CHARACTERIZATION OF PETROLEUM REFINERY LP RESULTS UNDER CONDITIONS OF DEGENERACY

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UNDER CONDITIONS OF DEGENERACY

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Nomenclature

A	$m \times n$ matrix
B	Optimal basis matrix
N	Number of constraints passing through the optimum
$\overline{a}_{ij}$	Incremental effect coefficients
b	Right hand side (R.H.S) of a constraint
c	Cost coefficient
c_B	Cost coefficient of basic variables
d	Directional vector
m	Number of equations
n	Number of decision variables
x	Primal variable of a LP problem
y	Dual variable of a LP problem
z_{max}	Optimal objective value

Subscripts

i	ith constraint in the LP problem and i varies from 1 to m
j	jth variable in the LP problem and j varies form 1 to n
q	Number of alternate optimal basis obtained by perturbing a single active constraint of a degenerate LP problem
r	Number of distinct alternate optimal basis obtained for a degenerate LP problem

Greek letter

δ	Small changes in R.H.S of a constraint or cost coefficient of a decision variable
λ	Parameter used in parametric programming

Acronyms

LINDO	Linear Interactive Discrete Optimization
PIMS	Process Industry Modeling System

Abbreviations

bbl	Barrel
GE	Greater than or equal to
LE	Less than or equal to
LL	Lower limit: implying a GE constraint is active
LP	Linear Programming
R.H.S	Right hand side
UL	Upper limit: implying an LE constraint is active

Refinery LP Variables

CCFO	Fuel oil flow rate from Catalytic Cracking Unit (CCU)
CCFODF	Catalytic cracking unit fuel oil flow rate for diesel fuel (DF) blending
CCFOFO	Catalytic cracking unit flow rate for fuel oil (FO) blending
CCG	Gasoline flow rate from CCU
CCGPG	Catalytic cracking unit gasoline flow rate for Premium Gasoline (PG) blending
CCGRG	Catalytic cracking unit gasoline flow rate for Regular Gasoline (RG) blending

CRUDE	Crude oil flow rate to the atmospheric crude distillation column (AD)
DF	No. 2 diesel fuel flow rate
FGAD	Fuel gas flow rate from AD
FGCC	Fuel gas oil flow rate from the CCU
FGRF	Fuel gas flow rate from the reformer (RF)
FO	No. 6 fuel oil flow rate
PG	Premium gasoline flow rate
RFG	Reformer gasoline flow rate
RFGPG	Reformer gasoline flow rate for PG blending
RFGRG	Reformer gasoline flow rate for RG blending
RG	Regular gasoline flow rate
SRDS	Straight-run distillate flow rate from AD
SRDSCC	Straight-run distillate flow rate to the catalytic cracking unit (CCU)
SRDSDF	Straight-run distillate flow rate for DF blending
SRDSFO	Straight-run distillate flow rate for FO blending
SRFO	Straight-run fuel oil flow rate from AD
SRFOCC	Straight-run fuel oil flow rate to the CCU
SRFODF	Straight-run fuel oil flow rate for DF blending
SRFOFO	Straight-run fuel oil flow rate for FO blending

SRG	Straight-run gasoline flow rate from AD
SRGPG	Straight-run gasoline flow rate for premium gasoline (PG) blending
SRGRG	Straight-run gasoline flow rate for regular gasoline (RG) blending
SRN	Straight-run naphtha flow rate from AD
SRNDF	Straight-run naphtha flow rate for diesel fuel (DF) blending
SRNPG	Straight-run naphtha flow rate for PG blending
SRNRF	Straight-run naphtha feed rate to the reformer (RF)
SRNRG	Straight-run naphtha flow rate for RG blending

CHAPTER 1

INTRODUCTION

Petroleum refineries are complex, large-scale manufacturing processes. The value of crude oil processed by a 200, 000 barrel per day facility exceeds ten million dollars a day, or four billion dollars a year. However, petroleum refining is a mature industry employing mature technology. Consequently, profit margins are low and economic optimization is essential to stay in business.

Linear Programs or LPs are key elements in the optimum planning and operations of a petroleum refinery. Refinery planners utilize custom LP software (e.g. AspenTech's PIMS program) to select among the many types of crude oil available for purchase. LPs are also used by the planners to identify optimum operating conditions for various refinery units. Examples include distillation cut points in the front-end atmospheric unit, reaction temperature in the Fluid Catalytic Cracking (FCC) unit and reformer (RF) feed rate to the alkylation unit, etc.

LPs are linear mathematical models of processes that are inherently non-linear. LP modelers are charged with creating and maintaining linear models that approximate refinery operation over the expected range of operation. The inputs to a refinery LP include crude oil availabilities and prices, product demand and prices, manufacturing cost information and constraints imposed by equipment, markets, regulations, and utilities. The output of a refinery LP includes the optimum daily profit, along with the associated refinery operating conditions and flow rates (activities in LP terms). The refinery manager uses this information to run the refinery on a day-to-day basis.

The LP also generates additional useful information in terms of incremental or marginal values of feeds, products, and the many intermediate streams produced in the refinery. This information provides insight regarding the economic impact of producing or consuming additional barrels relative to base operating plan defined by LP activities. In addition to predicting the marginal value (shadow price) of a feed, product or stream, the LP also provides incremental effect coefficients that predict the physical impact (changes in flow rates, temperatures, product properties, etc.) throughout the refinery if a decision is made to deviate slightly from the optimum conditions (activities) associated with the base operating plan.

Accurate interpretation of LP results is essential to perform optimization in a petroleum refinery. Misinterpretation of LP results can have significant impact on decision making and can lead to unexpected financial consequences.

A typical refinery LP model used for optimization has approximately 300-500 equations and 800-1,500 variables to optimize (Parkash, 2003). Interpretation of solutions for a refinery LP has to be made with prudence, because almost all practical size LP problems could be degenerate (Koltai and Terlaky, 2000; Kruse, 1993; Zornig, 1993).

The term degeneracy is frequently used to denote primal degenerate problems in the literature. In addition to primal degeneracy, an LP could also be dual degenerate (alternative optima). In order to be precise, the terms "primal degeneracy" and "dual degeneracy" will be used to represent different conditions of degeneracy. The state of primal degeneracy in LP often produces multiple optimal dual values with unique primal values (activity values) and unique objective function value. On the other hand, the state of dual degeneracy in LP produces multiple optimal primal values (activity values) with unique dual values and unique objective function value. Due to this phenomenon, the condition of degeneracy creates complications in choosing a specific solution for implementation.

The consequences of primal degeneracy are extensively discussed in the technical literature (Strum, 1969; Eilon and Flavell, 1974; Aucamp and Steinberg, 1982; Akgul, 1984; Knolmayer, 1984; Gal, 1993; Jansen et al., 1997; Koltai and Terlaky, 2000). However, many practitioners in the field of petroleum refinery optimization are not fully aware of the consequences of primal degeneracy.

Unlike primal degeneracy, the concept of dual degeneracy is rarely addressed in the literature. The main reason for this neglect is due to the fact that any set of multiple activity values obtained from alternate optimal solutions can be physically implemented to achieve the same objective function value. This flexibility of choosing a desired solution among multiple solutions is viewed favorably by several users (Paris, 1991).

Although implementing any of the multiple activities obtained for a dual degenerate LP produces optimal profit in the base case, optimal profit may not be sustained even for an infinitesimal change in the selling price or buying price of activities. Market price fluctuation is a common phenomenon in petroleum refinery operations. Therefore, a definitive approach must be developed to identify activity values that leads to optimal profit, despite market price fluctuations. The existence and impact of dual degeneracy are poorly understood by most refining planners.

The work reported in this document has two broad goals. The first is to provide a comprehensive methodology that allows a refinery planner to detect both types of degeneracy and correctly interpret the results from a single LP run. Most refinery planners are engineers with limited or no formal education in the field of LP theory or operations research. LP training is provided in-house by company experts or LP software training seminars. The methodology developed in this research is designed for users with this more practical rather than theoretical background. The second goal is to provide a more comprehensive treatment of dual degeneracy with an emphasis on interpretation in a petroleum refining context.

Degeneracy fundamentally implies the existence of multiple solutions in one fashion or another to the refinery LP problem. In contrast with the lack of a single, unique-in-all-respects solution, most planner decisions or recommendations are based on the output from a single solution.

Both of the goals described previously were motivated in part by this use of one-of-many-possible solutions. Successful application of the material presented in this document provides the basis to make better-informed business decisions while continuing to use LPs for petroleum refinery optimization.

1.1 Goal and Objectives

The goal of this research is to develop an approach to gauge the robustness of implementing a single LP optimal solution under conditions of dual degeneracy. The specific objectives of this research are to:

- Clarify and document the correct methods to detect degeneracy in a refinery LP.
- Provide techniques to distinguish between the unique and non-unique elements of a refinery LP.
- Provide an understanding of physically unrealizable results and provide the means to detect them.
- Explicitly identify the limitations associated with the output from a single LP run.
- Develop an approach to determine activity values for a dual degenerate LP that sustains optimality criteria, based on speculated market price fluctuations.
- Explain the economic implications of dual feasibility conditions for an LP solution in the context of petroleum refinery optimization.
- Develop an algorithm to determine alternate optimal solutions for a dual degenerate LP.
- Extend the primal incremental analysis approach developed by Aucamp and Steinberg (1982) to determine true incremental effect coefficients, in addition to determining true shadow prices.

In this research an innovative approach is developed to characterize LP solutions when the problem is dual degenerate. The categorization approach enables the user to implement specific solutions that maintain optimality criteria, under conditions of anticipated market price uncertainties. The relation between dual feasibility conditions and optimality of LP solutions is used to develop this approach. Furthermore, a novel perturbation technique implementing parametric programming is developed to determine alternate optimal solutions under conditions of dual degeneracy. The procedure and results will be illustrated for a simplified refinery LP model with 33 decision variables and 37 constraints.

1.2 Organization of Dissertation

Including the introduction chapter, this dissertation has six chapters. A chapter by chapter description of the dissertation follows.

Chapter 2 First, the basic notations and definitions involved in a LP problem are provided. After that, a simplified refinery LP is presented to explain in detail how a refinery LP is solved and its solutions are interpreted.

Chapter 3 Presents the geometric and algebraic solution of three different 2-D LPs: 1) non-degenerate, 2) primal degenerate, and 3) dual degenerate LP problem. This chapter discusses the consequences of degeneracy and provides strategies to identify different conditions of degeneracy from an optimal solution.

Chapter 4 Deals with the interpretation of LP results, when the LP is primal degenerate. The results and procedures are presented for a refinery LP.

Chapter 5 Provides a brief overview on the previous work related to dual degeneracy, and presents the novel approach and algorithm developed to treat dual degenerate problems. The results and procedures are illustrated for a refinery LP.

Chapter 6 Summarizes the contributions and future directions of this research.

CHAPTER 2

REFINERY LP FORMULATION AND SOLUTION

In this chapter, the basic definitions involved in describing an LP problem and its solution are provided. Then a simplified refinery LP obtained from literature (Pike, 1986) is presented to explain how a refinery LP is solved and its solutions are interpreted.

2.1 Notations and Definitions of the LP Problem

The information given in Gal (1986) is used as a guideline for writing this section. A petroleum refinery LP problem is presented in the form given by Equation (2.1)

$$\text{Maximize } z = c^T x \tag{2.1}$$

$$x \in X$$

with $X = \{x \in \Re^n | Ax \leq b, x \geq 0\}$ where $c = (c_1, \cdots, c_j, \cdots, c_n)^T \in \Re^n$, $b = (b_1, \cdots, b_j, \cdots, b_m)^T \in \Re^m$, $x = (x_1, \cdots, x_j, \cdots, x_n)^T \in \Re^n$, A an $(m \times n)$, matrix, $A = (a^1, \cdots, a^j, \cdots, a^n)$, $a^j = (a_{1j}, \cdots, a_{ij}, \cdots, a_{mj})^T$, $j = 1, \cdots, n$. The LP in general form given in Equation (2.1) is converted to standard form by introducing slacks and surplus $x_{n+i}, i = 1, \cdots, m$. The definition of variables $x = (x_1, \cdots, x_j, \cdots, x_n)^T$ and $x_{n+i}, i = 1, \cdots, m$ are given as follows:

Decision variables The set $\{x_1 \cdots x_n\}$ are called the decision variables. These variables represent barrels of crude, barrels of naphtha, barrels of gasoline, etc., in a refinery LP. Decision variables are not only limited to feed and production rates but also include physical properties (e.g. Reid vapor pressure) and operating conditions (e.g. reactor temperatures). The decision variables are adjustable “knobs” of the refining business. The refinery manager wants to know the best choice of values for these "knobs" or decision variables.

Slacks and Surplus The set $\{x_{n+1} \cdots x_{n+m}\}$ includes the slack and surplus variables. These variables are used to convert the LP problem to standard form. Slack variables are added to the less than or equal to (LE) constraints, and surplus variables are added to greater than or equal to (GE) constraints.

Now the problem is solved by computing an optimal basis with the characteristic basis-index $\rho = j_1, \cdots, j_m$ such that $x_{j1}, \cdots, x_{jm}$ are basic variables and after some rearrangement $j_1 = 1, \cdots, j_m = m$, the optimal solution for the LP in an expanded tableau or simplex tableau is given by Table 2.1 (Gal, 1986). The description of each entry in the tableau follows:

Table 2.1: Optimal Simplex Tableau (Gal, 1986)

Z	0	$\cdots$	0	$\overline{c}_{m+1}$	$\cdots$	$\overline{c}_n$	$\overline{y}_1$	$\cdots$	$\overline{y}_m$	z_{max}
1	1	$\cdots$	0	$\overline{a}_{1,m+1}$	$\cdots$	$\overline{a}_{1,n}$	$\overline{a}_{1,n+1}$	$\cdots$	$\overline{a}_{1,n+m}$	$\overline{b}_1$
$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$
m	0	$\cdots$	1	$\overline{a}_{m,m+1}$	$\cdots$	$\overline{a}_{m,n}$	$\overline{a}_{m,n+1}$	$\cdots$	$\overline{a}_{m,n+m}$	$\overline{b}_m$
	x_1	$\cdots$	x_m	x_{m+1}	$\cdots$	x_n	x_{n+1}	$\cdots$	x_{n+m}	x_B

Activity values Activity values are the values of decision variables, slack and surplus, in the optimal solution. The vector of activity values $x_B = \left[\overline{b}_1 \cdots \overline{b}_m \right]^T$ are called the primal solution or primal values. In Table 2.1, the column vector $\left[\overline{b}_1 \cdots \overline{b}_m \right]^T$ in the far-right column represents the activity values.

Dual Solution The set of values in the top row, labeled by Z in Table 2.1, represent the dual solution. Individual entries in the set are called dual values. This row is also referred to as the Z-row. All of the entries in the dual solution represent reduced costs or shadow prices.

Reduced cost The entries in the top row (Z-row) labeled by $\bar{c}_j$ for $j = 1, \cdots n$, are called opportunity costs, reduced costs or D-J or Delta-J value in PIMS (Process Industry Modeling System) convention (Parkash, 2003) . Reduced cost $\bar{c}_j$ is defined as the amount by which c_j must increase in order to enter x_j into the basis. In other words, in order to make the production or consumption of a resource x_j profitable, its cost coefficient c_j must be adjusted by an amount equal to $\bar{c}_j$.

Shadow price The entries in the $Z-$row labeled $\overline{y}_i$ for $j = n+1, \cdots, n+m$ are called shadow prices, marginal values or pi values in PIMS convention (Parkash, 2003), and are defined as the price for selling or buying one additional unit of the i-th resource. (i.e, $\overline{y}_i$ is the amount by which z_{max} changes on changing b_i by one unit.)

In addition to the above information, the incremental effect coefficients, $\overline{a}_{ij}$, available in the optimal simplex tableau are of considerable interest to practitioners in the field of petroleum refinery optimization. The following subsection explains the interpretation of incremental effect coefficients.

2.1.1 Incremental Effect Coefficients

Incremental effect coefficients are of two types: primal incremental effect coefficients and dual incremental effect coefficients. The column vector $[\overline{a}_{i,n+1} \cdots \overline{a}_{m,n+1}]^T$ below y_i (also defined for reduced cost, not just shadow prices) in Table 2.1 is called the primal incremental effect coefficients, and the row vector $[\overline{a}_{i,m+1} \cdots \overline{a}_{i,n} \; \overline{a}_{i,n+1} \cdots \overline{a}_{i,n+m}]$ left to b_i ignoring the identity structure in Table 2.1, is called the dual incremental effect coefficients.

Primal incremental effect coefficients direct the incremental change in activities when an active constraint is positively or negatively perturbed within a limited (sensitivity) range. For example, if the right hand side (R.H.S) of an active ith constraint is perturbed as $b_i + \delta$, where δ stands for small changes within a sensitivity range, the new activity values will change by an amount advocated by the primal incremental effect coefficients given by Equation (2.2).

$$x_{B,new} = \begin{pmatrix} \overline{b}_1 \\ \vdots \\ \overline{b}_m \end{pmatrix} + \delta \begin{pmatrix} \overline{a}_{i,n+1} \\ \vdots \\ \overline{a}_{m,n+1} \end{pmatrix} = \begin{pmatrix} \overline{b}_{1,new} \\ \vdots \\ \overline{b}_{m,new} \end{pmatrix} \tag{2.2}$$

Where $x_{B,new}$ is the new set of activity values obtained after the positive perturbation $b_i + \delta$, a similar analysis is also valid for a negative perturbation $b_i - \delta$. The above analysis will be referred to as the **primal incremental effect analysis** in this work.

The dual incremental effect coefficients direct the incremental change in dual values when the cost coefficient of a decision variable in the optimal basis is positively or negatively perturbed within a sensitivity range. For example, if the cost coefficient of the jth decision variable in the optimal basis is perturbed as $c_j + \delta$, the new dual values will change by an amount advocated by the dual incremental effect coefficients given by Equation (2.3).

$$(\overline{c}_{new} \; \overline{y}_{new}) = \begin{pmatrix} \overline{c}_{m+1} \\ \vdots \\ \overline{c}_n \\ \overline{y}_i \\ \vdots \\ \overline{y}_m \end{pmatrix}^T + \delta \begin{pmatrix} \overline{a}_{i,m+1} \\ \vdots \\ \overline{a}_{i,n} \\ \overline{a}_{i,n+1} \\ \vdots \\ \overline{a}_{i,n+m} \end{pmatrix}^T = \begin{pmatrix} \overline{c}_{m+1,new} \\ \vdots \\ \overline{c}_{n,new} \\ \overline{y}_{i,new} \\ \vdots \\ \overline{y}_{m,new} \end{pmatrix}^T \tag{2.3}$$

Where $(\overline{c}_{new} \; \overline{y}_{new})$ are the new dual values obtained after the positive perturbation $c_j + \delta$, a similar analysis is also valid for a negative perturbation $c_j - \delta$. The above analysis will be referred to as the **dual incremental effect analysis** in this work.

2.2 Solving and Interpreting Refinery LP Solutions

A petroleum refinery LP model adopted from Pike (1986) is used for case studies in this research. The flow sheet of the refinery LP model is shown in Figure 2.1. The expanded names for the process streams involved in the flow sheet are provided in Table 2.2.

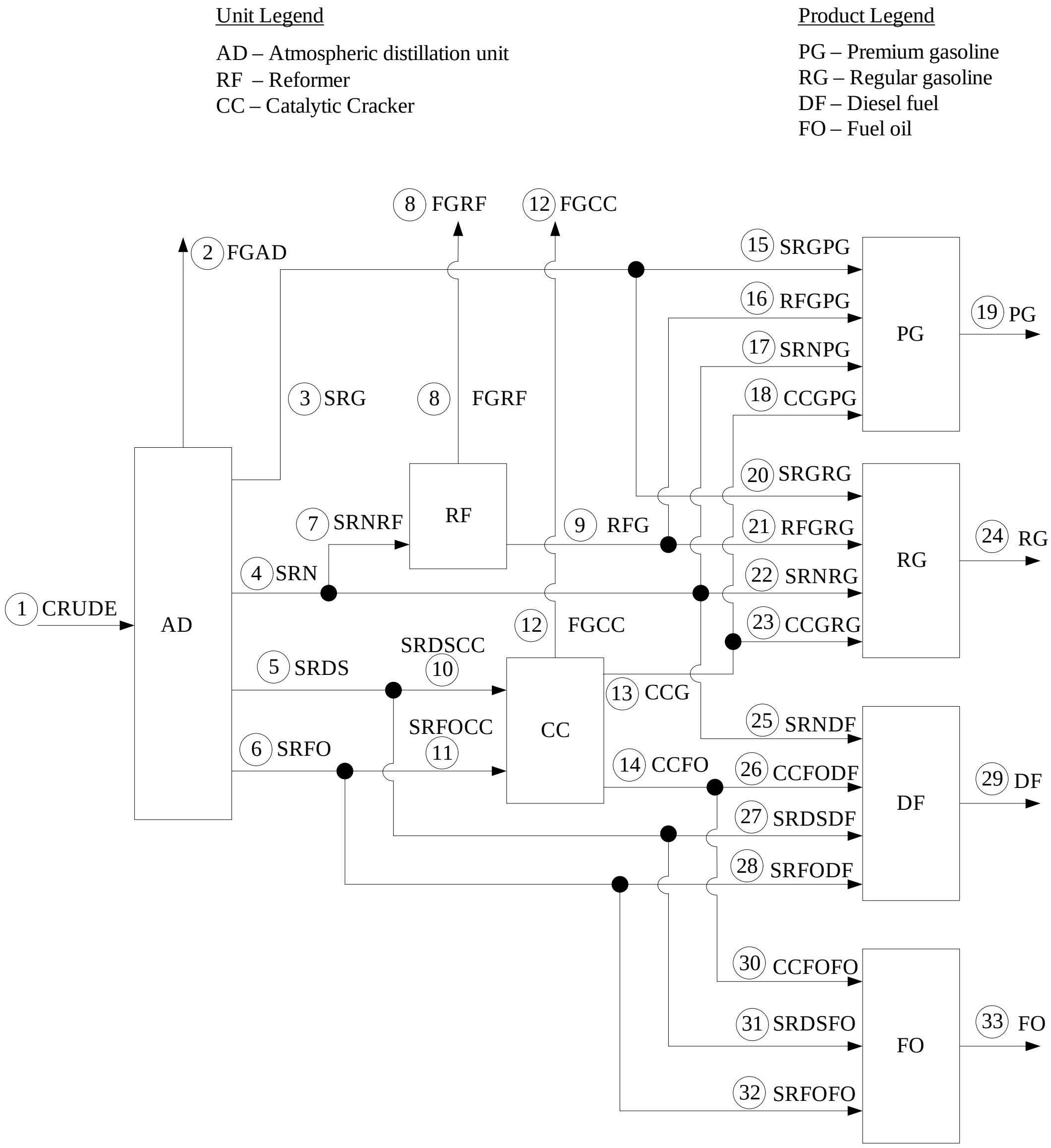


Figure 2.1: Process Flow Diagram for Pike's Refinery (Pike, 1986)

Table 2.2: Process Stream Description for Pike's Refinery LP Decision Variables (Pike, 1986)

NO.	Name	Definition (flow rates are in barrels per day)
1	CRUDE	Crude oil flow rate to the atmospheric crude distillation column (AD)
2	FGAD	Fuel gas flow rate from AD
3	SRG	Straight-run gasoline flow rate from AD
4	SRN	Straight-run naphtha flow rate from AD
5	SRDS	Straight-run distillate flow rate from AD
6	SRFO	Straight-run fuel oil flow rate from AD
7	SRNRF	Straight-run naphtha feed rate to the reformer (RF)
8	FGRF	Fuel gas flow rate from RF
9	RFG	Reformer gasoline flow rate
10	SRDSCC	Straight-run distillate flow rate to the catalytic cracking unit (CCU)
11	SRFOCC	Straight-run fuel oil flow rate to the CCU
12	FGCC	Fuel gas oil flow rate from the CCU
13	CCG	Gasoline flow rate from CCU
14	CCFO	Fuel oil flow rate from CCU
15	SRGPG	Straight-run gasoline flow rate for premium gasoline (PG) blending
16	RFGPG	Reformer gasoline flow rate for PG blending
17	SRNPG	Straight-run naphtha flow rate for PG blending
18	CCGPG	Catalytic cracking unit gasoline flow rate for PG blending
19	PG	Premium gasoline flow rate
20	SRGRG	Straight-run gasoline flow rate for regular gasoline (RG) blending
21	RFGRG	Reformer gasoline flow rate for RG blending
22	SRNRG	Straight-run naphtha flow rate for RG blending
23	CCGRG	Catalytic cracking unit gasoline flow rate for RG blending
24	RG	Regular gasoline flow rate
25	SRNDF	Straight-run naphtha flow rate for diesel fuel (DF) blending
26	CCFODF	Catalytic cracking unit fuel oil flow rate for diesel fuel (DF) blending
27	SRDSDF	Straight-run distillate flow rate for DF blending
28	SRFODF	Straight-run fuel oil flow rate for DF blending
29	DF	No. 2 diesel fuel flow rate
30	CCFOFO	Catalytic cracking unit flow rate for fuel oil(FO) blending
31	SRDSFO	Straight-run distillate flow rate for FO blending
32	SRFOFO	Straight-run fuel oil flow rate for FO blending
33	FO	No. 6 fuel oil flow rate

2.2.1 Refinery LP Formulation

Pike's refinery LP has 21 equality constraints, 16 inequality constraints and 33 decision variables. The 33 decision variables, along with their descriptions are provided in Table 2.2. The cost of inputs (crude oil), operating cost incurred in the units and the sales price of products produced in the refinery are listed in Table 2.3

Table 2.3: Crude Oil Cost, Product Sales Prices, and Operating Costs for the Petroleum Refinery Pike (1986)

S.I. No.	Variable	Cost Coefficient
1	CRUDE	Buying price of 33$/bb1
2	FGAD	Selling price of 0.01965 $/ft3
3	SRNRF	Reformer operating cost of 2.5$/bbl
4	FGRF	Selling price of 0.01965 $/ft3
5	SRDSCC	FCC operating cost of 2.2$/bbl
6	SRFOCC	FCC operating cost of 2.2$/bbl
7	FGCC	Selling price of 0.01965 $/ft3
8	PG	Selling price of 44.0813$/bbl
9	RG	Selling price of 43.68$/bbl
10	DF	Selling price of 40.32$/bbl
11	FO	Selling price of 13.14$/bbl

The LP for the Pike's problem is formulated as a maximization problem. The objective function is given as follows. The sales prices are shown as positive, and the cost are shown as negative in the objective function.

