

NEW IDEAS IN HIGGS PHYSICS

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TABLE OF CONTENTS

Chapter	Page
1 INTRODUCTION	1
2 Unfaithful Representations of Finite Groups and Tribimaximal Neutrino Mixing	7
2.1 The Discrete Symmetry	7
2.2 The Model	9
2.3 Derivation of Representations	14
2.4 Invariants Under the Discrete Symmetry	33
2.5 Calculation of the Neutrino Mass Matrix	38
2.6 Calculation of the Charged Lepton Mass Matrix	42
3 A 6D Higgsless Standard Model	45
3.1 The Model	45
3.2 Effective theory	49
3.3 Relation to EWPT	53
3.4 Non-oblique corrections and fermion masses	55
3.4.1 Improving the calculability	57
4 A New Two Higgs Doublet Model	61
4.1 Model and the Formalism	61
4.2 Phenomenological Implications	63
4.3 Cosmological Implications	68

5 CONCLUSIONS	71
BIBLIOGRAPHY	73

LIST OF TABLES

Table		Page
2.1	This table shows the matrices representing the generators in each irrep. of A_4 , in a certain basis.	8
2.2	This table shows the assignments of the fermions and Higgs fields under $SU(2)_L \times U(1)_Y \times [(S_3^4 \times Z_2^3) \rtimes A_4]$	11

LIST OF FIGURES

Figure	Page
<p>3.1 Symmetry breaking of $SU(2)_L \times U(1)_Y$ on the rectangle. At one boundary $y_1 = \pi R_1$, $SU(2)_L$ is broken to $U(1)_{I_3}$ while on the boundary $y_2 = \pi R_2$ the subgroup $U(1)_{I_3} \times U(1)_Y$ is broken to $U(1)_Q$, which leaves only $U(1)_Q$ unbroken on the entire rectangle. Locally, at the fixed point $(0, 0)$, $SU(2)_L \times U(1)_Y$ remains unbroken. The dashed arrows indicate the propagation of the lowest resonances of the gauge bosons.</p>	48
<p>3.2 Effect of the brane kinetic terms \mathcal{L}_0 on the KK spectrum of the gauge bosons (for the example of W^\pm). Solid lines represent massive excitations, the bottom dotted lines would correspond to the zero modes which have been removed by the BC's. Without the brane terms (a), the lowest KK excitations are of order $1/R \simeq 1 \text{ TeV}$. After switching on the dominant brane kinetic terms (b), the zero modes are approximately "restored" with a small mass $m_W \ll 1/R$ (dashed line), while the higher KK-levels receive small corrections to their masses (thin solid lines) and decouple below $\sim 1 \text{ TeV}$.</p>	52
<p>3.3 One-loop diagram for ϕ-ϕ scattering on S^1/Z_2. The total incoming momentum is (p, p'_5) and the total outgoing momentum is (p, p_5). Generally, it is possible that $p'_5 \neq p_5$, since the orbifold fixed points break 5D translational invariance.</p>	58
<p>4.1 Left panel: Branching ratio for $h \rightarrow \sigma\sigma$ as a function of m_h for the value of the parameter, $\lambda^* = 0.1$. Right panel: Branching ratio for $h \rightarrow \sigma\sigma$ as a function of λ^* for $m_h = 135 \text{ GeV}$.</p>	67

CHAPTER 1

INTRODUCTION

The Standard Model (SM) of electroweak interactions [1], based on the gauge symmetry group $SU(2)_L \times U(1)_Y$, provides a highly successful description of electroweak precision tests (EWPT) [2, 3]. One fundamental ingredient of the SM is the Higgs mechanism [4], which accomplishes electroweak symmetry breaking (EWSB) and at high energies unitarizes massive W^\pm and Z scattering through the presence of the scalar Higgs doublet [5]. Although the mass of the Higgs boson is not predicted by the SM, accurate measurements of the top quark and the W boson mass at the Tevatron, as well as the Z boson mass at LEP, have narrowed the SM Higgs boson mass between 80 and 200 GeV [3]. Failure to observe the SM Higgs boson at LEP2 has also placed a direct lower bound of 114 GeV on its mass [6]. The dominant decay modes of the SM Higgs boson are to $b\bar{b}$, WW , ZZ or $t\bar{t}$, depending on its mass. Extensions of the SM may avoid constraints on the Higgs mass, and may allow Higgs bosons with masses less than the above limits. The dominant decay modes of the Higgs bosons can also be altered in such extensions, thus transforming the discovery signals for the Higgs bosons at the Large Hadron Collider (LHC). However, there is as yet no direct evidence of the Higgs boson, so that the details of the Higgs sector, if it even exists, remain a mystery. Thus, it is important to explore alternative Higgs sector scenarios.

One interesting scenario involves the role of the Higgs sector in neutrino mixing. The existence of neutrino masses is now well established experimentally [7, 8]. At 1σ ,

the mass-squared differences and mixing angles are [7]:

$$\Delta m_{21}^2 = 7.65(+0.23/-0.20) \times 10^{-5} eV^2, \quad \Delta |m_{31}^2| = 2.40(+0.13/-0.11) \times 10^{-3} eV^2,$$

and

$$\sin^2 \theta_{23} = 0.50(+0.022/-0.016), \quad \sin^2 \theta_{12} = 0.341(+0.07/-0.06),$$

$$\sin^2 \theta_{13} < 0.035.$$

These values are in good agreement with a tribimaximal mixing pattern given by the mixing matrix [9,10]

$$U_{MNS} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix} P \quad (1.1)$$

where P is a diagonal phase matrix. This corresponds to

$$\sin^2 \theta_{23} = 1/2, \quad \sin^2 \theta_{12} = 1/3, \quad \sin^2 \theta_{13} = 0.$$

It has long been known that such a mixing pattern can be obtained using a finite family symmetry [10-25] such as A_4 [19-25]. In these models, A_4 is broken to a Z_2 subgroup in the neutrino sector by a triplet Higgs, with the VEV structure $(0, 1, 0)$ or some permutation thereof, and to a Z_3 subgroup in the charged lepton sector by a triplet Higgs, with the VEV structure $(1, 1, 1)$. However, there is a serious technical problem with this, in that couplings between the Higgs fields responsible for the symmetry breaking will force the VEV's to align, upsetting the desired breaking pattern [21-25]. To overcome this problem, one can introduce more complicated symmetries. In Section 2, we consider models where the SM lepton families belong to representations of the finite symmetry which are not faithful (that is, not every member of the group is represented by a distinct transformation). In effect, the

Higgs sector knows about the full symmetry while the lepton sector does not. We consider a renormalizable non-supersymmetric gauge theory with an additional finite symmetry that has the semi-direct product structure $G = (G_1 \times G_2) \rtimes A_4$, with $G_1 = S_3 \times S_3 \times S_3 \times S_3$ and $G_2 = Z_2 \times Z_2 \times Z_2$. A symmetry thus structured will contain G_1 , G_2 , and $G_1 \times G_2$ as invariant subgroups, so that G will have representations corresponding to the homomorphisms $G/(G_1 \times G_2) \sim A_4$, $G/G_1 \sim G_2 \rtimes A_4$, and $G/G_2 \sim G_1 \rtimes A_4$. SM leptons can then be assigned to representations of A_4 . Neutrino masses are generated by a Higgs field ϕ , belonging to a 16-dimensional representation of $G_1 \rtimes A_4$, while charged-lepton masses are generated by a Higgs field χ , belonging to a 6-dimensional representation of $G_2 \rtimes A_4$. The additional symmetries, G_1 and G_2 , prevent quadratic and cubic interactions between ϕ and χ and allow only a trivial quartic interaction (*i.e.*, the interaction is the product of quadratic invariants) that does not cause an alignment problem. In this way, the alignment problem is addressed without altering the desired properties of the family symmetry, so that neutrino mixing can be explained using only symmetries which are broken spontaneously by the Higgs mechanism.

However, no fundamental scalar particle has been observed yet in nature, and as long as there is no direct evidence for the existence of the Higgs boson, the actual mechanism of EWSB remains a mystery. In case the Higgs boson will also not be found at the Tevatron or the LHC, it will therefore be necessary to consider alternative ways to achieve EWSB without a Higgs. We explore this possibility in Section 3. It is well known, that in extra dimensions, gauge symmetries can also be broken by boundary conditions (BC's) on a compact space [27]. Here, a geometric "Higgs" mechanism ensures tree-level unitarity of longitudinal gauge boson scattering through a tower of Kaluza-Klein (KK) [28] excitations [29]. The original model for Higgsless EWSB [30] is an $SU(2)_L \times SU(2)_R \times U(1)_{B-L}$ gauge theory compactified on an interval $[0, \pi R]$ in five-dimensional (5D) flat space. At one end of the interval, $SU(2)_R \times$

$U(1)_{B-L}$ is broken to $U(1)_Y$. At the other end, $SU(2)_L \times SU(2)_R$ is broken to the diagonal subgroup $SU(2)_D$, thereby leaving only $U(1)_Q$ of electromagnetism unbroken in the effective four-dimensional (4D) theory. Although this model exhibited some similarities with the SM, the ρ parameter deviated from unity by $\sim 10\%$ and the lowest KK excitations of the W^\pm and Z were too light ($\sim 240 \text{ GeV}$) to be in agreement with experiment. These problems have later been resolved by considering the setup in warped space [33]. Based on the same gauge group, similar effects can be realized in 5D flat space [32], when 4D brane kinetic terms [34–36] dominate the contribution from the bulk. In 5D Higgsless models, a ρ parameter close to unity is achieved at the expense of enlarging the SM gauge group by an additional gauge group $SU(2)_R$, which introduces a gauged custodial symmetry in the bulk. However, it is possible to obtain consistent 6D Higgsless models of EWSB, which are based only on the SM gauge group $SU(2)_L \times U(1)_Y$ and allow the ρ parameter to be set equal to unity. We consider a Higgsless model for EWSB in six dimensions, which is based only on the SM gauge group $SU(2)_L \times U(1)_Y$, where the gauge bosons propagate in the bulk. The model is formulated in flat space with the two extra dimensions compactified on a rectangle and EWSB is achieved by imposing consistent BC's. The higher KK resonances of W^\pm and Z decouple below $\sim 1\text{TeV}$ through the presence of a dominant 4D brane induced gauge kinetic term. The ρ parameter is arbitrary and can be set exactly to one by an appropriate choice of the bulk gauge couplings and compactification scales. Unlike in the 5D theory, the mass scale of the lightest gauge bosons W and Z is solely set by the dimensionful bulk couplings, which (upon compactification via mixed BC's) are responsible for EWSB. We calculate the tree-level oblique corrections to the S, T , and U parameters and find that they are in better agreement with data than in proposed 5D warped and flat Higgsless models.

In Section 4, we present a model that includes a second Higgs doublet that provides an alternate explanation for the tiny masses of the SM neutrinos, as well as

possibilities for altering signals for discovery of the Higgs at the LHC. Our proposal is to extend the SM electroweak symmetry to $SU(2)_L \times U(1) \times Z_2$ and introduce three $SU(2) \times U(1)$ singlet right handed neutrinos, N_R , as well as an additional Higgs doublet, ϕ . While the SM symmetry is spontaneously broken by the VEV of an EW doublet χ at the 100 GeV scale, the discrete symmetry Z_2 is spontaneously broken by the tiny VEV of this additional doublet ϕ at a scale of 10^{-2} GeV. Thus in our model, tiny neutrino masses are related to this Z_2 breaking scale. We note that although our model has extreme fine tuning, that is no worse than the fine tuning problem in the usual GUT model. Many versions of the two Higgs doublet model have been extensively studied in the past [37]. The examples include: a) a supersymmetric two Higgs doublet model, b) non-supersymmetric two Higgs doublet models i) in which both Higgs doublets have vacuum expectation values (VEV's) with one doublet coupling to the up type quarks only, while the other coupling to the down type quarks only, ii) only one doublet coupling to the fermions, and iii) only one doublet having VEV's and coupling to the fermions [38]. What is new in our model is that one doublet couples to all the SM fermions except the neutrinos, and has a VEV which is same as the SM VEV, while the other Higgs doublet couples only to the neutrinos with a tiny VEV $\sim 10^{-2}$ eV. This latter involves the Yukawa coupling of the left-handed SM neutrinos with a singlet right-handed neutrino, N_R . The left-handed SM neutrinos combine with the singlet right-handed neutrinos to make massive Dirac neutrinos. The neutrino mass is so tiny because of the tiny VEV of the second Higgs doublet, which is responsible for the spontaneous breaking of the discrete symmetry, Z_2 . Note that in the neutrino sector, our model is very distinct from the sea-saw model. Lepton number is strictly conserved, and hence no $N_R N_R$ mass terms are allowed. Thus the neutrino is a Dirac particle, and there is no neutrino-less double β decay in our model. In the Higgs sector, in addition to the usual massive neutral scalar and pseudoscalar Higgs, and two charged Higgs, our model contains one essentially massless scalar

Higgs. We will show that this is still allowed by the current experimental data and can lead to an invisible decay mode of the SM-like Higgs boson, thus complicating the Higgs searches at the Tevatron and the LHC.

CHAPTER 2

Unfaithful Representations of Finite Groups and Tribimaximal Neutrino Mixing

2.1 The Discrete Symmetry

As described in the Introduction, we consider a renormalizable non-supersymmetric gauge theory with an additional finite symmetry given by the semi-direct product¹ $G = (G_1 \times G_2) \rtimes A_4$, with $G_1 = S_3 \times S_3 \times S_3 \times S_3$ and $G_2 = Z_2 \times Z_2 \times Z_2$. The group A_4 can be described using two generators obeying the relations,

$$X^2 = Y^3 = E, \quad XYX = Y^2XY^2, \quad (2.1)$$

where E is the identity. The irreducible representations are one real singlet, 1; two complex singlets, 1' and 1''; and one real triplet, 3. Table 1 gives X and Y in each of these representations for a certain choice of basis. The S_3 generators, A_i and B_i , and the Z_2 generators, C_i , obey

$$A_i^3 = B_i^2 = E, \quad B_i A_i B_i^{-1} = A_i^{-1}, \quad C_i^2 = E, \quad (2.2)$$

and C_i commutes with A_i and B_i . The remaining relations defining the full symmetry are

$$X A_1 X^{-1} = A_2, \quad X A_2 X^{-1} = A_1, \quad X A_3 X^{-1} = A_4, \quad X A_4 X^{-1} = A_3,$$

¹The semi-direct product, $N \rtimes H$, contains N and H as subgroups and obeys $hnh^{-1} \in N$ for all $n \in N$ and $h \in H$ [39]. Thus, N is an invariant subgroup. The number of elements in the group, denoted by $|N \rtimes H|$, is $|N||H|$. The semi-direct product exists when H has a factor group which is a subgroup of the automorphism group of N .

	X	Y
1	1	1
1'	1	ω
1''	1	ω^2
3	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

$$\omega = e^{2i\pi/3}$$

Table 2.1: This table shows the matrices representing the generators in each irrep. of A_4 , in a certain basis.

$$XB_1X^{-1} = B_2, \quad XB_2X^{-1} = B_1, \quad XB_3X^{-1} = B_4, \quad XB_4X^{-1} = B_3, \quad (2.3)$$

$$YA_1Y^{-1} = A_1, \quad YA_2Y^{-1} = A_3, \quad YA_3Y^{-1} = A_4, \quad YA_4Y^{-1} = A_2,$$

$$YB_1Y^{-1} = B_1, \quad YB_2Y^{-1} = B_3, \quad YB_3Y^{-1} = B_4, \quad YB_4Y^{-1} = B_2, \quad (2.4)$$

$$XC_1X^{-1} = C_1C_2C_2, \quad XC_2X^{-1} = C_3, \quad XC_3X^{-1} = C_2,$$

$$YC_1Y^{-1} = C_2, \quad YC_2Y^{-1} = C_3, \quad YC_3Y^{-1} = C_1. \quad (2.5)$$

It's easy to see that if C_1 , C_2 , and C_3 are all represented by the identity matrix, then (6) is trivially satisfied. So in this case, one need only find representations that respect Eqs. (2)-(5). But this is equivalent to finding representations of $G_1 \rtimes A_4$. The representations of this type that we will be using are a real 16-dimensional representation, a real 48-dimensional representation, and a real 8-dimensional representation. These will be referred to hereafter as 16_{AB} , 48_{AB} , and 8_{AB} . The matrices representing the remaining generators in each of these representations can be found in Section 2.3

below. Similarly, if $A_1, A_2, A_3, A_4, B_1, B_2, B_3,$ and B_4 are all represented by the identity matrix, then (4) and (5) are trivially satisfied. Finding these representations corresponds to finding representations of $G_2 \times A_4$. For this type, we will be using a real 6-dimensional representation, which we will call 6_C . The matrices representing the remaining generators in this representation can also be found in Section 2.3. Finally, if the A_i 's, B_i 's, and C_i 's are all represented by the identity matrix, then the only non-trivial relation is (2), corresponding to the representations of A_4 given in Table 1. These representations will be used for SM leptons.

2.2 The Model

The SM lepton assignments under A_4 are

$$e_{R1} \sim 1, e_{R2} \sim 1', e_{R3} \sim 1'', (L_1, L_2, L_3) \sim 3. \quad (2.6)$$

The finite symmetry is broken at a scale M_* , which is large compared to the weak scale, by two Higgs fields, ϕ and χ . Neutrino Dirac masses are generated by the real Higgs field ϕ belonging to 16_{AB} , while charged lepton masses are generated by the real Higgs field χ belonging to 6_C . Symmetry-invariant interactions between ϕ and χ must consist of products of G_1 invariants constructed from ϕ with G_2 invariants constructed from χ . The 16-dimensional representation to which ϕ belongs is $(2, 2, 2, 2)$ with respect to $G_1 = S_3 \times S_3 \times S_3 \times S_3$, so that there is only one quadratic G_1 invariant that can be constructed with ϕ , which is invariant under the full symmetry. Thus, there are no cubic invariants involving both ϕ and χ , and the only quartic invariant containing both is a trivial product of quadratic invariants, which does not generate a VEV alignment problem. Then the potential of ϕ and χ has the form

$$\begin{aligned} V_{\phi\chi} = & a_1 f_1(\phi, \phi) + a_2 f_2(\chi, \chi) + b_1 g_1(\phi, \phi, \phi) + b_2 g_2(\chi, \chi, \chi) + c_1 h_1(\phi) + c_2 h_2(\phi) \\ & + c_3 h_3(\chi) + c_4 h_4(\chi) + c_5 h_5(\chi) + c_6 h_6(\chi) + c_7 f_1(\phi, \phi) f_2(\chi, \chi), \end{aligned} \quad (2.7)$$

where the functions $f_1, f_2, g_1, g_2, h_1, h_2, h_3, h_4, h_5,$ and h_6 are given in Section 2.4 below.

The neutrino masses are generated from ϕ by integrating out multiplets of heavy right-handed neutrinos, with masses at a scale M_* which is large compared to the EW scale. These multiplets are $N \sim 3, N' \sim 48_{AB},$ and $N'' \sim 8_{AB}.$ If the Z_2 subgroup of A_4 generated by X is left unbroken by the VEV of ϕ (along with an additional accidental Z_2 that is actually part of $S_4,$ see [41]), the light neutrino mass matrix is forced to have the form

$$M_\nu = \begin{pmatrix} a_\nu & 0 & c_\nu \\ 0 & b_\nu & 0 \\ c_\nu & 0 & a_\nu \end{pmatrix}. \quad (2.8)$$

This matrix is diagonalized by

$$U_\nu = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & 1 \end{pmatrix} P_\nu, \quad (2.9)$$

where diagonal P_ν is a phase matrix. The charged lepton masses are generated from χ by integrating out multiplets of heavy vector-like fermions, whose masses are also at the high scale $M_*,$ with the same gauge quantum numbers as right-handed charged leptons. These are $E_{L,R} \sim 3$ and $E'_{L,R} \sim 6_C.$ If the Z_3 subgroup of A_4 generated by Y is left unbroken by the VEV of $\chi,$ the light left-handed charged lepton mass matrix is forced to have the form

$$\begin{aligned} M_e^\dagger M_e &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \begin{pmatrix} a_e & 0 & 0 \\ 0 & b_e & 0 \\ 0 & 0 & c_e \end{pmatrix} \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix} \\ &= U_L \begin{pmatrix} a_e & 0 & 0 \\ 0 & b_e & 0 \\ 0 & 0 & c_e \end{pmatrix} U_L^\dagger \end{aligned} \quad (2.10)$$

	$SU(2)_L$	$U(1)_Y$	$(S_3^4 \times Z_2^3) \rtimes A_4$
L	2	-1/2	3
e_{R1}	1	-1	1
e_{R2}	1	-1	1'
e_{R3}	1	-1	1''
N	1	0	3
N'	1	0	48_{AB}
N''	1	0	8_{AB}
E_L	1	-1	3
E_R	1	-1	3
E'_L	1	-1	6_C
E'_R	1	-1	6_C
ϕ	1	0	16_{AB}
χ	1	0	6_C
H	2	-1/2	1

Table 2.2: This table shows the assignments of the fermions and Higgs fields under $SU(2)_L \times U(1)_Y \times [(S_3^4 \times Z_2^3) \rtimes A_4]$

Eqs. (10) and (11) then give the desired form (1) for the mixing matrix $U_{MNS} = U_L^T U_\nu^*$. The symmetry assignments of the fermions and Higgs fields in the model are summarized in Table 2.

