# PRE-SERVICE ELEMENTARY TEACHERS'

# UNDERSTANDING OF PATTERN AND FUNCTION

By

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The task of completing this dissertation sent my thoughts back to a day in the fall of 2005 when this journey first began. I remember driving to Stillwater, Oklahoma to meet with my advisers before I would be beginning my graduate studies in the spring. Although I was excited about becoming a Cowboy, I was dreading the long evenings away from my family. I wanted to fast forward to the end result and walk off the stage with diploma in hand. However, now that graduation is in sight, I realize that the journey was the highlight of this endeavor. Completing this journey would not have been possible without the support and encouragement offered by my family, colleagues, fellow students, and advisers.

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## CHAPTER I

## INTRODUCTION

The battle cry of ‘algebra for all’ was spawned by the launch of Sputnik and gained momentum along with advances in technology and concerns over global competition. Proponents view mastery of algebra as the key to prosperity in a technological world, both for the individual and for the nation (Katz, 2007). Algebra is not only purported to be the gateway to success, it is also perceived by many as a civil rights issue (Kilpatrick & Iszák, 2008). Proponents cite statistics that detail how minorities have been underrepresented in the algebra classroom and, therefore, locked out of high paying careers in science and technology (Chazan, 2008; Katz, 2007). Sounding the battle cry, Chazan decried that “to suggest that not all students need to study algebra seems to be tantamount to suggesting that one does not see all students as capable thinkers or that one is willing to curtail the economic prospects of some…” (p. 21). Unfortunately, if the algebra that all students will be taking is the same algebra that so many students have struggled with in the past, then the efforts to reform school mathematics will be impeded. As Kaput (2000, p. 1) asserted, “…*this* algebra is the disease for which it purports to be the cure! It alienates even nominally successful students from genuine mathematical experience, prevents real reform, and acts as an engine of inequity…” (p. 1). Indeed, algebra will continue to fail students as long as it appears disconnected from meaningful mathematics and from the lives of the students taking it (Chazan, 2008).

The view that algebra is both the gateway to and the gatekeeper for success has led to efforts to reform the way algebra is taught in schools today. Research indicated that the lack of transition from arithmetic to algebra was responsible, in part, for the difficulties students encountered with algebra (Kaput, 2000; Smith, 2003). In response, the National Council of Teachers of Mathematics (NCTM) began a movement to incorporate algebra into the elementary curriculum (Kilpatrick & Izsák, 2008). The vision for school algebra promoted by NCTM seeks to integrate algebra throughout the school mathematics curriculum (Chazan, 2008). This vision, however, does not center on the commonly held belief that algebra consists primarily of symbolic manipulations one completes in order to determine the value of unknown quantities Instead, the vision of algebra supported by NCTM endorses algebra as a way of thinking versus something that one does (Kilpatrick & Izsák, 2008).

Algebraic thinking is a more global construct of mathematics than the commonly accepted version of symbolic algebra. According to Smith (2003), "…*algebraic thinking*, in contrast [to symbolic algebra], has been used in a broader sense to indicate the kinds of generalizing that precede or accompany the use of algebra…" (p. 138). Students are engaged in the process of algebraic thinking when they examine patterns and make predictions on how the pattern might be extended. Making note of how quantities in two data sets are related and formulating a rule that defines that relationship offers another example of thinking algebraically (Zarkis & Liljedahl, 2002). These dual processes of analyzing change and generalizing mathematical relationships form the basis of algebraic thinking (Billings, 2008). With this vision of algebraic thinking in mind, NCTM penned the Algebra Content Standard in the *Principles and Standards for School Mathematic.* This standard "…emphasizes relationships among quantities, including functions, ways of representing mathematical relationships, and the analysis of change" (NCTM, 2000, p. 37). The strands approach to curricular design allocates the extensive time required for students to develop these powerful habits of mind. Summarily, the integration of algebra throughout the pre-K to grade 12 mathematics curriculum promises to add fuel to the engine of math reform and to open the gateway to the powerful ideas of algebra (Kaput, 2000).

One of the most powerful tools for developing algebraic ways of thinking lies in the study of patterns (Smith, 2003). Algebraic thinking, according to Steele (2005), envelops "…the ability to analyze and recognize patterns, to represent the quantitative relationships between patterns, and to generalize these quantitative relationships" (p. 142). Patterning activities in the early grades can begin by having students describe sequences formed by skip counting. Recognizing, for example, that the sequence formed when counting by two's can be extended by adding two more to the last term facilitates the development of recursive thinking (Bezuska & Kenney, 2008). Pictorial growth patterns also afford the opportunity to analyze change, describe how a pattern changes and how it can be extended. These verbal descriptions can then serve as a launching point for finding the *nth* term in the pattern (Billings, 2008). As Zazkis and Liljedahl (2002) so eloquently stated, "…patterns are the heart and soul of math" (p. 379). Unfortunately, generalizing and formalizing patterns serves as one of the oft neglected big ideas of algebraic reasoning (Kaput, 2000).

In Kaput's view, there are five key aspects of algebra, including the side of algebra usually seen in the high school curriculum. Generalizing and analyzing patterns, along with "algebra as syntactically guided manipulation of (opaque) formalism…" (Kaput, 2000, p. 3) form the basis of algebra, whereas the remaining three aspects represent sub-strands or extensions of algebra. The first of these sub-strands, "…algebra as the study of structures abstracted from computations and relations" (Kaput, p. 3), primarily resides in the content of higher-level mathematics courses. The second sub-strand views "…algebra as the study of functions, relations, and joint variation" (Kaput, p. 3). The study of functional relationships has gained considerable favor in recent years, partially due to the role of technology in the mathematics classroom (Kilpatrick & Izsák, 2008). The predominant view of function is based on the definition of a function as a correspondence between two variables in which every value in the domain is paired with exactly one value in the domain. The alternative view of function as covariation focuses more on the relationship between two covarying sets (Smith, 2003). Patterning activities in the elementary and middle school classroom can be used to study the relationship between the position of a term and its value or shape to facilitate the development of functional thinking (NCTM, 2000). Smith (2003) asserted that a covariational approach has a greater potential of developing functional thinking than does the abstract notion of correspondence.

The remaining aspect, modeling, focuses on the language of algebra and its connections to the outside world. Kaput asserted that the study of patterns, along with the study of relationships and real-world applications of algebra, is dependent upon the ability to reason algebraically (Smith, 2003). Kaput (2000) argued that the integration of these three aspects of algebra illuminates the many connections algebra has with other branches of mathematics, as well as other disciplines. By beginning to develop these connections in elementary school, the transition between arithmetic and algebra is eased (Cai & Moyer, 2008; NCTM, 2000). However, until teachers are able to recognize and to support the development of this type of reasoning with their own students, what Kaput referred to as "Algebra the Institution" (p. 4) is likely to remain unchanged.

## Statement of the Problem

Revisualizing algebra as a way of thinking presents numerous challenges for elementary teachers (Kaput, 2000; Stephens, 2008). The first hurdle they must overcome is the regime of traditional algebra. The majority of elementary teachers view algebra as a set of rules and procedures used to solve equations (Billings, 2008; Stephens, 2008). This limited view of algebra creates a roadblock in reform efforts by impeding both the formation of connections between the big ideas of mathematics and the development of algebraic thinking (Stephens, 2008). Successful integration of algebra in the elementary curriculum also depends upon the ability of the teacher to recognize and cultivate the seeds of algebraic thinking (Kaput, 2000). However, this competency calls for teachers to adopt a view of algebra that they probably never experienced for themselves (Billings, 2008; Stephens, 2008). Before teachers are able to foster algebraic thinking in their own classrooms, they need to construct "… a personal understanding of what it means to think algebraically" (Billings, 2008, p. 279).

Adopting an algebraic way of thinking also requires a profound understanding of the connections between patterns, functions, and algebra. Smith (2003) voiced the complaint that these connections are not readily apparent to elementary teachers. A first grade teacher may incorporate patterning activities in her classroom, but may never realize that these engaging lessons in pattern recognition pave the way for future work with functions as a way to analyze change. Middle school teachers may provide opportunities for students to examine change in real-world situations, but not recognize how this connects to algebra (Smith, 2003). One the three main components of the Connection Standard authored by NCTM (2000) states that teachers should provide opportunities for students to "…understand how mathematical ideas interconnect and build on one another to produce a coherent whole" (p. 64). If teachers are unable to make these connections themselves, then surely they will not be able to help their students build them.

Building these connections can be accomplished through professional development, but most adherents to the visions of an integrated mathematics curriculum would agree that the foundation for this reform lies in the preparation of pre-service elementary teachers. The National Mathematics Advisory Panel, in a report issued in 2008, recommended that mathematics curriculum for pre-service elementary teachers include a focus on introductory algebra concepts. In the executive summary of the 2008 report on the quality of programs in elementary teacher preparation, the National Council on Teacher Quality (NCTQ) wrote that elementary teachers "…need to understand algebra as generalization of the arithmetic they address while studying numbers and operations, as well as algebra's connection to many of the patterns, properties, relationships, rules, and models that they will occupy their elementary students" (Greenberg & Walsh, 2008, p. 55). To meet this need, NCTQ recommended that mathematics curriculum developed for pre-service elementary teachers place a greater emphasis on algebra. Stump, Bishop, and Britton (2003) noted that this charge poses numerous challenges for the mathematics teacher educator.

The problem of adequately preparing pre-service elementary teachers to support the emergence of algebraic thinking rests on the shoulders of mathematics teacher educators. How can mathematics teacher educators foster the development of algebraic thinking in the pre-service elementary teachers' mathematics classroom in ways these future teachers can recognize the emergence themselves? Although there are numerous studies on how pre-service elementary teachers understand ideas about number, there is a lack of research on how these pre-service teachers conceptualize the big ideas of algebra (Stephens, 2008). This lack of research hinders the development of curriculum that would enable pre-service elementary teachers to adopt the vision of algebra as a way of thinking. Smith (2003) asserted that the study of patterns and generalization can build a bridge to functional understanding, as well as pave the way to algebraic thinking. The purpose of this study was to investigate how pre-service elementary teachers experience the concept of function, particularly through the study of patterns.

## Research Questions

Elementary teachers need to recognize and support the development of algebraic thinking in their students. Mathematics teacher educators, therefore, need to determine ways to foster the development of algebraic thinking in pre-service elementary teachers. The purpose of this study was to examine pre-service elementary teachers' understanding of pattern and function as a way to understand how to prepare then more effectively in supporting the development of algebraic thinking in their own students.

Research Questions:

1. How do pre-service elementary teachers communicate the idea of function while engaged in the study of patterns?
2. How do pre-service elementary teachers demonstrate their understanding of function while engaged in the study of patterns?
3. What is the nature of pre-service elementary teachers' understanding of pattern and function?

## Theoretical Perspective

The theoretical framework of this study is based upon the hermeneutic circle, as described by Brown (2001). Drawing from the works of Gadamer and Ricoeur, Brown proposed using this model as a means for interpreting how mathematical understanding evolves over time through the interplay between explaining and understanding. When an individual engages in a mathematical activity, one must interpret the meaning of the problem within the context of one's prior experiences. However, this interpretation results in new understandings of the problem at hand which, in turn, alters the original interpretation. The hermeneutic circle is formed as the elements of interpreting and explaining a problem enhance each other, creating a cycle of developing understanding. Brown suggested that this understanding is captured, in part, by the texts produced by the individual. These texts, whether in the form of conversation or written work, provide a means for understanding mathematical learning from the perspective of the learner (Brown, 2001). The task of analyzing these texts requires the researcher to engage in a circular hermeneutic process as well. The researcher enters into a circle of developing understanding while reading, describing, and interpreting the texts created by the participants. When applied to qualitative analysis, the hermeneutic circle opens a space for interpreting the experiences of others (Patton, 2002).

The quest to understand and interpret human phenomenon falls under the interpretative framework of hermeneutic phenomenology. Hermeneutic phenomenology combines the philosophies of both phenomenology and hermeneutics to describe and interpret lived experiences (Van Manen, 1990). This study sought to examine the development and communication of functional reasoning among pre-service elementary teachers while they were engaged in patterning activities. The questions called for the mining of data from the experiences of these individuals as they were occurring. Phenomenology seeks to describe the nature of lived experiences from the perspective of the individual (Brown, 1996; Van Manen, 1990). However, this study also endeavored to interpret how these phenomena were experienced by the individual participants. Hermeneutics adds another layer to the study through the task of interpretation (Van Manen, 1990).

According to Crotty (2003), well-grounded research consists of carefully chosen methods which are justified by the theoretical perspective from which the researcher views such issues as the nature of knowledge. Van Manen (1990) stated, "…that the question of knowledge always refers us back to our world, to our lives, to who we are, and to what makes us write, read, and talk together as educators: it is what stands iconically behind the words, the speaking and the language" (p. 46). The hermeneutic circle is manifested in the discourse produced by individuals as they interact within a particular context which opens a space for understanding others. Cobb (2007) described "…a classroom mathematical practice as an emergent phenomenon that is established jointly by the teacher and students in the course of their ongoing interactions" (p. 30). Based on these assumptions, this study took place in the mathematics classroom of a group of pre-service elementary teachers and focused on the interactions these pre-service teachers had with each other while doing mathematics.

## Significance of Study

Being able to recognize and support functional thinking among elementary students is a vital part of efforts to integrate algebra throughout the pre-K – 12 mathematics curriculum. The transformation of algebra from a formal sequence of courses taken primarily at the secondary level to a way of thinking that begins in the early grades necessitates corresponding changes in the curriculum of pre-service elementary teachers. The results of this study have the potential of affecting curricular choices made by mathematics teacher educators. In particular, the examination of how pre-service elementary teachers conceptualize and communicate the concept of function offers insight into how we, as mathematics teacher educators, can provide curricular opportunities that facilitate understanding.

## Limitations

Due to the qualitative nature of this study, the findings obtained are not generalizable or replicable. The descriptions of the experiences of this group of pre-service elementary teachers will be unique to them. Repeating the study with a different group of pre-service elementary teachers, or even with the same participants, would likely yield different descriptions of the phenomenon under investigation. As Van Manen (1990) stated, hermeneutic phenomenology is the "…*theory of the unique*, it is interested in what is essentially not replaceable" (p. 7). Nevertheless, the findings are meant to be a source of ideas; a resource for mathematics teacher educators.

## Definitions

A pattern will be understood as the repeating or changing structure of a sequence of numbers or shapes (Smith, 2003). Algebraic thinking encompasses the ability to identify and extend patterns, as well as the ability to recognize and generalize the quantitative relationships between patterns (Steele, 2005). A functional relationship exists when one associates the position of a term in a sequence with its shape or numeric value. A covariational approach to the concept of function focuses on how changes in the position of a term result in changes in its shape or value (Smith, 2003).

The hermeneutic circle emerges when individuals attempt to make sense of the mathematics they are involved with in the context in which this activity takes place. As the individual interprets the problem and attempts to explain it, either in symbols or in words, the explanations they offer changes their own understanding of the situation. This mathematical discourse creates an ever-evolving circle between explanation and understanding (Brown, 2001).

## Chapter Organization

In the next chapter, a review of the literature pertaining to the concept of function as well as research on the understanding of pattern and function among learners of mathematics is presented. Particular attention was paid to research involving the mathematical understanding of the pre-service elementary teacher. Chapter III provides a rationale and description of the chosen methodology along with a detailed explanation of the methods of data collection and data analysis. In chapter IV, the preliminary results of the study are presented along with descriptions of the pattern-finding experiences of the six primary participants in this investigation. The major themes and significant findings associated with the idea function are presented in Chapter V.  The final chapter, Chapter VI, relates the findings back to the literature to reveal what new insights have been gained from this study. This final chapter will conclude with a discussion of the implications of the study, along with suggestions for future considerations for research on the algebraic thinking of pre-service elementary teachers.

# CHAPTER II

## A REVIEW OF THE LITERATURE

The purpose of this study was to investigate how pre-service elementary teachers experience the concept of function, particularly through the study of patterns. This study of how pre-service elementary teachers conceptualize the idea of function is embedded within the relationship between patterns and functions. In particular, this study will examine how patterning activities provide a context for the conceptualization and communication of functional relationships. The research questions to be addressed are:

Research Questions:

1. How do pre-service elementary teachers communicate the idea of function while engaged in the study of patterns?
2. How do pre-service elementary teachers demonstrate their understanding of function while engaged in the study of patterns?
3. What is the nature of pre-service elementary teachers' understanding of pattern and function?

To develop the background necessary to address these questions, this review of the literature examined the role of language in the development of concepts and the various ways to conceptualize the idea of function. This chapter includes a summary the approaches taken to investigate students' conceptualizations of function and the results of this research investigation.

The first section examines the role of language in both the communication and construction of mathematical concepts. The works of Vygotsky and Brown are explored to support the idea that learning in mathematics is inherently connected to language. The section provides an argument for framing this study within the hermeneutic circle.

In the following section, the concept of function is presented in terms of its historical development as well as theories on how individuals conceptualize the idea of function. The theories provide a framework for examining how researchers have investigated the conceptualization of function by pre-service teachers and by students in school mathematics. The rationale for utilizing patterning activities to study the conceptualizations of function is established.

The final section will explore the body of research on the development of the concept of function. The theoretical frameworks used in previous studies on the conceptualization of function are presented as an organizing feature for the section. Particular interest is paid to studies which examine the function sense of pre-service teachers. The rationale for basing the present study on a quantitative view of function and placing it within a phenomenological perspective are made.

## Constructing Mathematical Concepts

The goal of this research study is to unpack the understandings pre-service elementary teachers hold concerning the concepts of pattern and function. Before attempting to examine how pre-service teachers conceptualize these ideas, the notion of understanding in mathematics is explored. Sierpinska (1992) utilized a theoretical framework of understanding based on the works of Locke (1985), Dewey (1988), and others (as cited by Sierpinska, 1992). Under this framework, acts of understanding can be broken down into four categories. The first of these acts, identification, occurs when an individual recognizes that an object is of special interest. In other words, something now stands out as different from other objects around it. The second act, discrimination, occurs when the individual distinguishes both the differences and commonalities between two objects in mathematics. Subsequently, the third act of understanding, generalization, is made possible as the individual expands the notion to other settings. In the fourth act, synthesis, a cohesive concept is formed as the individual merges the various properties and facts about objects together. Therefore, it is not possible for understanding to simply arise from reading the definition presented in the textbook. Instead, the understanding of a concept, Sierpinska (1992) explained, emerges after:

> …we have seen instances and non-instances of the object defined, when we can say what this object is and what it is not, when we have become aware of its relations with other concepts, when we have noticed that these relations are analogous to relations we are familiar with, when we have grasped the position that the object defined has inside a theory and what are its possible application." (p. 26)

In such a manner, understanding appears to evolve over time through engagement in mathematical activities.

*Mathematics as Hermeneutical Understanding*

Brown (2001) proposed a framework for illustrating how mathematical understanding continually evolves through the process of reconciling present experiences with prior understandings. When individuals engage in mathematical activities, they initially interpret the problem in terms of what they already understand about mathematics. Through their attempts to explain their understandings, the explanations offered changes what they initially understood about the situation. A process of reconciliation between explanation and understanding develops to form a recursive relationship referred to as the hermeneutic circle. Brown used the metaphor of the hermeneutic circle to describe this textual relationship between understanding and explanation that fuels the development of concepts in mathematics.

Brown's attempt to center mathematical understanding on language use is based primarily on the hermeneutics of Gadamer and Ricoeur (Brown, 2001). Hermeneutics was traditionally applied to the interpretation of Biblical texts, but more recent applications have moved beyond textual forms to include the interpretation of human experiences (Crotty, 2003). An underlying premise of hermeneutics resides in the role language plays in shaping all of life's experiences, including the ways in which we come to understand our world (Crotty, 2003). Textual accounts are created as an avenue for people to share their experiences and beliefs with each other. Social scientists have turned to hermeneutics as a framework for interpreting these texts in ways that lead to greater understandings. This framework typically includes the metaphor of the 'hermeneutic circle' which Crotty (2003) described as a cyclical process wherein one uses what one already understands about a concept in order to deepen that understanding.

Gadamer based his view of hermeneutics on the notions that understanding is historical and is mediated by language (Brown, 2001; Crotty, 2003). He believed that "hermeneutics must start from the position that a person seeking to understand something has a bond to that subject matter that comes into language through [tradition]" (Rundell, 1995, p. 32; as cited by Crotty, 2003, p. 100). Gadamer envisioned language to be a part of the historical traditions that surround us. These traditions, however, do not form some outer entity, but rather a learning environment in which we participate. Our understandings are a result of this participation, as we use an inherited set of symbols to grasp a concept (Brown, 2001; Crotty, 2003). Under the framework of hermeneutics, mathematical concepts are seen as cultural artifacts that are carried in the symbols used to communicate these ideas (Brown, 2001). An individual must make sense of the ideas using an inherited set of symbols when engaging in mathematical activity. The ways in which he or she approaches the problem are also dependent on prior experiences with these symbols (Brown, 2001). Gadamer viewed the issues of past traditions and present experiences as two opposite poles which must be reconciled in some manner to reach what he referred to as a 'fusion of horizons' (Crotty, 2003). The fusing of what is known from the past with what is experienced in the present results in a new understanding of a concept.

Brown (2001) drew on Ricoeur's description of the hermeneutical circle to explain how this fusion of horizons may take place. According to Ricoeur, the past experiences within a tradition serve as "the dialectical glue between a subject and the objects in her world" (Leonardo, 2003, p. 335). Through this medium of the past, the individual seeks to understand a concept and in the process, creates a discourse that partially captures these understandings (Brown, 2001; Leonardo, 2003). The explanations offered by the individual do not hold all that is understood by the person, since the understanding is ongoing and the explanations are frozen in time. However, the ongoing understanding may be transformed by the explanations offered which may lead to an even deeper understanding. This interaction between explanation and understanding, as described by Ricoeur, forms the hermeneutic circle (Brown, 2001; Crotty, 2003).

Brown (2001) stated that the process of learning mathematics “continuously evolves, oscillating between understanding and explanation” (p. 80). Drawing upon Ricoeur’s idea of the hermeneutic circle, Brown viewed understanding as a dynamic process which can be partially captured in the statements produced by the learner. Sierpinska (1992) also described understanding in mathematics as a hermeneutic process, but she rejected the cyclical nature implied by the metaphor of the hermeneutic circle. Sierpinska asserted that understanding in mathematics is more likely to be a discontinuous process, littered with instances of stagnation followed by giant leaps in understanding. She attributed these periods of stagnation to epistemological obstacles that occur due to misconceptions held by individuals or by certain societal groups. Conflicts between these misconceptions and new evidence that challenges them open a space for new understandings to develop. Sierpinska (1992) viewed the metaphor of the hermeneutic circle as a trap when the preexisting knowledge structures are incorrect or inadequate. She stated “…it is possible to escape the paradox if we abandon the metaphor of “circle” and bring forth the idea of spirality in describing cognitive processes” (p. 28).

Modifying the hermeneutic circle to incorporate the spiraling effect created by an evolutionary change in conceptual understanding aligns with Vygotsky’s notion of “a higher plane of thought” (1934/1986, p. 202). Vygotsky spoke of how generalizations lead to new levels of understanding that bring the individual to this higher plane. In this model, concept development is not simply a matter of acquiring a fixed body of knowledge.  As Vygotsky stated, a concept is “more than the sum of certain associative bonds formed by memory, more than a mere mental habit; it is a complex and genuine act of thought that cannot be taught by drilling” (p. 149). The word that represents the concept is but a generalization of the idea whose meaning evolves through the experiences that the individual has with the idea. Vygotsky asserted that these generalizations arise out of the need to communicate one’s thought processes. Without words, it is impossible to think in terms of concepts.

*Concept Development*

Thinking about a concept is part of one’s understanding and speech is used to explain this understanding to self and to others. The meaning of the word used to describe a concept evolves in the cycle between understanding and explaining that arise from experiences. Vygotsky (1934/1986) described concept development as a dynamic enterprise that engages the individual in problem-solving activities. In doing so, he rebuked the idea of studying concept development as a fixed course, instead viewing it as “…a live, thinking process” (p. 105).

Vygotsky (1934/1986) described three stages of concept development that arise out of the need to communicate complex ideas. In the first stage, objects are clumped into unorganized categories based primarily on trial and error. Through experiences with the objects, the individual formulates a set of rules for joining these objects into groups that Vygotsky referred to as complexes. In this second stage, the complexes formed evolve into “pseudoconcepts” that are held together by concrete facts the individual has derived from his or her experiences. The individual may adopt the same word or expression to describe the ideas as the teacher does, but his or her “…framework is purely situational with the word tied to something concrete and the adult's frame is conceptual" (Vygotsky, 1934/1986, p. 133). Vygotsky asserted that to arrive at the third level of a true concept, the individual must move beyond the concrete bonds of a pseudoconcept. The generalizations used to formulate a pseudoconcept need to be analyzed or separated into their constituent parts for conceptual understanding to occur. In this manner, Vygotsky stated, "the connections between concepts are neither associative nor structural, but are based on the *principle of the relations of generality*" (p. 204).

Vygotsky (1934/1986) illustrated the shift from the concrete to the conceptual with the leap taken from understanding arithmetic to comprehending algebra. The student of arithmetic derives concepts about number through experiences he or she has with objects. These arithmetic concepts, in turn, lead to generalizations about number that form the basis of algebra. For example, a student counting by twos might realize that the counting sequence could be represented in general terms by the formula $2n$. Viewing multiplication as repeated addition takes the individual to a higher level of understanding arithmetic because, as Vygotsky (1934/1986) stated, the one who understands the concepts of algebra gains "…a vantage point from which he sees concepts of arithmetic in a broader perspective" (p. 202).

The study of concept development cannot be broken down into a series of distinct steps. The symbols used to communicate mathematical thinking and the ways in which our experiences transform understandings complicates the analysis of concept development. Vygotsky (1934/1986) voiced the concern that "…to understand another's speech, it is not sufficient to understand his words – we must understand his thought" (p. 253). The image of the hermeneutic circle offers a means to access the thought processes of the individual engaged in mathematical activity (Brown, 2001). Under this image, the thinking process undertaken to understand a concept is partially revealed in the words used to explain these understandings while they are evolving. Brown (2001) asserted that such "…notions of hermeneutic understanding as applied to mathematics require a shift in emphasis from the learner focusing on mathematics as an externally created body of knowledge to be learnt, to this learner engaging in mathematical activity taking place over time" (p. 50).

This study focuses on the statements generated by a group of pre-service elementary teachers as they are engaging in a series of mathematical activities. The decision to center the study on mathematical activity is based on the view of mathematical learning presented in this section on concept development. In the next section, the various ways to conceptualize the idea of function will be explored in light of its historical development. The move from the idea of function as covariation to the more generalized view of function as a correspondence will be considered to evaluate the effect this switch has had on teaching the concept of function. In addition, theories on how learners think about functions will be presented.

## The Concept of Function

### *History of the Concept*

When asked to define a function, a student of algebra is apt to recite a textbook definition based on the relationship between members of two sets, referred to as the domain and range. According to this widely accepted definition, a "function from set *A* to set *B* is a correspondence from *A* to *B* in which each element of *A* is paired with one, and only one, element in set *B*" (Billstein, Libeskind, & Lott, 2001, p. 105). Although this correspondence view has been widely accepted since the 19th century, earlier views of a function were much less precise (Ponte, 1992). The word ‘function’ did not appear in print until the late 1600’s, however the idea of function permeated mathematics early on. Ponte offered examples such as counting procedures, which establish a correspondence between a given quantity and a number word. He also described ancient Babylonian tablets which contained representations of functions in the form of square and cube roots. Both of these examples illustrate how the notion of function was present long before the concept was defined.

The birth of analytical geometry during the 17th century led to a formalization of the concept of function (Burton, 2007; Ponte, 1992). Although Leibniz (1673) is credited with first using the word to refer to geometric objects such as the tangent to a curve, Euler (1748) is responsible for penning the first formal definition of a function (Burton, 2007; Ponte, 1992). Euler’s initial definition tied the concept of function to an analytical expression, but he later refined this definition to include any dependent relationship in which a given set of quantities covaries with another. Burton (2007) cited Euler’s definition as follows: “If therefore, $x$, denotes a variable quantity, then all quantities which depend upon $x$ in any way or are determined by it are called functions of it” (p. 611).

During the period of time that followed Euler’s definitions, great advances in analysis and the birth of set theory revealed inconsistencies in Euler’s definition of a function as covariation. Mathematicians of the nineteenth century pushed for a broader concept of function which led to the acceptance of Dirichlet’s (1837) correspondence view (Burton, 2007; Ponte, 1992). Similar to the textbook definition in use today, Dirichlet stated that "…*y* is a function of the variable *x*…if to every variable *x*…there corresponds a definite value of the variable *y*" (Burton, 2007, p. 612). The correspondence view granted mathematicians more flexibility when dealing with functions, but the very generality that was required for abstract algebra created a pedagogical nightmare in school mathematics (Ponte, 1992; Silverman, 2005; Smith, 2003). Students continue to struggle to make sense of the abstract concept of a correspondence and the symbolic expressions sometimes associated with it. Smith (2003) stated that this struggle perpetuates across generations of students because "…this approach omits building relationships through an understanding of covariation" (p. 141)

*Conceptualizations of Function*

The historical refinement of the concept of function offers two ways to conceptualize functional relationships (Confrey & Smith, 1995; Slavit, 1997; Smith, 2003; Billings, 2008). The first of these views is based upon Euler's definition of a function as a dependent relationship in which one data set covaries with another. Viewing a function as covariation places an emphasis on how changes in one variable result in changes with another. The second view draws upon the modern definition of function as a correspondence between two data sets (Slavit, 1997). Under this perspective, emphasis is placed on stating the relationship that maps members of one set, usually referred to as the domain, to members of another set, known as the range (Confrey & Smith, 1995; Slavit, 1997; Smith, 2003).

The idea of function as correspondence is the prevailing view presented in school algebra (Slavit, 1997; Smith, 2003). This perspective is often presented to early algebra learners as a function machine. Students examine how each input results in a particular output and try to write a rule for predicting outputs based on inputs. Smith (2003) offered the following example of how someone might analyze the functional relationship between two data sets using a correspondence approach. In this example, displayed here in Table 1, the individual would look across the table in an attempt to formulate the manner in which values in the first column, labeled *x*, are mapped to values in the second column, labeled *y*. Recognizing that each *y*- value is one more than three times its corresponding *x*- value might lead to the explicit formula $y = 1 + 3x$. The ability to generalize a pattern in this manner is a key component of algebraic thinking (Kaput, 2000; Smith, 2003). However, as Confrey and Smith (1995) argued, the correspondence approach fails to develop an understanding of function as covariation since the major focus is on writing an explicit rule.

Table 1

*Example of Functional Relationship*

| *x* | *y* |
|---|---|
| 0 | 1 |
| 1 | 4 |
| 2 | 7 |
| 3 | 10 |

*Note. Example appeared in Smith, 2003, p. 141*

Using the same example presented here in Table 1, Smith (2003) described how someone taking a covariational approach would look down the table instead of across the table, attending to the manner in which changes in the first column coordinate with changes in the second column. Using this approach to analyzing change, the individual would note that each unit increase in *x*-values corresponds to an increase of three in the value of *y*. This action differs from recursive thinking in that the individual focuses simultaneously on coordinating the rate of change in y with respect to x. Recursive thinking generally involves analyzing changes within a single data set, thus lacking the element of coordination between sets (Moss, Beatty, Barkin, & Shillolo, 2008). Saldanha and Thompson (1998) and others (Carlson, Jacobs, Coe, Larsen & Hsu, 2002) suggested that covariational analysis requires one to create an image of how two quantities change in relationship to each other. Thus, the emphasis in a covariational approach is placed on the dual actions used to create the function versus the static processes that relate $x$ to $y$ (correspondence approach) or the singular actions that generate the values within a set (recursive thinking). By focusing on the dual actions, one is able to see the repeated operations that created the relationship between two data sets (Confrey & Smith, 1995; Smith, 2003).

Table 2

*Developing a Correspondence through Covariation*

| $x$ | Process | $y$ |
|---|---|---|
| 0 | | 1 |
| 1 | 1 + 3 | 4 |
| 2 | 1 + 3 + 3 | 7 |
| 3 | 1 + 3 + 3+ 3 | 10 |
| 4 | 1 + 3(4) | 13 |
| $x$ | $1 + 3(x – 1)$ | $3x - 2$ |

Confrey and Smith (1995) asserted that recognition of these actions enables one to establish a correspondence between the two sets. For example, noting that the $y$ values increase by three with each unit increase in $x$, as in Table 1, can be used to rewrite each $y$ value in terms of the repeated addition used to create it. The results of this repeated addition is presented in Table 2 with the insertion of a process column. The repeated addition can then be abbreviated as multiplication, leading to the explicit rule of $y = 1 +$

$(x - 1)3$. Not only does the covariational approach offer a means of developing the correspondence between data sets, but it also has the potential of promoting the view of algebra as generalized arithmetic (Kaput, 2000; Smith, 2003).

Slavit (1997) and Smith (2003) argued that the covariational approach, with its emphasis on change, leads to a more complete understanding of function. However, Slavit (1997) asserted that each of these views is necessary for a full understanding of the idea of function. These two ideas of function, correspondence and covariation, are manifested in the ways individuals describe patterns (Smith, 2003). According to Smith (2003), a pattern can be defined in terms of its structure or it can be defined in terms of its change. Attending to the structure of a pattern places an emphasis on the invariant attributes of the pattern. On the other hand, examining change allows one to grasp the dynamic actions that create the growing pattern. Smith referred to these two as stasis and change, stating that both conceptualizations were necessary for the development of the idea of function.

Although the textbook definition of function seems simple enough, students still struggle to develop a full understanding of what is meant by a functional relationship (Ponte, 1992). Sierpinska (1992) identified several epistemological obstacles to developing an understanding of function, including the inability to analyze change and the tendency to think of functions only as equations. Combining the ideas of function as covariation and correspondence to create a more cohesive understanding of function was presented as one pathway to overcoming these epistemological obstacles (Slavit, 1997; Smith, 2003). In the next section, theories on how the concept of function develops are presented.

*Levels of Conceptualizations of Function*

The literature offers numerous frameworks for categorizing the development of the concept of function. The notion that the concept of function does not merge simply from exposure to the textbook definition permeated these theories. Instead, as Vinner and Dreyfus (1989) implied, a concept image is formed as a result of the experiences a student has while engaging in mathematical activities. The majority of the theories reviewed proposed that complex concepts such as the idea of function develop over several stages. Kaput (1992) based his framework on the historical development of algebra as a symbolic system. Dubinsky and Harel (1992), along with Breidenbach, Dubinsky, Hawks & Nichols (1992), proposed a framework based on the constructivist's views of Piaget. Others, such as Slavit, 1997, expanded on prior frameworks in an attempt to better understand how individuals conceptualize functions.

According to Kaput (1992), students develop the idea of function along the same lines as the historical development of algebra. He used the results of a study on how students constructed relationships between ordered pairs of numbers. The study called on students to write rules for linear functions based on inputs and outputs. Kaput noted that student understanding of function could be categorized as either pre-algebraic or algebraic. He classified a student's understanding of function as pre-algebraic if he or she was only able to express a functional relationship using everyday language or a type of symbolism that reflected what had been stated verbally. Often the rules written by these students were the product of guesses based on the last ordered pair in a data set. To Kaput, these students understood algebraic symbolism in the rhetorical sense, much like the early algebraists. On the other hand, a student's understanding was classified as algebraic if he or she could easily identify the nature of the relationship between data sets (i.e. linear) and write a rule using algebraic symbolism. Kaput cautioned that the distinction between the two classes is difficult to make when relatively easy tasks, such as linear functions, are used. This is due, perhaps, to the ease of connecting everyday language to the operations involved in creating linear patterns.

Many of these theories on the development of the concept of function were based on the constructivist views of Piaget, focusing on the cognitive stages experienced by the individual as a result of his or her experiences related to the concept at hand (Seldan & Seldan, 1992). Piaget (1977, as cited by Klanderman, 1996) described four stages in the development of the idea of function. In the first stage, the student fails to recognize any relationship between the two sets of variables. Once the student begins to see a relationship between individual pairs in the data set, they enter the second stage. This stage might correspond with what Vygotsky (1934/1986) described as the first stage toward concept development in which the learner tends to group objects into categories by trial and error. The relationship between the pairs of numbers forms a concept image of function in a haphazard fashion. The third stage is marked by the ability to view the relationship between the two data sets in qualitative terms. For example, the student in a problem involving rates may describe how the distance from a fixed point increases as time elapses. In the final stage, the student would be able to quantify this relationship (Piaget, 1977, as cited by Klanderman, 1996).

Building on this constructivist approach and the works of Bredienbach, Dubinsky, Hawks and Nichols (1992), Dubinsky and Harel (1992), proposed three stages of understanding which exist along a continuum of development. At the prefunction stage, an individual is generally unable to make sense of the concept in any fashion whatsoever. This stage could be said to align with the first stage described by Piaget. At the next level, an individual adopts an action conception of function and is able to work with the idea in a more procedural way. However, the individual struggles with functional relationships which are not easily described by algebraic formulas or those which require multiple steps to evaluate (i.e. composite functions or those with split domains). Once these struggles have been overcome, an individual can be said to hold a process conception of function. Instead of limiting one's focus on the steps taken to produce the outputs of a function, the individual with a process conception is able to think about the function in its entirety, combining it with other functions or reversing it. These stages exist along a continuum and are dependent upon the context or situation. An individual may present a process conception of function under certain conditions, such as when dealing with linear functions, but hold an action conception of function when encountering more complex functions. Therefore, Dubinsky and Harel (1992) cautioned that it is difficult, if not inappropriate, to attempt to categorize individuals' actions along this continuum.

Dubinsky and Harel (1992) cited a study conducted by Breidenbach, Dubinsky, Hawks, and Nichols (1992) which sought to refine their epistemological theory centered on three conceptualizations of function: prefunction, action, and process. The study involved 62 pre-service teachers majoring in mathematics and consisted of a pre-test, computer-based teaching intervention, and a post-test. Interviews were conducted during the teaching intervention to clarify understandings. A portion of the pre-test asked the pre-service teachers to explain what constitutes a function. Breidenbach, et al. (1992)

classified incomplete/missing responses at the prefunction level (40% of respondents). They grouped responses under the category of action if the description given failed to include the elements of input, transformation, and output (24%). Those responses which included all three of these elements were marked as exhibiting a process conceptualization of function (16%). Data gathered through interviews and the post-test indicated that the participants were able to transition from thinking of functions as actions to viewing functions as processes. Breidenbach, et al. (1992) asserted that the ability to adopt a process view of function is an essential step towards understanding the concept of function.

Dubinsky and Harel (1992) concurred with Sfard (1992) that a higher level of conception appears when the individual is able to conceive of the function as an object. The object view of function allows one to act upon the function in its entirety. Sfard asserted that without this ontological shift from process to object, or reification, the individual's "…approach will remain purely operational" (p. 64). The action, process, object progression described here is commonly referred to as the APO framework. Vygotsky's (1934/1986) notions of the development of complexes seems to fit along the continuum between action and process conceptions, with pseudoconcepts falling towards the right. However, his definition of a true concept would assumedly align with the object conception. At this juncture, the individual is able to generalize that which has been generalized before; to take that which was understood in the past and escalate it "…to a higher plane of thought" (p. 202).

Slavit (1997) embedded a property-based perspective within the APO framework. Under a property-oriented view of functions, students assimilate properties of functions through experiences with various classes of function. For example, analyzing functions from the vantage point of covariation can lead to the recognition of various growth properties associated with a class of functions. The recognition of these properties can be used to build a library of functions from which a more global understanding of function can be developed. Although Slavit emphasized the wealth of functional properties that can be derived from examining aspects of functional growth, he cautioned that other properties may need to be examined from a relational view. For this reason, Slavit professed, "…the covariance and correspondence views of function, under the property-oriented framework, should not be considered as contrasting or distinct viewpoints, but rather can be considered complementary ways of thinking about the concept of function as a mathematical object possessing various properties" (p. 270).

Each of the theories on concept development presented here emphasizes the fundamental role the act of generalization plays in this development. This is not surprising since functions can be thought of as the generalized relationship between two data sets (Blanton & Kaput, 2004; Warren, Cooper, & Lamb, 2006). The act of generalization can arise from examining how two data sets change in relationship to each other or by recognizing the correspondence between ordered pairs (Confrey & Smith, 1995; Smith, 2003). Covariational understanding is viewed as a bridge between thinking of a function as an action and conceptualizing function as a process (Silverman, 2005). The correspondence view aligns with Kaput's algebraic classification of functional understanding as well as the process conception detailed under the APO framework. Most argued that these two ideas of function are important building blocks of algebraic thinking (Kaput, 2000; Slavit, 1997; Smith, 2003, Warren, Cooper, & Lamb, 2006).

As this review has illustrated, the concept of function is multi-faceted and evolves as one engages in experiences with functional relationships. The task of understanding how learners conceptualize the idea of function is complex and multi-faceted as well. In the next section, approaches used by researchers to explore function understandings are presented, with particular attention given to studies involving the function understandings of pre-service teachers.

## Studies on Students' Understandings of Function

Studies on students' understandings of function can be organized under three general perspectives, as noted by Lobato and Bowers (2000). For example, researchers may focus on how multiple representations play a role in developing students' understanding of function. The emphasis here is placed on how students build connections between tables, graphs, verbal models, and the symbolic representations of functions. Alternatively, researchers have approached the study of functional understanding from a quantitative perspective using tables and/or patterns. These investigations were centered on the dual ideas of function as covariation and as a correspondence, with an emphasis on how students analyzed change in order to generalize patterns. However, Lobato and Bowers (2000) highlighted a third perspective from which researchers have investigated students' understanding of function. This perspective focuses not only on the mathematics that is being learned but also on the experiences of those engaged in the mathematical activity. These three perspectives are not mutually exclusive and can be combined to enhance the study of how students develop an understanding of function (Lobato & Bowers, 2000).

