PROSPECTIVE ELEMENTARY TEACHERS' DIVISION OF FRACTIONS UNDERSTANDIING: A MIXED METHODS STUDY

By

ADRIENNE REDMOND

Bachelor of Science in Elementary Education
Oklahoma State University
Stillwater, OK
2000

Master of Arts in Curriculum and Instruction
University of Phoenix
2003

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Dissertation Approved:

Dr. Juliana Utley

Dissertation Adviser

Dr. Patricia Lamphere-Jordan

Dr. Carolyn Beller

Dr. Karen High

Dr. A. Gordon Emslie

Dean of the Graduate College

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CHAPTER I

INTRODUCTION

A growing body of research describes the vital role that fractions play in the understanding of algebra in high school (Fennell, 2007; National Mathematics Panel, 2008). However, in the recently published National Mathematics Panel Report (2008), the National Mathematics Panel suggested that fractions continue to be an area of weakness for students and teachers. Without this vital understanding of fractions, and later algebra, students are less likely to attend and complete college (Brown, 2007; Evan, Gray, & Olchefske, 2006; Wu, 2001). Thus, a significant effort should be made to help students and teachers develop a profound understand fractions and fraction operations.

What does it mean to understand fractions, or even more generally, what does it mean to understand mathematics? This question has drawn much controversial discussion in the recent past. According to Schoenfeld (2007), understanding mathematics is having knowledge as well as knowing how to use and apply that knowledge. Schoenfeld highlighted the importance of "facts, concepts, procedures, definitions, and concepts," but stressed that mathematical proficiency is much more than regurgitating the content (Schoenfeld, 2007, p. 60). The reform efforts of the 20th century set out to specifically address student achievement and provided guidance for teachers to help their students develop mathematical proficiency.

Foundation of the Problem

The first major episode of the twentieth century that led to an outcry for change in mathematics education was the launch of Sputnik in 1957. There had been other calls for change, but none had the impact of the successful launch of the Russian Sputnik. The United States, while in the middle of the Cold War, felt as though they were perceived as a weak nation because they had been out maneuvered by the Soviet Union. The mathematics curriculum that was developed in the wake of this event was dubbed “new math” and included set theory, modular arithmetic, and symbolic logic (Schoenfeld, 2004). Teaching the new math was unsuccessful in helping the United States improve mathematical knowledge because the curriculum failed to provide the educational stakeholders with support (Schoenfeld, 2004). Teachers and parents felt disenfranchised by the curriculum, and this feeling made it difficult for teachers to implement the content with integrity. “By the 1970's, new math was dead and American schools went back to the basics" (Schoenfeld, 2004, p. 257). The “back-to-the-basics” curriculum focused on skills and procedures and often ignored problem solving or learning mathematics in context (Van de Walle, 2007).

In the 1980’s, a report entitled *A Nation at Risk* (Department of Education, 1983) was publicized. This call for educational reform and a national curriculum resounded throughout public schools in the United States. The report provided data that implied that American students were poor problem solvers, as well, and struggled with procedural knowledge as well. In response to this report, the National Research Council (NRC, 1989) released a report entitled *Everybody Counts* that discussed the inequity and low standards that plagued mathematics and society. This document also called for a national curriculum that had high standards for all students and not just the elite or college bound. *Everybody Counts* thus paved the way for the National Council of Teachers of Mathematics to publish the first set of national standards. The *Curriculum and Evaluation Standards* (NCTM, 1989) were developed to ensure that all students were active participants in their education, rather than passive listeners and required that "all" students be mathematically literate.

In the early 1990's, the National Science Foundation (NSF) decided to offer grants to those who were writers of reform curriculum. California was one of the first states to take advantage of this opportunity to be on the leading edge of change. Soon thereafter, parents began protesting against the reform-based curriculum entitling it "new-new math" or fuzzy math. This battle became heated and much politicized. By 1998, the Math Wars had turned into a national battle that is still being fought today. Those opposed to standards-based mathematics argued that the traditional education system works for all students.

The traditional educational system, originally designed for factory workers, has not changed much over the years. The job skills that were required involved following directions with minimal individual problem solving required (Papas & Tepe, 2002). The future job skills that students will need in the 21st Century workforce are drastically different. Students need to be able to problem solve, communicate, work collaboratively on teams, and be able to think and reason creatively. Without the opportunity to experience these skills in the classroom, this nation's children will be "left behind" (The Partnership for 21st Century Learning Skills, 2007). These are all very important skills that can be developed in the mathematics classroom (21st Century Learning, http://www.21stcenturyskills.org). Despite the reform efforts of the 20th Century, the latest results from the Program for International Student Assessment (PISA) show that America's 15 year-olds placed 27th out of 39 countries that participated (U.S. Department of Education, 2004).

Appalled by our dismal performance on international comparison studies, the No Child Left Behind Act (NCLB) was passed in 2001 to address some of the problems that our schools and nation were facing. The NCLB called for stronger state standards, stronger accountability systems, and the placement of highly qualified teachers in all classrooms. This act had the potential for helping to create smarter students and a highly prepared workforce for the future of America.

The NCLB act called for an accountability system to be established to monitor performance of all students in all subgroups. NCLB required that all students be at a proficient level or above in mathematics by the year 2014. Most students are currently performing below proficiency in mathematics by the end of fourth grade. The number of students scoring below proficiency in mathematics is even greater for students that are "at risk" with very few students having reached proficiency in mathematics by the end of their 12th grade year (Department of Education, 2004). The National Assessment of Education Progress (NAEP) has tracked the nation's mathematical progress since 1973. There have been statistically significant gains in all of the areas assessed, about 1.5% per year, but the rate of increase per year is still not enough to help students meet the goals prescribed in NCLB (Warfield & Kloosterman, 2006).

Fourth grade students' rational number concept performance on the *NAEP Report Card* has remained relatively stable since 2000 (Warfield & Kloosterman, 2006). While fourth grade students' ability to represent a fraction as a part of a whole has increased from 69% in 1992 to 83% in 2003, only 73% of eighth graders were able to represent a fraction as a part of a whole. When eighth grade students were provided with a scenario and asked to write a problem that would require them to use fraction division, only 12% could correctly write a problem to fit the situation. In 2003, only 64% of eighth graders could place the fraction ¾ correctly on a number line and when provided with a fraction division word problem, only 55% answered the problem correctly. Warfield and Kloosterman (2006) determined that American students are still struggling with rational number concepts.

When students first encounter instruction in rational number, they must reformulate their concepts of whole numbers in such a way that they can look at two numbers that are related to each other multiplicatively instead of additively (NRC, 2005, p. 310). Lamon (1999) found that some students make this leap from whole numbers to rational numbers easily while others struggle making the transition. Research also shows that some students that struggle with rational numbers do so because they continue to hold on to their whole number reasoning and apply it in situations where that thinking is not appropriate (Moss, 2005; Hiebert & Wearne, 1986; Hiebert & Behr, 1988) and, for the first time, many students begin to struggle with mathematics (Lamon, 1999; Lesh, Post, & Behr, 1988).

Fractions are often defined in textbooks and by prospective teachers as only a part of a whole (Carraher, 1996). This limited definition of fractions can have educational implications for teachers and students who are trying to make sense of fractions. In a seminal piece on the understanding of fractions, Kieren (1976) concluded there are seven interpretations of rational numbers: 1) fractions, 2) decimals, 3) equivalent classes of fractions, 4) ratio numbers, 5) multiplicative operators, 6) elements of an infinite ordered quotient field, and 7) measures or points on a number line. Kieren noted that these seven interpretations were neither an exhaustive list nor independent of one another. However, current researchers tend to collapse the seven interpretations down into five: 1) measure, 2) quotient, 3) ratio, 4) operator, and 5) part-whole (Lamon, 2007; Freudenthal, 1983; Behr, Lesh, & Silver, 1983; Behr, Harel, Post, & Lesh, 1992; Kieren, 1988). Due to the multiple interpretations, the teaching and learning of fractions is complicated and when students do not learn to make sense of these interpretations of fractions this lack of understanding continues to persist into adulthood (Lamon, 2005).

If adults do not learn fractions in school, when will this learning occur? Teachers often enter the teaching profession with a superficial knowledge of the mathematics they teach (Ma, 1999). "Effective mathematics teaching requires understanding what students know and need to learn and then challenging and supporting them to learn it well" (NCTM, 2000, p. 11). If teachers have not developed a deep understanding of the content that they teach, they are less likely to be able to teach for mathematical proficiency (Ball & Bass, 2003; Borko, Eisenhart, Brown, Underhill, Jones, & Agard, 1992; Ma, 1999; NRC, 2001). Many researchers attribute this lack of teacher knowledge to their own lack of appropriate schooling experiences (Schifter & Fosnot, 1993).

Researchers have classified teachers' understanding of fractions and fraction operations as disconnected and compartmentalized (Graeber, Tirosh, & Glover, 1989; Mewborn, 2001; NRC, 2001, 2005). Ball and Bass (2003) suggested that the answer to this problem does not lie in the number of mathematics courses that teachers take unless those courses are taught in a way that helps teachers to develop a deep understanding of content knowledge.

Statement of the Problem

Teachers that do not understand the interpretations of division of fractions often struggle to help their students make sense of that concept (Ball, 1990; Graeber, et al., 1999; Redmond & Utley, 2007). This is particularly troubling when teaching students to understand fraction division, where representations and conceptual understanding are vital to helping students make sense of division with fractions and the common invert and multiply and common denominator algorithms (Mewborn, 2001). Teachers also have trouble helping students make real-world connections for division of fractions due to their lack of ability to write word problems that accurately depict a real-world fraction division situation. Research shows that teachers and prospective elementary teachers often wrote multiplication problems when asked to write division problems or were unable to write a correct problem at all (Tirosh, 2000; Redmond & Utley, 2007).

In addition to having a fragile understanding of division of fractions, prospective elementary school teachers often come to their prospective programs with the belief that they know enough mathematics to teach mathematics effectively. They feel that teaching mathematics is simply a matter of explaining to children how to do a problem. These college students enter teacher education programs with a variety of formal and informal schooling experiences, which often influence their view of teaching and learning mathematics. The prospective teachers also enter teacher education programs with negative attitudes toward mathematics and these beliefs, in turn, can have a negative impact on their future students' attitudes and achievement. Teachers that have a negative disposition towards mathematics tend to teach in a more traditional manner (Swars, Daanem, & Giesen, 2006). However, Swars, Daanem, and Giesen (2006) found that the beliefs and personal teaching efficacy that teachers hold are quite resilient to change and must be addressed as early on in the career as possible.

In 2007, a pilot study was conducted to determine if a mathematics methods course could help increase the content knowledge, attitudes, confidence, and efficacy of prospective elementary teachers enrolled in their final mathematics methods course prior to student teaching. The researchers found that the division of fraction content knowledge and attitudes towards the teaching and learning of division of fractions of 61% of the participants actually increased throughout the duration of the mathematics methods course. However, while analyzing the solution strategies and interview data of the participants, the researchers found that many of the students who increased their content knowledge began to use a variety of representations throughout the posttest. The researchers suggested that future research should analyze why the students changed solution strategies as the problems became more complex (Redmond & Utley, 2007).

Numerous studies have examined prospective elementary and practicing elementary teachers' understanding of fractions and fraction division and have shown that teachers often have the same fragile understanding of fractions and fraction division as their students (Ma, 1999; 5; Redmond & Utley, 2007; Tirosh, 2000). However, few research studies have looked at the specific strategies that prospective elementary teachers use to solve division of fractions problems, how those strategies change after a mathematics methods course, and how prospective elementary teachers monitor their choice of strategies while solving division of fractions problems.

Purpose Statement

The purpose of this research study was to describe the content knowledge that prospective elementary teachers have at the beginning of their last mathematics methods course prior to student teaching and determined how that knowledge changed throughout the course of the semester. The research also examined and described the solution strategies that the prospective elementary teachers used at the end of their methods courses and how they monitored and chose those specific strategies. Finally, the researcher sought to determine what experiences in the mathematics methods course the prospective elementary teachers found had an effect on their own understanding of division of fractions and how they monitored their own behaviors when solving division of fractions problems.

Research Questions

The specific research questions guiding this study were:

1. What understanding of division of fractions do prospective elementary teachers bring to their final mathematics methods course?

2. What beliefs (confidence, anxiety, and personal teaching efficacy) do prospective elementary teachers bring about division of fractions to their final mathematics methods course?

3. Is there a difference in prospective elementary teachers' understanding of division of fractions after participation in their final mathematics methods course?

4. Is there a difference in prospective elementary teachers' beliefs (confidence, anxiety, and personal teaching efficacy) about the division of fractions after participation in their final mathematics methods course?

Results of this study contribute to the literature on the teaching and learning of fraction division and how to prepare teachers with a deep understanding of the mathematics that they teach. The results also help teacher educators to select effective methods to help prospective elementary teachers make sense of their thinking and monitoring of fraction division problems solutions in hopes that they will model those processes with their future students.

Assumptions

1. It is assumed that the participants responded honestly and thoughtfully in their interviews and surveys.
2. It was also assumed that the think aloud interview sessions provided a clear picture of what the participant was thinking and that probing and follow up questions helped to elicit how the student monitored their solution of problems.
3. It was assumed that metacognitive monitoring could be measured during a think aloud problem solving situation.

Limitations

1. The participants of this study were prospective elementary teachers enrolled in an intermediate mathematics methods course in a Midwestern university town.

Therefore, it was a sample of convenience and the findings may not be generalizable to the general population of all prospective elementary teachers.

2. The participants of this study were primarily female and Caucasian.
3. The researcher had prior experiences with the course; therefore, she brought with her some preconceived notions about the content knowledge and thinking strategies of the prospective elementary teachers.
4. The researcher had prior experience with the course participants in a previous mathematics methods course and thus has developed a relationship with the participants.

Definition of Terms

Prospective Elementary Teachers – Undergraduates who have declared a major in either elementary education.

Content Knowledge – The knowledge of the content that a teacher teaches. Mathematics content knowledge is knowledge of the mathematics that a teacher possesses. It is also referred to in the literature as subject matter knowledge.

Standards-Based Instruction: This refers to instruction that helps students develop both conceptual understanding and procedural fluency as recommended by the National Council of Teachers of Mathematics in *The Principles and Standards of School Mathematics* (2000). There is a focus on the five process standards during instruction (problem solving, communication, connections, reasoning and proof, and representation).

Conceptual Understanding of Mathematics: Conceptual understanding involves an understanding of the concepts, operations, and relations in mathematics (NRC, 2001).

Procedural Understanding of Mathematics: Procedural understanding is the knowledge of the "rules and procedures used in carrying out routine mathematical tasks and also the symbolism used to represent mathematics" (Van de Walle, 2007, p. 28).

Organization of the Study

This dissertation study is presented in the five chapter organizational format. Chapter I provided an introduction to the study, foundation of the problem, description of the problem, the purpose of the study, the research questions addressed in the study, assumptions and limitations, and definitions of terms that were used throughout the study. Chapter II includes a review of the literature related to the study as to provide "the reader with the results of other studies that are related to the study being reported. It also relates a study to the larger ongoing dialogue in the literature about the topic, and fills in the gaps and extends prior studies" (Cresswell, 2003, p. 30). In chapter III, the methodology of the study is discussed in detail so that future replications of the study will be possible. This section specifically addresses the participants, the design of the research, procedures used to collect data, instruments used to collect data, and data analysis procedures that were implemented. Chapter IV presents the analysis of the data and Chapter V presents the findings of the study as well as the conclusions, implications, and calls for additional research.

CHAPTER II

REVIEW OF LITERATURE

The purpose of this chapter is to report the relevant research related to the division of fractions content knowledge that prospective elementary teachers bring to their mathematics course and to see how their knowledge changes at the end of the class. The research questions guiding this review are:

1. What understanding of division of fractions do prospective elementary teachers bring to their final mathematics methods course?

2. What beliefs (confidence, anxiety, and personal teaching efficacy) do prospective elementary teachers bring about division of fractions to their final mathematics methods course?

3. Is there a difference in prospective elementary teachers' understanding of division of fractions after participation in their final mathematics methods course?

4. Is there a difference in prospective elementary teachers' beliefs (confidence, anxiety, and personal teaching efficacy) about the division of fractions after participation in their final mathematics methods course?

Several areas of research that are related to the current study will be addressed. Each area will be discussed and a summary of the important findings will be provided at the end of the chapter. The sections of this chapter include the cogent research on the following topics:

1. Environments in the Teaching and Learning of Mathematics
2. Teacher Knowledge
3. Teacher Knowledge and the Impact on Student Achievement
4. Research on the Teaching and Learning of Fraction Sense
5. Research on the Teaching and Learning of Division of Fractions

Environments in the Teaching and Learning of Mathematics

How Students Learn (NRC, 2005) provides a theoretical framework for looking at the effectiveness of teaching and learning of mathematics. In this framework, the authors stress the importance of looking at learner-centered environments, knowledge-centered environments, assessment-centered environments, and community-centered environments. Through each of these lenses, the authors bring out important characteristics of each environment. However, the researcher believes that it is essential to have aspects of all of these environments in order to teach mathematics effectively.

Learner- centered classroom environments

When classrooms use a learner-centered approach to teaching and learning, the teacher focuses on the students themselves. The teachers determine the preconceptions, ideas, attitudes, cultures, and backgrounds that students bring to the learning situation (NRC, 2005). Students do not come into the classroom as a blank slate. They bring with them a multitude of formal and informal experiences that they use as a connecting point for new knowledge. The learner-centered teacher is aware of this fact and attempts to assess students' misconceptions, build missing but necessary prior knowledge, and then use that knowledge to challenge students to make sense of the new knowledge (NRC, 2005; Van de Walle, 2007).

Knowledge-centered classroom environments

The knowledge-centered lens of teaching and learning looks at the content that is to be taught, why the content is taught, how it will be taught (curriculum), and to what extent the content will be taught (learning goals) (NRC, 2005, p. 14-15). The content, centered around the big ideas, should be learned in a clear, connected, and coherent way (NCTM, 2000; Van de Walle, 2007). The curriculum in the United States of America has often been described as shallow with many topics covered repeatedly without much depth (NCTM, 2006; Schmidt, McKnight, & Raizen, 1997; Van de Walle, 2007). This shallow coverage of many topics in such a short period has resulted in superficial learning of mathematics by American students (NRC, 2001). The National Council of Teachers of Mathematics (NCTM) recently published the *Curriculum Focal Points for Prekindergarten through Grade 8: A Quest for Coherence* to help teachers and curriculum developers create a curriculum that is focused and helps students to develop a deep and connected understanding of mathematics (2006).

> *"Organizing a curriculum around these described focal points, with a clear emphasis on the processes that Principles and Standards addresses in the Process Standards—communication, reasoning, representation, connections, and, particularly, problem solving—can provide students with a connected, coherent, ever expanding body of mathematical knowledge and ways of thinking" (p. 10).*

When students learn mathematics this way, they develop the mathematical proficiency that was called for in *Adding it Up* (2001).

Assessment-Centered Classroom Environments

An assessment-centered classroom consists of formative and summative assessments. Assessment should be ongoing and guide the teaching and learning of mathematics. Teachers in assessment-centered classrooms assess their students' preconceptions prior to learning and use that assessment information to choose learning tasks that will cause students to confront their misconceptions and construct new knowledge. During the learning process, the teachers monitor the students' progress to make sure that they are reorganizing their new knowledge effectively and monitoring their own learning processes. This metacognitive monitoring helps the students become validators of their own knowledge instead of relying on outside sources for answers. Researchers have found that when students monitor and assess their own learning achievement improves (Black & William, 1998; Lin & Lehman, 1999; National Research Council, 2000, 2005; White & Fredrickson, 1998).

In 1995, NCTM published the Assessment Standards for School Mathematics to address the need for formative assessment. NCTM identified six assessment standards that are important for the teaching and learning of mathematics. They suggested that assessment should (1) reflect the mathematics that students should know and do, (2) enhance mathematics learning, (3) promote equity, (4) be an open process, (5) promote valid inference, and (6) be a coherent process (NCTM, 2000, p. 21). This idea was further elaborated in the NCTM *Principles and Standards for School Mathematics* (PSSM) (2000). In the PSSM document, NCTM recommended the following components as part of the framework of the Assessment Principle: (a) assessment should enhance learning, and (b) assessment is a valuable tool for making instructional decisions. Teachers in an assessment-centered classroom constantly monitor their students' progress, but they also monitor their own teaching and use the information gleaned from assessments to modify their instructional decisions.

Community-Centered Classroom Environments

A community-centered classroom focuses on developing sociomathematical norms that encourage students to learn with and from one another. This type of classroom also encourages students to see real-world connections to the mathematics that they are learning. The teacher acts as a co-learner in the classroom instead of the authority. Students are encouraged to discuss and debate their findings and justify their thinking to the teacher and their peers. Students actively try to socially construct their knowledge through interactions with their peers.

According to Lampert and Cobb (2003), in order for all students to become successful in the mathematics classroom, the environment in which students learn must encourage them to participate in classroom discourse. This classroom discourse takes on many forms of communication, both written and verbal. Participating in this discourse can help students to make a personal connection to the mathematics they are learning (D'Ambrosio & Steffe, 1995). As students are discussing, debating, justifying, verifying, and writing about mathematics, the mathematical ideas that they are learning "become objects of reflection, refinement, discussion, and amendment" (NCTM, 2000, p. 60).

Mathematical discourse is essential in the classroom because it can help students to build meaning, reason mathematically, justify their thought processes, see multiple perspectives, and learn and use the language of mathematics by building on their informal knowledge (NCTM, 2000; Van de Walle, 2007). However, Cobb (1995) found for this communication to be useful, students must participate in small group discussions as well as whole class discussions. Students must learn to respect their classmates' findings and must make an attempt to see the other's point of view. Researchers caution that students might not always be able to see the other's perspective and may need real guidance to come to similar conclusions (NCTM, 2000; Lampert and Cobb, 2003).

Staples and Colonis (2007) described two specific types of discussions in the classroom: (1) sharing discussions and (2) collaborative discussions. When students participate in sharing discussions, they share their ideas but maintain a focus on their own reasoning and processes. Collaborative discussions often prompt students to make sense of the ideas of a classmate, make connections between their thinking and that of their classmates, and to build upon that those connections by developing meaning.

In collaborative discussions, an incorrect response can be used to generate discussions without making the child with the incorrect answer feel uncomfortable. The class is, however, not left to think that an incorrect answer is correct. When the students leave the classroom thinking an incorrect answer is correct, this can reinforce misconceptions, which are very hard to overcome. Teachers need to address the misconceptions the same day, if possible, by posing additional tasks that will bring out those misconceptions. If the teacher cannot address the misconceptions at that time, it is essential that he/she return to the misconceptions later. Collaborative discussions would provide the context for developing a deeper and more flexible understanding of the mathematics and, it appears, should occur in classrooms.

Foreman (1996) sees mathematics as an apprenticeship in which students participate. She argued that:

> *"The mathematical discourse that students are to master* is *a specialized type or genre of speech that she calls the mathematics register. In the course of their apprenticeship, students participate in the discourse in increasingly substantial ways as they come to understand the skills, norms values that are shared by mathematically literate adults"* (p. 239).

Community-centered classrooms provide students with more opportunities to participate in this mathematical discourse and thus allow students to communicate their ideas, to develop rich understandings of the mathematics in which they are learning, and to allow students to see mistakes that occur as learning opportunities and not points of shame (Brown & Campione, 1994; Cobb, Yackel, & Wood, 1992, NRC, 2005). Similarly, when students pose and answer challenging questions, they begin to reflect upon and analyze their peers' responses thus helping students to use metacognitive processes to monitor their own thought processes (NRC, 2005).

The three aforementioned pieces of literature, *Adding it Up*, *Principles and Standards for School Mathematics*, and *How Students Learn* share some of the same underlying themes: the way that mathematics is taught in the United States must change if our students are to compete in the future global environments and that teachers need to be prepared to teach their students by having a deep understanding themselves of the mathematics that they teach.

Teacher Knowledge

That teachers play an essential role in the education of students is an undisputed fact. However, the exact role that a teacher should play in helping students to understand mathematics has encountered much debate in the recent past (Schoenfeld, 2004). According to the National Council of Teachers of Mathematics (2000), "Effective mathematics teaching requires understanding what students know and need to learn and then challenging and supporting them to learn it well" (p. 16). To accomplish this task, teachers must have an understanding of how students learn as well as a deep understanding of the content that they are to teach.

Shulman and Grossman (1988) found that teacher knowledge is separated into two specific domains: subject matter knowledge and pedagogical content knowledge. Subject matter knowledge has been defined as "knowledge of the key facts, concepts, principles, and explanatory frameworks of a discipline, as well as the rules of evidence used to guide inquiry in the field" (Borko, et. al., 1992, p. 195). Pedagogical content knowledge is defined as the knowledge for teaching. "It represents the blending of content and pedagogy into an understanding of how particular topics, problems or issues are organized, represented, and adapted to the diverse interests and abilities of learners, and presented for instruction" (Shulman, 1987, p. 4). The interplay of these two domains of knowledge make up the specialized body of knowledge that a teacher needs to be able to pose appropriate tasks, ask thought-provoking questions, scaffold learning for the students, and provide rich and multiple representations to help students make sense of abstract mathematical ideas. Researchers have shown that this type of knowledge is not commonplace in American classrooms and teacher knowledge has often been categorized as fragmented (Ma, 1999; NCTM, 2000).

Many researchers attribute this fragmented knowledge to teachers' own formal schooling experiences (Schifter & Fosnot, 1993) but, according to Ma (1999), teachers' subject matter knowledge develops over three periods: schooling, teacher preparation, and teaching. However, researchers (Ball, 1990; Ma, 1999) have found that teachers in the United States do not typically develop a deep understanding of mathematics in school, during their pre-service programs, or during their teaching experiences. Therefore, it is vital that teachers have an opportunity to develop this understanding during their teacher preparation programs and continue developing that knowledge during their teaching careers.

Teacher Knowledge and Student Achievement

The research clearly shows that American teachers lack the mathematical knowledge to reach the vision of the NCTM Standards (2000); however, the research on the effects of teacher knowledge shows mixed results. Historically, research on the impact of teacher knowledge on students' achievement was measured by comparing the number of mathematics courses that teachers had taken and/or certification testing scores to students' achievement test scores (Darling-Hammond, 1999; Begle,1979; Monk, 1994; Sanders & Horn, 1998), but the results are mixed sometimes showing a relationship and other times showing none. In fact, Begle (1979) found no significant relationship between the number of mathematics courses the teacher had taken and student achievement, but did find a significant relationship between the number of mathematics education courses taken and student achievement. Monk (1994) also found similar results with secondary teachers suggesting that mathematics education courses have a greater impact on teacher knowledge than do mathematics content courses (Mewborn, 2003).

