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P-HARMONIC MORPHISMS, MINIMAL FOLIATIONS, AND CONFORMAL DEFORMATIONS OF METRICS

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P-HARMONIC MORPHISMS, MINIMAL FOLIATIONS, AND CONFORMAL DEFORMATIONS OF METRICS

A Dissertation APPROVED FOR THE

DEPARTMENT OF MATHEMATICS

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DEDICATION

to

My parents

Ou Yangwu and Liao Hanzhen

for

bringing me into this beautiful and challenging world

and

teaching me the value of learning

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INTRODUCTION

0.1 Background

For $p \in (1, \infty)$, a p-harmonic map is a map $\varphi : (M, g) \longrightarrow (N, h)$ between Riemannian manifolds that is a critical point of the p-energy functional. Locally, p-harmonic maps are solutions to a system of PDEs. When $p = 2$, they are called harmonic maps which include geodesics, harmonic functions, and minimal isometric immersions as special cases. A p-harmonic morphism is a map between Riemannian manifolds that preserves solutions of p-Laplace's equations in the sense that it pulls back germs of p-harmonic functions to germs of p-harmonic functions. When $p = 2$, they are called harmonic morphisms, and include holomorphic functions, Hopf fibrations, and Riemannian submersions with minimal fibers among others. In stochastic theory, harmonic morphisms are known to be Brownian path-preserving maps meaning that they send a Brownian motion to a Brownian motion. The study of harmonic morphisms has received wide attention from mathematicians since 1979 when Fuglede [19] and Ishihara [26] independently proved that a nonconstant harmonic morphism is equivalent to a horizontally weakly conformal harmonic map. For a detailed account and the developments of the study of harmonic morphisms see the recent book [8] by Baird and Wood. A regularly updated bibliography is available in [22]. Recently, Loubeau and Burel ([12] [35]) proved that the Fuglede-Ishihara type of characterization of harmonic morphisms is carried over to p-harmonic morphisms. Thus, p-harmonic morphisms make up a special subset of p-harmonic maps.

p-Harmonic maps of different p values have different regularity theory whilst p-Laplace operator of different p values have different applications in physics. Also,

p-harmonic morphisms of different p values have different geometry as shown by the following theorem which gives some interesting links among horizontal conformality, p-harmonicity and minimality of fibers of such maps:

Theorem 0.1. ([3], [4], [12], [53]) *Let $m > n \geq 2$ and $\varphi : (M^m, g) \to (N^n, h)$ be a horizontally conformal submersion.*

(I) *If $p = n$, then φ is p-harmonic (hence a p-harmonic morphism) if and only if $\{\varphi^{-1}(y)\}_{y \in N}$ is a minimal foliation of (M, g) of codimension n.*

(II) *If $p \neq n$, then any two of the following conditions imply the third:*

(a) *φ is p-harmonic (hence a p-harmonic morphism),*

(b) *$\{\varphi^{-1}(y)\}_{y \in N}$ is a minimal foliation of (M, g) of codimension n,*

(c) *φ is horizontally homothetic.*

Among others, the following two problems are fundamental in the study of p-harmonic morphisms.

> **Problem 1.** Given two model spaces (e.g., some nice space such as space forms or more general symmetric spaces) (M^m, g) and (N^n, h), does there exist a p-harmonic morphism $(M^m, g) \longrightarrow (N^n, h)$ for a particular value of $p > 1$?
>
> **Problem 2.** For fixed $p > 1$, and a fixed pair of model spaces (M^m, g) and (N^n, h), classify p-harmonic morphisms $(M^m, g) \longrightarrow (N^n, h)$ where it is possible.

Since the equation for a p-harmonic morphism is an over-determined system of PDEs, the existence of such maps is rare in general and it is even harder to find such maps from a compact Riemannian manifold. On the other hand, many interesting classification results are obtained: Baird-Wood ([6], [7]) classified all harmonic morphisms from simply connected 3-dimensional space forms to a Riemannian surface. Kasue and Washio ([29]) classified harmonic morphisms $\mathbb{R}^m \longrightarrow (N^n, h)\,(n \geq 3)$ with totally geodesic fibers. Eells and Yiu ([15]) proved that a harmonic morphism $S^m \longrightarrow S^n$ whose component functions are homogeneous polynomials of the same degree has to be one of Hopf fibrations. More recently, Bryant ([11]) obtained a classification of the submersive harmonic morphisms with one-dimensional fibers on a space form of dimension at least 4: there are just two types, those that arise from Killing fields and those with geodesic fibers orthogonal to a Riemannian foliation by totally umbilic hypersurfaces.

0.2 Main Results

In this section we describe the main results presented in this thesis. Some of them have appeared in [40], [41], [42], and [43]. For some related joint works see [36], [44] and [45].

The main results of this thesis can be put into the following four categories:

A. Constructions of nontrivial p-harmonic morphisms. Since there is no general way to solve the over-determined system of PDEs for a p-harmonic morphism, the geometric constructions of such maps become important. We find several methods to construct nontrivial p-harmonic morphisms from a given ones via conformal deformations of the metrics on the domain and/or the codomain manifold.

Theorem 1.8. Let $\varphi : (M^m, g) \longrightarrow (N^n, h)$ be a p_1-harmonic morphism with dilation λ and $p_1 \neq m$, and let $C_\varphi = \{x \in M : \lambda = 0\}$. Then for $p \in (1, \infty) \setminus \{m\}$, $\varphi : (M \setminus C_\varphi, (n\lambda^2)^{\frac{p_1 - p}{m-p}} g) \longrightarrow (N^n, h)$ is a p-harmonic morphism which is non-trivial if and only if the original p_1-harmonic morphism is non-trivial. Furthermore, if $\varphi : (M^m, g) \longrightarrow (N^n, h)$ is an m-harmonic morphism, then $\varphi : (M \setminus C_\varphi, n\lambda^2 g) \longrightarrow (N^n, h)$ is a p-harmonic morphism with constant dilation $1/\sqrt{n}$ for any $p \in (1, \infty)$, and hence it is a horizontally homothetic p-harmonic morphism.

Theorem 1.15. Let $\varphi : (M^m, g) \longrightarrow (N^n, h)$ be a horizontally homothetic harmonic morphism with dilation λ. Then for $p \in (1, \infty) \setminus \{m\}$ and for any non-constant function $\lambda_2 : N \longrightarrow \mathbb{R}^+$, $\varphi : (M^m, (\sqrt{n}\lambda\lambda_2 \circ \varphi)^{2\frac{n-p}{m-p}} g) \longrightarrow (N^n, \lambda_2^2 h)$ is a non-trivial p-harmonic morphism.

B. Some classifications of p-harmonic morphisms. We obtain a classification of horizontally homothetic submersions between Euclidean spaces which plays a crucial role in the classifications of p-harmonic and biharmonic morphisms. We also classify polynomial p-harmonic morphisms and holomorphic p-harmonic morphisms between Euclidean spaces generalizing Gudmundsson and Sigurdsson's result on the classification of holomorphic harmonic morphisms.

Theorem 2.2. Let $\varphi : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ $(m > n \geq 2)$ be a horizontally homothetic submersion. Then φ is a composition of an orthogonal projection $\mathbb{R}^m \longrightarrow \mathbb{R}^n$ followed by a homothety.

Theorem 2.3. 1) Let $m > n \geq 2,\ p \in (1, \infty)$. If $\varphi : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is a polynomial p-harmonic morphism, then either $p = 2$, and φ is a harmonic morphism; or $p \neq 2$, and φ is an orthogonal projection followed by a homothety.

2) For $n > 1$, $\varphi : \mathbb{C}^m \longrightarrow \mathbb{C}^n$ is a nonconstant holomorphic p-harmonic morphism for $p \in (1, \infty)$ if and only if φ is an orthogonal projection followed by a homothety.

C. Applications of p-harmonic morphisms. We find three applications of p-harmonic morphisms. The first application is a theorem that gives the existence of harmonic 3-spheres in general manifolds generalizing the result of Sacks and Uhlenbeck on the existence of harmonic 2-spheres. The second application is Theorem 3.5 (Page 35) that gives characterizations of twisted and warped products using p-harmonicity of their projection maps. Thirdly, we find applications of p-harmonic morphisms to the study of biharmonic morphisms.

Theorem 3.1. Let (N^n, h) be a Riemannian manifold whose universal covering space is not contractible. Then,

(a) There exists a homotopically non-trivial harmonic map $f_3 : S^3 \longrightarrow (N^n, h)$ which is nowhere conformal;

(b) There exist homotopically nontrivial harmonic maps $f_4 : (S^4, \lambda^2 g_4) \longrightarrow (N^n, h)$ and $f_7 : (S^7, \eta^2 g_7) \longrightarrow (N^n, h)$, where g_n denotes the standard metric on n-sphere.

Theorem 3.8. For $p \neq 4$, a submersive p-harmonic morphism $\varphi : (M^m, g) \longrightarrow (N^n, h)$ is also a biharmonic morphism if and only if φ is a horizontally homothetic harmonic morphism with harmonic energy density, i.e., $\triangle_g(n\lambda^2/2) = 0$.

Theorem 3.9. For $m > n \geq 2$, a polynomial map (i.e. a map whose component functions are polynomials) $\varphi : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is a biharmonic morphism if and only if it is a composition of an orthogonal projection followed by a homothety.

D. Constructions of minimal foliations via conformal deformations of metrics. We find links between p-harmonicity of a function and the minimality of its level hypersurfaces or its vertical graphs generalizing Baird-Eells and Burel-Loubeau's results. We show that the foliation by the level hypersurfaces of a submersive p-harmonic function or by the vertical graphs of a harmonic function can always be turned into a minimal foliation via a suitable conformal deformation of metric.

Theorem 4.6. Let $f : (M^m, g) \longrightarrow \mathbb{R}$ be a submersion. Then,

(I) f is a 1-harmonic function if and only if $\{f^{-1}(t)\}_{t\in\mathbb{R}}$ is a foliation of (M, g) by minimal hypersurfaces, or

(II) for $p \in (1, \infty)$, any two of the following conditions imply the third:

  (a) f is a p-harmonic function,

  (b) $\{f^{-1}(t)\}_{t\in\mathbb{R}}$ is a foliation of (M, g) by minimal hypersurfaces,

  (c) f is horizontally homothetic.

Theorem 4.10. Let $p \in (1, \infty)$, and $u : (M^m, g) \longrightarrow \mathbb{R}$ be a p-harmonic submersion. Then, $u : (M^m, \bar{g}) \longrightarrow \mathbb{R}$ is a 1-harmonic submersion and the foliation by the level hypersurfaces $\Gamma = \{u^{-1}(c) : c \in \mathbb{R}\}$ of u is a minimal foliation of $(M, \bar{g})$, where $\bar{g} = |\nabla u|^{2(p-1)/(m-1)}\, g$ is a Riemannian metric conformal to g.

Theorem 4.21. Let $u : M \longrightarrow \mathbb{R}$ be a submersion. Then any two of the following statements imply the third:

(a) u is a p-harmonic function for some $p \geq 1$,

(b) u is a solution of the MSE in (M, g), i.e., the vertical graph $\Gamma(u, c)$ is minimal in $(M \times \mathbb{R}, g + dt^2)$,

(c) u is horizontally homothetic.

Theorem 4.23. Let $u : (M^m, g) \longrightarrow \mathbb{R}$ be a harmonic function from a complete Riemannian manifold. Then, the vertical graphs $\mathcal{G} = \{(x, u(x)+c) \in M \times \mathbb{R} : c \in \mathbb{R}\}$ produce a foliation of the complete manifold $(M^m \times \mathbb{R}, (1 + |\nabla u|^2)^{1/m}(g + \mathrm{d}t^2))$ by minimal hypersurfaces.

CHAPTER 1

P-HARMONIC MORPHISMS

AND

CONFORMAL DEFORMATIONS OF METRICS

In this chapter, we first study the transformation of a p-harmonic morphism into a q-harmonic morphism via biconformal change of the domain metric and/or conformal change of the codomain metric. We then give several methods to construct non-trivial p-harmonic morphisms via conformal change of metrics.

In this thesis, we work in the category of smooth objects, so all manifolds, vector fields, and maps are assumed to be smooth unless otherwise stated.

1. 1 Preliminaries

In this section, we give the definitions of some basic notions involved in this thesis.

Definition 1.1. *For $p > 1$, a map $\varphi : (M, g) \longrightarrow (N, h)$ between Riemannian manifolds is called a* p**-harmonic map** *if $\varphi|\Omega$ is a critical point of the p-energy*

$$E_p(\varphi, \Omega) = \frac{1}{p}\int_{\Omega} |\mathrm{d}\varphi|^p \,\mathrm{d}x$$

for every compact subset Ω of M.

Using the first variational formula we see that p-harmonic maps are solutions of the following systems of PDEs:

$$\tau_p(\varphi) := |\mathrm{d}\varphi|_g{}^{p-2}\tau_2(\varphi) + (p-2)|\mathrm{d}\varphi|_g{}^{p-3}\mathrm{d}\varphi(\mathrm{grad}_g|\mathrm{d}\varphi|_g) = 0,$$

where $\tau_2(\varphi) = \mathrm{Trace}_g \nabla \mathrm{d}\varphi$ denotes the tension field of φ. Note that when $|\mathrm{d}\varphi| \neq 0$, we can write

$$\tau_p(\varphi) = |\mathrm{d}\varphi|_g^{\,p-2} \left[\tau_2(\varphi) + (p-2)\mathrm{d}\varphi(\mathrm{grad}_g(\ln |\mathrm{d}\varphi|_g))\right]. \tag{1}$$

For the case when the map is a real-valued function $u : (M, g) \longrightarrow \mathbb{R}$, the p-tension field operator agrees with the p-Laplace operator and we have:

$$\tau_p(u) = |\nabla u|^{p-2} \{\triangle u - g(\nabla u, \nabla \ln |\nabla u|)\} = \mathrm{div}(|\nabla u|^{p-2} \nabla u) = \triangle_p u. \tag{2}$$

Definition 1.2. *A map between Riemannian manifolds $\varphi : (M, g) \longrightarrow (N, h)$ is called a p-**harmonic morphism** if it preserves solutions of p-Laplace's equation in the the following sense: if for any function $f : U \longrightarrow \mathbb{R}$ defined on an open subset of N with $\varphi^{-1}(U)$ nonempty and $\triangle_p^N f = 0$, then $\triangle_p^M (f \circ \varphi) = 0$.*

A **horizontally weakly conformal** map $\varphi : (M, g) \longrightarrow (N, h)$ generalizes the notion of a Riemannian submersion in that for any $x \in M$ at which $\mathrm{d}\varphi_x \neq 0$, the restriction $\mathrm{d}\varphi_x|_{H_x} : H_x \longrightarrow T_{\varphi(x)}N$ is conformal and surjective, where the horizontal space H_x is the orthogonal complement of $V_x = \ker(\mathrm{d}\varphi_x)$ in T_xM. Thus it follows that there is a number $\lambda(x) \in (0, \infty)$ such that $h(\mathrm{d}\varphi(X), \mathrm{d}\varphi(Y)) = \lambda^2(x) g(X, Y)$ for any $X, Y \in H_x$. Note that at the point $x \in M$ where $\mathrm{d}\varphi_x = 0$ we can let $\lambda(x) = 0$ to obtain a continuous function $\lambda : M \longrightarrow R$ which is called the **dilation** of a horizontally weakly conformal map φ. A non-constant horizontally weakly conformal map φ is said to be **horizontally homothetic** if the gradient of $\lambda^2(x)$ is vertical, meaning that $X(\lambda^2) \equiv 0$ for any horizontal vector field X on M.

