THE UNIVERSITY OF OKLAHOMA

GRADUATE COLLEGE

The Casimir Energy and $\mathcal{PT}$-Symmetric Theories:

The Dielectric Cylinder and Unitarity of $\mathcal{PT}$-Symmetric QED

A Dissertation

SUBMITTED TO THE GRADUATE FACULTY

in partial fulfillment of the requirements
for the degree of

Doctor of Philosophy

by
INÉS CAVERO-PELÁEZ

Norman, Oklahoma
2005

UMI Number: 3178483

UMI Microform 3178483

ProQuest Information and Learning Company
300 North Zeeb Road
P.O. Box 1346
Ann Arbor, MI 48106-1346

The Casimir Energy and
$\mathcal{PT}$-Symmetric Theories:
The Dielectric Cylinder and Unitarity of $\mathcal{PT}$-Symmetric QED

A Dissertation Approved for the
Department of Physics and Astronomy

by

Kimball A. Milton (Chair)

Edward Baron

Phillip Gutierrez

Chung Kao

Darryl McCullough

Patrick Skubic

Acknowledgments

Just by mere chance, my first contact at OU was with Prof. Kimball A. Milton. He seemed to be very enthusiastic about physics and I was not wrong. When in my first day he drove me to the Quantum Field Theory class which had started ten minutes before, I still didn't know that he would become my advisor and I would have the luck to experience on first hand the view of one of those few *who could be called scientist.* I guess it is his fascination for discovering and will to learn that provides him with the fuel which drives him to an amazing reasoning and understanding. If Prof. Milton opens his mouth, you learn something!

I want to thank him not just for the physics he has taught me, but also for what I have learned by observing him. His critical thinking has been a great help to open my own mind. I also thank him because no matter how ridiculous my conclusions would be, he always gave them a thought and his comment was always productive. I would also like to thank his other students with whom we were having regular meetings; Kuloth Shajesh, who has been there since the beginning, Prachi Parashar who joined one year ago and Jeffrey Wagner who has been with us for few months. These meetings have been the best way to learn and make progress in our work.

I would like to mention Shankar Sachithanandam, just one desk away, with whom I had long conversations about physics and other matters.

I also want to thank the rest of my committee members. In particular, I would like to name Dr. Leonid Dickey whom I admire for his knowledge and interest in a wide variety of topics but, unfortunately, for an inevitable reason, he could not be present at the defense. Thanks to Dr. Darryl McCullough who kindly accepted a last minute call.

Finally I want to thank those I love, here and in Spain, who gave me continuous support. I don't need to mention their names because they know I am talking about them, but I cannot avoid to say that I have the most wonderful grandparents one can have, Miguel and Enar, and of course Magencio and Trini who are, and will always be, alive in my mind.

Contents

Chapter 1

About the Casimir Effect

1.1 Introduction

Since the beginning of the Casimir effect, the equivalence between the action at a distance between molecules (the van der Walls forces) and the local action of fields was established; the van der Walls interaction could be interpreted as a manifestation of the zero-point energy of the quantized fields.

In this fashion, Casimir calculated the interaction between neutral parallel conducting plates (1948). Let's say in the ideal case, when there are no real photons or any charges on the plates, only the disturbance of the vacuum of the electromagnetic field due to the boundaries is causing the interaction. The result of his calculation was a positive force between the plates, meaning that the plates attract each other [1].

However, not much importance was given to this effect until Boyer completed his calculation in 1968 [2]. Hoping to find an attractive force due to the vacuum fluctuations as suggested by Casimir in [1], he computed the Casimir force for a conducting sphere. The surprise came when he found that the self-energy in this case was repulsive. This was a very unexpected result and since then, nobody has been able to predict the outcome of the Casimir effect of a particular geometry without an explicit calculation. There is no general algorithm to predict the sign of the force.

The result for the sphere (as well as the parallel plates) has been reproduced several times and extended to the case of a dielectric sphere [3, 4], which result in the dilute limit is in perfect agreement with the one coming from the sum of the van de Walls interactions [5, 6].

The case of the cylinder has also been considered for the situation where the speed of light is the same on the inside and the outside of the body. However the case when c is differing on both media is more intractable. After the result obtained from the sum of the van der Walls interaction [7] it has been a challenge to be able to reproduce the zero Casimir energy for this configuration from the perspective of the fluctuations of the fields. That is the goal of the next pages.

We will give a brief overview of the Casimir effect and the zero-point energy but it is not the goal of the author to give a review of the subject; for that there are excellent sources [8, 9, 10, 11, 12].

The experimental development of the Casimir effect has been slower than the theoretical one. Different factors have made it a challenge to verify the existence of the Casimir force. These forces are very small and depend on the distance. For two parallel plates separated by a distance a, the force per unit area is given by

$$F = -\frac{\pi^2}{240}\frac{\hbar c}{a^4} = -1.30 \times 10^{-27} M\, m^2\, a^{-4}. \tag{1.1}$$

The small distances required imply high sensitivity in the experimental techniques and great accuracy in any measurement.

The first attempt to experimentally verify the results found by Casimir in 1948 took place ten years later. Sparnaay [13] set an experiment to measure the Casimir force between two parallel metal plates using a technique based on a spring balance. This was the first indication of the attraction force between the plates even though the precision of the measurements was not good enough and the results could be considered somewhat inconclusive. The first review reporting experimental developments in the Casimir effect was done by Sparnaay himself and Sarlemijn [14].

The first experiment considered successful in the new generation of experimental techniques was carried out by Lamoreaux [15]. He used a torsion pendulum and flat plates coated with Cu and Au. The results of this experiment were very successful; even though temperature corrections where not considered, there was no doubt left on the existence of the Casimir force. For details about this and several other experiments you can refer to the last part of the review by Bordag et al. [9].

1.2 The zero-point energy

Historically [16] the concept of zero-point energy first appeared with Max Planck in 1911. He hypothesized that the emission of radiation is discrete and the unit of emitted energy is $h\nu$. The average energy of a harmonic oscillator at temperature T was given by

$$\bar{E}(\nu, T) = \frac{h\nu}{2} + \frac{h\nu}{e^{\frac{h\nu}{kT}} - 1}, \tag{1.2}$$

where h is the Planck constant and ν is the frequency. This implied the existence of an energy of the harmonic oscillator at zero temperature. It was later, by means of the Heisenberg uncertainty principle in quantum mechanics, that the same idea would appear again, this time arising as a consequence of the theory itself. In quantum physics the uncertainty principle expresses a limitation on the accuracy of simultaneous measurement of observables

$$\Delta p \, \Delta q \geq \frac{\hbar}{2}. \tag{1.3}$$

This relation arises between any two observable quantities defined by non-commuting operators such as position and momentum,

$$[p, q] = \frac{\hbar}{i}. \tag{1.4}$$

Its immediate consequence is the inability in determining the state of a particle with precision in classical terms; trying to determine the position of a particle makes its momentum more uncertain. This gives rise to a discrete harmonic oscillator spectrum of energy

$$E_n = \hbar\omega(n + \frac{1}{2}) \qquad n = 0, 1, 2, ..., \tag{1.5}$$

which ground state $n = 0$ has non-zero energy[1]. It reflects the fact that if the particle is confined in a region of space (by the effect of a potential for example) by the uncertainty principle, its momentum cannot have a definite value, in particular the momentum cannot be zero since $\Delta q \sim \frac{\hbar}{2\Delta p}$, and therefore the energy cannot vanish in any state. The value of the ground state level has been called the *zero-point energy.*

[1]The constant spacing between successive levels, $\hbar\omega = h\nu$ is exactly what Planck had postulated.

This energy is indeed infinite since we are summing over an infinite number of states. However, it has always been considered as irrelevant since it is a constant and we can only measure changes in energy due to transitions between different quantum states. A simple shift in the origin of the energy is enough to disregard this infinity in the ground state. The spontaneous emission that Planck had hypothesized can be regarded as induced by the *zero-point oscillations* of the electromagnetic field; a purely quantum phenomenon.

We encounter the same in quantum field theory, this time in regard with the quantum fluctuations of the fields at each point. The ground state contribution of the energy $\frac{1}{2}\hbar\omega$, when summed over all modes, gives infinite energy. This is known as an ultraviolet divergence. The way to deal with this infinity in QFT is by renormalization techniques based on the idea described above that only differences in energy are measurable within a quantum system. However, *it may be that the system is subject to external conditions or boundaries and the value of the frequency ω depends upon them.*

When you confine the space between two parallel plates the modes of the fluctuations of the fields change, creating some pressure on the plates that gives rise to a measurable force; this effect is known as the Casimir Effect. The geometry of the boundary (parallel plates in the above example) plays an important role since it determines the new frequency states. The appearance of the new modes gives rise to a measurable energy after subtracting counterterms such as the ground-state energy of the fields in the absence of the boundary. For example, in the case of the parallel plates, the modes normal to the surface get discretized originating a vacuum energy density inside the plates that differs from that of the outside producing *an attractive* *net force* between the plates. If we talk about a conducting sphere, the same effect creates a *repulsive force.*

Later we will see in a detailed way how to deal with this kind of calculation in a cylindrical topology.

1.3 Approaches

We can compute the Casimir effect in many different ways. The most immediate one is to calculate half the sum of all the energy modes of the ground state subject to a particular topology and with the appropriate boundary conditions,

$$E_0 = \sum_{\text{modes}} \frac{1}{2}\hbar\omega, \tag{1.6}$$

obviously this is very divergent and therefore it seems clear that we need to find a procedure to regulate the vacuum energy.

There is not a consistent general-renormalization technique for removing the divergences occurring from the computation of the energy in a compact space subject to boundary conditions. For each specific geometry one has to develop a particular renormalization procedure. Next, we will briefly mention different mathematical methods to remove divergences in the Casimir calculations and illustrate them for the case of the self energy of the dielectric cylinder. Special attention is given to the calculation where the source theory is applied. By the use of the Green's functions we will derive the general expression for the dielectric cylinder with the speed of light differing on the inside and outside and the particular case for the dilute cylinder. The Green's functions give us the vacuum expectation value of the product of the fields that allows us to calculate the energy momentum tensor. Divergences appear because we need to compute expressions at the same space-time point; however, this approach gives a physically unambiguous understanding of the problem and after regulation we are able to subtract the finite contribution and identify the divergences. A whole section will be devoted to this.

1.3.1 The zeta function technique

This is essentially based on the definition of the Riemann zeta function,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad \mathrm{Re}\, s > 1. \tag{1.7}$$

An extensive study of the zeta-function can be found in [17]. In order to use this regularization technique we need to know the spectrum of the case under consideration. For simplicity, let's consider a scalar field in certain dimension N, satisfying the differential equation,

$$\boldsymbol{\mathcal{P}}\Phi_n(x) = \Lambda_n \Phi_n(x), \tag{1.8}$$

where n is the set $n = n_1, n_2, ..., n_N$ with $n_i = 1, 2, 3, ...$ and x is the N-dimension spatial coordinates $x = (x_1, x_2, ..., x_N)$. The operator $\boldsymbol{\mathcal{P}}$ is,

$$\boldsymbol{\mathcal{P}} = \mathbf{P_n} + V(x_n), \tag{1.9}$$

with $\mathbf{P_n}$ an elliptical differential operator in the spatial coordinates, for example the Laplacian.

If one is able to solve the differential equation

$$
\left(\mathbf{P_n} + V(x_n)\right)\Phi_n(x_n) = \Lambda_n \Phi_n(x_n), \tag{1.10}
$$

the vacuum energy takes the form,

$$
E_0 = \frac{\hbar}{2}\sum_n \Lambda_n, \tag{1.11}
$$

where $\Lambda_n = \lambda_{n_1} + ... + \lambda_{n_N}$ for the special case when the differential equation is separable in Cartesian coordinates. The above energy is in general very divergent but it can be regulated as [2]

$$
E_0(s) = \frac{\hbar}{2}\sum_n \Lambda_n^{-s}. \tag{1.12}
$$

This is the spectral zeta function which we can compute in the allowed region and then, by analytic continuation, extend the definition of the zeta function $\zeta(s)$ to the whole complex plane. We can conclude then that the vacuum energy is

$$
E_0 = \frac{\hbar}{2}\zeta(s=-1). \tag{1.13}
$$

To make this more transparent, let's assume we can separate the differential equation in Cartesian coordinates and that we are interested in just one direction, let's say in the z-direction. In this case we can Fourier transform the rest of the components so that

$$
\Phi(x) = \int_{-\infty}^{\infty} \frac{d\mathbf{k}_\perp}{(2\pi)^{N-1}} e^{i(\mathbf{k}_\perp . \mathbf{x}_\perp)}\phi(z), \tag{1.14}
$$

[2]We require that Λ_n never takes the value 0.

and the vacuum energy would now read,

$$E_0 = \frac{\hbar}{2}\sum_{n_z}\int_{-\infty}^{\infty}\frac{dk_1,...,dk_{N-1}}{(2\pi)^{N-1}}\sqrt{k_\perp^2+\lambda_{n_z}}, \tag{1.15}$$

where $k_\perp^2 = k_1^2 + k_2^2 + ... + k_{N-1}^2$, and $\phi(z)$ satisfies

$$\left(\frac{d^2}{dz^2} - k_\perp^2 + V(z)\right)\phi(z) = \lambda_{n_z}\phi(z). \tag{1.16}$$

The energy E_0 as it reads above diverges badly, therefore we regulate it. One way is using the zeta function. For that we do:

$$E_0(s) = \frac{\hbar}{2}\sum_{n_z}\int_{-\infty}^{\infty}\frac{dk_1...dk_{N-1}}{(2\pi)^{N-1}}(k_\perp^2+\lambda_{n_z})^{-s/2}. \tag{1.17}$$

We should be able to compute that integral by making use of the change $k_\perp^2 \to k_\perp^2\lambda_{n_z}$ and using polar coordinates. We write then,

$$\begin{aligned} E_0(s) &= \frac{\hbar}{2}\sum_{n_z}\int_{-\infty}^{\infty}\frac{(dk_\perp)}{(2\pi)^{N-1}}(k_\perp^2+1)^{-s/2}\lambda_{n_z}^{(N-1)/2-s/2} \\ &= \frac{\hbar}{2}\sum_{n_z}\int\frac{d^N\Omega}{(2\pi)^{N-1}}\int_0^{\infty}dk_\perp\, k_\perp^{N-2}(k_\perp^2+1)^{-s/2}\lambda_{n_z}^{(N-1-s)/2}, \end{aligned} \tag{1.18}$$

where $(dk_\perp) = dk_1dk_2...dk_{N-1}$. We arrive at a generalized zeta-function expression,

$$E_0(s) = \frac{\hbar}{2}A(s)\sum_{n_z}\lambda_{n_z}^{(N-1-s)/2}. \tag{1.19}$$

For example, for the case of the parallel plates with Dirichlet boundary conditions,

$\lambda = \frac{n^2\pi^2}{a^2}$ where a is the separation between the plates. For $N = 3$, equation (1.18) becomes,

$$\begin{aligned} E_0(s) &= -\frac{\hbar}{6\pi}\left(\frac{\pi}{a}\right)^{2-s}\sum_n \frac{1}{n^{s-2}} \\ &= -\frac{\hbar}{a^{2-s}}\frac{\pi^{1-s}}{6}\zeta(s-2), \end{aligned} \tag{1.20}$$

that evaluated at $s = -1$ and using $\zeta(-3) = 1/120$, we find that the energy per unit area is

$$E_0 = -\frac{\hbar}{a^3}\frac{\pi^2}{720}. \tag{1.21}$$

Casimir used an exponential regulator $e^{\delta\omega}$ with small ω to find the exact same value for the energy.

As noted above, in order to compute the mode summation we need to know what the eigenfrequencies for the specific case of interest are. Later we will see in detail how to calculate the function that gives us these for an infinite dielectric cylinder in a medium with permittivity and permeability ϵ', μ' while in the inside of the cylinder those are ϵ, μ, but for now we will just give the expression for the special case where the surrounding medium is vacuum, $\varepsilon' = 1, \mu' = 1$. With this topology, the eigenfrequencies of the electromagnetic oscillations are given by the roots of the equation [18]

$$f_m(k_z, \omega) = 0, \tag{1.22}$$

where

$$f_m(k_z, \omega) = -\Delta^2\left[(1-\varepsilon\mu)^2\frac{m^2k_z^2\omega^2}{\lambda^2\lambda'^2}J_m^2(\lambda a)H_m^2(\lambda' a) - D\tilde{D}\right]. \tag{1.23a}$$

$$D = \varepsilon\lambda' a J_m'(\lambda a) H_m(\lambda' a) - \lambda a H_m'(\lambda' a) J_m(\lambda a), \tag{1.23b}$$

$$\tilde{D} = \mu\lambda' a J_m'(\lambda a) H_m(\lambda' a) - \lambda a H_m'(\lambda' a) J_m(\lambda a). \tag{1.23c}$$

and

$$\lambda^2 = \varepsilon\mu\omega^2 - {k_z}^2, \tag{1.24}$$

with the equivalent expression for λ' with $\varepsilon' = 1$ and $\mu' = 1$.

In the above $\Delta = -\frac{2i}{\pi}$, the $D, \tilde{D}$ correspond to the transverse magnetic and electric modes respectively and J_m and H_m are Bessel function with H_m being the Hankel function of the first kind and $m = 0, \pm 1, \pm 2, \ldots$.

From (1.24) it is straight-forward to see that the frequencies are given by

$$\omega = a^{-1}(y^2 + k_z^2 a^2)^{1/2}, \quad y = \lambda' a, \tag{1.25}$$

and an easy expansion of $x = \lambda a$ in terms of $(\varepsilon - 1)$ allows us to write

$$x^2 = y^2 + (\varepsilon - 1)(y^2 + k_z^2 a^2). \tag{1.26}$$

Because of the presence of the Bessel functions, each root ω of (1.23a), characterizing the natural modes of propagation, has two indices. Each Bessel function of index m has an infinite number of discrete p. Therefore any root of (1.23a) can be designated as ω_{mp} and has dependence on k_z. The Casimir energy per unit length can then be given by

$$E = \frac{\hbar}{2}\int_{-\infty}^{\infty}\frac{dk_z}{2\pi}\sum_{m,p}\omega_{m,p}(k_z). \tag{1.27}$$

We can regulate this expression by adding a power $(-s)$ to the sum,

$$\begin{aligned} E(s) &= \frac{\hbar}{2}\int_{-\infty}^{\infty}\frac{dk_z}{2\pi}\sum_{m,p}\omega_{m,p}^{-s}(k_z) \\ &= \frac{\hbar}{2}a^s\int_{-\infty}^{\infty}\frac{dk_z}{2\pi}\sum_{m,p}(y_{m,p}^2+k_z^2a^2)^{-\frac{s}{2}}, \end{aligned} \tag{1.28}$$

so that the Casimir energy is expressed in terms of the zeta function

$$E(s) \propto \zeta(s/2=-1/2). \tag{1.29}$$

One of the sums in (1.28) can be turned into an integral by virtue of the argument principle,

$$E(s) = \frac{\hbar}{2}a^s\int_{-\infty}^{\infty}\frac{dk_z}{2\pi}\sum_m\frac{1}{2\pi i}\int_C dy(y^2+k_z^2a^2)^{-\frac{s}{2}}\frac{\partial}{\partial y}\ln f_m(y,k_z), \tag{1.30}$$

where the contour of integration C is chosen so that it encloses all the positive zeros of f_m in terms of y.

This energy turns out to be zero as predicted by the derivation from the Van der Walls interaction, and the details for computing the above integral can be seen in great detail in [19]. The same calculation for the case where the speed of light is the same in both media, inside and outside the cylinder, was calculated in [7].

1.3.2 The Heat kernel technique

The heat equation was first derived by Fourier in the theory of heat flow. The diffusion equation reads

$$v_t = k\nabla^2 v \tag{1.31}$$

with certain initial and boundary conditions. If we have a differential operator $\mathbf{P}$ so that $\mathbf{P}v + \partial_t v = 0$, formally $e^{-\mathbf{P}t}$ is a fundamental solution of that differential equation with boundary conditions $v(x,0) = \delta(0)$ and it is called the heat kernel of the operator $\mathbf{P}$.

If the eigenfunctions $\phi_n(x)$ of $\mathbf{P}$ form a complete set, with eigenvalues λ_n, the heat kernel can be written as

$$K(x,x';t) = \sum_n \phi_n(x)\phi_n^*(x')e^{-t\lambda_n}, \tag{1.32}$$

satisfying the equation

$$\left(\frac{\partial}{\partial t} + \mathbf{P}\right)K(x,x';t) = 0, \tag{1.33}$$

with initial conditions $K(x,y;0) = \delta(x-y)$, that describes the diffusion of heat from a point source at some time zero. The global heat kernel is the trace of the local heat kernel.

$$K(t) \sim \operatorname{Tr} K(x,x';t) = \int dx K(x,x;t), \tag{1.34a}$$

$$K(t) = \sum_n e^{-\lambda_n t}. \tag{1.34b}$$

This is an spectral function invariant of the operator $\mathbf{P}$ and it defines other spectral functions like the zeta-function. These two are related by

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty dt\, t^{s-1} K(t), \quad \mathrm{Re}\, s > \frac{d}{2}, \tag{1.35}$$

where d is the dimension.

The heat kernel is a very powerful tool. In particular the asymptotic expansion of the global heat kernel is of great importance. The coefficients of such an expansion are spectral invariants of the differential operator that describe the asymptotic properties of the spectrum. It should be made clear that the Casimir energy cannot be calculated from a heat kernel expansion[3] however, the divergent part of the regulated vacuum energy is associated with the heat kernel coefficients of the asymptotic expansion. If we expand the heat kernel for small t we get,

$$K(t) = \frac{1}{(4\pi t)^{\frac{d}{2}}} \sum_n a_{n/2} t^{n/2}, \tag{1.36}$$

where the $a_{n/2}$ are the heat kernel coefficients and they are related to the correspondent coefficients for the local heat kernel expansion,

$$a_{n/2} = \mathrm{Tr}\, a_{n/2}(x, x'). \tag{1.37}$$

This coefficients are functions of the geometric properties of the topology, like volume or surface, and its derivatives. More details can be found in [9] or in a differential

[3]The cylinder kernel however can predict the Casimir energy [20].

geometry textbook to get a more mathematical approach. We are interested in a_2 since in three dimensions the vanishing of this coefficient makes the vacuum energy of a massless field be unique. A general expression for the heat kernel coefficients can be found from (1.36) and (1.35),

$$a_{n/2} = (4\pi)^{d/2} \lim_{s \to \frac{(d-n)}{2}} \left(s - \frac{(d-n)}{2} \right) \Gamma(s) \zeta(s), \tag{1.38}$$

for n integer number. In three dimensions ($d = 3$) when $n = 4$ we have $s \to -\frac{1}{2}$. If a_2 is not zero, the zeta function has a pole at that particular value of s and the problem is unsolved; notice that we need $\zeta(-1/2)$ to get the Casimir energy, see equation (1.29).

To illustrate this we go back to the case of the dilute dielectric cylinder. The heat kernel coefficients for this case are calculated in [21]. A way to regulate the Casimir energy is by subtracting the Minkowski contribution of the entire space. The energy in (1.27) can be renormalized by

$$E = \frac{\hbar}{2} \int_{-\infty}^{\infty} dk_z \sum_{m,p} [\omega_{mp}(k_z) - \bar{\omega}_{mp}(k_z)], \tag{1.39}$$

where ω_{mp} are the eigenfrequencies of the electromagnetic oscillation with a cylindrical topology, and $\bar{\omega}_{mp}$ are the eigenfrequencies without the boundaries; those of the entire Minkowski space of either medium. As we did in the previous section, we can turn this sum into an integral,

$$\begin{aligned} E(s) &= \frac{\hbar}{2} a^s \int_{-\infty}^{\infty} \frac{dk_z}{2\pi} \sum_m \frac{1}{2\pi i} \int_C dy (y^2 + k_z^2 a^2)^{-\frac{s}{2}} \frac{\partial}{\partial y} \ln \frac{f_m(y, k_z)}{f_m^\infty(y, k_z)} \\ &= \frac{\hbar}{2} \zeta(s/2). \end{aligned} \tag{1.40}$$

the heat kernel coefficients are given by the residues of the zeta function,

$$a_n = (4\pi)^{3/2}\mathrm{Res}_{s=\frac{3}{2}-n}\Gamma(s)\zeta(s). \tag{1.41}$$

Bordag et al. calculate a_2 from the rotated version of (1.40) by using the asymptotic expansion of the modified Bessel functions and what they get is

$$a_2 = -\frac{9\pi}{88a^2}(\varepsilon - 1)^3, \tag{1.42}$$

the coefficient a_2 of the heat kernel expansion starts as $O((\varepsilon - 1)^3)$.

