FINITE REPRESENTATIONS OF A QUIVER ARISING FROM STRING THEORY AND THEIR CORRESPONDENCE WITH SEMI-STABLE SHEAVES

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1 Introduction

A quiver Γ is a directed graph. A representation V of a quiver Γ is an assignment to each vertex i of Γ a vector space $V(i)$, and to each directed edge $\overrightarrow{ij}$ (from vertex i to vertex j) of Γ, of a linear transformation $f_{ji} : V(i) \to V(j)$. Many problems in the representation theory of algebras, rings and Lie groups can be reduced to questions of representations of quivers [1]. Of particular importance are the quivers of finite representation type — those having only a finite number of non-isomorphic indecomposable representations.

In 1972, Gabriel [7] proved the following surprising result: The quiver Γ is of finite representation type if and only if its unoriented graph is one of the Dynkin diagrams A_n, D_n, E_6, E_7 or E_8.

Such a quiver is called an ADE quiver.

Many generalizations of Gabriel's Theorem have been given, ([5], [6] and [12].) In 1973, I.N. Bernstein, I.M. Gel'fand and V.A. Ponomarev [2] reproved this theorem, showing that it arises in a natural way via the use of systematic transformations of quiver representations, using roots, reflection functors, Coxeter functors and Weyl groups.

Recently, quiver theory has attracted the attention of physicists [4] because of its close relations with the study of D-branes and mirror symmetry. A special type of quiver arising from string theory, called an "$N = 1$ ADE quiver", was introduced in [4].

1.1 Describing N=1 ADE quivers

This requires some detailed explanation, mainly of the relations (1.1), which distinguish these from ADE quivers. To make our presentation intelligible to non-experts, we briefly recall some definitions and established facts. (Here all vectors are over a field k.)

A *quiver* $\Gamma = (V_\Gamma, E_\Gamma)$—without relations—is a directed graph.

A *representation* (V, f) of a quiver Γ is an assignment to each vertex $i \in V_\Gamma$ of a vector space $V(i)$, and to each directed edge $ij \in E_\Gamma$ of a linear transformation $f_{ji} : V(i) \to V(j)$.

A *morphism* $h : (V, f) \to (V', f')$ between representations of Γ over k is a collection $\{h_i : V(i) \to V'(i)\}_{i \in V_\Gamma}$ of k-linear maps such that for each edge $ij \in E_\Gamma$ the diagram

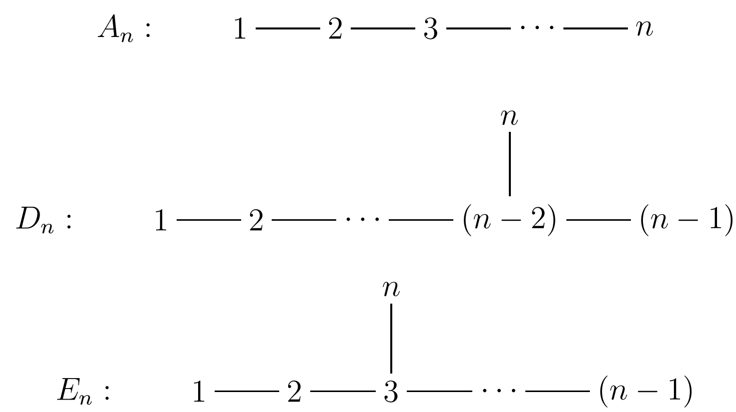
$$\begin{array}{ccc} V(i) & \xrightarrow{h_i} & V'(i) \\ {\scriptstyle f_{ji}}\big\downarrow & & {\scriptstyle f'_{ji}}\big\downarrow \\ V(j) & \xrightarrow{h_j} & V'(j) \end{array}$$

commutes. Compositions of morphisms are defined in the usual way. For a path $p : i_1 \to i_2 \to \cdots \to i_r$ in Γ, and a representation (V, f), we let f_p be the composition of the linear transformations $f_{i_{k+1} i_k} : V(i_k) \to V(i_{k+1})$, $1 \leq k < r$. And given vertices i, j in V_Γ, and paths $p_1, \cdots, p_n$ from i to j, a *relation* σ on quiver Γ is a linear combination $\sigma = a_1 p_1 + \cdots + a_n p_n$, $a_i \in k$. If (V, f) is a representation of Γ, we extend the f-notation by setting $f_\sigma = a_1 f_{p_1} + \cdots + a_n f_{p_n} : V(i) \to V(j)$. A *quiver with relations* is a pair (Γ, ρ), where $\rho = (\sigma_t)_{t \in T}$ is a set of relations on Γ. And a representation (V, f) of (Γ, ρ) is a representation (V, f) of Γ for which $f_\sigma = 0$ for all relations $\sigma \in \rho$. We can then define, in the obvious way, *subrepresentations* (V', f') of (V, f), the *sum* of two representations, and when a representation (V, f) of (Γ, ρ) is indecomposable, of finite representation type, and simple.

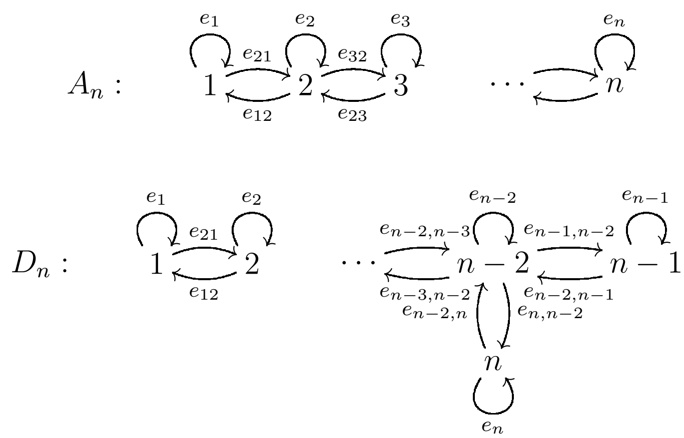
Definition 1.1. Given an *ADE* Dynkin diagram $\mathcal{D} = (V_\mathcal{D}, E_\mathcal{D})$ – an undirected graph– we let the associated quiver $\Gamma_\mathcal{D}$ be $\Gamma_\mathcal{D} = (V_{\Gamma_\mathcal{D}}, E_{\Gamma_\mathcal{D}})$ with: $V_{\Gamma_\mathcal{D}} := V_\mathcal{D}$, and

$$E_{\Gamma_\mathcal{D}} = \{(i,j), (j,i) \quad | \quad \{i,j\} \in E_\mathcal{D}\} \bigcup \{(i,i) \quad | \quad i \in V_\mathcal{D}\}$$

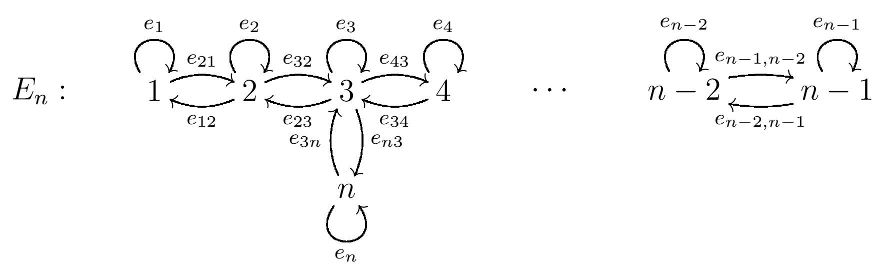
In other words, this is the standard digraph associated with graph Γ, except that we add a loop at each vertex. Recalling that ADE Dynkin diagram are, respectively,



The $N = 1$ ADE quivers are just the associated quivers to the above graphs, but with relations (1.1) below.



The quivers for E_n ($n = 6$, 7, 8) are,



The relation has the form

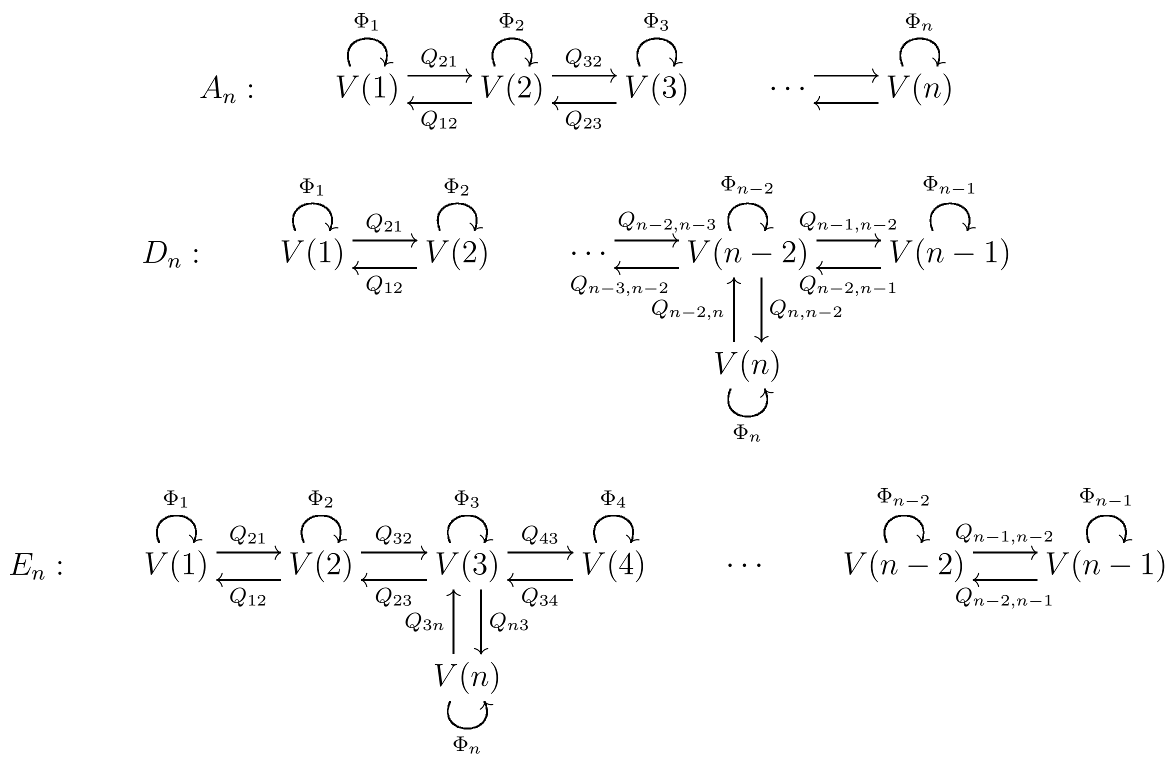
$$\sum_i s_{ij} e_{ji} e_{ij} + p'_j(e_j) = 0, \quad e_{ij} e_j = e_i e_{ij}. \tag{1.1}$$

where

$$\begin{cases} s_{ij} = 0 \quad \text{if} \quad i \quad \text{and} \quad j \quad \text{are not adjacent} \\ s_{ij} = 1 \quad \text{if} \quad i \quad \text{and} \quad j \quad \text{are adjacent and} \quad i > j \\ s_{ij} = -1 \quad \text{if} \quad i \quad \text{and} \quad j \quad \text{are adjacent and} \quad i < j \end{cases}$$

where $p'_j(x)$ is a certain fixed polynomial, $\forall j$.

If (V, f) is a representation of an $N = 1$ ADE quiver, the corresponding structures are



where we have write $Q_{ij} = f_{e_{ij}}$, $\Phi_j = f_{e_j}$. And the relation (1.1) becomes

$$\sum_i s_{ij} Q_{ji} Q_{ij} + p'_j(\Phi_j) = 0, \quad Q_{ij}\Phi_j = \Phi_i Q_{ij}.$$

Finally, we give a more technically precise statement of Gabriel's Theorem.

Theorem 1.1 (Gabriel). *[7] 1) Let* Γ *be a graph with orientation* Λ*. If in* $\mathrm{Rep}(\Gamma, \Lambda)$ *there are only finitely many non-isomorphic indecomposable objects, then* Γ *coincides with one of the graphs* A_n, D_n, E_6, E_7, E_8*.*

2)Let Γ *be a graph of one of the types* A_n, D_n, E_6, E_7, E_8*, and* Λ *some orientation on it. Then in* $\mathrm{Rep}(\Gamma, \Lambda)$ *there are only finitely many non-isomorphic indecomposable objects. In addition, the mapping*

$$V \to \dim V = (\dim V(i) : i \in \Gamma_0) \in \mathbf{R}^{|\mathbf{\Gamma_0}|}$$

sets up a one to one correspondence between classes of isomorphic indecomposable objects and positive roots in the root system of Γ*.*

1.2 Concerning my research

The primary goal of my thesis research has been to try to extend Gabriel's Theorem from ADE quivers to $N = 1$ ADE quivers. Because these new quivers are quivers with relations, they are more complex than quivers without relations. Nevertheless, I have made some success. The reader may also see the related work by Szendrői [21].

In Chapter 2, using a direct approach, I prove the finite representation type of $N = 1\, A_n$ quivers (Theorem 2.1), and of $N = 1\, D_n$ quivers (Theorem 2.2). The $N = 1\, E_6$ and E_7 cases are also considered, and partial results obtained.

In Chapter 3, by means of a different, unified approach, I prove the finite representation type of $N = 1$ ADE quivers, using the techniques of [2], but with modified reflection functors and Coxeter functors. Inspired by part 3 of Theorem 1 in Katz-Morrison [14], I also obtain a correspondence between indecomposable $N = 1$ representations and the rational curves in a Calabi-Yau 3-fold.

In Chapter 4, I consider the relationship between semi-stable sheaves and the indecomposable representation of $N = 1$ ADE quivers. I want to relate $N = 1$ ADE quiver theory to the deformation theory in Calabi-Yau 3-fold. The following conjecture is proved for the case of C a cA_n curve.

Conjecture 1.1. *There exists a natural one-to-one correspondence between the indecomposable representations of the $N = 1$ ADE quiver with the datum ρ described in* (1.1) *and a certain class of semistable quasi-coherent sheaves with support on a rational curve C in a Calabi–Yau 3-fold.*

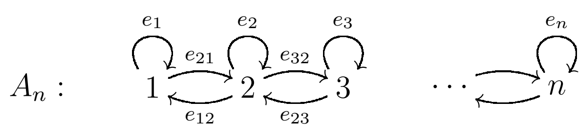
In Chapter 5, inspired by the work of Cachazo, Katz and Vafa in [4], we characterize the deformations of rational curves in Calabi-Yau 3-fold by field equation.

In Chapter 6, we generalize Reid's pagoda technique of [19] to give a characterization of rational curves in Calabi-Yau 3-fold via a sequence of semi-stable sheaves.

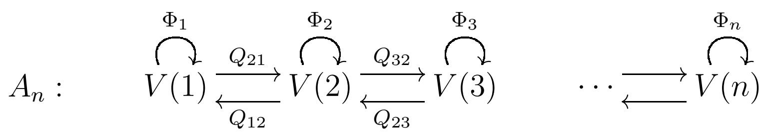
2 Direct proofs for indecomposable $N = 1$ ADE quiver representations

2.1 The $N = 1$ A_n case

In this chapter, for $N = 1$ A_n quiver,



we consider the representation of this $N = 1$ A_n quiver.



The representations of $N = 1$ A_n quiver should satisfy the relation (1.1). Explicitly, it satisfies the following relations,

$$Q_{12}Q_{21} + p_1'(\Phi_1) = 0$$

$$-Q_{21}Q_{12} + Q_{23}Q_{32} + p_2'(\Phi_2) = 0$$

$$\vdots$$

$$-Q_{i,i-1}Q_{i-1,i} + Q_{i,i+1}Q_{i+1,i} + p_i'(\Phi_i) = 0$$

$$\vdots$$

$$-Q_{n-1,n-2}Q_{n-2,n-1} + Q_{n-1,n}Q_{n,n-1} + p_{n-1}'(\Phi_{n-1}) = 0$$

$$-Q_{n,n-1}Q_{n-1,n} + p_n'(\Phi_n) = 0,$$

and

$$Q_{i,i+1}\Phi_{i+1} = \Phi_i Q_{i,i+1} \quad \Phi_{i+1}Q_{i+1,i} = Q_{i+1,i}\Phi_i \qquad i = 1,\dots,n-1.$$

where p' is a certain polynomial. We get Theorem 2.1.

Theorem 2.1. *Let $\mathcal{A} = \{rp'_i(x) \mid r \in \mathfrak{W}_{A_n}\}$, where $p'_i(x)$ are the polynomials in relation 1.1 and $\mathfrak{W}_{A_n}$ the Weyl group of A_n.*[1] *If no two positive elements in $\mathcal{A}$ have a common root and none of the polynomials in $\mathcal{A}$ are identically zero , then $N = 1$ A_n quiver is of finite representation type.*

We will give a proof of Theorem 2.1 on page 22. In this section, I will use A'_n to denote the $N = 1$ A_n quiver.

Lemma 2.1. *Let $(V, f) \in \mathrm{Rep(A'_n)}$. Let $a = \min\{i : V(i) \neq 0\}$. Let λ be an eigenvalue of $\Phi_a : V(a) \to V(a)$, then*

1. *there exists $b \geq a$, such that $\sum_{a\leq j\leq b} p'_j(\lambda) = 0$.*
2. *We can construct a simple sub-representation $(V_R, f) \subset (V, f)$ corresponding to $\sum_{a\leq j\leq b} p'_j(\lambda) = 0$.*
3. *Let $(W, f) \in \mathrm{Rep}(A'_n)$ be defined by*

$$W(i) = \begin{cases} \mathbf{C} & \text{for} \quad a \leq i \leq b \\ 0 & \text{otherwise} \end{cases}$$

For $x \in W(i)$, define $\Phi_i(x) = \lambda x$. Define $Q_{k-1,k}$ to be a scalar multiplication by $-\sum_a^{k-1} p'_i(\lambda)$, if $a < k \leq b$, and 0 otherwise. And define $Q_{k+1,k}$ to be the

---

[1] *By [14], page 461 and 463, we know W_{A_n} is generated by reflections r_i, which is defined by*

$$r_i(t_j) = t_{\sigma_i(j)} \quad \text{for} \quad 1 \leq i \leq n,$$

where σ_i is the simple transposition $(i, i+1)$ on the set $\{1,\dots,n\}$. By [4],page 3, we can write $p'_i(x)$ in the relation given in (1.1) as

$$A_n: \quad p'_i = t_i - t_{i+1} \quad i = 1,\dots,n$$

Then for a generator $r_k \in \mathfrak{W}_{A_n}$, we can define $r_k(p'_i)$ by linearity.

identity map if $a \le k < b$, 0 otherwise. The (V_R, f) defined in (2) is isomorphic to (W, f).

Proof. **Part (1):** Let v_a be an eigenvector of Φ_a corresponding to eigenvalue λ. Let

$$v_i = Q_{i,i-1}Q_{i-1,i-2}\dots Q_{a+2,a+1}Q_{a+1,a}v_a, \quad \text{for} \quad i \ge a$$

Let $b = \min\{i : v_{i+1} = 0\}$, (if $v_n \ne 0$, we let $b = n$.) Since $\Phi_j Q_{j,j-1}v_{j-1} = Q_{j,j-1}\Phi_{j-1}v_{j-1}$, v_j is an eigenvector of Φ_j corresponding to the same eigenvalue λ for $a \le j \le b$.