Note that in general, we have the following strict inclusion relations:

$$\{ \text{ Riemannian submersions}\} \subset \{ \text{ horizontally homothetic submersions}\}$$
$$\subset \{ \text{ horizontally conformal submersions}\}.$$

For a horizontally conformal submersion $\varphi : (M, g) \longrightarrow (N, h)$, we use $\mathcal{V}$ and $\mathcal{H}$ to denote the vertical and the horizontal distributions associated to φ. Thus, by definition of $\mathcal{V}$ and $\mathcal{H}$, we have the orthogonal decomposition $TM = \mathcal{H} \oplus \mathcal{V}$. Correspondingly, we decompose the metric g into horizontal and vertical parts: $g = g_h + g_v$. By a **biconformal change** of the metric g we mean the metric $\tilde{g} = \sigma^{-2}g_h + \rho^{-2}g_v$, where $\sigma, \rho : M \longrightarrow (0, \infty)$ are smooth functions on M. When $\sigma = \rho$, we have the usual notion of the **conformal change** of metric g with the conformal factor σ^{-2}.

Definition 1.3. *A* **codimension** n **foliation** $\mathcal{F}$ *of a manifold* M^m *is a decomposition of* M *into disjoint union of connected codimension* n *submanifolds* $M = \bigcup_{L \in \mathcal{F}} L$, *called leaves of the foliation, such that for each point* $p \in M$ *there is a neighborhood* U *of* M *and a submersion* $\varphi_U : U \longrightarrow \mathbb{R}^n$ *with* $\varphi^{-1}(y)$ *a leaf of* $\mathcal{F}|_U$, *the restriction of the foliation to* U *for each* $y \in \mathbb{R}^n$. *A foliation of a Riemannian manifold is called a* **minimal foliation** *if each leaf of the foliation is an immersed minimal submanifold.*

1.2 p-HARMONIC MORPHISMS UNDER BICONFORMAL CHANGES OF METRICS

In this section, we first derive a formula for the p-tension field of a horizontally conformal submersion under a biconformal change of the domain metric and a conformal change of the codomain metric. We then use it to obtain the conditions under which a p-harmonic morphism is transformed into a q-harmonic morphism under biconformal and/or conformal change of metrics.

Lemma 1.4. *Let $\varphi : (M^m, g) \longrightarrow (N^n, h)$ be a horizontally conformal submersion with dilation $\lambda : M \longrightarrow (0, \infty)$. Let $g = g_h + g_v$ be the decomposition of the metric g into horizontal and vertical components. Let $\tilde{g}$ be a biconformal change of g and $\tilde{h}$ a conformal change of h given by*

$$(a) \quad \tilde{g} = \sigma^{-2} g_h + \rho^{-2} g_v \qquad (b) \quad \tilde{h} = \bar{\nu}^{-2} h,$$

where $\sigma, \rho : M \longrightarrow (0, \infty)$ and $\bar{\nu} : N \longrightarrow (0, \infty)$ are smooth functions. Denote $\nu = \bar{\nu} \circ \varphi$. Then, the p-tension field of the map $\varphi : (M^m, \tilde{g}) \longrightarrow (N^n, \tilde{h})$ is given by

$$(3) \qquad \tilde{\tau}_p(\varphi) = n^{\frac{p-2}{2}} \lambda^{p-2} \sigma^p \nu^{2-p} \{\tau_2(\varphi) + \mathrm{d}\varphi(\mathrm{grad} \ln(\lambda^{p-2} \sigma^{p-n} \rho^{n-m} \nu^{n-p}))\}.$$

Proof. By Lemma 4.6.6 in [8], φ is a horizontally conformal submersion with respect to $\tilde{g}$ and $\tilde{h}$ on M and N with dilation $\tilde{\lambda} = \lambda \sigma \nu^{-1}$ and has tension field

$$(4) \qquad \tilde{\tau}_2(\varphi) = \sigma^2 \{\tau_2(\varphi) + \mathrm{d}\varphi(\mathrm{grad} \ln(\sigma^{2-n} \rho^{n-m} \nu^{n-2}))\}.$$

A direct computation using the fact that φ is a horizontally conformal submersion gives

$$(5) \qquad |\mathrm{d}\varphi|_{\tilde{g}}^{p-2} = n^{\frac{p-2}{2}} \lambda^{p-2} \sigma^{p-2} \nu^{2-p}.$$

On the other hand, one can check that the map φ with respect to the metrics $\tilde{g}$ and $\tilde{h}$ and the original map share the same horizontal and vertical spaces. Let $\{e_i\}$

(respectively $\{\tilde{e}_i\}$) be a local orthonormal frame of the horizontal distribution $\mathcal{H}$ with respect to metric g (respectively $\tilde{g}$). Then, $\tilde{e}_i = \sigma e_i$, and

$$\begin{aligned}
\text{(6)} \qquad \mathrm{d}\varphi(\mathrm{grad}_{\tilde{g}} f) =& \mathrm{d}\varphi((\mathrm{grad}_{\tilde{g}} f)_h + (\mathrm{grad}_{\tilde{g}} f)_v) \\
=& \mathrm{d}\varphi((\mathrm{grad}_{\tilde{g}} f)_h) = \mathrm{d}\varphi(\sum_{i=1}^{i=n} (\tilde{e}_i f)\tilde{e}_i) \\
=& \sigma^2 \mathrm{d}\varphi(\sum_{i=1}^{i=n} (e_i f)e_i) = \sigma^2 \mathrm{d}\varphi((\mathrm{grad} f)_h) \\
=& \sigma^2 \mathrm{d}\varphi(\mathrm{grad} f)
\end{aligned}$$

for any function f on M. Using (5) and (6) we have

$$\begin{aligned}
\text{(7)} \qquad (p-2)\mathrm{d}\varphi(\mathrm{grad}_{\tilde{g}} \ln|\mathrm{d}\varphi|_{\tilde{g}}) =& \mathrm{d}\varphi(\mathrm{grad}_{\tilde{g}} \ln|\mathrm{d}\varphi|_{\tilde{g}}^{p-2}) \\
=& \sigma^2 \mathrm{d}\varphi(\mathrm{grad}\ \ln(\lambda^{p-2}\sigma^{p-2}\nu^{2-p})).
\end{aligned}$$

Substituting Equations (4), (5), and (7) into the p-tension field formula (1) we obtain (3), which completes the proof of the lemma. □

As an immediate consequence, we have

Corollary 1.5. *$\varphi : (M^m, \tilde{g}) \longrightarrow (N^n, \tilde{h})$ is a p-harmonic morphism if and only if $\tau_2(\varphi) + \mathrm{d}\varphi(\mathrm{grad}\ \ln(\lambda^{p-2}\sigma^{p-n}\rho^{n-m}\nu^{n-p})) = 0$.*

Remark 1.6. *Note that when $\nu = 1$, $\sigma = \rho = \alpha^{-1}$, Corollary 1.5 reduces to Lemma 5.1 in* [12] *where the authors constructed some nontrivial p-harmonic morphism via a conformal change of the domain metric.*

Now we are ready to prove the following theorem which includes Proposition 4.6.8 in [8] as a special case.

Theorem 1.7. *For $p,\ q > 1$, let $\varphi : (M^m, g) \longrightarrow (N^n, h)$ be a submersive p-harmonic morphism with dilation λ. Let $\tilde{g}$ be a biconformal change of g and $\tilde{h}$ a conformal change of h given by (a) and (b) in Lemma 1.4. Then, the map $\varphi : (M^m, \tilde{g}) \longrightarrow (N^n, \tilde{h})$ is a q-harmonic morphism if and only if* grad $(\lambda^{q-p}\sigma^{q-n}\rho^{n-m}\nu^{n-q})$ *is vertical; equivalently, the function $\lambda^{q-p}\sigma^{q-n}\rho^{n-m}\nu^{n-q}$ is constant along horizontal curves.*

Proof. Suppose that $\varphi : (M^m, g) \longrightarrow (N^n, h)$ is a submersive p-harmonic morphism. Then, it is a p-harmonic horizontally conformal submersion. It follows that

$$\tau_p(\varphi) = |\mathrm{d}\varphi|^{p-2} \{\tau_2(\varphi) + (p-2)\mathrm{d}\varphi(\mathrm{grad}(\ln|\mathrm{d}\varphi|))\} = 0. \tag{8}$$

Noting that φ is a horizontally conformal submersion with dilation λ and $|\mathrm{d}\varphi|^2 = n\lambda^2 \neq 0$ we have, from (8),

$$\tau_2(\varphi) = \mathrm{d}\varphi(\mathrm{grad}(\ln\lambda^{2-p})). \tag{9}$$

Using (9) and (3) we obtain

$$\tilde{\tau}_q(\varphi) = n^{\frac{q-2}{2}}\lambda^{q-2}\sigma^q\nu^{2-q}\mathrm{d}\varphi(\mathrm{grad}\ \ln(\lambda^{q-p}\sigma^{q-n}\rho^{n-m}\nu^{n-q})). \tag{10}$$

Since φ is also a horizontally conformal submersion with respect to $\tilde{g}$ and $\tilde{h}$, we see from (10) that φ is a q-harmonic morphism with respect to $\tilde{g}$ and $\tilde{h}$ if and only if $\mathrm{d}\varphi(\mathrm{grad}\ \ln(\lambda^{q-p}\sigma^{q-n}\rho^{n-m}\nu^{n-q})) = 0$, which is equivalent to

$d\varphi(\text{grad }(\lambda^{q-p}\sigma^{q-n}\rho^{n-m}\nu^{n-q})) = 0$. This means that grad $(\lambda^{p-q}\sigma^{p-n}\rho^{n-m}\nu^{n-p})$ is vertical. Thus we obtain the theorem. $\square$

We will apply Theorem 1.7 to develop several methods to construct non-trivial p-harmonic morphisms via conformal changes of metrics in the next section. For more corollaries and applications of Theorem 1.7 see [42].

1.3 p-HARMONIC MORPHISMS VIA CONFORMAL DEFORMATIONS OF METRICS

We know from [12] that a horizontally homothetic harmonic morphism is always a p-harmonic morphism for any $p \in (1, \infty)$. From now on we call a p-harmonic morphism which is not horizontally homothetic a *non-trivial* p-harmonic morphism. It follows that a non-trivial p-harmonic morphism can not be a q-harmonic morphism for $q \neq p$. In particular, a non-trivial p-harmonic morphism with $p \neq 2$ is never a harmonic morphism. Since harmonic morphisms have been studied extensively (see [8]) it is important and interesting to study and, first of all, to find examples of non-trivial p-harmonic morphisms. Our next theorem provides a method to construct non-trivial p-harmonic morphisms via conformal changes of metrics.

Theorem 1.8. *Let $\varphi : (M^m, g) \longrightarrow (N^n, h)$ be a p_1-harmonic morphism with dilation λ and $p_1 \neq m$, and let $C_\varphi = \{x \in M : \lambda = 0\}$. Then for $p \in (1, \infty) \setminus \{m\}$, $\varphi : (M \setminus C_\varphi, (n\lambda^2)^{\frac{p_1-p}{m-p}} g) \longrightarrow (N^n, h)$ is a p-harmonic morphism which is non-trivial if and only if the original p_1-harmonic morphism is non-trivial. Furthermore, if $\varphi : (M^m, g) \longrightarrow (N^n, h)$ is an m-harmonic morphism, then $\varphi : (M \setminus C_\varphi, n\lambda^2 g) \longrightarrow$*

(N^n, h) is a p-harmonic morphism with constant dilation $1/\sqrt{n}$ for any $p \in (1, \infty)$, and hence it is a horizontally homothetic p-harmonic morphism.

Proof. If $\varphi : (M^m, g) \longrightarrow (N^n, h)$ is a p_1-harmonic morphism, then it is a horizontally weakly conformal p_1-harmonic map ([12], [19], [26], and [35]). In this case, we know that the dilation λ is given by $|d\varphi|^2 = n\lambda^2$. By the continuity of λ we see that C_φ is a closed subset and hence $M \setminus C_\varphi$ is an open subset. It is easy to see that the map $\varphi : (M \setminus C_\varphi, g) \longrightarrow (N^n, h)$ is a submersive p_1-harmonic morphism. Now applying Theorem 1.7 with $\sigma = \rho = |\mathrm{d}\varphi|^{-\alpha/2}$ we see that $\varphi : (M \setminus C_\varphi, |\mathrm{d}\varphi|^\alpha g) \longrightarrow (N^n, h)$ is a p-harmonic morphism if and only if grad $(\lambda^{p-p_1} |\mathrm{d}\varphi|^{\frac{\alpha(m-p)}{2}})$ is vertical. Using $|\mathrm{d}\varphi| = \sqrt{n}\lambda$, we have

$$\text{grad}\ (\lambda^{p-p_1} |\mathrm{d}\varphi|^{\frac{\alpha(m-p)}{2}}) = \text{grad}\ (C\lambda^{[\alpha(m-p)/2+(p-p_1)]}), \tag{11}$$

where C is a nonzero constant. It follows that if $\alpha = 2$ and $m = p_1$, then grad $(\lambda^{p-p_1} |\mathrm{d}\varphi|^{\frac{\alpha(m-p)}{2}})$ is always vertical i.e. $\varphi : (M \setminus C_\varphi, n\lambda^2 g) \longrightarrow (N^n, h)$ is a p-harmonic morphism for any p. It is easily checked that the dilation of $\varphi : (M \setminus C_\varphi, n\lambda^2 g) \longrightarrow (N^n, h)$ is $1/\sqrt{n}$. Thus we obtain the second statement of the theorem. To prove the first statement, notice that grad $(C\lambda^{[\alpha(m-p)/2+(p-p_1)]})$ is vertical if and only if grad λ is vertical or else $\alpha(m-p)/2 + (p - p_1) = 0$. It follows that if $p \neq m$ and $\alpha = \frac{2(p_1-p)}{m-p}$, then grad $(\lambda^{p-p_1} |\mathrm{d}\varphi|^{\frac{\alpha(m-p)}{2}})$ is vertical and hence $\varphi : (M \setminus C_\varphi, (n\lambda^2)^{\frac{(p_1-p)}{m-p}} g) \longrightarrow (N^n, h)$ is a p-harmonic morphism. We can check that the dilation λ_1 of the map $\varphi : (M \setminus C_\varphi, (n\lambda^2)^{\frac{p_1-p}{m-p}} g) \longrightarrow (N^n, h)$ is given by ${\lambda_1}^2 = C\lambda^{\frac{2(m-p_1)}{m-p}}$, where C is a nonzero constant. Since a conformal change of metric preserves the orthogonality, the horizontal subspace is invariant under the conformal change of metric, we can easily check that the new map is horizontally homothetic if and only if the original one is horizontally homothetic. Therefore the new map is a non-trivial p-harmonic morphism if and only if the original one is a non-trivial p_1-harmonic morphism. This completes the proof of the theorem. $\square$

Notice that the set C_φ of critical points of a horizontally weakly conformal map is empty when φ is submersive. Thus we have

Corollary 1.9. *If $\varphi : (M^m, g) \longrightarrow (N^n, h)$ is a submersive p_1-harmonic morphism with dilation λ. Then for $p \in (1, \infty) \setminus \{m\}$, $\varphi : (M, (n\lambda^2)^{\frac{p_1-p}{m-p}} g) \longrightarrow (N^n, h)$ is a p-harmonic morphism. Furthermore, if $\varphi : (M^m, g) \longrightarrow (N^n, h)$ is a submersive m-harmonic morphism, then $\varphi : (M, n\lambda^2 g) \longrightarrow (N^n, h)$ is a p-harmonic morphism with constant dilation $1/\sqrt{n}$ for any $p \in (1, \infty)$, and hence it is horizontally homothetic.*

In what follows we will see that Theorem 1.8 is useful not only in providing a method to construct many new and non-trivial examples of p-harmonic morphisms but also in offering a way to study the topological and the geometric properties of p-harmonic morphisms via harmonic morphisms at least in the case when the map is submersive.