More recently, researchers began to focus on the impact of teacher knowledge on students' achievement by measuring teacher knowledge using instruments that were designed to look at the depth and breadth of teacher knowledge (Ball, 1990; Fennema, et. al., 1996; Ma, 1999). In a seminal research study, Ma (1999) conducted a comparative study between American elementary teachers and Chinese teachers and found that American teachers were less likely to have the knowledge necessary to teach mathematics for understanding. More importantly, she described the characteristics of the teachers with a profound understanding of fundamental mathematics (PUFM). Those characteristics were summarized as follows: Teachers with a profound understanding of the mathematics necessary to teach elementary mathematics successfully were able to:

- Make connections among mathematical concepts and procedures. Therefore, the teachers' knowledge was not fragmented and thus they would be better able to help their students see the connections between and within mathematics.
- Solve problems in a variety of ways and were aware of which of those ways would be the most efficient.
- Value the beauty of mathematics and the simplicity and power of basic ideas. They often revisited these ideas in ways that guided their students to solve problems.
- Look at the structures of the mathematics that they teach and see connections to future learning as well as previous learning. They had a

> deep understanding of the entire elementary mathematics curriculum, not just the grade that they taught (p. 123).

Teachers with an integrated and conceptual understanding of the mathematics that they teach are more likely to structure their classes to help students to develop conceptual understanding as well (Brophy, 1991; Carlsen, 1991; Hashweh, 1986, Fennema & Franke, 2006). In fact, Sowder, Phillipp, Armstrong, and Shapelle (1998) conducted a study with middle grade teachers and found that as teachers' content knowledge increased, their teaching practices became more aligned with the *Standards*. The research shows that teacher knowledge does have an impact on students' achievement, however, the degree to which their knowledge has an effect is still yet to be determined (Fennema & Franke, 2006).

Research on the Teaching and Learning of Fraction Sense

As mentioned previously in this chapter, teacher knowledge is vital in helping students make sense of mathematics. This is no less true when discussing the teaching and learning of fractions. There is "no area of elementary school mathematics as mathematically rich, cognitively complicated, and difficult to teach as fractions, ratios and proportionality" (Smith, 2002. p. 3). When analyzing the most recent NAEP results (Department of Education, 2008), the 4th grade results are mixed. Fourth grade students score well when asked to determine which fractional part of a set is shaded and when asked to write the fraction demonstrated by a shaded figure; however, when students are asked to place the fraction ¾ on a number line, only 37% of the fourth grade students were successful. When asked write two fractions that are equivalent to a set of equivalent fractions (all representing various ways to write ½), only 58% of the 4th grade students were able to provide a correct response. Comparing fractions also showed poor results with only 58% of fourth grade students able to compare two unit fractions. Subtraction of fractions with a common denominator was also poor with only 53% of the fourth grade students answering the question correctly.

Eighth grade students' performance on fraction tasks was also dismal. When asked to choose a list of fractions that is listed in order from greatest to least, only 49% of the students successfully chose the correct answer. Fifty-three percent of eighth grade students answered a division of a whole number by a fraction contextual problem correctly and in 2003, only 64% could place ¾ on a number line. Twelfth grade students also struggled with fractions. Only 22% of twelfth grade students could solve a contextual problem involving subtraction, only 60% could multiply a whole number by a fraction correctly, and only 48% of students were able to multiply a mixed number by a fraction successfully.

Despite the recommendation by the *Standards* and mathematics educators, fractions continue to be an area of difficulty for students at all levels. The complexities of fractions are compounded by the multiple interpretations of fractions. Kieren (1976) concluded there are seven interpretations of rational numbers: 1) fractions, 2) decimals, 3) equivalent classes of fractions, 4) ratio numbers, 5) multiplicative operators, 6)elements of an infinite ordered quotient field, and 7) measures or points on a number line. Kieren noted that these seven interpretations were neither an exhaustive list nor independent of one another. However, current researchers tend to collapse the seven interpretations down into five: 1) measure, 2) quotient, 3) ratio, 4) operator, and 5) part-whole (Lamon, 2007; Freudenthal, 1983; Behr et al., 1983, 1992; Kieren, 1988).

A part-whole interpretation (see Figure 1) of 3/8 would be that you have three of the eight "equal sized shares" (Moss, 2005). This is the most common interpretation of a fraction used in primary and middle grades (Van de Walle, 2007). The measure interpretation (see figure 2) of $\frac{3}{8}$ would be that I have three one-eighths repeatedly. This differs from the part-whole interpretation because it is seen as a distance or length.

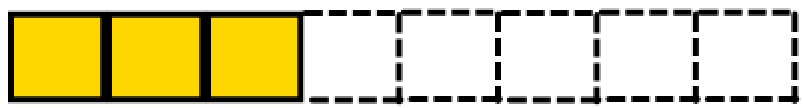

Figure 1. Example of Part-Whole Interpretation of Rational Numbers

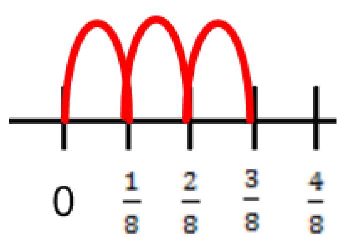

Figure 2. Example of Measurement Interpretation of Rational Numbers

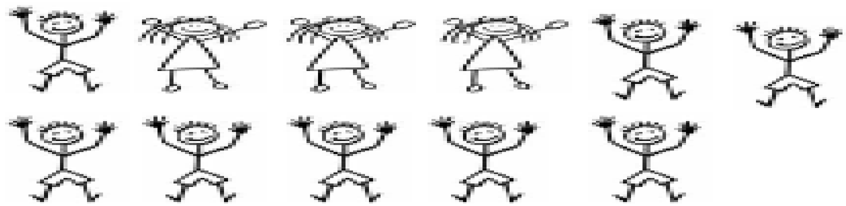

Figure 3. Example of Ratio Interpretation of Rational Numbers

The quotient interpretation of $\frac{3}{8}$ would be three divided by eight. Students are usually introduced to this interpretation in sixth or seventh grade. The ratio interpretation (see Figure 3) of $\frac{3}{8}$ would be that you have three girls for every eight boys in a class. This is also often confused with the part-whole interpretation, but it is very different. Instead of being 3 – one-eighth equal-sized pieces, it represents a multiplicative comparison of two quantities (not necessarily alike). Finally, the operator interpretation sees a fraction as a reducer or an enlarger of another quantity (also referred to as a stretcher or reducer in the literature).

When making sense of fractions, students also have to learn to reconceptualize the unit (Moss, 2005). For example, students that have constructed 3, see 3 as three individual units (o) (o) (o) and a unit of three simultaneously (ooo). They know that 3 is one less than 4 and one more than 2 on a number line. However, when students are required to make sense of $\frac{3}{8}$ they must change from seeing $\frac{3}{8}$ as 3 (ooo) and 8 (oooooooo) to a number less than one whole. The students have to begin to see $\frac{3}{8}$ as its own unit or three of eight fair shares (Behr et al, 1992; Moss, 2005). This becomes specifically important in understanding the division of fractions as will be discussed later in this study. Students often look at ¾ as three and four and fail to see that ¾ is a number less than one whole. When students learn to compare fractions, this misconception can be extremely troubling (Moss, 2005; Van de Walle, 2007; Tirosh, 2000). Students often believe that $\frac{3}{4}$ and $\frac{7}{8}$ are equivalent because they both have one piece missing. Students also believe that $\frac{1}{8}$ is larger than $\frac{1}{4}$ because eight is larger than four. Students also may suggest that when you cut a medium pizza into four unequal parts that one of those unequal parts is one fourth of the pizza (Van de Walle, 2007). Students need to make sense of and have experiences with the relative size of fractions in a variety of different contexts to make sense of rational numbers instead of focusing solely on the part-whole interpretation.

Carraher (1996) found that when teachers focus on the part-whole interpretation of fractions, several shortcomings in understanding result: (1) 'part –whole' fixation in which students only see fractions as a part of a whole which can impede their understanding of improper fractions, (2) 'cardinal sin' in which students focus on cardinal numbers instead of the ratio meaning of fractions, (3) 'missing links' in which fraction operations are not linked to whole number operations, ratios, proportions, or functions, and (4) 'no challenge' in which students do not see the connections to real-world application (p. 242 – 243). Therefore, teachers need to create learning opportunities for their students so they will be comfortable with all five interpretations of rational numbers and not just the part-whole interpretation.

Representations with Fractions

Other than focusing on the multiple interpretations of fractions, there has been extensive research done with elementary students to determine what types of materials will help them to make sense of fractions. The Rational Number Project – a National Science Foundation funded research project that looks at the teaching and learning of rational numbers – created fraction teaching materials based upon their research with children. This curriculum, designed for fourth and fifth grade students, utilizes concrete models and other forms of representations to help students develop an understanding of (a) part-whole model of fractions, (b) a flexible understanding of the unit, (c) order, (d) equivalency, (e) estimation of addition and subtraction of fractions, and (f) addition and subtraction of fractions with manipulative materials (Cramer, Post, & del Mas, 2002). Part of their research explored the use of continuous models (area models) and discrete models and the impact the use of these models have on students' understanding of fractions.

As part of their RNP research, Post, Wachsmuth, Lesh, and Behr (1985) found that for children to become more sophisticated in their understanding of rational numbers, they need to rely less and less on physical embodiments of fractions (physical models) and become more comfortable with the symbolic representations. The researchers suggested that this transition could be facilitated through the use of set models such as bingo chips. The use of set models requires the child to make a decision regarding what will be considered the whole set prior to comparing two fractional amounts, thus allowing them to compare amounts in a "coordinated way." Therefore, it is vital to introduce students to other models for fractions in addition to the area model (pie pieces or fraction circles). Post et al. (1985) noted that instruction using a variety of models takes longer than the traditional curriculum and thus the timeline must be restructured as to allow students an opportunity for these concepts to develop.

While researching the impact of discrete models on student learning, Behr, Wachsmuth & Post (1988), suggested that the set model (discrete set) and the area model (continuous model) require different cognitive demands. However, despite the difficulties that children first face when dealing with discrete sets, the children that had experiences with both discrete and continuous models answered more questions correctly on an assessment than students who used a traditional textbook program. This study reinforces the notion that students need the opportunity to use a variety of representations when learning fractions; however, the area model seems to be more appropriate in the initial stages of learning.

While examining the effectiveness of continuous and discrete models was the major focus of some of the early RNP research, more recent research is beginning to examine the effectiveness of virtual manipulatives in helping understanding of fractions. The dynamic way in which students can manipulate different fraction models has the potential for aiding students in constructing knowledge about fraction equivalency and fraction operations. Su and Heo (2005) conducted research with 3 groups of fifth-grade classrooms to determine if using virtual manipulatives would help the students to develop fraction sense. They found that the fifth grade students were able to create and test their conjectures for equivalent fractions more quickly than they would have been when using actual manipulatives. The students were also able to create their own rules for equivalent fractions by noticing that the partitions that were equivalent had either common multiples or common factors. After a whole class discussion, some students were even able to invent their own procedure for finding equivalent fractions based upon what they had noticed with the virtual manipulatives.

Effectiveness of RNP and other Standards Based Instruction on Student Learning

When learning to order fractions, typically students are taught to cross multiply or to find common denominators. These procedures are usually taught without meaning and those procedures are prone to error ((Behr et. al, 1984, 1992; Cramer et. al., 2002; Moss, 2005; NRC, 2001; Van de Walle, 2007). Behr et al., (1984) found that when students were allowed to develop an understanding of fractions using student-centered instruction, students were able to order fractions without using the standard rules and procedures for fractions. Students were able to rely on their fraction sense to order the fractions. Students' strategies tended to fall into four categories: (1) same numerator, (2) same denominator, (3) transitive, and (4) residual. Students utilized the same numerator strategy by looking at two fractions with the same numerator and recognizing that one fraction is larger than the other based on the size of the denominator. The students used the same denominator strategy when they compared fractions with the same denominator based on how many of those fractional parts are represented by the fraction. The students demonstrated a transitive strategy when they used benchmarks (such as a half) to compare two fractions, and a residual strategy when they compared the left over amounts of a fraction or the number of parts left to make a whole and then determine how many parts are left. Students that are taught the cross-multiply strategy or common denominator strategy without developing a meaning for these rules on their own were usually unable to use any other strategy.

Cramer, et. al. (2002) also compared the fraction understanding of 33 fourth-and-fifth grade classrooms that utilized a commercial curriculum (CC) to 33 fourth-and-fifth grade classrooms that used the Rational Number Project (RNP) curriculum and found that the RNP curriculum students significantly outperformed the CC students on the posttest as well as on the retention test given months later. An analysis of subscales also showed that the RNP "students had a stronger conceptual understanding of fractions, were better able to judge the relative sizes of two fractions, used this knowledge to estimate sums or differences, and were better able to transfer their understanding of fractions to tasks not directly taught to them" (Cramer, et al., 2002, p. 128). There were no differences found on addition and subtraction of fractions subscales; however, the CC students spent more time learning these topics than did the RNP groups. Taken together, these results suggest that the RNP curriculum has a greater influence on students' understanding of fraction concepts than the CC.

One argument against the use of student-centered instruction is that it is only beneficial for middle to high achieving students (Empson, 2003). Students that struggle mathematically can benefit from the RNP curriculum as well. Empson (2003) studied the fractional understanding of 2 low-achieving first-grade students and found that their success and failures may have been a function of their interactions in the classroom and not their individual abilities. When the teacher included the students in the class discussions and helped them to connect to the problems, they were able to accurately solve problems and participate in the class discussions, once again highlighting the importance of teacher knowledge and the ability for teachers to be prepared to pose questions and lead mathematical discussion.

Research on the Teaching and Learning of Division of Fractions

The research above shows the importance of developing fraction sense before introducing the rules and procedures for fractions operations and described ways to help students make sense of fractions. However, the teaching and learning of division of fractions is an even more complex issue. As mentioned above, less than half of 8th and 12th graders were successful in solving a simple contextual problem that involved the division of fractions. Part of the reason for this difficulty is the way in which fraction division is taught in schools (Kribs-Zaleta, 2006; Ma, 1999, Van de Walle, 2007). Teachers often teach this concept in a rule-based manner with very little opportunity for sense making. This rule-based instruction tends to cause students to follow meaningless procedures (the invert and multiply algorithm) that students often forget or complete incorrectly (Tirosh, 2000). Other than faulty procedures, a focus on rote procedures for division of fractions also leads to intuitively-based mistakes or mistakes based on lack of formal knowledge.

The two most common algorithms used to teach division of fractions are the ‘invert and multiply’ algorithm and the ‘common denominator’ algorithm. However, teachers rarely help students make sense of the procedures and research has shown that teachers, themselves, often do not understand the meaning behind the algorithm (Borko, et. al., 1992). Therefore, teachers often teach students the procedures void of meaning and this, in turn, limits students’ understanding of the procedure (Sharp & Adams, 2002).

The meaning of division with fractions is no different from division of whole numbers. There are two main interpretations of division that students encounter: (1) the measurement interpretation in which the size of the groups is known but the student is asked to find the number in each group and (2) the partitive interpretation in which the number of groups is known, but the student is asked to find the size of the groups (Gregg & Gregg, 2007). Ott, Snook, and Gibson (1991) suggested that making sense of these two interpretations is not necessarily natural for students and thus stressed the importance of providing students with concrete experiences during the initial stages of instruction.

Bulgar (2003) conducted a study with fourth-and-fifth grade students that had not had procedural instruction on division of fractions and wanted to determine if they would be able to create their own strategies for solving fraction division problems. She found that when students were given the opportunity to solve problems involving division of fractions, they were able to devise three types of solution strategies: (1) natural number strategies, (2) measurement strategies, and (3) fraction strategies. The students were able to construct their own understanding and solve the problems in ways that they could reproduce, even if they forgot their procedure.

Sharp and Adams (2002) also conducted research with fifth-grade students to see if they could solve division of fractions contextual problems prior to learning the common algorithm. The researchers provided the students with a simple context and some common strategies used were (1) adding up strategy, (2) repeated subtraction strategies, (3) and mental representations. In order to encourage students to think of fractions more procedurally, the researchers introduced fraction problems void of context. Students were guided to focus on the meaning of the problem. This encouraged some students to use symbolic representations of their answers while some students still relied on drawings. Finally, the researchers introduced contextual problems with remainders and found that children tended to deal with remainders in three ways: (1) the students referred to their whole number knowledge and just said that there was some left over, (2) some students just mentioned the fraction of the material that was left over, and (3) the remainder of the students actually referred to the specific unit and wrote the fraction of the unit correctly. To help students deal with the remainder appropriately, encouraged the students to focus on the unit and this helped most of the students deal appropriately with the remainder. At the conclusion of the study, six strategies emerged: (1) developed a common denominator algorithm with symbols, (2) used technical procedures but developed no general procedure, (3) attempted to use technical procedures but did so incorrectly, (4) developed a common denominator approach with pictures, (5) mental calculations, and (6) no accurate solution strategy. Interestingly enough, no student invented the invert and multiply algorithm for division of fractions.

Taken together, the research shows that students are able to develop their own strategies for solving division of fraction problems without knowing the invert and multiply algorithm. However, in order for teachers to provide their students with learning opportunities to construct knowledge of division of fractions, the teacher must have a deep understanding of division of fractions.

Teacher's Understanding of Division of Fractions

In 1999, Leiping Ma completed a study with 23 U.S. teachers and determined that only 43% of the teachers could correctly answer a division of fractions problem. She determined that the teachers' incomplete memories of the algorithm impeded their calculations. Only one of the teachers was able to come up with a representation of division of fractions. The remaining teachers confused division by ½ with division by 2, confounded division by ½ with multiplication by ½, confused all three concepts, or offered no representation at all. Ma also found that their pedagogical knowledge did not make up for gaps in knowledge about division of fractions (Ma, 1999).

Post, Harel, Behr, and Lesh (1991) conducted a research study to determine the rational number knowledge of in-service intermediate mathematics teachers. The participants were given two tests that required participants to solve fraction related problems. The tests included 14 division of fraction problems for the teachers to solve and then required the teachers to describe how they would teach the concepts to children. The teachers were then interviewed based on their results on the aforementioned tests. The researchers determined that only ten percent of the participants were able to give pedagogically sound explanations to their students and most teachers scored between 60 – 69% on the content knowledge test. Teachers' shallow and unstable understanding can have an impact on the knowledge that their students are able to develop. Based on the results of their study, the researchers concluded that teacher educators needed to help prospective teachers to build a deep understanding of rational number concepts before they leave their teacher preparation programs.

Prospective Teachers' Understanding of Fractions

Ball (1990) conducted a study of 217 elementary education majors and 35 secondary majors to determine the nature of the knowledge that they bring to their teacher education programs. The researcher provided the prospective teachers with a questionnaire and then conducted an interview with 35 (25 elementary and 10 secondary) of the participants to look deeper at their knowledge. During the interview, the teachers were provided with a division of fractions problem to solve and then were asked to determine a representation that could be used to help show conceptually how to divide fractions. She found that none of the elementary education majors and only forty percent of the secondary mathematics majors were able to provide an appropriate representation of a simple division of fractions problem. Forty percent of the elementary education majors and twenty percent of the secondary education majors provided an incorrect representation for division of fractions. The remaining students were unable to provide a representation at all. She concluded that the elementary education and secondary education majors had a fragile and limited understanding of division with fractions. In addition to superficial understanding of the content, prospective elementary teachers tended to have less confidence and more anxiety towards the teaching and learning of mathematics. Based on her findings, Ball (1990) suggests that teacher preparation programs should help prospective teachers to develop confidence and to build a deep knowledge of the content.

Building upon the work of Ball (1990), Borko, et. al. (1992) examined one prospective elementary teacher's beliefs and content knowledge of the division of fractions as a part of a larger study on teacher knowledge. Through semi-structured interviews and observations, the researchers determined that this specific prospective teacher had not developed strong pedagogical content knowledge or subject matter knowledge during her pre-service preparation and this lack of conceptual knowledge had an impact on her ability to teach division of fractions. The researchers described a specific instance where the teacher was reviewing the common algorithm for the division of fractions with her class of sixth graders and a student interrupted and asked why you invert and multiply. The prospective elementary teacher tried to draw a representation of the problem and instead showed a multiplication problem. After promising the student that she would have to get back to the student, she never revisited the topic for the child and told the researchers that she was pleased with the lesson. The researchers concluded that despite the experiences that the prospective teacher had in school and student teaching, she still reverted back to her own experiences as a learner of mathematics and taught in a way that was consistent with the way she had been taught. Based on these findings, the researchers suggested that prospective elementary teachers be given ample time to develop both conceptual and procedural knowledge in their teacher preparation courses. Methods courses must also provide opportunities for these budding educators to confront their own beliefs about the teaching and learning of mathematics.

Tirosh (2000) continued to explore the division of fractions content knowledge (both pedagogical content knowledge and subject matter knowledge) of prospective elementary education teachers almost ten years after the aforementioned studies. In this study, Tirosh specifically explored 30 prospective elementary teachers' understanding of students' mistakes and errors when solving division of fractions problems. She then encouraged the future teachers to think about the students' thinking and to become aware of types of errors that students make when solving division of fraction problems. She felt that an examination of misconceptions would prepare them to address students' misconceptions. Based on the results of her study, Tirosh determined that prospective elementary teachers' understanding of the errors that students might make was closely related to their subject matter knowledge. She concluded that more studies need to be conducted to determine if prospective elementary teachers are familiar with the informal understandings that students bring to their classrooms. Based on these findings, Tirosh also posited that it is important to focus not only on developing prospective elementary teachers' subject matter knowledge and pedagogical knowledge, but also attention should be paid to helping prospective teachers to understand their students informal understandings and common misconceptions that students bring to the learning situation.

Schram, Wilcox, Lanier, and Lapan (1988) found that when prospective elementary teachers were provided with experiences and opportunities to make sense of the common division algorithms their depth of knowledge increased. However, they found that many of the teachers did not carry those understandings with them into their first year of teaching (Mewborn, 2001).

Summary

The argument has been developed here that teachers need to have a deep understanding of the mathematics that they teach to help their students learn and to teach for understanding; however, research shows that in-service teachers and prospective elementary teachers often lack this essential understanding. Investigating the understanding and strategies the prospective elementary teachers use while solving division of fractions problems will help mathematics teacher educators to plan for and to put into practice more effective strategies to help prospective elementary teachers develop an understanding of division of fractions. Additionally, this will help prospective elementary teachers to enter the field of teaching with a deep understanding of division of fractions so that they will be prepared to provide learning opportunities for their students so they can learn to divide fractions with meaning.

In Chapter III, the Methodology utilized to explore these research questions will be discussed. First, the participants will be described. Next, a rich description of the setting will be discussed. In this description, a picture will be painted of the classroom in which this research took place and activities and instructional sequences will be described in detail. After describing the setting, instruments, procedures, and data analysis will be discussed. Finally, ethical considerations will be discussed.

CHAPTER III

METHODOLOGY

This sequential explanatory mixed methods research study used both qualitative data and quantitative data to examine the knowledge that prospective elementary teachers bring to their final mathematics methods course, determined whether that knowledge changed, and described what experiences the prospective elementary teachers felt contributed to that change. Chapter three describes the sampling procedures, participants, and setting as well as provides a detailed description of the instruments, design of the study and procedures used to collect and analyze the data. The study specifically addressed the following research questions:

1. What understanding of division of fractions do prospective elementary teachers bring to their final mathematics methods course?

2. What beliefs (confidence, anxiety, and personal teaching efficacy) do prospective elementary teachers bring about division of fractions to their final mathematics methods course?

3. Is there a difference in prospective elementary teachers' understanding of division of fractions after participation in their final mathematics methods course?

4. Is there a difference in prospective elementary teachers' beliefs (confidence, anxiety, and personal teaching efficacy) about the division of fractions after participation in their final mathematics methods course?

Research Design

The researcher conducted a mixed methods study instead of a purely qualitative or quantitative approach for pragmatic reasons. The research questions in this study sought to create a quantitative picture of the knowledge with which prospective elementary students entered their methods course. Additionally, the researcher sought to gain an in-depth understanding of the effects of the methods course on the strategies that prospective teachers used while solving fraction division problems. Thus, the research questions in this study guided the researcher to collect both quantitative data and qualitative data.

The research study employed a sequential explanatory design (Cresswell, 2003) that gave equal emphasis to qualitative and quantitative designs. Creswell describes this procedure as collecting the quantitative data, analyzing the data, and then switching to a qualitative design in future phases to explain the quantitative data in more depth. Neither quantitative nor qualitative data were given specific emphasis. Analyzing each data set helped the researcher develop an understanding of the questions under study.

In this study, the researcher used a phenomenological approach for the qualitative data. In that, she sought to capture and describe how participants experienced a phenomenon of interest (Patton, 2002). To get at the 'lived experiences', the participants must reflect on the experience after they have gone through it (Van Manen, 1990). The phenomenon was the research participants' lived experience in an intermediate mathematics methods course.

Participants

A convenience sample was used including 34 prospective elementary teachers enrolled in their second mathematics methods course at a Midwestern land-grant university. All participants were enrolled in 15 semester hours including an intermediate mathematics methods course. The participants spent one day per week in an elementary or middle school classroom interacting with students; additionally, as part of the mathematics methods courses they spent one hour per week for 11 weeks tutoring an elementary or middle school student in mathematics.

Prior to participating in the intermediate mathematics methods course, the prospective elementary teachers were required to complete 12 hours of college level mathematics. Within those 12 hours, six of the hours or 2 courses (Geometric Structures and Mathematical Structures) were designed specifically for prospective elementary teachers. Geometric Structures covers the fundamentals of plane geometry, transformations, polyhedra, and applications to measurement while Mathematical Structures covers the foundations of number (set theory, numeration, and the real number system), algebraic systems, functions and applications, and probability. The students had a variety of choices available for the remaining 6 hours of mathematics ranging from College Algebra to the 3-course Calculus Series.

The participants completed a primary (K-3) mathematics methods course that is geared toward helping prospective elementary teachers to view teaching and learning mathematics using a problem-centered approach. A bulk of the course is designed to focus on whole number concepts (including place value and the four operations), algebraic reasoning, geometry, measurement, data analysis, and probability. However, only minor emphasis was placed on teaching and learning rational number concepts. The participants conducted 5 clinical interviews with a child in the primary grades and analyzed how the student thought when solving mathematical tasks. The participants also spent a considerable amount of time reading and discussing the NCTM *Principles and Standards for School Mathematics* (2000). The instructors encouraged the prospective elementary teachers to utilize thinking strategies when solving problems instead of the generally accepted common algorithms and procedures. In essence, the course was designed to help the participants think of mathematics as "the science of pattern and order" and not just a collection of facts and procedures (Van de Walle, 2007).