Maximize z =

$$-33CRUDE + 0.01965FGAD - 2.5SRNRF + 0.01965FGRF - 2.2SRDSCC - 2.2SRFOCC + 0.01965FGCC + 45.36PG + 43.68RG + 40.32DF + 13.14FO$$

The constraints for the Pike's refinery LP are listed below.

Subject to

1) CRUDE $\leq$ 110,000 $\rightarrow$ Crude oil availability

Premium Gasoline (PG) blending

2) PG $\geq$ 10,000 $\rightarrow$ Minimum production requirement

3) SRGPG + RFGPG + SRNPG + CCGPG - PG = 0 $\rightarrow$ PG Blending material balance

4) 78.5SRGPG + 104RFGPG + 65SRNPG + 93.7CCGPG - 93PG $\geq$ 0 $\rightarrow$ PG Octane rating (physical property specification)

5) 18.4SRGPG + 2.57RFGPG + 6.54SRNPG + 6.9CCGPG - 12.7PG $\leq$ 0 $\rightarrow$ PG Vapor pressure (physical property specification)

Regular Gasoline (PG) blending

6) RG $\geq$ 10,000 $\rightarrow$ Minimum production requirement

7) SRGRG + RFGRG + SRNRG + CCGRG - RG = 0 $\rightarrow$ RG Blending material balance

8) 78.5SRGRG + 104RFGRG + 65SRNRG + 93.7CCGRG - 87RG $\geq$ 0 $\rightarrow$ RG Octane rating (physical property specification)

9) 18.4SRGRG + 2.57RFGRG + 6.54SRNRG + 6.9CCGRG - 12.7RG $\leq$ 0 $\rightarrow$ RG Vapor pressure (physical property specification)

Diesel Fuel (DF) blending

10) DF $\geq$ 10,000 $\rightarrow$ Minimum production requirement

11) SRNDF + CCFODF + SRDSDF + SRFODF - DF = 0 $\rightarrow$ DF Blending material balance

12) 272SRNDF + 294.4CCFODF + 292SRDSDF + 295SRFODF - 306DF $\leq$ 0 $\rightarrow$ DF Density specification (physical property specification)

13) 0.283SRNDF + 0.353CCFODF + 0.526SRDSDF + 0.980SRFODF - 0.5DF $\leq$ 0 $\rightarrow$ DF Sulfur specification (physical property specification)

Fuel Oil (FO) blending

14) FO $\geq$ 10,000 $\rightarrow$ Minimum production requirement

15) CCFOFO + SRDSFO + SRFOFO - FO = 0 $\rightarrow$ FO Blending material balance

16) 294.4CCFOFO + 292SRDSFO + 295SRFOFO - 352FO $\leq$ 0 $\rightarrow$ FO Density specification (physical property specification)

17) 0.353CCFOFO + 0.526SRDSFO + 0.980SRFOFO - 3FO $\leq$ 0 $\rightarrow$ FO Sulfur specification (physical property specification)

Atmospheric Distillation (AD) unit

18) CRUDE $\leq$ 100,000 $\rightarrow$ AD Equipment processing capacity

AD Unit Material Balance Constraints

19) 35.42CRUDE - FGAD = 0 $\rightarrow$ FGAD Yield

20) 0.27CRUDE - SRG = 0 $\rightarrow$ SRG Yield

21) 0.237CRUDE - SRN = 0 $\rightarrow$ SRN Yield

22) 0.087CRUDE - SRDS = 0 $\rightarrow$ SRDS Yield

23) 0.372CRUDE - SRFO = 0 $\rightarrow$ SRFO Yield

Catalytic Reformer (RF)

24) SRNRF $\leq$ 25,000 $\rightarrow$ RF Equipment processing capacity

RF Unit Material Balance Constraints

25) 158.7SRNRF - FGRF = 0 $\rightarrow$ FGRF Yield

26) 0.928SRNRF - RFG = 0 $\rightarrow$ RFG Yield

Catalytic cracking (FCC unit)

27) SRDSCC + SRFOCC $\leq$ 30,000 $\rightarrow$ FCC Equipment processing capacity

FCC Unit Material Balance Constraints

28) 336.9SRDSCC + 386.4SRFOCC - FGCC = 0 $\rightarrow$ FGCC Yield

29) 0.619SRDSCC + 0.688SRFOCC - CCG = 0 $\rightarrow$ CCG Yield

30) 0.189SRDSCC + 0.2197SRFOCC - CCFO = 0 $\rightarrow$ CCFO Yield

Stream splits (material balance constraints)

31) SRG - SRGPG - SRGRG = 0 $\rightarrow$ SRG Split

32) SRN - SRNRF - SRNPG - SRNRG - SRNDF = 0 $\rightarrow$ SRN Split

33) SRDS - SRDSCC - SRDSDF - SRDSFO = 0 $\rightarrow$ SRDS Split

34) SRFO - SRFOCC - SRFODF - SRFOFO = 0 $\rightarrow$ SRFO Split

35) RFG - RFGPG - RFGRG = 0 $\rightarrow$ RFG Split

36) CCG - CCGRG - CCGPG = 0 $\rightarrow$ CCG Split

37) CCFO - CCFODF - CCFOFO = 0 $\rightarrow$ CCG Split

Before attempting to solve Pike's refinery LP using the primal and dual simplex method, all the greater than or equal to (GE) constraints and equality constraints are algebraically manipulated to less than or equal to (LE) constraints. The GE constraint of the form given in Equation (2.4)

$$Ax \geq b \tag{2.4}$$

is multiplied by -1 and converted to LE form as given in Equation (2.5).

$$-Ax \leq -b \tag{2.5}$$

The equality constraints of the form given in Equation (2.6)

$$Ax = b \tag{2.6}$$

are initially converted to companion form given in Equation (2.7) by splitting into two inequalities.

$$\begin{aligned} Ax &\leq b \text{ (LE form)} \\ Ax &\geq b \text{ (GE form)} \end{aligned} \tag{2.7}$$

Then, the GE form in the above companion representation is converted to LE by multiplying it by -1 and given by Equation (2.8).

$$\begin{aligned} Ax &\leq b \text{ (LE form)} \\ -Ax &\leq -b \text{ (Modified GE)} \end{aligned} \tag{2.8}$$

After applying the above transformation to the refinery LP, the constraints are presented as follows:

1) CRUDE $\leq$ 110,000 → Crude oil availability

PG blending

2) - PG $\leq$ - 10,000 → Minimum production requirement

3) - PG + SRGPG + RFGPG + SRNPG + CCGPG $\leq$ 0 → PG Blending material balance

4) PG - SRGPG - RFGPG - SRNPG - CCGPG $\leq$ 0

5) 93PG - 78.5SRGPG - 104RFGPG - 65SRNPG - 93.7CCGPG $\leq$ 0 →PG Octane rating (physical property specification)

6) - 12.7PG + 18.4SRGPG + 2.57RFGPG + 6.54SRNPG + 6.9CCGPG $\leq$ 0 → PG Vapor pressure (physical property specification)

RG blending

7) - RG $\leq$ - 10,000 → Minimum production requirement

8) - RG + SRGRG + RFGRG + SRNRG + CCGRG $\leq$ 0 →RG Blending

9) RG - SRGRG - RFGRG - SRNRG - CCGRG $\leq$ 0

10) 87 RG - 78.5SRGRG - 104RFGRG - 65SRNRG - 93.7CCGRG $\leq$ 0 →RG Octane rating (physical property specification)

11) - 12.7RG + 18.4SRGRG + 2.57RFGRG + 6.54SRNRG + 6.9CCGRG $\leq$ 0 →RG Vapor pressure (physical property specification)

DF blending

12) - DF $\leq$ - 10,000 → Minimum production requirement

13) - DF + SRNDF + CCFODF + SRDSDF + SRFODF $\leq$ 0 →DF Blending material balance

14) DF - SRNDF - CCFODF - SRDSDF - SRFODF $\leq$ 0

15) - 306DF + 272SRNDF + 294.4CCFODF + 292SRDSDF + 295SRFODF ≤ 0 → DF Density specification (physical property specification)

16) - 0.5DF + 0.283SRNDF + 0.353CCFODF + 0.526SRDSDF + 0.98SRFODF≤ 0 → DF Sulfur specification (physical property specification)

FO blending

17) - FO ≤ - 10,000 →Minimum production requirement

18) - FO + CCFOFO + SRDSFO + SRFOFO ≤ 0 →FO Blending material balance

19) FO - CCFOFO - SRDSFO - SRFOFO ≤ 0

20) - 352FO + 294.4CCFOFO + 292SRDSFO + 295 SRFOFO ≤ 0 →FO Density specification (physical property specification)

21) - 3FO + 0.353CCFOFO + 0.526SRDSFO + 0.98SRFOFO ≤ 0 →FO Sulfur specification (physical property specification)

Crude Oil Atmospheric Distillation Column

22) CRUDE ≤ 100,000 → AD Equipment processing capacity

AD Unit Material Balance Constraints

23) 35.42CRUDE - FGAD ≤ 0 → FGAD Yield

24) - 35.42CRUDE + FGAD ≤ 0

25) 0.27CRUDE - SRG ≤ 0 →SRG Yield

26) - 0.27CRUDE + SRG ≤ 0

27) 0.237CRUDE - SRN ≤ 0 →SRN Yield

28) - 0.237CRUDE + SRN ≤ 0

29) 0.08699999CRUDE - SRDS ≤ 0 →SRDS Yield

30) - 0.08699999CRUDE + SRDS ≤ 0

31) 0.372CRUDE - SRFO ≤ 0 →SRFO Yield

32) - 0.372CRUDE + SRFO ≤ 0

Catalytic Reformer

33) SRNRF $\leq$ 25,000 → RF Equipment processing capacity

RF Unit Material Balance Constraints

34) 158.7SRNRF - FGRF $\leq$ 0 →FGRF Yield

35) - 158.7SRNRF + FGRF $\leq$ 0 36) 0.928SRNRF - RFG $\leq$ 0 →RFG Yield

37)- 0.928SRNRF + RFG $\leq$ 0

FCC Unit

38) SRDSCC + SRFOCC $\leq$ 30,000 → FCC Equipment processing capacity

FCC Unit Material Balance Constraints

39) 336.9SRDSCC + 386.4SRFOCC - FGCC $\leq$ 0 →FGCC Yield

40) - 336.9SRDSCC - 386.4SRFOCC + FGCC $\leq$ 0

41) 0.619SRDSCC + 0.688SRFOCC - CCG $\leq$ 0 →CCG Yield

42) - 0.619SRDSCC - 0.688SRFOCC + CCG $\leq$ 0

43) 0.189SRDSCC + 0.2197SRFOCC - CCFO $\leq$ 0 →CCFO Yield

44) - 0.189SRDSCC - 0.2197SRFOCC + CCFO $\leq$ 0

Stream Splits (material balance constraints)

45) - SRGPG - SRGRG + SRG $\leq$ 0 →SRG Split

46) SRGPG + SRGRG - SRG $\leq$ 0

47) - SRNRF - SRNPG - SRNRG - SRNDF + SRN $\leq$ 0 →SRN Split

48) SRNRF + SRNPG + SRNRG + SRNDF - SRN $\leq$ 0

49) - SRDSCC - SRDSDF - SRDSFO + SRDS $\leq$ 0 →SRDS Split

50) SRDSCC + SRDSDF + SRDSFO - SRDS $\leq$ 0

51) - SRFOCC - SRFODF - SRFOFO + SRFO $\leq$ 0 →SRFO Split

52) SRFOCC + SRFODF + SRFOFO - SRFO $\leq$ 0

53) - RFGPG - RFGRG + RFG $\leq$ 0 →RFG Split

54) RFGPG + RFGRG - RFG $\leq$ 0

55) - CCGPG - CCGRG + CCG $\leq$ 0 →CCG Split

56) CCGPG + CCGRG - CCG $\leq$ 0

57) - CCFODF - CCFOFO + CCFO $\leq$ 0 →CCFO Split

58) CCFODF + CCFOFO - CCFO $\leq$ 0

After adding slacks to the above formulation the standard form representation is given as

1) CRUDE + SLK1 = 110,000 →Crude oil availability

PG blending

2) - PG + SLK2 = - 10000 → Minimum production requirement

3) - PG + SRGPG + RFGPG + SRNPG + CCGPG + SLK3 = 0 →PG Blending material balance

4) PG - SRGPG - RFGPG - SRNPG - CCGPG + SLK4 = 0

5) 93PG - 78.5SRGPG - 104RFGPG - 65SRNPG - 93.7CCGPG + SLK5 = 0 →PG Octane rating (physical property specification)

6) - 12.7PG + 18.4SRGPG + 2.57RFGPG + 6.54SRNPG + 6.9CCGPG + SLK6 = 0 →PG Vapor pressure (physical property specification)

RG blending

7) - RG + SLK7 = - 10,000 →Minimum production requirement

8) - RG + SRGRG + RFGRG + SRNRG + CCGRG + SLK8 = 0 →RG Blending material balance

9) RG - SRGRG - RFGRG - SRNRG - CCGRG + SLK9 = 0

10) 87RG - 78.5SRGRG - 104RFGRG - 65SRNRG - 93.7CCGRG + SLK10 = 0 →RG Octane rating (physical property specification)

11) - 12.70RG + 18.40SRGRG + 2.570RFGRG + 6.540SRNRG + 6.90CCGRG + SLK11 =0→RG Vapor pressure (physical property specification)

DF blending

12) - DF + SLK12 = - 10000 →Minimum production rate

13) - DF + SRNDF + CCFODF + SRDSDF + SRFODF + SLK13 = 0 →DF Blending material balance

14) DF - SRNDF - CCFODF - SRDSDF - SRFODF + SLK14 = 0

15) - 306DF + 272SRNDF + 294.4 CCFODF + 292SRDSDF + 295SRFODF + SLK15 = 0 →DF Density specification (physical property specification)

16) - 0.5DF+ 0.283SRNDF+ 0.353CCFODF+ 0.526SRDSDF+ 0.98SRFODF + SLK16 = 0 →DF Sulfur specification (physical property specification)

FO blending

17) - FO + SLK17 = - 10000 →Minimum production rate

18) - FO + CCFOFO + SRDSFO + SRFOFO + SLK18 = 0 →FO Blending material balance

19) FO - CCFOFO - SRDSFO - SRFOFO + SLK19 = 0

20) - 352FO + 294.4CCFOFO + 292SRDSFO + 295SRFOFO + SLK20 = 0 →FO Density specification (physical property specification)

21) - 3FO + 0.353CCFOFO + 0.526SRDSFO + 0.98SRFOFO + SLK21 = 0 →FO Sulfur specification (physical property specification)

Atmospheric Distillation Column

22) CRUDE + SLK22 = 100,000 →AD equipment processing capacity

AD Unit Material Balance Constraints

23) 35.42CRUDE - FGAD + SLK23 = 0 →FGAD Yield

24) - 35.42CRUDE + FGAD + SLK24 = 0

25) 0.27CRUDE - SRG + SLK25 = 0 →SRG Yield

26) - 0.27CRUDE + SRG + SLK26 = 0

27) 0.237CRUDE - SRN + SLK27 = 0 →SRN Yield

28) - 0.237CRUDE + SRN + SLK28 = 0

29) 0.08699999CRUDE - SRDS + SLK29 = 0 →SRDS Yield

30) - 0.08699999CRUDE + SRDS + SLK30 =0

31) 0.372CRUDE - SRFO + SLK31 = 0 →SRFO Yield

32) - 0.372CRUDE + SRFO + SLK32 = 0

Catalytic Reformer

33) SRNRF + SLK33 = 25,000 →RF equipment processing capacity

RF Unit Material Balance Constraints

34) 158.7SRNRF - FGRF + SLK34 = 0 →FGRF Yield

35) - 158.7SRNRF + FGRF + SLK35 = 0

36) 0.928SRNRF - RFG + SLK36 = 0 →RFG Yield

37) - 0.928SRNRF + RFG + SLK37 = 0

FCC Unit

FCC Unit Material Balance Constraints

38) SRDSCC + SRFOCC + SLK38 = 30,000 →FCC Capacity

39) 336.9SRDSCC + 386.4SRFOCC - FGCC + SLK39 = 0 →FGCC Yield

40) - 336.9SRDSCC - 386.4SRFOCC + FGCC + SLK40 = 0

41) 0.619SRDSCC + 0.688SRFOCC - CCG + SLK41 = 0 →CCG Yield

42) - 0.619SRDSCC - 0.688SRFOCC + CCG + SLK42 = 0

43) 0.189SRDSCC + 0.2197SRFOCC - CCFO + SLK43 = 0 →CCFO Yield

44) - 0.189SRDSCC - 0.2197SRFOCC + CCFO + SLK44 = 0

Stream Splits (material balance constraints)

45) - SRGPG - SRGRG + SRG + SLK45 = 0 →SRG Split

46) SRGPG + SRGRG - SRG + SLK46 $=$ 0

47) - SRNRF - SRNPG - SRNRG - SRNDF + SRN + SLK47 $=$ 0 $\rightarrow$SRN Split

48) SRNRF + SRNPG + SRNRG + SRNDF - SRN + SLK48 $=$ 0

49) - SRDSCC - SRDSDF - SRDSFO + SRDS + SLK49 $=$ 0 $\rightarrow$SRDS Split

50) SRDSCC + SRDSDF + SRDSFO - SRDS + SLK50 $=$ 0

51) - SRFOCC - SRFODF - SRFOFO + SRFO + SLK51 $=$ 0 $\rightarrow$SRFO Split

52) SRFOCC + SRFODF + SRFOFO - SRFO + SLK52 $=$ 0

53) - RFGPG - RFGRG + RFG + SLK53 $=$ 0 $\rightarrow$RFG Split

54) RFGPG + RFGRG - RFG + SLK54 $=$ 0

55) - CCGPG - CCGRG + CCG + SLK55 $=$ 0 $\rightarrow$CCG Split

56) CCGPG + CCGRG - CCG + SLK56 $=$ 0

57) - CCFODF - CCFOFO + CCFO + SLK57 $=$ 0 $\rightarrow$CCFO Split

58) CCFODF + CCFOFO - CCFO + SLK58 $=$ 0

The original problem contains 33 decision variables (n), 21 material balance constraints (equality constraints), and 16 inequality constraints including capacity, sales, purchase or physical property constraints. The LP in the standard form has 58 constraints (m = 21 $\times$ 2 + 16) and 91 variables ($m + n$). All the variables are indexed in numerical order. The variables and their corresponding index are given in Table 2.4.

Table 2.4: Variable Index for the Refinery LP

Index	Variable	Index	Variable	Index	Variable	Index	Variable	Index	Variable
1	CRUDE	20	SRNDF	39	SLK6	58	SLK25	77	SLK44
2	FGAD	21	CCFODF	40	SLK7	59	SLK26	78	SLK45
3	SRNRF	22	SRDSDF	41	SLK8	60	SLK27	79	SLK46
4	FGRF	23	SRFODF	42	SLK9	61	SLK28	80	SLK47
5	SRDSCC	24	CCFOFO	43	SLK10	62	SLK29	81	SLK48
6	SRFOCC	25	SRDSFO	44	SLK11	63	SLK30	82	SLK49
7	FGCC	26	SRFOFO	45	SLK12	64	SLK31	83	SLK50
8	PG	27	SRG	46	SLK13	65	SLK32	84	SLK51
9	RG	28	SRN	47	SLK14	66	SLK33	85	SLK52
10	DF	29	SRDS	48	SLK15	67	SLK34	86	SLK53
11	FO	30	SRFO	49	SLK16	68	SLK35	87	SLK54
12	SRGPG	31	RFG	50	SLK17	69	SLK36	88	SLK55
13	RFGPG	32	CCG	51	SLK18	70	SLK37	89	SLK56
14	SRNPG	33	CCFO	52	SLK19	71	SLK38	90	SLK57
15	CCGPG	34	SLK1	53	SLK20	72	SLK39	91	SLK58
16	SRGRG	35	SLK2	54	SLK21	73	SLK40		
17	RFGRG	36	SLK3	55	SLK22	74	SLK41		
18	SRNRG	37	SLK4	56	SLK23	75	SLK42		
19	CCGRG	38	SLK5	57	SLK24	76	SLK43		

2.2.2 Refinery LP Solution

The given LP was solved using LINDO and the optimal solution determined. The LP optimal solution is usually represented in a tableau form called the optimal simplex tableau. The optimal simplex tableau is a 58×91 matrix. The entire optimal simplex tableau will not be presented due to its size. However, essential components of the solution matrix will be presented for interpretation.

Objective value The objective value or the optimal profit for this refinery was found to be z_{max} = \$701,823.43, which implies that for the given problem with the specified constraints the maximum profit that could be made is \$701,823.43.

Activity Values The optimal basis is given in Table 2.5. Since the problem has m = 58 equations there will be 58 variables in the optimal basis called the basic variables. The numerical value associated with a basic variable is interpreted as the activity values. For example, the activity of the Premium Gasoline (PG) decision variable is 47113.20 bbl/day in Table 2.5. This represents the amount of PG that has to be manufactured to attain the optimal profit of \$701,823.43.

Dual Values The dual values corresponding to the optimal basis are presented in Table 2.6. Since the problem has $n + m$ = 91 variables and m = 58 equations, there will be $n + m - m$ = 33 dual variables. The dual variables corresponding to the original decision variables in the $Z{-}row$ of optimal simplex tableau are referred to as reduced cost. For example, the reduced cost of Straight Run Naphtha for Regular Gasoline blending (SRNRG) in Table 2.6 is given as 8.05\$/bbl, implying that SRNRG stream is not manufactured in the process. In order to manufacture SRNRG in the process the selling price of SRNG has to be increased at least by \$8.05.

Table 2.5: Optimal Basis for the Refinery LP[†]

Index	Variable	Activity (bbl/day)	Index	Variable	Activity (bbl/day)	Index	Variable	Activity (bbl/day)
1	CRUDE	$100,000.00$	26	SRFOFO	$5,403.80$	54	SLK21	$22,286.68$
2	FGAD*	$3,542,000.00$	27	SRG	$27,000.00$	56	SLK23	0.00
3	SRNRF	$23,700.00$	28	SRN	$23,700.00$	58	SLK25	0.00
4	FGRF*	$3,761,190.00$	29	SRDS	$8,700.00$	60	SLK27	0.00
6	SRFOCC	$30,000.00$	30	SRFO	$37,200.00$	62	SLK29	0.00
7	FGCC*	$11,592,000.00$	31	RFG	$21,993.60$	64	SLK31	0.00
8	PG	$47,113.20$	32	CCG	$20,640.00$	66	SLK33	$1,300.00$
9	RG	$22,520.40$	33	CCFO	$6,591.00$	67	SLK34	0.00
10	DF	$12,491.00$	34	SLK1	$10,000.00$	69	SLK36	0.00
11	FO	$10,000.00$	35	SLK2	$37,113.20$	72	SLK39	0.00
12	SRGPG	$13,852.05$	36	SLK3	0.00	74	SLK41	0.00
13	RFGPG	$17,239.99$	39	SLK6	$188,607.17$	76	SLK43	0.00
15	CCGPG	$16,021.17$	40	SLK7	$12,520.40$	78	SLK45	0.00
16	SRGRG	$13,147.95$	41	SLK8	0.00	80	SLK47	0.00
17	RFGRG	$4,753.61$	45	SLK12	$2,491.00$	82	SLK49	0.00
19	CCGRG	$4,618.83$	46	SLK13	0.00	84	SLK51	0.00
21	CCFODF	$6,591.00$	48	SLK15	$153,666.99$	86	SLK53	0.00
22	SRDSDF	$4,103.80$	51	SLK18	0.00	88	SLK55	0.00
23	SRFODF	$1,796.20$	53	SLK20	$583,788.61$	90	SLK57	0.00
25	SRDSFO	$4,596.20$						

[†]There are a total of m = 58 decision and slack variables in the basis for the solution. The remaining 33 variables are in the set of non-basis variables.

$^*ft^3/day$.

The dual variables corresponding to slack and surplus in the $Z - row$ of the optimal simplex tableau are referred to as shadow prices. For example, the shadow price of Fuel Oil (FO) production constraint in Table 2.6 is given as -27.18 $/bbl. This implies manufacturing an additional barrel of FO in the process will reduce the objective function value by $27.18 .

Note that here the dual value of all GE constraints are multiplied by -1 because before solving the problem, all the GE constraints were converted to LE form.

Table 2.6: Optimal Dual Values for the Refinery LP†

Z	Dual Value	$\$/bbl$	Active at
SLK4	Premium gasoline blending	-19.32	LL (Equality)
SLK5	Premium gasoline octane rating	-0.28	LL
SLK9	Regular gasoline blending	-19.32	LL (Equality)
SLK10	Regular gasoline octane rating	-0.28	LL
SLK11	Regular gasoline vapor	0.00	LL
SLK14	Diesel fuel blending	-40.32	LL (Equality)
SLK16	Diesel fuel sulfur specification	0.00	LL
SLK17	Fuel oil production	-27.18	LL (Equality)
SLK19	Fuel oil blending	-40.32	LL (Equality)
SLK22	Atmospheric distillation unit capacity	8.15	UL
SLK24	Fuel gas yield from atmospheric distillation unit	-0.02	LL (Equality)
SLK26	Straight run gasoline yield from atmospheric distillation unit	-41.30	LL (Equality)
SLK28	Straight run naphtha yield from atmospheric distillation unit	-45.57	LL (Equality)
SLK30	Straight run distillate yield from atmospheric distillation unit	-40.32	LL (Equality)
SLK32	Straight run fuel oil yield from atmospheric distillation unit	-40.32	LL (Equality)
SLK35	Fuel gas yield from reformer unit	-0.02	LL (Equality)
SLK37	Reformed gasoline yield	-48.44	LL (Equality)
SLK38	Catalytic cracking (FCC) unit capacity	5.27	UL
SLK40	Fuel gas yield from catalytic cracking unit	-0.02	LL (Equality)
SLK42	Catalytic cracked gasoline yield	-45.56	LL (Equality)
SLK44	Catalytic cracked fuel oil yield	-40.32	LL (Equality)
SLK46	Straight run gasoline split	-41.30	LL (Equality)
SLK48	Straight run naphtha split	-45.57	LL (Equality)
SLK50	Straight run distillate split	-40.32	LL (Equality)
SLK52	Straight run fuel oil split	-40.32	LL (Equality)
SLK54	Reformed gasoline split	-48.44	LL (Equality)
SLK56	Catalytic cracked gasoline split	-45.56	LL (Equality)
SLK58	Catalytic cracked fuel oil split	-40.32	LL (Equality)
SLK5	Straight run distillate for catalytic cracking	5.35	Reduced cost
SLK14	Straight run naphtha for premium gasoline blending	8.05	Reduced cost
SLK18	Straight run naphtha for regular gasoline blending	8.05	Reduced cost
SLK 20	Straight run naphtha for diesel fuel blending	5.25	Reduced cost
SLK24	Catalytic cracked fuel oil for fuel oil blending	0.00	Reduced cost

†There are 33 variables in the set of non-basic variables. Depending on the type: decision variable, slack or surplus variable, the dual values represents either reduced cost or shadow prices. Negative dual values correspond to surplus variables (GE constraints).