From the matrices given in Section 2.3, it can be seen that the most general VEV structure for χ that leaves the Z_3 subgroup of A_4 generated by Y unbroken is

$$\langle \chi \rangle = (v_{\chi 1}, v_{\chi 2}, v_{\chi 1}, v_{\chi 2}, v_{\chi 1}, v_{\chi 2}). \quad (2.11)$$

Upon minimizing the potential, one finds that $v_{\chi 2} = 0$, $v_{\chi 1} \neq 0$ is allowed. Here, $C_1 C_2 C_3$ is left unbroken in addition to Y . Since the SM leptons do not transform under the C_i 's, these additional symmetries do not affect the light lepton mass matrices.

So the desired minimum is

$$\langle \chi \rangle = (v_\chi, 0, v_\chi, 0, v_\chi, 0). \quad (2.12)$$

Since $C_1 C_2 C_3$, and Y commute, the subgroup they generate is $Z_2 \times Z_3$. Of course, χ also trivially leaves all A_i 's and B_i 's unbroken.

The most general VEV structure for ϕ that leaves the Z_2 subgroup of A_4 generated by X unbroken is

$$\langle \phi \rangle = (v_{\phi_1}, v_{\phi_2}, v_{\phi_2}, v_{\phi_3}, v_{\phi_4}, v_{\phi_5}, v_{\phi_6}, v_{\phi_7}, v_{\phi_4}, v_{\phi_6}, v_{\phi_5}, v_{\phi_7}, v_{\phi_9}, v_{\phi_9}, v_{\phi_9}, v_{\phi_{10}}). \quad (2.13)$$

Upon minimizing the potential, we find that

$$\langle \phi \rangle = (0, 0, 0, 0, v_\phi, v_\phi, v_\phi, v_\phi, v_\phi, v_\phi, v_\phi, v_\phi, 0, 0, 0, 0) \quad (2.14)$$

is acceptable. In addition to X , this VEV leaves the generators $B_1, B_2, B_3 B_4$, and $A_3 A_4$ unbroken. These form the subgroup $D_4 \times S_3$, with D_4 generated by B_1, B_2 , and X and with S_3 generated by $A_3 A_4$ and $B_3 B_4$. Of course, ϕ also leaves all C_i 's unbroken. (To leave the accidental $Z_2 \subset S_4$ mentioned above unbroken requires $v_{\phi_5} = v_{\phi_6}$ in (14), which is satisfied in (15).)

From (8), we find that v_ϕ and v_χ in (13) and (15) must be solutions to

$$2a_1 + 3b_1 v_\phi + 2(c_1 + c_2)v_\phi^2 + 6c_7 v_\chi^2 = 0, \quad 2a_2 + 3b_2 v_\chi + 4(c_3 + c_5)v_\chi^2 + 16c_7 v_\phi^2 = 0.$$

Neutrino Dirac masses are generated through

$$\begin{aligned} \mathcal{L}_\nu = & \lambda(L_1 N_1 + L_2 N_2 + L_3 N_3)\tilde{H} + m_N(N_1^2 + N_2^2 + N_3^2) + m'_N f_3(N', N') + m''_N f_4(N'', N'') \\ & + \alpha_1 g_3(N, \phi, N') + \alpha_2 g_4(N'', \phi, N') + \beta g_5(\phi, N', N'), \end{aligned} \quad (2.15)$$

where the functions f_3, f_4, g_3, g_4 , and g_5 are given in Section 2.4. $N \sim 3$ is required because the SM Higgs H only breaks EW symmetry, so that it can only cause left-handed neutrinos to mix with a triplet. Since $3 \times 16_{AB} = 48_{AB}$, $\phi \sim 16_{AB}$ induces

mixing between N and $N' \sim 48_{AB}$. $N'' \sim 8_{AB}$ is needed to remove unwanted accidental symmetries. Upon integrating out the heavy right-handed neutrinos, the light neutrino mass matrix (9) is obtained (see Section 2.5). The light neutrino masses are found to be

$$m_1 = \left| \frac{\lambda^2 v^2}{2} \frac{m'_N m''_N - 4\alpha_2^2 v_\phi^2 + \beta v_\phi m''_N}{-2\alpha_1^2 v_\phi^2 m''_N + m_N (m'_N m''_N - 4\alpha_2^2 v_\phi^2 + \beta v_\phi m''_N)} \right|,$$

$$m_2 = \left| \frac{\lambda^2 v^2}{2} \frac{m'_N m''_N - 2\alpha_2^2 v_\phi^2 + \beta v_\phi m''_N}{-2\alpha_1^2 v_\phi^2 m''_N + m_N (m'_N m''_N - 2\alpha_2^2 v_\phi^2 + \beta v_\phi m''_N)} \right|,$$

$$m_3 = \left| \frac{\lambda^2 v^2}{2} \frac{m'_N + \beta v_\phi}{-2\alpha_1^2 v_\phi^2 + m_N (m'_N + \beta v_\phi)} \right|.$$

Charged lepton masses are generated through

$$\begin{aligned} \mathcal{L}_e = & \kappa (\bar{E}_{R1} L_1 + \bar{E}_{R2} L_2 + \bar{E}_{R3} L_3) H + m_E (\bar{E}_{R1} E_{L1} + \bar{E}_{R2} E_{L2} + \bar{E}_{R3} E_{L3}) + m'_E f_2 (\bar{E}'_R, E'_L) \\ & + \gamma_1 g_6 (\bar{E}_R, E'_L, \chi) + \gamma_2 g_6 (\bar{E}_L, E'_R, \chi) + \epsilon_1 \bar{e}_{R1} f_2 (E'_L, \chi) + \epsilon_2 g_7 (\bar{e}_{R2}, E'_L, \chi) + \epsilon_3 g_8 (\bar{e}_{R3}, E'_L, \chi) \\ & + \eta_1 g_2 (\bar{E}'_R, E'_L, \chi) + \eta_2 g_2 (\bar{E}'_L, E'_R, \chi) + c.c., \end{aligned} \quad (2.16)$$

where the functions g_6 , g_7 , and g_8 are once again given in Section 2.4. Upon integrating out the heavy fermions, the light charged lepton mass-squared matrix (11) is obtained (see Section 2.6). The masses are

$$m_e^2 = \frac{3|\kappa\epsilon_1\gamma_2 v_\chi^2 v|^2}{3|\epsilon_1 v_\chi (m_E + \gamma_2 v_\chi)|^2 + |m_E (m'_E + \eta_1 v_\chi + \eta_2 v_\chi) - \gamma_1 \gamma_2 v_\chi^2|^2},$$

$$m_\mu^2 = \frac{3|\kappa\epsilon_2\gamma_2 v_\chi^2 v|^2}{3|\epsilon_2 v_\chi (m_E + \omega\gamma_2 v_\chi)|^2 + |m_E (m'_E + \omega^2\eta_1 v_\chi + \omega\eta_2 v_\chi) - \gamma_1 \gamma_2 v_\chi^2|^2},$$

$$m_\tau^2 = \frac{3|\kappa\epsilon_3\gamma_2 v_\chi^2 v|^2}{3|\epsilon_3 v_\chi (m_E + \omega^2\gamma_2 v_\chi)|^2 + |m_E (m'_E + \omega\eta_1 v_\chi + \omega^2\eta_2 v_\chi) - \gamma_1 \gamma_2 v_\chi^2|^2}.$$

2.3 Derivation of Representations

Let $H \subset G$, and assume that we understand the representation theory of H . G can be decomposed into cosets of H ,

$$G = \sum_{i=1}^n s_i H = \sum_{i=1}^n \{s_i h \mid h \in H\}, \quad (2.17)$$

where the number n of cosets is equal to the number of elements in G divided by the number of elements in H . The coset decomposition is independent of the choice of the representative s_i for each coset. Let γ be a k -dimensional irreducible representation of H . It induces a representation γ_{\uparrow} of G given by

$$\gamma_{\uparrow}(g)_{ij} = \sum_{h \in H} \gamma(h) \delta(h, s_i^{-1} g s_j). \quad (2.18)$$

In other words, the ij sub-block of γ_{\uparrow} is $\gamma(s_i^{-1} g s_j)$ when $s_i^{-1} g s_j \in H$ and is zero otherwise. Note that the dimension of the induced representation is kn . In general, γ_{\uparrow} is reducible. Up to this point, it was not necessary to assume that H is invariant. Let us now do so,

$$h \in H \implies ghg^{-1} \in H, \quad \forall g \in G.$$

Then for each $g \in G$, we can define a new representation γ_g from γ

$$\gamma_g(h) = \gamma(ghg^{-1}). \quad (2.19)$$

For $g \in H$, γ_g is equivalent to γ (that is, γ_g is γ in a different basis),

$$\gamma_g(h) = \gamma(g)\gamma(h)\gamma^{-1}(g).$$

If g is outside of H , then γ_g may be either equivalent or inequivalent to γ . The set of all inequivalent irreducible representations that can be obtained from γ by the transformation (20) (including γ itself) is called the orbit O_{γ} of the representation γ . Note that the true singlet (i.e., $\gamma(h) = 1, \forall h \in H$) is always in its own orbit. Two

representations that belong to the same orbit have equivalent induced representations (19). If g_1 and g_2 belong to the same coset in (18), then they differ by a factor belonging to H . So, by an argument similar to that showing γ_g is equivalent to γ for $g \in H$, γ_{g_1} and γ_{g_2} are equivalent. Then, to identify the orbit, it suffices to consider how γ transforms under the coset representatives, s_i . Let H_γ be the set of all $g \in G$ such that γ_g is equivalent to γ . Then, not only does H_γ contain H , but it consists of a whole number of cosets from (18). H_γ is an invariant subgroup of G and is called the little group of the representation γ (or of the orbit O_γ). Note that the little group of the true singlet is always the entire group G . If each coset sends γ to an inequivalent representation, then the number of representations in O_γ is equal to the number n of cosets, and $H_\gamma = H$. In this case, the induced representation γ_\uparrow in (19) is irreducible. Otherwise, it is reducible.

As an example, consider the group $A_4 = (Z_2 \times Z_2) \rtimes Z_3$. Let X and Z be the Z_2 generators and Y be the Z_3 generator. Y cyclicly permutes the three Z_2 subgroups of $Z_2 \times Z_2$:

$$YXY^{-1} = Z, \quad YZY^{-1} = XZ, \quad Y(XZ)Y^{-1} = X. \quad (2.20)$$

Note that Z is not an independent generator ($Z = YXY^{-1}$). The decomposition into cosets of $Z_2 \times Z_2$ is

$$A_4 = \{E, X, Z, XZ\} + \{Y, YX, YZ, YXZ\} + \{Y^2, Y^2X, Y^2Z, Y^2XZ\}. \quad (2.21)$$

We can choose E, Y , and $Y^2 = Y^{-1}$ as coset representatives. Then the induced representation (19) takes the form

$$\gamma_\uparrow(X) = \begin{pmatrix} \gamma(X) & 0 & 0 \\ 0 & \gamma(Y^{-1}XY) & 0 \\ 0 & 0 & \gamma(YXY^{-1}) \end{pmatrix}, \quad \gamma_\uparrow(Z) = \begin{pmatrix} \gamma(Z) & 0 & 0 \\ 0 & \gamma(Y^{-1}ZY) & 0 \\ 0 & 0 & \gamma(YZY^{-1}) \end{pmatrix},$$

$$\gamma_{\uparrow}(Y) = \begin{pmatrix} 0 & 0 & \gamma(E) \\ \gamma(E) & 0 & 0 \\ 0 & \gamma(E) & 0 \end{pmatrix}. \quad (2.22)$$

There are four one-dimensional irreducible representations of $Z_2 \times Z_2$:

$$(a) \quad \gamma(X) = 1, \quad \gamma(Z) = 1$$

$$(b) \quad \gamma(X) = -1, \quad \gamma(Z) = -1$$

$$(c) \quad \gamma(X) = -1, \quad \gamma(Z) = 1$$

$$(d) \quad \gamma(X) = 1, \quad \gamma(Z) = -1$$

As always, the true singlet (a) belongs to its own orbit. The induced representation is

$$\gamma_{\uparrow}(X) = \gamma_{\uparrow}(Z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \gamma_{\uparrow}(Y) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

These matrices can be diagonalized simultaneously, yielding three one-dimensional representations:

$$1 : X = Z = 1, Y = 1;$$

$$1' : X = Z = 1, Y = \omega;$$

$$1'' : X = Z = 1, Y = \omega^*;$$

with $\omega = \exp(2i\pi/3)$. Using (21), we have for (b),

$$\gamma_X(X) = \gamma(YXY^{-1}) = \gamma(Z) = -1, \quad \gamma_Y(Z) = \gamma(YZY^{-1}) = \gamma(XZ) = 1,$$

which is (c), and

$$\gamma_{Y^{-1}}(X) = \gamma(Y^{-1}XY) = \gamma(XZ) = 1, \quad \gamma_{Y^{-1}}(Z) = \gamma(Y^{-1}ZY) = \gamma(X) = -1,$$

which is (d). Thus, (b), (c), and (d) make up a single orbit. Since the number of representations in this orbit is equal to the number of cosets in (22), the induced representation is irreducible. From (23), we have

$$3: X = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad Z = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

For every group G , there exists a maximal invariant subgroup H ; that is, there are no proper invariant subgroups that contain H . For this subgroup, the factor group G/H is simple. If G is itself a simple group then the maximal invariant subgroup is the trivial subgroup, $\{E\}$. There also exists a maximal invariant subgroup H' of H . So, we have a chain,

$$H_i \subset H_{i-1} \subset \dots \subset H_2 \subset H_1 = G,$$

where H_{j+1} is an invariant subgroup of H_j , and H_j/H_{j+1} is simple. This chain can be continued until the trivial subgroup $\{E\}$ is reached, but for our purposes it suffices to stop at the largest subgroup whose representation theory we already know. Then, if we know how to determine the representation theory of a group from that of its maximal invariant subgroup, we can apply this recursively. So, let G be a group, and let H be its maximal invariant subgroup. We will further assume that G/H is a cyclic group. Since the little group of a representation of H must be an invariant subgroup of G containing H , the little group for each representation must be either H or all of G . If the little group is H , the induced representation is irreducible. So we need only concern ourselves with the case where the little group is G .

Let us consider another example. The group S_4 is equal to $A_4 \rtimes Z_2$. The A_4 generators given above and the Z_2 generator, which we will denote by W , obey the

relations

$$WXW^{-1} = Z, WZW^{-1} = X, WYW^{-1} = Y^{-1}, \quad (2.23)$$

along with (21). The decomposition into cosets of A_4 is $S_4 = A_4 + WA_4$. Noting that $W^{-1} = W$, the induced representations have the form

$$\begin{aligned} \gamma_{\uparrow}(X) &= \begin{pmatrix} \gamma(X) & 0 \\ 0 & \gamma(WXW) \end{pmatrix}, \quad \gamma_{\uparrow}(Z) = \begin{pmatrix} \gamma(Z) & 0 \\ 0 & \gamma(WZW) \end{pmatrix}, \\ \gamma_{\uparrow}(Y) &= \begin{pmatrix} \gamma(Y) & 0 \\ 0 & \gamma(WYW) \end{pmatrix}, \quad \gamma_{\uparrow}(W) = \begin{pmatrix} 0 & \gamma(E) \\ \gamma(E) & 0 \end{pmatrix}. \end{aligned} \quad (2.24)$$

For the 1 of A_4 ,

$$\gamma_{\uparrow}(X) = \gamma_{\uparrow}(Z) = \gamma_{\uparrow}(Y) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \gamma_{\uparrow}(W) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Upon diagonalization, this yields

$$X = Z = Y = 1, W = 1;$$

$$X = Z = Y = 1, W = -1.$$

Without much difficulty, we see that $1'$ and $1''$ make up an orbit, so that the induced representation is irreducible,

$$X = Z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad Y = \begin{pmatrix} \omega & 0 \\ 0 & \omega^* \end{pmatrix}, \quad W = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The triplet 3 of A_4 ,

$$\gamma(X) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \gamma(Z) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \gamma(Y) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

must belong to its own orbit because there is no other possibility. We have

$$\gamma_W(X) = \gamma(WXW) = \gamma(Z) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\gamma_W(Z) = \gamma(WZW) = \gamma(X) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$\gamma_W(Y) = \gamma(WYW) = \gamma^{-1}(Y) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Since this representation must lie in its own orbit, there exists a matrix S such that

$$S\gamma_W(X)S^{-1} = \gamma(X), \quad S\gamma_W(Z)S^{-1} = \gamma(Z), \quad S\gamma_W(Y)S^{-1} = \gamma(Y).$$

Indeed, by inspection, we see that we can take

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Note that $S^{-1} = S$. The induced representation of S_4 can then be written

$$\gamma_{\uparrow}(X) = \begin{pmatrix} \gamma(X) & 0 \\ 0 & S\gamma(X)S \end{pmatrix}, \quad \gamma_{\uparrow}(Z) = \begin{pmatrix} \gamma(Z) & 0 \\ 0 & S\gamma(Z)S \end{pmatrix},$$

$$\gamma_{\uparrow}(Y) = \begin{pmatrix} \gamma(Y) & 0 \\ 0 & S\gamma(Y)S \end{pmatrix}, \quad \gamma_{\uparrow}(W) = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

Let

$$\mathcal{S} = \frac{1}{\sqrt{2}} \begin{pmatrix} I & I \\ I & -I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & S \end{pmatrix}.$$

Then

$$\mathcal{S}\gamma_{\uparrow}(X)\mathcal{S}^{-1} = \begin{pmatrix} \gamma(X) & 0 \\ 0 & \gamma(X) \end{pmatrix}, \quad \mathcal{S}\gamma_{\uparrow}(Z)\mathcal{S}^{-1} = \begin{pmatrix} \gamma(Z) & 0 \\ 0 & \gamma(Z) \end{pmatrix},$$

$$\mathcal{S}\gamma_{\uparrow}(Y)\mathcal{S}^{-1} = \begin{pmatrix} \gamma(Y) & 0 \\ 0 & \gamma(Y) \end{pmatrix}, \quad \mathcal{S}\gamma_{\uparrow}(W)\mathcal{S}^{-1} = \begin{pmatrix} S & 0 \\ 0 & -S \end{pmatrix}.$$

So there are two irreducible triplets of S_4 ,

$$X = \gamma(X), \quad Z = \gamma(Z), \quad Y = \gamma(Y), \quad W = S,$$

and

$$X = \gamma(X), \quad Z = \gamma(Z), \quad Y = \gamma(Y), \quad W = -S.$$

Let us now consider the group $(Z_2 \times Z_2 \times Z_2) \rtimes A_4$. Let C_i be the Z_2 generators.