Setting

The Intermediate Mathematics Methods course is designed to help prospective elementary teachers deepen their understanding of the mathematics content for grades 4 through 8 while learning appropriate pedagogical knowledge associated with the teaching and learning of intermediate mathematics. This class was taught using a standards-based approach. The instructor encouraged the students to problem solve, communicate their mathematical thinking, create and test conjectures, verify their thinking, use a variety of mathematical models and representations to solve problems, and to help them to see the connections within the mathematics domain as well as real-world connections and connections to other disciplines. The concepts covered during the semester align with the NCTM Content Standards as discussed in the *Principles and Standards for School Mathematics* (NCTM, 2000) with a third of the course focused on rational number concepts. Prospective elementary teachers learned pedagogically sound ways to teach and assess the content discussed in the course. The instructor worked to help the prospective elementary teachers develop deep subject matter and pedagogical content knowledge so that they will be better prepared to teach mathematics using a standards-based approach.

On the first two days of fraction instruction, the instructor began by helping the participants to develop fraction sense by providing a variety of activities designed to encourage a deep understanding of fractions. During these activities, the participants were encouraged to share their solution strategies with the class using a think, pair, and share strategy. Participants began the fraction instruction sequence by completing sharing tasks. These tasks are designed to encourage students to make sense of partitioning a quantity into fair shares. The participants used fraction manipulatives (fraction circles, fraction tiles, Cuisenaire rods, sets of counters, and pattern blocks) to solve sharing tasks of varying difficulty. Next, the participants began to explore how to introduce the language of fractions through activities. They also completed activities from the Marilyn Burns Fraction Kit and part-part whole tasks discussed in Van de Walle (2008). Finally, participants explored comparing and ordering of fractions using models and fraction benchmark fractions (see Van de Walle, 2008).

On the third day of instruction the participants solved a variety of contextual and computational fraction addition and subtraction problems by using models and invented strategies. The participants also began to use drawings and number lines to solve the problems. Each time, the participants shared their thinking strategies and discussed which strategies were valid, efficient, and generalizable. During this day of instruction, the participants began to write contextual problems that fit addition and subtraction computational problems. The participants discussed and shared their problems at great length. Many of the participants had a difficult time writing subtraction problems. The participants were also given a sheet with children's solution strategies – some correct and some incorrect – and were asked to determine if their methods were valid, efficient, and generalizable. The fourth class period focused on understanding of multiplication with fractions and was structured similarly to the third class period. The participants solved a variety of multiplication problems and shared solution strategies.

The final two days of fraction instruction focused specifically on the division of fractions. The instructor provided the participants with a whole number division problem and asked the participants to provide the two interpretations that students could use when solving the problem. After some probing, the participants decided on the measurement interpretation of division and partitioning (fair-share) interpretation of the problem. The instructor then provided the participants with a fraction division problem where a whole number is divided by a unit fraction. The participants then discussed at length the way you could use the whole number interpretations for division to make sense of fraction division. The participants then spent the rest of the fifth class period solving division of fractions problems with models and drawings where the dividend was larger than the divisor. The final day of fraction division instruction focused on writing fraction division contextual problems and solving division of fractions problems with a dividend that is smaller than the divisor.

Participants were always encouraged to share their strategies with their classmates. They would question each others' solution methods and most of the participants appeared to understand their classmates' strategies. At the conclusion of fraction instruction, participants seemed thankful to move on to the next concept, however, many said that they still did not understand the partitioning interpretation of division of fractions.

Instrumentation

The quantitative data sources utilized in this study were a pre/post fraction division content knowledge survey, pre/post division of fractions attitude survey, and demographic survey. The qualitative data sources will include field notes, subjects' mathematical thinking journals, and subject semi-structured interviews.

Quantitative Measures

Demographic Survey. The demographic survey (see Appendix A) was designed to collect demographic data on the participants such as: (1) age, (2) gender, (3) ethnicity, (4) college mathematics background, and (5) high school mathematics background. This information was collected for further analysis and descriptive statistics (i.e. mean number of mathematics courses, mean age, etc…) were calculated and reported.

Division of Fractions Understanding Test. The *pilot* survey contained ten questions that assessed the prospective elementary teachers' subject matter knowledge of the division of fractions. Six of the questions were computational problems and four were contextual problems. The contextual questions were adapted from Van de Walle (2007) and Gregg and Gregg (2007). The questions included a variety of problem types. A confidence scale was also included on each question. The participants circled how confident they were that their answer is correct based on a scale from 1 to 6 with one being not very confident and 6 being extremely confident.

During the pilot study, it was determined that two of the questions were problematic (question 7 and 9) and thus were dropped from the analysis. In order to address this situation, 4 additional contextual questions were created to replace the dropped items: 2 contextual problems with a measurement interpretation and 2 contextual problems with a partition interpretation. To choose the replacement items, the researcher piloted the four questions with a group of prospective elementary teachers that had completed the fraction division portion of their methods course. The researcher and two other mathematics education professors analyzed the responses to the questions to look for problems and then two replacement questions were chosen from the four-piloted questions. The revised instrument (see Appendix B) contains 10 questions that assessed the prospective elementary teachers' subject matter knowledge of the division of fractions. Six of the questions were computational problems and four contextual. The problems included (1) a whole number divided by a fraction, (2) a fraction divided by a fraction (with and without remainders), (3) a mixed number by a mixed number (with and without remainders), (4) a fraction divided by a whole number, (5) a fraction divided by a mixed number with a remainder, and (6) a whole number divided by a mixed number with a remainder. To insure the content validity of this instrument the researcher had two mathematics education faculty and one mathematics education Ph.D. student with experience teaching at the middle level examine the questions and insured that it would appropriately measure the content knowledge of prospective elementary education students.

Fraction Division Attitude Scale (FDAS). The FDAS (see Appendix C) was modified from the Fennema Sherman Attitude Survey (1976) as well as the Attitudes Towards Geometry Scales (Utley, 2004) and the Mathematics Teaching Efficacy Belief Instrument (Huinker & Enochs, 1995). The FDAS consisted of 15 questions and provided Likert-scaled responses from one to six. A response of one indicated that the prospective teacher strongly disagreed with the statement and a response of 6 meant that the prospective teacher strongly agreed with the statement. Negatively worded items were reverse coded and a total attitude score was calculated. The instrument contained 5 subscales with three questions in each subscale: Confidence to Teach Division of Fractions, Confidence to Learn Division of Fractions, Anxiety to Teach Division of Fractions, Anxiety to Learn Division of Fractions, and Personal Teaching Efficacy (Division of Fractions). Sample items on the instrument included "I often have trouble solving division of fractions problems" and "I understand division of fractions well enough to teach it." Two mathematics educators evaluated the face validity of the instrument and deemed the instrument valid. A Cronbach alpha was also calculated to determine the internal consistency of the scales.

Table 1

Internal Consistency FDAS

	PRE	POST
Total Attitude FDAS Survey	.929	.93
Confidence to teach Division of Fractions	.705	.807
Confidence to learn Division of Fractions	.742	.75
Anxiety to teach Division of Fractions	.904	.76
Anxiety to learn Division of Fractions	.618	.825
Personal Teaching Efficacy of Division of Fractions	.753	.722

Qualitative Measures

Observational Field Notes. The researcher conducted observations of 6 class periods in which fractions were taught. During the observations, the researcher took field notes concerning the way the class was taught, the questions students ask related to making sense of fractions and their teaching of fraction concepts, fraction activities in which they participated , misconceptions and thinking strategies the prospective elementary teachers used while solving fraction problems. This information was used to paint a picture of the interactions that occurred within the classroom and how the participants solved division of fractions problems. Additionally, the observational field notes were used to guide the semi-structured interviews.

Mathematical Thinking Journals. As part of their intermediate mathematics methods course, participants were required to solve mathematics problems throughout the semester and record their solution strategies in their mathematical thinking journal. The students' were encouraged to write about their solution strategies during the division of fractions portion of the class. Copies of thinking journal entries were obtained for each participant.

Interviews. Audio taped, semi-structured interviews with 5 participants were conducted. The interview protocol can be found in Appendix D. The choice of the 5 participants that were interviewed depended partly on who agreed to be interviewed and on the results of analysis of the content and attitude scale as well as observations of students during the fraction instruction. During the interview, the students were asked to solve division of fraction problems using a think-aloud strategy and concurrently will be asked open-ended questions that will tease out their content knowledge as well as the thinking strategies that they used during the problem solving process. The researcher asked participants why they chose each solution strategy.

In addition to solving problems, the researcher asked participants to describe what experiences they felt had an impact on their understanding of division of fractions. For example, the researcher asked the participants to describe specific experiences they had in the methods course that helped them to be better prepared to understand and teach division of fractions .

Procedures

This study was conducted in four phases. Table 1 provides an overview of these four phases. Prior to conducting the study, Institutional Review Board (IRB) approval was obtained. During phase one, the researcher solicited participants from the Intermediate Mathematics Methods course. Upon consent, the participants completed an attitude survey, demographic survey, and a content knowledge survey. Each survey was assigned a unique identifying number so that data could be paired during analysis. Only the course instructor had access to a code sheet containing participant identification numbers. The FDAS, demographic survey, and DOFUT survey were collected immediately. This quantitative phase of the study occurred prior to any discussion of rational number concepts.

The second phase of the research study involved observing and collecting field notes during the teaching of fractions section of the course. The researcher remained primarily an observer but interacted with the participants, thus the researcher took the role of observer as participant (Glesne & Peshkin, 1992). During this process the researcher took note of the discussions, questioning, activities, and thinking strategies that occur in order to paint a picture of the methods course as well as to help with specific questions to ask in the follow up interviews. During this phase of the research, mathematics thinking journals were collected and copied for further analysis.

During phase three of the research study, the researcher administered the post FDAS and post DOFUT survey. The researcher did some preliminary data analysis in order to aide in the choice of participants for interviews. Potentially interview participants were chosen based on the following criteria: High Understanding and Low Understanding. During phase four of the research study, the researcher conducted the semi-structured interviews.   These interviews lasted approximately 30 minutes and were audio taped.

Table 2

Phases of Data Collection

Phase 1	Phase 2	Phase 3	Phase 4
Demographic Survey Pre CK Survey Pre FDAS	Observations Mathematics Thinking Journals	Post CK Survey Post FDAS	Conduct Interviews

Data Analysis

Quantitative Data Analysis

The quantitative data were analyzed to determine the knowledge and attitudes that the prospective elementary teachers' had about the division of fractions. The data were analyzed using both descriptive (e.g. means, standard deviation, etc.) and inferential statistics (i.e. paired t-tests). The researcher collected data to describe the research participants' content knowledge and attitudes both before and after the methods course.

On the content knowledge survey, each question was scored by at least two independent researchers based on a scoring rubric (see Table 3) developed during the pilot study. Inter-rater reliability was 90%. The scores were discussed and rescored until 100% agreement was met.

Table 3

Scoring Rubric for DOFUT

	Attempted Procedural	Attempted Conceptual	Attempted Both Proc. & Conc.
Incorrect	1	1	1
Correct	3	3	4
PC/CC			2/2

The items that were considered both procedurally correct and conceptually correct were given a score of 4 because students showed both conceptual and procedural understanding in their responses. Items that were either procedurally correct or conceptually correct were given a score of 3 because they showed either procedural or conceptual understanding. When students made an attempt at a procedural and conceptual solution but did one incorrectly, they were given a score of 2 (bottom right cell of the table). When students made an attempt at a conceptual and/or procedural solution but did so incorrectly, they were given a score of 1. Finally, students that did not attempt to solve the problem or showed no apparent method were given a score of 0 on the problem.

A total content knowledge score (40 points possible) was calculated. Additionally scores were calculated on the two subscales: computation (24 points possible) and contextual (16 points possible). These items were compared pre/post using a paired samples t-test. Descriptive statistics were also calculated for each question and then compared pre/post for each question to look for changes and patterns in the data.

The FDAS was entered into an SPSS (16.0) file and negatively worded items were reverse coded and analyzed using a paired sample t-test. Scores on the five subscales of the FDAS and the instrument as a whole were calculated and analyzed pre/post using a paired sample t-test to determine if there was a significant change in prospective elementary teachers' fraction division attitudes.

Qualitative data were analyzed using a constant comparative method (Strauss & Corbin, 1998) to find emerging themes. The researcher analyzed specific statements and themes while searching for all possible meanings. "The researcher must also set aside all prejudgments, bracketing his or her own experiences and relying on intuition, imagination, and universal structures to obtain a picture of these experiences" (Cresswell, 1998, p. 52). Thus, using the constant comparative method, the researcher analyzed the data; look for categories, patterns and themes. This was used to understand what experiences the research participants felt had the most impact on their understanding of the teaching and learning of division of fractions. The participants' written work, thinking journal entries, and field notes were also triangulated with the interview data in order to ensure accuracy (Patton, 2002).

Ethical Considerations

In order to protect the identity of the research participants, all data were coded. Pseudonyms were used for all of the prospective elementary teachers to protect their identity, privacy, and confidentiality. The research participants were made aware of the risks and benefits of participation in advance and also signed consent forms that described in writing how confidentiality will be handled. The research participants were also given the opportunity to withdraw from the study at any time as outlined in the IRB.

Summary

A summary of the research questions being studied and the related research instruments and data analysis is provided below:

1. What understanding of division of fractions do prospective elementary teachers bring to their final mathematics methods course? This was measured with the Pre DOFUT and analyzed using descriptive statistics (means, standard deviations, and confidence intervals).

2. What beliefs (confidence, anxiety, and personal teaching efficacy) do prospective elementary teachers bring about division of fractions to their final mathematics methods course? This was measured with the Pre FDAS and journal prompts and analyzed using descriptive statistics (means, standard deviations, and confidence intervals) and the constant comparative method.

3. Is there a difference in prospective elementary teachers' understanding of division of fractions after participation in their final mathematics methods course? This was measured with the Pre/Post DOFUT and analyzed with both descriptive (means, standard deviations, and confidence intervals) and inferential statistics (paired samples t-test).

4. Is there a difference in prospective elementary teachers' beliefs (confidence, anxiety, and personal teaching efficacy) about the division of fractions after participation in their final mathematics methods course? This was measured with the Pre/Post FDAS, journal prompts, and interviews and analyzed with both descriptive (means, standard deviations, and confidence intervals), inferential statistics (paired samples t-test) and constant comparative method.

The results of the data analysis will be presented in Chapter IV and a discussion of those results will follow in Chapter V.

Chapter IV

Results

This sequential explanatory mixed methods research study combined qualitative and quantitative data to examine 34 prospective elementary teachers' beliefs about the teaching and learning of division of fractions, understanding of division of fractions, change in understanding during a methods course, and perceptions about experiences contributing to a change in their understanding. The specific research questions guiding this study were:

1. What understanding of division of fractions do prospective elementary teachers bring to their final mathematics methods course?

2. What beliefs (confidence, anxiety, and personal teaching efficacy) about division of fractions do prospective elementary teachers bring to their final mathematics methods course?

3. Is there a difference in prospective elementary teachers' understanding of division of fractions after participation in their final mathematics methods course?

4. Is there a difference in prospective elementary teachers' beliefs (confidence, anxiety, and personal teaching efficacy) about the division of fractions after participation in their final mathematics methods course?

This chapter will present findings from research data including pre/post attitude surveys, pre/post division of fractions understanding test, field notes, journal prompts, and interviews. First, results will be presented to highlight the understanding of and attitudes towards division of fractions that participants brought to their final mathematics methods course. Finally, the impact of the methods course on the prospective elementary teachers' understanding and attitudes towards division of fractions will be explored.

What Prospective Elementary Teachers Bring to their Final Math Methods Course

Participant Understanding.

In order to determine what understanding of division of fractions the prospective elementary teachers had before and after completing their final mathematics methods course, the researcher administered a Division of Fractions Understanding Test (DOFUT) and a Fraction Division Attitude Scale (FDAS). The ten problem DOFUT was scored using the 5-point (0 to 4) rubric found in Table 3, thus possible scores on this test could range from 0 to 40 with the computational problems accounting for 24 points and contextual problems accounting for 16 points. All data were entered into an SPSS 16.0 spreadsheet where descriptive statistics were calculated (see Table 4). The scores on the overall pre–division of fractions understanding test ranged from 7 to 33 with a mean of 22.20 (SD = 7.36).

Upon completion of scoring the DOFUT, the participants' scores on the DOFUT were sorted into three categories (good, moderate, or weak) based on their level of understanding. About one-third (35%) of participants demonstrated a weak understanding of division of fractions which included participants' whose scores ranged from 0 to 19, thus only receiving less than half of the points possible. Sixteen participants (47%) received 20 - 29 points on the DOFUT and were placed in the moderate understanding category. The remaining six participants' tests were placed in the good understanding pile because they obtained at least 30 points. Despite the fact that all of the participants had completed at least 4 college mathematics classes, only 18% of the participants coming into their final mathematics methods course demonstrated a good understanding of division of fractions.

The participants whose scores on the DOFUT showed a good understanding all used the common invert and multiply algorithm correctly to solve the problems on the test. While they have a good understanding of how to use the common invert and multiply algorithm correctly, it is difficult to understand from the DOFUT whether they have a conceptual understanding of division of fractions. Participants with a weak understanding of division of fractions had faulty procedures to solve the problems or had difficulty interpreting the contextual problems. The majority of participants in the moderate understanding category used the common invert and multiply algorithm (both correctly and incorrectly). Less frequently, these participants used area and measurement models (both correctly and incorrectly) to solve the problems or struggled with the contextual problems, thus lowering their total overall score. This highlights the importance of looking at the contextual and computational questions in more depth.

Table 4
Descriptive Statistics for Pre - Division of Fractions Understanding Test (N=34)

	Max Score	Mean	SD	Weak Understanding n (%)	Moderate Understanding n (%)	Good Understanding n (%)
Total Content	40	22.20	7.36	12 (35)	16 (47)	6 (18)
Computational	24	13.37	5.02	14 (41)	5 (15)	15 (44)
Contextual	16	8.83	3.38	9 (27)	15 (44)	10 (29)

Computational Problems. When examining the six computational problems from the division of fractions understanding test, scores ranged from 6 to 20 (maximum score = 24) with a mean of 13.37 (SD = 5.02). Participants' scores were sorted into three groups so that the researcher could examine their computational understanding of division of fractions in more depth. Fifteen (44%) participants scored from 18 to 24 points on the computation questions and were considered to have good computational understanding because they could correctly answer all 6 computational problems. Fourteen (41%) of the participants received less than 12 points, thus demonstrating a weak understanding when solving division of fractions computational problems. Participants (n = 5, 15%) with scores ranging from 12 to 17 demonstrated a moderate understanding of computational problems involving division of fractions. Therefore, these findings indicate that the participants tended to have either weak or good understanding of how to solve division of fractions computational problems at the beginning of their methods course.

In addition to looking at their total computational DOFUT scores, strategies participants used to solve division of fractions computation problems as well as common errors participants made were examined (see Table 5). Participants with correct solutions overwhelmingly used the common invert and multiply algorithm to solve the computational problems (see Appendix E). In addition to using the algorithm, a few other correct strategies were used by participants including the common denominator algorithm, an area model, a length/measurement model, and conceptual solutions that focused on the meaning of division.

Table 5
Pre – DOFUT Computational Solution Strategies by Problem (N = 34)

	Q#1 $6 \div \frac{1}{3}$ (n)	Q#2 $\frac{1}{2} \div \frac{1}{4}$ (n)	Q#3 $2\frac{1}{3} \div 1\frac{1}{6}$ (n)	Q#4 $\frac{3}{4} \div \frac{1}{2}$ (n)	Q#5 $\frac{1}{2} \div 4$ (n)	Q#6 $\frac{1}{6} \div 1\frac{1}{3}$ (n)
***Correct Solutions**	22	20	18	22	22	17
Invert and Multip	18	19	15	17	20	17
Common Denom	0	0	2	0	0	0
Area Model	2	1	0	1	2	0
Length Model	0	0	0	0	1	0
Conceptual Interp	1	1	0	1	0	0
Other	1	0	1	3	0	0
***Incorrect Solutions**	11	14	16	12	12	17
Invert and Multip	5	2	4	1	7	7
Cross Multip	5	9	10	7	4	8
Area Model	1	1	0	0	1	0
Length Model	0	1	0	0	0	0
Common Denom	0	0	2	2	0	2
Decimals	0	1	1	2	0	0
No Solution	1	0	0	0	0	0

*Columns and totals may not match. Some participants solved the problem in multiple ways.

The participants with incorrect solutions tried to solve the problems in a variety of ways (see Appendix F), but predominately used the common invert and multiply algorithm or the cross multiplication strategy. Participants using the common invert and multiply algorithm incorrectly did so by inverting the dividend instead of the divisor. Other common errors with the algorithm included finding the inverse of both the dividend and the divisor or making simple computation errors. When using the cross multiplication strategy, participants would multiply the numerator of the dividend by the denominator of the divisor and write this product as the denominator of the answer. Next, the participants attempted to multiply the denominator of the dividend by the numerator of the divisor and write this as the numerator of the answer. Less common errors included incorrect use of area and length/measurement models, mistakes while attempting to use the common denominator algorithm, combining the common invert and multiply algorithm with the cross multiplication strategy, or incorrectly changing a mixed number to an improper fraction. On problem three, one person in particular changed the mixed number to an improper fraction, found a common denominator, and then multiplied the numerators and kept the same denominator. There seemed to be more faulty procedures on problem three than on any other problem.

Upon completion of their K-12 school experience and their college mathematics content courses, it appears that prospective elementary teachers prefer to solve division of fractions computation problems with the common invert and multiply algorithm. At times, participants would miss a problem due to a minor computational error such as multiplying incorrectly. More commonly, participants would invert the dividend instead of the divisor or incorrectly cross multiply. Additionally, when problems became more difficult, even the participants that used alternative strategies on other problems were more likely to revert to using the common invert and multiply algorithm.

Contextual Problems. After examining the total fraction division test and the computational questions on the test, participants' scores on the four contextual questions from the division of fractions understanding test were examined. The contextual scores ranged from 1 to 13 with a mean of 8.83 (SD = 3.38) out of 16 possible points. The participants' DOFUT contextual scores were sorted into three categories: weak understanding (0-7), moderate understanding (8-11), and good understanding (12-16) of contextual division of fractions problems.

Those participants with a weak understanding (n = 9, 27%) of contextual division of fractions problems were not able to complete more than half of the contextual problems. Fifteen participants (44%) had a moderate understanding of the contextual problems while only 29% (n = 10) of the participants appeared to have a good understanding of the contextual problems (scores from 12 – 16). These participants were able to correctly answer every contextual question on the DOFUT, which suggests a good understanding of the division of fractions.

In order to comprehend why so many participants were still lacking a good understanding of division of fractions, strategies the participants used to solve the contextual problems were examined (see Table 6). A large percentage of participants used the common algorithm to correctly solve the contextual problems, however, other strategies varied by the context of the problem. For example, problem two asked the participants to determine how much pie would fit into $2\frac{1}{2}$ boxes. Six participants solved this problem correctly using an area model by drawing pictures to represent $\frac{5}{8}$ of a pie and squares to represent the boxes. They fairly shared the pieces of pie into each of the boxes to determine the amount of pie in each box (see Appendix G). Another common correct strategy participants used to solve contextual problems was the length/measurement model such as a drawing of a number line or ruler to solve problems. Overall, the majority of participants with correct solutions used the common invert and multiply algorithm to solve the contextual problems.

Table 6
Pre – DOFUT Contextual Solution Strategies by Problem

	Q#7 $10 \div \frac{2}{3}$ (n)	Q#8 $\frac{5}{8} \div 2\frac{1}{2}$ (n)	Q#9 $10\frac{2}{3} \div 3$ (n)	Q#10 $4\frac{1}{4} \div \frac{3}{4}$ (n)
***Correct Solutions**	26	21	16	12
Invert and Multiply	18	14	13	11
Area Model	1	6	0	0
Length Model	4	0	0	0
Other	3	1	3	1
***Incorrect Solutions**	7	11	16	19
Invert and Multiply	2	1	4	5
Cross Multiplication	3	3	4	2
Close Estimation	0	1	1	3
Area Model	0	4	0	3
Length Model	0	0	3	1
Symbols (repeated addition – couldn't deal with remainder)	0	0	0	4
Interpreted Backwards	2	2	1	0
Other	0	0	3	1
No Solution	1	2	2	3

*Columns and totals may not match. Some participants solved the problem in multiple ways.

The contextual questions appeared to be much more difficult for the participants than the computational problems. A few of the participants attempted to solve the problems using pictures and models, but the majority of the participants still used the common invert and multiply algorithm (see Appendix H). The participants often struggled with which number to use as the dividend and sometimes included an answer for both a ÷ b and b ÷ a because they were unsure of which number to choose first when setting up the algorithm. This was especially true when the problem required a fair-share interpretation of division (problems 8 and 9). When the problem involved dealing with a remainder, the participants that used conceptual solutions tended to struggle with determining what to do with the remainder. Errors in the common invert and multiply algorithm were consistent with errors on the computation questions with participants making simple computational errors or making procedural errors as well. More participants preferred not to attempt to solve a contextual problem than computational problems indicating that some participants are more comfortable solving computation problems than contextual ones.

Participants' description of how to divide fractions. After completing the DOFUT, the prospective elementary teachers were provided with a journal prompt. The specific prompt stated, "*Suppose you are talking to a friend about studying fractions in your mathematics methods course, they tell you they never understood the division of fractions. How would you go about explaining the concept of what it means to divide two fractions? Use the following problem to help you in your explanation,* $3\frac{1}{2} \div \frac{1}{4}$." This was administered to the participants to determine if they were able to solve division of fractions problems other than using the common invert and multiply algorithm. If they could not come up with a different method to solve the problem, did they demonstrate an understanding of why the procedure works in their discussion? All but seven participants (79%) correctly explained how to do the common invert and multiply algorithm as a step-by-step process without any explanation as to why the algorithm works (see figure 4). One participant provided a correct explanation that connected the meaning of whole number division to the meaning of fraction division. Incorrect explanations included drawings of how to solve the problem using an area model, a number line, incorrect conceptual solutions, and step by step descriptions of the algorithm. None of the participants attempted to explain why the common invert and multiply approach worked which might indicate a lack of understanding of why the procedure works.

Participants Beliefs

While looking at participants' understanding of division with fractions is important, it is also helpful to examine their attitudes about the teaching and learning of mathematics. In this section, the participants' division of fractions confidence (to solve, learn, and teach), anxiety (to learn and to teach) and personal teaching efficacy will be explored. In order to determine the participants' attitudes and beliefs about division of fractions, results from the DOFUT Confidence in Solutions Scale (CIS), FDAS, and journal prompts will be discussed.

