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Partial Regularity of Weak Solutions of Quasilinear Elliptic Systems and Weak Harnack Inequalities

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By

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Partial Regularity of Weak Solutions of Quasilinear Elliptic Systems and Weak Harnack Inequalities

A Dissertation APPROVED FOR THE DEPARTMENT OF MATHEMATICS

BY

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Abstract

Partial Regularity of Weak Solutions of Quasilinear Elliptic Systems and Weak Harnack Inequalities

by

Marina Borovikova

Chair: Rüdiger Landes,

In this thesis we study quasilinear elliptic systems of p-Laplacian type with a perturbation satisfying a natural (critical) growth condition. First, using test functions recently introduced by R. Landes we deduce Caccioppoli-type inequality for bounded weak solutions of such systems. Then we modify the classical approach of Giaquinta and Giusti to obtain higher integrability and as a consequence partial Hölder continuity of the above solution. Finally, we deduce weak Harnack inequalities for subsolutions and supersolutions for certain systems.

CHAPTER 1

Introduction

Problem 19 posted by D. Hilbert on the occasion of the 1900 International congress of Mathematicians in Paris was the following: *Are the solutions of regular problem in the Calculus of Variations always necessarily analytic?* This question has had a profound influence on many researchers and was a starting point of many great results. To show the connection of this question with our topic we consider the problem of minimizing the integral functional:

$$J(u) = \int_{\Omega} F(x, u, Du)\, dx$$

where $\Omega \subset \mathbb{R}^N$ and $F(x, \eta, \zeta)$ is a given functional on $\Omega \times \mathbb{R}^M \times \mathbb{R}^{MN}$ differentiable with respect to η and ζ. In the Calculus of variations such integral is called an energy functional. The goal is to prove that the minimizer of this functional is smooth. Problems of this type are related to elliptic systems in such way that a minimizer u is a weak solution of the associated Euler-Lagrange equation for the energy integrand. If a sufficiently smooth function u is a minimizer of $J(u)$, then Euler-Lagrange equation for this functional can be written as

$$-\operatorname{div} A(x,u,Du) + b(x,u,Du) = 0, \tag{1.0.1}$$

where A is a matrix and b is a vector defined by

$$A^i_\alpha(x,\eta,\zeta) = F_{\zeta^i_\alpha}(x,\eta,\zeta) \quad \text{and} \quad b^i(x,\eta,\zeta) = F_{\eta_i}(x,\eta,\zeta),$$

$\alpha = 1,\ldots,N$ and $i = 1,\ldots,M$.

About 50 years ago regularity theory for linear elliptic equations was mostly based on Schauder's estimates which guarantee that if leading coefficients of the equation are smooth, then solution is smooth. On the other hand, the existence theory had been developed with using more direct methods: if F is coercive, uniformly convex and satisfies the natural growth condition, then the minimization problem has a unique solution. In order to consider both the regularity and existence in the same context, the notion of solution had to be extended from regular to the Sobolev function. So the existence theory provided the existence of a solution (i.e. a minimizer) u in the Sobolev space $W^{1,p}$ and the missing step for the regularity problem to be solved was

$$u \in W^{1,p} \;\Rightarrow\; u \in C^{1,\alpha}$$

The problem for equations in the case $N = 2$ was solved by C.B.Morrey in 1938, but for $N \geq 3$ it remained open until De Georgi and J. Nash solved it independently in late 50's. In early 60's J. Moser in [Mos60] and [Mos61] developed a new method which allowed him to give a new proof of De Georgi's theorem and establish Harnack's inequality for linear elliptic equations. A remarkable fact about the Harnack inequality is that the Hölder continuity of the solution turns out to be a simple consequence of it.

The methods which De Giorgi, Moser and Nash used in their work about linear PDE's were in general nonlinear: they come from the structure assumption on the differential operator. This fact allowed for the extension to quasilinear equations such as p-Laplacian equation.

O.A. Ladyzhenskaya and N.N. Uraltseva [LU68] established the Hölder continuity of bounded weak solutions, extending De Giorgi's results, and about the same time J. Serrin [Ser64] and N. Trudinger [Tru67] obtained the Harnack inequality for bounded nonnegative solutions following Moser's idea. There are a lot of remarkable publications in this regard. We should mention here the classical books of C.B. Morrey, Jr. "Multiple integrals in the calculus of variations" [Mor66] and O.A. Ladyzhenskaya and N.N. Uraltseva "Linear and quasilinear elliptic equations" [LU68], ("the bible of elliptic equation"[Urb02])). For more history on these questions, see [Urb02].

The famous De Georgi's example in 1968 showed that regularity result can not be extended to systems even in a linear case. Modifying De Giorgi's example, Guisti and Miranda found a quasilinear elliptic system of type

$$\operatorname{div}(A(u)Du) = 0$$

with analytic coefficients which has a function $u = x/||x||$ as a bounded weak solution ($N \geq 3$) with singularity.

We can conclude that vector-valued minimizers or weak solutions of quasilinear elliptic systems are in general not regular and we can only establish a partial regularity, i.e., regularity outside a certain closed set (called the singular set). There are many open problems regarding

1) the size of the singular sets;

2) the conditions on A and B which can guarantee regularity of such solutions.

For the case $p = 2$, many authors have considered systems with the additional condition

$$a||u||_{L^\infty(\Omega)} < \theta\lambda$$

where $0 < \theta \leq 1$ and varies in different publications. Then the Hölder continuity of weak solutions has been proved by Ladyzhenskaya and Uraltseva, Hildebrand and Widman, Giaquinta and Giusti.

Concerning other results, K. Ulenbeck in 1977 obtained everywhere $C^{1,\mu}$-regularity for some type of quasilinear elliptic systems, and two years later P.A. Ivert generalized her result without the case of degeneration of ellipticity. In 1983 P. Tolksdorf derived everywhere-regularity for the bounded weak solutions of systems (1.0.1), where A is elliptic operator of the p-Laplacian type and the perturbation b satisfies the following growth condition:

$$|b(x, u, Du)| \leq c\,(1 + |Du|)^{p-1}.$$

As far as applications are concerned, it is more natural to consider case of a critical growth condition such as

$$|b(x, u, Du)| \leq c\,(1 + |Du|)^p.$$

Problem still remains open despite many attempts to find a full answer.

This thesis is devoted to studying the quasilinear elliptic systems of the p-Laplacian type with perturbations satisfying the natural (critical) growth condition. In Chapter 2 we assume that $1 < p < N$. In the first two sections of Chapter 2 we use specially constructed test functions recently introduced by R. Landes [Lan00] to prove the Caccioppoli type inequality for bounded weak solutions with the L^∞-bound depending on the maximal angle γ between the direction vector of the perturbation and direction vector of the solution. In Section 3 we discuss how the classical approach of Giaquinta and Guisti for $p = 2$, can be modified to obtain (with the help of the Caccioppoli estimate and the Inverse Hölder inequality) higher integrability property. As a result of this property in Section 4 we deduce partial Hölder continuity of bounded weak solutions and discuss the dimension of the singular set.

In Chapter 3 we prove weak Harnack inequalities for positive subsolutions and supersolutions of some p-Laplacian systems ($2 < p < N$). The proof is based on the Moser iteration method as it is presented in [Tru67] or [GT01].

1.1. Basic notations

Below we present some basic notations, inequalities and theorems which we use to state and prove our results.

$\mathbb{R}^N$ is N-dimensional Euclidean space.

Ω is a bounded domain in $\mathbb{R}^N$.

$B_\rho = B_\rho(x_o)$ stands for the ball in $\mathbb{R}^N$ with radius ρ centered at x_o.

$Q_R(x_0)$ is a cube with the center at x_0 and the sides parallel to the coordinate axes and of length $2R$.

For a Lebesgue measurable set E in $\mathbb{R}^N$ we use $|E|$ to denote its Lebesgue $N-$ measure.

For $1 \leq p < \infty$, with $L^p(\Omega)$ we denote the Banach space of bounded $p-$integrable functions on Ω with the norm

$$||u||_{L^p(\Omega)} = \Big(\int\limits_\Omega |u|^p \, dx\Big)^{\frac{1}{p}}.$$

$L^\infty(\Omega)$ stands for the Banach space of bounded functions on Ω with the norm

$$||u||_{L^\infty(\Omega)} = \operatorname*{ess\,sup}_\Omega |u|.$$

$W^{1,p}(\Omega; \mathbb{R}^M)$ is a Sobolev space, i.e., the space of vector-valued functions $u \in L^p(\Omega; \mathbb{R}^M)$ with distributional derivatives $D_\alpha u^i (\alpha = 1, \ldots, N; \imath = 1, \ldots, M)$

in $L^p(\Omega;\mathbb{R}^M)$. This is Hausdorff space with a norm

$$||u||_{p;\Omega} = (\int_\Omega (|u|^p + |Du|^p)\, dx)^{\frac{1}{p}},$$

where Du is the gradient of u, i.e., matrix $(D_\alpha u^i)_{\alpha=1,\ldots,N}^{i=1,\ldots,M}$.

$W_0^{1,p}(\Omega;\mathbb{R}^M)$ is the closure of the space C_0^∞ under the above norm.

$C^{0,\alpha}$ is the class of locally Hölder continuous functions with an exponent α.

u_R stands for the average of u over B_R, i.e.,

$$u_R = \fint_{B_R} u\, dx = \frac{1}{|B_R(x_\mathrm{o})|} \int_{B_R(x_\mathrm{o})} u\, dx.$$

1.2. Basic inequalities

Here we recall some classical inequalities which will be used for the various integral estimates in what follows.

(1) **Young's inequality** (in ϵ - form):

$$ab \leq \epsilon a^p + \epsilon^{q/p} b^q,$$

which holds for positive real numbers a, b, ϵ, p, q with p and q satisfying $1/p + 1/q = 1$.

(2) **Hölder's inequality**:

$$\int_\Omega uv\, dx \leq ||u||_{L^p(\Omega)} \cdot ||v||_{L^q(\Omega)},$$

where $u \in L^p(\Omega), v \in L^q(\Omega)$ with p and q are the same as for Young's inequality.

Next three inequalities can be found, for example, in [Eva98].

