q*;t) - z_d^2)$$
$$p(T) = 0 \quad p'(T) = k_1(u(q^*;T) - z_d^1). \tag{7.11}$$

System (7.11) can be written as

$$\mathcal{L}(q^*; \cos(u(q^*))p(q^*) = \begin{pmatrix} k_1 u(q^*;T) - z_d^1 \\ k_2 u(q^*;t) - z_d^2 \end{pmatrix} \in \mathcal{H}$$
$$p(T) = 0, \quad p'(T) = k_1(u(q^*;T) - z_d^1). \tag{7.12}$$

Since $k_2(u(q^*;t) - z_d^2) \in L^2(0,T;H)$, as shown in Chapter 4 problem in (7.11) has a unique weak solution. Using $p(q^*)$ in place of ϕ in (6.20) equation (7.2) can be written as

$$DJ(q^*; q - q^*) = 2\int_0^T \langle (\alpha^* - \alpha)u'(t;q^*) + (A_{\beta^*} - A_\beta)u(t;q^*)$$
$$+(\delta^* - \delta)\sin u(t;q^*), p(q^*)\rangle. \tag{7.13}$$

Thus we obtain the following result.

Theorem 7.2. *The Gâteaux derivative of the objective function $J(q)$ has the following representation*

$$DJ(q^*; q - q^*) = (\alpha^* - \alpha)a(q^*) + (\beta^* - \beta)b(q^*) + (\delta^* - \delta)c(q^*), \tag{7.14}$$

where

$$a = -\frac{\partial J}{\partial \alpha} = -2\int_0^T (u_t(t,x;q^*), p(t,x;q^*)), \tag{7.15}$$

$$c = -\frac{\partial J}{\partial \delta} = -2\int_0^T (\sin(u(t,x;q^*)), p(t,x;q^*)), \tag{7.16}$$

and

$$b = -\frac{\partial J}{\partial \beta} = -2\int_0^T (\nabla u(t,x), \nabla p(t,x)), \tag{7.17}$$

The optimality condition $DJ(q^*; q - q^*) \geq 0$ *for any* $q \in \mathcal{P}$ *is*

$$(\alpha^* - \alpha)a(q^*) + (\beta^* - \beta)b(q^*) + (\delta^* - \delta)c(q^*) \geq 0 \tag{7.18}$$

for any $(\alpha, \beta, \delta) \in P$.

In addition, the optimal coefficient $q^* \in \mathcal{P}$ for nonzero (a, b, c) can be compactly written as

$$\alpha^* = \frac{1}{2}\{\text{sign}(a) + 1\}\alpha_{max} - \frac{1}{2}\{\text{sign}(a) - 1\}\alpha_{min}, \tag{7.19}$$

$$\beta^* = \frac{1}{2}\{\text{sign}(b) + 1\}\beta_{max} - \frac{1}{2}\{\text{sign}(b) - 1\}\beta_{min}, \tag{7.20}$$

and

$$\delta^* = \frac{1}{2}\{\text{sign}(c) + 1\}\delta_{max} - \frac{1}{2}\{\text{sign}(c) - 1\}\delta_{min} \tag{7.21}$$

for more detail see [5].

Now we have the following Theorem

Theorem 7.3. *If the optimal coefficient* q^* *is located in the interior* $int\,\mathcal{P}$ *of the admissible set* $\mathcal{P}$*, then*

$$a = 0, \quad b = 0, \quad \text{and} \quad c = 0 \quad \text{in} \quad \Omega.$$

Proof. In the interior of $\mathcal{P}$, $\frac{\partial J}{\partial \alpha} = \frac{\partial J}{\partial \beta} = \frac{\partial J}{\partial \delta} = 0$. Thus $a = b = c = 0$. □

Theorem 7.4. *Consider the sine-Gordon equation (1.1) with a constant diffusion coefficient β. Let the admissible set be*

$$\mathcal{P} = [\alpha_{min}, \alpha_{max}] \times [\beta_{min}, \beta_{max}] \times [\delta_{min}, \delta_{max}]$$

with $\beta_{min} > 0$.