Since $Q_{b+1,b}v_b = 0$, we have

$$-Q_{b,b-1}Q_{b-1,b}v_b + p'_b(\lambda)v_b = 0$$

Since $v_b = Q_{b,b-1}v_{b-1}$, we have

$$-Q_{b,b-1}Q_{b-1,b}Q_{b,b-1}v_{b-1} + p'_b(\lambda)v_b = 0$$

Since

$$-Q_{b-1,b-2}Q_{b-2,b-1}v_{b-1} + Q_{b-1,b}Q_{b,b-1}v_{b-1} + p'_{b-1}(\lambda)v_{b-1} = 0$$

we have

$$-Q_{b,b-1}(Q_{b-1,b-2}Q_{b-2,b-1}v_{b-1} - p'_{b-1}(\lambda)v_{b-1}) + p'_b(\lambda)v_b = 0$$

It follows that

$$-Q_{b,b-1}Q_{b-1,b-2}Q_{b-2,b-1}v_{b-1} + p'_{b-1}(\lambda)v_b + p'_b(\lambda)v_b = 0 \tag{2.2}$$

Suppose that for $a < k \le j \le b$, we have

$$-Q_{b,b-1}Q_{b-1,b-2}\dots Q_{k,k-1}Q_{k-1,k}v_k + \sum_k^b p'_j(\lambda)v_b = 0, \quad (\dagger)$$

We want to show that for $a \le k-1 \le j \le b$,

$$-Q_{b,b-1}Q_{b-1,b-2}\dots Q_{k-1,k-2}Q_{k-2,k-1}v_{k-1} + \sum_{k-1}^{b} p'_j(\lambda)v_b = 0 \quad (\diamond)$$

The proof for $(\diamond)$ is the following calculation: In $(\dagger)$,

$$\begin{aligned}
& -Q_{b,b-1}Q_{b-1,b-2}\dots Q_{k,k-1}Q_{k-1,k}v_k \\
= \; & -Q_{b,b-1}Q_{b-1,b-2}\dots Q_{k,k-1}Q_{k-1,k}Q_{k,k-1}v_{k-1} \\
= \; & -Q_{b,b-1}Q_{b-1,b-2}\dots Q_{k,k-1}(Q_{k-1,k-2}Q_{k-2,k-1}v_{k-1} - p'_{k-1}(\lambda)v_{k-1}) \\
= \; & -Q_{b,b-1}Q_{b-1,b-2}\dots Q_{k,k-1}Q_{k-1,k-2}Q_{k-2,k-1}v_{k-1} + p'_{k-1}(\lambda)v_b
\end{aligned}$$

Inductively, we get

$$\sum_{a \le j \le b} p'_j(\lambda) = 0.$$

Part (2): Since $Q_{a,a+1}Q_{a+1,a}v_a + p'_a(\lambda)v_a = 0$, we get

$$-Q_{a+1,a}Q_{a,a+1}v_{a+1} = -Q_{a+1,a}Q_{a,a+1}Q_{a+1,a}v_a = p'_a(\lambda)v_{a+1}$$

Therefore,

$$Q_{a+1,a+2}Q_{a+2,a+1}v_{a+1} = -(p'_a(\lambda) + p'_{a+1}(\lambda))v_{a+1}.$$

If for $a \le j < b$,

$$-Q_{j,j-1}Q_{j-1,j}v_j = \sum_{a}^{j-1} p'_i(\lambda)v_i, \quad \text{and} \quad Q_{j,j+1}Q_{j+1,j}v_j = -\sum_{a}^{j} p'_i(\lambda)v_i,$$

then

$$-Q_{j+1,j}Q_{j,j+1}v_{j+1} = -Q_{j+1,j}Q_{j,j+1}Q_{j+1,j}v_j = \sum_{a}^{j} p'_i(\lambda)v_{j+1}$$

and

$$Q_{j+1,j+2}Q_{j+2,j+1}v_{j+1} = Q_{j+1,j}Q_{j,j+1}v_{j+1} - p'_{j+1}(\lambda)v_{j+1} = -\sum_{a}^{j+1} p'_i(\lambda)v_{j+1}.$$

Therefore, by induction, for any $a < k \le b$,

$$-Q_{k,k-1}Q_{k-1,k}v_k = \sum_{a}^{k-1} p'_i(\lambda)v_i, \quad Q_{k,k+1}Q_{k+1,k}v_k = -\sum_{a}^{k} p'_i(\lambda)v_k. \quad (\diamond)$$

By $(\diamond)$, we have

$$Q_{k-1,k}v_k = Q_{k-1,k}Q_{k,k-1}v_{k-1} = -\sum_{a}^{k-1} p'_i(\lambda)v_{k-1}$$

By definition of v_{k+1}, we have

$$v_{k+1} = Q_{k+1,k}v_k.$$

Therefore, we can define a simple sub-representation (V_R, f) of (V, f) by

$$V_R(i) = \begin{cases} \mathbf{C}v_i & \text{if} \quad a \le i \le b \\ 0 & \text{otherwise.} \end{cases}$$

Part (3) It easy to check that (W, f) satisfies $\sum_a^b p'_j(\lambda) = 0$. Since (V, f) is a one-dimensional representation, we can view each $V(i)$ as $\mathbf{C}$ for $a \le i \le b$. Then after changing the basis of $V(i)$, for $a \le i \le b$, we get $(V, f) \simeq (W, f)$. □

Lemma 2.2. *Let $V(1) = \mathbf{C}[x]/(x - \lambda_1)^n$, $V(2) = \mathbf{C}[x]/(x - \lambda_2)^m$, Φ_i=multiplication by x on $V(i)$ for $i = 1, 2$. Let $Q_{21} : V(1) \to V(2)$ and $Q_{12} : V(2) \to V(1)$ be $\mathbf{C}$-linear maps. Suppose $Q_{21}\Phi_1 = \Phi_2 Q_{21}$, $Q_{12}\Phi_2 = \Phi_1 Q_{12}$ and $\lambda_1 \ne \lambda_2$, then $Q_{21} = 0$ and $Q_{12} = 0$.*

Proof. Since $\Phi_2((x-\lambda_2)^{m-1}) = x(x-\lambda_2)^{m-1} = \lambda_2(x-\lambda_2)^{m-1}$, we have

$$Q_{12}\Phi_2((x-\lambda_2)^{m-1}) = Q_{12}\lambda_2(x-\lambda_2)^{m-1} = \lambda_2 Q_{12}((x-\lambda)^{m-1}))$$

Since $Q_{12}\Phi_2 = \Phi_1 Q_{12}$, we get

$$\Phi_1 Q_{12}((x-\lambda_2)^{m-1})) = \lambda_2 Q_{12}((x-\lambda_2)^{m-1})$$

Since the only eigenvalue for Φ_1 is λ_1, and $\lambda_1 \neq \lambda_2$, therefore we get

$$Q_{12}((x-\lambda_2)^{m-1}) = 0$$

Since

$$\Phi_2((x-\lambda_2)^{m-2}) = \lambda_2(x-\lambda_2)^{m-2} + (x-\lambda_2)^{m-1}$$

and $Q_{12}\Phi_2 = \Phi_1 Q_{12}$, we get

$$\Phi_1 Q_{12}((x-\lambda_2)^{m-2})) = \lambda_2 Q_{12}((x-\lambda_2)^{m-2})$$

As before, we get $Q_{12}((x-\lambda_2)^{m-2}) = 0$. Doing this recursively, we get $Q_{12}((x-\lambda_2)^i) = 0$ for $0 \leq i \leq m-1$.Therefore, we get $Q_{12} = 0$. Similarly, we get $Q_{21} = 0$. □

Remark 2.1. Given $(V, f) \in \mathrm{Rep}(A'_n)$, we can decompose (V, f) as the direct sum of sub-representations $(V_j, f_j) \in \mathrm{Rep}(A'_n)$ of (V, f), such that if Φ_i on $V_j(i)$ is not zero, then Φ_i has a single eigenvalue λ.

The reason is the following: First, by the Jordan decomposition Theorem, for each $V(i)$, we can choose a basis of $V(i)$, such that Φ_i on $V(i)$ has the Jordan Canonical form

$$\begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_k \end{pmatrix}$$

where each

$$J_l = \begin{pmatrix} B_{l1} & & & \\ & B_{l2} & & \\ & & \ddots & \\ & & & B_{lr_l} \end{pmatrix}$$

and where $B_{l1}, \ldots, B_{lr_l}$ are basic Jordan blocks belonging to λ_i. Notice

$$p_i'(\Phi_i) = \begin{pmatrix} p_i'(J_1) & & & \\ & p_i'(J_2) & & \\ & & \ddots & \\ & & & p_i'(J_k) \end{pmatrix}$$

and

$$p_i'(J_l) = \begin{pmatrix} p_i'(B_{l1}) & & & \\ & p_i'(B_{l2}) & & \\ & & \ddots & \\ & & & p_i'(B_{lr_l}) \end{pmatrix}$$

We know

$$-Q_{i,i-1}Q_{i-1,i} + Q_{i,i+1}Q_{i+1,i} + p_i'(\Phi_i) = 0$$

iff

$$(-Q_{i,i-1}Q_{i-1,i} + Q_{i,i+1}Q_{i+1,i} + p_i'(\Phi_i)) \mid_{V'(i)} = 0$$

for all Φ_i invariant subspaces $V'(i) \subset V(i)$ such that $\Phi_i \mid_{V'(i)}$ is a basic Jordan block belonging to a single eigenvalue λ_i. We know

$$Q_{i,i+1}Q_{i+1,i} \mid_{V'(i)} = \sum_j Q_{i,(i+1)_j} Q_{(i+1)_j,i}$$

where $Q_{(i+1)_j,i} : V'(i) \hookrightarrow V(i) \to V(i+1) \to (V(i+1))_j$, and $(V(i+1))_j$ is any Φ_{i+1} invariant subspace such that $\Phi_{i+1} \mid_{(V(i+1))_j}$ is a Jordan block belonging to a single eigenvalue λ'_{i+1}. Notice that if $\Phi_i \mid_{V'(i)}$ and $\Phi_{i+1} \mid_{(V(i+1))_j}$ have different eigenvalues, then by Lemma 2.2, $Q_{(i+1)_j,i} = 0$.

Remark 2.2. Given $(V, f) \in \mathrm{Rep}(A'_n)$, let $\mathcal{B} = \{i \mid V(i) \neq 0\}$. Then it is clear that necessary condition for (V, f) to be indecomposable is

1) $\mathcal{B}$ is a connected subgraph of A'_n,

2) for each $i \in \mathcal{B}$, there exists a constant λ, such that λ is an eigenvalue of Φ_i for all $i \in \mathcal{B}$. Moreover, for all $i \in \mathcal{B}$, the only eigenvalue of Φ_i is λ.

Lemma 2.3. *Let $(V, f) \in \mathrm{Rep}(A'_n)$ be an indecomposable representation, and let $\mathcal{B} = \{i \mid V(i) \neq 0\}$. Let $a = \min \mathcal{B}$, $b = \max \mathcal{B}$. Then $\sum_a^b p'_i(\lambda) = 0$.*

Proof. By Remark 2.1, we can assume that for (V, f), $\mathcal{B} = \{j : V(j) \neq 0\}$ is connected; for any two different $j_1, j_2 \in \mathcal{B}$, Φ_{j_1}, Φ_{j_2} have the same eigenvalue λ; for any $j \in \mathcal{B}$, the only eigenvalue for Φ_j is λ.

If $\sum_a^b p'_i(\lambda) \neq 0$, as in Lemma 2.1 part (1), there exist $c, d \in \mathcal{B}$, such that $a \leq c, \quad d \leq b$, $\sum_a^c p'_i(\lambda) = 0$, and $\sum_d^b p'_i(\lambda) = 0$. It follows that $\sum_a^c p'_i(x)$ and $\sum_d^b p'_i(x)$ have a common factor $(x - \lambda)$. Contradiction!

□

Lemma 2.4. *If $(V, f) \in \mathrm{Rep}(A'_n)$, then there exists a filtration*

$$0 \subset V^k \subset \ldots \subset V^1 \subset V^0 = V$$

of (V, f), such that $V^i/V^{i+1}$is simple.

Proof. By Remark 2.1, we can assume that for (V, f), $\mathcal{B} = \{j : V(j) \neq 0\}$ is connected; for any two different $j_1, j_2 \in \mathcal{B}$, Φ_{j_1}, Φ_{j_2} have the same eigenvalue λ; for any $j \in \mathcal{B}$, the only eigenvalue for Φ_j is λ. By Lemma 2.1, for $(V, f) \in \mathrm{Rep}(A'_n)$, we can construct an indecomposable subrepresentation

$$(V_R, f) \subset (V, f) \in \mathrm{Rep}(A'_n)$$

which is defined by

$$V_R(i) = \begin{cases} \mathbf{C}v_i & \text{for} \quad a \le i \le b \\ 0 & \text{otherwise} \end{cases}$$

and there is an equation $\sum_a^b p'_j(\lambda) = 0$ corresponding to (V_R, f), where v_i, a, b are defined in Lemma 2.1. By Lemma 2.3, $\mathcal{B} = \{i : a \le i \le b\}$. If not,let $c = \max \mathcal{B}$, let λ_1 be an eigenvalue of $V(c)$. Since we only have one eigenvalue λ of Φ_c on $V(c)$, $\lambda_1 = \lambda$. Then,as in Lemma 2.1 part (1), we get an equation $\sum_d^c p'_j(\lambda) = 0$ for some $d \le c$. Hence $\sum_d^c p'_j(x)$ and $\sum_a^b p'_j(x)$ have a common factor $(x-\lambda)$, Contradiction! Let $[(V, f)]^1 = \frac{(V,f)}{(V_R,f)}$, then $[(V, f)]^1 \in \mathrm{Rep}(A'_n)$. Let $a^1 = \min\{j \mid [(V, f)]^1(j) \neq 0\}$. If λ is an eigenvalue of Φ_{a^1} on $[(V, f)]^1(a^1)$, then we get an indecomposable subrepresentation $(V_{R_1}, f) \subset [(V, f)]^1$ which corresponds to an equation $\sum_{a^1}^{b^1} p'_j(\lambda) = 0$. We must have $a = a^1$, and $b = b^1$. Otherwise, $p_{ab}(x) := \sum_a^b p'_j(x)$ and $p_{a^1 b^1}(x) = \sum_{a^1}^{b^1} p'_j(x)$ have a common factor $(x - \lambda)$, but $p_{ab}(x) \neq p_{a^1 b^1}(x)$. Doing this repeatedly, we have $(V_{R_j}, f) \subset [(V, f)]^j$. Define

$$[(V, f)]^{j+1} = \frac{[(V, f)]^j}{(V_{R_j}, f)}.$$

Because max(dimV(i)) $< \infty$, there exists k , such that for all $1 \le l \le k$, λ is an eigenvalue of Φ_j on $[(V, f)]^{l+1}(j)$, and the indecomposable subrepresentation $(V_{R_l}, f) \subset [(V, f)]^l$ corresponds to the same equation $\sum_a^b p'_j(\lambda) = 0$, but for $m > k$, $(V_{R_m}, f) \subset [(V, f)]^m$ corresponds to a different equation

$$\sum_{a'}^{b'} p'_j(\lambda') = 0,$$

where $\lambda' \neq \lambda$. We claim such a λ' does not exist. Otherwise, let $a' = \min\{i : [(V,f)]^{k+1}(i) \neq 0\}$. Let λ_1 be an eigenvalue of $\Phi_{a'}$ on $[(V,f)]^{k+1}(a')$, we have $(V_{R_{k+1}}, f) \subset [(V,f)]^{k+1}$ corresponding to the equation $\sum_{a'}^{b'} p'_j(\lambda_1) = 0$, where $\{a' \leq u \leq b'\} \subset \{a \leq v \leq b\}$. Let $[v^j] \in (V_{R_j}, f)(a')$ be an eigenvector of $\Phi_{a'}$ on $[(V,f)]^j(a')$, let v^j be a pull back of $[v^j]$ to $(V,f)(a')$. Then we get $\Phi_{a'}$ on $V(a')$ to be

$$\begin{cases} \Phi_{a'}v^1 = \lambda v^1 \\ \Phi_{a'}v^2 = \lambda v^2 + a_{12}v^1 \\ \cdots\cdots\cdots \\ \Phi_{a'}v^k = \lambda v^k + a_{k-1,k}v^{k-1} + \ldots + a_{1,k}v^1 \\ \Phi_{a'}v^{k+1} = \lambda_1 v^{k+1} + a_{k,k+1}v^k + \ldots + a_{1,k+1}v^1 \\ \cdots\cdots\cdots \end{cases}$$

This implies that $\Phi_{a'}$ on $V(a')$ corresponds to a upper triangular matrix. It's easy to see that λ_1 is an eigenvalue of $\Phi_{a'}$ on $V(a')$.But the only eigenvalue of $\Phi_{a'}$ is λ. Contradiction! Then we get the following sequence

$$V(i) \to [V(i)]^1 \to [V(i)]^2 \to \ldots \to [V(i)]^k \to 0. \tag{2.3}$$

Define

$$V^j(i) := \mathrm{Ker}\left\{V(i) \to [V(i)]^1 \to \ldots \to [V(i)]^{k-j+1}\right\}$$

Then we get the following sequence

$$0 = V^{k+1} \subset V^k(i) \subset \ldots \subset V^1(i) \subset V^0(i) = V(i) \tag{2.4}$$

It follows that

$$\frac{V^j(i)}{V^{j+1}(i)} = \mathrm{Ker}\left([V(i)]^{k-j} \to [V(i)]^{k-j+1}\right) = V_{R_{k-j}}(i)$$

□

Lemma 2.5. *If* $(V, f) \in \text{Rep}(A'_n)$ *is an indecomposable object,then there exists a polynomial* $\sum_a^b p'_j(x) = (x-\lambda)^m g(x) \in \mathcal{A}$*, and* $1 \leq l \leq m$*, such that* $(x-\lambda)$ *is not a factor of* $g(x)$*, and* (V, f) *is defined by*

$$V(i) = \begin{cases} \mathbf{C}[x]/(x-\lambda)^l & a \leq i \leq b \\ 0 & otherwise \end{cases}$$

Proof. First, for each $i \in \{j : V(j) \neq 0\}$, by the proof of Lemma 2.4, the only eigenvalue of Φ_i on $V(i)$ is λ.

Let $V_R = V^k$ be defined by

$$p'_a(\lambda) + \ldots + p'_b(\lambda) = 0.$$

Let V_{R_1} be defined by

$$p'_{a'}(\lambda_1) + \ldots + p'_{b'}(\lambda_1) = 0.$$

By the argument of Lemma 2.4, we get $\lambda_1 = \lambda$, $a' = a$, and $b' = b$. Then $\Phi_j |_{V^{k-1}(j)} = \begin{pmatrix} \lambda & 0 \\ a_j & \lambda \end{pmatrix}$ for $a \leq j \leq b$ and some a_j.

We then can do a change of basis of $V^{k-1}(j)$, such that $a_j = 0$, or $a_j = 1$. Suppose a_j is not a constant, then there exists $a < c < b$, such that $\Phi_a |_{V^{k-1}(a)} = \ldots = \Phi_c |_{V^{k-1}(c)} \neq \Phi_{c+1} |_{V^{k-1}(c+1)}$.

<u>Case I</u> If $\Phi_a |_{V^{k-1}(a)} = \ldots = \Phi_c |_{V^{k-1}(c)} = \begin{pmatrix} \lambda & 0 \\ 1 & \lambda \end{pmatrix}$, then $\Phi_{c+1} |_{V^{k-1}(c+1)} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$. Let $Q_{i+1,i} = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$ for $1 \leq i \leq c-1$.[2] From $Q_{i+1,i}\Phi_i = \Phi_{i+1}Q_{i+1,i}$, we get $b_i = 0$, and $a_i = d_i$. From $Q_{i,i+1}\Phi_{i+1} = \Phi_i Q_{i,i+1}$, by the same argument, we get

[2]Here we abuse the notation,we use Q_{lj} to mean $Q_{lj} |_{V^{k-1}(i)}$. We continue this abusing of notation in the rest of proof.