(3) **Sobolev Inequality**: Let $1 \le p < N$. If $u \in W_0^{1,p}(\Omega;\mathbb{R}^M)$, then $u \in L^{\frac{np}{N-p}}(\Omega;\mathbb{R}^M)$ and there exists a constant $C = C(N,p)$ such that

$$||u||_{L^{\frac{np}{N-p}}(\Omega)} \le C||Du||_{L^p(\Omega)}.$$

(4) **Sobolev-Poincaré inequality**: Let $1 \le p < N$. If $u \in W^{1,p}(\Omega;\mathbb{R}^M)$, then there exists a constant $C = C(N,p)$ such that

$$||u - u_\Omega||_{L^{\frac{Np}{N-p}}(\Omega)} \le C||Du||_{L^p(\Omega)}.$$

(5) **Poincaré inequality**: If $1 < p < \infty$, $u \in W^{1,p}(B_R;\mathbb{R}^M)$, then we have

$$\int\limits_{B_R} |u - u_R|^p \, dx \le C(N,p)\; R^p \int\limits_{B_R} |Du|^p \, dx.$$

(6) **Dirichlet growth Theorem**[Gia83]: Let $u \in W^{1,p}(B_R(x_0);\mathbb{R}^M)$, $1 \le p \le N$. Suppose that for all $x \in B_R(x_0)$, all $r, 0 < r \le \delta(x) = R - |x - x_0|$,

$$\int\limits_{B_r(x)} |Du|^p \, dx \;\le\; L^p \big(r/\delta\big)^{N-p+p\mu}$$

holds with $0 < \mu \le 1$. Then $u \in C^{0,\mu}\,(B_\rho(x_0);\mathbb{R}^M)$ for all $\rho < R$.

(7) **John-Nirenberg Lemma**[GT01]: Let $u \in W^{1,1}(\Omega)$ where Ω is convex, and suppose there exists a constant K such that for all balls B_R

$$\int\limits_{\Omega \cap B_R} |Du| \, dx \le K\, R^{N-1}.$$

Then there exist positive constants μ and C depending only on N such that

$$\int_{\Omega} \exp\big(\frac{\mu}{K}|u-u_{\Omega}|\big)dx \leq C(\text{diam }\Omega)^{N},$$

where $\mu = \mu_0|\Omega|(\text{diam }\Omega)^{-N}$

(8) **Reverse Hölder inequality**: Let Q be an $N-$cube. Suppose

$$\fint_{Q_R(x_0)} g^q\,dx \leq c\big(\fint_{Q_{2R}(x_0)} g\,dx\big)^q + \fint_{Q_{2R}(x_0)} f^q\,dx + \theta \fint_{Q_{2R}(x_0)} g^q\,dx$$

for each $x_0 \in Q$ and each $R < \min\{\frac{1}{2}\,\text{dist}(x_0, \partial Q), R_0\}$, where R_0, b, θ are constants with $b > 1, R_0 > 0, 0 \leq \theta < 1$. Then $g \in L^p_{loc}(Q)$ for $p \in [q, q+\epsilon)$ and

$$\big(\fint_{Q_R} g^q\,dx\big)^{\frac{1}{p}} \leq c\big(\fint_{Q_{2R}} g\,dx\big)^{\frac{1}{q}} + c\big(\fint_{Q_{2R}} f^p\,dx\big)^{\frac{1}{p}}$$

for $Q_{2R} \subset Q, R < R_0$ where c and ϵ are positive constants depending only on b, θ, q and N.

We note here that the reverse Hölder inequality was originally proved by F.W. Gehring [Geh73] in a different setting. For purpose of this work we have cited the above version of this inequality from [Gia83].

Finally, we want to present the Theorem of P. Tolksdorf which will be used in Chapter 2. But first we need some background to present this result in a context that is applicable here. Let $1 < p < N$. We consider the following quasilinear elliptic systems:

$$
-\operatorname{div} A(x,u,Du) + b(x,u,Du) = f(x). \tag{1.2.1}
$$

We can rewrite (1.2.1) in a weak form as

$$
\int\limits_{\Omega} \sum_{\alpha=1}^{N} A^i_\alpha(x,u,Du)\ D_\alpha \varphi^i\,dx + \int\limits_{\Omega} b^i(x,u,Du)\ \varphi^i\,dx = \int\limits_{\Omega} f^i\ \varphi^i\,dx \tag{1.2.2}
$$

where $i = 1,\dots,M$, $\varphi \in C_0^\infty(\Omega)$ and coefficient functions $A^i_\alpha(x,\eta,\zeta)$ are subject to the hypothesis (A) consisting of

(A, i) $\displaystyle\sum_{\alpha=1}^{N}\sum_{i=1}^{M} A^i_\alpha(x,\eta,\zeta)\ \zeta^i_\alpha \ \geq\ \lambda\ |\zeta|^p\ ;$

(A, ii) $\displaystyle\sum_{\alpha=1}^{N}\left(\sum_{i=1}^{M} A^i_\alpha(x,\eta,\zeta)\ \mu^i\right)\left(\sum_{i=1}^{M}\mu^i\ \zeta^i_\alpha\right)\ \geq\ 0\ ;$

(A, iii) $|A^i_\alpha(x,\eta,\zeta)|\ \leq\ C|\zeta|^{p-1}.$

Condition (A, i) is the usual ellipticity condition. The structure condition (A, ii) is satisfied by systems in a "strict diagonal form" such as the p-Laplacian, for instance. For further discussion on this structure conditions the reader is referred to [Lan00, Section 5]. The perturbation

$$
B(u) = b(x,u,Du) = \big(b^i(x,u,Du)\big)_{k=1}^{M}
$$

is subject to the natural (critical) growth condition

$$
(B)\quad \big|b(x,\eta,\zeta)\big|\ \leq\ a\ \big(\ |\zeta|^p\ +\ 1\ \big).
$$

Here C, a and λ are positive constants. This growth condition is called "natural" since it is satisfied by the operator of Euler-Lagrange equation for functional of type

$$J(u) = \int\limits_{\Omega} h(u)|Du|^p \, dx$$

Also it is called "critical" since the growth exponent for the gradient is the same as the integration exponent of the Sobolev space. The inhomogeneity f is always at least in $L^1(\Omega)$.

Strictly speaking the above hypotheses only need to be satisfied for the actual range of

$$\eta = u(x) \in \mathbb{R}^N, \; \zeta = Du(x) \in \mathbb{R}^M \times \mathbb{R}^N \quad \text{and} \quad \mu = \mu(u(x)) \in \mathbb{R}^M, \; x \in \Omega \subset \mathbb{R}^N.$$

Since we are considering interior regularity, we define a *weak solution* u of (1.2.1) or (1.2.2) to be a function $u \in W^{1,p}(\Omega; \mathbb{R}^M)$ with the properties

$$A^i_\alpha(x, u(x), \; Du(x)) \in L^{p-1}(\Omega) \quad \text{and} \quad b^i(x, u(x), Du(x)) \in L^1(\Omega)$$

satisfying (1.2.2) for all

$$\varphi \in W^{1,p}_{\mathrm{o}}(\Omega; \mathbb{R}^M) \cap L^\infty(\Omega; \mathbb{R}^M).$$

We now present the above mentioned theorem of Tolksdorf.

THEOREM 1.2.3. *Let* B_R *be a ball with radius* $R \in (0, 1]$ *such that* $B_{3R} \subset \Omega$. *Then there are positive constants* c *and* μ *which depend only on* N, M, p, λ *and* C *such that*

$$M^p = ess \sup |Du|^p \leq c \, R^{-N} \fint\limits_{B_{3R}} (1 + |Du|)^p \, dx$$

and

$$|Du(x) - Du(x')| \leq c\,(1 + M)R^{-\mu}|x - x'|^{\mu}$$

for all solutions $u \in W^{1,p}(\Omega)$ *of (1.2.1) and all* $x, x' \in B_R$*.*

We should mention here that this theorem was stated for a wider class of systems [Tol83, p. 244]. h

CHAPTER 2

Caccioppoli Inequality and Partial Regularity of Weak Solutions

In this chapter we generalize some results of R. Landes [Lan] on the theory of quasilinear elliptic operators of the second order. The main tool here is specially constructed test functions introduced by R. Landes [Lan00]. Our goal for the first two sections is to establish Caccioppoli type inequality for bounded weak solutions of (1.2.1) with L^∞- norm depending on the maximal angle between perturbation and solution. As a consequence, in Section 3 we deduce with the help of Inverse Hölder inequality the higher integrability result for such solutions. In Section 4 we prove partial Hölder continuity result for above solutions [BL03].

2.1. Caccioppoli type inequality

To state our main result we introduce a function

$$M(\gamma) \;=\; \frac{\lambda}{a}\begin{cases} \left(\exp(-\gamma\ \cot\gamma)\ \sin\gamma\right)^{-1}, & if\ \gamma\ <\ \frac{\pi}{2}\,, \\ 1, & if\ \gamma\ \geq\ \frac{\pi}{2}\,. \end{cases}$$

THEOREM 2.1.1. ***(Caccioppoli inequality)***

Suppose that the hypotheses(A) and (B) are valid and suppose that a weak solution u of (1.2.1) is subject to the estimate

$$\|u\|_\infty \; < \; M(\gamma),$$

where γ is the maximal angle between the direction vectors of the solution and the perturbation, i.e., $\gamma = \sup\{\sphericalangle \big(u(x), b\big(x, u, D(u)\big)\big) \mid x \in \Omega\}$.

Then for $x_0 \in \Omega$, $B_R = B_R(x_\mathrm{o})$ and for some $R_\mathrm{o}(x_\mathrm{o}) > 0$ we have the Caccioppoli- type inequality

(2.1.2)
$$\int\limits_{B_R} |Du|^p \; dx \;\le\; \mathcal{K}_1 \; R^{-p} \int\limits_{B_{2R}} |u - u_{2R}|^p \; dx + \mathcal{K}_2 \int\limits_{B_{2R}} (|f| + a)\,|u - u_{2R}|\, dx$$

where the constants $\mathcal{K}_1$ and $\mathcal{K}_2$ do not depend on R, $0 < R \le R_\mathrm{o}$.

For the proof we need test functions constructed by projection onto convex sets. If K is a convex set of class C^2, then for a given function u we define the a modified function $u^{[K]}$ by

$$u^{[K]}(x) \;=\; \begin{cases} P\big(u(x)\big), & if \;\; u(x) \;\notin\; \overline{K}\;, \\ u(x), & if \;\; u(x) \;\in\; \overline{K}\;, \end{cases}$$

where $P(u)$ is the nearest point of $\overline{K}$ to u. Even though we do not know an explicit formula for $P(u)$ the derivative of P as a mapping from $\mathbb{R}^M \to \mathbb{R}^M$ can be determined in terms of u, $P(u)$, and the principal curvatures of the boundary ∂K at $P(u)$. For sets K with the property:

(K_1) *The boundary ∂K is a smooth manifold of class C^2 such that the minimal principal curvature is positive.*

It is shown in [Lan00] that $u^{[K]}(x)$ is in $W^{1,2}(\Omega)$ and satisfies the following lemma.