Confidence to solve division of fraction problems. In addition to solving each division of fractions problem, the participants were asked to circle on a 6-point Likert scale how confident they were in their solution to the problem. Participants circled a 1 to indicate that they were not confident at all and they circled a 6 if they were completely confident. A total confidence score, computational confidence score, and a contextual confidence

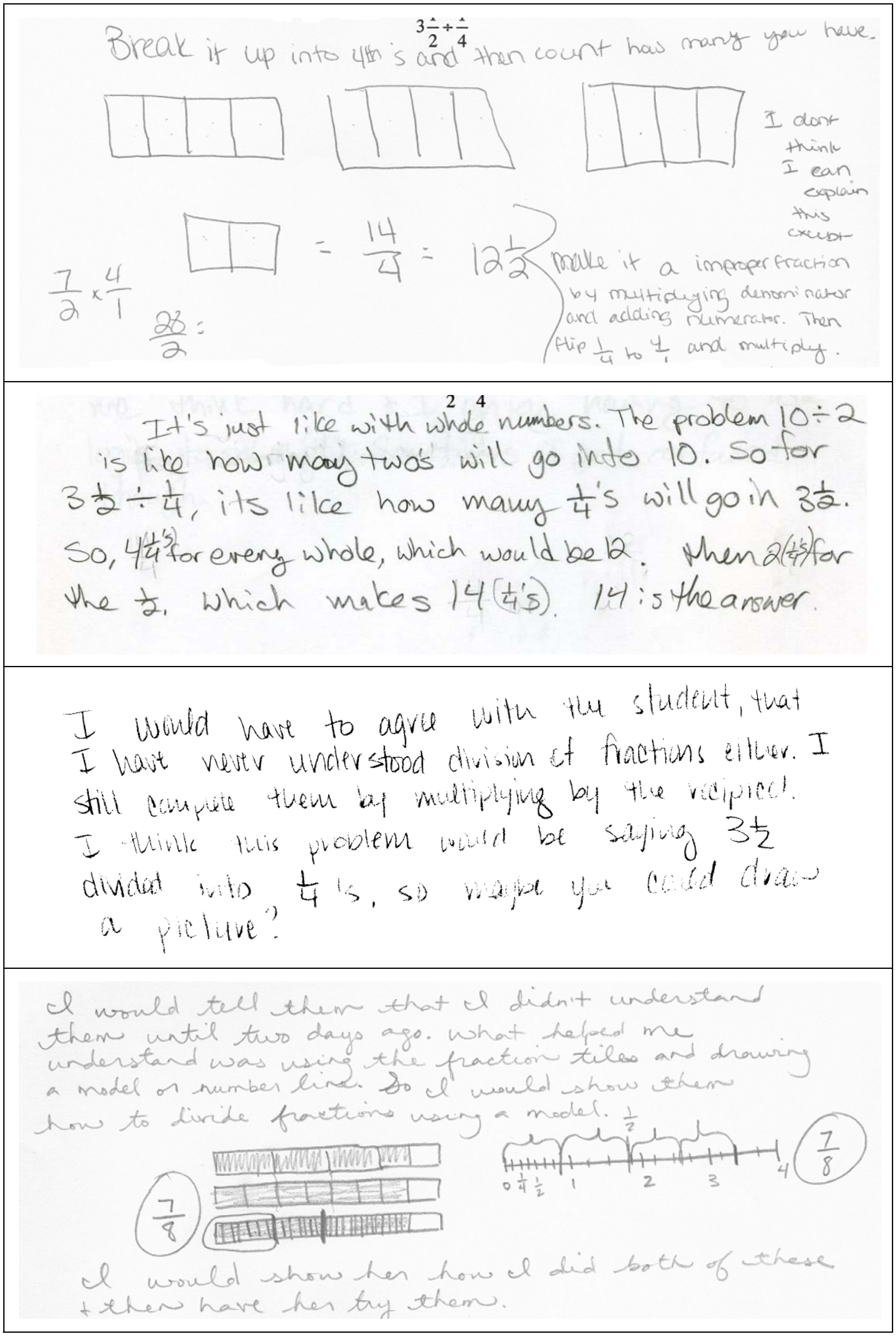

Figure 4. *Participants' descriptions of how to help a friend understand fraction division.*

score were calculated for each participant. Descriptive statistics were computed (See Table 7).

Table 7

Descriptive Statistics for Pre - Confidence in Solutions (CIS) (N=34)

	Max Score	Mean	SD	Low confidence (0-25) n(%)	Some Confidence (26-44) n(%)	High Confidence (45-60) n(%)
Confidence Total	60	34.94	15.52	14(41)	12(35)	8(33)
Computational	36	21.65	10.27	17(50)	7(21)	10(29)
Contextual	24	13.29	6.26	15(44)	11(32)	8(24)

Overall, participant ratings of their confidence to solve division of fractions problems revealed a mean score of 34.94 (SD = 15.52) with scores ranging from 0 to 60. Participants with confidence scores from 0 to 25 were considered to have low confidence in their solutions to division of fractions problems, which accounted for a little less than half (41%) of the participants in the study. Participants with some confidence in their solutions had scores from 26 to 44 with a little more than a third (35%) of the participants falling into this category. Similarly, a little less than a third (33%) of the participants indicated a high confidence in their solutions (scores from 45 to 60). While 33% of the participants were highly confident in their solutions, only 18% of the participants had a good understanding of the division of fractions.

After finding a difference between the number of participants with high confidence in solutions and those with a good understanding of division of division with fractions, the prospective elementary teachers' DOFUT total understanding score and their CIS score were compared (see Table 8). Four out of the six participants with a good total content knowledge chose that they were highly confident in their solutions on the pre-DOFUT while the other two participants had moderate confidence in their solutions on the DOFUT. The participants with moderate DOFUT scores had a tendency to obtain low to moderate CIS scores and those with weak DOFUT scores tended to have low CIS scores. The participants with moderate DOFUT scores had a variety of CIS scores, which suggests that participants at the extremes were more likely to have confidence scores that matched their understanding.

Table 8

Comparisons of Pre - CIS Scores with Pre - DOFUT Scores (N = 34)

	Weak Understanding n(%)	Moderate Understanding n(%)	Strong Understanding n(%)
Low CIS Score	9(26)	5(15)	0(0)
Moderate CIS Score	2(6)	8(24)	2(6)
High CIS Score	0(0)	4(12)	4(12)

Note: Percents add up to 100.

In addition to looking at overall confidence in their solutions, computational and contextual confidence was examined. Participants' computational CIS scores were sorted into three categories (low, moderate, and high confidence in solutions). Participants scoring at less than 18 points on the CIS scale were placed into the low confidence group because they received less than half of the confidence points possible. Results were quite similar to the total confidence scores with half of the participants claiming to not be confident (scores from 0 to 17), about a fifth of the participants claiming to have some confidence in their solutions (scores from 18 to 26), and the remaining 29% claiming to be highly confident (scores from 27 to 36) in their solutions. There was a disconnect between the computation confidence and level of understanding of division of fractions computation problems with only 29% of the participants claiming to have high confidence to solve computational problems and 44 % of the participants having a good understanding of the division of fractions.

Finally, contextual confidence scores were analyzed to determine how confident the participants were in their solutions to the contextual problems on the division of fractions understanding test. Forty-four percent of the participants scored from 0 to 11 suggesting that they lack confidence in their solutions to the contextual problems. Another 32% of the participants demonstrated some confidence in their solutions by scoring from 12 to 17. Twenty-four percent of the participants scored from 18 to 24 indicating a high level of confidence in their solutions.

Participants' Attitudes Toward Learning and Teaching. The CIS scale of the DOFUT provided an interesting view of the participants' confidence in their solutions to division of fractions problems. Next, the FDAS was administered to assess participants' attitudes (confidence, anxiety, and personal teaching efficacy) about the teaching and learning of division of fractions. The first two subscales of the FDAS examined the participants' confidence to learn and to teach the division of fractions. Means and standard deviations were calculated for the total attitude scale as well as for each subscale (see Table 9).

Table 9
Descriptive Statistics for Pre - Fraction Division Attitude Scale (FDAS) (N=34)

	Max Score	Mean	SD	Negative Attitudes n(%)	Neutral Attitudes n(%)	Positive Attitudes n(%)
Total Attitude Score	90	54.60	13.39	8(23)	11(31)	16(46)
Confidence to Learn	18	11.74	2.66	15(43)	13(37)	6(20)
Anxiety to Learn	18	10.40	3.26	11(31)	16(46)	7(23)
Confidence to Teach	18	9.26	3.31	3(9)	17(50)	14(41)
Anxiety to Teach	18	10.49	3.62	11(32)	12(36)	11(32)
Personal Teaching Efficacy	18	12.71	2.91	2(5)	16(46)	16(49)

The total attitude scores varied from 30 to 83 with a mean of 54.60 (SD = 13.39). This suggests that the typical participant had a somewhat positive overall attitude towards the teaching and learning of division with fractions. While looking at their overall attitude towards teaching and learning of division of fractions is important, examining the specific subscales offered a more thorough picture of these participants' beliefs.

The confidence to learn division of fractions subscale contained 3 questions where a low score indicated little to no confidence and a higher response indicated strong confidence. Scores ranged from 6 to 17 with a mean of 11.74 (SD = 2.66) out of 18 points possible suggesting that, on average, participants had some confidence when learning division of fractions. At the beginning of their final mathematics methods course, fifteen (43%) of the participants expressed little to no confidence when learning how to divide fractions (scores from 0 to 8). Only 20% (n = 7) of the participants indicated that they were highly confident when learning division of fractions (scores from 13 to 18). The other 12 (57%) participants indicated some confidence when learning division with fractions (scores from 9 to 12).

Results on the anxiety to learn the division of fractions subscale were quite similar to the confidence to learn subscale with scores ranging from 5 to 16 with a mean score of 10.40 (SD = 3.26). The anxiety to learn subscale consisted of three questions in which a low response indicates a high amount of anxiety and a high response indicates a low amount of anxiety. Almost half (n = 16, 46%) of the participants in the study indicated that they had a moderate amount of anxiety when learning division of fractions and obtained anxiety scores from 9 to 12. About a fourth (n = 11, 23%) of the participants denoted a low amount of anxiety when asked to solve a division of fractions problem (scores from 13 to 18). The remaining 11 participants (31%) indicated a high level of anxiety when solving the division of fractions problems (scores from 0 to 11). Taken together, these results show that there are still a substantial percentage of participants that are anxious and lack confidence when they solve division of fractions problems.

In addition to the quantitative data, qualitative journal prompts were assigned and collected to gain a more in depth picture of the participants' confidence and anxiety to learn division of fractions. The first prompt asked the participants to complete a statement in such a way that it conveyed how they felt when they saw a fraction division problem (see Appendix I). They were also encouraged to draw a picture of the metaphor used to convey their meaning and provide a description of why they chose the metaphor. These drawings were examined and grouped according to whether they indicated a positive, negative, or neutral reference. Some sample metaphors included: food references, having a tooth pulled, puzzle, tornado, a house with no foundation, monster in closet/bed, bottom of the ocean, and a dog learning new tricks. Fifty-three percent of the metaphors demonstrated a negative attitudes towards learning division of fractions, 12% of the responses demonstrated a neutral response, 24% demonstrated a positive response, and 12% of the responses didn't indicate an emotional response at all (they were procedural in their description). Some sample responses (see Appendix J) were:

- "Division of fractions is like a monster under my bed. It is scary and you don't want to go near it, but once you take time to look at it, it isn't as scary."
- "Division of fractions is like having a tooth pulled. To me, division of fractions is hard and painful! It makes my head hurt, just like having a tooth pulled used to."
- "Division of fractions is like a monster in the closet. Just like monsters in the closet, I know I am a big girl and can handle dividing fractions, but they scare me."

The majority of pictures evoked strong negative emotions towards the division of fractions, which seemed to confirm the quantitative data that suggested 54% of the participants had negative or neutral attitudes towards the learning of division of fractions.

In addition to the metaphor, the participants were also provided with a journal prompt that asked them to explain how they feel when they are first provided with a division of fractions question (see Appendix K). The participants offered a variety of responses to the prompt but the responses were similar to the metaphors provided above. Some sample responses from participants are provided below.

- *I am scared, anxious, confused.*
- *Oh No! What do I do? I immediately reverted back to my meaningless knowledge of algorithms and multiplied by the reciprocal.*
- *Nervous that I won't be able to correctly solve the problem.*
- *I think that since I was only taught the algorithm and never really developed a strong meaning of dividing fraction, I am uncertain and have difficulty solving fraction division problems.*
- *Do I need to rearrange the fractions (or changed to improper fractions) so the problem is easier to read? Or to make it less scary.*
- *Calm down. You can do this! It is not as hard as it looks, just take it step by step.*
- *My math teachers have always told me "you can do it!" So, I try and keep that in mind and not be overwhelmed by scary looking problems.*

Other participants provided a bulleted list of words such as scared, anxious, nauseated, panic, and fear. Approximately 80% of the responses to this prompt were negative and the remainder showed resiliency or a procedural solution, Figure 5 reveals a graphical representation of the words participants used and how often they occurred. The larger the size of the word indicates the word arose more frequently than words that are smaller. The words confused and nervous were the most common words used to describe how they would feel when solving a division of fraction problem.

Figure 5. Graphical Representation of Responses to Prompt (*created at Wordle.net*).

After examining the participants' beliefs about learning division of fractions, their beliefs about teaching division of fractions to their future students was explored. Data sources for examining the prospective elementary teachers' confidence and anxiety to teach and personal teaching efficacy (PTE) about teaching the division of fractions include the FDAS teaching confidence subscale, anxiety to teach subscale, and personal teaching efficacy subscale as well as journal prompts.

On the confidence and PTE subscales, a low score indicated a low amount of confidence or PTE and a high score suggested a high amount of PTE or confidence. On the anxiety to teach subscale, a low score indicated a high amount of anxiety and a high score indicated a low amount of anxiety. A score from 0 to 8 on the subscale questions suggest negative confidence, PTE or high anxiety towards teaching division of fractions.

If they received a score from 9 to 12, they were placed in the moderate category and if they had a score that ranged from 13 to 18, they were confident in their ability to solve division of fractions, strong PTE, or low anxiety toward teaching division of fractions.

First examined was the confidence to teach subscale of the *FDAS*. Scores ranged from 3 to 15 with a mean of 9.26 (SD = 3.31) out of 18 possible points. Half of the participants had a moderate feeling of confidence to teach the division of fractions. An additional two-fifths (n = 14) of the participants reported a high confidence towards teaching division of fractions. The remaining 9% of the participants indicated low confidence towards teaching division of fractions.

After analyzing the results of the confidence to teach subscale, the anxiety to teach subscale was also examined with scores ranging from 3 to 18 with a mean of 10.49 (SD = 3.62) out of 18 possible points. A little more than a third (n = 12, 36%) of the participants reported feeling somewhat anxious about teaching division of fractions. Thirty-two percent (n = 11) of the participants admitted to having a high amount of anxiety about teaching division of fractions while a little less than a third (n = 11, 32%) felt little to no anxiety to teach division of fractions. These results were very similar to the anxiety to learn subscale indicating that these prospective elementary teachers had similar anxiety to teach and learn the division of fractions (see Table 9).

While confidence and anxiety to teach division of fractions is important, the personal teaching efficacy is essential with research demonstrating the importance of a strong personal teaching efficacy on student achievement (Ashton & Webb, 1986; Moore & Esselman, 1992). Participants' PTE scores ranged from 7 to 18 with a mean of 12.71 (SD = 2.91) out of 18 possible points. Very few participants (n = 2, 5%) reported a low personal teaching efficacy prior to the beginning of their final mathematics methods course. The majority of the participants had a moderate PTE (n = 16, 46%) or a strong PTE (n = 16, 46%). This indicates that despite the prospective elementary teachers' lack of confidence to teach the division of fractions, they feel they can personally help their students understand the division of fractions.

After completion of their K-12 and college mathematics experiences, these prospective elementary teachers tended to have either neutral or negative feelings towards the learning of division of fractions but believe they will personally be able to help their students to understand the division of fractions despite these feelings.

How Their Understanding Changed

At the beginning of the study, participants solved a majority of problems with the common invert and multiply algorithm. During the methods course, participants experienced a variety of strategies they could use to help their future students make sense of division of fractions with understanding including using fraction circles, fraction tiles, sets of objects, number lines, and contextual problems. By participating in nine hours of student – centered activities with manipulatives, drawings, and symbols that connected to the meaning of whole number division, it was hoped that their understanding of division with fractions would increase. Data sources to examine the possible change in division of fraction understanding included pre/post DOFUT scores, journal prompts, field notes, and interviews.

To determine if there was a significant difference between prospective elementary teachers' understanding of division of fractions at the beginning and end of their intermediate mathematics methods course, paired samples t-tests were calculated for the total DOFUT score, the six computational questions and four contextual questions (see Table 10).

There was a significant difference [t (33) = -3.38, p=.002] between the prospective elementary teachers' total understanding of division of fractions scores from pre to post. The participants showed a significant increase in understanding of division of fractions at the end of the course. In addition to examining the total DOFUT score, paired samples t-tests were also conducted on the computational and contextual questions separately. Interestingly, there was a significant difference [t (33) = -3.54, p = .001] in computation scores from pre to post with the participants showing a significant increase in computation understanding but there was not a significant increase in the participants' contextual understanding of division of fractions.

After examining if the differences from pre-course to post-course were significant, percentage of correct responses on the DOFUT for each question was calculated and compared from pre to post (see Table 11). The participants' post DOFUT scores were again sorted into three categories (good, moderate, or weak understanding). Participants' whose scores ranged from 0 to 19 were placed in the weak understanding category. Receiving less than half of the points possible, 8% of participants demonstrated a weak understanding of division of fractions. This is a stark contrast to the pre DOFUT results where 35% of the participants fell into the weak understanding category.

When participants received between 20 – 29 of the points possible, their tests were placed in the moderate understanding category. These results were similar to the results on the pretest with slightly less than half (47% on the pre and 46% on the post-test) of the participants scoring in this range, showing they had a moderate understanding of division of fractions. Sixteen of the participants scored from 30 to 40 points and were placed in the good understanding category. While less than one-fifth (n = 6, 18%) demonstrated a strong understanding of division of fractions on the pre DOFUT, almost half (47%) of the participants now fell into this category on the post-test. Two participants increased from a weak understanding on the pre-DOFUT to a good understanding on the post-test. Seven participants had a gain of ten points or greater from pre-to post-test. None of the participants had a loss of more than five points from pre to post-test.

Table 10

Pre/Post DOFUT Comparisons (N = 34)

	Max Score	Mean (SD)				Confidence	
		Pre	Post	t	p-value	Lower	Upper
Total Understanding	40	22.20 (7.36)	26.66 (7.47)	-3.38	.002*	-7.13	-1.78
Computational	24	13.37 (5.02)	17.00 (5.02)	-3.54	.001*	-5.70	-1.55
Contextual	16	8.83 (3.38)	9.66 (2.95)	-1.54	.132	-1.92	.26

*Significant when $\alpha < .05$

Table 11

Descriptive Statistics for Division of Fractions Understanding Post - Test

	Max Score	Mean		SD		Weak Understanding		Moderate Understanding		Good Understanding	
		Pre	Post	Pre	Post	Pre n (%)	Post n (%)	Pre n (%)	Post n (%)	Pre n (%)	Post n (%)
Total Content	40	22.20	26.66	7.36	7.50	12 (35)	3(8)	16 (47)	15(46)	6 (18)	16(47)
Computational	24	13.37	17.00	5.02	5.0	14 (41)	3(8)	5 (15)	10(29)	15 (44)	21(62)
Contextual	16	8.83	9.70	3.38	2.90	9 (27)	5(14)	15 (44)	18(54)	10 (29)	11(32)

Computational Problems.

When examining only the six computational problems from the division of fractions understanding post - test, scores ranged from 5 to 24 with a mean of 17 (SD = 5.00). Participants scores were sorted based on their level of understanding (weak, moderate, and good) (see Table 12). Participants who scored from 18 to 24 points on the computational questions were placed in the good computational understanding category. Only 44% (n = 15) of the participants demonstrating a good computational understanding at the beginning of the course but the number increased to 21 (62%) at the conclusion of the course. On the pre-test, 41% (n = 14) of participants received between 0 to 11 points suggesting a weak understanding of division of fractions computational problems. However, on the post-test, this fell to only 8% (n = 3) of the participants indicating that participants computational understanding was strengthened by this course.

Participants with scores ranging from 12 to 17 demonstrated a moderate understanding of computational problems involving division of fractions. On the pre-test, 15% (n = 5) of the participants demonstrated a moderate understanding of computational problems involving division of fractions but on the post-test 29% (n = 10) of the participants scored at this level. In addition to looking at their total computational understanding scores, strategies participants used to solve division of fractions computation problems as well as common errors participants made while solving computation problems were examined.

Table 12
Pre and Post – DOFUT Computational Solution Strategies by Problem

	Q#1 $6 \div \frac{1}{3}$		Q#2 $\frac{1}{2} \div \frac{1}{4}$		Q#3 $2\frac{1}{3} \div 1\frac{1}{6}$		Q#4 $\frac{3}{4} \div \frac{1}{2}$		Q#5 $\frac{1}{2} \div 4$		Q#6 $\frac{1}{6} \div 1\frac{1}{3}$	
	Pre (n)	Post (n)	Pre (n)	Post (n)	Pre (n)	Post (n)	Pre (n)	Post (n)	Pre (n)	Post (n)	Pre (n)	Post (n)
***Correct Solutions**	22	30	20	30	18	28	22	23	22	28	17	17
Invert and Multiply	18	12	19	10	15	11	17	13	20	14	17	16
Common Denom	0	0	0	0	2	0	0	0	0	0	0	0
Area Model	2	2	1	1	0	0	1	1	2	2	0	0
Length Model	0	13	0	17	0	16	0	8	1	14	0	3
Conceptual Interpret.	1	5	1	3	0	0	1	0	0	0	0	0
Other	1	1	0	1	1	1	3	1	0	1	0	1
***Incorrect Solutions**	11	4	14	4	16	3	12	11	12	6	17	11
Invert and Multiply	5	2	2	1	4	3	1	0	7	1	7	6
Cross Multiplication	5	0	9	0	10	0	7	0	4	0	8	0
Area Model	1	1	1	1	0	0	0	2	1	0	0	2
Length Model	0	1	1	2	0	0	0	9	0	5	0	3
Common Denom	0	0	0	0	2	0	2	0	0	0	2	0
Decimals	0	0	1	0	1	0	2	0	0	0	0	0
No Solution	1	0	0	0	0	3	0	0	0	0	0	6

Examining the percentages of participants receiving a correct response on the DOFUT (see table 13) showed some differences in percent correct from pre to post test with increases in the percent of students getting a correct answer on five out of six computation questions. In the beginning of the study, the prospective elementary teachers had a tendency to predominately use the common invert and multiply algorithm on computational problems. The participants used the common invert and multiply algorithm or cross multiplication to solve almost every problem (see Table 12). At the conclusion of the study, the prospective teachers used a variety of methods to correctly solve division of fractions computational problems including use of number lines, drawings of fraction tiles, and the invert and multiply algorithm being the most prevalent method used (see Appendix L).

Participants seemed to abandon their faulty cross multiplication strategy that was prevalent on the pre-DOFUT and used more conceptual solutions. On questions 1, 2 and 3, the length/measurement model seemed to be the preferred correct strategy to use. All three of these problems had a whole number answer and seemed to be answered easily with a length/measurement model. When the answer required the participants to deal with a remainder such as question 3, 17 students began by using the number line to solve the problem, but only 8 participants dealt with the remainder correctly, the participants tended to revert to the common invert and multiply algorithm (see Appendix M). When the problem required participants to divide a smaller dividend by a larger divisor (question 6), 22 participants used the common invert and multiply algorithm to solve the problem, which was very similar to results from the Pre-DOFUT.

Table 13

Pre/Post DOFUT Comparisons with Solution Types (N = 34)

	Correct		Conceptual Solution		Procedural Solution		No response or Apparent Solution	
	Pre	Post	Pre	Post	Pre	Post	Pre	Post
Problem	n(%)	n(%)	n(%)	n(%)	n(%)	n(%)	n(%)	n(%)
1	22(64)	30(88)	6(17)	21(63)	28(80)	13(37)	1(3)	0(0)
2	20(58)	30(88)	7(20)	23(69)	28(80)	11(31)	0(0)	0(0)
3	18(53)	28(82)	5(14)	15(45)	29(86)	16(46)	0(0)	3(9)
4	22(64)	23(68)	4(11)	21(63)	30(89)	13(37)	0(0)	0(0)
5	22(64)	28(82)	4(11)	23(69)	30(89)	11(31)	0(0)	0(0)
6	17(50)	17(50)	0(0)	12(35)	34(100)	16(47)	0(0)	6(18)
7	26(77)	32(94)	5(15)	25(74)	28(82)	9(26)	1(3)	0(0)
8	21(62)	20(59)	11(32)	23(68)	21(68)	9(26)	2(6)	2(6)
9	16(46)	14(41)	8(24)	19(56)	24(70)	10(40)	2(6)	5(14)
10	12(35)	14(41)	10(40)	23(68)	21(51)	11(31)	3(9)	0(0)

To explore why the prospective elementary teachers preferred the length/measurement model and the common invert and multiply algorithm over other methods such as the area model or the set model, four interview participants (2 with weak understanding and 2 with a good understanding) were asked to solve 2 division of fractions problems ($4\frac{1}{2} \div \frac{5}{8}$ and $\frac{3}{4} \div 1\frac{1}{2}$) using a think aloud strategy. During the interview, the participants were asked to explain why they chose to use a particular strategy. On the first problem, the participants with a good understanding of division of fractions said in their interviews they tended to use the number line or the invert and multiply algorithm. When probed as to why she chose to use the traditional invert and multiply algorithm, Sarah, a participant with a good understanding, stated,

> "*Well, automatically, I would want to do it the traditional algorithm way that I was taught, so, I would convert the mixed number to an improper fraction, so I will do that which is 9 over 2. I know that I have to flip the second number so the problem becomes 9 halves times 8/5. I get 72 over ten and I know that 10 goes into 72, 7 times and so I would get 7 and 2/10 or 7 1/5.*"

*W*hen asked if she could solve it a different way, she chose to use the number line. She solved it incorrectly and did not deal with the remainder properly. She seemed perplexed that her number line answer was different from the answer she got when she used the common algorithm. She was unable to come to a conclusion as to why the answers were different.

> *Sarah:" I chose the number lines because it is a way that has clicked for me conceptually. I was never really introduced to number lines and I was never really taught how to use a number line and this semester, I was like, Oh, well that makes sense. I've noticed, using a number line to go back and check my work, usually results in a more accurate answer. If you would have asked me to solve it another way at the beginning of the semester, I would have not known how. I would have said, well that is the only way I know how to solve it."*

Despite having a good understanding and solving problems such as these in class, Sarah was still unable to deal with the remainder.