$Q_{i,i+1} = \begin{pmatrix} a_i' & 0 \\ c_i' & a_i' \end{pmatrix}$ for $1 \le i \le c-1$. Similar argument shows that $Q_{c+1,c} = \begin{pmatrix} u & 0 \\ v & 0 \end{pmatrix}$, and $Q_{c,c+1} = \begin{pmatrix} 0 & 0 \\ x & y \end{pmatrix}$. It follows that

$$Q_{i,i+1}Q_{i+1,i} \mid_{V^{k-1}(i)} = Q_{i+1,i}Q_{i,i+1} \mid_{V^{k-1}(i+1)}, \text{for} \quad a \le i \le c-1.$$

Then

$$\begin{cases} Q_{a,a+1}Q_{a+1,a} + p_a'(\Phi_a) = 0 \\ -Q_{a+1,a}Q_{a,a+1} + Q_{a+1,a+2}Q_{a+2,a+1} + p_{a+1}'(\Phi_{a+1}) = 0 \\ \qquad\qquad \vdots \\ -Q_{c,c-1}Q_{c-1,c} + Q_{c,c+1}Q_{c+1,c} + p_c'(\Phi_c) = 0 \end{cases}$$

We add all these equations together and get

$$\sum_a^c p_i'(\Phi) + Q_{c,c+1}Q_{c+1,c} = 0 \quad (A).$$

Checking the diagonal elements of left side of (A), we get $p_a'(\lambda) + \ldots + p_c'(\lambda) = 0$. Therefore, $\sum_a^c p_i'(x) = 0$ and $\sum_a^b p_i'(x) = 0$ have a common root. Contradiction !

<u>Case II</u> If $\Phi_a = \ldots = \Phi_c = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$, then $\Phi_{c+1} = \begin{pmatrix} \lambda & 0 \\ 1 & \lambda \end{pmatrix}$. It follows that $Q_{c+1,c} = \begin{pmatrix} 0 & 0 \\ u' & v' \end{pmatrix}$, and $Q_{c,c+1} = \begin{pmatrix} x' & 0 \\ y' & 0 \end{pmatrix}$. Since $p_a'(\lambda) \neq 0$, then from

$$Q_{a,a+1}Q_{a+1,a} + p_a'(\Phi_a) = Q_{a,a+1}Q_{a+1,a} + p_a'(\lambda)I = 0,$$

we get that $Q_{a,a+1}, Q_{a+1,a}$ are invertible matrices. It's easy to check that

$$Q_{a,a+1}Q_{a+1,a} = Q_{a+1,a}Q_{a,a+1}.$$

Since $p'_a(\lambda) + \ldots + p'_{i+1}(\lambda) \neq 0$ for $a \leq i \leq c-1$, and

$$-Q_{i+1,i}Q_{i,i+1} + Q_{i+1,i+2}Q_{i+2,i+1} + p'_{i+1}(\Phi_i) = 0,$$

we can argue as in the case $i = a$, and get

$$Q_{i,i+1}Q_{i+1,i} = Q_{i+1,i}Q_{i,i+1} \quad \text{for} \quad a \leq i \leq c-1.$$

Then

$$\begin{cases} Q_{a,a+1}Q_{a+1,a} + p'_a(\Phi_a) = 0 \\ -Q_{a+1,a}Q_{a,a+1} + Q_{a+1,a+2}Q_{a+2,a+1} + p'_{a+1}(\Phi_a) = 0 \\ \qquad\qquad\vdots \\ -Q_{c,c-1}Q_{c-1,c} + Q_{c,c+1}Q_{c+1,c} + p'_c(\Phi_c) = 0 \end{cases}$$

As in case I,we get $p'_a(\lambda) + \ldots + p'_c(\lambda) = 0$. Therefore, $p_{ac}(x) = 0$ and $p_{ab}(x) = 0$ have a common root. Contradiction !

Combining Case I and Case II, we get that a_j is a constant. If $a_j = 1$, then $p'_a(x) + \ldots + p'_b(x)$ has a factor $(x-\lambda)^2$.[3] If $m = 1$, this is a contradiction! If $2 \leq m$, this is OK.

Now let's consider the case V^{k-2}. Let $V_{R_2} \subset [(V, f)]^2$ again be defined by

$$p'_a(\lambda) + \ldots + p'_b(\lambda) = 0.$$

If Φ_j on $V^{k-1}(j)$ is defined by $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$,then Φ_j on $V^{k-2}(j)$ is defined by $\begin{pmatrix} \lambda & 0 & 0 \\ b_j & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$ for $a \leq j \leq b$ with $b_j = 0$ or $b_j = 1$. It follows that Φ_j on $V^{k-2}(j)/V^k(j)$ is defined by $\begin{pmatrix} \lambda & 0 \\ b_j & \lambda \end{pmatrix}$ for $a \leq j \leq b$ with $b_j = 0$ or $b_j = 1$. Arguing as in the case V^{k-1}, we

[3]This makes sense, since here we have $V^{k-1}(a) = \ldots = V^{k-1}(c) \simeq \mathbf{C}[x]/(x-\lambda)^2$, and $V^{k-1}(c+1) \simeq (\mathbf{C}[x]/(x-\lambda))^2$.

get $b_j = 0$ for $a \le j \le b$, or $b_j = 1$ for $a \le j \le b$. If Φ_j on $V^{k-1}(j)$ is defined by $\begin{pmatrix} \lambda & 0 \\ 1 & \lambda \end{pmatrix}$,then Φ_j on $V^{k-2}(j)$ is defined by $\begin{pmatrix} \lambda & 0 & 0 \\ b_j & \lambda & 0 \\ 0 & 1 & \lambda \end{pmatrix}$ for $a \le j \le b$ with $b_j = 0$ or $b_j = 1$. It follows that Φ_j on $V^{k-2}(j)/V^k(j)$ is defined by $\begin{pmatrix} \lambda & 0 \\ b_j & \lambda \end{pmatrix}$ for $a \le j \le b$ with $b_j = 0$ or $b_j = 1$. Again, arguing as in the case V^{k-1}, we get $b_j = 0$ for $a \le j \le b$, or $b_j = 1$ for $a \le j \le b$.

Repeating this process, we see that the Jordan canonical form for Φ_j with respect to the eigenvalue λ is

$$\begin{pmatrix} B_1 & & & \\ & B_2 & & \\ & & \ddots & \\ & & & B_k \end{pmatrix}$$

where B_i are the basic Jordan blocks belonging to the same eigenvalue λ, and the rank of the Jordan blocks is less than m. Notice that in the above argument, b_j is a constant each time. It follows that $\Phi_j = \Phi$ for $a \le j \le b$ is a constant.

Then we get the following system of equations for (V, f),

$$(B)\colon \begin{cases} Q_{a,a+1}Q_{a+1,a} + p'_a(\Phi) = 0 \\ -Q_{a+1,a}Q_{a,a+1} + Q_{a+1,a+2}Q_{a+2,a+1} + p'_{a+1}(\Phi) = 0 \\ \qquad\qquad\vdots \\ -Q_{b,b-1}Q_{b-1,b} + p'_b(\Phi) = 0 \end{cases}$$

with no $p_{cd}(\lambda) = 0$ except $p_{ab}(\lambda) = 0$. It follows that $Q_{a,a+1}Q_{a+1,a} = -p'_a(\Phi)$. Since $p'_a(\lambda) \neq 0$, $Q_{a,a+1}$, and $Q_{a+1,a}$ are invertible matrices. We get $Q_{a+1,a} = -(Q_{a,a+1})^{-1}p'_a(\Phi)$. Then we have,

$$Q_{a+1,a}Q_{a,a+1} = -(Q_{a,a+1})^{-1}p'_a(\Phi)Q_{a,a+1} = -(Q_{a,a+1})^{-1}Q_{a,a+1}p'_a(\Phi) = -p'_a(\Phi).$$

Hence

$$Q_{a,a+1}Q_{a+1,a} = Q_{a+1,a}Q_{a,a+1}$$

Since no $p_{cd}(\lambda) = 0$ except $p_{ab}(\lambda) = 0$, exactly as in the case of $i = a$, we get

$$Q_{i,i+1}Q_{i+1,i} = Q_{i+1,i}Q_{i,i+1}$$

for all $a \leq i \leq b-1$. Let $[v_i] \in V^i(a)/V^{i+1}(a)$ be an eigenvector with corresponding eigenvalue λ. Let $v_i \in V^i(a) \subset V(a)$ be a preimage of $[v_i]$. Then $\{v_i\}$ is linearly independent.(proof: Let $a_0v_0 + a_1v_1 + \ldots + a_kv_k = 0$, then $a_0v_0 = 0$ in V/V^1, this implies that $a_0 = 0$. Then $a_1v_1 = 0$, on V^1/V^2, it follows $a_1 = 0$. Repeat this process, we get $a_i = 0$ for $0 \leq i \leq k$.)

Since $Q_{a,a+1}Q_{a+1,a} + p'_a(\Phi) = 0$, and $p'_a(\lambda) \neq 0$, $Q_{a+1,a}$ is invertible. It follows that $\{Q_{a+1,a}v_i\}_{0\leq i\leq k}$ is linearly independent in $V(a+1)$.

Since $p_{cd}(\lambda) \neq 0$ except $p_{ab}(\lambda) = 0$, we get from the above system of equations (B) that $\{Q_{j,j-1}\ldots Q_{a+1,a}v_i\}_{0\leq i\leq k}$ is linearly independent in $V(j)$ for $a \leq j \leq b$.

Taking $\{Q_{j,j-1}\ldots Q_{a+1,a}v_i\}_{0\leq i\leq k}$ as a basis of $V(j)$ for $a \leq j \leq b$, we get that $Q_{i+1,i} = I$, and $Q_{i,i+1} = -(p'_a(\Phi) + \ldots + p'_i(\Phi))$ for $a \leq i \leq b$.

By the above argument, we get that each $Q_{i,i+1}$ and $Q_{i+1,i}$ must be generalized diagonal matrices for $a \leq i \leq b$, i.e, matrices of the form (A_{ij}), such that $A_{ij} = 0$ if $i \neq j$. Therefore,

$$V \simeq \oplus V_j,$$

where $\Phi\,|_{V_j(i)}$ is a basic Jordan block belonging to eigenvalue λ. Since (V, f) is indecomposable, we get (V, f) is defined by

$$V(i) = \begin{cases} \mathbf{C}[x]/(x-\lambda)^l & a \leq i \leq b \\ 0 & \text{otherwise} \end{cases}$$

for some $1 \leq l \leq m$. □

The proof of Theorem 2.1 Notice we have only finite number of elements in $\mathcal{A}$, each element $p(x) \in \mathcal{A}$ can be written as $\sum_{a_p}^{b_p} p_j'(x)$ for some a_p and b_p, and each element $\sum_a^b p_j'(x) \in \mathcal{A}$ only has finite number of distinct roots. If $\sum_a^b p_j'(x) = (x-\lambda)^m g(x)$ and $g(\lambda) \neq 0$, then by Lemma 2.1,Lemma 2.4,and Lemma 2.5, we get that there exist m indecomposable objects $(V, f) \in \mathrm{Rep}(A_n')$ corresponding to λ. Therefore, A_n' is of finite representation type. □

2.2 The $N = 1$ D_n case

By [14], page 461 and 463, we know W_{D_n} is generated by reflections r_i, for $1 \leq i \leq n-1$, together with r_n which is defined by

$$r_n(t_i) = \begin{cases} t_1 & \text{if } 1 \leq i \leq n-2 \\ -t_n & \text{if } i = n-1 \\ -t_{n-1} & \text{if } i = n \end{cases}$$

By [4],page 3, we can write $p_i'(x)$ in the relation given in (1.1) as

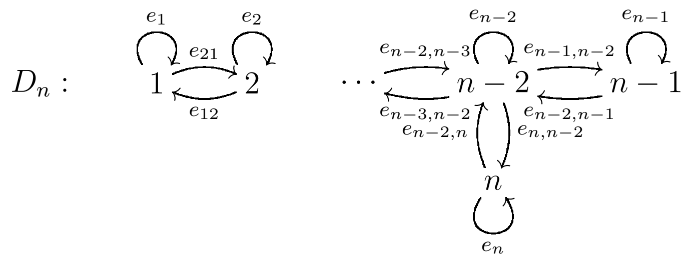
$$D_n: \quad p_i' = t_i - t_{i+1} \quad i = 1, \ldots, n-1$$

and

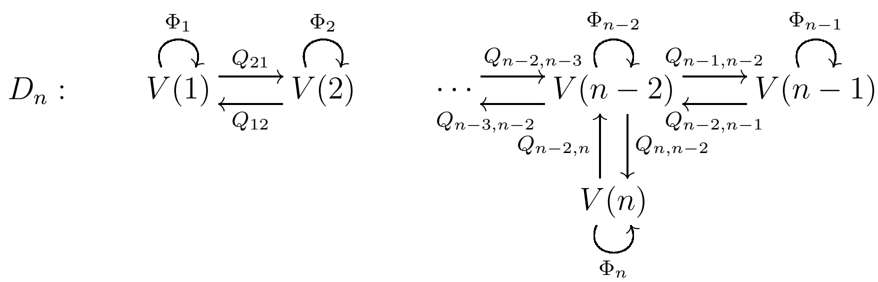
$$p_n' = t_{n-1} + t_n$$

Then for $r_k \in \mathfrak{W}_{D_n}$, we can define $r_k(p_i')$ by linearity.

In this section, we will consider the representations of the following $N = 1$ D_n quiver and we will use D'_n to denote the $N = 1$ D_n quiver.



The representations of the $N = 1$ D_n quiver



should satisfy the following relation (1.1)

$$\sum_i s_{ij} Q_{ji} Q_{ij} + p'_j(\Phi_j) = 0, \quad Q_{ij}\Phi_j = \Phi_i Q_{ij}.$$

We get the following Theorem 2.2.

Theorem 2.2. *Let $\mathcal{A} = \{rp'_i(x), r \in \mathfrak{W}_{D_n}\}$, where $p'_i(x)$ are the polynomials in relation 1.1 and $\mathfrak{W}_{D_n}$ the Weyl group of D_n. Suppose that none of elements in $\mathcal{A}$ has a multiple root and no two positive elements in $\mathcal{A}$ have a common root. Then D'_n is of finite representation type.*

We will prove this by means of a series of lemmas.

Lemma 2.6. *If V is a simple representation in $\mathbf{Rep}(D'_n)$, then $\dim V = (\dim V(i))_{i \in V_{D'_n}}$ is a positive root of D_n, where $V_{D'_n}$ denotes the set of vertices of D'_n.*

Proof. As in the A_n case, we can assume that $\mathcal{A} = \{m|V(m) \neq 0\}$ is connected. We can also assume that $V(n-1) \neq 0$ and $V(n) \neq 0$. Otherwise we are in the A_n case. Let $a = \min\{\text{n}|\text{V(n)} \neq 0\}$. Once again, we assume $a < n-2$. (Otherwise we are in the A_n case.) Let v_a be a λ-eigenvector of Φ_a on $V(a)$. Let $v_{a+1} = Q_{a+1,a}v_a$. From

$$Q_{a,a+1}Q_{a+1,a} + p_a'(\Phi_a) = 0,$$

we get

$$Q_{a,a+1}v_{a+1} = -p_a'(\lambda)v_a.$$

Similarly, from

$$-Q_{a+1,a}Q_{a,a+1}v_{a+1} + Q_{a+1,a+2}Q_{a+2,a+1}v_{a+1} + p_{a+1}'(\Phi_{a+1})v_{a+1} = 0,$$

we get

$$v_{a+2} = Q_{a+2,a+1}v_{a+1}, \quad \text{and} \quad Q_{a+1,a+2}v_{a+2} = -(p_a'(\lambda) + p_{a+1}'(\lambda))v_{a+1}.$$

If for all $j \leq k < k+1 \leq n-2$, we have

$$v_j = Q_{j,j-1}v_{j-1}, \quad \text{and} \quad Q_{j-1,j}v_j = -(p_a'(\lambda) + \cdots + p_j'(\lambda))v_{j-1}$$

then from

$$-Q_{k,k-1}Q_{k-1,k}v_k + Q_{k,k+1}Q_{k+1,k}v_k + p_k'(\lambda)v_k = 0$$

we get

$$v_{k+1} = Q_{k+1,k}v_k, \quad Q_{k,k+1}v_{k+1} = -(p_a'(\lambda) + \cdots + p_k'(\lambda))v_k.$$

Let $v_n = Q_{n,n-2}v_{n-2}$, $v_{n-1} = Q_{n-1,n-2}v_{n-2}$, $u_{n-2} = Q_{n-2,n-1}v_{n-1}$, and $w_{n-2} = Q_{n-2,n}v_n$. It follows that $Q_{n-1,n-2}u_{n-2} = p_{n-1}'(\lambda)v_{n-1}$ and $Q_{n,n-2}w_{n-2} = p_n'(\lambda)v_n$.

Then from

$$-Q_{n-2,n-3}Q_{n-3,n-2} + Q_{n-2,n-1}Q_{n-1,n-2} + Q_{n-2,n}Q_{n,n-2} + p'_{n-2}(\Phi_{n-2}) = 0 \quad (\dagger)$$

we get

$$(p'_a(\lambda) + \cdots + p'_{n-2}(\lambda))v_{n-2} + u_{n-2} + w_{n-2} = 0 \quad (\Diamond)$$

It follows that

$$Q_{n-1,n-2}w_{n-2} = -(p'_a(\lambda) + \cdots + p'_{n-1}(\lambda))v_{n-1}.$$

Similarly, we get

$$Q_{n,n-2}u_{n-2} = -(p'_a(\lambda) + \cdots + p'_{n-2}(\lambda) + p'_n(\lambda))v_n$$

Let $u_{n-3} = Q_{n-3,n-2}u_{n-2}$, from ($\dagger$), we get

$$Q_{n-2,n-3}u_{n-3} = (p'_{n-1}(\lambda) + p'_{n-2}(\lambda))u_{n-2} - (p'_a(\lambda) + \cdots + p'_{n-2}(\lambda) + p'_n(\lambda))w_{n-2}$$

Similarly, let $w_{n-3} = Q_{n-3,n-2}w_{n-2}$, from ($\dagger$), we get

$$Q_{n-2,n-3}w_{n-3} = (p'_n(\lambda) + p'_{n-2}(\lambda))w_{n-2} - (p'_a(\lambda) + \cdots + p'_{n-2}(\lambda) + p'_{n-1}(\lambda))u_{n-2}$$

More generally, define

$$u_{n-j} = Q_{n-j,n-j+1}u_{n-j+1}, \quad w_{n-j} = Q_{n-j,n-j+1}w_{n-j+1}$$

We can easily get the following fact: If $\forall l$ satisfying $3 \leq l \leq n-1$, we have

$$\begin{aligned} & Q_{n-l+1,n-l}u_{n-l} \\ = \; & (p'_{n-1}(\lambda) + \cdots + p'_{n-l+1}(\lambda))u_{n-l+1} - (p'_a(\lambda) + \cdots + p'_{n-2}(\lambda) + p'_n(\lambda))w_{n-l+1} \end{aligned}$$

and

$$
\begin{aligned}
& Q_{n-l+1,n-l}w_{n-l} \\
= \;& (p'_n(\lambda) + p'_{n-2}(\lambda) + \cdots + p'_{n-l+1}(\lambda))w_{n-l+1} - (p'_a(\lambda) + \cdots + p'_{n-2}(\lambda) + p'_{n-1}(\lambda))u_{n-l+1}
\end{aligned}
$$

then

$$
\begin{aligned}
& Q_{n-l,n-l-1}u_{n-l-1} \\
= \;& Q_{n-l,n-l+1}Q_{n-l+1,n-l}u_{n-l} + p'_{n-l}(\lambda)u_{n-l} \\
= \;& (p'_{n-1}(\lambda) + \cdots + p'_{n-l}(\lambda))u_{n-l} - (p'_a(\lambda) + \cdots + p'_{n-2}(\lambda) + p'_n(\lambda))w_{n-l}
\end{aligned}
$$

and

$$
\begin{aligned}
& Q_{n-l,n-l-1}w_{n-l-1} \\
= \;& Q_{n-l,n-l+1}Q_{n-l+1,n-l}w_{n-l} + p'_{n-l}(\lambda)w_{n-l} \\
= \;& (p'_n(\lambda) + p'_{n-2}(\lambda) + \cdots + p'_{n-l}(\lambda))w_{n-l} - (p'_a(\lambda) + \cdots + p'_{n-2}(\lambda) + p'_{n-1}(\lambda))u_{n-l}
\end{aligned}
$$

Assume that for some $a \leq j \leq n-1$, $u_{n-j} = 0$, or $w_{n-j} = 0$. Without loss of generality, assume $u_{n-j+1} \neq 0$, but $u_{n-j} = 0$. Then for all $k > j$, we have $u_{n-k} = 0$. From

$$-Q_{n-j+1,n-j}Q_{n-j,n-j+1}u_{n-j+1} + Q_{n-j+1,n_j+2}Q_{n-j+2,n-j+1}u_{n-j+1} + p'_{n-j+1}(\Phi_{n-j+1}) = 0$$

we get

$$Q_{n-j+1,n-j+2}Q_{n-j+2,n-j+1}u_{n-j+1} + p'_{n-j+1}(\lambda)u_{n-j+1} = 0$$

That is

$$(p'_{n-1}(\lambda)+\cdots+p'_{n-j+1}(\lambda))u_{n-j+1}-(p'_1(\lambda)+\cdots+p'_{n-2}(\lambda)+p'_n(\lambda))w_{n-j+1}=0 \quad (A)$$

From $(\Diamond)$, we can easily get

$$(-1)^{j-1}\prod_{k=2}^{j-1}\sum_{a}^{n-k}p'_l(\lambda)v_l+u_{n-j+1}+w_{n-j+1}=0 \quad (B)$$

From (A) and (B), we get

$$\dim V(a)=\cdots=\dim V(n-j)=1,$$

$$\dim V(n-j+1)=\cdots=\dim V(n-2)=2$$

and

$$\dim V(n-1)=\dim V(n)=1.$$

Hence V corresponds to positive root

$$p'_a(\lambda)+\cdots+p'_{n-j}(\lambda)+2p'_{n-j+1}(\lambda)+\cdots+2p'_{n-2}(\lambda)+p'_{n-1}(\lambda)+p'_n(\lambda)=0$$

□

Proposition 2.1. *There are only finitely many simple representation in* $\mathrm{Rep}(\mathrm{D}'_\mathrm{n})$.