LEMMA 2.1.3. *If $x \in \{y \in \Omega \ \mid \ u(y) \notin \overline{K}\}$, then we have the estimate*

$$\sum_{\alpha=1}^{N} \sum_{i=1}^{M} A_\alpha^i\big(x, u, D(u)\big)\big(Du(x) - Du^{[K]}(x)\big) \geq \lambda\tau(x) \ \big|Du(x)\big|^p \ ,$$

where

$$\tau(x) \ = \ 1 \ - \ \frac{1}{1 + \big|u(x) \ - \ P\big(u(x)\big)\big| \ \mu(x)} = \frac{\big|u(x) \ - \ P\big(u(x)\big)\big| \ \mu(x)}{1 \ + \ \big|u(x) \ - \ P\big(u(x)\big)\big| \ \mu(x)}$$

and $\mu(x)$ is the minimal principal curvature of ∂K at $P(u(x))$.

In the first step of our proof we need test functions obtained by projections onto sets K_γ, for $\gamma < \frac{\pi}{2}$, with the following property:

(K_2) *The angle between the position vector of a point v of the boundary and the outer normal at this point is less or equal to $\frac{\pi}{2} - \gamma$.*

In order to choose these sets as in the best manner possible we note that the elliptic spiral in the plane is the locus for which the position vector of the points of the curve has a constant angle with the normal direction at the points. Hence, in the x_1x_2-plane, say, we consider the curve $\mathcal{L}$, where $\mathcal{L}$ is given for nonnegative values of x_2 by two connected curves $\mathcal{L}_1$ and $\mathcal{L}_2$. The curve $\mathcal{L}_1$ is part of the logarithmic spiral

$$\mathcal{L}_1(t) \ = ||u||_\infty \ e^{-t \ \cot\gamma} \ (\cos t \ , \ \sin t), \quad for \quad 0 \ \le \ t \ \le \ \frac{\pi}{2} \ + \ \gamma$$

and $\mathcal{L}_2$ is the vertical line connecting

$$P_1 = ||u||_\infty e^{-\left(\frac{\pi}{2}+\gamma\right) \cot\gamma} \Big(\cos\big(\frac{\pi}{2}+\gamma\big), \ \ 0\Big)$$

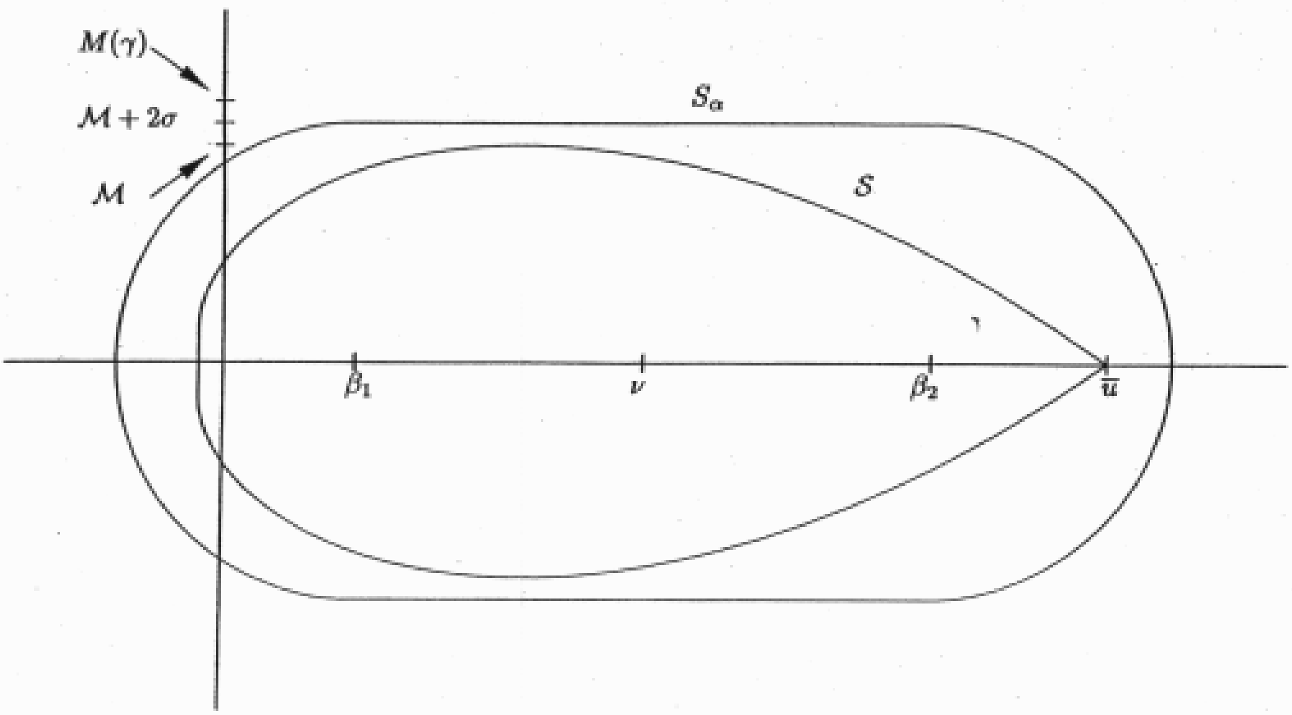
with

$$P_2 \ = ||u||_\infty e^{-\left(\frac{\pi}{2}+\gamma\right) \cot\gamma} \Big(\cos\big(\frac{\pi}{2}+\gamma\big), \sin \ \big(\frac{\pi}{2}+\gamma\big)\Big).$$

Then rotating $\mathcal{L}$ about the x_1-axis we obtain the boundary of a convex set. We rotate this set about the origin until its axis is parallel to $\overline{u}$ and denote it $\mathcal{S}$. It is elementary to see (but to verify the details is quite cumbersome) that there are sets K_γ containing $\mathcal{S}$ and satisfying K_1 and K_2, in any neighborhood of $\mathcal{S}$. In Fig. 1 on the next page the inner curve is an example for $\mathcal{L}$ with $\gamma = \frac{1}{5}\pi$. We further note that the maximal x_2-value $\mathcal{M}$ of $\mathcal{L}$ is given by

$$\mathcal{M} = ||u||_\infty e^{-\gamma\cot\gamma} \sin\gamma \ < \ M(\gamma) e^{-\gamma\cot\gamma}\sin\gamma,$$

and $\mathcal{M} \ < \ \frac{\lambda}{a}$.



$M(\gamma)$
$\mathcal{M}+2\sigma$
$\mathcal{M}$
S_α
$\mathcal{S}$
β_1
ν
β_2
$\overline{u}$

Fig. 1

2.2. Proof of the Caccioppoli inequality

In case $\gamma < \frac{\pi}{2}$ we obtain the first estimate for sets $B_{2R} \cap \Omega_\gamma$,

where $\Omega_\gamma \;=\; \left\{ x \in \Omega \;\; \middle| \;\; u(x) \notin \overline{K}_\gamma \right\}$.

We have the following proposition.

PROPOSITION 2.2.1. *For every $\delta > 0$ there are constants $\mathcal{K}_\delta$ not depending on R such that*

$$\int\limits_{\Omega_\gamma \cap B_{2R}} \left|Du\right|^p \eta^p \frac{\left|u - u^{[K_\gamma]}\right| \mu}{1 + \left|u - u^{[K_\gamma]}\right| \mu} \, dx$$
$$\leq \delta \int\limits_{B_{2R}} \eta^p |Du|^p dx \; + \; \mathcal{K}_\delta \frac{1}{R^p} \int\limits_{B_{2R}} |u - \overline{u}|^p dx \; + \frac{1}{\lambda} \int\limits_{B_{2R}} |f||u - \overline{u}| \, dx,$$

where η is a standard smooth cut-off function with support in B_{2R} , i.e.,

$0 \;\leq\; \eta \;\leq\; 1$, $\eta \,\big|_{B_R} \equiv\; 1$, *and* $|D\eta| \;\leq\; C_1/R$, *for some constant* C_1.

PROOF. Because of the assumption on the angle between the perturbation $b(x,u,Du)$ and the solution u we have $\left(B(u), u - u^{[K_\gamma]}\right) \;\geq\; 0$; further, $\left|u - u^{[K_\gamma]}\right| \leq |u - u_{2R}|$ since $u_{2R} \in \overline{K}_\gamma$. Using Lemma 2.1.3 and the Sobolev-Poincaré inequality the desired estimate follows as in [Lan] replacing 2 by p . Details follows.

□

Proposition 2.2.1 is not yet useful for our purpose since $|u - P(u)|$ is not bounded away from zero on Ω_γ. But we can choose sets K_γ such that $\mathrm{dist}\{\partial K_\gamma, \mathcal{S}\}$ becomes small enough, without μ going to zero. For instance, let

$$\Omega_{\rm o} \ = \big\{ x \in \Omega \ \mid \ \ \mathrm{dist}\{u(x), \mathcal{S}\} > \sigma \big\} \quad \text{with} \quad \sigma = \frac{1}{4}\big(\frac{\lambda}{a} - \mathcal{M}\big)$$

and choose K_γ such that $\mathrm{dist}\{\partial K_\gamma, \mathcal{S}\} < \frac{\sigma}{2}$, and there is a number $\epsilon > 0$ such that for $x \in \Omega_{\rm o}$ we have

$$\frac{\big|u - u^{[K_\gamma]}\big| \ \mu}{1 + \big|u - u^{[K_\gamma]}\big|\mu} > \epsilon.$$

Hence Proposition2.2.1 implies the following lemma.

LEMMA 2.2.2.

$$\int\limits_{\Omega_{\rm o} \cap B_{2R}} \eta^p |Du|^p \, dx \leq \delta \int\limits_{B_{2R}} |Du|^p \eta^p dx + \frac{1}{R^p}\mathcal{K}_{1,\delta} \int\limits_{B_{2R}} |u - u_{2R}|^p \, dx + \mathcal{K}_{2,\delta} \int\limits_{B_{2R}} |f| |u - \overline{u}| \, dx$$

for every $\delta > 0$. The constants $\mathcal{K}_{1,\delta}$ and $\mathcal{K}_{2,\delta}$ do not depend on $R, 0 < R < R_{\rm o}$.