Let the objective function be defined by

$$J(q) = k_1 |u(q;T) - z_d^1|^2 + k_2 \|u(q;t) - z_d^2\|^2_{L^2(0,T;H)}.$$

Then the mapping $q \to J(q)$ from $int\,\mathcal{P} \subset \mathbb{R}^3$ into $\mathbb{R}$ is differentiable. Its gradient $\nabla J(q) = (a,b,c)$, where a,b,c are defined in (7.22),(7.24), and (7.23). If the parameter $q^* \in int\,\mathcal{P}$ is optimal, then $\nabla J(q^*) = 0$.

Proof. To show that the mapping $q \to J(q)$ from $int\,\mathcal{P} \subset \mathbb{R}^3$ into $\mathbb{R}$ is differentiable it sufficies to show that $\nabla J(q) = (a,b,c)$ is continuous in $\mathcal{P}$ where

$$a = -\frac{\partial J}{\partial \alpha} = -2 \int_0^T (u_t(t,x;q^*), p(t,x;q^*)), \tag{7.22}$$

$$c = -\frac{\partial J}{\partial \delta} = -2 \int_0^T (\sin(u(t,x;q^*)), p(t,x;q^*)), \tag{7.23}$$

and

$$b = -\frac{\partial J}{\partial \beta} = -2 \int_0^T (\nabla u(t,x), \nabla p(t,x)), \tag{7.24}$$

Arguing as in Chapter 4, we can conclude that (7.11) has a unique weak solution $p \in W(0,T)$. Suppose $h(q^*) = \delta^* \cos(u(q^*)) - 1$ and $g(q^*) = k_2(u(q^*;t) - z_d^2)$. From Theorem 5.4 the mappings $q^* \to u(q^*)$, $q^* \to h(q^*)$, and $q^* \to g(q^*)$ from

$\mathcal{P}$ into $C([0,T]);V)$ are continuous, similarly the mapping $q^* \to u'(q^*)$ from $\mathcal{P}$ into $C([0,T]);H)$ is continuous. Continuity of $q^* \to p(q^*)$ $\mathcal{P}$ into $C([0,T]);V)$ and $q^* \to p'(q^*)$ $\mathcal{P}$ into $C([0,T]);H)$ can be proved similar as Theorem 5.4. Thus partial derivatives a, b, c defined in (7.22),(7.24), and (7.23) are continuous. Hence by [17] the mapping $q \to J(q)$ from $int\,\mathcal{P} \subset \mathbb{R}^3$ into $\mathbb{R}$ is differentiable. $\square$

Chapter 8

Computational Algorithm

In this chapter we discuss the computational algorithm to find the approximate solutions of (3.4). As mentioned in 2.5, let $\{w_j\}_{j=1}^{\infty}$ be eigenfunctions of $-\beta\Delta + I$ that form an orthonormal basis in H. Then $\{\frac{w_j}{\sqrt{\lambda_j}}\}_{j=1}^{\infty}$ is an orthonormal basis on V as in Chapter 3 . Fix $N \in \mathbb{N}$. Let $V_N = span\{w_1, w_2, ..., w_N\}$. Let $P_N : H \to V_N$ be the projection operator defined by $P_N v = \sum_{j=1}^{N}(v, w_j)w_j$ for any $v \in H$. As defined in Chapter 4, the approximate solution of (3.4) is

$$u_N(t,x) = \sum_{j=1}^{N} g_{jN}(t)w_j(x) \tag{8.1}$$

that satisfies

$$\begin{aligned} &\frac{d^2}{dt^2}(u_N, w_j) + \alpha\frac{d}{dt}(u_N, w_j) + a_\beta(u_N, w_j) + \delta(\sin(u_N), w_j) = (f, w_j) + (u, w_j) \\ &u_N(0) = P_N u_0 \quad \text{and} \quad \frac{d}{dt}u_N(0) = P_N u_1 \quad \text{for} \quad \text{any} \quad j \in \mathbb{N}. \end{aligned} \tag{8.2}$$