Proof. Notice that we have only a finite number of elements in $\mathcal{A}$, and each element $p\in\mathcal{A}$ has only a finite number of distinct roots. Lemma 2.6 says that for each $p\in\mathcal{A}$ and for each root λ of p, there exists a simple object $(V,f)\in\mathrm{Rep}(\mathrm{D}'_\mathrm{n})$ corresponding to (p,λ). Therefore, $\mathrm{Rep}(\mathrm{D}'_\mathrm{n})$ has only finite number of simple objects. □

In the remainder of this section, we try to show that each indecomposable object in $\mathrm{Rep}(\mathrm{D}'_\mathrm{n})$ is in fact simple, hence Theorem 2.2 would follow.

Lemma 2.7. *Let V be an $N = 1$ ADE quiver representation,let v_j be a λ- eigenvector of Φ_j. Then $Q_{ij}\Phi_j v_j$ is either a λ-eigenvector of Φ_i or 0.*

Proof. If v_j is an eigenvector of Φ_j corresponding to eigenvalue λ, then from (1.1), we get

$$Q_{ij}\Phi_j v_j = \Phi_i Q_{ij} v_j$$

which implies that

$$\lambda Q_{ij} v_j = \Phi_i Q_{ij} v_j$$

Hence, $Q_{ij}v_j$ is either an eigenvector of Φ_i corresponding to eigenvalue λ or a 0 vector. □

Lemma 2.8. *If (V, f) is a simple representation in $\mathbf{Rep}(D'_n)$, then $\Phi_i = \lambda I$*

Proof. Let $\mathcal{A} = \{d|V(d) \neq 0\}$. Then $\mathcal{A}$ is connected. Otherwise, V is not simple. Let $a = \min \mathcal{A}$, then Φ_a has a eigenvector v_a with eigenvalue λ. For $l \in \mathcal{A}$, let $U(l)$ be the λ-eigenvector space of Φ_l. By Lemma 2.7, it's easy to see that $(W, g) = \{U(l) : l \in \mathcal{A}\}$ is a sub-representation of V. Since V is simple, $(W, g) = V$, which proves the result.

□

Lemma 2.9. *Let $\mathcal{A} = \{rp'_i(x), r \in \mathfrak{W}_{D_n}\}$, where $p'_i(x)$ are the polynomials in relation 1.1 and $\mathfrak{W}_{D_n}$ the Weyl group of D_n. Suppose that none of elements in $\mathcal{A}$ has a multiple root and no two elements in $\mathcal{A}$ have a common root. If (V, f) is an indecomposable object in* $\mathrm{Rep}(\mathrm{D'_n})$, *then (V, f) is simple.*

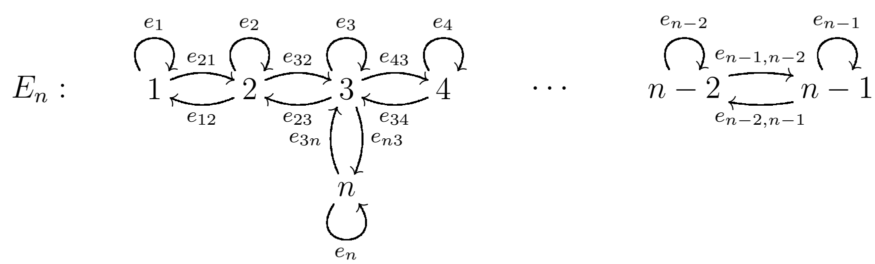
Proof. Let $a = \min\{i \mid V(i) \neq 0\}$. Let $v_{1a}, \cdots, v_{ka}$ be a basis of $V(a)$. For each v_{ia}, $1 \leq i \leq k$, we can construct a simple sub-representation V_i of V. By Lemma 2.6, V_i corresponds to a positive root $\sum \dim V_i(j) \cdot p'_j(\lambda) = 0$. By assumption, we get that $\sum \dim V_i(j) \cdot p'_j(x) = \sum a_j \cdot p'_j(x)$, i.e $\dim V_i(j) = a_j$ is independent of i. By assumption, we get that $V_s \cap V_t = \emptyset$ whenever $s \neq t$, $1 \leq s \leq k$ and $1 \leq t \leq k$. (If $v \in V_s(c) \cap V_t(c)$ for some c, then we can construct a simple representation W such that $v \in W(c)$. It follows that $W \subset V_s$ and $W \subset V_t$. But since V_s and V_t are simple, we get $W = V_s = V_t$. This is a contradiction since $s \neq t$.) If there exists $v \in V(a+1) \setminus \cup_{1 \leq i \leq k} V_i(k+1)$, then we can construct a simple representation W which corresponds to a polynomial $\sum b_i \cdot p_i'(x)$ different from $\sum a_j \cdot p_j'(x)$ since $b_a = 0$. This contradicts the assumption. Since $v_{1a}, \cdots, v_{ka}$ is a basis of $V(a)$, it is easy to get that $(\oplus_{i \neq j} V_i) \cap V_j = \emptyset$ for $1 \leq j \leq k$. It follows that $V = \oplus_{i=1}^k V_i$. Since V is indecomposable, then there exists an i, such that $V = V_i$. □

Corollary 2.1. *$N = 1$ ADE quiver is of finite representation type.*

Proof. This follows from Proposition 2.6 and Lemma 2.9. □

2.3 The $N = 1$ E_n case

In this section, we will study the representations of the following $N = 1$ E_n quivers for $n = 6, 7, 8$.



We use E_n' to denote the $N = 1$ E_n quiver.

Example 2.1. For E_6, the root types are $e_i - e_j$,$e_0 - e_i - e_j - e_k$ and $2e_0 - \sum_{j=1}^6 e_{i_j}$.

For $e_i - e_j$,we get the following curves. $C_1, C_2, C_3, C_4, C_5, C_1 + C_2, C_2 + C_3, C_3 + C_4, C_4 + C_5, C_1 + C_2 + C_3, C_2 + C_3 + C_4, C_3 + C_4 + C_5, C_1 + C_2 + C_3 + C_4, C_2 + C_3 + C_4 + C_5, C_1 + C_2 + C_3 + C_4 + C_5$

For $e_0 - e_i - e_j - e_k$,we get the following table.

Type	curve
(000111)	$C_0 + C_1 + 2C_2 + 3C_3 + 2C_4 + C_5$
(001011)	$C_0 + C_1 + 2C_2 + 2C_3 + 2C_4 + C_5$
(001101)	$C_0 + C_1 + 2C_2 + 2C_3 + C_4 + C_5$
(001110)	$C_0 + C_1 + 2C_2 + 2C_3 + C_4$
(010011)	$C_0 + C_1 + C_2 + 2C_3 + 2C_4 + C_5$
(010101)	$C_0 + C_1 + C_2 + 2C_3 + C_4 + C_5$
(010110)	$C_0 + C_1 + C_2 + 2C_3 + C_4$
(011001)	$C_0 + C_1 + C_2 + C_3 + C_4 + C_5$
(011010)	$C_0 + C_1 + C_2 + C_3 + C_4$
(011100)	$C_0 + C_1 + C_2 + C_3$
(100011)	$C_0 + C_2 + 2C_3 + 2C_4 + C_5$
(100101)	$C_0 + C_2 + 2C_3 + C_4 + C_5$
(100110)	$C_0 + C_2 + 2C_3 + C_4$
(101001)	$C_0 + C_2 + C_3 + C_4 + C_5$
(101010)	$C_0 + C_2 + C_3 + C_4$
(101100)	$C_0 + C_2 + C_3$
(110001)	$C_0 + C_3 + C_4 + C_5$
(110010)	$C_0 + C_3 + C_4$
(110100)	$C_0 + C_3$
(111000)	C_0

For $2e_0 - \sum_{j=1}^{6} e_{i_j}$, we only get one curve, $2C_0 + C_1 + 2C_2 + 3C_3 + 2C_4 + C_5$.

Lemma 2.10. *Let* (V, f) *be a simple representation in* (E_6). *If* $\dim V(i) \leq 3$, *then* (V, f) *must correspond to a positive root of* E_6.

Proof. If $V(1) = 0$, then we are in the same case as D_5. Assume $V(1) \neq 0$. Let v_1 be an eigenvector of Φ_1 in $V(1)$. Define $v_2 = Q_{21}v_1$, $v_3 = Q_{32}v_2$, $v_4 = Q_{43}v_3$, $v_5 = Q_{54}v_4$, and $v_6 = Q_{63}v_3$. Then

$$v_1, Q_{12}v_2 = -p_1'(\lambda)v_1.$$

$$v_2, Q_{23}v_3 = -(p_1'(\lambda) + p_2'(\lambda))v_2.$$

$$v_3, Q_{34}v_4 = -(p_1'(\lambda) + p_2'(\lambda) + p_3'(\lambda))v_3 - Q_{36}v_6,$$

$$v_4, Q_{45}v_5 = -(p_1'(\lambda) + p_2'(\lambda) + p_3'(\lambda) + p_4'(\lambda))v_4 - Q_{43}Q_{36}v_6$$

$$v_5, Q_{54}Q_{43}Q_{36}v_6 = -(p_1'(\lambda) + p_2'(\lambda) + p_3'(\lambda) + p_4'(\lambda) + p_5'(\lambda))v_5.$$

Let $u_4 = Q_{45}v_5$. Define $u_3 = Q_{34}u_4$, $u_2 = Q_{23}u_3$, $u_1 = Q_{12}u_2$, and $u_6 = Q_{63}u_3$. Then

$$Q_{54}u_4 = p_5'(\lambda)v_5$$

$$Q_{43}u_3 = (p_5'(\lambda) + p_4'(\lambda))u_4$$

$$Q_{32}u_2 = (p_5'(\lambda) + p_4'(\lambda) + p_3'(\lambda))u_3 + Q_{36}u_6$$

$$Q_{21}u_1 = (p_5'(\lambda) + p_4'(\lambda) + p_3'(\lambda) + p_2'(\lambda))u_2 + Q_{23}Q_{36}u_6$$

$$Q_{12}Q_{23}Q_{36}Q_{63}u_3 + (p_5'(\lambda) + p_4'(\lambda) + p_3'(\lambda) + p_2'(\lambda) + p_1'(\lambda))u_1 = 0$$

Let $w_3 = Q_{34}v_4$. Define $w_2 = Q_{23}w_3$, $w_1 = Q_{12}w_2$, and $w_6 = Q_{63}w_3$. Then we obtain

$$
\begin{aligned}
& w_6 = Q_{63}w_3 = Q_{63}Q_{34}v_4 = -(p_1'(\lambda) + p_2'(\lambda) + p_3'(\lambda) + p_6'(\lambda))v_6 \\
& Q_{43}w_3 = Q_{43}Q_{34}v_4 = Q_{45}Q_{54}v_4 + p_4'(\lambda)v_4 = u_4 + p_4'(\lambda)v_4. \\
& Q_{36}w_6 = Q_{36}Q_{63}w_3 \\
= \;& Q_{36}Q_{63}Q_{43}v_4 \\
= \;& -(p_1'(\lambda) + p_2'(\lambda) + p_3'(\lambda) + p_6'(\lambda))Q_{36}v_6 \\
= \;& (p_1'(\lambda) + p_2'(\lambda) + p_3'(\lambda) + p_6'(\lambda))((p_1'(\lambda) + p_2'(\lambda) + p_3'(\lambda))v_3 + w_3) \\
& Q_{32}w_2 = Q_{32}Q_{23}w_3 \\
= \;& Q_{34}Q_{43}w_3 + Q_{36}Q_{63}w_3 + p_3'(\lambda)w_3 \\
= \;& Q_{34}(u_4 + p_4'(\lambda)v_4) + Q_{36}Q_{63}w_3 + p_3'(\lambda)w_3 \\
= \;& u_3 + p_4'(\lambda)w_3 + Q_{36}Q_{63}w_3 + p_3'(\lambda)w_3 \\
& Q_{21}w_1 = Q_{21}Q_{12}w_2 \\
= \;& Q_{23}Q_{32}w_2 + p_2'(\lambda)w_2 \\
= \;& Q_{23}[u_3 + (p_3'(\lambda) + p_4'(\lambda))w_3 + Q_{36}w_6] + p_2'(\lambda)w_2 \\
= \;& u_2 + (p_3'(\lambda) + p_4'(\lambda) + p_2'(\lambda))w_2 + Q_{23}Q_{36}w_6 \\
= \;& u_2 + (p_3'(\lambda) + p_4'(\lambda) + p_2'(\lambda))w_2 \\
+ \;& (p_1'(\lambda) + p_2'(\lambda) + p_3'(\lambda) + p_6'(\lambda))(-(p_1'(\lambda) + p_2'(\lambda) + p_3'(\lambda))(p_1'(\lambda) + p_2'(\lambda))v_2 + w_2)
\end{aligned}
$$

Since

$$
Q_{12}Q_{21}w_1 + p_1'(\lambda)w_1 = 0
$$

we get

$$\begin{aligned} & u_1 + (p_1'(\lambda) + p_2'(\lambda) + p_3'(\lambda) + p_4'(\lambda))w_1 \\ + \quad & (p_1'(\lambda) + p_2'(\lambda) + p_3'(\lambda) + p_6'(\lambda))(p_1'(\lambda)(p_1'(\lambda) + p_2'(\lambda)) \\ & (p_1'(\lambda) + p_2'(\lambda) + p_3'(\lambda))v_1 + w_1) = 0 \qquad (1') \end{aligned}$$

Simplifying $(1')$, we obtain

$$\begin{aligned} & u_1 + (2p_1'(\lambda) + 2p_2'(\lambda) + 2p_3'(\lambda) + p_4'(\lambda) + p_6'(\lambda))w_1 \\ + \quad & (p_1'(\lambda) + p_2'(\lambda) + p_3'(\lambda) + p_6'(\lambda))(p_1'(\lambda)(p_1'(\lambda) + p_2'(\lambda)) \\ & (p_1'(\lambda) + p_2'(\lambda) + p_3'(\lambda))v_1 = 0 \qquad (1) \end{aligned}$$

Therefore,

$$Q_{21}u_1 = dv_2 + eu_2 + fw_2 \quad \dagger$$

Where

$$\begin{aligned} d &= (p_1'(\lambda) + 2p_2'(\lambda) + 2p_3'(\lambda) + p_4'(\lambda) + p_6'(\lambda))(p_1'(\lambda) + p_2'(\lambda) + p_3'(\lambda)) \\ &. \ (p_1'(\lambda) + p_2'(\lambda) + p_3'(\lambda) + p_6'(\lambda))(p_1'(\lambda) + p_2'(\lambda)) \\ e &= -(2p_1'(\lambda) + 2p_2'(\lambda) + 2p_3'(\lambda) + p_4'(\lambda) + p_6'(\lambda)) \\ f &= -(2p_1'(\lambda) + 2p_2'(\lambda) + 2p_3'(\lambda) + p_4'(\lambda) + p_6'(\lambda)) \\ &. \ (p_1'(\lambda) + 2p_2'(\lambda) + 2p_3'(\lambda) + p_4'(\lambda) + p_6'(\lambda)) \end{aligned}$$

Let $Q_{36}u_6 = av_3 + bu_3 + cw_3$. Assume v_3, u_3, and w_3 are linearly independent. Let's first do the following calculations of $-Q_{32}Q_{23}Q_{36}v_6$ and $Q_{34}Q_{43}Q_{36}v_6$.

$$
\begin{aligned}
& -Q_{32}Q_{23}Q_{36}u_6 \\
= & -Q_{32}Q_{23}(av_3 + bu_3 + cw_3) \\
= & \ a(p_1'(\lambda) + p_2'(\lambda))v_3 - bQ_{32}u_2 - cQ_{32}w_2 \\
= & \ a(p_1'(\lambda) + p_2'(\lambda))v_3 - b[(p_5'(\lambda) + p_4'(\lambda) + p_3'(\lambda))u_3 + av_3 + bu_3 + cw_3] \\
- & \ c[u_3 + (p_4'(\lambda) + p_3'(\lambda))w_3 + (p_1'(\lambda) + p_2'(\lambda) + p_3'(\lambda) + p_6'(\lambda)) \\
. & \ ((p_1'(\lambda) + p_2'(\lambda) + p_3'(\lambda))v_3 + w_3)] \\
= & \ A_1v_3 + B_1u_3 + C_1w_3
\end{aligned}
$$

Where

$$
\begin{aligned}
A_1 &= a(p_1'(\lambda) + p_2'(\lambda)) - ba \\
&- c(p_1'(\lambda) + p_2'(\lambda) + p_3'(\lambda) + p_6'(\lambda))(p_1'(\lambda) + p_2'(\lambda) + p_3'(\lambda)) \\
B_1 &= -b(p_5'(\lambda) + p_4'(\lambda) + p_3'(\lambda)) - b^2 - c \\
C_1 &= -bc - c(p_4'(\lambda) + p_3'(\lambda)) - c(p_1'(\lambda) + p_2'(\lambda) + p_3'(\lambda) + p_6'(\lambda))
\end{aligned}
$$

$$
\begin{aligned}
& Q_{34}Q_{43}Q_{36}u_6 \\
= & \ Q_{34}Q_{43}(av_3 + bu_3 + cw_3) \\
= & \ aQ_{34}v_4 + bQ_{34}Q_{43}u_3 + cQ_{34}Q_{43}w_3 \\
= & \ aw_3 + b(p_5'(\lambda) + p_4'(\lambda))Q_{34}u_4 + cQ_{34}(u_4 + p_4'(\lambda)v_4) \\
= & \ aw_3 + b(p_5'(\lambda) + p_4'(\lambda))u_3 + cu_3 + cp_4'(\lambda)w_3 \\
= & \ A_1'v_3 + B_1'u_3 + C_1'w_3
\end{aligned}
$$