Proposition 1.10. *let $\varphi : (M^m, g) \longrightarrow (N^n, h)$ be a C^2 submersive p-harmonic morphism for $p \in (1, \infty)$. Then*

(i) *φ is C^∞,*

(ii) *φ is an open map,*

(iii) *If M is compact then $\varphi(M) = N$.*

Proof. It follows from Corollary 1.9 that any submersive p-harmonic morphism becomes a harmonic morphism by a conformal change of the metric on the domain manifold. The Proposition follows from the facts that a conformal change of the metric does not change the topological and differentiable structures of the manifold and that a harmonic morphism enjoys the Properties (i), (ii), (iii) (cf. [19]). □

Remark 1.11. *It can be seen from Theorems 1.8 and Proposition 1.10 that as far as the related topology is concerned submersive p-harmonic morphisms of different p values make no difference since the existence of a p-harmonic morphism for one particular value of p implies the existence of p-harmonic morphisms for all the other values of p. Furthermore, if one takes a conformal geometry point of view, submersive p-harmonic morphisms of different p values still have the same geometry. However, if one takes the stand of Riemannian geometry, then as we have already seen from Theorems 0.1 that p-harmonic morphisms of different p values make striking difference.*

Now we use Theorem 1.8 to construct some non-trivial p-harmonic morphisms from some well known harmonic morphisms.

Example 1.12. *Let* $f : R^n \times R^n \longrightarrow R^n, f(x,y) = xy$, $n = 1, 2, 4,$ *or* 8, *denote the standard multiplications in the real algebras of real, complex, quaternionic and Cayley numbers. Then it is well-known (see e.g., [8]) that the Hopf construction maps* $F : R^n \times R^n \longrightarrow R^{n+1}$ *with*

$$F(x,y) = (|x|^2 - |y|^2, 2f(x,y))$$

is a harmonic morphism with dilation λ given by $\lambda^2 = 4\left|(x,y)\right|^2$ for any $(x,y) \in R^n \times R^n$. Since the only critical point is the origin $0 \in R^{2n}$ we know from Theorem 1.8 that for $p \in (1,\infty)\backslash\{2n\}$, the map $F : (R^{2n}\backslash\{0\}, (4(n+1)|X|^2)^{\frac{2-p}{2n-p}}\delta_{ij}) \longrightarrow R^{n+1}$ is a p-harmonic morphism defined by homogeneous polynomials of degree two, where $X \in R^{2n} \setminus \{0\}$, and δ_{ij} denotes the standard Euclidean metric. In particular, let $p = n+1 = 3$, then we obtain a 3-harmonic morphism $F : (R^4\backslash\{0\}, \frac{1}{12|X|^2}\delta_{ij}) \longrightarrow R^3$ given by $F(x_1,x_2,x_3,x_4) = (x_1^2+x_2^2-x_3^2-x_4^2,\ 2x_1x_3-2x_2x_4,\ 2x_1x_4+2x_2x_3)$. This 3-harmonic morphism has minimal fibers by Theorem 0.1. Thus it follows that for any $y_0 = (c_1,c_2,c_3) \in R^3$, the fiber $F^{-1}(y_0)$, which is the intersection of three quadratic hypersurfaces

$$\begin{cases} x_1^2 + x_2^2 - x_3^2 - x_4^2 = c_1 \\ 2x_1x_3 - 2x_2x_4 = c_2 \\ 2x_1x_4 + 2x_2x_3 = c_3, \end{cases}$$

is a geodesic in $(\mathbb{R}^4 \setminus \{0\}, \frac{1}{12|X|^2}\delta_{ij})$.

The following proposition shows that there are abundant examples of non-trivial p-harmonic morphisms between conformally flat spaces.

Proposition 1.13. *For any $n \in \mathbb{N}$, and the pairs $(m(n),n)$ with values listed in the following Table 1, there exist non-trivial p-harmonic morphism $\varphi : (\mathbb{R}^{2m(n)} \setminus \{0\}, (4(n+1)|X|^2)^{\frac{2-p}{2m(n)-p}}\delta_{ij}) \longrightarrow \mathbb{R}^{n+1}$, where $p \in (1,\infty) \setminus \{2m(n)\}$ and $X \in \mathbb{R}^{2m(n)} \setminus \{0\}$.*

Table 1

n	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	...	*n+8*
$m(n)$	*1*	*2*	*4*	*4*	*8*	*8*	*8*	*8*	...	*16m(n)*

Proof. By Theorem 4.6 in [38], we know that for the $(m(n), n)$ values listed in Table 1, there exist domain minimal umbilical quadratic harmonic morphisms $\varphi : \mathbb{R}^{2m(n)} \longrightarrow \mathbb{R}^{n+1}$. Since these quadratic harmonic morphisms are given by Clifford systems and have dilation given by $\lambda^2 = 4\,|X|^2$, it follows that $0 \in \mathbb{R}^{2m(n)}$ is the only critical point. Therefore, by Theorem 1.8, we obtain the proposition. $\square$

Remark 1.14. *We will see from Theorem 2.3 in Chapter 2 that for p $\neq 2$, the only polynomial p-harmonic morphism between Euclidean spaces is a composition of an orthogonal projection followed by a homothety. Proposition 1.13 implies that there are many polynomial p-harmonic morphisms if we give the domain space a suitable conformally flat metric.*

1.4 NON-TRIVIAL P-HARMONIC MORPHISMS FROM HORIZONTALLY HOMOTHETIC HARMONIC MORPHISMS

In [45] we use the composition law for p-harmonic morphisms to prove that pre-composition or post-composition a weakly conformal map to a horizontally homothetic harmonic morphism gives an m- or n-harmonic morphism. In the following we will provide a method to construct non-trivial p-harmonic morphisms by conformal changes of metrics on both the domain and the target manifolds of a horizontally homothetic harmonic morphism. For the conditions on the invariance of p-harmonic morphism under the conformal change of the metric on the domain and the target manifolds see [42] and Section 4.6 in [8]. First we note that horizontally homothetic harmonic morphisms is a large class of submersive harmonic morphisms which includes any Riemannian submersion with minimal fibers. However, if we apply Corollary 1.9 directly to a horizontally homothetic harmonic morphism $\varphi : (M^m, g) \longrightarrow (N^n, h)$ with dilation λ to get a p-harmonic morphism $\varphi : (M^m, (n\lambda^2)^{\frac{2-p}{m-p}} g) \longrightarrow (N^n, h)$, then it follows from Theorem 1.8 that the resulting p-harmonic morphism is a trivial one. We overcome this by proving the following theorem.

Theorem 1.15. *Let $\varphi : (M^m, g) \longrightarrow (N^n, h)$ be a horizontally homothetic harmonic morphism with dilation λ. Then for $p \in (1, \infty) \setminus \{m\}$ and for any non-constant function $\lambda_2 : N \longrightarrow \mathbb{R}^+$, $\varphi : (M^m, (\sqrt{n}\lambda\lambda_2 \circ \varphi)^{2\frac{n-p}{m-p}} g) \longrightarrow (N^n, \lambda_2^2 h)$ is a non-trivial p-harmonic morphism.*

Proof. We first note that a theorem in [20] implies that φ has to be a submersion since it is horizontally homothetic harmonic morphism. Let $\lambda_2 : N \longrightarrow R^+$ be a non-constant function. Consider the metric $\lambda_2^2 h$ on N. It follows from Theorem 4.1 in [45] that $\varphi : (M^m, g) \longrightarrow (N^n, \lambda_2^2 h)$ is an n-harmonic morphism with dilation λ_1 given by $\lambda_1^2 = \lambda^2(\lambda_2 \circ \varphi)^2$, where λ is the dilation of $\varphi : (M^m, g) \longrightarrow (N^n, h)$. Now by Theorem 1.8 we see that $\phi : (M^m, (\sqrt{n}\lambda\lambda_2 \circ \varphi)^{2\frac{n-p}{m-p}} g) \longrightarrow (N^n, \lambda_2^2 h)$ is a p-harmonic morphism for $p \in (1, \infty) \setminus \{m\}$, where $\phi = \varphi : M \longrightarrow N$ as a map between manifolds without metrics. It remains to check that ϕ is non-trivial, i.e., it is not horizontally homothetic. We can easily check that the dilation λ_3 of ϕ is given by ${\lambda_3}^2 = C\lambda^{2\frac{m-n}{m-p}}(\lambda_2 \circ \varphi)^{2\frac{m-n}{m-p}}$, where C is a nonzero constant, and λ is the dilation of φ. Note that λ^2 is vertical since φ is horizontally homothetic. It follows that to check that ϕ is horizontally homothetic it is enough to verify that $(\lambda_2 \circ \varphi)^{2\frac{m-n}{m-p}}$ is vertical. Since λ_2 is not constant by our choice, there exists a vector field Y on N such that $Y(\lambda_2)$ does not vanish identically. It is well known (cf. e.g. [4]) that there is a horizontal vector field X on M which is φ-related to Y. Therefore, $X((\lambda_2 \circ \varphi)^{2\frac{m-n}{m-p}}) = 2\frac{m-n}{m-p}\lambda_2^{2\frac{m-n}{m-p}-1}\varphi_* X(\lambda_2) = 2\frac{m-n}{m-p}\lambda_2^{2\frac{m-n}{m-p}-1}Y(\lambda_2)$, which is not identically zero since λ_2 is non-constant by the assumption. Thus ϕ is not horizontally homothetic and hence it is a non-trivial p-harmonic morphism. $\square$

Noting that a Riemannian submersion with minimal fibers is a horizontally homothetic harmonic morphism we immediately have

Corollary 1.16. *Let $\varphi : (M^m, g) \longrightarrow (N^n, h)$ be a Riemannian submersion with minimal fibers. Then for $p \in (1, \infty) \setminus \{m\}$, φ becomes a non-trivial p-harmonic morphism after a suitable conformal change of the metric on the domain and/or target manifold is made.*

Theorem 1.17. *Let $h_{2n-1} : (S^{2n-1}, g_{2n-1}) \longrightarrow (S^n, g_n)$ ($n = 2, 4,$ or 8) be the Hopf fibration with the standard metrics induced from the corresponding Euclidean spaces. Let $\sigma : (S^n, g_n) \longrightarrow (S^n, g_n)$ be a conformal diffeomorphism such that $\sigma^* g_n = \lambda^2 g_n$ with λ a nonconstant positive function on S^n. Then, for $p \in (1, \infty) \setminus \{2n-1\}$, $\sigma \circ h_{2n-1} : (S^{2n-1}, (4n\lambda^2 \circ h_{2n-1})^{\frac{n-p}{2n-1-p}} g_{2n-1}) \longrightarrow (S^n, g_n)$ is a non-trivial p-harmonic morphism. In particular, with respect to suitably chosen conformally flat metrics on* *the domain and the target spheres, the Hopf fibration* $h_{2n-1} : S^{2n-1} \longrightarrow S^n$ *becomes a non-trivial p-harmonic morphism for* $p \in (1, \infty)$.

Proof. The proof of the theorem is similar to that of Theorem 1.15 and is omitted.

$\square$

Example 1.18. *Let* $\hat{R}^2 = R^2 \cup \{\infty\}$. *If we identify* (S^2, g_2) *with* $(\hat{R}^2, \frac{4({du_1}^2+{du_2}^2)}{(1+|u|^2)^2})$ *through the stereographic projection, then it is not difficult to check that* $\sigma : (S^2, g_2) \longrightarrow (S^2, g_2)$ *given by* $\sigma(u_1, u_2) = (u_1 - u_2, u_1 + u_2)$ *is a conformal diffeomorphism such that* $\sigma^* g_2 = \lambda^2 g_2$, *where* $\lambda = \frac{\sqrt{2}(1+u_1^2+u_2^2)}{1+2u_1^2+2u_2^2}$. *Therefore, by Theorem 1.17, we have a non-trivial p-harmonic morphism* $\sigma \circ h_3 : (S^3, (\frac{4(1+u_1^2+u_2^2)}{1+2u_1^2+2u_2^2} \circ h_3)^{2\frac{2-p}{3-p}} g_3) \longrightarrow (S^2, g_2)$ *for* $p \in (1, \infty) \setminus \{3\}$.

Remark 1.19. *We remark that in their effort to construct examples of non-trivial p-harmonic morphisms, the authors in* [12] *start with a horizontally weakly conformal map* $\phi : (S^3, g_3) \longrightarrow (S^2, g_2)$ *given by*

$$(\cos s e^{ia}, \sin s e^{ib}) \longrightarrow (\cos \alpha(s), \sin \alpha(s) e^{i(ka+lb)}),$$

where the function $\alpha(s)$ *is chosen such that* ϕ *is horizontally weakly conformal, where* $s \in [0, \pi/2], a, b \in [0, 2\pi[$. *Then they reduce the partial differential equations for p-harmonic map to an ordinary differential equation which is solved to produce p-harmonic morphisms* $\phi : (S^3, (\frac{\sin \alpha}{\sin^2 2s})^{2(p-2)/(p-3)}(k^2 \sin^2 s + l^2 \cos^2 s)^{(p-1)/(p-3)} g_3) \longrightarrow (S^2, g_2)$ (*compare it with Example 1.18*). *Notice that in their examples, if* $p \neq 2$ *then the metric and hence the map is not globally defined. By comparison, we start with a horizontally homothetic harmonic morphism, a better raw material, then it turns out* *that we can construct non-trivial p-harmonic morphisms without having to solve any differential equations but just performing a suitable conformal change of the metric on the domain sphere. As a result, our examples are all globally defined p-harmonic morphisms between compact spaces.*

1.5 P-HARMONIC MORPHISMS FROM TWISTED PRODUCTS

Our next theorem gives a method to construct non-trivial p-harmonic morphisms from the projection of a twisted product and conformal mappings. Recall that the **doubly twisted product** of Riemannian manifolds (M, g) and (N, h) with twisting functions $\alpha,\ \beta : M \times N \longrightarrow (0, \infty)$ is referred to the Riemannian manifold $(M \times N, \alpha^2 g + \beta^2 h)$ which is denoted by ${}_{\alpha^2}M \times_{\beta^2} N$. When $\alpha \equiv 1$ we have a **twisted product** with twisting function $\beta(x, y)$. When $\alpha \equiv 1$ and β depends only on the points on M we have a **warped product** with the warping function $\beta(x)$. For more study on the geometry of doubly twisted products we refer to [47]. Propositions 2.4.26 and 4.5.11 in [8] characterize warped products as special harmonic morphisms. As a generalization, we give the following characterization of twisted products as special n-harmonic morphisms.