The fact that the dilute approximation gives $a_2 = 0$ in second order in $\varepsilon - 1$ means that the value of the Casimir energy in this order is unique and finite.

1.4 Derivation for the Casimir Energy for a Dielectric Cylinder

The following is the work of Kimball A. Milton and myself [22]; we calculate the Casimir pressure on the walls of an infinite circular dielectric-diamagnetic cylinder with electric permittivity ε and magnetic permeability μ inside the cylinder and surrounded by vacuum with permittivity 1 and permeability 1 so that $\varepsilon\mu \neq 1$. We will show that the corresponding Casimir energy per unit length is divergent as expected but, for $\mu = 1$, the finite coefficient of $(\varepsilon - 1)^2/a^2$ in the expansion for the dilute approximation yields the surprising zero result found by summing the van der Waals energies between the molecules that make up the material [7]. The latter calculation was verified by a perturbative calculation by Barton [23].

Although there should be divergences in the energy proportional to $(\varepsilon - 1)^2 a$ and $(\varepsilon - 1)^2/a$, the coefficient of $(\varepsilon - 1)^2/a^2$ is unique and finite as we have stated above, consequence of having the coefficient $a_2 = 0$ in the heat kernel expansion [21].

We will first calculate the dyadic Green's functions that will allow us to compute the one-loop vacuum expectation values of the quadratic field products. This enables us to calculate the vacuum expectation value of the stress tensor, the discontinuity of which across the surface gives the stress on the cylinder. We detail the calculation of the bulk Casimir stress, which would be present if either medium filled all space and must be subtracted from the stress found previously. Finally, the case of a dilute dielectric cylinder is considered and by detailed analytic and numerical calculations, it is shown that the Casimir stress vanishes both in order $\varepsilon - 1$ and $(\varepsilon - 1)^2$. The significance of divergences encountered in the calculation is discussed.

1.4.1 Green's Function Derivation of the Casimir Energy

In a medium of constant electric permittivity ε' and magnetic permeability μ' we insert an infinitely long cylinder of radius a with permittivity and permeability ε and μ. The product of these parameters is different than that of the outside parameters. There are no real charges of any kind present in the problem, $\rho = \mathbf{J} = 0$ and since we work at a fixed frequency we can Fourier transform the electric and magnetic fields,

$$\mathbf{E}(\mathbf{r}, t) = \int_{\infty}^{\infty} \frac{d\omega}{2\pi} \mathbf{E}(\mathbf{r}, \omega) e^{-i\omega t}, \tag{1.43a}$$

$$\mathbf{B}(\mathbf{r}, t) = \int_{\infty}^{\infty} \frac{d\omega}{2\pi} \mathbf{B}(\mathbf{r}, \omega) e^{-i\omega t}, \tag{1.43b}$$

and the corresponding Maxwell's equations are

$$\nabla \times \mathbf{E} = i\omega\mu\mathbf{H}, \qquad \nabla \cdot \mathbf{D} = 0, \tag{1.44a}$$

$$\nabla \times \mathbf{H} = -i\omega\varepsilon\mathbf{E}, \qquad \nabla \cdot \mathbf{B} = 0. \tag{1.44b}$$

In order to write down the Green's dyadic equations, we introduce a polarization source $\mathbf{P}$. The first equation in (1.44b) and the second one in (1.44a) get then changed to,

$$\nabla \times \mathbf{H} = -i\omega\varepsilon\mathbf{E} - i\omega\mathbf{P}, \tag{1.45a}$$

$$\nabla \cdot \mathbf{D} = -\nabla \cdot \mathbf{P}. \tag{1.45b}$$

The linear relation of polarization source with the electric field defines the Green's dyadic as

$$\mathbf{E}(x) = \int (dx')\mathbf{\Gamma}(x, x') \cdot \mathbf{P}(x'). \tag{1.46}$$

Since the response is translationally invariant in time, we also introduce the Fourier transform of the dyadic at a given frequency ω,

$$\mathbf{\Gamma}(x, x') = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \exp\left[-\mathrm{i}\omega(t - t')\right]\mathbf{\Gamma}(\mathbf{r}, \mathbf{r}', \omega). \tag{1.47}$$

We can then, by simple substitution in (1.45a) and (1.44a), write the dyadic Maxwell's equations in a medium characterized by a dielectric constant ε and a permeability μ, both of which may be functions of frequency:

$$\begin{aligned} \boldsymbol{\nabla} \times \boldsymbol{\Gamma} - \mathrm{i}\omega\mu\boldsymbol{\Phi} &= 0 \qquad (1.48a) \\ -\boldsymbol{\nabla} \times \boldsymbol{\Phi} - \mathrm{i}\omega\varepsilon\boldsymbol{\Gamma} &= \mathrm{i}\omega\mathbf{1}\delta(\mathbf{r}-\mathbf{r}'). \qquad (1.48b) \end{aligned}$$

Since it is more convenient to have divergenceless Green dyadics, we redefine the electric Green's dyadic in the following way,

$$\boldsymbol{\Gamma}'(\mathbf{r},\mathbf{r}',\omega) = \boldsymbol{\Gamma}(\mathbf{r},\mathbf{r}',\omega) + \frac{\mathbf{1}}{\varepsilon(\omega)}\delta(\mathbf{r}-\mathbf{r}'), \qquad (1.49)$$

so that the dyadic Maxwell's equations can now be written

$$\begin{aligned} \boldsymbol{\nabla} \times \boldsymbol{\Gamma}' - \mathrm{i}\omega\mu\boldsymbol{\Phi} &= \frac{1}{\varepsilon}\boldsymbol{\nabla} \times \mathbf{1}, \qquad \boldsymbol{\nabla} \cdot \boldsymbol{\Phi} = \mathbf{0}, \qquad (1.50a) \\ -\boldsymbol{\nabla} \times \boldsymbol{\Phi} - \mathrm{i}\omega\varepsilon\boldsymbol{\Gamma}' &= \mathbf{0}, \qquad \boldsymbol{\nabla} \cdot \boldsymbol{\Gamma}' = \mathbf{0}. \qquad (1.50b) \end{aligned}$$

and where the unit dyadic $\mathbf{1}$ includes a three-dimensional δ function,

$$\mathbf{1} = \mathbf{1}\delta(\mathbf{r}-\mathbf{r}'). \qquad (1.51)$$

The corresponding second order equations are

$$\begin{aligned} (\nabla^2 + \omega^2\varepsilon\mu)\boldsymbol{\Gamma}' &= -\frac{\mathbf{1}}{\varepsilon}\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{1}), \qquad (1.52a) \\ (\nabla^2 + \omega^2\varepsilon\mu)\boldsymbol{\Phi} &= \mathrm{i}\omega\boldsymbol{\nabla} \times \mathbf{1}. \qquad (1.52b) \end{aligned}$$

Quantum mechanically, these Green's dyadics give the one-loop vacuum expectation values of the product of fields at a given frequency ω,

$$
\begin{aligned}
\langle \mathbf{E}(\mathbf{r})\mathbf{E}(\mathbf{r}')\rangle &= \frac{\hbar}{\mathrm{i}}\boldsymbol{\Gamma}(\mathbf{r},\mathbf{r}'), &\text{(1.53a)}\\
\langle \mathbf{H}(\mathbf{r})\mathbf{H}(\mathbf{r}')\rangle &= -\frac{\hbar}{\mathrm{i}}\frac{1}{\omega^2\mu^2}\overrightarrow{\boldsymbol{\nabla}}\times\boldsymbol{\Gamma}(\mathbf{r},\mathbf{r}')\times\overleftarrow{\boldsymbol{\nabla}}'. &\text{(1.53b)}
\end{aligned}
$$

Notice that from (1.48a) the magnetic dyadic is the cross product of $\boldsymbol{\nabla}$ and the electric dyadic.

Thus, from the knowledge of the classical Green's dyadics, we can calculate the vacuum energy or stress.

Since the TE and TM modes do not separate, we cannot use the general waveguide decomposition of modes into those of TE and TM type. However we can introduce the appropriate partial wave decomposition for a cylinder, in terms of cylindrical coordinates (r,θ,z):

$$
\begin{aligned}
\boldsymbol{\Gamma}'(\mathbf{r},\mathbf{r}';\omega) &= \sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\bigg\{(\boldsymbol{\nabla}\times\hat{\mathbf{z}})f_m(r;k,\omega)\chi_{mk}(\theta,z)\\
&\qquad + \frac{\mathrm{i}}{\omega\varepsilon}\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\hat{\mathbf{z}})g_m(r;k,\omega)\chi_{mk}(\theta,z)\bigg\}, &\text{(1.54a)}\\
\boldsymbol{\Phi}(\mathbf{r},\mathbf{r}';\omega) &= \sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\bigg\{(\boldsymbol{\nabla}\times\hat{\mathbf{z}})\tilde{g}_m(r;k,\omega)\chi_{mk}(\theta,z)\\
&\qquad - \frac{\mathrm{i}\varepsilon}{\omega\mu}\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\hat{\mathbf{z}})\tilde{f}_m(r;k,\omega)\chi_{mk}(\theta,z)\bigg\}, &\text{(1.54b)}
\end{aligned}
$$

where the cylindrical harmonics are

$$\chi(\theta, z) = \frac{1}{\sqrt{2\pi}} e^{\mathrm{i}m\theta} e^{\mathrm{i}kz}, \tag{1.55}$$

and the dependence of f_m etc. on $\mathbf{r}'$ is implicit. Notice that these are vectors in the second tensor index. Because of the presence of these harmonics we have

$$\boldsymbol{\nabla} \times \hat{\mathbf{z}} \quad \rightarrow \quad \hat{\mathbf{r}} \frac{\mathrm{i}m}{r} - \hat{\boldsymbol{\theta}} \frac{\partial}{\partial r} \equiv \boldsymbol{\mathcal{M}}, \tag{1.56a}$$

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \hat{\mathbf{z}}) \quad \rightarrow \quad \hat{\mathbf{r}} \mathrm{i}k \frac{\partial}{\partial r} - \hat{\boldsymbol{\theta}} \frac{mk}{r} - \hat{\mathbf{z}} d_m \equiv \boldsymbol{\mathcal{N}}, \tag{1.56b}$$

in terms of the cylinder operator

$$d_m = \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} - \frac{m^2}{r^2}. \tag{1.57}$$

It is trivial to see that the divergence of (1.54a) and (1.54b) is zero, satisfying immediately two the the dyadic Maxwell's equations. Now, if we use the Maxwell equation (1.50b) we conclude[4]

$$\tilde{g}_m \quad = \quad g_m, \tag{1.58a}$$

$$(d_m - k^2) \tilde{f}_m \quad = \quad -\omega^2 \mu f_m. \tag{1.58b}$$

[4]The ambiguity in solving for these equations is absorbed in the definition of subsequent constants of integration.

More elaborate work is needed to get a condition form the other Maxwell equation (1.50a). Using the above we can write (1.50a) as,

$$\sum_m \int_{-\infty}^{\infty} \frac{dk}{2\pi} \left\{ -\boldsymbol{\mathcal{M}} \frac{(d_m - k^2)}{\omega^2 \mu} \tilde{f}_m - \frac{\mathrm{i}}{\omega \varepsilon}(d_m - k^2) \boldsymbol{\mathcal{N}} g_m \right\} \quad \chi_{mk}(\theta, z) =$$
$$\sum_m \int_{-\infty}^{\infty} \frac{dk}{2\pi} \left\{ \mathrm{i}\omega\mu \boldsymbol{\mathcal{N}} g_m + \varepsilon \boldsymbol{\mathcal{M}} \tilde{f}_m \right\} \chi_{mk}(\theta, z) + \frac{1}{\varepsilon} \boldsymbol{\nabla} \times \mathbf{1} \quad . \tag{1.59}$$

if we multiply the above by $\int_0^{2\pi} \int_{-\infty}^{\infty} d\theta dz \chi^*_{m'k'}(\theta, z)$ and apply

$$\int_0^{2\pi} \int_{-\infty}^{\infty} d\theta dz \chi^*_{m'k'}(\theta, z) \chi_{mk}(\theta, z) = 2\pi \delta(k - k') \delta_{mm'}, \tag{1.60}$$

we find

$$-\frac{1}{\omega^2 \mu} \boldsymbol{\mathcal{N}} (d_m - k^2 + \omega^2 \mu \varepsilon) \tilde{f}_m - \frac{i}{\omega \varepsilon} \boldsymbol{\mathcal{M}} (d_m - k^2 + \omega^2 \mu \varepsilon) g_m =$$
$$\frac{1}{\varepsilon} \int_0^{2\pi} \int_{-\infty}^{\infty} d\theta dz \chi^*_{mk}(\theta, z) (\boldsymbol{\nabla} \times \mathbf{1}) \frac{1}{r} \delta(r - r') \delta(\theta - \theta') \delta(z - z'), \tag{1.61}$$

where the delta functions are now made explicit. By dotting this expression with $\hat{\mathbf{z}}$ we notice that $\hat{\mathbf{z}} \cdot \boldsymbol{\mathcal{M}} = 0$ and $\hat{\mathbf{z}} \cdot \boldsymbol{\mathcal{N}} = -d_m$ and after a little manipulation we get to the fourth order differential equation:

$$d_m \mathcal{D}_m \tilde{\mathbf{f}}_m(r; r', \theta', z') = \frac{\omega^2 \mu}{\varepsilon} \boldsymbol{\mathcal{M}}'^* \frac{1}{r} \delta(r - r') \chi^*_{mk}(\theta', z'). \tag{1.62}$$

If we now dot it with $(\boldsymbol{\nabla} \times \hat{\mathbf{z}})$, we don't get a contribution from $\boldsymbol{\mathcal{N}}$ and we learn that a similar equation holds for g_m:

$$d_m \mathcal{D}_m \mathbf{g}_m(r; r', \theta', z') = -\mathrm{i}\omega \boldsymbol{\mathcal{N}}'^{*} \frac{1}{r} \delta(r - r') \chi^{*}_{mk}(\theta', z'), \tag{1.63}$$

where we have made the second, previously suppressed, position arguments explicit and the prime on the differential operator signifies action on the second primed argument. The Bessel operator appears,

$$\mathcal{D}_m = d_m + \lambda^2, \qquad \lambda^2 = \omega^2 \varepsilon \mu - k^2. \tag{1.64}$$

In order to solve those equations, we separate variables in the second argument,

$$\tilde{\mathbf{f}}_m(r, \mathbf{r}') = \left[\boldsymbol{\mathcal{M}}'^{*} F_m(r, r'; k, \omega) + \frac{1}{\omega} \boldsymbol{\mathcal{N}}'^{*} \tilde{F}_m(r, r'; k, \omega) \right] \chi^{*}_{mk}(\theta', z'), \tag{1.65a}$$

$$\mathbf{g}_m(r, \mathbf{r}') = \left[-\frac{\mathrm{i}}{\omega} \boldsymbol{\mathcal{N}}'^{*} G_m(r, r'; k, \omega) - \mathrm{i} \boldsymbol{\mathcal{M}}'^{*} \tilde{G}_m(r, r'; k, \omega) \right] \chi^{*}_{mk}(\theta', z'), \tag{1.65b}$$

where we have introduced the two scalar Green's functions F_m, G_m, which satisfy

$$d_m \mathcal{D}_m F_m(r, r') = \frac{\omega^2 \mu}{\varepsilon} \frac{1}{r} \delta(r - r'), \tag{1.66a}$$

$$d_m \mathcal{D}_m G_m(r, r') = \omega^2 \frac{1}{r} \delta(r - r'), \tag{1.66b}$$

while $\tilde{F}_m$ and $\tilde{G}_m$ are annihilated by the operator $d_m \mathcal{D}_m$,

$$d_m \mathcal{D}_m \tilde{F}(r, r') = d_m \mathcal{D}_m \tilde{G}(r, r') = 0. \tag{1.67}$$

The Green's dyadics have now the form:

$$
\begin{aligned}
\boldsymbol{\Gamma}'(\mathbf{r},\mathbf{r}';\omega) &= \sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\bigg\{\boldsymbol{\mathcal{M}}\boldsymbol{\mathcal{M}}'^{*}\left(-\frac{d_m-k^2}{\omega^2\mu}\right)F_m(r,r') \\
&\quad+\frac{1}{\omega}\boldsymbol{\mathcal{M}}\boldsymbol{\mathcal{N}}'^{*}\left(-\frac{d_m-k^2}{\omega^2\mu}\right)\tilde{F}_m(r,r')+\boldsymbol{\mathcal{N}}\boldsymbol{\mathcal{N}}'^{*}\frac{1}{\omega^2\varepsilon}G_m(r,r') \\
&\quad+\frac{1}{\omega\varepsilon}\boldsymbol{\mathcal{N}}\boldsymbol{\mathcal{M}}'^{*}\tilde{G}_m(r,r')\bigg\}\chi_{mk}(\theta,z)\chi^{*}_{mk}(\theta',z'), \qquad (1.68a)\\
\boldsymbol{\Phi}(\mathbf{r},\mathbf{r}';\omega) &= \sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\bigg\{-\frac{\mathrm{i}}{\omega}\boldsymbol{\mathcal{M}}\boldsymbol{\mathcal{N}}'^{*}G_m(r,r')-\mathrm{i}\boldsymbol{\mathcal{M}}\boldsymbol{\mathcal{M}}'^{*}\tilde{G}_m(r,r') \\
&\quad-\frac{\mathrm{i}\varepsilon}{\omega\mu}\boldsymbol{\mathcal{N}}\boldsymbol{\mathcal{M}}'^{*}F_m(r,r')-\frac{\mathrm{i}\varepsilon}{\omega^2\mu}\boldsymbol{\mathcal{N}}\boldsymbol{\mathcal{N}}'^{*}\tilde{F}_m(r,r')\bigg\} \\
&\quad\times\chi_{mk}(\theta,z)\chi^{*}_{mk}(\theta',z'). \qquad (1.68b)
\end{aligned}
$$

In the following, we will apply these equations to a dielectric-diamagnetic cylinder of radius a, where the interior of the cylinder is characterized by a permittivity ε and permeability μ, while the outside is vacuum, so $\varepsilon=\mu=1$ there. Let us consider the case that the source point is outside, $r'>a$. If the field point is also outside, $r,r'>a$, the scalar Green's functions $F'_m, G'_m, \tilde{F}', \tilde{G}'$ that make up the above Green's dyadics (we designate with primes the outside scalar Green's functions or constants) obey the differential equations (1.66a), (1.66b), and (1.67) with $\varepsilon=\mu=1$. To solve these fourth-order differential equations we introduce auxiliary Green's functions $\mathcal{G}_m^{F'(G')}(r,r')$ and $\mathcal{G}_m^{\tilde{F}'(\tilde{G}')}(r,r')$,

$$d_m\mathcal{D}_mF'=\omega^2 d_m\mathcal{G}_m^{F'},\qquad d_m\mathcal{D}_mF'=\omega^2 d_m\mathcal{G}_m^{\tilde{F}'}, \qquad (1.69)$$

and similar expressions for $\mathcal{G}^{G'(\tilde{G}')}$ satisfying ($m\neq 0$)

$$d_m \mathcal{G}_m^{F'(G')}(r,r') = \frac{1}{r}\delta(r-r'), \tag{1.70a}$$

$$d_m \mathcal{G}_m^{\tilde{F}'(\tilde{G}')}(r,r') = 0, \tag{1.70b}$$

which therefore have the general form

$$\mathcal{G}_m^{F'(G')}(r,r') = a_m'^{F(G)}(r')\frac{1}{r^{|m|}} - \frac{1}{2|m|}\left(\frac{r_<}{r_>}\right)^{|m|}, \tag{1.71a}$$

$$\mathcal{G}_m^{\tilde{F}'(\tilde{G}')}(r,r') = a_m'^{\tilde{F}(\tilde{G})}(r')\frac{1}{r^{|m|}}, \tag{1.71b}$$

where $r_<(r_>)$ is the lesser (greater) of r, r' and we discarded a possible $r^{|m|}$ term because we seek a solution which vanishes at infinity. Thus $F'_m, G'_m, \tilde{F}'$ and $\tilde{G}'$ satisfy the second-order differential equations

$$\mathcal{D}_m F'_m = \omega^2 \mathcal{G}_m^{F'}, \qquad \mathcal{D}_m G'_m = \omega^2 \mathcal{G}_m^{G'}, \tag{1.72a}$$

$$\mathcal{D}_m \tilde{F}'_m = \omega^2 \mathcal{G}_m^{\tilde{F}'}, \qquad \mathcal{D}_m \tilde{G}'_m = \omega^2 \mathcal{G}_m^{\tilde{G}'}. \tag{1.72b}$$

Now, from (1.72a) and the first identity in (1.64) we can write the above as ($\lambda'^2 = \omega^2 - k^2$)

$$d_m(F'_m - \frac{\omega^2}{\lambda'^2}\mathcal{G}_m^{F'}) = -\lambda'^2(F'_m - \frac{\omega^2}{\lambda'^2}\mathcal{G}_m^{F'}), \tag{1.73}$$

and the same holds for $G'_m, \tilde{F}'_m$ and $\tilde{G}'_m$. The solutions to these equations are Bessel functions:

$$F'_m - \frac{\omega^2}{\lambda'^2}\mathcal{G}^{F'}_m = A'^F_m(r')H_m(\lambda' r) - \frac{\omega^2}{\lambda'^2}\frac{\pi}{2\mathrm{i}}J_m(\lambda' r_<)H_m(\lambda' r_>), \tag{1.74}$$

while G'_m obeys a similar expression with the replacement $F \to G$ [5]. Similarly, from (1.72b)

$$\tilde{F}'_m - \frac{\omega^2}{\lambda'}\mathcal{G}^{\tilde{F}}_m = A'^{\tilde{F}}_m(r')H_m(\lambda' r), \tag{1.75}$$

and for $\tilde{G}'_m$ replace $F \to G$. Here, to have the appropriate outgoing-wave boundary condition at infinity, we have used $H_m(\lambda' r) = H^{(1)}_m(\lambda' r)$.