With the A_4 generators in (21), they obey

$$XC_1X^{-1} = C_1C_2C_2, \quad XC_2X^{-1} = C_3, \quad XC_3X^{-1} = C_2, \quad (2.25)$$

$$ZC_1Z^{-1} = C_3, \quad ZC_2Z^{-1} = C_1C_2C_3, \quad ZC_3Z^{-1} = C_2, \quad (2.26)$$

$$YC_1Y^{-1} = C_2, \quad YC_2Y^{-1} = C_3, \quad YC_3Y^{-1} = C_1. \quad (2.27)$$

We have the chain

$$Z_2^3 \subset (Z_2^3) \rtimes Z_2 \subset (Z_2^3) \rtimes (Z_2 \times Z_2) \subset (Z_2^3) \rtimes A_4$$

of invariant subgroups. Without much difficulty, we can see that the representations

$$(a) \quad C_1 = 1, \quad C_2 = -1, \quad C_3 = -1$$

$$(b) C_1 = -1, C_2 = 1, C_3 = 1$$

$$(c) C_1 = -1, C_2 = 1, C_3 = -1$$

$$(d) C_1 = 1, C_2 = -1, C_3 = 1$$

$$(e) C_1 = -1, C_2 = -1, C_3 = 1$$

$$(f) C_1 = 1, C_2 = 1, C_3 = -1,$$

lie in the same orbit with respect to $(Z_2^3) \rtimes A_4$. Now consider the subgroup $(Z_2^3) \rtimes Z_2$, where the last Z_2 is generated by X . From (26), under X

$$C_1 \rightarrow C_1 C_2 C_3, C_2 \leftrightarrow C_3,$$

so that

$$(a) \leftrightarrow (a), (b) \leftrightarrow (b), (c) \leftrightarrow (d), (e) \leftrightarrow (f).$$

So (a) and (b) each give two one-dimensional representations of $(Z_2^4) \rtimes Z_2$:

$$(a) C_1 = 1, C_2 = -1, C_3 = -1, X = 1$$

$$(a') C_1 = 1, C_2 = -1, C_3 = -1, X = -1$$

$$(b) C_1 = -1, C_2 = 1, C_3 = 1, X = 1$$

$$(b') C_1 = -1, C_2 = 1, C_3 = 1, X = -1,$$

while (c/d) and (e/f) each give two-dimensional irreducible representations:

$$(c/d) C_2 = M_1, C_1 = C_3 = M_2, X = S$$

$$(e/f) C_3 = M_1, C_1 = C_2 = M_2, X = S,$$

where

$$M_1 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, M_2 \equiv \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, S \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (2.28)$$

Now add Z to obtain $(Z_2^3) \rtimes (Z_2 \times Z_2)$. Under Z

$$C_1 \rightarrow C_3, C_2 \rightarrow C_1 C_2 C_3, X \leftrightarrow X,$$

so that

$$(a) \leftrightarrow (b), (a') \leftrightarrow (b'), (c/d) \leftrightarrow (c/d), (e/f) \leftrightarrow (e/f).$$

Now (a/b) and (a'/b') each give two-dimensional irreducible representations:

$$(a/b) C_1 = M_1, C_2 = C_3 = M_2, X = I, Z = S$$

$$(a'/b') C_1 = M_1, C_2 = C_3 = M_2, X = -I, Z = S$$

For (c/d) , the induced representation is

$$C_2 = \begin{pmatrix} M_1 & 0 \\ 0 & M_1 \end{pmatrix}, C_1 = C_3 = \begin{pmatrix} M_2 & 0 \\ 0 & M_2 \end{pmatrix}, X = \begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix}, Z = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

This can be block-diagonalized by inspection,

$$(c/d) C_2 = M_1, C_1 = C_3 = M_2, X = S, Z = I$$

$$(c'/d') C_2 = M_1, C_1 = C_3 = M_2, X = S, Z = -I.$$

For (e/f) , the induced representation is

$$C_1 = C_4 = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}, C_2 = C_3 = \begin{pmatrix} M_2 & 0 \\ 0 & M_1 \end{pmatrix}, X = \begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix}, Z = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

Noting that $M_2 = SM_1S$, we can block-diagonalize this using the same method that was used in the S_4 example for the triplet orbit. This gives

$$(e/f) C_1 = C_4 = M_1, C_2 = C_3 = M_2, X = S, Z = S$$

$$(e'/f') C_1 = C_4 = M_1, C_2 = C_3 = M_2, X = S, Z = -S.$$

Finally, we add Y . With a little effort, it can be seen that (a/b) , (c/d) , and (e/f) lie in one orbit, and (a'/b') , (c'/d') , and (e'/f') lie in another. Then, the induced representations are irreducible. So, we finally end up with two six-dimensional irreducible representations of $(Z_2^3) \rtimes A_4$:

$$C_1 = \begin{pmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_2 \end{pmatrix}, C_2 = \begin{pmatrix} M_2 & 0 & 0 \\ 0 & M_1 & 0 \\ 0 & 0 & M_2 \end{pmatrix}, C_3 = \begin{pmatrix} M_2 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_1 \end{pmatrix},$$

$$X = \begin{pmatrix} I & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & S \end{pmatrix}, Z = \begin{pmatrix} S & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & S \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 & I \\ I & 0 & 0 \\ 0 & I & 0 \end{pmatrix},$$

and

$$C_1 = \begin{pmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_2 \end{pmatrix}, C_2 = \begin{pmatrix} M_2 & 0 & 0 \\ 0 & M_1 & 0 \\ 0 & 0 & M_2 \end{pmatrix}, C_3 = \begin{pmatrix} M_2 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_1 \end{pmatrix},$$

$$X = \begin{pmatrix} -I & 0 & 0 \\ 0 & -S & 0 \\ 0 & 0 & S \end{pmatrix}, Z = \begin{pmatrix} S & 0 & 0 \\ 0 & -I & 0 \\ 0 & 0 & -S \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 & I \\ I & 0 & 0 \\ 0 & I & 0 \end{pmatrix}.$$

It can be checked directly that these matrices respect all of the relations (21), (26), (27), and (28).

Now consider the group $(S_3 \times S_3 \times S_3 \times S_3) \rtimes A_4$. Let A_i and B_i be the S_4 generators. They obey

$$A_i^3 = B_i^2 = E, \quad B_i A_i B_i^{-1} = A_i^{-1}. \quad (2.29)$$

The irreducible representations of S_3 are two one-dimensional representations given by

$$A_i = 1, \quad B_i = 1,$$

$$A_i = 1, \quad B_i = -1,$$

and one two-dimensional representation given by

$$A_i = M_A \equiv \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, \quad B_i = M_B \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

With the A_4 generators in (4), A_i and B_i respect the relations

$$\begin{aligned} X A_1 X^{-1} &= A_2, & X A_2 X^{-1} &= A_1, & X A_3 X^{-1} &= A_4, & X A_4 X^{-1} &= A_3, \\ X B_1 X^{-1} &= B_2, & X B_2 X^{-1} &= B_1, & X B_3 X^{-1} &= B_4, & X B_4 X^{-1} &= B_3, \end{aligned} \quad (2.30)$$

$$\begin{aligned} Z A_1 Z^{-1} &= A_3, & Z A_2 Z^{-1} &= A_4, & Z A_3 Z^{-1} &= A_1, & Z A_4 Z^{-1} &= A_2, \\ Z B_1 Z^{-1} &= B_3, & Z B_2 Z^{-1} &= B_4, & Z B_3 Z^{-1} &= B_1, & Z B_4 Z^{-1} &= B_2, \end{aligned} \quad (2.31)$$

$$\begin{aligned} Y A_1 Y^{-1} &= A_1, & Y A_2 Y^{-1} &= A_3, & Y A_3 Y^{-1} &= A_4, & Y A_4 Y^{-1} &= A_2, \\ Y B_1 Y^{-1} &= B_1, & Y B_2 Y^{-1} &= B_3, & Y B_3 Y^{-1} &= B_4, & Y B_4 Y^{-1} &= B_2. \end{aligned} \quad (2.32)$$

As in the last example, we have a chain of invariant subgroups,

$$S_3^4 \subset (S_3^4) \rtimes Z_2 \subset (S_3^4) \rtimes (Z_2 \times Z_2) \subset (S_3^4) \rtimes A_4.$$

The representation $(2, 2, 2, 2)$ under $S_3 \times S_3 \times S_3 \times S_3$ lies in its own orbit. This representation can be written in terms of 16-dimensional matrices,

$$\gamma(A_1) = M_A \otimes I \otimes I \otimes I, \quad \gamma(B_1) = M_B \otimes I \otimes I \otimes I,$$

$$\gamma(A_2) = I \otimes M_A \otimes I \otimes I, \quad \gamma(B_2) = I \otimes M_B \otimes I \otimes I,$$

$$\gamma(A_3) = I \otimes I \otimes M_A \otimes I, \quad \gamma(B_3) = I \otimes I \otimes M_B \otimes I,$$

$$\gamma(A_4) = I \otimes I \otimes I \otimes M_A, \quad \gamma(B_4) = I \otimes I \otimes I \otimes M_B.$$

Now add X to obtain the subgroup $(S_3^4) \rtimes Z_2$. From (14), under X

$$A_1 \leftrightarrow A_2, \quad A_3 \leftrightarrow A_4,$$

$$B_1 \leftrightarrow B_2, \quad B_3 \leftrightarrow B_4.$$

Consider how this rearranges the eigenvalues of each of the 16 basis states under the diagonal generators (A_1, A_2, A_3, A_4) ,

$$(1) (\omega, \omega, \omega, \omega) \longrightarrow (\omega, \omega, \omega, \omega) \sim (1)$$

$$(2) (\omega^2, \omega, \omega, \omega) \longrightarrow (\omega, \omega^2, \omega, \omega) \sim (3)$$

$$(3) (\omega, \omega^2, \omega, \omega) \longrightarrow (\omega^2, \omega, \omega, \omega) \sim (2)$$

$$(4) (\omega^2, \omega^2, \omega, \omega) \longrightarrow (\omega^2, \omega^2, \omega, \omega) \sim (4)$$

$$(5) (\omega, \omega, \omega^2, \omega) \longrightarrow (\omega, \omega, \omega, \omega^2) \sim (9)$$

$$(6) (\omega^2, \omega, \omega^2, \omega) \longrightarrow (\omega, \omega^2, \omega, \omega^2) \sim (11)$$

$$(7) (\omega, \omega^2, \omega^2, \omega) \longrightarrow (\omega^2, \omega, \omega, \omega^2) \sim (10)$$

$$(8) (\omega^2, \omega^2, \omega^2, \omega) \longrightarrow (\omega^2, \omega^2, \omega, \omega^2) \sim (12)$$

$$(9) (\omega, \omega, \omega, \omega^2) \longrightarrow (\omega, \omega, \omega^2, \omega) \sim (5)$$

$$(10) (\omega^2, \omega, \omega, \omega^2) \longrightarrow (\omega, \omega^2, \omega^2, \omega) \sim (7)$$

$$(11) (\omega, \omega^2, \omega, \omega^2) \longrightarrow (\omega^2, \omega, \omega^2, \omega) \sim (6)$$

$$(12) (\omega^2, \omega^2, \omega, \omega^2) \longrightarrow (\omega^2, \omega^2, \omega^2, \omega) \sim (8)$$

$$(13) (\omega, \omega, \omega^2, \omega^2) \longrightarrow (\omega, \omega, \omega^2, \omega^2) \sim (13)$$

$$(14) (\omega^2, \omega, \omega^2, \omega^2) \longrightarrow (\omega, \omega^2, \omega^2, \omega^2) \sim (15)$$

$$(15) (\omega, \omega^2, \omega^2, \omega^2) \longrightarrow (\omega^2, \omega, \omega^2, \omega^2) \sim (14)$$

$$(16) (\omega^2, \omega^2, \omega^2, \omega^2) \longrightarrow (\omega^2, \omega^2, \omega^2, \omega^2) \sim (16).$$

This yields the permutation matrix

$$P_X \equiv \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Note that $P_X^{-1} = P_X$. We can now check directly that

$$P_X \gamma(B_1) P_X^{-1} = \gamma(B_2), \quad P_X \gamma(B_2) P_X^{-1} = \gamma(B_1),$$

$$P_X \gamma(B_3) P_X^{-1} = \gamma(B_4), \quad P_X \gamma(B_4) P_X^{-1} = \gamma(B_3).$$

So, we obtain two 16-dimensional irreducible representations of $(S_3^4) \rtimes Z_2$,

$$(a) \quad A_i = \gamma(A_i), \quad B_i = \gamma(B_i), \quad X = P_X$$

$$(b) \quad A_i = \gamma(A_i), \quad B_i = \gamma(B_i), \quad X = -P_X.$$

Next add Z to obtain the subgroup $(S_3^4) \rtimes (Z_2 \times Z_2)$. Proceeding as is the previous step yields

$$P_Z \equiv \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Again note that $P_Z^{-1} = P_Z$. This gives four 16-dimensional irreducible representations of $(S_3^4) \rtimes (Z_2 \times Z_2)$,

$$(a) A_i = \gamma(A_i), B_i = \gamma(B_i), X = P_X, Z = P_Z$$

$$(a') A_i = \gamma(A_i), B_i = \gamma(B_i), X = P_X, Z = -P_Z$$

$$(b) A_i = \gamma(A_i), B_i = \gamma(B_i), X = -P_X, Z = P_Z$$

$$(b') A_i = \gamma(A_i), B_i = \gamma(B_i), X = -P_X, Z = -P_Z.$$

Finally, add Y to obtain $(S_3^4) \times A_4$. Then, (a) lies in its own orbit, while (a'), (b), and (b') lie in another orbit. First, consider the orbit of (a). We find that the permutation matrix

$$P_Y \equiv \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

respects the relations

$$P_Y \gamma(A_1) P_Y^{-1} = \gamma(A_1), \quad P_Y \gamma(A_2) P_Y^{-1} = \gamma(A_3), \quad P_Y \gamma(A_3) P_Y^{-1} = \gamma(A_4), \quad P_Y \gamma(A_4) P_Y^{-1} = \gamma(A_2),$$

$$P_Y \gamma(B_1) P_Y^{-1} = \gamma(B_1), \quad P_Y \gamma(B_2) P_Y^{-1} = \gamma(B_3), \quad P_Y \gamma(B_3) P_Y^{-1} = \gamma(B_4), \quad P_Y \gamma(B_4) P_Y^{-1} = \gamma(B_2),$$

$$P_Y P_X P_Y^{-1} = P_Z, \quad P_Y P_Z P_Y^{-1} = P_X P_Z, \quad P_Y (P_X P_Z) P_Y^{-1} = P_X.$$

(Note that $P_Y^{-1} = P_Y^2$.) The induced representation can then be written

$$\begin{aligned} \gamma_{\uparrow}(A_i) &= \begin{pmatrix} \gamma(A_i) & 0 & 0 \\ 0 & P_Y^2 \gamma(A_i) P_Y & 0 \\ 0 & 0 & P_Y \gamma(A_i) P_Y^2 \end{pmatrix}, \quad \gamma_{\uparrow}(B_i) = \begin{pmatrix} \gamma(B_i) & 0 & 0 \\ 0 & P_Y^2 \gamma(B_i) P_Y & 0 \\ 0 & 0 & P_Y \gamma(B_i) P_Y^2 \end{pmatrix} \\ \gamma_{\uparrow}(X) &= \begin{pmatrix} \gamma(X) & 0 & 0 \\ 0 & P_Y^2 \gamma(X) P_Y & 0 \\ 0 & 0 & P_Y \gamma(X) P_Y^2 \end{pmatrix}, \quad \gamma_{\uparrow}(Z) = \begin{pmatrix} \gamma(Z) & 0 & 0 \\ 0 & P_Y^2 \gamma(Z) P_Y & 0 \\ 0 & 0 & P_Y \gamma(Z) P_Y^2 \end{pmatrix}, \\ \gamma_{\uparrow}(Y) &= \begin{pmatrix} 0 & 0 & I \\ I & 0 & 0 \\ 0 & I & 0 \end{pmatrix}. \end{aligned}$$

Let

$$\mathcal{P} = \frac{1}{\sqrt{3}} \begin{pmatrix} I & I & I \\ I & \omega I & \omega^2 I \\ I & \omega^2 I & \omega I \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ 0 & P_Y & 0 \\ 0 & 0 & P_Y^2 \end{pmatrix}.$$

Then

$$\begin{aligned} \mathcal{P} \gamma_{\uparrow}(A_i) \mathcal{P}^{-1} &= \begin{pmatrix} \gamma(A_i) & 0 & 0 \\ 0 & \gamma(A_i) & 0 \\ 0 & 0 & \gamma(A_i) \end{pmatrix}, \quad \mathcal{P} \gamma_{\uparrow}(B_i) \mathcal{P}^{-1} = \begin{pmatrix} \gamma(B_i) & 0 & 0 \\ 0 & \gamma(B_i) & 0 \\ 0 & 0 & \gamma(B_i) \end{pmatrix}, \\ \mathcal{P} \gamma_{\uparrow}(X) \mathcal{P}^{-1} &= \begin{pmatrix} \gamma(X) & 0 & 0 \\ 0 & \gamma(X) & 0 \\ 0 & 0 & \gamma(X) \end{pmatrix}, \quad \mathcal{P} \gamma_{\uparrow}(Z) \mathcal{P}^{-1} = \begin{pmatrix} \gamma(Z) & 0 & 0 \\ 0 & \gamma(Z) & 0 \\ 0 & 0 & \gamma(Z) \end{pmatrix}, \\ \mathcal{P} \gamma_{\uparrow}(Y) \mathcal{P}^{-1} &= \begin{pmatrix} P_Y & 0 & 0 \\ 0 & \omega P_Y & 0 \\ 0 & 0 & \omega^2 P_Y \end{pmatrix}. \end{aligned}$$

So the result is three 16-dimensional irreducible representations of $(S_3^4) \times A_4$. For the other orbit, the induced representation is irreducible. It is given by

$$\begin{aligned} \gamma_{\uparrow}(A_1) &= \begin{pmatrix} \gamma(A_1) & 0 & 0 \\ 0 & \gamma(A_1) & 0 \\ 0 & 0 & \gamma(A_1) \end{pmatrix}, \quad \gamma_{\uparrow}(B_1) = \begin{pmatrix} \gamma(B_1) & 0 & 0 \\ 0 & \gamma(B_1) & 0 \\ 0 & 0 & \gamma(B_1) \end{pmatrix}, \\ \gamma_{\uparrow}(A_2) &= \begin{pmatrix} \gamma(A_2) & 0 & 0 \\ 0 & \gamma(A_4) & 0 \\ 0 & 0 & \gamma(A_3) \end{pmatrix}, \quad \gamma_{\uparrow}(B_2) = \begin{pmatrix} \gamma(B_2) & 0 & 0 \\ 0 & \gamma(B_4) & 0 \\ 0 & 0 & \gamma(B_3) \end{pmatrix}, \\ \gamma_{\uparrow}(A_3) &= \begin{pmatrix} \gamma(A_3) & 0 & 0 \\ 0 & \gamma(A_2) & 0 \\ 0 & 0 & \gamma(A_4) \end{pmatrix}, \quad \gamma_{\uparrow}(B_3) = \begin{pmatrix} \gamma(B_3) & 0 & 0 \\ 0 & \gamma(B_2) & 0 \\ 0 & 0 & \gamma(B_4) \end{pmatrix}, \\ \gamma_{\uparrow}(A_4) &= \begin{pmatrix} \gamma(A_4) & 0 & 0 \\ 0 & \gamma(A_3) & 0 \\ 0 & 0 & \gamma(A_2) \end{pmatrix}, \quad \gamma_{\uparrow}(B_4) = \begin{pmatrix} \gamma(B_4) & 0 & 0 \\ 0 & \gamma(B_3) & 0 \\ 0 & 0 & \gamma(B_2) \end{pmatrix}, \\ \gamma_{\uparrow}(X) &= \begin{pmatrix} P_X & 0 & 0 \\ 0 & -P_X P_Z & 0 \\ 0 & 0 & -P_Z \end{pmatrix}, \quad \gamma_{\uparrow}(Z) = \begin{pmatrix} -P_Z & 0 & 0 \\ 0 & P_X & 0 \\ 0 & 0 & -P_X P_Z \end{pmatrix}, \quad Y^{(48)} = \begin{pmatrix} 0 & 0 & I \\ I & 0 & 0 \\ 0 & I & 0 \end{pmatrix}. \end{aligned}$$

We can also see that the representations $(2, 1, 1, 1)$, $(1, 2, 1, 1)$, $(1, 1, 2, 1)$, and $(1, 1, 1, 2)$ of $S_3 \times S_3 \times S_3 \times S_3$ make up an orbit. The 8-dimensional representation of $(S_3^4) \times A_4$ this orbit gives rise to is given by

$$A_1 = \text{diag}(\omega, \omega^2, 1, 1, 1, 1, 1, 1), \quad A_2 = \text{diag}(1, 1, \omega, \omega^2, 1, 1, 1, 1),$$

$$A_3 = \text{diag}(1, 1, 1, 1, \omega, \omega^2, 1, 1), \quad A_4 = \text{diag}(1, 1, 1, 1, 1, 1, \omega, \omega^2),$$

$$Y = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

2.4 Invariants Under the Discrete Symmetry

In this section, we give the symmetry invariants which are used in our model. These can be computed directly from the matrices given in the previous section.