Proposition 1.20. (1) *The projection $\pi_2 : M^m \times {}_{f^2}N^n \longrightarrow (N^n, h)$ of the twisted product onto its second factor is a submersive n-harmonic morphism with totally geodesic fibers and integrable horizontal distribution.*
(2) *Conversely, any submersive n-harmonic morphism $\varphi : (M^m, g) \longrightarrow (N^n, h)$ with totally geodesic fibers and integrable horizontal distribution is locally the projection of a twisted product.*

Proof. It is easily checked that the projection $\pi_2 : M \times_{f^2} N \longrightarrow (N, h)$ of the twisted product onto its second factor is a horizontally conformal submersion with dilation $1/f(x, y)$. We know from [47] that the foliation of the twisted product $M \times_{f^2} N$ with leaves $M \times \{y\}$ is a totally geodesic foliation. Since the leaves are nothing but the fibers of π_2, we conclude that π_2 is a horizontally conformal submersion with totally geodesic fibers. Now it follows from Theorem 0.1 that π_2 is an n-harmonic morphism. Clearly, π_2 has integrable horizontal distribution. Thus we obtain Statement (1). To prove Statement (2), let φ be a submersive n-harmonic morphism, then it is a horizontally conformal submersion with dilation $\lambda > 0$. Let $\mathcal{V}$ and $\mathcal{H}$ denote the vertical and horizontal distributions of φ. We can decompose the metric g into its vertical and horizontal parts as $g = g_v + g_h$. It is easy to check that $\varphi : (M^m, \bar{g}) \longrightarrow (N^n, h)$ becomes a Riemannian submersion with totally geodesic fibers for $\bar{g} = g_v + \lambda^2 g_h$. It follows from de Rham theorem that $(M, \bar{g})$ is locally isometric to a Riemannian product so that φ becomes the projection onto the second factor. Therefore, locally, (M, g) is isometric to a twisted product in such a way that φ is the projection onto the second factor. Thus, we complete the proof of the proposition. $\square$

Now we are in a position to give the following method of constructing p-harmonic morphisms via twisted products and conformal mappings.

Theorem 1.21. *Let (M^m, g) be a Riemannian manifold and $\varphi : (N^n, h) \longrightarrow (Q^n, k)$ be a conformal mapping with dilation λ. Let $f : M \times N \longrightarrow (0, \infty)$ be a smooth function such that $f(x, y) \neq \lambda(y)\eta(x)$ for $\eta : M \longrightarrow (0, \infty)$. Let $\alpha,\ \beta : M \times N \longrightarrow$*

$(0,\infty)$ *be given by*

$$\alpha^2 = (\sqrt{n}\lambda(y)/f(x,y))^{2\frac{n-p}{m+n-p}}, \textit{and} \;\; \beta^2 = (f(x,y))^{\frac{2m}{m+n-p}}(\sqrt{n}\lambda(y))^{2\frac{n-p}{m+n-p}}.$$

Then, for $p \in (0,\infty) \setminus \{m+n\}$*, the projection* $\pi_2 : {}_{\alpha^2}M^m \times_{\beta^2} N^n \longrightarrow N^n$ *of a doubly twisted product onto its second factor followed by the conformal mapping* φ *is a non-trivial* p*-harmonic morphism.*

Proof. It follows from Proposition 1.20 that $\pi_2 : M^m \times_{f^2} N^n \longrightarrow N^n$ is a submersive n-harmonic morphism. On the other hand, we know from [45] that a conformal mapping between manifolds of same dimension is an n-harmonic morphism. Therefore, by the composition law of p-harmonic morphisms, $\varphi \circ \pi_2 : M \times_{f^2} N \longrightarrow Q$ is a submersive n-harmonic morphism with dilation $\lambda(y)/f(x,y)$ for $(x,y) \in M \times N$. Since $f(x,y) \neq \lambda(y)\eta(x)$, the dilation $\lambda(y)/f(x,y)$ must depend on y, so it is not constant along horizontal curves. It follows that the n-harmonic morphism is not horizontally homothetic and hence non-trivial. Now applying Corollary 1.9 we see that

$$\varphi \circ \pi_2 : (M \times N, (\sqrt{n}\lambda(y)/f(x,y))^{2\frac{n-p}{m+n-p}}(g + f(x,y)^2 h) \longrightarrow (Q,k)$$

is a p-harmonic morphism for $p \in (0,\infty) \setminus \{m+n\}$. By Theorem 1.8, the p-harmonic morphism $\varphi \circ \pi_2$ is non-trivial since the original n-harmonic morphism is non-trivial. Finally, a simple computation gives the twisting functions α and β. $\square$

When $f(x) \equiv 1$ we have

Corollary 1.22. *Let* $\varphi : (N^n, g) \longrightarrow (Q^n, k)$ *be a conformal mapping with non-constant conformal factor* λ*, and* (M^m, g) *be any Riemannian manifold. Then, for* $p \in (1, \infty) \setminus \{m+n\}$*, the map*

$$\varphi \circ \pi_2 : (M^m \times N^n, (n\lambda^2)^{(n-p)/(m+n-p)}(g+h)) \longrightarrow (Q^n, k)$$

is a non-trivial p-harmonic morphism.

Remark 1.23. *When* $p = 2$ *Theorem 1.21 and Corollary 1.22 provide a method that can be applied to construct infinitely many new harmonic morphisms.*

Example 1.24. *Let* $\pi_2 : R^2 \times R^3 \longrightarrow R^3$ *be the orthogonal projection, and let* $\sigma : R^3 \longrightarrow S^3$ *be the inverse of the stereographic projection. It is well-known that* σ *is a conformal mapping with dilation* $\lambda = 2/(1+|y|^2)$ *for* $y \in R^3$*. Then, by Corollary 1.22, we have a non-trivial harmonic morphism from conformally flat space to the standard sphere*

$$\varphi = \sigma \circ \pi_2 : (R^5 \equiv R^2 \times R^3, (12/(1+|y|^2)^2)^{1/3}\delta_{\alpha\beta}) \longrightarrow (S^3, g_3).$$

Note that the fibers of φ *are 2-planes in* R^5 *but they are no longer minimal with respect to the conformally flat metric. To see this, suppose the fibers of* φ *were minimal, then it follows from Theorem 0.1 that* φ *is horizontally homothetic which means that* φ *is a trivial harmonic morphism. This is impossible since we know that* φ *is not trivial.*

CHAPTER 2

SOME CLASSIFICATIONS OF P-HARMONIC MORPHISMS

In this chapter we first give a classification of horizontally homothetic submersions between Euclidean spaces. We then use it to classify holomorphic p-harmonic morphisms and polynomial p-harmonic morphisms between Euclidean spaces.

2. 1 A classification of horizontally homothetic submersions

Lemma 2.1. *Let $\varphi : (M^m, g) \longrightarrow (N^n, h)$ $(m > n \geq 2)$ be a harmonic morphism of warped product type from a complete and simply connected manifold (M^m, g) with sectional curvature $K_M \geq 0$, then (M, g) is isometric to the Riemannian product $F \times \tilde{N}$ of a fiber F of φ and the universal covering space $\tilde{N}$ of N, and φ is the projection onto $\tilde{N}$ followed by the universal covering map which is a homothety.*

Proof. By [52], we know that (M, g) is isometric to the warped product $F \times_{\lambda^{-2}} \tilde{N}$ of a fiber F of φ and the universal covering space $\tilde{N}$ of N, and φ is the projection onto $\tilde{N}$ followed by the universal covering map. It follows from [55] that a complete warped product with non-negative sectional curvature must be a Riemannian product. From this we obtain the lemma. $\square$

We remark that the curvature condition in Lemma 2.1 is essential, for example, consider the hyperbolic space H^m of sectional curvature -1 in the upper half-space model. We know (see [8]) that the projection of H^m onto its boundary R^{m-1} is a harmonic morphism of warped product type from a complete and simply connected space whilst H^m is not isometric to the Riemannian product of a fiber and R^{m-1}.

Now we give the following theorem which will be used in several places in the rest of this thesis.

Theorem 2.2. *Let $\varphi : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ $(m > n \geq 2)$ be a horizontally homothetic submersion. Then φ is a composition of an orthogonal projection $\mathbb{R}^m \longrightarrow \mathbb{R}^n$ followed by a homothety.*

Proof. By [21] we know that for a horizontally homothetic submersion $\varphi : (M^m, g) \longrightarrow (N^n, h)$ $(m > n \geq 2)$ with dilation λ we have

$$K_M(X,Y) = \lambda^2 K_N(\tilde{X},\tilde{Y}) - \frac{3}{4}|[X,Y]^\nu|^2 - \frac{\lambda^4}{4}|\mathrm{grad}_\nu(\frac{1}{\lambda^2})|^2,$$

where X and Y are orthonormal horizontal vector fields on M, $\tilde{X}$ and $\tilde{Y}$ are corresponding φ-related vector fields on N, and $\mathrm{grad}_\nu(1/\lambda^2)$ denotes the vertical part of the gradient of $1/\lambda^2$. Since the sectional curvature $K_M = K_N = 0$ we see from the above equation that $[X,Y]^\nu = 0$ and $\mathrm{grad}_\nu(1/\lambda^2) = 0$ i.e., the horizontal distribution is integrable and λ is constant. Therefore, φ is, up to a homothety, a Riemannian submersion with integrable horizontal distribution. Therefore, the foliation of $\mathbb{R}^m$ determined by the fibers of φ is a Riemannian foliation. Applying Theorem 1.3 in [55] we conclude that the principal curvatures of the leaves (i.e., the fibers of φ) are zero and hence the latter are totally geodesic. Now the statement that φ is a composition of an orthogonal projection followed by a homothety follows from Lemma 2.1. $\square$

2.2 POLYNOMIAL AND HOLOMORPHIC P-HARMONIC MORPHISMS

In recent years, much work has been done (see [8] and [22]) in constructing and classifying harmonic morphisms between certain model spaces. For example, it is proved in [29] that if $\varphi : \mathbb{R}^m \longrightarrow (N^n, h)$ $(n \geq 3)$ is a nonconstant harmonic morphism with totally geodesic fibers, then (N, h) is isometric to $\mathbb{R}^n$ and φ is an orthogonal projection $\mathbb{R}^m \longrightarrow \mathbb{R}^n$ followed by a homothety. In [21] it is showed that $\varphi : U \longrightarrow \mathbb{R}^n (n \geq 3)$ is a horizontally homothetic harmonic morphism with totally geodesic fibers from a connected open subset of $\mathbb{R}^m$, then φ is the restriction of an orthogonal projection $\mathbb{R}^m \longrightarrow \mathbb{R}^n$ followed by a homothety. It is also proved in [23] that any non-constant holomorphic harmonic morphism $\varphi : U \longrightarrow \mathbb{C}^n (n \geq 2)$ from an open and connected subset U of $\mathbb{C}^m$ is the restriction of an orthogonal projection followed by a homothety.

In this section we give classifications of polynomial p-harmonic morphisms and holomorphic p-harmonic morphisms between Euclidean spaces generalizing Gudmundsson and Sigurdsson's classification of holomorphic harmonic morphisms.

Theorem 2.3. 1) *Let $m > n \geq 2$, $p \in (1, \infty)$. If $\varphi : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is a polynomial p-harmonic morphism, then either $p = 2$, and φ is a harmonic morphism; or $p \neq 2$, and φ is an orthogonal projection followed by a homothety.*
2) *For $n > 1$, $\varphi : \mathbb{C}^m \longrightarrow \mathbb{C}^n$ is a nonconstant holomorphic p-harmonic morphism for $p \in (1, \infty)$ if and only if φ is an orthogonal projection followed by a homothety.*

Proof. For 1), notice that if φ is a p-harmonic morphism then it follows from [35] that it is a horizontally weakly conformal p-harmonic map. On the other hand, since φ is polynomial map, and it is proved in [1] that any horizontally weakly conformal polynomial map $\varphi : \mathbb{R}^m \longrightarrow \mathbb{R}^n$, $(m > n \geq 2)$ has to be harmonic and hence a harmonic morphism. Therefore if $p \neq 2$, then φ is both a p-harmonic and a 2-harmonic morphism, which, by [35], is possible if and only if φ is a horizontally homothetic map. Now using Theorem 2.2 we obtain the statement. For Statement 2), if $p = 2$ the proof was given in [23]. If $\varphi : \mathbb{C}^m \longrightarrow \mathbb{C}^n$ is a p-harmonic morphism for $p \neq 2$, then it is a horizontally weakly conformal p-harmonic map. Noting that a holomorphic map $\varphi : \mathbb{C}^m \longrightarrow \mathbb{C}^n$ is automatically harmonic we conclude that φ is also a harmonic morphism because it is a horizontally weakly conformal harmonic map. A similar argument shows that φ is horizontally homothetic and it is an orthogonal projection followed by a homothety by Theorem 2.2. □

Remark 2.4. *Note that locally, a p-harmonic morphism $\varphi : (M^m, g) \longrightarrow (N^n, h)$ is a solution to the following over-determined system of partial differential equations:*

$$\operatorname{div}\left(|\mathrm{d}\varphi|^{p-2}\,\mathrm{d}\varphi\right) = 0$$

$$g^{ij}\frac{\partial\varphi^\alpha}{\partial x_i}\frac{\partial\varphi^\beta}{\partial x_j} = \lambda^2(x)h^{\alpha\beta}.$$

For p = 2, we know (cf. [2], [38], [46]) that many polynomial maps $\varphi : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ including the maps given by multiplications of real, complex, quaternionic, and Cayley numbers solve the above equations. Our Theorem 2.3 implies that for p $\neq$ 2, the only polynomial solution is the special linear map.

CHAPTER 3

SOME APPLICATIONS OF P-HARMONIC MORPHISMS

In this chapter we give three applications of p-harmonic morphisms. The first application shows the existence of harmonic 3-spheres in general manifolds generalizing the result of Sacks and Uhlenbeck on the existence of harmonic 2-spheres. The second application is a theorem that gives characterizations of twisted and warped products using p-harmonicity of their projection maps. Thirdly, we find applications of p-harmonic morphisms in the study of biharmonic morphisms.

3. 1 The existence of harmonic spheres

An interesting problem in the study of harmonic maps concerns the existence and nonexistence of harmonic maps from standard spheres into a manifold. For example, in their series of papers [32], [33], and [34], Lin and Wang study approximable harmonic maps based on the existence and nonexistence of nonconstant harmonic maps $S^2 \longrightarrow N$ called harmonic 2-spheres. One of their conjecture is (see [32]): any weakly harmonic map of finite energy from M^m into N is smooth if there are no harmonic spheres S^k in N for $2 \leq k \leq m-1$. A well-known theorem of Sacks and Uhlenbeck [49] guarantees the existence of harmonic 2-sphere in N if the universal covering space of N is not contractible. However, Sacks and Uhlenbeck' technique, as they pointed out in their paper, does not extend to give the existence of higher dimensional harmonic spheres. To the author's knowledge (see also remark in [32]), there has not been any general statement concerning the existence of higher dimensional harmonic spheres in the literature besides Sacks and Uhlenbeck's result.

Now using our knowledge on p-harmonic morphisms we have the following general existence of harmonic and conformal harmonic spheres.

Theorem 3.1. *Let (N^n, h) be a Riemannian manifold whose universal covering space is not contractible. Then,*

(a) There exists a homotopically non-trivial harmonic map $f_3 : S^3 \longrightarrow (N^n, h)$ which is nowhere conformal;

(b) There exist homotopically nontrivial harmonic maps $f_4 : (S^4, \lambda^2 g_4) \longrightarrow (N^n, h)$ and $f_7 : (S^7, \eta^2 g_7) \longrightarrow (N^n, h)$, where g_n denotes the standard metric on n-sphere.