The dependence of the constants on the second variable r' can be deduced by noticing that, naturally, the Green's dyadics have to satisfy Maxwell's equations in their second variable. Thus, by imposing the Helmholtz equations in the second variable together with the boundary conditions at $r' = \infty$, it is easy to see that

$$a'^F_m(r') = a'^F_m\frac{1}{r'^{|m|}} + b'^F_m H_m(\lambda' r'), \tag{1.76a}$$

$$A'^F_m(r') = A'^F_m\frac{1}{r'^{|m|}} + B'^F_m H_m(\lambda' r'), \tag{1.76b}$$

and with similar relations for $a'^G_m(r'), A'^G_m(r'), a'^{\tilde{G}}_m(r')$, and so on. Then, the outside Green's functions have the form

$$\begin{aligned} F'_m(r,r') = {} & \frac{\omega^2}{\lambda'^2}\left[\frac{a'^F_m}{r'^{|m|}} + b'^F_m H_m(\lambda' r')\right] r^{-|m|} - \frac{\omega^2}{\lambda'^2}\frac{1}{2|m|}\left(\frac{r_<}{r_>}\right)^{|m|} \\ & + \left[\frac{A'^F_m}{r'^{|m|}} + B'^F_m H_m(\lambda' r')\right] H_m(\lambda' r) - \frac{\omega^2}{\lambda'^2}\frac{\pi}{2\mathrm{i}}J_m(\lambda' r_<)H_m(\lambda' r_>), \end{aligned} \tag{1.77}$$

[5] Notice that in principle we could also have a term proportional to $J_m(\lambda' r)$, but it doesn't happen because of boundary conditions. The primes represent functions or coordinates outside the cylinder

while G'_m has the same form with the constants $a'^F_m, b'^F_m, A'^F_m, B'^F_m$ replaced by a'^G_m, b'^G_m, A'^G_m and B'^G_m, respectively. The homogeneous differential equations have solutions

$$\tilde{F}'_m(r,r') = \frac{\omega^2}{\lambda'^2}\left[\frac{a'^{\tilde{F}}_m}{r'^{|m|}} + b'^{\tilde{F}}_m H_m(\lambda' r')\right] r^{-|m|} + \left[\frac{A'^{\tilde{F}}_m}{r'^{|m|}} + B'^{\tilde{F}}_m H_m(\lambda' r')\right] H_m(\lambda' r), \quad (1.78)$$

while in $\tilde{G}'_m$ we replace $a'^{\tilde{F}} \to a'^{\tilde{G}}$, etc.

When the source point is outside and the field point is inside, all the Green's functions satisfy the homogeneous equations (1.67) with ε, $\mu \neq 1$, and then, following the above scheme we have that

$$d_m \mathcal{G}^F_m = d_m \mathcal{G}^G_m = d_m \mathcal{G}^{\tilde{F}}_m = d_m \mathcal{G}^{\tilde{G}}_m = 0, \quad (1.79)$$

and

$$\mathcal{G}^F_m(r,r') = a^F_m(r') r^{|m|}, \quad (1.80)$$

since now r can be 0. Also $\mathcal{D}_m F_m = \omega^2 \mathcal{G}^F_m$ and a expression similar to (1.73) holds. We find that[6]

$$F_m - \frac{\omega^2}{\lambda^2}\mathcal{G}^F_m = A^F_m(r') J_m(\lambda r). \quad (1.81)$$

$G_m, \tilde{F}_m$ and $\tilde{G}_m$ have the same form, and the constants $a^F_m(r'), A^F_m(r')$, etc. follow the pattern in (1.76a) and (1.76b). Now, we may write for $r < a, r' > a$

$$F_m(r,r') = \frac{\omega^2}{\lambda^2}\left[\frac{a^F_m}{r'^{|m|}} + b^F_m H_m(\lambda' r')\right] r^{|m|} + \left[\frac{A^F_m}{r'^{|m|}} + B^F_m H_m(\lambda' r')\right] J_m(\lambda r), \quad (1.82)$$

and similarly for $G_m, \tilde{F}_m, \tilde{G}_m$, with the corresponding change of constants. In all of

[6] Now we have the $J_m(\lambda r)$ since it is convergent at $r = 0$ and the Hankel function is missing.

the above, the outside and inside forms of λ are given by

$$\lambda'^2 = \omega^2 - k^2, \qquad \lambda^2 = \omega^2\mu\varepsilon - k^2. \tag{1.83}$$

The various constants are to be determined, as far as possible, by the boundary conditions at $r = a$. The boundary conditions at the surface of the dielectric cylinder are the continuity of tangential components of the electric field, of the normal component of the electric displacement, of the normal component of the magnetic induction, and of the tangential components of the magnetic field (we assume that there are no surface charges or currents):

$$\begin{aligned} &\mathbf{E}_t \text{ is continuous}, \qquad && \varepsilon E_n \text{ is continuous}, \\ &\mathbf{H}_t \text{ is continuous}, \qquad && \mu H_n \text{ is continuous}. \end{aligned} \tag{1.84}$$

These conditions are redundant, but we will impose all of them as a check of consistency. In terms of the Green's dyadics, the conditions read

$$\hat{\theta} \cdot \mathbf{\Gamma}' \Big|_{r=a-}^{r=a+} = \mathbf{0}, \tag{1.85a}$$

$$\hat{\mathbf{z}} \cdot \mathbf{\Gamma}' \Big|_{r=a-}^{r=a+} = \mathbf{0}, \tag{1.85b}$$

$$\hat{\mathbf{r}} \cdot \varepsilon\mathbf{\Gamma}' \Big|_{r=a-}^{r=a+} = \mathbf{0}, \tag{1.85c}$$

$$\hat{\mathbf{r}} \cdot \mu\mathbf{\Phi} \Big|_{r=a-}^{r=a+} = \mathbf{0}, \tag{1.85d}$$

$$\hat{\theta} \cdot \mathbf{\Phi} \Big|_{r=a-}^{r=a+} = \mathbf{0}, \tag{1.85e}$$

$$\hat{\mathbf{z}} \cdot \mathbf{\Phi} \Big|_{r=a-}^{r=a+} = \mathbf{0}. \tag{1.85f}$$

We can also impose the Helmholtz equations (1.52a) and (1.52b). From those we learn that the coefficients of terms with powers of r are related in the following way

$$\hat{a}'^{F} + \hat{a}'^{G} = 0, \tag{1.86a}$$

$$b'^{G} - (\mathrm{sgn} m)\frac{k}{\omega} b'^{\tilde{F}} = 0, \tag{1.86b}$$

$$b'^{\tilde{G}} - (\mathrm{sgn} m)\frac{k}{\omega} b'^{F} = 0, \tag{1.86c}$$

for the Green's dyadics outside the cylinder and equivalent expressions for the inside (no primes). To illustrate this we apply the $\hat{\theta}\hat{\theta}$ component of the magnetic Helmholtz equation (in a similar manner it can be seen that all components of the second order differential Maxwell's equations hold), $(\nabla^2 + \omega^2\varepsilon\mu)\Phi_{\theta\theta} = 0$. Plugging in the $\hat{\theta}\hat{\theta}$ component of $\mathbf{\Phi}$ (equation (1.68b)) and using (1.82) we find,

$$\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\Bigg\{-\mathrm{i}\frac{\mathrm{i}|m|}{\omega}\frac{mk}{r'}\left[\frac{a_m^G}{r'^{|m|}} + b_m^G H_m(\lambda' r')\right] r^{|m|-1} - \mathrm{i}|m|\frac{\partial}{\partial r'}\left[\frac{a_m^{\tilde{G}}}{r'^{|m|}} + b_m^{\tilde{G}} H_m(\lambda' r')\right] r^{|m|-1} - \frac{\mathrm{i}\varepsilon}{\omega\mu} mk \frac{\partial}{\partial r'}\left[\frac{a_m^F}{r'^{|m|}} + b_m^F H_m(\lambda' r')\right] r^{|m|-1} - \frac{\mathrm{i}\varepsilon}{\omega^2\mu}\frac{m^2k^2}{r'} \times \left[\frac{a_m^{\tilde{F}}}{r'^{|m|}} + b_m^{\tilde{F}} H_m(\lambda' r')\right] r^{|m|-1}\Bigg\}\chi_{mk}\chi'_{mk} = 0. \tag{1.87}$$

From here we obtain expression for the inside constants; the equivalent conditions of those found in equations (1.86a),(1.86c) and (1.86c) for the outside,

$$\frac{\varepsilon}{\mu}\hat{a}^{F} - \hat{a}^{G} = 0, \tag{1.88a}$$

$$b^{G} + (\mathrm{sgn} m)\frac{\varepsilon}{\mu}\frac{k}{\omega} b^{\tilde{F}} = 0, \tag{1.88b}$$

$$b^{\tilde{G}} + (\mathrm{sgn} m)\frac{\varepsilon}{\mu}\frac{k}{\omega} b^{F} = 0, \tag{1.88c}$$

where we have introduced the abbreviations for any constant K

$$\hat{K}^F = K^F - (\text{sgn}m)\frac{k}{\omega}K^{\tilde{F}}, \qquad \hat{K}^G = K^G - (\text{sgn}m)\frac{\omega}{k}K^{\tilde{G}}, \tag{1.89}$$

and the same for $\hat{K}'^F$ and $\hat{K}'^G$ (the outside). Then, from the boundary conditions we can solve for the remaining constants. Notice that, due to the tensorial character of the Green's dyadics, each of the above six boundary conditions (1.85a), (1.85b), (1.85c), (1.85d), (1.85e), (1.85f) are in fact three equations corresponding to the three prime coordinates. For example, from (1.85a) we obtain,

$$\begin{aligned}
&\sum_m \int \frac{dk}{2\pi}\Big\{\frac{d_m - k^2}{\omega^2\mu}\frac{\partial}{\partial r}\boldsymbol{\mathcal{M}}'^* F_m + \frac{1}{\omega}\frac{d_m - k^2}{\omega^2\mu}\frac{\partial}{\partial r}\boldsymbol{\mathcal{N}}'^* \tilde{F}_m - \\
&\qquad \frac{1}{\omega^2\varepsilon}\frac{mk}{r}\boldsymbol{\mathcal{N}}'^* G_m - \frac{1}{\omega\varepsilon}\frac{mk}{r}\boldsymbol{\mathcal{M}}'^* \tilde{G}_m\Big\}\chi_{mk}\chi'^*_{mk}\big|_{a-} = \\
&\sum_m \int \frac{dk}{2\pi}\Big\{\frac{d_m - k^2}{\omega^2\mu}\frac{\partial}{\partial r}\boldsymbol{\mathcal{M}}'^* F'_m + \frac{1}{\omega}\frac{d_m - k^2}{\omega^2}\frac{\partial}{\partial r}\boldsymbol{\mathcal{N}}'^* \tilde{F}'_m - \\
&\qquad \frac{1}{\omega^2}\frac{mk}{r}\boldsymbol{\mathcal{N}}'^* G'_m - \frac{1}{\omega}\frac{mk}{r}\boldsymbol{\mathcal{M}}'^* \tilde{G}'_m\Big\}\chi_{mk}\chi'^*_{mk}\big|_{a+}.
\end{aligned} \tag{1.90}$$

This is a vector equation in the prime argument and to solve for the constants we have to look at the different components of the equation. The $\hat{\mathbf{z}}'$ component gives us

$$\begin{aligned}
&\frac{1}{\omega}\frac{d_m - k^2}{\omega^2\mu}\frac{\partial}{\partial r}(-d'_m)\tilde{F}_m - \frac{1}{\omega^2\varepsilon}\frac{mk}{r}(-d'_m)G_m\big|_{a-} = \\
&\frac{1}{\omega}\frac{d_m - k^2}{\omega^2}\frac{\partial}{\partial r}(-d'_m)\tilde{F}'_m - \frac{1}{\omega^2}\frac{mk}{r}(-d'_m)G'_m\big|_{a+}.
\end{aligned} \tag{1.91}$$

The $\hat{\theta}'$ component is,

$$
-\frac{d_m - k^2}{\omega^2\mu}\frac{\partial}{\partial r}\frac{\partial}{\partial r'}F_m - \frac{1}{\omega}\frac{d_m - k^2}{\omega^2\mu}\frac{\partial}{\partial r}\frac{mk}{r'}\tilde{F}_m + \frac{1}{\omega^2\varepsilon}\frac{mk}{r}\frac{mk}{r'}G_m +
\frac{1}{\omega\varepsilon}\frac{mk}{r}\frac{\partial}{\partial r'}\tilde{G}_m\Big|_{a-} = -\frac{d_m - k^2}{\omega^2\mu}\frac{\partial}{\partial r}\frac{\partial}{\partial r'}F'_m - \frac{1}{\omega}\frac{d_m - k^2}{\omega^2}\frac{\partial}{\partial r}\frac{mk}{r'}\tilde{F}'_m +
\frac{1}{\omega^2}\frac{mk}{r}\frac{mk}{r'}G'_m + \frac{1}{\omega}\frac{mk}{r}\frac{\partial}{\partial r'}\tilde{G}'_m\Big|_{a+}, \tag{1.92}
$$

and the $\hat{\mathbf{r}}'$ component,

$$
-\frac{d_m - k^2}{\omega^2\mu}\frac{\partial}{\partial r}\frac{\mathrm{i}m}{r'}F_m - \frac{1}{\omega}\frac{d_m - k^2}{\omega^2\mu}\frac{\partial}{\partial r}\mathrm{i}k\frac{\partial}{\partial r'}\tilde{F}_m + \frac{1}{\omega^2\varepsilon}\frac{mk}{r}\mathrm{i}k\frac{\partial}{\partial r'}G_m +
\frac{1}{\omega\varepsilon}\frac{mk}{r}\frac{\mathrm{i}m}{r'}\tilde{G}_m\Big|_{a-} = -\frac{d_m - k^2}{\omega^2\mu}\frac{\partial}{\partial r}\frac{im}{r'}F'_m - \frac{1}{\omega}\frac{d_m - k^2}{\omega^2}\frac{\partial}{\partial r}\mathrm{i}k\frac{\partial}{\partial r'}\tilde{F}'_m +
\frac{1}{\omega^2}\frac{mk}{r}\mathrm{i}k\frac{\partial}{\partial r'}G'_m + \frac{1}{\omega}\frac{mk}{r}\frac{\mathrm{i}m}{r'}\tilde{G}'_m\Big|_{a+}. \tag{1.93}
$$

Manipulating these equations we will be able to group the coefficients of $r'^{-|m|+1}$, $\frac{\partial H_m(\lambda' r')}{\partial r'}$ and $\frac{H_m(\lambda' r')}{r'}$ and extract the three equations:

$$
-\varepsilon\lambda a J'_m(\lambda a)B_m^{\tilde{F}} - \frac{mk}{\omega\varepsilon}J_m(\lambda a)B_m^G = -\lambda' a H'_m(\lambda' a)B_m'^{\tilde{F}}
-\frac{mk}{\omega}H_m(\lambda' a)B_m'^G + \frac{mk\omega}{\lambda'^2}\frac{\pi}{2\mathrm{i}}J_m(\lambda' a), \tag{1.94a}
$$

$$
-\varepsilon|m|\lambda a J'_m(\lambda a)A_m^F + \frac{mk\varepsilon}{\omega}\lambda a J'_m(\lambda a)A_m^{\tilde{F}} + \frac{m^2k^2}{\omega^2\varepsilon}J_m(\lambda a)A_m^G
-\frac{m|m|k}{\omega\varepsilon}J_m(\lambda a)A_m^{\tilde{G}} = -|m|\lambda' a H'_m(\lambda' a)A_m'^F + \frac{mk}{\omega}\lambda' a H'_m(\lambda' a)A_m'^{\tilde{F}}
+\frac{m^2k^2}{\omega^2}H_m(\lambda' a)A_m'^G - \frac{m|m|k}{\omega}H_m(\lambda' a)A_m'^{\tilde{G}}, \tag{1.94b}
$$

$$
\varepsilon\lambda a J'_m(\lambda a)B_m^F + \frac{mk}{\omega\varepsilon}J_m(\lambda a)B_m^{\tilde{G}} = \lambda' a H'_m(\lambda' a)B_m'^F
-\frac{\omega^2}{\lambda'^2}\frac{\pi}{2\mathrm{i}}\lambda' a J'_m(\lambda' a) + \frac{mk}{\omega}H_m(\lambda' a)B_m'^{\tilde{G}}. \tag{1.94c}
$$

The same mechanism can be applied to the rest of the boundary conditions. The three equations following from (1.85b) are:

$$B_m^G = \varepsilon \left(\frac{\lambda'}{\lambda}\right)^2 \left[B_m'^G \frac{H_m(\lambda' a)}{J_m(\lambda a)} - \frac{\omega^2}{\lambda'^2}\frac{\pi}{2\mathrm{i}}\frac{J_m(\lambda' a)}{J_m(\lambda a)}\right], \tag{1.95a}$$

$$|m|A_m^G - m\frac{\omega}{k}A_m^{\tilde{G}} = \varepsilon \left(\frac{\lambda'}{\lambda}\right)^2 \frac{H_m(\lambda' a)}{J_m(\lambda a)} \left[|m|A_m'^G - m\frac{\omega}{k}A_m'^{\tilde{G}}\right], \tag{1.95b}$$

$$B_m^{\tilde{G}} = \varepsilon \left(\frac{\lambda'}{\lambda}\right)^2 \frac{H_m(\lambda' a)}{J_m(\lambda a)} B_m'^{\tilde{G}}, \tag{1.95c}$$

and those coming from (1.85c) are:

$$m\varepsilon^2 J_m(\lambda a)B_m^{\tilde{F}} + \frac{k}{\omega}\lambda a J_m'(\lambda a)B_m^G = mH_m(\lambda' a)B_m'^{\tilde{F}} + \frac{k}{\omega}\lambda' a H_m'(\lambda' a)B_m'^G$$
$$-\frac{k\omega}{\lambda'^2}\frac{\pi}{2\mathrm{i}}\lambda' a J_m'(\lambda' a), \tag{1.96a}$$

$$\varepsilon^2 m J_m(\lambda a)B_m^F + \frac{k}{\omega}\lambda a J_m'(\lambda a)B_m^{\tilde{G}} = mH_m(\lambda' a)B_m'^F - m\frac{\omega^2}{\lambda'^2}\frac{\pi}{2\mathrm{i}}J_m(\lambda' a)$$
$$+\frac{k}{\omega}\lambda' a H_m'(\lambda' a)B_m'^{\tilde{G}}. \tag{1.96b}$$

$$\varepsilon^2 m^2 J_m(\lambda a)A_m^F - \frac{m|m|k}{\omega}\varepsilon^2 J_m(\lambda a)A_m^{\tilde{F}} - \frac{k^2|m|}{\omega^2}\lambda a J_m'(\lambda a)A_m^G$$
$$+\frac{mk}{\omega}\lambda a J_m'(\lambda a)A_m^{\tilde{G}} = m^2 H_m(\lambda' a)A_m'^F - \frac{m|m|k}{\omega}H_m(\lambda' a)A_m'^{\tilde{F}}$$
$$-\frac{|m|k^2}{\omega^2}\lambda' a H_m'(\lambda' a)A_m'^G + \frac{mk}{\omega}\lambda' a H_m'(\lambda' a)A_m'^{\tilde{G}}, \tag{1.96c}$$

From the set of equations involving the magnetic part Φ, we find that (1.85d) gives

$$\mu m J_m(\lambda a) B_m^G + \frac{\varepsilon k}{\omega} \lambda a J_m'(\lambda a) B_m^{\tilde{F}} = m H_m(\lambda' a) B_m'^G$$
$$-m \frac{\omega^2}{\lambda'^2} \frac{\pi}{2\mathrm{i}} J_m(\lambda' a) + \frac{k}{\omega} \lambda' a H_m'(\lambda' a) B_m'^{\tilde{F}}, \tag{1.97a}$$

$$-\mu m J_m(\lambda a) B_m^{\tilde{G}} - \frac{\epsilon k}{\omega} \lambda a J_m'(\lambda a) B_m^F = -m H_m(\lambda' a) B_m'^{\tilde{G}}$$
$$+\frac{k\omega}{\lambda'^2} \frac{\pi}{2\mathrm{i}} \lambda' a J_m'(\lambda' a) - \frac{k}{\omega} \lambda' a H_m'(\lambda' a) B_m'^F. \tag{1.97b}$$

$$\frac{\mu m|m|k}{\omega} J_m(\lambda a) A_m^G - \mu m^2 J_m(\lambda a) A_m^{\tilde{G}} - \frac{\varepsilon k m}{\omega} \lambda a J_m'(\lambda a) A_m^F$$
$$+\frac{\varepsilon |m| k^2}{\omega^2} \lambda a J_m'(\lambda a) A_m^{\tilde{F}} = \frac{m|m|k}{\omega} H_m(\lambda' a) A_m'^G - m^2 H_m(\lambda' a) A_m'^{\tilde{G}}$$
$$-\frac{mk}{\omega} \lambda' a H_m'(\lambda' a) A_m'^F + \frac{|m|k^2}{\omega^2} \lambda' a H_m'(\lambda' a) A_m'^{\tilde{F}}. \tag{1.97c}$$

By imposing (1.85e) we get the conditions

$$\lambda a J_m'(\lambda a) B_m^G + \frac{\varepsilon m k}{\omega \mu} J_m(\lambda a) B_m^{\tilde{F}} = \lambda' a H_m'(\lambda' a) B_m'^G$$
$$+\frac{mk}{\omega} H_m(\lambda' a) B_m'^{\tilde{F}} - \frac{\omega^2}{\lambda'^2} \frac{\pi}{2\mathrm{i}} \lambda' a J_m'(\lambda' a), \tag{1.98a}$$

$$-\frac{|m|k}{\omega} \lambda a J_m'(\lambda a) A_m^G + m \lambda a J_m'(\lambda a) A_m^{\tilde{G}} + \frac{m^2 k \varepsilon}{\omega \mu} J_m(\lambda a) A_m^F$$
$$-\frac{m|m|\varepsilon k^2}{\omega^2 \mu} J_m(\lambda a) A_m^{\tilde{F}} = -\frac{|m|k}{\omega} \lambda' a H_m'(\lambda' a) A_m'^G + m \lambda' a H_m'(\lambda' a) A_m'^{\tilde{G}}$$
$$+\frac{m^2 k}{\omega} H_m(\lambda' a) A_m'^F - \frac{m|m|k^2}{\omega^2} H_m(\lambda' a) A_m'^{\tilde{F}}, \tag{1.98b}$$

$$\lambda a J_m'(\lambda a) B_m^{\tilde{G}} + \frac{m k \varepsilon}{\omega \mu} J_m(\lambda a) B_m^F = \lambda' a H_m'(\lambda' a) B_m'^{\tilde{G}}$$
$$-\frac{\omega m k}{\lambda'^2} \frac{\pi}{2\mathrm{i}} J_m(\lambda' a) + \frac{mk}{\omega} H_m(\lambda' a) B_m'^F. \tag{1.98c}$$

And finally (1.85f) gives us

$$
\begin{aligned}
B_m^{\tilde{F}} &= \frac{\mu}{\varepsilon}\left(\frac{\lambda'}{\lambda}\right)^2 \frac{H_m(\lambda' a)}{J_m(\lambda a)} B_m'^{\tilde{F}}, && (1.99a)\\
-A_m^F + \frac{k}{\omega}\frac{|m|}{m} A_m^{\tilde{F}} &= \frac{\mu}{\varepsilon}\left(\frac{\lambda'}{\lambda}\right)^2 \frac{H_m(\lambda' a)}{J_m(\lambda a)} \left[-A_m'^F + \frac{k}{\omega}\frac{|m|}{m} A_m'^{\tilde{F}}\right], && (1.99b)\\
B_m^F &= \frac{\mu}{\varepsilon}\left(\frac{\lambda'}{\lambda}\right)^2 \left[B_m'^F \frac{H_m(\lambda' a)}{J_m(\lambda a)} - \frac{\omega^2}{\lambda'^2}\frac{\pi}{2\mathrm{i}}\frac{J_m(\lambda' a)}{J_m(\lambda a)}\right]. && (1.99c)
\end{aligned}
$$

By combining these equations we find the remaining constants, but the equations are not all independent. First, from (1.95b), (1.99b), (1.94b) and (1.96c) we learn that the coefficients of terms involving Bessel functions and $r'^{-|m|}$ cancel among themselves in a way such that the ones from the outside do not mix with those from the inside:

$$
\begin{aligned}
\hat{A}_m^F = \hat{A}_m^G &= 0, && (1.100a)\\
\hat{A}_m'^F = \hat{A}_m'^G &= 0. && (1.100b)
\end{aligned}
$$

The same can be found if we use (1.98b) and (1.97c) instead of (1.94b) and (1.96c).