$16_{AB} \times 16_{AB}$ invariant ($x_i, x'_j \sim 16_{AB}$):

$$\begin{aligned} f_1(x_i, x'_j) &= x_1x'_{16} + x_2x'_{15} + x_3x'_{14} + x_4x'_{13} + x_5x'_{12} + x_6x'_{11} + x_7x'_{10} + x_8x'_9 \\ &\quad + x_9x'_8 + x_{10}x'_7 + x_{11}x'_6 + x_{12}x'_5 + x_{13}x'_4 + x_{14}x'_3 + x_{15}x'_2 + x_{16}x'_1 \end{aligned}$$

$6_C \times 6_C$ invariant ($w_i, w'_j \sim 6_C$):

$$f_2(w_i, w'_j) = w_1w'_1 + w_2w'_2 + w_3w'_3 + w_4w'_4 + w_5w'_5 + w_6w'_6.$$

$48_{AB} \times 48_{AB}$ invariant ($y_i, y'_j \sim 48_{AB}$):

$$f_3(y_i, y'_j) = y_1y'_{16} + y_2y'_{15} + y_3y'_{14} + y_4y'_{13} + y_5y'_{12} + y_6y'_{11} + y_7y'_{10} + y_8y'_9$$

$$\begin{aligned}
& y_9 y'_8 + y_{10} y'_7 + y_{11} y'_6 + y_{12} y'_5 + y_{13} y'_4 + y_{14} y'_3 + y_{15} y'_2 + y_{16} y'_1 + y_{17} y'_{32} + y_{18} y'_{31} \\
& + y_{19} y'_{30} + y_{20} y'_{29} + y_{21} y'_{28} + y_{22} y'_{27} + y_{23} y'_{26} + y_{24} y'_{25} + y_{25} y'_{24} + y_{26} y'_{23} + y_{27} y'_{22} + y_{28} y'_{21} \\
& + y_{29} y'_{20} + y_{30} y'_{19} + y_{31} y'_{18} + y_{32} y'_{17} + y_{33} y'_{16} + y_{34} y'_{15} + y_{35} y'_{14} + y_{36} y'_{13} + y_{37} y'_{12} + y_{38} y'_{11} \\
& + y_{39} y'_{10} + y_{40} y'_{9} + y_{41} y'_{8} + y_{42} y'_{7} + y_{43} y'_{6} + y_{44} y'_{5} + y_{45} y'_{4} + y_{46} y'_{3} + y_{47} y'_{2} + y_{48} y'_{1}
\end{aligned}$$

$8_{AB} \times 8_{AB}$ invariant ($z_i, z'_j \sim 8_{AB}$):

$$f_4(z_i, z'_j) = z_1 z'_2 + z_2 z'_1 + z_3 z'_4 + z_4 z'_3 + z_5 z'_6 + z_6 z'_5 + z_7 z'_8 + z_8 z'_7$$

$16_{AB} \times 16_{AB} \times 16_{AB}$ invariant ($x_i, x'_j, x''_k \sim 16_{AB}$):

$$\begin{aligned}
g_1(x_i, x'_j, x''_k) = & x_1 x'_1 x''_1 + x_2 x'_2 x''_2 + x_3 x'_3 x''_3 + x_4 x'_4 x''_4 + x_5 x'_5 x''_5 + x_6 x'_6 x''_6 + x_7 x'_7 x''_7 + x_8 x'_8 x''_8 \\
& + x_9 x'_9 x''_9 + x_{10} x'_{10} x''_{10} + x_{11} x'_{11} x''_{11} + x_{12} x'_{12} x''_{12} + x_{13} x'_{13} x''_{13} + x_{14} x'_{14} x''_{14} + x_{15} x'_{15} x''_{15} + x_{16} x'_{16} x''_{16}
\end{aligned}$$

$6_C \times 6_C \times 6_C$ invariant ($w_i, w'_j, w''_k \sim 6_C$):

$$\begin{aligned}
g_2(w_i, w'_j, w''_k) = & w_1 w'_3 w''_5 + w_5 w'_1 w''_3 + w_3 w'_5 w''_1 + w_1 w'_4 w''_6 + w_6 w'_1 w''_4 + w_4 w'_6 w''_1 \\
& + w_2 w'_3 w''_6 + w_6 w'_2 w''_3 + w_3 w'_6 w''_2 + w_2 w'_4 w''_5 + w_5 w'_2 w''_4 + w_4 w'_5 w''_2
\end{aligned}$$

$3 \times 16_{AB} \times 48_{AB}$ invariant ($t_i \sim 3, x_j \sim 16_{AB}, y_k \sim 48_{AB}$):

$$g_3(t_i, x_j, y_k) = t_1 (x_{16} y_{33} + x_{15} y_{34} + x_8 y_{35} + x_7 y_{36} + x_{14} y_{37} + x_{13} y_{38} + x_6 y_{39} + x_5 y_{40})$$

$$+x_{12}y_{41} + x_{11}y_{42} + x_4y_{43} + x_3y_{44} + x_{10}y_{45} + x_9y_{46} + x_2y_{47} + x_1y_{48})$$

$$t_2(x_{16}y_1 + x_7y_{10} + x_6y_{11} + x_5y_{12} + x_4y_{13} + x_3y_{14} + x_2y_{15} + x_1y_{16}$$

$$+x_{15}y_2 + x_{14}y_3 + x_{13}y_4 + x_{12}y_5 + x_{11}y_6 + x_{10}y_7 + x_9y_8 + x_8y_9)$$

$$t_3(x_{16}y_{17} + x_{15}y_{18} + x_{12}y_{19} + x_{11}y_{20} + x_8y_{21} + x_7y_{22} + x_4y_{23} + x_3y_{24}$$

$$+x_{14}y_{25} + x_{13}y_{26} + x_{10}y_{27} + x_9y_{28} + x_6y_{29} + x_5y_{30} + x_2y_{31} + x_1y_{32})$$

$8_{AB} \times 16_{AB} \times 48_{AB}$ invariant ($z_i \sim 8_{AB}$, $x_j \sim 16_{AB}$, $y_k \sim 48_{AB}$):

$$g_4(z_i, x_k, y_j) =$$

$$z_1(x_{15}y_1 + x_5y_{11} + x_3y_{13} + x_1y_{15} + x_{15}y_{17} + x_{11}y_{19} + x_7y_{21} + x_3y_{23} + x_{13}y_{25} + x_9y_{27} + x_5y_{29} + x_{13}y_3$$

$$+x_1y_{31} + x_{15}y_{33} + x_7y_{35} + x_{13}y_{37} + x_5y_{39} + x_{11}y_{41} + x_3y_{43} + x_9y_{45} + x_1y_{47} + x_{11}y_5 + x_9y_7 + x_7y_9)$$

$$+z_2(x_8y_{10} + x_6y_{12} + x_4y_{14} + x_2y_{16} + x_{16}y_{18} + x_{16}y_2 + x_{12}y_{20} + x_8y_{22} + x_4y_{24} + x_{14}y_{26} + x_{10}y_{28} + x_6y_{30}$$

$$+x_2y_{32} + x_{16}y_{34} + x_8y_{36} + x_{14}y_{38} + x_{14}y_4 + x_6y_{40} + x_{12}y_{42} + x_4y_{44} + x_{10}y_{46} + x_2y_{48} + x_{12}y_6 + x_{10}y_8)$$

$$+z_3(x_{14}y_1 + x_5y_{10} + x_2y_{13} + x_1y_{14} - x_{14}y_{17} - x_{13}y_{18} - x_{10}y_{19} + x_{13}y_2 - x_9y_{20} - x_6y_{21} - x_5y_{22} - x_2y_{23}$$

$$-x_1y_{24} - x_{14}y_{33} - x_{13}y_{34} - x_6y_{35} - x_5y_{36} - x_{10}y_{41} - x_9y_{42} - x_2y_{43} - x_1y_{44} + x_{10}y_5 + x_9y_6 + x_6y_9)$$

$$+z_4(x_8y_{11} + x_7y_{12} + x_4y_{15} + x_3y_{16} - x_{16}y_{25} - x_{15}y_{26} - x_{12}y_{27} - x_{11}y_{28} - x_8y_{29} + x_{16}y_3 - x_7y_{30} - x_4y_{31}$$

$$-x_3y_{32} - x_{16}y_{37} - x_{15}y_{38} - x_8y_{39} + x_{15}y_4 - x_7y_{40} - x_{12}y_{45} - x_{11}y_{46} - x_4y_{47} - x_3y_{48} + x_{12}y_7 + x_{11}y_8)$$

$$+z_5(-x_{12}y_1 - x_3y_{10} - x_2y_{11} - x_1y_{12} + x_{12}y_{17} + x_{11}y_{18} - x_{11}y_2 + x_4y_{21} + x_3y_{22} + x_{10}y_{25} + x_9y_{26} + x_2y_{29}$$

$$-x_{10}y_3 + x_1y_{30} - x_{12}y_{33} - x_{11}y_{34} - x_4y_{35} - x_3y_{36} - x_{10}y_{37} - x_9y_{38} - x_2y_{39} - x_9y_4 - x_1y_{40} - x_4y_9)$$

$$+z_6(-x_8y_{13} - x_7y_{14} - x_6y_{15} - x_5y_{16} + x_{16}y_{19} + x_{15}y_{20} + x_8y_{23} + x_7y_{24} + x_{14}y_{27} + x_{13}y_{28} + x_6y_{31} + x_5y_{32}$$

$$-x_{16}y_{41} - x_{15}y_{42} - x_8y_{43} - x_7y_{44} - x_{14}y_{45} - x_{13}y_{46} - x_6y_{47} - x_5y_{48} - x_{16}y_5 - x_{15}y_6 - x_{14}y_7 - x_{13}y_8)$$

$$+z_7(-x_8y_1 - x_8y_{17} - x_7y_{18} - x_4y_{19} - x_7y_2 - x_3y_{20} - x_6y_{25} - x_5y_{26} - x_2y_{27} - x_1y_{28} - x_6y_3 + x_8y_{33}$$

$$+x_7y_{34} + x_6y_{37} + x_5y_{38} - x_5y_4 + x_4y_{41} + x_3y_{42} + x_2y_{45} + x_1y_{46} - x_4y_5 - x_3y_6 - x_2y_7 - x_1y_8)$$

$$+z_8(x_{15}y_{36} - x_{14}y_{11} - x_{13}y_{12} - x_{12}y_{13} - x_{11}y_{14} - x_{10}y_{15} - x_9y_{16} - x_{16}y_{21} - x_{15}y_{22} - x_{12}y_{23} - x_{11}y_{24} - x_{14}y_{29}$$

$$-x_{13}y_{30} - x_{10}y_{31} - x_9y_{32} + x_{16}y_{35} - x_{15}y_{10} + x_{14}y_{39} + x_{13}y_{40} + x_{12}y_{43} + x_{11}y_{44} + x_{10}y_{47} + x_9y_{48} - x_{16}y_9)$$

$16_{AB} \times 48_{AB} \times 48_{AB}$ invariant ($x_i \sim 16_{AB}; y_j, y'_k \sim 48_{AB}$):

$$\begin{aligned}
g_5(x_i, y_j, y'_k) = & x_1 y_1 y'_1 + x_2 y_2 y'_2 + x_3 y_3 y'_3 + x_4 y_4 y'_4 + x_5 y_5 y'_5 + x_6 y_6 y'_6 + x_7 y_7 y'_7 + x_8 y_8 y'_8 \\
& + x_9 y_9 y'_9 + x_{10} y_{10} y'_{10} + x_{11} y_{11} y'_{11} + x_{12} y_{12} y'_{12} + x_{13} y_{13} y'_{13} + x_{14} y_{14} y'_{14} + x_{15} y_{15} y'_{15} + x_{16} y_{16} y'_{16} \\
& + x_{17} y_{17} y'_{17} + x_{18} y_{18} y'_{18} + x_{19} y_{19} y'_{19} + x_{20} y_{20} y'_{20} + x_{21} y_{21} y'_{21} + x_{22} y_{22} y'_{22} + x_{23} y_{23} y'_{23} + x_{24} y_{24} y'_{24} \\
& + x_{25} y_{25} y'_{25} + x_{26} y_{26} y'_{26} + x_{27} y_{27} y'_{27} + x_{28} y_{28} y'_{28} + x_{29} y_{29} y'_{29} + x_{30} y_{30} y'_{30} + x_{31} y_{31} y'_{31} + x_{32} y_{32} y'_{32} \\
& + x_{33} y_{33} y'_{33} + x_{34} y_{34} y'_{34} + x_{35} y_{35} y'_{35} + x_{36} y_{36} y'_{36} + x_{37} y_{37} y'_{37} + x_{38} y_{38} y'_{38} + x_{39} y_{39} y'_{39} + x_{40} y_{40} y'_{40} \\
& + x_{41} y_{41} y'_{41} + x_{42} y_{42} y'_{42} + x_{43} y_{43} y'_{43} + x_{44} y_{44} y'_{44} + x_{45} y_{45} y'_{45} + x_{46} y_{46} y'_{46} + x_{47} y_{47} y'_{47} + x_{48} y_{48} y'_{48}
\end{aligned}$$

$3 \times 6_C \times 6_C$ invariant ($t_i \sim 3; w_j, w'_k \sim 6_C$):

$$g_6(t_i, w_j, w'_k) = t_1(w_5 w'_5 - w_6 w'_6) + t_2(w_1 w'_1 - w_2 w'_2) + t_3(w_3 w'_3 - w_4 w'_4)$$

$1' \times 6_C \times 6_C$ invariant ($s' \sim 1'; w_i, w'_j \sim 6_C$):

$$g_7(s', w_i, w'_j) = s'(w_1 w'_1 + w_2 w'_2 + \omega^2 w_3 w'_3 + \omega^2 w_4 w'_4 + \omega w_5 w'_5 + \omega w_6 w'_6)$$

$1'' \times 6_C \times 6_C$ invariant ($s'' \sim 1''; w_i, w'_j \sim 6_C$):

$$g_8(s'', w_i, w'_j) = s''(w_1 w'_1 + w_2 w'_2 + \omega w_3 w'_3 + \omega w_4 w'_4 + \omega^2 w_5 w'_5 + \omega^2 w_6 w'_6)$$

For our purposes, it suffices to have the $16_{AB} \times 16_{AB} \times 16_{AB} \times 16_{AB}$ and $6_C \times 6_C \times 6_C \times 6_C$

invariants for the case where all four fields are the same.

$16_{AB} \times 16_{AB} \times 16_{AB} \times 16_{AB}$ invariants ($x_i \sim 16_{AB}$):

$$h_1(x_i) = x_1^2 x_{16}^2 + x_2^2 x_{15}^2 + x_3^2 x_{14}^2 + x_4^2 x_{13}^2 + x_5^2 x_{12}^2 + x_6^2 x_{11}^2 + x_7^2 x_{10}^2 + x_8^2 x_9^2,$$

$$h_2(x_i) = x_1 x_2 x_{15} x_{16} + x_1 x_3 x_{14} x_{16} + x_2 x_4 x_{13} x_{15} + x_3 x_4 x_{13} x_{14} + x_1 x_5 x_{12} x_{16} + x_4 x_5 x_{12} x_{13}$$

$$+ x_2 x_6 x_{11} x_{15} + x_3 x_6 x_{11} x_{14} + x_5 x_6 x_{11} x_{12} + x_2 x_7 x_{10} x_{15} + x_3 x_7 x_{10} x_{14} + x_5 x_7 x_{10} x_{12}$$

$$+ x_1 x_8 x_9 x_{16} + x_4 x_8 x_9 x_{13} + x_6 x_8 x_9 x_{11} + x_7 x_8 x_9 x_{10}$$

$6_C \times 6_C \times 6_C \times 6_C$ invariants ($w_i \sim 6_C$):

$$h_3(w_i) = w_1^4 + w_2^4 + w_3^4 + w_4^4 + w_5^4 + w_6^4,$$

$$h_4(w_i) = w_1^2 w_2^2 + w_3^2 w_4^2 + w_5^2 w_6^2,$$

$$h_5(w_i) = w_1^2 w_3^2 + w_1^2 w_4^2 + w_1^2 w_5^2 + w_1^2 w_6^2 + w_2^2 w_3^2 + w_2^2 w_4^2 + w_2^2 w_5^2 + w_2^2 w_6^2$$

$$+ w_3^2 w_5^2 + w_3^2 w_6^2 + w_4^2 w_5^2 + w_4^2 w_6^2,$$

$$h_6(w_i) = w_1 w_2 w_3 w_4 + w_1 w_2 w_5 w_6 + w_3 w_4 w_5 w_6$$

2.5 Calculation of the Neutrino Mass Matrix

In this section, we show how the neutrino mass matrix is computed. From Section 2.4, the term in Eq. (13) that mixes N and N' is²

$$\begin{aligned}
g_3(N, \langle \phi \rangle, N') &= v_\phi N_1 (N'_{35} + N'_{36} + N'_{39} + N'_{40} + N'_{41} + N'_{42} + N'_{45} + N'_{46}) \\
&+ v_\phi N_2 (N'_5 + N'_6 + N'_7 + N'_8 + N'_9 + N'_{10} + N'_{11} + N'_{12}) \\
&+ v_\phi N_3 (N'_{19} + N'_{20} + N'_{21} + N'_{22} + N'_{27} + N'_{28} + N'_{29} + N'_{30}).
\end{aligned}$$

The term that mixes N' and N'' is

$$\begin{aligned}
g_4(N'', \langle \phi \rangle, N') &= v_\phi N''_1 (N'_5 + N'_7 + N'_9 + N'_{11}) + v_\phi N''_2 (N'_6 + N'_8 + N'_{10} + N'_{12}) \\
&+ v_\phi N''_3 (N'_5 + N'_6 + N'_9 + N'_{10}) + v_\phi N''_4 (N'_7 + N'_8 + N'_{11} + N'_{12}) \\
&- v_\phi N''_5 (N'_1 + N'_2 + N'_3 + N'_4) - v_\phi N''_6 (N'_{13} + N'_{14} + N'_{15} + N'_{16}) \\
&- v_\phi N''_7 (N'_1 + N'_2 + N'_3 + N'_4) - v_\phi N''_8 (N'_{13} + N'_{14} + N'_{15} + N'_{16}) \\
&+ v_\phi N''_1 (N'_{19} + N'_{21} + N'_{27} + N'_{29}) + v_\phi N''_2 (N'_{20} + N'_{22} + N'_{28} + N'_{30}) \\
&- v_\phi N''_3 (N'_{19} + N'_{20} + N'_{21} + N'_{22}) - v_\phi N''_4 (N'_{27} + N'_{28} + N'_{29} + N'_{30}) \\
&+ v_\phi N''_5 (N'_{17} + N'_{18} + N'_{25} + N'_{26}) + v_\phi N''_6 (N'_{23} + N'_{29} + N'_{31} + N'_{32}) \\
&- v_\phi N''_7 (N'_{17} + N'_{18} + N'_{25} + N'_{26}) - v_\phi N''_8 (N'_{23} + N'_{29} + N'_{31} + N'_{32}) \\
&+ v_\phi N''_1 (N'_{35} + N'_{39} + N'_{41} + N'_{45}) + v_\phi N''_2 (N'_{36} + N'_{40} + N'_{42} + N'_{46})
\end{aligned}$$

²Note that, in the 16_{AB} basis used here, $\phi_{17-i} = \phi_i^*$, $i = 1 - 8$.

$$\begin{aligned}
& -v_\phi N_3''(N'_{35} + N'_{36} + N'_{41} + N'_{42}) - v_\phi N_4''(N'_{39} + N'_{40} + N'_{45} + N'_{46}) \\
& + v_\phi N_5''(N'_{33} + N'_{34} + N'_{37} + N'_{38}) + v_\phi N_6''(N'_{43} + N'_{44} + N'_{47} + N'_{48}) \\
& - v_\phi N_7''(N'_{33} + N'_{34} + N'_{37} + N'_{38}) - v_\phi N_8''(N'_{43} + N'_{44} + N'_{47} + N'_{48})
\end{aligned}$$