In order to set up the finite induction we define sets $Z(r, \nu) \subset \mathbb{R}^M$ as cylinders of radius r with a half ball of the same radius attached to their faces. The rotation axis of the cylinder is on the line through the origin with the direction of u_{2R}. The center of the cylinder is at $\frac{\nu}{|u_{2R}|} u_{2R}$, and the centers of the half ball are at $\frac{\beta_1}{|u_{2R}|} u_{2R}$, and $\frac{\beta_2}{|u_{2R}|} u_{2R}$, respectively, where

$$\beta_1 = m(\gamma) + r \quad \text{and} \quad \beta_2 = M(\gamma) - r$$

with $M(\gamma)$ as defined above and $m(\gamma)$ some number less than

$$||u||_\infty e^{-\left(\frac{\pi}{2} + \gamma\right) \cot \gamma} \cos\big(\frac{\pi}{2} + \gamma\big).$$

In case $|\overline{u}| = 0$ the direction of the axis can be chosen arbitrarily. Fig.1 shows $\mathcal{S}$ and S_α. For $\gamma \geq \frac{\pi}{2}$, we set $m(\gamma) = -M(\gamma) = -\dfrac{\lambda}{a}$ and define $\Omega_\mathrm{o} = \phi$, then we have for all γ and $\alpha = \dfrac{m(\gamma) + M(\gamma)}{2}$ that $\big(\Omega \setminus u^{-1}(Z(\mathcal{M} + \sigma, \alpha)\big) \subset \Omega_\mathrm{o}$. Since the sets $Z(r, \nu)$ are not of class C^2 we cannot use them for the construction of the test functions directly. Instead we use convex sets S_ν of class C^2 containing $Z(\mathcal{M} + 2\sigma, \nu)$ such that the boundary of the half ball coincides with the boundary of S_ν for those points which have a distance of $(\mathcal{M} + \sigma)$ or less to the axis of $Z(\mathcal{M} + 2\sigma, \nu)$. The idea of the proof now is to construct test functions with S_α moving it step by step up and down the with step length β, say. Roughly speaking, we will estimate $|Du|^p$ on the preimages of the sets cut successively from $\mathcal{S}$ by $S_{\alpha \pm j\beta}$ as long as $\overline{u}$ is in $S_{\alpha \pm (j+1)\beta}$. Adjusting if necessary the step length in the last steps the remaining set will be so small that the usual smallness argument can be applied. We define

$$\Omega_j = \Omega_\mathrm{o} \cup \big\{ x \in \Omega \mid u(x) \notin S_{\alpha + j\beta} \big\}$$

for all γ, where β is some fixed number with $0 < \beta < \sigma$, and get the following proposition.

Proposition 2.2.3.

For every $\delta > 0$ *we have the estimate*

$$\int\limits_{B_{2R} \cap (\Omega_2 \setminus \Omega_0)} \frac{|u - u^{[S_{\alpha+2\beta}]}| \ \big(\mathcal{M} + 2\sigma\big)^{-1}}{1 \ + \ |u - u^{[S_{\alpha+2\beta}]}| \ \big(\mathcal{M} + 2\sigma\big)^{-1}} | \ Du|^p \eta^p dx$$

$$\leq \delta \int_{B_{2R}} |Du|^p \eta^p dx + \frac{1}{R^p}\mathcal{K}_{1,\delta} \int_{B_{2R}} |u - u_{2R}|^p\, dx + \mathcal{K}_{2,\delta} \int_{B_{2R}} (|f| + a)\, |u - \overline{u}|\, dx.$$

PROOF. We test the equation (E) with $\eta^p\big(u - u^{[S_{\alpha+2\beta}]}\big)$ and obtain the estimate observing that on $\Omega_2 \setminus \Omega_{\mathrm{o}}$ we have $\big|u(x) - u^{[S_{\alpha+2\beta}]}(x)\big| < 2\beta$, and hence the inequality

$$\frac{(\mathcal{M}+2\sigma)^{-1}}{1 + |u - u^{[S_{\alpha+2\beta}]}|\ (\mathcal{M}+2\sigma)^{-1}} > \frac{1}{\mathcal{M}+4\sigma} = \frac{a}{\lambda},$$

provides the result in a similar manner as in [Lan]. $\square$

Proof of the Theorem.

First we note that there is a positive constant $\epsilon > 0$ such that for all

$x \in \Omega_1 \setminus \Omega_0$ we have $\dfrac{|u - u^{[S_{\alpha+2\beta}]}|\ \big(\mathcal{M}+2\sigma\big)^{-1}}{1 \ + \ |u - u^{[S_{\alpha+2\beta}]}|\ \big(\mathcal{M}+2\sigma\big)^{-1}} \geq \epsilon,$

therefore Proposition 2.1 yields the Caccioppoli estimate for this set.

For every $\delta > 0$ we have

$$\int_{B_{2R}\cap(\Omega_1\ \cup\ \Omega_{\mathrm{o}})} \eta^p |Du|^p dx$$

$$\leq \delta \int_{B_{2R}} |Du|^p \eta^p dx + \frac{1}{R^p}\mathcal{K}_{1,\delta} \int_{B_{2R}} |u - u_{2R}|^p\, dx + \mathcal{K}_{2,\delta} \int_{B_{2R}} (|f| + a)\, |u - \overline{u}|\, dx.$$

The estimate for the whole set B_{2R} now proceeds with a finite induction. Suppose that the Caccioppoli estimate holds for $(\Omega_k \cup \Omega_0) \ \cap \ B_{2R}$. Then testing with $\left(u - u^{[S_{\alpha+(k+2)\beta}]}\right)$ yields the estimate for $(\Omega_{k+1} \cup \Omega_0) \ \cap \ B_{2R}$. Likewise we obtain the estimate for $(\Omega_{-(k+1)} \cup \Omega_0) \ \cap \ B_{2R}$ from the one for $(\Omega_{-k} \cup \Omega_0) \ \cap \ B_{2R}$. After finitely many steps, m_+ and m_- say, we have $u((\Omega_{m_+} \cup \Omega_{m_-} \cup \Omega_0) \cap B_{2R}) \subset B_\epsilon(\overline{u})$, for some arbitrarily small ϵ, adjusting the step length β in the last steps, if necessary.

Finally, we use $\eta|u - \overline{u}|$ as test function to estimate $\eta^p|Du|^p$ on $u^{-1}(B_\epsilon(\overline{u}))$ with the usual smallness argument. $\square$

2.3. Higher Integrability

In this section we get the higher integrability result without further restrictions on the structure conditions on $A(u)$ and $B(u)$. Caccioppoli inequality together with the inverse Holder inequality serves as a basic tool here.

THEOREM 2.3.1. *Let $f \in L^l(\Omega)$ with $l > p/t$, $t = p - 1 + p/N$ and $N \geq 3$. If u satisfies the hypothesis of Theorem 2.1.1, then there are positive constants ϵ and $\mathcal{K}$ not depending on R such that $u \in W^{1,q}_{loc}(\Omega)$ for all $q \in [p, p+\epsilon)$ and*

$$(2.3.2)\quad \Big(\fint_{B_{R/2}} |Du|^q dx\Big)^{\frac{1}{q}} \leq \mathcal{K}\Big\{ \Big(\fint_{B_R} |Du|^p dx\Big)^{\frac{1}{p}} + \Big(R\Big[\fint_{B_R} (|f| + a)^{\frac{q}{t}}\, dx\Big]^{\frac{t}{q}}\Big)^{\frac{1}{p-1}} \Big\}.$$

PROOF. The proof is based on application of the reverse Hölder inequality (see, for instance,[Gia83, p.122]). From the Caccioppoli inequality and Sobolev - Poincaré inequality it follows that

$$\fint_{B_{R/2}} |Du|^p \, dx \le \mathcal{K}_3 \Big(\fint_{B_R} |Du|^{\frac{Np}{N+p}} \, dx \Big)^{\frac{N+p}{N}} + \mathcal{K}_2 \fint_{B_R} (|f| + a)|u - \bar{u}| \, dx \, .$$

Let $g = |Du|^{\frac{Np}{N+p}}$, then we can rewrite the last inequality in the form

$$\fint_{B_{R/2}} g^{\frac{Np}{N+p}} \, dx \le \mathcal{K}_3 \Big(\fint_{B_R} g \, dx \Big)^{\frac{N+p}{N}} + \mathcal{K}_2 \fint_{B_R} (|f| + a)|u - \bar{u}| \, dx \, .$$

Applying at first the Hölder inequality, then the Sobolev - Poincaré inequality and Young's inequality to the last integral and setting $r = Np/(N-p)$, we get

$$\begin{aligned}
\mathcal{K}_2 \int_{B_R} (|f| + a)|u - \bar{u}| \, dx &\le \mathcal{K}_2 \Big(\int_{B_R} |u - \bar{u}|^r \, dx \Big)^{\frac{1}{r}} \Big(\int_{B_R} (|f| + a)^{\frac{r}{r-1}} \, dx \Big)^{\frac{r-1}{r}} \\
&\le \mathcal{K}_4 \Big(\int_{B_R} |Du|^p \, dx \Big)^{\frac{1}{p}} \Big(\int_{B_R} (|f| + a)^{\frac{r}{r-1}} \, dx \Big)^{\frac{r-1}{r}} \\
&\le \theta \int_{B_R} |Du|^p \, dx + \mathcal{K}_5(\theta) \Big(\int_{B_R} (|f| + a)^{\frac{r}{r-1}} \, dx \Big)^{\frac{r-1}{r} \frac{p}{p-1}} \\
&= \theta \int_{B_R} g^{\frac{N+p}{N}} \, dx + \int_{B_R} F^{\frac{N+p}{N}} \, dx,
\end{aligned}$$

where $0 < \theta < 1$, and the function F is given by

$$\begin{aligned}
F \;&=\; \Big[\mathcal{K}_5 (|f| + a)^{\frac{r}{r-1}} \Big(\int_{B_R} (|f| + a)^{\frac{r}{r-1}} \, dx \Big)^{\frac{r-1}{r} \frac{p}{p-1} - 1} \Big]^{\frac{N}{N+p}} \\
&=\; \mathcal{K}_6 (|f| + a)^{\frac{q}{st}} \Big(\int_{B_R} (|f| + a)^{\frac{p}{t}} \, dx \Big)^{\frac{N}{N+p} (\frac{t}{p-1} - 1)}
\end{aligned}$$

for s and q with $q = \dfrac{Nps}{N+p}$. Consequently,

$$\fint_{B_{R/2}} g^{\frac{N+p}{N}} \, dx \;\le\; \mathcal{K}_3 \Big(\fint_{B_R} g \, dx \Big)^{\frac{N+p}{N}} + \theta \fint_{B_R} g^{\frac{N+p}{N}} \, dx + \fint_{B_R} F^{\frac{N+p}{N}} \, dx.$$