Next we determine the coefficients of functions involving just Bessel functions. From (1.98c) and (1.96b) we find using (1.99c) and (1.95c) that

$$
\begin{aligned}
B_m^{\tilde{G}} &= -\frac{\varepsilon^2}{\mu}(1-\varepsilon\mu)\frac{mk\omega}{\lambda\lambda' D} J_m(\lambda a) H_m(\lambda' a) B_m^F, && (1.101a)\\
B_m'^{\tilde{G}} &= -\left(\frac{\lambda}{\lambda'}\right)^2 \frac{\varepsilon}{\mu}(1-\varepsilon\mu)\frac{mk\omega}{\lambda\lambda' D} J_m^2(\lambda a) B_m^F, && (1.101b)\\
B_m'^F &= \frac{\omega^2}{\lambda'^2}\frac{\pi}{2\mathrm{i}}\frac{J_m(\lambda' a)}{H_m(\lambda' a)} + \left(\frac{\lambda}{\lambda'}\right)^2 \frac{\varepsilon}{\mu}\frac{J_m(\lambda a)}{H_m(\lambda' a)} B_m^F, && (1.101c)
\end{aligned}
$$

all in terms of

$$B_m^F = -\frac{\mu}{\varepsilon}\frac{\omega^2}{\lambda\lambda'}\frac{D}{\Xi}, \tag{1.102}$$

found by subtracting $\frac{k}{\omega}$ times equation (1.96b) from (1.98c) and using (1.97b)[7]. The denominators occurring here are[8]

$$\begin{aligned}
\Xi &= (1-\varepsilon\mu)^2\frac{m^2k^2\omega^2}{\lambda^2\lambda'^2}J_m^2(\lambda a)H_m^2(\lambda' a) - D\tilde{D}, &\text{(1.103a)}\\
D &= \varepsilon\lambda' a J_m'(\lambda a)H_m(\lambda' a) - \lambda a H_m'(\lambda' a)J_m(\lambda a), &\text{(1.103b)}\\
\tilde{D} &= \mu\lambda' a J_m'(\lambda a)H_m(\lambda' a) - \lambda a H_m'(\lambda' a)J_m(\lambda a). &\text{(1.103c)}
\end{aligned}$$

The second set of constants is found using (1.97a), (1.94a), (1.99a) and (1.95a):

$$\begin{aligned}
B_m^{\tilde{F}} &= -\frac{\mu}{\varepsilon^2}(1-\varepsilon\mu)\frac{mk\omega}{\lambda\lambda'\tilde{D}}J_m(\lambda a)H_m(\lambda' a)B_m^G, &\text{(1.104a)}\\
B_m'^{\tilde{F}} &= -\left(\frac{\lambda}{\lambda'}\right)^2\frac{1}{\varepsilon}(1-\varepsilon\mu)\frac{mk\omega}{\lambda\lambda'\tilde{D}}J_m^2(\lambda a)B_m^G, &\text{(1.104b)}\\
B_m'^G &= \frac{\omega^2}{\lambda'^2}\frac{\pi}{2\mathrm{i}}\frac{J_m(\lambda' a)}{H_m(\lambda' a)} + \left(\frac{\lambda}{\lambda'}\right)^2\frac{1}{\varepsilon}\frac{J_m(\lambda a)}{H_m(\lambda' a)}B_m^G, &\text{(1.104c)}
\end{aligned}$$

in terms of

$$B_m^G = -\varepsilon\frac{\omega^2}{\lambda\lambda'}\frac{\tilde{D}}{\Xi}, \tag{1.105}$$

coming from (1.104b) and (1.96a)[9].

---

[7](1.97b) is the same equation as (1.94c), which can easily be seen by using (1.99c).

[8]The denominator structure appearing in Ξ is precisely that given by Stratton [18] and appearing in (1.23a).

[9]By using (1.99a) it can be seen that this equation is the same as (1.98a).

It might be thought that $m = 0$ is a special case, and indeed

$$\frac{1}{2|m|}\left(\frac{r_<}{r_>}\right)^{|m|} \to \frac{1}{2}\ln\frac{r_<}{r_>}, \tag{1.106}$$

but just as the latter is correctly interpreted as the limit as $|m| \to 0$, so the coefficients in the Green's functions turn out to be just the $m = 0$ limits for those given above, so the $m = 0$ case is properly incorporated.

It is now easy to check that, as a result of the conditions (1.86a), (1.86b), (1.86c), (1.88a), (1.88b), (1.88c), (1.100a), and (1.100b), the terms in the Green's functions that involve powers of r or r' do not contribute to the electric or magnetic fields. So, even though we are not able to determine all the constants, (notice that there is some ambiguity in these since they cannot be uniquely determined) it is not an issue since the energy will be well defined. These constants enter always in the same form and therefore their individual value is not relevant. As we might have anticipated, only the pure Bessel function terms contribute.

1.5 Stress on the Cylinder

We are now in a position to calculate the pressure on the surface of the cylinder from the radial-radial component of the stress tensor

$$P = \langle T_{rr}\rangle(a-) - \langle T_{rr}\rangle(a+) \tag{1.107}$$

where

$$T_{rr} = \frac{1}{2}\left[\varepsilon(E_\theta^2 + E_z^2 - E_r^2) + \mu(H_\theta^2 + H_z^2 - H_r^2)\right]. \tag{1.108}$$

As a result of the boundary conditions (1.84), the pressure on the cylindrical walls are given by the expectation value of the squares of field components just outside the cylinder, therefore

$$\begin{aligned} T_{rr}\Big|_{r=a-} - T^{rr}\Big|_{r=a+} &= \frac{\varepsilon-1}{2}\left(E_\theta^2 + E_z^2 + \frac{E_r^2}{\varepsilon}\right)\Bigg|_{r=a+} \\ &\quad + \frac{\mu-1}{2}\left(H_\theta^2 + H_z^2 + \frac{H_r^2}{\mu}\right)\Bigg|_{r=a+}. \end{aligned} \tag{1.109}$$

These expectation values are given by (1.53a), (1.53b), where the latter may also be written as

$$\langle \mathbf{H}(\mathbf{r})\,\mathbf{H}(\mathbf{r}')\rangle = -\frac{\hbar}{\omega\mu}\boldsymbol{\Phi}(\mathbf{r},\mathbf{r}') \times \overleftarrow{\boldsymbol{\nabla}}'. \tag{1.110}$$

It is quite straightforward to write the vacuum expectation values of the fields occurring here outside the cylinder in terms of the Green's functions,

$$\begin{aligned} \langle E_r(r)E_r(r')\rangle &= \frac{\hbar}{\mathrm{i}}\Gamma_{rr'} = \frac{\hbar}{2\pi\mathrm{i}}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\Bigg\{-\frac{m^2}{rr'}\frac{d_m - k^2}{\omega^2}F'_m(r,r') \\ &\quad - \frac{mk}{\omega r}\frac{\partial}{\partial r'}\frac{d_m-k^2}{\omega^2}\tilde{F}'_m(r,r') + \frac{k^2}{\omega^2}\frac{\partial}{\partial r}\frac{\partial}{\partial r'}G'_m(r,r) \\ &\quad + \frac{km}{\omega r'}\frac{\partial}{\partial r}\tilde{G}'_m(r,r')\Bigg\}, \end{aligned} \tag{1.111a}$$

$$\langle E_z(r)E_z(r')\rangle = \frac{\hbar}{\mathrm{i}}\Gamma_{zz'} = \frac{\hbar}{2\pi\mathrm{i}}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\frac{1}{\omega^2}d_m d'_m G'_m(r,r'), \tag{1.111b}$$

$$
\begin{aligned}
\langle E_\theta(r)E_\theta(r')\rangle &= \frac{\hbar}{\mathrm{i}}\Gamma_{\theta\theta'} = \frac{\hbar}{2\pi\mathrm{i}}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\biggl\{-\frac{\partial}{\partial r}\frac{\partial}{\partial r'}\frac{d_m-k^2}{\omega^2}F'_m(r,r') \\
&\quad - \frac{mk}{\omega r'}\frac{\partial}{\partial r}\frac{d_m-k^2}{\omega^2}\tilde{F}'_m(r,r') + \frac{m^2k^2}{\omega^2 rr'}G'_m(r,r') \\
&\quad + \frac{mk}{\omega r}\frac{\partial}{\partial r'}\tilde{G}'_m(r,r')\biggr\}. \qquad (1.111c)
\end{aligned}
$$

According to (1.110) the magnetic field expectation values can be written as follows,

$$
\begin{aligned}
\langle H_r(r)H_r(r')\rangle &= -\frac{\hbar}{2\pi}\frac{\mathrm{i}}{\omega}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\biggl\{-\frac{m^2}{\omega rr'}(d'_m-k^2)G'_m(r,r') \\
&\quad + \frac{mk}{r}\frac{\partial}{\partial r'}\tilde{G}'_m(r,r') + \frac{k^2}{\omega}\frac{\partial}{\partial r}\frac{\partial}{\partial r'}F'_m(r,r) \\
&\quad - \frac{km}{\omega^2 r'}\frac{\partial}{\partial r}(d'_m-k^2)\tilde{F}'_m(r,r')\biggr\}, \qquad (1.112a)
\end{aligned}
$$

$$
\langle H_z(r)H_z(r')\rangle = -\frac{\hbar}{2\pi}\frac{\mathrm{i}}{\omega}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\frac{1}{\omega}d_m d'_m F'_m(r,r'), \qquad (1.112b)
$$

$$
\begin{aligned}
\langle H_\theta(r)H_\theta(r')\rangle &= -\frac{\hbar}{2\pi}\frac{\mathrm{i}}{\omega}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\biggl\{-\frac{1}{\omega}\frac{\partial}{\partial r}\frac{\partial}{\partial r'}(d'_m-k^2)G'_m(r,r') \\
&\quad + \frac{mk}{r'}\frac{\partial}{\partial r}\tilde{G}'_m(r,r') + \frac{m^2k^2}{\omega rr'}F'_m(r,r') \\
&\quad - \frac{mk}{\omega^2 r}\frac{\partial}{\partial r'}(d_m-k^2)\tilde{F}'_m(r,r')\biggr\}. \qquad (1.112c)
\end{aligned}
$$

When these vacuum expectation values are substituted into the stress expression (1.109), and the property of d_m exploited,

$$
d_m r^{\pm|m|} = 0, \qquad d_m J_m(\lambda r) = -\lambda^2 J_m(\lambda r), \qquad (1.113)
$$

(of course, the later formula holds for H_m as well and the same for d'_m acting on the primed coordinate), we obtain the pressure on the cylinder as

$$\begin{aligned}
P \;=\; & \hbar\frac{\varepsilon-1}{4\pi\mathrm{i}}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\frac{\lambda^2}{\Xi}\Bigg\{H_m'^2(x')J_m(x)J_m'(x)\lambda\lambda' x'(\omega^2\mu+k^2) \\
& + H_m'(x')J_m^2(x)H_m(x')\Bigg[\frac{m^2k^2\omega^2}{x'\varepsilon}\Bigg((2\varepsilon+2)(1-\varepsilon\mu) \\
& + \frac{\omega^2\varepsilon+k^2}{\lambda^2}(1-\varepsilon\mu)^2\Bigg) + x\lambda\lambda'\left(\frac{m^2}{x'^2}\left(k^2+\frac{\omega^2}{\varepsilon}\right)+\lambda'^2\right)\Bigg] \\
& - H_m'(x')J_m'^2(x)H_m(x')\mu\lambda'^2x'(\omega^2\varepsilon+k^2) \\
& - J_m(x)J_m'(x)H_m^2(x')\lambda\lambda' x'\left[\frac{m^2}{x'^2}(k^2\mu+\omega^2)+\lambda'^2\mu\right]\Bigg\} \\
& + \hbar\frac{\mu-1}{4\pi\mathrm{i}}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\frac{\lambda^2}{\Xi}\Big\{(\varepsilon\longleftrightarrow\mu)\Big\},
\end{aligned} \tag{1.114}$$

where $x=\lambda a$, $x'=\lambda' a$ and the last bracket indicates that the expression there is similar to the one for the electric part by switching ε and μ, showing manifest symmetry between the electric and magnetic parts. In order to simplify this expression, we make an Euclidean rotation [24],

$$\omega\to\mathrm{i}\zeta \qquad \lambda\to\mathrm{i}\kappa, \tag{1.115}$$

so that the Bessel functions are replaced by the modified Bessel functions,

$$J_m(x)H_m(x')\to\frac{2}{\pi\mathrm{i}}I_m(y)K_m(y'), \tag{1.116a}$$

$$J_m'(x)H_m(x')\to-\frac{2}{\pi}I_m'(y)K_m(y'), \tag{1.116b}$$

$$J_m(x)H_m'(x')\to-\frac{2}{\pi}I_m(y)K_m'(y'), \tag{1.116c}$$

$$J_m'(x)H_m'(x')\to\frac{2\mathrm{i}}{\pi}I_m'(y)K_m'(y'), \tag{1.116d}$$

where $y=\kappa a$ and $y'=\kappa' a$. Then (1.114) becomes

$$
\begin{aligned}
P &= \frac{\varepsilon-1}{16\pi^3 a^4}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty} d\zeta a\, dka \frac{\hbar}{\tilde{\Xi}}\Big\{K_m'^2(y')I_m(y)I_m'(y)y(k^2a^2-\zeta^2a^2\mu)\\
&\quad - K_m'(y')I_m^2(y)K_m(y')\left[\frac{m^2k^2a^2\zeta^2a^2}{y'^3\varepsilon}\bigg(-2(\varepsilon+1)(1-\varepsilon\mu)\right.\\
&\quad \left. + \frac{k^2a^2-\zeta^2a^2\varepsilon}{y^2}(1-\varepsilon\mu)^2\bigg) - \frac{y^2}{y'}\left(\frac{m^2}{y'^2}\left(k^2a^2-\frac{\zeta^2a^2}{\varepsilon}\right)+y'^2\right)\right]\\
&\quad - K_m'(y')I_m'^2(y)K_m(y')\mu y'(k^2a^2-\zeta^2a^2\varepsilon)\\
&\quad - I_m(y)I_m'(y)K_m^2(y')y\left[\frac{m^2}{y'^2}(k^2a^2\mu-\zeta^2a^2)+y'^2\mu\right]\Big\} + (\varepsilon\leftrightarrow\mu),
\end{aligned}
\tag{1.117}
$$

where

$$
\tilde{\Xi} = \frac{m^2k^2a^2\zeta^2a^2}{y^2y'^2}I_m^2(y)K_m^2(y')(1-\varepsilon\mu)^2 + \Delta\tilde{\Delta}, \tag{1.118a}
$$

$$
\Delta = \varepsilon y' I_m'(y)K_m(y') - yK_m'(y')I_m(y), \tag{1.118b}
$$

$$
\tilde{\Delta} = \mu y' I_m'(y)K_m(y') - yK_m'(y')I_m(y). \tag{1.118c}
$$

This result reduces to the well-known expression for the Casimir pressure when the speed of light is the same inside and outside the cylinder, that is, when $\varepsilon\mu = 1$. Then, it is easy to see that the denominator reduces to

$$
\tilde{\Xi} = \Delta\tilde{\Delta} = \frac{(\varepsilon+1)^2}{4\varepsilon}\left[1-\xi^2y^2[(I_mK_m)']^2\right], \tag{1.119}
$$

where $\xi = (\varepsilon-1)/(\varepsilon+1)$. In the numerator introduce polar coordinates,

$$
y^2 = k^2a^2+\zeta^2a^2, \qquad ka = y\sin\theta, \qquad \zeta a = y\cos\theta, \tag{1.120}
$$

and carry out the trivial integral over θ. The result is

$$P = -\frac{1}{8\pi^2 a^4}\int_0^\infty dy\, y^2 \sum_{m=-\infty}^{\infty} \frac{d}{dy}\ln\left(1-\xi^2[y(K_m I_m)']^2\right), \tag{1.121}$$

which is exactly the finite result derived in Ref. [7].

1.6 Bulk Casimir Stress

The expression derived above, (1.117), is incomplete. It contains an unobservable "bulk" energy contribution, which the formalism would give if either medium, that of the interior with dielectric constant ε and permeability μ, or that of the exterior with dielectric constant and permeability unity, fills all the space [4]. The corresponding stresses are computed from the free Green's functions which satisfy (1.66a) and (1.66b),

$$\begin{aligned} d_m \mathcal{D}_m F_m(r,r') &= \frac{\omega^2\mu}{\varepsilon}\frac{1}{r}\delta(r-r'), && \text{(1.122a)} \\ d_m \mathcal{D}_m G_m(r,r') &= \omega^2\frac{1}{r}\delta(r-r'), && \text{(1.122b)} \end{aligned}$$

therefore

$$F_m^{(0)}(r,r') = \frac{\mu}{\varepsilon}G_m^{(0)}(r,r') = -\frac{\omega^2\mu}{\varepsilon\lambda^2}\left[\frac{1}{2|m|}\left(\frac{r_<}{r_>}\right)^{|m|} + \frac{\pi}{2\mathrm{i}}J_m(\lambda r_<)H_m(\lambda r_>)\right], \tag{1.123}$$

where $0 < r, r' < \infty$.

Notice that in this case, both $\tilde{F}_m^{(0)}$ and $\tilde{G}_m^{(0)}$ are zero and the Green's dyadics are given by

$$\begin{aligned}
\mathbf{\Gamma}^{(0)\prime}(\mathbf{r},\mathbf{r}';\omega) &= \sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\bigg\{\boldsymbol{\mathcal{M}}\boldsymbol{\mathcal{M}}'^{*}\left(-\frac{d_m-k^2}{\omega^2\mu}\right)F_m^{(0)}(r,r') \\
&\quad +\frac{1}{\omega^2\varepsilon}\boldsymbol{\mathcal{N}}\boldsymbol{\mathcal{N}}'^{*}G_m^{(0)}(r,r')\bigg\}\chi_{mk}(\theta,z)\chi^{*}_{mk}(\theta',z'), &(1.124a)\\
\mathbf{\Phi}^{(0)}(\mathbf{r},\mathbf{r}';\omega) &= \sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\bigg\{-\frac{\mathrm{i}}{\omega}\boldsymbol{\mathcal{M}}\boldsymbol{\mathcal{N}}'^{*}G_m^{(0)}(r,r') \\
&\quad -\frac{\mathrm{i}\varepsilon}{\omega\mu}\boldsymbol{\mathcal{N}}\boldsymbol{\mathcal{M}}'^{*}F_m^{(0)}(r,r')\bigg\}\chi_{mk}(\theta,z)\chi^{*}_{mk}(\theta',z'). &(1.124b)
\end{aligned}$$

It should be noticed that such Green's dyadics do not satisfy the appropriate boundary conditions, and therefore we cannot use (1.109), but rather one must compute the interior and exterior stresses individually by using (1.108). Because the two scalar Green's functions differ only by a factor of μ/ε in this case, for the electric part the inside stress tensor is

$$\begin{aligned}
T_{rr}^{(0)}(a-) &= \frac{\hbar}{2\pi\mathrm{i}}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\frac{1}{\omega^2\varepsilon}\bigg[\frac{\partial}{\partial r}\frac{\partial}{\partial r'}(-d_m G_m^{(0)}) \\
&\quad +\left(-d_m'-\frac{m^2}{rr'}\right)(-d_m G_m^{(0)})\bigg]\bigg|_{r=r'=a-}, &(1.125)
\end{aligned}$$

while the outside bulk stress is given by the same expression with $\lambda\to\lambda'=\omega^2-k^2$ and $\varepsilon=\mu=1$. When we substitute the appropriate interior and exterior Green's functions given in (1.123) we find a rather simple formula,

$$P^b = T_{rr}^{(0)}(a-)-T_{rr}^{(0)}(a+), \tag{1.126}$$

$$P^b = \frac{\hbar}{16\pi^2 a^2}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty} d\omega\, dk \Big\{ x^2 J_m'(x) H_m'(x) + (x^2 - m^2) J_m(x) H_m(x)$$
$$- x'^2 J_m'(x') H_m'(x') - (x'^2 - m^2) J_m(x') H_m(x') \Big\}. \tag{1.127}$$

After performing the Euclidean rotation, $\omega \to i\zeta$, we find that the bulk contribution to the pressure is

$$P^b = \frac{\hbar}{16\pi^3 a^4}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty} d\zeta a\, dka \Big\{ y^2 I_m'(y) K_m'(y) - (y^2 + m^2) I_m(y) K_m(y)$$
$$- y'^2 I_m'(y') K_m'(y') + (y'^2 + m^2) I_m(y') K_m(y') \Big\}. \tag{1.128}$$

This term must be subtracted from the pressure given in (1.117). Note that $P^b = 0$ in the special case $\varepsilon\mu = 1$ as it should be.

In the following, we are going to be interested in dilute dielectric media, where $\mu = 1$ and $|\varepsilon - 1| \ll 1$. This suggests that we can expand in powers of $(\varepsilon - 1)$. To do that we notice that from (1.83) and $x = \lambda a, x' = \lambda' a$ we can write the rotated ones,

$$y^2 = \zeta^2 a^2 \mu\varepsilon + k^2 a^2 \equiv x^2, \tag{1.129a}$$

$$y'^2 = \zeta^2 a^2 + k^2 a^2 \equiv y^2. \tag{1.129b}$$

The inside parameters can be expressed in terms of the outside ones, $x^2 = y^2 + \zeta^2 a^2(\varepsilon - 1)$, so that we can expand,[10]

$$x \sim y + \frac{1}{2}\frac{\zeta^2 a^2}{y}(\varepsilon - 1) - \frac{1}{8}\frac{\zeta^4 a^4}{y^3}(\varepsilon - 1)^2 + ... \tag{1.130}$$

[10] Notice that for convenience we are changing the notation, from here we are referring to $\zeta^2 a^2 \mu\varepsilon + k^2 a^2$ as x^2 and the equivalent expression for outside as y^2.