Since the symmetries B_1 , B_2 , B_3B_4 , and A_3A_4 are unbroken, components of N' and N'' that transform under these symmetries cannot mix with the light neutrinos. This leaves

$$\begin{aligned}
p_1 &= \frac{N'_5 + N'_6 + N'_7 + N'_8 + N'_9 + N'_{10} + N'_{11} + N'_{12}}{\sqrt{8}}, \\
p_2 &= \frac{N'_{19} + N'_{20} + N'_{21} + N'_{22} + N'_{27} + N'_{28} + N'_{29} + N'_{30}}{\sqrt{8}}, \\
p_3 &= \frac{N'_{35} + N'_{36} + N'_{39} + N'_{40} + N'_{41} + N'_{42} + N'_{45} + N'_{46}}{\sqrt{8}}, \\
q_1 &= \frac{N''_1 + N''_2}{\sqrt{2}}, \quad q_2 = \frac{N''_3 + N''_4}{\sqrt{2}}.
\end{aligned}$$

We now have

$$g_3(N, \langle \phi \rangle, N') = \sqrt{8}v_\phi(N_1p_3 + N_2p_1 + N_3p_2),$$

$$g_4(N'', \langle \phi \rangle, N') = 2v_\phi(q_1p_1 + q_2p_1 + q_1p_2 - q_2p_2 + q_1p_3 - q_2p_3) + \dots,$$

where the ellipses in the second equation refer to terms involving only decoupled components. The mass matrix for $(\nu_1, \nu_2, \nu_3, N_1, N_2, N_3, p_1, p_2, p_3, q_1, q_2)$ has the form

$$\frac{1}{2}\mathcal{M}_\nu = \begin{pmatrix} 0 & m \\ m^T & M \end{pmatrix},$$

with

$$m = \begin{pmatrix} \frac{1}{2}\lambda v & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}\lambda v & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}\lambda v & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$M = \begin{pmatrix} m_N & 0 & 0 & 0 & 0 & \sqrt{2}\alpha_1 v_\phi & 0 & 0 \\ 0 & m_N & 0 & \sqrt{2}\alpha_1 v_\phi & 0 & 0 & 0 & 0 \\ 0 & 0 & m_N & 0 & \sqrt{2}\alpha_1 v_\phi & 0 & 0 & 0 \\ 0 & \sqrt{2}\alpha_1 v_\phi & 0 & m'_N + \beta v_\phi & 0 & 0 & \alpha_2 v_\phi & \alpha_2 v_\phi \\ 0 & 0 & \sqrt{2}\alpha_1 v_\phi & 0 & m'_N + \beta v_\phi & 0 & \alpha_2 v_\phi & -\alpha_2 v_\phi \\ \sqrt{2}\alpha_1 v_\phi & 0 & 0 & 0 & 0 & m'_N + \beta v_\phi & \alpha_2 v_\phi & -\alpha_2 v_\phi \\ 0 & 0 & 0 & \alpha_2 v_\phi & \alpha_2 v_\phi & \alpha_2 v_\phi & m''_N & 0 \\ 0 & 0 & 0 & \alpha_2 v_\phi & -\alpha_2 v_\phi & -\alpha_2 v_\phi & 0 & m''_N \end{pmatrix}.$$

Here, m only contains entries at the EW scale, while M contains entries at the higher scale M_* . To order M_W^2/M_*^2 , \mathcal{M}_ν is block-diagonalized by

$$\mathcal{U}_\nu = \begin{pmatrix} I & -mM^{-1} \\ M^{-1}m^T & I \end{pmatrix}.$$

The light neutrino mass matrix M_ν is given by the upper-left block of $\mathcal{U}_\nu \mathcal{M}_\nu \mathcal{U}_\nu^T$,

$$\frac{1}{2}M_\nu = -mM^{-1}m^T.$$

Let

$$S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Then,

$$S^{-1}MS = \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix},$$

with

$$A = \begin{pmatrix} m_N & \sqrt{2}\alpha_1 v_\phi & 0 \\ \sqrt{2}\alpha_1 v_\phi & m'_N + \beta v_\phi & 2\alpha_2 v_\phi \\ 0 & 2\alpha_2 v_\phi & m''_N \end{pmatrix},$$

$$B = \begin{pmatrix} m_N & \sqrt{2}\alpha_1 v_\phi & 0 \\ \sqrt{2}\alpha_1 v_\phi & m'_N + \beta v_\phi & \sqrt{2}\alpha_2 v_\phi \\ 0 & \sqrt{2}\alpha_2 v_\phi & m''_N \end{pmatrix},$$

$$C = \begin{pmatrix} m_N & -\sqrt{2}\alpha_1 v_\phi \\ -\sqrt{2}\alpha_1 v_\phi & m'_N + \beta v_\phi \end{pmatrix}.$$

So, we can write

$$\begin{aligned} \frac{1}{2}M_\nu &= -mS \begin{pmatrix} A^{-1} & 0 & 0 \\ 0 & B^{-1} & 0 \\ 0 & 0 & C^{-1} \end{pmatrix} S^{-1}m^T \\ &= -\frac{\lambda^2 v^2}{8} \begin{pmatrix} (A^{-1})_{11} + (C^{-1})_{11} & 0 & (A^{-1})_{11} - (C^{-1})_{11} \\ 0 & 2(B^{-1})_{11} & 0 \\ (A^{-1})_{11} - (C^{-1})_{11} & 0 & (A^{-1})_{11} + (C^{-1})_{11} \end{pmatrix} \end{aligned}$$

This mass matrix is diagonalized by (10), and the masses are given by

$$m_1 = \left| \frac{\lambda^2 v^2}{2} (A^{-1})_{11} \right|, \quad m_2 = \left| \frac{\lambda^2 v^2}{2} (B^{-1})_{11} \right|, \quad m_3 = \left| \frac{\lambda^2 v^2}{2} (C^{-1})_{11} \right|.$$

2.6 Calculation of the Charged Lepton Mass Matrix

In this section, we show how the charged lepton mass matrix is computed. From Section 2.4, the terms in Eq. (14) that mix e_{R1} , e_{R2} , and e_{R3} with E'_L are

$$\bar{e}_{R1}f_2(E'_L, \langle\chi\rangle) + c.c. = v_\chi\bar{e}_{R1}(E'_{L1} + E'_{L3} + E'_{L5}) + c.c.,$$

$$g_7(\bar{e}_{R2}, E'_L, \langle\chi\rangle) + c.c. = v_\chi\bar{e}_{R2}(E'_{L1} + \omega^2 E'_{L3} + \omega E'_{L5}) + c.c.,$$

$$g_8(\bar{e}_{R3}, E'_L, \langle\chi\rangle) + c.c. = v_\chi\bar{e}_{R3}(E'_{L1} + \omega E'_{L3} + \omega^2 E'_{L5}) + c.c..$$

The term that mixes E_R and E'_L is

$$g_6(\bar{E}_R, E'_L, \langle\chi\rangle) + c.c. = v_\chi(\bar{E}_{R1}E'_{L5} + \bar{E}_{R2}E'_{L1} + \bar{E}_{R3}E'_{L3}) + c.c.,$$

with a similar result for the term that mixes E_L and E'_R . In the basis with $(\bar{e}_{L1}, \bar{e}_{L2}, \bar{e}_{L3}, \bar{E}_{L1}, \bar{E}_{L2}, \bar{E}_{L3}, \bar{E}'_{L1}, \bar{E}'_{L3}, \bar{E}'_{L5})$ on the left and $(e_{R1}, e_{R2}, e_{R3}, E_{R1}, E_{R2}, E_{R3}, E'_{R1}, E'_{R3}, E'_{R5})$ on the right, the mass matrix has the form

$$\mathcal{M}_e = \begin{pmatrix} 0 & M' \\ m & M \end{pmatrix},$$

with

$$m = \begin{pmatrix} \kappa v & 0 & 0 \\ 0 & \kappa v & 0 \\ 0 & 0 & \kappa v \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$M' = \begin{pmatrix} 0 & 0 & 0 & \epsilon_1 v_\chi & \epsilon_1 v_\chi & \epsilon_1 v_\chi \\ 0 & 0 & 0 & \epsilon_2 v_\chi & \omega^2 \epsilon_2 v_\chi & \omega \epsilon_2 v_\chi \\ 0 & 0 & 0 & \epsilon_3 v_\chi & \omega \epsilon_3 v_\chi & \omega^2 \epsilon_3 v_\chi \end{pmatrix},$$

and

$$M = \begin{pmatrix} m_E & 0 & 0 & 0 & 0 & \gamma_1 v_\chi \\ 0 & m_E & 0 & \gamma_1 v_\chi & 0 & 0 \\ 0 & 0 & m_E & 0 & \gamma_1 v_\chi & 0 \\ 0 & \gamma_2 v_\chi & 0 & m'_E & \eta_2 v_\chi & \eta_1 v_\chi \\ 0 & 0 & \gamma_2 v_\chi & \eta_1 v_\chi & m'_E & \eta_2 v_\chi \\ \gamma_2 v_\chi & 0 & 0 & \eta_2 v_\chi & \eta_1 v_\chi & m'_E \end{pmatrix}.$$

Here, m only contains entries at the EW scale, while M and M' contain entries at the higher scale M_* . To order M_W^2/M_*^2 , the left-handed mass-squared matrix $\mathcal{M}_e^\dagger \mathcal{M}_e$ is block-diagonalized by

$$\mathcal{U}_L = \begin{pmatrix} I & m^\dagger M (M^\dagger M + M'^\dagger M')^{-1} \\ (M^\dagger M + M'^\dagger M')^{-1} M^\dagger m & I \end{pmatrix}.$$

The upper left entry of $\mathcal{U}_L \mathcal{M}_e^\dagger \mathcal{M}_e \mathcal{U}_L^\dagger$ is the light left-handed mass-squared matrix

$$M_e^\dagger M_e = m^\dagger m - m^\dagger M (M^\dagger M + M'^\dagger M')^{-1} M^\dagger m$$

Let

$$S = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & \omega & 0 & \omega^2 & 0 \\ 1 & 0 & \omega^2 & 0 & \omega & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & \omega & 0 & \omega^2 \\ 0 & 1 & 0 & \omega^2 & 0 & \omega \end{pmatrix}.$$

Then $S^\dagger M S$ and $S^\dagger M'^\dagger M' S$ are both block diagonal (three 2×2 blocks each). So we have

$$m^\dagger M (M^\dagger M + M'^\dagger M')^{-1} M^\dagger m = m^\dagger S \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix} S^\dagger m.$$

$$= \frac{|\kappa v|^2}{3} \begin{pmatrix} A_{11} + B_{11} + C_{11} & A_{11} + \omega^2 B_{11} + \omega C_{11} & A_{11} + \omega B_{11} + \omega^2 C_{11} \\ A_{11} + \omega B_{11} + \omega^2 C_{11} & A_{11} + B_{11} + C_{11} & A_{11} + \omega^2 B_{11} + \omega C_{11} \\ A_{11} + \omega^2 B_{11} + \omega C_{11} & A_{11} + \omega B_{11} + \omega^2 C_{11} & A_{11} + B_{11} + C_{11} \end{pmatrix}$$

This has the form (11). The masses are given by

$$m_e^2 = |\kappa v|^2(1 - A_{11}), \quad m_\mu^2 = |\kappa v|^2(1 - B_{11}), \quad m_\tau^2 = |\kappa v|^2(1 - C_{11}).$$

CHAPTER 3

A 6D Higgsless Standard Model

3.1 The Model

Let us consider a 6D $SU(2)_L \times U(1)_Y$ gauge theory in a flat space-time background, where the two extra spatial dimensions are compactified on a rectangle¹. The coordinates in the 6D space are written as $z_M = (x_\mu, y_m)$, where the 6D Lorentz indices are denoted by capital Roman letters $M = 0, 1, 2, 3, 5, 6$, while the usual 4D Lorentz indices are symbolized by Greek letters $\mu = 0, 1, 2, 3$, and the coordinates y_m ($m = 1, 2$) describe the fifth and sixth dimension.² The physical space is thus defined by $0 \leq y_1 \leq \pi R_1$ and $0 \leq y_2 \leq \pi R_2$, where R_1 and R_2 are the compactification radii of a torus T^2 , which is obtained by identifying the points of the two-dimensional plane R^2 under the actions $T_5 : (y_1, y_2) \rightarrow (y_1 + 2\pi R_1, y_2)$ and $T_6 : (y_1, y_2) \rightarrow (y_1, y_2 + 2\pi R_2)$. We denote the $SU(2)_L$ and $U(1)_Y$ gauge bosons in the bulk respectively by $A_M^a(z_M)$ ($a = 1, 2, 3$ is the gauge index) and $B_M(z_M)$. The action of the gauge fields in our model is given by

$$\mathcal{S} = \int d^4x \int_0^{\pi R_1} dy_1 \int_0^{\pi R_2} dy_2 (\mathcal{L}_6 + \delta(y_1)\delta(y_2)\mathcal{L}_0), \quad (3.1)$$

where \mathcal{L}_6 is a 6D bulk gauge kinetic term and \mathcal{L}_0 is a 4D brane gauge kinetic term localized at $(y_1, y_2) = (0, 0)$, which read respectively

$$\mathcal{L}_6 = -\frac{M_L^2}{4} F_{MN}^a F^{MN a} - \frac{M_Y^2}{4} B_{MN} B^{MN}, \quad \mathcal{L}_0 = -\frac{1}{4g^2} F_{\mu\nu}^a F^{\mu\nu a} - \frac{1}{4g'^2} B_{\mu\nu} B^{\mu\nu}, \quad (3.2)$$

¹Chiral compactification on a square has recently been considered in Ref. [42].

²For the metric we choose a signature $(+, -, -, -, -, -)$.

with field strengths $F_{MN}^a = \partial_M A_N^a - \partial_N A_M^a + f^{abc} A_M^b A_N^c$ (f^{abc} is the structure constant) and $B_{MN} = \partial_M B_N - \partial_N B_M$. In Eqs. (3.2), the quantities M_L and M_Y have mass dimension +1, while g and g' are dimensionless. Since the boundaries of the manifold break translational invariance and are "singled out" with respect to the points in the interior of the rectangle, brane terms like \mathcal{L}_0 can be produced by quantum loop effects [34, 35] or arise from classical singularities in the limit of vanishing brane thickness [36].

Unlike in five dimensions (for a discussion of the $\xi \rightarrow \infty$ limit in generalized 5D R_ξ gauges see, *e.g.*, Ref. [43] and also Ref. [30]), we cannot go to a unitary gauge where all fields $A_{5,6}^a$ ($a = 1, 2, 3$) and $B_{5,6}$ are identically set to zero. Instead, there will remain after dimensional reduction one combination of physical scalar fields in the spectrum³. To make these scalars sufficiently heavier than the Lee-Quigg-Thacker bound of $\approx 2 \text{ TeV}$, we can assume, *e.g.*, a seventh dimension compactified on \mathcal{S}^1/Z_2 with compactification radius $R_3 \lesssim R_1, R_2$. By setting $A_{5,6,7}^a = B_{5,6,7} = 0$ (A_7^a and B_7 are the seventh components of the gauge fields) on all boundaries of this manifold, the associated scalars can acquire for compactification scales $R_1^{-1}, R_2^{-1} \simeq 1 - 2 \text{ TeV}$, masses well above 2 TeV . Therefore, at low energies $\lesssim 2 - 3 \text{ TeV}$, we have a model without any *light* scalars and will, in what follows, neglect the heavy scalar degrees of freedom.

Since the Lagrangian in Eq. (3.2) does not contain any explicit gauge symmetry breaking, we can obtain consistent new BC's on the boundaries by requiring the variation of the action to be zero. Variation of the action in Eq. (3.2) yields after

³We thank H. Murayama and M. Serone for pointing out this fact.

partial integration

$$\begin{aligned}
\delta S &= \int d^4x \int_{y_1=0}^{\pi R_1} dy_1 \int_{y_2=0}^{\pi R_2} dy_2 [M_L^2 (\partial_M F^{aM\mu} - f^{abc} F^{bM\mu} A_M^c) \delta A_\mu^a + M_Y^2 \partial_M B^{M\mu} \delta B_\mu] \\
&+ \int d^4x \int_{y_2=0}^{\pi R_2} dy_2 [M_L^2 F_{5\mu}^a \delta A^{a\mu} + M_Y^2 B_{5\mu} \delta B^\mu]_{y_1=0}^{\pi R_1} \\
&+ \int d^4x \int_{y_1=0}^{\pi R_1} dy_1 [M_L^2 F_{6\mu}^a \delta A^{a\mu} + M_Y^2 B_{6\mu} \delta B^\mu]_{y_2=0}^{\pi R_2} \\
&+ \int d^4x \left[\frac{1}{g^2} (\partial_\mu F^{a\mu\nu} - f^{abc} F^{b\mu\nu} A_\mu^c) \delta A_\nu^a + \frac{1}{g'^2} \partial_\mu B^{\mu\nu} \delta B_\nu \right]_{(y_1, y_2)=(0,0)} = 0, \quad (3.3)
\end{aligned}$$

where we have (as usual) assumed that the gauge fields and their derivatives go to zero for $x_\mu \rightarrow \infty$. The bulk terms in the first line in Eq. (3.3), lead to the familiar bulk equations of motion. Moreover, since the minimization of the action requires the boundary terms to vanish as well, we obtain from the second and third line in Eq. (3.3) a set of consistent BC's for the bulk fields.

We break the electroweak symmetry $SU(2)_L \times U(1)_Y \rightarrow U(1)_Q$ by imposing on two of the boundaries following BC's:

$$\text{at } y_1 = \pi R_1 : A_\mu^1 = 0, \quad A_\mu^2 = 0, \quad (3.4a)$$

$$\text{at } y_2 = \pi R_2 : \partial_{y_2} (M_L^2 A_\mu^3 + M_Y^2 B_\mu) = 0, \quad A_\mu^3 - B_\mu = 0. \quad (3.4b)$$

The Dirichlet BC's in Eq. (3.4a) break $SU(2)_L \rightarrow U(1)_{I_3}$, where $U(1)_{I_3}$ is the $U(1)$ subgroup associated with the third component of weak isospin I_3 . The BC's in Eq. (3.4b) break $U(1)_{I_3} \times U(1)_Y \rightarrow U(1)_Q$, leaving only $U(1)_Q$ unbroken on the entire rectangle (see Fig. 3.1). Note, in Eq. (3.4b), that the first BC involving the derivative with respect to y_2 actually follows from the second BC $\delta A_\mu^3 = \delta B_\mu$ by minimization of the action. The gauge groups $U(1)_{I_3}$ and $U(1)_{I_3} \times U(1)_Y$ remain unbroken at the boundaries $y_1 = 0$ and $y_2 = 0$, respectively. Locally, at the fixed point $(y_1, y_2) = (0, 0)$, $SU(2)_L \times U(1)_Y$ is unbroken. We can restrict ourselves, for simplicity, to the solutions which are relevant to EWSB, by imposing on the other

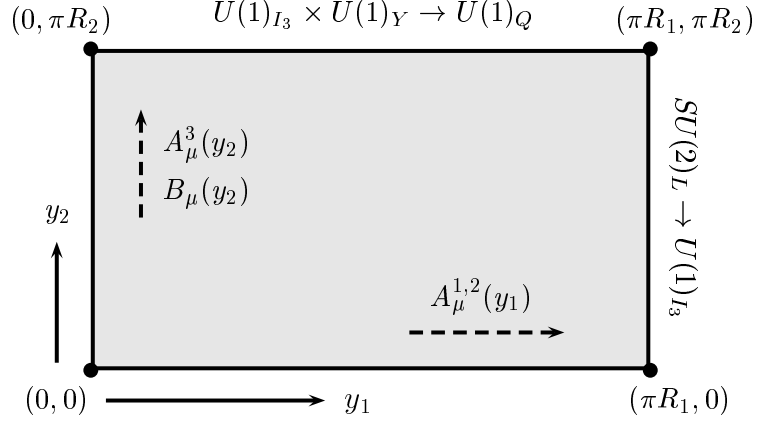


Figure 3.1: Symmetry breaking of $SU(2)_L \times U(1)_Y$ on the rectangle. At one boundary $y_1 = \pi R_1$, $SU(2)_L$ is broken to $U(1)_{I_3}$ while on the boundary $y_2 = \pi R_2$ the subgroup $U(1)_{I_3} \times U(1)_Y$ is broken to $U(1)_Q$, which leaves only $U(1)_Q$ unbroken on the entire rectangle. Locally, at the fixed point $(0, 0)$, $SU(2)_L \times U(1)_Y$ remains unbroken. The dashed arrows indicate the propagation of the lowest resonances of the gauge bosons.

two boundaries the following Dirichlet BC's:

$$\text{at } y_1 = 0 \quad : \quad A_\mu^{1,2}(z_M) = \bar{A}_\mu^{1,2}(x_\mu), \quad (3.5a)$$

$$\text{at } y_2 = 0 \quad : \quad A_\mu^3(z_M) = \bar{A}_\mu^3(x_\mu), \quad B_\mu(z_M) = \bar{B}_\mu(x_\mu), \quad (3.5b)$$

where the bar indicates a boundary field. The Dirichlet BC's in Eqs.(3.5) require $A_\mu^{1,2}$ to be independent of y_2 , while A_μ^3 and B_μ become independent of y_1 , such that we can generally write $A_\mu^{1,2} = A_\mu^{1,2}(x_\mu, y_1)$, $A_\mu^3 = A_\mu^3(x_\mu, y_2)$, and $B_\mu = B_\mu(x_\mu, y_2)$. For the transverse⁴ components of the gauge fields the bulk equations of motion then take the forms

$$(p^2 + \partial_{y_1}^2)A_\mu^{1,2}(x_\mu, y_1) = 0, \quad (p^2 + \partial_{y_2}^2)A_\mu^3(x_\mu, y_2) = 0, \quad (p^2 + \partial_{y_2}^2)B_\mu(x_\mu, y_2) = 0, \quad (3.6)$$

where $p^2 = p_\mu p^\mu$ and $p_\mu = i\partial_\mu$ is the momentum in the uncompactified 4D space. Since we assume all the gauge couplings to be small, we will, in what follows, treat

⁴Note that $\partial_M F^{aM\mu} = p^2 P_{\mu\nu}(p)A^{a\mu} + (\partial_{y_1}^2 + \partial_{y_2}^2)A_\nu^a = 0$, where $P_{\mu\nu}(p) = g_{\mu\nu} - p_\mu p_\nu / p^2$ is the operator projecting onto transverse states.