Where

$$
\begin{aligned}
A_1' &= 0 \\
B_1' &= b(p_5'(\lambda) + p_4'(\lambda)) + c \\
C_1' &= a + cp_4'(\lambda)
\end{aligned}
$$

From

$$-Q_{32}Q_{23}Q_{36}u_6 + Q_{34}Q_{43}Q_{36}u_6 + Q_{36}Q_{63}Q_{36}u_6 + p_3'(\lambda)Q_{36}u_6 = 0$$

or equivalently,

$$-Q_{32}Q_{23}Q_{36}u_6 + Q_{34}Q_{43}Q_{36}u_6 + (p_6'(\lambda) + p_3'(\lambda)Q_{36}u_6 = 0 \quad \star$$

we get

$$(A_1+A_1'+a(p_6'(\lambda)+p_3'(\lambda)))v_3+(B_1+B_1'+b(p_6'(\lambda)+p_3'(\lambda))u_3+(C_1+C_1'+c(p_6'(\lambda)+p_3'(\lambda))w_3 = 0$$

Since v_3, u_3 and w_3 are linearly independent, we obtain

$$
\begin{aligned}
A_1 + A_1' + a(p_6'(\lambda) + p_3'(\lambda)) &= 0 \\
B_1 + B_1' + b(p_6'(\lambda) + p_3'(\lambda)) &= 0 \\
C_1 + C_1' + c(p_6'(\lambda) + p_3'(\lambda)) &= 0
\end{aligned}
$$

That is

$$\begin{aligned}
& a(p_1'(\lambda) + p_2'(\lambda) + p_3'(\lambda) + p_6'(\lambda)) - ba \\
- \quad & c(p_1'(\lambda) + p_2'(\lambda) + p_3'(\lambda) + p_6'(\lambda))(p_1'(\lambda) + p_2'(\lambda) + p_3'(\lambda)) = 0 \quad (I) \\
& -b(p_5'(\lambda) + p_4'(\lambda) + p_3'(\lambda)) - b^2 - c \\
+ \quad & b(p_5'(\lambda) + p_4'(\lambda)) + c + b(p_6'(\lambda) + p_3'(\lambda)) = 0 \quad (II) \\
& -bc - c(p_4'(\lambda) + p_3'(\lambda)) - c(p_1'(\lambda) + p_2'(\lambda) + p_3'(\lambda) + p_6'(\lambda)) \\
+ \quad & a + cp_4'(\lambda) + c(p_3'(\lambda) + p_6'(\lambda)) = 0 \quad (III)
\end{aligned}$$

From (II), we get $b = p_6'(\lambda)$. From (I) and (III), we get

$$a = c(p_1'(\lambda) + p_2'(\lambda) + p_3'(\lambda) + p_6'(\lambda))$$

It follows that

$$\begin{aligned}
Q_{21}u_1 &= (p_5'(\lambda) + p_4'(\lambda) + p_3'(\lambda) + p_2'(\lambda))u_2 + Q_{23}Q_{36}u_6 \\
&= (p_5'(\lambda) + p_4'(\lambda) + p_3'(\lambda) + p_2'(\lambda))u_2 - a(p_1'(\lambda) + p_2'(\lambda))v_2 + bu_2 + cw_2 \quad \ddagger
\end{aligned}$$

Compare †, and ‡, we get

$$a'v_2 + b'u_2 + c'w_2 = 0 \quad \diamondsuit$$

Where

$$
\begin{aligned}
a' &= d + a(p_1'(\lambda) + p_2'(\lambda)) \\
&= (p_1'(\lambda) + p_2'(\lambda) + p_3'(\lambda) + p_6'(\lambda))(p_1'(\lambda) + p_2'(\lambda)) \\
&. \quad [(p_1'(\lambda) + 2p_2'(\lambda) + 2p_3'(\lambda) + p_4'(\lambda) + p_6'(\lambda))(p_1'(\lambda) + p_2'(\lambda) + p_3'(\lambda)) + c] \\
b' &= e - (p_5'(\lambda) + p_4'(\lambda) + p_3'(\lambda) + p_2'(\lambda) + p_6'(\lambda)) \\
&= -(2p_1'(\lambda) + 3p_2'(\lambda) + 3p_3'(\lambda) + 2p_4'(\lambda) + p_5'(\lambda) + p_6'(\lambda)) \\
c' &= f - c \\
&= -(2p_1'(\lambda) + 2p_2'(\lambda) + 2p_3'(\lambda) + p_4'(\lambda) + p_6'(\lambda)) \\
&. \quad (p_1'(\lambda) + 2p_2'(\lambda) + 2p_3'(\lambda) + p_4'(\lambda) + p_6'(\lambda)) - c
\end{aligned}
$$

Thus, we conclude that $\dim V(2) \leq 2$. From $\Diamond$, we get

$$-a'p_1'(\lambda)v_1 + b'u_1 + c'w_1 = 0 \quad (2)$$

Next we will show that equation (1) is not a multiple of equation (2). If

$$
\begin{aligned}
c' &= (2p_1'(\lambda) + 2p_2'(\lambda) + 2p_3'(\lambda) + p_4'(\lambda) + p_6'(\lambda))b' \\
&= -(2p_1'(\lambda) + 2p_2'(\lambda) + 2p_3'(\lambda) + p_4'(\lambda) + p_6'(\lambda)) \\
&. \quad (2p_1'(\lambda) + 3p_2'(\lambda) + 3p_3'(\lambda) + 2p_4'(\lambda) + p_5'(\lambda) + p_6'(\lambda))
\end{aligned}
$$

we get

$$c = (2p_1'(\lambda) + 2p_2'(\lambda) + 2p_3'(\lambda) + p_4'(\lambda) + p_6'(\lambda))(p_1'(\lambda) + p_2'(\lambda) + p_3'(\lambda) + p_4'(\lambda) + p_5'(\lambda))$$

If

$$
\begin{aligned}
-a_1'p_1'(\lambda) &= (p_1'(\lambda)+p_2'(\lambda)+p_3'(\lambda)+p_6'(\lambda))(p_1'(\lambda)(p_1'(\lambda)+p_2'(\lambda)) \\
&\cdot\ (p_1'(\lambda)+p_2'(\lambda)+p_3'(\lambda))b'
\end{aligned}
$$

we get

$$c = (p_1'(\lambda)+p_2'(\lambda)+p_3'(\lambda))(p_1'(\lambda)+p_2'(\lambda)+p_3'(\lambda)+p_4'(\lambda)+p_5'(\lambda))$$

If

$$p_1'(\lambda)+p_2'(\lambda)+p_3'(\lambda)+p_6'(\lambda) \neq 0$$
$$p_1'(\lambda)+p_2'(\lambda)+p_3'(\lambda)+p_4'(\lambda)+p_5'(\lambda) \neq 0$$

then equation (1) is not a multiple of equation (2). Then we combine (1) and (2) to conclude that $\dim V(1) \leq 1$. □

Lemma 2.11. *Let (V, f) be a simple representation in (E_7). If $\dim V(i) \leq 4$, then (V, f) must correspond to a positive root of E_7.*

Proof.

$$v_1, Q_{12}v_2 = -p_1'(\lambda)v_1.$$
$$v_2, Q_{23}v_3 = -(p_1'(\lambda)+p_2'(\lambda))v_2.$$
$$v_3, Q_{34}v_4 = -(p_1'(\lambda)+p_2'(\lambda)+p_3'(\lambda))v_3 - Q_{37}v_7,$$
$$v_4, Q_{45}v_5 = -(p_1'(\lambda)+p_2'(\lambda)+p_3'(\lambda)+p_4'(\lambda))v_4 - Q_{43}Q_{37}v_7$$
$$v_5, Q_{56}v_6 = -(p_1'(\lambda)+p_2'(\lambda)+p_3'(\lambda)+p_4'(\lambda)+p_5'(\lambda))v_5 - Q_{54}Q_{43}Q_{37}v_7$$
$$v_6, Q_{65}Q_{54}Q_{43}Q_{37}v_7 = -(p_1'(\lambda)+p_2'(\lambda)+p_3'(\lambda)+p_4'(\lambda)+p_5'(\lambda)+p_6'(\lambda))v_6$$

Let $u_5 = Q_{56}v_6$. Then

$$Q_{65}u_5 = p_6'(\lambda)v_6$$

$$Q_{54}u_4 = (p_6'(\lambda) + p_5'(\lambda))v_5$$

$$Q_{43}u_3 = (p_6'(\lambda) + p_5'(\lambda) + p_4'(\lambda))u_4$$

$$Q_{32}u_2 = (p_6'(\lambda) + p_5'(\lambda) + p_4'(\lambda) + p_3'(\lambda))u_3 + Q_{37}u_7$$

$$Q_{21}u_1 = (p_6'(\lambda) + p_5'(\lambda) + p_4'(\lambda) + p_3'(\lambda) + p_2'(\lambda))u_2 + Q_{23}Q_{37}u_7$$

$$Q_{12}Q_{23}Q_{37}Q_{73}u_3 + (p_6'(\lambda) + p_5'(\lambda) + p_4'(\lambda) + p_3'(\lambda) + p_2'(\lambda) + p_1'(\lambda))u_1 = 0$$

Let $w_4 = Q_{45}v_5$.

$$Q_{54}w_4 = Q_{54}Q_{45}v_5 = u_5 + p_5'(\lambda)v_5$$

$$Q_{43}w_3 = Q_{43}Q_{34}w_4 = Q_{45}Q_{54}w_4 + p_4'(\lambda)w_4 = u_4 + (p_5'(\lambda) + p_4'(\lambda)w_4.$$

$$\begin{aligned}
& Q_{32}w_2 = Q_{32}Q_{23}w_3 \\
= \quad & Q_{34}Q_{43}w_3 + Q_{37}Q_{73}w_3 + p_3'(\lambda)w_3 \\
= \quad & Q_{34}(u_4 + (p_5'(\lambda) + p_4'(\lambda))w_4) + Q_{37}Q_{73}w_3 + p_3'(\lambda)w_3 \\
= \quad & u_3 + (p_3'(\lambda) + p_4'(\lambda) + p_5'(\lambda))w_3 + Q_{37}w_7 \\
& Q_{21}w_1 = Q_{21}Q_{12}w_2 \\
= \quad & Q_{23}Q_{32}w_2 + p_2'(\lambda)w_2 \\
= \quad & Q_{23}[u_3 + (p_3'(\lambda) + p_4'(\lambda) + p_5'(\lambda))w_3 + Q_{37}w_7] + p_2'(\lambda)w_2 \\
= \quad & u_2 + (p_2'(\lambda) + p_3'(\lambda) + p_4'(\lambda) + p_5'(\lambda))w_2 + Q_{23}Q_{37}w_7 \quad (E)
\end{aligned}$$

Let $s_3 = Q_{34}v_4$. Then

$$Q_{43}s_3 = Q_{43}Q_{34}v_4 = Q_{45}Q_{54}v_4 + p_4'(\lambda)v_4 = w_4 + p_4'(\lambda)v_4$$

$$Q_{32}s_2 = Q_{34}Q_{43}s_3 + Q_{37}Q_{73}s_3 + p_3'(\lambda)s_3 = w_3 + (p_4'(\lambda) + p_3'(\lambda))s_3 + Q_{37}s_7$$

$$s_3 = -(p_1'(\lambda) + p_2'(\lambda) + p_3'(\lambda))v_3 - Q_{37}v_7$$

$$s_7 = -(p_1'(\lambda) + p_2'(\lambda) + p_3'(\lambda) + p_7'(\lambda))v_7$$

$$Q_{37}v_7 = -(p_1'(\lambda) + p_2'(\lambda) + p_3'(\lambda))v_3 - s_3$$

$$Q_{21}s_1 = Q_{23}Q_{32}s_2 + p_2'(\lambda)s_2 = av_2 + bw_2 + cs_2$$

where

$$\begin{aligned} a &= -(p_1'(\lambda) + p_2'(\lambda))(p_1'(\lambda) + p_2'(\lambda) + p_3'(\lambda))(p_1'(\lambda) + p_2'(\lambda) + p_3'(\lambda) + p_7'(\lambda)) \\ b &= 1 \\ c &= (p_1'(\lambda) + 2p_2'(\lambda) + 2p_3'(\lambda) + p_4'(\lambda) + p_7'(\lambda)) \end{aligned}$$

Since

$$Q_{12}Q_{21}s_1 + p_1'(\lambda)s_1 = 0$$

we get

$$-ap_1'(\lambda)v_1 + bw_1 + (c + p_1'(\lambda))s_1 = 0$$

$$
\begin{aligned}
Q_{21}w_1 &= (p_1'(\lambda)+p_2'(\lambda))(p_1'(\lambda)+p_2'(\lambda)+p_3'(\lambda))(p_1'(\lambda)+p_2'(\lambda)+p_3'(\lambda)+p_7'(\lambda))v_2 \\
&- (2p_1'(\lambda)+2p_2'(\lambda)+2p_3'(\lambda)+p_4'(\lambda)+p_7'(\lambda))w_2 \\
&- [(2p_1'(\lambda)+2p_2'(\lambda)+2p_3'(\lambda)+p_4'(\lambda)+p_7'(\lambda)) \\
&\times (p_1'(\lambda)+2p_2'(\lambda)+2p_3'(\lambda)+p_4'(\lambda)+p_7'(\lambda))s_2] \quad (E')
\end{aligned}
$$

Let

$$
\begin{aligned}
Q_{37}w_7 &= Av_3+Bu_3+Cw_3+Ds_3 \\
Q_{37}u_7 &= A'v_3+B'u_3+C'w_3+D's_3
\end{aligned}
$$

$$
\begin{aligned}
& -Q_{32}Q_{23}Q_{37}w_7 \\
= \ & -Q_{32}Q_{23}(Av_3+Bu_3+Cw_3+Ds_3) \\
= \ & A(p_1'(\lambda)+p_2'(\lambda))v_3 - B[(p_6'(\lambda)+p_5'(\lambda)+p_4'(\lambda)+p_3'(\lambda))u_3+Q_{37}u_7] \\
- \ & C[u_3+(p_3'(\lambda)+p_4'(\lambda)+p_5'(\lambda))w_3+Q_{37}w_7] \\
- \ & D[w_3+(p_4'(\lambda)+p_3'(\lambda))s_3+Q_{37}s_7] \\
= \ & a_1v_3+b_1u_3+c_1w_3+d_1s_3
\end{aligned}
$$

where

$$
\begin{aligned}
a_1 &= A(p_1'(\lambda) + p_2'(\lambda)) - BA' - CA \\
&- D(p_1'(\lambda) + p_2'(\lambda) + p_3'(\lambda))(p_1'(\lambda) + p_2'(\lambda) + p_3'(\lambda) + p_7'(\lambda)) \\
b_1 &= -B(p_6'(\lambda) + p_5'(\lambda) + p_4'(\lambda) + p_3'(\lambda)) - BB' - C - CB \\
c_1 &= -BC' - C(p_3'(\lambda) + p_4'(\lambda) + p_5'(\lambda)) - CC - D \\
d_1 &= BD' - CD - D(p_4'(\lambda) + p_3'(\lambda)) - D(p_1'(\lambda) + p_2'(\lambda) + p_3'(\lambda) + p_7'(\lambda))
\end{aligned}
$$

Also

$$
\begin{aligned}
& Q_{34}Q_{43}Q_{37}w_7 \\
=\ & Q_{34}Q_{43}(Av_3 + Bu_3 + Cw_3 + Ds_3) \\
=\ & As_3 + BQ_{34}[p_6'(\lambda) + p_5'(\lambda) + p_4'(\lambda)]u_4 \\
+\ & CQ_{34}[u_4 + (p_5'(\lambda) + p_4'(\lambda))w_4] + DQ_{34}[w_4 + p_4'(\lambda)v_4] \\
=\ & As_3 + B[p_6'(\lambda) + p_5'(\lambda) + p_4'(\lambda)]u_3 \\
+\ & C[u_3 + (p_5'(\lambda) + p_4'(\lambda))w_3] + Dw_3 + Dp_4'(\lambda)s_3 \\
=\ & a_2v_3 + b_2u_3 + c_2w_3 + d_2s_3
\end{aligned}
$$

where

$$
\begin{aligned}
a_2 &= 0 \\
b_2 &= B(p_6'(\lambda) + p_5'(\lambda) + p_4'(\lambda)) + C \\
c_2 &= C(p_5'(\lambda) + p_4'(\lambda)) + D \\
d_2 &= A + Dp_4'(\lambda)
\end{aligned}
$$

Since we have

$$-Q_{32}Q_{23}Q_{37}w_7 + Q_{34}Q_{43}Q_{37}w_7 + (p_7'(\lambda) + p_3'(\lambda))Q_{37}w_7 = 0$$

we get

$$\begin{aligned}
a_1 + a_2 + A(p_7'(\lambda) + p_3'(\lambda)) &= 0 \\
b_1 + b_2 + B(p_7'(\lambda) + p_3'(\lambda)) &= 0 \\
c_1 + c_2 + C(p_7'(\lambda) + p_3'(\lambda)) &= 0 \\
d_1 + d_2 + D(p_7'(\lambda) + p_3'(\lambda)) &= 0
\end{aligned}$$

That is,

$$\begin{aligned}
& A(p_1'(\lambda) + p_2'(\lambda)) - BA' - CA - [D(p_1'(\lambda) + p_2'(\lambda) + p_3'(\lambda)) \\
\times \quad & (p_1'(\lambda) + p_2'(\lambda) + p_3'(\lambda) + p_7'(\lambda))] + A(p_7'(\lambda) + p_3'(\lambda)) = 0 \quad (I) \\
& -B(p_6'(\lambda) + p_5'(\lambda) + p_4'(\lambda) + p_3'(\lambda)) - BB' - C - CB \\
+ \quad & B(p_6'(\lambda) + p_5'(\lambda) + p_4'(\lambda)) + C + B(p_7'(\lambda) + p_3'(\lambda)) = 0 \quad (II) \\
& -BC' - C(p_3'(\lambda) + p_4'(\lambda) + p_5'(\lambda)) - CC - D \\
+ \quad & C(p_5'(\lambda) + p_4'(\lambda)) + D + C(p_7'(\lambda) + p_3'(\lambda)) = 0 \quad (III) \\
& BD' - CD - D(p_4'(\lambda) + p_3'(\lambda)) - D(p_1'(\lambda) + p_2'(\lambda) + p_3'(\lambda) + p_7'(\lambda)) \\
+ \quad & A + Dp_4'(\lambda) + D(p_7'(\lambda) + p_3'(\lambda)) = 0 \quad (IV)
\end{aligned}$$

Similarly, since $Q_{37}w_7$ and $Q_{37}u_7$ have the same form, we can also get the following equations.