Consequently the modified Bessel functions can be expanded as,

$$
\begin{aligned}
I_m(x) &\sim I_m(y) + (\varepsilon-1)\frac{1}{2}\frac{\zeta^2 a^2}{y} I'_m(y) + \\
&\quad (\varepsilon-1)^2\left[-\frac{1}{4}\frac{\zeta^4 a^4}{y^3} I'_m(y) + \left(1+\frac{m^2}{y^2}\right)\frac{1}{8}\frac{\zeta^4 a^4}{y^2} I_m(y)\right] + \ldots, \qquad (1.131\text{a})\\
I'_m(x) &\sim I'_m(y) + (\varepsilon-1)\frac{1}{2}\frac{\zeta^2 a^2}{y}\left[-\frac{1}{y} I'_m(y) + \left(1+\frac{m^2}{y^2}\right) I_m(y)\right] - \\
&\quad (\varepsilon-1)^2\frac{\zeta^4 a^4}{8y^2}\left[-\left(1+\frac{m^2+3}{y^2}\right) I'_m(y) + \frac{2}{y}\left(1+\frac{2m^2}{y^2}\right) I_m(y)\right],
\end{aligned}
$$

(1.131b)

where we have used the modified Bessel equation,

$$
I''(y) = -\frac{1}{y} I'_m(y) + \left(1+\frac{m^2}{y^2}\right) I_m(y). \qquad (1.132)
$$

The same expansion is true for the Hankel function $K_m(x)$. When we expand the integrand in (1.128) in this manner, the leading terms yield

$$
\begin{aligned}
P^b &= -\frac{\hbar}{8\pi^3 a^4}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty} dka \int_{-\infty}^{\infty} d\zeta a \Big\{ (\varepsilon-1)\zeta^2 a^2 I_m(y) K_m(y) \\
&\quad + \frac{1}{4}(\varepsilon-1)^2 \frac{(\zeta a)^4}{y}[I_m(y)K_m(y)]' + \mathrm{O}\big((\varepsilon-1)^3\big)\Big]. \qquad (1.133)
\end{aligned}
$$

When we introduce polar coordinates as in (1.120) we find,

$$
\begin{aligned}
P^b &= -\frac{\hbar}{8\pi^3 a^4}\sum_{m=-\infty}^{\infty}\int_{0}^{\infty} dy\, y^3 \int_{0}^{2\pi} d\theta \Big[(\varepsilon-1) cos^2\theta I_m(y) K_m(y) \\
&\quad + \frac{(\varepsilon-1)^2}{4} y\, cos^4\theta [I_m(y)K_m(y)]' + \mathrm{O}\big((\varepsilon-1)^3\big)\Big], \qquad (1.134)
\end{aligned}
$$

$$
\begin{aligned}
P^b \; = \; & -\frac{\hbar}{8\pi^3 a^4} \sum_{m=-\infty}^{\infty} \int_0^\infty dy\, y^3 \Big[(\varepsilon - 1) I_m(y) K_m(y) \\
& + \frac{3(\varepsilon-1)^2}{16} y [I_m(y) K_m(y)]' + \mathrm{O}\big((\varepsilon-1)^3\big) \Big]. \qquad (1.135)
\end{aligned}
$$

1.7 Dilute Dielectric Cylinder

We now turn to the case of a dilute dielectric medium filling the cylinder, that is, set $\mu = 1$ and consider $\varepsilon - 1$ as small. We can then expand the integrand in (1.117) in powers of $(\varepsilon - 1)$ and, because the expression is already proportional to that factor, we need only expand the integrand to first order. Let us write it as

$$
P \approx \frac{(\varepsilon-1)\hbar}{16\pi^3 a^4} \int_{-\infty}^\infty d\zeta a \int_{-\infty}^\infty dka \sum_{m=-\infty}^\infty \frac{N}{\Delta\tilde{\Delta}}, \qquad (1.136)
$$

where we have noted that the $(\varepsilon - 1)^2$ in $\tilde{\Xi}$ (1.118a) can be dropped. Expanding the numerator and denominator according to

$$
N = N^{(0)} + (\varepsilon - 1) N^{(1)} + \dots, \qquad \Delta\tilde{\Delta} = 1 + (\varepsilon - 1)\Delta^{(1)} + \dots, \qquad (1.137)
$$

we can write

$$
P \approx \frac{(\varepsilon-1)\hbar}{16\pi^3 a^4} \int_{-\infty}^\infty d\zeta a \int_{-\infty}^\infty dka \sum_{m=-\infty}^\infty \Big\{ N^{(0)} + (\varepsilon-1)\big(N^{(1)} - N^{(0)}\Delta^{(1)}\big) + \dots \Big\}, \qquad (1.138)
$$

where

$$
\begin{aligned}
N^{(0)} &= -(k^2a^2-\zeta^2a^2)K'_m(y)I'_m(y) \\
&\quad -\left[\frac{m^2}{y^2}(k^2a^2-\zeta^2a^2)+y^2\right]K_m(y)I_m(y), &(1.139\text{a})\\
N^{(1)} &= \frac{\zeta^2a^2}{2}\left(1+\frac{m^2}{y^2}\right)(k^2a^2-\zeta^2a^2)K'^2_m(y)I^2_m(y) \\
&\quad +\frac{\zeta^2a^2}{2}(k^2a^2-\zeta^2a^2)K'^2_m(y)I'^2_m(y) \\
&\quad -\frac{\zeta^2a^2}{2}\left[\frac{m^2}{y^2}(k^2a^2-\zeta^2a^2)+y^2\right]K^2_m(y)I'^2_m(y) \\
&\quad -\frac{\zeta^2a^2}{2}\left(1+\frac{m^2}{y^2}\right)\left[\frac{m^2}{y^2}(k^2a^2-\zeta^2a^2)+y^2\right]K^2_m(y)I^2_m(y) \\
&\quad +\zeta^2a^2\left[y\left(1+\frac{m^2}{y^2}\right)+\frac{m^2}{y^3}(k^2a^2-\zeta^2a^2)-\frac{4}{y^3}m^2k^2a^2\right] \\
&\quad \times K'_m(y)K_m(y)I^2_m(y) \\
&\quad +\left[y^2\zeta^2a^2-\zeta^2a^2(k^2a^2-\zeta^2a^2)\right]K_m(y)K'_m(y)I_m(y)I'_m(y) \\
&\quad +\left[y\zeta^2a^2+\frac{\zeta^2a^2}{y}(k^2a^2-\zeta^2a^2)\right]K_m(y)K'_m(y)I'^2_m(y), &(1.139\text{b})\\
\Delta^{(1)} &= -\frac{1}{y}\zeta^2a^2[I_m(y)K_m(y)]'+yI'_m(y)K_m(y)-\zeta^2a^2I'_m(y)K'_m(y) \\
&\quad +\zeta^2a^2\left(1+\frac{m^2}{y^2}\right)I_m(y)K_m(y). &(1.139\text{c})
\end{aligned}
$$

When we introduce polar coordinates as in (1.120), the possible angular dependent terms that we encounter are:

$$k^2a^2 \rightarrow y^2\sin^2\theta, \tag{1.140a}$$

$$\zeta^2a^2 \rightarrow y^2\cos^2\theta, \tag{1.140b}$$

$$(k^2a^2-\zeta^2a^2) \rightarrow y^2(1-2\cos^2\theta), \tag{1.140c}$$

$$\zeta^2 a^2(k^2a^2-\zeta^2a^2) \quad\to\quad y^4\cos^2(1-2\cos^2)=y^4(\cos^2\theta-2\cos^4\theta), \tag{1.140d}$$

$$\zeta^2a^2\,k^2a^2 \quad\to\quad y^4\cos^2\theta\sin^2\theta=y^4(\cos^2\theta-\cos^4\theta), \tag{1.140e}$$

so that the only angular integrals occurring are the trivial integrals,

$$\int_0^{2\pi} d\theta\cos^2\theta=\pi, \tag{1.141a}$$

$$\int_0^{2\pi} d\theta\sin^2\theta=\pi, \tag{1.141b}$$

$$\int_0^{2\pi} d\theta\cos^4\theta=\frac{3\pi}{4}. \tag{1.141c}$$

Then, the straightforward reduction of (1.138) is

$$\begin{aligned}
P \;\approx\; & -\frac{\hbar}{8\pi^2a^4}(\varepsilon-1)\sum_{m=-\infty}^{\infty}\int_0^\infty dy\bigg\{y^3K_m(y)I_m(y) \\
& -(\varepsilon-1)\frac{y^4}{2}\bigg[\frac{1}{2}K_m'^2(y)I_m'(y)I_m(y) \\
& +K_m'(y)I_m'^2(y)K_m(y)+K_m'^2(y)I_m'^2(y)\frac{y}{4} \\
& -K_m'^2(y)I_m^2(y)\frac{y}{4}\left(1+\frac{m^2}{y^2}\right)+K_m^2(y)I_m^2(y)\frac{y}{2}\left(1+\frac{m^2}{y^2}\right)\left(1-\frac{m^2}{2y^2}\right) \\
& +K_m^2(y)I_m'(y)I_m(y)\left(1+\frac{m^2}{2y^2}\right)-K_m^2(y)I_m'^2(y)\frac{y}{2}\left(1-\frac{m^2}{2y^2}\right)\bigg]\bigg\}.
\end{aligned} \tag{1.142}$$

The leading term in the pressure,

$$P^{(1)}=-\frac{\hbar}{8\pi^2a^4}(\varepsilon-1)\sum_{m=-\infty}^{\infty}\int_0^\infty dy\,y^3K_m(y)I_m(y), \tag{1.143}$$

can also be obtained from (1.117) by setting $\varepsilon = \mu = 1$ everywhere in the integrand, and the denominator $\tilde{\Xi}$ is then unity. This is also exactly what is obtained to leading order $\mathrm{O}[(\varepsilon-1)^1]$ from the bulk stress (1.135). Thus the total stress vanishes in leading order:

$$P^{(1)} - P^b = \mathrm{O}[(\varepsilon-1)^2], \tag{1.144}$$

which is consistent with the interpretation of the Casimir energy as arising from the pairwise interaction of dilutely distributed molecules.

1.8 Evaluation of the $(\varepsilon - 1)^2$ term

We now turn to the considerably more complex evaluation of the $(\varepsilon - 1)^2$ term in (1.142).

1.8.1 Summation method

As a first approach to evaluating this second-order term, we first carry out the sum on m by use of the addition theorem

$$K_0(kP) = \sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')} K_m(k\rho) I_m(k\rho'), \quad \rho > \rho', \tag{1.145}$$

where $P = \sqrt{\rho^2 + \rho'^2 - 2\rho\rho' \cos(\phi - \phi')}$. Then by squaring this addition theorem and applying suitable differential operators, in the singular limit $\rho' \to \rho$ we obtain the following formal results:

$$\sum_{m=-\infty}^{\infty} K_m^2(k\rho)I_m^2(k\rho) = \int_0^{2\pi}\frac{d\phi}{2\pi}K_0^2(z), \tag{1.146a}$$

$$\sum_{m=-\infty}^{\infty} m^2K_m^2(k\rho)I_m^2(k\rho) = \int_0^{2\pi}\frac{d\phi}{2\pi}[K_0'(z)]^2(k\rho)^2\cos^2\frac{\phi}{2}, \tag{1.146b}$$

$$\sum_{m=-\infty}^{\infty} m^4K_m^2(k\rho)I_m^2(k\rho) = \int_0^{2\pi}\frac{d\phi}{2\pi}\left[K_0'(z)\frac{z}{4}-K_0''(z)(k\rho)^2\cos^2\frac{\phi}{2}\right]^2, \tag{1.146c}$$

$$\sum_{m=-\infty}^{\infty} K_m^2(k\rho)I_m(k\rho)I_m'(k\rho) = \int_0^{2\pi}\frac{d\phi}{2\pi}K_0(z)K_0'(z)\sin\frac{\phi}{2}, \tag{1.146d}$$

$$\sum_{m=-\infty}^{\infty} m^2K_m^2(k\rho)I_m(k\rho)I_m'(k\rho) = \int_0^{2\pi}\frac{d\phi}{2\pi}\frac{z}{2}K_0(z)K_0'(z)k\rho\cos^2\frac{\phi}{2}, \tag{1.146e}$$

$$\sum_{m=-\infty}^{\infty} K_m'^2(k\rho)I_m^2(k\rho) = \int_0^{2\pi}\frac{d\phi}{2\pi}\left[K_0'(z)\right]^2\sin^2\frac{\phi}{2}, \tag{1.146f}$$

$$\sum_{m=-\infty}^{\infty} m^2K_m^2(k\rho)I_m'^2(k\rho) = \int_0^{2\pi}\frac{d\phi}{2\pi}\frac{z^2}{4}K_0^2(z)\cos^2\frac{\phi}{2}, \tag{1.146g}$$

$$\begin{aligned}\sum_{m=-\infty}^{\infty} I_m(k\rho)I_m'(k\rho)K_m'^2(k\rho) &= \int_0^{2\pi}\frac{d\phi}{2\pi}K_0'(z)\sin\frac{\phi}{2}\\ &\quad\times\left[K_0(z)\sin^2\frac{\phi}{2}-\frac{K_0'(z)}{z}\right],\end{aligned} \tag{1.146h}$$

$$\sum_{m=-\infty}^{\infty} I_m'^2(k\rho)K_m'^2(k\rho) = \int_0^{2\pi}\frac{d\phi}{2\pi}\left[K_0(z)\sin^2\frac{\phi}{2}-\frac{K_0'(z)}{z}\right]^2. \tag{1.146i}$$

Here $z = 2k\rho\sin\frac{\phi}{2}$, and we recognize that in this singular limit (which omits delta functions, i.e., contact terms) terms with I_m and K_m interchanged in the sum have the same values. (For further discussion of this, see Ref. [19].)

When we put this all together, we obtain the following expression for the pressure at second order:

$$\begin{aligned}
P^{(2)} &= \frac{(\varepsilon-1)^2}{4096\pi^2 a^4}\int_0^\infty dz\, z^5 \int_0^{2\pi}\frac{d\phi}{2\pi}\left\{\frac{K_0'^2(z)+K_0^2(z)(1-4/z^2)}{\sin^6\phi/2}\right. \\
&\qquad \left. +\, 2\frac{(1-8/z^2)K_0^2(z)-2(1+3/z^2)K_0'^2(z)}{\sin^4\phi/2} - 16\frac{K_0^2(z)}{z^4\sin^2\phi/2}\right\} \\
&= \frac{(\varepsilon-1)^2}{15360\pi^2 a^4}\int_0^{2\pi}\frac{d\phi}{2\pi}\left[\frac{5}{\sin^6\phi/2}-\frac{66}{\sin^4\phi/2}-\frac{20}{\sin^2\phi/2}\right]. \qquad (1.147)
\end{aligned}$$

Of course, the ϕ integrals in (1.147) are divergent. However, we will regulate them by continuing from the region where the integrals converge:

$$\int_0^{2\pi} d\phi \left(\sin\frac{\phi}{2}\right)^s = \frac{2\sqrt{\pi}\,\Gamma\left(\frac{1+s}{2}\right)}{\Gamma\left(1+\frac{s}{2}\right)}, \qquad (1.148)$$

which is valid for $\mathrm{Re}\, s > -1$. We will take the right side of (1.148) to define the angular integral for negative s. Then we see that those integrals vanish when $s = -2n$ where n is a positive integer. Thus, this analytic continuation procedure says that the result (1.147) is zero. As for the bulk term, the addition theorem (1.145) implies that the y integral in the second term in (1.135) reduces to

$$\sum_{m=-\infty}^{\infty}\int_0^\infty dy\, y^4\,(I_m(y)K_m(y))' = \int_0^\infty dy\, y^4 \frac{d}{dy}K_0(0) = 0. \qquad (1.149)$$

This argument is exactly that given in Ref. [8] to show that the Casimir energy of a dilute dielectric-diamagnetic cylinder with $\varepsilon\mu = 1$ vanishes. However, it is not very convincing, because it seems to show no relevance of cancellations between various terms in the expressions for the pressure. That relevance will be established in the method which follows.

1.8.2 Numerical analysis

We now turn to a detailed numerical treatment of the second-order terms in (1.142) and (1.135). It is based on use of the uniform asymptotic or Debye expansions for the Bessel functions, $m \gg 1$:

$$
\begin{aligned}
I_m(y) &\sim \frac{1}{\sqrt{2\pi m}} t^{1/2} e^{m\eta} \left(1 + \sum_{k=1}^{\infty} \frac{u_k(t)}{m^k}\right), && \text{(1.150a)}\\
K_m(y) &\sim \sqrt{\frac{\pi}{2m}} t^{1/2} e^{-m\eta} \left(1 + \sum_{k=1}^{\infty} (-1)^k \frac{u_k(t)}{m^k}\right), && \text{(1.150b)}\\
I'_m(y) &\sim \frac{1}{\sqrt{2\pi m}} \frac{1}{z} t^{-1/2} e^{m\eta} \left(1 + \sum_{k=1}^{\infty} \frac{v_k(t)}{m^k}\right), && \text{(1.150c)}\\
K'_m(y) &\sim -\sqrt{\frac{\pi}{2m}} \frac{1}{z} t^{-1/2} e^{-m\eta} \left(1 + \sum_{k=1}^{\infty} (-1)^k \frac{v_k(t)}{m^k}\right), && \text{(1.150d)}
\end{aligned}
$$

where $y = mz$ and $t = 1/\sqrt{1+z^2}$. (The value of η is irrelevant here.) The polynomials in t appearing here are generated by

$$
\begin{aligned}
u_0(t) &= 1, \quad v_0(t) = 1, && \text{(1.151a)}\\
u_k(t) &= \frac{1}{2} t^2 (1-t^2) u'_{k-1}(t) + \frac{1}{8} \int_0^t dx\, (1-5x^2) u_{k-1}(x), && \text{(1.151b)}\\
v_k(t) &= u_k(t) + t(t^2-1) \left(\frac{1}{2} u_{k-1}(t) + t u'_{k-1}(t)\right). && \text{(1.151c)}
\end{aligned}
$$

If we use the above expansions of the modified Bessel functions we can write the second-order expression for the pressure as

$$
P = \frac{(\varepsilon - 1)^2}{16\pi^2 a^4} \sum_{m=0}^{\infty}{}' \int_0^{\infty} dy\, y^4 g_m(y), \tag{1.152}
$$

where the prime means that the term $m = 0$ is counted with half weight. The explicit form for $g_m(y)$ can be immediately read off from (1.142) and (1.135), and the prime on the summation sign means that the $m = 0$ term is counted with half weight. We have recognized that the summand is even in m. Let us subtract and add the first five terms in the uniform asymptotic expansion for g_m, $m \gg 1$:

$$g_m(y) \sim \frac{1}{2m^2} \sum_{k=1}^{5} \frac{1}{m^k} f_k(z), \tag{1.153}$$

where $z = y/m$ and

$$\begin{aligned}
f_1(z) &= \frac{4 + z^2}{4z(1+z^2)^3}, & (1.154a)\\
f_2(z) &= \frac{-8 + 8z^2 + z^4}{8z(1+z^2)^{7/2}}, & (1.154b)\\
f_3(z) &= \frac{16 - 84z^2 + 84z^4 - 16z^6 - 5z^8}{16z(1+z^2)^6}, & (1.154c)\\
f_4(z) &= \frac{-64 + 1024z^2 - 1864z^4 + 504z^6 - 9z^8}{64z(1+z^2)^{13/2}}, & (1.154d)\\
f_5(z) &= \frac{64 - 2416z^2 + 11808z^4 - 15696z^6 + 6856z^8 - 555z^{10} - 15z^{12}}{64z(1+z^2)^9}. & (1.154e)
\end{aligned}$$

We note that when these functions are inserted into (1.152) in place of g_m, the first three f_k give divergent integrals, logarithmically so for f_1 and f_3, and linearly divergent for f_2. We also note the crucial fact that

$$\int_0^\infty dz\, z^4 f_4(z) = 0, \tag{1.155}$$

which means that $\zeta(1)$, which would indicate an unremovable divergence, does not occur in the summation over m. This is the content of the proof that the Casimir energy for a dilute dielectric cylinder is finite in this order, given by Bordag and Pirozhenko [21]. We also note that when the divergent part is removed from the f_2 integration we again get zero,

$$\int_0^\infty dz \left(z^4 f_2(z) - \frac{1}{8} \right) = 0. \tag{1.156}$$

The suggestion is that this term may be simply omitted as a contact term. (But see below.)

However, the two logarithmically divergent terms, corresponding to f_1 and f_3, give finite contributions, because they are multiplied by formally zero values of the Riemann zeta function. The first one may be regulated by a small change in the power:

$$\lim_{s\to 0} \sum_{m=1}^\infty m^{2-s} \int_0^\infty dz\, z^{4-s} f_1(z) = \lim_{s\to 0} \frac{1}{4}\zeta(-2+s)\frac{1}{s} = -\frac{\zeta(3)}{16\pi^2}. \tag{1.157a}$$

The f_3 term gives similarly

$$\lim_{s\to 0} \sum_{m=0}^\infty{}' m^{-s} \int_0^\infty dz\, z^{4-s} f_3(z) = \zeta'(0)\left(-\frac{5}{16}\right) = \frac{5}{32}\ln 2\pi. \tag{1.157b}$$

Although it would appear that a finite term would emerge from f_4, that term vanishes because remarkably

$$\int_0^\infty dz\, z^4 \,\ln z\, f_4(z) = 0. \tag{1.158}$$

The f_5 term is completely finite:

$$\sum_{m=1}^{\infty} \frac{1}{m^2} \int_0^\infty dz\, z^4 f_5(z) = \frac{19\pi^2}{7680}. \tag{1.159}$$

Following the above prescription, we arrive at the following entirely finite expression for the pressure on the cylinder:

$$\begin{aligned} P &= \frac{(\varepsilon-1)^2}{32\pi^2 a^4}\Bigg\{ -\frac{\zeta(3)}{16\pi^2} + \frac{5}{32}\ln 2\pi + \frac{19\pi^2}{7680} \\ &\quad + 2\sum_{m=1}^{\infty}\int_0^\infty dy\, y^4 \left[g_m(y) - \frac{1}{2m^2}\sum_{k=1}^{5}\frac{1}{m^k} f_k(y/m)\right] \\ &\quad + \int_0^\infty dy\, y^4 \left[g_0(y) - \frac{1}{16}\frac{1}{y^4} - \frac{1}{2} f_3(y)\right]\Bigg\}. \end{aligned} \tag{1.160}$$

Here in g_0 we have subtracted a linearly divergent term, which when combined with that removed in (1.156) gives

$$\frac{1}{8}\sum_{m=0}^{\infty}{}' \int dy. \tag{1.161}$$

We regard this, rather cavalierly, as a contact term, which we simply omit. In the next section we will give the correct treatment of this f_2 term. All that remains is to do the integrals numerically. We do so for m from 0 through 4, after which we use the next nonzero term in the uniform asymptotic expansion,

$$\sum_{m=5}^{\infty}\int_0^\infty dz\, z^4 \left[\frac{1}{m^3} f_6(z) + \frac{1}{m^4} f_7(z)\right] = -\frac{209}{64512}\sum_{m=5}^{\infty}\frac{1}{m^4}, \tag{1.162}$$

because, again, the integral over f_6 vanishes.

When all the above is included, to 6 decimal places, we obtain

$$
\begin{aligned}
P &= \frac{(\varepsilon-1)^2}{32\pi^2 a^4}(-0.007612 + 0.287168 + 0.024417 - 0.002371 - 0.000012 \\
&\qquad - 0.301590) = 0.000000,
\end{aligned}
\tag{1.163}
$$

where the successive terms come from (1.157a), (1.157b), (1.159), the numerical integral over the first 4 subtracted g_ms ($m > 0$), the remainder (1.162), and the numerical integral over the subtracted g_0, respectively. This constitutes a convincing demonstration of the vanishing of the Casimir pressure in this case. It is similar to the numerical demonstration [7] of the seemingly coincidental vanishing of the Casimir energy for a dilute dielectric-diamagnetic cylinder, obtained by expanding (1.121) to order ξ^2.

1.8.3 Exponential regulator

Although the calculation in the previous subsection is quite standard, and undoubtedly correct, the reader might rightly object that zeta-function regulation has been employed, and infinite terms simply omitted. Therefore, and to make contact with known results, let us insert a regulator to make all the sums and integrals completely finite. It would be best, as in Ref. [25], to insert such a regulator before rotating the frequency in the complex plane. However, this is much more complicated here than in that reference; and because the expressions here are formally much more divergent, the regulator adopted there appears insufficient. It will suffice for the present purposes to simply insert by hand an exponential regulator into the expression (1.152):

$$P_{\text{reg}} = \frac{(\varepsilon-1)^2}{16\pi^2 a^4} \sum_{m=0}^{\infty}{}' \int_0^\infty dy\, y^4 g_m(y) e^{-\delta y}, \tag{1.164}$$

where $\delta \to 0+$ at the end of the calculation. Then it is easy to repeat the calculation of the previous subsection. One has only to carry out the sum

$$\sum_{m=1}^{\infty} e^{-\delta m z} = \frac{1}{e^{\delta z}-1}. \tag{1.165}$$

Then the f_1 term, instead of (1.157a), is

$$\int_0^\infty dz\, z^4 f_1(z) \frac{d^2}{d\delta^2 z^2} \frac{1}{e^{\delta z}-1} = \frac{13\pi}{32\delta^3} - \frac{\zeta(3)}{16\pi^2}. \tag{1.166}$$

The f_2 term has no finite part:

$$-\int_0^\infty dz\, z^4 f_2(z) \frac{d}{d\delta z} \frac{1}{e^{\delta z}-1} = -\frac{1}{16\delta}, \tag{1.167}$$

where the reader should note that no ad hoc subtraction as in (1.156) has been employed. The evaluation of (1.167) uses the fact that

$$\int_0^\infty dz\, z^2\, f_2(z) = 0. \tag{1.168}$$

The f_3 term is, instead of (1.157b),

$$\int_0^\infty dz\, z^4\, f_3(z) \left(\frac{1}{e^{\delta z}-1} + \frac{1}{2} e^{-\delta z} \right) = -\frac{315\pi}{8192\delta} + \frac{5}{32} \ln 2\pi. \tag{1.169}$$

Here we have subtracted a term from the $m = 0$ contribution:

$$\int_0^\infty dy\, y^4 \left[g_0(y) - \frac{1}{2} f_3(y) \right] e^{-\delta y} = -0.301590 + \frac{1}{16\delta}. \tag{1.170}$$

The divergent term here cancels that in (1.167), and the finite part is the value of the last integral in (1.160). Thus we recover exactly the same numerical result (1.163) found in the previous subsection, plus two divergent terms

$$P_{\rm div} = \frac{(\varepsilon - 1)^2}{32\pi^2 a^4} \left(\frac{13\pi^2}{32\delta^3} - \frac{315\pi}{8192\delta} \right). \tag{1.171}$$

The form of the divergences is exactly as expected [21, 23]. In particular, there is no $1/\delta^2$ divergence, because of the identity (1.168).