Proof. According to the main theorem of Sacks and Uhlenbeck in [49], there exists a non-trivial harmonic map $\sigma_2 : S^2 \longrightarrow (N^n, h)$ which is in fact a conformal minimal immersion off finite number of isolated points on S^2. Let $h_3 : S^3 \longrightarrow S^2$ denote the Hopf fibration. It is well-known (see e.g. [8]) that h_3 is a harmonic morphism. Using the fact that a harmonic morphism pulls back harmonic maps to harmonic maps we obtain a harmonic 3-sphere $f_3 = \sigma_2 \circ h_3 : S^3 \longrightarrow (N^n, h)$. Clearly, f_3 is homotopically non-trivial. It is nowhere conformal since it does not preserve the angle making by a vertical and a horizontal vector defined by the Hopf fibration h_3. Thus, we obtain Statement (a). For Statement (b), we know from [8] that there exists a harmonic morphism $\sigma_4 : (S^4, \bar{g_4}) \longrightarrow S^2$ with $\bar{g_4} = \lambda^2 g_4$, so $f_4 = \sigma_2 \circ \sigma_4$ gives the required nontrivial harmonic 4-sphere in N with respect to a conformal Euclidean metric $\bar{g_4}$ on S^4. To obtain the nontrivial 7-sphere f_7, we first use Theorem 1.15 to get a harmonic morphism $H_7 : (S^7, (4\lambda \circ h_7)^{4/5} g_7) \longrightarrow (S^4, \lambda^2 g_4)$, where h_7 denotes the Hopf fibration $S^7 \longrightarrow S^4$. Now using H_7 to pull back f_4 we get the harmonic 7-sphere f_7, which completes the proof of the theorem. $\square$

Remark 3.2. *Note that for $n \geq 3$, although the harmonic map f_3 in Theorem 3.1 is neither a horizontally weakly conformal nor a weakly conformal map it preserves the angles of horizontal curves of the Hopf fibration at each point $x \in S^3$. Also, compare with Sacks and Uhlenbeck's minimizing harmonic 2-spheres the higher dimensional harmonic spheres can not even be stable (see* [56]*).*

For the existence of p-harmonic spheres we have the following results.

Proposition 3.3. (1) *Let N be a manifold whose universal covering is not contractible. Then, for $p \in (1, \infty) \setminus \{3\}$, there exists a p-harmonic map $(S^3 \setminus \{\Gamma\}, \bar{g}) \longrightarrow (N^n, h)$, where Γ denotes a disjoint union of finite number of great circles in S^3, and $\bar{g}$ is a metric conformal to the standard metric g_3 on S^3.*

(2) *For $p \in (1, \infty)$ and $k \in \mathbb{N}$, there exist full p-harmonic maps $f_{2k} : S^3 \longrightarrow S^{2k}$ with constant p-energy; Furthermore, there exists a metric conformal to the standard metric on 3-sphere so that f_{2k} becomes a p-harmonic map with nonconstant p-energy.*

(3) *For $p \in (1, \infty)$, $n = 4$, and $7 \leq n \leq 13$, there exist full p-harmonic maps $f_n : S^7 \longrightarrow S^n$ with constant p-energy; Furthermore, there exists a metric conformal to the standard metric on 7-sphere so that f_n becomes a p-harmonic map with nonconstant p-energy.*

Proof. For Statement (1), we can easily check that for the harmonic map $f_3 = \sigma_2 \circ h_3$ defined in Theorem 3.1, $|df_3|^2 = 4|d\sigma_2|^2 \circ h_3$. Noting that there are finite number of points $q_1, ..., q_k \in S^2$ at which $|d\sigma_2| = 0$ and that the Hopf fibration h_3 is a submersion we see that $\Gamma = \cup_{i=1}^{i=k} {h_3}^{-1}(q_i)$, a disjoint union of finite number of the great circles in S^3, contains all the points at which $|df_3| = 0$. Now it follows from [53] that the map $f_3 : (S^3 \backslash \{\Gamma\}, (4|d\sigma_2|^2)^{(2-p)/(3-p)} g_3) \longrightarrow (N^n, h)$ is a p-harmonic map. For Statements (2) and (3), we recall that an eigenmap between spheres is a harmonic map with constant energy density. It is easily checked that an eigenmap is a p-harmonic map for any p. Notice that the standard minimal immersion $f_k : S^2 \longrightarrow S^{2k}$ is a full eigenmap (see e.g. [13]) and that there exists full λ_2 eigenmap $\sigma_4 : S^4 \longrightarrow S^n$ with $n = 4$ and $7 \leq n \leq 13$ (see [25]). Therefore, using the fact that a p-harmonic morphism pulls back p-harmonic maps to p-harmonic maps we can use the usual Hopf fibrations or the non-trivial p-harmonic morphism $\sigma \circ h_{2n-1}$ $(n = 2, 4)$ constructed in Theorem 1.17 to pull back these eigenmaps to obtain the required p-harmonic maps. □

3.2 CHARACTERIZATIONS OF TWISTED AND WARPED PRODUCTS

In this section we give a theorem that characterizes a twisted product among doubly twisted products and a warped product among twisted products using p-harmonicity of their projection maps (for curvature conditions for a twisted product to be a warped product see [16]). In proving the theorem we need the following lemma.

Lemma 3.4. *The projection* ${}_{\alpha^2}M^m \times_{\beta^2} N^n \longrightarrow (N^n, h),\ \varphi(x,y) = y$, *of a doubly twisted product onto its second factor is a p-harmonic morphism if and only if* $\alpha^m \beta^{n-p} = f(x)$ *for some function* $f : M \longrightarrow (0,\infty)$.

Proof. Consider the projection of the Riemannian product

$$(M^m \times N^n, G = g + h) \longrightarrow (N, h), \quad \varphi(x,y) = y. \tag{12}$$

It is a Riemannian submersion with totally geodesic fibers and hence a harmonic morphism with dilation $\lambda = 1$. Note that the horizontal space at the point (x,y) can be identified with T_yN and hence $G_h = h,\ G_v = g$. Applying Theorem 1.7 with $p = 2$, $q = p$, $\lambda = 1$, $\nu = 1$, $\sigma = \beta^{-1}$ and $\rho = \alpha^{-1}$ we conclude that the projection ${}_{\alpha^2}M^m \times_{\beta^2} N^n \longrightarrow (N^n, h),\ \varphi(x,y) = y$, of a doubly twisted product onto its second factor is a p-harmonic morphism if and only if $\mathrm{grad}_G(\alpha^m\beta^{n-p})$ is vertical. This, in local coordinates, is equivalent to

$$h^{ij}\frac{\partial}{\partial y^i}(\alpha^m\beta^{n-p})\frac{\partial}{\partial y^j} = 0.$$

It follows that

$$h^{ij}\frac{\partial}{\partial y^i}(\alpha^m\beta^{n-p}) = 0 \tag{13}$$

for any $j = 1, \ldots, n$. Since the metric h is positive definite we see from Equation (13) that $\frac{\partial}{\partial y^i}(\alpha^m\beta^{n-p}) = 0$ for any i hence the function $\alpha^m\beta^{n-p}$ does not depend on the points in N. Thus we obtain the lemma. □

Theorem 3.5. *Let* $\alpha,\ \beta : M^m \times N^n \longrightarrow (0,\infty)$ *be two functions. Then,*

(1) *the projection* $\varphi : {}_{\alpha^2}M^m \times_{\beta^2} N^n \longrightarrow (N^n, h),\ \varphi(x,y) = y,$ *of a doubly twisted product onto its second factor is an* n*-harmonic morphism if and only if* ${}_{\alpha^2}M^m \times_{\beta^2} N^n$ *can be written as a twisted product;*

(2) *the projection* $\varphi : M^m \times_{\beta^2} N^n \longrightarrow (N^n, h),\ \varphi(x,y) = y,$ *of a twisted product onto its second factor is a* p*-harmonic morphism with* $p \neq n$ *if and only if* $M^m \times_{\beta^2} N^n$ *can be written as a warped product;*

(3) *the projection* $\varphi : M^m \times_{\beta^2} N^n \longrightarrow (M, g),\ \varphi(x,y) = x,$ *of a twisted product onto its first factor is a* p*-harmonic morphism if and only if* $M^m \times_{\beta^2} N^n$ *can be written as a Riemannian product.*

Proof. For Statement (1), we know from Proposition 1.20 in Chapter 1 that the projection of a twisted product onto its second factor is an n-harmonic morphism. Now suppose the projection $\varphi : {}_{\alpha^2}M^m \times_{\beta^2} N^n \longrightarrow (N^n, h),\ \varphi(x,y) = y$, of a doubly twisted product onto its second factor is an n-harmonic morphism. Then, by Lemma 3.4, $\alpha = (1/f(x))^{1/m}$ for some function $f : M \longrightarrow (0, \infty)$. It follows that the doubly twisted product ${}_{\alpha^2}M^m \times_{\beta^2} N^n$ can be written as a twisted product of $(M^m, \bar{g})$ and (N^n, h) with the twisting function β, where $\bar{g} = \alpha^2(x)g$ is a metric on M conformal to g.

For Statement (2), we know from [52] (see also [8], Proposition 2.4.26) that the projection of a warped product onto its second factor is a horizontally homothetic harmonic morphism hence a p-harmonic morphism for any $p > 1$ by [12]. Conversely, suppose the projection $\varphi : M^m \times_{\beta^2} N^n \longrightarrow (N^n, h),\ \varphi(x,y) = y$, of a twisted product onto its second factor is a p-harmonic morphism with $p \neq n$. Using Lemma

3.4 with $\alpha = 1$ and the fact that $p \neq n$ we conclude that the twisting function $\beta = (f(x))^{1/(n-p)}$ for some function $f : M \longrightarrow (o, \infty)$. It follows that β depends only on the points in M, so the twisted product $M^m \times_{\beta^2} N^n$ is in fact a warped product.

To prove Statement (3), we note that the horizontal and vertical distributions of φ are $\mathcal{H} = TM$ and $\mathcal{V} = TN$ respectively. Let $G = g + \beta^2 h$, then $G_h = g$ and $G_v = \beta^2 h$. Since the projection $\varphi : M^m \times N^n \longrightarrow (M, g)$, $\varphi(x, y) = x$ is a Riemannian submersion with totally geodesic fibers, it is a harmonic morphism with dilation $\lambda = 1$. Noting that $M^m \times_{\beta^2} N^n$ is isometric to ${}_{\beta^2}N^n \times M^m$, we can apply Theorem 1.7 with $p = 2$, $q = p$, $\lambda = \nu = 1$, $\sigma = 1$ and $\rho = \beta^{-1}$ to conclude that the projection $\varphi : M^m \times_{\beta^2} N^n \longrightarrow (M, g)$, $\varphi(x, y) = x$, of a twisted product onto its first factor is a p-harmonic morphism if and only if $\mathrm{grad}_G(\beta^m)$ is vertical. This, together with the fact that the horizontal distribution is integrable, implies that the twisting function β does not depend on the points in M. Thus, the twisted product $M^m \times_{\beta^2} N^n$ can be written as a Riemannian product of (M^m, g) and $(N^n, \bar{h})$, where $\bar{h}$ is a metric conformal to h on N. This ends the proof of the theorem. $\square$

From Statement (3) of Theorem 3.5 we can easily deduce the following corollary.

Corollary 3.6. *For any $p > 1$, the projection $\varphi : M^m \times_{\beta^2} N^n \longrightarrow (M, g)$, $\varphi(x, y) = x$, of a warped product onto its first factor is a p-harmonic morphism if and only if β is a constant and hence $M^m \times_{\beta^2} N^n$ is in fact a Riemannian product up to a homothety.*

Remark 3.7. *Yun proved in* [57]*, Theorem 2.4, that the projection of a warped product onto its first factor is harmonic (hence a harmonic morphism) if and only if the warping function is a constant. Clearly, Corollary 3.6 includes Yun's result as a special case.*

3.3 APPLICATIONS TO THE STUDY OF BIHARMONIC MORPHISMS

A **biharmonic morphism** (see [39] for precise definition and background) is a map between Riemannian manifolds that pulls back local biharmonic functions to local biharmonic functions. These maps are characterized as a special subclass of horizontally weakly conformal biharmonic maps ([36], [39]). In this section, we prove a theorem that characterize those p-harmonic morphisms which are also biharmonic morphisms. We then use this to give a complete classification of polynomial biharmonic morphisms between Euclidean spaces.

Theorem 3.8. *For $p \neq 4$, a submersive p-harmonic morphism $\varphi : (M^m, g) \longrightarrow (N^n, h)$ is also a biharmonic morphism if and only if φ is a horizontally homothetic harmonic morphism with harmonic energy density, i.e., $\triangle_g(n\lambda^2/2) = 0$.*

Proof. If φ is a horizontally homothetic harmonic morphism, then it is a submersion by [20], and it is a p-harmonic morphism for any $p > 1$ by [12]. If, in addition, φ has harmonic energy density, then it is also a biharmonic morphism by Theorem 3.8 in [39]. Thus we obtain the "if part" of the theorem. For the "only if part", suppose φ is a submersive p-harmonic morphism. Then, it is a p-harmonic horizontally conformal submersion with dilation λ such that $|\mathrm{d}\varphi|^2 = n\lambda^2$. Using (1) we have

$$\tau_2(\varphi) + (p-2)\mathrm{d}\varphi(\mathrm{grad}_g(\ln\lambda)) = 0. \tag{14}$$

On the other hand, if φ is also a biharmonic morphism, then, by Theorem 4.1 in [36], we have

$$\lambda^2\tau_2(\varphi) + \mathrm{d}\varphi(\mathrm{grad}_g\lambda^2) = 0,$$

which can be written as

$$\tau_2(\varphi) + 2\mathrm{d}\varphi(\mathrm{grad}_g(\ln\lambda)) = 0. \tag{15}$$

It follows from (14) and (15) that

$$(p-4)\mathrm{d}\varphi(\mathrm{grad}_g(\ln\lambda)) = 0. \tag{16}$$

Thus, if $p \neq 4$, then $\mathrm{d}\varphi(\mathrm{grad}_g(\ln\lambda)) = 0$, which means that φ is a horizontally homothetic submersion. It follows from [12] that φ is a p-harmonic morphism for any $p > 1$ and in particular a horizontally homothetic harmonic morphism. It follows then from Theorem 3.8 of [39] that φ must have harmonic energy density. Thus, we complete the proof of the theorem. $\square$

Theorem 3.9. *For $m > n \geq 2$, a polynomial map (i.e. a map whose component functions are polynomials) $\varphi : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is a biharmonic morphism if and only if it is a composition of an orthogonal projection followed by a homothety.*

Proof. It is well-known (see e.g. [8]) that the composition of an orthogonal projection followed by a homothety is a horizontally homothetic harmonic morphism with constant energy density. Thus, by Theorem 3.8, it is also a biharmonic morphism.