Primal Incremental Effects The column vector below each of the shadow prices in the optimal simplex tableau contain the primal incremental effect coefficients. The shadow price of FO along with the corresponding primal incremental effect coefficients are listed in Table 2.7. The associated base objective function value and the activity values are also listed. The FO production constraint is a GE constraint and written in LE form before solving the problem. Therefore, the primal incremental coefficients are multiplied by -1 and presented in Table 2.7.

Table 2.7: Primal Incremental Effect Coefficients Associated with Fuel Oil Production Constraint†

		FO Production	Original solution	+1 FO Production increase	-1 FO Production decrease
	Z	$-27.18\ \frac{\$}{bbl}$	$701,823.43$	$701,796.25$	$701,850.61$
Index	Solution basis	Incremental effect ($\overline{a}_{ij}$)	Activity bbl/day	Activity bbl/day	Activity bbl/day
10	DF	-1.00	$12,491.00$	$12,490.00$	$12,492.00$
11	FO	1.00	$10,000.00$	$10,001.00$	$9,999.00$
22	SRDSDF	-1.06	$4,103.80$	$4,102.74$	$4,104.85$
23	SRFODF	0.06	$1,796.20$	$1,796.26$	$1,796.15$
25	SRDSFO	1.06	$4,596.20$	$4,597.26$	$4,595.15$
26	SRFOFO	-0.06	$5,403.80$	$5,403.74$	$5,403.85$
45	SLK12	-1.00	$2,491.00$	$2,490.00$	$2,492.00$
48	SLK15	-14.17	$153,666.99$	$153,652.81$	$153,681.16$
53	SLK20	60.17	$583,788.61$	$583,848.79$	$583,728.44$
54	SLK21	2.50	$22,286.68$	$22,289.18$	$22,284.18$

†Basic variables with zero primal incremental effect coefficients are omitted

As demonstrated in Table 2.7, when the FO production (constrained at the minimum in the optimal solution) is increased by one unit, the objective function value is reduced by the shadow price of FO constraint, and the change in activity values are determined by the primal incremental effect coefficients. Similar analysis is also valid for reducing the FO production by one unit. This analysis is referred to as primal incremental effect analysis.

Dual Incremental Effects The row vector corresponding to an activity value (ignoring the identity structure of the matrix) represents the dual incremental effect coefficients. The PG activity along with the dual incremental effects coefficients in the transpose form (column format) are given in Table 2.8. The associated dual values and the objective function value in the base case are also provided.

As evident from Table 2.8, when the selling price of PG is increased by one unit, the activity value governs the change in objective function value and the dual incremental effect coefficient values dictate the change in dual values. Similar analysis is valid for decrease in the PG price by one unit. This analysis is also referred to as dual incremental effect analysis.

Table 2.8: Dual Incremental Effect Coefficients Associated with Premium Gasoline Activity Value†

			Base case	$c_j + 1$	$c_j - 1$
		PG activity (bbl)	Objective value ($)	Objective value ($)	Objective value ($)
		$47,113.20$	$701,823.43$	$748,936.63$	$654,710.23$
Z	Dual variable*	Incremental effect ($\overline{a}_{ij}$)	Dual value ($/bbl)	Dual value ($/bbl)	Dual value ($/bbl)
SLK4	PGblend(2)	14.50	-19.32	-4.82	-33.82
SLK5	PGoctane	-0.17	-0.28	-0.45	-0.11
SLK9	RGblend(2)	14.50	-19.32	-4.82	-33.82
SLK10	RGoctane	-0.17	-0.28	-0.45	-0.11
SLK22	ADcapacity	0.24	8.15	8.39	7.91
SLK26	SRGyield(2)	1.42	-41.30	-39.88	-42.72
SLK28	SRNyield(2)	-2.63	-45.57	-48.20	-42.94
SLK37	RFGyield(2)	-2.83	-48.44	-51.27	-45.61
SLK38	CCcapacity	0.77	5.27	6.04	4.51
SLK42	CCGyield(2)	-1.12	-45.56	-46.67	-44.44
SLK46	SRGsplit(2)	1.42	-41.30	-39.88	-42.72
SLK48	SRNsplit(2)	-2.63	-45.57	-48.20	-42.94
SLK54	RFGsplit(2)	-2.83	-48.44	-51.27	-45.61
SLK56	CCGsplit(2)	-1.12	-45.56	-46.67	-44.44
SRDSCC	Reduced cost	0.08	5.35	5.43	5.28
SRNPG	Reduced cost	6.30	8.05	14.35	1.75
SRNRG	Reduced cost	6.30	8.05	14.35	1.75
SRNDF	Reduced cost	2.63	5.25	7.88	2.62

†Dual variables with zero dual incremental effect coefficients are omitted

*Description of dual variables are given in Table B.1 of Appendix B

2.3 Summary

This chapter explained the basic notations and definitions for an LP problem. Furthermore, these definitions and notations are illustrated for the simplified refinery LP model. This model will be used throughout the remainder of the document to illustrate the different concepts that are developed. The next chapter of this dissertation will introduce the different conditions of degeneracy in LP problems.

CHAPTER 3

DEGENERACY IN LP

An LP problem could be non-degenerate, primal degenerate, or dual degenerate. These three different conditions of LP problems are explained geometrically and algebraically in this section. Moreover, some of the background information related to degeneracy is also provided.

3.1 Non-Degenerate LP Problem

Definition: *An LP problem is considered to be non-degenerate if the optimal solution is uniquely determined by a single corner point with exactly n constraints passing through it.*

A 2-D LP obtained from Taha (2006), represented by Equation (3.1), is used to demonstrate this phenomenon.

$$\text{Maximize } z = 5x_1 + 4x_2 \tag{3.1}$$

Subject to

$6x_1 + 4x_2 \leq 24$	Constraint $\#1$
$x_1 + 2x_2 \leq 6$	Constraint $\#2$
$-x_1 + x_2 \leq 1$	Constraint $\#3$
$x_2 \leq 2$	Constraint $\#4$
$x_1, x_2 \geq 0$	Non-negativity

The above problem in the general form has n = 2 variables and m = 4 equations. The geometric solution of the 2-D LP is illustrated in Figure 3.1. As shown in Figure 3.1, the optimal vertex C for this 2-D LP is determined by a unique basis, because no more than

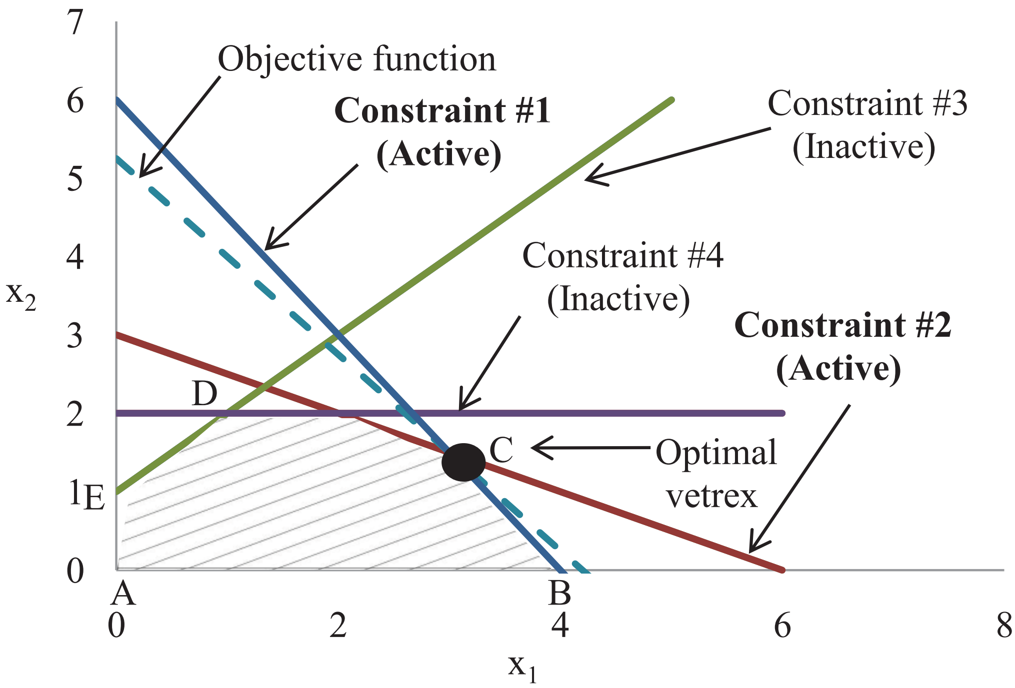


Figure 3.1: Graphical Solution for the 2-D Non-Degenerate LP

(n) two constraints, constraint #1 and constraint #2, pass through the optimum. When the optimum is represented by a single point C, the dimension of the "optimal face" is zero. These are the essential geometric characteristics of a non-degenerate LP problem.

Now the LP problem is solved algebraically using the simplex method. The resulting optimal simplex tableau is shown in Table 3.1. According to Table 3.1 none of the basic

Table 3.1: Optimal Tableau for a 2-D Non-Degenerate LP

Basis	x_1	x_2	s_1	s_2	s_3	s_4	RHS
z	0	0	3/4	1/2	0	0	21
x_1	1	0	1/4	-1/2	0	0	3
x_2	0	1	-1/8	3/4	0	0	3/2
s_3	0	0	3/8	-5/4	1	0	5/2
s_4	0	0	1/8	-3/4	0	1	1/2

(x_1, x_2, s_3, s_4) and non-basic variables (s_1, s_2) have zero primal or dual values, respectively, in the optimal solution. This is an indication that the LP is non-degenerate. In this case, the solution is unique, implying that there is a unique value for every activity, dual value and incremental effect coefficient.

3.2 Primal Degeneracy

***Definition**: A basic solution $x \in \Re^n$ is said to be primal degenerate if more than n of the constraints are active at x* (Bertsimas and Tsitsiklis, 1997, p.58).

In this section, the concept of primal degeneracy will be explained geometrically and algebraically using a modified version of a 2-D LP problem obtained from Taha (2006). The 2-D primal degenerate LP in general form is given in Equation (3.2).

$$\text{Maximize } z = 5x_1 + 4x_2 \tag{3.2}$$

Subject to

$$6x_1 + 4x_2 \leq 20 \qquad \text{Constraint } \#1$$

$$x_1 + 2x_2 \leq 6 \qquad \text{Constraint } \#2$$

$$-x_1 + x_2 \leq 1 \qquad \text{Constraint } \#3$$

$$x_2 \leq 2 \qquad \text{Constraint } \#4$$

$$x_1, x_2 \geq 0 \qquad \text{Non-negativity}$$

The geometric solution of the above 2-D LP is given in Figure 3.2. The shaded region, ABCDE, is the feasible space and the optimum is given by the vertex, point C. The given LP is primal degenerate, because for this 2-D LP problem, only two constraints are required to define the optimum. However, the optimum vertex, point C, is over-determined with three constraints: constraint #1, constraint #2, and constraint #4. Therefore, at this optimum vertex point C, three solutions can be generated with two constraints active at a time based on the combination formula given by Equation (3.3).

$$C_n^N = \binom{N}{n} = \frac{N!}{n!(N-n)!} \tag{3.3}$$

where, N is the number of constraints passing through the optimal point and n is the dimension of the problem (number of original decision variables).

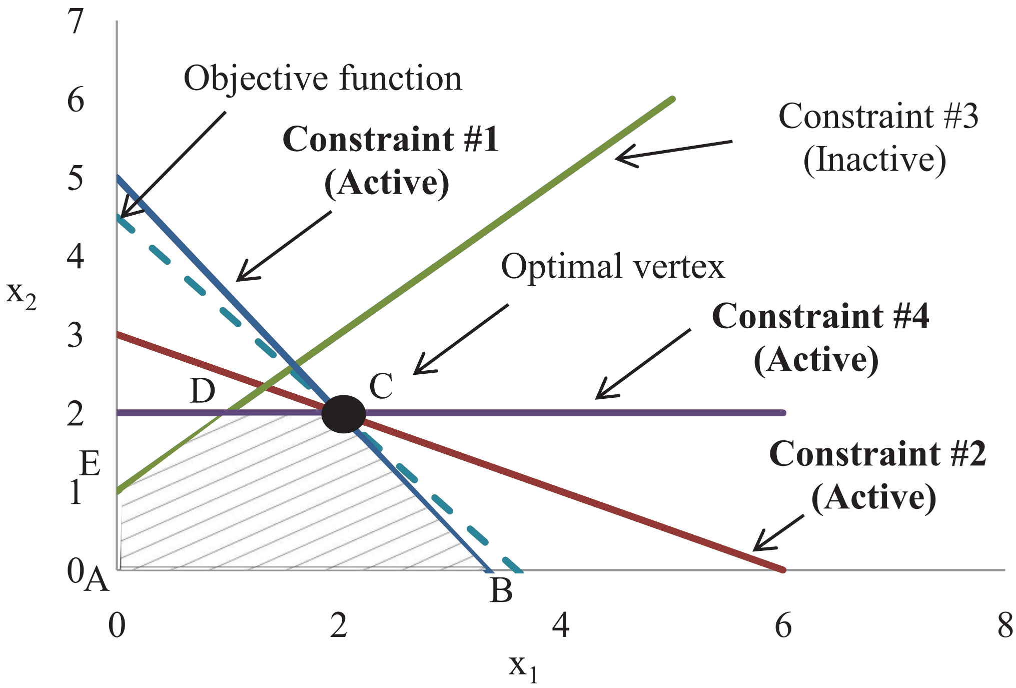


Figure 3.2: Graphical Solution for the 2-D Primal Degenerate LP

The multiple bases associated with the optimal vertex point C are generated algebraically using the simplex method with two constraints active at a time. The three bases are given by Table 3.2, Table 3.3, and Table 3.4. It is well known that, not all bases associated with a primal degenerate optimal vertex are dual feasible (optimal). The simplex tableau given by Table 3.4, generated with constraint #2 and constraint #4 active, is dual infeasible and non-optimal because of the "-6" dual value for constraint #4. However, it can be pivoted further to get the simplex tableau given by Table 3.2. Therefore, the two optimal bases possible for this primal degenerate vertex C are given by Table 3.2 and Table 3.3.

Table 3.2: Primal Degenerate Solution for the 2-D LP with Constraints #1 and #2 Active

Basis	x_1	x_2	s_1	s_2	s_3	s_4	RHS
z	0	0	3/4	1/2	0	0	18
x_1	1	0	1/4	-1/2	0	0	2
x_2	0	1	-1/8	3/4	0	0	2
s_3	0	0	3/8	-5/4	1	0	1
s_4	0	0	1/8	-3/4	0	1	0

Table 3.3: Primal Degenerate Solution for the 2-D LP with Constraints #1and #4 Active

Basis	x_1	x_2	s_1	s_2	s_3	s_4	RHS
z	0	0	5/6	0	0	2/3	18
x_1	1	0	1/6	0	0	-2/3	2
x_2	0	1	0	0	0	1	2
s_2	0	0	-1/6	1	0	-4/3	0
s_3	0	0	1/6	0	1	-5/3	1

Table 3.4: Primal Degenerate Solution for the 2-D LP with Constraints #2 and #4 Active

Basis	x_1	x_2	s_1	s_2	s_3	s_4	RHS
z	0	0	0	5	0	-6	18
x_1	1	0	0	1	0	-2	2
x_2	0	1	0	0	0	1	2
s_1	0	0	1	-6	0	8	0
s_3	0	0	0	1	1	-3	1

Notice that both the optimal simplex tableaux (Table 3.2 and Table 3.3) have a unique objective function value and primal solution (activities), but different dual solutions (reduced cost and shadow prices). In both cases, at least one of the basic variables has a zero activity value. In the solution presented in Table 3.2, the basic variable s_4 has an activity of zero. In Table 3.3, basic variable s_1 has an activity value of zero. These observations are an indication of primal degeneracy and are one of the distinguishing characteristics of a primal degenerate LP. As discussed later in Chapter 4, an activity of zero does not indicate primal degeneracy under certain conditions.

3.2.1 Consequence of Primal Degeneracy

Interpretation of LP optimal solutions under primal degeneracy becomes difficult, because primal degeneracy results in multiple dual solutions and unique primal solutions. The optimal dual value of an LP problem is interpreted either as reduced cost of a decision variable or as the shadow price of a constraint and has significant managerial interest. Many authors define shadow price based on managerial requirement. The most widely accepted definition of shadow price is given as follows: shadow price, p_i, of the ith resource, b_i, is the achievable rate of increase in the objective function per unit increase in resource (Aucamp and Steinberg, 1982). Mathematically, the definition of shadow price is given by Equation (3.4), when the partial derivative exists.

$$p_i = \frac{\partial z_{max}}{\partial b_i}\ ,\ 1 \leq i \leq m \tag{3.4}$$

where z_{max} is the optimal objective function value as a function of R.H.S of the constraint b_i, and p_i is the shadow price if ith constraint. Here 'p' stands for price. Different versions of the definitions of shadow price associated with different managerial interpretations can be found in Goyal and Soni (1984), Goh (1996), and Ronen (1982).

Since primal degeneracy produces multiple optimal dual solutions, the "true" shadow price values among the multiple optimal dual solutions must be identified to make correct business decisions. The most cited reference for the identification of true shadow price is Aucamp and Steinberg (1982). They make the case that all the optimal dual variables y_i^* do not necessarily correspond to shadow price. From a petroleum refining standpoint, this implies that not all dual values are physically realizable. The process of characterizing and interpreting dual values from a refinery LP that is primal degenerate is discussed in detail in the next chapter.

3.3 Dual Degeneracy

An LP is said to be dual degenerate or have alternative optima if every basic optimal solution to the dual is degenerate. This study chooses to use the term "dual degeneracy" to refer to LP that has alternative optima. The present study defines dual degeneracy in LP as follows:

Definition*: An LP problem is said to be dual degenerate or have multiple optima if the dimension of the optimal face is larger than zero.*

The phenomenon of dual degeneracy is explained geometrically and algebraically using a 2-D LP problem obtained from Taha (2006). The 2-D dual degenerate LP in general form is given in Equation (3.5).

$$
\begin{aligned}
&\text{Maximize } z = 2x_1 + 4x_2 \\
&\text{Subject to} \\
&x_1 + 2x_2 \leq 5 && \text{Constraint } \#1 \\
&x_1 + x_2 \leq 4 && \text{Constraint } \#2 \\
&x_1, x_2 \geq 0 && \text{Non-negativity}
\end{aligned}
\tag{3.5}
$$

The geometric solution of the above 2-D LP is given in Figure 3.3. The feasible space for this problem is denoted by the shaded region ABCD. By inspection of Figure 3.3, one of the active constraints (constraint #1) is parallel to the objective function line. Therefore, the entire line segment DC in Figure 3.3 is considered to be optimum. A line has a dimension of one in hyperspace. Since the dimension of the optimal face is larger than zero; the problem is dual degenerate and does not have a unique solution. All of the solutions have the same objective function value, but the activities defined by the coordinate values of every solution point are different.

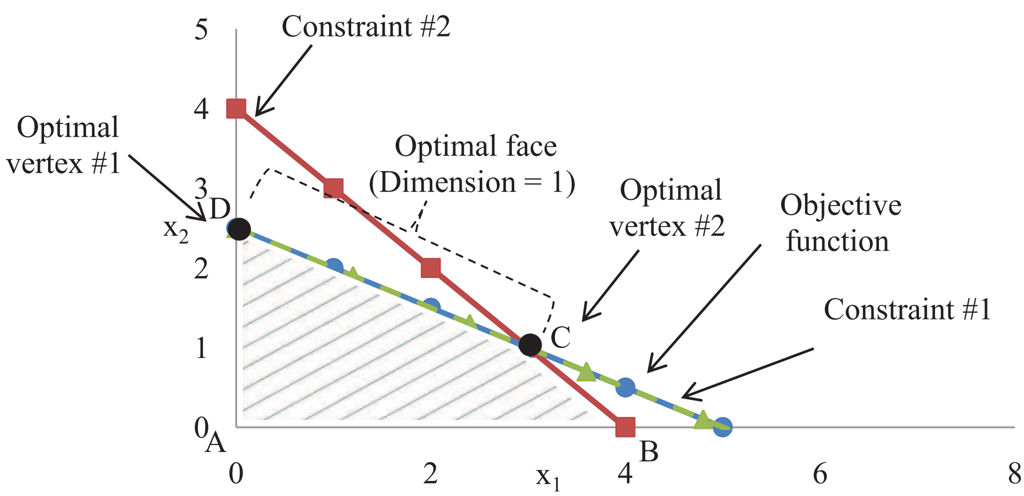


Figure 3.3: Graphical Solution for the 2-D Dual Degenerate LP

There are two distinct solutions at corner points D ($x_1 = 0, x_2 = 5/2$) and C ($x_1 = 3, x_2 = 1$). Apart from these two corner point solutions, from the line segment DC, an infinite number of optimal solutions with the same objective function value can be generated using the convex combination formula given by Equation (3.6).

$$x_1 = \alpha \times (0) + (1 - \alpha) \times (3)$$

$$x_2 = \alpha \times (5/2) + (1 - \alpha) \times (1) \qquad (3.6)$$

where $0 \leq \alpha \leq 1$.

3.3.1 Consequences of Dual Degeneracy

The 2-D LP problem given in the above section is solved by the simplex method. This method is capable of determining solutions only at the two corner points C and D. The two optimal simplex tableaux corresponding to the corner points C and D are given in Table 3.5 and Table 3.6.

Table 3.5: Optimal Solution #1 for the 2-D Dual Degenerate LP

Basis	x_1	x_2	s_1	s_2	RHS
z	0	0	2	0	10
x_2	1/2	1	1/2	0	5/2
s_2	1/2	0	-1/2	1	3/2

Table 3.6: Optimal Solution #2 for the 2-D Dual Degenerate LP

Basis	x_1	x_2	s_1	s_2	RHS
z	0	0	2	0	10
x_2	0	1	1	-1	1
x_1	1	0	-1	2	3

Mathematically, a dual degenerate LP is identified by the presence of dual variables having zero values in the optimum. In this case, the presence of a zero reduced cost for x_1 in Table 3.5 and a zero shadow price for s_2 in Table 3.6 are indications that the problem is dual degenerate.

Also, as evident from both the optimal tableaux in Table 3.5 and Table 3.6, the problem has an unique objective function value and unique dual solution, but multiple (non-unique) primal solutions (activity values) and multiple (non-unique) incremental effect coefficients. This is a defining characteristic of an LP problem when it is dual degenerate.

The existence of multiple activity values and multiple incremental effect coefficients creates confusion in choosing a specific solution for implementation in the actual process. This effect subsequently causes complications in decision making. A methodology to eliminate ambiguity and mistakes when interpreting a dual degenerate LP is presented in Chapter 5.

CHAPTER 4

CHARACTERIZATION OF LP RESULTS UNDER CONDITIONS OF PRIMAL DEGENERACY

Primal degenerate LP often produces multiple optimal dual values and incremental effect coefficients with unique objective function value and unique activity values. Determination of true shadow price values among multiple dual values under conditions of primal degeneracy is well established in literature. However, practitioners in the field of petroleum refinery optimization are not fully aware of the consequences of primal degeneracy. This chapter illustrates the results and procedures for a primal degenerate refinery LP. The Refinery LP presented in Section 2.2 is selected for case study. The original LP problem is not primal degenerate. Therefore, the LP is converted to a primal degenerate problem by changing the Fluid Catalytic Cracking (FCC) unit capacity from 30,000 to 21,055 bbl/day.

4.1 Check for Primal Degeneracy

The modified LP is solved using LINDO, and an optimal solution is found. The optimal basis, the corresponding basis index, and activity values are given in Table 4.1. Inspection of Table 4.1 reveals that 22 of the 58 basic variables have zero values. Among the 22 variables, 21 are the slacks associated with material balance equality constraints. If only these 21 variables have zero values, primal degeneracy is caused only by a specific representation of the problem (Bertsimas and Tsitsiklis, 1997). Because, as demonstrated from Section 2.2.1, before solving the refinery LP, the material balance equality constraints of the form given in Equation (4.1) are converted to the companion form given in Equation (4.2).

Table 4.1: Optimal Basis #1 for the Primal Degenerate Refinery LP

Index	Variable	Activity (bbl/day)	Index	Variable	Activity (bbl/day)	Index	Variable	Activity (bbl/day)
1	CRUDE	$100,000.00$	27	SRG	$27,000.00$	54	SLK21	$20,200.00$
2	FGAD*	$3,542,000.00$	28	SRN	$23,700.00$	56	SLK23	0.00
3	SRNRF	$12,198.57$	29	SRDS	$8,700.00$	58	SLK25	0.00
4	FGRF*	$1,935,913.41$	30	SRFO	$37,200.00$	60	SLK27	0.00
6	SRFOCC	$21,055.00$	31	RFG	$11,320.28$	62	SLK29	0.00
7	FGCC*	$8,135,652.00$	32	CCG	$14,485.84$	64	SLK31	0.00
8	PG	$9,999.97$	33	CCFO	$4,625.78$	66	SLK33	$12,801.43$
9	RG	$42,806.15$	34	SLK1	$10,000.00$	67	SLK34	0.00
10	DF	$30,972.21$	35	SLK2	0.00	69	SLK36	0.00
11	FO	$10,000.00$	36	SLK3	0.00	72	SLK39	0.00
12	SRGPG	$4,313.71$	39	SLK6	$33,013.62$	74	SLK41	0.00
13	RFGPG	$5,686.26$	40	SLK7	$32,806.15$	76	SLK43	0.00
16	SRGRG	$22,686.29$	41	SLK8	0.00	78	SLK45	0.00
17	RFGRG	$5,634.02$	44	SLK11	$11,778.64$	80	SLK47	0.00
19	CCGRG	$14,485.84$	45	SLK12	$20,972.21$	82	SLK49	0.00
20	SRNDF	$11,501.43$	46	SLK13	0.00	84	SLK51	0.00
21	CCFODF	$4,625.78$	48	SLK15	$634,102.63$	86	SLK53	0.00
22	SRDSDF	$8,700.00$	51	SLK18	0.00	88	SLK55	0.00
23	SRFODF	$6,145.00$	53	SLK20	$570,000.00$	90	SLK57	0.00
26	SRFOFO	$10,000.00$						

$^*ft^3/day$.

$$Ax = b \tag{4.1}$$

$$Ax \leq b \tag{4.2}$$
$$-Ax \leq -b$$

Therefore, for each of the equality constraints, one of the two inequalities will be in the solution basis and the other will be non-basic. Consequently, every optimal solution will always contain at least 21 variables with primal values (activities) of zero. In order for the LP to be truly primal degenerate, one or more basic variables not associated with an equality constraint must have a primal value of zero. In this case the basic variable SLK4, which is the slack variable associated with PG production constraint, has zero value. Therefore, the LP is primal degenerate.