1.8.4 Interpretation of divergences

In the previous section we computed divergent contributions to the Casimir pressure for a dilute cylinder. For simplicity, we chose an exponential regulator with a small dimensionless parameter $\delta \to 0+$. How do we interpret these terms? It is perhaps easiest to imagine that δ as given in terms of a proper-time cutoff, $\delta = \tau/a$, $\tau \to 0+$. Then if we consider the energy, rather than the pressure, the divergent terms have the form

$$E_{\rm div} = e_3 \frac{aL}{\tau^3} + e_1 \frac{L}{a} \frac{1}{\tau}. \tag{1.172}$$

Here L is the (large) length of the cylinder. Thus, the leading divergence corresponds to an energy term proportional to the surface of the cylinder, and it therefore appears sensible to absorb it into a renormalized surface energy which enters into a phenomenological description of the material system. The $1/\tau$ divergence is more problematic. It is proportional to the ratio of the length to the diameter of the cylinder, so it seems likely that this would be interpretable as an energy term referring to the shape of the body. If one could compute the Casimir energy of an extremely elongated ellipsoid, we would expect an energy term proportional to the ratio of curvatures. (Of course, a cylinder has zero curvature.) This appears to be exactly of the form of a surface integral [26]

$$\int dS\, \kappa_1 \kappa_2, \tag{1.173}$$

in terms of the principal curvatures κ_i, $i = 1, 2$. Such terms are well known not to contribute to the observable energy. Had a divergent term proportional to δ^{-2} appeared in the pressure, it would have implied a divergent energy of the form $e_2(\ln a)/\tau^2$, which would have been impossible to remove. (For the dielectric sphere the situation is simpler, in that divergences are all associated with positive powers of the sphere's radius [5].)

In any case, although the structure of the divergences is universal, the coefficients of those divergences depend in detail upon the particular regularization scheme adopted. In contrast, the term proportional to $(\varepsilon - 1)^2/a^2$ is unique. Thus, of course, it could not have been any other than that zero value given by the van der Waals calculations [7].

1.9 Conclusions

Since the beginning of the subject, the identity of the Casimir force with van der Waals forces between individual molecules has been evident [27, 1]. It is essentially just a change of perspective from action at a distance to local field fluctuations. So it was no surprise that the retarded dispersion force between molecules, the Casimir-Polder force, could be derived from the Lifshitz force between parallel dielectric surfaces [28, 29]. However, the identity is not really that trivial, because both the van der Waals and the Casimir energies contain divergent contributions. This is particularly crucial when one is considering the self-energy of a single body rather than the energy of interaction of distinct bodies. Thus it was nontrivial when it was proved that the Casimir energy of a dilute dielectric sphere [5] coincided with that obtained by summing the van der Waals energies of the constituent molecules [6].

When it was shortly thereafter discovered that the sum of van der Waals forces vanished for a dielectric cylinder [30, 7] it was universally believed that the corresponding Casimir energy, in the dilute approximation, must also vanish. This result was verified by a perturbative calculation [23]. Here we have proved this by a full Casimir calculation. The importance of this finding is difficult to evaluate at this point; a zero value suggests some underlying symmetry, which is certainly far from apparent. It probably has technological implications, for example in the physics of nanotubes.

The nature of divergences in such Casimir calculations is still under active study [31, 20, 10]. The universality of the finite Casimir term makes it hard not to think it has some real significance. As an example of how subtle interpretation of divergences can be, we recall that it has now been proved that the total Casimir energy for electromagnetic modes interior and exterior to an arbitrarily shaped smooth infinitesimally thin closed perfectly conducting surface is finite [32]. This is hard to reconcile with the existence of local divergences in the energy density near the surface proportional to $(\kappa_1 - \kappa_2)^2$. Presumably, these divergences belong to the surface itself and have nothing to do with the global Casimir energy [20, 10]. But the open questions are profound and challenging. Outgoing work on this matter by Milton, Wagner and myself has just been submitted [33].

Bibliography

[1] H. B. G. Casimir. *Proc. Kon. Ned. Akad. Wetensh.*, 51:793,1948.

[2] T. H. Boyer, Phys. Rev. **174**, 1764 (1968).

[3] K. A. Milton, Ann. Phys. (N. Y.) **127**, 4 (1980).

[4] K. A. Milton and Y. J. Ng, Phys. Rev. E **55**, 4207 (1997) [arXiv: hep-th/9607186].

[5] I. Brevik, V. N. Marachevsky, and K. A. Milton Phys. Rev. Lett. **82**, 3948 (1999) [arXiv: hep-th/9810062]; G. Barton, J. Phys. A **32**, 525 (1999); J. S. Høye and I. Brevik, J. Stat. Phys. **100**, 223 (2000) [arXiv: quant-ph/9903086]; M. Bordag, K. Kirsten, and D. Vassilevich, Phys. Rev. D **59**, 085011 (1999) [arXiv: hep-th/9811015].

[6] K. A. Milton and Y. J. Ng, Phys. Rev. E, **57**, 5504 (1998) [arXiv: hep-th/9707122].

[7] K. A. Milton, A. V. Nesterenko, and V. V. Nesterenko, Phys. Rev. D **59**, 105009 (1999) [arXiv:hep-th/9711168, v3].

[8] K. A. Milton, *The Casimir Effect: Physical Manifestations of Zero-Point Energy*, World Scientific, Singapore, 2001.

[9] M. Bordag, U. Mohideen, and V. M. Mostepanenko, Phys. Rep. **353**, 1 (2001) [arXiv: quant-ph/0106045].

[10] K. A. Milton, J. Phys. A **37**, R209 (2004) [arXiv: hep-th/0406024].

[11] V. V. Nesteranko, G. Lambiase and G. Scarpetta, arXiv:hep-th/0503100.

[12] S. K. Lamoreaux, arXiv:quant-ph/9907076.

[13] M. J. Sparnaay, Physica 24, 751 (1958).

[14] A. Sarlemijn and M. J. Sparnaay, *Physics in the Making: Essays on Developments in 20th Century Physics in Honor o H. B. G. Casimir on the occasion of his 80th Birthday* North Holland, Amsterdam, 1989) 235-246.

[15] S. K. Lamoreaux, Phys. Rev. Lett. 78, 5 (1987).

[16] D. W. Sciama *The Physical Significance of the Vacuum State of a Quantum Field. "The Philosophy of vacuum" pg 137.* Simon Saunders and Harvey R. Brown.

[17] E. Elizalde, S. D. Odintsov, A. Romeo, A. A. Bytsenk and S. Zerbini *Zeta Regularization Techniques with Applications.* Wordl Scientific, 1994.

[18] J. A. Stratton, *Electromagnetic Theory.* McGraw-Hill, New York, 1941.

[19] A. Romeo and K. A. Milton, Phys. Lett. B 621, 309 (2005) [arXiv:hep-th/0504207].

[20] S. A. Fulling, J. Phys. A **36**, 6529 (2003) [arXiv: quant-ph/0302117].

[21] M. Bordag and I. G. Pirozhenko, Phys. Rev. D **64**, 025019 (2001) [arXiv: hep-th/0102193].

[22] Inés Cavero-Peláez and Kimball A. Milton, Annals of Physics (in press) [arXiv:hep-th/0412135].

[23] G. Barton, J. Phys. A **34**, 4083 (2001).

[24] I. Brevik, B. Jensen, and K. A. Milton, Phys. Rev. D **64**, 088701 (2001) [arXiv: hep-th/0004041].

[25] L. L. DeRaad, Jr. and K. A. Milton, Ann. Phys. (N.Y.) **136**, 229 (1981).

[26] D. Deutsch and P. Candelas, Phys. Rev. D **20**, 3063 (1979); P. Candelas, Ann. Phys. (N.Y.) **143**, 241 (1982); ibid. **167**, 257 (1986).

[27] H. B. G. Casimir and D. Polder, Phys. Rev. **73**, 360 (1948).

[28] E. M. Lifshitz, Zh. Eksp. Teor. Fiz. **29**, 94 (1956) [English translation: Soviet Phys. JETP **2**, 73 (1956)]; I. D. Dzyaloshinskii, E. M. Lifshitz, and L. P. Pitavskii, Usp. Fiz. Nauk **73**, 381 (1961) [English translation: Soviet Phys. Usp. **4**, 153 (1961)]; L. D. Landau and E. M. Lifshitz, *Electrodynamics of Continuous Media*, Pergamon, Oxford, 1960.

[29] J. Schwinger, L. L. DeRaad, Jr., and K. A. Milton, Ann. Phys. (N.Y.) **115**, 1 (1978).

[30] A. Romeo, private communication with K. A. Milton.

[31] N. Graham, R. L. Jaffe, V. Khemani, M. Quandt, O. Schroeder, and H. Weigel, Nucl. Phys. B **677**, 379 (2004) [arXiv: hep-th/0309130], and references therein.

[32] F. Bernasconi, G. M. Graf, and D. Hasler, Ann. Inst. H. Poincaré **4**, 1001 [arXiv: math-ph/0302035].

[33] I. Cavero-Peláez, K. A. Milton, and J. Wagner, arXiv: hep-th/0508001. Submitted to Phy. Rev. D.

Chapter 2

About $\mathcal{PT}$-Symmetric Quantum Theories

2.1 Introduction

Traditionally, a quantum system is defined by a Hermitian Hamiltonian; we borrow the notion of Hamiltonian from classical mechanics and we quantize it. The position and momentum of the particle become operators,

$$\mathcal{H}(x,p) \to H(X,P) \equiv \mathcal{H}(x \to X, p \to P), \tag{2.1}$$

and ordering becomes important since X and P do not commute. The dynamics of the quantum system is then defined by the Schrödinger equation

$$\mathrm{i}\hbar \frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle, \tag{2.2}$$

where $|\psi(t)\rangle$ is a vector state representing the state of a particle and can be expanded on a basis of orthonormal eigenvectors $|\alpha_i\rangle$

$$|\psi(t)\rangle = \sum_i c_i(t)|\alpha_i\rangle, \tag{2.3}$$

with eigenvalues E_i,

$$H|\alpha_i\rangle = E_i|\alpha_i\rangle. \tag{2.4}$$

The eigenvalues E_i are just numbers, they are the result of measuring the Hamiltonian operator and they represent the energy of the state $|\alpha_i\rangle$. Experimentally these energy levels can be observed and measured. In order to assure that the eigenvalues are real numbers we impose the condition on the Hamiltonian that this is Hermitian,

$$H = H^\dagger. \tag{2.5}$$

The hermiticity condition restricts the number of Hamiltonians which can possibly manifest in nature as we know it so far, and excludes any complex Hamiltonian.

Despite this, in the late 1970's[1], some people observed that a non-Hermitian Hamiltonian of the kind

$$H = P^2 + \mathrm{i}X^3, \tag{2.6}$$

had real eigenvalues [1, 2]. This finding didn't create a revolution at the time since the price they had to pay for it seemed to be too high. A Hamiltonian of the kind (2.5) possesses orthogonal eigenvectors, $\langle\alpha_i|\alpha_j\rangle = \delta_{ij}$ where $|\alpha_i\rangle$ is considered to be

[1] Even years before that, there had been some interest [1].

normalized, however the eigenstates of (2.6) are not mutually orthogonal and the time evolution generated by the Hamiltonian is not unitary. These negative features resulted in the non-Hermitian Hamiltonians being disregarded even though they were giving a real spectrum.

It was almost twenty years later [3], in 1996, when Bender and Milton applied the delta expansion to the kind of Hamiltonians in (2.6), and in 1998 Bender et al. noticed that the non-Hermitian Hamiltonians with real spectrum, all had space-time reflection symmetry [4]. They called it $\mathcal{PT}$-Symmetric Quantum Mechanics.

The operators $\mathcal{P}$, parity reflection, and $\mathcal{T}$, time reversal, are defined by the transformations:

$$\mathcal{P} \;:\; x \longrightarrow -x, \qquad p \longrightarrow -p, \qquad \mathrm{i} \longrightarrow \mathrm{i}, \tag{2.7a}$$

$$\mathcal{T} \;:\; x \longrightarrow x, \qquad p \longrightarrow -p, \qquad \mathrm{i} \longrightarrow -\mathrm{i}. \tag{2.7b}$$

What Bender et al. observed was that the Hamiltonian in (2.6), even though it is not invariant under $\mathcal{P}$ transformations or $\mathcal{T}$ transformations independently, it is indeed invariant under the combination $\mathcal{PT}$,

$$\mathcal{P} \;:\; H = P^2 + \mathrm{i}X^3 \longrightarrow H^{\mathcal{P}} = P^2 - \mathrm{i}X^3, \tag{2.8a}$$

$$\mathcal{T} \;:\; H = P^2 + \mathrm{i}X^3 \longrightarrow H^{\mathcal{T}} = P^2 - \mathrm{i}X^3, \tag{2.8b}$$

$$\mathcal{PT} \;:\; H = P^2 + \mathrm{i}X^3 \longrightarrow H^{\mathcal{PT}} = P^2 + \mathrm{i}X^3. \tag{2.8c}$$

They throughly studied several cases. For example, the harmonic oscillator[2]

$$H = p^2 + x^2 \tag{2.9}$$

has $\mathcal{PT}$ symmetry and well-known real eigenvalues, $E_n = 2n + 1$. If we

- add $\mathrm{i}x$ to (2.9) the resultant Hamiltonian does not break $\mathcal{PT}$ symmetry,

$$p^2 + x^2 + ix \longrightarrow p^2 + (-x)^2 + (-\mathrm{i})(-x) = p^2 + x^2 + ix, \tag{2.10}$$

  and it has indeed real eigenvalues, $E_n = 2n + 5/4$.

- add $\mathrm{i}x - x$ we get a Hamiltonian that is not $\mathcal{PT}$ invariant,

$$p^2 + x^2 + ix - x \longrightarrow p^2 + x^2 + \mathrm{i}x + x, \tag{2.11}$$

  and its eigenvalues are not real, $E_n = 2n + 1 + \frac{\mathrm{i}}{2}$.

This is a good motivation for further studying whether this symmetry can define a well-behaved quantum theory. If that is the case, we should be able to guarantee that when substituting the mathematical-looking condition of Hermiticity $H = H^\dagger$ by the more physical condition of $\mathcal{PT}$-Symmetry, $H = H^{\mathcal{PT}}$, the spectrum of the system under consideration remains real and has unitary time evolution.

[2]We will write small letters for momentum and position but it is implicit that these are operators

2.2 Foundations of the $\mathcal{PT}$-Symmetric Quantum Mechanics

A new theory of quantum mechanics should reflect all the characteristics of a quantum theory that has a Hermitian Hamiltonian. In particular,

1. The state of a particle is represented by a vector in the Hilbert space. The eigenvalues of the Hamiltonian $H|\psi_n\rangle = E_n|\psi\rangle$ are the energy levels of the system, they are real and form a complete set in the Hilbert space of physical state vectors. A norm should be well-defined in the Hilbert space so that the probability of a particle $|\alpha\rangle$ to have energy E_n is a positive real number,

$$\mathcal{P}(E_n) \propto |\langle\psi_n|\alpha\rangle|^2 > 0. \tag{2.12}$$

2. The time evolution operator should be unitary, so that the norm is preserved in time. For a state that undergoes a time translation $U = \mathcal{U}(t, t_0)$ we can write,

$$|\alpha\rangle \to U|\alpha\rangle. \tag{2.13}$$

   In ordinary quantum mechanics the time evolution of the inner product would be

$$\langle\beta|\alpha\rangle \to \langle\beta|U^\dagger U|\alpha\rangle = \langle\beta|\alpha\rangle, \tag{2.14}$$

   where $U^\dagger$ is the adjoint of the operator U; $\langle f, Ug\rangle = \langle U^\dagger f, g\rangle$ for any f and g in the Hilbert space. In $\mathcal{PT}$-symmetric theories it should be true that the time evolution operator U is such that

$$\langle\beta|\alpha\rangle \rightarrow \langle\beta|U^{\mathcal{PT}}U|\alpha\rangle = \langle\beta|\alpha\rangle, \tag{2.15}$$

for certain $U^{\mathcal{PT}}$ yet to be defined and such that it will ensure that the probability remains constant in time.

3. The Hamiltonian incorporates the symmetries of the theory. If the physical system is symmetric under transformations carried by the operator O, this commutes with the Hamiltonian, $[H, O] = 0$, and has the same eigenvectors.

   In the new theory, the Hamiltonians under consideration commute with $\mathcal{PT}$ since the space-time reflection symmetry is preserved,

$$[H, \mathcal{PT}] = 0. \tag{2.16}$$

Traditionally if (2.16) is satisfied, both operators have the same eigenfunctions. However, the fact that $\mathcal{PT}$ is not a linear operator[3] imposes an extra condition, *the $\mathcal{PT}$-Symmetry of the Hamiltonian must be unbroken* (more details below). This assures that the eigenfunctions of H are simultaneously eigenfunctions of the operator $\mathcal{PT}$ and it guarantees the existence of a real energy spectrum. In [5] the proof goes like: Since H has unbroken $\mathcal{PT}$-symmetry, the operators H and $\mathcal{PT}$ have the same eigenstates,

$$H\psi = E\psi, \quad \text{and simultaneously} \quad \mathcal{PT}\psi = \lambda\psi, \tag{2.17}$$

[3]Notice that $\mathcal{T}$ changes i into $-$i

and λ is of the form $e^{\mathrm{i}\theta}$ since recalling that $(\mathcal{PT})^2 = 1$ and multiplying the second equation in (2.17) by $\mathcal{PT}$ we get,

$$1 = \lambda^* \lambda, \qquad \text{therefore} \qquad \lambda = e^{\mathrm{i}\theta}. \tag{2.18}$$

Now it is trivial to see that the energies E are real. If we multiply $\mathcal{PT}$ from the left to the first equation in (2.17) and use (2.16) we find,

$$e^{\mathrm{i}\theta} E\psi = E^* e^{\mathrm{i}\theta}\psi, \qquad \text{then} \qquad E = E^*. \tag{2.19}$$

As an example of the importance of the unbroken $\mathcal{PT}$-symmetry, let's look at the Hamiltonian,

$$H = p^2 + x^2(\mathrm{i}x)^\epsilon, \tag{2.20}$$

where ϵ is a real number. This Hamiltonian is analyzed in detail by Bender et al. in [6]. It turns out that for $\epsilon \geq 0$ the eigenfunctions of H are also eigenfunctions of $\mathcal{PT}$, which is illustrated in the above reference for the cases when $\epsilon = 1, 2, 5$ and for the general case of $-1 < \epsilon < 0$. However, when $\epsilon < 0$ the Hamiltonian spontaneously breaks the $\mathcal{PT}$ symmetry and, even though the two operators commute with each other, their eigenfunctions are not all simultaneously the same for H and $\mathcal{PT}$. Moreover, the energy of these eigenfunctions is complex. This implies that a transition occurs at $\epsilon = 0$, below this value complex values of the energy start appearing [6]. The transition point happens to be the harmonic oscillator. The spontaneous breaking of $\mathcal{PT}$ and the appearance of complex eigenvalues can have important consequences. In [7] they show that this feature is a manifestation of an explicit breaking of supersymmetry.

For an unbroken $\mathcal{PT}$-symmetric Hamiltonian, the completeness condition of the eigenfunctions $\psi_n(x)$ of real argument x reads as,

$$\sum_{n=0}^{\infty}(-1)^n\psi_n(x)\psi_n(x') = \delta(x - x'). \tag{2.21}$$

This result has been confirmed numerically with excellent accuracy [8, 9]. The natural way to extend the definition of inner product of two functions from the Hermitian formulation to the $\mathcal{PT}$-symmetric formulation would be:

$$(\psi_m(x), \psi_n(x)) \equiv \int dx\, [\mathcal{PT}\psi_m(x)]\, \psi_n(x). \tag{2.22}$$

with $\mathcal{PT}\psi_m(x) = [\psi_m(-x)]^* = \psi_m(x)$, since the phase can be absorbed into the eigenfunction. However, with respect to this inner product we have,

$$(\psi_m(x), \psi_n(x)) = (-1)^n\delta_{nm}. \tag{2.23}$$

In other words, the eigenstates are orthogonal to each other only if $m \neq n$. With this formulation it seems that the energy eigenvalues span the Hilbert space into eigenstates half of which the norm is positive and half negative. This means that the norm defined by $\mathcal{PT}$ in (2.22) is indefinite and it is not acceptable from the probabilistic point of view of a good quantum theory.

We resolve this problem by observing that associated with the $\mathcal{PT}$ norm there exist an additional symmetry since there are equal number of positive and negative norm states. To give a measure of this *signature of the $\mathcal{PT}$ norm states* we introduce a new operator $\mathcal{C}$. The construction of $\mathcal{C}$ is in terms of the energy eigenstates of the Hamiltonian,

$$\mathcal{C}(x, y) = \sum_{n} \psi_n(x)\psi_m(y). \tag{2.24}$$

If we can obtain the linear operator $\mathcal{C}$, we can define, for two arbitrary functions $f(x)$ and $g(x)$, a new inner product with positive define signature,

$$(f(x), g(x)) \equiv \int dx \, [\mathcal{CPT} f(x)] \, g(x). \tag{2.25}$$

2.3 The $\mathcal{C}$ operator

In the last part of the previous section we learn that the $\mathcal{C}$ operator is of fundamental importance in order to construct a consistent $\mathcal{PT}$-symmetric theory [10]. From the construction of the $\mathcal{C}$ operator in (2.24) we can deduce several of its properties:

- The eigenvalues of $\mathcal{C}$ are ± 1.

$$\mathcal{C}\psi_n(x) = \int dy \, \mathcal{C}(x, y)\psi_n(y) = \int dy \sum_{m=0}^{\infty} \psi_m(x)\psi_m(y)\psi_n(y) \tag{2.26}$$

  using (2.23) we find that

$$\mathcal{C}\psi_n(x) = (-1)^n \psi_n(x). \tag{2.27}$$

- $\mathcal{C}$ commutes with the Hamiltonian.

$$
\begin{aligned}
[\mathcal{C}, H]\psi_n(x) &= \mathcal{C}H\psi_n(x) - H\mathcal{C}\psi_n(x) \\
&= \mathcal{C}E\psi_n(x) - H(-1)^n\psi_n(x) \\
&= E(-1)^n\psi_n(x) - (-1)^n E\psi_n(x) = 0. \qquad (2.28)
\end{aligned}
$$

- $\mathcal{C}$ commutes with $\mathcal{PT}$.[4]

$$
\begin{aligned}
[\mathcal{C}, \mathcal{PT}]\psi_n(x) &= \mathcal{C}\psi_n(x) - \mathcal{PT}(-1)^n\psi_n(x) \\
&= (-1)^n\psi_n(x) - (-1)^n\psi_n(x) = 0. \qquad (2.29)
\end{aligned}
$$

Once we have constructed the operator $\mathcal{C}$ we can define an inner product that is constant in time and positive definite,

$$(f(x), g(x)) \equiv \int_C dx\, [\mathcal{CPT}f(x)]\, g(x). \qquad (2.30)$$

like in (2.25). And in terms of $\mathcal{CPT}$ the eigenstates $\psi_n(X)$ are complete under

$$\sum_n \psi_n(x)[\mathcal{CPT}\psi_n(y)] = \delta(x-y). \qquad (2.31)$$

It is interesting to notice that the inner product defined in this way depends on the choice of the Hamiltonian, making it in this way a dynamical characteristic. Notice that since $\mathcal{CPT}$ commutes with the Hamiltonian H, the eigenstate $\psi_0(x)$ evolves in

[4] Remember that the phase can be absorbed in the eigenstate

time in the ordinary way,

$$\psi_t(x) = e^{-iHt}\psi_0(x), \tag{2.32}$$

and its norm with respect to (2.25) is preserved in time.

$$\begin{aligned}
(\psi_t, \psi_t) &= \int_C dx[\mathcal{CPT}\psi_t(x)]\psi_t(x) \\
&= \int_C dx[\mathcal{CPT}e^{-iHt}\psi_0(x)]e^{-iHt}\psi_0(x) \\
&= \int_C dx\, e^{iHt}[\mathcal{CPT}\psi_0(x)]e^{-iHt}\psi_0(x) = (\psi_0, \psi_0).
\end{aligned} \tag{2.33}$$

The main difficulty in the construction of the operator $\mathcal{C}$ is that you need to compute all of the energy eigenstates. This is not an easy task in quantum mechanics and it makes it impossible to generalize for a quantum field theory, where we do not have an equivalent Schrödinger equation for eigenstates. A more reasonable way to calculate $\mathcal{C}$ is by making use of the properties above [11].