The interview participants with weak division of fractions understanding tended to use the number line and fraction tiles to solve the first division of fractions problem provided. They used the measurement interpretation, mentioning they were looking for the number of groups of five-eighths in $4\frac{1}{2}$. Both of these interviewed participants failed to deal with the remainder correctly on this problem. This was consistent with observations during instruction and analysis of post-content DOFUT's. When asked why she chose to use the number line, Lauren mentioned, "*It is just the easiest method for me to use, rather than like the array. It is easier for me to see the numbers and I can divide it however I want. I mean I don't know if I did this one correctly but it's the easiest method visually for me to use*." When asked if she could solve the problem in a different way, Lauren said she could use the common algorithm but she always got confused with which fraction to invert. Despite strength of understanding, the number line appeared to be a preferred strategy due to its flexibility and ease of use. Using the number line also fits in well with the measurement interpretation of the problem, which appeared as the most common interpretation used on the post-DOFUT.

On the post-DOFUT, most of the participants were able to answer division of fraction problems when the dividend was larger than the divisor. However, when the dividend was smaller than the divisor, several participants were unable to solve the problems correctly by using the measurement interpretation, perhaps suggesting it is more difficult for students to use this interpretation when the dividend is smaller than the divisor.

To explore this further, the interviewed participants were asked by to solve $\frac{3}{4} \div 1\frac{1}{2}$. Only one of the interview participants was able to correctly solve this problem and she did so using the common invert and multiply algorithm. When asked to solve the problem in another manner, she was unable to solve the problem. Maureen said,

> "*With this one, I am dividing a larger fraction into a smaller fraction. I don't think of it in terms of .. With the traditional way, flipping the fractions then it is kind of the same procedure. I don't know how to put that into terms.. Um, when you are [using the traditional algorithm] you are multiplying. Multiplication seems easier to me than division. So in the traditional way, I change it to multiplication rather than division. So technically, I am not doing division, I am doing multiplication still. This one is a little more difficult to think of in terms of modeling it. I guess or figuring out another way. I honestly never understood how to do this in class."*

When the dividend was smaller than the divisor, students with weak understanding rarely had a strategy for solving this type of problem. On a similar problem on the post test, 18% (n = 6) of participants didn't attempt to solve the problem and another 32% (n = 11) of the participants solved the problem incorrectly using a number line or algorithm. During class observations, this was evident in the strategies that the participants chose to solve the problem. They often tried to interpret the problem in reverse and solved $1\frac{1}{2} \div \frac{3}{4}$ instead of $\frac{3}{4} \div 1\frac{1}{2}$. Similarly, during the interviews, Sarah was asked to solve the problem in any way she could. After drawing a number line, she said,

> "*Um, I honestly don't know. These ones [where the dividend is larger than the division], it happens in every class, um, it's harder because I would think of it as switched around (1 ½ divided by ¾) and have the big one, like you are dividing*

the big one by ¾. I don't know. It's just like, all different cause like 1 ½ divided by ¾ is 2 but your taking the smaller one and dividing it by the bigger one so that is where I always get stuck. I am a very visual person and I would like see the smaller part and if I am dividing it by a bigger number, how am I supposed to divide it by what I don't have? So if I had to take a guess and not think about it, I would have just said 2, but that would be my guess. From 0 to ¾ and jump another ¾ to 1 ½. But, that's where it gets confusing for me.

She was able to think of a solution to the problem by asking herself how many groups of one number are in the dividend, but could not make sense of the problem when the dividend was smaller than the divisor.

During class, one of the participants with weak understanding also had a very difficult time making sense of her peers' solutions in class when her peer used the fair share/partition interpretation of division. This student mentioned, "*I don't even know why we are looking at this solution. It doesn't make sense to me that way. Why don't you just do it using the measurement interpretation? It is how many groups of* $\frac{1}{4}$ *are in* $\frac{5}{8}$*. Why would you even ask that question*?" A classmate replied, "*Because, some students might see it that way. You should be familiar with both interpretations [fair share and measurement]. Your students may need to think about it like that*." This participant with weak understanding failed to see the importance of trying to make sense of her peer's solutions and never really understood the fair share interpretation of division. She was only able to make sense of her solution and thus became frustrated by another participant's interpretation.

Upon completion of their final mathematics methods course, it appears that prospective elementary teachers have a variety of strategies to solve the division of fractions computation problems, predominately the invert and multiply algorithm, number lines and fraction tiles. However, participants tended to use more conceptual solutions first, before using the algorithm. A few errors in the invert and multiply algorithm were still evident on the post-test because this was not addressed during the course. Participants would still invert the dividend instead of the divisor. Additionally, when problems became more difficult, even the participants that used alternative strategies on other problems were more likely to revert to using the common invert and multiply algorithm.

Contextual Problems. After exploring the total fraction division post-test and the computational questions on the test, participants' scores on the four contextual questions from the DOFUT were examined. The contextual scores ranged from 4 to 16 with a mean of 9.70 (SD = 2.90). Once again, the participants' DOFUT contextual scores were sorted into three groups: weak understanding (0-7), moderate understanding (8-11), and good understanding (12-16) of contextual division of fractions problems.

On the pre-test, 27% (n = 9) of the participants demonstrated a weak understanding of the division of fractions contextual problems, but on the post – test, only five (14%) prospective elementary teachers demonstrated a weak understanding. Nineteen (54%) participants demonstrated a moderate understanding of division with fractions on the post-DOFUT as compared to 44% (n = 15) on the pre-DOFUT. Lastly, only 29% (n = 10) of the participants appeared to have a good understanding of the contextual problems on the pre-test with 32% (n = 11) of the participants demonstrating a good understanding of division of fractions on the post-test.

At the beginning of the course, a few participants attempted to solve the contextual problems conceptually with some attempting to use a number line or an area model. At the conclusion of the study, prospective elementary teachers used a variety of methods to correctly (see Appendix N) and incorrectly (see Appendix O) solve division of fractions problems. Participants seemed to choose solution strategies for contextual problems based on the context of problem (see Table 14). The contextual problems were chosen specifically to include 2 measurement (problems 7 and 10) and 2 fair share/partition (problems 8 and 9) interpretations of division. When solving problems with a measurement interpretation, participants tended to use a number line strategy or the common invert and multiply algorithm. When the problem utilized a partition/fair share interpretation, the participants were more likely to use an area model, such as drawings of fraction circles to represent a pie being shared fairly.

Table 14

Pre and Post – DOFUT Contextual Solution Strategies by Problem (N = 34)

	Q#7 $10 \div \frac{2}{3}$		Q#8 $\frac{5}{8} \div 2\frac{1}{2}$		Q#9 $10\frac{2}{3} \div 3$		Q#10 $4\frac{1}{4} \div \frac{3}{4}$	
	Pre (n)	Post (n)	Pre (n)	Post (n)	Pre (n)	Post (n)	Pre (n)	Post (n)
***Correct Solutions**	26	32	21	20	16	14	12	14
Invert and Multiply	18	9	14	8	13	13	11	8
Area Model	1	2	6	11	0	0	0	1
Length Model	4	19	0	2	0	1	0	4
Other	3	2	1	2	3	3	1	1
***Incorrect Solutions**	7	2	11	12	16	15	19	20
Invert and Multiply	2	0	1	1	4	2	5	4
Cross Multiplication	3	0	3	0	4	0	2	0
Close Estimation	0	0	1	0	1	0	3	1
Area Model	0	0	4	5	0	2	3	4
Length Model	0	1	0	3	3	8	1	10
Symbols (repeated addition – couldn't deal with remainder)	0	0	0	0	0	3	4	2
Interpreted Backwards	2	0	2	2	1	1	0	0
Other	0	1	0	1	3	2	1	1
No Solution	1	0	2	2	2	5	3	0

**Columns and totals may not match. Some participants solved the problem in multiple ways.

Participants Beliefs at the Conclusion of the Course

The prospective elementary teachers increased their understanding of division of fractions from pre-to post-test. While this understanding is important, beliefs and attitudes towards mathematics can have an impact on teachers' ability to teach mathematics effectively. In this section, how participants' division of fractions confidence (to solve, learn, and teach), anxiety (to learn and to teach) and personal teaching efficacy changed over the course of the semester will be reported. Data sources include pre/post results from the DOFUT Confidence in Solutions Scale (CIS), pre/post FDAS, and interviews.

Confidence to solve division of fraction problems. The six-point Likert CIS scale was included on the DOFUT to determine if participants were confident in their solutions to each question. Participants circled a number from 1 to 6 with 1 indicating that the participant was not confident at all in their solution and a 6 representing complete confidence in their solution. On the CIS scale, the participants' confidence in their solutions responses to the total content knowledge tests [$t(33) = -3.93$, $p = .000$], computation questions [$t(33) = -3.74$, $p = .001$], and contextual questions [$t(33) = -3.04$, $p = .005$] all showed a significant increase from pre-to post-test (see Table 15). This indicates that the participants were more confident in their solutions to the division of fractions questions at the conclusion of the course than at the beginning.

Table 15

Pre/Post CIS Comparisons (N = 34)

	Max Score	Mean (SD)				Confidence	
		Pre	Post	t	p-value	Lower	Upper
Total Confidence	60	34.94 (15.52)	44.68 (10.14)	-3.93	.000*	-14.80	-4.68
Computational Confidence	36	21.65 (10.27)	28.26 (6.18)	-3.74	.001*	-10.22	-3.00
Contextual Confidence	24	13.29 (6.26)	16.42 (4.87)	-3.04	.005*	-5.22	-1.02

*Significant when α < .05

To further explain the differences in scores from pre-test to post-test, four prospective elementary teachers were interviewed. During the follow-up interviews, the participants attributed the gain in understanding and confidence to their new knowledge of solution strategies and manipulatives. However, participants with a weak understanding and low confidence believed that they did not have enough time to develop a strong conceptual understanding of challenging problems or not enough practice of what they had learned in class. When asked how she would solve a division of fractions problem, Maureen offered this comment:

> *I don't really conceptually know how to do this one…I am just beginning to understand how to do it conceptually. I feel like one semester of getting it to click is definitely not enough. I have the basics of why I should solve it like that and why I should understand it but it still doesn't fully make sense to me. So, whenever I try to do it with manipulatives or a model it is kind of, I second guess*

> *myself more. It was definitely eye opening as a lot of things did click but we didn't have enough time to necessarily practice on those methods and that new knowledge that we had gotten so it was more like, here is another way to do it but if it doesn't work for you, you don't have to use it.*

Maureen's comment echoed statements by several participants during classroom observations that more time was needed when learning how to solve and teach the division of fractions.

In addition to examining how the participants' confidence in their solution strategies changed over the course of the semester, it was also important to determine if their beliefs changed as well. The FDAS was administered to the participants at the beginning and end of the course to determine if their attitudes towards the division of fractions changed as a result of participating in this course. To determine if there was a change in prospective elementary teachers beliefs (confidence, anxiety, and PTE), a paired-samples t-test was conducted on the total anxiety scale, the confidence to learn subscale, the anxiety to learn subscale, confidence to teach subscale, anxiety to teach subscale, and the PTE subscale (see Table 16). Results indicate that there were no significant differences in total attitude score, learning confidence, learning anxiety, teaching anxiety, teaching confidence, or PTE.

The total attitude scores varied from 30 to 83 with a mean of 53.26 (SD = 16.12). This mean score indicates, on average, participants had neutral overall attitude towards the teaching and learning of division of fractions at the conclusion of the course. There was not much change in the mean score from pre to post. Since there was very little difference in the total attitude scores, each subscale was examined separately.

Table 16

Pre-Post Fraction DAS Attitude Survey Comparisons (N = 34)

	Max Score	Mean (SD)		t	p-value	Confidence	
		Pre	Post			Lower	Upper
Total Attitude	90	54.6 (13.39)	53.26 (16.12)	.46	.644	-4.503	7.189
Confidence to Learn	18	9.26 (3.31)	10.00 (3.22)	-1.09	.283	-2.125	.640
Anxiety to learn	18	10.40 (3.26)	9.89 (3.91)	.71	.477	-.939	1.968
Confidence to Teach	18	11.74 (2.66)	10.86 (3.36)	1.42	.162	-.374	2.146
Anxiety to Teach	18	10.49 (3.63)	10.03 (3.85)	.683	.499	-.903	1.817
PTE	18	12.71 (2.92)	12.49 (3.54)	.370	.714	-1.027	1.484

Table 17
Pre/Post Comparison of FDAS Descriptive Statistics (N=34)

	Max Score	Mean (SD) Pre	Mean (SD) Post	Negative Attitudes Pre n(%)	Negative Attitudes Post n(%)	Neutral Attitudes Pre n(%)	Neutral Attitudes Post n(%)	Positive Attitudes Pre n(%)	Positive Attitudes Post n(%)
Total Attitude Score	90	54.60 (13.39)	53.26 (16.12)	8(23)	11(32)	11(31)	11(32)	16(46)	12(34)
Confidence to Learn	18	11.74 (2.66)	10.00 (3.22)	15(43)	11(32)	13(37)	12(36)	6(20)	11(32)
Anxiety to Learn	18	10.40 (3.26)	9.89 (3.91)	11(31)	14(40)	16(46)	10(28)	7(23)	11(32)
Confidence to Teach	18	9.26 (3.31)	10.86 (3.36)	3(9)	4(11)	17(50)	21(63)	14(41)	12(36)
Anxiety to Teach	18	10.49 (3.62)	10.03 (3.85)	11(32)	10(31)	12(36)	16(46)	11(32)	7(23)
Personal Teaching Efficacy	18	12.71 (2.91)	12.49 (3.54)	2(5)	1(2)	16(46)	17(49)	16(49)	16(49)

Participants' Confidence to Learn. The confidence to learn subscale was inspected first. The scores ranged from 6 to 17 with a mean of 10.00 (SD = 3.22) suggesting that on average, participants had some confidence when learning the division of fractions with a score from 9 to 12 indicating a moderate response to the confidence questions. At the conclusion of their final mathematics methods course, 32% (n = 11) of participants fell within this range. On the pre-survey, 43% (n = 15) reported feeling neutral to the learning of division of fractions. A little less than a third (n = 11, 32%) of the participants scored between 0 and 8 thus demonstrating a lack of confidence in their ability to learn the division of fractions. Only 32% (n = 11) of the participants indicated that they were completely confident when learning the division of fractions (scores from 13 to 18) at the conclusion of the course, up from 20% (n = 7) at the beginning of the course. These findings suggest that the course could have a positive impact on prospective elementary teachers' confidence to learn division with fractions, however, not a significant difference (see Table 17).

Participants' Anxiety to Learn. Results on the anxiety to learn the division of fractions subscale were quite similar to the confidence to learn subscale with scores ranging from 3 to 16 and a mean score of 9.89 (SD = 3.91), down slightly from the pre-survey mean of 10.40. Only ten (28%) participants scored in the moderate anxiety category when learning division of fractions (scores from 9 to 12). About a third (n= 11, 32%) of participants indicated that they had a low amount of anxiety when asked to solve a division of fractions problem (scores from 13 to 18). Another 40% (n = 14) indicated a high level of anxiety when solving the division of fractions problems (scores from 0 to 11) which increased from 31% on the pre-survey. These results suggest the methods course did have some impact on the prospective elementary teachers' beliefs about learning the division of fractions; however, the impact appears to be small with more teachers feeling anxious to learn the division of fractions (see Table 17).

Participants' Confidence to Teach. At the conclusion of this course, these prospective elementary teachers will be teaching in the public schools as intern teachers. Therefore, it is important to examine their beliefs about teaching the division of fractions to their potential students. The participants were asked questions related to their confidence to teach the division of fractions. Participants with little confidence to teach the division of fractions had scores ranging from 0 to 11. Despite the fact that 32% (n = 11) of the participants were not confident about learning division of fractions, only 11% (n = 4) were not confident to teach the division with fractions (see Table 17). This was explored during the interviews where all but one of the participants agreed they would be able to look back at their notes from class and would work through problems before teaching the class, therefore, believed they would do a good job. Only one participant expressed any negative confidence about teaching the division of fractions to a group of fifth graders. Marley stated,

> *"To fifth graders, no. When it comes to math, I am not very confident. It has never been one of my best subjects. But, once I find something like the number line that works for me, then I feel better about it and in the upper grades I just don't feel confident to do math because I just get nervous and I can't get the answer or they're going to know it more than me or something like that so I feel like a lot more confident with the younger grades."*

It appeared that her lack of confidence to learn mathematics effected her confidence to teach the division of fractions to her potential students.

Participants' Anxiety to Teach. Next, the prospective elementary teachers' responses to the anxiety to teach the division of fractions were analyzed (see Table 17). Forty percent (n = 14) of the participants expressed a strong feeling of anxiety towards teaching the division of fractions. Another 46% (n = 16) had moderate feelings about teaching the division of fractions, which increased from 36% (n = 12) on the pre-survey. A little less than a fourth (n = 7, 23%) of the participants expressed that they were not anxious at all to teach the division of fractions. During the interviews, all of the participants, despite strength of understanding of division of fractions, expressed a feeling of anxiety if they were to teach the division of fractions to a group of fifth graders today. Maureen stated,

> *"I feel like I could teach it with several resources but I would have to really do research before teaching it. I could not just go right in and teach division of fractions. I would probably have to sit down, do every problem before I actually presented it to my class, and make sure that I conceptually understood it as well as a way that I could… If they were to come to me and show me their strategies then we would be able to learn from each other, so I think I would also learn that way."*

As she explained her feelings during the interview, she began fidgeting in her seat and rolling her chair around. It was clear she was anxious about the thought of teaching this to students.

Participants' Personal Teaching Efficacy. Finally, the participants' personal teaching efficacy of division of fractions was examined (see Table 17). Almost all of the participants had moderate (n = 16, 49%) to good (n = 16, 49%) personal teaching efficacy. Only two participants expressed that they had low personal teaching efficacy (PTE) towards the division of fractions on the post-survey, down from 5% (n = 5) expressing a low PTE on the pre-survey.

Conclusion

This study examined thirty-four prospective elementary teachers' understanding of and beliefs about the teaching and learning of division of fractions at the beginning of their final mathematics methods course. Changes in participants understanding, beliefs, and strategy use were also explored.

At the beginning of the course, quantitative data showed that participants tended to enter their final mathematics methods course with a weak to moderate understanding of division with fractions. These prospective elementary teachers primarily used the common invert and multiply algorithm to solve division of fractions problems (both correctly and incorrectly). The quantitative data also suggested that participants' understanding was procedural in nature and prone to procedural errors.

Both quantitative and qualitative data were collected to examine the prospective elementary teachers' beliefs about the teaching and learning of division with fractions at the beginning of the course. The quantitative data showed that these participants tended to have moderate attitudes about the teaching and learning of division of fractions. The qualitative data suggests that the participants were anxious about learning and teaching division and saw division of fractions as a step-by-step procedure instead of a concept to be understood.

At the conclusion of the methods course, participants' computational understanding and total understanding of division of fractions were significantly higher than they were at the beginning of the course. However, there were no significant differences in contextual understanding at the conclusion of the course. Their strategic competence increased by the end of the semester with students choosing to solve problems with area models and measurement length models in addition to using the common invert and multiply algorithm. Qualitative data showed that participants were able to understand the measurement interpretation of division better than they were able to answer problems utilizing a fair share/partitioning interpretation. Participants also suggested that they struggled with these types of problems because they did not have enough time to develop a strong understanding of the concepts.

Changes in attitudes and beliefs about the teaching and learning of division of fractions were explored as well. Quantitative data showed significant differences in participant's confidence in their own solutions to division of fractions problems. There were not any significant differences in the prospective elementary teachers' confidence to learn, anxiety to learn, confidence to teach, anxiety to teach, and personal teaching efficacy of division with fractions. Qualitative data showed that participants were confident about teaching division of fractions, but still felt anxious to teach without having access to their notes. They all felt the course helped them to understand division with fractions on a deeper level than they previously held.

In Chapter V, a summary of the results as well as conclusions, implications, and recommendations for future research will be discussed.

CHAPTER V

SUMMARY, CONCLUSIONS AND RECOMMENDATIONS

The difficulty of teaching division of fractions is well documented in literature with Ma (1999) referring to the teaching and learning of division of fractions as the most difficult concept to learn and teach. Research has also shown that practicing teachers' understanding of division of fractions is often procedural in nature and riddled with the same common errors and misconceptions as students (Tirosh, 2000). Teachers of mathematics need to have a strong understanding of the mathematics they teach in order to teach for understanding (NCTM, 2000). If teachers are not developing this deep understanding of division of fractions in their K-12 education or in their college coursework, when will they develop the depth of understanding needed to impact student achievement?

This research study set out to explore the understanding of division of fractions and beliefs about the teaching and learning of division of fractions that prospective elementary teachers bring to their final mathematics methods course. Additionally, how those understandings and beliefs change over the course of the semester, what experiences the participants felt helped to increase their understandings of the teaching and learning of division of fractions was explored. The questions guiding this study were:

1. What understanding of division of fractions do prospective elementary teachers bring to their final mathematics methods course?

2. What beliefs (confidence, anxiety, and personal teaching efficacy) do prospective elementary teachers bring about division of fractions to their final mathematics methods course?

3. Is there a difference in prospective elementary teachers' understanding of division of fractions after participation in their final mathematics methods course?

4. Is there a difference in prospective elementary teachers' beliefs (confidence, anxiety, and personal teaching efficacy) about the division of fractions after participation in their final mathematics methods course?

Both quantitative and qualitative data were collected using a sequential explanatory mixed methods design. Thirty-four prospective elementary teachers - in their final mathematics methods course prior to student teaching - completed a pre/post Fraction Division Attitude Survey (FDAS), pre/post Division of Fractions Understanding Test (DOFUT) and a demographic survey. During the course, three journal prompts were collected and field notes were taken during six observations. Additionally, a semi-structured interview was conducted at the conclusion of the course with 4 participants. Results from both quantitative and qualitative data, were analyzed in order to determine the understanding of and beliefs about the teaching and learning of division of fractions that these thirty-four prospective elementary teachers had prior to and at the conclusion of their final mathematics methods course.

What Prospective Elementary Teachers Bring to their Final Math Methods Course

Understanding. The first research question sought to explore the understandings about division with fractions prospective elementary teachers brought to their final mathematics methods course. Quantitative and qualitative data were collected and analyzed. Means, standard deviations and percentages were calculated for the overall Division of Fractions Understanding Test (DOFUT), the computation questions, contextual questions, and confidence in solution (CIS) scale. Additionally, solution strategies were examined and common strategies and error patterns were determined from the data. Finally, the journal prompt asking students to explain how they would explain division with fraction to a friend (see Appendix I) was analyzed for emerging themes.

Data revealed that about a third of the prospective elementary teachers had a weak overall understanding of division of fractions at the beginning of their final mathematics methods course. The students with moderate or good understanding of division of fractions tended to use the common invert and multiply algorithm despite the difficulty or type of problem, although only about half used this algorithm correctly. When the algorithm was used incorrectly, the participants tended to find the reciprocal of the dividend instead of the divisor or used a cross multiplication procedure that resulted in an incorrect answer. On the contextual problems, participants demonstrated a lack of understanding of division of fractions in context and often set up the problem incorrectly. Occasionally, some participants would try to use number lines or area models to solve the contextual problems but very few were successful in doing so. These results are similar to findings from Tirosh (2000) and Ball (1999) who found that prospective teachers often have the same misconceptions and "buggy" algorithms as their future students.

Examination of journal prompts showed similar results. When the participants were asked to help a friend to understand division of fractions, only one participant was able to provide a conceptually correct solution. The remaining participants provided either incorrect drawings or offered a step-by-step description of how to do the common invert and multiply algorithm (both correctly and incorrectly). None of the participants that used a step-by-step explanation provided justification as to why their process worked. This is troubling because the prompt mentioned that the friend did not understand the procedure in the first place and the majority of participants just tried to explain the algorithm by providing step-by-step directions. This seems to suggest the majority of participants saw division of fractions solely as a collection of facts or steps and had not developed a conceptual understanding or strategic competence for solving division of fractions, two vital strands of mathematical proficiency (NRC, 2001).

Strategic competence is a crucial strand of mathematical proficiency, especially when solving problems because it is necessary to use strategic competence when deciding which strategy to use, monitoring the use of the strategy and transferring knowledge to new situations (Kilpatrick, Swafford, & Findell, 2001). Without strategic competence, prospective elementary teachers will struggle to lead a class discussion based upon students' solution strategies (Lampert, 2001; Silver, Ghousseini, Gosen, Charalambous, Font – Strawhun, 2005) and may revert back to teaching the invert-and-multiply procedure without meaning. Wheatley and Reynolds (1999) point out the possible impedance of understanding that can occur when procedures are taught prior to conceptual understanding. Teaching procedures prior to understanding can lead to teachers and their students seeing mathematics only as a collection of facts and procedures and not patterns and relationships. Taken together, these findings suggest that at the beginning of their final mathematics methods courses, these prospective elementary teachers lacked the conceptual understanding and strategic competence that is necessary to effectively teach the division of fractions to children, especially a classroom that aligns with the *NCTM Principles and Standards for School Mathematics* (2000).

In addition to examining participant understanding, confidence in solutions were examined. Forty-one percent of the participants lacked confidence in their solutions to division of fractions problems. Participants with a weak understanding or good understanding were more likely to have confidence scores that matched their level of understanding. Participants with a moderate understanding of division of fractions had confidence scores in each of the three levels, predominately within the moderate category. These findings suggest that participants with weak or moderate understanding of fractions struggle with adaptive reasoning. Where strategic competence is essential during the problem solving process, adaptive reasoning is important for justifying and verifying the accuracy of solution strategies (NRC, 2001). Teachers that lack adaptive reasoning may struggle to determine if their students' solution strategies are correct and may not be able to effectively teach in a Standards – based classroom (NCTM, 2000; NRC, 2001).

Participants' Beliefs about the Teaching and Learning of Division of Fractions

The second research question wanted to examine what beliefs about teaching and learning of division of fractions these prospective elementary teachers brought with them to their final mathematics methods course. In order to assess participants' beliefs, means, standard deviations and percentages from the FDAS were computed and analyzed. Overall, the majority of participants had a positive (46%) or neutral (31%) attitudes towards the teaching and learning of division of fractions with a mean score of 54.6 (SD = 13.39) out of 90 possible points. However, when the 5 subscales were analyzed, the results were varied.

When examining the learning subscales (both confidence and anxiety to learn division of fractions), results suggested the majority of participants had either neutral (37%) or low (43%) confidence and either neutral (46%) to high (31%) anxiety when presented with fraction division problems. Very few participants responded with high confidence or low anxiety when faced with learning to divide fractions. Qualitative data revealed similar results. In response to drawing a metaphor to represent how they felt about dividing with fractions, over half (53%) of the participants indicated a negative feeling. When asked how they felt when asked to solve a division of fractions problem, about 80% of the participants indicated that they felt anxious or had negative feelings.