Let $\bar{g}_N = \{g_{jN}\}_{j=1}^N \in \mathbb{R}^N$. We can rewrite (8.2) as the following vector differential equation

$$\bar{g}''_N(t) + \alpha \bar{g}'_N(t) + \beta \Lambda \bar{g}_N(t) = \bar{F}(t, \bar{g}_N) \tag{8.3}$$

with the initial data

$$\vec{g}_N(0) = \begin{bmatrix} (P_N u_0, w_1) \\ (P_N u_0, w_2) \\ . \\ . \\ . \\ (P_N u_0, w_N) \end{bmatrix} = \begin{bmatrix} \int_0^1 u_0 dx \\ \sqrt{2} \int_0^1 u_0 \cos(\pi x) dx \\ . \\ . \\ . \\ \sqrt{2} \int_0^1 u_0 \cos((N-1)\pi x) dx \end{bmatrix}.$$

and

$$\vec{g}'_N(0) = \begin{bmatrix} (P_N u_1, w_1) \\ (P_N u_1, w_2) \\ . \\ . \\ . \\ (P_N u_1, w_N) \end{bmatrix} = \begin{bmatrix} \int_0^1 u_1 dx \\ \sqrt{2} \int_0^1 u_1 \cos(\pi x) dx \\ . \\ . \\ . \\ \sqrt{2} \int_0^1 u_1 \cos((N-1)\pi x) dx \end{bmatrix}.$$

where $u_0 \in L^2(0,T;V)$ and $u_1 \in L^2(0,T;H)$.

Here,

$$\vec{g}_N(t) = \begin{bmatrix} g_{1N}(t) \\ g_{2N}(t) \\ . \\ . \\ . \\ g_{NN}(t) \end{bmatrix} \in \mathbb{R}^N.$$

As in Chapter 4, define

$$\vec{F}(t, \bar{g}_N) = \begin{bmatrix} (f(t), w_1) + (u_N, w_1) - \delta(\sin(u_N), w_1) \\ (f(t), w_2) + (u_N, w_2) - \delta(\sin(u_N), w_2) \\ . \\ . \\ . \\ (f(t), w_N) + (u_N, w_N) - \delta(\sin(u_N), w_N) \end{bmatrix}.$$

Write

$\vec{F}(t, \bar{u}_M) = \bar{U} + \bar{V} - \bar{W}$, where

$$\vec{U} = \begin{bmatrix} (f(t), w_1) \\ (f(t), w_2) \\ . \\ . \\ . \\ (f(t), w_N) \end{bmatrix} = \begin{bmatrix} \int_0^1 f(t)dx \\ \sqrt{2}\int_0^1 f(t)\cos(\pi x)dx \\ . \\ . \\ . \\ \sqrt{2}\int_0^1 f(t)\cos((N-1)\pi x)dx \end{bmatrix}$$

$$\vec{V} = \begin{bmatrix} (u_N, w_1) \\ (u_N, w_2) \\ . \\ . \\ . \\ (u_N, w_N) \end{bmatrix} = \begin{bmatrix} g_{1N}(t) \\ g_{2N}(t) \\ . \\ . \\ . \\ g_{NN}(t) \end{bmatrix}$$

and

$$\vec{W} = \begin{bmatrix} \delta(\sin\ u_N, w_1) \\ \delta(\sin\ u_N, w_2) \\ . \\ . \\ . \\ \delta(\sin\ u_N, w_N) \end{bmatrix} = \begin{bmatrix} \delta\int_0^1 \sin\ (\sum_{j=1}^N g_{jN}(t)w_j(x))w_1(x)dx \\ \delta\int_0^1 \sin\ (\sum_{j=1}^N g_{jN}(t)w_j(x))w_2(x)dx \\ . \\ . \\ . \\ \delta\int_0^1 \sin\ (\sum_{j=1}^N w_{jN}(t)w_j(x))w_N(x)dx \end{bmatrix},$$

$$\Lambda = \begin{bmatrix} 1 & 0 & 0 & . & . & . & 0 \\ 0 & 1+(\pi)^2 & 0 & . & . & . & 0 \\ 0 & 0 & 1+(2\pi)^2 & & & & \\ . & . & . & & & & \\ 0 & 0 & 0 & . & . & . & 1+((N-1)\pi)^2 \end{bmatrix}.$$