By solving the Schrödinger equation of the $\mathcal{PT}$-symmetric Hamiltonian

$$\begin{aligned}
H &= H_0 + \epsilon H_1 \\
&= \frac{1}{2}p^2 + \frac{1}{2}x^2 + i\epsilon x^3,
\end{aligned} \tag{2.34}$$

the operator $\mathcal{C}$ is calculated in [12] in a perturbative way up to order three in powers of ϵ. They calculate the eigenfunctions and construct the $\mathcal{C}$ operator like in (2.24). The result they find can be expressed as an exponential

$$\mathcal{C} = e^{Q}\mathcal{P}, \qquad Q = \epsilon Q_1 + \epsilon^3 Q_3 + \dots \tag{2.35}$$

where Q real function of x and p and $\mathcal{P}$ the parity operator.The good news is that this can be generalized to quantum field theory[5] .

This representation of $\mathcal{C}$ is very convenient because it allows us to construct the operator just from the commutation relations with the Hamiltonian and it makes it possible to extend the concept to QFT. If we assume we can construct (2.35) the operator Q must be such that $\mathcal{C}$ satisfies the appropriate conditions, (2.27), (2.28) and (2.29). In [11] this is applied to the case of quantum field theory with cubic interaction, and in [13] the $\mathcal{C}$ operator for a $\mathcal{PT}$-symmetric quantum electrodynamics is calculated.

We look for $\mathcal{C}$ of the form

$$\mathcal{C} = e^{Q}\mathcal{P}, \tag{2.36}$$

subject to the mentioned conditions. Then

- $\mathcal{C}^2 = 1$, then we can write

$$1 = e^{Q(x,p)}\mathcal{P}e^{Q(x,p)}\mathcal{P} = e^{Q(x,p)}e^{Q(-x,-p)}, \tag{2.37}$$

  and conclude that

$$Q(x,p) = -Q(-x,-p). \tag{2.38}$$

- $[\mathcal{C},\mathcal{PT}] = 0$. Therefore $e^{Q}\mathcal{PPT} = \mathcal{PT}e^{Q}\mathcal{P}$ and since $\mathcal{P}^2 = 1$ if we multiply by $\mathcal{T}$ form the right and apply $\mathcal{T}^2 = 1$, we get

$$\begin{aligned} e^{Q(x,p)} &= \mathcal{PT}e^{Q(x,p)}\mathcal{PT} \\ &= e^{Q(-x,p)}, \end{aligned} \tag{2.39}$$

[5]The above series is constructed with only odd powers of ϵ since as it is noted in [11] the Q_{2n}'s can be derived from equations arising from $Q_{2n-1}, Q_{2n-3}, ...Q_1$.

since Q is real.

- $[\mathcal{C}, H] = 0$ This condition depends on the Hamiltonian of the system since we remember that H has to be invariant under the product $\mathcal{PT}$ but not necessarily under parity itself. To first order in ϵ, for the Hamiltonian in (2.34) we have,

$$\begin{aligned} 0 &= [e^{Q}\mathcal{P}, (H_0 + \epsilon H_1)] \\ &= [\epsilon Q_1 \mathcal{P}, H_0] + [\mathcal{P}, \epsilon H_1]. \end{aligned} \tag{2.40}$$

In the last equality the first commutator is,

$$[Q_1\mathcal{P}, H_0] = [Q_1, H_0]\mathcal{P}, \tag{2.41}$$

where we have used the fact that $\mathcal{P}$ commutes with H_0. Noticing that H_1 anti-commutes with $\mathcal{P}$,

$$[\mathcal{P}, H_1] = -2H_1\mathcal{P}, \tag{2.42}$$

we conclude that,

$$[Q_1, H_0] = 2H_1. \tag{2.43}$$

We can find similar expressions for all the Q_{2n+1} by substituting the expansion of Q in (2.40) and grouping coefficients of ϵ.

We can summarize by saying that Q is odd with respect to both $\mathcal{P}$ and $\mathcal{T}$, since the first condition tells us that $1 = e^{Q}\mathcal{P}e^{Q}\mathcal{P}$ and multiplying this equation by e^{-Q}, we have that $e^{-Q} = \mathcal{P}e^{Q}\mathcal{P}$ and therefore $Q = -\mathcal{P}Q\mathcal{P}$.

From the second condition $e^Q\mathcal{PT} = \mathcal{PT}e^Q$. If we multiply by $\mathcal{P}$ from the left and consider that $\mathcal{T}^2 = 1$ and Q is odd with respect to $\mathcal{P}$ we have that $\mathcal{P}e^Q\mathcal{PT} = \mathcal{T}e^Q$; $e^{-Q} = \mathcal{T}e^Q\mathcal{T}$. That is to say

$$Q = -\mathcal{T}Q\mathcal{T}, \tag{2.44}$$

and the relation with the Hamiltonian is determined by the dynamics of the system itself.

The calculation of the operator $\mathcal{C}$ for a quantum field theory with cubic interaction is presented in [11], where the complex field Hamiltonian is

$$H = \int d\mathbf{x} \left[\frac{1}{2}\pi_{\mathbf{x}}^2 + \frac{1}{2}(\nabla\phi_{\mathbf{x}})^2 + \frac{1}{2}\mu^2\phi_{\mathbf{x}}^2 + i\epsilon\phi_{\mathbf{x}}^3\right], \tag{2.45}$$

in some dimension D. This quantum field theory has already appeared in the literature in studies of the Lee-Yang edge singularity [14] and in the Reggeon field theory [15]. The construction of the $\mathcal{C}$ operator for the $i\phi^3$ field theory shows that this quantum field theory is a fully acceptable unitary quantum theory and not just an interesting but unrealistic mathematical curiosity.

Furthermore, an exact construction of the $\mathcal{C}$ operator [16] was carried out for the Lee model, a cubic quantum field theory in which mass, wave-function, and coupling-constant renormalization can be done exactly [17]. The construction of the $\mathcal{C}$ operator for the Lee model explains a long-standing puzzle. It is known that there is a critical value of the renormalized coupling constant g for the Lee model and that when g exceeds this critical value, the unrenormalized coupling constant becomes pure imaginary, and hence the Hamiltonian becomes non-Hermitian. As a consequence, a ghost state having negative Hermitian norm appears when $g > g_{\rm crit}$, and the presence of this ghost state causes the S matrix to be nonunitary. By constructing the $\mathcal{C}$ operator we can reinterpret the Hilbert space for the theory. By using a $\mathcal{CPT}$ inner product, the ghost state now has a positive norm and the Lee model becomes a consistent unitary quantum field theory. This physical reinterpretation of the Lee model was anticipated by F. Kleefeld in a series of papers [18].

Next we show the construction of a unitary $\mathcal{PT}$-symmetric Quantum Electrodynamics [13].

2.4 $\mathcal{PT}$-Symmetric Quantum Electrodynamics

In [13] we examine $\mathcal{PT}$-symmetric quantum electrodynamics, a very interesting non-Hermitian quantum field theory. Unlike the scalar $i\phi^3$ field theory, $\mathcal{PT}$-symmetric quantum electrodynamics possesses many of the features of conventional quantum electrodynamics, including Abelian gauge invariance. The Hamiltonian for quantum electrodynamics becomes non-Hermitian if the unrenormalized electric charge e is taken to be imaginary. However, if one also specifies that the potential A^μ in such a theory transforms as a pseudovector rather than a vector, then the Hamiltonian becomes $\mathcal{PT}$ symmetric. The construction of the $\mathcal{C}$ operator provides strong evidence that $\mathcal{PT}$-symmetric quantum electrodynamics is a viable and consistent unitary quantum field theory.

While $\mathcal{PT}$-symmetric quantum electrodynamics is similar to an $i\phi^3$ field theory because its interaction Hamiltonian is cubic and its coupling constant is pure imaginary, this quantum field theory is especially interesting because, like a $\mathcal{PT}$-symmetric $-\phi^4$ scalar quantum field theory in four dimensions, $\mathcal{PT}$-symmetric electrodynamics is asymptotically free [19]. The only asymptotically free quantum field theories described by Hermitian Hamiltonians are those that possess a *non-Abelian* gauge invariance; $\mathcal{PT}$ symmetry allows for new kinds of asymptotically free theories that do not have to possess a non-Abelian gauge invariance.

In order to formulate a Lorentz covariant quantum field theory one begins by specifying the Lorentz transformation properties of the quantum fields under the proper orthochronous Lorentz group. (For example, one can specify that the field $\phi(\mathbf{x},t)$ transforms as a scalar). In addition, one is free to specify the transformation properties of the fields under parity reflection. (For example, one can specify that $\phi(\mathbf{x},t)$ transforms as a scalar, so that it does not change sign under $\mathcal{P}$, or that it transforms as a pseudoscalar, so that it changes sign under $\mathcal{P}$). Having fully specified the transformation properties of the fields, one then formulates the (scalar) Lagrangian in terms of these fields.

A non-Hermitian but $\mathcal{PT}$-symmetric version of electrodynamics can be constructed by assuming that the four-vector potential transforms as an *axial* vector [20]. As a consequence, the electromagnetic fields transform under parity reflection like

$$\mathcal{P}: \quad \mathbf{E} \to \mathbf{E}, \quad \mathbf{B} \to -\mathbf{B}, \quad \mathbf{A} \to \mathbf{A}, \quad A^0 \to -A^0, \tag{2.46}$$

as opposite to their transformation properties in the ordinary quantum electrodynamics where under $\mathcal{P}$, $\mathbf{E} \to -\mathbf{E}$, $\mathbf{B} \to \mathbf{B}$, $\mathbf{A} \to -\mathbf{A}$, and $A^0 \to A^0$. Under time reversal, the transformations are assumed to be conventional:

$$\mathcal{T}: \quad \mathbf{E} \to \mathbf{E}, \quad \mathbf{B} \to -\mathbf{B}, \quad \mathbf{A} \to -\mathbf{A}, \quad A^0 \to A^0. \tag{2.47}$$

The Lagrangian of the theory then possesses an imaginary coupling constant in order that it be invariant under the product of these two symmetries:

$$\mathcal{L} = -\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + \tfrac{1}{2}\psi^\dagger\gamma^0\gamma^\mu\tfrac{1}{i}\partial_\mu\psi + \tfrac{1}{2}m\psi^\dagger\gamma^0\psi + ie\psi^\dagger\gamma^0\gamma^\mu\psi A_\mu. \tag{2.48}$$

The corresponding Hamiltonian density is then

$$\mathcal{H} = \tfrac{1}{2}(E^2 + B^2) + \psi^\dagger\left[\gamma^0\gamma^k\left(-i\nabla_k + ieA_k\right) + m\gamma^0\right]\psi. \tag{2.49}$$

The electric current appearing in both the Lagrangian and Hamiltonian densities, $j^\mu = \psi^\dagger\gamma^0\gamma^\mu\psi$, transforms conventionally under both $\mathcal{P}$ and $\mathcal{T}$:

$$\mathcal{P}j^\mu(\mathbf{x},t)\mathcal{P} = \begin{pmatrix} j^0 \\ -\mathbf{j} \end{pmatrix}(-\mathbf{x},t), \tag{2.50a}$$

$$\mathcal{T}j^\mu(\mathbf{x},t)\mathcal{T} = \begin{pmatrix} j^0 \\ -\mathbf{j} \end{pmatrix}(\mathbf{x},-t). \tag{2.50b}$$

Just as in the case of ordinary quantum electrodynamics, $\mathcal{PT}$-symmetric electrodynamics has an Abelian gauge invariance. In this paper we choose to work in the Coulomb gauge, $\boldsymbol{\nabla}\cdot\mathbf{A}=0$, so the nonzero canonical equal-time commutation relations are

$$\{\psi_a(\mathbf{x},t),\psi_b^\dagger(\mathbf{y},t)\} = \delta_{ab}\delta(\mathbf{x}-\mathbf{y}), \tag{2.51a}$$

$$[A_i^T(\mathbf{x}),E_j^T(\mathbf{y})] = -i\left[\delta_{ij}-\frac{\nabla_i\nabla_j}{\nabla^2}\right]\delta(\mathbf{x}-\mathbf{y}), \tag{2.51b}$$

where T denotes the transverse part,

$$\boldsymbol{\nabla}\cdot\mathbf{A}^T=\boldsymbol{\nabla}\cdot\mathbf{E}^T=0. \tag{2.52}$$

In the following, the symbols $\mathbf{E}$ and $\mathbf{B}$ represent the transverse parts of the electromagnetic fields, so

$$\boldsymbol{\nabla}\cdot\mathbf{E}=\boldsymbol{\nabla}\cdot\mathbf{B}=0. \tag{2.53}$$

2.5 Calculation of the $\mathcal{C}$ Operator

As in quantum-mechanical systems and scalar quantum field theories, we seek a $\mathcal{C}$ operator in the form [11]

$$\mathcal{C}=\mathrm{e}^Q\mathcal{P}, \tag{2.54}$$

where $\mathcal{P}$ is the parity operator, and our objective will be to calculate the operator Q [21].

Because $\mathcal{C}$ must satisfy the three defining properties

$$\begin{aligned} \mathcal{C}^2 &= 1, & \text{(2.55a)} \\ [\mathcal{C}, \mathcal{PT}] &= 0, & \text{(2.55b)} \\ [\mathcal{C}, H] &= 0, & \text{(2.55c)} \end{aligned}$$

we know that Q has to satisfy (2.3) and (2.44). In particular we infer from Eq. (2.55a) that

$$e^Q \mathcal{P} e^Q \mathcal{P} = 1, \qquad Q = -\mathcal{P} Q \mathcal{P}, \tag{2.56}$$

and because $\mathcal{PT} = \mathcal{TP}$ and $\mathcal{T}^2 = \mathcal{P}^2 = 1$, we infer from (2.55b) that

$$0 = e^Q \mathcal{PPT} - \mathcal{PT} e^Q \mathcal{P} = e^Q \mathcal{T} - \mathcal{T} e^{-Q}, \tag{2.57}$$

then

$$Q = -\mathcal{T} Q \mathcal{T}. \tag{2.58}$$

The two equations (2.55a) and (2.55b) can be thought of as kinematical constraints on Q.

The third equation (2.55c), which can be thought of as a dynamical condition on Q, allows us to determine Q perturbatively. If we separate the interaction part of the Hamiltonian from the free part,

$$H = H_0 + eH_1, \tag{2.59}$$

and seek Q in the form of a power series

$$Q = eQ_1 + e^3 Q_3 + \cdots , \tag{2.60}$$

then the first contribution to the Q operator is determined by

$$[Q_1, H_0] = 2H_1. \tag{2.61}$$

To use Eq. (2.61) to determine the operator Q_1, we construct the most general nonlocal *ansatz* for the operator Q_1 in terms of the sixteen independent Dirac tensors. There is no condition of gauge invariance on this operator because we have chosen to work in the Coulomb gauge. There are sixteen tensor functions in principle, which we take to be defined by

$$\begin{aligned}
Q_1 \;=\; \int d\mathbf{x}\, d\mathbf{y}\, d\mathbf{z} \Big\{ & \left[f_+^{kl}(\mathbf{x},\mathbf{y},\mathbf{z}) E^k(\mathbf{x}) + f_-^{kl}(\mathbf{x},\mathbf{y},\mathbf{z}) B^k(\mathbf{x}) \right] \psi^\dagger(\mathbf{y}) \gamma^0 \gamma^l \psi(\mathbf{z}) \\
& + \left[g_+^{k}(\mathbf{x},\mathbf{y},\mathbf{z}) E^k(\mathbf{x}) + g_-^{k}(\mathbf{x},\mathbf{y},\mathbf{z}) B^k(\mathbf{x}) \right] \psi^\dagger(\mathbf{y}) \gamma^0 \gamma^5 \psi(\mathbf{z}) \\
& + \left[h_+^{k}(\mathbf{x},\mathbf{y},\mathbf{z}) E^k(\mathbf{x}) + h_-^{k}(\mathbf{x},\mathbf{y},\mathbf{z}) B^k(\mathbf{x}) \right] \psi^\dagger(\mathbf{y}) \gamma^5 \psi(\mathbf{z}) \\
& + \left[j_+^{kl}(\mathbf{x},\mathbf{y},\mathbf{z}) E^k(\mathbf{x}) + j_-^{kl}(\mathbf{x},\mathbf{y},\mathbf{z}) B^k(\mathbf{x}) \right] \psi^\dagger(\mathbf{y}) \gamma^l \psi(\mathbf{z}) \\
& + \left[F_+^{kl}(\mathbf{x},\mathbf{y},\mathbf{z}) B^k(\mathbf{x}) + F_-^{kl}(\mathbf{x},\mathbf{y},\mathbf{z}) E^k(\mathbf{x}) \right] \psi^\dagger(\mathbf{y}) \gamma^0 \gamma^5 \gamma^l \psi(\mathbf{z}) \\
& + \left[G_+^{k}(\mathbf{x},\mathbf{y},\mathbf{z}) B^k(\mathbf{x}) + G_-^{k}(\mathbf{x},\mathbf{y},\mathbf{z}) E^k(\mathbf{x}) \right] \psi^\dagger(\mathbf{y}) \gamma^0 \psi(\mathbf{z}) \\
& + \left[H_+^{k}(\mathbf{x},\mathbf{y},\mathbf{z}) B^k(\mathbf{x}) + H_-^{k}(\mathbf{x},\mathbf{y},\mathbf{z}) E^k(\mathbf{x}) \right] \psi^\dagger(\mathbf{y}) \psi(\mathbf{z}) \\
& + [J_+^{kl}(\mathbf{x},\mathbf{y},\mathbf{z}) B^k(\mathbf{x}) + J_-^{kl}(\mathbf{x},\mathbf{y},\mathbf{z}) E^k(\mathbf{x})] \psi^\dagger(\mathbf{y}) \gamma^5 \gamma^l \psi(\mathbf{z}) \Big\}.
\end{aligned} \tag{2.62}$$

In Eq. (2.62) we have taken into account the fact that the electric and magnetic fields are transverse, $\boldsymbol{\nabla}\cdot\mathbf{E} = \boldsymbol{\nabla}\cdot\mathbf{B} = 0$ [see Eq. (2.53)]. The parity constraint (2.56) is already satisfied by this form of Q because $f_\pm$, $g_\pm$, $\cdots$, are respectively even and odd functions:

$$f_\pm(\mathbf{x},\mathbf{y},\mathbf{z}) = \pm f_\pm(-\mathbf{x},-\mathbf{y},-\mathbf{z}). \tag{2.63}$$

For example, the first term in (2.62) is proportional to $\psi^\dagger(\mathbf{y})\gamma^0\gamma^l\psi(\mathbf{z})$, which is a vector and therefore it changes sign under parity. If this term is multiplied by the electric field (that does not change sign under $\mathcal{P}$) the coefficient has to be an even function, f_+. But if it is multiplied by the magnetic field that changes sign under $\mathcal{P}$ the coefficient has to be an odd function f_-. We will see that with this construction the time-reversal constraint (2.58) is automatically satisfied by Q_1 in (2.62).