Conversely, suppose φ is a biharmonic morphism, then, by [39], it is a special horizontally weakly conformal biharmonic map. Since φ is assumed to be a polynomial map, it is harmonic by a theorem in [1], which states that a horizontally weakly conformal polynomial map between Euclidean spaces is harmonic. It follows that φ is a harmonic morphism since it is a horizontally weakly conformal harmonic map ([19], [26]). By Theorem 3.8 in [39], φ is a horizontally homothetic harmonic morphism with harmonic energy density being both a harmonic morphism and a biharmonic morphism. It follows from [20] that φ is a submersion since it is a nonconstant horizontally homothetic harmonic map. Finally, using the classification of horizontally homothetic maps between Euclidean spaces (Theorem 2.2) we conclude that φ is a composition of an orthogonal projection followed by a homothety. $\square$

Remark 3.10. *Note that there are many polynomial harmonic morphisms between Euclidean spaces (for classifications of quadratic harmonic morphisms see* [38], [46]*). However, as indicated by Theorem 3.9, the only polynomial biharmonic morphism between Euclidean spaces is a composition of an orthogonal projection followed by a homothety. This is also true for polynomial p-harmonic morphism with $p \neq 2$ (see Theorem 2.3 in Chapter 2).*

CHAPTER 4

MINIMAL FOLIATIONS

AND

CONFORMAL DEFORMATIONS OF METRICS

One fundamental question raised by Harvey and Lawson ([24]) in the study of foliations is: given a foliation $\mathcal{F}$ of a manifold M, when can one find a Riemannian metric on M so that all the leaves of $\mathcal{F}$ are minimal submanifolds? In this chapter, we prove that for the foliation $\mathcal{F}$ defined by the fibers of a submersive p-harmonic morphism $\varphi : (M^m, g) \longrightarrow (N^n, h)$ we can always find a metric $\bar{g}$ in the conformal class of g so that $\bar{g}$ turns $\mathcal{F}$ into a minimal foliation. In the case of codimension one, we find links between the p-harmonicity and the minimality of the level hypersurfaces or the vertical graphs a submersive function $u : (M^m, g) \longrightarrow \mathbb{R}$, generalizing the results of Baird-Eells and Burel-Loubeau (Theorem 0.1).

4. 1 Minimal foliations of codimension ≥ 2

Minimal foliations are also called harmonic foliations by some authors (see e.g. [27] and [28]). For minimal foliations of Nil and Sol spaces, and for the characterization of the hyperbolic space H^m by the existence of minimal coordinate plane foliations see [40]. In [51], the author studies the regularity and some global properties of codimension one minimal foliations of R^m with the standard Euclidean metric. In this section, we first show how to turn the fiber foliation of a p-harmonic morphism into a minimal foliation, then we prove that many Euclidean spaces R^m provided with a suitable conformally flat metric admit minimal foliations of codimension greater than two.

We know from Theorem 0.1 that the fibers of a submersive p-harmonic morphism φ determine a minimal foliation only when $p = n$, the dimension of the target manifold, or when φ is trivial. As another application of Theorem 1.8 we have

Theorem 4.1. (*Minimal foliation via p-harmonic morphisms*) *Let* $m > n \geq 2,\ p \in (1, \infty)$*, and let* $\varphi : (M^m, g) \longrightarrow (N^n, h)$ *be a p-harmonic morphism with dilation* λ*. Then, the fibers of* φ *determine a minimal foliation of* $(M \setminus \{C_\varphi\}, (n\lambda^2)^{(p-n)/(m-n)} g)$ *of dimension* $(m - n)$*.*

Proof. For $p = n$, the statement follows from Theorem 0.1; for $p \neq n$, one applies Theorems 1.8 and 0.1 to get the result. □

Theorem 4.2. *For* $n \in \mathbb{N}$*, and the pairs* $(m(n), n)$ *with values listed in Table 1, then* $R^{2m(n)+n+1}$ *admits a minimal foliation of codimension* $(n + 1)$ *with respect to a conformally flat metric.*

Proof. We know from Theorem 4.6 in [38] that for the pairs $(m(n), n)$ with values listed in Table 1, there exist domain minimal umbilical quadratic harmonic morphisms. Let $\varphi : R^{2m(n)} \longrightarrow R^{n+1}$ be one of those quadratic harmonic morphism. Since φ is given by a Clifford system [38] we can check that it has dilation λ given by $\lambda^2 = 4 |X|^2$ for $X \in R^{2m(n)}$. Let $\phi : R^{n+1} \longrightarrow R^{n+1}$ denote the identity map which is clearly a harmonic morphism. It follows from [41] that the direct sum $\varphi \oplus \phi : R^{2m(n)} \times R^{n+1} \longrightarrow R^{n+1}$ defined by $(\varphi \oplus \phi)(X, Y) = \varphi(X) + \phi(Y)$ is also a harmonic morphism. Since ϕ is the identity map we can easily check that the rank of $\varphi \oplus \phi$ is $n+1$ at any point and hence it is a submersive harmonic morphism. Also, a simple computation shows that $\varphi \oplus \phi$ has dilation λ_1 given by $\lambda_1^2 = 1 + 4\,|X|^2$. Therefore, by Theorem 4.1, we see that the fibers of $\varphi \oplus \phi$ give a minimal foliation of $\left(R^{2m(n)+n+1}, [(n+1)(1+4|X|^2)]^{-\frac{n-1}{2m(n)}}\delta_{ij}\right)$. Therefore we obtain the theorem. $\square$

Example 4.3. *Let* $\varphi : R^4 \longrightarrow R^3$ *with* $\varphi(x_1, x_2, x_3, x_4) = (x_1^2 + x_2^2 - x_3^2 - x_4^2,\ 2x_1x_3 + 2x_2x_4,\ 2x_2x_3 - 2x_1x_4)$ *be the Hopf construction map. It is well-known to be a harmonic morphism with dilation* λ *given by* $\lambda^2 = 4\,|X|^2$ *for* $X \in R^4$. *Let* $\phi : R^3 \longrightarrow R^3$ *with* $\phi(y_1, y_2, y_3) = -(y_1, y_2, y_3)$, *the identity map with a negative sign. Then the direct sum* $\varphi \oplus \phi : R^4 \times R^3 \longrightarrow R^3$ *is given by* $(\varphi \oplus \phi)(X, Y) = \varphi(X) + \phi(Y) = (x_1^2 + x_2^2 - x_3^2 - x_4^2 - y_1,\ 2x_1x_3 + 2x_2x_4 - y_2,\ 2x_2x_3 - 2x_1x_4 - y_3)$. *Therefore, it follows from Theorem 4.2 that the fibers of* $\varphi \oplus \phi$ *give a codimension three minimal foliation* $\{(\varphi \oplus \phi)^{-1}(c) |\ c \in R^3\}$ *of* $\left(R^7, [3(1+4|X|^2)]^{-1/4}\delta_{ij}\right)$. *Note that each leave* $(\varphi \oplus \phi)^{-1}(c)$ *of the foliation is given by the points* $(X, Y) \in R^4 \times R^3$ *satisfying the following equations*

$$\begin{cases} y_1 = x_1^2 + x_2^2 - x_3^2 - x_4^2 + c_1 \\ y_2 = 2x_1x_3 + 2x_2x_4 + c_2 \\ y_3 = 2x_2x_3 - 2x_1x_4 + c_3. \end{cases}$$

Noting that the fiber $(\varphi \oplus \phi)^{-1}(0)$ *can be viewed as the graph of* φ *in* $R^4 \times R^3$: $G : R^4 \longrightarrow R^4 \times R^3$ $G(X) = (X, \varphi(X))$ *we see that the isometric immersion*

$$G : R^4 \longrightarrow \left(R^4 \times R^3,\ [3(1+4|X|^2)]^{-1/4}\delta_{ij}\right),\ \ G(X) = (X, \varphi(X))$$

is a harmonic map because it is minimal.

4.2 P-HARMONIC FUNCTIONS AND THE MINIMAL FOLIATIONS BY LEVEL HYPERSURFACES

In this section we first characterize those p-harmonic functions $u : (M, g) \longrightarrow \mathbb{R}$ whose level hypersurfaces produce minimal foliations of (M, g). We then prove that the foliation by the level hypersurfaces of a submersive p-harmonic function can be turned into a minimal foliation via a suitable conformal deformation of metric.

Let (M, g) be a Riemannian manifold and $\bar{g}$ another metric on M. In this and the subsequent sections we will use ∇, $|.|$, $\triangle$, and div (respectively, $\bar{\nabla}$, $|.|_{\bar{g}}$, $\bar{\triangle}$ and $\mathrm{div}_{\bar{g}}$) to denote the gradient, the norm, the Laplacian, and the divergence taken with respect to g (respectively, $\bar{g}$). We use the convention on the Laplacian so that on $\mathbb{R}^m$, $\triangle u = \sum_{i=1}^{m} \partial^2 u / \partial {x_i}^2$.

As noted in [12], for $p < 1$, the p-energy E_p is not a norm and $W^{1,p}$ is not a Banach space; besides, for $p = 1$, although $W^{1,1}$ becomes a Banach space, it is impossible to derive a Euler-Lagrange equation corresponding to critical points of 1-energy. So in general, it is assumed that $p > 1$ when p-harmonic maps are studied. However, in the case $u : (M, g) \longrightarrow \mathbb{R}$ we do have the 1-energy functional

$$E_1(u) = \int_M |\nabla u| \, \mathrm{d}x \tag{17}$$

as in [9], where the authors defined the functional for a certain class of functions defined on a domain in a Euclidean space. In fact, Bombieri et al. in [9] called the functions which are critical points of the functional the functions of least gradient and they show that the level hypersurfaces of such a function are minimal. This leads to a construction of minimal graphs which are not hyperplanes in $\mathbb{R}^m$ for $m \geq 9$ and thereby solves the famous Bernstein's problem. In conformity with the language of the p-energy and p-harmonic maps we adopt the following

Definition 4.4. *A submersion $u : (M, g) \longrightarrow \mathbb{R}$ is said be* **1-harmonic** *if it is a critical point of the* 1*-energy functional (17) defined on all functions on M which are submersions.*

Lemma 4.5. *A submersion $u : (M, g) \longrightarrow \mathbb{R}$ is* 1*-harmonic if and only if the* 1*-tension field $\tau_1(u) \equiv 0$, where*

$$\tau_1(u) = |\nabla u|^{-1} \{\triangle u - g(\nabla u, \nabla \ln |\nabla u|)\}. \tag{18}$$

Proof. It is well-known (see e.g. [10]) that the Euler-Lagrange equation of the functional (17) is $\operatorname{div}(\nabla u/|\nabla u|) = 0$. It is easily checked that

$$\operatorname{div}(\nabla u/|\nabla u|) = |\nabla u|^{-1} \{\triangle u - g(\nabla u, \nabla \ln |\nabla u|)\} = \tau_1(u). \tag{19}$$

Thus we obtain the Lemma. □

Notice that by combining (2) and (18) we have a unified form of formula of p-tension field of a submersion $u : (M, g) \longrightarrow \mathbb{R}$ including the $p = 1$ case,

$$\text{for } p \geq 1, \quad \tau_p(u) = \triangle_p u = |\nabla u|^{p-2} \{\triangle u - g(\nabla u, \nabla \ln |\nabla u|)\}. \tag{20}$$

Now we give the following theorem which generalizes Theorem 0.1 to the real-valued function case.

Theorem 4.6. *Let $u : (M^m, g) \longrightarrow \mathbb{R}$ be a submersion. Then,*

(I) *u is a 1-harmonic function if and only if $\{u^{-1}(t)\}_{t\in\mathbb{R}}$ is a foliation of (M,g) by minimal hypersurfaces, or*

(II) *for $p \in (1,\infty)$, any two of the following conditions imply the third:*

(a) *u is a p-harmonic function,*

(b) *$\{u^{-1}(t)\}_{t\in\mathbb{R}}$ is a foliation of (M,g) by minimal hypersurfaces,*

(c) *u is horizontally homothetic.*

To prove Theorem 4.6 we need the following lemmas.

Lemma 4.7. *Let $u : (M^m, g) \longrightarrow \mathbb{R}$ be a submersion and $\eta = \nabla u / |\nabla u|$ denote the unit normal vector field of the level hypersurfaces of u. Then the following statements are equivalent:*

(i) *u is horizontally homothetic,*

(ii) $\mathrm{d}u(\mathrm{grad}(\ln|\nabla u|)) = g(\nabla u, \nabla \ln|\nabla u|) = 0$,

(iii) $\mathrm{Hess}_u(\eta,\eta) = 0$,

where $\mathrm{Hess}_u(X,Y)$ *denotes the Hessian of u.*

Proof. Since the target manifold is of dimension one and u is a submersion we see that u is a horizontally conformal submersion with dilation λ given by $\lambda^2 = |\mathrm{d}u|^2 = |\nabla u|^2 = g(\nabla u, \nabla u)$. Using local coordinates $\{x^i\}$ on M we have

$$\begin{aligned}
\mathrm{d}u(\mathrm{grad}(\ln|\nabla u|)) &= (u_k \mathrm{d}x^k)(g^{ij}\partial_i(\ln|\nabla u|)\partial_j) \\
&= \{g^{ik}\partial_i(|\nabla u|)u_k\} / |\nabla u| \\
&= g(\nabla u, \nabla|\nabla u|) / |\nabla u|,
\end{aligned}$$

where $\partial_i = \partial/\partial x_i,\ u_k = \partial u/\partial x_k$, and Einstein convention of summation is used. On the other hand, it follows from [54](p.106) that

$$\mathrm{Hess}_u(\eta, \eta) = g(\nabla u, \nabla \lambda)/\lambda = g(\nabla u, \nabla \left|\nabla u\right|)/\left|\nabla u\right|.$$

From the above two equations we have

$$\mathrm{d}u(\mathrm{grad}(\ln \left|\nabla u\right|)) = \mathrm{Hess}_u(\eta, \eta) = g(\nabla u, \nabla \ln \left|\nabla u\right|). \tag{21}$$

Note that, in general, a horizontally weakly conformal map $\varphi : (M^m, g) \longrightarrow (N^n, h)$ with dilation λ given by $\lambda^2 = g^{ij}\frac{\partial \varphi^\alpha}{\partial x_i}\frac{\partial \varphi^\beta}{\partial x_j}h_{\alpha\beta} = \left|\mathrm{d}\varphi\right|^2/n$ is horizontally homothetic if $X(\lambda^2) = 0$ for any horizontal vector field X on M. One can easily check that this is equivalent to $\mathrm{d}\varphi(\mathrm{grad}(\ln \left|\mathrm{d}\varphi\right|)) = 0$. This, together with Equation (21), proves the lemma. $\square$

Corollary 4.8. *For $p, q \in [1, \infty)$, a p-harmonic submersion $u : (M^m, g) \longrightarrow \mathbb{R}$ is also a q-harmonic submersion for $p \neq q$ if and only if u is horizontally homothetic in which case it is p_1-harmonic for any $p_1 \geq 1$.*

Proof. We know from (20) that a p-harmonic submersion u is also a q-harmonic submersion if and only if

$$\triangle u + (p-2)\mathrm{d}u(\mathrm{grad}(\ln \left|\nabla u\right|)) = 0$$

$$\triangle u + (q-2)\mathrm{d}u(\mathrm{grad}(\ln \left|\nabla u\right|)) = 0.$$

It follows that $\mathrm{d}u(\mathrm{grad}(\ln \left|\nabla u\right|)) = 0$ since $p \neq q$. By Lemma 4.7, u is horizontally homothetic. Conversely, if u is a horizontally homothetic p-harmonic submersion, then, by (20), $\triangle u = 0$, i.e., u is also a harmonic submersion. Using (20) again we see that u is a p_1-harmonic submersion for any $p_1 \in [1, \infty)$ hence in particular it is also a q-harmonic submersion. $\square$

Lemma 4.9. *Let $u : (M^m, g) \longrightarrow \mathbb{R}$ be a submersion and $\eta = \nabla u / |\nabla u|$ be the unit normal vector field of the level hypersurfaces of u. Let $H(\eta)$ denote the mean curvature of the level hypersurfaces. Then we have*

$$(m-1)H(\eta) = -\tau_1(u) = -\{\triangle u - \mathrm{d}u(\mathrm{grad}(\ln |\nabla u|))\} / |\nabla u| . \tag{22}$$

Proof. By our convention on Laplace operator $\triangle$, Equation (8.7) in [54] reads

$$(m-1)H(\eta) = \{-\triangle u + \mathrm{Hess}_u(\eta, \eta)\} / |\nabla u| .$$

Using Equations (18) and (21) we obtain the lemma. $\square$

Now we proceed to prove Theorem 4.6.