While the majority of prospective elementary teachers had negative or neutral attitudes about learning fraction division, their attitudes towards teaching division of fractions tended to be much more positive. Ninety-one percent of participants indicated a moderate or high level of confidence to teach division of fractions, while 68% marked a moderate or low level of anxiety towards teaching division with fractions. Additionally, 95% of participants indicated a moderate or high personal teaching efficacy related to teaching division of fractions. At the beginning of their final mathematics methods course, these results indicate that while these teachers are anxious and lack confidence to learn division of fractions, they tend to be confident in their ability to teach the division of fractions to their future students. However, it is important to note that participants were predominately procedural in the way they explained division of fractions, which may suggest that prospective elementary teachers feel more comfortable teaching fraction division because they see teaching mathematics as a collection of procedures. A productive disposition is defined as a “habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one’s own efficacy” (Kilpatrick, Swafford, & Findell, 2001). These results suggest that prospective elementary teachers tend to enter their final mathematics with a lack of productive disposition, one of the five strands of mathematical proficiency.

The findings of this research are in line with research that suggests that mathematics anxiety interrupts cognitive processing and can lead to lower mathematics competence, confidence, and achievement (Ashcroft, 2002). If these prospective elementary teachers have negative attitudes towards the teaching and learning of division of fractions, it could impact their ability to effectively teach this concept to their future students (Aiken, 1972).

Changes in Participants’ Understanding

Participants’ Understanding. The third research question investigated how conceptually based instruction in division of fractions influenced prospective elementary teachers’ understanding of division of fractions. To assess changes in understanding, both descriptive statistics (means and standard deviations) and inferential statistics (paired-sample t-tests) were analyzed. On the DOFUT, the computational questions, contextual questions, and CIS scale were similarly analyzed for change from pre-course to post-course. In addition to quantitative results, transcripts from interviews were examined and used to provide a more in depth picture of prospective elementary teachers at the conclusion of the methods course.

Data suggests that the mathematics methods course did have a positive impact on the participants overall understanding of division of fractions with this difference from pre-test to post test being significant [$t(33) = -3.38$, $p=.002$]. There was also a significant difference [$t(33) = -3.54$, $p = .001$] in participants' scores on the computational problems from pre to post-test, however, there were no significant differences in contextual scores from pre to post. Similarly, Ball (1990) found that prospective teachers struggled with contextual problems involving division of fractions. These findings highlight the need for further professional development for teachers on understanding and writing contextual division of fractions problems.

The most notable difference from the pre – test to the post - test was the increase in variety of strategies used by the participants. On the pre-test, the majority of participants used the invert-and-multiply algorithm to solve the division of fractions problems. However, on the post-test, the participants used a variety of strategies to solve division of fractions problems often choosing a number line or fraction tiles model prior to using the common invert-and-multiply algorithm. This indicated that the prospective teachers had begun to develop a deeper understanding for division of fractions. However, errors still occurred when participants chose to use the common invert and multiply algorithm. This could possibly be due to the fact that the algorithm was not specifically addressed during instruction.

In addition to looking at strategy use, participants' confidence in their solutions was also examined. Participants were more confident in their solutions at the conclusion of the course with the differences from pre to post being significant for total content knowledge tests [t (33) = -3.93, p = .000], computation questions [t (33) = -3.74, p = .001], and contextual questions [t (33) = -3.04, p = .005]. This might suggest that when participants' understanding and knowledge of solution strategies increases, they may be more confident in their solutions.

The qualitative data supported the quantitative data by providing a more in depth look at participants understanding, choice of solution strategies, and confidence in solutions. When asked about their gains in understanding scores, participants tended to attribute gains to their new knowledge of solution strategies and use of manipulatives. Losses in understanding and confidence were accredited to not having enough time to develop a strong conceptual understanding of problems that are more challenging or not enough practice of what they had learned in class. Participants specifically struggled with problems involving remainders and problems that required participants to use the fair share/partitioning interpretation of division. The prospective elementary teachers seemed much more comfortable using models and drawings to solve problems that could easily use a measurement interpretation of division. When the difficulty of the problem increased, the participants were more likely to revert to their common invert-and-multiply algorithm.

***Changes in Beliefs About Teaching and Learning of Division of Fractions*.**

The fourth research question examined how participants' beliefs about teaching and learning division of fractions changed over the course of the semester. While the descriptive statistics show a slight improvement in beliefs about teaching and learning of division of fractions overall as well as the specific subscales, there were no significant differences demonstrated at the conclusion of the course. Since the FDAS only had three questions per subscale, it may have not been sensitive enough to measure changes in attitude from pre to post-test. Researchers often suggest that beliefs and attitudes are difficult to measure with Likert-type surveys (Utley, 2004; Tschannen-Moran & Hoy, 2001) thus highlighting the need for a mixed-methods approach to studying the problem.

Since the quantitative data showed no significant differences from pre – test to post - test, qualitative data were then used to explain the results obtained from the FDAS. All but one of the interview participants claimed that they would be effective teaching division of fractions to their future students. They claimed they would be able to look back on notes, texts, problems, and solution strategies learned in class prior to teaching their suture students and would then be effective at teaching division of fractions. One interviewed participant claimed that she would not be effective teaching division of fractions due to a lack of confidence and high anxiety towards all mathematics. This supports research (Charalambos, Philippou & Kyriakides, 2002; Ernest, 2000, Gibson & Dembo, 1984; McDevitt, Heikkinen, Alcorn, Ambrosio, & Gardner, 1993) that suggest teachers' attitudes and beliefs can have a negative impact on their ability to teach mathematics.

Implications

The results of this study have several implications for teacher education and teacher professional development. First, this study found that prospective elementary teachers often enter their methods courses with a shallow and often procedural understanding of division with fractions. This understanding often includes misconceptions and procedural errors that persist from their own schooling experiences. This demonstrates that prospective elementary teachers are not leaving their middle school, high school, and college mathematics experiences with a deep understanding of the mathematics that they are to teach. Despite the research findings that suggest this profound understanding of mathematics is needed to teach in a way suggested by the NCTM *Principles and Standards for School Mathematics* (Ball, 1990; Ma, 1999; NCTM, 2000), prospective elementary teachers are not prepared to teach for mathematics proficiency. This deep understanding could be developed in a college mathematics course that focused heavily on rational number concepts and this understanding could then be enhanced in the mathematics methods courses.

Second, over the course of the semester, significant gains in understanding, confidence in solutions, and increased awareness of solution strategies were obtained. This indicates that focused instruction with a conceptual emphasis can help prospective elementary teachers to develop proficiency with division of fractions. However, instruction did not significantly improve prospective elementary teachers' understanding of contextual problems. This indicates that mathematics educators need to focus more specifically on writing and solving contextual problems.

Third, results indicated these prospective elementary teachers entered their mathematics methods course with neutral to negative dispositions towards the teaching and learning of division of fractions. These feelings did not significantly change over the course of the semester. Swars and Giesen (2006) found that beliefs and attitudes are resistant to change. Teacher educators need to be aware of this when providing professional development for classroom teachers. Those negative and neutral beliefs must be examined and addressed as beliefs and attitudes are related to student achievement.

Recommendations for Future Research

While data revealed some interesting findings about prospective teachers' understanding of and attitudes toward division of fractions, further research is needed on how to help prospective elementary teachers to develop mathematical proficiency. Recommendations for future research from this study leads to the following future research studies:

- Additional research is needed to determine why prospective elementary teachers enter their methods courses with negative to neutral attitudes towards the learning of division of fractions but have positive attitudes towards their ability to teach division of fractions.
- It was determined in this study that the participants' understanding of computational division of fractions problems improved significantly but their contextual understanding did not improve significantly. Future research

should examine this phenomenon and explore ways to improve teachers understanding of contextual problems.

- The participants in the study developed a variety of strategies to use to help their future students develop an understanding of division of fractions, however, when they encountered a computational or contextual problem with a dividend that is smaller than the divisor, they could not find a useful strategy to help them solve the problem. Additional research should investigate how educators can help these teachers to develop strategies for solving division of fractions problems when the dividend is smaller than the divisor. It would be interesting to see if a mathematics course designed to enrich prospective elementary teachers' understanding of rational numbers would produce even more gains in understanding of division with fractions.
- At the conclusion of the course, several participants still used the common invert and multiply algorithm to solve division of fractions problems (some correctly and some incorrectly). Future classes should not only develop a conceptual understanding of division of fractions, but should also provide instruction on procedural fluency as well to address the misconceptions that participants may have with the common algorithms. Additionally, participants need to develop an understanding of the common invert and multiply algorithm if they are going to use it to solve problems.
- While this class did show small but significant gains in an understanding of division with fractions, the understanding was still not a profound understanding of mathematics (Ma, 1999). There is a need for longitudinal

studies that follow prospective elementary teachers through their mathematics courses, methods courses, and professional development to determine the most effective way to develop this profound understanding of division with fractions. An interesting question to ask might be: What experiences do prospective elementary teachers and in-service elementary teachers find useful in helping to develop their own understanding of teaching and learning division of fractions? Additionally, longitudinal research needs to be conducted to examine how prospective elementary teachers' level of understanding of division of fractions affects their own future students' understanding of division of fractions.

Despite the fact that there is a long history of research in the area of division of fractions, these participants still struggled with solving contextual problems. Future research needs to examine connections and relationships between teachers' understanding of division with fractions and their students understanding. Students' attitudes are influenced by their own teachers' attitudes about mathematics. In order for students' understanding to increase, we have to help their teachers understandings and beliefs to increase as well.

Concluding Remarks

The need for highly qualified teachers has been well documented in the literature (Ball, 1990; Ma, 1999). However, teachers are still entering classrooms ill prepared to teach for mathematics proficiency (NRC, 2001). This research study provides evidence that prospective elementary teachers can develop a conceptual understanding of division with fractions by participating in hands-on inquiry based activities designed to help upper elementary students to develop an understanding of fraction division. However, changing beliefs about the teaching and learning of division with fractions is harder to accomplish. Teacher preparation programs as well as professional development programs should work to improve teachers' attitudes and beliefs about mathematics. Without this important component of professional development, teachers will continue to pass on their negative beliefs about the teaching and learning of division of fractions to their future students.

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APPENDICES

Appendix A

Demographic Survey

Please respond to each of the following questions:

1. Gender (Circle One): Male Female

2. Ethnicity (Circle One): African American Asian Caucasian

Hispanic Native American Other: ____________________________

3. What is your age: ____________________ years

4**. What is your college mathematics background? (Circle each course you have taken):**

College Algebra	Functions	Applications of Modern Math
Trigonometry	Statistics	Calculus I
Calculus II	Calculus III	Geometric Structures
Mathematical Structures	Other: ____________________________	

5. What is your high school mathematics background? (circle each course you have taken):

Algebra I	Algebra II	Algebra III
Geometry	Trigonometry	Pre-Calculus
Math Analysis	Statistics	Calculus

Other: ____________________________

Appendix B

Division of Fractions Understanding Test (DOFUT)

Pseudonym:__________________________ Date:__________________________

Solve each of the following fraction problems in a way that makes sense to you. Explain how you solved each problem. Indicate the level of confidence you have in your solution.

1. $6 \div \frac{1}{3}$

On the following scale, indicate your level of confidence in your solution to the question above.

Not Confident					Completely Confident
1	2	3	4	5	6

2. $\frac{1}{2} \div \frac{1}{4}$

On the following scale, indicate your level of confidence in your solution to the question above.

Not Confident					Completely Confident
1	2	3	4	5	6

3. $2\frac{1}{3} \div 1\frac{1}{6}$

On the following scale, indicate your level of confidence in your solution to the question above.

Not Confident					Completely Confident
1	2	3	4	5	6

4. $\frac{3}{4} \div \frac{1}{2}$

On the following scale, indicate your level of confidence in your solution to the question above.

Not Confident					Completely Confident
1	2	3	4	5	6

5. $\frac{1}{2} \div 4$

On the following scale, indicate your level of confidence in your solution to the question above.

Not Confident					Completely Confident
1	2	3	4	5	6

6. $\frac{1}{6} \div 1\frac{1}{3}$

On the following scale, indicate your level of confidence in your solution to the question above.

Not Confident					Completely Confident
1	2	3	4	5	6

7. Marcy volunteered to make spirit bows for the spirit squad. She has 10 yards of ribbon. Each spirit bow requires $\frac{2}{3}$ yard of ribbon. How many bows will she be able to make?

On the following scale, indicate your level of confidence in your solution to the question above.

Not Confident					Completely Confident
1	2	3	4	5	6

8. I have $\frac{5}{8}$ of a whole pie. It fills up exactly $2\frac{1}{2}$ boxes. How much will be in each box?

On the following scale, indicate your level of confidence in your solution to the question above.

Not Confident					Completely Confident
1	2	3	4	5	6

9. Margo wants to walk $10\frac{2}{3}$ miles to prepare for her school walkathon. If she walks the same amount each day, how many miles will she need to walk in three days to prepare for the walkathon?

On the following scale, indicate your level of confidence in your solution to the question above.

Not Confident					Completely Confident
1	2	3	4	5	6

10. Devon needs $4\frac{1}{4}$ pounds of beans to make chili. How many three fourths pound bags does she need?

On the following scale, indicate your level of confidence in your solution to the question above.

Not Confident					Completely Confident
1	2	3	4	5	6

Appendix C

Fraction Division Attitude Scale (FDAS)

Pseudonym: ______________________ Date: ___________________

As you respond to the following statements, put yourself into the role of a learner and/or teacher of mathematics and circle your level of agreement. A response of 1 (one) means you strongly disagree with the statement and a response of 6 (six) means you strongly agree with the statement.

	Strongly Disagree					Strongly Agree
1. I often have trouble solving division of fractions problems.	1	2	3	4	5	6
2 . I do not get nervous when I am asked to solve a division of fractions problems.	1	2	3	4	5	6
3. I understand the division of fractions well enough to teach it.	1	2	3	4	5	6
4. I am nervous about having to teach my future students how to divide fractions.	1	2	3	4	5	6
5. I become tense when I think about having to teach my future students how to divide fractions.	1	2	3	4	5	6
6. I am sure of myself when I solve a division of. fractions problem	1	2	3	4	5	6
7. When I solve division of fractions problems, I do not get anxious.	1	2	3	4	5	6
8. I will not be very effective when I will have to teach my future students how to divide fractions.	1	2	3	4	5	6
9. I am not anxious about teaching the division of fractions.	1	2	3	4	5	6
10. I will generally teach the division of fractions ineffectively.	1	2	3	4	5	6
11. When I see a division of fractions problem, I feel uneasy. .	1	2	3	4	5	6
12. I am sure that I will be able to teach the division of fractions to my future students.	1	2	3	4	5	6
13. It will be easy to teach division of fractions to my. future students	1	2	3	4	5	6
14. I am not for sure if I will be able to teach students how to divide fractions.	1	2	3	4	5	6
15. Solving division of fraction problems is simple.	1	2	3	4	5	6

Appendix D
Interview Protocol

Interviewer will say the following: "Good morning _______________. I appreciate you agreeing to be interviewed. I am going to ask you a series of questions based upon the division of fractions and your experiences with division of fractions in this course. You may choose to not answer a question at anytime. Does this sound okay to you?" (Wait for response and answer questions they may have) "I am planning on videotaping our conversation. The recording will be transcribed verbatim and used as part of this research project. No one but the researchers will have access to your responses and all data will be reported in general such that your name will not be attached to any of your comments. Do I have your permission to video record our conversation?" (Wait for a response and start recorder only after permission has been granted)
Provide the student with the problem 4 ½ divided by 5/8 and ask the student to solve the problem in a way that makes sense to them. Ask the following questions after the student has completed the problem.

- Why did you choose to solve the problem this way?
- What does the answer mean?
- Is there another way to solve the problem?

How do you think you would have solved this problem in January before talking about fractions this semester?
Provide the student with the problem ¾ divided by 1 ½ and ask the student to solve the problem in a way that makes sense to them. Ask the following questions after the student has completed the problem.

- Why did you choose to solve the problem this way?
- What does the answer mean?
- Is there another way to solve the problem?

Do you believe that you know how to teach division of fractions effectively to fifth graders? Talk to me about that.
Do you feel that the experiences that you have had during this class have changed your ability to solve or teach division of fractions?

If so, what experiences do you feel had the most effect on your learning? Talk to me a little about that.
Based on the content knowledge post test results, I will ask the following questions:
I noticed on problem "____" that you solved the problem using "_____________________" but on problem "_____" you used "________" strategy. Can you talk to me about that? Why did you chose to use that strategy?
I also noticed that on the pre-test you tended to use "__________" strategy on questions " _____" through " ______" but on the post-test you used "_____" strategy. Talk to me a little about that.

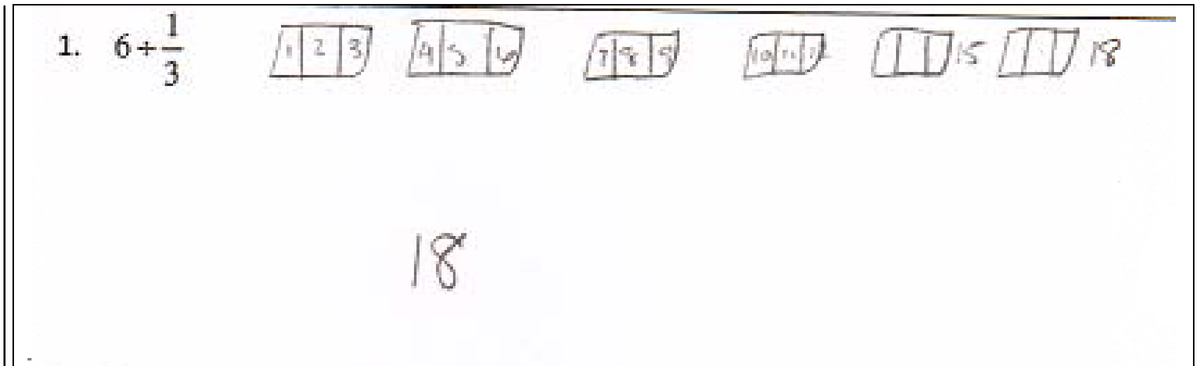
Based on the attitude survey results, I will ask the following question:

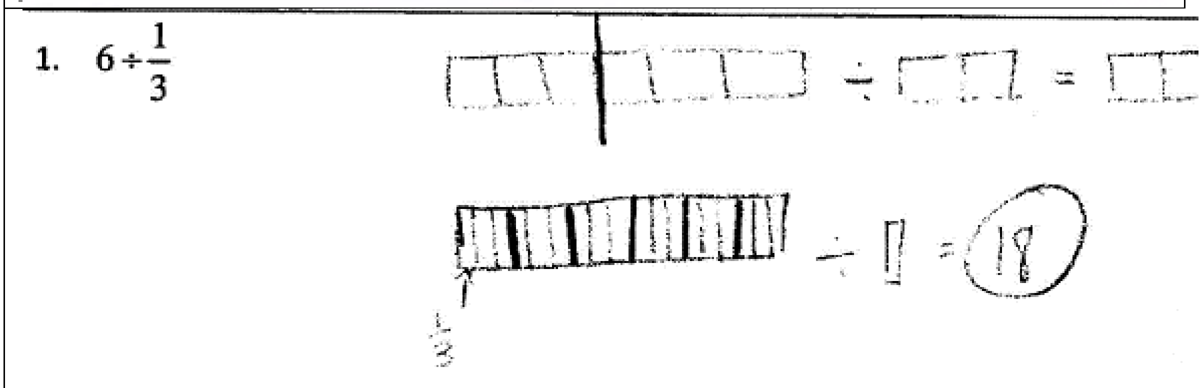
I noticed on the pre-survey your attitude towards fractions tended to be "_______" and on the post-survey, your attitudes towards the division of fractions tended to be "_____".
Talk to me a little about that.
Possible probing questions:

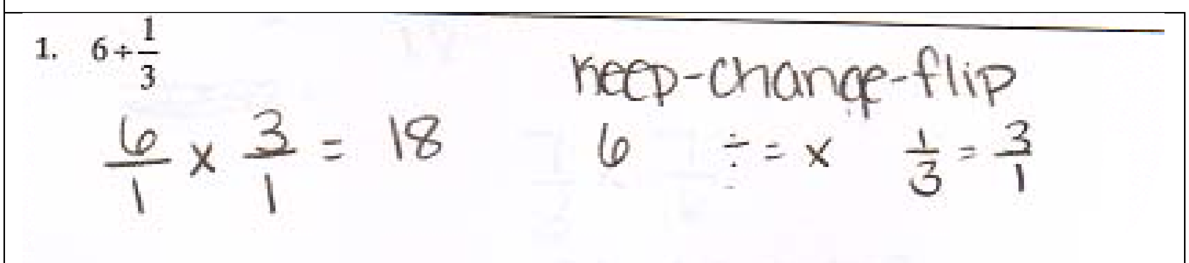
What do you mean by ______________?
Could you give me an example of ______________?
You mentioned that ____________________.
How did you feel about __________________?

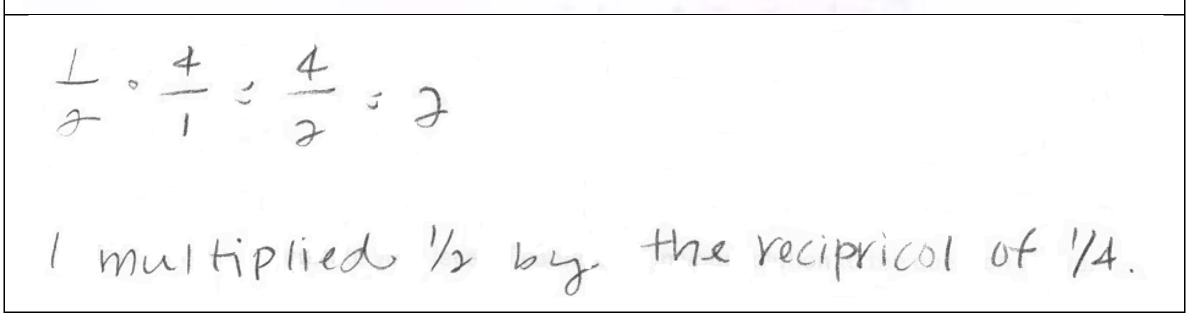
Appendix E

Correct Solutions to Pre – DOFUT Computational Problems



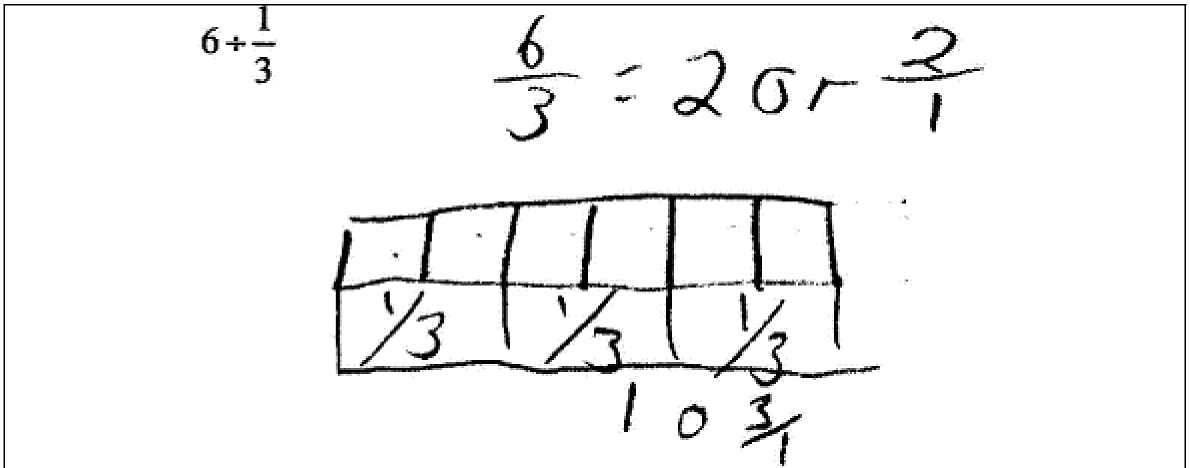


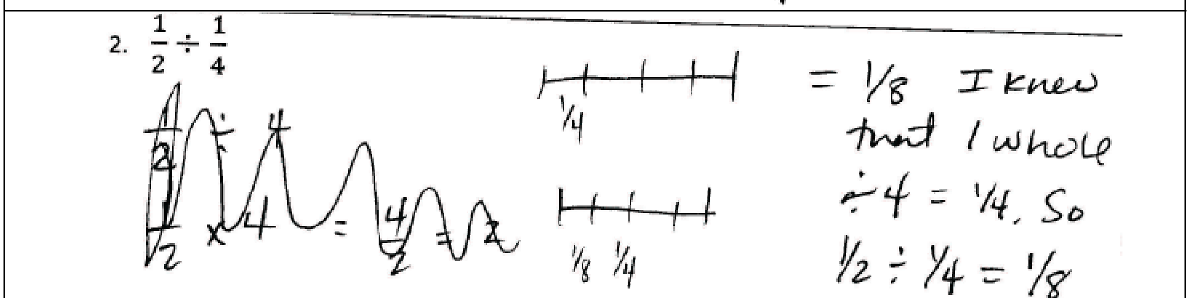


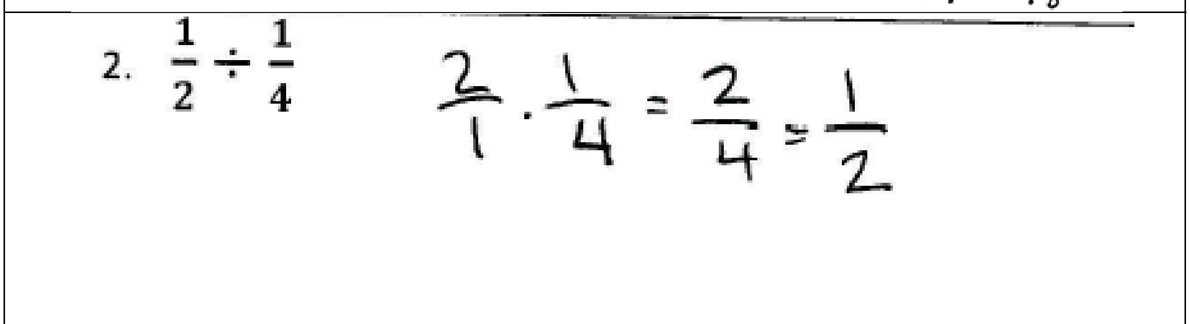


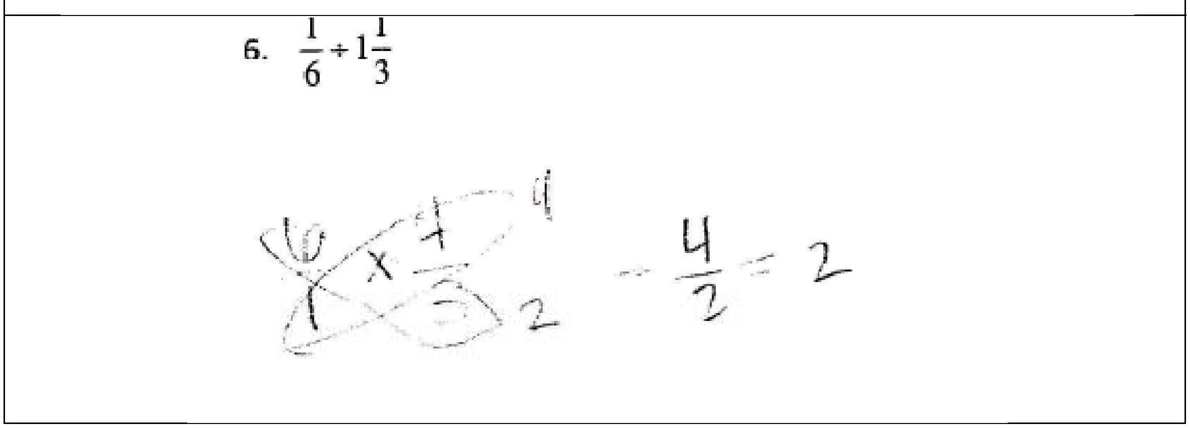
Appendix F

Incorrect Solutions to Pre – DOFUT Computational Problems







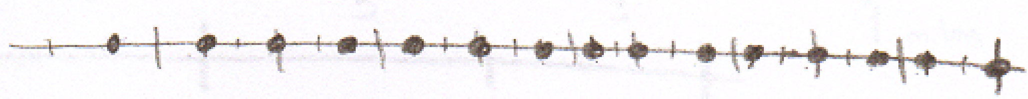


Appendix G

Correct Solutions to Pre – DOFUT Contextual Problems

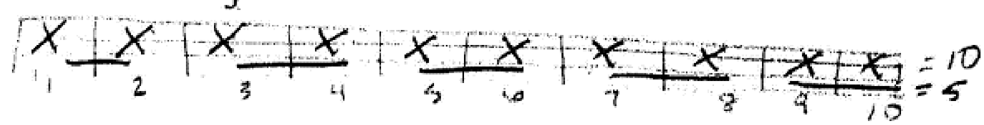
Marcy volunteered to make spirit bows for the spirit squad. She has 10 yards of rib

spirit bow requires $\frac{2}{3}$ yard of ribbon. How many ribbons will she be able to make?