4.2 Analyzing Single Optimal Solution

In current petroleum refinery optimization practice, only a single optimal solution is generated. This section provides a systematic approach to categorize dual values and primal incremental effect coefficient of a single optimal solution. The task involved in the categorization strategy is two-fold: one is to characterize dual values as unique or non-unique, and the other is to categorize dual values as p^+ shadow price, p^- shadow price, or $p^{invalid}$ shadow price. The corresponding primal incremental effect coefficients will be categorized as $\overline{a}_{ij}^+$, $\overline{a}_{ij}^-$, and $\overline{a}_{ij}^{invalid}$.

The primal incremental analysis described in Section 2.1.1 will be applied for this classification strategy.

The dual value y_i of an active constraint $\sum_{j=1}^{n} a_{ij}x_j \leq b_i$ is called the p^+ shadow price if on positively perturbing the R.H.S b_i of this active constraint by a small amount, δ yields a primal feasible solution. Alternatively, the change in activity values for smaller perturbation δ given by Equation (4.3) yields a primal feasible solution, implying that all the entries in $x_{B,new}$ are positive. The primal incremental effect coefficients corresponding to this shadow price are referred to as: $\overline{a}_{ij}^{+}$.

$$x_{B,new} = \begin{pmatrix} \bar{b}_1 \\ \vdots \\ \bar{b}_m \end{pmatrix} + \delta \begin{pmatrix} \overline{a}_{i,n+1} \\ \vdots \\ \overline{a}_{m,n+1} \end{pmatrix} = \begin{pmatrix} \bar{b}_{1,new} \\ \vdots \\ \bar{b}_{m,new} \end{pmatrix} \tag{4.3}$$

The dual value y_i of an active constraint $\sum_{j=1}^{n} a_{ij}x_j \leq b_i$ is called the p^- shadow price if on negatively perturbing the R.H.S b_i of this active constraint by a small amount, δ yields a primal feasible solution. Alternatively, the change in activity values for smaller perturbation δ given by Equation (4.4) yields a primal feasible solution, implying that all the entries in $x_{B,new}$ are positive. The primal incremental effect coefficients corresponding to this shadow price are referred to as: $\overline{a}_{ij}^{-}$.

$$x_{B,new} = \begin{pmatrix} \bar{b}_1 \\ \vdots \\ \bar{b}_m \end{pmatrix} - \delta \begin{pmatrix} \overline{a}_{i,n+1} \\ \vdots \\ \overline{a}_{m,n+1} \end{pmatrix} = \begin{pmatrix} \bar{b}_{1,new} \\ \vdots \\ \bar{b}_{m,new} \end{pmatrix} \tag{4.4}$$

The dual value y_i of an active constraint $\sum_{j=1}^{n} a_{ij}x_j \leq b_i$ is called the $p^{invalid}$ shadow price if both the operations in Equation (4.3) and Equation (4.4) yield primal infeasible solutions, implying that at least one entry in $x_{B,new}$ is negative. The primal incremental effect coefficients corresponding to this shadow price are referred to as: $\overline{a}_{ij}^{invalid}$.

Also from the above primal incremental effect analysis results, the dual value y_i is called unique if $p^+ = p^-$; otherwise, it is considered to be non-unique. For each of the non-unique dual values y_i, the unavailable p^+ or p^- can only be determined by generating alternate optimal solutions.

Now the above approach will be implemented for the single optimal solution determined in Table 4.1. A value of δ = 1 will be used in Equation (4.3) and Equation (4.4) for the purpose of demonstration. The dual values corresponding to this single optimal solution are given in Table 4.2.

Table 4.2: Optimal Dual Values #1 for the Primal Degenerate Refinery LP

Z	Constraint	$\$/bbl$	Active at
SLK2	Premium gasoline production	-1.94	LL
SLK3	Premium gasoline blending	-8.77	LL (Equality)
SLK5	Premium gasoline octane rating	-0.60	LL
SLK8	Regular gasoline blend	-8.77	LL (Equality)
SLK10	Regular gasoline octane rating	-0.60	LL
SLK14	Diesel fuel blending	-64.16	LL (Equality)
SLK16	Diesel fuel sulfur specification	47.67	UL
SLK17	Fuel oil production	-4.30	LL (Equality)
SLK19	Fuel oil blending	-17.44	LL (Equality)
SLK24	Fuel gas yield from atmospheric distillation unit	-0.02	LL (Equality)
SLK26	Straight run gasoline yield from atmospheric distillation unit	-38.56	LL (Equality)
SLK28	Straight run naphtha yield from atmospheric distillation unit	-50.67	LL (Equality)
SLK30	Straight run distillate yield from atmospheric distillation unit	-39.08	LL (Equality)
SLK32	Straight run fuel oil yield from atmospheric distillation unit	-17.44	LL (Equality)
SLK35	Fuel gas yield from reformer unit	-0.02	LL (Equality)
SLK37	Reformed gasoline yield	-53.93	LL (Equality)
SLK38	Catalytic cracking (FCC) unit capacity	31.19	UL
SLK40	Fuel gas yield from catalytic cracking unit	-0.02	LL (Equality)
SLK42	Catalytic cracked gasoline yield	-47.72	LL (Equality)
SLK44	Catalytic cracked fuel oil yield	-47.33	LL (Equality)
SLK46	Straight run gasoline split	-38.56	LL (Equality)
SLK48	Straight run naphtha split	-50.67	LL (Equality)
SLK50	Straight run distillate split	-39.08	LL (Equality)
SLK52	Straight run fuel oil split	-17.44	LL (Equality)
SLK54	Reformed gasoline split	-53.93	LL (Equality)
SLK56	Catalytic cracked gasoline split	-47.72	LL (Equality)
SLK58	Catalytic cracked fuel oil split	-47.33	LL (Equality)

LL $\rightarrow$ Lower Limit: implying the GE constraint is active.

UL $\rightarrow$ Upper Limit: implying the LE constraint is active.

To categorize dual values, primal incremental analysis is performed for one of the dual values listed in Table 4.2. The dual value for the Fluid Catalytic Cracking (FCC) capacity constraint is selected for analysis. The FCC capacity is a less than or equal to (LE) constraint with a dual value of 31.19 $/bbl.

In this case, for the refinery LP, there are 58 activities in the optimal basis, which is obvious in Table 4.1. Equivalently, there will be 58 elements in the column vector of primal incremental effect coefficients under the FCC capacity dual value of 31.19 $/bbl in the optimal simplex tableau. The dual value along with the primal incremental effect coefficient data are listed in Table 4.3. Only the primal incremental effect coefficients having non-zero value in the optimal basis are included in Table 4.3.

The dual value of 31.19 $/bbl is first checked to determine if it is a p^+ shadow price. Therefore, the R.H.S of the FCC capacity constraint is positively incremented by +1 bbl from 20,155 to 20,156 bbl/day. Based on primal incremental effect analysis, the effect of this variation on the activity values are determined by adding the set of original activity values (primal values) with the incremental effect coefficient values. The results are listed under +1 FCC capacity increase in Table 4.3. For the positive FCC capacity increment, all the activities in Table 4.3 except the slack associated with the atmospheric distillation capacity remained positive. As seen in Table 4.3, the distillation slack activity (SLK22) changed from 0 bbl/day to -3.58 bbl/day. This indicates that the new solution is primal infeasible as negative flow rates are physically unachievable. Consequently, the dual value 31.19$/bbl for the FCC capacity constraint is not a p^+ shadow price.

Table 4.3: Primal Incremental Effect Analysis for the LE constraint†

		FCC capacity	Original solution	+1 FCC capacity increase	-1 FCC capacity decrease
	Z	$31.19 \frac{\$}{bbl}$	$\$594,259.67$	$\$594,290.86$	$\$594,228.49$
Index	Solution basis	Incremental effect ($\overline{a}_{ij}$)	Activity bbl/day	Activity bbl/day	Activity bbl/day
1	CRUDE	3.58	100,000.00	100,003.58*	99996.42
2	FGAD[1]	126.69	354,1999.41	354,2126.09	354,1872.72
3	SRNRF	0.23	12,198.58	12,198.81	12,198.35
4	FGRF[1]	36.20	1,935,915.00	1,935,951.20	1,935,878.80
6	SRFOCC	1.00	21,055.00	21,056.00	21,054.00
7	FGCC[1]	386.40	8,135,652.00	8,136,038.40	8,135,265.60
9	RG	1.87	42,806.12	42,807.99	42,804.25
10	DF	1.48	30,972.19	30,973.67	30,970.71
16	SRGRG	0.97	22,686.27	22,687.24	22,685.30
17	RFGRG	0.21	5,634.01	5,634.22	5,633.80
19	CCGRG	0.69	14,485.84	14,486.53	14,485.15
20	SRNDF	0.62	11,501.41	11,502.03	11,500.79
21	CCFODF	0.22	4,625.78	4,626.00	4,625.56
22	SRDSDF	0.31	8,700.00	8,700.31	8,699.69
23	SRFODF	0.33	6,144.99	6,145.32	6,144.66
27	SRG	0.97	27,000.00	27,000.96	26,999.03
28	SRN	0.85	23,700.00	23,700.84	23,699.15
29	SRDS	0.31	8,700.00	8,700.31	8,699.69
30	SRFO	1.33	37,199.99	37,201.32	37,198.66
31	RFG	0.21	11,320.28	11,320.50	11,320.07
32	CCG	0.69	14,485.84	14,486.53	14,485.15
33	CCFO	0.22	4,625.78	4,626.00	4,625.56
34	SLK1	−3.58	10,000.00	9,996.42	10,003.58
40	SLK7	1.87	32,806.12	32,807.99	32,804.25
44	SLK11	0.63	11,778.65	11,779.28	11,778.02
45	SLK12	1.48	20,972.19	20,973.67	20,970.71
48	SLK15	31.61	634,102.07	634,133.67	634,070.46
55	SLK22	−3.58	0.00	−3.58*	3.58
66	SLK33	−0.23	12,801.42	12,801.19	12,801.65

*Not physically realizable.

†Basic variables with primal incremental effect coefficients of zero values have been omitted.

[1] ft^3/day.

In a petroleum refinery LP, most of the activities will be in terms of flow rate (bbl/day) and therefore the activities should always be a positive quantity for it to be physically realizable or to be physically implementable in the process. Therefore, the -3.58 bbl/day distillation slack activity will not be physically realized or cannot be physically implemented in the actual process. Furthermore, this negative slack -3.58 bbl/day demands a crude distillation capacity of 100,003.58 bbl/day, which is 3.38 bbl/day more than the 100,000 bbl/day of crude distillation capacity physically available in the process. Consequently, the dual value 31.19$/bbl is not physically realizable for a positive increase in FCC capacity.

Based on the above argument, in this research the term "true shadow price" will be replaced with "physically realizable shadow price" and the term "primal feasibility" will be replaced with "physically realizable activities". These are more explicit and improve comprehension by engineers who use LP results but are not familiar with mathematical LP nomenclature.

Now, the FCC dual value of 31.19$/bbl is evaluated to determine whether it is a p^- shadow price. The FCC capacity constraint is negatively decremented from 20,155 to 20,154 bbl/day and the primal incremental effect analysis is repeated by subtracting the incremental effect coefficient from the original activity (primal value). Results are shown in the last column of Table 4.3. All activities remained positive, which implies the resultant activities are primal feasible. Thus, the dual value 31.19$/bbl is physically realizable for a negative perturbation of the FCC constraint and referred to as p^- shadow price. Since the dual value 31.19$/bbl for the FCC constraint is a p^- and not p^+, this implies $p^+ \neq p^-$. Consequently, the dual value of FCC constraint is not unique.

Similar primal incremental effect analysis is performed for all the dual values listed in Table 4.2 and the resultant categorization is presented in Table 4.4.

In order to determine the missing p^+ or p^- shadow price of constraints in Table 4.4, alternate optimal solutions have to be generated.

Table 4.4: Classification of Dual Values for the Primal Degenerate Refinery LP Obtained from the Single Optimal Solution

Constraint	Dual Value $\$/bbl$	Category I	Category II
Premium gasoline production	-1.94	p^-	Non-unique
Premium gasoline blending	-8.77	p^-	Non-unique
Premium gasoline octane rating	-0.60	p^-	Non-unique
Regular gasoline blend	-8.77	p^-	Non-unique
Regular gasoline octane rating	-0.60	p^-	Non-unique
Diesel fuel blending	-64.16	p^+	Non-unique
Diesel fuel sulfur specification	47.67	p^-	Non-unique
Fuel oil production	-4.30	p^+	Non-unique
Fuel oil blending	-17.44	p^-	Non-unique
Fuel gas yield from atmospheric distillation unit	-0.02	$p^+ = p^-$	Unique
Straight run gasoline yield from atmospheric distillation unit	-38.56	p^-	Non-unique
Straight run naphtha yield from atmospheric distillation unit	-50.67	p^+	Non-unique
Straight run distillate yield from atmospheric distillation unit	-39.08	p^-	Non-unique
Straight run fuel oil yield from atmospheric distillation unit	-17.44	p^-	Non-unique
Fuel gas yield from reformer unit	-0.02	$p^+ = p^-$	Unique
Reformed gasoline yield	-53.93	p^+	Non-unique
Catalytic cracking unit capacity	31.19	p^+	Non-unique
Fuel gas yield from catalytic cracking unit	-0.02	$p^+ = p^-$	Unique
Catalytic cracked gasoline yield	-47.72	p^+	Non-unique
Catalytic cracked fuel oil yield	-47.33	p^+	Non-unique
Straight run gasoline split	-38.56	p^-	Non-unique
Straight run naphtha split	-50.67	p^+	Non-unique
Straight run distillate split	-39.08	p^-	Non-unique
Straight run fuel oil split	-17.44	p^-	Non-unique
Reformed gasoline split	-53.93	p^+	Non-unique
Catalytic cracked gasoline split	-47.72	p^+	Non-unique
Catalytic cracked fuel oil split	-47.33	p^+	Non-unique

4.3 Determining Alternate Optimal Solutions

A new perturbation technique implementing parametric programming is developed at Oklahoma State University (OSU) to determine alternate optimal solutions. Initially, the step-by-step procedure of this algorithm will be explained. Then this algorithm will be implemented to the primal degenerate refinery LP to generate alternate optimal solutions.

4.3.1 New Perturbation Technique Implementing Parametric Programming

This algorithm is developed based on the suggestions given by Akgul (1984). When the LP is primal degenerate, the optimal basis is geometrically characterized by a unique vertex. However, more than n constraints pass through the optimum, where n is the dimension of the problem. From this geometric visualization (Figure 3.2), it is apparent that the problem will have a unique primal solution; conversely, it has alternate dual solutions. The rationale for using the parametric programming approach for determining alternate optimal basis corresponding to a primal degenerate vertex is described in Section A.1 of Appendix A

The algorithm exploits the fact that if all constraints passing through the optimum are parametrically varied one by one, all alternate bases corresponding to the primal degenerate vertex can be generated with unique primal solutions and alternate dual solutions. This algorithm can be applied for any single arbitrary optimal solution obtained from an LP solver.

The steps in the algorithm follow:

Step 1 The set of all active constraints in an optimal solution are determined. Active constraints are those constraint whose slack or surplus is maintained at zero value in the optimal solution. The definition of an active constraint in mathematical notation is given as follows:

A constraint in an LP problem in general form is written as given in Equation (4.5).

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i \tag{4.5}$$

After adding a slack s_i, the above constraint can be written in the standard form as given in Equation (4.6).

$$\sum_{j=1}^{n} a_{ij}x_j + s_i = b_i \tag{4.6}$$

The constraint, $\sum_{j=1}^{n} a_{ij}x_j \leq b$, is active if $s_i = 0$ in the optimal solution.

Step 2 All the active constraints are parametrically perturbed one at a time using parametric programming. This perturbation technique is explained as follows: if b_i corresponds to an active constraint, the R.H.S is parametrically varied as $b_i + \lambda d$ using parametric programming, where λ is the parameter and d is the directional vector. The parametric variation will generate alternate optimal basis corresponding to the primal degenerate optimal vertex. The alternate optimal basis obtained by parametrically varying this constraint is listed as $\{B_{1,1} \cdots B_{1,q}\}$, where the index $1, q$ is the number of alternate basis obtained varying constraint number one.

Step 3 Similar perturbation using parametric programming is performed for all other active constraints. After doing this, the possible alternate basis obtained by this process is listed as:$\{B_{1,1} \cdots B_{1,q}, B_{2,1} \cdots B_{2,q}, \cdots, \cdots B_{N,1} \cdots B_{N,q}\}$, where N is the number of active constraints.

Step 4 The different set of basis obtained in step 3 is compared to each other and the unique basis among them are determined and listed as: $\{B_1 \cdots B_r\}$, where r is the number of distinct basis.

Step 5 The unique set of alternate basis obtained in step 4 is used to create the set of multiple optimal simplex tableaux corresponding to the primal degenerate vertex as given in Table 4.5

Table 4.5: Optimal Tableau, Inverse Matrix Format

$c^T - c_B B^{-1}1A$	$-c_B B^{-1}$	$c_B B^{-1}b$
$B^{-1}A$	B^{-1}	$B^{-1}b$

where c is the cost coefficient of decision variables, c_B is the cost coefficient of basic variables, B is the optimal basis matrix, and A is the $m \times n$ matrix.

Step 6 Using the alternate dual solutions obtained in step 5, the p^+ shadow price of a constraint is determined as given in Equation (4.7)

$$p^+ = min\left\{y_1 \cdots y_m\right\} \tag{4.7}$$

and the p^- shadow price of a constraint is determined as given in Equation (4.8).

$$p^- = max\left\{y_1 \cdots y_m\right\} \tag{4.8}$$

The proof for the above development is given in Aucamp (1984) and the above claim is valid only if alternate optimal solutions including p^+ and p^- are produced. The most reliable method to verify whether a dual value is a p^+ or p^- is by primal incremental analysis approach discussed in Section 4.2. Furthermore the primal incremental effect coefficients associated with p^+ and p^- are identified as: $\overline{a}_{ij}^+$ and $\overline{a}_{ij}^-$.

Demonstration of this algorithm for a 2-D primal degenerate LP is given in Appendix A.

4.3.2 Implementation

Parametric perturbation technique is applied to the primal degenerate refinery LP. Results showed that in addition to the optimal bases given in Table 4.1, three other optimal bases are attainable. The multiple optimal bases are given in Table 4.6, Table 4.8, and Table 4.10. Their corresponding dual values are listed in Table 4.7, Table 4.9, and Table 4.11.

Table 4.6: Optimal Basis #2 for the Primal Degenerate Refinery LP

Index	Variable	Activity (bbl/day)	Index	Variable	Activity (bbl/day)	Index	Variable	Activity (bbl/day)
1	CRUDE	$100,000.00$	27	SRG	$27,000.00$	54	SLK21	$20,200.00$
2	FGAD*	$3,542,000.00$	28	SRN	$23,700.00$	56	SLK23	0.00
3	SRNRF	$12,198.57$	29	SRDS	$8,700.00$	58	SLK25	0.00
4	FGRF*	$1,935,913.41$	30	SRFO	$37,200.00$	60	SLK27	0.00
6	SRFOCC	21055.00	31	RFG	$11,320.28$	62	SLK29	0.00
7	FGCC*	$8,135,652.00$	32	CCG	$14,485.84$	64	SLK31	0.00
8	PG	$9,999.97$	33	CCFO	$4,625.78$	66	SLK33	$12,801.43$
9	RG	$42,806.15$	34	SLK1	$10,000.00$	67	SLK34	0.00
10	DF	$30,972.21$	35	SLK2	0.00	69	SLK36	0.00
11	FO	$10,000.00$	36	SLK3	0.00	72	SLK39	0.00
12	SRGPG	$4,313.71$	39	SLK6	$33,013.62$	74	SLK41	0.00
13	RFGPG	$5,686.26$	40	SLK7	$32,806.15$	76	SLK43	0.00
16	SRGRG	$22,686.29$	41	SLK8	0.00	78	SLK45	0.00
17	RFGRG	$5,634.02$	44	SLK11	$11,778.64$	80	SLK47	0.00
19	CCGRG	$14,485.84$	45	SLK12	$20,972.21$	82	SLK49	0.00
20	SRNDF	$11,501.43$	46	SLK13	0.00	84	SLK51	0.00
21	CCFODF	$4,625.78$	48	SLK15	$634,102.63$	86	SLK53	0.00
22	SRDSDF	$8,700.00$	51	SLK18	0.00	88	SLK55	0.00
23	SRFODF	$6,145.00$	53	SLK20	$570,000.00$	90	SLK57	0.00
26	SRFOFO	$10,000.00$						

$^{*}ft^3/day$.

Table 4.7: Optimal Dual Values #2 for the Primal Degenerate Refinery LP

Z	Constraint	$\$/bbl$	Active at
SLK4	Premium gasoline blending	-19.32	LL (Equality)
SLK5	Premium gasoline octane rating	-0.28	LL
SLK9	Regular gasoline blending	-19.32	LL (Equality)
SLK10	Regular gasoline octane rating	-0.28	LL
SLK14	Diesel fuel blending	-52.42	LL (Equality)
SLK16	Diesel fuel sulfur specification	24.20	UL
SLK17	Fuel oil production	-15.57	LL (Equality)
SLK19	Fuel oil blending	-28.71	LL (Equality)
SLK22	Atmospheric distillation unit capacity	3.78	UL
SLK24	Fuel gas yield from atmospheric distillation unit	-0.02	LL (Equality)
SLK26	Straight run gasoline yield from atmospheric distillation unit	-41.30	LL (Equality)
SLK28	Straight run naphtha yield from atmospheric distillation unit	-45.57	LL (Equality)
SLK30	Straight run distillate yield from atmospheric distillation unit	-39.69	LL (Equality)
SLK32	Straight run fuel oil yield from atmospheric distillation unit	-28.71	LL (Equality)
SLK35	Fuel gas yield from reformer unit	-0.02	LL (Equality)
SLK37	Reformed gasoline yield	-48.44	LL (Equality)
SLK38	Catalytic cracking unit capacity	17.67	UL
SLK40	Fuel gas yield from catalytic cracking unit	-0.02	LL (Equality)
SLK42	Catalytic cracked gasoline yield	-45.56	LL (Equality)
SLK44	Catalytic cracked fuel oil yield	-43.88	LL (Equality)
SLK46	Straight run gasoline split	-41.30	LL (Equality)
SLK48	Straight run naphtha split	-45.57	LL (Equality)
SLK50	Straight run distillate split	-39.69	LL (Equality)
SLK52	Straight run fuel oil split	-28.71	LL (Equality)
SLK54	Reformed gasoline split	-48.44	LL (Equality)
SLK56	Catalytic cracked gasoline split	-45.56	LL (Equality)
SLK58	Catalytic cracked fuel oil split	-43.88	LL (Equality)

LL $\rightarrow$ Lower Limit: implying the GE constraint is active.

UL $\rightarrow$ Upper Limit: implying the LE constraint is active.

Table 4.8: Optimal Basis #3 for the Primal Degenerate Refinery LP

Index	Variable	Activity (bbl/day)	Index	Variable	Activity (bbl/day)	Index	Variable	Activity (bbl/day)
1	CRUDE	$100,000.00$	27	SRG	$27,000.00$	54	SLK21	$20,200.00$
2	FGAD*	$3,542,000.00$	28	SRN	$23,700.00$	56	SLK23	0.00
3	SRNRF	$12,198.57$	29	SRDS	$8,700.00$	58	SLK25	0.00
4	FGRF*	$1,935,913.41$	30	SRFO	$37,200.00$	60	SLK27	0.00
6	SRFOCC	$21,055.00$	31	RFG	$11,320.28$	62	SLK29	0.00
7	FGCC*	$8,135,652.00$	32	CCG	$14,485.84$	64	SLK31	0.00
8	PG	$10,000.00$	33	CCFO	$4,625.78$	66	SLK33	$12,801.43$
9	RG	$42,806.11$	34	SLK1	$10,000.00$	67	SLK34	0.00
10	DF	$30,972.21$	36	SLK3	0.00	69	SLK36	0.00
11	FO	$10,000.00$	37	SLK4	0.00	72	SLK39	0.00
12	SRGPG	$4,313.73$	39	SLK6	$33,013.57$	74	SLK41	0.00
13	RFGPG	$5,686.27$	40	SLK7	$32,806.11$	76	SLK43	0.00
16	SRGRG	$22,686.27$	41	SLK8	0.00	78	SLK45	0.00
17	RFGRG	$5,634.01$	44	SLK11	$11,778.66$	80	SLK47	0.00
19	CCGRG	$14,485.84$	45	SLK12	$20,972.21$	82	SLK49	0.00
20	SRNDF	$11,501.43$	46	SLK13	0.00	84	SLK51	0.00
21	CCFODF	$4,625.78$	48	SLK15	$634,102.63$	86	SLK53	0.00
22	SRDSDF	$8,700.00$	51	SLK18	0.00	88	SLK55	0.00
23	SRFODF	$6,145.00$	53	SLK20	$570,000.00$	90	SLK57	0.00
26	SRFOFO	$10,000.00$						

$^{*}ft^3/day.$

Table 4.9: Optimal Dual Values #3 for the Primal Degenerate Refinery LP

Z	Constraint	$\$/bbl$	Active at
SLK2	Premium gasoline production	-1.33	LL
SLK5	Premium gasoline octane rating	-0.50	LL
SLK9	Regular gasoline blending	0.00	LL (Equality)
SLK10	Regular gasoline octane rating	-0.50	LL
SLK14	Diesel fuel blending	-60.49	LL (Equality)
SLK16	Diesel fuel sulfur specification	40.34	UL
SLK17	Fuel oil production	-7.82	LL
SLK19	Fuel oil blending	-20.96	LL (Equality)
SLK22	Atmospheric distillation unit capacity	1.18	UL
SLK24	Fuel gas yield from atmospheric distillation unit	-0.02	LL (Equality)
SLK26	Straight run gasoline yield from atmospheric distillation unit	-39.41	LL (Equality)
SLK28	Straight run naphtha yield from atmospheric distillation unit	-49.07	LL (Equality)
SLK30	Straight run distillate yield from atmospheric distillation unit	-39.27	LL (Equality)
SLK32	Straight run fuel oil yield from atmospheric distillation unit	-20.96	LL (Equality)
SLK35	Fuel gas yield from reformer unit	-0.02	LL (Equality)
SLK37	Reformed gasoline yield	-52.22	LL (Equality)
SLK38	Catalytic cracking unit capacity	26.96	UL
SLK40	Fuel gas yield from catalytic cracking unit	-0.02	LL (Equality)
SLK42	Catalytic cracked gasoline yield	-47.04	LL (Equality)
SLK44	Catalytic cracked fuel oil yield	-46.25	LL (Equality)
SLK46	Straight run gasoline split	-39.41	LL (Equality)
SLK48	Straight run naphtha split	-49.07	LL (Equality)
SLK50	Straight run distillate split	-39.27	LL (Equality)
SLK52	Straight run fuel oil split	-20.96	LL (Equality)
SLK54	Reformed gasoline split	-52.22	LL (Equality)
SLK56	Catalytic cracked gasoline split	-47.04	LL (Equality)
SLK58	Catalytic cracked fuel oil split	-46.25	LL (Equality)

LL $\rightarrow$ Lower Limit: implying the GE constraint is active.