### *Representational Perspective*

Functions can be represented using various formats, each of which is thought to provide accessibility to understanding the concept of function. Experiences creating tables and graphs, writing symbolic rules, and analyzing real-world phenomena are believed to play a crucial role in developing this understanding (NCTM, 2000). Therefore, researchers adopting this viewpoint investigated how students make connections between the various forms of a function (Lobato & Bowers, 2000). The focus of the of studies using this viewpoint and reviewed for this investigation tended to be on secondary and higher level mathematics, therefore only those pertaining to pre-service teachers will be presented in this review (Klanderman, 1996; Haciomeroglu, 2006).

Klanderman (1996) utilized the APO framework (action, process, object progression) in his study on pre-service teachers' understanding of functions. The purpose of his study was to evaluate levels of understanding within multiple representations of functional relationships. The participants in the study included 19 pre-service elementary teachers and 6 pre-service secondary teachers. Klanderman created an instrument to evaluate levels of understanding and combined the results of this survey with analyses of videotaped interviews and journal entries. The instrument consisted of five problems which presented a functional relationship within either a real-world setting (3) or in a tabular format (2). The three real-world problems represented one linear, one exponential, and one quadratic relationship. Only linear and exponential relationships were presented in tabular form. Pre-service teachers were asked to find the next value, determine an out-of-sequence value, write a rule, and then graph the relationship.

Klanderman (1996) found that the majority of pre-service elementary teachers were able to extend the patterns, but many had difficulties generalizing the relationship. Pre-service teachers had the most success writing equations to represent the linear relationship within the real-world setting (74%), however, only 16% were successful writing a rule when the relationship was presented in tabular form. The two problems involving exponential relationships created the most difficulties for pre-service elementary teachers. Only 32% of these participants were able to generalize the relationship inherent in the real-world setting and only 16% were successful with the tabular problem. The primary cause of this problem was the tendency to linearize the exponential relationships. Klanderman classified nine of the pre-service teachers' understanding as mode dependent, noting that the real-world setting facilitated their ability to extend and generalize a pattern. In addition, he suggested that a high level of understanding functions may be dependent upon a sound understanding of the concept of variable.

Haciomeroglu (2006) also examined pre-service teachers' understanding of the concept of function within multiple representations. She focused her investigation on evaluating both the subject matter and pedagogical content knowledge of the concept of function held by two prospective secondary teachers. In her discussion, Haciomeroglu stated that the two participants had difficulty identifying functions when presented either verbally, in the form of word problems, or numerically in tables. She attributed these difficulties to deficiencies in subject matter knowledge and noted that pre-service teachers should be able to work flexibly with all functional representations. However, this study centered only on the subject matter knowledge needed to teach the concept of function at the secondary level.

Welder (2007) addressed the question of what mathematical content knowledge is needed to enable pre-service elementary teachers to become effective teachers of pre-algebraic concepts. She identified nine areas of concern, involving both number concepts and concepts related to equations and functions. Welder listed the skill set needed for individuals to be able to understand the functional relationship between two data sets. These skills included the ability to produce outputs, analyze rates of change, and make connections between different representations of functional relationships. In addition, Welder believed that the ability to generalize patterns should be part of the repertoire of skills needed by pre-service elementary teachers. For her investigation, Welder developed a 51-item quantitative instrument to measure the depth of pre-service elementary teachers' content knowledge of the concepts of number and function. Forty-eight pre-service elementary teachers enrolled in a mathematics content course participated in the study by taking the pre-and post-tests created by Welder. Participants in the course made significant gains (mean standardized difference of .3906, $p < .001$) in mathematical content knowledge of function-related concepts. However, Welder identified the ability to work flexibly with functions within multiple formats as a key problem area. In particular, Welder found that these pre-service teachers had difficulties generalizing pictorial patterns and identifying functional relationships presented in a word-problem format.

The typical library of functions recommended for K – $8^{th}$ grade level mathematics includes linear, exponential, and quadratic functions (NCTM, 2000). Stump and Bishop

(2002; also cited in Stump, Bishop, & Britton, 2003) examined pre-service elementary teachers' ability to analyze these three functions within multiple representations. The 30 teachers participating in the study completed activities which focused on analyzing change within real-world functional relationships. After completing these activities, the pre-service teachers answered a post-test consisting of a series of questions involving linear, exponential, and quadratic functions. Stump and Bishop (2002) used the results of this post-test to analyze the participants' understanding of function within multiple representations.

Although the majority of the pre-service elementary teachers could recognize a linear relationship when presented in tabular, symbolic, or graphic form, Stump and Bishop (2002) found that more than a third of them were unable to clearly explain how to go about this task. When dealing with exponential equations, most of the pre-service elementary teachers were able to write a rule to match an exponential situation, but only a third could describe the pattern of change. Interestingly, the majority of the pre-service elementary teachers were able to recognize and describe quadratic relationships when presented in tabular or graphic form, but approximately one third of those participating had difficulties working with quadratic patterns created by geometric figures. Moreover, only half of the participants could write a rule to match the quadratic geometric pattern. Stump and Bishop (2002) voiced concern over the difficulties pre-service teachers had with communicating their understanding of functional relationships. They stated that "…before they can successfully promote algebraic thinking in their own classrooms, pre-service teachers need to understand algebra as a way of thinking, a way of working with the patterns that occur every day" (p. 1912).

The four studies presented here highlight several areas of concern dealing with pre-service teachers' abilities to generalize functional relationships (Haciomeroglu, 2006; Klanderman, 1996; Stump & Bishop, 2002, Stump, Bishop, & Britton, 2003; Welder, 2007). Quantitative data presented in either tabular (Haciomeroglu, 2006; Klanderman, 1996) or pictorial formats (Stump & Bishop, 2002, Stump, Bishop, & Britton, 2003; Welder, 2007) presented the most difficulty for the pre-service teachers participating in these studies. The most common types of functions studied by early algebra students are linear, exponential, and quadratic (NCTM, 2000). However, these pre-service teachers struggled to generalize exponential patterns (Klanderman, 1996; Stump & Bishop, 2002, Stump, Bishop, & Britton, 2003) and geometric representations of quadratic relationships (Stump & Bishop, 2002, Stump, Bishop, & Britton, 2003). These reported struggles with generalizing functional relationships point to a breach in the ability of pre-service teachers to make the connection between quantitative data and symbolic representations.

*Quantitative Perspective*

The majority of studies examining functional understanding from a quantitative perspective used tables and/or patterns as a venue. NCTM (2000) suggested that Pre-K through grade 12 curricular activities designed to promote an understanding of function include the study of patterns and the analysis of change. Smith (2003) reiterated this suggestion by emphasizing the importance of understanding change and the impact such understanding has on the ability to analyze functional relationships. Several studies reviewed in preparation for this investigation examined the extent to which elementary students are capable of functional thinking as well as the effectiveness of instructional materials (Blanton & Kaput, 2004; Warren, Cooper, & Lamb, 2006). The idea of using patterning activities as a venue for studying the pre-service elementary teachers' understanding of function lies at the heart of this study. For this reason, particular attention will be paid to presenting the results of these types of studies.

*Studies with elementary/middle school students.* Blanton and Kaput (2004) explored ways in which a group of elementary students developed and communicated their understandings of function. Based on data collected from one problem situation visited by a group of students in grades pre-K through 5, Blanton and Kaput looked for ways in which students organized data, analyzed change, and communicated their understandings in the classroom. They found that students as early as kindergarten were capable of coordinating covarying quantities and could verbalize a correspondence between data sets as early as first grade. By third grade, students were capable of expressing these verbal rules using formal symbols. Blanton and Kaput asserted that the results of their study support the view that elementary students are capable of functional reasoning. However, they noted that the teachers in the study tended to emphasize change within singular data sets which is a less effective route to functional understanding. Blanton and Kaput (2004) argued that "…a fundamental conceptual shift…must occur in teachers' thinking in order to move from analyses of single variable data to those attending to two or more quantities simultaneously" (p. 141).

The results of Warren, Cooper, and Lamb's (2006) investigation into the functional thinking capacity of elementary students supported Blanton and Kaput's (2004) findings. They examined the efficacy of an instructional strategy which represented arithmetic as change through the incorporation of a function machine. Warren, Cooper, and Lamb videotaped the classroom experiences of 45 nine-year olds and their teachers. They also conducted pre- and post-tests on student ability to analyze change and determine outputs produced by a function machine. Prior to instruction, 46% of the students could describe the mechanism of change produced by the machine and 50% could determine outputs based on specified inputs. Following instructions, these percentages increased significantly to 58% and 90% respectively. Warren, Cooper, and Lamb (2006) noted that the ordering of inputs played an important role in developing functional reasoning skills. When the inputs were placed in numerical order, students tended to focus on the sequential change in outputs. However, by placing the inputs in random order, students were forced to look across the table for a relationship between the two data sets.

Although recursive thinking plays an important role in analyzing change and developing algebraic thinking (Bezuska & Kenney, 2008), many students tend to rely heavily on recursive reasoning at the expense of covariational thinking (Moss, Beatty, Barkin, & Shillilo (2008). In a teaching experiment involving 34 fourth graders, Moss, Beatty, Barkin, & Shillilo (2008) found that the majority of these students used recursive strategies in their attempts to generalize patterns. Besides identifying the over-reliance on recursive thinking when analyzing patterns, Moss, et al. also remarked that the students had a tendency to apply proportional reasoning, or whole object reasoning, to linear situations involving both a constant and a rate of change. A portion of their teaching experiment included the creation of an online learning environment between two fourth grade classrooms. Within this collaborative classroom, the students learned to communicate their understandings of functional relationships and develop strategies for developing a correspondence between data sets.

Pictorial growth patterns, or geometric patterns, were commonly used in studies based on the quantitative relationships inherent in functions. One such study, conducted by Billings, Tiedt, and Slater (2007), investigated how experiences with pictorial growth patterns encouraged the development of algebraic thinking in young children. They defined a pictorial growth pattern as "…a pattern made from a sequence of figures that change from one term to the next in a predictable way" (p. 303). This qualitative study analyzed the responses of eight $2^{nd}/3^{rd}$ grade students engaged in two task-based interviews. Billings, Tiedt, and Slater offered a summary of the key processes used to complete the patterning tasks based on these students' responses. Under the model, the students' analysis of change progressed from a covariational analysis to one of correspondence. The first stage of covariational analysis began with a description of the sequential change between figures followed by the application of this description in the construction of the next possible figure in the pattern. This stage culminated with the ability to "identify what stays the same and what changes in the pattern" (Billings, Tiedt, & Slater, 2007, p. 304). Building upon this knowledge, the students were able to progress to a correspondence analysis of change by connecting what changes in the pattern with the position of the figure in the pattern sequence. Ultimately, some of the students were able to take that connection and think of it in more general terms. These students were able to visualize the construction of any figure in the sequence based on its position (Billings, Tiedt, & Slater, 2007).

Billings, Tiedt, and Slater (2007) detailed how several aspects of pictorial growth patterns and the types of questions they posed to the students facilitated the transition from covariational analysis to correspondence. Encouraging students to attend to the structure of the figure by asking them questions about the next figure, i.e. what parts of the figure will stay the same and what parts will change, helps them to extend the pattern on to later figures in the sequence. In addition, Billings, Tiedt, and Slater stated the importance of going on to ask the students to build or describe the construction of future figures, such as the 25th one, to discourage the use of recursive strategies. They also remarked that attending to the physical structure of a pattern was a key predictor of success in the task of extending and generalizing growth patterns.

The connection between the structure of figures and success in generalizing patterns was also noted in a study conducted by Steele (2005). She analyzed the effects of a teaching experiment involving a group of 7th graders, focusing her analysis on the written work of eight of these students. As part of a cycle, students were first asked to work independently to solve a linear or quadratic patterning problem and record, in writing, the paths they took to generalize the patterns. Following this independent practice, the students shared their strategies in small groups. Steele combined the written reflections, transcripts of group conversations, and a series of individual interviews to assess the types of knowledge students used to solve these problems. Although the details of the knowledge framework she utilized in her study are beyond the scope of this discussion, portions of the design of her study and the implications stated are pertinent. In particular, the use of written reflections on problem solving does provide an entry for researchers to access students' understanding (Steele, 2005). Additionally, Steele's study illustrated how textual descriptions centered on the structural make-up of concrete representations may enable students to successfully generalize patterns.

These five studies explored the functional understandings of students in the early and middle grades (Billings, Tiedt, & Slater, 2007; Blanton & Kaput, 2004, Moss, Beatty, Barkin, & Shillilo, 2008; Steele, 2005; Warren, Cooper, & Lamb, 2006). Evidence that even students in the early grades were capable of functional thinking was presented (Blanton & Kaput, 2004; Warren, Cooper, & Lamb, 2006). The promotion of functional thinking was shown to be facilitated by the study of patterns. In particular, the examination of the structure of a pattern in terms of how each figure stays the same and how it changes from the previous figure enabled students to generalize patterns (Billings, Tiedt, & Slater, 2007; Steele, 2005). However, the tendency to think of change only in recursive terms impeded growth in the understanding of function as two covarying quantities (Blanton & Kaput, 2004; Moss, et al., 2008). There is a need for teachers to have experiences describing patterns of change between two data sets to create a shift away from recursive thinking (Blanton & Kaput, 2004).

*Studies with pre-service teachers.* Billings (2008) and Smith (2003) summarized the processes used by elementary teachers to generalize pictorial growth patterns. Both reports were based on informal observations of teachers' actions as they were engaged in patterning activities. Although not presented as research, these summaries did offer a few examples of how these teachers approached the process of generalizing patterns. One such approach utilized the physical structure as a tool for generalizing the functional relationships represented in pictorial growth patterns (Billings, 2008). The utilization of structure was also reported to be a valuable tool used by elementary and middle school students (Billings, Tiedt, & Slater (2007; Steele, 2005). Smith (2003) noted that the rules written by teachers still retained the actions used to physically construct the figures in a pattern. Additionally, the use of variables was mentioned as a way to keep track of how the structure of a figure changes in a predictable way (Billings, 2008). Smith (2003) emphasized how this focus on stasis (structure) and change are important tools for understanding functional relationships. Billings (2008) reiterated this, stating that the analysis of change is an important tool in generalizing patterns.

Neither of these summaries was presented as research and a search revealed few research-based studies on pre-service elementary teachers' understandings of the concept of function as a quantitative relationship. However, one noteworthy study, offered by Zazkis and Liljedahl (2002) explored the pathways taken by a group of pre-service elementary teachers in their attempts to generalize a complex number pattern. The pattern consisted of an array of numbers whose general term could be expressed using a piece-wise defined function. The researchers reported that the pre-service teachers experienced more success extending the pattern than locating the general term. Zazkis and Liljedahl attributed these difficulties in part to the use of recursive reasoning. They stated that the persistent use of recursive reasoning and additive thinking prohibited the pre-service teachers from seeing the overall structure of the problem. In addition, they noted that the pre-service teachers were able to write rules for patterns generated by repeated addition if the patterns lacked a constant component (i.e. $3n$), but not if a constant was also required. Zazkis and Liljedahl also identified a disconnect between the participants' ability to verbally describe the pattern and their ability to write a symbolic rule. However, the researchers considered the verbal descriptions to be the product of algebraic thinking. They argued against the push to use symbolism, stating that students "…should have the opportunity to engage in situations that promote such thinking without the restraints of formal symbolism" (p. 400). In conclusion, Zazkis and Liljedahl (2002) asserted that pre-service teachers should have opportunities to engage in the study of patterns so they may see there are other ways to communicate algebraically.

There were few studies on pre-service teachers' ability to analyze the quantitative relationships exhibited in pictorial growth patterns. The summaries presented here revealed that pre-service teachers experienced many of the same difficulties elementary and middle school students experienced when analyzing quantitative relationships. For example, the tendency to reason recursively was identified as a hindrance for both groups. Blanton and Kaput (2004) noted that teachers need to shift away from looking at change within one data set to the view of change in two data sets so they can support algebraic thinking in their own classrooms. Zazkis and Liljedahl (2002) proposed that the study of patterns provides the opportunity for pre-service teachers to communicate algebraically in ways other than through the use of formal symbols. The present study proposes to fill the gap in research on how pre-service teachers conceptualize and communicate the idea of function through patterning activities. A key part of this question is dependent on describing the experiences pre-service teachers have with the concept of function while analyzing patterns. In the following section of the paper, studies focusing on the experience of understanding function are presented.

*Phenomenological Perspective*

Researchers placing an emphasis on the phenomenology of understanding function consider how past and present experiences with the idea contribute to the development of an individual's concept image (Lobato & Bowers, 2000). Whereas the focus of analysis for research based on either a representational or quantitative perspective is the outcome of the experiment, analysis based on a phenomenological perspective is focused on the journey. In other works, the researcher is interested in the process of coming to know a concept (Lobato & Bowers, 2000). The phenomenological approach adds another layer to the understandings of a phenomenon by considering how individuals construct meaning within a social setting. Language, gestures, the individual's experiences, and social interactions are all elements that play a role in developing an understanding of a concept (Lobato & Bowers, 2000; Radford, Bardini, & Sabena, 2007).

Lobato & Bowers (2000) offered two examples of research they were pursuing from a phenomenological perspective. The information provided on both examples was geared more towards explaining the design of the studies in lieu of results since both studies were ongoing on that time. Bowers had combined a representational perspective with a phenomenological emphasis to describe how a group of 7$^{th}$ graders "come to develop understandings of linear functions as they interact with the software [SimCalc] in the social setting of a classroom" (Lobato & Bowers, 2000, p. 12). Note this research question is centered on the process of developing an understanding and not on the level of that understanding. Lobato also offered an example of research she was conducting with a small group of high school students. Her research was framed within both a quantitative and a phenomenological perspective in an attempt to understand how students' experiences with quantitative relationships contributed to their understanding of covariation. Lobato noted that many of the students held misconceptions of speed based on their personal experiences. For example, some were convinced that speed was a function of how fast your legs were moving based on their experiences walking alongside a taller person (Lobato & Bowers, 2000).

Radford, Bardini, and Sabena (2007) embedded their study on students' algebraic thinking within the framework of social semiotics. Social semiotics, or semiotic cultural theory, combines the works of Vygotsky and Leont'ev with the theoretical perspective of phenomenology. Within this theory, thinking is said to be made visible in the words, gestures, and other socio-cultural signs adopted by the individual as he or she attempts to make sense of a phenomenon (Radford, Bardini, & Sabena, 2007). In their investigation, Radford, Bardini, and Sabena sought to describe how individuals come to generalize patterns by diverting their attention away from what they referred to as the particular so as to imagine the general. They analyzed the verbal, kinesthetic, and symbolic artifacts produced by the participants while they were engaged in the process of pattern finding so to paint a picture of the "…anatomy of the genesis of students' generalizations" (p. 509).

Radford, Bardini, and Sabena (2007) videotaped the conversations and actions of two small groups of 9th graders as they worked on the task of generalizing a dot pattern. The first figure consisted of two rows of circles; two circles on the top and three on the bottom. Each consecutive figure added a circle to each row, so the second figure consisted of seven circles and the third given was made up of nine circles. Radford, Bardini, and Sabena detailed several processes through which the participants communicated their thinking. For example, one of the participants communicated his understanding through the use of gestures and body rhythms while another student primarily used verbal signs. Radford, Bardini, and Sabena explored the various ways their body actions and words enabled them to transcend the particular. For example, spatial terms or gestures referring to the top or bottom of the figure illustrated how the individual was now thinking in terms of the general structure of the pattern. Use of words related to time, such as ‘always’, was also indicative of proceeding to the general term in the sequence. Radford, Bardini, and Sabena stated that…

> …because the crux of the generalization of patterns lies in the fact that it predicates something that holds for *all* elements in a class based on the study of a *few* of them, the spatial and temporal nature of the particular has to be overcome in the ontogenetic construction of generalization. (p. 515)

A third, less obvious strategy used to divert attention away from the visual figure was made evident in what was not said. In the example given by Radford, Bardini, and Sabena, the individual avoided labeling the imagined by pausing whenever he referred to the general.

Radford, Bardini, and Sabena (2007) also gathered the written work produced by the students in the study. Analysis of this data combined with the multi-semiotic analysis of the videotaped data revealed the manner in which the written formulas produced by the students reflected their experiences with the activity. For example, one pair of students had noticed that you always add one more than the figure number to obtain the number of small circles on top, two more to obtain the quantity on the bottom row, and then add the two amounts together to find the total number of circles for any figure in the sequence. The parenthesis in the formula written, $(n + 1) + (n + 2)$, illustrated the path they took to generalize the pattern. The other group in the study developed the formula, $n \times 2 + 3$, which illustrated a fourth scheme for transcending the general. In this case, the students attended to the structure of the pattern, noting the parts of the figure that stayed the same and the parts that change. Radford, Bardini, and Sabena (2007) stated that "…in order to perceive the general, the students…have to bring to the fore some aspects of the figure (*emphasis)* and leave other aspects behind (*de-emphasis*)" (p. 522).

The studies described by Lobato and Bowers (2000) and by Radford, Bardini, and Sabena (2007) illustrate how a phenomenological perspective can be used to add another layer to our understanding of how students develop and communicate an understanding of function. Embedding a study based on a representational and/or a quantitative viewpoint within a phenomenological perspective is one way researchers can add to the body of knowledge of concept development (Lobato & Bowers, 2000). The purpose of this study was to investigate how pre-service teachers experience the concept of function through the study of patterns. In keeping with this purpose, the phenomenological perspective was combined with the quantitative perspective so as to provide a more complete picture of this phenomenon. The final summary section of this chapter presents a discussion of how the body of literature reviewed here was used to frame the study. In addition, a theoretical framework based on the hermeneutic circle will be proposed.

## Chapter Summary

The two concepts of function used for this study are based on the definitions of function as covariation and as a correspondence. Sierpinska (1992) stated that "…the notion of function can be regarded as a result of the human endeavor to come to terms with changes observed and experienced in the surrounding world" (p. 31). She emphasized that the basis for understanding functions lies in the ability to identify such changes and the relationships between them. In mathematics, this basis of change translates to the covariation definition of function wherein changes in one variable are associated with changes in another variable (Smith, 2003). The ability to view the idea of function as covariation is believed to be a crucial step in the development of algebraic reasoning (Smith, 2003). However, the view of function as correspondence is also believed to be a key stepping stone to understanding the concept of function (Slavit, 1997). With this view, one is readily able to see the relationship between two data sets.

Smith (2003) proposed using the study of patterns as a context for experiencing function as both covariation and correspondence. He discussed two approaches to analyzing pictorial patterns which can be used to build a framework for developing connections between patterns, functions, and algebra. The first of these approaches focuses on how the pattern is constructed, which he referred to as stasis. Examining how a pattern is constructed and noting the constant pattern between the position of the figure in the sequence and how it is constructed allows one to establish a relationship or correspondence. The dual focus on stasis and change within the context of patterns potentially leads to generalization, a key feature of algebraic thinking (Smith, 2003). If teachers are going to support algebraic thinking, they need to be able to identify and generalize change (Bezuska & Kenney, 2008). In response to this problem, the design of this study focused on how the ideas of stasis and change can be used to generalize patterns. The pre-service teachers in this study were invited to experience the idea of function by engaging in conversations about how they construct, extend, and ultimately generalize patterns.

The issues explored in this literature review; the interrelationship between communication and conceptualization in mathematics, the concept of function, and the supporting role patterns play in developing algebraic thinking form the framework for this study. This framework draws upon the idea of the hermeneutic circle in the learning of mathematics. A diagram of this framework is presented in Figure 1. The study of patterns and function is embedded within the matrix of algebraic thinking which is represented by the shaded rectangle. The study of patterns provides a context for developing the idea of function. Through these patterning activities, the participants have the opportunity to communicate and demonstrate their understanding of function. The circle of arrows represents the hermeneutic circle that evolves between the processes of explaining and understanding.

The idea of mathematics as hermeneutic understanding was explored in this chapter. The key characteristic of this idea lies in the view that mathematical understanding is communicated through language. Sierpinska (1992) stated that understanding in mathematics can be linked to Ricoeur's hermeneutics because of the "…relationship between the symbol side of the mathematical concept and the object side" (p. 30). For the purpose of this study, this is interpreted to mean an individual uses traditional words or symbols related to a concept as well as the image he or she has of that particular concept in order to communicate with others. This image is continually shaped and refined by the experiences the individual has with the concept or idea both in and outside of the mathematics classroom (Vinner, 1992). Brown (2001) had suggested that mathematics "…can be seen as a subject of hermeneutic understanding if the emphasis is placed on interpreting mathematical activity, which itself might embrace the generation of mathematical statements" (p. 50). Thus, determining how an individual conceptualizes a mathematical idea entails taking snapshots of this image in the statements produced during mathematical activity (Brown, 2001).

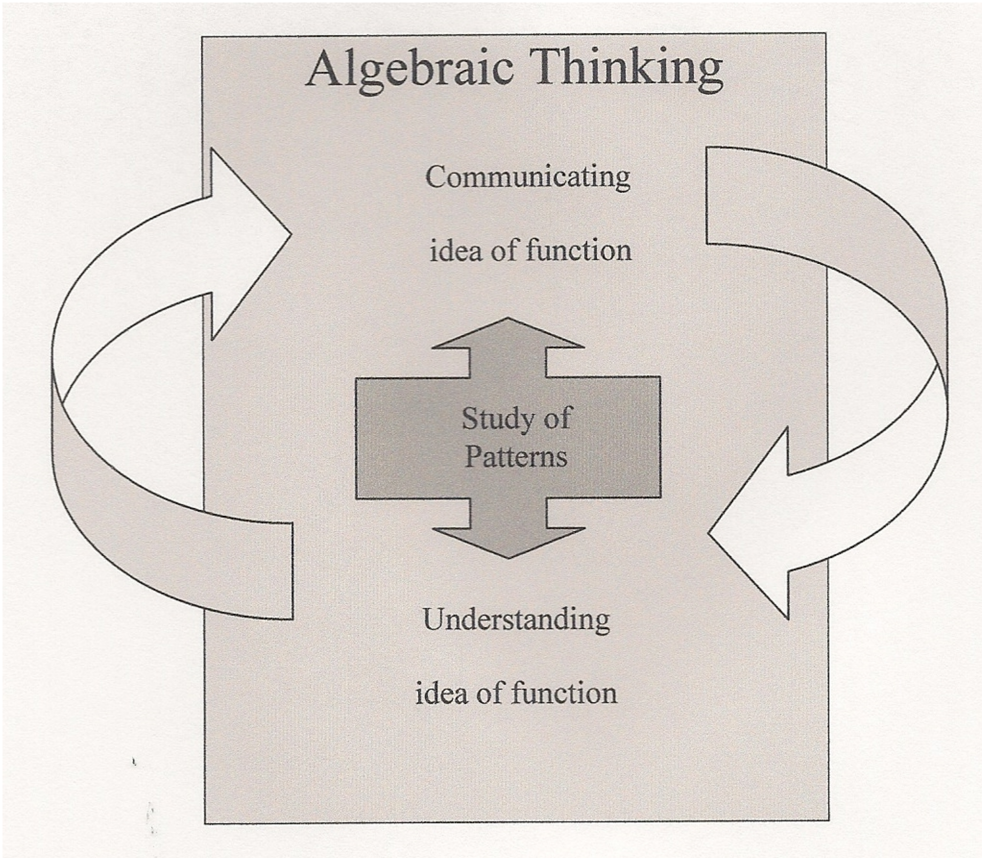



*Figure 1:* Diagram of framework for context of study based on a model used by Paterson & Higgs (2005) which was adapted from the works of Bontekoe (1996).

In Chapter III, the choice of methodology and methods that were used in this study will be explained. The chosen methods were intended to capture the discourse produced by participating pre-service teachers during pattern solving activities. Because the concept image of function, or any other mathematical idea, evolves over time, a series of snapshots were taken throughout the study. By collecting the snapshots across time, it was hoped that the research results would formulate a clearer description of how pre-service elementary teachers conceptualize the idea of function.

# CHAPTER III

## METHODOLOGY

The purpose of this investigation was to understand how pre-service elementary teachers communicate their understanding of function through the study of patterns. In particular, this study endeavored to describe the nature of functional reasoning among pre-service elementary teachers and answer the following questions:

1. How do pre-service elementary teachers conceptualize the idea of function?
2. How do pre-service elementary teachers engage in the process of generalizing patterns?
3. What is the nature of pre-service elementary teachers' understanding of pattern and function?

The choice of methodology and methods used in this investigation were guided by both the purpose of this study and the nature of these research questions. An in-depth discussion of the theoretical perspective that guided these choices is presented in the first section of this chapter. This discussion provides a rationale for choosing a qualitative approach in the investigation. This chapter also provides a detailed description of the research setting and participants, the overall research design, the methods of data collection and data analysis, ethical considerations, and issues of trustworthiness.

## Theoretical Perspective

Lobato and Bowers (2000) presented three views through which research on functional reasoning has been approached. The first of these views centers on the various forms used to describe functional relationships. Labeled as a "multi-representational perspective" (p. 1), this view examines how learners conceptualize functions when presented with different representational forms. We can consider the relationship $f(x) = 2x + 1$ to illustrate this perspective. Besides the symbolic form given, the relationship between the independent variable, $x$, and the dependent variable, $f(x)$, can be represented in a table that displays a finite set of ordered pairs. The function can also be presented as the graph of a line with slope equal to two and *y*-intercept of one. Studies based on this theoretical frame would examine a student's ability to move flexibly from one form to another (Lobato & Bowers, 2000). For example, Klanderman (1996) examined pre-service teachers' understanding of function under different representational situations. Klanderman asserted that opportunities to view functions under varying representations allows students to form the types of connections necessary for conceptual understanding to develop.

Lobato and Bowers (2000) criticized the multi-representational perspective for failing to assess student understanding of the quantitative relationship between the domain and range of a function. By taking a quantitative approach, researchers can focus on how students analyze change in functional relationships. The quantitative perspective, Thompson and Thompson (1995) argued, forms the foundation of algebraic reasoning. Failure "…to ground the development of algebraic thinking …on understandings of quantities and quantitative reasoning in dynamic situations, is like building a house on the second floor. The house will not stand" (Thompson & Thompson, 1995, p. 98). Conceptually, the quantitative perspective aligns with the Smith's notions of stasis and change. Billings (2008) adopted a quantitative perspective in her analysis of the ways in which pre-service elementary teachers generalize pictorial growth patterns. She posited that the analysis of change is an important tool in establishing a functional relationship.

The first two approaches described are based on widely-held beliefs about the nature of functions and have the tendency to ignore the meanings the individual student assigns to the concept of function (Lobato & Bowers, 2000). A phenomenological perspective, on the other hand, focuses on the student's personal understanding of function as it develops through his or her experiences with mathematics. The goal of this approach is not to classify or quantify an individual's understanding, but rather to describe the nature of this understanding as it develops (Lobato & Bowers, 2000; Van Manen, 1990). The phenomenological perspective, as Van Manen stated, "… is discovery oriented. It wants to find out what a certain phenomenon means and how it is experienced" (p. 29). Radford, Bardini, and Sabena (2007) adopted this perspective in their research on both verbal and nonverbal cues students utilize in the process of generalizing patterns. Their study illustrated how the use of a phenomenological perspective opens a space for the researcher to examine how the individual communicates an understanding of the concept of function.

As previously stated, the quantitative approach focuses on how quantities change in relation to one another, a key component of algebraic thinking. Recognizing and promoting algebraic thinking lies at the heart of the purpose for this study. However, this study was also interested in the nature of functional reasoning as it is experienced by pre-service elementary teachers. Therefore, the quantitative perspective of function has been combined with the phenomenological perspective to examine how pre-service teachers come to understand the concept of function. In the next section, a rationale for using a phenomenological approach for this study will be presented by examining how such an approach can be applied to the realm of mathematics education. An explanation of how this approach has been combined with hermeneutics to provide an avenue for interpreting the experiences of others through discourse analysis will also be stated.

*A Phenomenological Perspective*

Phenomenology has been defined as both a theoretical perspective and as a research methodology (Patton, 2002). Developed by Husserl in the early twentieth century, phenomenology focuses on how people describe their experiences (Crotty, 2003; Patton, 2002). As a general theoretical perspective, phenomenology guides the researcher to locate a study in the midst of those who have experienced the phenomenon under investigation (Patton, 2002). Patton (2002) pointed out the differences between this philosophical application of phenomenology and a methodological approach that would center on uncovering the common threads of shared experiences. Patton (2002) argued that "…one can employ a general phenomenological perspective to elucidate the importance of using methods that capture people's experiences of the world without conducting a phenomenological study that focuses on the essence of shared experience" (p. 107). This particular study is grounded within the phenomenological perspective without adhering to the specific approaches of phenomenology as a research methodology.

Brown (1996, 2001) asserted that Husserl's phenomenology provides a framework for characterizing how an individual conceptualizes mathematical ideas. Under this framework, emphasis shifts away from seeing mathematics as a concrete set of ideas. Instead, mathematics is viewed as a human activity that is shaped by the experiences of the participant (Brown, 2001; Cobb, 2007). Describing how an individual experiences a mathematical concept requires that the researcher attend to the perspectives the individual brings to the classroom and how these perspectives shape his or her understandings. Phenomenology focuses on these individual reflections and the experiential world of the participants in an attempt to describe the nature of the learning experience (Van Manen, 1990). The phenomenological perspective does not rely on "…an expert overview of mathematics…since it is not available to the learner" (Brown, 1996, p. 120). Instead, the phenomenological perspective gains its insight from the interpretations supplied by the learner as he or she engages in mathematical activity (Lobato & Bowers, 2000).

*Hermeneutic Phenomenology*

Focusing on how individuals interpret their experiences necessarily turns to a question of how language is used to communicate our understanding. Vygotsky (1934/1986) described language and other psychological tools of discourse as a medium in which individuals construct meaning through their social interactions with others. Under this construct, access to the meanings an individual assigns to mathematical concepts is gained through the discourses he or she participates in (Sfard, 2001). Hermeneutics opens a space for the interpretation of the texts produced in these social interactions (Crotty, 2003; Van Manen, 1990). Van Manen justified the combined approach of hermeneutics and phenomenology, stating that all accounts of an experience are captured in texts and thus the added interpretative flavor of hermeneutics complements the descriptive mode of phenomenology.

Hermeneutics was first applied to the interpretation of Biblical texts, but the notion has been broadened to include applications to other forms of discourse. These forms of discourse include written texts, verbal communications, and experiential situations on the basis that all understanding is created and shared through language (Crotty, 2003). According to Patton (2002), "…hermeneutics provides a theoretical framework for interpretive understanding, or meaning, with particular attention to context and original purpose" (p. 114). Within this framework, it is assumed that understanding the experiences of others requires interpretation of the texts they produce. The key to developing this interpretation lies in the idea of the hermeneutic circle (Brown, 2001; Crotty, 2003). Crotty described the hermeneutic circle as a recurring process of coming to understand something by employing what one already knows. These prior understandings fuel our interpretation of what we are reading or experiencing, thus leading to deeper understandings.

This study sought to understand the nature and communication of functional reasoning among pre-service elementary teachers while they were engaged in the experiences of pattern finding. The theoretical perspective employed in this investigation should also support the research process, including the choice of methods (Crotty, 2003). As stated previously in this section, the analysis of the texts produced by a group of pre-service elementary teachers as they engaged in mathematical activity was used to answer the research questions posed. Hermeneutic phenomenology offered a suitable lens through which to read and interpret these texts. The use of this lens brings with it the assumption of the relationship between explanation and understanding, referred to as the hermeneutic circle (Brown, 2001). A thorough discussion of the research design utilized in this study and how it adhered to these assumptions is presented in the following section.

## Research Design

### *Research Setting and Participants*

#### *The Setting*

Based on the assumptions of a phenomenological perspective, this study took place in the mathematics classroom of a group of pre-service elementary teachers and focused on the experiences these prospective teachers shared while they were engaged in mathematical activities. In qualitative inquiry, the context, or setting, in which a study takes place, plays a key role in the interpretation of the experiences of the participants (Patton, 2002). Whereas in quantitative studies, a researcher may wish to downplay the role of context by controlling the environment, a researcher using qualitative methods seeks to pull data from the everyday world of the researched (Patton, 2002; Van Manen, 1990). This study was interested in the experiences of pre-service elementary teachers as they were learning about functions. Therefore, this study took place in their mathematics classroom during the normal course of small group problem solving.

Agee (2002) described a setting as a bounded environment in which the inhabitants have co-created rules of behavior and membership requirements. Settings may be embedded within other settings much like Russian nesting dolls. This mathematics classroom was embedded within the larger setting of a satellite campus, which in turn is part of a Midwestern regional university. The nontraditional student is more the norm on this campus and many of the students hold full-time or part-time jobs while juggling the demands of family and school. The students in this classroom were either seeking certification in elementary education, special education, or early childhood education. Students in these three programs are required to take 12 hours of mathematics content courses, of which 9 hours are specialized courses for the elementary major. The curriculum used is designed to build conceptual understanding of elementary level mathematics through concrete models and meaningful problem-solving tasks. This combined approach is believed to build both mathematical and pedagogical knowledge.

*Teacher as Researcher*

My interest in how pre-service teachers communicate their mathematical understandings was sparked by the everyday experiences I have in the classroom as a mathematics teacher educator. One purpose of this study was to inform my own teaching practices by providing insight into how pre-service elementary teachers conceptualize and communicate the idea of function. This purpose arises from the call, as a teacher, "…to understand, through observation and inquiry, the various kinds of knowledge individuals construct as they engage with real phenomena" (Cochran-Smith & Lytle, 1999, p. 16). Therefore, the participants selected for this study were all students enrolled in one of the mathematics content courses I was teaching during the spring of 2009. Precautions were taken to help alleviate some of the issues that are inherent in the teacher-as-researcher relationship. These precautions are detailed in the following descriptions of the selection of the participants and the methods of data collection.

*The Participants*

The pre-service teachers who participated in this study were all enrolled in the researcher's section of Math 3433: Modeling Numeration and Operations during the spring of 2009. This section met once a week for approximately two hours and forty minutes. According to the catalog description, this course is a broad overview of numeration and operations along with a focus on problem solving, logic, relations and their properties, set and axiomatic concepts of numbers, whole number operations and properties. Course content is explored from a modeling perspective. This is one of three required mathematics courses beyond College Algebra (or its equivalent) for students in Early Childhood, Elementary, and Special Education programs. The courses may be taken in any particular order; therefore a number of the students were taking their first of the three required courses, whereas others were taking their last. The study took place near the end of the spring semester of 2009 and it may be assumed that a stable relationship existed between the students in the class and the teacher/researcher.

Forty-one students were enrolled in this course at the time of data collection. Prior to beginning the study, approval was gained from the two institutional review boards (IRB) associated with the researcher and the proposed participants. The purpose of the study was explained the week before beginning a unit on patterns. On the day of data collection, an invitation to participate in the investigation was extended to the 30 students in attendance. Each student was given an informed consent form which described the duration of the study, the type of data to be collected, how the data would be collected, recorded, and stored, as well as precautions taken to insure the confidentiality of each participant. They were informed that all data collection would occur during the context of the regularly scheduled classroom with the exception of follow-up interviews. All students were expected to participate in the patterning activities whether or not they signed the consent form, since these activities were part of the curriculum for this course. An attempt was to be made to audiotape all group conversations regardless of whether or not all members of the group gave their consent to participate in the study. No data record was to be retained on any individual choosing not to take part in the study. In keeping with regular course procedures, none of the data collected was to be used in the grading process with the exception of written reflections. As noted in the course syllabus, the students write a series of reflections during the semester which comprise 6% of their grade, but do not receive points for activities completed in class. An alternative prompt was offered to those students in the class who did not participate in the study. These reflections received a completion grade of 10 points each.

These detailed instructions were provided to communicate the fact that participating in the study was voluntary and would not play a role in evaluating course performance. Peer pressure to participate was eased by requiring all students to take part in the activities and to record their group conversations. The fact that these activities were part of normal class activities also satisfied IRB concerns regarding pressures to participate (Byrdon-Miller & Greenwood, 2006). Additional measures were taken to control the potential coercive nature of the teacher/researcher relationship with students in the class. The lack of grades associated with the activities was one way to ease the effects of coercion. Additionally, each prospective participant was given time to read the consent form and to place the consent form, whether signed of unsigned, in an envelope. As the researcher, I did not access these consent forms until time to select a subgroup of participants for the purpose of follow-up interviews. However, these interviews had to take place before final grades were due to both insure availability of interviewee and to safeguard against lack of recall. To address this issue, each participant in the study was given a second consent form to sign that indicated participation (or lack of participation) would not play a role in determining their final grade in this course. These promises were all made in good faith and were communicated to the participants both verbally and in writing. I must acknowledge, however, that the participants had to accept these promises out of trust.

All 30 students present on the day of data collection agreed to participate in the study. These students completed an anonymous survey of demographic information prior to beginning the patterning activities. From the results of this survey, it was determined that the mean age of participants was 28.37 years with a median age of 25. Only two of the thirty students were male. Twenty-four (80%) of the participants identified themselves as White; the remaining participants identifying themselves as either Hispanic ($n$ = 3; 10%) or Native American ($n$ = 3; 10%). The majority of the participants were Elementary Education majors ($n$ = 22; 73.3%). Six of the participants (20%) indicated that they were majoring in Early Childhood Education (Pre-K through Grade 3) and the remaining two (6.7%) stated that they were Special Education majors. A summary of the college-level mathematics completed by the participants is displayed in Table 3.