By Proposition 1.1[Gia83, p.122], there is an $\epsilon > 0$ not depending on R such that $g \in L^s_{loc}(\Omega)$ for $s \in \left[\frac{N+p}{N}, \frac{N+p}{N} + \epsilon\right)$ and

$$\Big(\fint_{B_{R/2}} g^s \, dx \Big)^{\frac{1}{s}} \le \mathcal{K}_7 \Big\{ \Big(\fint_{B_R} g^{\frac{N+p}{N}} \, dx \Big)^{\frac{N}{N+p}} + \Big(\fint_{B_R} F^s \, dx \Big)^{\frac{1}{s}} \Big\}.$$

The latter can rewritten in the terms of Du as

$$\Big(\fint_{B_{R/2}} |Du|^q \, dx \Big)^{\frac{1}{q} \frac{Np}{N+p}} \le \mathcal{K}_7 \Big\{ \Big(\fint_{B_R} |Du|^p \, dx \Big)^{\frac{N}{N+p}} + \Big(\fint_{B_R} F^s \, dx \Big)^{\frac{1}{s}} \Big\}.$$

From this inequality it immediately follows that

$$\Big(\fint_{B_{R/2}} |Du|^q \, dx \Big)^{\frac{1}{q}} \le \mathcal{K}_8 \Big\{ \Big(\fint_{B_R} |Du|^p \, dx \Big)^{\frac{1}{p}} + \Big(\fint_{B_R} F^s \, dx \Big)^{\frac{1}{q}} \Big\}.$$

Now we estimate the last integral using the Hölder inequality:

$$\begin{aligned}
\fint_{B_R} F^s \, dx &= \mathcal{K}_6^s \fint_{B_R} (|f| + a)^{\frac{q}{t}} \, dx \, \Big(\int_{B_R} (|f| + a)^{\frac{p}{t}} \, dx \Big)^{\frac{q}{p}(\frac{t}{p-1} - 1)} \\
&\le \mathcal{K}_6^s \fint_{B_R} (|f| + a)^{\frac{q}{t}} \, dx \Big\{ \Big[\int_{B_R} (|f| + a)^{\frac{q}{t}} \, dx \Big]^{\frac{p}{q}} \Big(\frac{R^N}{R^N}\Big)^{\frac{p}{q}} \Big(\int_{B_R} dx \Big)^{1 - \frac{p}{q}} \Big\}^{\frac{q}{p}(\frac{t}{p-1} - 1)} \\
&\le \mathcal{K}_9 \Big\{ \Big[\int_{B_R} (|f| + a)^{\frac{q}{t}} \, dx \Big]^{\frac{t}{q}} R \Big\}^{\frac{q}{p-1}}
\end{aligned}$$

and the Proposition follows.

□

2.4. Partial Regularity of Bounded Weak Solutions

In order to obtain the Hölder continuity for $p = 2$ the solution locally is compared to the solution of the unperturbed system with constant coefficients for which regularity properties are known from the classical theory [Gia83, p.167]. The argument is based on the fact that for systems with constant coefficients the ellipticity implies an inequality of the type

$$|D(u-v)|^p \le \sum_{i=1}^{N} \sum_{k=1}^{M} \big(A_i^k\big(x,u,D(u)\big) - A_i^k\big(x,v,D(v)\big)\big)\big(Du(x) - Dv(x)(x)\big),$$

with $p = 2$. Such a condition is often referred to as a strict monotonicity condition. However, it is not satisfied even for the p-Laplacian, if $p < 2$. Instead of employing such a strict monotonicity condition we have an argument based on the assumption of the convexity of potential of the elliptic operator. That, of course, is satisfied by the p-Laplacian for all $p > 1$. In order not to introduce further technical details we assume in the following that $A(u)$ actually is the p-Laplacian. Our argument immediately applies to more general operators as long as the regularity result of Tolksdorf is available [Tol83]. We have the following theorem.

THEOREM 2.4.1. *Let u be a weak solution of the p-Laplace system:*

$$-\operatorname{div}(|Du|^{p-2}Du) + B = f$$

with $u, B,$ and f as in Theorem 2.3.1. If, moreover, $f \in L^s(\Omega)$ for $s > N/p$, then for every $0 < \rho < R < R_{\mathrm{o}}(x_{\mathrm{o}})$ for some $R_{\mathrm{o}}(x_{\mathrm{o}})$ we have

$$\begin{aligned} \int_{B_\rho} (1+|Du|^p)\,dx \le C\Big\{\Big[\Big(\frac{\rho}{R}\Big)^N + \chi(x_{\mathrm{o}},R)\Big]\int_{B_R}(1+|Du|^p)dx \\ + \|f\|_{L^\sigma(\Omega)} R^{N-p+p\alpha}\big[1+\chi(x_{\mathrm{o}},R)\big]\Big\} \end{aligned} \tag{2.4.2}$$

with

$$\sigma = \max\{\frac{q}{t}, s\}, \qquad \alpha = \min\{1 - \frac{1}{p-1}\big(\frac{N}{\sigma} - 1\big), 1 - \frac{N}{\sigma p}\} > 0$$

and

$$\chi(x_o, R) = (R^{p-N} \int\limits_{B_R} |Du|^p \, dx)^{\frac{q-p}{pq}}.$$

PROOF. Since (2.4.2) is obvious for $\rho \geq R/2$, we assume $\rho < R/2$ and consider a weak solution of the unperturbed system

$$\text{div } (|Dv|^{p-2} Dv) = 0, \quad \text{on } B_{R/2},$$

$$v = \text{u}, \quad \text{on } \partial B_{R/2}.$$

For the p-Laplacian we have the maximum principle that the values of the solution are in the convex hull of its boundary values, hence $\|v\|_{L^\infty(B_{R/2})} \leq \|u\|_{L^\infty(\Omega)}$ (see, [Lan]). Since a weak solution of the homogeneous problem is the unique minimum of the associated functional we have

$$\int\limits_{B_{R/2}} |Dv|^p \, dx \leq \int\limits_{B_{R/2}} |Du|^p \, dx$$

and from [Tol83] we infer

$$\int\limits_{B_\rho} |Dv|^p \, dx \leq \text{ const } (\rho/R)^N \int\limits_{B_{R/2}} (1 + |Dv|^p) \, dx \, .$$

For $w = u - v$, we have $w \in W_0^{1,p}(B_{R/2})$, and

$$\int\limits_{B_{R/2}} |Dw|^p \, dx \leq \text{ const } \int\limits_{B_{R/2}} (|Du|^p + |Dv|^p) \, dx \leq \text{ const } \int\limits_{B_{R/2}} |Du|^p \, dx \, .$$

Setting $\tilde{\alpha} = 1 - \frac{1}{p-1}(\frac{N}{\sigma} - 1)$ we want to show now that

$$
\begin{aligned}
&\int\limits_{B_{R/2}} (1 + |Du|^p)|w|\, dx \\
(2.4.3) \qquad &\leq \text{const } \chi(x_0, R)[\int\limits_{B_R} (1 + |Du|^p)\, dx + R^{N-p+p\tilde{\alpha}}\, \|f\|_{L^\sigma(\Omega)}] \,.
\end{aligned}
$$

Indeed, applying at first Hölder inequality and then Proposition 2.3.1, we get

$$
\begin{aligned}
\fint\limits_{B_{R/2}} (1 + |Du|^p)|w|\, dx &\leq \Big(\fint\limits_{B_{R/2}} |w|^{\frac{q}{q-p}}\, dx\Big)^{\frac{q-p}{q}} \big[(\fint\limits_{B_{R/2}} dx)^{\frac{p}{q}} + (\fint\limits_{B_{R/2}} |Du|^q\, dx)^{\frac{p}{q}}\big] \\
&\leq c_1\Big(\fint\limits_{B_{R/2}} |w|^{\frac{q}{q-p}}\, dx\Big)^{\frac{q-p}{q}} \big[1 + \fint\limits_{B_R} |Du|^p\, dx + \{\, R(\fint\limits_{B_R} (|f| + a)^{\frac{q}{t}}\, dx)^{\frac{t}{q}}\}^{\frac{p}{p-1}}\big].
\end{aligned}
$$

Since $\|w\|_{L^\infty(B_{R/2})} \leq\ 2\|u\|_{L^\infty(\Omega)}$, we can write

$$
|w|^{\frac{q}{q-p}} = |w|^{\frac{p}{q-p}} |w| \leq \text{const}|w|
$$

and obtain (2.4.3) with the help of Poincare's inequality:

$$
\Bigg(\fint\limits_{B_{R/2}} |w|^{\frac{q}{q-p}}\, dx\Bigg)^{\frac{q-p}{q}} \leq c_2 \Bigg(\fint\limits_{B_{R/2}} |w|^p dx\Bigg)^{\frac{q-p}{pq}} \leq c_3 \Bigg(R^p \fint\limits_{B_{R/2}} |Du|^p\, dx\Bigg)^{\frac{q-p}{pq}} .
$$

To prove (2.4.2), we use the convexity of the map $\xi \to |\xi|^p$, as in [LM] to get

$$
\begin{aligned}
\int\limits_{B_\rho} |Du|^p\, dx &\leq \frac{1}{2} \int\limits_{B_\rho} |Du|^p\, dx + c \int\limits_{B_\rho} |Dv|^p\, dx \\
&\qquad + p \sum_{\alpha=1}^{N} \sum_{i=1}^{M} \int\limits_{B_{R/2}} |Du|^{p-2}\, D_\alpha u^i D_\alpha (u - v)^i\, dx
\end{aligned}
$$

Since the trivial extension of $w = u - v$ also can be used as a test function for the original equation, that yields

$$\int\limits_{B_\rho} |Du|^p \, dx \le \frac{1}{2} \int\limits_{B_\rho} |Du|^p \, dx + c \int\limits_{B_\rho} |Dv|^p \, dx$$

$$+ ap \int\limits_{B_{R/2}} (1 + |Du|^p)|w| \, dx + \int\limits_{B_{R/2}} |f||w| \, dx \, .$$

We further note

$$\int\limits_{B_{R/2}} |f||w| \, dx \le \|f\|_{L^\sigma(\Omega)} \, R^{N(1-1/\sigma)} = \|f\|_{L^\sigma(\Omega)} \, R^{N-p+p(1-N/\sigma p)}.$$

Using the facts gathered at the beginning of the proof we conclude that

$$\int\limits_{B_\rho} |Du|^p \, dx \le \text{const}\Big\{ (\frac{\rho}{R})^n \int\limits_{B_R} (1 + |Du|^p) \, dx + \chi(x_o, R) \big[\int\limits_{B_R} (1 + |Du|^p) \, dx$$

$$+ R^{N-p+p\alpha} \|f\|_{L^\sigma(\Omega)} \big] + \|f\|_{L^\sigma(\Omega)} R^{N-p+p\alpha} \Big\} .$$

$\square$

Now from Theorem 2.4.1 we obtain the local Hölder continuity.