Proof of Theorem 4.6: Statement (I) follows immediately from Equation (22). To prove Statement (II), we first note that the statement is true for $p = 2$ by Equation (22). For $p \in (1, \infty) \setminus \{2\}$, we proceed as follows:

$(a) + (b) \Rightarrow (c)$: Suppose that u is a p-harmonic function with $p \in (1, \infty) \setminus \{2\}$ and that $\{u^{-1}(t)\}_{t \in \mathbb{R}}$ is a foliation of (M, g) by minimal hypersurfaces. Then, it follows from Statement (I) that u is a 1-harmonic submersion. Since $p \neq 1$ by assumption, we apply (II) of Corollary 4.8 to conclude that u is horizontally homothetic.

$(a) + (c) \Rightarrow (b)$: Suppose that u is p-harmonic for $p \neq 1$ and that u is horizontally homothetic. It follows from (II) of Corollary 4.8 that u is also a 1-harmonic submersion. Applying Statement (I) we obtain (b).

$(b)+(c) \Rightarrow (a)$: It follows from (b) and Statement (I) that u is a 1-harmonic submersion; this, together with (c) and (II) of Corollary 4.8, shows that u is also a p-harmonic function for any p. This yields (a), and completes the proof of Theorem 4.6.

Our next theorem shows that we can always find a metric with respect to which the foliation by the level hypersurfaces of a submersive p-harmonic function becomes a minimal foliation.

Theorem 4.10. *Let $p \in (1, \infty)$, and $u : (M^m, g) \longrightarrow \mathbb{R}$ be a p-harmonic submersion. Then, $u : (M^m, \bar{g}) \longrightarrow \mathbb{R}$ is a 1-harmonic submersion and the foliation by the level hypersurfaces $\Gamma = \{u^{-1}(c) : c \in \mathbb{R}\}$ of u is a minimal foliation of $(M, \bar{g})$, where $\bar{g} = |\nabla u|^{2(p-1)/(m-1)}\, g$ is a Riemannian metric conformal to g.*

Proof. By Theorem 4.6, we only need to prove that $u : (M^m, \bar{g}) \longrightarrow \mathbb{R}$ is a 1-harmonic submersion. This is equivalent to showing that

$$\tau_1(u, \bar{g}) = \left|\bar{\nabla} u\right|_{\bar{g}}^{-1} \{\bar{\triangle} u - \bar{g}(\bar{\nabla} u, \bar{\nabla} \ln \left|\bar{\nabla} u\right|_{\bar{g}})\} \equiv 0. \tag{23}$$

Let $F = |\nabla u|^{(1-p)/(m-1)}$, then $\bar{g} = F^{-2} g$. A direct computation gives

$$\begin{cases} \left|\bar{\nabla} u\right|_{\bar{g}} = F\left|\nabla u\right|, \quad \sqrt{\det(\bar{g}_{ij})} = F^{-m}\sqrt{\det(g_{ij})}, \\ \bar{\triangle} u = F^2 \triangle u + (2-m) F g(\nabla F, \nabla u), \\ \bar{g}(\bar{\nabla} u, \bar{\nabla} \ln \left|\bar{\nabla} u\right|_{\bar{g}}) = F^2 g(\nabla u, \nabla \ln |\nabla u|) + F g(\nabla F, \nabla u). \end{cases} \tag{24}$$

Using (24) and the first equality in (23) we have

$$(25)\qquad \tau_1(u,\bar{g}) =$$

$$(F\,|\nabla u|)^{-1}\{F^2\triangle u + (1-m)Fg(\nabla F,\nabla u) - F^2|\nabla u|^{-1}g(\nabla u,\nabla\,|\nabla u|)\}.$$

Substituting $F = |\nabla u|^{(1-p)/(m-1)}$ into (25) we obtain

$$\tau_1(u,\bar{g}) = |\nabla u|^{m(1-p)/(m-1)}\cdot|\nabla u|^{p-2}\,\{\triangle u + (p-2)g(\nabla u,\nabla \ln|\nabla u|)\},$$

from which, together with (3), we obtain

$$(26)\qquad \tau_1(u,\bar{g}) = |\nabla u|^{m(1-p)/(m-1)}\,\tau_p(u,g).$$

It follows from (26) that $u:(M^m,g)\longrightarrow\mathbb{R}$ is a p-harmonic submersion if and only if $u:(M^m,\bar{g})\longrightarrow\mathbb{R}$ if a 1-harmonic submersion. This completes the proof of the theorem. $\square$

In [14] Proposition 1, Chruściel proved that a function $u:(M^3,g)\longrightarrow\mathbb{R}$ is a 3-harmonic submersion if and only if there exists a conformal metric $\lambda^2 g$ such that $u:(M^m,\lambda^2 g)\longrightarrow\mathbb{R}$ is a harmonic Riemannian submersion in which case $\Gamma=\{u^{-1}(c): c\in\mathbb{R}\}$ is a minimal foliation of $(M,\lambda^2 g)$ by the level hypersurfaces of u. Now we give the following generalization.

Proposition 4.11. *A function $u:(M^m,g)\longrightarrow\mathbb{R}$ is an m-harmonic submersion if and only if $u:(M^m,|\nabla u|^2\,g)\longrightarrow\mathbb{R}$ is a harmonic Riemannian submersion in which case $\Gamma=\{u^{-1}(c): c\in\mathbb{R}\}$ is a minimal foliation of $(M,|\nabla u|^2\,g)$ by the level hypersurfaces of u.*

Proof. Applying Theorem 4.10 to u with $p = m$ we conclude that $u : (M^m, |\nabla u|^2 g) \longrightarrow \mathbb{R}$ is a 1-harmonic submersion and $\Gamma = \{u^{-1}(c) : c \in \mathbb{R}\}$ is a minimal foliation of $(M, |\nabla u|^2 g)$ by level hypersurfaces of u. On the other hand, it is easily checked that $u : (M^m, |\nabla u|^2 g) \longrightarrow \mathbb{R}$ is a Riemannian submersion and an m-harmonic function since m-harmonicity is invariant under the conformal change of the metric on the domain manifold. It follows from Corollary 4.8 that $u : (M^m, |\nabla u|^2 g) \longrightarrow \mathbb{R}$ is horizontally homothetic since it is p-harmonic for two different p values. In this case, it is p-harmonic for any $p \in [1, \infty)$. In particular, $u : (M^m, |\nabla u|^2 g) \longrightarrow \mathbb{R}$ is a harmonic Riemannian submersion. Conversely, if $u : (M^m, |\nabla u|^2 g) \longrightarrow \mathbb{R}$ is a harmonic Riemannian submersion, then, by Corollary 4.8, it is p-harmonic for any $p \in [1, \infty)$, and in particular, it is m-harmonic. Therefore, $u : (M^m, g) \longrightarrow \mathbb{R}$ is also an m-harmonic submersion since g is conformal to $|\nabla u|^2 g$ and m-harmonicity is invariant under the conformal change of metric on the domain manifold. $\square$

Proposition 4.12. *Let $m \geq 3$, and $u : (M^m, g) \longrightarrow \mathbb{R}$ be a submersive harmonic function whose level hypersurfaces are not minimal in (M^m, g). Let $\bar{g} = |\nabla u|^{2/(m-1)} g$. Then, $u : (M^m, \bar{g}) \longrightarrow \mathbb{R}$ is a nontrivial 1-harmonic submersion whose level hypersurfaces are minimal in $(M^m, \bar{g})$.*

Proof. Applying Theorem 4.10 to u with $p = 2$ we see that, with respect to the metric $\bar{g}$, u is a 1-harmonic submersion. By Corollary 4.8, $u : (M^m, \bar{g}) \longrightarrow \mathbb{R}$ is a trivial 1-harmonic submersion if and only if it is horizontally homothetic. Thus, to see that $u : (M^m, \bar{g}) \longrightarrow \mathbb{R}$ is a nontrivial 1-harmonic submersion it is enough to show that it is not horizontally homothetic i.e.

$$\bar{g}(\bar{\nabla} u, \bar{\nabla}\ln \left|\bar{\nabla} u\right|_{\bar{g}}) \neq 0 \quad \text{for some point.} \tag{27}$$

Since u is assumed to be a submersive harmonic function whose level hypersurfaces are not minimal in (M, g), we see from (22) that

$$g(\nabla u, \nabla\ln |\nabla u|) \neq 0 \quad \text{for some point.} \tag{28}$$

Substituting $F = |\nabla u|^{-1/(m-1)}$ into the last equation of (24) and performing a further calculation yields

$$\bar{g}(\bar{\nabla} u, \bar{\nabla}\ln \left|\bar{\nabla} u\right|_{\bar{g}}) = (m-2)(m-1)^{-1} |\nabla u|^{\frac{-2}{m-1}} g(\nabla u, \nabla\ln |\nabla u|).$$

This, together with the assumptions that $m \geq 3$, $|\nabla u| \neq 0$ and (28), shows that (27) holds, hence $u : (M^m, \bar{g}) \longrightarrow \mathbb{R}$ is a nontrivial 1-harmonic submersion. Thus, we complete the proof of the proposition. $\square$

4.3 THE MINIMAL SURFACE EQUATION IN A RIEMANNIAN MANIFOLD

In this section we study the minimal surface equation in a Riemannian manifold (M, g). After explaining the geometric meaning of the solutions and giving some entire solutions of the minimal surface equation in Nil space and in a hyperbolic space we study the transformation of the minimal surface equation under the conformal change of metrics.

Definition 4.13. *By the minimal surface equation (MSE) in a Riemannian manifold (M, g) we mean the following PDE*

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0, \tag{29}$$

or, equivalently,

$$\triangle u - \frac{|\nabla u|}{1+|\nabla u|^2} g(\nabla u, \nabla |\nabla u|) = 0, \tag{30}$$

where $u : M \supseteq \Omega \longrightarrow \mathbb{R}$ is a function.

Remark 4.14. *(i) When (M^m, g) is Euclidean space $\mathbb{R}^m$ with the standard metric δ_{ij}, then the minimal surface equation (29) gives the well-known minimal surface equation in a Euclidean domain which often appears in the form*

$$\sum_{i=1}^{m}\left(\delta_{ij} - \frac{u_i u_j}{1+|\nabla u|^2}\right) u_{ij} = 0. \tag{31}$$

The study of this equation is the main contribution to the progress of nonlinear elliptic PDE theory in the last century. Indeed, most early works on nonlinear elliptic problems focused on this equation. For the beautiful theorems on existence, uniqueness and regularity of the solutions of the MSE in Euclidean domain see, e.g., [30], [50] *and the references therein.*

(ii) When (M, g) is the 2-dimensional hyperbolic space (B^2, g^H) with the open disk model, then we obtain the minimal surface equation in (B^2, g^H) (see Example 4.18) which has been studied by Nelli and Rosenberg in their recent papers [37], [48].

(iii) It follows from Lemma 4.7 that a submersion $u : (M, g) \longrightarrow \mathbb{R}$ is a horizontally homothetic if and only if $g(\nabla u, \nabla |\nabla u|) = 0$. This and Equation (30) imply that any

horizontally homothetic harmonic submersion $u : (M, g) \longrightarrow \mathbb{R}$ *is an entire solution of the minimal surface equation in* (M, g).

Example 4.15. *Let* $(\mathbb{R}^3, g_{Nil})$ *denote Nil space, one of the eight three-dimensional geometries, where the metric with respect to the standard coordinates* (x, y, z) *in* $\mathbb{R}^3$ *can be written as* $g_{Nil} = \mathrm{d}x^2 + \mathrm{d}y^2 + (\mathrm{d}z - x\mathrm{d}y)^2$. *Then, it follows from Theorem 3.11 in* [40] *that the function* $u : (\mathbb{R}^3, g_{Nil}) \longrightarrow \mathbb{R}$ *defined by* $u(x, y, z) = Ax + By + C(z - xy/2)$ *is a horizontally homothetic harmonic submersion, where* A, B, C *are constants and not all of them are zero. So, by (iii) of Remark (4.14),* u *produces a family of entire solution of the minimal surface equation in Nil space.*

The following theorem gives the geometric meaning of the minimal surface equation in a Riemannian manifold.

Theorem 4.16. *Let* $u : (M^m, g) \longrightarrow \mathbb{R}$ *be a function, and let* $\Gamma(u, c) = \{(x, u(x) + c) \in M \times \mathbb{R} : x \in M\}$ *be the vertical graph of* u *at the height* c. *Then, the vertical graphs* $\{\Gamma(u, c) : c \in \mathbb{R}\}$ *form a foliation of* $(M \times \mathbb{R}, g + dt^2)$ *by minimal hypersurfaces if and only if* u *is a solution of the minimal surface equation in* (M, g).

Proof. Consider $f : (M \times \mathbb{R}, g + dt^2) \longrightarrow \mathbb{R}$ defined by $f(x, t) = u(x) - t$. Let $\bar{g} = g + dt^2$ denote the product metric, then a direct computation yields

$$\bar{\nabla} f = \nabla u - \frac{\partial}{\partial t}, \qquad \left|\bar{\nabla} f\right|_{\bar{g}} = \sqrt{1 + |\nabla u|^2}, \qquad \bar{\triangle} f = \triangle u. \tag{32}$$

Note that f is a submersion and its level hypersurface $f^{-1}(-c) = \{(x, t) \in M \times \mathbb{R} : t = u(x) + c\}$ is the vertical graph $\Gamma(u, c)$ of u at the height c. It follows from [40] that the foliation $\{\Gamma(u,c) : c \in \mathbb{R}\}$ of $(M \times \mathbb{R}, \bar{g})$ by the level hypersurfaces of f is a minimal foliation if and only if f is 1-harmonic submersion, i.e., f is a solution of

$$\operatorname{div}_{\bar{g}} \left(\left| \bar{\nabla} f \right|_{\bar{g}}^{-1} \bar{\nabla} f \right) = 0. \tag{33}$$

Using (18), (32) and the fact that $\bar{g} = g + dt^2$ is the product metric on $M \times \mathbb{R}$ and $\ln\sqrt{1+|\nabla u|^2}$ does not depend on t we have

$$\begin{aligned} \operatorname{div}_{\bar{g}} \left(\left| \bar{\nabla} f \right|_{\bar{g}}^{-1} \bar{\nabla} f \right) &= \{\triangle u - g(\nabla u, \nabla \ln\sqrt{1+|\nabla u|^2}\,)\}/\sqrt{1+|\nabla u|^2} \\ &= \operatorname{div} \left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \right). \end{aligned} \tag{34}$$

It follows from (34) that f is a solution of Equation (33) if and only if u is a solution of the minimal surface equation (29). Thus we obtain the theorem. $\square$

Note that if $u : M \longrightarrow \mathbb{R}$ is a constant function then it solves the MSE in (M, g) trivially; in fact, the foliation by vertical graphs of u is the canonical foliation $\{M \times \{t\} : t \in \mathbb{R}\}$ which is well-known to be a totally geodesic foliation.