An attempt was made to group the students in pairs for the shared activities; however one student had to leave prior to participating in the shared activities. The odd number of students remaining for these activities necessitated the formation of a one group of three. One student arrived late from break after the groups were already formed.

Another student opted to work alone during this time, so the late student was allowed to join his table mates to form a second group of three. In all, there were eleven groups of two and two groups of three on the day of data collection.

Table 3

*Previous College-Level Mathematics*

| Course Name | Frequency | Percent |
|---|---|---|
| College Algebra | 22 | 73.3 |
| Math Structures | 8 | 26.7 |
| Trigonometry | 4 | 13.3 |
| Math 3414: Geometry & Measurement | 19 | 63.3 |
| Math 3443: Real Numbers & Statistics | 4 | 13.3 |
| Other | 3 | 10 |

*Primary Participants*

The individual responses on the first stage of data collection were used to identify potential primary participants since these questions were answered prior to any group discussions. Responses to these two problems involving linear patterns were analyzed based on whether or not the individual successfully extended the pattern to determine the value of the 15$^{th}$ term and whether or not the individual wrote a verbal or symbolic rule that could be used to find any term in the sequence. A four-point scale was used to categorize the level of generalization with a one awarded if the individual did not successfully extend the pattern or write a rule; a two if the individual extended the pattern but was did not write a rule; a three if the individual extended the pattern and wrote a verbal rule; and a four if the individual extended the pattern and wrote a symbolic rule. The precise details of this selection process are presented in chapter four.

The following week, all students in attendance who had participated in the prior week's patterning activities (*n* = 29) were given a consent form describing the purpose and nature of the follow-up interview process. Of the 23 out of the 29 students present, 15 indicated their availability and willingness to participate in an interview by signing a second informed consent form. Seven of these 15 students were extended an invitation via email based on the distribution of responses on the two patterning activities completed during the first 'think' session. The intent was to draw a sample that would represent the range of understandings within this group of pre-service elementary teachers. The advantages of selecting a heterogeneous sample include the possibility of not only uncovering the unique, but also unveiling commonalities that all participants shared (Patton, 2002). Although this type of purposeful selection should contribute to the credibility of this study, it will not lead to generalizable results due to the small sample size and the limited population from which it was drawn (Patton, 2002). All seven participated in the follow-up interview; however the data record of one of these individuals was incomplete. Therefore, the sample was limited to six primary participants. Descriptors of the six primary participants are displayed in Table 4. A more in-depth description of the primary participants is presented in chapter four.

Table 4

*Descriptors of Primary Participants*

| Pseudo-name | Age | Gender | Ethnicity | Major |
|---|---|---|---|---|
| Tara | 27 | F | White | EE |
| Cathy | 28 | F | White | EE |
| Jill | 25 | F | White | EE |
| Matt | 34 | M | White | EE |
| Ashley | 22 | F | White | EE |
| Shelly | 24 | F | White | EE |

*Note: EE – Elementary Education*

## *Data Collection*

Under the theoretical perspective of hermeneutic phenomenology, the source of data should come from the lived experiences of the participants. This involves attempts to either capture textual data while the phenomenon is occurring or to obtain descriptions of the event from the individuals that experienced it (Van Manen, 1990). This study sought to describe how pre-service elementary teachers conceptualize and communicate their understanding of function while experiencing the concept through pattern-finding activities. The data collected for this study came from three different sources: group conversations, written documents, and individual interviews. All three of these sources were used to create an ongoing account of the experiences these participants had while working with patterns in the mathematics classroom. This ongoing process of data collection was followed to take snapshots of the evolving mathematical understandings of the participants.

The primary source of data was obtained by audio-taping the group conversations of the students as they were engaged in pattern-finding activities. The original intent was to have all groups tape their conversations, but to only review those recordings in which all members had consented to participate in the study. On the day of data collection, all students present (30 out of 41) agreed to participate, therefore there was no reason to exclude any of the data collected. The written documents consisted of demographic information, individual work on two different pattern finding tasks, student work on group activities, as well as a reflection written immediately following the patterning activities. In addition, lecture notes, field notes, and my personal reflections as the teacher/researcher were included in this written data file. Selected students were interviewed to obtain their descriptions about the pattern-finding activities they participated in. These interviews were audio-taped and transcribed. However, Van Manen (1990) cautioned that "…all recollections of experiences, reflections on experiences, descriptions of experiences, taped interviews about experience, or transcribed conversations about experiences are already transformations of these experiences" (p. 54). By obtaining multiple data sources, I was able to look for both the consistencies and inconsistencies that arose in these accounts. The employment of data triangulation should contribute to the credibility of this qualitative study (Patton, 2002).

*Data Collection Format*

The basic format employed on the day of data collection resembled the 'think-pair-share' instructional strategy advocated by McTighe and Lyman (1988). In the typical 'think-pair-share' strategy, students are given time to ponder a question or problem posited by the teacher. After sufficient time is spent working independently on the problem, the students pair up and discuss their strategies. Closure is then achieved by having the pairs come together as a group and share their understandings with the entire class (McTighe & Lyman, 1988). This strategy was modified by replacing the group sharing session with a cooperative problem solving activity. The students first worked independently on two pattern finding tasks, then paired up to share their strategies. Immediately after sharing their strategies, the students worked together to solve four problems of a similar nature. Closure was achieved by asking the students to write a reflection on their experiences during the patterning activities. This reflective element was also utilized by Steele (2005) in her study on the use of writing to develop algebraic thinking. This four-stage strategy will be referred to as 'think-pair-together-reflect'.

*The 'think' session.* During the think stage, the participants examined two patterning problems involving arithmetic (linear) sequences. The first problem used geometric figures (small squares) to represent an arithmetic sequence and the second problem presented a numerical sequence in tabular form. A copy of the 'think' activity is included in Appendix A. In general, the participants were asked to determine the next figure or term in each pattern, find the fifteenth term, and then write a rule for the *nth* figure or term in the sequence. One purpose of the 'think' session was to produce a written text; therefore, the participants were asked to explain how they completed each of these tasks. The 'think' session was also designed to provide a snapshot of the prior understandings each individual brought to the classroom. This stage occurred before any class instruction or group activities on these particular types of patterning activities had taken place. As Steele (2005) asserted, this individual stage not only gives the researcher insight into how each student approached the problems, but also helps "…the students fix firmly their own thinking about the problems" (p. 145).

The participants were allowed approximately thirty minutes to complete the two problems. After completing the problems, the participants turned in their written work and were allowed a break before beginning the 'pair' session. During this break, a copy of each participant's work was made and the original work was returned to the participants for use during the 'pair' stage. This permitted the participants to record shared understandings on their papers while still preserving the original record.

*The 'pair' session.* Each pair or triad was responsible for recording their conversations while sharing the strategies employed during the previous 'think' activity. The participants were asked to explain their strategies and to make note of any differences in approaches and/or answers. In general, the participants followed the same line of questioning as printed on the assignment. This stage was designed to obtain a snapshot of their understandings which were embedded within the statements offered in their explanations.

*The 'together' session.* After sharing their individual work, the participants were given four similar pictorial growth patterns to analyze together. The 'together' session provided a unique opportunity to capture the participants' evolving understanding of the patterns in the explanations that they offered to one another. This snapshot was taken to preserve the process of learning, which, as Brown (2001) stated, "…continuously evolves, oscillating between understanding and explanation; that is, between an on-going learning process and statements generated within this process…" (p. 80). As in the 'pair' stage, each group was responsible for recording their own conversations. Copies of the problem sets used in this session are included in Appendix B.

*The 'reflect' session.* At the conclusion of the 'together' session, the participants were asked to respond in writing to a series of prompts. The prompts served two general purposes: 1) to gain insight into how the participant experienced the pattern-finding activities and 2) to determine what connections the participant might be making between this experience and their past experiences with patterns and functions. These reflections were written in class immediately after the conclusion of the 'together' session. A copy of the participant response sheet with prompts is included in Appendix C.

*Additional Sources of Data*

*Interviews with Primary Participants.* As described previously, individual interviews were conducted with seven purposively-selected participants after in-class data collection was completed. The interviews were structured around the think-pair-together-reflect activities completed in class. The participants were asked to interpret each problem and explain how they solved it. In addition, participants were asked to complete a similar problem that they had not worked previously. An interview guide was used to provide an element of consistency across all seven interviews. As Patton (2002) stated, the use of a guide grants the interviewer the freedom "…to build a conversation within a particular subject area, to word questions spontaneously, and to establish a conversational style…" (p. 343). A semi-structured approach meets the purpose of qualitative research by inviting the participants to use their own words to describe their experiences (Patton, 2002). A copy of the interview guide is included in Appendix D. These interviews were audio taped and transcribed to transfer the data into written form.

Written work completed by the participant during the interviews was also collected. Following each interview, a memo was written to record my observations, immediate recollections about the participant's reactions to the questions, and other reflective notes.

*Demographic survey.* A demographic survey was used to create a description of the participants in the study. Students were asked to provide information on their age, gender, ethnicity, major field of study, and prior mathematical background. Questions on mathematical background included the name of courses taken at both the high school and college level. The students completed these forms prior to beginning the patterning activities. They were asked to report this information anonymously. The demographic information obtained was tabulated and stored in an SPSS data file on a password protected, desktop computer. The primary participants in this study were asked the same demographic questions during the follow-up interview phase. This demographic information was used only to describe each participant as the use of this information to explain why a participant exhibited or failed to exhibit a certain characteristic of the phenomena under investigation would have been out of line with the theoretical perspective adopted for this study.

*Researcher's Memos.* During the course of this investigation, memos were written after each of the two data collection session and each interview. Copies of the instructional lesson plans and these memos were included in the collection of written documents. Brown (1996) remarked that "…in engaging in educational research, we are invariably engaged in the task of capturing the experiences of the research process in some tangible and collectable form" (p. 262). Thus, the reflexive activity of writing contributed to the formation of the researcher's circle of understanding.

*Lessons Learned on Data Collection*

The primary issue faced during data collection was directly related to time. My dual roles as teacher and researcher battled with the constraints time placed on both instructional and data collection activities. In addition, data collection took place near the end of the semester and time was at premium. Although the ‘together’ activities were already part of the curriculum, I did have to take time away from other instructional efforts to make room for the other three stages of data collection in the classroom. Saving time became a driving force during data collection, leading to several decisions that affected what data was collected. For example, students who were not in attendance on the first day of data collection were not invited to participate in the study. This decision was made because there was insufficient time to adequately explain the nature of the investigation. I also chose not to take the time to explain how to operate the recording devises used by the participants. We used a variety of digital recorders, several standard cassette recorders, and two mini-cassette recorders which made it impossible to provide instructions to everyone at the same time. This decision also led to missing data during the group conversations. One pair of students stopped their digital recorder every time there was a lapse in conversation. This practice left no means of capturing the length of periods of silence or time spent on a particular task. Another pair of students did not record their conversations during the ‘pair’ and ‘together’ sessions, possibly because they were unfamiliar with the tape recorder they were using. I also missed out on the chance to interview one of the primary participants selected due to the fact that this stage of the study took place during the close of the semester. These lost opportunities could have been avoided if more time had been available.

*Summary of Data Collection*

In keeping with the theoretical perspective of hermeneutic phenomenology, an attempt was made to identify sources of data that would yield textual accounts of the experiences of pre-service teachers engaging in pattern activities. The data obtained in this study was used to address the questions of how pre-service elementary teachers conceptualize and communicate the idea of function while completing these activities. Multiple data sources were identified to strengthen the design of this study through the process of triangulation.

The group conversations took place in the classroom while the participants were actually experiencing the phenomenon under investigation. The rationale for using this type of data is based on the assumption that the participants would communicate their ideas about patterns and functions while working on these activities. The intent was to identify and describe the relationship between explanation and understanding as manifested in the hermeneutic circle.

Mathematical understandings are not only communicated with spoken words, but also with the words and symbols used to our thinking on paper. The written sources described in this section provided access to these understandings, but did leave interpretation of the intent of the author up to the reader. The follow-up interviews offered the possibility of reconciling the differences between the intended meanings of the author and my interpretations. A summary of the sources of data collected is presented in Table 5. In the next section of this chapter, a description of how the data was analyzed is discussed.

Table 5

*Summary of Data Collection*

| Source | Description |
| --- | --- |
| Documents | Demographic Survey<br>Written work completed during 'think' session<br>Written work from during 'together' session<br>Written work completed during interviews<br>Student reflections<br>Lesson Plans<br>Researcher's memos |
| Transcribed Data | Audio-taped group conversations during 'pair' and 'together' sessions<br>Audio-taped interviews with individual participants |

## Data Analysis

Data analysis was guided by the purpose of the study and the theoretical perspective that informed the research process (Patton, 2002). This research endeavor employed the interpretive lens of hermeneutic phenomenology in an attempt to understand how pre-service elementary teachers conceptualize and communicate the idea of function. The use of this paradigm includes the assumption that access to understanding is gained by reading and interpreting the texts of those who have experienced the phenomenon under investigation (Van Manen, 1990). Hermeneutic analysis provides a systematic framework for interpreting these texts in light of the purpose of the study (Patterson & Williams, 2002).

*Hermeneutic Analysis*

According to Brown (2001), hermeneutics views texts as an avenue to understanding the perspectives of others. The researcher navigates this avenue by engaging in a circular dialogue with the text, referred to as the hermeneutic circle (Patterson & Williams, 2002). The hermeneutic circle is a metaphor for an analytical process that entails "…understanding the whole through grasping its parts, and comprehending the meaning of parts through divining the whole" (Crotty, 2003, p. 92). In research, the whole is constituted by the phenomenon under investigation with the individual data making up the parts. Analysis of the individual parts is made possible by the researcher's understanding of the phenomenon. These new-found interpretations are then integrated with the holistic understandings such that "…by circuitously viewing a phenomenon as a whole and as a sum of individual parts, the researcher gains knowledge to build increasing understanding of the experience" (vonZweek, Paterson, & Pentland, 2008, p. 119).

Analysis begins with the understandings the researcher brings to the phenomenon under investigation. These understandings may have come from prior experiences and/or a review of the literature pertaining to the phenomenon (Patton, 2002; Patterson & Williams, 2002). Patterson and Williams described these prior understandings as the "…scaffolding upon which knowledge is built" (p. 23). They referred to Heidegger's 'forestructures of understanding' as a metaphor to describe how knowledge is shaped by prior exposure to the phenomenon. Analysis thus begins by examining the parts or data with the illuminating lens of what the researcher already understands about the research problem (Patton, 2002). However, it is important to understand that this analysis does not take place at the conclusion of data collection as might occur in a quantitative study. Data analysis commences with the gathering of the first text and shapes future inquiries (Patterson & Williams, 2002; Patton, 2002).

The initial analysis focuses on the individual level since hermeneutic phenomenology is essentially "…a philosophy or *theory of the unique*" (Van Manen. 1990, p. 7). The researcher starts the analytical process, Patterson and Williams (2002) stated, by reading the texts produced by the individual to understand how this relates to the phenomenon under investigation. These understandings are then used to re-examine the text in closer detail as the next stage in the part-whole analysis. The circle of understanding at this individual level is used to identify themes and create a written interpretation of the text. Subsequent analyses of the texts produced by other individuals may produce similar themes or ones that had not been recognized before; therefore the researcher should reexamine previously interpreted texts as appropriate. Ultimately, cross-case analysis may be pursued if shared meanings are uncovered. In summary, Patterson and Williams (2002) stated that "…hermeneutics is an empirical enterprise characterized by critical and "meaningful" thought beginning with a particular perspective (the forestructure of understanding) progressing through a rigorous an systematic cyclical analysis (the hermeneutic circle) in which interpretations are evaluated and modified on the basis of the data that is then presented as evidence of the warrants for conclusions" (p. 36). In the next section, I will explain how I intend to follow this systematic process of analysis.

*Steps Taken in Data Analysis*

Patterson and Williams (2002) stated that the hermeneutic research process begins with the conceptual framework, or forestructures of understanding. This study sought to understand how pre-service elementary teachers communicate their understanding of function through the study of patterns. The researcher's prior experiences teaching pre-service elementary teachers and the literature review conducted for this study led to the following assumptions:

1. Understanding in mathematics is a hermeneutic process.
2. We communicate our understandings through language; therefore, access to these understandings is gained by interpreting texts produced by the individual.
3. The dual characteristics of patterns in mathematics, stasis and change, support the development of algebraic thinking and functional reasoning.
4. The concept of function may be viewed as a correspondence between two data sets, e.g. by looking across the table of values or as a covariational relationship between two data sets by looking down both sides of the table simultaneously.

These assumptions formed the whole of what was understood about how to interpret the conceptualization and communication of the idea of function in pre-service elementary teachers.

The texts collected for this investigation were analyzed in three stages. The first stage was conducted to identify a pool of primary participants. This was the briefest of the three stages and took place before the follow-up interviews were conducted. The second stage of analysis focused on the experiences of the six primary participants. This in-depth analysis was cyclical in nature and, subsequently, was the most time-consuming.

The final stage involved re-integrating the parts with the whole through cross-case analysis. This analysis made it possible to clarify the commonalities and differences in how these six participants made sense of the ideas of pattern and function. Each of the three stages will be described in the following section of this report.

*Preliminary analysis.* The written texts gathered from the 'think' session were analyzed for the purpose of selecting the primary participants in this investigation. This preliminary analysis took place during the week after collecting the data. The participants had completed three tasks during the 'think' session, two tasks involving patterns, and one follow-up question concerning the idea of function. Both patterning tasks involved finding the next term in a sequence as well as the $15^{th}$ and *nth* terms. Participants were asked to explain how they completed each of these three steps in the patterning tasks. The third 'think' task asked the participant to explain the meaning of function as represented by these two patterning tasks.

A scoring system was used to describe the level of completion on each of the two patterning task. A response was scored a four if the participant successfully completed each step and wrote a symbolic rule to describe the *nth* term. The same level of completion was awarded a three if the rule was described in verbal terms instead of symbolic. If only the first two steps in the task were completed correctly, then a score of two was applied. A score of one was given if only the first task was completed successfully. The scoring system was applied to both patterning tasks resulting in two scores per individual.

The third question posed asked the participants to explain the idea of function represented in the two patterning problems they had completed. The responses to this question were copied and taped onto index cards to make it possible to uncover the commonalities and differences between these explanations. Each individual response was coded and then these codes were sorted into several broader categories. This process made it possible to group the responses under several categories or themes associated with the idea of function.

The selection of the primary participants was then based on the results of this initial stage of analysis. The next phase of analysis focused on the texts produced by these primary participants during all four stages of data collection ('think-pair-together-reflect) as well as data harvested from the interview process.

*Analysis at the individual level.* The second stage of analysis considered only the experiences of the six primary participants while they were engaged in the pattern-finding activities. The data analyzed for this stage included the individual written work completed during the 'think-pair-together-reflect' sessions and transcriptions from the 'pair-together' and individual interview sessions. The analysis of data followed the same order as the patterning activities, in that texts associated with the 'think' session were analyzed before those texts revolving around the patterning tasks completed during the 'together' session. In addition, the analysis took place at the individual level and followed a somewhat circuitous route by considering each participant's responses associated with one patterning task before considering the next task. This section describes how that route was followed.

Under the framework of hermeneutic analysis, transcribing the data initiates the conversation the researcher will be having with the text (Patterson & Williams, 2002). Effort was taken to transcribe the data verbatim and to identify each speaker by a pseudo-name. In addition, each sentence in the transcription was numbered to create an indexing system, as recommended by Patterson and Williams (2002). Transcribed data was indexed by line number, the identifying code for the primary participant, and by source. For example, a line taken from the paired conversations might be labeled *line 57_7TG* to indicate that this was the 57$^{th}$ line on the transcript of the conversations between participant number seven and his or her partner. In contrast, an item from the interview might be labeled *line 103_7I* to indicate that this line came from the transcript of the interview with participant number seven. A similar indexing system was employed to incorporate the written data sources into the individual data records of the participants. These records were identified first by session, participant number, and then by source. For example, a written response by participant number seven from think pattern one, question 1(b), was labeled *T1b_7W*. Written responses to the two reflection prompts were labeled in the same manner.

The records associated with the first patterning task in the 'think' session were analyzed first. Each individual data record associated with this task, once assembled and indexed, was read and re-read to identify specific sequences that were connected to a particular aspect of the research problem. For example, a segment of the interview may have been highlighted because it provided insight into how the interviewee explained the construction of the next geometric figure in a problem. Each segment was then coded, based on how the text described one or more aspects of functional reasoning, patterning, and/or algebraic thinking. Codes generally pertained to how the individual related the structure of the geometric figure to the general term or how he or she used everyday language to explain the relationship between the position of each term and its shape or value. This process of identifying and coding segments was applied to the records associated with the first think pattern at the individual level before proceeding to the next step.

After coding each segment for one individual, the next step was to create an interpretive memo. Reflective writing is an integral part of the hermeneutic process and enables the researcher to connect the parts back to the whole (Brown, 2001; Van Manen, 1990). This piece of writing described each aspect that had been identified through the careful reading of the individual's texts. Excerpts from the transcript file and/or examples of written work were incorporated into the memo and presented as evidence for others to examine. This process was repeated with the texts produced by each of the primary participants, resulting in six interpretive memos associated with the analysis of responses to think pattern one. These six memos were then reviewed and written as a cohesive description of the experiences this group of pre-service teachers had while completing the first patterning task.

This systematic process was repeated with the texts associated with the remaining pattern-finding activities. Each cycle produced a series of interpretive memos that were then reviewed and rewritten as understanding of the research phenomenon continued to evolve in the process of explanation. At this point in the analysis, organizing the various codes and looking for interrelationships between them became a crucial step in the interpretive process (Patterson & Williams, 2002). This third stage of analysis, began with an attempt to sort these codes under the initial framework derived from the participants' descriptions of the idea of function. This attempt was made to ascertain how this group of pre-service elementary teachers would communicate their understanding of these four ideas. The possibility of additional themes was also considered during this sorting process.

*Integrating parts with whole.* Hermeneutic analysis requires the researcher to take a circuitous path while interpreting the data. Deciding when and how to break free of this circular path and come to a conclusion can be a difficult task for the researcher. The final stage of analysis was taken with the idea of illuminating the findings obtained in this investigation so as to exit the hermeneutic circle. A cross-case analysis was conducted after identifying the major themes associated with how this group of pre-service elementary teachers understood the ideas of pattern and function. Each interpretive memo was re-analyzed in light of these four themes and the associated sub-themes, resulting in a third account of how pre-service elementary teachers engage in the process of generalizing patterns. This analytical procedure added another layer to the interpretation and led to a more holistic picture of how this group of pre-service elementary teachers conceptualized the idea of function.

## Issues of Trustworthiness

Steps were taken during this investigation to address the issues of trustworthiness. Lincoln and Guba (1985) identified the following concerns with respect to the trustworthiness of a study: credibility, transferability, dependability, and confirmability. This section explains how these four issues were addressed during this investigation.

### *Credibility*

According to Lincoln and Guba (1985), the issue of credibility can be addressed, in part, by employing procedures that will help convince others that the results are believable. Prolonged engagement is one such procedure that adds credibility to the findings of qualitative inquiry. Lincoln and Guba (1985) defined prolonged engagement as "…the investment of sufficient time to achieve certain purposes: learning the "culture," testing for misinformation introduced by distortions either of the self or of the respondents, and building trust" (p. 301). Although the actual data collection took place within a single class period, the participants and the researcher had been in close contact with each other as students and instructor. The researcher also had spent the previous four years teaching mathematics courses designed specifically for pre-service elementary teachers. This experience provided an opportunity to develop an understanding of the culture of the pre-service elementary teachers' mathematics classroom.

The triangulation of data sources employed should also increase the credibility of the findings in this investigation. Triangulation was achieved by gathering multiple sources of data related to the same experience. The sources included written work on patterning activities, paired conversations related to each task, and follow-up interviews. Each source provided a way to clarify a participant's response.

The interpretations of the researcher were also exposed to critique through peer debriefing. Two mathematics educators volunteered to review the interpretations of the data and offer their opinions. The debriefing sessions provided the opportunity to bounce ideas around and consider alternative interpretations. Our discussions on the emerging findings also helped to clarify my understanding of how these pre-service elementary teachers demonstrated their understanding of pattern and function.

*Transferability*

Transferability refers to the extent to which the findings in this study are applicable to other settings (Lincoln & Guba, 1985; Patton, 2002). The issue of transferability was addressed by providing a thick description of the setting, the patterning tasks, the primary participants, and their experiences. This thick description incorporated the words of the participants to communicate how they went about the process of generalizing patterns. These detailed accounts should permit the reader to judge whether these findings are, indeed, transferable.

*Dependability*

The dependability of qualitative research is analogous to the issue of reliability in quantitative inquiry (Lincoln & Guba, 1985). The dependability of this investigation is supported, in part, by the detailed explanations offered concerning the steps taken in this endeavor. This chapter presented a thorough discussion of the research design as well as an explanation of the steps taken in data analysis. By doing so, others have the opportunity to employ these procedures in a similar setting to investigate pre-service elementary teachers' understanding of pattern and function. However, as is the case with qualitative inquiry, even if the same participants took part in this process again, their experiences would likely not be the same (Patton, 2002).

*Confirmability*

Confirmability refers to the objectivity of the data presented (Lincoln & Guba, 1985). Assuring confirmability can be achieved, in part, through the detailed explanation of the methodology and through the practice of reflexivity (Patton, 2002). The details of the methodology adhered to during this investigation have been presented in this chapter. The choice of methodology was guided by both the purpose of this study and by the theoretical perspective of hermeneutic phenomenology, as Crotty (2003) suggested. These two considerations necessitated the mining of data from the experiences of the participants, a task that has been explained in this chapter. Multiple sources of data pertaining to the same experiences were gathered to assist in the interpretation and understanding of the texts produced by the participants during their pattern finding experiences.

An explanation of the systematic process of data analysis was also detailed in this chapter. Threats to confirmability may arise during this stage if the researcher permits bias to enter into the process unchecked (Patton, 2002). Patton remarked that researchers can deal with the issue of bias "…through conscious and committed reflexivity – entering the *hermeneutical circle of* interpretation and therein reflecting on and analyzing how their perspective interacts with the perspectives they encounter" (p. 570). A key part of the analytical process employed during this study entailed the writing of interpretive memos at various stages of analysis. The writing of these memos provided an opportunity to reflect on the emerging understandings associated with the research problem and to explore how these new understandings conflicted or affirmed what was already understood.

## Summary

The results of this study described aspects of algebraic thinking associated with pattern recognition and functional reasoning that were present in the discourses of the pre-service elementary teachers participating in this study. These results were used to describe the nature of functional reasoning among pre-service elementary teachers and to answer the following questions:

1. How do pre-service elementary teachers communicate the idea of function while engaged in the study of patterns?

2. How do pre-service elementary teachers demonstrate their understanding of function while engaged in the study of patterns?
3. What is the nature of pre-service elementary teachers' understanding of pattern and function?

The interpretive descriptions of these results are presented in the next two chapters of this dissertation. These descriptions were subjected to critique and further analysis in light of the literature, thus providing another opportunity to relate parts to whole. A discussion of the relationship of these findings to the literature is provided in the final chapter. Although the experiences described were unique to this group of individuals, the descriptions and interpretations created through this research endeavor provides insight into the broader problem of how to support the development of algebraic thinking through the study of patterns.

# CHAPTER IV

## ANALYSIS OF DATA

The purpose of this study was to learn more about how pre-service elementary teachers communicate their understanding of patterns and function while engaged in the process of generalizing patterns. This purpose necessitated the gathering of texts produced by a group of pre-service elementary teachers while they were actually investigating linear patterns. The subsequent analysis of these texts was guided by the research questions in this investigation:

1. How do pre-service elementary teachers communicate the idea of function while engaged in the study of patterns?
2. How do pre-service elementary teachers demonstrate their understanding of function while engaged in the study of patterns?
3. What is the nature of pre-service elementary teachers' understanding of pattern and function?

The texts collected in this investigation were analyzed in three stages. The initial stage involved a preliminary analysis of the written texts associated with the 'think' session to identify a pool of potential primary participants. The primary participants selected from this pool took part in a follow-up interview; which represented the final phase of data collection. The second stage of analysis focused on the experiences the primary participants had while engaged in these pattern-finding activities. The final stage returned to the data set as a whole, including the body of literature, to create a synthesis of the results of this investigation. This chapter begins with the presentation of the findings from the first stage of analysis, followed by an introduction to the six primary participants in this investigation. This chapter will conclude with a presentation of the results of the second stage of analysis by presenting a summary of the pattern-finding experiences of the primary participants.

## Results from First Stage of Analysis

A preliminary examination of the written texts produced during the ‘think’ session was completed before the interview stage of data collection to identify the primary participants. The selection of the primary participants was based on the range of possible understandings evidenced during the ‘think’ session. This session took place before any group discussions (either paired or whole class), therefore the texts produced most likely represented the prior understandings these participants brought to the classroom.

The participants ($n$ = 29) were asked to complete two patterning activities involving arithmetic (linear) sequences. After completing the two patterning activities, the participants were asked to describe the idea of function as represented by the problems they had just encountered. The first patterning activity was presented as a sequence of geometric figures formed by joining small squares together. Participants were asked to draw the fourth figure in the model and then describe how the model changes or grows. They were also asked to describe the $15^{th}$ figure without drawing it and then to write a rule for describing the *nth* figure in the sequence. The second pattern was presented in a t-table with sequential values of *n* provided up to the $4^{th}$ term. The participants were asked to complete the table by finding the next term ($5^{th}$), as well as the $15^{th}$ and *nth* terms in the sequence. In both activities, the participants were asked to explain how they determined these values and rules. A copy of the form completed by the participants during the 'think' session is available in Appendix A.

A scoring system was developed to describe the four possible levels of responses. The scoring system was based on the final outcome of each problem. For example, a response on the first patterning problem was awarded a two if a correct description of how many squares it would take to draw the fifteenth figure was provided but no correct rule (verbal or symbolic) was stated. A summary of the scoring system is displayed in Table 6.

Table 6

*Scoring System for Linear 'Think' Problems*

| 1 | 2 | 3 | 4 |
|---|---|---|---|
| Pattern was not extended or was extended incorrectly with incorrect or missing explanation. No rule or an incorrect rule provided. | Pattern was extended correctly or, if incorrect, the explanation was correct. No rule or an incorrect rule provided. | Pattern was extended correctly, or if incorrect, the explanation was correct. Supplied a correct rule using words instead of symbols. | Pattern was extended correctly, or if incorrect, the explanation was correct. Supplied a correct rule using symbols. |

*Note. There was no case in which a participant wrote a correct rule, but failed to extend the pattern.*

*The First 'Think' Pattern*

The first of the two patterning activities required the analysis of a pictorial growth pattern. The first figure in the pattern resembled a symmetric cross with one center square and one square adjoined to each of the four sides of the center square. The figure grew by adding one square to each arm of the cross for a total of four squares. The participants were asked to draw the $4^{th}$ figure in the pattern. All 29 were successful with this task. If students were able to correctly identify the number of squares needed to construct the $15^{th}$ figure (61) or verbally describe this quantity, then they were classified as 'able to extend the pattern.' All but three of the 29 participants were successful at extending the pattern in this manner. Errors in arithmetic or failing to include the center square in the description were considered to be 'successes' instead of failures since the description of 'how you know' made it clear that these errors were unrelated to ability to extend the pattern. For example, one participant wrote 15 x 4 = 80 and then offered 81 as the number of squares. This is obviously an arithmetic error and does not indicate inability to extend the pattern. Another student wrote 60 squares, explaining that "$1^{st}$ figure had 1 square each leg, $4^{th}$ figure 4, $15^{th}$ would have 15 each leg…15 x 4 = 60."

After extending the pattern, the participants were asked to write a rule for determining the *nth* term of the sequence. This question was somewhat open-ended in that it did not specify the format for a rule. Sixteen of the 29 participants wrote a rule that could be used to correctly determine the number of squares needed to construct any figure in the pattern. Of these 16 rules, three were written in verbal form and the remaining thirteen were written in symbolic form. Ten of the participants who were successful at extending the pattern either did not write a rule or gave an incorrect rule. The remaining three who did not write a correct rule had not been successful extending the pattern to find the $15^{th}$ term.

*The Second 'Think' Pattern*

The second 'think' pattern presented the first four terms of an arithmetic (linear) sequence in a t-table. The participants were required to complete the table by determining the $5^{th}$, $15^{th}$, and *nth* terms. All but one participant correctly identified the value of the $5^{th}$ term as 12. This participant did not complete any portion of the table nor did she answer any of the questions associated with the table. Five of the participants (including the one describe preciously) did not extend the table to find the $15^{th}$ term or offered an incorrect response with an incorrect explanation. The remaining 24 offered either a correct response or an incorrect response with a correct explanation. Fifteen participants completed the table by writing a rule that could be used to determine the value of the *nth* term. Two wrote a verbal rule and the remaining 13 wrote symbolic rules. A summary of these results are compared with the results from the first pattern problem in Table 7.

Table 7

*Cross-Tabulation of End-Result*

| | | Think 2 | | | | |
|---|---|---|---|---|---|---|
| | Score | Did not extend | Extended Pattern | Verbal Rule | Symbolic Rule | Total |
| Think 1 | Did not extend | 0 | 2 | 1 | 0 | 3 |
| | Extended Pattern | 4 | 5 | 0 | 1 | 10 |
| | Verbal Rule | 0 | 2 | 0 | 1 | 3 |
| | Symbolic Rule | 1 | 0 | 1 | 11 | 13 |
| | Total | 5 | 9 | 2 | 13 | 29 |

*Note. The first linear problem appears on the left, labeled as Think 1. The second linear problem appears along the top of the table, labeled as Think 2.*

*The Idea of Function*

After completing the two patterning activities, the participants were asked to write a response to the following prompt:

> Each of these problems represented functional relationships. How would you explain what a function is to someone who may have not have had experiences with these types of problems?

Eleven of the 29 participants either did not respond to this prompt or simply wrote "not sure" or "don't know" in the space provided. The eighteen valid responses were copied and taped onto index cards before being read/re-read to obtain a feel for how the idea of function was represented in each written response. The responses were sorted into four categories after several different attempts.

*Function as a pattern.* Seven of the respondents described a function as a pattern. This response was not unexpected since they had just completed the two patterning activities when this question was posed. The responses classified as 'function as a pattern' are listed below.

> "Functions are patterns that you have to determine to finish sequential problems."
>
> "It's a pattern you must find then put into an equation so you could find any number to the pattern."
>
> "A function is a pattern that is used to solve the rest of the problems, where the problems are related to each other somehow."
>
> "I would tell them that you have to find the pattern in the problem so you know how to find the sum."

“In these problems the relationship is a pattern made up of mathematical functions repeated over with each change of the variable.”

“A function would be a pattern…” (*Comment*: First part of a particular response)

*Function as a rule.* Responses in which the author made direct reference to a function as a rule or as an equation were grouped under the category of ‘Function as a rule’. Three of the eighteen responses were placed in this category.

“A function is a formula that represents a pattern.”

“A function is a basic rule that show us how things work. It helps to understand how problems are solved. It also makes it easier to calculate answers to one problem using a variety of different numbers.”

“A function is an algebraic equation that you sometimes have to figure out. You put a number into an equation and get an answer.”

*Function as a relation.* Responses in which the author referred to a function as a relationship between two data sets were grouped under the category of ‘Function as a relation’. Four of the eighteen responses were placed in this category.

“I would explain that there is an input and output number and somehow they are related.” “I’d ask them to take a look at the numbers to see if they notice any patterns, if they do what it is. Functions are fun and hard to explain any other way than by modeling.”

“I think the best way to show this would be a physical representation, such as using apples and oranges.”

“I would lay out blocks to demonstrate the concept and explain after they have looked over examples. Then reinforce the relationship.”

> “…when something goes with another.” (*Comment*: Last part of a particular response)

*Function as a process*. The final category, ‘function as a process’, included those responses that referred to a function as how the relationship or pattern was created instead of referring to the idea as the thing (relationship, rule, or pattern) itself. Six of the responses were grouped under this category.

> “A function is the course numbers take when put in relations with other numbers.”
>
> “A function is a way of figuring out a pattern for a set of values. To find the equation in the pattern, examine the first few figures or numbers in the set and evaluate similarities and differences to arrive at an equation for continuing the pattern.”
>
> “You can look at the problem and realize what these numbers have in common (how each number advances).”
>
> “Functions are steps that you use to get answers to different steps in a problem. You have to be able to answer each step in order to get a final answer.”
>
> “I would explain that you have to experiment with numbers to see how they relate to each other.”
>
> “A functional relationship is where you apply the same action and get a similar result. Such as (n x 2) plus 2.

## The Primary Participants

Selection of primary participants was based on the reported results to the three questions associated with the ‘think’ session. The results were not analyzed in light of the research questions at this time. Instead, the individual records of the six primary participants were examined to address the research questions of how pre-service teachers conceptualize and communicate functional relationships as represented in arithmetic sequences.

*Descriptions of the Primary Participants*

*Tara.* Tara was 27 years old at the time of this study and attending college on a full-time basis. She was a junior, majoring in elementary education. Tara had completed one of the three required courses in mathematics for elementary teachers during the fall semester (Modeling: Geometry and Measurement) and had received credit for college algebra while still in high school. During our follow-up interview, I asked Tara if she had ever had experiences working with pictorial growth patterns and she replied, "Not that I can remember…maybe a long time ago…back in high school or 7$^{th}$ grade or something." Assumedly, the experiences she brought to the classroom were not specific to these types of problems although they may have supported the processes she used to generalize the patterns. On the day of data collection, Tara was paired with two other female students in the class. This triad of students had worked together as a team for most of the semester.

*Cathy.* Cathy also wrote symbolic rules for both patterning problems. As a 28-year old elementary education major, Cathy had an unusually strong background in mathematics. She had originally planned to major in accounting and had completed a course in business calculus. She also had taken two semesters of regular calculus because she 'just really liked math" as well as one of the three required mathematics courses for elementary majors. Cathy related her past experiences with patterning activities during our follow-up interview, stating, "Well, I definitely did some in high school…but other than that it's been ten years." She did not recall having worked with physical models or tables, stating that "they were more sequences." Cathy's past experiences in the mathematics classroom assumedly provided exposure to both the ideas of patterns and function, as well as the related concept of rates of change. Like Tara, the opportunity to work with pictorial growth patterns was a new experience for Cathy. During the shared activities, Cathy teamed up with the two other students that she was used to working with.

*Ashley*. Both Tara and Cathy wrote a symbolic rule for each of the patterns presented in the 'think' session. Ashley, another female student, had also written symbolic rules for these two patterns. At 22, Ashley was one of the younger students in the class and had recently transferred to the university from out of state. She had taken three mathematics courses for elementary majors at her former university, but had been required to also take the three analogous courses at this institution. One of the three courses she had already taken was very similar in nature to the present course (Modeling: Numbers and Operations) and had included the study of patterns. These past experiences benefited Ashley during the patterning activities completed in class, for as Ashley stated during the interview, "that's also why some of this stuff came easy to me….I already had been through some of the struggles." Ashley also explained the gains she experienced from re-taking the course.

> Sometimes when it comes to math like I can't really talk it out…I have to think it out so when it comes to when I have to explain it to someone…it's hard…and so then that's something that I did learn this semester with my group is that how do I tell them without just giving them the answer…like how do I teach them…to think of some examples to use…so that was a tough…but good experience.

*Matt.* Ashley was partnered with Matt, another primary participant in the study. One of only two males enrolled in the course, Matt was 34 years old at the time of the study and majoring in early childhood education. When asked if he had ever worked with pictorial patterns before, Matt replied, “no…probably just Tetris.” Matt also professed to having had difficulties with algebra, stating, “…yeah when I took algebra in the ninth grade…I didn’t do so well in algebra.” He reported having taken both Algebra 1 and Geometry in high school, but substituted a mathematics course in critical thinking for College Algebra. Matt successfully wrote a verbal rule to describe the *nth* figure in the first pattern, but failed to write a rule for the second pattern of the ‘think’ session. Matt confessed that he struggled with writing symbolic rules because he “…can’t do the ‘a + b’ type thing.”

*Jill.* Jill, a 25-year old elementary major, also wrote a verbal rule to describe the *nth* figure in the first ‘think’ pattern; however the rule she wrote was incomplete. When asked if she had analyzed pictorial growth patterns in her prior mathematics courses, Jill responded by saying, “I remember seeing things like this before but it hasn’t been in my college experience…not since I’ve had…you know like public school math classes have I seen anything like this…so it…so I was a little rusty.” When asked if she preferred having the opportunity to explore patterns with shapes, Jill explained why she preferred to work with the pictorial patterns.

> Yes…I definitely prefer having the shapes there…because it helps…I think I’m just very much a visual learner and when I can see how the pattern is growing it does so much more for me than just seeing a set of numbers…sometimes it you

know something really simple will you know…I'll fail to catch it…you know when it's just numbers…so I definitely prefer the shapes.

Although Jill had completed three years of high school mathematics (Algebra 1, Geometry, and Algebra 2) she admitted that she "never really understood Algebra 2." After having taken a course in intermediate algebra, Jill explained, "…I don't think I was really ready for college algebra…but the college algebra teacher just really went above and beyond…and he was really good…and I used to say I didn't like math but I found that I am really kind of good at it." Jill and her partner worked together on a regular basis in this mathematics classroom.