UL $\rightarrow$ Upper Limit: implying the LE constraint is active.

Table 4.10: Optimal Basis #4 for the Primal Degenerate Refinery LP

Index	Variable	Activity (bbl/day)	Index	Variable	Activity (bbl/day)	Index	Variable	Activity (bbl/day)
1	CRUDE	$100,000.00$	27	SRG	$27,000.00$	54	SLK21	$20,200.00$
2	FGAD*	$3,542,000.00$	28	SRN	$23,700.00$	56	SLK23	0.00
3	SRNRF	$12,198.57$	29	SRDS	$8,700.00$	58	SLK25	0.00
4	FGRF*	$1,935,913.41$	30	SRFO	$37,200.00$	60	SLK27	0.00
6	SRFOCC	$21,055.00$	31	RFG	$11,320.28$	62	SLK29	0.00
7	FGCC*	$8,135,652.00$	32	CCG	$14,485.84$	64	SLK31	0.00
8	PG	$10,000.00$	33	CCFO	$4,625.78$	66	SLK33	$12,801.43$
9	RG	$42,806.11$	34	SLK1	$10,000.00$	67	SLK34	0.00
10	DF	$30,972.21$	37	SLK4	0.00	69	SLK36	0.00
11	FO	$10,000.00$	39	SLK6	$33,013.73$	72	SLK39	0.00
12	SRGPG	$4,313.73$	40	SLK7	$32,806.11$	74	SLK41	0.00
13	RFGPG	$5,686.27$	41	SLK8	0.00	76	SLK43	0.00
16	SRGRG	$22,686.27$	42	SLK9	0.00	78	SLK45	0.00
17	RFGRG	$5,634.00$	44	SLK11	$11,778.50$	80	SLK47	0.00
19	CCGRG	$14,485.84$	45	SLK12	$20,972.21$	82	SLK49	0.00
20	SRNDF	$11,501.43$	46	SLK13	0.00	84	SLK51	0.00
21	CCFODF	$4,625.78$	48	SLK15	$634,102.63$	86	SLK53	0.00
22	SRDSDF	$8,700.00$	51	SLK18	0.00	88	SLK55	0.00
23	SRFODF	$6,145.00$	53	SLK20	$570,000.00$	90	SLK57	0.00
26	SRFOFO	$10,000.00$						

$^{*}ft^3/day$.

Table 4.11: Optimal Dual Values #4 for the Primal Degenerate Refinery LP

Z	Constraint	$\$/bbl$	Active at
SLK2	Premium gasoline production	-1.33	LL
SLK3	Premium gasoline blending	0.00	LL (Equality)
SLK5	Premium gasoline octane rating	-0.50	LL
SLK10	Regular gasoline octane rating	-0.50	LL
SLK14	Diesel fuel blending	-60.49	LL (Equality)
SLK16	Diesel fuel sulfur specification	40.34	UL
SLK17	Fuel oil production	-7.82	LL
SLK19	Fuel oil blending	-20.96	LL (Equality)
SLK22	Atmospheric distillation unit capacity	1.18	UL
SLK24	Fuel gas yield from atmospheric distillation unit	-0.02	LL (Equality)
SLK26	Straight run gasoline yield from atmospheric distillation unit	-39.41	LL (Equality)
SLK28	Straight run naphtha yield from atmospheric distillation unit	-49.07	LL (Equality)
SLK30	Straight run distillate yield from atmospheric distillation unit	-39.27	LL (Equality)
SLK32	Straight run fuel oil yield from atmospheric distillation unit	-20.96	LL (Equality)
SLK35	Fuel gas yield from reformer unit	-0.02	LL (Equality)
SLK37	Reformed gasoline yield	-52.22	LL (Equality)
SLK38	Catalytic cracking unit capacity	26.96	UL
SLK40	Fuel gas yield from catalytic cracking unit	-0.02	LL (Equality)
SLK42	Catalytic cracked gasoline yield	-47.04	LL (Equality)
SLK44	Catalytic cracked fuel oil yield	-46.25	LL (Equality)
SLK46	Straight run gasoline split	-39.41	LL (Equality)
SLK48	Straight run naphtha split	-49.07	LL (Equality)
SLK50	Straight run distillate split	-39.27	LL (Equality)
SLK52	Straight run fuel oil split	-20.96	LL (Equality)
SLK54	Reformed gasoline split	-52.22	LL (Equality)
SLK56	Catalytic cracked gasoline split	-47.04	LL (Equality)
SLK58	Catalytic cracked fuel oil split	-46.25	LL (Equality)

LL $\rightarrow$ Lower Limit: implying the GE constraint is active.

UL $\rightarrow$ Upper Limit: implying the LE constraint is active.

Once all the optimal bases are found, the p^+ shadow price of a constraint is determined as given in Equation (4.7) and the p^- shadow price of a constraint is determined as given in Equation (4.8). The dual values obtained for the Premium Gasoline (PG) production constraint is considered to demonstrate this procedure.

The dual value of PG production constraint corresponding to optimal basis #1 is given in Table 4.2 as: -1.94 $\$/bbl$. In case of optimal basis #2 (Table 4.6), the slack variable SLK2 associated with the PG constraint is in the basis and maintained at zero value. Therefore, the dual value is 0 $\$/bbl$. The dual value corresponding to both optimal basis #3 and optimal basis #4 is -1.33 $\$/bbl$ (Table 4.9 and Table 4.11). From these dual values, the p^+ shadow price of PG constraint is determined as given in Equation (4.9).

$$p^+ = min\{-1.94, 0, -1.33, -1.33\} = -1.94 \tag{4.9}$$

and the p^- shadow price of PG constraint is determined as given in Equation (4.10).

$$p^- = max\{-1.94, 0, -1.33, -1.33\} = 0 \tag{4.10}$$

The p^+ and p^- shadow price for all other constraints are determined for the refinery LP by completing a similar analysis, the results are tabulated in Table 4.12, and the corresponding primal incremental effect coefficients are determined as $\overline{a}_{ij}^+$ and $\overline{a}_{ij}^-$, respectively.

The reporting guidelines given by Ho (2000) are followed to generate Table 4.12. The p^+ and p^- shadow price is given in terms of rate of change of objective function when the right hand side (R.H.S) of the constraint is perturbed. For example, as evident from Table 4.12, the p^+ shadow price of PG production constraint is -1.94 $\$/bbl$, meaning that the objective function will decrease by \$1.94 when the R.H.S of the PG production constraint is increased by 1. Similarly, the p^- shadow price of PG production constraints is given as 0 $\$/bbl$, implying that the objective function will not change when the R.H.S of this constraint is decreased by 1.

Table 4.12: Physically Realizable Shadow Prices for the Primal Degenerate LP

Z	Constraints	p^+ (\$/bbl)	p^- \$/bbl)
SLK2	Premium gasoline production	-1.94	0.00
SLK3	Premium gasoline blend	-19.32	0.00
SLK5	Premium gasoline octane rating	-0.60	0.28
SLK8	Regular gasoline blend	-19.32	0.00
SLK10	Regular gasoline octane rating	-0.60	0.28
SLK14	Diesel fuel blend	-64.16	52.42
SLK16	Diesel fuel sulfur	24.20	-47.67
SLK17	Fuel oil production	-15.57	4.30
SLK19	Fuel oil blend	-28.71	17.44
SLK22	Distillation capacity	0.00	-3.78
SLK24	Fuel gas yield from atmospheric distillation unit	-0.02	0.02
SLK26	Straight run gasoline yield from atmospheric distillation unit	-41.30	38.56
SLK28	Straight run naphtha yield from atmospheric distillation unit	-50.67	45.57
SLK30	Straight run distillate yield from atmospheric distillation unit	-39.69	39.08
SLK32	Straight run fuel oil yield from atmospheric distillation unit	-28.71	17.44
SLK35	Fuel gas yield from reformer unit	-0.02	0.02
SLK37	Reformed gasoline yield	-53.93	48.44
SLK38	Catalytic cracking unit capacity	17.67	-31.19
SLK40	Fuel gas yield from catalytic cracking unit	-0.02	0.02
SLK42	Catalytic cracked gasoline yield	-47.72	45.56
SLK44	Catalytic cracked fuel oil yield	-47.33	43.88
SLK46	Straight run gasoline split	-41.30	38.56
SLK48	Straight run naphtha split	-50.67	45.57
SLK50	Straight run distillate split	-39.69	39.08
SLK52	Straight run fuel oil split	-28.71	17.44
SLK54	Reformed gasoline split	-53.93	48.44
SLK56	Catalytic cracked gasoline split	-47.72	45.56
SLK58	Catalytic cracked fuel oil split	-47.33	43.88

4.4 Summary

This chapter examined the condition of primal degeneracy for a refinery LP and implemented the primal incremental analysis approach developed by Aucamp and Steinberg (1982) to determine true shadow prices. In addition to determining true shadow prices, this study has extended the primal incremental effect analysis method to determine true incremental effect coefficients.

In current refinery practice only a single optima solution is produced. The user may not be aware that the LP is primal degenerate with multiple dual values. This study has utilized the primal incremental analysis approach to characterize dual values obtained from a single optimal solution as unique or non-unique dual value and p^{+}, p^{-} or $p^{invalid}$ shadow prices.

CHAPTER 5

CHARACTERIZATION OF LP RESULTS UNDER CONDITIONS OF DUAL DEGENERACY

The state of dual degeneracy in LP produces alternate optimal solutions with multiple activity values, unique dual values and unique objective function value. Unlike primal degeneracy, it appears that a definitive approach for choosing a specific solution among the multiple solutions is not developed so far. This work has developed a truly unique approach to distinguish the significance of implementing one solution to the other based on a business logic.

In this chapter some of the previous work related to dual degeneracy is introduced. Then, the novel methodologies developed in this work are presented along with the results obtained for the simplified refinery LP model.

5.1 Background

In this section, first, an overview of previous work related to dual degeneracy is discussed in detail. Finally, motivation for this research is stated based on the gaps found in literature.

5.1.1 Dual Degeneracy and Interpretation of LP Solution

Initial studies on this topic were done in the field of farm planning. According to Powell (1969) "linear programming is an advisory aid and may be used to generate some of the sub-optimal and alternate optimal solutions based on the significant preference expressed by the farmers. From the set of solutions, the farmer can select a farm plan which most satisfactorily corresponds to his real planning objectives". Therefore, based on the opinion given by Powell (1969), when alternate optima exist, any solution that meets the needs of the farmer can be implemented. Furthermore, Powell (1969) recommends to use even a suboptimal solution if that is the preference of the farmer.

The topic of alternative optima has gained prominence and has led to interesting debates since the publications of Paris (Paris, 1981, 1983, 1985). “For many years, LP users have regarded multiple solutions either as an exceptional event or as a nuisance to be avoided. Indeed, in many circles, multiple optimal solutions are a source of embarrassment and often the main goal of researchers is to define sufficient conditions for unique solutions” (Paris, 1985).

In a discussion provided by Paris (1991)[p.227], alternative optima is viewed favorably because the existence of multiple optimal solutions makes the final selection strategy a real problem of choice to be determined with criteria other than mathematical programming. Paris (1991) expressed that when an LP problem exhibits multiple optimal solutions, it means that the problem at hand provides potentially more flexible implementation options than a similar problem that exhibits unique optimal solutions.

An algorithmic approach on choosing among multiple optimal solutions is also given by Paris (1991)[p.229-223]. The procedure suggests solving the overall optimization problem in two stages. First, the optimal linear programming solution maximizing the primary objective should be sought. Second, if there are multiple solutions, then the extreme points of the solution space should be determined, and quadratic programming should be used to search for a unique linear combination of these extreme points that minimizes the sum of squares of deviations of optimal activity from the real world activity levels. This algorithm, proposed to determine a unique solution among multiple optimal solutions, is slightly modified by McCarl and Nelson (1983). The modified algorithm does not require the determination of all the extreme points corresponding to alternate optima.

Miller (1985) was not convinced by the methodologies developed by Paris (1991)[p.229-223] and McCarl and Nelson (1983). Because, when one moves away from basic optimal solutions, a number of the usual primal-dual properties are disturbed. This fact must be taken into account, when the optimal solution to the dual problem is also important to the analyst. Furthermore, in addition to the above methods, Drynan (1986) proposed that it may be best to solve an initial LP and if there are multiple optima, select the best in the second stage. Alternatively, it may be best to subjectively evaluate many near-optimal and optimal solutions. Finally, it may be best to solve a comprehensive LP, in which case the tradeoffs between goals need to be represented by a set of prespecified weights.

In summary, based on the above discussion, choosing among multiple optimal solutions is based on the planner's preference, which is purely based on experience and should be based on a specific logic.

5.1.2 Theoretical Studies on Dual Degeneracy

Apart from developing algorithms on interpreting multiple optimal solutions, several studies have been done to give more theoretical insight to the concept of dual degeneracy. Pioneering research in this field began by studying the uniqueness of solutions in linear programming problems (Mangasarian, 1979). A normal form of an optimal solution of an LP problem is defined and an algorithm is proposed to reduce the optimal solution to its normal form. This algorithm enables one to describe the optimal solution set dimension (Kantor, 1993).

In a significant development, Sierksma and Tijssen (2003) developed a relatively simple procedure to determine the dimension of the optimal solution set and degeneracy degree to add more insight into understanding dual degeneracy. The theorem developed to determine the dimension of the optimal solution set is given as follows: In a primal-dual pair of general LP-models with finite solution, the degeneracy degree of the primal (dual) optimal face is equal to the dimension of the dual (primal) optimal face.

In the study, given in Appa (2002), dual degeneracy does not always contribute to multiple solutions. This claim is also substantiated by providing a simple 2-D LP example.

The coefficients of objective function selected for this example 2-D LP have zero values. Therefore, this problem cannot be ideally considered as an optimization problem.

Based on the above survey, most of the studies provided methodologies to determine the dimension of the optimal face when the problem is dual degenerate.

5.1.3 Analysis of Needs and Gaps

In literature, the occurrence of multiple optimal solutions for an LP is viewed favorably, because this gives flexibility for the user to choose the desired solution. This phenomenon might be favorable for performing optimization for the base case where even smaller changes in market prices are not present. Petroleum refinery optimization is characterized by market price changes for feed stocks and finished products like gasoline, diesel and kerosene. In such cases, the existence of multiple solutions adds more confusion and choosing a specific optimal solution for implementation from the multiple solutions should remain optimal for smaller changes in market price. The wide array of literature discussing alternate optimal solutions for an LP under dual degeneracy did not derive enough emphasis on choosing specific solutions that maintain optimality for smaller changes in market price.

Algorithms to determine a single desired solution from the set of multiple solutions are given in McCarl and Nelson (1983) and Paris (1991). Concerns have been raised about these methodologies by Miller (1985), because the solution produced using this approach will not have the shadow price and incremental effect coefficient information. Concerns raised by Miller (1985) are also a concern for this research because this study is not only interested on the set of activities and objective function values, but also on incremental effect coefficients and shadow prices as well.

Market price fluctuation is a common phenomenon in petroleum refinery operations. Therefore, implementing a specific solution among alternate optimal solution should sustain optimal profit despite market price fluctuations. Sound economic justifications have to be established on choosing a specific basic solution when alternate optima are present to resolve this issue. Moreover, when a specific solution is selected for implementation, other information such as shadow price and incremental effect coefficient must be readily available for managerial interpretations.

5.2 Methodology

The state of dual degeneracy in LP produces alternate optimal solutions. A truly innovative approach is developed in this research to choose among multiple optimal solutions for implementation in actual petroleum refining processes. The principal logic underlying this novel approach begs the questions: what solution among the multiple solutions has to be implemented if the price of a commodity is going to increase in the market? or, what solution among the multiple optimal solutions has to be implemented if the price of a commodity is going to decrease? This novel solution approach can be used to analyze the multiple optimal solutions of a refinery LP.

The solution approach is based on a systematic classification process in which the activity values are classified into three classes. With reference to optimal simplex tableau given in Table 2.1, the activity value of a jth decision variable in the optimal basis is categorized as $c^+_{optimal}$, $c^-_{optimal}$, or $c_{suboptimal}$.

The dual incremental analysis given in Section 2.1.1 of Chapter 2 will be applied for this classification strategy.

The activity value $\overline{b}_i$ for the jth decision variable in the optimal basis is called $c^+_{optimal}$, if the operation in Equation (5.1) yields a dual feasible solution, implying that all the entries in $(\overline{c}_{new}\ \overline{y}_{new})$ are dual feasible. $c^+_{optimal}$ activity value leads to optimal objective function value when the price of this activity increases within a sensitivity range.

$$(\overline{c}_{new}\ \overline{y}_{new}) = \begin{pmatrix} \overline{c}_{m+1} \\ \vdots \\ \overline{c}_{n} \\ \overline{y}_{i} \\ \vdots \\ \overline{y}_{m} \end{pmatrix}^T + \delta \begin{pmatrix} \overline{a}_{i,m+1} \\ \vdots \\ \overline{a}_{i,n} \\ \overline{a}_{i,n+1} \\ \vdots \\ \overline{a}_{i,n+m} \end{pmatrix}^T = \begin{pmatrix} \overline{c}_{m+1,new} \\ \vdots \\ \overline{c}_{n,new} \\ \overline{y}_{i,new} \\ \vdots \\ \overline{y}_{m,new} \end{pmatrix}^T \tag{5.1}$$

The activity value $\overline{b}_i$ for the jth decision variable in the optimal basis is called the $c^{-}_{optimal}$, if the operation in Equation (5.2) yields a dual feasible solution, implying that all the entries in $(\overline{c}_{new}\ \overline{y}_{new})$ are dual feasible. $c^{-}_{optimal}$ activity value leads to optimal objective function value when the price of this activity decreases within a sensitivity range.

$$(\overline{c}_{new}\ \overline{y}_{new}) = \begin{pmatrix} \overline{c}_{m+1} \\ \vdots \\ \overline{c}_{n} \\ \overline{y}_{i} \\ \vdots \\ \overline{y}_{m} \end{pmatrix}^T - \delta \begin{pmatrix} \overline{a}_{i,m+1} \\ \vdots \\ \overline{a}_{i,n} \\ \overline{a}_{i,n+1} \\ \vdots \\ \overline{a}_{i,n+m} \end{pmatrix}^T = \begin{pmatrix} \overline{c}_{m+1,new} \\ \vdots \\ \overline{c}_{n,new} \\ \overline{y}_{i,new} \\ \vdots \\ \overline{y}_{m,new} \end{pmatrix}^T \tag{5.2}$$

The activity value $\overline{b}_i$ for the jth decision variable in the optimal basis is called $c_{suboptimal}$, if both the operation in Equation (5.1) and Equation (5.2) yields dual infeasible solution, implying that at least one entry in $(\overline{c}_{new}\ \overline{y}_{new})$ is dual infeasible. $c_{suboptimal}$ activity leads to non-optimal objective function value when the price of this activity either increases or decreases.

As an additional finding from the above dual incremental effect analysis, an activity value $\overline{b}_i$ for the jth decision variable in the optimal basis is considered to be unique if $c^{+}_{optimal} = c^{-}_{optimal}$. Otherwise it is considered to be non-unique. For each of the non-unique activity values, the unavailable $c^{+}_{optimal}$ or $c^{-}_{optimal}$ can be determined by generating alternate optimal solutions. Before introducing the algorithms to determine alternate optimal solutions, the dual feasibility condition of an LP problem will be discussed.

The foundation of the above dual incremental effect analysis methodology is based on the dual feasibility conditions of LP solution. The dual feasibility conditions has to be clearly understood in relation to petroleum refinery operations to appreciate the validity of this approach. The following section explains the dual feasibility conditions of LP solution in the context of petroleum refining process applications.

5.3 Dual Feasibility Condition

Refinery LP is a maximization LP. Therefore, this section deals only with the feasibility conditions of a maximization LP. Most of the LP texts present the dual feasibility conditions of a maximization LP as given in Table 5.1. As inferred from Table 5.1, for a maximization LP to be dual feasible, the shadow price of a less than or equal to (LE) constraint must be positive, the shadow price of a greater than or equal to (GE) constraint must be negative, shadow price of an equality constraint must be free, and the reduced cost (dual surplus) must be positive. This condition is universally accepted and is one of the requirements for the LP solution to be optimal.

Table 5.1: Dual Feasibility Conditions for a Maximization LP

Dual value	Sign of dual value
Shadow price of a LE constraint	Positive
Shadow price of a GE constraint	Negative
Shadow price of a equality constraint	Free
Reduced cost (Dual surplus)	Positive

Shadow price has units of ($\$/bbl$). This prompts questions such as: Why do the shadow price of LE constraints have to be positive? Why do the the shadow price of GE constraints have to be negative? Why do the the reduced costs have to be positive in the optimal solution? These questions are answered geometrically by Rardin (1997) and Winston (1991). According to them for a maximization LP, increasing the R.H.S of an active LE constraint is considered as relaxing the constraint, and this adds points to the feasible space or the size of the feasible region increases. Consequently, the objective function value increases. Thus, the shadow price associated with the LE constraint in a maximization LP is positive. Conversely, for a maximization LP, increasing the R.H.S of an active GE is considered as tightening the constraint and this removes points from the feasible space, or the size of the feasible region reduces. Consequently, the objective function value reduces. Thus, the shadow price associated with the GE constraint in a maximization LP is negative.

The above explanation is purely intuitive or geometric, and the understanding on dual feasibility in conjunction with petroleum refinery application is required. Although several books (Paris, 1991; Dorfman et al., 1958; Wagner, 1975; Geary and McCarthy, 1964) are completely devoted to the economic interpretation of LP solution, it seems that a comprehensive explanation of dual feasibility criteria in relation to the manufacturing industry is absent. This section extends the explanation provided by Rardin (1997) and Winston (1991) to understand the feasibility criteria of dual values applied to petroleum refining process.

5.3.1 Dual Feasibility of an LE Constraint

The atmospheric distillation capacity constraint of the refinery LP is selected to provide a process explanation for the sign convention associated with the dual value of an LE constraint. The distillation capacity constraint is given by Equation (5.3).

$$\text{CRUDE} \leq 100,000 \tag{5.3}$$

This capacity constraint stipulates that not more than 100, 000 bbl of distillation capacity is available in the petroleum refinery. The optimizer makes use of this capacity only if it is able to produce valuable products that could contribute to the profitability of a petroleum refinery. Otherwise, this resource will not be used in the process and it is left as a slack.

When this constraint becomes binding (constrained) in the process, the optimizer has exhausted the use of this valuable resource, contributing to the increase in the objective function value. As a result, increasing the R.H.S of this constrained LE constraint provides more of this valuable resource, and thus, increases the objective value. Therefore, for a maximization LP, the shadow price associated with an LE constraint is always positive.

Negative shadow price for an LE constraints implies that the objective function of a maximization problem can be decreased by increasing the R.H.S of an active LE constraint. This result is not economically viable and therefore dual infeasible.

5.3.2 Dual Feasibility of a GE constraint

The Premium Gasoline (PG) production constraint of the refinery LP is selected for demonstration. The PG production constraint is given in Equation (5.4).

$$\text{PG} \geq 10,000 \tag{5.4}$$

This constraint demands the optimizer to produce at least 10,000 bbl of premium gasoline in the refining process. The same constraint can also be viewed from a different perspective, if the PG production is profitable or contributes to the increase in the profit margin of the refinery. Based on the stipulation of the constraint in Equation (5.4) the optimizer has the liberty to produce more than 10,000 bbl of PG. Another crucial question is: What implication would it make when this PG production constraint (or this GE) becomes constrained?

When this constraint becomes constrained in the maximization LP, the optimizer determined that producing more than 10,000 bbl of PG was not profitable and is going to reduce the profit function (objective value). Consequently, the optimizer limits the production of PG to 10,000 bbl. Therefore, when the R.H.S of this active production constraint is increased, the objective value is definitely going to decrease.

Positive shadow price for a GE constraint implies that the objective function of a maximization problem can be increased by increasing the R.H.S of an active GE constraint. This result is not economically meaningful and is therefore dual infeasible.

5.3.3 Dual Feasibility of an Equality Constraint

The sign of shadow price of an equality constraint can be either positive or negative. An equality constraint of the form $Ax = b$ can be written in the companion form as: $Ax \leq b$ and $Ax \geq b$. In the optimal solution one of these constraints will be active based on the LP model. If the $Ax \leq b$ constraint is active then the shadow price will be positive. On the other hand, if the $Ax \geq b$ is active, then shadow price will be negative.

5.3.4 Dual Feasibility of Reduced Cost

Reduced cost is also referred to as dual surplus. A dual constraint is written in the form given in Equation (5.5).

$$\text{Imputed price} \geq \text{Market price} \tag{5.5}$$

The above constraint implies that if the manager decides to sell the resource available instead of manufacturing a certain product, the available resource has to be sold at a price at least equal to the market price of the product for the business to be profitable.