A_μ^a approximately as a "free" field (*i.e.*, without self interaction) and drop all cubic and quartic terms in A_μ^a .

We assume that the fermions, in the first approximation, are localized on the brane at $(y_1, y_2) = (0, 0)$, away from the walls of electroweak symmetry breaking. This choice will avoid any unwanted non-oblique corrections to the electroweak precision parameters.

3.2 Effective theory

The total effective 4D Lagrangian in the compactified theory $\mathcal{L}_{\text{total}}$ can be written as $\mathcal{L}_{\text{total}} = \mathcal{L}_0 + \mathcal{L}_{\text{eff}}$, where $\mathcal{L}_{\text{eff}} = \int_0^{\pi R_1} dy_1 \int_0^{\pi R_2} dy_2 \mathcal{L}_6$ denotes the contribution from the bulk, which follows from integrating out the extra dimensions. After partial integration along the y_1 and y_2 directions, we obtain for \mathcal{L}_{eff} the non-vanishing boundary term

$$\mathcal{L}_{\text{eff}} = -M_L^2 \pi R_2 \left[\bar{A}_\mu^1 \partial_{y_1} A^{1\mu} + \bar{A}_\mu^2 \partial_{y_1} A^{2\mu} \right]_{y_1=0} - \pi R_1 \left[M_L^2 \bar{A}_\mu^3 \partial_{y_2} A^{3\mu} + M_Y^2 \bar{B}_\mu \partial_{y_2} B^\mu \right]_{y_2=0}, \quad (3.7)$$

where we have applied the bulk equations of motion and eliminated the terms from the boundaries at $y_1 = \pi R_1$ and $y_2 = \pi R_2$ by virtue of the BC's in Eqs. (3.4). Notice, that in arriving at Eq. (3.7) we have redefined the bulk gauge fields as $A_\mu \rightarrow A'_\mu \equiv A_\mu / \sqrt{2}$ to canonically normalize the kinetic energy terms of the KK modes. In order to determine $\mathcal{L}_{\text{total}}$ explicitly, we first solve the equations of motion in Eq. (3.6) and insert the solutions into the expression for \mathcal{L}_{eff} in Eq. (3.7). The most general solutions for Eqs. (3.6) can be written as

$$A_\mu^{1,2}(x_\mu, y_1) = \bar{A}_\mu^{1,2}(x_\mu) \cos(py_1) + b_\mu^{1,2}(x_\mu) \sin(py_1), \quad (3.8a)$$

$$A_\mu^3(x_\mu, y_2) = \bar{A}_\mu^3(x_\mu) \cos(py_2) + b_\mu^3(x_\mu) \sin(py_2), \quad (3.8b)$$

$$B_\mu(x_\mu, y_2) = \bar{B}_\mu(x_\mu) \cos(py_2) + b_\mu^Y(x_\mu) \sin(py_2), \quad (3.8c)$$

where $p = \sqrt{p_\mu p^\mu}$ and we have already applied the BC's in Eq. (3.5). The coefficients $b_\mu^a(x_\mu)$ and $b_\mu^Y(x_\mu)$ are then determined from the BC's in Eqs. (3.4). For $b_\mu^{1,2}(x_\mu)$, *e.g.*, we find from the BC's in Eq. (3.4a) that $b_\mu^{1,2}(x_\mu) = -\bar{A}_\mu^{1,2}(x_\mu) \cot(p\pi R_1)$ and hence one obtains

$$A_\mu^{1,2}(x_\mu, y_1) = \bar{A}_\mu^{1,2}(x_\mu) [\cos(py_1) - \cot(p\pi R_1) \sin(py_1)]. \quad (3.9a)$$

In a similar way, one arrives after some calculation at the solutions

$$\begin{aligned} A_\mu^3(x_\mu, y_2) &= \bar{A}_\mu^3(x_\mu) \left[\cos(py_2) + \frac{M_L^2 \tan(p\pi R_2) - M_Y^2 \cot(p\pi R_2)}{M_L^2 + M_Y^2} \sin(py_2) \right] \\ &+ \bar{B}_\mu(x_\mu) \frac{M_Y^2 \tan(p\pi R_2) + M_Y^2 \cot(p\pi R_2)}{M_L^2 + M_Y^2} \sin(py_2), \end{aligned} \quad (3.9b)$$

$$\begin{aligned} B_\mu(x_\mu, y_2) &= \bar{A}_\mu^3(x_\mu) \frac{M_L^2 \tan(p\pi R_2) + M_L^2 \cot(p\pi R_2)}{M_L^2 + M_Y^2} \sin(py_2) \\ &+ \bar{B}_\mu(x_\mu) \left[\cos(py_2) + \frac{M_Y^2 \tan(p\pi R_2) - M_L^2 \cot(p\pi R_2)}{M_L^2 + M_Y^2} \sin(py_2) \right] \end{aligned} \quad (3.9c)$$

Inserting the wavefunctions in Eqs. (3.9) into the effective Lagrangian in Eq. (3.7), we can rewrite \mathcal{L}_{eff} as

$$\mathcal{L}_{\text{eff}} = \bar{A}_\mu^a \Sigma_{aa}(p^2) \bar{A}^{a\mu} + \bar{A}_\mu^3 \Sigma_{3B}(p^2) \bar{B}^\mu + \bar{B}_\mu \Sigma_{BB}(p^2) \bar{B}^\mu, \quad (3.10)$$

where $(aa) = (11), (22)$, and (33) and the momentum-dependent coefficients Σ are given by

$$\begin{aligned} \Sigma_{11}(p^2) &= \Sigma_{22}(p^2) = \pi R_2 M_L^2 p \cot(p\pi R_1), \\ \Sigma_{33}(p^2) &= -\pi R_1 M_L^2 p \frac{M_L^2 \tan(p\pi R_2) - M_Y^2 \cot(p\pi R_2)}{M_L^2 + M_Y^2}, \\ \Sigma_{3B}(p^2) &= -2\pi R_1 M_L^2 M_Y^2 p \frac{\tan(p\pi R_2) + \cot(p\pi R_2)}{M_L^2 + M_Y^2}, \\ \Sigma_{BB}(p^2) &= -\pi R_1 M_Y^2 p \frac{M_Y^2 \tan(p\pi R_2) - M_L^2 \cot(p\pi R_2)}{M_L^2 + M_Y^2}. \end{aligned} \quad (3.11)$$

The Σ 's can be viewed as the electroweak vacuum polarization amplitudes which summarize in the low energy theory the effect of the symmetry breaking sector. The

presence of these terms leads at tree level to oblique corrections (as opposed to vertex corrections and box diagrams) of the gauge boson propagators and affects electroweak precision measurements [44,45]. Since \mathcal{L}_{eff} in Eq. (3.7) generates effective mass terms for the gauge bosons in the 4D theory⁵, the KK masses of the W^\pm bosons are found from the zeros of the inverse propagator as given by the solutions of the equation

$$\Sigma_{11}(p^2) - \frac{p^2}{2g^2} = 0. \quad (3.12)$$

To determine the KK masses of the gauge bosons, we will from now on assume that the brane terms \mathcal{L}_0 dominate the bulk kinetic terms, *i.e.*, we take $1/g^2, 1/g'^2 \gg (M_{L,Y\pi})^2 R_1 R_2$. As a result, we find for the W^\pm 's the mass spectrum

$$\begin{aligned} m_n &= \frac{n}{R_1} \left(1 + \frac{2g^2 M_L^2 R_1 R_2}{n^2} + \dots \right), \quad n = 1, 2, \dots, \\ m_0^2 &= \frac{2g^2 M_L^2 R_2}{R_1} + \mathcal{O}(g^4 M_L^4 R_2^2) = m_W^2, \end{aligned} \quad (3.13)$$

where we identify the lightest state with mass m_0 with the W^\pm . Observe in Eq. (3.13), that the inclusion of the brane kinetic terms \mathcal{L}_0 for $1/R_1, 1/R_2 \gtrsim \mathcal{O}(TeV)$ leads to a decoupling of the higher KK-modes with masses m_n ($n > 0$) from the electroweak scale, leaving only the W^\pm states with a small mass m_0 in the low-energy theory (see Fig. 3.2). Note that a similar effect has been found for warped models in Ref. [47].

The calculation of the mass of the Z boson goes along the same lines as for W^\pm , but requires, due to the mixing of \bar{A}_μ^3 with \bar{B}_μ in Eq. (3.10), the diagonalization of the kinetic matrix

$$M_{\text{kin}} = \begin{pmatrix} \Sigma_{33}(p^2) - \frac{p^2}{2g^2} & \frac{1}{2}\Sigma_{3B}(p^2) \\ \frac{1}{2}\Sigma_{3B}(p^2) & \Sigma_{BB}(p^2) - \frac{p^2}{2g'^2} \end{pmatrix}, \quad (3.14)$$

which has the eigenvalues

$$\begin{aligned} \lambda_\pm(p^2) &= \frac{1}{2} \left(\Sigma_{33}(p^2) - \frac{p^2}{2g^2} + \Sigma_{BB}(p^2) - \frac{p^2}{2g'^2} \right) \\ &\pm \frac{1}{2} \sqrt{\left(\Sigma_{33}(p^2) - \frac{p^2}{2g^2} - \Sigma_{BB}(p^2) + \frac{p^2}{2g'^2} \right)^2 + \Sigma_{3B}^2(p^2)}, \end{aligned} \quad (3.15)$$

⁵For an effective field theory approach to oblique corrections see, *e.g.*, Ref. [46].

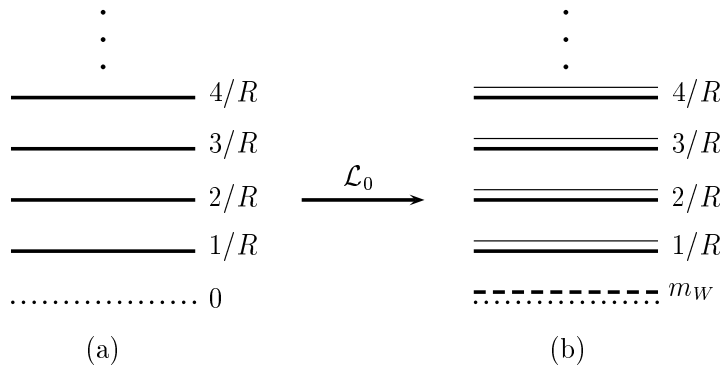


Figure 3.2: Effect of the brane kinetic terms \mathcal{L}_0 on the KK spectrum of the gauge bosons (for the example of W^\pm). Solid lines represent massive excitations, the bottom dotted lines would correspond to the zero modes which have been removed by the BC's. Without the brane terms (a), the lowest KK excitations are of order $1/R \simeq 1 \text{ TeV}$. After switching on the dominant brane kinetic terms (b), the zero modes are approximately “restored” with a small mass $m_W \ll 1/R$ (dashed line), while the higher KK-levels receive small corrections to their masses (thin solid lines) and decouple below $\sim 1 \text{ TeV}$.

where the KK towers of the γ and Z are given by the solutions of the equations $\lambda_-(p^2) = 0$ (for γ) and $\lambda_+(p^2) = 0$ (for Z), respectively. By taking in Eq. (3.15) the limit $p^2 \rightarrow 0$, it is easily seen that $\lambda_-(p^2) = 0$ has a solution with $p^2 = 0$, which we identify with the massless γ of the SM, corresponding to the unbroken gauge group $U(1)_Q$. The lowest excitation in the tower of solutions to $\lambda_+(p^2) = 0$ has a mass-squared

$$m_Z^2 = \frac{2(g^2 + g'^2)M_L^2 M_Y^2 R_1}{(M_L^2 + M_Y^2)R_2} + \mathcal{O}(g^4 M_L^4 R_2^2), \quad (3.16)$$

which we identify with the Z of the SM. All other KK modes of the γ and Z have masses of order $\gtrsim 1/R_2$ and thus decouple for $1/R_1, 1/R_2 \gtrsim \mathcal{O}(\text{TeV})$, leaving only a massless γ and a Z with mass m_Z in the low-energy theory.

3.3 Relation to EWPT

One important constraint on any model for EWSB results from the measurement of the ρ parameter, which is experimentally known to satisfy the relation $\rho = 1$ to better than 1% [2]. In our model, we find from Eqs. (3.13) and (3.16) a fit of the natural zeroth-order SM relation for the ρ parameter in terms of

$$\rho \equiv \frac{m_W^2}{m_Z^2 \cos^2 \theta_W} = \frac{g^2}{g^2 + g'^2} \frac{M_L^2 + M_Y^2}{M_Y^2} \left(\frac{R_2}{R_1} \right)^2 \frac{1}{\cos^2 \theta_W} = 1, \quad (3.17)$$

where $\theta_W \approx 28.8^\circ$ is the Weinberg angle of the SM. For definiteness, we will choose in the following the 4D brane couplings g and g' to satisfy the usual SM relation $g^2/(g^2 + g'^2) = \cos^2 \theta_W \approx 0.77$. Defining $\rho = 1 + \Delta\rho$, we then obtain from Eq. (3.17) that $\Delta\rho = 0$ if the bulk kinetic couplings and compactification radii satisfy the relation

$$(M_L^2 + M_Y^2)/M_Y^2 = R_1^2/R_2^2. \quad (3.18)$$

Although we can thus set $\Delta\rho = 0$ by appropriately dialing the gauge couplings and the size of the extra dimensions, we observe in Eq. (3.10) that \mathcal{L}_{eff} introduces a manifest breaking of custodial symmetry (which transforms the three gauge bosons A_μ^a among themselves) and will thus contribute to EWPT via oblique corrections to the SM parameters.⁶

To estimate the effect of the oblique corrections in our model let us consider in the 4D effective theory a general vacuum polarization tensor $\Pi_{AB}^{\mu\nu}(p^2)$ between two gauge fields A and B which can (for canonically normalized fields) be expanded as [46]

$$i\Pi_{\mu\nu}^{AB}(p^2) = ig_A g_B \left[\Pi_{AB}^{(0)} + p^2 \Pi_{AB}^{(1)} \right] g_{\mu\nu} + p_\mu p_\nu \text{ terms}, \quad (3.19)$$

where g_A and g_B are the couplings corresponding to the gauge fields A and B , respectively. After going in \mathcal{L}_{eff} back to canonical normalization by redefining $A_\mu^a \rightarrow A'_\mu \equiv A_\mu^a/g$ and $B_\mu \rightarrow B'_\mu \equiv B_\mu/g'$, we identify $\Sigma_{aa}(p^2) \simeq \frac{1}{2}[\Pi_{aa}^{(0)} + p^2 \Pi_{aa}^{(1)}]$, for

⁶Note, however, that in the limit $p^2 \rightarrow 0$, we have $\Sigma_{11} = \Sigma_{33}$, which restores custodial symmetry.

(aa) = (11), (22), (33), (BB), while $\Sigma_{3B}(p^2) \simeq \Pi_{3B}^{(0)} + p^2\Pi_{3B}^{(1)}$. From Eqs. (3.11) we then obtain the polarization amplitudes

$$\begin{aligned}\Pi_{11}^{(0)} &= \Pi_{22}^{(0)} = 2M_L^2 \frac{R_2}{R_1}, & \Pi_{11}^{(1)} &= \Pi_{22}^{(1)} = -2 \frac{\pi^2 M_L^2}{3} R_1 R_2, \\ \Pi_{33}^{(0)} &= 2 \frac{M_L^2 M_Y^2}{M_L^2 + M_Y^2} \frac{R_1}{R_2}, & \Pi_{33}^{(1)} &= -2 \frac{\pi^2 M_L^2 R_1 R_2}{M_L^2 + M_Y^2} (M_L^2 + \frac{1}{3} M_Y^2), \\ \Pi_{3B}^{(0)} &= -2 \frac{M_L^2 M_Y^2}{M_L^2 + M_Y^2} \frac{R_1}{R_2}, & \Pi_{3B}^{(1)} &= -\frac{4}{3} \frac{\pi^2 M_L^2 M_Y^2}{M_L^2 + M_Y^2} R_1 R_2.\end{aligned}\quad (3.20)$$

A wide range of effects from new physics on EWPT can be parameterized in the ϵ_1 , ϵ_2 , and ϵ_3 framework [45], which is related to the S, T , and U formalism of Ref. [44] by $\epsilon_1 = \alpha T$, $\epsilon_2 = -\alpha U/4 \sin^2\theta_W$, and $\epsilon_3 = \alpha S/4 \sin^2\theta_W$. The experimental bounds on the relative shifts with respect to the SM expectations are roughly of the order $\epsilon_1, \epsilon_2, \epsilon_3 \lesssim 3 \cdot 10^{-3}$ [48]. From Eq. (3.20) we then obtain for these parameters explicitly

$$\epsilon_1 = g^2(\Pi_{11}^{(0)} - \Pi_{33}^{(0)})/m_W^2 = -2g^2 \frac{M_L^2 R_1}{m_W^2 R_2} (M_Y^2/(M_L^2 + M_Y^2) - (R_2/R_1)^2) \quad (3.21a)$$

$$\epsilon_2 = g^2(\Pi_{33}^{(1)} - \Pi_{11}^{(1)}) = -g^2 \frac{4\pi^2}{3} \frac{M_L^4}{M_L^2 + M_Y^2} R_1 R_2, \quad (3.21b)$$

$$\epsilon_3 = -g^2 \Pi_{3B}^{(1)} = g^2 \frac{4\pi^2}{3} \frac{M_L^2 M_Y^2}{M_L^2 + M_Y^2} R_1 R_2, \quad (3.21c)$$

where we have used in the last equation that $-\epsilon_3/(gg') = \Pi_{3\gamma}^{(1)}/\sin^2\theta_W - \Pi_{33}^{(1)} = \cot\theta_W \Pi_{3B}^{(1)}$ [45]. Note in Eq. (3.21a), that for our choice of parameters we have $\epsilon_1 = \Delta\rho = 0$. The quantities $|\epsilon_2|$ and $|\epsilon_3|$, on the other hand, are bounded from below by the requirement of having sufficiently many KK modes below the strong coupling (or cutoff) scale of the theory. Using ‘‘naive dimensional analysis’’ (NDA) [49,50], one obtains for the strong coupling scale Λ of a D -dimensional gauge theory [51] roughly $\Lambda^{D-4} \simeq (4\pi)^{D/2} \Gamma(D/2)/g_D^2$, where g_D is the bulk gauge coupling. In our 6D model, we would therefore have $\Lambda \simeq \sqrt{2}(4\pi)^{3/2} M_{L,Y}$ which leads for $M_{L,Y} \simeq 10^2 \text{ GeV}$ to a cutoff $\Lambda \simeq 6 \text{ TeV}$. Assuming for simplicity $M_L = M_Y$, it follows from Eq. (3.18) that $R_2 = R_1/\sqrt{2}$, and using Eqs. (3.21b) and (3.21c) we obtain

$$\epsilon_3 \simeq \frac{g^2}{96\sqrt{2}\pi} (\Lambda R_2)^2 \simeq 2.3 \times 10^{-3} \times (g\Lambda R_2)^2, \quad (3.22)$$

while $\epsilon_2 \simeq \epsilon_3$. It is instructive to compare the value for ϵ_3 in our 6D setup as given by Eq. (3.22) with the corresponding result of the 5D model in Ref. [32]. We find that by going from 5D to 6D, the strong coupling scale of the theory is lowered from $\sim 10 \text{ TeV}$ down to $\sim 6 \text{ TeV}$. Despite the lowering of the cutoff scale, however, the parameter ϵ_3 is in the 6D model by $\sim 15\%$ smaller than the corresponding 5D value⁷. This is due to the fact that in the 6D model the bulk gauge kinetic couplings satisfy $M_L = M_Y \simeq 100 \text{ GeV}$, while they take in 5D the values $M_L \simeq M_Y \simeq 10 \text{ GeV}$, which is one order of magnitude below the electroweak scale. From Eq. (3.22) we then conclude that one can take for the inverse loop expansion parameter $\Lambda R_2 \simeq 1/g \approx 1.6$ in agreement with EWPT. Like in the 5D case, however, the 6D model seems not to admit a loop expansion parameter in the regime $\Lambda R_2 \gg 1$ as required for the model to be calculable.