*Shelly.* The sixth primary participant, Shelly, was a 24-year old elementary education major. Shelly was teamed with another female student for the 'pair – together' stages of data collection. About her partner, Shelly stated, "I've never seen her before…ever…ever in our class…so…um…I can't…I can't even think of her name." However, Shelly stated during our interview that the lack of familiarity did not have a negative effect on the learning experience. She remarked, "…for the most part …the majority we were able to figure out…and then if I didn't understand something she would understand it or vice versa."

At the time of the study, Shelly was taking the second of the three required courses in mathematics for elementary teachers. She admitted to having only taken the minimum number of required mathematics courses in high school and had to take two remediation courses in college before taking College Algebra. Shelly explained the reasons behind the need for remediation during the follow-up interview.

339. S: Well in high school I uh didn't really take a whole lot of math…I took what was required basically

340. R: okay

341. S: and I took…um…I lost a lot of it because I um…I'm bad at math…I'm awful at math…as you can probably tell.

342. R: …and that's why you made an A (in this course)

343. S: I am…I mean I'm awful at math…I just ah…I just don't like math…and so um…it just takes me longer to understand something than most people I guess…but…um so in high school I just took mainly what was required and then like...

The courses Shelly took at the community college assumedly included the study of arithmetic sequences and linear functions. During our interview, I asked her if she had ever had experiences with pattern finding problems before. She replied, "Not necessarily writing a rule but I remember having to write like the next numeral that would go in sequence with the others." Shelly did not write a rule for either of the two patterns presented during the 'think' session. She extended the first pattern to find the 15$^{th}$ term, but did not find the correct entry for the 15$^{th}$ term in the table on Think Pattern 2.

## Results from the Second Stage of Analysis

The next stage of analysis centered on the texts produced by the six primary participants while engaged in the process of pattern finding. This stage involved the hermeneutic analysis of the texts produced by the primary participants. During this analysis, a series of interpretive memos were written which described the pattern-finding experiences of each participant. For organizational purposes, the memos were integrated to form two detailed accounts of the experiences these six pre-service elementary teachers had while engaged in pattern-finding activities. The first of these described the pattern-finding experiences of Tara, Cathy, and Ashley and is included in Appendix E. All three of these participants wrote symbolic rules for each of the patterns they analyzed. The second account, located in Appendix F, presents detailed descriptions of the experiences of Shelly, Matt, and Jill; the three who did not always make use of the symbolic. Summaries of the pattern-finding experiences of the six participants are presented in this section.

*The Pattern Finding Experiences of Tara, Cathy, and Ashley*

Three of the primary participants (Tara, Cathy, and Ashley) wrote symbolic rules for both of the patterning problems completed during the 'think' session of data collection. In addition, these three played an active role in the generalization of the patterns encountered during the 'together' session. The 'together' session consisted of four patterning problems which combined both a geometric form and a t-table to represent an arithmetic sequence. Participants were generally asked to extend the pattern to determine the number of cubes needed to construct the next term, two or three non-sequential terms, as well as the *nth*. These problems are included in Appendix B. Summary descriptions of how these three went about the process of generalizing the six arithmetic sequences used in this study are presented next.

*Summary of responses to 'think' pattern one.* The pathway Tara took en route to generalizing the first of these patterning problems (Appendix A) began with the process of attending to the structure of the figures illustrated in the table. While examining the structure, she began to analyze the changes that occurred in the construction of sequential figures. Through this analysis of change, Tara noticed how the figure changed and how the figure stayed the same. She used these dual ideas of stasis and change to explain how she drew the fourth figure in the table. When asked to describe the $15^{th}$ figure though, Tara noticed the connection between how the figure is constructed and the position it has in the sequence. By linking the position of the term to its value, Tara was able to derive an efficient way of finding the $15^{th}$ (or any term) in the sequence of figures. She was able to make this connection by thinking about what she was thinking. Tara then used a variable and linked the variable to the position to arrive at the general term, or a rule that she understood has to work all the time. Tara effectively used a specific term (in this case, the $15^{th}$ figure) to perceive the general.

Cathy communicated her understanding of the general term by explaining the arithmetic required to find out many squares it would take to build any figure in the sequence. She connected the rate of change to the coefficient of the variable, *n*, and the constant to the middle square. In a similar fashion, Ashley made use of repeated addition to explain the connection between her rule and the construction of the figure ("and then the four…each time you have the four sides and each time… you add four more…so two times four…three times four…yeah."

*Summary of responses to 'think' pattern two.* The strategies these three pre-service elementary teachers used to generalize the second think pattern (Appendix A) were similar to the processes they utilized on the first one. All three used the analysis of change to locate the fifth term in the table, but began to look for a relationship across the table to find the $15^{th}$ term. Each sought to establish a correspondence across the table, because, as Tara stated, they "knew to make a rule it had to pertain to the *nth* figure."

Ashley's desire to find a shorter way to find the 15$^{th}$ term provided the push for her to look for a correspondence across the table. She uncovered that route by inserting a column of differences between the two data sets. Tara and Cathy said they were just able to "see" the numerical relationship between the two data sets. All three were successful, in part, because they knew that to make a rule, "it had to pertain to the *nth* figure."

*Summary of Tara's experiences during the 'together' session.* Tara was the more vocal member of her group and also the first one to state a way to describe the *nth* term of the patterns the trio analyzed together (Appendix B). The second patterning problem seemed to be the most challenging for the group based on the amount of time spent as compared to time spent on the other two problems completed. Although Tara noted how the model was changing with the addition of each red cube, she actively sought to find a pattern that would connect the number of red cubes to the number of yellow cubes in each figure. Once she arrived at a verbal way to connect the two parts of the model, she struggled to find a way to express that relationship symbolically. However, through the process of explaining how to apply her verbal rule, Tara noticed a connection between the number of red cubes and the corresponding number of sides "all the way around the red cubes".

*Summary of Cathy's Together Session.* Cathy adopted the role of leader throughout the 'together' session with her two partners. Initially, Cathy tried to help her partners, Jesus and Daniela 'see' the relationship between the two data sets by noticing a pattern in what arithmetic could be used to connect the two data sets. This strategy of visualizing the arithmetic relationship between the two data sets had proven to be a successful route for Cathy until she encountered Identifying Patterns 2. This time, the arithmetic of six more than twice the number was not as easy to ‘see.’ By examining the construction of the physical representation of this sequence, Cathy noticed the constant presence of the six cubes on the sides. She wrote in her reflection that once she “…realized that the six blocks on each end were constant in every figure, this became much easier to solve.” Cathy then connected the rate of change to the variable part of the model to arrive at the rule, $2n + 6$. From that point forward, Cathy encouraged her partners to determine the constant feature of the model as a way to visualize the general term.

*Summary of Ashley’s ‘together’ session.* For her reflection, Ashley chose to write about the third pattern (Identifying Patterns 3).

> On the chair model, there are 3 squares that are always the same no matter how high the legs are. Therefore +3 is a constant. Next there are 3 factors that change, the legs (2) and the back (1). Those are affected by n. $3n + 3$.

Like Cathy, Ashley tuned into the relationship between how the model stayed the same (the constant) and how the model changed (the variable part). What is not known is whether or not she noticed this connection after Matt explained how he found the $85^{th}$ term in the second pattern. Ashley admitted that Identifying Patterns 2 presented a greater challenge for her. This might have been due in part to the fact that she concentrated on one of the figures in the model as she sought to establish a relationship between the number of red blocks and yellow blocks. This focused view on only one figure in the model may have made it difficult to identify the part of the model that never changes or as Cathy phrased it, the part that’s “always going to be there.”

*Tara’s explanations of function.* Tara initially described a function as a pattern but later on combined the ideas of function as a relation and as a process to describe the idea as "…how this was related to this figure." Tara responded to the 'think' question on the idea of function by stating that "…a function could be the pattern that is used to solve the rest of the problems, where the problems are related to each other somehow." While sharing her response with her partners, Tara professed that she "…wasn't for sure what a function would be." At the conclusion of the patterning activities though, Tara wrote the following in response to the second reflection prompt (Appendix C):

> This activity helped me learn the meaning of a functional relationship! I had to analyze & compare each figure & numbers in the chart to figure out their relationship to each other. It helped me understand patterns, relations, & functions in a hands-on-way.

She explained her response during the interview, stating, "Like I said… we had a hard time deciding what a functional relationship was…and you know I just kind of figured out that it was how this was related to this figure."

*Cathy's explanations of function.* Cathy's explanation of function was fixed upon the idea of function as a rule. She explained that "…a function is a formula that represents a pattern" in her written response to the third 'think' question. Her reply to the second reflection prompt reiterates this idea of function:

> I think this activity helps students see that every pattern can be expressed as a formula. If students have significant opportunities to analyze different patterns and understand the relationship, it will become easier for them to see the relationships.

During the interview, Cathy was asked to explain what is meant by a functional relationship in everyday terms. She responded to this question by describing a functional relationship as objects that are "…related…that you couldn't have one with the other." However, her mathematical definition refined this idea by restricting a function to "…a formula that expresses the relationship between two points."

*Ashley's explanations of function.* Ashley also linked a function with the idea of a rule, describing a function as "…an algebraic equation that you sometimes have to figure out. You put a number into an equation and get an answer." She used a similar description of a mathematical function during the interview, stating that "…a function is like…I don't know…it's like an equation and you have to figure out what goes into the equation." Her response to the second reflection prompt brings clarity to this explanation of function.

> To understand finding a pattern, you have to see the relationship between numbers. We were given a starting number (*n*) and a value and we had to figure out, analyze how we got that. Which is a function.

*Summary.* The successes experienced by Tara, Cathy, and Ashley did not come without some struggles. All three ran into difficulties generalizing the second pattern in the 'together' session despite having had no problems prior to this. Success seemed to be dependent in part on their intent to forge a relationship between the two data sets, whether they were examining that relationship in terms of the physical construction of concrete models or in terms of the numerical relationship between ordered pairs.

*The Pattern Finding Experiences of Shelly, Matt, and Jill*

This section will present descriptions of the experiences the other three participants had during these pattern-finding activities. These three participants, Shelly, Matt, and Jill, had varying degrees of success in the process of generalizing patterns.

*Summary of responses to think pattern one.* Shelly evaluated how the model changed from one term to the next in 'think' pattern one (Appendix A). She applied her understanding of how the model grew to jump from the fourth to the fifteenth term. However, she did not perform the arithmetic required to answer the question of how many squares it would take to actually construct that figure. Shelly also applied the process of change to write a recursive rule for constructing any figure in the pattern. She admitted to being confused about what was meant by the *nth* term of a sequence, stating that she didn't understand the concept "to the fullest extent."

Matt began the process of generalizing the first 'think' pattern by analyzing the incremental changes occurring from the first figure to the fourth. When asked to explain how the figure grows, Matt identified how the figure stayed the same and then described how the figure changed by linking the position of the figure to its construction. The description Matt wrote about how the model changed contains much of the same language as the verbal rule he wrote for the *nth* figure. He effectively used the visual construction of the figure to describe the general term, but did not make use of the symbolic. Matt had explained, in part, why he chose to write a verbal rule during the follow-up interview. In particular, Matt was asked to comment on the last statement that he wrote concerning the first patterning activity.

34. R: …you said that the figure is a visual…it's a visual of it…where the rule is an explanation…do you feel the same way about the symbolic one…is that still an explanation? Like her $4n + 1$?
35. M: Yes but I think that just some people would have more trouble understanding that (symbolic rule) than just reading it (verbal rule).

36. R: uh huh……than reading it…okay…they would wonder maybe…what would be the problems…what would be the trouble that someone might have?

37. M: Well someone may not recognize what n is…

Jill also wrote a verbal rule that explained how to construct any figure in the model. Her description of the fifteenth figure was very similar to Shelly's in that she did not complete the arithmetic necessary to state the total number of squares used in the construction of that figure. However, she did make the link between the position of the figure and the number of squares on each leg, as did Matt. Her verbal rule for any figure in the model is similar to her description of the 15th term. She linked the number of squares on each leg to the position, *n*, but does not include the presence of the center square. The missing center square and the lack of arithmetic resulted in an incomplete picture of the general term.

*Summary of responses to think pattern two.* Shelly sought to establish a correspondence between the two data sets in 'think' pattern two (Appendix A) by locating a pattern of differences. However, she did not coordinate the changes occurring in the values of *n* (from 5 to 15) while applying this pattern. This lack of coordination, along with an arithmetic error, made it difficult to find "a pattern in the numbers." Her experience with think pattern 2 illustrates how looking across the table (seeking a correspondence) can actually be a roadblock in the process of generalizing patterns if the individual does not also coordinate the changes in each data set with one another.

Matt was also unable to write a rule for the *nth* term in the table, despite having successfully written a verbal rule to describe the first pattern. The second think pattern forced the individual to use a variable by requesting the value of a quantity on the right when the quantity on the left was *n*. Matt had been able to use the visual representation of the first think pattern to write a verbal rule, but struggled with the second pattern. Matt's experiences with the second think pattern illustrate how the use of a variable can present a roadblock in the process of generalizing patterns for individuals who "may not recognize what *n* is."

Jill, on the other hand, was more successful with the second think pattern than she had been with think pattern 1. Although she seemed to stumble upon a rule for connecting the two data sets, she was able to express that rule in symbolic terms whereas her rule for think pattern 1 had been written using everyday language. Jill explained how she "only saw the relation for each column and not how the columns related to each other." However, Jill persisted in her pursuit to figure out the correspondence between the two data sets, or as she stated, "what do I do to this to get these."

*Summary of Shelly's experiences in the 'together' session.* Shelly approached the majority of the problems in the 'together' session (Appendix B) by focusing on the numerical data displayed in the table instead of using the physical models to look for a connection between position and construction. When faced with the task of jumping ahead to a non-sequential term, Shelly would look for some type of numerical relationship between the known values in both data sets. Whether she applied proportional reasoning, as in Identifying Pattern 4, or the rate of change, as in Identifying Pattern 2, Shelly usually limited her analysis to one specific pair in the data set.

Shelly frequently sought verification for the answers or suggestions she made during both the 'together' session and during the interview. For example, while analyzing Identifying Pattern 4 in the interview, Shelly asked, "is that right" after providing an answer for the number of squares needed to construct the 8th bridge in the pattern. She also asked, "Does that make sense?" after providing an explanation for how she got that answer.

*Summary of Matt's experiences in the 'together' session.* Matt described himself as a visual problem solver and was successful in the task of generalizing these patterns when he made use of the pictorial representation. In Identifying Patterns 1, Matt departed from the physical representation and focused on the values recorded in the t-table. He attempted to find a numerical relationship of some sort and, in that search, tried to use proportional reasoning to determine the value of a non-sequential term. He would have noticed that the relationship between the two data sets was non-proportional if he had examined the relationship between known data pairs.

Matt did make use of the physical models in Identifying Patterns 2 and 4. Besides noting what parts of the model stayed the same, Matt was also able to connect the variable part of the model to its position. Matt was then able to use the relationship between the position of the figure and its construction to describe a non-sequential figure in the model. Although he referred to a specific term in each sequence, he was just one step away from describing the general term.

While analyzing the pattern of H's (Figure XX), Matt demonstrated that he understood that a general rule would be dependent on the position of the figure ("probably want to take…um… the numbers consistent with eight." Later on in the interview, Matt explained how his recent experiences with patterning helped him learn to think algebraically.

389. M: seeing the whole picture first…you gotta take it step by step…you know look at the pattern and then go from there…well obviously like this I took 8 plus 2 plus 8 times 4 and then I took 15 plus 2 plus 15 times 4 and then I took it to the *n*…algebraically…

390. R: Yeah…you can see it algebraically…so what's the…what do you think was the difficulty for you getting the symbolic rule…why was that hard?

391. M: Well it goes back to adding numbers with letters…

392. R: …numbers and letters…yeah

393. M: You know it was just the turning point when you were learning math…you know you were so used to adding, subtracting, dividing, multiplying…then somewhere along the way growing up…now we're going to put *a* and *b* in this…and it's like you already learned it in English (*both laugh*)

394. R: So it's getting that variable…throwing in that variable.

395. M: Yeah…

*Summary of Jill's experiences in 'together' session.* For Identifying Patterns 1, Jill described how she used the physical model to determine the number of faces to paint on an *n*-cube high tower. However, Jill struggled with Identifying Patterns 2, perhaps because she "could…see how this one changes and…see how this one changes but…" not how the two sets changed together. When faced with this type of situation, Jill usually resorted to the method of guess and check so as to find some rule that would work all the time.

*Shelly's explanation of function.* Shelly described the idea of function as both a pattern and a relation. In response to the third think question, Shelly wrote that "…a function would be a pattern or when something goes with another." During the follow-up interview, Shelly explained a function in an everyday way as "…how something works maybe…how something functions…how something…um…how something goes together maybe." At that time, Shelly also professed that her mathematical definition of a function would be"…kind of like my everyday definition…how something goes together." She wrote in her reflection that being with a group helped her to "…understand the patterns, relation, and function of the problem." Shelly also mentioned how the activity strengthened her "understanding of these concepts by having examples to see and go by or follow before actually beginning the activity."

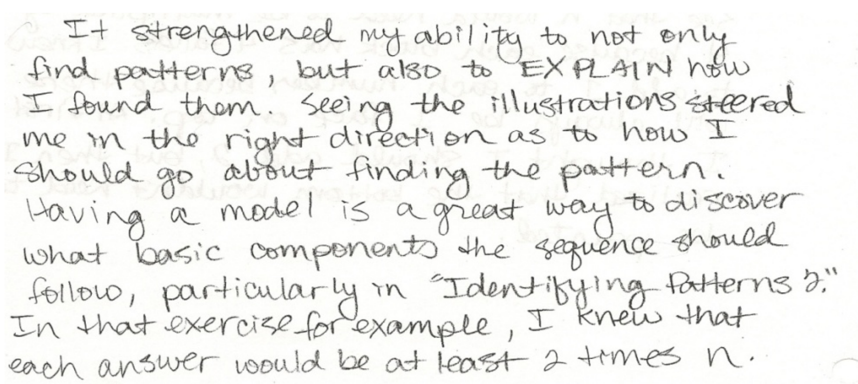
*Matt's explanations of function.* Matt also aligned the everyday use of the word function with its mathematical meaning by referring to a function as something that "works together." During the 'pair' session, Matt described a function as "…a physical representation such as using…say apples and oranges." His intent was to use this representation as a way to explain the idea to someone who may not have had experiences with functions. At the time of the interview, I asked Matt to explain what is meant by a functional relationship in everyday life. He responded that when something is a function of something else "…it means it's clicking…it's working…it's…you're working together …you know you're understanding it." Matt went on to define a function in mathematical terms, stating that "…it would be about like one number plus another number equals…it gives you the answer so obviously the function works together."

*Jill's explanation of function.* While sharing her response to the third think question during the 'pair' session, Jill said, "I would explain that you have to experiment with numbers to see how they related to each other just because that's how I found it…I had to sit there…mess around with it until it came." During the interview, Jill explained a functional relationship outside the world of math as "…something that works…that would be my personal definition of it anytime somebody said you know the function of this…it's how it works…it's kind of how I see it." When asked to explain her understanding of function in the mathematical sense, Jill replied:

> I guess….the way I kind of understand it as sort of like a formula but it applies to a pattern I guess…it's kind of how I see it you know…it's just sort of like a way you can put in a certain number and get out a certain number.

In some ways, her mathematical definition is in agreement with her everyday view of a function as "something that works."

The second question posed in the 'reflect' session prompted the participants to explain how the experiences they had just had with pattern finding strengthened their understanding of patterns, relations, and functions. Jill's written response is presented in Figure 2.



It strengthened my ability to not only
find patterns, but also to EXPLAIN how
I found them. Seeing the illustrations steered
me in the right direction as to how I
should go about finding the pattern.
Having a model is a great way to discover
what basic components the sequence should
follow, particularly in "Identifying Patterns 2."
In that exercise for example, I knew that
each answer would be at least 2 times n.

*Figure 2. Jill's reflective response to prompt two.*

Jill had explained how she was a 'little rusty' at generalizing patterns. However, through these experiences and the need to explain her reasoning to her partner Kristen, Jill surmised that she had a better understanding of the behavior of these types of patterns. She talked about the value of these experiences during the follow-up interview.

> Having us talk with our peers…definitely made me realize that maybe I didn't understand it as well as I thought I did because when somebody would ask me how'd you get that…I don't know…I just did…and so actually having to verbalize it made me…it made me understand it better because I had to show somebody else you know…

*Summary.* This section presented the experiences of Shelly, Matt, and Jill as they engaged in the process of pattern finding. Unlike the experiences of Tara, Cathy, and Ashley, these three were not always successful in the task of generalizing patterns, especially when it came to writing a symbolic rule.

## Chapter Summary

This chapter presented a summary of the first two stages of analysis. The first stage involved a preliminary analysis of the written responses of all 29 participants in the study. This preliminary analysis was subsequently used to select the six primary participants. A summary of the experiences these six primary participants had while engaged in the process of pattern finding was also presented in this chapter. Their experiences were used to gain insight into how preservice elementary teachers go about the process of generalizing patterns.

The process of generalization was assisted by various strategies or factors employed by this group of pre-service elementary teachers. These factors, or catalysts, will be summarized in the next chapter. In addition, the failure to describe the general term could be explained, in part, by other factors or strategies that effectively served as roadblocks in the process of generalization. These roadblocks will be summarized in Chapter V as well. The elements of catalysts and roadblocks will serve as a framework describing the nature of pre-service teachers' understanding of pattern and function.

# CHAPTER V

## SUMMARY OF RESULTS

The purpose of this study was to describe the nature of preservice elementary teachers' understanding of pattern and function. The texts produced by six pre-service elementary teachers engaged in the process of generalizing patterns were analyzed to address this purpose. Particular attention was paid to the ways this group of pre-service elementary teachers communicated their understanding of function while analyzing patterns. Descriptions of their experiences with pattern finding were presented in the previous two chapters. This chapter presents a summary of the results by describing the major themes associated with the research questions posed in this investigation. The primary research question focused on the nature of pre-service elementary teachers' understanding of pattern and function. This research question was addressed through the positing of two sub-questions that examined the pattern-finding experiences of pre-service elementary teachers.

Research Questions:

1. How do pre-service elementary teachers communicate the idea of function while engaged in the study of patterns?
2. How do pre-service elementary teachers demonstrate their understanding of function while engaged in the study of patterns?

3. What is the nature of pre-service elementary teachers' understanding of pattern and function?

This chapter begins by presenting the idea of function as communicated by the participants in this investigation followed by the approaches taken by this group of pre-service elementary teachers while engaged in the process of generalizing patterns. The chapter concludes with descriptions of how the conceptualizations of function and the approaches adopted by these pre-service elementary teachers served as either catalysts or roadblocks in the process of generalizing patterns.

## Communicating the Idea of Function

The ideas of function generated by the original group of pre-service elementary teachers (18 of 29) in response to the third think question formed the initial framework for how preservice elementary teachers describe the idea of function. The four ideas of function, as presented in chapter IV, included the idea of function as a pattern, a rule, a relation, and a dynamic process.

A system of open coding was used during the initial analysis of the statements generated by the primary participants. After coding the statements, an attempt was made to sort these codes under the initial framework derived from the participants' descriptions of the idea of function. This attempt was made to ascertain how this group of preservice elementary teachers would communicate their understanding of these four ideas. However, the possibility of additional themes was considered during this sorting process. The following section presents the definitions of each of these themes and the identifying codes that fell under each theme. Exemplary statements from each of the six primary participants are provided to illustrate each theme and/or sub-theme.

*The Idea of Pattern: A function is a pattern you must find*

Seven of the eighteen respondents to the third think question associated the idea of function with "…a pattern you must find." For the purpose of this study, a pattern was defined as the repeating or changing structure of a sequence of numbers or shapes. The theme of *pattern* emerged from the statements made by the six primary participants which communicated the constant features of the sequences they examined. Since these were all linear patterns of a non-proportional type, they involved both a constant rate of change and the presence of a constant term. In other words, *the idea of pattern* encompassed both how the model changed and how the model stayed the same. These two ideas of pattern emerged as descriptions of the changing structure of the figures used in this study ("how many are we adding each time") and the repeating part that is "…always going to be there."

*References to "How many are we adding each time."* While examining Identifying Patterns 4, Cathy asked her partners, "How many are we adding each time?" All six of these participants, at one time or another, made statements that expressed how a particular model was changing. These statements usually included the words "each time" or "each figure" which implied that this rate *always* applied to this particular sequence of figures or numbers. A sample of statements associated with 'how many are added each time' is provided.

> Tara: "…so one red block and two yellow blocks each time you go down."
>
> Cathy: "…and then the four that you're multiplying by is the number that it increases with for each term."

Ashley: "…each time you have the four sides and each time… you add four more…so two times four…three times four…yeah…"

Shelly: "One to the top…one to the bottom…and one over to the other leg. I think that's three blocks each time…yeah."

Matt: "The chang (*sic)* was adding two blocks to the left and right of each side for the model to balance correctly."

Jill: "There is 1 block added to each leg of the shape each step."

*References to "It's always going to be there."* The constant part of a non-proportional linear situation was visible in the pictorial representations of these sequences. The following statements express ways this group of preservice elementary teachers described the constant term. Each of these statements references the part of the model that was "always going to be there…those there that don't change."

Tara: "And it's off your base…like you've got your middle…your center base all the way across…"

Cathy: "…that's the constant…it doesn't change"

"You're always going to have three in the center…so on the same principle you're always going to have those there that don't change"

Ashley: "On the chair model, there are 3 squares that are always the same no matter how high the legs are. Therefore +3 is a constant."

Shelly: "yeah…there's always four…because the block has four sides"

Matt: "In model number 4, the four blocks stayed consistence (*sic*) across the top of the model"

Jill: "Um…I think usually the four is going to represent the row that's already there"

A Function is a Relation: *When something goes with another*

Four of the eighteen responses to the third think question were initially categorized as 'function as a relation.' In mathematics, a relation is defined as an ordered pair while in everyday language, a relation may be considered to be a set of objects that somehow go together. The ways that these preservice elementary teachers communicated the idea of function as a relation were by describing a function as "basically a pair" or by describing the relationship between the position of a term and its corresponding construction or value.

*References to "Basically a pair."* Shelly explained that a function was "…when something goes with another." References to a function as a relationship between two sets were coded as 'basically a pair.' A sampling of statements made by the six primary participants coded under this category is presented.

Cathy: "I guess I would explain that they were related…that you couldn't have one without the other."

Ashley: "…there has to be a variable there…there has to be another part."

Shelly: "…a function…when something goes with another…so basically a pair."

Matt: "I just said it was a physical relationship such as using say apples and oranges."

Jill: "…something that relates to both things."

*References to "the figure number"* Ryan had described how the construction of each figure in the first think pattern was based on the position of the figure in the sequence, or figure number. Statements associated with pictorial patterns often made specific references to the connection between the position of each figure and how it was constructed. These types of statements could be broken down into two categories; references to a specific term and references to the general term. Examples are given for statements coded under each of these two categories.

*References to the specific:*

Tara: "…like in the first figure there was one in the middle…with one on each side…the second figure there was one in the middle with two on each side…and then one in the middle with three on each side."

Cathy: "…well the first one…1 and 5 is what? It's one times four plus one. The second one…2 and 9…so that would be two times four plus 1…"

Matt: "85 times 2 plus 6…because you have 85 red blocks so you have one on the top and one on the bottom…"

Jill: "15 squares on each leg. I know this because there is 1 in the first figure, 2 in the second, and so on."

*References to "whatever"* Additional references made while describing the relationship across the table were stated in more general terms. The individual either made an explicit reference to the position of the figure or used words such as 'whatever' or "however many" to indicate position. The following statements illustrate how each participant described the construction or value of 'whatever' term in the sequence.

Tara: "Well the n was whatever number figure…like the first figure and then times four would be four sides that you're adding on to."

Cathy: "…so it's red times that and that"

Ashley: "I'm trying to figure out what this has to do with this."

Shelly: "Well it would have two legs and two in the middle."

Matt: "You have a center tile and with each figure you add tiles based on the number of the figure to the top bottom left and right of the center."

Jill: "…however many blocks you have you multiply that by 4."

The Idea of Rule: *A function is a basic rule*

Three of the eighteen respondents to the third think question had described a function as a rule or formula. For example, Ashley explained that "a function is an algebraic equation that you sometimes have to figure out. You put a number into an equation and get an answer." There were two sub-themes under the idea of function as a rule; references to a formula and references to how the respondent understood a rule as something that "always works."

*References to "a formula."* The idea of function as a rule was often communicated by the direct reference to a rule or algebraic equation. The reference to function as a rule could be communicated using everyday language, such as when Matt referred to a function as "one number plus another number", or it could be expressed using a combination of symbols and words.

Tara: "…and then my rule was one plus n times four equals the number of squares in the model."

Cathy: "…a function is a formula that expresses…the relationship…between two points."

Ashley: "A function is like an equation…and you have to figure out what goes into the equation and what comes out."

Matt: " I said that the model is like a visual and in a way the rule is just an explanation written down."

Jill: "The way I kind of understand it as sort of like a formula but it applies to a pattern I guess…"

*References to "it always works."* A rule can be defined as an equation or method that, when applied, produces a specific outcome. Jill had described a function as "…sort of like a way that you put in a certain number and get out a certain number." The idea of a function as a rule, therefore, encompassed the understanding that a rule has to work all the time. Several references were made to how the rule used to describe the pattern had to always work. To verify whether or not a rule would work, the individual would substitute known values into the proposed equation to see if the expected output was obtained. Examples of statements made that were coded as references to how "it always works" include the following:

Tara: "…just kind a back tracked to figure out how we could get it…to work…each time…"

Cathy: "So…let me make sure that works out. So two times 4 plus 6 is 14. Okay. so…2 times 85 plus 6…"

Ashley: "So… 3n plus 3…times 3 plus 3…so…3 times one…but that doesn't work."

Jill: "…you know then try it out for each one and it always works and I'll say Yes!…There we have it."

Shelly: "I don't think um 36 times three is correct because if you multiply it by 3 each time…this doesn't work..see 2 times 3 is this over here…3 times 3 is that….3 times 4 is that…so that…that's not going to work."

A Function as a Dynamic Process: *How do we get a pattern?*

Six of the eighteen responses to the third think session were classified under the idea of function as a process on the basis that these responses referenced *how* the pattern, rule, or relationship was forged versus the presence of these three features. Understandably, the idea of function as a process overlapped with the ideas of function as pattern, rule, or relationship. General statements associated with the idea of function as a process as uttered by the primary participants are provided.

Tara: "…so what could be our pattern…if we have n red blocks…how do we get a pattern?"

"Okay we got that but how do you get a relationship…for like a rule…from your red block to your yellow block."

Ashley: "We were given a starting number (n) and a value and had to figure out, analyze how we got that. Which is a function."

Shelly: "…you have to find a pattern in the numbers."

*Summary*

The four ideas of function (pattern, rule, relation, and process) were presented in this section as a way of describing how preservice elementary teachers communicate the idea of function. The words of the participants were used to illustrate each of these four ideas. All four ideas played a role in the process of generalizing linear patterns used by the six primary participants in this investigation. In addition, this group of preservice elementary teachers also employed numerous strategies as they went about the process of generalizing patterns. These approaches are introduced in the next section as sub-themes associated with the idea of function as a process.

## Process of Generalizing Patterns

The task of generalizing patterns was accompanied by the employment of numerous approaches which, along with the four ideas of function described in the previous section, served as catalysts or roadblocks in this task. A total of seven sub-themes within the idea of function as a process were uncovered. Each of these sub-themes is described in this section.

### *The Process of Visualization: Do you see it?*

The visual representations of the sequences analyzed in this investigation provided a visual that Jill explained "…did everything (for her)…because it really shows you why it's going to be that way." Matt also espoused the virtues of having a physical model, stating that this representation made it possible for him to "…see them all visually." The participants either communicated how they were *seeing the relationship* or how the relationship or pattern was *hard to see*.

*Seeing the relationship.* Several of the participants referred to how they were able to visualize the quantitative relationships between data pairs. Usually the process of visualization involved looking for the pattern or relationship between the terms of a sequence.

> Tara: "I looked at the first figure and I could see there's four on the outside of the middle and if you add one all the way around the outside then it goes to two on

the outside and so if you add one all the way around each side as you go across that becomes your pattern."

Cathy: "Um…I just kind of saw it"

Ashley: "To understand finding a pattern you have to see the relationship between numbers."

Matt: "…seeing the whole picture first…you got to take it step by step…you know look at the pattern and then go from there."

Jill: "…somehow just looking at how the pattern is growing it just kind of made you see that it was going to grow by 4 each time."

Jill: "You know the whole thing of seeing the relation between two sets of numbers"

*It's hard to see.* The participants also described how they struggled to see the relationship between the position of a term and its value or construction. Jill explained how she struggled with seeing a pattern, stating, "I can see how that one changes…I see how this one changes but I don't see how they change together."

Tara: "…staring at it you know we were kind of getting somewhere but we could not figure out the rule and I think we were getting frustrated as to the pattern…"

Cathy: "I'm not seeing it"

Ashley: "It's hard to see…like I mean obviously you know what changes…but it's hard to think of the formula for that like…how do you come about what makes that change or whatever…"

Shelly: "And um…I'm very visual…so it's hard to see that…this is really red or this is supposed to be green or whatever."

Jill: "…a lot of times what happens is I can see the relationship between these numbers and I can see the relationship between these but I can't really see it between the two of them."

*The Process of "thinking"*

At times, a participant explained how the process of thinking about it made it possible to arrive at a way to describe the general term. These explanations either referred to the singular act of thinking or to the act of thinking about what one was thinking.

Tara: "Well we came to that and then we kind a back tracked…and then we thought okay now how did I get that…"

Tara: 'just thought about it and compared the numbers in the chart to each other."

Matt: "I just thought well 5 goes into 15 three times so I just took twelve and times that by three."

*The Process of looking for a pattern of differences*

Three of the participants inserted a column of differences between terms in a table to try and establish a relationship between the corresponding entries. This particular approach examined the quantitative relationship between the two data sets instead of making use of the visual. The successful application of this particular strategy was somewhat dependent upon the individual's ability to generalize the relationship within this column of differences.

Ashley: "…the difference…so one plus two equals three which is the difference between four and one. See that's how I did it because 5 + 2 equals 7 so 5 plus 7 equals 12."

Shelly: "Four is six away from ten, so twelve will be seven away from five."

Matt: “Starting with one, if you add 3 you get 4…”

*The process of using proportional reasoning*

The application of proportional reasoning was another approach taken by some of the primary participants in this study. Since these patterns were all linear sequences with both a variable and a constant part, the use of proportional reasoning should only be applied to the part of the model that was changing. Without first separating the contribution made by the constant, the use was proportional reasoning was not appropriate. Statements which indicated the application of proportional reasoning are provided.

Cathy: “…so it’s red times that and that”

Shelly: “So it would double…cause one to four is double…6…12…6 plus 6 is 12…so 4 plus 4 is 8 and 12 plus 12 is 24.”

Matt: “Because 4 goes into 36 nine times…so 9 times 17”

Matt: “…because you have 85 red blocks so you have one on the top and one on the bottom…”

*The process of using the rate of change*

In a similar fashion as the two previous strategies, applying the rate of change could be viewed as a roadblock or as catalyst in the process of generalizing patterns. Once again, success was somewhat dependent on the ability to recognize the contribution of the constant feature in the model as well as the ability to coordinate the changes in both data sets.

Tara: “R plus one…equals yellow plus two…”

Ashley: “and each time you have the four sides and each time…you add four more…so two times four….three times four….that’s why it’s always four”

Shelly: “…the 4th figure has 4…blocks on each side…so you would add one for…on each one for each figure…so you add 11 for the 15th…”

Matt: “So to get the 5th one…since number 4 had a ten next to it…I added two to ten to get the twelve.”

Jill: “There was a 10-number jump between 5 and 15, so I assumed there would be a 20-number-jump between 12 and the next number since the value column seems to go by twos”

*The process of “guessing”*

Jill described the process of generalizing patterns as being a “guess and check thing.” The success of guessing was dependent upon understanding the idea of function as a rule that “always works.” Other contributions made by Jill placed under this category are provided.

Jill: “…the only reason I even came about it…was how…um…just kind of by luck…I started doubling things”

Jill: “…a lot of times it’s a guess and check thing…I’ll say you know what do I do to this to get these”

*The process of using a variable: “the a + b thing”*

At some point in the process of generalizing patterns, the participants usually attempted to make use of the variable, *n.* In most cases, the variable was used to represent the position of the figure in the sequence. However, attempts to understand and make use of the symbolic were also described as an area of struggle. For example, Matt commented on his struggle to make use of the symbolic, stating that "…most of the…the math problems she (*Ashley*) came up…especially with the *n*…you know the adding of the *n*'s…" Examples of the ways these participants used a variable are provided.

Tara: "1 + (*n* x 4) = # of squares where *n* is the number of the figure."

Cathy: "…two times the red plus six and if the red is *n*…then it would be 2*n* + 6."

Ashley: "Replace 1 with *n*, second # is *n* + 2. Therefore, the answer is *n* + (*n* + 2)."

Shelly: "I didn't really understand what the *nth* term meant until I asked you right before the test…and so I don't think that I completely understood it to the fullest extent."

Matt: "I can't do that *a* + *b* thing."

Jill: "Okay. I think it's 3*n* + 3."

*Summary*

These seven subthemes under the idea of function as a process illustrate the ways "of figuring out a pattern for a set of values" employed by the participants in this study. These approaches, coupled with the ideas of function presented in the previous section, could serve as either catalysts or roadblocks in the process of generalizing patterns. The final section of this chapter describes the relationship between these ideas of function and the approaches adopted by this group of pre-service elementary teachers while they were engaged in pattern-finding activities.

## Summary of the Pattern Finding Process

The elements involved in the process of pattern finding were categorized under four ideas of function and seven ways pre-service elementary teachers approached the task of generalizing a pattern. These elements acted in conjunction to serve as catalysts or roadblocks in the process of generalizing patterns. Those factors which contributed to success were labeled catalysts and factors which impeded progress towards generalization were labeled roadblocks. The final section of this chapter provides examples of the catalysts and roadblocks experienced by the six primary participants in this investigation.

*Catalysts and Roadblocks in Generalizing Patterns*

An approach was labeled a catalyst if its use seemed to direct the individual's thinking towards the general term. These catalysts were not stand-alone ways to go about the process of generalizing patterns. In most instances, the successful application of a catalyst was dependent upon the understanding of one or more ideas of function. Without this coupling, the approach employed often acted as a roadblock to generalizing a pattern.

*The process of visualization.* The process of visualization served as a catalyst when the individual was able to "see the relationship" between position and construction of a term in the sequence. In Identifying Patterns 1, Cathy initiated the task of pattern finding with her partners by asking them if they "…could see any sort of relationship…between the number of faces and the numbers of cubes…" Cathy demonstrated her understanding of function as a relation and a rule by explaining how "…1 and 5 is one times four plus one…3 and 13…would be three times ….four plus 1. So your formula is 4*n* plus 1."

However, the process of visualizing the relationship was somewhat of a roadblock in Identifying Patterns 2 because, as Ashley described it, the relationship was "hard to see." While examining this particular pattern, Cathy admitted that she was "not seeing it."

The breakthrough came when she "realized that the 6 blocks on each end were constant in every figure." Understanding the idea of function as pattern helped Cathy focus on what part of the model stayed the same and to separate the variable part of the model from the constant. She was then able to use the two parts of the model to write the rule, $2n + 6$, which reflected the facts that "for every red block that was added, 2 yellow blocks were added, $2n$, and the 6 constant yellow blocks on the end."

*The Process of Thinking About It.* Tara explained that she was able to write a rule after she thought about how she determined the $15^{th}$ term. The process of 'thinking about it' helped her to generalize the relationship between the position of each figure in the first think pattern and the manner in which it was constructed. In this way, 'thinking about it' served as a catalyst along with the idea of function as a relation.

*The process of looking for a pattern of differences.* Ashley, Shelly, and Matt all looked for a pattern of differences between the two data sets displayed in tabular form in the second think pattern. Ashley was able to use this pattern as a catalyst because she understood both the ideas of function as relation and as a rule that "always work." She described how to generalize the pattern of differences in her written response to the last question on think pattern 2: "Replace 1 with $n$, second # is $n + 2$. Therefore, the answer is $n + (n + 2)$."

Shelly and Matt were not able to locate the general term in Think Pattern 2 using the same pattern of differences. Shelly failed to coordinate the changes in both data sets while determining the differences and Matt applied the pattern of differences after trying to use proportional reasoning to find the $15^{th}$ term. Although he tried to describe the relationship across the table, his rule for the *nth* term only applied to changes in the values of *n*. This also indicated a misunderstanding of function as a pattern due to a lack of coordination of the changes in both data sets.

*The process of using proportional reasoning.* The use of proportional reasoning served as a catalyst when coupled with the idea of function as a pattern. To make use of the proportional relationship imbedded in these linear patterns, the individual had to focus on the variable part of the model. Cathy achieved this goal by first identifying which part of the model stayed the same. As she explained, "I think it's easier to find the constant first…and I think once you realize…once you find out what's not changing…it's easier to find out what is changing." The connection between the variable part of the model and the position of the figure could then be explained using proportional reasoning, as Matt applied to Identifying Patterns 2: "…because you have 85 red blocks so you have one on the top and one on the bottom."

Proportional reasoning became a roadblock when applied to the quantitative relationship represented in a table. Both Matt and Shelly attempted to use a known pair of quantities in the table to determine the missing value for a specific, non-sequential term. For example, Matt tried to use the last known ordered pair in the table displayed in Think Pattern 2 (5, 12) to determine the 15$^{th}$ term, stating, "I just thought well 5 goes into 15 three times so I just took twelve and times that by three." His way of thinking did not apply to the non-proportional relationship between the two data sets. Ashley described the process Matt applied as, "Okay so basically you were doing like this type of thing…um… 15 equals *x* kinda like …well kind of setting it up that way…and then what does it take to get to here…it takes three and so it has to take three to get to there." Although Matt was considering the idea of function as a relation, he did not verify that the proportional relationship would work for all pairs in the relationship (function as a rule).