Before we state a corollary we need to recall the definition of a **Hausdorff measure**. Let X be a metric space and $\mathcal{F}$ be a family of subsets of X with $\emptyset$ in it. Let $h : \mathcal{F} \longrightarrow [0, \infty]$ be a function such that $h(\emptyset) = 0$. For any positive ϵ and any $E \subset X$ we define

$$\mu_\epsilon(E) = \inf \big\{ \sum_{j=0}^{\infty} h(F_j) \big| E \subseteq \bigcup_{0}^{\infty} F_j, F_j \in \mathcal{F}, h(F_j) < \epsilon \big\}.$$

Since $\mu_\epsilon > \mu_\delta$ for $0 < \epsilon < \delta$,

$$\mu(E) = \lim_{\epsilon \to 0^+} \mu_\epsilon(E) = \sup_{\epsilon > 0} \mu_\epsilon(E).$$

The set functions μ is called the **Carathéodori constraction** for $\mathcal{F}$ and h. It is easy to see that μ is an outer measure for which all Borel sets are measerable.

To define k- dimensional Hausdorff measure in $\mathbb{R}^N$, let $X = \mathbb{R}^N$ and $\mathcal{F}$ the family all open sets in $\mathbb{R}^N$.

$h(F) = h_k(F) = 2^{-k}\omega_k(\operatorname{diam} F)^k$ where ω_k is the Lebesgue measure of the unit ball in $\mathbb{R}^k$ and $k = 0, 1, \dots$. The Carathéodory constraction μ for the choice of F and h_k is called the **k-dimensional Hausdorff measure in** $\mathbb{R}^N$ which we denote here by H^k. We define the **Hausdorff dimension** of E by

$$\dim_H E = \inf\{k > 0 \big| H^k(E) = 0\}.$$

We will use the following theorem:

THEOREM 2.4.4. *Let Ω be an open set in $\mathbb{R}^N$. Let $v \in L^1_{loc}(\Omega), 0 \leq \beta < N$ and*

$$E = \Big\{a \in \Omega \Big| \limsup_{R \to 0^+} R^{-\beta} \int\limits_{B_R(a)} |v| \, dx > 0\Big\}.$$

Then $H^\beta(E) = 0$ and hence $\dim_H E \leq \beta$.

We say here that u is **partially regular** if u is Hölder continuous in an open subset $\Omega_0 \subset \Omega$ such that the Hausdorff measure $H^{N-q}(\Omega \backslash \Omega_0) = 0$ for some $q > p$.

Corollary 2.4.5. ***(Partial Regularity)***

With the assumptions of Theorem 2.4.1, there is an open set $\Omega_{\rm o} \subset \Omega$ *such that* $u \in C^{{\rm o},\alpha}(\Omega_{\rm o})$, *where* γ *is the same as in Theorem 2.4.1 and* $(N-q)$*-dimensional Hausdorff measure* $H^{N-q}(\Omega \setminus \Omega_{\rm o}) = 0$ *for some* $q > p$.

Proof. Let

$$\phi(R) = R^{p-N} \int\limits_{B_R} (1 + |Du|^p)\, dx$$

and $\rho = \tau R$ with $0 < \tau < 1$. It follows from (2.4.3) that

$$\phi(\tau R) \leq \mathcal{C}(\tau R)^{p-N} \big\{ [\tau^N + \chi(R)]\, R^{N-p} \phi(R) + \|f\|_{L^\sigma(\Omega)} \big(1 + \chi(R)\big)\, R^{N-p+p\alpha} \big\}$$

$$= \mathcal{C} \big\{ (\tau^p + \chi(R)\, \tau^{p-N})\, \phi(R) + \tau^{p-N} \|f\|_{L^\sigma(\Omega)} \big(1 + \chi(R)\big)\, R^{p\alpha} \big\}$$

Let now $\alpha < \beta < 1$ and choose τ in such a way that $2\,\mathcal{C}\tau^{p-p\beta} = 1$ (we may assume $2\,\mathcal{C} > 1$ and so $\tau < 1$.) Defining

$$\Omega_0 = \big\{ x_0 \in \Omega \mid \exists R_0 < \min\{1, \mathrm{dist}(x_0, \partial\Omega)\} \colon \sup_{R<R_0} \chi(x_o, R) < \tau^N \big\}\,,$$

we get for $x_0 \in \Omega_0$ and $R < R_0$, analogous to those in [Gia83, p170] the estimate

$$\phi(\tau R) \leq \mathcal{C} \big\{ (\tau^p + \tau^p) \phi(R) + \|f\|_{L^\alpha(\Omega)} (1 + \tau^N)\, R^{p\alpha} \big\}$$

$$\leq 2\,\mathcal{C}\, \tau^{p-p\beta} \tau^{p\beta}\, \phi(R) + 2\,\mathcal{C} \|f\|_{L^\alpha(\Omega)}\, R^{p\gamma} = \tau^{p\beta}\, \phi(R) + \mathcal{C} R^{p\alpha}.$$

By iteration we obtain

$$\phi(\tau^k R) = \phi\big(\tau(\tau^{k-1} R)\big) \leq \tau^{p\beta} \phi(\tau^{k-1} R) + \mathcal{C}(\tau^{k-1} R)^{p\alpha}$$

$$\le \tau^{kp\beta}\phi(R) + \mathcal{C}\,(\tau^{k-1}R)^{p\alpha}\,\sum_{l=0}^{k}\tau^{lp(\beta-\gamma)}$$

$$\le \left\{\phi(R) + \mathcal{C}R^{p\alpha}/(\tau^{p\alpha}-\tau^{p\beta})\right\}\tau^{kp\alpha}$$

$$\le \mathrm{const}(\tau^{k})^{p\gamma} = \mathrm{const}(\rho/R)^{p\alpha},$$

and hence for any $\rho < R$, we have $\phi(x_0,\rho) \le \mathrm{const}\,(\rho/R)^{p\alpha}$, yielding

$$\rho^{p-N}\int_{B_\rho}(1+|Du|^p)\,dx \le \mathrm{const}\,\rho^{p\alpha}$$

or

$$\int_{B_\rho}|Du|^p\,dx \le \mathrm{const}\,\rho^{N-p+p\alpha}.$$

Morrey's classical criterion which we stated above as Dirichlet Growth Theorem provides the local Hölder continuity with exponent α. $\square$

CHAPTER 3

Weak Harnack Inequalities

In this chapter we consider quasilinear elliptic systems of the p-Laplacian type $(2 < p < N)$ with the perturbation satisfying the natural growth condition. We prove weak Harnack type inequalities for weak subsolutions and supersolutions of this system.

3.1. Preliminaries and the main results

Classical Harnack inequality for nonnegative harmonic function u in an open set $\Omega \subset \mathbb{R}^N$ says that for every $x_0 \in \Omega$ and for every ball $B_r(x_o)$ with $3r < \operatorname{dist}\{x_o, \partial\Omega\}$ there exists a positive constant C independent of r such that

$$\sup_{B_r(x_o)} u \leq C \inf_{B_r(x_o)} u$$

It is natural to try to extend this result to a wider class of equations and even systems. In 1961 J. Moser proved Harnack's inequality for linear elliptic equations [Mos61] which made no use of the traditional proof of the Hölder continuity of the solution. It was a significant contribution since continuity became a consequence of this inequality as Moser showed in his paper. In 1967 N.Trudinger was able to successfully apply Moser's method to quasilinear elliptic equations, but for systems there is no such result yet. In this work we were able to prove only weak Harnack inequality for subsolutions. We also proved weak Harnack inequality for subsolutions but in a rather special case.

Here we consider systems of the p-Laplacian type, i.e.,

$$-\operatorname{div}(|Du|^{p-2}Du) + b = 0 \tag{3.1.1}$$

with perturbation b satisfying for some constant a the following growth condition

$$|b| \le a|Du|^p. \tag{3.1.2}$$

In the case of a single equation ($N = 1$) N. Trudinger proved that any bounded weak solution of such an equation satisfies the strong form of Harnack inequality [Tru67]. In the case of systems we could prove only weak Harnack inequalities under additional conditions on perturbation b.