3.4 Non-oblique corrections and fermion masses

In the previous discussion, we have assumed that the fermions are (approximately) localized at $(y_1, y_2) = (0, 0)$. This would make the fermions exactly massless, since they have no access to the EWSB at $y_1 = \pi R_1$ and $y_2 = \pi R_2$. In this limiting case, the effects on the electroweak precision parameters $(\epsilon_1, \epsilon_2, \epsilon_3/S, T, U)$ come from the oblique corrections due to the vector self energies as given by Eq. (3.10). A more realistic case will be to extend the fermion wave functions to the bulk, *i.e.*, to the walls of EWSB, where fermion mass operators of the form $C\bar{\Psi}_L\Psi_R$ (C is some appropriate mass parameter) can be written. Thus, although the fermion wave functions will be dominantly localized at $(0, 0)$, the profile of the wavefunctions in the bulk will be such that it will have small contributions from the symmetry breaking walls, giving rise to fermion masses. The hierarchy of fermion masses would then be accommodated by

⁷Notice that in Ref. [32], the strong coupling scale is defined by $1/\Lambda = 1/\Lambda_L + 1/\Lambda_R$, while we assume for $M_L = M_Y$ that $\Lambda = \Lambda_L = \Lambda_Y$.

some suitable choice of the parameters C [52].

To make the incorporation of heavy fermions in our model explicit, let us introduce the 6D chiral quark fields \mathcal{Q}_i , \mathcal{U}_i , and \mathcal{D}_i ($i = 1, 2, 3$ is the generation index), where \mathcal{Q}_i are the isodoublet quarks, while \mathcal{U}_i and \mathcal{D}_i denote the isosinglet up and down quarks, respectively. For the cancellation of the $SU(3)_C \times SU(2)_L \times U(1)_Y$ gauge and gravitational anomalies we assume that \mathcal{Q}_i have positive and $\mathcal{U}_i, \mathcal{D}_i$ have negative $SO(1, 5)$ chiralities [53]. Next, we consider the action of the top quark fields with zero bulk mass, which is given by

$$\begin{aligned} \mathcal{S}_{\text{fermion}} &= \int dx^4 \int_0^{\pi R_1} dy_1 \int_0^{\pi R_2} dy_2 i(\bar{\mathcal{Q}}_3 \Gamma^M D_M \mathcal{Q}_3 + \bar{\mathcal{U}}_3 \Gamma^M D_M \mathcal{U}_3) \\ &+ \int dx^4 \int_0^{\pi R_1} dy_1 \int_0^{\pi R_2} dy_2 K \delta(y_1) \delta(y_2) i[\bar{\mathcal{Q}}_3 \Gamma^\mu D_\mu \mathcal{Q}_3 + \bar{\mathcal{U}}_3 \Gamma^\mu D_\mu \mathcal{U}_3] \\ &+ \int dx^4 \int_0^{\pi R_1} dy_1 \int_0^{\pi R_2} dy_2 C \delta(y_1 - \pi R_1) \delta(y_2 - \pi R_2) \bar{\mathcal{Q}}_{3L} \mathcal{U}_{3R} + \text{h.c.} \end{aligned} \quad (3.23)$$

where we have added in the second line 4D brane kinetic terms with a (common) gauge kinetic parameter $K = [m]^{-2}$ at $(y_1, y_2) = (0, 0)$ and in the third line we included a boundary mass term with coefficient $C = [m]^{-1}$, which mixes \mathcal{Q}_{3L} and \mathcal{U}_{3R} at $(y_1, y_2) = (\pi R_1, \pi R_2)$. Note, that the addition of the boundary mass term in the last line of Eq. (3.23) is consistent with gauge invariance, since $U(1)_Q$ the only gauge group surviving at $(y_1, y_2) = (\pi R_1, \pi R_2)$. Consider now first the limit of a vanishing brane kinetic term $K \rightarrow 0$. Like in the 5D case [31], appropriate Dirichlet and Neumann BC's for $\mathcal{Q}_{3L,R}$ and $\mathcal{U}_{3L,R}$ would give, in the KK tower corresponding to the top quark, a lowest mass eigenstate, which is a Dirac fermion with mass m_t of the order $m_t \sim C/R^2$, where we have defined the length scale $R \sim R_1 \sim R_2$. Next, by analogy with the generation of the W^\pm and Z masses, switching on a dominant brane kinetic term $K/R^2 \gg 1$, ensures an approximate localization of \mathcal{Q}_{3L} and \mathcal{U}_{3R} at $(y_1, y_2) = (0, 0)$ and leads to $m_t \sim C/K$ [32]. Now, the typical values of non-oblique corrections to the SM gauge couplings coming from the bulk are⁸ $\sim CR/K \sim$

⁸The factor C becomes obvious when treating the brane fields in Eq. (3.23) as 4D fields, in which

$m_t/(1/R)$ and keeping these contributions under control, the compactification scale $1/R$ must be sufficiently large. Like in 5D models, this generally introduces a possible tension between the 3rd generation quark masses and the coupling of the Z to the bottom quark. Replacing in the above discussion $\mathcal{U}_{3L,R}$ with $\mathcal{D}_{3L,R}$ and m_t by the bottom quark mass $m_b(m_Z) \approx 3 \text{ GeV}$, we thus estimate for $1/R \sim 1 \text{ TeV}$ a shift of the SM $Z \rightarrow \bar{b}_L b_L$ coupling by roughly $\sim 0.3\%$, which is of the order of current experimental uncertainties⁹. Similarly, we predict in our model the coupling of the Z to the top quark to deviate by $\sim 10\%$ from the SM value, which can be checked in the electroweak production of single top in the Tevatron Run 2. It can also be tested in the $t\bar{t}$ pair production in a possible future linear collider.

3.4.1 Improving the calculability

To improve the calculability of the model, it seems necessary to raise (for given $1/g_D^2$) the strong coupling scale Λ , which would allow the appearance of more KK modes below the cutoff. In fact, it has recently been argued that the compactification of a 5D gauge theory on an orbifold S^1/Z_2 gives a cutoff which is by a factor of 2 larger than the NDA estimate obtained for an uncompactified space [48]. Let us now demonstrate this effect explicitly by repeating the NDA calculation of Ref. [49] on an orbifold following the methods of Refs. [35] and [54]. For this purpose, consider a 5D scalar field $\phi(x_\mu, y)$ (where we have defined $y = y_1$), propagating in an S^1/Z_2 orbifold extra dimension. The radius of the 5th dimension is R and periodicity implies $y + 2\pi R \sim y$. As a consequence, the momentum in the fifth dimension is quantized as $p_5 = n/R$ for integer n . Under the Z_2 action $y \rightarrow -y$ the scalar transforms as $\phi(x_\mu, y) = \pm\phi(x_\mu, -y)$, where the $+$ ($-$) sign corresponds to ϕ being even (odd) under

case $C = [m]^{+1}$ and $K = [m]^0$.

⁹The LEP/SLC fit of $\Gamma_b/\Gamma_{\text{had}}$ in Z decay requires the shift of the $Z \rightarrow \bar{b}_L b_L$ coupling to be $\lesssim 0.3\%$ [3].

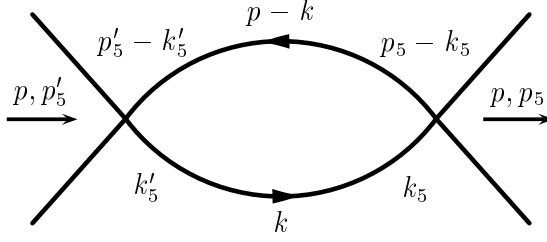


Figure 3.3: One-loop diagram for ϕ - ϕ scattering on S^1/Z_2 . The total incoming momentum is (p, p'_5) and the total outgoing momentum is (p, p_5) . Generally, it is possible that $|p'_5| \neq |p_5|$, since the orbifold fixed points break 5D translational invariance.

Z_2 . The scalar propagator on this space is given by [35, 54]

$$D(p, p_5, p'_5) = \frac{i}{2} \left\{ \frac{\delta_{p_5, p'_5} \pm \delta_{-p_5, p'_5}}{p^2 - p_5^2} \right\}, \quad (3.24)$$

where the additional factor $1/2$ takes into account that the physical space is only half of the periodicity. Consider now the one-loop ϕ - ϕ scattering diagram in Fig. 3.3. The total incoming momentum is (p, p'_5) and the total outgoing momentum is (p, p_5) , which can in general be different, since 5D translation invariance is broken by the orbifold boundaries. Locally, however, momentum is conserved at the vertices. The diagram then reads

$$i\Sigma = \frac{1}{4} \frac{\lambda^2}{2} \frac{1}{2\pi R} \sum_{k_5, k'_5} \int \frac{d^4 k}{(2\pi)^4} \left\{ \frac{\delta_{k_5, k'_5} \pm \delta_{-k_5, k'_5}}{k^2 - k_5^2} \right\} \left\{ \frac{\delta_{(p_5 - k_5), (p'_5 - k'_5)} \pm \delta_{-(p_5 - k_5), (p'_5 - k'_5)}}{(p - k)^2 - (p_5 - k_5)^2} \right\}, \quad (3.25)$$

where λ is the quartic coupling and the additional factor $1/4$ results from working on S^1/Z_2 . After summing over k'_5 , the integrand can be written as

$$F(k_5) = \frac{1}{(k^2 - k_5^2) [(p - k)^2 - (p_5 - k_5)^2]} \left\{ \delta_{p_5, p'_5} + \delta_{p_5, -p'_5} \pm \delta_{2k_5, (p_5 + p'_5)} \pm \delta_{2k_5, (p_5 - p'_5)} \right\}. \quad (3.26)$$

In Eq. (3.26), the first two terms in the bracket conserve $|p'_5|$ and contribute to the bulk kinetic terms of the scalar. The last two terms, on the other hand, violate $|p'_5|$ conservation and thus lead to a renormalization of the brane couplings [35]. Note

that these brane terms lead in Eq. (3.25) to a logarithmic divergence. Applying, on the other hand, to the bulk terms the Poisson resummation identity

$$\frac{1}{2\pi R} \sum_{m=-\infty}^{\infty} F(m/R) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-2\pi i k R n} F(k), \quad (3.27)$$

we obtain a sum of momentum space integrals, where the “local” $n = 0$ term diverges linearly like in 5D uncompactified space. This term contributes a linear divergence to the diagram such that the scattering amplitude becomes under order one rescalings of the random renormalization point for the external momenta of the order

$$i\Sigma \rightarrow \frac{\lambda^2}{4} \int \frac{d^5 k}{(2\pi)^5} [k^2(p-k)^2]^{-1} \simeq \frac{\lambda^2}{2} \frac{\Lambda}{(4\pi)^{5/2} \Gamma(5/2)}, \quad (3.28)$$

where Λ is an ultraviolet cutoff. On S^1/Z_2 , we thus indeed obtain for the strong coupling scale $\Lambda \simeq 48\pi^3 \lambda^{-2}$, which is two times larger than the NDA value obtained in 5D uncompactified space. This is also in agreement with the definition of Λ for a 5D gauge theory on an interval given in Ref. [48].

Similarly, when the 5th dimension is compactified on $S^1/(Z_2 \times Z'_2)$ [55], we expect a raising of Λ by a factor of 4 with respect to the uncompactified case. Let us briefly estimate how far this could improve the calculability of our 6D model. To this end, we assume, besides the two extra dimensions compactified on the rectangle, two additional extra dimensions with radii R_3 and R_4 , each of which has been compactified on $S^1/(Z_2 \times Z'_2)$. We assume that the gauge bosons are even under the actions of the $Z_2 \times Z'_2$ groups. Moreover, we take for the bulk kinetic coefficients in eight dimensions $M_L^4 = M_Y^4$ and set $R_3 = R_4 = R_2 = R_1/\sqrt{2}$. From the expression analogous to Eq. (3.21c), we then obtain the estimate $\epsilon_3 \simeq g^2(\pi M_L R_2)^4/3\sqrt{2}$, where the relative factor $(\pi R_2/2)^2$, arises from integrating over the physical space on each circle, which is only 1/4 of the circumference. With respect to the NDA value $\Lambda^4 \simeq (4\pi)^4 \Gamma(4) M_L^4$ in uncompactified space, the cutoff gets now modified as $\Lambda^4 \rightarrow 16 \cdot \Lambda^4$, implying that

$$\epsilon_3 \simeq \frac{g^2}{192\sqrt{2}} (\Lambda R_2/4)^4 \simeq 1.3 \times 10^{-3} \times (\Lambda R_2/4)^4. \quad (3.29)$$

In agreement with EWPT, the loop expansion parameter could therefore assume here a value $(\Lambda R_2)^{-1} \simeq 0.25$, corresponding to the appearance of 4 KK modes per extra dimension below the cutoff. Taking also a possible additional raising of Λ by a factor of $\sqrt{2}$ due to the reduced physical space on the rectangle into account, one could have $(\Lambda R_2)^{-1} \simeq 0.2$ with 5 KK modes per extra dimension below the cutoff. In conclusion, this demonstrates that by going beyond five dimensions, the calculability of Higgsless models could be improved by factors related to the geometry.

CHAPTER 4

A New Two Higgs Doublet Model

4.1 Model and the Formalism

Our proposed model is based on the symmetry group $SU(3)_c \times SU(2)_L \times U(1) \times Z_2$. In addition to the usual SM fermions, we have three EW singlet right-handed neutrinos, $N_{Ri}, i = 1 - 3$, one for each family of fermions. The model has two Higgs doublets, χ and ϕ . All the SM fermions and the Higgs doublet χ , are even under the discrete symmetry, Z_2 , while the RH neutrinos and the Higgs doublet ϕ are odd under Z_2 . Thus all the SM fermions except the left-handed neutrinos, couple only to χ . The SM left-handed neutrinos, together with the right-handed neutrinos, couple only to the Higgs doublet ϕ . The gauge symmetry $SU(2) \times U(1)$ is broken spontaneously at the EW scale by the VEV of χ , while the discrete symmetry Z_2 is broken by a VEV of ϕ , and we take $\langle \phi \rangle \sim 10^{-2} eV$. Thus, in our model, the origin of the neutrino masses is due to the spontaneous breaking of the discrete symmetry Z_2 . The neutrinos are massless in the limit of exact Z_2 symmetry. Through their Yukawa interactions with the Higgs field ϕ , the neutrinos acquire masses much smaller than those of the quarks and charged leptons due to the tiny VEV of ϕ .

The Yukawa interactions of the Higgs fields with the leptons are

$$\mathcal{L}_y = y_l \bar{\Psi}_L^l l_R \chi + y_\nu \bar{\Psi}_L^l N_R \tilde{\phi} + h.c., \quad (4.1)$$

where $\bar{\Psi}_L^l = (\bar{\nu}_l, \bar{l})_L$ is the usual lepton doublet and l_R is the charged lepton singlet. The first term gives rise to the mass of the charged leptons, while the second term gives a tiny neutrino mass. The interactions with the quarks are the same as in the

Standard Model with χ playing the role of the SM Higgs doublet. Note that in our model, a SM left-handed neutrino, ν_L combines with a right handed neutrino, N_R , to make a massive Dirac neutrino with a mass $\sim 10^{-2}$ eV, the scale of Z_2 symmetry breaking.

For simplicity, we do not consider CP violation in the Higgs sector. (Note that in this model, spontaneous CP violation would be highly suppressed by the small VEV ratio and could thus be neglected. However, one could still consider explicit CP violation). The most general Higgs potential consistent with the $SM \times Z_2$ symmetry is [56]

$$V = -\mu_1^2 \chi^\dagger \chi - \mu_2^2 \phi^\dagger \phi + \lambda_1 (\chi^\dagger \chi)^2 + \lambda_2 (\phi^\dagger \phi)^2 + \lambda_3 (\chi^\dagger \chi) (\phi^\dagger \phi) - \lambda_4 |\chi^\dagger \phi|^2 - \frac{1}{2} \lambda_5 [(\chi^\dagger \phi)^2 + (\phi^\dagger \chi)^2]. \quad (4.2)$$

The physical Higgs fields are a charged field H , two neutral scalar fields h and σ , and a neutral pseudoscalar field ρ . In the unitary gauge, the two doublets can be written

$$\chi = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2}(V_\phi/V)H^+ \\ h_0 + i(V_\phi/V)\rho + V_\chi \end{pmatrix},$$

$$\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} -\sqrt{2}(V_\chi/V)H^+ \\ \sigma_0 - i(V_\chi/V)\rho + V_\phi \end{pmatrix}, \quad (4.3)$$

where $V_\chi = \langle \chi \rangle$, $V_\phi = \langle \phi \rangle$, and $V^2 = V_\chi^2 + V_\phi^2$. The particle masses are

$$m_W^2 = \frac{1}{4}g^2V^2, \quad m_H^2 = \frac{1}{2}(\lambda_4 + \lambda_5)V^2, \quad m_\rho^2 = \lambda_5V^2,$$

$$m_{h,\sigma}^2 = (\lambda_1V_\chi^2 + \lambda_2V_\phi^2) \pm \sqrt{(\lambda_1V_\chi^2 - \lambda_2V_\phi^2)^2 + (\lambda_3 - \lambda_4 - \lambda_5)^2V_\chi^2V_\phi^2}. \quad (4.4)$$

An immediate consequence of the scenario under consideration is a very light scalar σ with mass

$$m_\sigma^2 = 2\lambda_2 V_\phi^2 [1 + O(V_\phi/V_\chi)]. \quad (4.5)$$

The mass eigenstates h, σ are related to the weak eigenstates h_0, σ_0 by

$$h_0 = ch + s\sigma, \quad \sigma_0 = -sh + c\sigma, \quad (4.6)$$

where c and s denotes the cosine and sine of the mixing angles, and are given by

$$c = 1 + O(V_\phi^2/V_\chi^2),$$

$$s = -\frac{\lambda_3 - \lambda_4 - \lambda_5}{2\lambda_1} (V_\phi/V_\chi) + O(V_\phi^2/V_\chi^2). \quad (4.7)$$

Since $V_\phi \sim 10^{-2}$ eV and $V_\chi \sim 250$ GeV, this mixing is extremely small, and can be neglected. Hence, we see that h behaves essentially like the SM Higgs (except of course in interactions with the neutrinos).