*The process of using the rate of change.* Using the rate of change as a way to generalize a pattern was a catalyst when acting along with the idea of function as a pattern. The recognition of repeated addition as multiplication, for example, enabled the individual to see "…the algebra…the math in it" as Jill's partner had explained. Ashley also made the connection between the constant rate of change and the rule she had written for Think Pattern 1, stating "…and each time you have the four sides and each time…you add four more…so two times four….three times four….that's why it's always four."

Shelly successfully applied the rate of change to write a recursively-defined rule for Think Pattern 1, but did not describe how to find the general term. Shelly focused on how the model changed from one term to the next but did not apply the idea of function as a relation to look across the table for a way to link the rate of change with the position of a term. During the 'together' session (Identifying Patterns 3), Shelly attempted to use the position of the 35$^{th}$ term to determine the value of the 36$^{th}$ entry. She explained her idea for using the 35$^{th}$ term to find the 36$^{th}$ to her partner, stating, "…cause we're trying to go up to 36 so we could go one number lower…wouldn't this be 38?...because three plus 35 is 38." Using the rate of change can be a roadblock when the individual doesn't understand how to describe the relationship across the table.

*The process of guessing.* Jill explained how she was often able to arrive at a way to describe the general term "just kind of by luck." The idea of function as a rule that 'always works' caused her to test the "little theories" she came up with until she found one that "worked all the time." The idea of function as a relation was also necessary for the successful application of guessing. Although Jill had complained that she "only saw the relationship for each column but not how the columns related to each other," she was trying to figure out what to "…do to this to get these." For Jill, the process of guessing served as a catalyst because she also understood the ideas of function as a relation and as a rule that "always" works.

*The process of using a variable.* Tara, Cathy, Ashley, and Jill consistently used a variable to represent the position of a term in a sequence. Even when Jill wrote a verbal rule for think Pattern 1, she used *n* to represent "…the number of boxes on each leg." The process of using a variable was a successful catalyst to writing a symbolic rule when the individual understood the idea of function as a relation, that is, they "…knew that to make the rule it had to pertain to the *nth* figure."

Shelly professed that she "…didn't really understand what the *nth* term meant." The use of a variable presented a roadblock for her because she didn't really understand the connection between the variable *n* and the *nth* term. This lack of understanding the idea of function as a relation between position and value prevented her from experiencing consistent success in generalizing these patterns.

Although the use of variable was a roadblock for Shelly, Matt achieved a degree of success in generalizing patterns despite not being able to "do that $a + b$ thing." His success was limited to those concrete models where Matt was able "…see them all visually." Matt explained the process as, "…seeing the whole picture first…you got to take it step by step…you know look at the pattern and then go from there." He was able to make use of the idea of function as a pattern to describe how each model was constructed and in turn, describe the construction of any specific figure in the sequence without making use of the variable, *n*.

*Summary*

The examination of approaches used while attempting to generalize a pattern revealed that an approach represented a roadblock when it was applied without also considering the ideas of function as a pattern, relation, or a rule. An approach served as a catalyst when coupled with these other ideas of function. For example, the use of proportional reasoning led to a rule for describing a relation only when the individual separated the variable part of the model from the constant. Without the analysis of the patterns of constant and change, the use of proportional reasoning served as a roadblock to generalizing patterns.

The overall process of generalizing a pattern followed a similar route and was either assisted by catalysts or hindered by roadblocks. The individual would first describe how the model was changing and would usually apply that rate of change to find the next term in the sequence. The quest to locate a non-sequential term usually drew attention to the relationship across the table. At this point, Matt and Shelly would usually focus on describing the construction or value of that specific term. The others generally tried to find a way to describe any term in the sequence and then apply their rule to complete the table.

## Chapter Summary

This chapter presented the findings as an integration of the individual experiences this group of pre-service teachers had while they were engaged in patterning activities. Cross-case analysis revealed four common ideas of function and seven approaches for generalizing patterns. The dependent relationship between these ideas of function and the approaches adopted to generalize patterns was explored in the concluding section of this chapter.

The final chapter of this dissertation discusses these findings in light of prior research on the understanding of the ideas of pattern and function. A model for describing the levels of understanding pattern and function is proposed based on the findings presented in Chapter VI. The chapter concludes with a description of the implications raised from this research investigation along with suggestions for future research in the area of pre-service elementary teachers' understanding of pattern and function.

# CHAPTER VI

## DISCUSSION OF FINDINGS

The purpose of this study was to unpack the understandings pre-service teachers have pertaining to the ideas of pattern and function. The intent was to bring insight into how mathematics teacher educators can use patterning activities to prepare pre-service elementary teacher to support the development of algebraic thinking in their future students. The research questions guiding this endeavor were as follows:

1. How do pre-service elementary teachers communicate the idea of function while engaged in the study of patterns?
2. How do pre-service elementary teachers demonstrate their understanding of function while engaged in the study of patterns?
3. What is the nature of pre-service elementary teachers' understanding of pattern and function?

The previous chapter summarized the findings related to the ideas of pattern and function as experienced by six pre-service elementary teachers engaged in the study of patterns. Four themes associated with the idea of function were described using the words of the participants. In addition, seven approaches or subthemes involved in the process of generalizing patterns were identified. The ways in which these approaches served as either roadblocks or catalysts in the process of generalizing patterns was examined.

The first section of this discussion on the significance of these findings addresses the first research question on how pre-service elementary teachers communicate the idea of function. A model of the relationship between the different ideas associated with the concept of function is explored. The next section addresses the question of how pre-service elementary teachers demonstrate their understanding of function through the process of generalizing patterns. The interplay between the ideas of function and the approaches used is explained. The answers to these two questions were then applied to offer an explanation of the nature of pre-service elementary teachers' understanding of pattern and function.

## Communicating the Idea of Function

The primary participants in this investigation communicated four ideas of function; function as a pattern, function as a relation, function as a rule, and function as a dynamic process. These four ideas overlapped to form the idea of function as experienced by these pre-service elementary teachers. This section summarizes these four ideas and relates each back to the concepts of function reviewed for this study. The section concludes with an illustration of the concept image formed by these four overlapping ideas of function.

### *The idea of function as a pattern*

The participants communicated the idea of function as "a pattern you must find" by referencing how the model changed and how the model stayed the same from one term to the next. In general, statements associated with change, "how many are added each time," were made before the individual noted the constant part of the model that is "always going to be there." The identification of the constant part often made it possible to generalize the relationship between position of the term and the variable part of the model.

Cathy explained the advantage of identifying the constant part by stating that "…once you find out what's not changing…it's easier to find out what is changing." Sierpinska (1992) cited the identification of the source of change as a central component of understanding the idea of function. She also asserted that a roadblock to understanding change emerges when one focuses "…on how things change" (p. 36) instead of what changes. The application of recursive reasoning builds on the idea of how things change and was described by numerous researchers as a roadblock to generalizing patterns (Blanton & Kaput, 2004; Moss, Beatty, Barkin, & Shillilo, 2008; Zazkis & Liljedahl, 2002).

Saldanha and Thompson (1998) stressed the need to coordinate the changes in both data sets instead of relying solely on recursive reasoning. They explained this coordination as the ability to note the changes in one data set without losing sight of how the other set was also changing. Shelly's approach to finding the 36th term in Identifying Patterns 3 exemplifies how the failure to coordinate change interferes with the process of generalizing patterns. She proposed to add the rate of change to the 35th term so as to find the value of the 36th. Since the 35th term was also an unknown quantity, Shelly mistakenly assumed that it would be 35 and proceeded to add three more to come up with a value of 38 for the 36th term. Not only did she seem confused about the meaning of the 35th term, but she also failed to consider how the jump from the last known pair of quantities (4, 15) would come into play.

The theme of pattern centered on the idea of covariational analysis that Smith (2003) avowed was central to the development of the concept of function. Smith explained that individuals began the process of pattern finding by first examining the changes exhibited as one moved down the table. If the individual attended to both sides of the table simultaneously, he or she might begin to note patterns in variation across the table (Saldanha & Thompson, 1998; Smith, 2003). In such a manner, the analysis of covariation potentially facilitates the formation of a correspondence between the two data sets. Tara began to analyze the sequence of figures in Identifying pattern 2 by noting how "you add 1 red block and you add two yellow blocks each time you…you go down." Her analysis of the rate of change corresponds with the idea of function as covariation. Tara's later comment on "…what could be our pattern…if we have *n* red blocks…how do we get a pattern?" illustrates the transition from looking down the table to looking across the table for a pattern of correspondence.

*The Idea of Function as Relation*

The idea of function as a relation was manifested in references to a function as "basically a pair" or in attempts to relate the position of a term to its corresponding construction or value. The attempt to formulate this correspondence often made it possible to describe the construction or value of the general term. For example, in Think Pattern 1, Matt described how to construct any term in the sequence by stating that "…you have a center tile and with each figure you add tiles based on the number of the figure to the top, bottom, right, and left of the center."

Billings, Tiedt, and Slater (2007) described the progression from an analysis of change to the analysis of correspondence achieved by a group of second and third graders. They noted that the first stage, analysis of change, culminated with the ability to describe how the figures in a pictorial growth model changed and what parts of the figures stayed the same. This particular phenomenon was described in the discussion of the idea of function as a pattern. The transition to the analysis of correspondence was marked by the connection of position to a particular figure, as Matt had achieved with Think Pattern 1.

The idea of function as a relation aligns with the correspondence view of function in that the emphasis is on the relationship between the two data sets (Slavit, 1997; Smith, 2003). The analysis of correspondence seeks to describe the relationship that maps the members of one set to members of another set (Confrey & Smith, 1995; Slavit, 1997; Smith, 2003). While analyzing the first pattern presented in the 'together' session, Cathy asked her partners to see "…the relationship between the number of faces and the number of cubes." The recognition of the correspondence between two data sets leads to the ability to perceive the general term (Radford, Bardini, & Sabena, 2007; Smith, 2003), as Cathy attempted to help her partners do.

Smith (2003) described the subsequent ability to generalize a pattern as a key component of algebraic thinking. For some of the participants, the relationship between position and value was something that they "just thought about" or "just saw." For example, while sharing her answers to Think Pattern 2, Tara explained that "…to get the relationship between the *n* and the value… (she)…realized you have to go n times 2 plus 2 to get the value." For others, the correspondence between the two data sets was not so transparent. Jill described this lack of transparency, stating that "…a lot of times what happens is I can see the relationship between these numbers and I can see the relationship between these but I can't really see it between the two of them." Ponte (1992) and Smith (2003) stated that the struggle to see the relationship was due to the abstract nature of the concept of correspondence, along with the symbolic expressions sometimes associated with it.

*The Idea of Function as a Rule*

Ashley described a function as "…an equation…and you have to figure out what goes into the equation and what comes out." This idea of function as an equation or a rule was communicated through direct references to a rule or through statements that indicated the individual understood that a rule "always works." The idea of function as a rule that always works often operated in conjunction with the idea of function as a relation. Jill applied both ideas to test her little theories on the relationship between position and value in a table. This potential connection between the idea of function as a rule and as a relation was also exemplified in Tara's statement that she "…knew to make a rule, it had to relate to the *nth* term." Recursive rules, like the one Shelly wrote to describe think pattern 1 ("You always draw or add an additional square to each side to complete the next figure"), are also rules that "always work but these rules do not consider the idea of function as a relation.

Jill had described a function as "…sort of like a way that you put in a certain number and get out a certain number." The idea of function as a rule describes the action of a function machine, a model that is often used to introduce the concept of function in school mathematics. Breidenbach, Dubinsky, Hawks, and Nichols (1992) defined an action as "…any repeatable physical or mental manipulation that transforms objects (e.g., numbers, geometric figures, sets) to obtain objects" (p. 249). Smith (2003) noted that the rules written by teachers retained the actions used to construct the figures in pictorial growth patterns. This phenomenon was most pronounced in the rules written to generalize the second 'together' pattern. Matt and Ashley worked independently on this particular pattern and both seemed to arrive at a rule around the same time. Ashley was in the process of stating a rule based on the third term in the sequence ($6 + (n - 2)2$ plus 4) when Matt described a way to find the 85$^{th}$ term (85 times 2 plus 6). After listening to Matt's explanation, Ashley exclaimed, "That is so simple now!" Each had focused on different aspects of the same model to arrive at equivalent expressions for the general term.  Matt viewed the constant part of the pattern (the two sets of three blocks on each end) in the same way as Cathy who had stated the rule '$2n + 6$'. His partner, Ashley, on the other hand, noticed that the four corner blocks around the middle were always there. These differences in perceiving which part of the model stayed the same led to different ways of expressing the general term.

*The Idea of Function as a Process*

The idea of function as a process was communicated through references to how an individual might find a pattern, rule, or relation. For example, after Tara had described the rate of change in Identifying Patterns 2, she asked, "…what could be our pattern…if we have *n* red blocks…how do we get a pattern?" Her question of how to get a pattern was not referencing how to find the rate of change, instead Tara was referring to "how to get a relationship…like a rule…from (the) red block to (the) yellow blocks." The approaches taken by these pre-service elementary teachers in their quest to "…get a pattern…get a relationship…get a rule…" constituted the idea of function as a process.

As they engaged in the process of generalizing patterns, the pre-service elementary teachers identified patterns of stasis and change and then used these patterns to extend the sequence and locate a non-sequential term. Using the idea of pattern also opened a space for viewing the relationship across the table and made it possible to write a rule for describing this relationship. All of the acts these pre-service teachers engaged in while pattern finding are included under Smith's (2003) definition of algebraic thinking. Smith defined algebraic thinking as "…the kinds of generalizing that precede or accompany the use of algebra…" (p. 138). Zazkis and Liljedahl (2002) explained that students are engaged in the process of algebraic thinking when they examine and extend patterns, make note of how two data sets are related, and formulate a rule that defines this relationship. These explanations make it possible to view the idea of function as a process as a form of algebraic thinking.

*Summary: How Pre-service Elementary Teachers Communicate the Idea of Function*

Examining the idea of function as a process revealed the interdependency of all four ideas communicated by this group of pre-service elementary teachers. The illustration of the interdependency of these four ideas of function, displayed in Figure 3, can serve as a model for understanding the nature of algebraic thinking. The idea of function as a process is represented by the outer circle and was equated with the process of algebraic thinking. The approaches used while generalizing patterns, combined with the other three ideas of function constitute this idea of function as a dynamic process. The process of examining patterns usually began with the analysis of change. This analysis of change is depicted by the largest of the three inner circles and represents the idea of function as a pattern. The user operating under the idea of function as a pattern considers how changes in position relate to changes in the figures used to represent the sequence. This coordination of change, or analysis of covariation, often led to the idea of function as a relation.

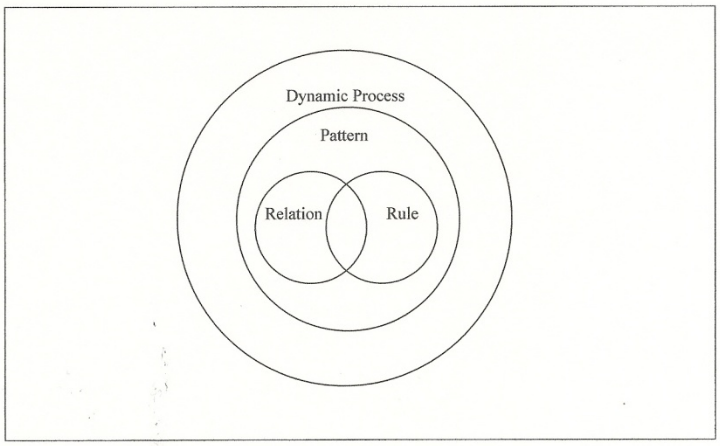



*Figure 3. Model of the interdependency of the ideas of function as represented in algebraic thinking.*

The two overlapping, inner circles represent the ideas of function as relation and as a rule. The attempt to forge a relation between the two data sets did not always result in a rule, such as in the misapplication of proportional reasoning sometimes used by Matt. In the same manner, not every rule described the relationship across the table, as in the recursively-defined rule Shelly wrote for think pattern 1. Therefore, these two ideas of function, though not disjoint, are not subsets of each other. The region where all four ideas intersect represents the kernel of algebraic thinking, the idea of generalization as stated by Kaput (2000) and Smith (2003). Radford, Bardini, and Sabena (2007) described the "…idea of generalization as a shift of attention that leads one to *see* the general *in* and *through* the particular" (p. 525 – 526).

The idea of generalization incorporates the ability to describe the value or construction of any term in a sequence. With pictorial growth patterns, this would mean one is able to visualize the construction of any figure based on its position. With quantitative relationships, one would need to be able to visualize the arithmetic or, as Jill's partner described it, see "…the math in it." The next section discusses the idea of generalization by explaining the approaches adopted by this group of pre-service teachers that either enabled or blocked them from visualizing the general term.

## The Process of Generalizing Patterns

The pre-service elementary teachers participating in this study demonstrated their understanding of function by engaging in the process of generalizing patterns. The specific approaches they adopted in this process were examined in the previous chapter so that comparisons could be drawn between successful and unsuccessful applications of a specific approach. Successful applications were labeled catalysts in the process of generalizing patterns whereas unsuccessful uses were determined to represent roadblocks. This section categorizes the application of catalysts and roadblocks as involving either an integrated approach or an incomplete approach to generalizing patterns.

### *An Integrated Approach to Generalizing Patterns*

The process of generalizing pictorial growth patterns usually began with the analysis of how the model was changing from one term to the next. These changes were often described as a unit rate of change such as in Tara's description of the pattern of change in Identifying Patterns 2 ("…so one red block and two yellow blocks each time you go down"). The impetus to switch from analyzing sequential change to an analysis of correspondence was provided by the push to find an "easier way" to locate a non-sequential term. The primary participants in this study described looking for an easier way as "what to do to this to get these."

Formulating a relationship across the table was achieved using various approaches. Jill resorted to checking “little theories” for connecting the values across the table. Her approach was based on the idea of function as a relation and operated as a catalyst because Jill understood that her rule had to work for all data pairs. Her partner sometimes looked for a pattern of differences across the table, as did Ashley. They were successful because they were able to visualize the connection between the changing pattern of differences and the position of the term in the table. Both of these examples illustrate how the successful application of a specific approach was dependent upon an understanding of function as a pattern, as a relation, and as a rule.

With pictorial growth patterns, the pre-service elementary teachers usually attended to the construction of the model. They analyzed how the figures changed from one term to the next and how they stayed the same. By identifying how the model stayed the same, they were able to separate the variable part of the model from the part that “was always there.” They could then look for a connection between the position of the figure and the part that always varied. For example, Tara thought about how she had determined the 15$^{th}$ term in think pattern 1 to visualize what any term in the sequence would look like. Others applied proportional reasoning to describe the variable part of the model as evident by Cathy’s explanation that “it’s red times that and that” in regards to Identifying Pattern 2.

The analysis of the structure of a pattern as a way to visualize the general term was described by Radford, Bardini, and Sabena (2007) as one of four potential schemes the participants in their study used for transcending the general. The researchers explained that “…in order to perceive the general, the students…have to bring to the fore some aspects of the figure (*emphasis*) and leave other aspects behind (*de-emphasis*)" (p. 522). The verbal rule that Jill wrote for the first think pattern omitted the presence of the constant center square, perhaps because she was focusing on the variable part of the model. While describing the 8th term in the pattern of H's, Jill explained that she was "…just kind of forgetting about these guys…that's where the little plus two would come from if you're writing the rule." Her focus was on the unit of change and not on the constant part of part of the model.

*Summary.* There were numerous other examples in the literature linking the analysis of the construction of a figure to the successful generalization of patterns (Billings, Tiedt, Slater, 2007; Billings, 2008; Smith, 2003; Steele, 2005). Smith (2003) and Billings (2008) emphasized how this focus on structure (stasis) and patterns of change are important tools in generalizing patterns. The analysis of stasis and change often resulted in the identification of the differences and similarities between each figure. This analysis incorporated the idea of function as a pattern. The quest to forge a connection between position and construction involved looking for a relationship across the table. This process enveloped the idea of function as a relation and sometimes a rule. The application of an integrated approach to analyzing change and the connection between position and construction made it possible for these pre-service elementary teachers to describe the general term.

*An Incomplete Approach to Generalizing Patterns*

Several of the strategies employed under the idea of function as a process were labeled as both catalysts and roadblocks to generalization. The label of roadblock was marked by the failure to incorporate the ideas of function as both a pattern and a relation. For example, Shelly's rule for think pattern 1 generalized the pattern of change but not the relationship across the table ("You always draw or add an additional square to each side to complete the figure."). Her recursively-defined rule emerged from the way she determined the construction of the 15th term in the table through the repeated application of the rate of change. In the second think pattern, Shelly attempted to look across the table to describe a pattern of differences between the two data sets. However, she omitted the idea of function as a pattern by neglecting to coordinate the changes in both data sets.

The over-reliance on recursive reasoning was identified as a roadblock to generalizing patterns in the literature (Moss, Beatty, Barkin, & Shillilo, 2008). Shelly's application of recursive reasoning was present in her description of think pattern 1, as well as in her attempt to generate the 36th term in Identifying Patterns 3 described in the previous chapter. Although Shelly utilized the idea of pattern as a rate of change, she did not "…really understand what the *nth* term meant…to the fullest extent." As a result, her approach to generalizing pattern 3 lacked the idea of function as a relation. Smith (2003) and others (Ponte, 1992) identified the abstract nature of the idea of function as a correspondence between two points as a source of struggle for students of algebra. Smith (2003) asserted that this struggle is perpetuated by the emphasis on function as a correspondence at the expense of developing an understanding of function as covariation.

Shelly's lack of understanding what the *nth* meant was often marked by the steps she followed while analyzing these patterns. Shelly would first describe the sequential terms in the table and then focus on finding the value of the non-sequential term that appeared next. In her attempts to find this value, Shelly focused on the quantitative relationships represented in the table and did not analyze the general construction of the figures used to represent these relationships. This difficulty relating quantitative relationships to other representations was identified as a common roadblock to generalizing patterns experienced by preservice teachers (Stump & Bishop, 2002; Stump, Bishop, & Britton, 2003).

Matt's use of proportional reasoning was a roadblock in some instances, but a catalyst under other circumstances. Like Shelly, Matt's intent was not to describe the general term but to find a way to produce the requested non-sequential term in the table. When Matt focused on the physical construction of the figures in a pattern he usually found a direct way to describe a non-sequential term. By making use of the visual, Matt was able to identify the constant and varying parts of the model. He was then able to apply proportional reasoning to forge a connection between the position of the figure and the part of the model that was changing. When Matt successfully applied proportional reasoning to describe a specific term in the sequence, his description was easily transcribed into a general rule. Although Ashley usually re-stated his descriptions as a general rule, Matt did demonstrate his understanding of any figure in the sequence could be constructed. Billings, Tiedt, and Slater (2007) identified how attending to the physical structure of a model was a key predictor of success in the process of extending and generalizing patterns.

On the other hand, when Matt only focused on the quantitative relationship represented in a t-table, his tendency to apply proportional reasoning failed him. In these cases, he would simply work with the last known pair in the table in his attempt to find a non-sequential term, a tendency Kaput associated with a pre-algebraic understanding of function. Although Matt was working across the table to establish a relationship between position and value, he did not consider how the patterns of stasis and change did not create a simple proportional relationship. His application of whole object reasoning was identified as a common tendency among fourth graders in their attempts to analyze linear situations such as this which involved both a constant and a rate of change.

*Summary.* Both Matt and Shelly usually began the task of generalizing patterns with the intent to describe a specific term instead of the general term. This was not necessarily a roadblock to generalization if they were able to use their description of a specific term to explain the construction of any term in the sequence. Matt accomplished this task on his own with the first think pattern, primarily because he was able to link the position of the figure with its construction. Shelly also extended this pattern to find a specific term, but her use of the single process of analyzing change did not result in a general rule. This incomplete approach to generalizing patterns lacked the contributions of the ideas of function as a pattern and as a relation. As a result, the individual was not able to effectively use a specific term in their attempt to explain the general.

*Summary: How Pre-service Elementary Teachers Demonstrate Their Understanding of Function*

This group of pre-service elementary teachers demonstrated their understanding of function through the idea of function as generalization. This idea of function was formed by the overlapping ideas of function as a process, a pattern, a relation, and a rule. The successful application of the idea of function as a process to generalize a pattern was dependent, primarily, on the understanding of function as a pattern and a relation. The idea of function as a rule that “always works” often acted in conjunction with function as a relation to explain the construction of the general term.

An integrated approach entailed the coupling of a strategy for generalizing a pattern with the ideas of function as a pattern, a relation, and sometimes a rule. The incomplete approach omitted one or more of these three ideas of function, most notably the ideas of function as a pattern and function as a relation. These two ideas of function, as a pattern and as a relation, correspond with the concept definitions of function as covariation and correspondence, respectively. Slavit (1997) and Smith (2003) asserted that the combination of these two concepts of function create a more cohesive understanding of function.

Smith (2003) explained that a covariational approach places an emphasis on the dynamic actions that create a pattern. Attending to these actions makes it possible to visualize the repeated operations that created the relationship between two data sets (Confrey & Smith, 1995; Smith, 2003). On the other hand, attending to the invariant features of a pattern makes it possible to consider the similarities between terms in a sequence. The fusion of these two ideas of function works in tandem to flush out the image of the general term which can then be described either through the use of symbols or words. The acts of understanding the ideas of pattern and function were visible in the pathways taken to describe the general term. The next section describes these acts and connects them to theory on the nature of understanding in mathematics.

## Pre-service Elementary Teachers' understanding of Pattern and Function

The primary goal of this study was to explain the nature of pre-service elementary teachers' understanding of function as manifested in the ways they communicated and demonstrated their understandings while engaged in pattern-finding activities. The decision to use pictorial growth patterns in the investigation was based on their known potential for supporting the development of algebraic thinking (Billings, Tiedt, & Slater; 2007; Smith, 2003; Steele, 2005). The analysis of how pre-service elementary teachers communicate and demonstrate their understanding of function while studying patterns revealed the presence of a major avenue towards generalizing a pattern. Progress along this avenue was either assisted by catalysts in an integrated approach or hindered by the roadblock of an incomplete approach.

*Identification and Discrimination: Analysis of Covariation*

The entry point along the avenue began with the analysis of change. Through the analysis of change, the participant was usually able to make note of how the shape changed from one figure to the next. For example, Tara explained the change in Identifying Patterns 2 as "…one red block and two yellow blocks each time you go down." This identification of a pattern of differences usually led to the examination of the commonalities between all figures in the sequence, as when Tara noted that "…the corners are always going to be four." Their attention shifted to the overall structure of each figure as they began to consider how they could construct a non-sequential term. This shift initiated the analysis of the relationship between position and construction, thus keeping them on track towards generalizing a pattern.

*Generalization and Synthesis: Analysis of Correspondence*

The analysis of relationship, or correspondence, was independent of time or sequencing, unlike the analysis of change. Instead, the focus was on how the overall structure related to the position of the figure in the sequence. Progress along the avenue was made possible by employing an integrated approach which enabled the user to discriminate how the ways in which the model changed and how it stayed the same contributed to the construction of any figure. Through this act of discrimination, the individual was able to describe the general term.

Although some of the participants, like Matt, described this relationship through a particular term, others (e.g., Tara, Cathy, and Ashley) expressed their thinking in a more general way.

Billings, Tiedt, and Slater (2007) described a similar progression towards generalizing pictorial growth pattern in their study of elementary students' analyses of patterns. They explained how these students began with a description of the sequential change between figures which they used to describe the construction of the next figure in the pattern. This stage, which they equated with the analysis of covariation, culminated with the description of the differences and similarities between each figure. Building upon this knowledge, the students were able to progress to the analysis of the correspondence between position and construction of the figures in the pattern. This analysis made it possible to take that connection and express it in more general terms.

*Four Acts of Understanding Linear Patterns*

The progression along this avenue towards generalizing a pattern was marked by three key levels of understanding the nature of the relationship represented by the pattern. The first level of understanding involved the identification of the pattern of change which often led to the second level of discriminating between how the model changed and how the model stayed the same. This level of discrimination pointed towards the structure of the relationship, culminating with the generalization of the pattern. These levels of understanding a pattern were described by Sierpinska (1992) as representative of the nature of understanding in mathematics.

According to Sierpinska (1992), there are four acts of understanding in mathematics. The first act of understanding, identification, occurs when an individual recognizes that an object is of special interest. In the case of exploring linear patterns, the individuals in this study first attended to the constant rate of change. The second act, discrimination, occurs when the individual distinguishes both the differences and commonalities between two objects in mathematics. The two objects, in the case of linear patterns, represented the elements of stasis and change. Sierpinska noted that the third act of understanding, generalization, was made possible as the individual expands these notions to other settings. This act of understanding a linear pattern was marked by the shift from recognizing the elements of stasis and change in specific terms to the perception of how these elements were represented in the general. Sierpinska included a fourth level of understanding in mathematics; the level of synthesis. At this level, the individual formulates a cohesive concept by noticing the properties shared by all of the objects under study. Someone who recognized the properties shared by all of these linear patterns and applied them in their analysis of patterns, would have developed a cohesive concept of linear patterns and function.

In the pictorial models used in these patterning activities, the constant part of a linear equation was visible and could be separated from the variable part. By identifying the constant and the rate of change, the individual at the synthesis level could quickly generalize the pattern using the properties of a linear function. Cathy progressed from the level of generalization to that of synthesis after her experiences with Identifying Pattern 2. She explained how she arrived at that point in her written response to the first reflection prompt:

> "Once I realized that the 6 blocks on each end were constant in every figure, this became much easier to solve. I then realized that for every red block that was added, 2 yellow blocks were added, *2n*, and the 6 constant yellow blocks on the end, making the formula, $2n + 6$.

After that point, she stopped trying to get her partners to just see the relationship between position and construction and instead asked them to identify the constant and the rate of change. She then used the constant, *b*, and the rate of change, *m*, to help them write the general rule for a linear pattern as the rate of change times the term number plus the constant.

Under Slavit's (1997) property-oriented view of functions, students assimilate properties of functions through their experiences with different classes of functions. The recognition of these properties creates a library of functions which contributes to the formation of a more complete understanding of the concept of function. The experiences this group of pre-service elementary teachers had with linear patterns initiated the assimilation of the properties of linear functions, one of the basic building blocks in the library of functions. In many ways, the assimilation of these properties emerged through the interplay between explanation and understanding as represented by the hermeneutic circle. Jill explained this phenomenon in her response to the second reflection prompt (Figure 22), stating how her experiences not only strengthened her ability to find patterns, "…but also to EXPLAIN how…(she)…found them." She went on to offer the example of how she discovered "what basic components the sequence should follow." Like others in the group, Jill had begun to notice the properties of a linear sequence.

*Summary*

Understanding a linear pattern begins with the identification of the pattern of change. The shift from the analysis of change to the study of structure led to the ability to coordinate both the variable and invariant features of the model. This act of discrimination often led to the ability to generalize the pattern of stasis and change. One of the secondary participants described this progression from identification to generalization in her definition of a function: “To find the equation in the pattern, examine the first few figures or numbers in the set and evaluate similarities and differences to arrive at an equation for continuing the pattern.” Through their multiple experiences with linear patterns, some the pre-service elementary teachers engaged in these activities were also able to recognize the properties of the class of linear patterns. The assimilation of these properties could be assembled so as to create the concept of linear function.

## Summary of Findings

The purpose of this study was to examine pre-service elementary teachers’ understating of pattern and function so as to better understand how to prepare then for supporting the development of algebraic thinking in their own students. To address this purpose, the texts produced by six pre-service elementary teachers while they were engaged in pattern finding were collected and analyzed. The design of this study was based on the assumption that understanding in mathematics is a hermeneutical process that evolves through the recursive relationship between explanation and comprehension. This assumption implied that access to pre-service elementary teachers’ understanding of pattern and function could be obtained, at least in part, by analyzing the explanations each offered while engaged in pattern-finding activities. These texts were analyzed under the same assumptions of hermeneutics, so that the researcher's understanding evolved over time through the interplay between explaining what was understood and interpreting what was explained.

The first research question asked how pre-service elementary teachers conceptualize and communicate the idea of function while engaged in pattern-finding activities. Four overlapping ideas of function emerged through the analysis of their conversations and written texts associated with the task of pattern finding. Further examination of these four ideas revealed that the intersection of all four represented the kernel of algebraic thinking, the idea of generalization. All five of these ideas (process, pattern, relation, rule, generalization) formed a model of algebraic thinking.

The second research question addressed how pre-service elementary teachers demonstrated their understanding of function while engaged in the process of generalizing patterns. The approaches taken by the six primary participants in this study were identified and categorized as either catalysts or roadblocks. An approach or strategy served as a catalyst when coupled with the ideas of function as pattern (covariation), relation (correspondence), and rule. This combination of a strategy with two or more other ideas of function was labeled an integrated approach to pattern finding. Roadblocks occurred when one of these ideas was missing form the explanations offered by an individual, particular the ideas of function as a pattern or a relation. The employment of a roadblock was labeled an incomplete approach to pattern finding due to the missing elements of pattern and relation.  In summary, successful generalization appeared to be predicated upon the adoption of an integrated approach versus an incomplete approach.

The primary intent of this investigation was to explain the nature of pre-service elementary teachers' understanding of pattern and function. This intent was accomplished, in part, through the analysis of how pre-service elementary teachers communicate and demonstrate their understanding of function while engaged in the process of pattern finding. The understanding of linear patterns followed a route made possible by the understanding of function as a pattern (covariation) and a relation (correspondence). The first act of understanding a pattern involved the identification of the pattern of change. This was often followed by the discrimination between what changes and what remains the same within the structure of each figure in a pattern. Through the coordination of these two features of a pattern, the individual is able to visualize the construction of the general term. The repeated act of generalizing patterns makes it possible for the individual to begin to notice the properties that connect all linear patterns. In this final act of synthesis, the individual formulates a cohesive understanding of liner patterns.

The interpretations offered in this discussion paint a picture of how pre-service elementary teachers communicate their understanding of pattern and function while analyzing linear patterns. This picture includes a model of how the ideas of functions communicated by this group of pre-service elementary teachers incorporated the important features of algebraic thinking. A description of how pre-service elementary teachers demonstrated their understanding of functions while analyzing patterns was also offered, including a critique of the approaches taken to complete the task. The final part of this picture described four acts of understanding linear patterns, as experienced by this group of pre-service elementary teachers. The next section of this chapter considers the implications of the findings discussed here.

## Implications

The purpose of this research endeavor was to develop an understanding of how to better prepare pre-service elementary teachers for the task of supporting the development of algebraic thinking in their own students. Stump and Bishop (2002) had warned that the success of this task was dependent upon pre-service teachers' understanding of "…algebra as a way of thinking, a way of working with patterns that occur every day" (p. 1912). The idea that experiences with patterns would lead to an understanding of algebra as a way of thinking was one of the assumptions made while designing this study. The study first sought to understand how pre-service elementary teachers communicate and demonstrate their understanding of function while engaged in pattern-finding activities. The findings were then used to describe the nature of pre-service elementary teachers' understanding of pattern and function.

Much of what was learned from this investigation confirmed what other researchers have noted about the processes elementary students employ to generalize patterns. The common pitfalls to generalization that were identified as roadblocks in this investigation were also identified as problem areas for the students these pre-service elementary teachers will have in their future classrooms. The fact that not all pre-service elementary teachers are equipped with the profound understanding of function that is necessary to support algebraic thinking has also been documented by other research.

However, the results of this study did lead to a potential model of the components of algebraic thinking that are involved in the study of patterns. This model can be used as a curriculum guide for designing productive experiences with patterns and function for the pre-service elementary teacher. The model could also be used as a way to represent the relationships between the ideas of function that are involved in pattern-finding activities. For example, this illustration of algebraic thinking might be used as tool in a mathematics methods class to openly discuss the ways algebraic thinking can be supported in the elementary classroom. Smith (2003) noted that elementary teaches often fail to connect the study of pattern with the idea of function and other big ideas of algebra. This model provides a way to make bring these connections forward and discuss how to support their formation in the elementary classroom.

The four acts of understanding linear patterns may also be used to guide curricular decisions. Progression through these four acts was assisted by the use of pictorial growth patterns and made possible by the coordination of both the changing and the invariant features of the models used to represent a linear pattern. Therefore, the study of patterns undertaken as part of the pre-service elementary teachers' mathematics content courses should include opportunities to work with physical representations of numeric sequences. Too often, the study of patterns includes identifying classes of sequences and using memorized formulas to describe the general term. Not only is the application of substituting values into a pre-specified formula beyond the scope of elementary mathematics, the procedures used do little to develop conceptual understanding of patterns and function.

The progression through these acts of understanding also forms a guide for the types of questions one might pose while students are engaged in pattern-finding activities. For example, Billings, Tiedt and Slater (2007) suggested asking questions about the construction of the next figure in the model and how it will differ or be the same as the previous figure. They also suggested that teachers ask students to build a non-sequential term without applying recursive strategies. Asking these types of questions can help the student find a way to describe the construction of any figure in the pattern. However, to assist the pre-service elementary teacher in the task of building a library of functions, it would also be important to ask them to notice the properties that all patterns of a particular class had in common. By synthesizing the properties of various classes of functions, they will be prepared to help their students make the same connections for themselves.

Finally, the experiences these pre-service elementary teachers had while analyzing the second pattern in the 'together' session indicate a need to vary the level of difficulty in patterning tasks. This particular problem posed a challenge for this group of pre-service elementary teachers because the patterns of stasis and change were "kind of hard to see." As a result, the participants arrived at multiple ways of describing the general term. The use of pictorial growth problems that encourage multiple interpretations can become a source for discussion of topics such as equivalency of expressions and the relationship between the constant and the rate of change in a linear expression.

## Recommendations for Further Study

The results of this investigation open the door to more questions about how the study of patterns can be used to support the development of algebraic thinking, particularly in regards to understanding the idea of function. This present study only considered the ways pre-service elementary teachers communicate and demonstrate their understanding of function while examining linear patterns. Other classes of function were not included in this investigation due to possible interactions between the type of function and level of understanding. This study also adopted a quantitative perspective and did not consider how different representations of the same class of function might influence the process of generalizing patterns.

Further research on how pre-service elementary teachers assimilate properties of functions is recommended. For example, the question of whether or not the four acts of understanding identified here would apply to the study of other classes of function, such as exponential and quadratic, should be investigated. If the process of understanding other classes of functions does follow a similar route, then the interplay between type of function and level of understanding could also be examined.

The pictorial growth patterns employed in this study made it possible for the individual to identify both a constant and a variable part of a model. The question of whether or not the same four acts of understanding would be distinguishable given a different representation of the same sequence is one that should be pursued. For example, are there differences in the approaches taken to generalize a pattern when the data is only presented in a table or when the situation is represented in the form of a word problem instead?

The question of whether or not the traditional approach to studying sequences taken in most algebra textbooks equips pre-service elementary teachers with the tools to support algebraic thinking in their own classrooms was raised in the discussion of these research findings. The answer to that question could be pursued through a comparative study between groups using a traditional approach to studying sequences and groups using a multi-representational approach that includes geometric figures.

These suggestions for future research are made with the goal of finding better ways to prepare pre-service elementary teachers for the task of supporting the development of algebraic thinking in the elementary grades. The layer-cake approach to studying algebra has been blamed for blocking the way to future careers in the high paying fields of science and technology for too many students (Chazan, 2008; Kaput, 2000; Katz, 2007; Kilpatrick & Iszák, 2008). The solution to this dilemma lies in the development of algebra as a way of thinking beginning in the early grades (Chazan, 2008; Kilpatrick & Iszák, 2008). The success of this solution lies in the ability of elementary teachers to abandon the traditional view of algebra and adopt the view of algebra as a way of thinking. This view requires a profound understanding of the connections between patterns, functions, and algebra (Smith, 2003). Therefore, the questions raised here should be pursued by those who are instructed with the task of preparing the pre-service elementary teacher of mathematics.

## Conclusion

The task of preparing pre-service teachers to become effective teachers of mathematics is a daunting one. Identifying what mathematics content knowledge is necessary to teach elementary mathematics, along with the depth of pedagogical content knowledge required, are hot topics among teacher researcher charged with this responsibility. This present study explored these two issues in regards to the ideas of pattern and function through the lens of hermeneutic phenomenology. The use of this perspective entailed the mining of data while a group of pre-service elementary teachers where actively engaged in the process of generalizing patterns. The analysis of the texts they produced while engaged in these activities yielded a rich portrait of how pre-service elementary teachers communicate and demonstrate their understanding of pattern and function.

This portrait of pre-service elementary teachers' understanding evolved over time as individual records were analyzed and interpreted by the researcher. Each new interpretation was examined in light of what was understood about how others in the group had described their experiences with patterning. This integration of parts and whole was presented in chapter VI in the form of common themes associated with the idea of function and common approaches to generalizing patterns. However, the final analysis did not take place until the researcher re-considered what others had said about how individuals conceptualize the ideas of pattern and function. This integration of the parts understood from the present study with the whole of what is understood by others made it possible to complete the picture of pre-service elementary teachers' understanding of pattern and function.

A model of how the ideas of function generated by this group of pre-service teachers overlap to form the basis of algebraic thinking composed part of this portrait of understanding. The re-examination of the literature made it possible to recognize the core of algebraic thinking, the idea of generalization, as represented by the overlapping ideas of function communicated by these pre-service elementary teachers. Similarly, the acts of understanding patterns and function were made explicit by reviewing how others had described the nature of understanding in mathematics. This final step in the hermeneutical analysis of the texts produced by these six participants made it possible to escape the trap of the hermeneutic circle and bring this particular piece of research to a conclusion.