Let $\bar{Z}_1(t) = \bar{g}_N(t)$ and $\bar{Z}_2(t) = \bar{g}'_N(t)$. Then the initial value problem (8.3) can be reduced into the following system of first order ODEs

$$\begin{aligned} &\bar{Z}'_1(t) = \bar{Z}_2(t) \\ &\bar{Z}'_2(t) = -\alpha \bar{Z}_2(t) - \beta \Lambda \bar{Z}_1(t) + \bar{F}(t, \bar{u}_N) \\ &\bar{Z}_1(0) = \bar{g}_N(0), \quad \bar{Z}_2(0) = \bar{g}'_N(0). \end{aligned} \tag{8.4}$$

The approximate solution of (3.6) is

$$u_N(t,x) = \sum_{j=1}^{N} g_{jN}(t) \sqrt{2} \cos((j-1)\pi x). \tag{8.5}$$

Now we compute the approximate solution of the adjoint system

$$\begin{aligned} &p'' - \alpha^* p' + A^*_\beta p + (\delta^* \cos(u(q^*) - 1))p = k_2(u(q^*;t) - z_d^2) \\ &p(T) = 0, \quad p'(T) = k_1(u(q^*;T) - z_d^1). \end{aligned} \tag{8.6}$$

Let $p(T-s,x) = w(s,x)$ for any $x \in (0,1)$, then we have $p_t(T-s,x) = -w_s(s,x)$

and $p_{tt}(T-s,x) = w_{ss}(s,x)$. The adjoint system (8.6) can be written as

$$\begin{aligned} &w'' + \alpha w' + A_\beta w + (\delta\cos(u(q)-1))w = k_2(u(q;t) - z_d^2) \\ &w(0,x) = 0 \quad w'(0,x) = k_1(u(q^*;T) - z_d^1). \end{aligned} \tag{8.7}$$

The approximate solution of the adjoint system (8.7) is given by

$$\begin{aligned} &\langle y_N'', w_k\rangle + \alpha(y_N', w_k) + (A_\beta y_N, w_k) + \delta(P_N\cos(u_N(q))y_N, w_k) \\ &= (k_2 P_N(u_N(q;t) - z_d^2), w_k) + (y_N, w_k) \\ &y_N(0) = Q_N 0, \quad y_N'(0) = P_N k_1(u(q^*;T) - z_d^1) \end{aligned} \tag{8.8}$$

where $y_N = \sum_{j=1}^N h_j(t) w_j(x)$.

Equation (8.8) is equivalent to the following vector differential equation

$$\bar{h}_N''(s) + \alpha\bar{h}_N'(s) + \beta\Lambda\bar{h}_N(s) = \bar{H}(s, \bar{h}_N) \tag{8.9}$$

with the initial data

$$\vec{h}_N(0) = \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ . \\ 0 \end{bmatrix}$$

and

$$\vec{h}'_N(0) = \begin{bmatrix} (P_N k_1(u(q;T) - z_d^1), w_1) \\ (P_N k_1(u(q;T) - z_d^1), w_2) \\ . \\ . \\ . \\ (P_N k_1(u(q;T) - z_d^1), w_N) \end{bmatrix} = \begin{bmatrix} \int_0^1 (u(q;T) - z_d^1) dx \\ \sqrt{2} \int_0^1 (u(q;T) - z_d^1) \cos(\pi x) dx \\ . \\ . \\ . \\ \sqrt{2} \int_0^1 (u(q;T) - z_d^1) \cos((N-1)\pi x) dx \end{bmatrix}.$$

Here,

$$\vec{h}_N(s) = \begin{bmatrix} h_1(s) \\ h_2(s) \\ . \\ . \\ . \\ h_N(s) \end{bmatrix} \in \mathbb{R}^N.$$