We say that a function $u \in W^{1,p}(\Omega; \mathbb{R}^M)$ is a *subsolution* (*supersolution*) of (3.1.1) if it satisfies the following inequality

$$\int_\Omega |Du|^p \sum_{i=1}^{M} \sum_{\alpha=1}^{N} D_\alpha u^i \; D_\alpha \varphi^i \, dx + \int_\Omega \sum_{i=1}^{M} b^i \varphi^i \, dx \le 0 \quad (\ge 0)$$

for all

$$\varphi^i \in W_\mathrm{o}^{1,p}(\Omega) \cap L^\infty(\Omega), \quad \varphi^i \ge 0, \; i = 1, \ldots, M.$$

THEOREM 3.1.3. *If* $u \in W^{1,p}(\Omega, \mathbb{R}^M) \cap L^\infty(\Omega, \mathbb{R}^M)$ *is a weak nonnegative subsolution of (3.1.1) in* $B(2R) \subset \Omega$ *and if for some positive constant* γ

$$b \cdot u \le (1-\gamma)|Du|^p, \tag{3.1.4}$$

then

$$\sup_{B(R)} |u| \le c\|u\|_{q,B(2R)} \cdot R^{-N/q} \tag{3.1.5}$$

for all $q \ge p$*, where* c *is a constant that does not depend on* R*.*

THEOREM 3.1.6. *If* $u \in W^{1,p}(\Omega, \mathbb{R}^M) \cap L^\infty(\Omega, \mathbb{R}^M)$ *is a weak nonnegative supersolution of (3.1.1) in* $B(4R) \subset \Omega$ *and if for some positive constant* γ

$$b \cdot u \ge (1+\gamma)|Du|^p, \tag{3.1.7}$$

then

$$\inf_{B(R)} |u| \ge c\|u\|_{q,B(2R)} \cdot R^{-N/q} \tag{3.1.8}$$

for all $q \le pN/(N-p)$*, where* c *is a constant that does not depend on* R*.*

3.2. Proof of the main results

PROOF. We assume initially that $R = 1$, $B = B(2)$ for subsolutions and $B = B(4)$ for supersolutions. Let $\varphi^i = (u^i + k)\varphi$, k is a positive constant,

$$\varphi \in W^{1,p}(B) \cap L^\infty(B),\ \varphi \ge 0 \quad \text{and} \quad \bar{k} = (k, ..., k) \in \mathbb{R}^M.$$

When u is a subsolution of (3.1.1), we have

$$\frac{1}{2}\int\limits_B |Du|^{p-2}\sum_{\alpha=1}^{N} D_\alpha|u+\bar{k}|^2 D_\alpha\varphi\ dx \le -\int\limits_B |Du|^p\varphi\ dx + \int\limits_B b\cdot(u+\bar{k})\varphi\ dx$$

$$= -\int\limits_B |Du|^p\varphi\ dx + \int\limits_B (b\cdot u)\varphi\ dx + \int\limits_B (b\cdot \bar{k})\varphi\ dx$$

$$\le -\int\limits_B |Du|^p\varphi\ dx + (1-\gamma)\int\limits_B |Du|^p\varphi\ dx + (ak\sqrt{m})\int\limits_B |Du|^p\varphi\, dx$$

$$= -(\gamma - ak\sqrt{m})\int\limits_B |Du|^p\varphi\, dx.$$

If we choose k so small that $ak\sqrt{M} < \gamma$, we get

$$\int\limits_B |Du|^{p-2}|u+\bar{k}|\sum_{\alpha=1}^{N} D_\alpha|u+\bar{k}|^2 D_\alpha\varphi\ dx \le -\delta\int\limits_B |Du|^p\varphi\ dx \tag{3.2.1}$$

with $\delta = \gamma - ak\sqrt{M}$.

Similarly, if u is a nonnegative supersolution of (3.1.1), we deduce that

$$\int\limits_B |Du|^{p-2}|u+\bar{k}|\sum_{\alpha=1}^{N} D_\alpha|u+\bar{k}|^2 D_\alpha\varphi\ dx \ge \delta\int\limits_B |Du|^p\varphi\ dx \tag{3.2.2}$$

with the same $\bar{k}$ and δ as above.

We define for non-negative function $\eta \in C_0^1(B)$ the test function $\varphi = |u|^\beta\eta^p$, where $\beta \ge 0$ for subsolutions and $\beta \le 0$ for supersolutions. Then it follows from (3.2.1) and (3.2.2) that

$$|\beta|\int\limits_B |Du|^{p-2}|D|u+\bar{k}|^2|u+\bar{k}|^\beta\eta^p\ dx + \delta\int\limits_B |Du|^p|u+\bar{k}|^\beta\eta^p\ dx$$
$$\le p\int\limits_B |Du|^{p-2}|D|u+\bar{k}||u+\bar{k}|^{\beta+1}\eta^{p-1}|D\eta|\ dx$$

and hence

$$\delta \int\limits_B |Du|^p |u+\bar{k}|^\beta \eta^p \; dx$$

$$\begin{aligned}
&\le p \int\limits_B \; |Du|^{p-2} |D|u+\bar{k}||u+\bar{k}|^{\beta+1} \eta^{p-1} |D\eta| \; dx \\
&= \int\limits_B \; (|Du|^{p-1} |\eta|^{p-1} |u+\bar{k}|^{\beta\frac{p-1}{p}})(p|D\eta|u^{\frac{\beta+p}{p}}) \; dx \\
&\le \frac{\delta}{2} \int\limits_B |Du|^p |u+\bar{k}|^\beta \eta^p \; dx + K(\delta,p) \int\limits_B |u+\bar{k}|^{\beta+p} |D\eta|^p \; dx.
\end{aligned}$$

Then

$$\int\limits_B |Du|^p |u+\bar{k}|^\beta \eta^p \; dx \le \frac{2K(\delta,p)}{\delta} \int\limits_B |u+\bar{k}|^{\beta+p} |D\eta|^p \; dx$$

and since $|D|u+\bar{k}|| \le |D(u+\bar{k})| = |Du|$, we have

$$\int\limits_B |Du|^p |u+\bar{k}|^\beta \eta^p \; dx \le c_1 \int\limits_B |u+\bar{k}|^{\beta+p} |D\eta|^p \; dx \tag{3.2.3}$$

with $c_1 = 2K(\delta,p)/\delta$.

We'll follow Moser iteration method as in [Tru67] and [GT01]. Define

$$w \;=\; \begin{cases} |u+\bar{k}|^{\frac{\beta+p}{p}}\,, & if \;\; \beta \;\neq\; -p \;, \\ \ln|u+\bar{k}|, & if \;\; \beta \;=\; -p \;. \end{cases}$$

Letting $\sigma = \beta + p$, it follows from (3.2.3) that

$$\int\limits_B |\eta Dw|^p \; dx \;\le\; \begin{cases} c_1 |\sigma|^p \displaystyle\int\limits_B |w^p D\eta|^p \; dx\,, & if \;\; \beta \;\neq\; -p \;, \\ c_1 \displaystyle\int\limits_B |D\eta|^p \; dx, & if \;\; \beta \;=\; -p \;. \end{cases} \tag{3.2.4}$$

By Sobolev inequality we have

$$\|\eta w\|_{\frac{Np}{N-p}} \le \text{const}\|D(\eta w)\|_p$$

and hence

$$\|\eta w\|^p_{\frac{Np}{N-p}} \le c_2 \int_B (|\eta Dw|^p + |wD\eta|^p)\, dx.$$

Choose $\eta \in C_0^\infty(B(r_2))$ as a standart cut-off function for $B(r_1)$,

where $1 \le r_1 < r_2 < 2$ for subsolutions and $1 \le r_1 < r_2 < 4$ for supersolutions,

$\eta \equiv 1$ on $B(r_1), \eta \equiv 0$ outside $B(r_2)$ and $|D\eta| \le \text{const}/(r_2 - r_1)$. Then

$$\Big(\int_{B(r_1)} w^{\frac{\sqrt{M}}{N-p}}\, dx\Big)^{\frac{N-p}{N}} \le \|\eta w\|^p_{\frac{Np}{N-p},\ B(r_2)} \le c_3|\sigma|^p \int_{B(r_2)} w^p|D\eta|^p\, dx$$

$$\le \Big(\frac{c_4|\sigma|}{r_2 - r_1}\Big)^p \|w\|^p_{p,\ B(r_2)}.$$

Set $\chi = \frac{N}{N-p}$. Then

(3.2.5) $$\|w\|_{\chi p,\ B(r_1)} \le \frac{c_4|\sigma|}{r_2 - r_1}\|w\|_{p,\ B(r_2)}.$$

For $r < 2$ for subsolutions or $r < 4$ for supersolutions we introduce the quantity

(3.2.6) $$\Phi(q, r) = \Big(\int_{B(r)} |u + \bar{k}|^q\, dx\Big)^{1/q}.$$

If $\sigma > 0$, then from (3.2.5) we deduce that

$$\Big(\int_{B(r_1)} |u + \bar{k}|^{\chi\sigma}\, dx\Big)^{\frac{1}{\chi p}} \le \frac{c_4\ \sigma}{r_2 - r_1}\Big(\int_{B(r_2)} |u + \bar{k}|^{\sigma}\, dx\Big)^{\frac{1}{p}}.$$

Then

$$\Big(\int_{B(r_1)} |u + \bar{k}|^{\chi\sigma}\, dx\Big)^{\frac{1}{\chi\sigma}} \le \Big(\frac{c_4\ \sigma}{r_2 - r_1}\Big)^{\frac{p}{\sigma}}\Big(\int_{B(r_2)} |u + \bar{k}|^{\sigma}\, dx\Big)^{\frac{1}{\sigma}}$$

and hence

(3.2.7) $$\Phi(\chi\sigma, r_1) \le \Big(\frac{c_4|\sigma|}{r_2 - r_1}\Big)^{\frac{p}{\sigma}}\ \Phi(\sigma, r_2).$$

For $\sigma < 0$ we have

$$\Big(\int\limits_{B(r_1)} |u+\bar{k}|^{\chi\sigma}\, dx\Big)^{\frac{1}{\chi\sigma}} \geq \Big(\frac{c_4|\sigma|}{r_2-r_1}\Big)^{\frac{p}{\sigma}} \Big(\int\limits_{B(r_2)} |u+\bar{k}|^{\sigma}\, dx\Big)^{\frac{1}{\sigma}}$$

and hence

$$\Phi(\sigma, r_2) \leq \big(\frac{c_4|\sigma|}{r_2-r_1}\big)^{\frac{p}{|\sigma|}}\ \Phi(\chi\sigma, r_1). \tag{3.2.8}$$

We are going to iterate inequalities (3.2.5) and (3.2.6). When u is a subsolution, then $\beta \geq 0$ and hence $\sigma \geq p$. Hence by taking any q, $q \geq p$, we set $\sigma_i = \chi^i p$ and $r_i = 1 + 2^{-i}$. Then from (3.2.5) we deduce that

$$\begin{aligned}\Phi(\chi^{i+1}, r_{i+1}) &\leq \big(2^{i+1}c_4\chi^i q\big)^{\frac{p}{\chi^i q}}\ \Phi(\chi^i q, r_i) = \big(c_5(2\chi)^i\big)^{\frac{p}{\chi^i q}}\ \Phi(\chi^i q, r_i)\\ &\leq c_5^{\frac{p}{q}\chi^{-i}} (2\chi)^{\frac{p}{q}\ i\chi^{-i}}\ c_5^{\frac{p}{q}\chi^{-(i-1)}} (2\chi)^{\frac{p}{q}(i-1)\chi^{-(i-1)}} \Phi(\chi^{i-1}, r_{i-1})\\ &\leq c_5^{\frac{p}{q}\sum\limits_{j=1}^{i}\chi^{-j}} (2\chi)^{\frac{p}{q}\sum\limits_{j=1}^{i} j\chi^{-j}}\ \Phi(q,2).\end{aligned}$$

Since $\chi = \frac{N}{N-p}$,

$$\sum_{j=1}^{\infty}\chi^{-j} \qquad \text{and} \qquad \sum_{j=1}^{\infty} j\chi^{-j}$$

are bounded and hence

$$\sup_{B(1)} |u+\bar{k}| = \Phi(\infty, 1) \leq \text{const}\|u+\bar{k}\|_{q,B(2)}. \tag{3.2.9}$$

Since the last estimate is valid for any $k < \frac{\gamma}{a\sqrt{M}}$, then letting $k \to 0$ for $q \geq p$ we obtain the following inequality

$$\sup_{B(1)} |u| \leq \text{const}\|u\|_{q,B(2)}. \tag{3.2.10}$$

Now, using transformation $x \longmapsto \frac{x}{R}$, we can show that (3.1.5) follows from (3.2.10).