The solution of Eq. (2.61) is obtained by using the canonical commutation relations (2.51a) and (2.51b), to find the required commutators. These imply that

$$\left[E^k(\mathbf{x}), \frac{1}{2}\int d\mathbf{w} B^2(\mathbf{w})\right] = \int d\mathbf{w}\,\left[E^k(\mathbf{x}), B^n(\mathbf{w})\right] B^l(\mathbf{w}), \tag{2.64a}$$

since $B^l(\mathbf{w})$ is a c-number. Using $B^n(\mathbf{w}) = \epsilon^{lmn}\nabla^m A^n(\mathbf{w})$ we have,

$$\begin{aligned}\left[E^k(\mathbf{x}), \frac{1}{2}\int d\mathbf{w} B^2(\mathbf{w})\right] &= \epsilon^{lmn}\int d\mathbf{w} B^l(\mathbf{w})\nabla^m_{\mathbf{w}}\left[E^k(\mathbf{x}), A^n(\mathbf{x})\right] \\ &= \epsilon^{lmn}\int d\mathbf{w} B^l(\mathbf{w})\nabla^m_{\mathbf{w}} i(\delta^{kn} - \frac{\nabla^l\nabla^k}{\nabla^2})\delta(\mathbf{x}-\mathbf{w}).\end{aligned} \tag{2.64b}$$

The second term in the parenthesis does not contribute because it is multiplied by ϵ^{lmn}, then the above becomes,

$$
\begin{aligned}
& i\epsilon^{lmn}\int d\mathbf{w}B^l(\mathbf{w})\nabla^m_{\mathbf{w}}\delta(\mathbf{x}-\mathbf{w}) = -i\epsilon^{lmn}\int d\mathbf{w}B^l(\mathbf{w})\nabla^m_{\mathbf{x}}\delta(\mathbf{x}-\mathbf{w}) \\
= \;& -i\epsilon^{klm}\nabla^m B^l(\mathbf{x}) = i(\boldsymbol{\nabla}\times\mathbf{B})_k(\mathbf{x}) \qquad (2.64c)
\end{aligned}
$$

Doing the same for $\left[B^k(\mathbf{x}), \frac{1}{2}\int d\mathbf{w}E^2(\mathbf{w})\right]$ we find

$$
\begin{aligned}
\left[E^k(\mathbf{x}), \frac{1}{2}\int d\mathbf{w}B^2(\mathbf{w})\right] &= i(\boldsymbol{\nabla}\times\mathbf{B})_k(\mathbf{x}), \qquad (2.65a) \\
\left[B^k(\mathbf{x}), \frac{1}{2}\int d\mathbf{w}E^2(\mathbf{w})\right] &= -i(\boldsymbol{\nabla}\times\mathbf{E})_k(\mathbf{x}), \qquad (2.65b)
\end{aligned}
$$

Let's look at the Dirac part now. There are two terms,

$$
\begin{aligned}
&\left[\int d\mathbf{y}\, d\mathbf{z}\, \phi(\mathbf{y},\mathbf{z})\psi^\dagger(\mathbf{y})\Gamma\psi(\mathbf{z}), \int d\mathbf{w}\, \psi^\dagger(\mathbf{w})\gamma^0\gamma^k\frac{1}{i}\nabla_k\psi(\mathbf{w})\right] \\
&\quad = \frac{i}{2}\int d\mathbf{y}\, d\mathbf{z}\big[(\nabla^z_k + \nabla^y_k)\phi(\mathbf{y},\mathbf{z})\psi^\dagger(\mathbf{y})\{\Gamma, \gamma^0\gamma^k\}\psi(\mathbf{z}) \\
&\qquad + (\nabla^z_k - \nabla^y_k)\phi(\mathbf{y},\mathbf{z})\psi^\dagger(\mathbf{y})[\Gamma, \gamma^0\gamma^k]\psi(\mathbf{z})\big], \qquad (2.65c)
\end{aligned}
$$

$$
\begin{aligned}
&\left[\int d\mathbf{y}\, d\mathbf{z}\, \phi(\mathbf{y},\mathbf{z})\psi^\dagger(\mathbf{y})\Gamma\psi(\mathbf{z}), m\int d\mathbf{w}\, \psi^\dagger(\mathbf{w})\gamma^0\psi(\mathbf{w})\right] \\
&\quad = m\int d\mathbf{y}\, d\mathbf{z}\, \phi(\mathbf{y},\mathbf{z})\psi^\dagger(\mathbf{y})[\Gamma, \gamma^0]\psi(\mathbf{z}). \qquad (2.65d)
\end{aligned}
$$

When we use this to compute (2.61), we find that there are sixteen resulting equations for the tensor coefficients, which break up into two independent sets of eight equations each. The following lists the coefficients of each Dirac tensor:

$$\psi^\dagger(\mathbf{y})\gamma^0\gamma^5\psi(\mathbf{z}) \; : \; g_+^k i(\boldsymbol{\nabla}\times\boldsymbol{B})_k - g_-^k i(\boldsymbol{\nabla}\times\boldsymbol{E})_k - i(\nabla_l^z - \nabla_l^y)\left[J_-^{kl}E^k + J_+^{kl}B^k\right]$$
$$-2m\left[h_+^k E^k + h_-^k B^k\right] = 0 \tag{2.66a}$$

$$\psi^\dagger(\mathbf{y})\gamma^5\psi(\mathbf{z}) \; : \; h_+^k i(\boldsymbol{\nabla}\times\boldsymbol{B})_k - h_-^k i(\boldsymbol{\nabla}\times\boldsymbol{E})_k - i(\nabla_l^z + \nabla_l^y)\left[F_-^{kl}E^k + F_+^{kl}B^k\right]$$
$$-2m\left[g_+^k E^k + g_-^k B^k\right] = 0 \tag{2.66b}$$

$$\psi^\dagger(\mathbf{y})\gamma^l\psi(\mathbf{z}) \; : \; j_+^{kl} i(\boldsymbol{\nabla}\times\boldsymbol{B})_k - j_-^{kl} i(\boldsymbol{\nabla}\times\boldsymbol{E})_k + i(\nabla_l^z - \nabla_l^y)\left[G_-^k E^k + G_+^k B^k\right]$$
$$+\epsilon_{mpl}(\nabla_p^z + \nabla_p^y)\left[J_-^{km}E^k + J_+^{km}B^k\right] - 2m\left[f_+^{kl}E^k + f_-^{kl}B^k\right]$$
$$= 0 \tag{2.66c}$$

$$\psi^\dagger(\mathbf{y})\gamma^0\gamma^5\gamma^l\psi(\mathbf{z}) \; : \; F_-^{kl} i(\boldsymbol{\nabla}\times\boldsymbol{B})_k - F_+^{kl} i(\boldsymbol{\nabla}\times\boldsymbol{E})_k + (\nabla_p^z - \nabla_p^y)\epsilon_{mpl}\left[f_+^{km}E^k\right.$$
$$\left.+f_-^{km}B^k\right] - i(\nabla_l^z + \nabla_l^y)\left[h_+^k E^k + h_-^k B^k\right] = 0 \tag{2.66d}$$

$$\psi^\dagger(\mathbf{y})\gamma^0\psi(\mathbf{z}) \; : \; G_-^k i(\boldsymbol{\nabla}\times\boldsymbol{B})_k - G_+^k i(\boldsymbol{\nabla}\times\boldsymbol{E})_k + i(\nabla_l^z - \nabla_l^y)\left[j_+^{kl}E^k + j_-^{kl}B^k\right]$$
$$= 0 \tag{2.66e}$$

$$\psi^\dagger(\mathbf{y})\gamma^5\gamma^l\psi(\mathbf{z}) \; : \; J_-^{kl} i(\boldsymbol{\nabla}\times\boldsymbol{B})_k - J_+^{kl} i(\boldsymbol{\nabla}\times\boldsymbol{E}) - i(\nabla_l^z - \nabla_l^y)\left[g_+^k E^k + g_-^k B^k\right]$$
$$+\epsilon_{mpl}(\nabla_p^z + \nabla_p^y)\left[j_+^{km}E^k + j_-^{km}B^k\right] = 0 \tag{2.66f}$$

$$\psi^\dagger(\mathbf{y})\psi(\mathbf{z}) \; : \; H_-^k i(\boldsymbol{\nabla}\times\boldsymbol{B})_k - H_+^k i(\boldsymbol{\nabla}\times\boldsymbol{E})_k + i(\nabla_l^z + \nabla_l^y)\left[f_+^{kl}E^k + f_-^{kl}B^k\right]$$
$$= 0 \tag{2.66g}$$

$$\psi^\dagger(\mathbf{y})\gamma^0\gamma^l\psi(\mathbf{z}) \; : \; f_+^{kl} i(\boldsymbol{\nabla}\times\boldsymbol{B})_k - f_-^{kl} i(\boldsymbol{\nabla}\times\boldsymbol{E})_k + i(\nabla_l^z + \nabla_l^y)\left[H_-^k E^k + H_+^k B^k\right]$$
$$+\epsilon_{mpl}(\nabla_p^z - \nabla_p^y)\left[F_-^{km}E^k + F_+^{km}B^k\right] - 2m\left[j_+^{kl}E^k + j_-^{kl}B^k\right]$$
$$= -2i\delta(\mathbf{x}-\mathbf{y})\delta(\mathbf{x}-\mathbf{z})\frac{1}{\nabla^2}(\boldsymbol{\nabla}\times\boldsymbol{B})_l(\mathbf{x}). \tag{2.66h}$$

Since there is only one inhomogeneous term, this means that the coefficients that satisfy the set of equations with no driving term must vanish. These are $f_-, g_+, h_-, j_+, F_-, G_+, H_-$, and J_+. The remaining equations are most conveniently written in momentum space, where the Fourier transform is defined by

$$\tilde{f}(\mathbf{p}) = \int d\mathbf{x}\, \mathrm{e}^{-i\mathbf{p}\cdot\mathbf{x}} f(\mathbf{x}). \tag{2.67}$$

If the momenta corresponding to the coordinates $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are $\mathbf{p}, \mathbf{q}, \mathbf{r}$, then as a result of translational invariance there is an overall momentum-conserving delta function, which sets $\mathbf{p} + \mathbf{q} + \mathbf{r} = \mathbf{0}$. Using dyadic notation, it is not hard to show that these equations are, in terms of the two independent vectors $\mathbf{p}$ and $\mathbf{t} = \mathbf{r} - \mathbf{q}$, given by

$$\begin{aligned}
\mathbf{p} \times \tilde{\mathbf{g}}_- + \tilde{\mathbf{J}}_- \cdot \mathbf{t} - 2m\tilde{\mathbf{h}}_+ &= \mathbf{0}, && \text{(2.68a)}\\
\mathbf{p} \times \tilde{\mathbf{h}}_+ + \tilde{\mathbf{F}}_+ \cdot \mathbf{p} + 2m\tilde{\mathbf{g}}_- &= \mathbf{0}, && \text{(2.68b)}\\
\mathbf{p} \times \tilde{\mathbf{j}}_- - i\tilde{\mathbf{J}}_- \times \mathbf{p} - \tilde{\mathbf{G}}_-\mathbf{t} - 2m\tilde{\mathbf{f}}_+ &= \mathbf{0}, && \text{(2.68c)}\\
\mathbf{p} \times \tilde{\mathbf{F}}_+ - \tilde{\mathbf{h}}_+\mathbf{p} + i\tilde{\mathbf{f}}_+ \times \mathbf{t} &= \mathbf{0}, && \text{(2.68d)}\\
\mathbf{p} \times \tilde{\mathbf{G}}_- + \tilde{\mathbf{j}}_- \cdot \mathbf{t} &= \mathbf{0}, && \text{(2.68e)}\\
\mathbf{p} \times \tilde{\mathbf{J}}_- - \tilde{\mathbf{g}}_-\mathbf{t} + i\tilde{\mathbf{j}}_- \times \mathbf{p} &= \mathbf{0}, && \text{(2.68f)}\\
\mathbf{p} \times \tilde{\mathbf{H}}_+ + \tilde{\mathbf{f}}_+ \cdot \mathbf{p} &= \mathbf{0}, && \text{(2.68g)}\\
\mathbf{p} \times \tilde{\mathbf{f}}_+ - i\tilde{\mathbf{F}}_+ \times \mathbf{t} - \tilde{\mathbf{H}}_+\mathbf{p} + 2m\tilde{\mathbf{j}}_- &= \frac{2}{p^2}\mathbf{1} \times \mathbf{p}. && \text{(2.68h)}
\end{aligned}$$

We may take all the coefficient tensors to be transverse to $\mathbf{p}$ in the first index,

$$\mathbf{p} \cdot \tilde{\mathbf{f}}_+ = \mathbf{0}, \quad \mathbf{p} \cdot \tilde{\mathbf{F}}_+ = \mathbf{0}, \quad \mathbf{p} \cdot \tilde{\mathbf{g}}_- = 0, \tag{2.69}$$

and so on, which is consistent with the transversality of the electric and magnetic fields appearing in the construction (2.62) of Q_1. This property then allows us to solve Eqs. (2.68d), (2.68e), (2.68f) and (2.68g) for $\tilde{\mathbf{F}}_+$, $\tilde{\mathbf{G}}_-$, $\tilde{\mathbf{H}}_+$, and $\tilde{\mathbf{J}}_-$ in terms of $\tilde{\mathbf{f}}_+$, $\tilde{\mathbf{g}}_-$, $\tilde{\mathbf{h}}_+$, and $\tilde{\mathbf{j}}_-$:

$$\begin{aligned}
\tilde{\mathbf{F}}_+ &= \frac{1}{p^2}\left(-\mathbf{p} \times \tilde{\mathbf{h}}_+\mathbf{p} + i\mathbf{p} \times \tilde{\mathbf{f}}_+ \times \mathbf{t}\right), & (2.70a)\\
\tilde{\mathbf{G}}_- &= \frac{1}{p^2}\mathbf{p} \times \tilde{\mathbf{j}}_- \cdot \mathbf{t}, & (2.70b)\\
\tilde{\mathbf{J}}_- &= -\frac{1}{p^2}\left(\mathbf{p} \times \tilde{\mathbf{g}}_-\mathbf{t} - i\mathbf{p} \times \tilde{\mathbf{j}}_- \times \mathbf{p}\right), & (2.70c)\\
\tilde{\mathbf{H}}_+ &= \frac{1}{p^2}\mathbf{p} \times \tilde{\mathbf{f}}_+ \cdot \mathbf{p}. & (2.70d)
\end{aligned}$$

The remaining four equations then imply that

$$\begin{aligned}
\mathbf{p} \times \tilde{\mathbf{g}}_-\left(p^2 - t^2\right) + i\mathbf{p} \times \tilde{\mathbf{j}}_- \cdot (\mathbf{p} \times \mathbf{t}) - 2mp^2\tilde{\mathbf{h}}_+ &= \mathbf{0}, & (2.71a)\\
i\mathbf{p} \times \tilde{\mathbf{f}}_+ \cdot (\mathbf{p} \times \mathbf{t}) - 2mp^2\tilde{\mathbf{g}}_- &= \mathbf{0}, & (2.71b)\\
\mathbf{p} \times \tilde{\mathbf{j}}_- \cdot (\mathbf{pp} - \mathbf{tt}) - i\mathbf{p} \times \tilde{\mathbf{g}}_-\mathbf{p} \times \mathbf{t} - 2mp^2\tilde{\mathbf{f}}_+ &= \mathbf{0}, & (2.71c)\\
\mathbf{p} \times \tilde{\mathbf{f}}_+ \cdot \left[(\mathbf{tt} - \mathbf{1}t^2) - (\mathbf{pp} - \mathbf{1}p^2)\right] + i\mathbf{p} \times \tilde{\mathbf{h}}_+\mathbf{p} \times \mathbf{t} + 2mp^2\tilde{\mathbf{j}}_- &= 2(\mathbf{1} \times \mathbf{p}). & (2.71d)
\end{aligned}$$

Equations (2.71b) and (2.71a) allow us to solve immediately for $\tilde{\mathbf{g}}_-$ and $\tilde{\mathbf{h}}_+$ in terms of $\tilde{\mathbf{j}}_-$ and $\tilde{\mathbf{f}}_+$:

$$\tilde{\mathbf{g}}_- = \frac{1}{2mp^2} i\mathbf{p} \times \tilde{\mathbf{f}}_+ \cdot (\mathbf{p} \times \mathbf{t}), \tag{2.72a}$$

$$\tilde{\mathbf{h}}_+ = \frac{i}{2mp^2} \left[\mathbf{p} \times \tilde{\mathbf{j}}_- \cdot (\mathbf{p} \times \mathbf{t}) + (t^2 - p^2) \frac{1}{2m} \tilde{\mathbf{f}}_+ \cdot (\mathbf{p} \times \mathbf{t}) \right], \tag{2.72b}$$

and then from Eqs. (2.71c) and (2.71d) we obtain two equations for $\tilde{\mathbf{j}}_-$ and $\tilde{\mathbf{f}}_+$:

$$\mathbf{p} \times \tilde{\mathbf{j}}_- \cdot (\mathbf{tt} - \mathbf{pp}) + 2mp^2 \tilde{\mathbf{f}}_+ \cdot \left[\mathbf{1} + \frac{(\mathbf{p} \times \mathbf{t})(\mathbf{p} \times \mathbf{t})}{4m^2p^2} \right] = \mathbf{0}, \tag{2.73a}$$

$$\begin{aligned} \mathbf{p} \times \tilde{\mathbf{f}}_+ \cdot & \left[(\mathbf{tt} - \mathbf{1}t^2) - (\mathbf{pp} - \mathbf{1}p^2) + \frac{t^2 - p^2}{4m^2p^2} (\mathbf{p} \times \mathbf{t})(\mathbf{p} \times \mathbf{t}) \right] \\ & + 2mp^2 \tilde{\mathbf{j}}_- \cdot \left[\mathbf{1} + \frac{(\mathbf{p} \times \mathbf{t})(\mathbf{p} \times \mathbf{t})}{4m^2p^2} \right] = 2(\mathbf{1} \times \mathbf{p}). \end{aligned} \tag{2.73b}$$

From Eq. (2.73a) we see that

$$\tilde{\mathbf{f}}_+ \cdot (\mathbf{t} \times \mathbf{p}) = \mathbf{0}. \tag{2.74}$$

Then we can solve Eq. (2.73a) for $\mathbf{f}_+$ in terms of $\mathbf{j}_-$, which when substituted into Eq. (2.73b) yields an equation that can be solved easily for $\mathbf{j}_-$.

In this way it is straightforward to solve for all the coefficient tensors. In terms of the denominator

$$\Delta = 4m^2p^2 + k^2, \tag{2.75}$$

where $\mathbf{k} = \mathbf{p} \times \mathbf{t}$, the nonzero tensor coefficients in Q_1 are

$$\tilde{\mathbf{F}}_+ = \frac{2i}{p^2\Delta} \mathbf{p} \times \mathbf{k}\, \mathbf{p}, \tag{2.76a}$$

$$\tilde{\mathbf{f}}_+ = -\frac{2}{p^2\Delta} \mathbf{p} \times \mathbf{k}\, \mathbf{t}, \tag{2.76b}$$

$$\tilde{\mathbf{j}}_- = \frac{4m}{\Delta}\mathbf{1} \times \mathbf{p}, \tag{2.76c}$$

$$\tilde{\mathbf{J}}_- = -i\mathbf{j}_-, \tag{2.76d}$$

$$\tilde{\mathbf{h}}_+ = -\frac{2i}{\Delta}\mathbf{k}, \tag{2.76e}$$

$$\tilde{\mathbf{H}}_+ = 2\frac{\mathbf{p}\cdot\mathbf{t}}{p^2}\frac{\mathbf{k}}{\Delta}, \tag{2.76f}$$

$$\tilde{\mathbf{g}}_- = \mathbf{0}, \tag{2.76g}$$

$$\tilde{\mathbf{G}}_- = \frac{4m}{p^2\Delta}\mathbf{p} \times \mathbf{k}. \tag{2.76h}$$

Note that the parity constraint (2.63) is satisfied because the $+$ quantities are even under $\mathbf{p} \to -\mathbf{p}$, $\mathbf{t} \to -\mathbf{t}$, while the $-$ quantities are odd. The time-reversal constraint (2.58) is satisfied because of the presence of i in $\tilde{\mathbf{F}}_+$, $\tilde{\mathbf{J}}_-$, and $\tilde{\mathbf{h}}_+$, owing to $\mathcal{T}$ being an antiunitary operator. The odd functions undergo another sign change under $\mathcal{T}$ because all momenta change sign [see Eq. (2.67)].

2.6 Conclusion

We have no doubt that the $\mathcal{PT}$ symmetric quantum theories are rather unconventional, however the interest in such theories is increasing. This new kind of quantum theories opens new and interesting possibilities. The theory is relatively new, still under development and deeper understanding is necessary specially in its quantum field theory version. The $\mathcal{PT}$-symmetric theories are rather dynamical theories since the fundamental definitions, like the norm or inner product, depend on the Hamiltonian of the theory itself.

One of the keys for developing a successful $\mathcal{PT}$-symmetric theory is the construction of the $\mathcal{C}$ operator. It can be constructed perturbatively in quantum field theories. In particular, we have showed how to calculate it in $\mathcal{PT}$ quantum electrodynamics. With the construction of the first-order term in the Q operator and thus the leading approximation to the $\mathcal{C}$ operator, we provide convincing evidence that the $\mathcal{PT}$-symmetric quantum electrodynamics originally proposed in Ref. [20] is unitary and that this construction enables us to obtain a unitary S matrix for the theory. Therefore, there can be little doubt that such a $\mathcal{PT}$-symmetric theory is self-consistent and one should now investigate whether such a theory may be used to describe natural phenomena. Indeed, this theory provides an interesting test of Gell-Mann's *Totalitarian Principle*, which states that "Everything which is not forbidden is compulsory" [22].

Bibliography

[1] T. T. Wu, Phys. Rev. 115, 1390 (1959).

[2] R. Brower, M. Furman and M. Moshe, Phys. Lett. B76,213 (1978)

[3] Carl M. Bender and Kimball A. Milton. Phys. Rev. D55 (1997) 3255-3259. [arXiv:hep-th/960848].

[4] C. M. Bender and S. Boettcher, Phys. Rev. Lett. 80, 5243 (1998),[arXiv:physics/9712001 v3].

[5] Carl M. Bender, arXiv:quant-ph/0501052.

[6] Carl M.Bender, Stefan Boettcher, and Peter N. Meisinger, J. Math. Phys.40; 2201-2229 (1999), [arXiv:quant-ph/9809072].

[7] P. Dorey, C. Dunning and R. Tateo, J. Phys. A **34**, L391 (2001); *ibid.* **34**, 5679 (2001).

[8] C. M. Bender, S. Boettcher, P. N. Meisinger, and Q. Wang, Phys. Lett. A 302, 286 (2002).

[9] G. A. Mezincescu, J. Phys. A: Math. Gen. 33, 4911(2000).

[10] C. M. Bender, D. C. Brody, and H. F. Jones, Phys. Rev.Lett. **89**, 270401 (2002) and Am. J. Phys. **71**, 1095 (2003) [quant-ph/0208076].

[11] C. M. Bender, D. C. Brody, and H. F. Jones, Phys. Rev. Lett., 93, 251601-1 4 (2004) [arXiv:hep-th/0402011 v3]; Phys. Rev.D70, 025001-1 19 [hep-th/0402183].

[12] C. M. Bender, P. N. Meisinger, and Q. Wang, J. Phys. A **36**, 1973 (2003) [quant-ph/0211166].

[13] Carl M. Bender, Inés Cavero-Peláez, Kimball A. Milton, and K. V. Shajesh, Phys. Lett. B613 (2005) 97-104 [arXiv:hep-th/0402011].

[14] M. E. Fisher, Phys. Rev. Lett. **40**, 1610 (1978); J. L. Cardy, *ibid.* **54**, 1345 (1985); J. L. Cardy and G. Mussardo, Phys. Lett. B **225**, 275 (1989); A. B. Zamolodchikov, Nucl. Phys. B **348**, 619 (1991).

[15] H. D. I. Abarbanel, J. D. Bronzan, R. L. Sugar, and A. R. White, Phys. Rep. **21**, 119 (1975); R. Brower, M. Furman, and M. Moshe, Phys. Lett. B **76**, 213 (1978); B. Harms, S. Jones, and C.-I Tan, Nucl. Phys. **171** 392 (1980) and Phys. Lett. B **91B**, 291 (1980).

[16] C. M. Bender, S. F. Brandt, J.-H. Chen, and Q. Wang, preprint hep-th/0411064, Phys. Rev. D, in press.

[17] T. D. Lee, Phys. Rev. **95**, 1329 (1954); G. Källén and W. Pauli, Dan. Mat. Fys. Medd. **30**, No. 7 (1955); S. S. Schweber, *An Introduction to Relativistic Quantum Field Theory*, (Row, Peterson and Co., Evanston, 1961), Chap. 12; G. Barton, *Introduction to Advanced Field Theory*, (John Wiley & Sons, New York, 1963),

Chap. 12; S. Weinberg, Phys. Rev. **102**, 285 (1956); R. Amado, Phys. Rev. **122**, 696 (1961); M. T. Vaughn, Nuovo Cimento **40**, 803 (1965); V. Glaser and G. Källén, Nucl. Phys. **2**, 706 (1956).

[18] F. Kleefeld, hep-th/0408028 and hep-th/0408097.

[19] C. M. Bender, K. A. Milton, and V. M. Savage, Phys. Rev. D **62**, 85001 (2000). "Solution of Schwinger-Dyson Equations for $\mathcal{PT}$-Symmetric Quantum Field Theory".

[20] K. A. Milton, Czech. J. Phys. **54**, 85 (2004), hep-th/0308035.

[21] In Refs. [16, 23] it was shown that the correct representation of the $\mathcal{C}$ operator has the form $\mathcal{C} = e^Q\mathcal{P}_\mathrm{I}$, where $\mathcal{P}_\mathrm{I}$ is the *intrinsic* parity reflection operator. (The difference between $\mathcal{P}$ and $\mathcal{P}_\mathrm{I}$ is that $\mathcal{P}_\mathrm{I}$ does not reflect the spatial arguments of the fields.) However, this is a technical distinction for the case of a cubic interaction Hamiltonian because it does not affect the final result for the Q operator.

[22] R. Kane, Erkenntnis **24**, 115 (1986).

[23] C. M. Bender, S. F. Brandt, J.-H. Chen, and Q. Wang, Phys. Rev. D, preprint hep-th/0412316.

Appendix A

About other wonders

Nadear

Aunque anochezca estaré pescando
y tú podrás venir a verme
a conversar con el río
tú o cualquiera que entre mis sombras
distinga mi sombra
bajo la luz de la luna acariciando el río.

Mecido por el canto del búho
no dormiré
aunque aparezca en mi semblante
el principio azul
de la bella durmiente
es que fabrico sueños

en mi factoría de papel

junto al río.

Lo que hago allí es no esperarte

no esperar

fluir con el agua

con la facilidad de la sonrisa

fluir y mirar

la cabellera de la vida

cómo en silencio inunda mi mirada

cómo me lleva

incesantemente aquí sentado

mojándome en el río

el río, el río.

Javier Sanz Seral