As in Chapter 4, define

$$\vec{H}(s, \bar{h}_N) = \begin{bmatrix} (P_N k_2(u_N(q;t) - z_d^2), w_1) + (h_N, w_1) - (P_N \delta(\cos(u_N) h_N, w_1) \\ (P_N k_2(u_N(q;t) - z_d^2), w_2) + (h_N, w_2) - (P_N \delta(\cos(u_N) h_N, w_1) \\ . \\ . \\ . \\ (P_N k_2(u_N(q;t) - z_d^2), w_N) + (h_N, w_N) - (P_N \delta(\cos(u_N) h_N, w_1) \end{bmatrix},$$

write

$$\vec{H}(t, \bar{w}_N) = \bar{A} + \bar{B} + \bar{C},$$

where

$$\vec{A} = \begin{bmatrix} (P_N k_2(u(q;t) - z_d^2), w_1) \\ (P_N k_2(u(q;t) - z_d^2), w_2) \\ \cdot \\ \cdot \\ \cdot \\ (P_N k_2(u(q;t) - z_d^2), w_N) \end{bmatrix} = \begin{bmatrix} \int_0^1 (u(q;t) - z_d^2) dx \\ \sqrt{2} \int_0^1 (u(q;t) - z_d^2) \cos(\pi x) dx \\ \cdot \\ \cdot \\ \cdot \\ \sqrt{2} \int_0^1 (u(q;t) - z_d^2) \cos((N-1)\pi x) dx \end{bmatrix},$$

$$\vec{B} = \begin{bmatrix} (y_N, w_1) \\ (y_N, w_2) \\ \cdot \\ \cdot \\ \cdot \\ (y_N, w_N) \end{bmatrix} = \begin{bmatrix} h_1(s) \\ h_2(s) \\ \cdot \\ \cdot \\ \cdot \\ h_N(s) \end{bmatrix},$$

and

$$\vec{C} = \begin{bmatrix} (P_N\delta(\cos\ u_N)y_N, w_1) \\ (P_N\delta(\cos\ u_N)y_N, w_2) \\ . \\ . \\ . \\ (P_N\delta(\cos\ u_N)y_N, w_N) \end{bmatrix} = \begin{bmatrix} \delta\cos\ u_N h_1 \\ \delta\cos\ u_N h_2 \\ . \\ . \\ . \\ \delta\cos\ u_N h_N \end{bmatrix}.$$

Chapter 9

Numerical results

For our numerical experiments we choose to use a Fourier series method for the solution of the sine-Gordon equation (1.1), and MATLAB function *fminicon* for the minimization of the cost functional. As described in Chapter 2 eigenfunctions of the operator A_β, $w_j = \cos(\pi(j-1)x), j = 1, 2, ...$, are chosen as an orthonormal basis in H. As described in Chapter 8, let $P_N : H \to V_N$ be the projection operator defined from H onto $V_N = \text{span}\{w_1, w_2, ..., w_N\}$. Expanding the functions in (4.13) into the Fourier cosine series we have

$$\begin{aligned} &g_k'' + \alpha g_k' + \beta_k g_k + \delta S_k = F_k \\ &g_k(0) = P_N u_0, \quad g_k'(0) = P_N u_1, \end{aligned} \tag{9.1}$$

where $\beta_k = \beta[1 + (\pi(k-1))^2]$, $g_k(t), F_k(t), P_N u_0$ and $P_k u_1$ are the Fourier coefficients of the solution $u_N(t)$ in (4.13). Similarly $S_k(t)$ is the Fourier cosine coefficient of $P_N \sin(u_N)(t)$. The cost functional $J_N(q)$ can be written as

$$J_N(q) = k_2 \sum_{i=1}^{M} \sum_{k=1}^{N} [Y_k(q;t_i) - Z(t_i)]^2 + k_1 \sum_{k=1}^{N} [Y_k(q;T) - Z(T)]^2, \tag{9.2}$$

where $k_1 + k_2 > 0$ and $Z(t_i)$ for $i = 1, 2, ..., T$ are observations for the parameter set $\bar{q} = (\bar{\alpha}, \bar{\beta}, \bar{\delta})$.