The constraint given in Equation (5.5) can be converted to an equality by adding a surplus. The modified version is given in Equation (5.6).

$$\text{Imputed price} - \text{surplus} = \text{Market price} \tag{5.6}$$

The constraint given in Equation (5.6) being inactive implies that the surplus will be non-zero and the product will have a reduced cost. This implies that this specific product will not be manufactured in the process because the optimizer determined that manufacturing this product will incur a loss. To profitably manufacture this product, the market price has to be increased by at least an amount equal to the surplus value.

Negative surplus implies that reducing the market price by the amount of the surplus will actually make the production of the specific product profitable. This result is not economically meaningful and is therefore dual infeasible.

In summary, this section provided a comprehensive description on interpreting the dual feasibility condition with respect to petroleum refinery process application. The next section will provide the algorithm to determine alternate optimal solutions when the LP is dual degenerate.

5.4 Algorithm

When an LP is dual degenerate, infinite number of activity values lead to the same objective function value. In this research, only the extreme point solutions will be analyzed for implementation. A perturbation technique implementing parametric programming is developed to determine alternate optimal solutions.

The perturbation technique developed in this section is similar to the algorithm developed in Section 4.3 of Chapter 4 to determine alternate optimal basis when the LP is primal degenerate.

A primal degenerate LP has a unique optimal vertex, this property enables utilization of the perturbation technique to generate multiple optimal bases for a primal degenerate LP. In case of dual degenerate LP, the dimension of the optimal face is larger than zero and will have multiple optimal vertices. This property makes it complicated to use the perturbation technique for the dual degenerate LP. However, a new strategy is developed in this research to deal with this complication.

When an LP has alternative optimal solutions or is dual degenerate, it will be primal degenerate in the dual space. This property of the dual degenerate LP problem is exploited in this research to implement the perturbation technique for generating multiple optimal solutions. This property is also illustrated graphically to gain more understanding. A 2-D dual degenerate LP given in Equation (5.7) is selected for this illustration.

$$\text{Maximize } z = 2x_1 + 4x_2 \tag{5.7}$$

Subject to

$$x_1 + 2x_2 \leq 5 \qquad \text{Constraint } \#1$$

$$x_1 + x_2 \leq 4 \qquad \text{Constraint } \#2$$

$$x_1, x_2 \geq 0 \qquad \text{Non-negativity}$$

The graphical solution of the 2-D LP is given in Figure 5.1(a). As noticed in Figure 5.1(a), the optimal face has two vertices, D and C. This one-dimensional optimal face can be converted to a single vertex by transforming this primal problem to a dual problem (Dantzig and Thapa, 2003). The dual form of the 2-D LP is given in Equation (5.8),

$$\text{Minimize } z = 5y_1 + 4y_2 \tag{5.8}$$

Subject to

$$y_1 + y_2 \geq 2 \qquad \text{Constraint } \#1$$

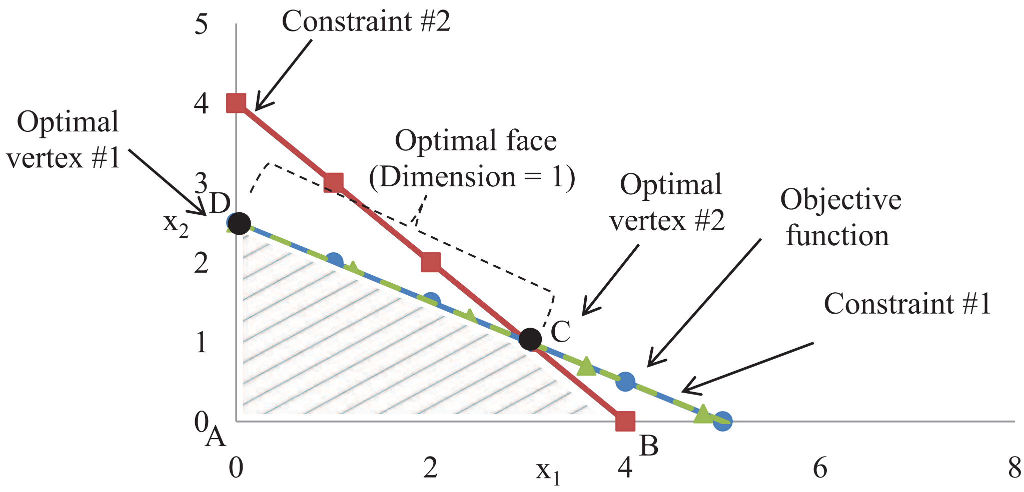
$$2y_1 + y_2 \geq 4 \qquad \text{Constraint } \#2$$

$$y_1, y_2 \geq 0 \qquad \text{Non-negativity}$$

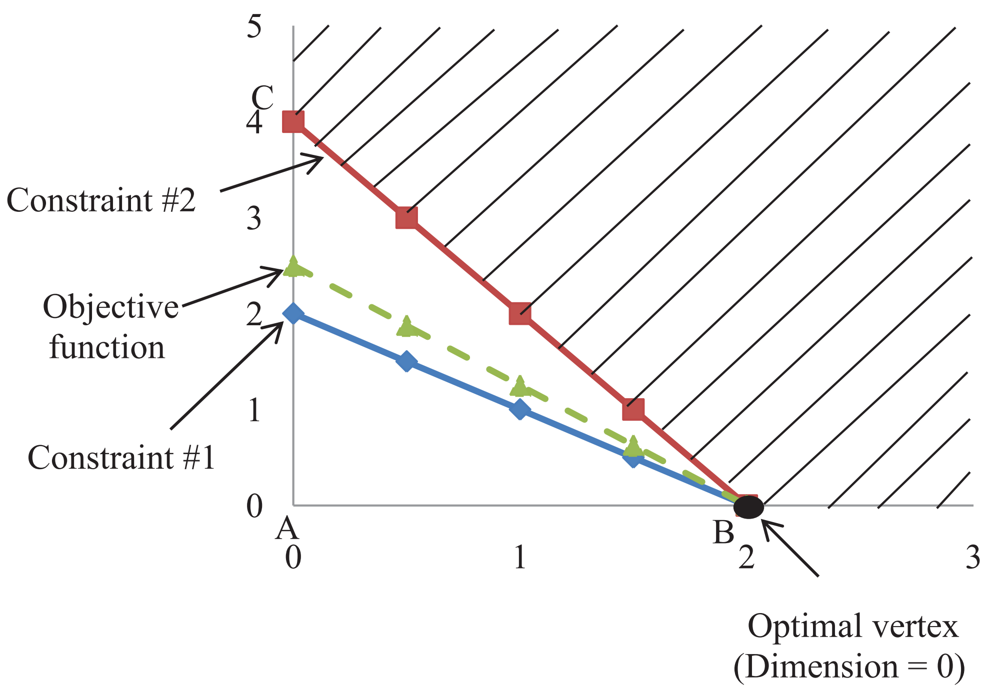
and the graphical solution of this dual LP is given in Figure 5.1(b). Visual observation indicated that the one-dimensional optimal face is converted to a unique vertex. This unique vertex gives the advantage of implementing the perturbation technique. The procedure followed in this algorithm is given as follows:

Step 1 The given primal LP is converted to a dual LP. This dual is solved to generate a single optimal solution. The dual of the refinery LP is given in Appendix B.

Step 2 The set of all active constraints in the optimal solution of the dual space are deter-



(a) Primal Space



(b) Dual Space

Figure 5.1: Geometry of Dual Degenerate LP in Primal and Dual Space

mined. Active constraints are those constraints whose slack or surplus are maintained at zero value in the optimal solution.

Step 3 All the active constraints in the dual space are parametrically perturbed one at a time using parametric programming. This perturbation technique is explained as follows: if $\sum_{i=1}^{m} y_i^T a_{ij} \leq c_j$ is one of the active constraints, the R.H.S, c_j of this active constraint is parametrically varied as $c_j + \lambda d$ using parametric programming, where λ is the parameter and d is the directional vector. The parametric variation will generate alternate optimal basis corresponding to the primal degenerate optimal vertex in the dual space. From the alternate optimal basis obtained in the dual space the corresponding basic variable in the primal basis can be easily determined, because the basic variables in the dual space will be non-basic variables in the primal space. The basic variables obtained by this procedure are listed as $\{B_{1,1} \cdots B_{1,q}\}$, where the index $1, q$ represents the number of alternate basis obtained by varying constraint number one.

Step 4 Similar perturbation using parametric programming for all other active constraints is performed. Subsequently, primal basic variables are determined using the dual basic variable information available. After doing this, the possible alternate basis obtained by this process is listed as $\{B_{1,1} \cdots B_{1,q}, B_{2,1} \cdots B_{2,q}, \cdots, \cdots B_{N,1} \cdots B_{N,q}\}$, where N is the number of active constraints.

Step 5 The set of different basis obtained in step 3 are compared to each other and the unique basis among them are determined and listed as: $\{B_1 \cdots B_r\}$, where r is the number of distinct basis.

Step 6 The unique set of alternate basis obtained in step 5 is used to create the set of optimal simplex tableaux corresponding to the primal degenerate vertex by applying the formulas given in Table 5.2.

Table 5.2: Optimal Tableau with Formula

$c^T - c_B B^-1A$	$-c_B B^{-1}$	$c_B B^{-1} b$
$B^{-1} A$	B^{-1}	$B^{-1} b$

5.5 Results and Discussion

This section illustrates the results and procedures for a dual degenerate LP. The refinery LP presented in Section 2.2.1 of Chapter 2 is selected for case study. The refinery LP is inherently dual degenerate and has multiple activity values for decision variables that have zero cost coefficients. In order to have multiple activity values for decision variables that also have non-zero cost coefficients, the LP problem is modified such that the cost coefficient of Premium Gasoline (PG) \$45.36 in the original LP is changed to \$44.0813.

5.5.1 Check for Dual Degeneracy

The LP problem is solved using LINDO and an optimal solution is found. The optimal basis, the corresponding basis index and activity values at optimum are given in Table 5.3. The associated dual values in the optimum are given in Table 5.4. Observation from Table 5.4 indicated that some of the dual values (non-basic variables) have zero values. This condition confirms that the LP is dual degenerate. The dual values that are maintained at zero are the shadow price of Regular Gasoline (RG) vapor pressure constraint, the shadow price of Diesel Fuel (DF) sulfur specification constraint, the reduced cost of Straight Run Naphtha for Premium Gasoline blending (SRNPG), the reduced cost of Straight Run Naphtha for Regular Gasoline blending (SRNRG), and the reduced cost of Catalytic Cracked Fuel Oil for Diesel Fuel blending (CCFODF).

Geometrically, when the LP is dual degenerate, the dimension of the primal optimal face will be larger than zero. For the refinery LP considered for this study, the dimension of the primal optimal face was found to be five, because there are five dual variables (non-basic) that have zero value in the optimal solution. This procedure for determining the dimension of the optimal face is given in Tijssen and Sierksma (1998) and Gonzaga (2007).

5.5.2 Analyzing Single Optimal Solution

In current petroleum refinery optimization practice, only a single optimal solution is generated. This section provides a systematic approach to categorize activity values for a single optimal solution. The dual incremental analysis discussed in Section 5.2 is implemented. The purpose of the categorization strategy is two-fold: one is to characterize activity values as unique or non-unique, and the other is to categorize activity values as $c^+_{optimal}$, or $c^-_{optimal}$, or $c_{suboptimal}$.

The dual incremental analysis is succinctly represented by Equation (5.1) and Equation (5.2) in Section 5.2. A value of δ = 1 will be used in Equation (5.1) and Equation (5.2) for demonstration purpose. Initially, the Regular Gasoline (RG) production activity value of 22,520 bbl/day in Table 5.3 is selected for classification.

The row of dual incremental effect coefficients corresponding to the RG activity value 22,520 bbl/day, excluding the identity structure of the optimal tableau, is presented in a transpose form (column format) in Table 5.5.

In the classification process, the cost coefficient \$43.68 for the RG decision variable is perturbed from \$43.68 to \$42.68. As observed in Table 5.5, all the dual values remained feasible. Therefore, this RG activity value is considered as $c^-_{optimal}$. On the other hand, when the cost coefficient is changed from \$43.68 to \$44.68, the resultant solution has some infeasible dual values. In this case the shadow price 0.10 \$/bb1 for RG octane and 0.10 \$/bbl for PG octane constraints are infeasible. Besides, the reduced costs -6.37 \$/bbl for Straight Run Naphtha for PG Blending (SRNPG) and -6.37 \$/bbl for Straight Run Naphtha for RG blending (SRNRG) are dual infeasible. Therefore this activity value is not $c^+_{optimal}$. This inference also implies $c^+_{optimal} \neq c^-_{optimal}$. Consequently, the activity value 22,520 bbl/day for RG production is non-unique.

Table 5.3: Optimal Basis #1 for the Dual Degenerate Refinery LP

Index	Variable	Activity (bbl/day)	Index	Variable	Activity (bbl/day)	Index	Variable	Activity (bbl/day)
1	CRUDE	$100,000.00$	26	SRFOFO	$5,403.80$	54	SLK21	$22,286.68$
2	FGAD*	$3,542,000.00$	27	SRG	$27,000.00$	56	SLK23	0.00
3	SRNRF	$23,700.00$	28	SRN	$23,700.00$	58	SLK25	0.00
4	FGRF*	$3,761,190.00$	29	SRDS	$8,700.00$	60	SLK27	0.00
6	SRFOCC	$30,000.00$	30	SRFO	$37,200.00$	62	SLK29	0.00
7	FGCC*	$11,592,000.00$	31	RFG	$21,993.60$	64	SLK31	0.00
8	PG	$47,113.20$	32	CCG	$20,640.00$	66	SLK33	$1,300.00$
9	RG	$22,520.40$	33	CCFO	$6,591.00$	67	SLK34	0.00
10	DF	$12,491.00$	34	SLK1	$10,000.00$	69	SLK36	0.00
11	FO	$10,000.00$	35	SLK2	$37,113.20$	72	SLK39	0.00
12	SRGPG	$13,852.05$	36	SLK3	0.00	74	SLK41	0.00
13	RFGPG	$17,239.99$	39	SLK6	$188,607.17$	76	SLK43	0.00
15	CCGPG	$16,021.17$	40	SLK7	$12,520.40$	78	SLK45	0.00
16	SRGRG	$13,147.95$	41	SLK8	0.00	80	SLK47	0.00
17	RFGRG	$4,753.61$	45	SLK12	$2,491.00$	82	SLK49	0.00
19	CCGRG	$4,618.83$	46	SLK13	0.00	84	SLK51	0.00
21	CCFODF	$6,591.00$	48	SLK15	$153,666.99$	86	SLK53	0.00
22	SRDSDF	$4,103.80$	51	SLK18	0.00	88	SLK55	0.00
23	SRFODF	$1,796.20$	53	SLK20	$583,788.61$	90	SLK57	0.00
25	SRDSFO	$4,596.20$						

$^*ft^3/day$.

Table 5.4: Optimal Dual Values #1 for the Dual Degenerate Refinery LP

Z	Dual Value	$\$/bbl$	Active at
SLK4	Premium Gasoline production	-37.86	LL
SLK5	Premium Gasoline octane rating	-0.07	LL
SLK9	Regular Gasoline blending	-37.86	LL (Equality)
SLK10	Regular Gasoline octane rating	-0.07	LL
SLK11	Regular Gasoline vapor	0.00	UL
SLK14	Diesel fuel blending	-40.32	LL (Equality)
SLK16	Diesel fuel sulfur specification	0.00	UL
SLK17	Fuel oil production	-27.18	LL
SLK19	Fuel oil blending	-40.32	LL (Equality)
SLK22	Atmospheric distillation unit capacity	7.85	UL
SLK24	Fuel gas yield from atmospheric distillation unit	-0.02	LL (Equality)
SLK26	Straight run gasoline yield from atmospheric distillation unit	-43.11	LL (Equality)
SLK28	Straight run naphtha yield from atmospheric distillation unit	-42.21	LL (Equality)
SLK30	Straight run distillate yield from atmospheric distillation unit	-40.32	LL (Equality)
SLK32	Straight run fuel oil yield from atmospheric distillation unit	-40.32	LL (Equality)
SLK35	Fuel gas yield from reformer unit	-0.02	LL (Equality)
SLK37	Reformed gasoline yield	-44.82	LL (Equality)
SLK38	Catalytic cracking unit capacity	4.29	UL
SLK40	Fuel gas yield from catalytic cracking unit	-0.02	LL (Equality)
SLK42	Catalytic cracked gasoline yield	-44.13	LL (Equality)
SLK44	Catalytic cracked fuel oil yield	-40.32	LL (Equality)
SLK46	Straight run gasoline split	-43.11	LL (Equality)
SLK48	Straight run naphtha split	-42.21	LL (Equality)
SLK50	Straight run distillate split	-40.32	LL (Equality)
SLK52	Straight run fuel oil split	-40.32	LL (Equality)
SLK54	Reformed gasoline split	-44.82	LL (Equality)
SLK56	Catalytic cracked gasoline split	-44.13	LL (Equality)
SLK58	Catalytic cracked fuel oil split	-40.32	LL (Equality)
SRDSCC	Straight run distillate for catalytic cracking	5.26	Reduced cost
SRNPG	Straight run naphtha for premium gasoline blending	0.00	Reduced cost
SRNRG	Straight run naphtha for regular gasoline blending	0.00	Reduced cost
SRNDF	Straight run naphtha for diesel fuel blending	1.89	Reduced cost
CCFOFO	Catalytic cracked fuel oil for fuel oil blending	0.00	Reduced cost

LL $\rightarrow$ Lower Limit: implying the GE constraint is active.

UL $\rightarrow$ Upper Limit: implying the LE constraint is active.

Table 5.5: Dual Incremental Effect Analysis for the RG Activity

			Base case	$c_j + 1$	$c_j - 1$
		RG activity (bbl)	Objective value ($)	Objective value ($)	Objective value ($)
		$22,520.40$	$641,579.78$	$664,100.18$	$619,059.38$
Z	Dual variable*	Incremental effect ($\overline{a}_{ij}$)	Dual value ($/bbl)	Dual value ($/bbl)	Dual value ($/bbl)
SLK5	PGoctane	0.17	-0.07	$0.10^{\dagger}$	-0.23
SLK9	RGblend(2)	-15.50	-37.86	-53.36	-22.36
SLK10	RGoctane	0.17	-0.07	$0.10^{\dagger}$	-0.23
SLK11	RGvapor	0.00	0.00	0.00	0.00
SLK14	DFblend(2)	0.00	-40.32	-40.32	-40.32
SLK16	DFsulfur	0.00	0.00	0.00	0.00
SLK17	FOproduction	0.00	-27.18	-27.18	-27.18
SLK19	FOblend(2)	0.00	-40.32	-40.32	-40.32
SLK22	ADcapacity	0.25	7.85	8.10	7.60
SLK24	FGADyield(2)	0.00	-0.02	-0.02	-0.02
SLK26	SRGyield(2)	-2.42	-43.11	-45.53	-40.69
SLK28	SRNyield(2)	1.70	-42.21	-40.51	-43.91
SLK30	SRDSyield(2)	0.00	-40.32	-40.32	-40.32
SLK32	SRFOyield(2)	0.00	-40.32	-40.32	-40.32
SLK35	FGRFyield(2)	0.00	-0.02	-0.02	-0.02
SLK37	RFGyield(2)	1.83	-44.82	-42.98	-46.65
SLK38	CCcapacity	0.08	4.29	4.37	4.21
SLK40	FGCCyield(2)	0.00	-0.02	-0.02	-0.02
SLK42	CCGyield(2)	0.12	-44.13	-44.01	-44.24
SLK44	CCFOyield(2)	0.00	-40.32	-40.32	-40.32
SLK46	SRGsplit(2)	-2.42	-43.11	-45.53	-40.69
SLK48	SRNsplit(2)	1.70	-42.21	-40.51	-43.91
SLK50	SRDSsplit(2)	0.00	-40.32	-40.32	-40.32
SLK52	SRFOsplit(2)	0.00	-40.32	-40.32	-40.32
SLK54	RFGsplit(2)	1.83	-44.82	-42.98	-46.65
SLK56	CCGsplit(2)	0.12	-44.13	-44.01	-44.24
SLK58	CCFOsplit(2)	0.00	-40.32	-40.32	-40.32
SRDSCC	Reduced cost	-0.01	5.26	5.25	5.26
SRNPG	Reduced cost	-6.37	0.00	$-6.37^{\dagger}$	6.37
SRNRG	Reduced cost	-6.37	0.00	$-6.37^{\dagger}$	6.37
SRNDF	Reduced cost	-1.70	1.89	0.19	3.59
CCFOFO	Reduced cost	0.00	0.00	0.00	0.00

†Dual infeasible.

*Description of dual variables are given in Table B.1 of Appendix B.

Similar analysis is performed for the activity values of all decision variables of the single optimal solution listed in Table 5.3 and the classification is summarized in Table 5.6.

Table 5.6: Classification of Activity Values for the Dual Degenerate Refinery LP Obtained from the Single Optimal Solution

Decision variable	Activity value (bbl/day)	Category I	Category II
CRUDE	$100,000.00$	$c^{+}_{optimal} = c^{-}_{optimal}$	Unique
FGAD (ft^3)	$3,542,000.00$	$c^{+}_{optimal} = c^{-}_{optimal}$	Unique
SRNRF	$23,700.00$	$c^{+}_{optimal}$	Non-unique
FGRF (ft^3)	$3,761,190.00$	$c^{+}_{optimal}$	Non-unique
SRFOCC	$30,000.00$	$c^{+}_{optimal} = c^{-}_{optimal}$	Unique
FGCC (ft^3)	$11,592,000.00$	$c^{+}_{optimal} = c^{-}_{optimal}$	Unique
PG	$47,113.20$	$c^{+}_{optimal}$	Non-unique
RG	$22,520.40$	$c^{-}_{optimal}$	Non-unique
DF	$12,491.00$	$c^{+}_{optimal} = c^{-}_{optimal}$	Unique
FO	$10,000.00$	$c^{+}_{optimal} = c^{-}_{optimal}$	Unique
SRGPG	$13,852.05$	$c_{suboptimal}$	Non-unique
RFGPG	$17,239.99$	$c_{suboptimal}$	Non-unique
CCGPG	$16,021.17$	$c_{suboptimal}$	Non-unique
SRGRG	$13,147.95$	$c_{suboptimal}$	Non-unique
RFGRG	$4,753.61$	$c_{suboptimal}$	Non-unique
CCGRG	$4,618.83$	$c_{suboptimal}$	Non-unique
CCFODF	$6,591.00$	$c^{+}_{optimal}$	Non-unique
SRDSDF	$4,103.80$	$c_{suboptimal}$	Non-unique
SRFODF	$1,796.20$	$c^{+}_{optimal}$	Non-unique
SRDSFO	$4,596.20$	$c^{+}_{optimal}$	Non-unique
SRFOFO	$5,403.80$	$c^{-}_{optimal}$	Non-unique
SRG	$27,000.00$	$c^{+}_{optimal} = c^{-}_{optimal}$	Unique
SRN	$23,700.00$	$c^{+}_{optimal} = c^{-}_{optimal}$	Unique
SRDS	$8,700.00$	$c^{+}_{optimal} = c^{-}_{optimal}$	Unique
SRFO	$37,200.00$	$c^{+}_{optimal} = c^{-}_{optimal}$	Unique
RFG	$21,993.60$	$c^{+}_{optimal}$	Non-unique

5.5.3 Variables of Interest to the User

The dual incremental analysis applied to the dual degenerate refinery LP suggested that the development of this method is based on market price uncertainty. However, not all activities involved in the refinery have a market price associated with them. The activities in the optimal solution of a refinery LP typically involve decision variables, slack variables, and surplus variables. By default, slack and surplus have zero cost coefficients. The decision variables are of three types: raw materials, finished product and intermediate products. Almost all of the intermediate products produced in the refinery do not have cost coefficients because they are not exposed to the market. In some instances, the intermediate products do have cost coefficients associated with them in the form of operating cost. In all instances, the raw materials have a buying price and the finished products have a selling price.

Under conditions of dual degeneracy, this research will focus only on analyzing activities that have a cost coefficient associated with them. For the refinery LP considered in this study, only 11 of the 33 decision variables have cost coefficients. The 11 variables along with their cost coefficients are given in Table 5.7.

The next section will generate alternate optimal solutions for the dual degenerate LP considered in this case study. Multiple activity values will be classified based on the business significance associated with them.

Table 5.7: Refinery LP Decision Variables Containing Cost Coefficients

S.I. No.	Variable	Cost Coefficient
1	CRUDE	Buying price of 33$/bb1
2	FGAD	Selling price of 0.01965 $/ft3
3	SRNRF	Operating cost of 2.5$/bbl
4	FGRF	Selling price of 0.01965 $/ft3
5	SRDSCC	Operating cost of 2.2$/bbl
6	SRFOCC	Operating cost of 2.2$/bbl
7	FGCC	Selling price of 0.01965 $/ft3
8	PG	Selling price of 44.0813$/bbl
9	RG	Selling price of 43.68$/bbl
10	DF	Selling price of 40.32$/bbl
11	FO	Selling price of 13.14$/bbl

5.5.4 Determining Alternate Optimal Solutions

For the LP considered in this study, results obtained in Section 5.5.2 confirmed that 19 decision variables have multiple activity values. Explanations provided in Section 5.5.3 demonstrated that it is adequate to analyze multiple activities that have cost coefficients associated with them.

In this section, the parametric perturbation technique developed in Section 5.4 is used to generate multiple optimal solutions for the dual degenerate LP. In addition to the single optimal solution obtained in Section 5.5.1, 12 more alternate optimal solutions are produced and listed in Table 5.8. For completeness, the multiple activity values obtained in each of the optimal basis for the entire 19 decision variables are presented in Table 5.8.

As evident from Table 5.8, of the 13 alternate optimal basic solutions produced, the Premium Gasoline (PG) activity has 10 distinct values. Initially, these values will be analyzed and categorized as $c^{+}_{optimal}$, $c^{-}_{optimal}$, and $c_{suboptimal}$. In this section, three PG activity values: 47,113 bbl/day, 43,692 bbl/day and 10,000 bbl/day in Table 5.8 are selected for analysis.

The PG activity value 47,113 bbl/day along with the associated row of dual incremental effect coefficients excluding the identity structure of the optimal tableau is presented in a transpose form (column format) in Table 5.9. In the classification process, the cost coefficient 44.0813$/bbl for the PG decision variable is perturbed from 44.0813$/bbl to 45.0813$/bbl. In doing so, as observed from Table 5.9, all the dual values remained feasible. Since this positive perturbation of PG cost coefficient produced dual feasible solution, the PG activity value 47,113.20 bbl/day is determined as $c^{+}_{optimal}$ implying, the user must implement this activity value in the actual process in order to attain optimal profit if the market price of PG is speculated to increase.