The interactions of the neutral Higgs fields with the Z are given by

$$\begin{aligned} \mathcal{L}_{gauge} = & \frac{\bar{g}}{2V} (cV_\phi + sV_\chi) (\rho\partial^\mu h - h\partial^\mu \rho) Z_\mu + \frac{\bar{g}}{2V} (sV_\phi - cV_\chi) (\rho\partial^\mu \sigma - \sigma\partial^\mu \rho) Z_\mu \\ & + \frac{\bar{g}^2}{4} (sV_\phi - cV_\chi) h Z^\mu Z_\mu + \frac{\bar{g}^2}{4} (cV_\phi + sV_\chi) \sigma Z^\mu Z_\mu + \frac{\bar{g}^2}{8} (h^2 + \sigma^2 + \rho^2) Z^\mu Z_\mu \end{aligned} \quad (4.8)$$

where $\bar{g}^2 = g^2 + g'^2$, and V_χ and V_ϕ are the two VEV's.

4.2 Phenomenological Implications

We now consider the phenomenological implications of this model. There are several interesting phenomenological implications which can be tested in the upcoming

neutrino experiments and high energy colliders. The light neutrinos in our model are Dirac particles. So neutrino-less double beta decay is not allowed in our model. This is a very distinctive feature of our model for the neutrino masses compared to the traditional see-saw mechanism. In the see-saw model, light neutrinos are Majorana particles, and thus neutrino-less double beta decay is allowed. The current limit on the double beta decay is $m_{ee} \sim 0.3 \text{ eV}$. This limit is expected to go down to about $m_{ee} \sim 0.01 \text{ eV}$ in future experiments [57]. If no neutrino-less double beta decay is observed to that limit, that will cast serious doubts on the see-saw model. In our model, of course, it is not allowed at any level.

Next, we consider the implications of our model for high energy colliders. First we consider the production of the light scalar σ in e^+e^- collisions. The only possible decay modes of this particle are a diphoton mode, $\sigma \rightarrow \gamma\gamma$ which can occur at the one-loop level and, if it has enough mass, a $\sigma \rightarrow \nu\bar{\nu}$ mode. The one loop decay to two photons takes place with quarks, W bosons, or charged Higgs bosons in the loop. The largest contribution to this decay mode is $\sim e^8 m_\sigma^5 / m_q^4$. This gives the lifetime of σ to be $\sim 10^{20}$ years, which is much larger than the age of the universe. Thus σ essentially behaves like a stable particle, and its production at the colliders will lead to missing energy in the event. The couplings of σ to quarks and charged leptons takes place only through mixing which is highly suppressed (proportional to the ratio V_ϕ/V_χ). Thus we need only consider its production via its interactions with gauge bosons. The $ZZ\sigma$ coupling is also highly suppressed, so that processes such as $e^+e^- \rightarrow Z^* \rightarrow Z\sigma$ and $Z \rightarrow Z^*\sigma \rightarrow f\bar{f}\sigma$ are negligible. However, no such suppression occurs for the $ZZ\sigma\sigma$ coupling. Consider the Z decay process $Z \rightarrow Z^*\sigma\sigma \rightarrow f\bar{f}\sigma\sigma$. A direct calculation yields the width (neglecting the σ and fermion masses),

$$\Gamma(Z \rightarrow f\bar{f}\sigma\sigma) = \frac{G_F^3 m_Z^5 (g_V^2 + g_A^2)}{2\sqrt{2}(2\pi)^5} \int_0^{m_Z/2} dE_1 \int_0^{m_Z/2} dE_2 \\ \times \int_{-1}^1 d(\cos\theta) \frac{E_1^2 E_2^2 (3 - \cos\theta)}{(2E_1 E_2 - 2E_1 E_2 \cos\theta - m_Z^2)^2 + m_Z^2 \Gamma_Z^2}, \quad (4.9)$$

where $g_V = T_3 - 2Q \sin^2 \theta_W$ and $g_A = T_3$. This gives

$$\sum_f \Gamma(Z \rightarrow f\bar{f}\sigma\sigma) \simeq 2.5 \times 10^{-7} \text{ GeV}. \quad (4.10)$$

For the 1.7×10^7 Z 's observed at resonance at LEP1 [58], this gives an expectation of only about two such events.

Now we consider the production of the heavy Higgs particles in our model. Since the charged Higgs H^\pm and the pseudoscalar, ρ can be produced along with the light scalar σ , there will be stricter mass bound on these particles than in a typical two Higgs doublet model. Let us consider the pseudoscalar ρ , and assume $m_\rho < m_Z$. Then the Z can decay via $Z \rightarrow \sigma\rho$. Since ρ couples negligibly to all SM fermions except the neutrinos, here we need only consider its decay to $\nu\bar{\nu}$ (or $\sigma\sigma$ if we consider CP violation), so this process contributes to the invisible decay width of the Z . The width for this process is

$$\Gamma = \frac{G_F m_Z^3}{24\sqrt{2}\pi} \left(1 - \frac{m_\rho^2}{m_Z^2}\right)^3 \quad (4.11)$$

This is less than the experimental uncertainty in the invisible Z width for $m_\rho \gtrsim 78 \text{ GeV}$. (The experimental value of the invisible Z width is $499.0 \pm 1.5 \text{ MeV}$ [59].)

For $m_\rho > m_Z$, real pseudoscalar ρ can be produced via $e^+e^- \rightarrow Z^* \rightarrow \rho\sigma$. The total cross section for this process is

$$\sigma = \frac{G_F^2 m_Z^4 (g_V^2 + g_A^2) s}{24\pi} \left(\frac{1}{s - m_Z^2}\right)^2 \left(1 - \frac{m_\rho^2}{s}\right)^3. \quad (4.12)$$

For LEP2, $\sqrt{s} \simeq 200 \text{ GeV}$, we find that less than one event is expected in $\simeq 3000 \text{ pb}^{-1}$ [6] of data for $m_\rho \gtrsim 95 \text{ GeV}$. Note that the bound on the ρ mass we obtain is much less than the mass for which the Higgs potential becomes strongly coupled ($\lambda_5 \leq 2\sqrt{\pi}$ which gives $m_\rho \leq 470 \text{ GeV}$).

For $m_\rho > m_Z$, the Z can still decay invisibly through $Z \rightarrow \rho^* \sigma \rightarrow \nu \bar{\nu} \sigma$. The width for this decay is

$$inv_{width} \Gamma = \frac{G_F m_Z^2 y_{\nu_l}^2}{3\sqrt{2}(2\pi)^3} \int_0^{m_Z/2} dE \frac{E^3(m_Z - 2E)}{(m_Z^2 - 2m_Z E - m_\rho^2)^2}. \quad (4.13)$$

Summing over generations, this gives

$$\Gamma(m_\rho = 100 \text{ GeV}) \simeq (0.1 \text{ MeV}) \left(\frac{1}{3} \sum_l y_{\nu_l}^2 \right)$$

$$\Gamma(m_\rho = 200 \text{ GeV}) \simeq (4 \times 10^{-3} \text{ MeV}) \left(\frac{1}{3} \sum_l y_{\nu_l}^2 \right). \quad (4.14)$$

Even if we take $\frac{1}{3} \sum y_{\nu_l}^2 \sim 1$, these values are well within the experimental uncertainty in the invisible Z width of 1.5 MeV . Note that if we allow explicit CP violation in the Higgs sector, the invisible decay $Z \rightarrow \rho \sigma \rightarrow \sigma \sigma \sigma$ will also occur.

Our model has very interesting implications for the discovery signals of the Higgs boson at the high energy colliders, such as the Tevatron and LHC. Note that since V_ϕ is extremely small compared to V_χ , the neutral Higgs boson, h is like the SM Higgs boson so far its decays to fermions and to W and Z bosons are concerned. However, in our model, h has new decay modes, such as $h \rightarrow \sigma \sigma$ which is invisible. This could change the Higgs signal at the colliders dramatically. The width for this invisible decay mode $h \rightarrow \sigma \sigma$ is given by

$$\Gamma(h \rightarrow \sigma \sigma) = \frac{(\lambda_3 + \lambda_4 + \lambda_5)^2 V_\chi^2}{32\pi m_h}. \quad (4.15)$$

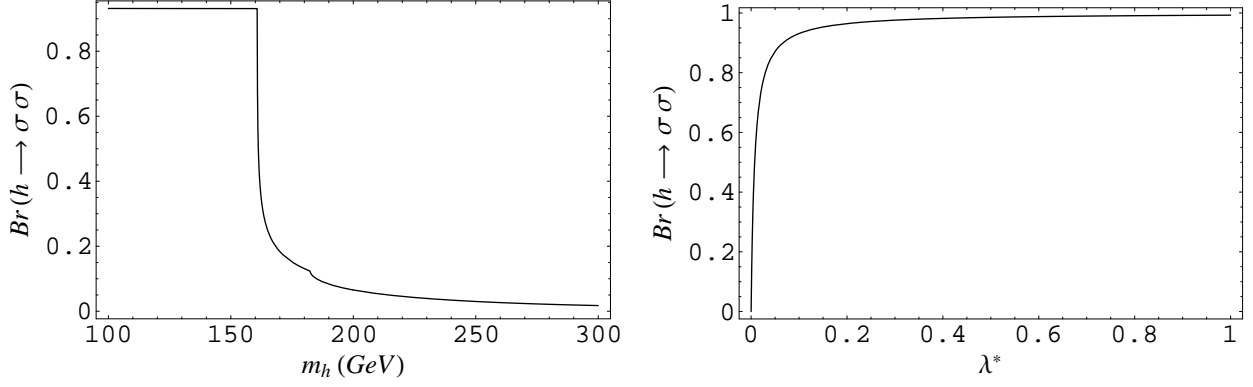


Figure 4.1: Left panel: Branching ratio for $h \rightarrow \sigma\sigma$ as a function of m_h for the value of the parameter, $\lambda^* = 0.1$. Right panel: Branching ratio for $h \rightarrow \sigma\sigma$ as a function of λ^* for $m_h = 135 \text{ GeV}$.

Using

$$m_h^2 = 2\lambda_1 V_\chi^2 + O(V_\phi^2/V_\chi^2), \quad (4.16)$$

this can be written

$$\Gamma(h \rightarrow \sigma\sigma) = \frac{(\lambda_3 + \lambda_4 + \lambda_5)^2 m_h}{64\pi\lambda_1}. \quad (4.17)$$

Depending on the parameters, it is possible for the dominant decay mode of h to be this invisible mode. The branching ratios for the Higgs decay to this invisible mode are shown in Fig. 4 (left panel), for the Higgs mass range from 100 to 300 GeV , for the choice of the value of the parameter, λ^* equal to 0.1 where λ^* is defined to be equal to $\frac{(\lambda_3 + \lambda_4 + \lambda_5)^2}{\lambda_1}$. The right panel in Fig. 4 shows how this branching ratio depends on this parameter for a Higgs mass of 135 GeV . (The results for the branching ratio is essentially the same for other values of the Higgs mass between 120 and 160 GeV). We see that for a wide range of this parameter, for the Higgs mass up to about 160 GeV , the invisible decay mode dominates, thus changing the Higgs search strategy at the Tevatron Run 2 and the LHC. The production rate of the neutral scalar Higgs h in our model are essentially the same as in the SM. This implies that the Higgs mass bound from LEP is not significantly altered. (The L3 collaboration set a bound of

$m_h \geq 112.3 \text{ GeV}$ for an invisibly decaying Higgs with the SM production rate [60]). However, because of the dominance of the invisible decay mode, it will be very difficult to observe a signal at the LHC in the usual production and decay channels such as $qqh \rightarrow qqWW$, $qqh \rightarrow qq\tau\tau$, $h \rightarrow \gamma\gamma$, $h \rightarrow ZZ \rightarrow 4l$, $t\bar{t}h$ (with $h \rightarrow b\bar{b}$) and $h \rightarrow WW \rightarrow l\nu l\nu$ [61]. However, a signal with such an invisible decay mode of the Higgs (as in our model) can be easily observed at the LHC through the weak boson fusion processes, $qq \rightarrow qqW^+W^- \rightarrow qqH$ and $qq \rightarrow qqZZ \rightarrow qqH$ [62] if appropriate trigger could be designed for the ATLAS and CMS detector. For example, with only 10 fb^{-1} of data at the LHC, such a signal can be observed at the 95 percent CL with an invisible branching ratio of 31 percent or less for a Higgs mass of upto 400 GeV [62]. Thus our model can be easily tested at the LHC for a large region of the Higgs mass. Of course, establishing that this signal is from the Higgs boson production will be very difficult at the LHC. For the Higgs search at the Tevatron, the usual signal from the Wh production, and the subsequent decays of h to WW^* or $b\bar{b}$ will be absent. The most promising mode in our model will be the production of ZH , with Z decaying to $l^\pm l^\pm$ ($l = e, \mu$) and the Higgs decaying invisibly. There will be a peak in the missing energy distribution in the final state with a Z . We urge the Tevatron collaborations to look for such a signal.

4.3 Cosmological Implications

Our model has several interesting astrophysical and cosmological implications. Firstly, there is a problem with primordial nucleosynthesis [63]. This occurs because the relatively strong interactions between left- and right-handed neutrinos and the light scalar σ will keep right-handed neutrinos and σ in thermal equilibrium with left-handed neutrinos during nucleosynthesis. So, the effective number of light degrees of freedom, $g_* = g_b + \frac{7}{8}g_f$ (g_b and g_f are the numbers of bosonic and fermionic spin degrees of freedom respectively), is

$$g_* = (g_*)_{SM} + 1 + \frac{7}{8}(6) = 17. \quad (4.18)$$

(Equivalently, the effective number of neutrinos is $N_\nu = 6 + \frac{4}{7}$.) This increases the expansion rate of the universe, which is proportional to $\sqrt{g_*}$. As a result, reactions which interconvert protons and neutrons freeze out of thermal equilibrium at a higher temperature, increasing the ratio of neutrons to protons during nucleosynthesis. This increase alters the abundances of light elements produced in subsequent nucleosynthesis reactions, most notably, helium-4 is greatly overproduced. The mass fraction of helium-4 obtained here is $\simeq 0.3$ compared to the observed fraction $\simeq 0.25$. To solve this problem, our model requires a non-standard nucleosynthesis scenario. One possibility is a large neutrino degeneracy. It is assumed in standard nucleosynthesis that the chemical potential of neutrinos $\mu_\nu \simeq 0$. However, since relic neutrinos are not observed, this is not required by observation. A large value of μ_ν alters the equilibrium ratio of neutrons to protons,

$$\frac{n}{p} = e^{-\mu_\nu/T} \left(\frac{n}{p} \right)_{\mu_\nu=0}, \quad (4.19)$$

leading to an alteration of light element abundances. Our problem can be solved with $\mu_\nu \sim 0.1 \text{ MeV}$. In depth studies have been conducted, where the effective number of neutrinos, neutrino degeneracy and the density of baryons are allowed to vary, in order to find the most general values consistent with BBN and WMAP [65](as well as studies which fix $N_\nu = 3$, leading to much stronger bounds on neutrino degeneracy [66]). These studies find upper bounds on N_ν from 7.1 to 8.7, depending on how conservative an interpretation of the data is used. Another possible solution could be the existence of massive particle species that decay after nucleosynthesis. Energetic decay products of these particles interact with background nuclei, causing non-thermal nuclear reactions, such as helium-4 dissociation, that reset light element

abundances [64]. (We also note that in the above analysis, we have taken three right-handed neutrinos. For the oscillation experiments, as well as for direct measurements, the lightest neutrino mass can be zero. So, only two right-handed neutrinos are strictly required. This could make the Big Bang nucleosynthesis problem somewhat milder.)

There are also bounds on the effective number of neutrinos coming from astrophysical observations other than light element abundances. For example, data from WMAP and the Sloan Digital Sky Survey (SDSS) power spectrum of luminous red galaxies, give a bound $0.8 < N_\nu < 7.6$ [67]. The authors of [68] claim that data from the SDSS Lyman- α forest power spectrum, along with cosmic microwave background, supernova, and galaxy clustering data, seem to require $N_\nu > 3$.

Additionally, the $\nu\bar{\nu}\sigma$ interaction can affect supernova explosion dynamics, and since this interaction can be fairly strong it may bind $\nu\bar{\nu}$, giving rise to the possibility of $\nu\bar{\nu}$ atoms and a new kind of star formation.

Also, the spontaneous breaking of the discrete global symmetry Z_2 will lead to the formation of cosmological domain walls. These walls will have energy per unit area $\eta \sim V_\phi^3$, so their effect will be small. The resulting temperature anisotropies are

$$\frac{\delta T}{T} \simeq G\eta H_0^{-1} \sim 10^{-20}, \quad (4.20)$$

where G is Newton's gravitational constant and H_0 is the present Hubble parameter. The observed level of CMB temperature anisotropies is 10^{-5} [59], so this is not a problem.

CHAPTER 5

CONCLUSIONS

We have presented several scenarios that alter the Higgs sector from that of the SM.

First, we presented a renormalizable non-supersymmetric model based on the finite symmetry $G = (G_1 \times G_2) \rtimes A_4$, with $G_1 = S_3 \times S_3 \times S_3 \times S_3$ and $G_2 = Z_2 \times Z_2 \times Z_2$, with SM leptons assigned to representations of A_4 . Neutrino masses are generated by a Higgs field ϕ belonging to a 16-dimensional representation of $G_1 \rtimes A_4$ while charged-lepton masses are generated by a Higgs field χ belonging to a 6-dimensional representation of $G_2 \rtimes A_4$. The additional symmetries, G_1 and G_2 , prevent quadratic and cubic interactions between ϕ and χ and allow only a trivial quartic interaction that does not cause an alignment problem, addressing the alignment problem without altering the desired properties of the family symmetry. In this way, we are able to explain all aspects of neutrino mixing using only symmetries which are spontaneously broken by the Higgs mechanism.

Next, we have considered a 6D Higgsless model for EWSB based only on the SM gauge group $SU(2)_L \times U(1)_Y$. The model is formulated in flat space with the two extra dimensions compactified on a rectangle of size $\sim (TeV)^{-2}$. EWSB is achieved by imposing consistent BC's on the edges of the rectangle. The higher KK resonances of W^\pm and Z decouple below $\sim 1 TeV$ through the presence of a dominant 4D brane induced gauge kinetic term at the point where $SU(2)_L \times U(1)_Y$ remains unbroken. The ρ parameter is arbitrary and can be set exactly to unity by appropriately choosing the bulk gauge couplings and compactification scales. The resulting gauge couplings

in the effective 4D theory arise essentially from the brane couplings, slightly modified (at the level of one percent) by the bulk interaction. Thus, the main role played by the bulk interactions is to break the electroweak gauge symmetry. We calculate the tree-level oblique corrections to the S , T , and U parameters and find them to be consistent with current data.

Finally, we have presented a simple extension of the Standard Model supplemented by a discrete symmetry, Z_2 . We have also added three right-handed neutrinos, one for each family of fermions, and one additional Higgs doublet. While the electroweak symmetry is spontaneously broken at the usual 100 GeV scale, the discrete symmetry, Z_2 remains unbroken to a scale of about 10^{-2} eV . The spontaneous breaking of this Z_2 symmetry by the VEV of the second Higgs doublet generates tiny masses for the neutrinos. The neutral heavy Higgs in our model is very similar to the SM Higgs in its couplings to the gauge bosons and fermions, but it also couples to a very light scalar Higgs present in our model. This light scalar Higgs, σ , is essentially stable, or decays to $\nu\bar{\nu}$. Thus the production of this σ at the high energy colliders leads to missing energy. The SM-like Higgs, for a mass up to about 160 GeV dominantly decays to the invisible mode $h \rightarrow \sigma\sigma$. Thus the Higgs signals at high energy hadron colliders are dramatically altered in our model. Our model also has interesting implications for astrophysics and cosmology.

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The Higgs mechanism, which is responsible for electroweak symmetry breaking and unitarization of massive W^\pm and Z scattering, is a fundamental ingredient of the Standard Model (SM). However, there is as yet no direct evidence of the Higgs boson, so that the details of the Higgs sector, if it even exists, remain a mystery. Here, we explore several scenarios that alter the Higgs sector from that of the SM. The first uses additional symmetries of the Higgs sector to address certain issues of neutrino mixing, the second uses extra dimensional boundary conditions to avoid the need for a Higgs entirely, and the last uses additional Higgs fields to provide an alternative explanation for tiny neutrino masses.

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