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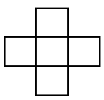
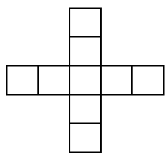
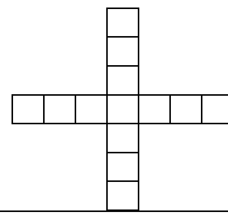
## APPENDIX A

**Title of Project: Preservice Elementary Teachers' understanding of Pattern and Function**
**Principal Investigator(s):** Valerie Sharon, Northeastern State University
**Academic Advisor:** Dr. Patricia Jordan, Oklahoma State University

### Think #1: Pattern Recognition

1. Examine the pattern of small squares below.

| $1^{st}$ Figure | $2^{nd}$ figure | $3^{rd}$ Figure | $4^{th}$ Figure |
|---|---|---|---|
|  |  |  | |

a) Draw the $4^{th}$ figure in the pattern in the space provided.

b) Describe how the model changes (grows)?

c) If you were to draw the $15^{th}$ figure in this pattern, how many squares would you need to draw? Explain how you know.

d) Write a rule for determining the nth term of the sequence.

e) What is the connection between how each figure in the model is constructed and the rule that you wrote (1c)?

**Think #2: Pattern Recognition**

2. Complete the table.

| *n* | Value |
|---|---|
| 1 | 4 |
| 2 | 6 |
| 3 | 8 |
| 4 | 10 |
| 5 | |
| 15 | |
| *n* | |

a) Explain how you determined the $5^{th}$ value in the table (when $n = 5$).

b) Explain how you determined the $15^{th}$ value in the table (when $n = 15$).

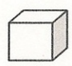
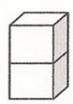
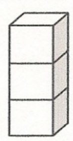
c) Explain how you determined the rule for finding the *nth* term in the table.

**Think #3: What is a function?**

3. Each of these three problems represented functional relationships. How would you explain what a function is to someone who may not have had experiences with these types of problems

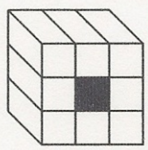
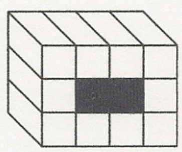
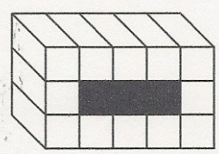
## APPENDIX B

### Identifying Patterns 1

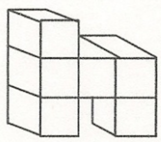
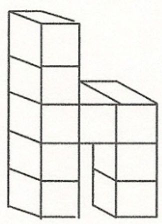
| Term Number (# of cubes) | Model | Written Description | Process Column | Numerical Value (# of faces to paint) |
|---|---|---|---|---|
| 1 |  | A 1 cube-high tower has 5 faces to paint. | | 5 |
| 2 |  | A 2 cube-high tower has 9 faces to paint. | | 9 |
| 3 |  | A 3-cube high tower has ? faces to paint. | | ? |
| 4 | | | | |
| 36 | | | | |
| 100 | | | | |
| *n* | | | | |

Adapted from material received at *Algebra for All* workshop, Fall 1999.

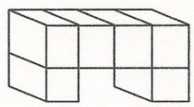
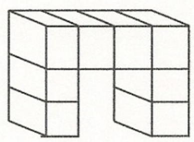
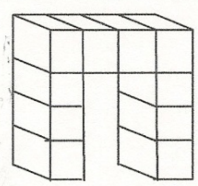
## Identifying Patterns 2

| Term Number (# of red blocks) | Model | Written Description | Process Column | Numerical Value (# of yellow blocks) |
|---|---|---|---|---|
| 1 |  | 8 yellow blocks are needed to surround 1 red block | | 8 |
| 2 |  | 10 yellow blocks are needed to surround 2 red blocks | | 10 |
| 3 |  | 12 yellow blocks are needed to surround 3 red blocks | | 12 |
| 4 | | | | |
| 85 | | | | |
| 100 | | | | |
| *n* | | | | |

# Identifying Patterns 3

| Term Number (# of chairs) | Model | Written Description | Process Column | Numerical Value (# of blocks in chair) |
|---|---|---|---|---|
| 1 |  | Chair with 1 block high legs and 1 block high back. | | 6 |
| 2 |  | Chair with 2 block high legs and 2 block high back. | | |
| 3 | | | | ? |
| 4 | | | | |
| 36 | | | | |
| 100 | | | | |
| $n$ | | | | |

# Identifying Patterns 4

| Term Number (# of bridges) | Model | Written Description | Process Column | Numerical Value (# of blocks in bridges) |
|---|---|---|---|---|
| 1 |  | | | |
| 2 |  | | | |
| 3 |  | | | |
| 4 | | | | |
| 58 | | | | |
| 100 | | | | |
| *n* | | | | |

## APPENDIX C

**Title of Project: Preservice Elementary Teachers' understanding of Pattern and Function**

**Principal Investigator(s):** Valerie Sharon, Northeastern State University
Contact Information: 918.449.6501 or sharon@nsuok.edu

**Academic Advisor:** Dr. Patricia Jordan, Oklahoma State University
Contact Information: 405.744.8142

**Reflect #1:**

Please write a reflection on the activities that you and the members of your group completed today. Before writing your reflection, please read each of the prompts provided and then write a response to each prompt. You will receive a completion grade of 10 points for your reflection per the syllabus. If you agreed to participate in the study, then I, Valerie Sharon, will make a copy of your reflection before returning it to you. Your name will be covered before any photocopies are made. An identifying number will be recorded on the copy and the original will be returned to you.

Prompt 1: Choose one of the problems you worked on today in class. Explain how you were able to determine a rule for finding the *n*th term in the sequence. Include a discussion of any difficulties you and/or the members of your group experienced while working on this problem.

Prompt 2: In reference to the remaining prompts: NCTM (2000) states the following standard for Algebra in the *Principles and Standards for School Mathematics:*

> Instructional programs from pre-kindergarten through grade 12 should enable all students to – Understand patterns, relations, and functions; represent and analyze mathematical situations and structures using algebraic symbols; use mathematical models to represent and understand quantitative relationships; and analyze change in various contexts (p. 158).

In what ways did the activity strengthen your understanding of the concepts included in this standard?

## APPENDIX D

**Project Title:** ***Preservice Teachers' Understanding of Pattern and Function***

Primary Investigator: Valerie Sharon
Faculty sponsor: Dr. Patricia Jordan

**Protocol: Interview**

1. What prior experiences have you had working with patterns like the ones that you completed in these patterning activities?
2. Which types of patterns did you think was the easiest one to work?
3. What features about these patterns made them easier than the others?

Let's look at the problems that you completed on the survey.

4. Explain how you were able to determine the next figure in this pattern.
5. Explain how you were able to determine the fifteenth entry in this model.
6. *If able to determine explicit rule, ask:*

    a) What features about this model made it possible for you to determine a rule or formula? And/or: Explain how you were able to write a rule for finding the nth entry in this model.

    *If not, ask:*

    b) What features about this model made it difficult for you to determine a formula for the nth entry?

7. The activity focused on patterns that formed arithmetic sequences

- Explain how you and/or the members of your group were able to work each of these problems.
- What strategies did you use to determine the pattern rule?
- What connections do you see between the rule you wrote and how each figure is constructed?

8. Explain the methods you used in looking for patterns and writing algebraic expressions to generalize your findings.
9. Describe how you felt when you are asked to work on problems like these?
10. How would you define a functional relationship in everyday terms?
11. How would you define a functional relationship in mathematical terms?
12. How are these two ways of thinking about functions related?
13. How can a teacher build on students understanding of function in everyday terms to develop a concept of function?

Algebraic thinking includes the ability to identify and extend patterns, as well as the ability to recognize and generalize the quantitative relationships between patterns.

14. What experiences during this unit on patterns did you find most helpful for developing algebraic thinking? Explain.
15. How did the experiences you had with patterns contribute to your understanding of functions?

Let's look at the other problems that you worked on:

16. *If able to determine extend the pattern, ask*:

    a) How were you able to find the next term?

*If not, ask:*

b) What features about this model made it difficult for you to find the next term?

17. *If able to determine the fifteenth term, ask*:

How were you able to find the fifteenth term?

18. What features about this model made it difficult for you to find the next term?

19. *If able to determine explicit rule, ask:*

a) What features about this model made it possible for you to determine a rule or formula? And/or: Explain how you were able to write a rule for finding the nth entry in this model.

*If not, ask:*

b) What features about this model made it difficult for you to determine a formula?

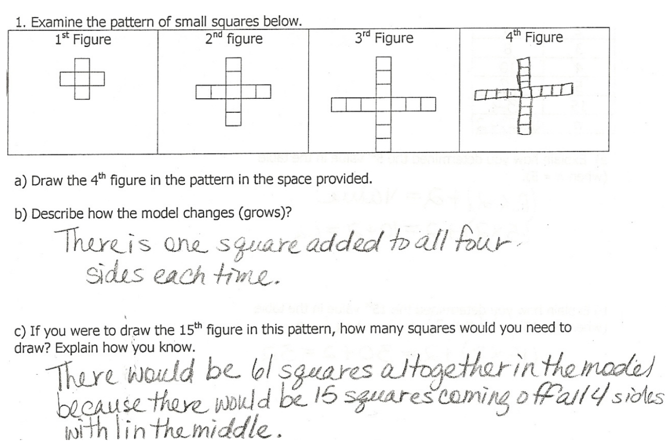
## APPENDIX E

Descriptions of the Pattern-finding Experiences of Tara, Cathy, and Ashley

Detailed descriptions of the pattern-finding experiences of Tara, Cathy, and Ashley are presented in this section. This group of participants wrote symbolic rules for the linear patterns they examined during this investigation. Summaries of their experiences were presented in Chapter IV.

The first think problem required the analysis of an arithmetic sequence represented by a collection of small squares arranged to form a ‘plus sign’. The participants were asked to draw the next figure and then describe how the model changes. Following this description, the participants were requested to predict how many squares it would take to draw the 15$^{th}$ figure. The participants were also asked to write a rule for determining the *nth* term in the sequence and then to explain the connection between their rule and how each figure in the model was constructed.

*Tara’s Responses to Think Pattern 1*


1. Examine the pattern of small squares below.

1$^{st}$ Figure | 2$^{nd}$ figure | 3$^{rd}$ Figure | 4$^{th}$ Figure

a) Draw the 4$^{th}$ figure in the pattern in the space provided.

b) Describe how the model changes (grows)?

There is one square added to all four sides each time.

c) If you were to draw the 15$^{th}$ figure in this pattern, how many squares would you need to draw? Explain how you know.

There would be 61 squares altogether in the model because there would be 15 squares coming off all 4 sides with 1 in the middle.

*Figure 4.* *Tara’s written response to first ‘think’ pattern*

Tara extended the pattern to determine the number of squares it would take to draw the 15th figure and wrote a symbolic rule for determining the *nth* term. Her written response to the first three questions is displayed in Figure 4. Tara's descriptions of how the model grows focused on the sequential changes in the model in terms of its structure. Tara described how she uncovered this pattern during our follow-up interview:

> "Okay…um…I looked at the first figure and I could see there's four on the outside of the middle and if you add one all the way around the outside then it goes to two on the outside and so if you add one all the way around each side as you go across that becomes your pattern…" (*line 45_17I*)

Tara's description illustrates how she analyzed sequential change in terms of how the figure varies from one to the next. In addition, Tara also tuned in on the one constant feature of the model or what she referred to as "the middle."

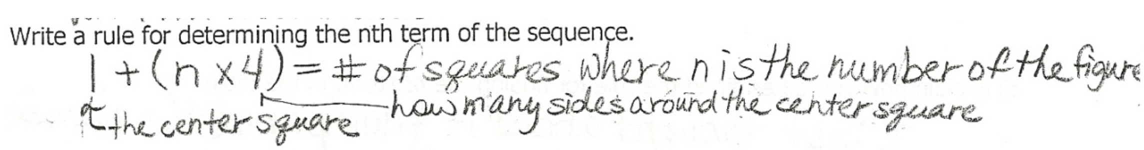
Tara's written response to the third question does not indicate the route she took to determine how many squares it would take to draw the 15th figure in the model. Although she indicated there would be 15 squares coming off each side of the center square, it is unclear whether she made that connection based on the position of the figure or if she simply noted that there would be 11 additional squares added to each end of the figure from the 4th to the 15th. The interview provided an opportunity to clarify how she arrived at the answer.

> Okay…now that one there we didn't know your…uh formula at the time…to plug in…so we were thinking that um…there was automatically going to be…like in the first figure there was one in the middle…with one on each side…the second figure there was one in the middle with two on each side…and then one in the

middle with three on each side. So we were thinking on the 15th figure there would be one in the middle with 15 on each side. (*lines 49, 51, 53, 55-58_17I*).

Her statements indicate that Tara developed a correspondence between the position of the figure and the shape while coordinating the change in structure from one figure to the next. She commented on the presence of the constant one in the middle and stated the connection between the position of the figure in the sequence and the number of squares on each end of the center square.

Her written rule (Figure 5) makes the connection between the variable *n* and the position of the figure in the sequence explicit. In addition, Tara also stated the connection between the structure of the figures and her choices for both the constant and the coefficient in her explanation of the rule she wrote. Thus far, Tara described the construction of the figures, noting which parts of the model changed from one figure to the next and which parts stayed the same. She was then able to connect the pattern of change to the position of each figure in the model and made appropriate use of the variable *n* to write a symbolic rule. However, her description of how she arrived at that rule offers insight into two additional factors that made generalization possible.


Write a rule for determining the nth term of the sequence.

$1 + (n \times 4)$ = # of squares where n is the number of the figure

↑the center square how many sides around the center square

*Figure 5. Tara's symbolic rule for think pattern 1*

During the interview, Tara responded to the question of how building the 15th figure helped her go on to write a rule for the *nth* term.

> Well we came to that and then we kind a back tracked…and then we thought okay now how did I get that…and we did one…which is like the center square…and

> then we did plus and then parenthesis *n* which would be your…figure…your *nth* figure…and then times four cause you're going to be adding four around…one on each side. (*lines 66-70_17I*)

Tara explained how thinking about how she determined the 15th term helped her to write a rule for the *nth* term in the sequence. Tara had also referred to the act of thinking when she described how she determined the number of squares it would take to build the 15th figure ("…so we were thinking on the 15th figure there would be one in the middle with 15 on each side"). Tara went on to clarify the rule the rule she had written during our interview, stating:

> And then…then we tested it to see if it would work…So the one was always there because the center one and then however many times four. Does that make sense? We just kind a back tracked to figure out how we could get it…to work…each time…" (*lines 72, 74, 80, 82_17I*)

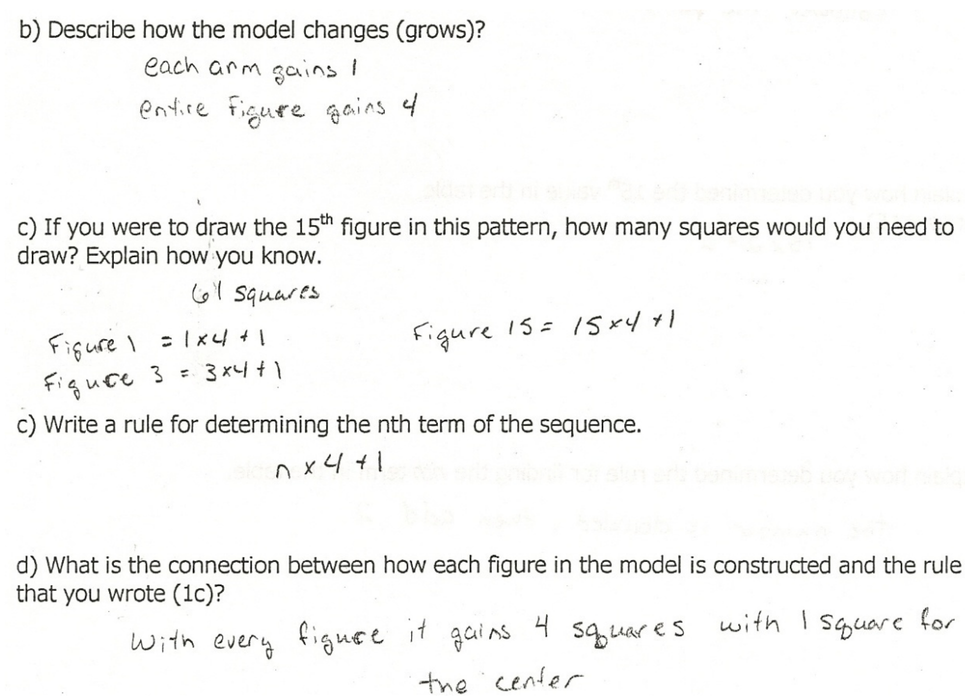
Along with metacognition, the idea of 'testing it to see if it would work each time' played a role in the process Tara used to generalize this pattern.

An interesting side note emerged from the descriptions Tara offered during the interview. At this point in the interview, only her written work over the first think questions was being reviewed. This written work was recorded prior to any group interactions and thus represented her efforts alone at generalizing this pattern. However, in the interview, Tara switched agency from the singular voice to the collective 'we'. She continued to use the collective voice as she explained how she succeeded in writing a rule for the *nth* term of the sequence. The interview took place approximately three weeks after she had completed the first 'think' problems; therefore, it is understandable that time may have blurred the distinctions between which problems she completed on her own and which problems were completed with her partners.

*Cathy and Ashley's Responses to Think Pattern 1*

Like Tara, Cathy and Ashley made use of the variable, *n*¸ to write a symbolic rule by directly linking the variable to the position of each term in the pattern. There were a few differences and/or distinctions in how they went about the process of forming this correspondence that are worth noting.

Cathy's responses to the questions associated with the first 'think' pattern are displayed in Figure 6. In her explanation of how many squares it would take to draw the $15^{th}$ figure in the model, Cathy recorded the arithmetic required to determine the number of squares it would take to build the first, third, and fifteenth figures in the sequence. The rule she wrote is symbolic and is directly linked to the arithmetic shown. Cathy's explanation of the connection between the rule she wrote and how the model was constructed breaks the rule down into two parts: a variable part based on how the figure changed (incremental change) and a constant part based on how the figure stayed the same. She summarized the connection between the two parts of her formula during the follow-up interview, stating "Well…the plus one is the initial center square…that is repeated every time and then the four that you're multiplying by is the number that it increases with for each term" (*lines 24, 26_18I*). This explicit connection to the rate of change (slope) and constant of the linear pattern was also evident in the explanations she offered to her partners, as Cathy stated, "…and then the connection was you add four squares for every figure and then one square for the center" (*line 27_18TG*).



b) Describe how the model changes (grows)?

each arm gains 1

entire Figure gains 4

c) If you were to draw the 15th figure in this pattern, how many squares would you need to draw? Explain how you know.

61 squares

Figure 1 = 1x4+1

Figure 3 = 3x4+1

Figure 15 = 15x4+1

c) Write a rule for determining the nth term of the sequence.

n x 4 +1

d) What is the connection between how each figure in the model is constructed and the rule that you wrote (1c)?

With every figure it gains 4 squares with 1 square for the center

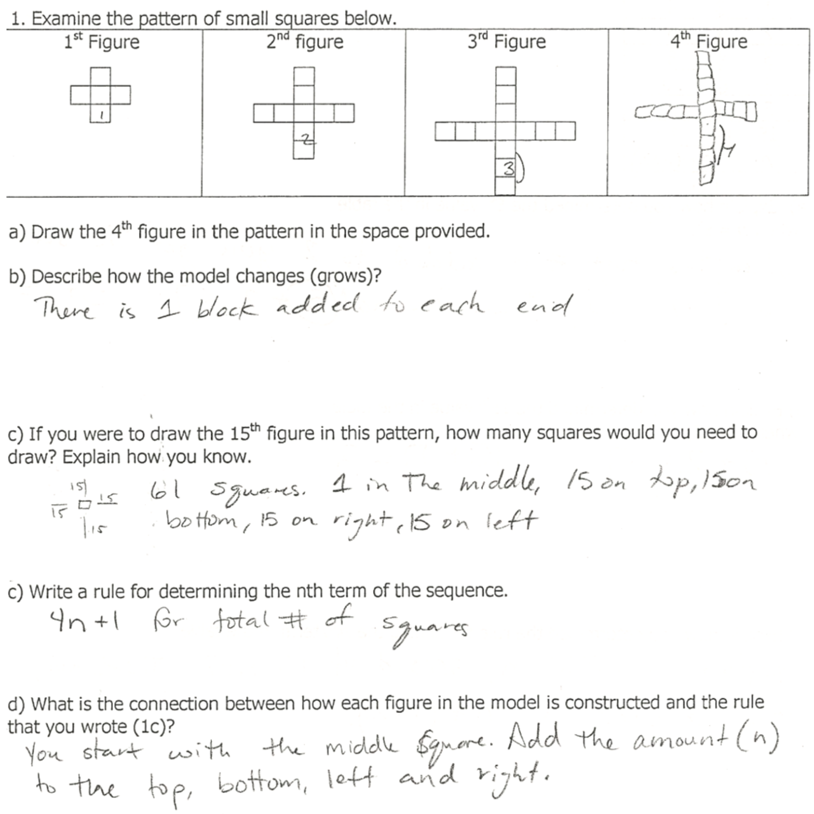
*Figure 6. Cathy's written responses to think pattern 1*

Ashley's responses to the questions associated with the first 'think' pattern are displayed in Figure 7. Her explanation of how to draw the 15$^{th}$ figure is more specific in terms of spatial directions than those offered by Tara or Cathy. Ashley drew a diagram of the fifteenth figure to explain her response to how many squares it would take to build that figure. She also used the general structure of the figure to explain how she determined the rule. Ashley expounded upon her explanation in the follow-up interview.

> Let's see…well the plus one is because you always have the one in the middle…um and then the four…each time you have the four sides and each time… you add four more…so two times four…three times four…yeah…(*lines 23, 25_25I*).

The responses of Cathy and Ashley illustrate how the process of determining a specific term can lead to the development of a rule for finding the *nth* term. Cathy actually recorded the arithmetic used to determine the number of squares needed to construct the

15th figure and then replaced the variable part of the expression used with the symbol *n*. Ashley used a labeled diagram for the 15th figure which included the number of squares on each end of the center square. The rule she wrote embodies this diagram by replacing the quantity of squares with a variable. Their success was fueled by their illustrations of the entire process needed to construct a specific figure.


1. Examine the pattern of small squares below.

1st Figure | 2nd figure | 3rd Figure | 4th Figure

a) Draw the 4th figure in the pattern in the space provided.

b) Describe how the model changes (grows)?

There is 1 block added to each end

c) If you were to draw the 15th figure in this pattern, how many squares would you need to draw? Explain how you know.

61 squares. 1 in The middle, 15 on top, 15 on bottom, 15 on right, 15 on left

c) Write a rule for determining the nth term of the sequence.

4n+1 for total # of squares

d) What is the connection between how each figure in the model is constructed and the rule that you wrote (1c)?

You start with the middle square. Add the amount (n) to the top, bottom, left and right.

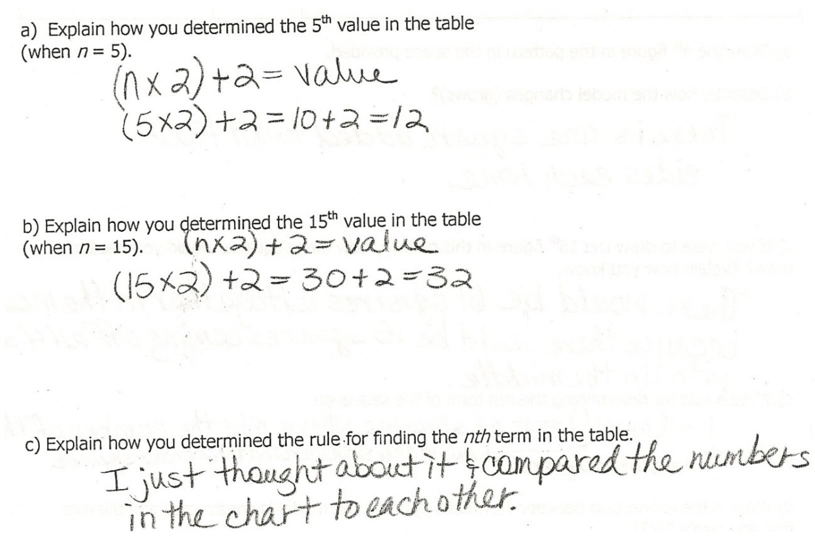
*Figure 7. Ashley's written responses to think pattern 1*

*Descriptions of Experiences: The Second Think Pattern*

The second 'think' problem displayed an arithmetic sequence in a t-table. Participants were asked to determine the value of the next term (5th), as well as the value of the 15th and *nth* terms of the sequence. A description of Tara, Cathy, and Ashley's responses in regards to this patterning activity is presented in this section.

*Tara’s Responses to Think Pattern 2*

Tara’s written responses to the questions associated with the table are displayed in Figure 8. Her written work shows the application of the general term to find the 5th and 15th terms of the sequence.



a) Explain how you determined the 5th value in the table (when $n = 5$).

$(n \times 2) + 2 = \text{value}$

$(5 \times 2) + 2 = 10 + 2 = 12$

b) Explain how you determined the 15th value in the table (when $n = 15$).

$(n \times 2) + 2 = \text{value}$

$(15 \times 2) + 2 = 30 + 2 = 32$

c) Explain how you determined the rule for finding the *nth* term in the table.

I just thought about it & compared the numbers in the chart to each other.

*Figure 8. Tara’s written responses to think pattern 2*

From Tara’s written responses, it is clear that she established a relationship between the position of the term in the table and its value. However, the reasons why she sought to establish this relationship had to be reconstructed from her conversations during the ‘pair’ session and the follow-up interview. For example, when describing her thinking to her partners, she said:

> Okay I got mine…by… I first tried out doing 4 + 2 is 6 and then 6 + 2 is 8 but then to get the relationship between the *n* and the value I realized you have to go *n* times 2 plus 2 to get the value. So one times 2 is 2 plus 2 is four or five times 2 is 10 plus 2 is 12… or *n* times 2 plus 2 is your value (*Line 4_17TG*).

Her response indicates that initially Tara noted the changes that were occurring as she moved down the table along one data set. However, as she moved down the table, she realized that she had to “get a relationship between the *n* and the value.”  To come up with a rule, Tara wrote that she ‘just thought about it and compared the numbers in the chart to each other.”  The interview offered an opportunity to ask her to explain why she started to look across the table instead of just down the table.

123. R: yeah…Can you kind of see …can you remember what maybe made you realize that it’s …how to …how to make that go across the table?

124. T: Um….Okay…

125. R: how you go from here to here…why you said that you have to double it and add two.

126. T: right…um because we knew that to write a rule…or…I was assuming to write a rule it had to have something to do with the *nth* figure

127. R: Okay…

128. T: so I was trying to make it make sense to do it with the *nth* figure so and how they would all work out each time…so I was thinking okay 4 times 2 plus 2 works and then 3 times 2 plus 2 works and I think I just reasoned it out that way because I knew that to make the rule it had to pertain to the *nth* figure…

129. R: right…it had to pertain back to the *nth* figure

130. T:  So I had to be able to……and this way I couldn’t make a rule…

131. R: No

132. T: To say 6 plus 2 and 8 plus 2…I was saying that but I couldn’t get the rule so I had to see that it related this way

133. R: Uh huh…yes…so I assume you noticed that this is like two more than double?

134. T: yeah

As in the ‘think’ pattern 1, the knowledge that a rule had to “work out each time” played a role in the process of generalizing the pattern in ‘think’ pattern 2. Tara abandoned the task of looking down the table because, as she stated, that way she “couldn’t make a rule.” Instead, she began to look for the relationship across the table as she “knew that to make a rule it had to pertain to the *nth* figure.” As in the first ‘think’ problem, Tara linked the position of the term with its value by using specific ordered pairs, i.e. the second and third terms, to determine a correspondence. She then made appropriate use of the variable, *n*, to write a symbolic rule for the general term.

*Cathy and Ashley’s Responses to ‘Think’ Pattern 2*

Cathy and Ashley also used algebraic symbolism to write a rule for the *nth* term in the second ‘think’ pattern. Cathy did not share the process she used to arrive at a rule for describing the second think pattern with her partners, Jesus and Daniela. However, she did try to help Daniela complete the table based on the pattern of differences between terms in the table that Daniela had recorded.

37. C: Okay well if you look at (laughs)…okay you…you’re going in the right direction but what is seventeen in relationship to the fifteen?

38. D: Two.

39. C: Yeah…so you’re adding two…so you’re taking the original number and you’re adding itself.

40. D: Uh huh.

41. C: And then you're adding two…so if you put that as an equation and simplify it then you get two times the original number…plus two.

During the interview, Cathy tried to explain how she came up with the rule for this pattern during the 'think' session.

28. C: yeah (laughs) um well the…the term number is multiplied by 2 and then you just add two

29. R: okay…do you think you just kind of played around with that a little bit or just saw that it was two more than double

30. C: I kind of just saw it

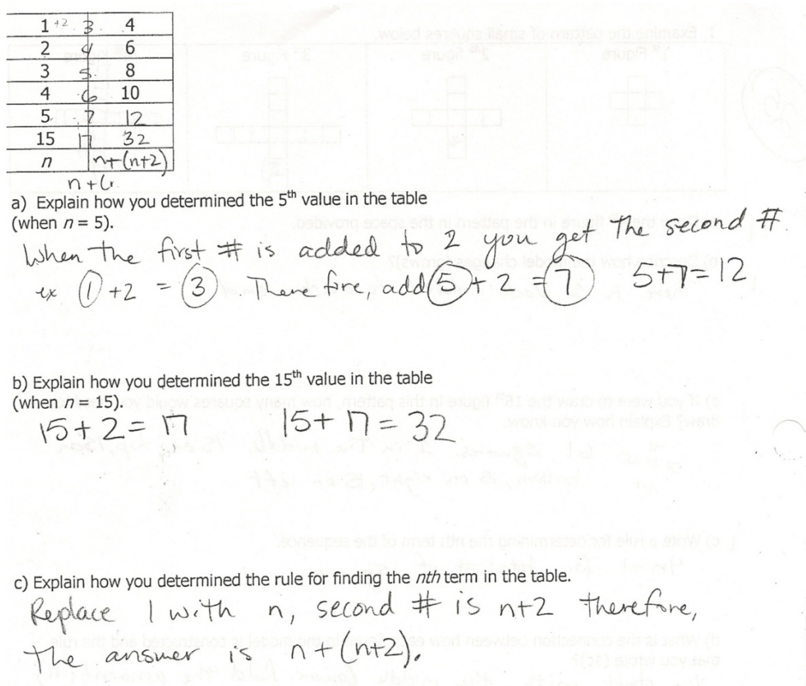
31. R: okay…so good number sense.

Perhaps Cathy was able to arrive at her rule of $2n + 2$ by trial and error or perhaps by 'good number sense'.

Ashley looked for a way to connect the values on the left to those on the right as well. Her written responses to the second think pattern are displayed in Figure 9. She clarified her response to the first question of how she determined the $5^{th}$ value in the table during her paired conversation with Matt.

> Okay. Um…I wrote that when the first number is added to two you get the second number…mmm…I wrote that weird. I should have said the difference …the difference… so one plus two equals three which is the difference between four and one. See that's how I did it because 5 + 2 is…equals to 7 so 5 + 7 equals 12 (*line 33_25TG*).

Instead of looking for a direct way to connect the values in each data set like Tara and Cathy, Ashley inserted a column of differences in the center of the table that linked each corresponding pair together.



| | | |
|---|---|---|
| 1 +2 | 3 | 4 |
| 2 | 4 | 6 |
| 3 | 5 | 8 |
| 4 | 6 | 10 |
| 5 | 7 | 12 |
| 15 | 17 | 32 |
| $n$ | | n+(n+2) |

n+6

a) Explain how you determined the 5th value in the table (when $n$ = 5).

When the first # is added to 2 you get the second #.
ex 1 +2 = 3. There fire, add 5 + 2 = 7 5+7=12

b) Explain how you determined the 15th value in the table (when $n$ = 15).

15+2=17 15+17=32

c) Explain how you determined the rule for finding the *nth* term in the table.

Replace 1 with n, second # is n+2 therefore,
The answer is n+(n+2).

*Figure 9. Ashley's written responses to think pattern 2*

In her description of how she determined the rule, Ashley used a specific term, in this case, the first, to explain her rule. Further insight into how she arrived at the general term may be obtained from a segment of the follow-up interview when she described how she filled in the table.

40. R: when you just had to get the fifth one did you do it the same way that you were doing it…

41. A: I think I just added two

42. R: just added two…which would be the obvious thing to do

43. A: yeah

44. R: okay…so was it trying to find the 15th one…

45. A:  yeah

46. R: that made you jump to this other idea…of looking

47. A: yes

48. R: across the table…

49. A: yes (laughs)

50. R: okay…all right…sounds good

51. A: I didn't want to be doing this to get the $15^{th}$…there has to be a shorter way

52. R: okay…and you kind of showed me that there…yeah instead of you doing the brute force method which you could have done.

53. A: yeah

54. R: just add two and add two and add two…okay…and I guess you could kind of tell me where this formula comes form

55. A: oh the $n + n + 2$?

56. R: Yes…

57. A: yeah…cause $n$ is the 1 2 3 4…

58. R: mmhmm

59. A: then there was see….I just subtracted one from 4 and two from six and three from 8 and I found that these were also going up by two.

60. R: Oh! Okay!

61. A: Okay so I took $n + 2$ which was three and then I added $n$… $n$ plus $n$ plus 2…so when I add these two numbers…that gets me this number…

Ashley sought a correspondence across the table because she knew there has to be a shorter way. Knowing to look for a shortcut provided the impetus to look across the table for a relationship between the two data sets. The pattern that Ashley finds is a pattern of differences and is based on the recognition that the pattern is consistent for all values in the table ("it works all the time"). Once she established the pattern, she used the variable $n$ in place of the specific value on the left to generalize the arithmetic.

*The 'Together' Session: Tara*

Tara and her partners, Anna and Christy, completed three of the four patterning problems during the time available (approximately 30 minutes). Identifying Patterns 1 asked the participants to determine the number of faces they would have to paint on a tower made of a varying quantity of cubes. The first tower consisted of one cube; therefore one would need to paint five faces of the cube. The second tower consisted of two cubes meaning there would be nine faces to paint. Tara described a rule for the general term shortly after the group began taping this session (within the first 30 seconds). Tara simply stated, "Okay on identifying patterns one…if you take…each amount of cubes and you times it times four plus one you get the value…" She asked her partners, "Do you see that too?" Her partners did not question her response and she offered no further explanation on how she arrived at the rule. The rule she recorded on paper, $(4 \times n) + 1 = v$, is similar to the verbal rule she stated to her partners. The group spent approximately six minutes completing the table before moving on to the second pattern (Figure 10)

Identifying Patterns 2

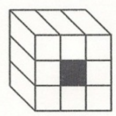
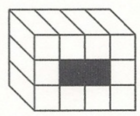
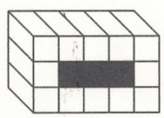
| Term Number (# of red blocks) | Model | Written Description | Process Column | Numerical Value (# of yellow blocks) |
|---|---|---|---|---|
| 1 |  | 8 yellow blocks are needed to surround 1 red block | | 8 |
| 2 |  | 10 yellow blocks are needed to surround 2 red blocks | | 10 |
| 3 |  | 12 yellow blocks are needed to surround 3 red blocks | | 12 |
| | | | | |

*Figure 10. Identifying patterns 2*

In contrast to the first problem, Tara and her partners spent almost 15 minutes tackling Identifying Pattern 2, with the first nine minutes devoted to determining a way to describe the general term. Tara initiated the task by explaining how the model was changing.

> If you have two in the middle so you add those to make 10 and when you have three in the middle…so you add 1 red block and you add two yellow blocks each time…right?
>
> So one red block and two yellow blocks each time you go down…so what could be our pattern…if we have…n red blocks…how do we get a pattern…hang on…let's back up…maybe we should do the first one first (laughs)…trying to jump ahead here (*lines 93, 94_17TG*).

Tara described the rate of change as 'one red block and two yellow blocks each time you go down', but indicated that this is not the pattern she is looking for.

One of her partners, Anna, came up with an idea about three minutes into the discussion.

107. A: Or you just add two to the outside of it. Two look…

108. T: Two columns to the outside…

109. A: yeah…

110. T: yeah…

111. A: Cause you'd have to scoot that over… cause this one would be shaded in and then this one wouldn't. And then you just add two blocks to the outside.

112. T: Mmhmm…two yellow blocks. Okay so we got that but how do you get a relationship…for like a rule…from your red block to your yellow block. What is the rule for that?

At this point, Tara demonstrated her understanding of function as correspondence by stating "how do you get a relationship…like a rule…from your red block to your yellow block"? She communicated her understanding as a 'relationship…like a rule' that would map one set to another set, in this case, red blocks to yellow blocks. In contrast, her partner, Anna, continued to consider how the rate of change might help the trio analyze the pattern.

120. T: Let me think…

121. A: You're gonna add two…you're adding 2 to each

122. T: It's R plus 1…equals yellow plus two…I don't know…I don't know how to…

123. A: There's gotta be a simpler… easier…R plus 1 plus….

Tara asked for time to think, but her partner persisted with the idea of using the rate of change to establish a rule. Tara incorporated that idea in the relationship she described in line 122. This rule mimics her initial description of the rate of change as "one red block and two yellow blocks each time you go down" (line 98). Anna pushed for a simpler way to describe the pattern.

126. T: Okay…so if we have *n* red blocks…n times…oh…hello! Figured it out…look right here 1..2 ..3..4..5..6..7..8..9..10..11..12..13..14… okay….hang on….eight…

127. A: Wouldn't this one be like this one right here?

128. T: It's the number of sides all the way around the red…

Earlier in the conversation, Anna had tried to compare the second pattern to the previous tower problem, but quickly realized that this pattern was not the same. Here she pointed out the similarity between this particular pattern and the second 'think' pattern as both sequences grew by the addition of two.

134. A: …it's the same thing that this one's doing…you have to add two in between …

135. T: so *n* times 2?

136. A: so if your number is the number of red blocks then you're adding two to get your value.

137. C: Right.

138. T: But there's already some around it though.

139. A: I know…but look…there… but say this is your original number...your number of red blocks. And let's say there's 8 …you have to add 2 to 8 to get 10…

140. C: (to get ten). Yeah…

141. T: yeah…but what's your rule for *y* though? I mean…

142. A: Well *y*…the number of yellow blocks is your *n* in this thing and the value is your number of yellow blocks around it…your red blocks is *n* and then your value is this so it would be the same thing as *n* times 2 plus 2…

143. T: *n* times 2 plus 2?

144. A: yeah…*n* times 2 plus 2.

145. T: Okay…so let's try it out…if we have 2 red blocks that would be 2 times two that's four so *n* times 2 plus 2 is only 6 and it's actually 10 yellow blocks.

146. A: Okay…never mind…(whispered)

Anna tried to apply the same rule that worked on the second think pattern since the rate of change was the same. Tara tested to see if the rule would work when the number of red blocks was two, but quickly discovered the rule did not apply. She returned to her idea that the rule had something to do with the number of sides around the red blocks in the center.

147. T: I'm sorry…I know…I was thinking along those lines but I was trying to count the yellow blocks but it extends…but it's all the way around the sides plus four…I think…see if you do 1...2...3...4...5...6 7..8.. 9..10 plus four whereas if you do this 1.. 2 ..3..4..5..6..7..8 plus four is twelve. That's it. It's the number of sides all the way around the red cube plus four.

148. A: Oh…hmm

149. T: See what I mean? If you try the two there's 1...2...3...4...5...6 plus four for the corners… is ten.

150. A: so how do you put that in a thing…it would be *n* plus…

151. T: *n* is the number of red blocks so it's the number of sides around the red block…so um…how would you word it or like in a formula for the number of sides around *n*

Tara has arrived at the general term in a verbal way but struggled to express her rule symbolically. Anna tried to assist Tara make use of the symbolic.

152. A: You'd put *r*…ur…*y*…you're saying that the number of red blocks you'd have *r*…

153. C: right

154. A: Plus the number of yellow blocks…that would be

155. T: No I'm not saying the number of red blocks…I'm saying the number of sides outside the red blocks. See there's one red blocks it's got four sides…there's two red blocks it has six sides…oh but that's plus two…six..seven..eight plus 2…that's it! It's um…you were really close…look at that. It's your *r* times 2 plus 2 in parentheses plus 4. Try it out. Let's go. Or let's do *n*…cause it's gonna go *n* times 2 in parentheses plus 2 in brackets plus 4. Let's try it. Okay…so *n* would be 4 red blocks…right?

156. A: uh huh

157. T: times 2 is 8 plus 2 is 10 plus 4 is 14. Yeah!

158. A: Yeah

159. C: yeah

160. T: We got it…that's it! (Pause) Right?

161. C: I think so. Your *n* is your number of…

162. T: red blocks…times 2 plus 2 plus 4 equals your number of yellow blocks.

Tara made specific reference to the first figure to describe the construction of any term in the sequence when stating her verbal rule of "…the number of sides all the way around the red cube plus four" (*line 117_17TG*). The additional four yellow blocks, Tara explained in the follow-up interview, were necessary because "…the corners are always going to be four" (*line 244_17I*). Although the rule is still dependent upon the construction of a particular figure, it does separate the variable part of the model from the constant part. However, Tara struggled to state the rule in a more general form. In the process of explaining her rule to Anna and Christy, Tara offered specific examples, noting that when "there's one red block it's got four sides…there's two red blocks it has six sides…" (*line 155_17TG*). In this process of explaining, Tara realized a way to connect the number of red cubes to the "number of sides all the way around." She states this quantity as "*r* times 2 plus 2 in parenthesis."