Table 5.8: Alternate Optimal Solutions Obtained for the Dual Degenerate Refinery LP

	Solution 1 Activity (bbl/day)	Solution 2 Activity (bbl/day)	Solution 3 Activity (bbl/day)	Solution 4 Activity (bbl/day)	Solution 5 Activity (bbl/day)	Solution 6 Activity (bbl/day)	Solution 7 Activity (bbl/day)
SRNRF	23,700	23,157	20,556	19,974	18,913	17,805	17,805
FGRF	3,761,190	3,674,954	3,262,242	3,169,893	3,001,458	2,825,697	2,825,697
PG	47,113	43,692	27,319	23,655	16,973	10,000	10,000
RG	22,520	25,981	42,541	46,247	53,005	60,058	60,058
SRGPG	13,852	10,511	0	0	0	0	4,314
RFGPG	17,240	12,541	4,425	8,774	12,186	7,179	5,686
SRNPG	0	0	2,254	3,726	4,787	2,821	0
CCGPG	16,021	20,640	20,640	11,155	0	0	0
SRGRG	13,148	16,489	27,000	27,000	27,000	27,000	22,686
RFGRG	4,754	8,948	14,651	9,762	5,365	9,344	10,837
SRNRG	0	543	890	0	0	3,074	5,895
CCGRG	4,619	0	0	9,485	20,640	20,640	20,640
CCFODF	6,591	6,591	6,591	6,591	6,591	6,591	3,791
SRDSDF	4,104	4,104	4,104	4,104	4,104	4,104	8,700
SRFODF	1,796	1,796	1,796	1,796	1,796	1,796	0
CCFOFO	0	0	0	0	0	0	2,800
SRDSFO	4,596	4,596	4,596	4,596	4,596	4,596	0
SRFOFO	5,404	5,404	5,404	5,404	5,404	5,404	7,200
RFG	21,994	21,489	19,076	18,536	17,551	16,523	16,523

	Solution 8 Activity (bbl/day)	Solution 9 Activity (bbl/day)	Solution 10 Activity (bbl/day)	Solution 11 Activity (bbl/day)	Solution 12 Activity (bbl/day)	Solution 13 Activity (bbl/day)	Number of distinct solutions
SRNRF	23,700	19,577	21,892	19,654	21,440	23,700	10
FGRF	3,761,190	3,106,900	3,474,183	3,119,010	3,402,600	3,761,190	10
PG	47,113	21,156	35,727	21,636	32,887	47,113	10
RG	22,520	48,774	34,037	48,289	36,909	22,520	10
SRGPG	17,073	0	15,412	996	10,731	13,852	8
RFGPG	21,994	0	20,315	0	19,897	17,240	11
SRNPG	0	516	0	0	2,260	0	7
CCGPG	8,046	20,640	0	20,640	0	16,021	5
SRGRG	9,927	27,000	11,588	26,004	16,269	13,148	8
RFGRG	0	18,168	0	18,238	0	4,754	10
SRNRG	0	3,607	1,808	4,047	0	0	8
CCGRG	12,594	0	20,640	0	20,640	4,619	5
CCFODF	6,591	6,591	6,591	6,591	3,791	3,263	3
SRDSDF	5,900	4,104	4,104	4,104	8,700	8,700	3
SRFODF	0	1,796	1,796	1,796	0	528	3
CCFOFO	0	0	0	0	2,800	3,328	3
SRDSFO	2,800	4,596	4,596	4,596	0	0	3
SRFOFO	7,200	5,404	5,404	5,404	7,200	6,672	3
RFG	21,994	18,168	20,315	18,238	19,897	21,994	10

Analogously, when the cost coefficient of PG is changed from 44.0813$/bbl to 43.0813-$/bbl, the resultant solution has some infeasible dual values. In this case, the shadow price 0.10$/bb1 for the PG octane and 0.10$/bbl for the Regular Gasoline (RG) constraints are infeasible. Besides, the reduced costs -6.30$/bbl, -6.30$/bbl, and -0.74$/bbl for Straight Run Naphtha for Premium Gasoline Blending (SRNPG), Straight Run Naphtha for Regular Gasoline blending (SRNRG), and Straight Run Naphtha for Diesel Fuel blending (SRNDF) are dual infeasible. Since this negative perturbation of PG cost coefficient produced dual infeasible solution, the PG activity value 47,113.20 bbl/day is not $c^{-}_{optimal}$, implying that the user must not use this activity value to attain optimal profit if the market price of PG is expected to decrease. Also, here the user has the flexibility to use solution #1, solution #8, or solution #13 in Table 5.8 for implementation because these solution sets have the PG activity as 47,113.20 bbl/day.

Secondly, the PG activity value 43,692 bbl/day is analyzed. Required data is given in Table 5.10. As observed from Table 5.10, dual incremental effect analysis showed that both positive and negative perturbation of PG cost coefficient yielded dual infeasible solutions. Therefore, this activity value for PG is categorized as $c_{suboptimal}$. Thus, the user cannot achieve optimal objective function value by implementing this solution for an increase or decrease in the market price of PG.

Finally, the PG activity value 10,000 bbl/day is analyzed. Required data is given in Table 5.11. As observed in Table 5.11 dual incremental effect analysis resulted in dual feasible solutions for a negative perturbation of the cost coefficient and dual infeasible solution for a positive perturbation. Consequently, the PG activity value 10,000 bbl/day is categorized as $c^{-}_{optimal}$. This PG activity value 10,000 bbl/day must be implemented to attain optimal profit if the market price of PG is expected to decrease. Moreover, here the user has the flexibility to use solution #6, solution #7, or solution #13 in Table 5.8 for implementation because these solution sets have the PG activity as 10,000 bbl/day.

Table 5.9: Dual Incremental Effect Analysis for the PG Activity 47,113.20 bbl/day

			Base case	$c_j + 1$	$c_j - 1$
		PG activity (bbl)	Objective value ($)	Objective value ($)	Objective value ($)
		47113.20	641579.78	688692.98	594466.58
Z	Dual variable*	Incremental effect ($\overline{a}_{ij}$)	Dual value ($/bbl)	Dual value ($/bbl)	Dual value ($/bbl)
SLK4	PGblend(2)	14.50	−37.86	−23.36	−52.36
SLK5	PGoctane	−0.17	−0.07	−0.23	0.10†
SLK9	RGblend(2)	14.50	−37.86	−23.36	−52.36
SLK10	RGoctane	−0.17	−0.07	−0.23	0.10†
SLK11	RGvapor	0.00	0.00	0.00	0.00
SLK14	DFblend(2)	0.00	−40.32	−40.32	−40.32
SLK16	DFsulfur	0.00	0.00	0.00	0.00
SLK17	FOproduction	0.00	−27.18	−27.18	−27.18
SLK19	FOblend(2)	0.00	−40.32	−40.32	−40.32
SLK22	ADcapacity	0.24	7.85	8.09	7.61
SLK24	FGADyield(2)	0.00	−0.02	−0.02	−0.02
SLK26	SRGyield(2)	1.42	−43.11	−41.69	−44.53
SLK28	SRNyield(2)	−2.63	−42.21	−44.84	−39.58
SLK30	SRDSyield(2)	0.00	−40.32	−40.32	−40.32
SLK32	SRFOyield(2)	0.00	−40.32	−40.32	−40.32
SLK35	FGRFyield(2)	0.00	−0.02	−0.02	−0.02
SLK37	RFGyield(2)	−2.83	−44.82	−47.65	−41.98
SLK38	CCcapacity	0.77	4.29	5.06	3.52
SLK40	FGCCyield(2)	0.00	−0.02	−0.02	−0.02
SLK42	CCGyield(2)	−1.12	−44.13	−45.24	−43.01
SLK44	CCFOyield(2)	0.00	−40.32	−40.32	−40.32
SLK46	SRGsplit(2)	1.42	−43.11	−41.69	−44.53
SLK48	SRNsplit(2)	−2.63	−42.21	−44.84	−39.58
SLK50	SRDSsplit(2)	0.00	−40.32	−40.32	−40.32
SLK52	SRFOsplit(2)	0.00	−40.32	−40.32	−40.32
SLK54	RFGsplit(2)	−2.83	−44.82	−47.65	−41.98
SLK56	CCGsplit(2)	−1.12	−44.13	−45.24	−43.01
SLK58	CCFOsplit(2)	0.00	−40.32	−40.32	−40.32
SRDSCC	Reduced cost	0.08	5.26	5.33	5.18
SRNPG	Reduced cost	6.30	0.00	6.30	−6.30†
SRNRG	Reduced cost	6.30	0.00	6.30	−6.30†
SRNDF	Reduced cost	2.63	1.89	4.52	−0.74†
CCFOFO	Reduced cost	0.00	0.00	0.00	0.00

†Dual infeasible

*Description of dual variables are given in Table B.1 of Appendix B

Table 5.10: Dual Incremental Effect Analysis for the PG Activity 43,692 bbl/day

			Base case	$c_j + 1$	$c_j - 1$
		PG activity (bbl)	Objective value ($)	Objective value ($)	Objective value ($)
		$43,692.03$	$641,579.74$	$685,271.76$	$597,887.71$
Z	Dual variable*	Incremental effect ($\overline{a}_{ij}$)	Dual value ($/bbl)	Dual value ($/bbl)	Dual value ($/bbl)
SLK4	PGblend(2)	16.85	−37.86	−21.01	−54.72
SLK5	PGoctane	−0.19	−0.07	−0.26	0.13†
SLK9	RGblend(2)	−7.24	−37.86	−45.10	−30.63
SLK10	RGoctane	0.03	−0.07	−0.04	−0.10
SLK11	RGvapor	0.36	0.00	0.36	−0.36†
SLK14	DFblend(2)	0.00	−40.32	−40.32	−40.32
SLK16	DFsulfur	0.00	0.00	0.00	0.00
SLK17	FOproduction	0.00	−27.18	−27.18	−27.18
SLK19	FOblend(2)	0.00	−40.32	−40.32	−40.32
SLK22	ADcapacity	0.20	7.85	8.05	7.64
SLK24	FGADyield(2)	0.00	−0.02	−0.02	−0.02
SLK26	SRGyield(2)	1.78	−43.11	−41.33	−44.90
SLK28	SRNyield(2)	−2.89	−42.21	−45.10	−39.32
SLK30	SRDSyield(2)	0.00	−40.32	−40.32	−40.32
SLK32	SRFOyield(2)	0.00	−40.32	−40.32	−40.32
SLK35	FGRFyield(2)	0.00	−0.02	−0.02	−0.02
SLK37	RFGyield(2)	−3.11	−44.82	−47.93	−41.71
SLK38	CCcapacity	−0.78	−4.29	−5.07	−3.51
SLK40	FGCCyield(2)	0.00	−0.02	−0.02	−0.02
SLK42	CCGyield(2)	−1.13	−44.13	−45.26	−42.99
SLK44	CCFOyield(2)	0.00	−40.32	−40.32	−40.32
SLK46	SRGsplit(2)	1.78	−43.11	−41.33	−44.90
SLK48	SRNsplit(2)	−2.89	−42.21	−45.10	−39.32
SLK50	SRDSsplit(2)	0.00	−40.32	−40.32	−40.32
SLK52	SRFOsplit(2)	0.00	−40.32	−40.32	−40.32
SLK54	RFGsplit(2)	−3.11	−44.82	−47.93	−41.71
SLK56	CCGsplit(2)	−1.13	−44.13	−45.26	−42.99
SLK58	CCFOsplit(2)	0.00	−40.32	−40.32	−40.32
SRDSCC	Reduced cost	0.08	5.26	5.33	5.18
SRNPG	Reduced cost	7.26	0.00	7.26	−7.26†
CCGRG	Reduced cost	−0.74	0.00	−0.74	0.74
SRNDF	Reduced cost	2.89	1.89	4.78	−1.00†
CCFOFO	Reduced cost	0.00	0.00	0.00	0.00

†Dual infeasible

*Description of dual variables are given in Table B.1 of Appendix B

Table 5.11: Dual Incremental Effect Analysis for the PG Activity 10,000 bbl/day

			Base case	$c_j + 1$	$c_j - 1$
		PG activity (bbl)	Objective value ($)	Objective value ($)	Objective value ($)
		$10,000.00$	$641,579.31$	$651,579.31$	$631,579.31$
Z	Dual variable*	Incremental effect ($\overline{a}_{ij}$)	Dual value ($/bbl)	Dual value ($/bbl)	Dual value ($/bbl)
SLK2	PGproduction	1.00	0.00	$1.00^{\dagger}$	-1.00
SLK4	PGblend(2)	0.00	-37.86	-37.86	-37.86
SLK5	PGoctane	0.00	-0.07	-0.07	-0.07
SLK9	RGblend(2)	0.00	-37.86	-37.86	-37.86
SLK10	RGoctane	0.00	-0.07	-0.07	-0.07
SLK14	DFblend(2)	0.00	-40.32	-40.32	-40.32
SLK17	FOproduction	0.00	-27.18	-27.18	-27.18
SLK19	FOblend(2)	0.00	-40.32	-40.32	-40.32
SLK22	ADcapacity	0.00	7.85	7.85	7.85
SLK24	FGADyield(2)	0.00	-0.02	-0.02	-0.02
SLK26	SRGyield(2)	0.00	-43.11	-43.11	-43.11
SLK28	SRNyield(2)	0.00	-42.21	-42.21	-42.21
SLK30	SRDSyield(2)	0.00	-40.32	-40.32	-40.32
SLK32	SRFOyield(2)	0.00	-40.32	-40.32	-40.32
SLK35	FGRFyield(2)	0.00	-0.02	-0.02	-0.02
SLK37	RFGyield(2)	0.00	-44.82	-44.82	-44.82
SLK38	CCcapacity	0.00	-4.29	-4.29	-4.29
SLK40	FGCCyield(2)	0.00	-0.02	-0.02	-0.02
SLK42	CCGyield(2)	0.00	-44.13	-44.13	-44.13
SLK44	CCFOyield(2)	0.00	-40.32	-40.32	-40.32
SLK46	SRGsplit(2)	0.00	-43.11	-43.11	-43.11
SLK48	SRNsplit(2)	0.00	-42.21	-42.21	-42.21
SLK50	SRDSsplit(2)	0.00	-40.32	-40.32	-40.32
SLK52	SRFOsplit(2)	0.00	-40.32	-40.32	-40.32
SLK54	RFGsplit(2)	0.00	-44.82	-44.82	-44.82
SLK56	CCGsplit(2)	0.00	-44.13	-44.13	-44.13
SLK58	CCFOsplit(2)	0.00	-40.32	-40.32	-40.32
SRDSCC	Reduced cost	0.00	5.26	5.26	5.26
SRNPG	Reduced cost	0.00	0.00	0.00	0.00
CCGPG	Reduced cost	0.00	0.00	0.00	0.00
SRNDF	Reduced cost	0.00	1.89	1.89	1.89
SRFODF	Reduced cost	0.00	0.00	0.00	0.00
SRDSFO	Reduced cost	0.00	0.00	0.00	0.00

†Dual infeasible

*Description of dual variables are given in Table B.1 of Appendix B

Among the 10 distinct PG activity values obtained in Table 5.8, three were analyzed. Based on analysis, 47,113.20 bbl/day was determined as $c^{+}_{optimal}$, 10,000 bbl/day was determined as $c^{-}_{optimal}$, and 43,692 bbl/day was determined as $c_{suboptimal}$. Without further detailed analysis, the rest of the seven activity values listed in Table 5.8 can be determined as $c_{suboptimal}$, because once $c^{+}_{optimal}$ and $c^{-}_{optimal}$ for an activity value is determined, other activity values will be suboptimal. The proof for this claim is obvious from the explanation given in Aucamp (1984). Furthermore, this immediate conclusion can be verified based on a simple calculation of determining the change in objective function value with respect to these activity values when the cost coefficient of PG is perturbed both positively and negatively.

The simple calculation is demonstrated in Table 5.12. As viewed from Table 5.12, all the activity values contributed the same objective function value $641,580 in the base case. However, when the cost coefficient of PG is changed from 44.0813$/bbl to 45.0813$/bbl, the activity value 47,113.20 bbl/day yielded the maximum profit $688,693. Therefore, this value is called $c^{+}_{optimal}$. On the other hand, for a negative perturbation 44.0813$/bbl to 43.0813$/bbl, the activity value 10,000 bbl/day produced the maximum objective function value $631,580. Therefore, this activity value is called the $c^{-}_{optimal}$. Table 5.12 demonstrates that all other activities resulted in a suboptimal objective function value for both positive and negative perturbation. Succinctly, the $c^{+}_{optimal}$ value of an activity can be determined using Equation (5.9) and $c^{-}_{optimal}$ value of an activity can be determined using Equation (5.10). The proof for these equations can be derived based on the proof given in Aucamp (1984).

$$c^{+}_{optimal} = max\left\{\bar{b}_1, \cdots, \bar{b}_k\right\} \tag{5.9}$$

$$c^{-}_{optimal} = min\left\{\bar{b}_1, \cdots, \bar{b}_k\right\} \tag{5.10}$$

where $\left\{\bar{b}_1, \cdots, \bar{b}_k\right\}$ are all possible distinct activity values.

The above claim by Equation (5.9) and Equation (5.10) will be valid only if alternate optimal activity values including $c^+_{optimal}$ and $c^-_{optimal}$ are generated. The dual incremental effect analysis approach is the most reliable method to conclude whether an activity value is $c^+_{optimal}$, $c^-_{optimal}$, or $c_{suboptimal}$.

Table 5.12: Multiple Activity Analysis for PG

		Base case	$c_j + 1$	$c_j - 1$
S.I. No.	PG activity (bbl/day)	Objective value ($)	Objective value ($)	Objective value ($)
1	$47,113$	$641,580$	$688,693$	$594,467$
2	$43,692$	$641,580$	$685,272$	$597,888$
3	$35,727$	$641,580$	$677,307$	$605,853$
4	$32,887$	$641,580$	$674,467$	$608,693$
5	$27,319$	$641,580$	$668,899$	$614,261$
6	$23,655$	$641,580$	$665,235$	$617,925$
7	$21,636$	$641,580$	$663,216$	$619,943$
8	$21,156$	$641,580$	$662,736$	$620,424$
9	$16,973$	$641,580$	$658,553$	$624,607$
10	$10,000$	$641,580$	$651,580$	$631,580$

The $c^+_{optimal}$ and $c^-_{optimal}$ for all other activities that have cost coefficients are determined for the refinery LP by completing a similar analysis and are tabulated in Table 5.13.

Table 5.13: $c^+_{optimal}$ and $c^-_{optimal}$ Activity Values for the Dual Degenerate Refinery LP

	$c^+_{optimal}$	$c^-_{optimal}$
	bbl/day	bbl/day
SRNRF	$23,700$	$17,805$
FGRF	$3,761,190$	$2,825,697$
PG	$47,113$	$10,000$
RG	$60,058$	$22,520$

5.6 Caveats

In this chapter, a well defined approach to choose a unique solution among multiple solutions for a dual degenerate problem was discussed. The methodology considered only corner point solutions for analysis. In some instances these corner points in the dual degenerate optimal face could be primal degenerate as well. In such instances, after choosing the desired corner point solution corresponding to a $c^{+}_{optimal}$ or $c^{-}_{optimal}$ activity value, the true shadow price and true incremental effect coefficients corresponding to this corner point have to be determined based on the procedures described in Chapter 4. This assures accurate interpretation of LP results for optimization.

5.7 Summary

This chapter investigated the condition of dual degeneracy for a refinery LP and implemented a truly innovative approach called dual incremental effect analysis to determine activity values that assure optimal profit, despite market price fluctuations.

In current refinery practice only a single optima solution is produced. The user may not be aware that the LP is dual degenerate with multiple dual values. This study has utilized the dual incremental analysis approach to characterize activity values obtained from a single optimal solution as: unique or non-unique and $c^{+}_{optimal}$, $c^{-}_{optimal}$ or $c_{suboptimal}$. Furthermore, a perturbation technique implementing parametric programming was developed to generate alternate optimal solutions.

CHAPTER 6

CONCLUSIONS

The summary of the findings of this research, contributions made and the future direction of this research applied to primal degeneracy and dual degeneracy are discussed under two sections.

6.1 Primal Degeneracy

6.1.1 Summary

This study investigated the phenomenon of primal degeneracy in refinery LP. The findings of this research suggested that interpreting only the single optimal solution produced for a primal degenerate LP will lead to fallible business decisions with negative economic impacts. For example, for the primal degenerate refinery LP considered in this research, the FCC constraint has three dual values: 17.67 $/bbl, 26.96 $/bbl and 31.19 $/bbl. The FCC unit is an economic driver in the refinery and processes thousands of barrels of crude every day. From the different solutions obtained in each of the different optimal bases, it is clear that using erroneous shadow price information for this constraint will lead to significant economic losses.

6.1.2 Contributions

Often, an LP optimal solution is considered to be primal degenerate when some of the basic variables have a zero value. This is not a sufficient condition to conclude that the LP is actually primal degenerate. Sometimes primal degeneracy is created due to a particular representation of the LP model. The methodology to verify whether the LP optimal solution is actually primal degenerate or is primal degenerate just due to a specific representation of the LP model has been clearly explained in this research.

In this research, the concept of true shadow price is absolutely correlated with process implications in refinery operations. The term true shadow price is explained clearly in the context of petroleum refinery optimization, for ease of comprehension and implementation in the actual refinery operation. Furthermore, the term true shadow price is replaced with physically realizable shadow price to receive attention from refinery optimization practitioners. For industrial practitioners whose capability is limited to producing a single optimal solution, an approach to categorize optimal dual values as p^+ shadow price, p^- shadow price, or $p^{invalid}$ shadow price was also developed. A perturbation technique incorporating parametric programming is developed to determine alternate optimal dual solutions when the LP is primal degenerate.

6.1.3 Recommendations and Future Work

When an LP problem is primal degenerate, three phenomena are observed: some of the constraints have p^+ shadow price equal to p^- shadow price, some of the constraints have p^+ shadow price not equal to p^- shadow price, and some other constraints have p^+ shadow price equal to zero value and p^- shadow price equal to a non-zero value. Understanding the cause of this phenomena will provide more flexibility in developing the LP model.

This task could be accomplished by classifying the constraints in the given primal LP based on its properties as strongly binding, weakly binding, and implicit equalities. The definitions for this classification of constraints are given in Karwan. et al. (1983). Algorithms to determine properties of constraints are given in Gal (1992), Telgen (1983), Thompson et al. (1966), Dula (1994), and Goberna et al. (2006). In the task, these algorithms could be implemented to determine properties of constraints involved in a refinery LP model. Once the constraint properties are identified, they can be correlated with their respective p^+ and p^- shadow price found in the optimal solution. This approach can assist in determining the cause for the constraints having different kinds of shadow price values.

6.2 Dual Degeneracy

6.2.1 Summary

This study examined the condition of dual degeneracy in LP. Findings of this study indicated that the magnitude of difference among activity values obtained for alternate optimal solutions is significant. In this study for the dual degenerate LP considered, the activity value for Premium Gasoline (PG) production varied between 47,113 bbl/day and 10,000 bbl/day. Although implementing any activity value obtained within this range produced the same objective function value in the base case, not all solutions produced the optimal profit when the market price of PG either decreases or increases.

For example, consider a situation in which the user is not aware that the LP is dual degenerate and has only a single optimal solution that suggests manufacturing 10,000 bbl/day of PG. If the user implemented this plan, and the market price of PG increased by a dollar, the resultant profit would be $37,113 less compared to implementing the activity value of 47,113 bbl/day. The above example illustrated the business impact of implementing one solution over the other. Therefore, when the LP is dual degenerate, alternate optimal solutions have to be analyzed appropriately to achieve optimal profit.

6.2.2 Contributions

Under conditions of dual degeneracy, a truly novel approach called the dual incremental effect analysis method has been developed to categorize multiple activities so that the user can implement specific activity values that sustain optimal profit despite market price fluctuations. Furthermore, from a single optimal solution, the dual incremental effect analysis approach was also used to determine activities that can have multiple values.

The dual feasibility condition of LP was presented in the context of petroleum refinery operation, for determining whether the new solution obtained after a change in the market price of an activity is optimal or not. Also, a novel perturbation technique for implementing parametric programming was developed to generate alternate optimal solutions when the LP is dual degenerate.

6.2.3 Recommendations and Future Work

In this study, a single variable sensitivity analysis approach was used to analyze multiple optimal solutions under conditions of degeneracy. In actual refinery operations market price of two or more commodities may vary simultaneously. Therefore, this study could be extended to provide the largest sensitivity region of any single or simultaneous change of cost coefficients of decision variables in the objective function. The methodology provided in Arsham (2007) provided some leads for this type of analysis.

When the problem is dual degenerate and produces multiple optimal solutions, not all the variables produce multiple activity values; some variables have unique activity values. Understanding the cause for this behavior will provide more flexibility in developing the LP model (Cheng, 1985). This task could be accomplished by classifying the variable in the given primal LP model based on its properties as strongly extraneous, weakly extraneous, free, essential, or inessential. The definitions for this classification of variables are given in Karwan. et al. (1983). Algorithms to determine properties of constraints are given in Gal (1992), Gal (1975), Telgen (1983), Thompson et al. (1966), Dula (1994), Caron et al. (1989) and Goberna et al. (2006). These algorithms can be applied to classify variables by transforming the given primal problem to a dual problem.

Although the parametric perturbation technique developed in this research is capable of determining all the possible alternate optimal solutions, it is computationally laborious and does not include a stopping criteria to guarantee that all possible alternate optimal solutions are generated. A pivoting type algorithm with less computational effort with efficient stopping criteria can be developed to resolve this issue. Currently literatures are available to determine the dimension of the optimal face of a dual degenerate LP (Gal, 1985; Kantor, 1993; Kruse, 1993; Zornig and Gal, 1996; Zornig, 1993; Zornig and Gal, 1996; Gonzaga, 2007). However, these studies have not quantified the number of extreme points possible for this multi-dimensional optimal face. If the possible number of extreme points was determined this would serve as a useful stopping criteria.

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APPENDIX A

PARAMETRIC PERTURBATION TECHNIQUE

Conventionally parametric programming is used for sensitivity analysis. However, in this research parametric programming is used to determine alternate optimal solution when the LP is degenerate. In this appendix, initially the reason for using parametric programming to determine alternate optimal solution is presented. Then the algorithm described in Section 4.3 is demonstrated for a 2-D LP.