$$
\begin{aligned}
& A'(p_1'(\lambda)+p_2'(\lambda)) - B'A' - C'A - [D'(p_1'(\lambda)+p_2'(\lambda)+p_3'(\lambda)) \\
\times \quad & (p_1'(\lambda)+p_2'(\lambda)+p_3'(\lambda)+p_7'(\lambda))] + A'(p_7'(\lambda)+p_3'(\lambda)) = 0 \quad (I') \\
& -B'(p_6'(\lambda)+p_5'(\lambda)+p_4'(\lambda)+p_3'(\lambda)) - B'B' - C' - C'B \\
+ \quad & B'(p_6'(\lambda)+p_5'(\lambda)+p_4'(\lambda)) + C' + B'(p_7'(\lambda)+p_3'(\lambda)) = 0 \quad (II') \\
& -B'C' - C'(p_3'(\lambda)+p_4'(\lambda)+p_5'(\lambda)) - C'C - D' \\
+ \quad & C'(p_5'(\lambda)+p_4'(\lambda)) + D' + C'(p_7'(\lambda)+p_3'(\lambda)) = 0 \quad (III') \\
& B'D' - C'D - D'(p_4'(\lambda)+p_3'(\lambda)) - D'(p_1'(\lambda)+p_2'(\lambda)+p_3'(\lambda)+p_7'(\lambda)) \\
+ \quad & A' + D'p_4'(\lambda) + D'(p_7'(\lambda)+p_3'(\lambda)) = 0 \quad (IV')
\end{aligned}
$$

From (II), we get

$$B' + C = p_7'(\lambda)$$

From (III), we get

$$BC' + CC = Cp_7'(\lambda)$$

From (IV), we get

$$BD' - CD - D(p_1'(\lambda)+p_2'(\lambda)+p_3'(\lambda)) + A = 0$$

From (II'), we get

$$B'B' + C'B = p_7'(\lambda)B'$$

From (III'), we get

$$B'C' + CC' = C'p_7'(\lambda)$$

From (IV'), we get

$$B'D' - C'D - D'(p_1'(\lambda) + p_2'(\lambda) + p_3'(\lambda)) + A' = 0$$

We get $B' = B$ and $C' = C$

$$A - A' = (p_1'(\lambda) + p_2'(\lambda) + p_3'(\lambda))(D - D')$$

Then from (E), we obtain

$$\begin{aligned}
& Q_{21}w_1 \\
= \;& u_2 + (p_2'(\lambda) + p_3'(\lambda) + p_4'(\lambda) + p_5'(\lambda))w_2 + Q_{23}Q_{37}w_7 \\
= \;& u_2 + (p_2'(\lambda) + p_3'(\lambda) + p_4'(\lambda) + p_5'(\lambda))w_2 - A(p_1'(\lambda) + p_2'(\lambda))v_2 \\
+ \;& Bu_2 + Cw_2 + Ds_2 \quad (E'')
\end{aligned}$$

Comparing (E') and (E''), we get

$$\begin{aligned}
& (p_1'(\lambda) + p_2'(\lambda))(p_1'(\lambda) + p_2'(\lambda) + p_3'(\lambda))(p_1'(\lambda) + p_2'(\lambda) + p_3'(\lambda) + p_7'(\lambda))v_2 \\
- \;& (2p_1'(\lambda) + 2p_2'(\lambda) + 2p_3'(\lambda) + p_4'(\lambda) + p_7'(\lambda))w_2 \\
- \;& [(2p_1'(\lambda) + 2p_2'(\lambda) + 2p_3'(\lambda) + p_4'(\lambda) + p_7'(\lambda)) \\
\times \;& (p_1'(\lambda) + 2p_2'(\lambda) + 2p_3'(\lambda) + p_4'(\lambda) + p_7'(\lambda))s_2] \\
= \;& u_2 + (p_2'(\lambda) + p_3'(\lambda) + p_4'(\lambda) + p_5'(\lambda))w_2 - A(p_1'(\lambda) + p_2'(\lambda))v_2 \\
+ \;& Bu_2 + Cw_2 + Ds_2
\end{aligned}$$

That is, we get

$$A''v_2 + B''u_2 + C''w_2 + D''s_2 = 0$$

where

$$
\begin{aligned}
A'' &= (p_1'(\lambda) + p_2'(\lambda)) \\
&\times [(p_1'(\lambda) + p_2'(\lambda) + p_3'(\lambda))(p_1'(\lambda) + p_2'(\lambda) + p_3'(\lambda) + p_7'(\lambda)) + A] \\
B'' &= -B - 1 \\
C'' &= -(2p_1'(\lambda) + 3p_2'(\lambda) + 3p_3'(\lambda) + 2p_4'(\lambda) + p_5'(\lambda) + C) \\
D'' &= -[(2p_1'(\lambda) + 2p_2'(\lambda) + 2p_3'(\lambda) + p_4'(\lambda) + p_7'(\lambda)) \\
&\times (p_1'(\lambda) + 2p_2'(\lambda) + 2p_3'(\lambda) + p_4'(\lambda) + p_7'(\lambda)) + D]
\end{aligned}
$$

□

3 A unified proof using reflection functors

In this chapter, a unified construction, under certain conditions, of a finite-to-one correspondence between the simple representations of an $N = 1$ ADE quiver and the positive roots of the usual ADE quiver has been given. This matches the physicists' predictions.

The "reflection functors" which were used in [2] to reprove Gabriel's Theorem provide us a way to attack this problem. In this chapter, we first modify the "reflection functors" in [2], and then apply our modified reflectors to get our Main Theorem in Section 3.3.2.

3.1 A quick review of the geometry of threefolds for a general ADE fibration over C.

In this section, we refer the reader to [4] and [14]. Let $C \subset Y$ be a rational curve (not necessarily irreducible) in a 3-fold Y with K_Y trivial in a neighborhood of C and $\pi : Y \to X$ a birational morphism such that $\pi(C) = p \in X$ and $\pi|_{Y \backslash C}$ is an isomorphism onto $X \backslash p$. We consider an analytic neighborhood of p (still denoted X) and its inverse image under π (still denoted Y). By a lemma of Reid [19] (1.1, 1.14), the generic hyperplane section through p is a surface X_0 with an isolated rational double point, and the proper transform of X_0 is a partial resolution $Y_0 \to X_0$ (i.e. the minimal resolution $Z_0 \to X_0$ factors through $Y_0 \to X_0$).

The partial resolution $Y_0 \to X_0$ determines combinatorial data $\Gamma_0 \subset \Gamma$ consisting of an ADE Dynkin diagram Γ (the type of the singularity p) and a subgraph Γ_0 (the dual graph of the exceptional set of Y_0).

Let $\mathcal{Z} \to \mathrm{Def}(Z_0)$, $\mathcal{Y} \to \mathrm{Def}(Y_0)$, and $\mathcal{X} \to \mathrm{Def}(X_0)$ be semi-universal deformations of Z_0, Y_0, and X_0. Following [14], there are identifications

$$\mathrm{Def}(Z_0) \cong V =: \mathrm{Res}(\Gamma) \tag{3.5}$$

$$\mathrm{Def}(Y_0) \cong V/\mathfrak{W}_0 =: \mathrm{PRes}(\Gamma, \Gamma_0) \tag{3.6}$$

$$\mathrm{Def}(X_0) \cong V/\mathfrak{W} =: \mathrm{Def}(\Gamma) \tag{3.7}$$

where V is the complex root space associated to Γ and $\mathfrak{W}$ is its Weyl group. $\mathfrak{W}_0 \subset \mathfrak{W}$ is the subgroup generated by reflections of the simple roots corresponding to $\Gamma - \Gamma_0$. Deformations of Z_0 or Y_0 can be blown down to give deformations of X_0 ([22] Theorem 1.4) and the induced classifying maps are given by the natural maps $V \to V/\mathfrak{W}$ and $V/\mathfrak{W}_0 \to V/\mathfrak{W}$ under the above identifications.

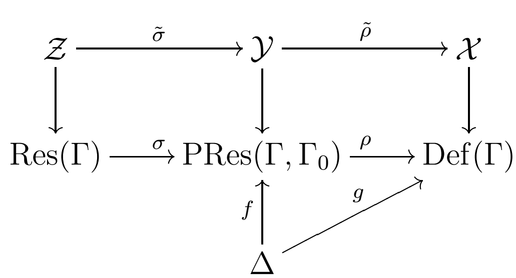
We can view X as the total space of a 1-parameter family X_t defined by the classifying map

$$g : \Delta \to \mathrm{Def}(\Gamma).$$

Similarly, we get the compatible family Y_t given by a map

$$f : \Delta \to \mathrm{PRes}(\Gamma, \Gamma_0).$$

That is, we get the diagram



By [14], $\mathcal{Y}$ is a blowup of $\mathcal{X} \times_{V/\mathfrak{W}} V/\mathfrak{W}_0$ and $\mathcal{Z}$ is a blowup of $\mathcal{X} \times_{V/\mathfrak{W}} V$. By construction, Y is the pullback of $\mathcal{Y}$ by f and X is the pullback of $\mathcal{X}$ by g.

3.1.1 The geometry of threefolds with small resolutions for a general ADE fibration over C

Let $X \subset \mathbf{C} \times \mathbf{C}^3$ be an ADE fibration over $\mathbf{C}$. Let $t_i : \mathbf{C} \to V$ is a map ($1 \le i \le n+1$ in A_n case, $1 \le i \le n$ in D_n and E_n case), where V is the complex root space defined on (3.5). We consider the A_n case first. Then $X \subset \mathbf{C} \times \mathbf{C}^3$ is defined by the equation

$$xy = z^{n+1} + \alpha_2(\omega)z^{n-1} + \cdots + \alpha_{n+1}(\omega).$$

We write $h : \mathbf{C} \to V \subset \mathbf{C}^{n+1}$ as

$$h = (t_1, \cdots, t_{n+1}) : \mathbf{C} \to \mathbf{C}^{n+1}, \sum_{i=1}^{n+1} t_i = 0.$$

Referring to [14], $\alpha_1, \cdots, \alpha_{n+1}$ are elementary symmetric functions in $t_i, \cdots, t_{n+1}$.

Let Z be the closure of the graph of the rational map

$$X \to (\mathbf{P^1})^{\mathbf{n}},\ (\mathbf{x}, \mathbf{y}, \mathbf{z}, \omega) \to \left[\mathbf{x}, \prod_{\mathbf{j=1}}^{\mathbf{i}} (\mathbf{z} + \mathbf{t_j}(\omega))\right]_{\mathbf{i}}.$$

and let (u_i, v_i) be coordinates of the i-th $\mathbf{P^1}$ in $(\mathbf{P^1})^{\mathbf{n}}$. Using the identities

$$[x, z + t_1(\omega)] = [(z + t_2(\omega)) \cdots (z + t_n(\omega)), -y],$$

we get

$$xv_j = u_j \prod_{i=1}^{j} (z + t_i(\omega))\, (1 \le j \le n),$$

and

$$\prod_{i=k+1}^{j} (z + t_i(\omega)) u_j v_k = u_k v_j\, (1 \le k < j \le n).$$

We refer the reader to [14] for the more complicated fibrations of the D and E cases. We list the defining equation of their deformations as follows:

$$D_n : \; x^2 + y^2 z + \frac{\prod_{i=1}^{n}(z + t_i^2(\omega)) - \prod_{i=1}^{n} t_i^2(\omega)}{z} + 2\prod_{i=1}^{n} t_i(\omega) y$$

$$E_6 : \; x^2 + z^4 + y^3 + \epsilon_2 y z^2 + \epsilon_5 y z + \epsilon_6 z^2 + \epsilon_8 y + \epsilon_9 z + \epsilon_{12}$$

$$E_7 : \; x^2 + y^3 + y z^3 + \epsilon_2 y^2 z + \epsilon_6 y^2 + \epsilon_8 y z + \epsilon_{10} z^2 + \epsilon_{12} y + \epsilon_{14} z + \epsilon_{18}$$

$$E_8 : \; x^2 + y^3 + z^5 + \epsilon_2 y z^3 + \epsilon_8 y z^2 + \epsilon_{12} z^3 + \epsilon_{14} y z + \epsilon_{18} z^2 + \epsilon_{20} y + \epsilon_{24} z + \epsilon_{30}$$

where ϵ_i are complicated homogeneous polynomials in the $t_j' s$ of degree i and invariant under the permutation of the $t_j' s$. We define entire functions $\{p_i'(t)\}$ as follows,

$$A_n : p_i' = t_i - t_{i+1} \quad i = 1, \cdots, n \tag{3.8}$$

$$D_n : p_i' = t_i - t_{i+1} \quad i = 1, \cdots, n-1 \quad \text{and} \quad p_n' = t_{n-1} + t_n \tag{3.9}$$

$$E_n : p_i' = t_i - t_{i+1} \quad i = 1, \cdots, n-1 \quad \text{and} \quad p_n' = -t_1 - t_2 - t_3 \tag{3.10}$$

3.2 A description of the reflection functors

3.2.1 Reflection functors

Given an $N = 1$ ADE quiver Γ and $k \in V_\Gamma$, denote by Γ_k^+ the quiver defined by dropping all arrows starting from k, and denote by Γ_k^- the quiver defined from Γ by dropping all arrows ending at k.

Given a representation V of an $N = 1$ ADE quiver Γ, we can define a representation which we still denote it as V, of Γ_k^+ by forgetting all maps which have domain $V(k)$. Similarly, we define a representation which we still denote it by V, of Γ_k^- by forgetting all maps which has range $V(k)$. Then we can apply the reflection functor F_k^+ in [2] to the representation V of Γ_k^+ and apply the reflection functor F_k^- in [2] to the representation V of Γ_k^-.

In the following definition 3.1, we modify the reflection functors in [2] for the purpose of this thesis.

Definition 3.1. Let Γ be an $N=1$ ADE quiver and k a vertex of Γ. Let

$$\Gamma^k = \{i \mid i \text{ adjacent to } k\}$$

For a quiver representation W of Γ_k^+, define a representation $F_k^+(W)$ of Γ_k^- by

$$F_k^+(W)(i) = \begin{cases} W(i) & \text{if} \quad i \neq k \\ \ker h & \text{if} \quad i = k \end{cases} \tag{3.11}$$

where

$$h : \oplus_{i\in\Gamma^k} W(i) \to W(k)$$

is defined by

$$h\left((x_i)_{i\in\Gamma^k}\right) = \sum_{i\in\Gamma^k} s_{ik} Q_{ki} x_i$$

If i, $j \neq k$, we define $Q'_{ij} = Q_{ij} : W(j) \to W(i)$. If $i \in \Gamma^k$, define $Q'_{ik} : F_k^+(W)(k) \to W(i)$ by

$$Q'_{ik}(x_j)_{j\in\Gamma^k} = -s_{ki} x_i \tag{3.12}$$

For a quiver representation U of Γ_k^-, define a representation $F_k^-(U)$ of Γ_k^+ by

$$F_k^-(U)(i) = \begin{cases} U(i) & \text{if} \quad i \neq k \\ \operatorname{coker} g & \text{if} \quad i = k \end{cases} \tag{3.13}$$

where

$$g : U(k) \to \oplus_{i\in\Gamma^k} U(i)$$

is defined by

$$g(x) = (Q_{ik} x)_{i\in\Gamma^k}$$

and define $Q'_{ki} : U(i) \to F_k^-(U)(k)$ by the natural composition of

$$U(i) \to \oplus_{j \in \Gamma^k} U(j) \to F_k^-(U)(k) \tag{3.14}$$

Remark 3.1. The definitions of the $F_k^+(W)$ and Q'_{ik} in 3.1 are different than the corresponding definitions in [2], while $F_k^-(U)$ and Q'_{ki} in 3.1 are the same as the corresponding definitions in [2].

3.2.2 The action of the Weyl group on $\{p'_i\}$, $1 \le i \le n$

By [14], pp 461 and 463, we know the Weyl group $\mathfrak{W}_{A_n}$ of A_n is generated by reflections $r_1, \cdots, r_n$, which act as permutations of $t_1, \cdots, t_{n+1}$, where t_i is defined on Section 3.1.1.

In the A_n case, we can write $p'_i(x)$ in the relation given in (1.1) as

$$A_n : \quad p'_i = t_i - t_{i+1} \quad i = 1, \ldots, n$$

By [14], pp 461 and 463, we know the Weyl group $\mathfrak{W}_{D_n}$ of D_n is generated by reflections r_i, for $1 \le i \le n-1$, which act as permutations of $t_1, \cdots, t_n$, together with r_n which is defined by

$$r_n(t_i) = \begin{cases} t_1 & \text{if } 1 \le i \le n-2 \\ -t_n & \text{if } i = n-1 \\ -t_{n-1} & \text{if } i = n \end{cases}$$

In the D_n case, we can write $p'_i(x)$ in the relation given in (1.1) as

$$D_n : \quad p'_i = t_i - t_{i+1} \quad i = 1, \ldots, n-1$$

and

$$p'_n = t_{n-1} + t_n$$

By [14], pp 461 and 463, we know that the Weyl group $\mathfrak{W}_{E_n}$ of E_n is generated by reflections r_i for $1 \leq i \leq n-1$, which act as permutations of $t_1, \cdots, t_n$, together with r_n, which is defined by

$$r_n(t_i) = \begin{cases} t_i - \frac{2}{3}(t_1 + t_2 + t_3) & \text{if} \quad 1 \leq i \leq 3 \\ t_i + \frac{1}{3}(t_1 + t_2 + t_3) & \text{if} \quad 4 \leq i \leq n \end{cases}$$

In the E_n case, we can write $p_i'(x)$ in the relation given in (1.1) as

$$E_n: \quad p_i' = t_i - t_{i+1} \quad i = 1, \ldots, n \quad \text{and} \quad p_n' = -t_1 - t_2 - t_3$$

Based on these definitions of r_i, $1 \leq i \leq n$, one can easily get the following Lemma 3.1.

Lemma 3.1. *Let $\mathfrak{W}_\Gamma$ be the Weyl group of the Dynkin diagram Γ and let $r_i \in \mathfrak{W}_\Gamma$ ($1 \leq i \leq n$) be a set of generators of reflections. If j is distinct from i and not adjacent to i, then $r_i(p_j'(\Phi_j)) = p_j'(\Phi_j)$. If j is adjacent to i and $j \neq i$, then $r_i(p_j'(\Phi_j)) = p_j'(\Phi_j) + p_i'(\Phi_j)$. Finally, $r_i(p_i'(\Phi_i)) = -p_i'(\Phi_i)$.*

3.3 Finite-to-one correspondence

In this section, we will give a proof, using reflection functors, that in the case of simple and distinct roots, the irreducible quiver representations are in finite-to-one correspondence with the contractible curves in the threefold.

3.3.1 Applying the reflection functors to $N = 1$ ADE quiver representations

Let Γ be an N=1 ADE quiver. Let

$$\mathcal{A}_\Gamma = \left\{ \sum_i n_i p_i' | n_i \in Z, \text{not all} \quad n_i \quad \text{zero} \right\}$$

where p_i', $1 \leq i \leq n$, are the polynomials in relation (1.1)

(*) Suppose no two elements $\sum n_i p_i'$, $\sum m_i p_i'$ of the set $\mathcal{A}_\Gamma$ have a common root unless there is a constant c with $m_i = cn_i$ for all i.

Lemma 3.2. *$(*)$ holds for any generic collection of polynomials p_i' of positive degree.*

Proof. Let $X = \{(p_i')_{1\le i\le n} \mid \deg p_i' \le k_i\}$. Then $X \cong \mathbf{C}^{\sum(k_i+1)}$.

We want to find polynomials $\{f_i\}$, $1 \le i \le n$, such that $\deg f_i = k_i$, and in the set

$$\mathcal{A}_\Gamma = \left\{\sum n_i f_i \mid n_i \in \mathbf{Z}, \text{ not all } \quad n_i \quad \text{zero}\right\}$$

$(*)$ no two elements have a common root. Then this $\{f_i\}$ corresponds to a point in X.

For any two elements (f_i), $(g_i) \in X$, $\sum f_i$ and $\sum g_i$ have common roots $\iff$ $Res(\sum f_i, \sum g_i) = 0$ and $Res(\sum f_i, \sum g_i) \equiv 0 \iff (f_i) = m(g_i)$ for some non-zero constant m.