15 ribbons

7. Marcy volunteered to make spirit bows for the spirit squad. She has 10 yards of ribbon. Each spirit bow requires $\frac{2}{3}$ yard of ribbon. How many ribbons will she be able to make?



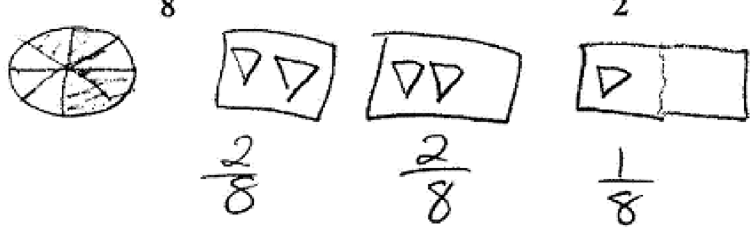
= 15

8. I have $\frac{5}{8}$ of a whole pie. It fills up exactly $2\frac{1}{2}$ boxes. How much will be in each box?

$$\frac{5}{8} \div 2\frac{1}{2} = \frac{5}{8} \div \frac{5}{2} = \frac{5}{8} \times \frac{2}{5} = \frac{2}{8} = \frac{1}{4}$$

= 1/4

8. I have $\frac{5}{8}$ of a whole pie. It fills up exactly $2\frac{1}{2}$ boxes. How much will be in each box?



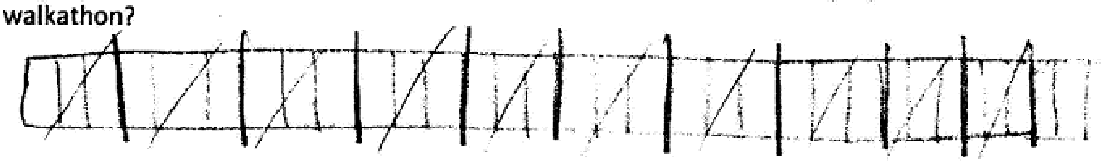
Appendix H

Incorrect Solutions to Pre – DOFUT Contextual Problems

9. Margo wants to walk $10\frac{2}{3}$ miles to prepare for her school walkathon. If she walks the same amount each day, how many miles will she need to walk in three days to prepare for the walkathon?

$$10\frac{2}{3} + 10\frac{2}{3} + 10\frac{2}{3} = 30\frac{6}{3} = 30.2 \text{ miles}$$

9. Margo wants to walk $10\frac{2}{3}$ miles to prepare for her school walkathon. If she walks the same amount each day, how many miles will she need to walk in three days to prepare for the walkathon?



① $111 + \frac{1}{3} +$

② $111 + \frac{1}{3} +$

③ $111 + \frac{1}{3} +$

8. I have $\frac{5}{8}$ of a whole pie. It fills up exactly $2\frac{1}{2}$ boxes. How much will be in each box?

$$2\frac{1}{2} \div \frac{5}{8}$$

$$\frac{5}{2} \times \frac{8}{5} = \frac{40}{10} = 4$$

1) change from mixed to improper.
2) mult. by reciprical
3) reduce

Appendix I

Journal Prompt Number One – Helping a Friend to Solve Division of Fractions

1. Suppose you are talking to a friend about studying fractions in your mathematics methods course, they tell you that they have never understood division of fractions. How would you go about explaining the concept of what it means to divide two fractions? Use the following problem to help you in your explanation. $3\frac{1}{2} \div \frac{1}{4}$

It's just like with whole numbers. The problem 10÷2 is like how many two's will go into 10. So for $3\frac{1}{2} \div \frac{1}{4}$, it's like how many $\frac{1}{4}$'s will go in $3\frac{1}{2}$. So, 4 $\frac{1}{4}$'s for every whole, which would be 12. Then 2($\frac{1}{4}$'s) for the $\frac{1}{2}$, which makes 14 $\frac{1}{4}$'s). 14 is the answer.

---

1. Suppose you are talking to a friend about studying fractions in your mathematics methods course, they tell you that they have never understood division of fractions. How would you go about explaining the concept of what it means to divide two fractions? Use the following problem to help you in your explanation. $3\frac{1}{2} \div \frac{1}{4}$

① $3\frac{1}{2} \div \frac{1}{4}$

② Change mixed number to improper fraction

$3\frac{1}{2}$ $(3 \times 2) + 1 = 7$

improper $= \frac{7}{2}$

③ $\frac{7}{2} \div \frac{1}{4}$

④ Change 2nd fraction to reciprocal + multiply strait across to get an

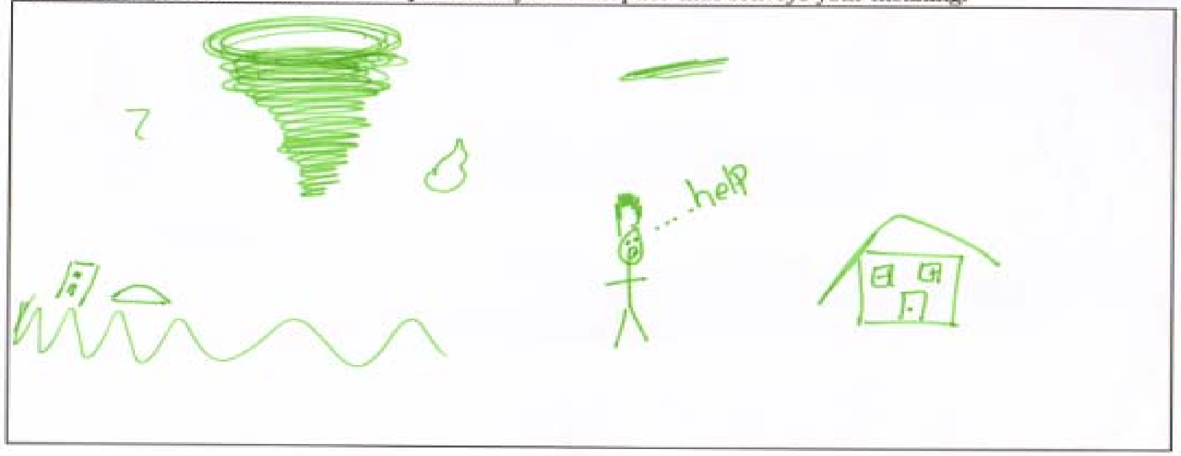
$\frac{7}{2} \times \frac{4}{1} = \frac{28}{2} = 14$

Appendix J

Journal Prompt Two – Metaphor Sample Responses

Division of Fractions is like a tornado.

In the space provided below: Draw a picture of your metaphor that conveys your meaning.



Description: (Provide a sentence or two that provides the reader with why you chose this particular metaphor)

Because division of fractions are confising + send me into a dizzy. They also flip things around, and you have to flip something when dividing fractions

Division of Fractions is like a monster in the closet.

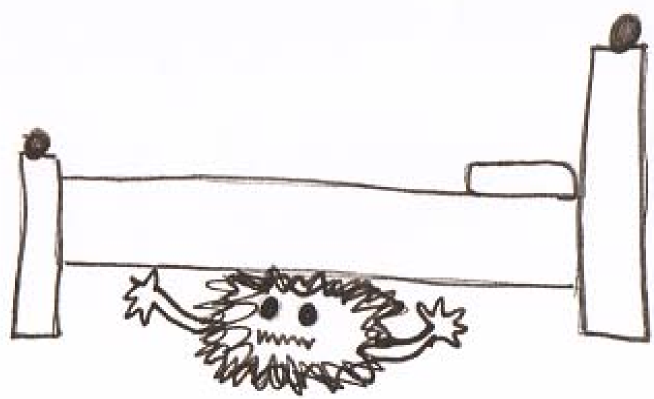
In the space provided below: Draw a picture of your metaphor that conveys your meaning.

Description: (Provide a sentence or two that provides the reader with why you chose this particular metaphor)

Just like monsters in the closet, I know I am a big girl and can handle dividing fractions; but... they scare me.

Division of Fractions is like a monster under my bed.

In the space provided below: Draw a picture of your metaphor that conveys your meaning.

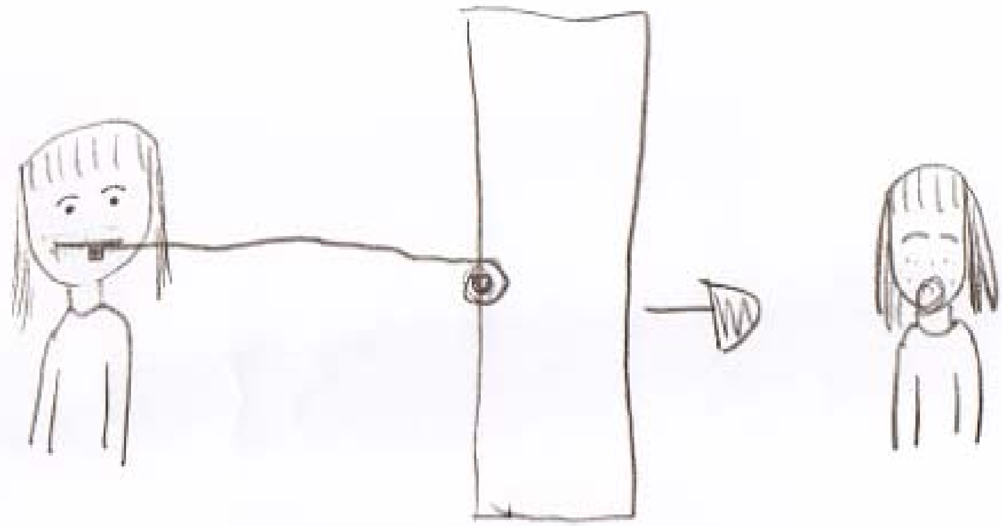


Description: (Provide a sentence or two that provides the reader with why you chose this particular metaphor)

It is scary and you don't want to go near it but once you take time & look at it, it isn't as scary

Division of Fractions is like having a tooth pulled. .

In the space provided below: Draw a picture of your metaphor that conveys your meaning.



Description: (Provide a sentence or two that provides the reader with why you chose this particular metaphor)

To me, division of fractions is hard and painful! It makes my head hurt just like having a tooth pulled used to!

Appendix K

Journal Prompt Three – Feelings Towards Solving Division of Fractions

3. What words/phrases might you use to describe your feeling at that moment you realize you need to solve a division of fractions problem?

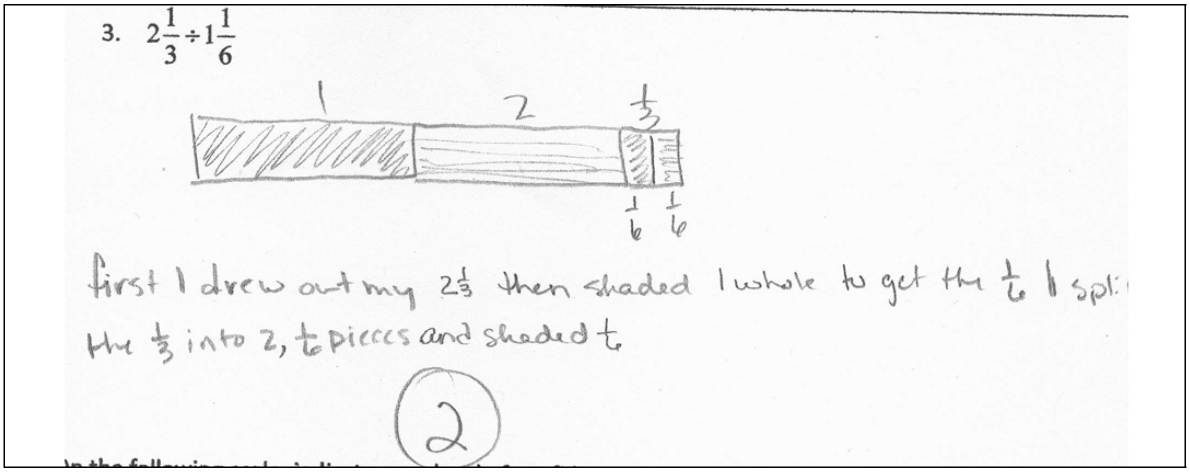
Nervous that I won't be able to correctly solve the problem.

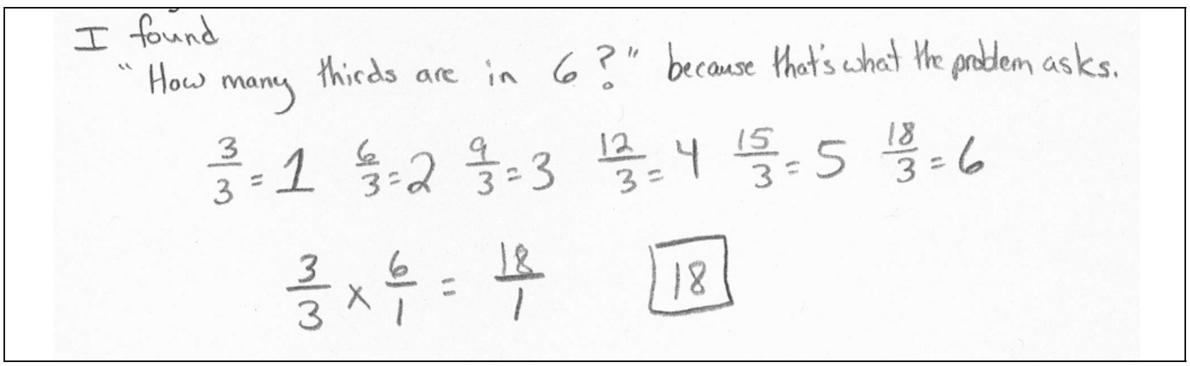
Calm down... you can do this!
It is not as hard as it looks, just take step by step!

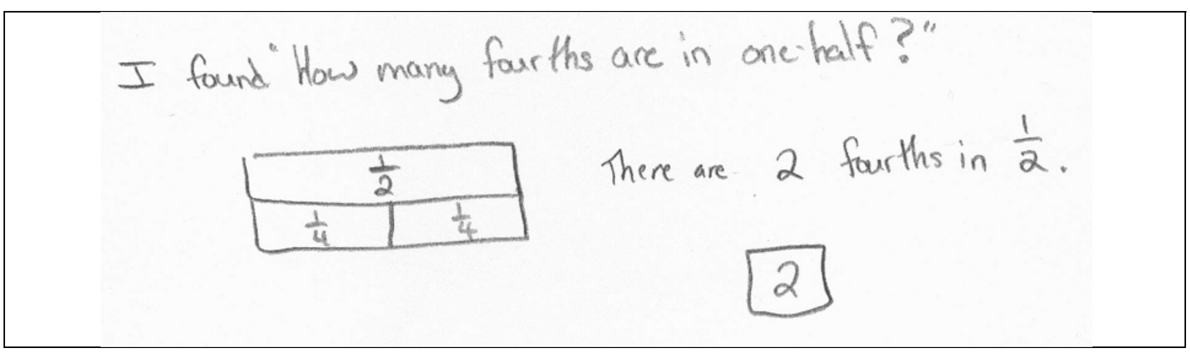
Queasy
Stressed
Nervous
Uncertain
Concerned
Lost
Confused
Frustrated