Now we prove the following proposition which gives the minimality of vertical graphs with respect to a conformally deformed metric.

Proposition 4.17. *The MSE in* $(M^m, F^{-2}g)$ (*called the conformal minimal surface equation in* (M, g)) *is given by*

$$\operatorname{div} \left(\frac{\nabla u}{\sqrt{1+F^2|\nabla u|^2}} \right) - \frac{(m-2)g(\nabla u, \nabla F)}{F\sqrt{1+F^2|\nabla u|^2}} = 0. \tag{35}$$

Proof. Let $\bar{g} = F^{-2}g$, then a direct computation gives

$$\begin{cases} \bar{\nabla} u = F^2 \nabla u, \quad \left|\bar{\nabla} u\right|_{\bar{g}} = F\left|\nabla u\right|, \\ \\ \mathrm{div}_{\bar{g}}(X) = \mathrm{div}(X) - mF^{-1}X(F), \end{cases} \tag{36}$$

from which we have

$$\mathrm{div}_{\bar{g}}\left(\frac{\bar{\nabla} u}{\sqrt{1+\left|\bar{\nabla} u\right|_{\bar{g}}^2}}\right) = F^2 \mathrm{div}\left(\frac{\nabla u}{\sqrt{1+F^2\left|\nabla u\right|^2}}\right) - \frac{(m-2)Fg(\nabla u, \nabla F)}{\sqrt{1+F^2\left|\nabla u\right|^2}}.$$

This and the MSE in $(M, \bar{g})$ give the conformal minimal surface equation (35) and hence the proposition follows. $\square$

In the following example we will use ∇, $|.|$, and div to denote the gradient, the norm, and the divergence taken with respect to the standard Euclidean metric δ_{ij}.

Example 4.18. (*The minimal surface equation in hyperbolic space*) *Let* (B^m, g^H) *be the* m*-dimensional hyperbolic space with open-ball model, where* $B^m = \{x \in \mathbb{R}^m : |x| < 1\}$ *and* $g^H = F^{-2}\delta_{ij}$ *with* $F = 2^{-1}(1-|x|^2)$*. Then, the minimal surface equation in the hyperbolic space* (B^m, g^H) *is the conformal minimal surface equation in the Euclidean space* $(\mathbb{R}^m, \delta_{ij})$*, which, by Proposition 4.17, can be written as*

$$\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+F^2\left|\nabla u\right|^2}}\right) + \frac{(m-2) < x, \nabla u >}{F\sqrt{1+F^2\left|\nabla u\right|^2}} = 0, \quad x \in \mathbb{R}^m \tag{37}$$

When $m = 2$*, the minimal surface equation in* B^2 *becomes*

$$\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+F^2\left|\nabla u\right|^2}}\right) = 0. \tag{38}$$

A solution of this equation is a function $u : B^2 \supseteq \Omega \longrightarrow \mathbb{R}$ *whose vertical graph is a minimal surface in* $B^2 \times \mathbb{R}$*.*

Note that recently, Nelli and Rosenberg [37] [48] derived the MSE (38) by computing the mean curvature of the graph of u using a special adapted orthonormal frame along the graph of u. They also proved that for any rectifiable Jordan curve at infinity of $B^2 \times \mathbb{R}$ there is a minimal graph over B^2 whose asymptotic boundary is the given curve. In the next theorem we give a class of entire solutions of the MSE in an m-dimensional hyperbolic space.

Theorem 4.19. *In hyperbolic space (B^m, g^H) of the open-ball model as in Example 4.18, the function $u(x) = (a_1x_1 + ... + a_{m-1}x_{m-1})(1+|x|^2-2x_m)^{-1}$ is an entire solution of the MSE, where $a_1, ..., a_{m-1}$ are constant. Furthermore, the vertical graphs of u produce a foliation by minimal hypersurfaces none of which is totally geodesic.*

Proof. Consider the hyperbolic space in the upper-half space model $(\mathbb{R}^m_+, g^H_+)$, where $\mathbb{R}^m_+ = \{(y_1, ..., y_m) \in \mathbb{R}^m : y_m > 0\}$ and $g^H_+ = F^{-2}\delta_{ij}$ with $F(y) = y_m$. Let $f : (\mathbb{R}^m_+, g^H_+) \longrightarrow \mathbb{R}$ be a function defined by $f(y) = a_1y_1 + ... + a_{m-1}y_{m-1} =< A, y >$, where $A = (a_1, ..., a_{m-1}, 0), y = (y_1, ..., y_m) \in \mathbb{R}^m_+$. Let $\bar{g} = F^{-2}\delta_{ij}$. Then a straightforward computation using (24) shows that

$$\bar{\triangle} f = F^2 \triangle f + (2-m)F\langle \nabla F, \nabla f\rangle = 0, \text{ and}$$

$$\bar{g}(\bar{\nabla} f, \bar{\nabla} \ln \left|\bar{\nabla} f\right|_{\bar{g}}) = F^2\langle \nabla f, \nabla \ln |\nabla f|\rangle + F\langle \nabla F, \nabla f\rangle = 0.$$

It follows that f is a horizontally homothetic harmonic submersion in hyperbolic space $(\mathbb{R}^m_+, g^H_+)$ and hence, by (iii) of Remark 4.14, f is an entire solution of the minimal surface equation in hyperbolic space $(\mathbb{R}^m_+, g^H_+)$. The foliation of the vertical graphs of f is not a totally geodesic foliation for otherwise, by (iii) of Theorem

4.16, we would have $\left|\bar{\nabla} f\right|_{\bar{g}} = y_m |A| =$ constant, which is impossible. It is well-known that hyperbolic spaces (B^m, g^H) and $(\mathbb{R}^m_+, g^H_+)$ are isometric to each other. In fact, it is easily checked that the map $\varphi : B^m \longrightarrow \mathbb{R}^m_+$ with $\varphi(x) = (1 + |x|^2 - 2x_m)^{-1}(2x_1, ..., 2x_{m-1}, 1 - |x|^2)$ is an isometry. Since the MSE is invariant under isometries we conclude that $u = f \circ \varphi : (B^m, g^H) \longrightarrow \mathbb{R}$ with $u(x) = (2a_1x_1 + ... + 2a_{m-1}x_{m-1})(1 + |x|^2 - 2x_m)^{-1}$ is an entire solution of the MSE in (B^m, g^H), which gives the theorem. $\square$

4.4 *P*-HARMONIC FUNCTIONS AND THE MINIMAL FOLIATION BY VERTICAL GRAPHS

In this final section we give a link among p-harmonicity, horizontal homothety, and the minimality of the vertical graphs of a function. We also show that the foliation by the vertical graphs of a harmonic function can always be turned into a minimal foliation via a conformal deformation of metric.

First, we prove the following theorem which can be viewed as a dual to Theorem 4.6 in some sense.

Theorem 4.20. *Let $u : M \longrightarrow \mathbb{R}$ be a submersion. Then any two of the following statements imply the third:*

(a) *u is a p-harmonic function for some $p \geq 1$,*

(b) *u is a solution of the MSE in (M, g), i.e., the vertical graph $\Gamma(u, c)$ is minimal in $(M \times \mathbb{R}, g + dt^2)$,*

(c) *u is horizontally homothetic.*

Proof. $(a)+(b) \Rightarrow (c)$: Since u is a p-harmonic submersion we use (20) to have

$$\triangle u + (p-2)g(\nabla u, \nabla \ln |\nabla u|) = 0.$$

Combining this equation and the MSE (30) we obtain

$$(\frac{|\nabla u|^2}{1+|\nabla u|^2} + p - 2)g(\nabla u, \nabla \ln |\nabla u|) = 0. \tag{39}$$

If $p = 1$, then (39) implies that $g(\nabla u, \nabla \ln |\nabla u|) = 0$ identically since the first factor on the left hand side of (39) is never zero. It follows that u is horizontally homothetic. If $p \neq 1$, we claim that $g(\nabla u, \nabla \ln |\nabla u|) = 0$ identically and hence u is also horizontally homothetic in this case. Suppose otherwise, then there exists a point $x \in M$ such that

$$g_x(\nabla u, \nabla \ln |\nabla u|) \neq 0. \tag{40}$$

By continuity, there exists a neighborhood W of x on which the inequality (40) holds. It follows from (39) that $\frac{|\nabla u|^2}{1+|\nabla u|^2} + p - 2 = 0$ for any point in W, which implies that $|\nabla u| = \pm\sqrt{(p-2)/(1-p)}$ is a constant on W. It follows that $\nabla \ln |\nabla u| = 0$ and hence $g(\nabla u, \nabla \ln |\nabla u|) = 0$ on W, a contradiction.

$(a)+(c) \Rightarrow (b)$: Since u is a horizontally homothetic p-harmonic submersion, by Corollary 3.6 in [40], it is q-harmonic for any $q \geq 1$. In particular, it is a horizontally homothetic harmonic submersion. Thus, as in Remark 4.14 (iii), u is a solution of the MSE in (M, g) and the vertical graphs of u are minimal.

$(b)+(c) \Rightarrow (a)$: Since u is horizontally homothetic, $g(\nabla u, \nabla |\nabla u|) = 0$. This, together with the MSE (30), gives $\triangle u = 0$. Thus, u is a horizontally homothetic harmonic submersion and, by Corollary 3.6 in [40] again, it is a p-harmonic submersion for any $p \geq 1$. $\square$

Corollary 4.21. *A submersion $u : M \longrightarrow \mathbb{R}$ has minimal level hypersurfaces and minimal vertical graphs if and only if it is a horizontally homothetic p-harmonic function for some $p \geq 1$.*

Proof. If submersion u has minimal level hypersurfaces, then it is 1-harmonic by Theorem 4.6. If the vertical graphs of u are minimal, then Theorem 4.20 implies that u is horizontally homothetic. By Corollary 4.8, a horizontally homothetic 1-harmonic submersion is p-harmonic for any $p \geq 1$. Conversely, if u is a horizontally homothetic p-harmonic submersion for some $p \geq 1$, then, by Corollary 4.8, u is also a 1-harmonic submersion and hence the level hypersurfaces are minimal. On the other hand, Theorem 4.20 implies that the vertical graphs of a horizontally homothetic p-harmonic submersion are minimal. This completes the proof of the corollary. $\square$

The next theorem shows that we can always turn harmonic graphs into minimal hypersurfaces by a suitable choice of a metric.

Theorem 4.22. *Let $u : (M^m, g) \longrightarrow \mathbb{R}$ be a harmonic function from a complete Riemannian manifold. Then, the vertical graphs $\mathcal{G} = \{(x, u(x)+c) \in M \times \mathbb{R} : c \in \mathbb{R}\}$ produce a foliation of the complete manifold $(M^m \times \mathbb{R}, (1 + |\nabla u|^2)^{1/m}(g + \mathrm{d}t^2))$ by minimal hypersurfaces.*

Proof. Consider the function $f : (M^m \times \mathbb{R}, g + \mathrm{d}t^2) \longrightarrow \mathbb{R}$ given by $f(x, t) = t - u(x)$. The level hypersurface of f is $f^{-1}(c) = \{(x, u(x) + c) \in M \times \mathbb{R} : c \in \mathbb{R}\}$. Thus, the foliation by the vertical graphs of $u(x)$ is the foliation by the level hypersurfaces of $f(x,t)$. Let $\bar{g}$ denote the product metric $g + \mathrm{d}t^2$. Then, by (32), we have $\left|\bar{\nabla} f\right|_{\bar{g}}^2 = 1 + |\nabla u|^2$, and $\bar{\triangle} f = \triangle u \equiv 0$ since u is assumed to be harmonic. Thus $f : (M^m \times \mathbb{R}, g + \mathrm{d}t^2) \longrightarrow \mathbb{R}$ is a harmonic submersion. Applying Theorem 4.10 to f with $p = 2$ we obtain the Theorem except for the completeness of the metric $(1 + |\nabla u|^2)^{1/m}(g + \mathrm{d}t^2)$. Note that the product metric $\bar{g} = g + \mathrm{d}t^2$ is complete since g is assumed to be complete. It follows from Theorem 4.2 in [18] that the pointwise conformally deformed metric $(1 + |\nabla u|^2)^{1/m}(g + \mathrm{d}t^2)$ is compete since $(1 + |\nabla u|^2)^{1/m} \geq 1 > 0$. □

The following corollary shows that there are many minimal hypersurfaces in complete conformally flat spaces.

Corollary 4.23. *For any harmonic function $u : \mathbb{R}^m \longrightarrow \mathbb{R}$ with $m \geq 2$, the foliation $\mathcal{F} = \{(x, u(x)+c) \in \mathbb{R}^m \times \mathbb{R} : c \in \mathbb{R}\}$ of $\mathbb{R}^{m+1}$ by the parallel graphs of u is a minimal foliation with respect to the complete conformally flat metric $(1 + |\nabla u|^2)^{1/m}\ \delta_{ij}$ on $\mathbb{R}^{m+1}$. Furthermore, each graph is a homologically area-minimizing hypersurface.*

Proof. The first part of the corollary follows immediately form Theorem 4.22. That each graph is a homologically area-minimizing hypersurface follows from the fact that $\mathbb{R}^{m+1}$ is orientable and Corollary 3.9 in [40]. □

By Fischer-Colbrie and Schoen's classification Theorem [17](see also the recent paper of Li and Wang [31]), a complete, oriented, stable, minimal surface in a complete 3-manifold of nonnegative scalar curvature must be either conformally a plane $\mathbb{R}^2$ or conformally a cylinder $\mathbb{R} \times S^1$. The following proposition gives some examples of the same type of minimal surfaces in a complete conformally flat space of strictly negative scalar curvature.

Proposition 4.24. *With respect to the complete conformally flat metric* $\frac{1}{2}\sqrt{4+x^2+y^2}\,\delta_{ij}$ *of strictly negative scalar curvature on* $\mathbb{R}^3$*, (i) the foliation of* $\mathbb{R}^3$ *by the parallel hyperbolic paraboloids* $\mathcal{F}=\{z=\frac{1}{2}xy+c : c\in\mathbb{R}\}$ *is a non-totally geodesic minimal foliation with each leaf a complete, orientable, stable, minimal surface; (ii) the foliation by the parallel planes* $\mathcal{F}_3=\{(x,y,z)\mathbb{R}^3 : z=c, c\in\mathbb{R}\}$ *is a totally geodesic foliation with each leaf a complete, orientable, stable, minimal surface which is conformally a Euclidean plane* $\mathbb{R}^2$*.*

Proof. See [43] □

Remark 4.25. *It was proved in* [40] *that the foliation of* $\mathbb{R}^3$ *by the parallel hyperbolic paraboloids* $\mathcal{F}=\{z=\frac{1}{2}xy+c : c\in\mathbb{R}\}$ *is also a non-totally geodesic minimal foliation with respect to Nil metric* $g_{Nil}=dx^2+dy^2+(dz-xdy)^2$ *on* $\mathbb{R}^3$*. Therefore, we have an example of a foliation of* $\mathbb{R}^3$ *which is a non-totally geodesic minimal foliation with respect to two different Riemannian metrics.*

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