The trio took less than six minutes to complete the third patterning problem, but did not have time to tackle Identifying Pattern 4. In the interview, Tara explained why she thought the second problem was more difficult than the other two the group had completed.

> Well because that one you can see that you automatically add one to each end (references Pattern 3)…similar to the first…but this one right here was a lot more difficult because you're adding to the center and you're still making it go all the

> way around…so we had a harder time coming up with that rule…and that…that whole thing was more challenging…it took us a lot more time. *(line 209_17I*).

She also chose to write about the second patterning problem in response to the first prompt on the reflection piece. The participants were to explain how they were able to determine a rule for one of the patterning problems, including a discussion of any difficulties they experienced while working on this problem. Tara responded to the prompt, writing:

> Out of the problems that I worked on today, I felt like the most challenging one to get the rule for finding the *nth* term in the sequence was the Identifying Patterns 2. I could figure out how to get the yellow blocks fast & easier by just adding 2 all the way down, but to actually get the rule for the *nth* term it took more thinking. We determined that [(n x 2) = 2] = 4 would get it and then we shortened it (& it still worked) for the rule to be (n x 2) + 6 equals the number of yellow blocks, where n equals the number of red blocks. I really enjoyed these problems. They made me think & they were fun! I think that these would be great to use in an upper elementary classroom.

*The ‘Together’ Session: Cathy*

Cathy was also partnered with two pre-service teachers in the classroom for the ‘together’ session. One of her partners, Jesus, had successfully written a rule for the first ‘think’ pattern, but not for the second. The other partner, Daniela, had not written a rule for either of the two patterns. When the trio first paired up to share their responses to the two think patterns, Daniela suggested that Cathy go first, stating “…you’re the smart one.” Cathy adopted the role of leader, replying, “Okay, so what did we think about the first one?” While engaged in the pattern-finding activities of the ‘together’ session, Cathy continued in this leadership role.

As the trio began to analyze the first pattern, Cathy initially directed attention to the rate of change.

56. C: Okay…okay we all know a cube has 6 faces…right?

57. J: Yeah

58. C: And apparently we’re not painting the bottom face….so that’s five for one.

59. J: Oh…okay.

60. C: When you add a cube…you’re adding???

61. J: Four faces

After the trio determined the number of faces to paint on a four-cube high tower, Cathy then directed their attention to the relationship across the table.

75. C: Okay…are you seeing any sort of relationship?

76. J: Yeah…we keep adding four…

77. C: Okay.

78. J: for every cube we add

79. C: …but are you seeing the relationship between the number of faces and the number of cubes?

Jesus correctly described the relationship between the two data sets as he moved down the table, but Cathy wanted him to consider the relationship across the table. She communicated this by asking Jesus to look for a relationship between the number of faces to paint and the number of cubes. When both Jesus and Daniela indicated that they did not see a relationship between the two data sets, Cathy took the initiative to explain what she was seeing.

82. C: Um…well the first one… 1 and 5 is what?...is 1 times (pause) 4 plus 1. The second one is 2 and 9 so that would be 2 times (pause) 4 plus 1. The third one…3 and 13… so it would be 3 times

83. J: 4 plus 1

84. C: Right. Three times 4 plus 1. So your formula is 4n + 1.

Cathy used specific examples to explain how to generate the 'formula' for Identifying Patterns 1. Much like her explanation on the first think pattern, she detailed the arithmetic used to generate the first four terms of the sequence and then generalized that arithmetic to write the rule.

Identifying Patterns 2 took the trio approximately 5 minutes to describe the general term, about twice as long as the previous problem. There were numerous gaps in the conversation when either statements made were incomplete or no one was speaking. About a third of the way into the conversation, Cathy asked the group, "Are you seeing anything here?"

112. J: I don't know. That's like the two and the five…go in by five

113. C: Okay…

114. J: And the three and the four goes by four. I don't know what that means.

115. C: So…Okay…um…Well we can know the number of red blocks because the number of red blocks is given. (pause of 29 sec)

116. J: Yeah

117. D: Umm (pause of 21 sec )

118. C: I'm not seeing it.(pause)

119. J: $n + 2$?

120. C: $n + 2$…

121. D: Okay…

122. J: (laughs)…

123. C: Okay…so…(8 sec) so…okay…okay (5 sec)…three times (5 sec)…2 times …..6 (muttering under breath)

Jesus suggested a rule that is built on the rate of change ($n + 2$) which Cathy seemed to ignore as she continued to study the pattern. Unlike the previous problem they had just completed, Cathy complained that she was "...not seeing" the relationship between the number of red blocks and the number of yellow blocks in each figure. After having spent almost five minutes pondering the problem, Cathy softly stated, "So it's red times…that and that plus…six….so it's…" Her voice then rose as she blurted out, "Okay…red times two plus 6. Okay …now let me explain my reasoning." (*lines 133, 135_18TG*)

Cathy first described the variable part of the relationship as "red times that and that" which she then translated into "red times two." She used the phrase "that and that" to indicate the row of yellow cubes above and below the center row of red cubes. Although Cathy is referring to a particular figure in the model, she used general terms to describe the relationship instead of specifying quantities as she did in her description of how to write a rule for the previous pattern. Once she is satisfied with her verbal rule, she explained her reasoning to her partners.

124. C: On the first one…um…you have three on each side.

125. J: Uh huh

126. C: And then three in the center. One of them is red. …kinda seeing what I'm doing?

127. J: yeah.

128. C: Like…you're always gonna have those…the ones in the middle that are above and below the red…are always gonna be two times the red…but then you're always gonna have those six on the end that don't really…do you see it?

129. J: So red…

130. D: Okay… Say it again.

131. C: Like the ones above and below …the red…

132. D: Okay

133. C: Like this is the red…it's always gonna be two times the red.

134. D: Okay…

135. C: Cause you always have one above and one below the red.

136. D: Yes.

137. C: And then you're always gonna have…one group of three on each end.

138. D: Yes

139. C: Which is six.

140. D: Okay.

141. C: So it would be… two times the red plus six and if the red is n

142. D: Okay.

143. C: then it would be 2n + 6. So…let me make sure that works out. So two times 4 plus 6 is 14. Okay. so…2 times 85 plus 6…

As she explained her reasoning, Cathy pointed out the way the model changes ("you're always gonna have those…the ones in the middle that are above and below the red…are always gonna be two times the red") and the way the model stayed the same ("you're always gonna have those six on the end that don't really…"). She used this pattern of stasis and change to describe how she arrived at the rule, "two times the red plus six." Cathy described how she was able to make the connection between these two parts of the model in her reflection.

> Once I realized that the 6 blocks on each end were constant in every figure, this became much easier to solve. I then realized that for every red block that was added, 2 yellow blocks were added, 2n, and the 6 constant yellow blocks on the end, making the formula 2n + 6.

The trio turned the page to begin analyzing the third of the four patterning problems. The third pattern was shaped like a chair. The first chair has a seat constructed with three cubes and one cube on each of the two legs as well as the back of the chair. The seat remains constant and the pattern grows by the addition of one cube to each of the two legs as well as the back. Jesus began describing how the model was changing from one chair to the next.

181. J: Aren't we just like basically adding one to each leg…to each leg

182. C: Yeah…we're adding one to each leg and one to…the back

183. J: (back) So this would be…there'd be 12 for the three.

184. D: What about two? Did you do two?

185. J: Yeah… nine.

186. C: Nine.

187. J: So it would be something like…3 n plus one?

188. C: Look at what's not changing. The set of blocks that doesn't really factor into what changes. Do you see it?

189. J: I do.

190. C: Yeah.

191. J: Okay

192. C: So you added 3 n + …

193. J: plus 3.

194. C: Right. 3n + 3.

195. J: Alright…Mr. Jesus…you're smart

In the first two patterning problems the trio tackled together, Cathy focused their attention on trying to *see* the relationship between the two data sets. After her experience with Identifying Pattern 2, Cathy, broke that relationship down by asking Jesus to "look at what's not changing." Evidently, Jesus sees that the seat of the chair will always have three blocks and completes the rule he had initiated.

Identifying Pattern 4 was described as a bridge or table (Appendix B). The first bridge was one block high and four blocks wide. The bridge grew only in height by the addition of two blocks, one to each leg of the bridge. After having just described how the $100^{th}$ chair (Identifying Patterns 3) would take 303 blocks to build, Jesus jokingly stated, "So…for a big chair we need a big table and here we go" (*line 234_18TG*).

237. C: Okay…so what's the constant?

238. J: Oh here we go. See that…

239. C: Do you see the constant?

240. J: yeah…there

241. C: Right. So that's what we're going to be adding to…at the end. That's what we're going to be adding. That's the constant…it doesn't change.

242. D: yep.

243. C: It's always going to be there. There's always going to be four blocks. If you didn't have any legs there would always just be those four blocks.

This is the first time that Cathy used the word 'constant' to describe the part of the model that never changes. After her partners identify this part of the model, Cathy pressed for them to explain how the model changes.

251. C: And how many are we adding each time?

252. D: Two.

253. C: Right. So what is our formula?

254. D: 2n + 4.

Although Daniela had struggled with writing rules up to this point in the activity, she now comes up with the last one with just a little guidance from Cathy. Right before the trio started working on the 'together' patterns, Daniela had voiced the opinion that Cathy was "…really good with formulas in math"(*line 52_18TG*) and used that opinion to explain why Cathy had experienced more success with the 'think' problems than either she or Jesus had ("…so these worked perfectly for you" (*line 54_18TG*)). By the end of the together session, Daniela was experiencing the same success.

Part of Daniela's success may be attributable to the assistance Cathy offered to both of her partners during the 'together' session. During the interview, Cathy admitted that she was trying to be helpful.

72. R: …it seemed to me like you were trying to help…to help them along…do you remember doing that
73. C: yes
74. R:Okay… cause you were trying to point those things out and you were successful I thought…
75. C: I was?
76. R: that was working so um…I just I think once you got past this one in particular (Identifying pattern 2)…when you moved on to these last two…
77. C: yeah
78. R: you started pointing out you know parts of it…do you remember what you were using….what rationale…or what you were trying to call their attention to
79. C: the constant
80. R: the constant…okay…and that helps to…what role does that constant end up playing…say in a formula like this one
81. C: well whatever term you have you're always gonna add that constant to it
82. R: okay
83. C: I think it's easier to find the constant first…
84. R: Mmh
85. C: and I think once you realize…once you find out what's not changing…it's easier to find out what is changing

*The 'Together' Session: Ashley*

On identifying patterns one, Ashley and her partner, Matt, quickly recognized that the addition of one cube translated to "another four…another four" (*line 75_13 &25_TG*)

faces to paint. They used this information to find the number of faces to paint in a tower four cubes high.

54. M: Three…It would be four to 17 wouldn't it?

55. A: yes…So if we have 36…I'm not going to use 36…this is where we'd have to figure out…

56. M: the pattern

As in the second 'think' pattern, Ashley knew "there has to be a shorter way." When Matt suggested the 36th term would be 153, Ashley takes that into consideration.

> Well…let's work with this…thirty-six is 153…There has to be a pattern. It's really that you already decided here you're adding…. four…four…four…I'm trying to figure out…what this has to do with this…Oh!…three…Okay so 4 times whatever this is plus 1. Yeah…four times n plus one.

Ashley remarked on how they had already figured out how the pattern of change, but now she sought to find the pattern in "what this has to with this." In this case, her idea of pattern has to do with the relationship across the table instead of down the table.

They spent less than a minute studying the second patterning problem before moving on to Identifying Pattern 3. Ashley stated a possible rule for the chair problem shortly after the pair turned the page ("…three…nine…twelve…fifteen…so what is three times three?…So… 3n plus 3…times 3 plus 3" (*line 119_13 &25TG*). She approached the fourth problem in the 'together session" by noting "Okay…so it's just the legs that are different. So the one leg…the one high whatever you call it …you have 1…2…3…4…5… 6…6 blocks…so the two high leg…you have 4…eight…" (*line 129_13 & 25TG*). Matt suggested they could determine the number of blocks needed to build the $58^{th}$ bridge by taking "…like 58 times 2 plus a …plus 4…" (*line 136_13 & 25TG*). Ashley responded to his suggestion, stating, "Oh…okay so 3 times 2 is 6 plus 4 is 10…so 4 times 2 is 8 plus 4 is 18…I mean 12…so 2n + 4. So …2 times 58 plus 4?" In this response, she verified the rule would work for known values of *n* and then re-stated his expression for finding the $58^{th}$ term in a general way.

The pair returned to Identifying Pattern 2. During this time, Ashley can be heard muttering numbers under her breath but Ryan is silent. As Ashley verbalized the pathway she was taking towards finding the general term, Matt suddenly blurted how to determine the number of cubes needed to build the $85^{th}$ figure.

149. A: 6 plus…n – 2…so…3 is 6 plus 2 plus 4…equals 12....How do we do 4? 6 plus 4 – 2 times 2 plus 4…2…4…so there's fourteen…

150. M: 85 times 2 + 6

151. A: 6 plus…minus 2 plus 4…plus 4 is 8…okay…

152. M: 85 times 2 plus 6

153. A: Eighty-five times 2 plus 6?

154. M: uh huh

155. A: 85 times 2 plus 6? Why do you say that?

156. M: Cause you have 85 red blocks so you have one on the top and one on the bottom…and it takes 6 to build your sides…three on each side.

157. A: three times two is 6 plus 6 is 12…or three …one times 2…oh…That is sooo simple now.

Ashley restated his rule using the variable, *n*, in place of the specific quantity, 58, that Matt had used to describe his rule. As she compared his rule to hers ("six plus n minus 2 times 2 plus 4" (*line 167_13 & 25TG*), she is amazed at how simple it is now. She attempted to explain her thinking to Matt.

> So you got the four corners…which is where you get the plus four from...I guess. And then...so…I just add the two sides…there's 6…that's where the six comes from and the middle…however many blocks umm…three minus two…or like n – 2 because the two blocks stay the same…and then this times two… That sounds so confusing now that I've seen yours…This is really confusing. Now that I know I never would have done it this way..okay…so 6

During the interview, Ashley tried again to explain how she came up with the rule for Identifying Pattern 2. When asked if she worked with the numbers in the table or used the pictorial representations, Ashley replied, "I think I worked with the model" (*line 105_25I*). She pointed to the third figure in the model and said, "I think that's what I used…this one…and I looked back and checked it with the other two" (*line 119_25I*). Ashley went on to explain why the second pattern was more difficult to generalize than the other three patterning problems.

> I think just finding the formula to like get you know 85 and 100 and n…I don't…cause…it's hard to see…like…I mean obviously you know what changes….…but it's hard to think of the formula for that like…how do you come about what makes that change or whatever…well these ones…these ones…the change is obvious…like it's just right in front of you like… right…like….hey that's what changing it's…

# APPENDIX F

## PATTERN FINDING EXPERIENCES: SHELLY, MATT, AND JILL

Detailed descriptions of the pattern-finding experiences of Shelly, Matt, and Jill are presented in this section. This group of participants experienced some roadblocks with generalizing patterns due, in part, to the use of variables. Summaries of their experiences were included in Chapter IV.

The ‘think’ session took place prior to any discussion on patterns or sequences in this particular classroom. The participants worked independently as they responded in writing to a series of questions associated with two arithmetic sequences. Immediately afterwards, the participants shared their responses with one or two partners. The researcher also conducted follow-up interviews to clarify the written and shared responses. The assumption was made by the researcher that these explanations associated with the ‘think’ session best represented the understandings this group of pre-service elementary teachers brought to their mathematics classroom. Descriptions of the explanations offered by Shelly, Matt, and Jill are presented in the first section.

*Shelly’s Responses to Think Pattern 1*

Shelly’s responses to the first three questions on Think Pattern 1 are displayed in Figure 11. Shelly used the construction of the figure to describe how the model grows, stating that “it expands by one block on each side.” She did not state the total number of squares needed to construct the fifteenth figure, but did provide sufficient information to demonstrate that she could extend the pattern by building this particular figure. Her explanation of how she knew included reference to the sequential change from one figure to the next, but it is unclear whether or not she was making a connection between the position of the figure and the total number of squares on each end of the center square or applying recursive reasoning to determine that quantity.  The interview data provided insight into the question of whether she was linking the position of the figure to its value or analyzing sequential change.

> Okay…well…um…like…um…three four…you would add…um…so it would be 4 plus…no…I don't know…um…the 4th figure has 4….4…blocks…on each side….So you add one for…on each one for each figure…So you would add 11 for the 15th…is that correct? (*lines 19-23_20I*)

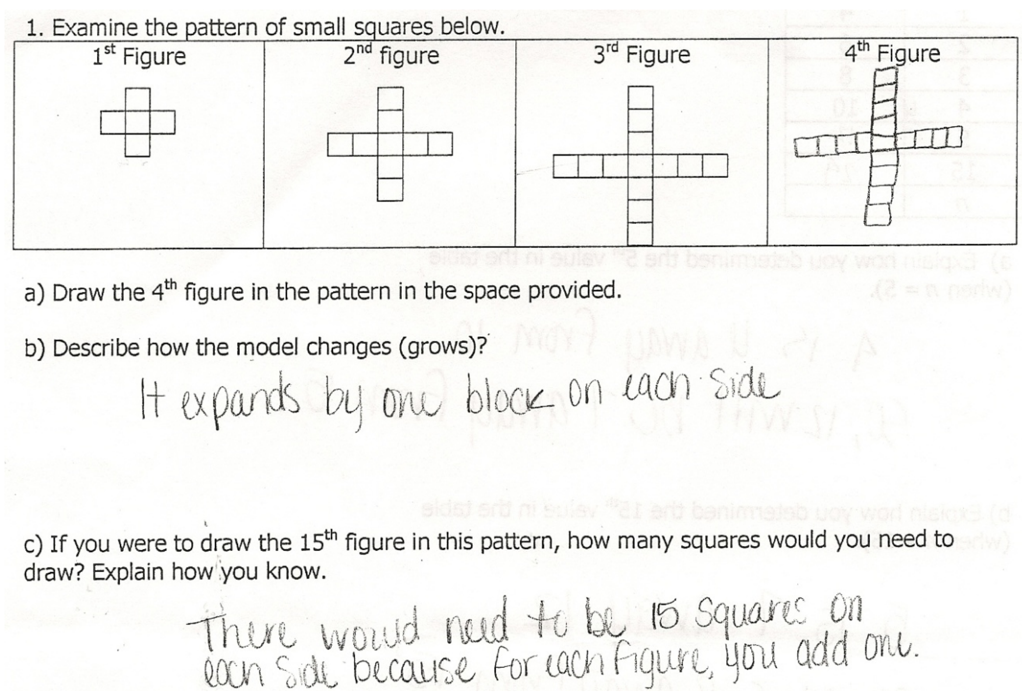
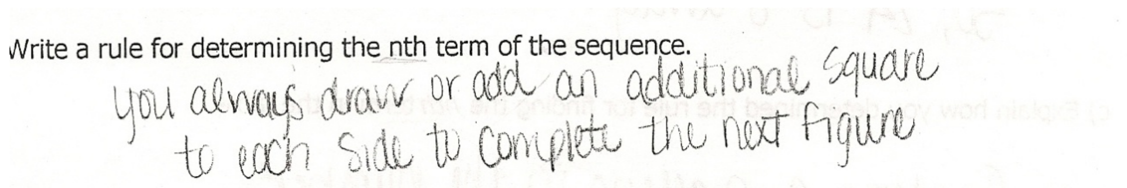



*Figure 11. Shelly's written responses to first think pattern.*

Shelly generalized the pattern of change between sequential terms instead of linking the position of the figure with the number of squares on each end of the center square. She applied recursive reasoning to make the jump from the 4th to the 15th. She applied the same type of reasoning to write a verbal rule to describe the *nth* term in the sequence (Figure 12). The rule Shelly wrote makes reference to the sequential change from one figure to the next and is built on recursive reasoning. Although she coordinated the change in both data sets by noting that ‘you always draw or add an additional square to each side to complete the next figure,’ she did not analyze these changes in terms of correspondence.



Write a rule for determining the nth term of the sequence.

you always draw or add an additional square to each side to complete the next figure

*Figure 12. Shelly’s rule for first think pattern.*

The request to write a rule was worded in a somewhat open manner in that the type of rule was not specified. Shelly wrote a verbal rule, but her rule cannot be applied to find any term in the sequence. The question of why she wrote a recursive rule instead of a general rule for the *nth* term was addressed during the follow-up interview.

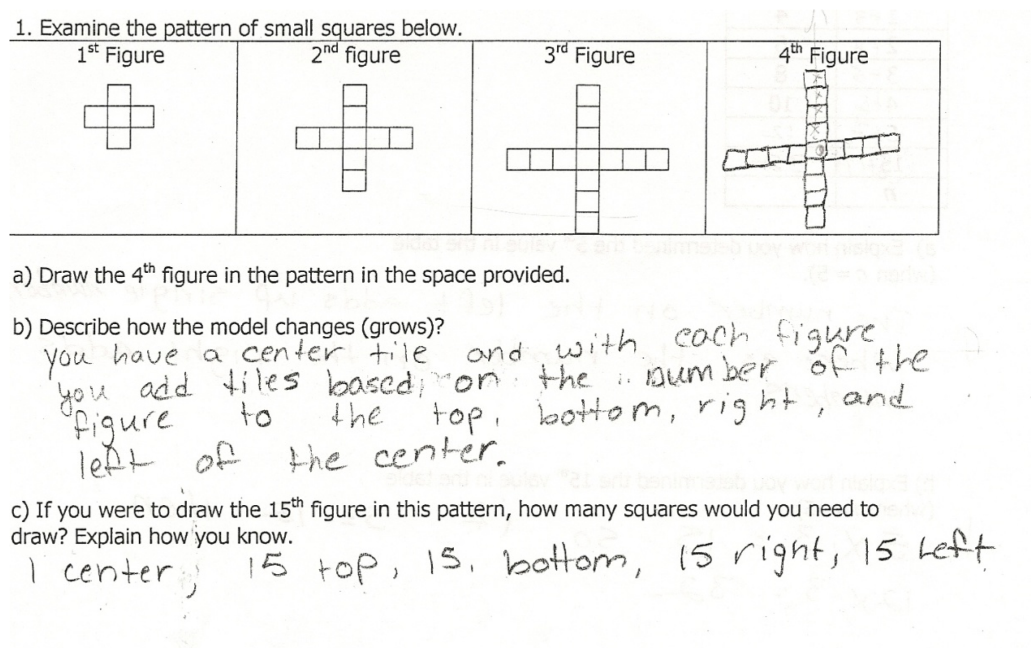
31. R: Okay when you were asked to write a rule…let’s see you wrote “you always draw or add an additional one to each side to complete the next figure”…which is um…which is a rule for finding the next one
32. S: but you’re wanting… like n…
33. R: Yeah…
34. S: You’re wanting the…n…
35. R: Yes…was it the question…were you not sure what I was asking for or was that what you thought I meant at the time?

36. S:  Well I think um…I don't know…I didn't really understand what the *nth* term meant until I asked you right before the test…
37. R: Okay
38. S:  And So I don't think that I completely understood it to the fullest extent so um…
39. R: Okay
40. S:  the *nth* term would that be n…um…I don't know…n plus one? I mean I don't think that's it…that  doesn't sound right…cause you're adding one to every…
41. R: To every part…
42. S:  Right.

Shelly confessed that she was confused about what the *nth* term meant. She attempted to write a rule during the interview (line 40: $n + 1$), but quickly recognized that the rule would not work since the figure grows by four squares each time. Shelly admittedly did not fully understand what the *nth* term meant.

*Matt's Responses to Think Pattern 1*

Matt did not make use of the symbolic when writing a rule for the *nth* figure in the first pattern problem, but did write a verbal rule for describing the construction of the general term. His written responses to the first three questions in Think Pattern 1 are presented in Figure 13.


1. Examine the pattern of small squares below.

| 1st Figure | 2nd figure | 3rd Figure | 4th Figure |
|---|---|---|---|

a) Draw the 4th figure in the pattern in the space provided.

b) Describe how the model changes (grows)?

you have a center tile and with each figure you add tiles based on the number of the figure to the top, bottom, right, and left of the center.

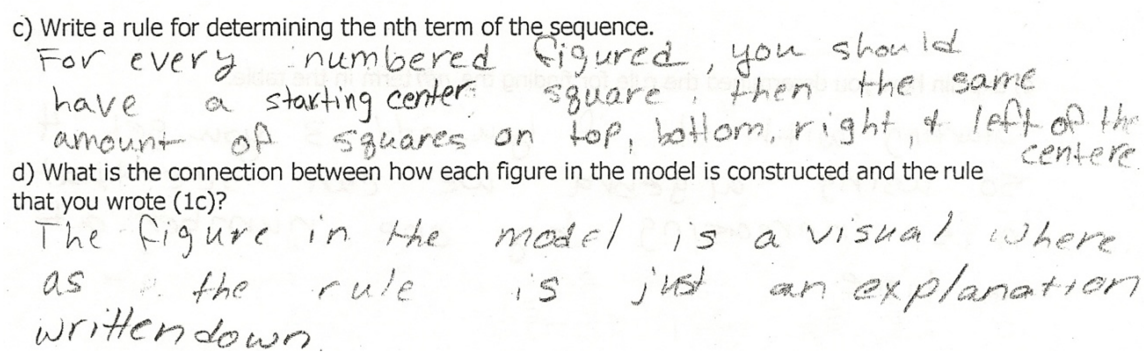
c) If you were to draw the 15th figure in this pattern, how many squares would you need to draw? Explain how you know.

1 center, 15 top, 15 bottom, 15 right, 15 left

*Figure 13. Matt's written responses to think pattern 1*

During the 'pair' stage of pattern-finding activities, Matt and his partner Ashley discussed how they went about each task. For the first task of drawing the fourth figure, Matt explained to his partner that "…basically I did an extra tile to the top bottom left and right (*line 10_13TG*)." From this response, it can be inferred that Matt initially analyzed the changes that were occurring from one figure to the next. In his written description of how the model grows, Matt made specific references to the structure of the figure, noting the presence of the constant center tile. He used spatial terms such as top and bottom to indicate where the additional tiles were located around the center square. Most notably, Matt made a direct link between the position of the figure and number of tiles needed to construct it.

Matt's description of how to draw the 15th figure further illustrates how he applied his understanding of how the model changes and how the model stays the same. He made note of the constant center tile and then linked the variable part of the model to the position of the figure in the pattern. All along, Matt referred to how the figure is constructed using spatial terms. Matt made these connections explicit in the verbal rule he wrote in response to next question on the activity (Figure 14).


c) Write a rule for determining the nth term of the sequence.

For every numbered figured, you should have a starting center square, then the same amount of squares on top, bottom, right, & left of the centere

d) What is the connection between how each figure in the model is constructed and the rule that you wrote (1c)?

The figure in the model is a visual where as the rule is just an explanation writtendown.

*Figure 14. Matt's rule for first think pattern.*

Matt clarified what he meant by 'the same amount of squares' in our interview.

19. M: So for every…you know…it's just like that…for every number figure you should…you should have a starting center square…
20. R: Okay.
21. M: …and then the same amount of squares on top bottom left…you know.
22. R: And how do you know how many go on top…why did you have four up here?
23. M: Because of the corresponding figure…the first one started with first and then there was one square on the top and on the right…and left and then on the bottom.
24. R: Okay
25. M: But you had your center square and it just continued on the second so obviously on the fourth one you have your center with four tiles on the top right left and bottom.
26. R: So if I had the hundredth figure I would the square here and how many going up this way?

27. M: you would have a hundred on top…

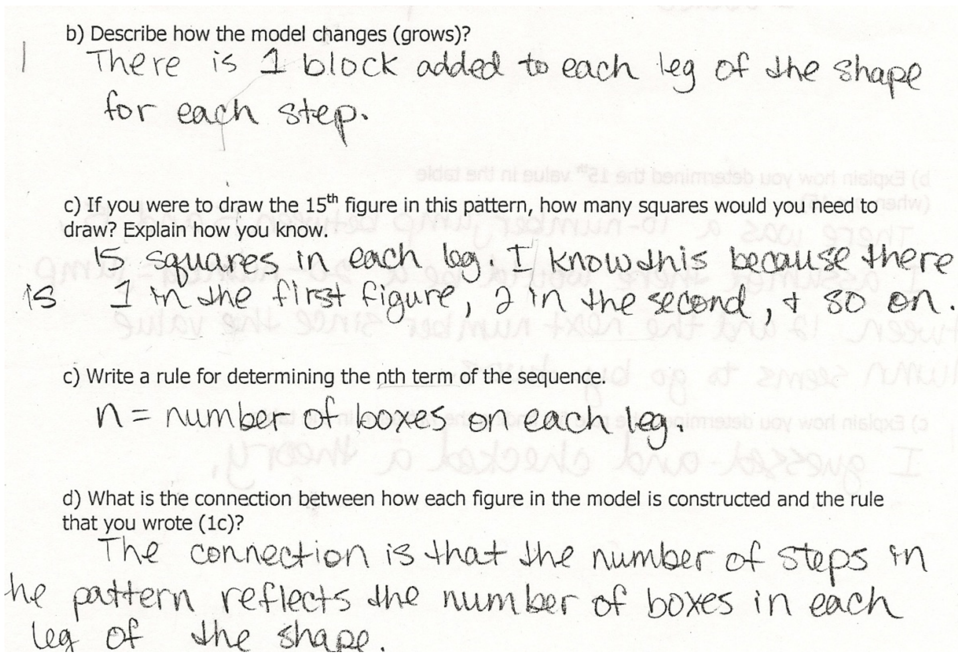
28. R: Okay

29. M: hundred on the bottom…and a hundred on the left a hundred on the right.

Matt was able to write an explicit, verbal rule that will work all the time. He was able to apply his rule to determine the construction of any figure in the sequence, as evidenced by our conversation about the 100th figure. During our interview, Matt explained why he chose to write a verbal rule unlike his partner, stating, "I can't do the a + b thing…but I can try and explain it into words instead…it's basically where that comes from" (*lines 15, 17_13I*).

*Jill's Responses to Think Pattern 1*

Jill's written response to how many squares it would take to draw the 15th figure is incomplete (Figure 15). Jill made the connection between the position of the figure in the sequence and its construction, but does not apply the arithmetic needed to completely answer the question. Her response even omits the presence of the center square, as if the assumption is made that it is always present or perhaps a 'hole' in the shape. Her verbal rule also omits the presence of the center square. The rule is incomplete as it would require one to be familiar with the structure of the model for it to be used to correctly construct any figure in the pattern.



b) Describe how the model changes (grows)?

There is 1 block added to each leg of the shape for each step.

c) If you were to draw the 15th figure in this pattern, how many squares would you need to draw? Explain how you know.

15 squares in each leg. I know this because there is 1 in the first figure, 2 in the second, + so on.

c) Write a rule for determining the nth term of the sequence.

n = number of boxes on each leg.

d) What is the connection between how each figure in the model is constructed and the rule that you wrote (1c)?

The connection is that the number of steps in the pattern reflects the number of boxes in each leg of the shape.

*Figure 15. Jill's written responses to first think pattern.*

During her paired conversation, Jill made note of the fact she did not complete the problem.

5. J: Okay…on the front side…you know it's funny is you know how you  said that you add four for each step
6. P: uh huh
7. J: I totally didn't even do that.
8. P: yes.
9. J: I didn't…I said that you add one to each leg for each step that you do.
10. P: Oh see like I did that but then I was just like..oh you just add…you're adding four
11. J: yeah…I never said anything about four…I just said one
12. K : I saw one but then I realized okay it's that and then I realized it just adds up too so that's how I got this 15 times 4 equals 60 plus one for the middle one

13. J: yeah…yeah…Yep…I didn't do that…I just said um add one block for each leg for each step
14. P: (didn't do that)…uh huh…that's what I said too…just add a square each time to each end
15. J: You just took it further
16. P: (Laughing) Yeah…
17. J: …which equals four!
18. P: and then I saw the algebra….Yep I saw the math in it.
19. J: Okay…. Wow….Bout all I've got to say on that one.
20. P: yeah…then I got my equation and I was done.

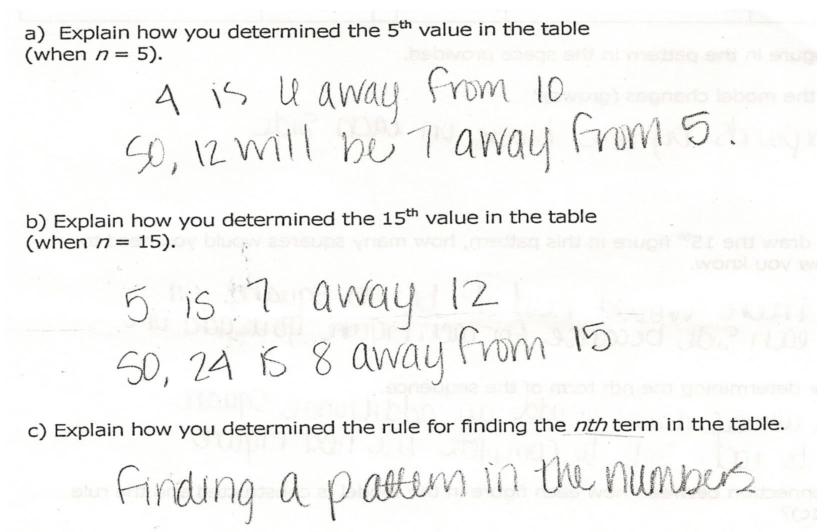
Her partner generalized the arithmetic she had used to calculate the number of squares it would take to build the $15^{th}$ figure so as to arrive at the general term. The arithmetic she recorded on paper provided a pathway to write the rule using symbols by allowing her to see "the math in it" (*line 18_7TG*). Jill, on the other hand, confessed that she "totally didn't even do" the arithmetic that would have allowed her to visualize the algebra.

## Descriptions of Experiences: The Second Think Pattern

The second think pattern lacked the visual representation provided in the first problem. An arithmetic sequence was displayed in a table and the participants were asked to complete the table by filling in the values of the $5^{th}$, $15^{th}$, and *nth* terms of the sequence. As with the first patterning problem, participants were also asked to explain the processes they used to complete the table. Descriptions of the explanations offered by Shelly, Matt, and Jill are presented in this section.

*Shelly's Responses to Think Pattern 2*

Shelly's responses to the second think problem are displayed in Figure 16. She was able to find the next term in the table, but did not locate the 15th term correctly or write a rule for the *nth* term. Shelly used a pattern of differences to determine the 5th term, but when she applied this same pattern of differences to locate the 15th term she made two errors. The first mistake is an arithmetic error. She corrected this mistake while sharing her responses with her partner, Pam.



a) Explain how you determined the 5th value in the table (when $n = 5$).

4 is 6 away from 10
So, 12 will be 7 away from 5.

b) Explain how you determined the 15th value in the table (when $n = 15$).

5 is 7 away 12
So, 24 is 8 away from 15

c) Explain how you determined the rule for finding the *nth* term in the table.

Finding a pattern in the numbers

*Figure 16. Shelly's written responses to second think pattern.*

26. P: um…how did you determine the 15th value in the table…because you got …different answer than I got.
27. S: Okay… I'm not sure how to explain it…um…there are….seven numbers in between 5 and 12…so there's seven spaces in between 5 and 12 so… I put …24 next to 15 because there's eight spaces in between 15 and 24…Does that make sense?...Eight spaces means like eight numbers.

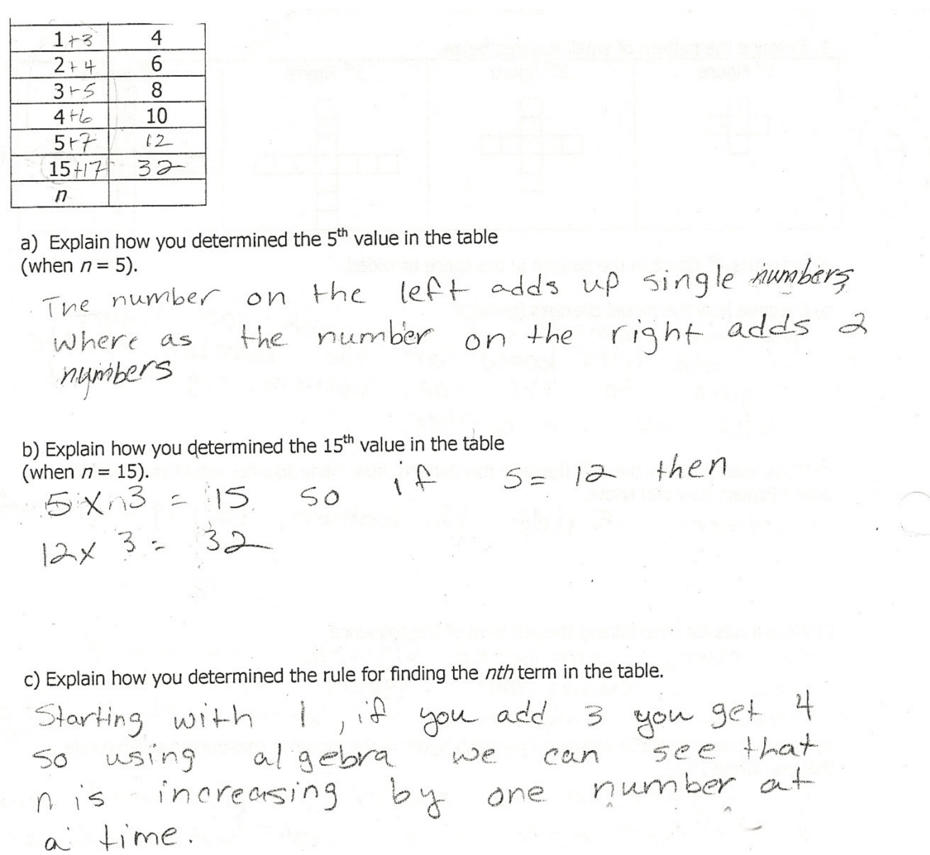
28. P: And then you added it together? I kind of did the same thing but it confused me when it did a jump because …I was adding two to the value which would also be the same as if you like for your n column here.
29. S: I just continued the pattern …see…it goes 3, 4, 5, 6, 7 and then I did 8…8 spaces in between 15 … like…it would be the same thing…don't you think?
30. P: yeah…what I…I assumed that they were just skipping so if you added two to five…to six through 14…
31. S: If you added 8 to 15.
32. P: yeah…
33. S: that would be 23…23…yeah…I'm sorry

Unlike her partner, Shelly did not take into consideration the jump from the 5$^{th}$ term to the 15$^{th}$ term. Although she did look across the table to establish a correspondence between the two data sets, she failed to coordinate the changes that were simultaneously occurring in the vertical direction. Shelly did not complete the table by finding the *nth* term, but did indicate that one could determine a rule by "finding a pattern in the numbers." (Figure 16) During the 'pair' session, Shelly remarked to her partner, "…really…it was one of those questions I didn't know how to explain so I put…um…you have to find a pattern in the numbers" (*line 40_20TG*).

*Matt's Responses to Think Pattern 2*

Matt wrote a verbal rule for the first pattern that was based on the physical construction of the figures. His responses to the questions associated with the table in think pattern 2 are displayed in Figure 17. Matt's response to the first question (a) indicates that he looked down both sides of the table to analyze change in each data set.

He clarified how he extended the pattern to find the 5$^{th}$ term during the paired conversation with his partner, Ashley, by stating, "…so to get the 5$^{th}$ one…since number 4 had a 10 next to it…I took five and I added two to ten to get the twelve" (*line 32_13TG).*



| | |
|---|---|
| 1+3 | 4 |
| 2+4 | 6 |
| 3+5 | 8 |
| 4+6 | 10 |
| 5+7 | 12 |
| 15+17 | 32 |
| *n* | |

a) Explain how you determined the 5$^{th}$ value in the table (when $n = 5$).

The number on the left adds up single numbers where as the number on the right adds 2 numbers

b) Explain how you determined the 15$^{th}$ value in the table (when $n = 15$).

5 x 3 = 15 so if 5 = 12 then 12 x 3 = 32

c) Explain how you determined the rule for finding the *nth* term in the table.

Starting with 1, if you add 3 you get 4 so using algebra we can see that n is increasing by one number at a time.

*Figure 17. Matt's written responses to second think pattern.*

Matt inserted a column of differences between the two data sets. This column of differences could have been used to determine the 15$^{th}$ term in the table, but Matt described something quite different in his response to question b. He explained his reasoning to his partner, Ashley, during the 'pair' session.

39. M: Well because 5 times 3 is 15 okay so that's… that's… you know so I took okay and then that gave me that…so five….so five equals that…so 12 times 3

40. A: Okay so basically you were doing like this type of thing…um… 15 equals x kinda like …well kind of setting it up that way…and then what does it take to get to here…it takes three and so it has to take three to get to there. So you can use that for…Wow that is way complex thinking. I just said that 15 + 2 equals 17 so 15 +17 is 32.
41. M: Well because it tripled…here see…(both laugh)
42. A: I really don't understand. I mean I understand **now** where you got it from but
43. M: I just thought well 5 goes into 15 three times so I just took twelve and times that by three.
44. A: I get you now but I don't think I'd ever…ever done that…I don't think I would ever do it like that.

His use of proportional reasoning is not only inappropriate in this case, but he also made an error in the arithmetic. Whether Matt inserted the column of differences after he applied proportional reasoning to find the $15^{th}$ term, or whether he simply came up with an incorrect explanation to verify the results recorded in the table was unclear.