A.1 Rationale for Using Parametric Programming

A 2-D primal degenerate LP in general form is given in Equation (A.1)

$$\text{Maximize } z = 5x_1 + 4x_2 \tag{A.1}$$

Subject to

$$6x_1 + 4x_2 \leq 20 \qquad \text{Constraint } \#1$$

$$x_1 + 2x_2 \leq 6 \qquad \text{Constraint } \#2$$

$$-x_1 + x_2 \leq 1 \qquad \text{Constraint } \#3$$

$$x_2 \leq 2 \qquad \text{Constraint } \#4$$

$$x_1, x_2 \geq 0 \qquad \text{Non-negativity}$$

The geometric solution is illustrated in Figure A.1. As evident from Figure A.1 the LP is primal degenerate because three constraints pass through the optimum vertex C for this 2-D problem. Therefore, based on the combination formula given by Equation (3.3) three solutions are possible at the vertex C. One approach to generate all the three solutions is

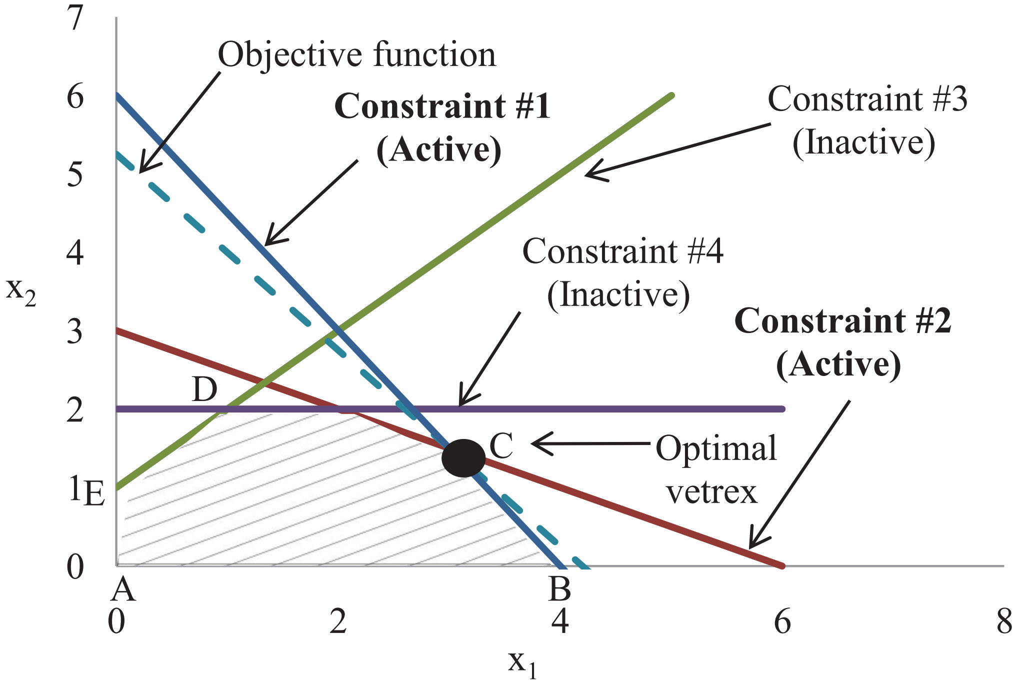

Figure A.1: Graphical Solution for the 2-D Non-Degenerate LP

to solve three different non-degenerate LPs with two constraints active at a time. Based on this idea the geometric and the algebraic solution for the three non-degenerate problems are given as follows:

A.1.1 Solution #1

Initially one of the solutions possible at the vertex C in Figure A.1 is generated with constraints #1 and #2 active. The geometric solution is given in Figure A.2 and the algebraic solution is given in Table A.1

A.1.2 Solution #2

Now one other solution possible at vertex C in Figure A.1 is generated with constraints #1 and #4 active. The geometric solution is given in Figure A.3 and the algebraic solution is given in Table A.2

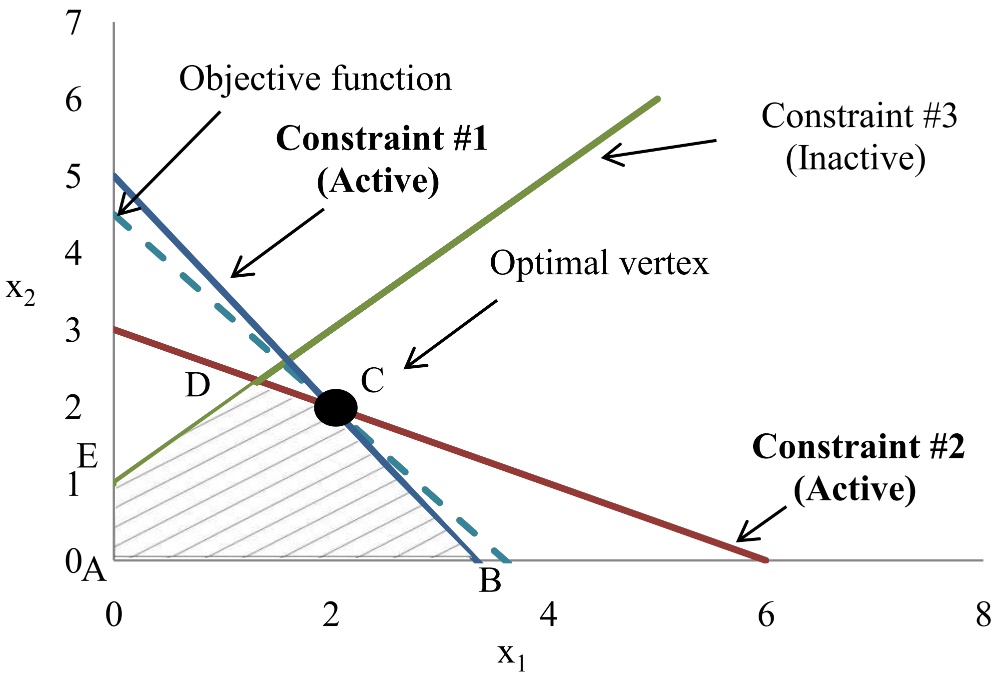

Figure A.2: Graphical Solution with Constraints #1 and #2 Active

Table A.1: Solution with Constraints #1 and #2 Active

Basis	x_1	x_2	s_1	s_2	s_3	s_4	RHS
z	0	0	3/4	1/2	0	0	18
x_1	1	0	1/4	-1/2	0	0	2
x_2	0	1	-1/8	3/4	0	0	2
s_3	0	0	3/8	-5/4	1	0	1
s_4	0	0	1/8	-3/4	0	1	0

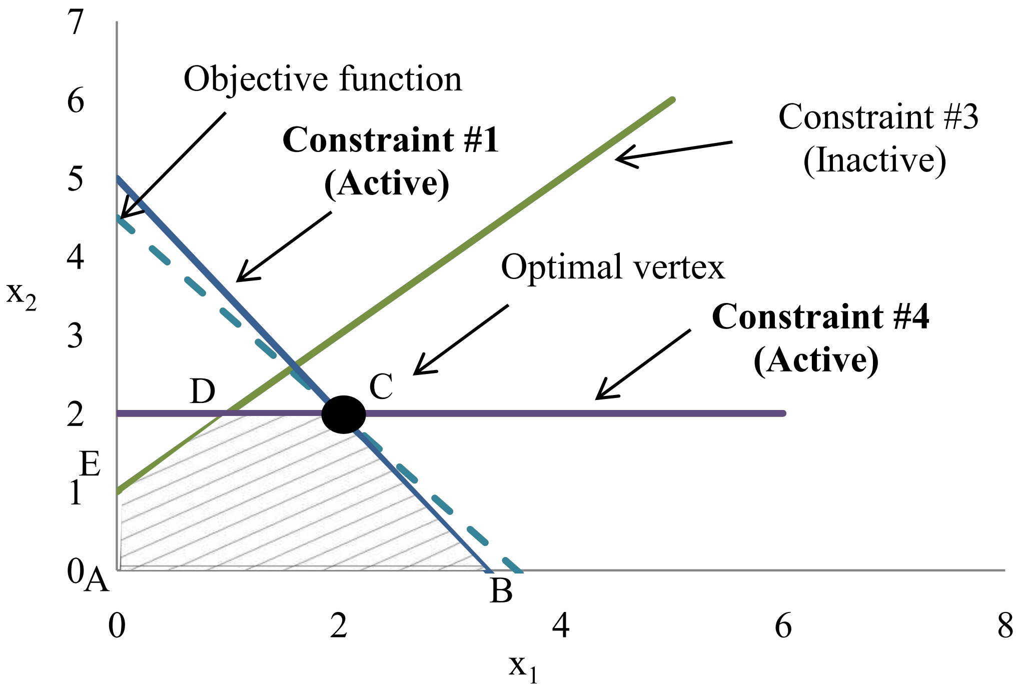

Figure A.3: Graphical Solution with Constraints #1and #4 Active

Table A.2: Solution with Constraints #1and #4 Active

Basis	x_1	x_2	s_1	s_2	s_3	s_4	RHS
z	0	0	5/6	0	0	2/3	18
x_1	1	0	1/6	0	0	-2/3	2
x_2	0	1	0	0	0	1	2
s_2	0	0	-1/6	1	0	-4/3	0
s_3	0	0	1/6	0	1	-5/3	1

A.1.3 Solution #3

Finally, the other solution possible at vertex C in Figure A.1 is generated with constraints #2 and #4 active. The geometric solution is given in Figure A.4 and the algebraic solution is given in Table A.3

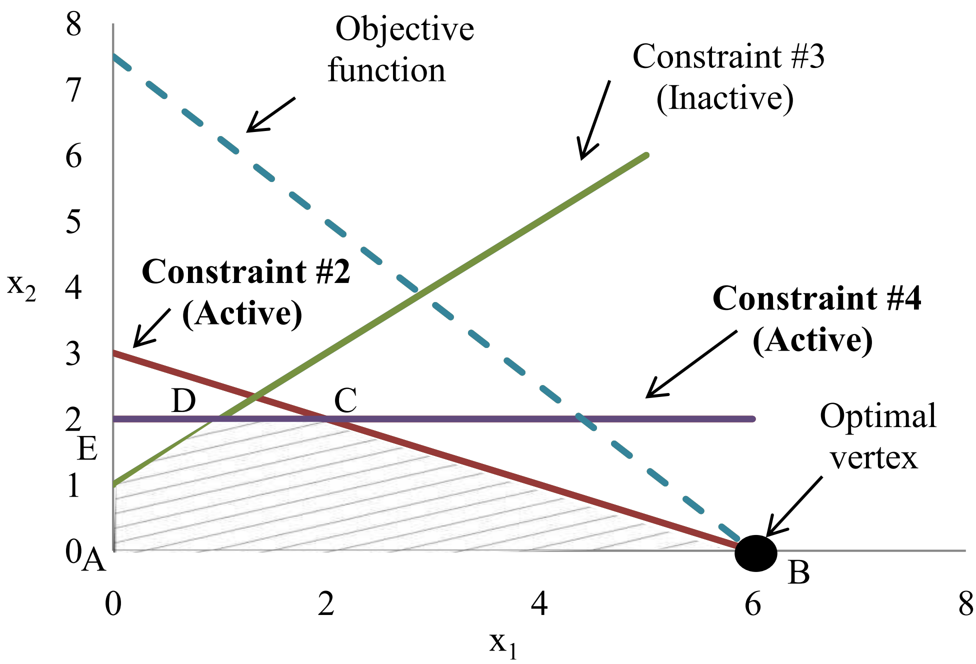

Figure A.4: Graphical Solution with Constraints #2 and #4 Active

Table A.3: Solution with Constraints #2 and #4 Active

Basis	x_1	x_2	s_1	s_2	s_3	s_4	RHS
z	0	0	0	5	0	-6	18
x_1	1	0	0	1	0	-2	2
x_2	0	1	0	0	0	1	2
s_1	0	0	1	-6	0	8	0
s_3	0	0	0	1	1	-3	1

The solution obtained in this case (solution #3) is non-optimal because of the negative dual value“-6” for constraint #4 in Table A.3. Furthermore, as observed from Figure A.4 the optimum is shifted from vertex C to B because vertex C is no longer optimal with only constraints #2 and #4 active.

From the above analysis it is obvious that all the three possible solutions generated by the combination formula approach are not optimal and only two are optimal in this case. Therefore, the combination formula approach may require computing solutions that are non-optimal; consequently this approach could be computationally intense. To reduce the computational effort, an approach that determines only the optimal solutions at a degenerate vertex have to be developed. Parametric programming which is traditionally used to perform sensitivity analysis can be used to generate the entire possible optimal basis at a primal degenerate point by parametrically varying each of the active constraints. As a result, in this research, parametric programming is used to reduce the computation effort while generating alternate optimal basis corresponding to a primal degenerate vertex.

A.2 Demonstration of Algorithm

Parametric perturbation technique to determine alternate optimal basis corresponding to a primal degenerate vertex is demonstrated in this section. The 2-D primal degenerate LP given by Equation (A.1) in Section A.1 is selected as an example. The variables in the 2-D LP problem and their corresponding index is given in Table A.4

The single optimal solution obtained initially by solving the 2-D primal degenerate LP using LINDO is presented in Table A.1. The step by step procedure to determine alternate optimal basis for this 2-D primal degenerate LP follows:

Step1 Inspection of single optimal solution given by Table A.4 showed that constraint #1, constraint #2 and constraint #4 are active.

Step2 Initially the R.H.S of constraint #1 is parametrically varied using the software pack-

Table A.4: Variable Index for the 2-D LP

Index	Variable
x1	1
x2	2
s1	3
s2	4
s3	5
s4	6

age LINDO and the alternate optimal basis corresponding to the original R.H.S value 20 are determined as $B_{1,1} = \{1, 2, 5, 6\}$ and $B_{1,2} = \{1, 2, 4, 5\}$.

Step3 Similarly, the R.H.S of other active constraints #2 and #4 are parametrically varied. The alternate optimal basis obtained by varying constraint #2 is determined as $B_{2,1} = \{1, 2, 5, 6\}$ and $B_{2,2} = \{1, 2, 4, 5\}$. The alternate optimal basis obtained by varying constraint #4 is determined as $B_{4,1} = \{1, 2, 5, 6\}$ and $B_{4,2} = \{1, 2, 4, 5\}$. As a result, including the basis obtained in this step and step 2, there are a total of of six optimal basis: $\{B_{1,1}, B_{1,2}, B_{2,1}, B_{2,2}, B_{4,1}, B_{4,2}\}$.

Step4 The set of bases obtained in step 3 is compared to each other and the unique basis among them is determined as: $B_1 = \{1, 2, 5, 6\}$ and $B_2 = \{1, 2, 4, 5\}$.

Step5 The optimal simplex tableaux corresponding to the unique basis obtained in step 4 is generated using the formula given in Table 4.5 and presented in Table A.1 and Table A.2.

Step6 The p^+ and p^- shadow price of constraint #1 is determined as $p^+ = min\{3/4, 5/6\} = 3/4$ and $p^- = max\{3/4, 5/6\} = 5/6$. Based on a similar evaluation the p^+ and p^- of constraint #2 is determined as 0 and $1/2$. The p^+ and p^- of constraint #4 is determined as 0 and $2/3$.

APPENDIX B

DUAL FORMULATION OF THE REFINERY LP

The transformed LP with all constraints in the less than or equal to (LE) form in Section 2.2 of Chapter 2 has 33 variables and 58 constraints. This problem is converted to a dual problem based on the procedure given in Dantzig and Thapa (2003).

The dual formulation will have 58 variables for each of the primal constraints and 33 constraints for each of the primal variables. The dual variables for the 58 constraints are defined as $Y_1 \cdots Y_{58}$. The dual formulation is given as follows:

Minimize z = 110,000Y_1 - 10,000Y_2 - 10,000Y_7 - 10,000Y_{12} - 10,000 $Y_1$7 + 10,0000Y_{22} + 25,000Y_{33} - 30,000Y_{38}

Subject to

1) -Y_1 - Y_{22} - 35.42Y_{23} + 35.42Y_{24} - 0.27Y_{25} + 0.27Y_{26} - 0.237Y_{27} + 0.237Y_{28} - 0.087Y_{29} + 0.087Y_{30} - 0.372Y_{31}+0.372Y_{32} $\leq$ 33 $\rightarrow$ CRUDE

2) Y_{23} - Y_{24} $\leq$ -0.01965 $\rightarrow$ FGAD

3) -Y_{33} - 158.7Y_{34} + 158.7Y_{35} - 0.928Y_{36} + 0.928Y_{37} + Y_{47} - Y_{48} $\leq$ 2.5 $\rightarrow$ SRNRF

4) Y_{34} - Y_{35} $\leq$ -0.01965 $\rightarrow$ FGRF

5) Y_{38} - 336.90Y_{39} + 336.90Y_{40} - 0.619Y_{41} + 0.619Y_{42} - 0.189Y_{43} + 0.189Y_{44} + Y_{49} - Y_{50} $\leq$ 2.2 $\rightarrow$ SRDSCC

6) -Y_{38} - 386.40Y_{39} + 386.4Y_{40} - 0.688Y_{41} + 0.688Y_{42} - 0.2197Y_{43} + 0.2197Y_{44} + Y_{51} - Y_{52} $\leq$ 2.2 $\rightarrow$ SRFOCC

7) Y_{39} - $Y_{40} \leq$ -0.01965 → FGCC

8) Y_2 + Y_3 - Y_4 - 93Y_5 + 12.7$Y_6 \leq$ -45.36 → PG

9) Y_7 + Y_8 - Y_9 - 87Y_{10} + 12.7$Y_{11} \leq$ -43.68 → RG

10) Y_{12} + Y_{13} - Y_{14} + 306Y_{15} + 0.5$Y_{16} \leq$ -40.32 → DF

11) Y_{17} + Y_{18} - Y_{19} + 352Y_{20} + 3$Y_{21} \leq$ -13.14 → FO

12) -Y_3 + Y_4 + 78.5Y_5 - 18.4Y_6 + 1Y_{45} - $Y_{46} \leq 0$ → SRGPG

13) -Y_3 + Y_4 + 104Y_5 - 2.57Y_6 + 1Y_{53} - $Y_{54} \leq 0$ → RFGPG

14) -Y_3 + Y_4 + 65Y_5 - 6.54Y_6 + Y_{47} - $Y_{48} \leq 0$ → SRNPG

15) -Y_3 + Y_4 + 93.7Y_5 - 6.9Y_6 + Y_{55} - $Y_{56} \leq 0$ → CCGPG

16) -Y_8 + Y_9 + 78.5Y_{10} - 18.4Y_{11} + Y_{45} - $Y_{46} \leq 0$ → SRGRG

17) -Y_8 + Y_9 + 104Y_{10} - 2.57Y_{11} + Y_{53} - $Y_{54} \leq 0$ → RFGRG

18) -Y_8 + Y_9 + 65Y_{10} - 6.54Y_{11} + Y_{47} - $Y_{48} \leq 0$ → SRNRG

19) -Y_8 + Y_9 + 93.7Y_{10} - 6.9Y_{11} + Y_{55} - $Y_{56} \leq 0$ → CCGRG

20) -Y_{13} + Y_{14} - 272Y_{15} - 0.283Y_{16} + Y_{47} - $Y_{48} \leq 0$ → SRNDF

21) -Y_{13}+Y_{14}-294.4Y_{15}-0.353Y_{16}+Y_{57}-$Y_{58} \leq 0$ → CCFODF

22) -Y_{13}+Y_{14}-292Y_{15}-0.526Y_{16}+Y_{49}-$Y_{50} \leq 0$ → SRDSDF

23) -Y_{13}+Y_{14}-295Y_{15}-0.98Y_{16}+Y_{51}-$Y_{52} \leq 0$ → SRFODF

24) -Y_{18}+Y_{19}-294.4Y_{20}-0.353Y_{21}+Y_{57}-$Y_{58} \leq 0$ → CCFOFO

25) -Y_{18}+Y_{19}-292Y_{20}-0.526Y_{21}+Y_{49}-$Y_{50} \leq 0$ → SRDSFO

26) -Y_{18}+Y_{19}-295Y_{20}-0.98Y_{21}+Y_{51}-$Y_{52} \leq 0$ → SRFOFO

27) Y_{25}-Y_{26}-Y_{45}+$Y_{46} \leq 0$ → SRG

28) Y_{27}-Y_{28}-Y_{47}+$Y_{48} \leq 0$ → SRN

29) Y_{29}-Y_{30}-Y_{49}+$Y_{50} \leq 0$ → SRDS

30) Y_{31}-Y_{32}-Y_{51}+$Y_{52} \leq 0$ → SRFO

31) Y_{36}-Y_{37}-Y_{53}+$Y_{54} \leq 0$ → RFG

32) Y_{41}-Y_{42}-Y_{55}+$Y_{56} \leq 0$ → CCG

33) Y_{43}-Y_{44}-Y_{57}+$Y_{58} \leq 0$ → CCFO

Each of the dual variables in the above formulation is represented in an abbreviated form for better comprehension. The abbreviated version and the detailed description of each of the dual variables is given in Table B.1.

Table B.1: Description of Dual Variables for the Refinery LP

Dual variable	Abbreviated form	Constraint description	Dual variable	Abbreviated form	Constraint description
Y_1	CRUDEavail	Crude availability	Y_{30}	SRDSyield(2)	SRDS Yield , GE
Y_2	PGproduction	PG Production	Y_{31}	SRFOyield(1)	SRFO Yield , LE
Y_3	PGblend(1)	PG Blending, LE	Y_{32}	SRFOyield(2)	SRFO Yield , GE
Y_4	PGblend(2)	PG Blending, GE	Y_{33}	RFcapacity	RF Capacity
Y_5	PGoctane	PG Octane rating	Y_{34}	FGRFyield(1)	FGRF Yield , LE
Y_6	PGvapor	PG Vapor pressure	Y_{35}	FGRFyield(2)	FGRF Yield , GE
Y_7	RGproduction	RG Production	Y_{36}	RFGyield(1)	RFG Yield , LE
Y_8	RGblend(1)	RG Blending, LE	Y_{37}	RFGyield(2)	RFG Yield , GE
Y_9	RGblend(2)	RG Blending, GE	Y_{38}	CCcapacity	FCC Capacity
Y_{10}	RGoctane	RG Octane rating	Y_{39}	FGCCyield(1)	FGCC Yield , LE
Y_{11}	RGvapor	RG vapor pressure	Y_{40}	FGCCyield(2)	FGCC Yield , GE
Y_{12}	DFproduction	DF production	Y_{41}	CCGyield(1)	CCG Yield , LE
Y_{13}	DFblend(1)	DF blending, LE	Y_{42}	CCGyield(2)	CCG Yield , GE
Y_{14}	DFblend(2)	DF blending, GE	Y_{43}	CCFOyield(1)	CCFO Yield , LE
Y_{15}	DFdensity	DF density specification	Y_{44}	CCFOyield(2)	CCFO Yield , GE
Y_{16}	DFsulfur	DF Sulfur specification	Y_{45}	SRGsplit(1)	SRG Split , LE
Y_{17}	FOproduction	FO production	Y_{46}	SRGsplit(2)	SRG Split , GE
Y_{18}	FOblend(1)	FO blending, LE	Y_{47}	SRNsplit(1)	SRN Split , LE
Y_{19}	FOblend(2)	FO blending, GE	Y_{48}	SRNsplit(2)	SRN Split , GE
Y_{20}	FOdensity	FO density specification	Y_{49}	SRDSsplit(1)	SRDS Split , LE
Y_{21}	FOsulfur	FO Sulfur specification	Y_{50}	SRDSsplit(2)	SRDS Split , GE
Y_{22}	ADcapacity	AD Capacity	Y_{51}	SRFOsplit(1)	SRFO Split , LE
Y_{23}	FGADyield(1)	FGAD Yield , LE	Y_{52}	SRFOsplit(2)	SRFO Split , GE
Y_{24}	FGADyield(2)	FGAD Yield , GE	Y_{53}	RFGsplit(1)	RFG Split , LE
Y_{25}	SRGyield(1)	SRG Yield , LE	Y_{54}	RFGsplit(2)	RFG Split , GE
Y_{26}	SRGyield(2)	SRG Yield , GE	Y_{55}	CCGsplit(1)	CCG Split , LE
Y_{27}	SRNyield(1)	SRN Yield , LE	Y_{56}	CCGsplit(2)	CCG Split , GE
Y_{28}	SRNyield(2)	SRN Yield , GE	Y_{57}	CCFOsplit(1)	CCFO Split , LE
Y_{29}	SRDSyield(1)	SRDS Yield , LE	Y_{58}	CCFOsplit(2)	CCFO Split , GE

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Scope and Method of Study

The phenomenon of degeneracy inevitably occurs in most large LP models. An LP could be primal degenerate, dual degenerate, or both primal and dual degenerate. Primal degeneracy of LP and its solution interpretation is well established in literature, but the notion of dual degeneracy (alternative optima) has received less attention. The condition of dual degeneracy or alternative optima leads to multiple optimal bases with multiple activity values or multiple primal solutions. Current refinery optimization practitioners are not fully aware of the consequences of degeneracy and business decisions are made using a single LP run.

The purpose of this study is to investigate the effects of dual degeneracy in the context of petroleum refinery optimization and simultaneously to develop strategies to select a specific set of activity values for implementation based on business logic. When an LP has alternate optimal solutions or is dual degenerate, it will be primal degenerate in the dual space. This property of the dual degenerate problem is exploited in this research to derive business logic on the interpretation of LP solutions produced by a dual degenerate LP.

This study developed a novel dual incremental analysis approach to choose a desired set of activity values based on small changes in the market price of activities when the LP is dual degenerate. Furthermore, a perturbation technique implementing parametric programming is developed to generate multiple optimal bases when the LP is dual degenerate. Results are presented, along with a simplified refinery model containing 33 decision variables and 37 constraints.

Findings and Conclusion

Findings of this study indicated that for the dual degenerate refinery LP the magnitude of the difference among activity values obtained for each of the alternate optimal solutions is significant. Although the optimality criteria (primal and dual feasible) for the LP is satisfied for each of the alternate optimal solutions in the base case, the optimality criteria may not be satisfied even for an infinitesimal change in the market price of activities. The dual incremental analysis approach and the underlying business logic developed in this research serves two purposes for a dual degenerate LP: 1) characterize each of the activity values obtained for a single LP run, and 2) choose a desired set of activity values for implementation among multiple optimal solutions generated.

ADVISOR’S APPROVAL: ______________________