For $a = (a_i) \in \mathbf{Z}^n$, $b = (b_i) \in \mathbf{Z}^n$, let $f_a = \sum a_i p_i'$, $g_b = \sum b_i p_i'$. Then f_a corresponds with $(a_i p_i') \in X$ and g_b corresponds with $(b_i p_i') \in X$. Let

$$U = X - \bigcup_{a\in\mathbf{Z}^n,\, b\in\mathbf{Z}^n} Z\left(Res(f_a, g_b)\right)$$

Then any points in U should satisfy condition $(*)$. □

Lemma 3.3. *Let V be an $N = 1$ ADE quiver representation,let v_j be a λ- eigenvector of Φ_j. Then $Q_{ij}\Phi_j v_j$ is either a λ-eigenvector of Φ_i or 0.*

Proof. If v_j is an eigenvector of Φ_j corresponding to eigenvalue λ, then from (1.1), we get

$$Q_{ij}\Phi_j v_j = \Phi_i Q_{ij} v_j$$

which implies that

$$\lambda Q_{ij} v_j = \Phi_i Q_{ij} v_j$$

Hence, $Q_{ij}v_j$ is either an eigenvector of Φ_i corresponding to eigenvalue λ or a 0 vector. □

Lemma 3.4. *Let V be a simple representation of an $N = 1$ ADE quiver. Then there exists λ such that if $v_i \in V(i) \neq 0$, then $\Phi_i v_i = \lambda v_i$.*

Proof. Let $\mathcal{A} = \{d | V(d) \neq 0\}$. Then $\mathcal{A}$ is connected. Otherwise, V is not simple. Let $a = \min \mathcal{A}$, then Φ_a has a eigenvector v_a with eigenvalue λ. For $l \in \mathcal{A}$, let $U(l)$ be the λ-eigenvector space of Φ_l. By Lemma 3.3, it's easy to see that $(W, g) = \{U(l) : l \in \mathcal{A}\}$ is a sub-representation of V. Since V is simple, $(W, g) = V$, which proves the result. □

Therefore, to show that we have only finitely many simple representations, it suffices to consider representations V for which there exists a λ such that if $0 \neq v_d \in V(d)$, then $\Phi_d v_d = \lambda v_d$. *In the rest of this section, we only consider quiver representations V with this property.*

Lemma 3.5. *Let V be a simple representation of an $N = 1$ ADE quiver. Suppose V is not concentrated at vertex k. Then*

$$\dim \left(F_k^+(V)\right)_k = \sum_{i \in \Gamma^k} \dim (V(i)) - \dim (V(k))$$

Proof. We know that $\left(F_k^+(V)\right)(k) = \ker h$, where $h : \oplus_{i \in \Gamma^k} V(i) \to V(k)$ is defined by

$$h(x_i)_{i \in \Gamma^k} = \sum_{i \in \Gamma^k} h_{ki} x_i$$

with

$$h_{ki} = s_{ki} Q_{ik}$$

Proving the lemma is equivalent to proving that h is surjective.

Case I: $V(k) \neq 0$. If h is not surjective and $h \neq 0$, then we can replace $V(k)$ by $h\left(\oplus_{i \in \Gamma^k} V(i)\right)$ to get a sub-representation of V. But this contradicts the simplicity of V. Case II: $V(k) = 0$. We get that h is surjective since $h \equiv 0$ in this case. □

Lemma 3.6. *Let V be a simple representation of an $N = 1$ ADE quiver. Suppose V is not concentrated at vertex k. Then*

$$\dim \left(F_k^-(V)\right)_k = \sum_{i \in \Gamma^k} \dim (V(i)) - \dim (V(k))$$

Proof. We know that $\left(F_k^-(V)\right)(k) = \operatorname{coker} g$, where $g : V(k) \to \oplus_{i \in \Gamma^k} V(i)$ is defined by $g(x) = (Q_{ik}x)_{i \in \Gamma^k}$. To prove the lemma is equivalent to prove that g is injective.

Case I: $V(k) \neq 0$. If $\ker g \neq 0$, then we can define a simple sub-representation which concentrated at vertex k. This contradicts the simplicity of V.

Case II: $V(k) = 0$. We get that g is injective since $g \equiv 0$ in this case. □

Lemma 3.7. *Let V be a simple representation of an $N = 1$ ADE quiver. Suppose V is not concentrated at vertex k. If $p'_k(\lambda) \neq 0$, then there is a natural isomorphism φ between $F_k^+(V)(k)$ and $F_k^-(V)(k)$.*

Proof. Let $g : V(k) \to \oplus_{i \in \Gamma^k} V(i)$ be defined by

$$g(x) = (Q_{ik}x)_{i \in \Gamma^k}$$

and $h : \oplus_{i \in \Gamma^k} V(i) \to V(k)$ be defined by

$$h(x_i)_{i \in \Gamma^k} = \sum_{i \in \Gamma^k} h_{ki} x_i$$

where

$$h_{ki} = s_{ik} Q_{ki}$$

We have

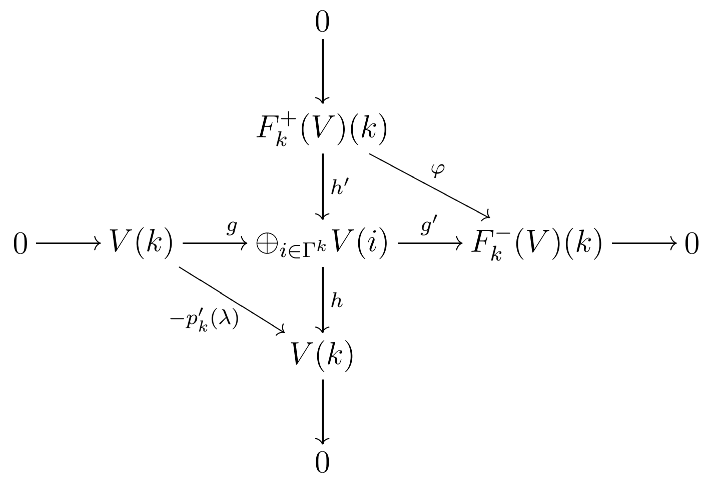
$$F_k^-(V)(k) = \operatorname{coker} g$$

and

$$F_k^+(V)(k) = \ker h$$

Since V is simple and not concentrated at k, g is injective and h is surjective. We have

$$\dim F_k^+(V)(k) = \dim F_k^-(V)(k) = \sum_{i\in\Gamma^k} \dim V(i) - \dim V(k)$$



Since $p_k'(\lambda) \neq 0$, $\operatorname{im} g \cap F_k^+(V)(k) = \{0\}$. Let $g' : \oplus_{i\in\Gamma^k}V(i) \to F_k^-(V)(k)$ be the natural surjective map induced by g and let $h' : F_k^+(V)(k) \to \oplus_{i\in\Gamma^k}V(i)$ be the natural inclusion map induced by h. Then $\varphi = g' \circ h' : F_k^+(V)(k) \to F_k^-(V)(k)$ is a natural isomorphism (Since $\dim F_k^+(V)(k) = \dim F_k^-(V)(k)$ and φ is injective by $\operatorname{im} g \cap F_k^+(V)(k) = \{0\}$.) $\square$

Definition 3.2. By Lemma 3.7, if $p_k'(\lambda) \neq 0$, we can construct a new representation $F_k(V)$ of Γ by

$$F_k(V)(i) = \begin{cases} V(i) & \text{if} \quad i \neq k \\ F_k^+(V)(k) & \text{if} \quad i = k \end{cases}$$

defining Q_{lk}' as it is defined in $F_k^+(V)(k)$ and defining Q_{km}' as the composition map $p_k'(\lambda)\cdot\varphi^{-1}\circ \underline{Q_{km}'} : V(m) \to F_k^+(V)(k)$, where $\underline{Q_{km}'} : V(m) \to F_k^-(V)(k)$ is the natural map defined in $F_k^-(V)$ and $\varphi : F_k^+(V)(k) \to F_k^-(V)(k)$ is the isomorphism defined in Lemma 3.7.

If V is simple, we define

$$\Phi_i' : F_k(V)(i) \to F_k(V)(i)$$

by $\Phi_i'(x) = \lambda x$, where λ is the eigenvalue of Φ on $V(i)$ that appeared in the representation of V. Abusing notation, we still denote Φ_i' as Φ_i.

Lemma 3.8. *If V is a simple representation of $N = 1$ ADE quiver, then*

$$\sum_i \dim(V(i)) \cdot p_i'(\lambda) = 0$$

Proof. This follows from the fact that $\forall$ pair i and j, Tr $Q_{ij}Q_{ji} =$ Tr $Q_{ji}Q_{ij}$, and $\forall$, i, Tr $\Phi_i = \lambda \cdot \dim V(i)$, where λ is an eigenvalue for all Φ_i. Now take trace operation to relations (1.1) and then sum the resulting equations. The result follows. □

Lemma 3.9. *Let V be a simple representation of an $N = 1$ ADE quiver Γ, not concentrated at vertex k. Then*

$$\sum \dim\left(F_k(V)\right)(i) r_k\left(p_i'(\lambda)\right) = \sum \dim V(i) p_i'(\lambda)$$

Proof.

$$\begin{aligned}
& \sum \dim\left(F_k(V)\right)(i) r_k\left(p_i'(\lambda)\right) \\
= & \sum_{i\in\Gamma^k} \dim V(i)\left(p_k'(\lambda) + p_i'(\lambda)\right) + \sum_{j\in\Gamma-\Gamma^k} \dim V(j) p_j'(\lambda) \\
+ & \left(-\dim V(k) + \sum_{i\in\Gamma^k} \dim V(i)\right)\left(-p_k'(\lambda)\right) \\
= & \sum \dim V(i) p_i'(\lambda)
\end{aligned}$$

□

Proposition 3.1. *Let V be a simple representation of an $N = 1$ ADE quiver which is not concentrated at vertex k. If $p'_k(\lambda) \neq 0$, then $F_k(V)$ satisfies the following new relations*

$$\sum_i s_{ij} Q'_{ji} Q'_{ij} + r_k \left(p'_j(\Phi_j) \right) = 0, \quad Q'_{ij}\Phi_j = \Phi_i Q'_{ij}. \tag{3.15}$$

Proof. If $i \notin \Gamma^k$ and $i \neq k$, where i is a vertex of Γ such that $V(i) \neq 0$, there is nothing to prove. For $j \in \Gamma^k \cup \{k\}$, $b \in (F_k(V))_j$, we have

$$Q'_{ij}\Phi_j b = \lambda Q'_{ij} b = \Phi_i Q'_{ij} b$$

For $i \in \Gamma^k$ and $x \in V(i)$, by Definition 3.2, we know that

$$Q'_{ki}x = p'_k(\lambda) \cdot \varphi^{-1} \circ \underline{Q'_{ki}}x$$

where $\underline{Q'_{ki}}x = [(x_j)_{j \in \Gamma^k}] \in F_k^-(V)(k)$, and

$$x_j = \begin{cases} 0 & \text{if} \quad j \neq i \\ x & \text{if} \quad j = i \end{cases}$$

After a short computation, we see that

$$Q'_{ki}x = (y_j)_{j \in \Gamma^k}$$

where

$$y_j = \begin{cases} p'_k(\lambda)x + s_{ik}Q_{ik}Q_{ki}x & \text{if} \quad j = i \\ Q_{jk}s_{ik}Q_{ki}x & \text{if} \quad j \neq i \end{cases}$$

It follows that

$$s_{ki}Q'_{ik}Q'_{ki}x = s_{ki}Q'_{ik}(y_j)_{j \in \Gamma^k} = -p'_k(\lambda)x - Q_{ik}s_{ik}Q_{ki}x$$

Hence for $i \in \Gamma^k$ we have

$$\begin{aligned}
& \sum_j s_{ji} Q'_{ij} Q'_{ji} x + r_k \left(p'_i(\lambda)\right) x \\
= & \sum_j s_{ji} Q'_{ij} Q'_{ji} x + r_k \left(p'_i(\lambda)\right) x \\
= & \sum_j s_{ji} Q'_{ij} Q'_{ji} x + p'_i(\lambda) x + p'_k(\lambda) x \\
= & \sum_{j \neq k} s_{ji} Q_{ij} Q_{ji} x + s_{ki} Q'_{ik} Q'_{ki} x + p'_i(\lambda) x + p'_k(\lambda) x \\
= & \sum_{j \neq k} s_{ji} Q_{ij} Q_{ji} x - p'_k(\lambda) x - Q_{ik} s_{ik} Q_{ki} x + p'_i(\lambda) x + p'_k(\lambda) x \\
= & \; 0
\end{aligned}$$

Let $(x_i)_{i \in \Gamma^k} \in F_k^+(V)(k)$. Then

$$s_{ik} Q'_{ki} Q'_{ik} (x_i)_{i \in \Gamma^k} = Q'_{ki} x_i = \left(x_{i_j}\right)_{j \in \Gamma^k}$$

where

$$x_{i_j} = \begin{cases} p'_k(\lambda) x_i + Q_{ik} s_{ik} Q_{ki} x_i & \text{if} \quad j = i \\ Q_{jk} s_{ik} Q_{ki} x_i & \text{if} \quad j \neq i \end{cases}$$

Hence we have

$$\begin{aligned}
& \sum_{i \in \Gamma^k} s_{ik} Q'_{ki} Q'_{ik} (x_i)_{i \in \Gamma^k} + r_k(p'_k(\lambda))(x_i)_{i \in \Gamma^k} \\
= & \sum_{i \in \Gamma^k} s_{ik} Q'_{ki} Q'_{ik} (x_i)_{i \in \Gamma^k} - p'_k(\lambda)(x_i)_{i \in \Gamma^k} \\
= & \sum_{i \in \Gamma^k} \left(x_{i_j}\right)_{j \in \Gamma^k} - p'_k(\lambda)(x_i)_{i \in \Gamma^k} \\
= & \; 0
\end{aligned}$$

□

Lemma 3.10. *If V is a simple representation of an $N = 1$ ADE quiver which is not concentrated at vertex k and if $(*)$ holds, then $F_kF_k(V) \cong V$. Consequently, $F_k(V)$ is a simple representation.*

Proof. We know that $Q'_{ki} : V(i) \to F_k(V)(k)$ is defined by $Q'_{ki}x_i = p'_k(\lambda)\varphi^{-1}\underline{Q_{ki}}x_i$, where $\underline{Q_{ki}} : V(i) \to F_k^-(V)(k)$ is the composition of $V(i) \to \oplus_{i\in\Gamma^k}V(i)$ and $\oplus_{i\in\Gamma^k}V(i) \to F_k^-(V)(k)$ (See Definition 3.2). We also know that

$$F_kF_k(V)(k) = \left\{(x_i) \in \oplus_{i\in\Gamma^k}V(i) \mid \sum_{i\in\Gamma^k} s_{ik}Q'_{ki}x_i = 0\right\}$$

We have

$$\sum_{i\in\Gamma^k} s_{ik}Q'_{ki}x_i = p'_k(\lambda)\varphi^{-1}\sum_{i\in\Gamma^k} s_{ik}\underline{Q_{ki}}x_i$$

Since $p'_k(\lambda) \neq 0$ and φ is an isomorphism, we get

$$F_kF_k(V)(k) = \{(-s_{ki}Q_{ik}x) \mid x \in V(k)\}$$

Let $g : V \to F_kF_k(V)$ be defined in the following way:

$$g_i = \begin{cases} i : V(i) \to F_kF_k(V)(i) = V(i) & \text{if } i \neq k \\ (-s_{ki}Q_{ik}) & \text{if } i = k \end{cases}$$

where $i : V(i) \to F_kF_k(V)(i) = V(i)$ is the identity map.

Then it is clear that (3.16) is commutative.

$$\begin{array}{ccc} V(k) & \xrightarrow{Q_{ik}} & V(i) \\ \downarrow{\scriptstyle g_k} & & \downarrow{\scriptstyle g_i} \\ F_kF_k(V)(k) & \xrightarrow{Q''_{ik}} & V(i) \end{array} \tag{3.16}$$

Let's check the commutativity of (3.17).

$$
\begin{array}{ccc}
V(i) & \xrightarrow{Q_{ik}} & V(k) \\
\downarrow{\scriptstyle g_i} & & \downarrow{\scriptstyle g_k} \\
V(i) & \xrightarrow{Q''_{ki}} & F_kF_k(V)(k)
\end{array}
\tag{3.17}
$$

Let $(Q''_{ki}x_i)_j$ (resp. $(Q'_{ki}x_i)_j$) denote the $j-$th coordinate of $Q''_{ki}x_i$ (resp. $Q'_{ki}x_i$). We know that

$$
(Q''_{ki}x_i)_j = \begin{cases} -p'_k(\lambda)x_i + Q'_{ik}s_{ik}Q'_{ki}x_i \\ Q'_{jk}s_{ik}Q'_{ki}x_i \quad \text{if } j \neq i \end{cases}
$$

where

$$
(Q'_{ki}x_i)_j = \begin{cases} p'_k(\lambda)x_i + Q_{ik}s_{ik}Q_{ki}x_i \\ Q_{jk}s_{ik}Q_{ki}x_i \quad \text{if } j \neq i \end{cases}
$$

If $i > k$, then we have

$$
\begin{aligned}
(Q''_{ki}x_i)_i &= -p'_k(\lambda)x_i + Q'_{ik}s_{ik}Q'_{ki}x_i \\
&= -p'_k(\lambda)x_i + p'_k(\lambda)x_i + Q_{ik}s_{ik}Q_{ki}x_i \\
&= Q_{ik}s_{ik}Q_{ki}x_i = Q_{ik}Q_{ki}x_i \\
&= Q_{ik}Q_{ki}x_i = -s_{ki}Q_{ik}Q_{ki}x_i
\end{aligned}
$$

If $i > k$ and $j > k$, then we have

$$
(Q''_{ki}x_i)_j = Q'_{jk}s_{ik}Q'_{ki}x_i = Q_{jk}s_{ik}Q_{ki}x_i = Q_{jk}Q_{ki}x_i = -s_{kj}Q_{jk}Q_{ki}x_i
$$

If $i > k$ and $j < k$, then we have

$$
(Q''_{ki}x_i)_j = Q'_{jk}s_{ik}Q'_{ki}x_i = Q'_{jk}Q'_{ki}x_i = -Q_{jk}Q_{ki}x_i = -s_{kj}Q_{jk}Q_{ki}x_i
$$

If $i < k$, then we have

$$\begin{aligned}(Q''_{ki}x_i)_i &= -p'_k(\lambda)x_i + Q'_{ik}h'_{ki}x_i \\ &= -p'_k(\lambda)x_i - Q'_{ik}Q'_{ki}x_i \\ &= -p'_k(\lambda)x_i + (p'_k(\lambda)x_i + Q_{ik}h_{ki}x_i) \\ &= Q_{ik}h_{ki}x_i = -Q_{ik}Q_{ki}x_i \\ &= -s_{ki}Q_{ik}Q_{ki}x_i\end{aligned}$$

If $i < k$ and $j > k$, then we have

$$(Q''_{ki}x_i)_j = Q'_{jk}s_{ik}Q'_{ki}x_i = -Q'_{jk}Q'_{ki}x_i = -Q_{jk}s_{ik}Q_{ki}x_i = Q_{jk}Q_{ki}x_i = -s_{kj}Q_{jk}Q_{ki}x_i$$

If $i < k$ and $j < k$, then we have

$$(Q''_{ki}x_i)_j = Q'_{jk}s_{ik}Q'_{ki}x_i = -Q'_{jk}Q'_{ki}x_i = Q_{jk}s_{ik}Q_{ki}x_i = -Q_{jk}Q_{ki}x_i = -s_{kj}Q_{jk}Q_{ki}x_i$$

Therefore the diagram (3.17) is commutative.