Matt attempted to write a verbal rule to describe how to find the *nth* term in the table (Figure 17). From Matt's rule, it appears he was attempting to generalize the pattern of differences between the two data sets. However, the conclusion he stated simply described how the values in the column on the left were changing. Matt tried to explain how to apply his rule to his partner, Ashley.

45. M: Starting with one, if you add three you get 4 so using algebra we can see that n is increasing by one number at a time.

46. A: Yeah, but how did you find the value…

47. M: Well…I uh I added. One plus three equals four

48. A: But what was your…the value of your *nth* term

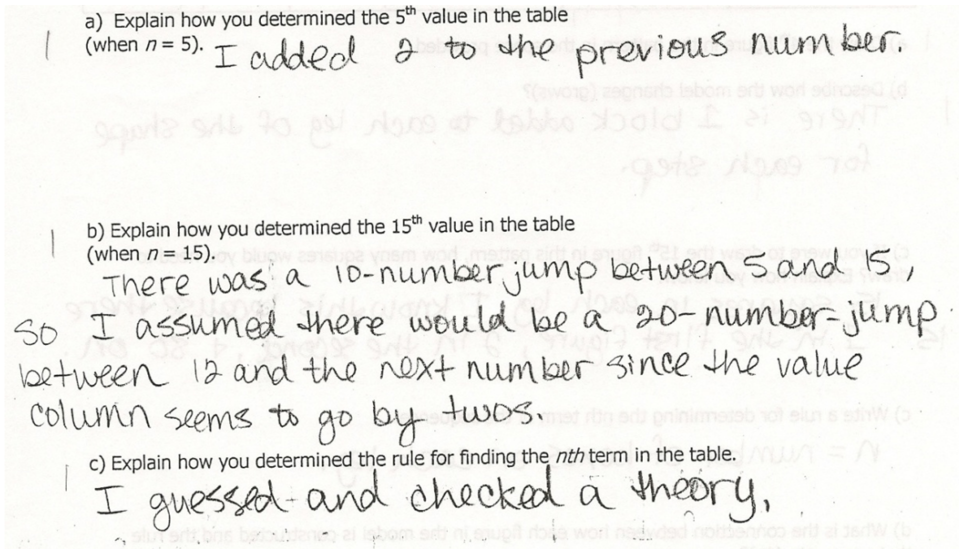
49. M: Well, n would equal 1. Well n would equal … corresponding number starting with 3…4…5… sort of in increasing order well I don't know…

50. A: That's where I used this rule…the n plus n plus 2 …and that's how you'd get it… I don't know…I mean that's how…cause yours …you'd have to know what n **is** so you could compare it to something. You'd have to compare it to something. I mean it works if you have the numbers…

Matt's success at generalizing the second think pattern was quite limited. He extended the pattern to locate the $15^{th}$ term, but wrote an ineffective verbal rule. Matt's rule simply described the changes occurring in the data set on the left and does not include a method for determining corresponding values on the right.

*Jill's Response to Think Pattern 2*

Jill had written an incomplete verbal rule for the first pattern but wrote a symbolic rule for think pattern 2. Her written responses to think pattern 2 are displayed in Figure 18.


a) Explain how you determined the 5th value in the table (when $n = 5$). I added 2 to the previous number.

b) Explain how you determined the 15th value in the table (when $n = 15$). There was a 10-number jump between 5 and 15, so I assumed there would be a 20-number-jump between 12 and the next number since the value column seems to go by twos.

c) Explain how you determined the rule for finding the *nth* term in the table. I guessed and checked a theory.

*Figure 18. Jill's written responses to second think pattern.*

Notice that she began by applying recursive reasoning to find the 5th term and then coordinates the changes in both data sets to jump to the 15th term. In her written description, Jill explained that she "guessed and checked a theory" to determine the rule for the *nth* term in the table. Jill described her process of guess and check to her partner, Kristen, during the 'pair' session.

25. J: Um…I didn't get it at first.
26. K: So what did you get?
27. J: I only saw the relation for each column but not how the columns related to each other and then it just clicked halfway through (laughs) and I was surprised (laughs). It's all the sudden like Ohhh!…(laughing)
28. K: Well…like with me whenever I…I see a pattern I automatically start adding.
29. J: yeah.
30. K: So I say what did this add…like a complete habit
31. J: Yeah

32. K: Like I don't even say multiply or anything I have to say add.

33. J: Yeah… I divided…like I made it way more difficult than it had to be. I made them into fractions and said how was 1 over 4 similar (K: Wow) to 2 over 6 and I was dividing to see how the uh decimals came out (K: mmhmm) and they had… they were absolutely random you know (K laughts). I mean they were getting…the decimals were getting bigger (K: yeah)…and so I noticed that. But…but it was like you know point four one three seven five and

34. K: And you were like whatever that means…

35. J: yeah…yeah…

Jill explained that she was struggling to describe the relationship between the two data sets when "it just clicked halfway through" (*line 27_7TG*). She explained this to her partner:

> The only reason I even came about it halfway through after writing this description about how I didn't get it was how um…just kind a by luck…I was… I started doubling things…and then I was like okay it's that times two plus two…pretty much. And …That's… it was just luck. (*lines 45, 47_7TG)*

During the interview, Jill repeated the problems she was having in her attempts to forge a correspondence between the two data sets, stating, "Well a lot of times what happens is I can see the relationship between these numbers (right column) and I can see the relationship between these (left) but I can't see it between the two of them." (*line 69_7I*) During the follow-up interview, Jill expanded on her method of guessing and checking a theory.

69. J: and a lot of times it's kind of a guess and check thing…I'll say you know what do I do to this to get these and then you know it's all of the sudden I'll try something out and I'll realize you know that if you double this and add two that's exactly what it is and you know then try it out for each one and it always works and I'll say Yes…there we have it.

70. R: that was good…that's interesting…and I noticed you erased this at the top…

71. J: yeah…

72. R: and you erased all this stuff

73. J: yeah…sometimes I get you know little theories and they're just not…I realize that after the fact you know so I guess that's kind of a downfall to just kind of taking a stab in the dark it would the relation could be wrong sometimes

74. R: yeah…but this time it was just perfectly right

Jill made comments about how she tried so many little theories and none of them seemed to work. She first tried to uncover a pattern by dividing pairs of numbers, but when the relationship proved to be non-proportional, she tried other little theories until she found one that "always works."

*Summary of Responses to Think Pattern 2*

Shelly sought to establish a correspondence between the two data sets by locating a pattern of differences. However, she did not coordinate the changes occurring in the values of *n* (from 5 to 15) while applying this pattern. This lack of coordination, along with an arithmetic error, made it difficult to find "a pattern in the numbers" (Figure 18).

Her experience with think pattern 2 illustrates how looking across the table (seeking a correspondence) can actually be a roadblock in the process of generalizing patterns if the individual does not also coordinate the changes in each data set with one another.

Matt was also unable to write a rule for the *nth* term in the table, despite having successfully written a verbal rule to describe the first pattern. Matt had explained, in part, why he chose to write a verbal rule during the follow-up interview. In particular, Matt was asked to comment on the last statement that he wrote concerning the first patterning activity (Figure 14).

38. R: …you said that the figure is a visual…it's a visual of it…where the rule is an explanation…do you feel the same way about the symbolic one…is that still an explanation? Like her $4n + 1$?
39. M: Yes but I think that just some people would have more trouble understanding that (symbolic rule) than just reading it (verbal rule).
40. R: uh huh……than reading it…okay…they would wonder maybe…what would be the problems…what would be the trouble that someone might have?
41. M: Well someone may not recognize what n is…

The second think pattern forced the individual to use a variable by requesting the value of a quantity on the right when the quantity on the left was *n*. Matt had been able to use the visual representation of the first think pattern to write a verbal rule, but struggled with the second pattern. Matt's experiences with the second think pattern illustrate how the use of a variable can present a roadblock in the process of generalizing patterns for individuals who "may not recognize what *n* is" (*line 37_13I*).

Jill, on the other hand, was more successful with the second think pattern than she had been with think pattern 1. Although she seemed to stumble upon a rule for connecting the two data sets, she was able to express that rule in symbolic terms whereas her rule for think pattern 1 had been written using everyday language. Jill explained how she "only saw the relation for each column and not how the columns related to each other" (*line 27_7TG*). However, Jill persisted in her pursuit to figure out the correspondence between the two data sets, or as she stated, "what do I do to this to get these" (*line 69_7I*).

Descriptions of Experiences: The 'Together' Session

The 'together' session offered an opportunity to examine how pre-service elementary teachers communicate their understanding of function while engaged in pattern-finding activities. Each pair or trio joined forces to complete four patterning activities that combined both a pictorial representation and a t-table for displaying an arithmetic sequence. These activities are included in Appendix B. Descriptions of the experiences Shelly, Matt, and Jill had while engaged in these patterning activities are presented in this section.

*The 'Together' Session: Shelly*

Shelly and her partner completed two of the four patterning activities in this session; the first and the third. They attempted to complete the second problem, but encountered difficulties when asked to describe the $85^{th}$ term. The first pattern consisted of a tower built out of a single column of cubes (Appendix B). Participants were asked to determine the number of faces they would have to paint on towers constructed of a varying quantity of cubes. Shelly and her partner seemed to work through the problem in conjunction.

52. P: Okay…it's a tower though…that's what it is…so you can't paint the bottom
53. S: ah…
54. P: cause it's on the ground. So there's fours sides and a top
55. S: yes that's right
56. P: okay…so a tower two cubes high would have…nine…yeah…because you're covering up one of the tops…yeah.
57. P: ….thirty-six… So if you took….so it would be 36 times 4 right?…36 times 4?…
58. S: So 144 sides and one top…is that right?

The pair broke the arithmetic needed to determine the number of faces to paint into two steps by separating the number of lateral faces to paint (which varied based on the number of cubes) from the constant top of the tower. They were then able to find both the $36^{th}$ term and the $100^{th}$ term in the table without having to apply recursive reasoning. The next value on the left, after 100, was represented by the symbol, *n*.

59. P: so it would be 145. So 100 cube high would be 400 sides and one top that would be 401 total.
60. S: Is that 11?
61. P: …I think it's *n*…
62. S: n?
63. P: yeah…so basically…. it would be …it would be *n* times 4 …plus 1
64. S: n times 4? I don' know how you're getting that.

65. P: Because basically they just want you to put *n* in place of a number. So instead a 100…they're just saying it could be any number…it could be 105 or 120 …

66. S: Okay.

67. P: …but either way we would always multiply it by 4 and add one

68. S: Oh…okay…yeah that makes sense.

Although Shelly had been following along with her partner while determining the number of faces to paint as the tower grew, she was caught off guard when her partner offered the rule, n times 4 plus 1. Her partner interpreted her question of how she arrived at the formula as a question on the role of the variable *n*. Her partner described this role as "any number" (*line 78_20TG*); an explanation that Shelly agreed "makes sense."

The two briefly worked on the second pattern, Identifying Patterns (Appendix B). Shelly offered an initial analysis of the problem to her partner before they moved on to Identifying pattern 3.

> If three were in the middle…3… 6…12? Then…14…yeah so you increase it by 2 for every red block you add in the middle. Now they want to do the big numbers…and we have to figure out how we can do that…so that would be…I always have problems figuring out how to write it…(*lines 90, 92_20TG*)

Shelly correctly identified the rate of change in the number of yellow blocks as the number of red blocks increases by one. She recognized that she must now make a jump from describing sequential terms to the 85$^{th}$ one and admits to having problems figuring that out. In her reflection, Shelly identified the issue as a problem for her and her partner.

The most problem that we had as a pair or group, was when the number jumped from 5 to 36 or a really low number such as a single digit number to a two digit number in which it took longer to find the pattern (*R1_20W*)

She explained why they decided to move on to the third pattern during our interview.

I don't remember….I think we had a hard time because we… we didn't have enough of the right colored blocks…So……um it was kind of hard to see when you had a couple of yellow…a couple of green …red… or whatever…And um…Yeah…I'm very visual…so it's hard to see that…this is really red or this is supposed to be green or whatever…(*lines 132 – 136_20I*)

The third pattern was shaped like a chair. The first chair has a seat constructed with three cubes and one cube on each of the two legs as well as the back of the chair. The seat remained constant and the pattern grew by the addition of one cube to each of the two legs as well as the back. Shelly described how the model changed to her partner.

122. S: One to the top…one to the bottom…and one over to the other leg. I think that's three blocks each time…yeah.

123. P: Okay…So the next one would be 15….(pause)

124. S: okay….(pause)

Shelly communicated her understanding of how the model changes from one term to the next by stating, "…that's three blocks each time." The use of the word "each" indicates that she knew she could always add three to find the next term in the sequence. She then applied this understanding in an attempt to jump from the 4$^{th}$ term to the 36th.

125. S: okay…I've got a question. If the number was 35 and we added three legs to the 35 it would be 39… right?

126. P: Mm hhmm

127. S: So would this number be 39…no it would be 38….right? If the number was 35…and you added 3 legs…it would be 38 maybe?

128. P: you lost me.

129. S: If this number here was 35 instead of thirty-six…

130. P: mmhmm

131. S: cause we're trying to go up to 36 so we could go one number lower…wouldn't this be 38?...because three plus 35 is 38?

Shelly communicated her understanding of the recursive relationship between terms in the table by attempting to go backwards to determine how many squares it would take to build the 36th chair in the model. The incomplete nature of Shelly's understanding of function is visible in her explanation of how one might go about the process of determining the number of squares it would take to construct the 36th chair. Using the single process of analyzing change, Shelly tried to build on from a previous term (in this case the 35th) to describe the next chair. However, she used the term number itself as the value on the left side of the table in the process. Her partner tried to point out why this process did not make sense to her.

138. P: I don't think so…because… those are way too close together…see…. on the third one it was 12 and the fourth one was 15…so we need to be a lot bigger.

139. S: Oh…that's right

140. P: I think we should be able to multiply by something

Although Shelly agreed with her partner's concerns about the application of recursive reasoning, she returned to the idea of backtracking while re-examining the pattern.

144. S: Okay…so…if there was…three……nine…fifteen…that's right…wait a minute…you add three each time.

145. P: Yes…yes

146. S: So if our number was 35 up here… and we add 3 to that …it would be 38. See how it goes……one two …

147. P: Huh?

148. S: If this was thirty five and you went to 36…36 is really close together so I don't know if that's correct or not

149. P: 36 times 3…108.

150. S: Wait…wouldn't we do 36 times…4? Because there's…oh…oh well

151. P: No…that wouldn't be right cause it's times three every time

152. S: So it would be 108?

153. P: I don't know…yeah… Okay…let's see if that works…let's try like five…If we did five…then it would be… five blocks…

154. S: I don't think… um…36 times three is correct because if you multiply it by 3 each time…this doesn't work..see 2 times 3 is this over here…3 times 3 is that….3 times 4 is that…so that…that's not going to work

155. P: three times four is that…so it's three numbers more than that…hold on

156. S: so it would be 111.

157. P: What?

158. S: Yeah…because 3 times 4 is 12 …plus 3 is 15 and that's that number

159. P:Okay so if we took 36 times 3 and add 3

160. S: it should be 111.

161. P: So 36 times 3 plus 3 is 108 plus 3…111

162. S: Like do we know where the three is from?

163. P: it's just part of the pattern…we don't really see it…like on the chair right

164. S: Um…I don't know

165. P: Okay…so 100 times 3…plus 3…so that would be 303?

166. S: right

167. P: so that would be *n* times 3 plus 3.

The two arrived at a description for the general term after Shelly's partner came up with the idea to multiply the term number by three. Shelly recognized that multiplying by three did not work for the given values and the two recognize the need to add three additional cubes. Her partner described the general rule, *n* times 3 plus 3. However, neither of them connected the need to add three more than n times three to the constant presence of the chair's seat.

Immediately after the together session, the participants wrote responses to two prompts in reflection over the patterning activities. In response to the second prompt, Shelly wrote about the advantages of being paired with a partner.

> Being with a group, I was able to better understand the patterns, relations, and function of the problem. I was also able to better understand further in detail the sequence and functionality of the problem by being paired up and having a partner (*R2_20W*).

The paired conversations offered an opportunity to capture the understandings of the speakers in the explanations that they offered to each other, but it is difficult to truly separate these understandings from one speaker to the next. The interview provided an opportunity for Shelly to demonstrate her understanding of patterns at that particular point in time while hopefully limiting the effects of someone else's statements. Since the two did not have time to analyze the 4th pattern, I asked Shelly to take a look at this pattern (Figure 19). She first described the pattern in terms of its construction and then began to look for a way to describe the 8th term without building the preceding terms.

Identifying Patterns 4

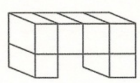
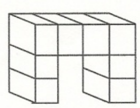
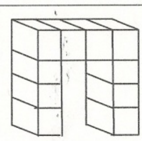
| Term Number (# of bridges) | Model | Written Description | Process Column | Numerical Value (# of blocks in bridges) |
|---|---|---|---|---|
| 1 |  | | | |
| 2 |  | | | |
| 3 |  | | | |

*Figure 19. Identifying patterns 4*

194. S: So you're wanting me to tell you how many blocks it would take to build the 8th one?

194. R: yeah…but I want to build it…

195. S: Okay

196. R: So you're going to have to tell me what it looks like…

197. S: Like how many blocks it would have in it?

198. R: Yeah…just what it looks like…what its structure would look like

199. S: Well it would have two legs and two in the middle

200. R: Okay…and so I have two in the middle and then how long are the legs?

201. S: Oh…

202. R: How many blocks would it take to build each leg?

203. S: Let's see here…for the 8th one…I keep looking back at these…

204. R: Well you can probably figure it out from those…

205. S: Um…

206. R: Remember I can't see them is the only thing

207. S: So it would be double…so it would be 24…

208. R: 24?

209. S: Yeah…is that right?...because…

210. R: So like…

211. S: No…because one to four is double… 6…12…6 plus 6 is 12

212. R: Okay

213. S: Does that make sense?

Shelly offered a short-cut for finding a non-sequential term that does not yield the correct answer, 20. She recognized that the number of squares used to construct the bridge doubled from the first to the fourth and made the assumption that the number would double again from the 4th to the 8th. Although Shelly did look down both sides of the table in search of a short-cut, she did not look across the table to forge a connection.

*Summary of Shelly's experiences.* The previous example illustrates the way Shelly approached the majority of the problems in these patterning activities. Instead of using the physical models to look for a connection between position and construction, she generally focused on the numerical data displayed in the table. When faced with the task of jumping ahead to a non-sequential term, Shelly would look for some type of numerical relationship between the known values in both data sets. Whether she applied proportional reasoning, as in Identifying Pattern 4, or the rate of change, as in Identifying Pattern 2, Shelly usually limited her analysis to one specific pair in the data set.

Shelly frequently sought verification for the answers or suggestions she made during both the 'together' session and during the interview. For example, while analyzing Identifying Pattern 4 in the interview, Shelly asked, "is that right" after providing an answer for the number of squares needed to construct the $8^{th}$ bridge in the pattern. She also asked, "Does that make sense?" after providing an explanation for how she got that answer.

*The 'Together' Session: Matt*

Matt wrote a verbal rule for the first think pattern, but did not generalize the second think pattern. The first think pattern had been presented as a pictorial growth pattern whereas the second only displayed the sequence in a numerical fashion. The four patterns to be analyzed during the 'together' session combined both a geometric representation of the pattern and a table. The first of these patterns was based on the number of exposed faces on a n-cube high tower. Shortly after beginning this session, Matt offered a short-cut for determining how many faces to paint on a tower that was 36 cubes high.

57. M: I get 153.

58. A: How'd you come up with that?

59. M: Because it’s 4 goes into 36 nine times.

60. A: Nine times?

61. M: So nine times 17  (laughs)

62. A: Okay….sooo….okay…so 17 times nine is what?

63. M: 153…Don’t quote me on this though.

Matt applied proportional reasoning to jump from the fourth term to the $36^{th}$. He apparently worked down the table, concentrating only on how the values on the left might be related and did verify whether or not his idea would work for known pairs in the relationship.

As described in the previous chapter, Ashley considered his answer but persisted in looking across the table to find a way to connect the values in both data sets. She stated a rule for Identifying Pattern 1, $4n + 1$, and demonstrated to Matt that the rule worked for the first four pairs in the table. The duo briefly attempted to complete the second patterning problem before moving on to the last two patterns. Ashley quickly determined the rule for the third pattern, but Matt was the first to find a way to connect a value on the left with its output on Identifying Patterns 4 (Figure 19).

129. M: so the first one has one down and four across?

130. A: yes…so

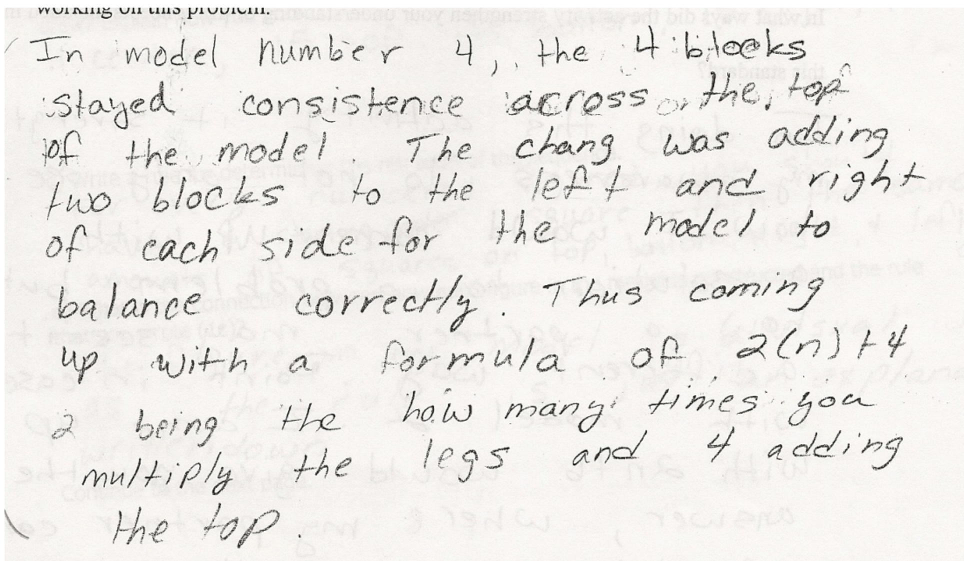
131. A: Four… its six is four..so 12 I mean 18..

132. M: Well..I think…I think

133. A: It has to be…

134. M: Would it be like 58 times 2 plus a …plus 4…

In response to the first prompt in the ‘reflect’ session, Matt explained how he arrived at this conclusion (Figure 20). Matt explained how he used the physical representation of the pattern to determine the construction of the 58th figure. He noted that “the first one has one down and four across” and realized that the 58th one would have 58 cubes on each end and four across. Although Ashley had verbalized the rule in terms of the *nth* figure as 2n plus 4, Matt explained that the coefficient represented ‘how many times you multiply the legs’ and the constant, 4, represented the top of the bridge.


working on this problem.

In model number 4, the 4 blocks stayed consistence across the top of the model. The chang was adding two blocks to the left and right of each side for the model to balance correctly. Thus coming up with a formula of 2(n)+4 2 being the how many times you multiply the legs and 4 adding the top.

*Figure 20. Matt’s reflective response to prompt one.*

When the duo returned to Identifying Patterns 2, Matt used the general construction of the model to describe the 85th term.

153. M: 85 times 2 plus 6

154. A: Eighty-five times 2 plus 6?

155. M: uh huh

156. A: 85 times 2 plus 6? Why do you say that?

157. M: Cause you have 85 yellow blocks so you have one on the top and one on the bottom…and it takes 6 to build your sides…three on each side.

As on Identifying Patterns 4, Ashley substituted the variable, *n*, for the specific value Matt used, stating "…so n times 2 plus 6" (*line 163_13TG*).

Whether or not Matt would have made use of the symbolic on his own is difficult to say due to his interaction with his partner. In the interview, Matt explained Ashley's contributions to the problem solving process.

> I can see them all visually um…most of the…the math problems she came up…especially with the n…you know the adding of the n's you know and …: except for that one…(pointing to identifying patterns 2)…Where she had this elaborate idea drawn out and I was like…okay…simplicity here…Um…cause…(laughs)…I…I don't do longevity…and so um…it was just a matter of talking them all out and just seeing the formula…but most of them like I said she came up with (*lines 229 - 237_13I*).

The interview offered the opportunity for Matt to work on a pattern finding activity without Ashley's input. He was shown the first three figures in a pattern of H's (Figure 21) and asked to describe the construction of the 8$^{th}$ figure.

> so…since it's saying constructing the 8$^{th}$ figure…and it looks like the pattern there was one two three…so you have the 8 in the middle… and then obviously you'll have your ends so it would make a total of nine going horizontally…no…8…9…you would have 10 going horizontally…with one on each end…but your 8 are still going to remain in the middle…and then…you have….8 more on the top of the…on each end….and 8 more on the bottom…(*lines 338 – 348_13I*)

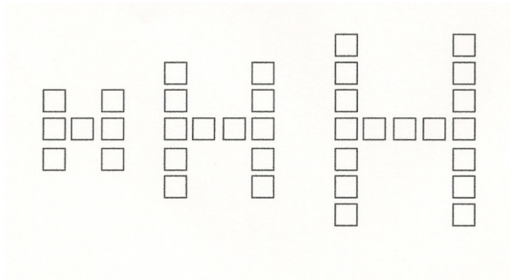


*Figure 21. Pattern of H’s*

After describing the construction, Matt determines that it would take 42 cubes to construct the 8th ‘H’ in the pattern.

355. R: 42…okay…all right…so you’d have 42…so let’s see…now they ask for the 15th …but I’m okay with that…I think you can explain the 15th…so what would the rule be?

356. M: Probably want to take um…the numbers consistent with the 8

357. R: mmhmm

358. M: (silence) I’m not sure how I would write that.

359. R: well why don’t you write down what you did with the 8…you had…the 8 plus 2….so go ahead and write that out…anywhere you want to.…it doesn’t matter

360. M: 8 plus 2…

361. R: and then you described how you got that part of it…so what did you have there

362. M: are we looking for the total or are we looking for the…

363. R: well how did you get those though…because you told me verbally…verbally you told me you would have 8 here and 8 here and 8 here and 8 here and then we took what?

364. M: 8 times 4…

365. R: yeah…8 times 4 so I would write that out here….8 times 4…and so…can you now…can you think about…what if I was doing the 15th one?

366. R: let's go ahead and do that one…what would that same stuff look like?

367. M: well there you have 15 plus 2

368. R: uh huh

369. M: and then 15 times 4

370. R: okay…so what if you were doing the *nth* one? Or any one?

371. M: silence(writing rule)

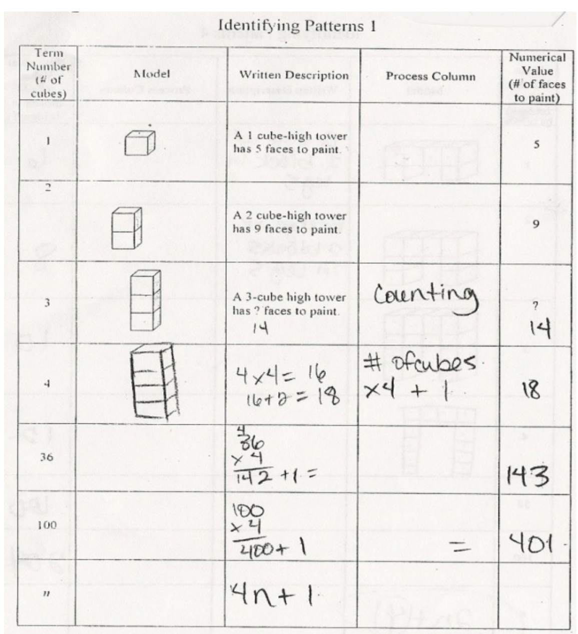
372. R: there you go!

373. M: n plus 2 plus n times 4

Matt recognized that he would need to connect the rule to the position of the figure, but is unsure how to write that out. With some prodding, he wrote down the arithmetic that he used to determine the number of cubes it would take to build the 8th and the 15th terms. He then used this arithmetic to write a rule for any figure in the sequence of H's. Matt's rule reflected his verbal description of particular figures in the model. He used the 3rd to describe how to construct the 8th and knew that the rule would "take the numbers consistent with 8." With additional prodding, Matt backtracked to describe the 'numbers consistent with 8' as the '8 plus 2' for the horizontal span and '4 times 8' for the four vertical sections of the model. The symbolic rule he wrote, "n plus 2 plus n times 4"communicates his understanding of the relationship between the position of the figure and the value it takes on. Moreover, the symbolic rule is entrenched in the everyday words he used to describe the construction of specific figures in the model.

*The 'Together' Session: Jill*

Jill had written a verbal rule for the *nth* term in pattern one, even making use of the variable n in her description (n = the number of squares on each leg). Her rule was incomplete in that she failed to include the constant part of the model (the central square). When her partner, Kristen, explained her rule for pattern one, Jill remarked that she "totally didn't even do that" (*line 7_7TG*) and recognized how Kristen "just took it further" (*line 17_7TG*). Jill did notice the need to take "it further" when faced with the familiar t-table in Think Pattern 2. She arrived at a rule for describing the relationship between the two data sets in the table by a process of guess and check. As they progressed to the 'together' session, the need to write a symbolic rule for the *nth* term became more explicit. The problems in this session combined the pictorial representation of a pattern along with a table. Jill's written response to the first patterning problem is displayed in Figure 22.



Identifying Patterns 1

| Term Number (# of cubes) | Model | Written Description | Process Column | Numerical Value (# of faces to paint) |
|---|---|---|---|---|
| 1 | | A 1 cube-high tower has 5 faces to paint. | | 5 |
| 2 | | A 2 cube-high tower has 9 faces to paint. | | 9 |
| 3 | | A 3-cube high tower has ? faces to paint. 14 | Counting | ? 14 |
| 4 | | 4×4 = 16 16+2 = 18 | # of cubes ×4 + 1 | 18 |
| 36 | | 36 ×4 142 +1 = | | 143 |
| 100 | | 100 ×4 400+1 | = | 401 |
| n | | 4n+1 | | |

*Figure 22. Jill's written response to Identifying Patterns 1*

Note in the process column, Jill has written out instructions for finding the number of cubes needed to build the $4^{th}$ tower in the model in general terms. She communicated her understanding of function as a general way of expressing a relationship between two data sets using a combination of words and symbols (# of cubes x 4 + 1). She then replaced the words to write a symbolic rule of $4n + 1$. Originally, Jill and Kristen had thought they would be painting the top and the bottom of the tower, but then Kristen noticed that they were building a tower and would not need to paint the bottom. Jill made a 'minor adjustment" (*line 84)* to account for the missing face on the tower. She responded to her partner's inquiry of "how did you do it?" (*line 79_7TG)*

> Well what I got was that however many blocks you have you multiply that by 4 for however many blocks you have and then you add two because that's to account for the top and the bottom. So you just multiply by 4 and add 1 if you're just doing the top…so it's the same thing only with one instead of two. (*lines 80, 82_7TG)*

Jill's explanation of how she determined the rule is stated in terms of the structure of the model which illustrates how she made use of the visual to connect the two data sets.

The two of them had completed work on Identifying Pattern 1 within the first 4 minutes of sharing. The pair began to analyze the second pattern (Figure 10) and right away, Jill shared with her partner, "I don't even know with that one. I've tried some little theories but they're wrong (*line 88_7TG*). Jill figured out that the rate of change and used that to determine how many squares it would take to build the $4^{th}$ model, stating, "It's 2…and then plus 2 more would be 14" (*line 101_7TG*). The two then began the task of describing a non-sequential term.

103. J: so it's just gonna be two more every time (pause).

104. J: Hmm…so that's kinda the same number I came up with the other one. I see how that one changes …I see how this one changes and I see how this one changes but I don't see how they change together

105. K: mmhmm

106. J: Like the equation that you'd write. So…so then for 85 of those bad boys how many would you need?

107. K: So we have…so I have to find a relationship still between those two.

108. J:Hmm

Although Jill made the comment about not seeing how they change together, she was not referring to the rate of change because she acknowledged to her partner that "it's just gonna be two more every time" (*line 103_7TG*). Instead, Jill was communicating the problems she had with analyzing the relationship across the table. She explained these difficulties in reference to this patterning problem (Identifying Patterns 2) in the interview.

> Yeah…that happens to me a lot of times…you know I can see how this one…you know…each term is going to be one higher than the next…and I can the pattern in this one but a lot of times I just have a…I will be completely blocked off from seeing what does one have to do with eight…you know and it will really bother me…you know…um.. so that's when I will just have to start guessing at it…and I'm sure that other people have different methods of going about it but usually whenever it's just not really plain and obvious to me to start with… I just have to start…oh let's try this (*line 132_7I*)

After several long pauses in the conversation, Kristen comes up with the rule $n + (n+ 2) + 6$ which she built by looking for a pattern of differences between the number of red cubes and the number of yellow cubes used to construct the figures. During the interview, Jill explained why the second patterning problem was not as easy as the first one.

> I think because even though they're both three-dimensional, this one seems more so… like it's definitely got more meat to it…instead of just being a single row…it's the whole huge…you know…the whole cube is growing…And I think that made it…maybe made it harder to visualize…you know the numbers are increasing more rapidly or something like that to where you know…like hold on a second…and so it seemed a little you know slower to come to a conclusion on that one. (*lines 108, 110_7I*)

When the two move on to pattern 3, Jill comes up with the rule before her partner.

109. J: Okay I think it's 3n + 3.

110. K: Huh?

111. J: It works for all the other ones

112. K: Okay

113. K: Number 6…nine is plus three…yeah it is.  The number of times…I made the three connection too.

114. J: Yep! I think it's good!

Jill recognized that her rule was correct because it worked "for all of the other ones," but she doesn't explain if she used the physical model to write the rule or if she looked for and applied a process of guess and check to connect the data pairs in the table.

The interview offered another opportunity for Jill to demonstrate how she went about the process of analyzing these patterns. She completed the same patterning activity that Matt had worked on during his interview session (Figure 21).

185. J: okay I would tell you to start…let's see…okay…with um…8 blocks in the middle

186. R:okay

187. J: and…let's see…and then one more on either side of the middle and then eight on each leg of the H

188. R: okay

189. J: …I think… because I'm just going by the fact that this is the third term and it has three in the middle and three here, three there, three there, and three here. And I guess I'm just kind of forgetting about these guys…that's where the little plus two would come from if you're writing the rule…probably

She goes on to state the rule, 5n plus 2, and remarked, "Awesome…that one came real easily" (*lines 201, 203_7I*). Jill thus communicated her understanding of function as correspondence by making the connection between how the position of the term relates to its construction. She used a specific figure, the third, to make this connection and then applies that to her instructions on how to build the eighth figure. By doing so, Jill communicated her understanding of the relationship across the table by determining "…what do I do to this to get these" (*line 69_7I*).

## Oklahoma State University Institutional Review Board

| | |
|---|---|
| Date: | Wednesday, April 01, 2009 |
| IRB Application No | ED0959 |
| Proposal Title: | Preservice Elementary Teachers Understanding of Pattern and Function |
| Reviewed and Processed as: | Exempt |

**Status Recommended by Reviewer(s): Approved   Protocol Expires: 3/31/2010**

Principal Investigator(s):

| | |
|---|---|
| Valerie Sharon | Patricia Jordan |
| 8310 S. 69th E. Ave. | 247 Willard |
| Tulsa, OK 74133 | Stillwater, OK 74078 |

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The IRB application referenced above has been approved. It is the judgment of the reviewers that the rights and welfare of individuals who may be asked to participate in this study will be respected, and that the research will be conducted in a manner consistent with the IRB requirements as outlined in section 45 CFR 46.

☒ The final versions of any printed recruitment, consent and assent documents bearing the IRB approval stamp are attached to this letter. These are the versions that must be used during the study.

As Principal Investigator, it is your responsibility to do the following:

1. Conduct this study exactly as it has been approved. Any modifications to the research protocol must be submitted with the appropriate signatures for IRB approval.
2. Submit a request for continuation if the study extends beyond the approval period of one calendar year. This continuation must receive IRB review and approval before the research can continue.
3. Report any adverse events to the IRB Chair promptly. Adverse events are those which are unanticipated and impact the subjects during the course of this research; and
4. Notify the IRB office in writing when your research project is complete.

Please note that approved protocols are subject to monitoring by the IRB and that the IRB office has the authority to inspect research records associated with this protocol at any time. If you have questions about the IRB procedures or need any assistance from the Board, please contact Beth McTernan in 219 Cordell North (phone: 405-744-5700, beth.mcternan@okstate.edu).

Sincerely,

Shelia Kennison, Chair
Institutional Review Board

NORTHEASTERN STATE UNIVERSITY
Tahlequah Muskogee Broken Arrow

Institutional Review Board

Human Subjects Review
Date: April 1, 2009

Proposal Title: Preservice Elementary Teachers' Understanding of Pattern and Function

IRB #: 09-217

Dear Mrs. Sharon and Dr. Jordan

Your research proposal has been approved by the Institutional Review Board at Northeastern State University. It is the IRB's opinion that you have provided adequate safeguards for the welfare of the participants in this study.

You are authorized to begin your research and implement this study as of the start date listed in your application, or the date of this communication if you listed the start date as "As soon as possible", "Upon IRB Approval", or a similar phrase. This authorization is valid until the end date listed in your application, or for one year after approval of your study, whichever is earlier. After this authorization runs out, you are required to submit a continuation or renewal request for board approval.

This approval is granted with the understanding that the research will be conducted within the published guidelines of the NSU Institutional Review Board and as described in your application. Any changes or modifications to the approved protocols should be submitted to the IRB for approval if they could substantially affect the safety, rights, and welfare of the participants in your study.

Thank you for sending us your application for research involving human subjects. In doing so, you safeguard the welfare of participants in your study and federal funding of our university.

Signed: E Bell

Ernst Bekkering

Chair, Institutional Review Board

601 North Grand Ave. • Tahlequah, Oklahoma 74464-2399 • p: 918.456.5511, ext. 2220 • f: 918.444.2295

VITA

Valerie Vinyard Sharon

Candidate for the Degree of

Doctor of Philosophy/Education

Thesis: PRESERVICE ELEMENTARY TEACHERS' UNDERSTANDING OF PATTERN AND FUNCTION

Major Field: Professional Education Studies with focus on Mathematics

Biographical:

Education:

Completed the requirements for the Doctor of Philosophy/Education in Professional Education Studies at Oklahoma State University, Stillwater, Oklahoma in May, 2010.

Completed the requirements for the Master of Science in College Teaching with emphasis in Mathematics at Northeastern State University in Tahlequah, Oklahoma in May, 2004.

Completed the requirements for the Bachelor of Arts in Biology at Trinity University, San Antonio, Texas in August, 1979.

Experience: Assistant Professor in Curriculum and Instruction; Instructor of Mathematics; High School Mathematics Teacher; Adjunct Instructor of Mathematics; Middle School Mathematics Teacher; Mathematics and Biology Tutor.

Certifications: Oklahoma Standard Teaching Certificate in Algebra, Analysis, General Mathematics, Geometry, Statistics, Trigonometry, Anatomy/Physiology, Biology, Botany, General Science, Zoology (7 – 12); Texas Provisional Secondary Biology and Mathematics, (6 – 12).

Professional Memberships: The Honor Society of Phi Kappa Phi
Kappa Mu Epsilon National Mathematics Honor Society
Kappa Delta Pi National Honor Society
National Council of Teachers of Mathematics (NCTM)
Research Council on Mathematics Learning (RCML)

Name: Valerie Vinyard Sharon
Date of Degree: May, 2010

Institution: Oklahoma State University
Location: Stillwater, Oklahoma

Title of Study: PRESERVICE ELEMENTARY TEACHERS' UNDERSTANDING OF PATTERN AND FUNCTION

Pages in Study: 257
Candidate for the Degree of Doctor of Philosophy

Major Field: Professional Education Studies with focus on Mathematics

Scope and Method of Study: The purpose of this study was to unpack the understandings pre-service elementary teachers have pertaining to the ideas of pattern and function. The intent was to bring insight into how mathematics teacher educators can use patterning activities to prepare pre-service elementary teachers to support the development of algebraic thinking in their future students. The assumption that learning in mathematics is a hermeneutical process indicated that access to pre-service elementary teachers' understanding of pattern and function could be obtained by analyzing the explanations offered while they were engaged in pattern-finding activities. Based on this assumption, the texts produced by six pre-service elementary teachers while they were engaged in the process of generalizing linear patterns were collected and analyzed. The methodological framework of hermeneutic phenomenology guided the research design and the analysis of the textual accounts collected for this study.

Findings and Conclusions: Four overlapping ideas of function emerged through the analysis of the conversations and written texts associated with the task of pattern-finding: the ideas of function as a pattern, as a relation, as a rule, and as a process. The region where all four ideas overlapped represented the kernel of algebraic thinking, the idea of function as generalization. When these pre-service elementary teachers combined these four ideas in an integrated approach to pattern-finding, they were able to generate a way to describe the general term in a linear pattern. The omission of one of these four ideas, particularly the ideas of function as a pattern or as a relation, presented a roadblock to generalization. In conclusion, the results of this study pointed to a four-stage process to understanding linear functions. The first stage begins with the identification of the pattern of change. This was followed by a stage of discrimination, in which the individual noted both the variable and invariant features of a linear model. The act of discrimination often led to the ability to generalize the pattern. Ultimately, a few of the participants began to notice those properties that all linear patterns had in common. This final stage of synthesis resulted in a more complete understanding of the concepts of pattern and function.

ADVISER'S APPROVAL: Dr. Patricia Jordan