In all the numerical experiments we used observation times $t_j = T.j/K$ where $j = 0, 1, 2, ..., K$ and $T = 4$. The model values are specified in the following table

Table 9.1: Parameter values for numerical simulations

Time and spatial intervals	$[0, T] \times [0, 1] = [0, 4] \times [0, 1]$
Admissible set	$\mathcal{P}_{ad} = [0.1, 1] \times [0.1, 1] \times [0, 2]$
Initial conditions	$u_0(x) = \sin(\pi x), \quad u_1(x) = x$
Forcing function	$f(t, x) = 1$
Dimension of system of ODE = N	64
Number of Partitions in [0,4] = M	64
Number of Partitions in [0,1] = K	128

To simulate the data $z_d^1(T, x)$ and $z_d^2(t, x)$, let $\bar{q} = (.2, .2, .3) \in \mathcal{P}_{ad}$ be the set of test parameters. Numerical solution of (1.1) is computed by using 4th order Runge-Kutta method. Since real data always contain some noise, we set

$$z_d(t, x) = u(\bar{q}; t, x) + \epsilon\gamma(x), \tag{9.3}$$

where ϵ is noise level and $\gamma(x)$ is a random variable uniformly distributed on interval [-.5,.5].

Let $q_0 \in \mathcal{P}_{ad}$ be an arbitrary chosen set of parameters. A MATLAB function called *fminicon* is used for minimization of the cost functional J_N. The minimizers q_N^*, minimum values of functional $J_N(q_N^*)$, and error

$$E = \frac{||q^* - \bar{q}||_{\mathbb{R}^3}}{||\bar{q}||_{\mathbb{R}^3}}$$

at different noise levels ϵ are given in the following tables. The first row of each table shows that the identification algorithm is successful for data z_d without noise, whereas the precision of the identification decreases with the increasing noise level. Without loss of generalities we can assume that $k_2 = 1$ in all the examples. Our experiments revealed that for $\epsilon = 0$, identification algorithm is successful for any k_1. For $\epsilon = 0.001$, the best identification is achieved for $k_1 = 1$, and for $\epsilon = 0.01$, the best identification is achieved for $k_1 = 2$.

Table 9.2: Identification results for $k_1 = 0$ and $k_2 = 1$

ϵ	q_N^*	$J_N(q_N^*)$	E
0	(0.1998, 0.1996, 0.3017)	9.7130e-008	0.0041
0.001	(0.1945, 0.1991, 0.2726)	0.0029	0.0679
0.01	(0.2737, 0.2751, 0.1910)	0.3458	0.3674

Table 9.3: Identification results for $k_1 = 1$ and $k_2 = 1$

ϵ	q_N^*	$J_N(q_N^*)$	E
0	(0.2001, 0.2001, 0.3000)	1.7996e-007	2.1820e-004
0.001	(0.2056, 0.2040, 0.3031)	0.0155	0.0182
0.01	(0.1218, 0.1470, 0.2870)	1.6254	0.2312

Table 9.4: Identification results for $k_1 = 2$ and $k_2 = 1$

ϵ	q_N^*	$J_N(q_N^*)$	E
0	(0.2000, 0.2000, 0.3000)	2.7806e-007	1.2957e-004
0.001	(0.2017, 0.1997, 0.3100)	0.0293	0.0245
0.01	(0.2077, 0.2096, 0.2745)	3.1094	0.0687

Table 9.5: Identification results for $k_1 = 25$ and $k_2 = 1$

ϵ	q_N^*	$J_N(q_N^*)$	E
0	(0.2000, 0.2000, 0.3000)	2.2272e-007	7.4062e-005
0.001	(0.2013, 0.2026, 0.2905)	0.1534	0.0242
0.01	(0.1901, 0.1887, 0.3541)	14.0577	0.1362

Table 9.6: Identification results for $k_1 = 50$ and $k_2 = 1$

ϵ	q_N^*	$J_N(q_N^*)$	E
0	(0.2000, 0.2000, 0.3000)	2.3466e-007	5.4141e-005
0.001	(0.2001, 0.2022, 0.2925)	0.3265	0.0190
0.01	(0.1735, 0.1713, 0.3546)	31.3486	0.1628