Indeed, for any R such that $B(2R) \subset \Omega$ and $\varphi^i \in W_0^p(B(2R))$, we may rewrite (3.1.1) in a weak form

$$(3.2.11) \qquad \int\limits_{B(2R)} |Du|^{p-2} \sum_{\alpha=1}^{N} D_\alpha u^i D_\alpha \varphi \, dx = \int\limits_{B(2R)} b^i \varphi^i \, dx \qquad (i = 1, \ldots, m).$$

Let $x = Ry$ and $v(y) = u(Ry)$. Since $D_{x_\alpha} v^i(y) = \frac{1}{R} D_{y_\alpha} v^i(y)$, we have

$$\int\limits_{B(2R)} |Du|^{p-2} \sum_{\alpha=1}^{N} D_\alpha u^i D_\alpha \varphi \, dx = \int\limits_{B(2)} \frac{1}{R^p} |D_y v|^{p-2} \sum_{\alpha=1}^{N} D_{y_\alpha} v^i \; D_{y_\alpha} \varphi(Ry) \, cR^N \, dy.$$

On the other hand,

$$\int\limits_{B(2R)} b^i \varphi^i \, dx = \int\limits_{B(2)} b^i(Ry) \varphi^i(Ry) cR^N \, dy.$$

Hence, letting $\widetilde{\varphi}^i(y) = \varphi^i(Ry)$, we obtain that (3.2.11) is equivalent to

$$(3.2.12) \qquad \int\limits_{B(2)} |Dv|^{p-2} \sum_{\alpha=1}^{N} D_\alpha v^i \; D_\alpha \widetilde{\varphi} \, dy = \int\limits_{B(2)} \widetilde{b^i} \widetilde{\varphi^i} \, dy,$$

where $\widetilde{b}(y) = R^p b(Ry)$. Hence u is a subsolution of (3.2.11) if and only if v is subsolution of (3.2.12). Moreover

$$\begin{aligned} \widetilde{b} \cdot v = R^p b(Ry) \cdot u(Ry) &\le R^p (1-\gamma) |D_x u(Ry)|^p \\ &= R^p (1-\gamma) 1/R^p |D_y v|^p = (1-\gamma)|D_y v|^p \end{aligned}$$

and

$$|\widetilde{b}| = R^p |b(Ry)| \le a R^p \; |D_x u(Ry)|^p = a |D_y v|^p.$$

By (3.2.10) we have

$$\sup_{B(1)} |v(Ry)| \le c \|v(Ry)\|_{q, B(2)} \qquad \text{for} \qquad q \ge p.$$

Since

$$\sup_{B(1)} |v(Ry)| = \sup_{B(R)} |u(x)|$$

and

$$\|v(Ry)\|_{q,B(2)}^p = \int_{B(2)} |u(Ry)|^q \, dy = \int_{B(2R)} |u(x)|^q \, (c_7/R^N) \, dx = c_7 R^{-N} \|u\|_{q,B(2R)}^q,$$

we obtain (3.1.5).

For the case when u is a supersolution, that is when $\beta \leq 0$ and $\sigma \leq p$ we will next show that for any q and q_0 with $0 < q_0 < q \leq p\chi = \frac{Np}{N-p}$ the following inequalities hold.

$$\Phi(q, 2) \leq c\Phi(q_0, 3) \tag{3.2.13}$$

and

$$\Phi(-q_0, 3) \leq c\Phi(-\infty, 1), \tag{3.2.14}$$

where $c = c(N, p, q, q_0, \|u\|_\infty)$. To prove (3.2.13), we take $\sigma = q_0$. Then there exists $l \in \mathbb{Z}_+$ such that $\chi^{l-1}q_0 < q \leq \chi^l q_0$. Letting $r_i = 2 + 1/2^i$ and using (3.2.5), we have

$$\begin{aligned}\Phi(q, 2) &= \Phi(\chi^l q_0 (\frac{q}{\chi^l q_0}), 2) \leq c_8 \big(\Phi(\chi^l q_0, 2)\big)^{\chi^l q_0} \leq c_9 \Phi(\chi^l q_0, 2) \leq c_9 \Phi(\chi^l q_0, r_l) \\ &\leq c_9 (c_5 \chi)^{\frac{p}{q_0} \sum_{i=1}^{l-1} i\chi^{-i}} \Phi(q_0, 3) \leq c\Phi(q_0, 3),\end{aligned}$$

where c depends on N, p, q, q_0 and $\|u\|_\infty$.

To prove (3.2.14), we take $\sigma = \chi^i q_0, r_0 = 3$ and $r_i = 1+1/2^{i-1},\ i = 1, 2 \ldots .$

Then it follows from (3.2.6) that

$$\begin{aligned}
\Phi(-q_0,3) &\le \Big(\frac{c_4 q_0}{3-2}\Big)^{\frac{p}{q_0}} \le (c_4q_0)^{\frac{p}{q_0}}(2c_4\chi q_0)^{\frac{p}{\chi q_0}}\Phi(-\chi^2 q_0, 3/2) \le \ldots \\
&\le (c_4q_0)^{\frac{p}{q_0}}(2c_4\chi q_0)^{\frac{p}{\chi q_0}} \ldots (2^i c_4 \chi^i q_0^{\frac{p}{\chi^i q_0}}\Phi(-\chi^{i+1}q_0, r_{i+1}) \\
&\le (c_4q_0)^{\frac{p}{q_0}}(2c_4\chi q_0)^{\frac{p}{\chi q_0}} \ldots (2^i c_4 \chi^i q_0)^{\frac{p}{\chi^i q_0}}\Phi(-\chi^{i+1}q_0, r_{i+1}) \\
&\le (c_4q_0)^{\frac{p}{q_0}\sum_{j=1}^{i}\chi^{-j}}(2\chi)^{\frac{p}{q_0}\sum_{j=1}^{i} j\chi^{-j}}\Phi(-\chi^{i+1}q_0, r_{i+1}).
\end{aligned}$$

By letting i tend to ∞ in the above estimate and since

$$\Phi(-\infty, r) = \inf_{B(r)}|u+\bar{k}|, \quad \sum_{j=1}^{\infty}\chi^{-j} \quad \text{and} \quad \sum_{j=1}^{\infty} j\chi^{-j}$$

are bounded, we conclude that (3.2.14) is valid.

We will finish the proof of Theorem 3.1.6 if we can show that

$$\Phi(q_0, 3) \le c\ \Phi(-q_0, 3)$$

for some $q_0 > 0$. But this follows from the John-Nirenberg Lemma.

Indeed, to apply this Lemma it is enough to show that for any ball $B(2R) \in B$ the following estimate is valid

$$\int_{B(R)} |Dw|\ dx \le \text{const}\ R^{N-1}.$$

For $\sigma = 0$ in (3.2.4) we have the estimate

$$\int_B |\eta Dw|^p\ dx \le c_1 \int_B |D\eta|^p\ dx \tag{3.2.15}$$

with $B = B(4)$ and $w = \ln|u+\bar{k}|$.

Choose the cut-off function η so that $\eta \equiv 1$ in $B(R)$, $\eta \equiv 0$ outside $B(2R)$ and $|D\eta| \le$

c/R. Then from (3.2.15) with the help of Hölder inequality we obtain

$$\int\limits_{B(R)} |Dw|\ dx \leq \int\limits_{B(2R)} \eta|Dw|\ dx \leq c_2\Big(\int\limits_{B(2R)} |\eta Dw|^p\ dx\Big)^{\frac{1}{p}}\ R^{N(1-\frac{1}{p})}$$

$$\leq c_1 c_2\Big(\int\limits_{B(2r)} |\ D\eta|^p\ dx\Big)^{\frac{1}{p}}\ R^{N-\frac{N}{p}} \leq c_3 R^{\frac{N-p}{p}}\ R^{N-\frac{N}{p}} = c_3 R^{N-1}$$

.

By John-Nirenberg Lemma, there is a positive constant $q_0 = \mu/c_3$ such that

$$\int\limits_{B(3)} e^{q_0|w-w_0|}\, dx \leq C$$

where

$$w_0 = \fint\limits_{B_3} w\ dx.$$

Hence we have the following inequalities:

$$\int\limits_{B(3)} e^{q_0(w-w_0)}\, dx \leq C$$

and

$$\int\limits_{B(3)} e^{q_0(w_0-w)}\, dx \leq C.$$

Multiplying two last inequalities, we get

$$\int\limits_{B(3)} e^{q_0 w}\, dx \int\limits_{B(3)} e^{-q_0 w}\, dx \leq C_1$$

and since $e^{q_0 w} = |u+\bar{k}|^{q_0}$ and $e^{-q_0 w} = |u+\bar{k}|^{-q_0}$,

$$\Big(\int\limits_{B_3} |u+\bar{k}|^{q_0}\ dx\Big)^{\frac{1}{q_0}} \leq C_2\Big(\int\limits_{B_3} |u+\bar{k}|^{-q_0}\ dx\Big)^{\frac{1}{-q_0}}$$

or equivalently

$$\Phi(q_0, 3) \leq c\ \Phi(-q_0, 3)$$

as required. $\square$

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