Appendix L

Correct Solutions to Post – Computational Problems

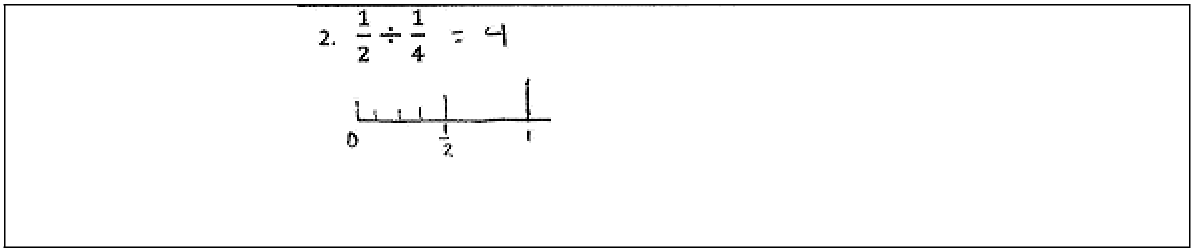


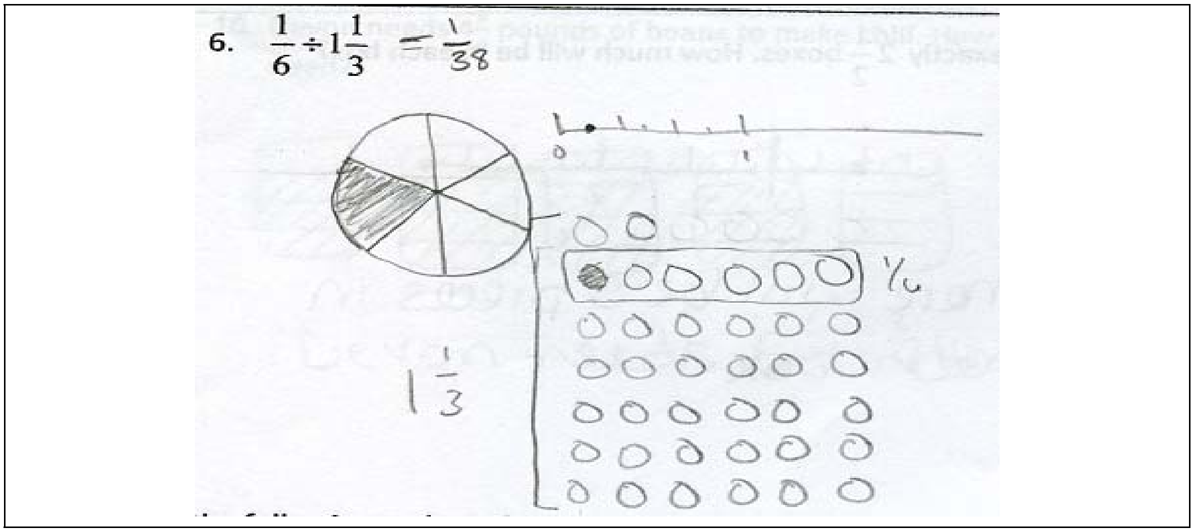


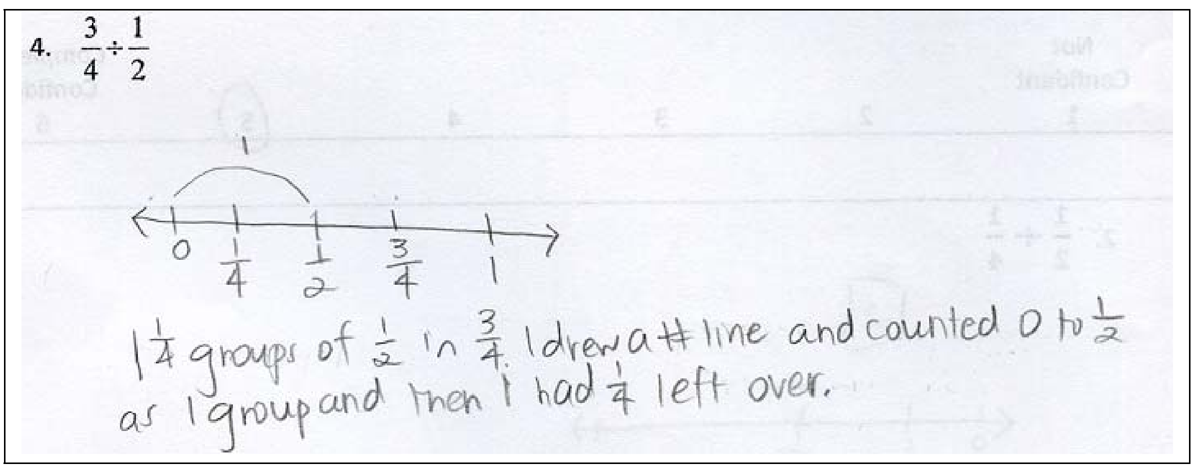


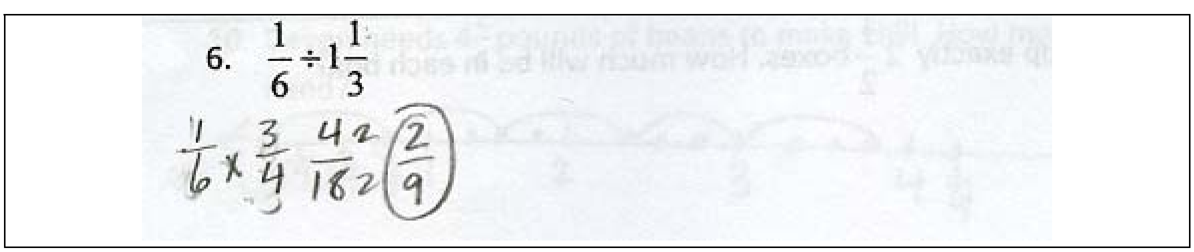
Appendix M

Incorrect Solutions to Post – DOFUT Computational Problems



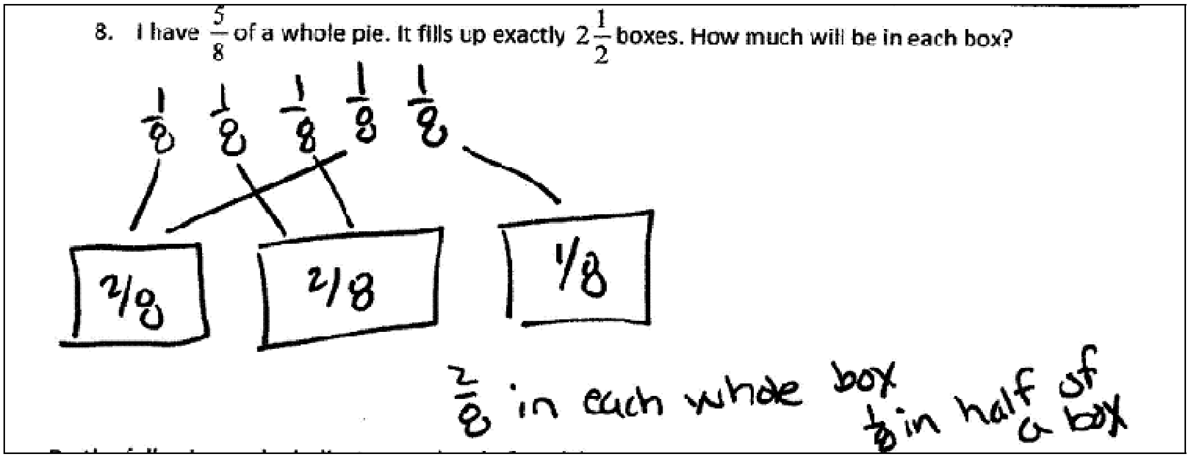


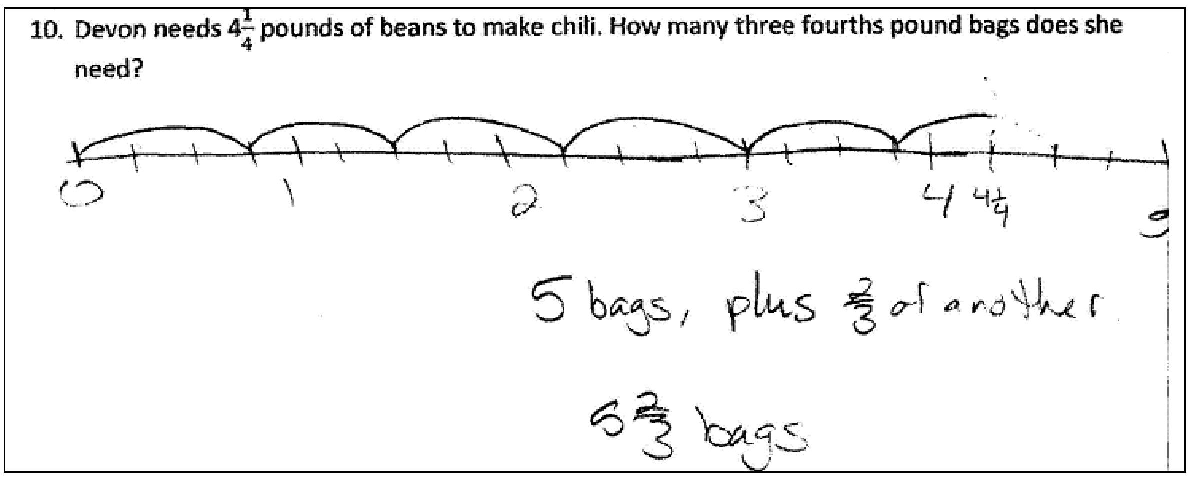


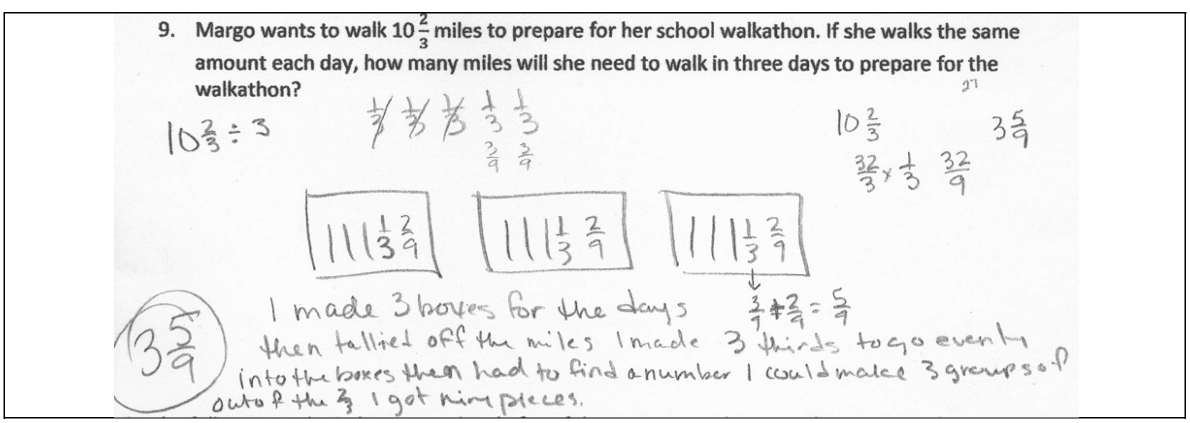


Appendix N

Correct Solutions to Post – DOFUT Contextual Problems

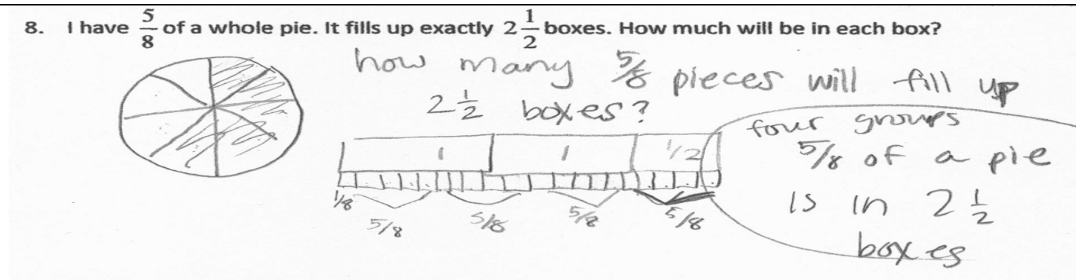




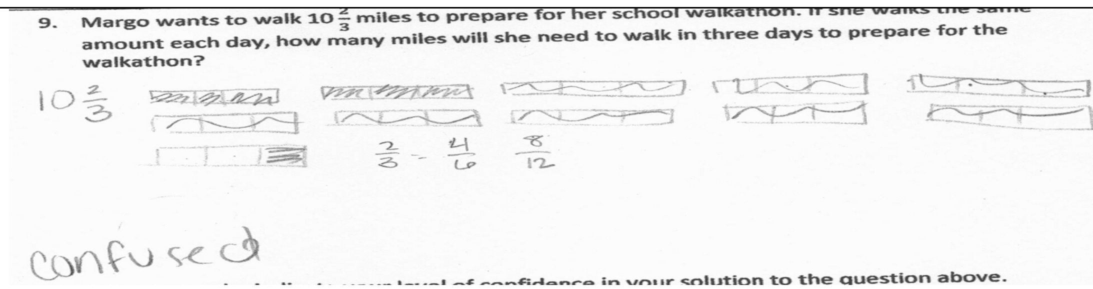


Appendix O

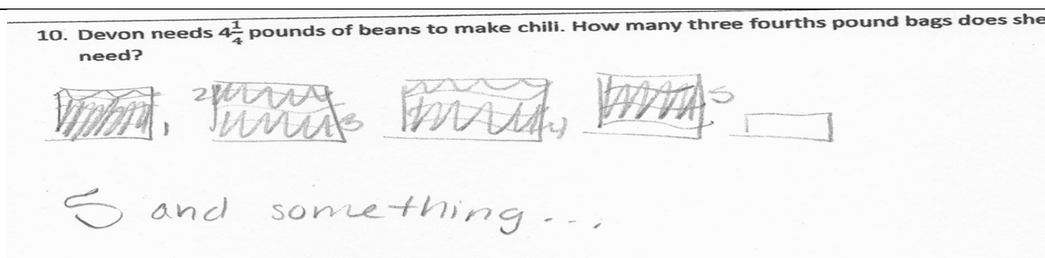
Incorrect Solutions to Post – DOFUT Contextual Problems



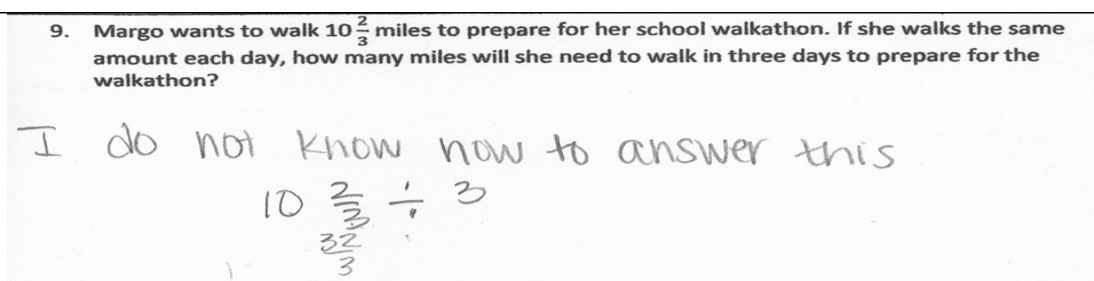
8. I have $\frac{5}{8}$ of a whole pie. It fills up exactly $2\frac{1}{2}$ boxes. How much will be in each box?



9. Margo wants to walk $10\frac{2}{3}$ miles to prepare for her school walkathon. If she walks the same amount each day, how many miles will she need to walk in three days to prepare for the walkathon?



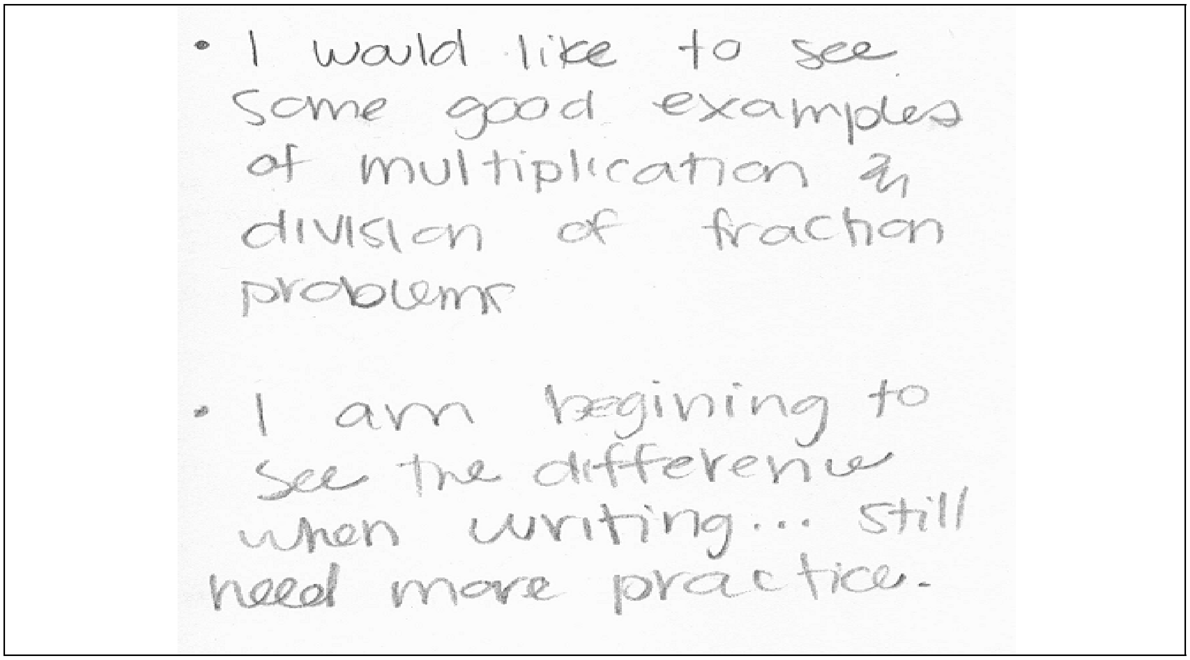
10. Devon needs $4\frac{1}{4}$ pounds of beans to make chili. How many three fourths pound bags does she need?



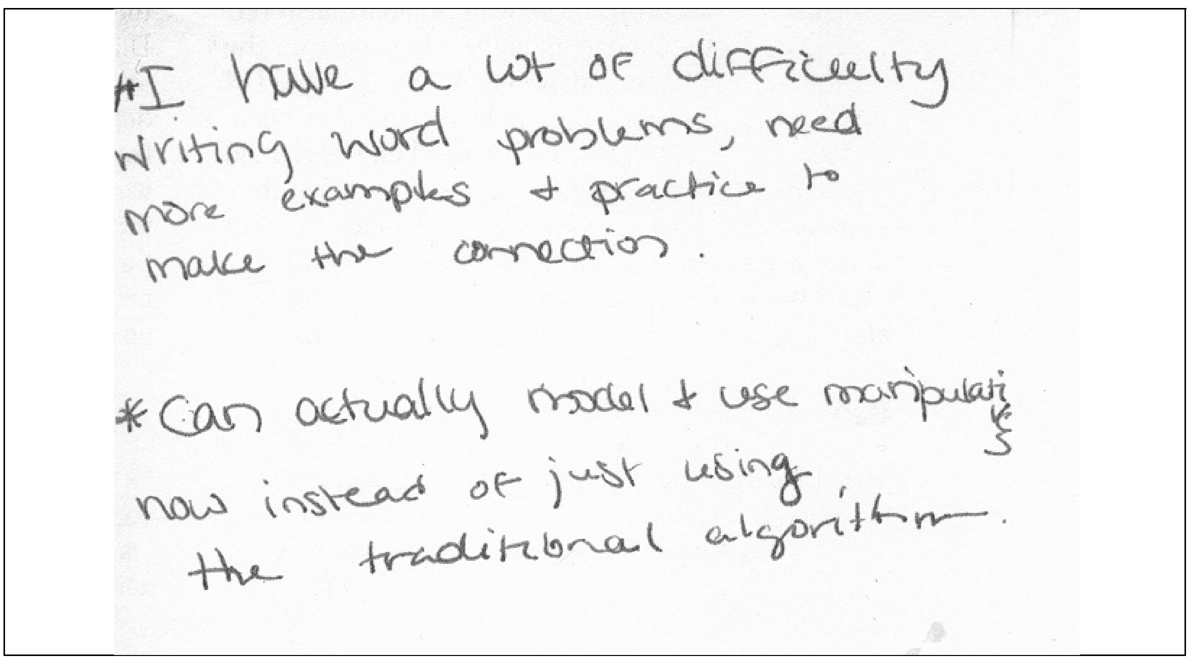
9. Margo wants to walk $10\frac{2}{3}$ miles to prepare for her school walkathon. If she walks the same amount each day, how many miles will she need to walk in three days to prepare for the walkathon?

Appendix P

Sample Response to Exit Slips



- I would like to see some good examples of multiplication & division of fraction problems
- I am begining to see the difference when writing... still need more practice.



*I have a lot of difficulty writing word problems, need more examples + practice to make the connection.

*Can actually model + use manipulatives now instead of just using the traditional algorithm.

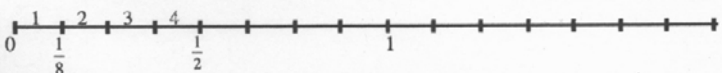
Appendix Q

Handouts/Student Response Sheets

Division of Fraction Strategies

The following students solved $\frac{1}{2} \div 4 = ?$ in a variety of ways. Examine and describe each student's method. Then, respond to the questions that follow.

Sally's Method



I know that you need to split the $\frac{1}{2}$ into 4 equal parts. So, I drew a numberline and found 0, $\frac{1}{2}$, and 1 on my numberline. Then I split each half into 4 equal parts. I can then see that each of the parts is $\frac{1}{8}$ of the whole.

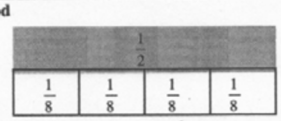
So, $\frac{1}{2} \div 4 = \frac{1}{8}$.

Describe Sally's thinking:

She started w/ her benchmark fractions & divided the number line into main #'s. Then, she divided the 0 to 1 part into 4 equal part & saw that there were 4 equal parts of $\frac{1}{8}$ in $\frac{1}{2}$.

1/.

Jose's Method

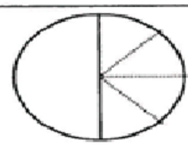


I used fraction tiles to see that if I split $\frac{1}{2}$ evenly 4 times I would get 4 pieces that each had a size of $\frac{1}{8}$.

Describe Jose's thinking:

He used visual aid to see that whatever fraction it would be, would have to go in equally 4 parts. In return, that meant only $\frac{1}{8}$th would go in 4 times evenly.

Martin's Method



If I had $\frac{1}{2}$ of a cookie and split it equally between 4 friends, each friend would get $\frac{1}{4}$ of $\frac{1}{2}$ which is $\frac{1}{8}$ of the cookie.

Describe Martin's thinking:

What do student's need to understand about division of fractions in order to solve problems in this way?

Are the methods mathematically valid? (Do they result in accurate answers? Do the procedures make good mathematical sense?) **Explain.**

3. **Is each method generalizable?** (Can it be applied effectively to other problems of the sa type?) **Explain.**

4. **Is each method efficient?** Does it take a long time to do? Can the procedure be streamli or represented in a different way to take less time but still record mathematical thinking?) **Explain.**

5. **Do you see these student's strategies as "good" strategies? Explain.**

Write a word problem for each of the following fraction problems. Solve each problem using the methods stated below.

3. $2\frac{1}{2} \div \frac{1}{4}$

Problem:____

____

____

____

____

A. Solve using an area model

B. Solve using a set model

C. Solve using a measurement model

D. Solve using an algorithm

Write a word problem for each of the following fraction problems. Solve each problem using the methods stated below.

4. $2\frac{1}{4} \div \frac{2}{3}$

Problem:____

____

____

____

____

A. Solve using an area model

B. Solve using a set model

C. Solve using a measurement model

D. Solve using an algorithm

Appendix R

Presentation Used During Division of Fractions Instruction

Review

2.

$\frac{4}{4}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{8}$	$\frac{1}{10}$
10		$3\frac{1}{3}$				

a) $\frac{2}{3}$ of 10 = ______ b) $\frac{3}{4}$ x 10 is ______

c) $\frac{2}{5}$ of 10 = ______ d) $\frac{3}{5}$ x 10 is ______

e) $5\frac{1}{4}$ of 10 = ______ f) $\frac{3}{8}$ x 10 is ______

g) $\frac{5}{8}$ of 10 = ______ h) $\frac{7}{8}$ x 10 is ______

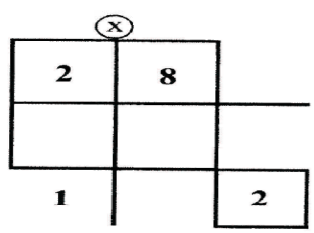
i) $1\frac{1}{2}$ of 10 = ______ j) $\frac{15}{10}$ x 10 is ______

Make up two of your own.

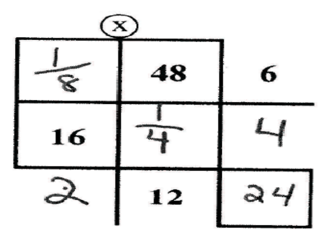
k) ______ of 10 = ______ l) ______ x 10 is ______

Fraction Two Ways

3.



4.



Before students can understand multiplication and division of fractions, what is the prerequisite knowledge that they need to have obtained?

They need to:

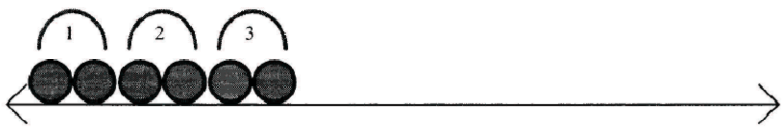
- Know what multiplication means?
- Know what division means?
- Know how to model whole number multiplication and division.
- Understand equivalent fractions. (Not just the algorithm)
- Have strong fraction and number sense.
- Know how to model fractions
- Have had some experiences with part-part-whole tasks, sharing tasks, and combining and taking apart fractions (preferably with models and symbols).
- Be able to communicate and represent their mathematical thinking.

Division of Fraction Instruction

- Should involve both measurement and partitive situations.
- Include both computational and contextual problems.
- Should involve conceptual instruction before procedural instruction.
- Follow a logical sequence of instruction.
    - Whole number divided by a fraction: even division
    - Whole number divided by a fraction: uneven division
    - Fraction divided by a whole number
    - Fraction divided by a fraction
    - Mixed number divided by a fraction
    - Mixed number divided by a mixed number

For division, you want to start by focusing on the meaning of division. What are the possible meanings of 6 ÷ 2?

Measurement interpretation: How many groups of 2 are in 6?



Partitive or Fair sharing model of division: If there are 2 groups, how many are in each group?

What is the meaning of $6 \div \frac{1}{2}$?

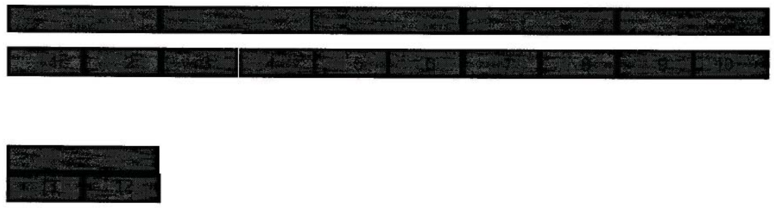
How many $\frac{1}{2}$s are in 6?
measur

If there are 6 groups how many $\frac{1}{2}$ ÷ 6 $\frac{1}{2}$ are in each grou

part
If there are $\frac{1}{2}$ groups, how many are in each group.

What is the meaning of $6 \div \frac{1}{2}$?

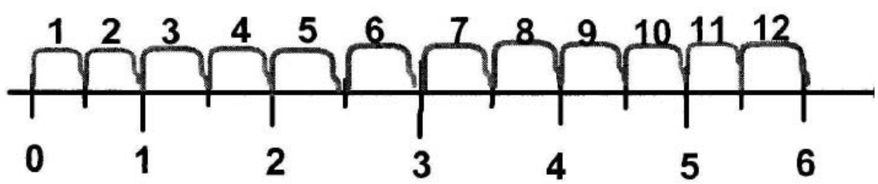
How many halves are in 6?



There are 12 halves in 6.

What is the meaning of $6 \div \frac{1}{2}$?

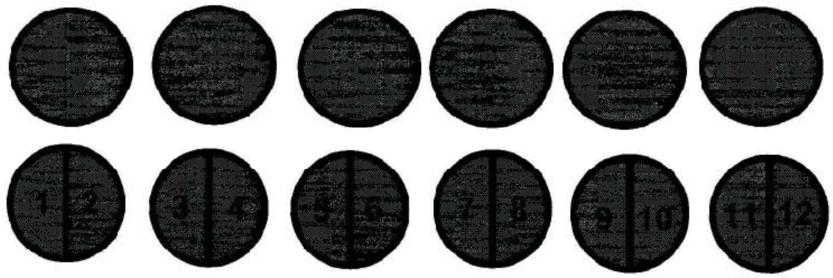
How many halves are in 6?



There are 12 halves in 6.

What is the meaning of $6 \div \frac{1}{2}$?

How many halves are in 6?



There are 12 halves in 6.

$2 \div \frac{1}{3} =$

Meaning: (1) How many $\frac{1}{3}$rds are in 2?

Model: If you have 2 things shared (2) between $\frac{1}{3}$ group, how many are in each group?

There are 6 things in each group

6 $\frac{1}{3}$rd are in 2

Answer:

$1\frac{1}{2} \div \frac{1}{3} =$

Meaning: How many groups of $\frac{1}{3}$ are in $1\frac{1}{2}$?

Model: If you have $1\frac{1}{2}$ things shared between $\frac{1}{3}$ groups, how many are in each group?

There are $4\frac{1}{2}$ one-thirds in $1\frac{1}{2}$

Remainder Goes into

Answer: There $4\frac{1}{2}$ thing in each group

Elliott's Thinking 16.mov

$\frac{1}{2} \div 2 =$

Meaning: How many groups of 2 are in $\frac{1}{2}$?

Model: If you have $\frac{1}{2}$ thing shared between 2 groups, how many are in each group?

"John is building a patio. Each section requires 2/3 of a cubic yard of concrete. The concrete truck holds 2 1/4 cubic yards of concrete. If there is not enough for a full section at the end, John can put in a divider and make a partial section. How many sections can John make with the concrete in the truck?" (Van de Walle, 2007)

Shaylee has 10 1/2 yards of ribbon to make bows for her class project. If each bow uses 3/4 yard of ribbon, how many bows can she make for her party?

1. Provide each pair of children with a fraction division question similar to $6 \div \frac{1}{2}$ to solve by drawing a diagram. Have pairs of children check each other's work.

Examples:

$$6 \div \tfrac{1}{2} = \qquad 5 \div \tfrac{1}{2} = \qquad 3 \div \tfrac{1}{3} =$$
$$6 \div \tfrac{1}{5} = \qquad 4 \div \tfrac{1}{3} = \qquad 5 \div \tfrac{1}{4} =$$

- When all pairs of children have solved the problems, write their number sentences on the board.
- Invite children to examine the number sentences and write about their observations.
- Ask children to share their observations. For example, they may notice that the whole number (dividend) and the denominator of the divisor are multiplied to get the result.
- Ask children to write a multiplication equation for each division equation.

2. Follow the procedures above but use a different set of questions.

Examples:

$$5 \div \tfrac{3}{4} = \tfrac{20}{3} \text{ or } 6\tfrac{2}{3} \qquad 3 \div \tfrac{2}{3} = \tfrac{9}{2} \qquad 4 \div \tfrac{2}{3} = \tfrac{12}{2}$$

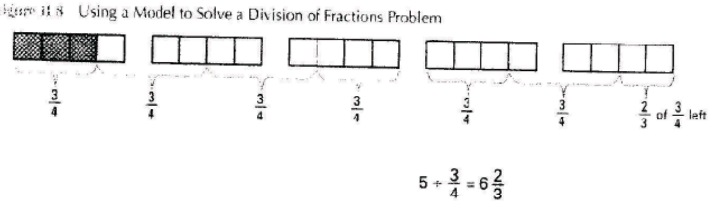
Question: How can you get a result of $6\frac{2}{3}$ from the problem $5 \div \frac{3}{4}$? See Figure 11-8. Observation: Multiply the dividend (5) and the denominator of the divisor (4), then divide this number by the numerator of the divisor (3).

The rule may be verbalized: Change the problem to a multiplication problem and invert the divisor, and then multiply the two numbers.

Number sentence : $5 \div \frac{3}{4} = 5 \times \frac{4}{3} = \frac{20}{3}$ or $6\frac{2}{3}$

Children can then test whether this procedure works whenever fractions are divided.

Figure 11-8 Using a Model to Solve a Division of Fractions Problem



Excerpt from IMAP Methods Text

Developing the Algorithms

Gregg, J. & Gregg, D. U. (2007). Measurement and fair-sharing models for dividing fractions. *Mathematics Teaching in the Middle School, 12*(9), 491 - 496.

Common Denominator Approach

Invert and Multiply Approach

Thinking of division as the inverse of multiplication

VITA

Adrienne Anne Redmond

Candidate for the Degree of

Doctor of Philosophy

Dissertation: PROSPECTIVE ELEMENTARY TEACHERS' DIVISION OF FRACTIONS UNDERSTANDING: A MIXED METHODS STUDY

Major Field: Professional Education/Mathematics Education

Education: Completed the requirements for the Doctor of Philosophy in Professional Education with an Emphasis in Mathematics Education at Oklahoma State University, Stillwater, Oklahoma in December, 2009.

Masters of Arts in Curriculum and Instruction, University of Phoenix, May, 2003.

Bachelor's of Science in Elementary Education, Oklahoma State University, Stillwater, Oklahoma, May, 2000.

Experience:

2009 Adjunct Professor, College of Education, Oklahoma State University – Stillwater Campus. Courses taught: Teaching Mathematics at the Primary Level, Teaching Mathematics with Technology

2008 Adjunct Professor, College of Education, Oklahoma University – Tulsa Campus. Undergraduate courses taught: Problem-Centered Learning

2005 - Present Research Assistant and Graduate Teaching Associate, School of Teaching and Curriculum Leadership & School Science and Mathematics Association, Oklahoma State University, Stillwater, OK. Undergraduate courses taught: August Experience, Elementary Science Methods Lab, Teaching Mathematics at the Intermediate Level, Teaching Mathematics at the Primary Level.

2000-2005 Middle School Mathematics Teacher, Riverside Middle School, Fort Worth Independent School District, Fort Worth, Texas. Courses taught: Sixth Grade Mathematics & Eighth Grade Mathematics.

Professional Memberships:

Member, National Council of Teachers of Mathematics
Member, Research Council on Mathematics Learning
Member, School Science and Mathematics Association

Name: Adrienne Redmond Date of Degree: December, 2009

Institution: Oklahoma State University Location: Stillwater, Oklahoma

Title of Study: PROSPECTIVE ELEMENTARY TEACHERS' DIVISION OF FRACTIONS UNDERSTANDING: A MIXED METHODS STUDY

Pages in Study: 165 Candidate for the Degree of Doctor of Philosophy

Major Field: Professional Education/Mathematics Education

Scope and Method of Study:

This sequential explanatory mixed methods study explored 34 prospective elementary teachers understanding of division of fractions at the beginning and end of their final mathematics methods course. Participants' beliefs on the teaching and learning of division with fractions were also explored.

Findings and Conclusions:

At the beginning of the course, the participants had a procedural and oftentimes weak to moderate understanding of division of fractions. Very few participants were able to demonstrate a good understanding of fractions division. However, at the conclusion of the course, participants significantly improved their computational understanding and overall understanding of fraction division. They did not significantly show an increase in an understanding of contextual problems at the conclusion of the course. Participants' tended to have low to neutral confidence in their solutions to division with fractions prior to taking the course, but made significant gains in confidence by the end of the semester. However, there were no significant increases in attitudes (confidence to learn, confidence to teach, anxiety to learn, anxiety to teach, and personal teaching efficacy) about teaching and learning division with fractions.

ADVISER'S APPROVAL: Dr. Juliana Utley