Figure 9.1: Data z_d for noise level $\epsilon = 0.00$

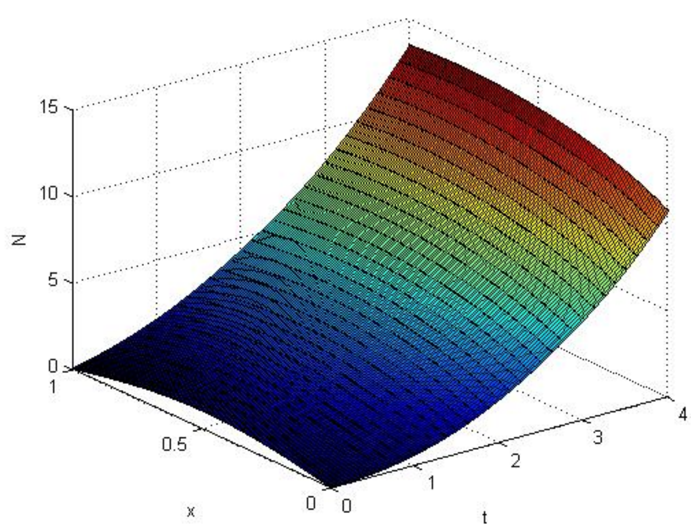
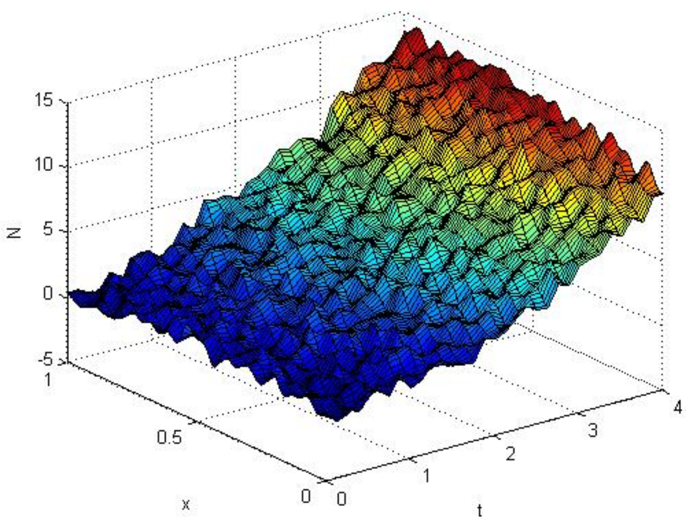


Figure 9.2: Data z_d for noise level $\epsilon = 0.01$



Chapter 10

Conclusions

In this thesis we proved existence and uniqueness of the weak solution of damped sine-Gordon equation with Neumann boundary condition. We showed that the weak solution is continuous with respect to the parameters. Weak Gâteaux differentiability of the solution is established by using the method of transposition by Lions and Magenes [8]. Weak Gâteaux differentiability of the solution map is used to establish the Gâteaux differentiability of the cost functional J. An adjoint system is established and used to represent the Gâteaux derivative of the cost functional J. We proved that the partial derivatives $\frac{\partial J}{\partial \alpha}$, $\frac{\partial J}{\partial \beta}$, and $\frac{\partial J}{\partial \delta}$ are 0 when optimal parameter $q^* \in int\mathcal{P}$. Continuity of partial derivatives with respect to α, β, and δ is used to prove differentiability of cost functional J on the admissible set of parameters $\mathcal{P}_{ad}$.

In addition, we developed a computational algorithm for approximate solutions of the adjoint system. A Fourier method is used to compute numerical solution of the sine-Gordon equation (1.1). MATLAB function *fminicon* is used for the minimization of the cost functional J. Our experiments showed that the identification algorithm is successful for data without noise, whereas the precision of identification decreases with the increasing noise level. In addition, our experiments revealed that for $\epsilon = 0$, identification algorithm is successful for any k_1. For $\epsilon = 0.001$, the best identification is achieved for $k_1 = 1$, and for $\epsilon = 0.01$, the best identification is achieved for $k_1 = 2$.

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