Diagram (3.18) is commutative since λ is a common eigenvalue of $V(k)$ and $F_kF_k(V)(k)$.

$$\begin{array}{ccc} V(k) & \xrightarrow{\lambda} & V(k) \\ \downarrow g_k & & \downarrow g_k \\ F_kF_k(V)(k) & \xrightarrow{\lambda} & F_kF_k(V)(k) \end{array} \tag{3.18}$$

We prove the later part of the Lemma here. Since V is not concentrated at vertex k, $\exists m \neq k$ such that $V(m) \neq 0$. It follows that $F_k(V)(m) = V(m) \neq 0$. Let $v \in F_k(V)(m)$ be an nonzero element. If $F_k(V)$ is not simple, then there exists, actually we can construct a simple sub-representation W of $F_k(V)$, such that $v \in W(m)$. It follows that $F_k(W)$ is a proper sub-representation of $F_kF_k(V)$. Since $F_kF_k(V) \cong V$ and V is simple, this leads a contradiction.

$\square$

Corollary 3.1. *Let Γ be an $N = 1$ ADE quiver. Let*

$$\mathcal{B}_\Gamma = \{r_i(p'_j(x)) | r_i \in \mathfrak{W}_\Gamma\}$$

where $\mathfrak{W}_\Gamma$ is the Weyl group of Γ and p'_j is the polynomial defined on relation (1.1). Suppose $(*)$ holds and each element in $\mathcal{B}_\Gamma$ has simple roots. If V is a simple representation, then either $F_k(V)$ is simple or $V \cong L_k$, where L_k is a simple representation concentrated at vertex k.

Proof. If V is simple and concentrated at vertex k, then $V \cong L_k$, where L_k is a simple representation concentrated at vertex k. Assume V is not concentrated at vertex k. Since V is simple, by Lemma 3.5 and Lemma 3.6, we can apply F_k to V. Then $F_k(V)$ is simple by the later part of Lemma 3.10. $\square$

3.3.2 A proof of the Main Theorem

Let Γ be a quiver. Following [2], for a representation V, we define $\dim(V) = (\dim V(i))_{i \in V_\Gamma}$. Denote by $\mathcal{C}_\Gamma = \{x = (x_\alpha) \mid x_\alpha \in \mathbf{Q}, \alpha \in V_\Gamma\}$, where $\mathbf{Q}$ denotes the set of rational numbers. We call a vector $x = (x_\alpha)$ positive (written $x > 0$) if $x \neq 0$ and $x_\alpha \geq 0$ for all $\alpha \in V_\Gamma$. For each $\beta \in V_\Gamma$, denote by σ_β the linear transformation in $\mathcal{C}_\Gamma$ defined by the formula $(\sigma_\beta x)_\gamma = x_\gamma$ for $\gamma \neq \beta$, $(\sigma_\beta x)_\beta = -x_\beta + \sum_{l \in \Gamma^\beta} x_l$, where $l \in \Gamma^\beta$ is the set of vertices adjacent to β.

For each vertex $\alpha \in V_\Gamma$ we denoted by Γ_α the set of edges containing α. Let Λ be an orientation of the graph Γ. We denote by $\sigma_\alpha\Lambda$ the orientation obtained from Λ by changing the directions of all edges $l \in \Gamma_\alpha$. Following [2], we say that a vertex i of a quiver Γ with orientation Λ is (-)-accessible (resp. (+)-accessible) if for any edge e having i as a vertex, we have the final vertex of $f(e)$ of e satisfying $f(e) \neq i$ (resp. the initial vertex $i(e)$ of e satisfying $i(e) \neq i$.) We say that a sequence of vertices

$\alpha_1, \alpha_2, \cdots, \alpha_k$ is (+)-accessible with respect to Λ if α_1 is (+)-accessible with respect to Λ, α_2 is (+)-accessible with respect to $\sigma_{\alpha_1}\Lambda$, α_3 is (+)-accessible with respect to $\sigma_{\alpha_2}\sigma_{\alpha_1}\Lambda$, and so on. We define a (-)accessible sequence similarly.

Definition 3.3. Let Γ be a graph without loops. We denote by $\mathscr{C}_\Gamma$ the linear space over $\mathbf{Q}$ consisting of sets $x = (x_\alpha)$ of rational numbers x_α $(\alpha \in \Gamma_V)$. We call a vector $x = (x_\alpha)$ *positive* (written $x > 0$) if $x \neq 0$ and $x_\alpha \geq 0$ for all $\alpha \in \Gamma_V$. We denote by B the quadratic form on the space $\mathscr{C}_\Gamma$ defined by the formula $B(x) = \sum x_\alpha^2 - \sum_{l \in \mathcal{E}_\Gamma} x_{r_1(l)} x_{r_2(l)}$, where $r_1(l)$ and $r_2(l)$ are the ends of the edge l. We denote by $<,>$ the corresponding symmetric bilinear form.

Lemma 3.11. *[2, Lemma 2.3] Suppose that the form B for the graph Γ is positive definite. Let $c = \sigma_n \cdots \sigma_2 \sigma_1$. If $x \in \mathscr{C}_\Gamma$, $x \neq 0$, then for some i the vector $c^i x$ is not positive.*

Lemma 3.12. *Let V be a simple representation of N=1 ADE quiver Γ. Then $\dim V = (\dim V(i))$ corresponds to a positive root of Γ.*

Proof. By Lemma 3.11, we know that if we repeatedly apply the reflection functors to a simple representation, then at some stage we will get a simple representation concentrated at a single vertex. The dimension for the simple representation is 1. For any $g(x) \in \mathcal{B}_\Gamma$, $r_k(g(x)) \in \mathcal{B}_\Gamma$. Then the conclusion follows from Lemma 3.9. □

Main Theorem. *Let Γ be an $N = 1$ ADE quiver. Let $\mathcal{B}_\Gamma = \{r_i(p'_j(x))\}$, where $r_i \in \mathfrak{W}_\Gamma$ and p'_j, $j \in V_\Gamma$ are the polynomials defined in relation (1.1). Assume no element in $\mathcal{B}_\Gamma$ has a multiple root. If $(*)$ holds, then $N = 1$ ADE quivers have finite representation type.*

Proof. Let V be a simple representation of an $N = 1$ ADE quiver. Let $\mathcal{A} = \{i | V(i) \neq 0\}$. We can assume that $\mathcal{A}$ is connected. Otherwise, V would be decomposable. We apply the forgetful functors to V to get an (+)-accessible (resp. (-)-accessible) diagram (no loop)

$$\begin{array}{ccccc} V(1) \longleftarrow \cdots \longleftarrow & V(k) & \longleftarrow \cdots \longleftarrow V(n) \\ & \uparrow & \\ & V(l) & \end{array}$$

$$\begin{array}{ccccc} V(1) \longrightarrow \cdots \longrightarrow & V(k) & \longrightarrow \cdots \longrightarrow V(n) \\ & \downarrow & \\ & V(l) & \end{array}$$

(For the type A case, $V(l) = 0$.)

Let $c = \sigma_n \cdots \sigma_2 \sigma_1$. By [2], there exists k such that $c^k(\dim V) \not> 0$. By $(*)$, Lemma 3.8 and Lemma 3.9, we know that $\sum_i \dim V(i).p'_i(x)$ is the only element in $\mathcal{A}_\Gamma$ which vanishes at λ. By Corollary 3.1 and Proposition 3.1, this implies that there exist $\beta_1, \cdots, \beta_l$ and a simple representation $L_{\beta_{l+1}}$ which is concentrated at a vertex of Γ such that

$$V = F_{\beta_1} \cdots F_{\beta_k}(L_{\beta_{k+1}})$$

V corresponds to the positive root

$$\dim V = \sigma_{\beta_1} \cdots \sigma_{\beta_k}(\overline{\beta_{k+1}})$$

where

$$(\overline{\beta_{k+1}}) = \begin{cases} 0 & \text{if} \quad i \neq k+1 \\ 1 & \text{if} \quad i = k+1 \end{cases}$$

Since the usual ADE quiver only has finitely many positive roots, $N = 1$ ADE quivers have finite representation type. This finishes the proof of the theorem. □

From the above Main Theorem, one can get the following Proposition 3.2.

Corollary 3.2. *Let Γ be an $N = 1$ ADE quiver. Let $\mathcal{B}_\Gamma = \{r_i(p'_j(x)) | r_i \in \mathfrak{W}_\Gamma\}$, where $\mathfrak{W}_\Gamma$ is the Weyl group of Γ and p'_j is the polynomial defined on relation (1.1). Assume each element in $\mathcal{B}_\Gamma$ has simple roots. If $(*)$ holds, then there is a finite-to-one* *correspondence between simple representations of $N = 1$ ADE quivers and the positive roots of ADE Dynkin diagram.*

Proof. We know that $\mathcal{B}_\Gamma$ has only finitely many elements. Each element of $\mathcal{B}_\Gamma$ which is in fact a polynomial has only finitely many simple roots. By our Main Theorem, each root of an element in $\mathcal{B}_\Gamma$ corresponds with a simple representation. Hence, the desired result follows. □

3.4 A correspondence between indecomposable representations and ADE configuration of curves.

An "ADE configuration of curves" in Y is a 1 dimensional connected projective scheme $C \subset Y$, such that

1. $\exists \bar{S} \subset Y$, $C \subset \bar{S}$
2. letting $S = \pi(\bar{S})$, then $\bar{S} \to S$ is a resolution of ADE singularities with exceptional scheme C.

We need the following proposition which is essentially part 3 of Theorem 1 in [14].

Proposition 3.2. *The irreducible components of the discriminant divisor $\mathfrak{D} \subset \mathrm{Res}(\Gamma)$ are in one to one correspondence with the positive roots of Γ. Under the identification of $\mathrm{Res}(\Gamma)$ with the complex root space U, the component $\mathfrak{D}_v$ corresponding to the positive root $v = \sum_{i=1}^n a_i e_i$ is $v^\perp \subset U$, ie the hyperplane perpendicular to v.*

Moreover, $\mathfrak{D}_v$ corresponds exactly to those deformations of Z_0 in $\mathcal{Z}$ to which the curve

$$C_v := \bigcup_{i=1}^n a_i C_{e_i}$$

lifts. For a generic point $t \in \mathfrak{D}_v$, the corresponding surface $\mathcal{Z}_t$ has a single smooth -2 curve in the class $\sum_{i=1}^n a_i [C_{e_i}]$ thus there is a small neighborhood B of t such that the restriction of $\mathcal{Z}$ to B is isomorphic to a product of $\mathbb{C}^{n-1}$ with the semi-universal family over $\mathrm{Res}(A_1)$.

Theorem 3.1. *Let X be a ADE fibration corresponding to Γ, with base $\mathbb{C}$. Let Y be a small resolution of X. Let $\mathcal{B}_\Gamma = \{r_i(p'_j(x)) | r_i \in \mathfrak{W}_\Gamma\}$, where $\mathfrak{W}_\Gamma$ is the Weyl group of Γ and p'_j is the polynomial defined in relation (1.1). Assume no element in $\mathcal{B}_\Gamma$ has multiple roots and assume $(*)$ holds. Then there exists a 1-1 correspondence between the indecomposable representations of the $N = 1$ ADE quiver and the ADE configuration of curves in Y.*

Proof. By Pinkham [17] and Katz-Morrison [14], we have the following commutative diagram

$$\begin{array}{ccc} Y & \longrightarrow & \mathcal{Y} \\ \big\downarrow{\scriptstyle \pi} & & \big\downarrow \\ X & \longrightarrow & \mathcal{X} \\ \big\downarrow{\scriptstyle \varphi} & & \big\downarrow \\ \mathbb{C} & \xrightarrow{f} & \mathbb{C}^n \end{array}$$

where $\mathbb{C}$ denotes the set of complex numbers and $\mathcal{Y}$ denotes the $\mathbb{C}^*$-equivariant simultaneous resolution $\mathcal{Y} \to \mathcal{X}$ inducing $Y_0 \to X_0$. For an indecomposable representation V of the $N = 1$ ADE quiver Γ, we have

$$\sum \dim V(i) \cdot p'_i(\lambda) = 0 \tag{3.19}$$

for some λ. The dimension vector $(\dim V(i))_{i \in V_\Gamma}$ will correspond to a positive root ρ. By (3.8), (3.9), and (3.10), we can express $p'_i(x)$, $i = 1, \cdots, n$ in terms of t_i, $i = 1, \cdots, n$. By Proposition 3.2 or part 3 of Theorem 1 in Katz-Morrison [14, pp. 467], (3.19) will give an equation for $\rho^\perp$. Hence $f(\lambda) = (t_i(\lambda))_{i \in V_\Gamma} \in \rho^\perp$. It follows from Proposition 3.2 that there exists an ADE configuration of curves $C_\rho \subset \pi^{-1}(\lambda) \subset Y$.

Conversely, for an ADE configuration of curves $C \subset Y$, we have that $\varphi \circ \pi(C) = \lambda \in \mathbb{C}$ (Since π is projective, $\varphi \circ \pi(C)$ is projective in $\mathbb{C}$. It follows that $\varphi \circ \pi(C)$ is a finite subset of $\mathbb{C}$. Since C is connected, $\varphi \circ \pi(C)$ is connected in $\mathbb{C}$. Hence $\varphi \circ \pi$ is a point in $\mathbb{C}$.) Moreover, $\pi(C)$ is a point in X (By Katz-Morrison [14], we know that $\mathcal{X}$ is affine. Hence $\pi(C)$ is a point in X.) By Proposition 3.2, we know that $f(\lambda) \in \rho^{\perp}$ for some positive root ρ. Since we assume that each element in $\mathcal{B}_\Gamma$ has simple roots and $(*)$ holds, C corresponds to a unique positive root ρ. We can express ρ as $\rho = \sum a_i \cdot \rho_i$ where ρ_i is a simple positive root. From our Main Theorem, we can construct a simple representation V of $N = 1$ ADE quiver Γ which corresponds to the positive root ρ by applying the reflection functors. This finishes the proof of Theorem 3.1.

□

Example 3.1.

$$V(1) = V(2) = \mathbb{C}$$

$$(p_1' + p_2')(\lambda) = 0$$

$$C \xrightarrow{f} \qquad \mathrm{Def}(A_2) = \{t \in \mathbb{C}^3, \sum t_i = 0\}$$

$$t_i = f_i(t)$$

$$p_1'(t) = f_2(t) - f_1(t)$$

$$p_2'(t) = f_3(t) - f_2(t)$$

$$\begin{aligned}(p_1' + p_2')(\lambda) = 0 \quad &\Longleftrightarrow \quad f_1(t) = f_3(t) \\ &\Longleftrightarrow \quad f(\lambda) \in \rho^{\perp} \\ &\Longleftrightarrow \quad \text{have curve}\, C_\rho \subset Y\end{aligned}$$

Conversely, if have a curve $C_\rho \subset Y$, $\pi : Y \to \mathbb{C}$. $\pi(C_\rho) = \lambda, \Rightarrow f(\lambda) \in \rho^{\perp}$. Suppose $\rho = \sum r_i p_i'$. We can use Theorem 2.2 to construct a quiver representation V, such that, $\dim(V_i) = r_i$.

Example 3.2.

$$\mathrm{Def}(D_4) = \mathbb{C}^4$$

Equations of roots

$$\rho_1^{\perp} \quad t_1 - t_2 \tag{3.20}$$

$$\rho_2^{\perp} \quad t_2 - t_3 \tag{3.21}$$

$$\rho_3^{\perp} \quad t_3 - t_4 \tag{3.22}$$

$$\rho_4^{\perp} \quad t_3 + t_4 \tag{3.23}$$

$$\rho^{\perp} = (\rho_1 + 2\rho_2 + \rho_3 + \rho_4)^{\perp} \quad t_1 + t_2 \tag{3.24}$$

$$(p_1' + 2p_2' + p_3' + p_4')(\lambda) = 0$$

We can use Theorem 2.2 to construct a quiver representation V, such that

$$V(1) = V(3) = V(4) = \mathbb{C}, \quad V(2) = \mathbb{C}^2$$

4 Semi-stable sheaves whose reduced support is a rational curve

In this chapter, we focus on the proof of Conjecture 1.2 in page 6. For convenience, I copy this conjecture here.

Conjecture 1.2. *There exists a natural one-to-one correspondence between the indecomposable representations of the $N = 1$ ADE quiver with the datum ρ described in* (1.1) *and a certain class of semistable quasi-coherent sheaves with support on a rational curve C in a Calabi–Yau 3-fold.*

4.1 Preparation

In this section, we briefly recall some definitions and established facts.

Definition 4.1. (c.f. [10]) Let X be a Noetherian scheme. Let $\mathcal{E}$ be a coherent sheaf on X. The *support* of $\mathcal{E}$ is the closed set $\mathrm{Supp}(\mathcal{E}) = \{x \in X | \mathcal{E}_x \neq 0\}$. Its dimension is called the dimension of the sheaf on $\mathcal{E}$ and is denoted by $\dim(\mathcal{E})$.

The annihilator ideal sheaf of $\mathcal{E}$, i.e. the kernel of $\mathcal{O}_X \to \mathcal{E}nd(\mathcal{E})$, defines a subscheme structure on $\mathrm{Supp}(\mathcal{E})$.

Definition 4.2. (c.f. Simpson [20]) Let X be a projective scheme over $S = \mathrm{Spec}(\mathtt{C})$ with a very ample invertible sheaf $\mathcal{O}_X(1)$. For any coherent sheaf $\mathcal{E}$ on X, there is a polynomial in n with rational coefficients $P(\mathcal{E}, n)$ called the *Hilbert polynomial* of $\mathcal{E}$. It is defined by the condition that $P(\mathcal{E}, n) = \dim H^0(X, \mathcal{E}(n))$ for $n \gg 0$. Let $d = d(\mathcal{E})$ denote the dimension of the support of $\mathcal{E}$. It is equal to the degree of the Hilbert polynomial. The coefficient of the leading term is $r/d!$ where $r = r(\mathcal{E})$ is an integer which we call the *rank* of $\mathcal{E}$. Denote the coefficient of the next term by $a(\mathcal{E})/(d-1)!$. Thus

$$P(\mathcal{E}, n) = rn^d/d! + an^{d-1}/(d-1)! + \cdots$$

where $a = a(\mathcal{E})$. Let $\mu(\mathcal{E})$, the *slope* of $\mathcal{E}$, denote the quotient a/r. We will call the quotient $p = P/r$ the *normalized Hilbert polynomial of* $\mathcal{E}$. A coherent sheaf $\mathcal{E}$ is of *pure dimension* $d = d(\mathcal{E})$ if for any nonzero subsheaf $\mathcal{F} \subset \mathcal{E}$, we have $d(\mathcal{F}) = d(\mathcal{E})$. A coherent sheaf $\mathcal{E}$ is *p-semistable* (resp. *p-stable*) if it is of pure dimension, and if for any subsheaf $\mathcal{F} \subset \mathcal{E}$, there exists an N such that

$$p(\mathcal{F}, n) \leq p(\mathcal{E}, n)$$

(resp. $<$) for $n \geq N$. A coherent sheaf $\mathcal{E}$ is *μ-semistable*(resp. *μ-stable*) if it is pure dimension d and if for any subsheaf $\mathcal{F} \subset \mathcal{E}$, we have $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$ (resp. $<$). Note that p-semistability implies μ-semistability, whereas μ-stability implies p-stability.

Remark 4.1. For sheaves of dimension 1, p and μ semistability are equivalent.

Remark 4.2. Here are some elementary properties, which have the same proofs as for vector bundles. Any sheaf $\mathscr{E}$ of pure dimension d has a unique filtration

$$0 = \mathscr{E}_0 \subset \mathscr{E}_1 \subset \cdots \subset \mathscr{E}_k = \mathscr{E}$$

such that the quotients $\mathscr{E}_i/\mathscr{E}_{i-1}$ are p-semistable of pure dimension d and such that the normalized Hilbert polynomials $P(\mathscr{E}_i/\mathscr{E}_{i-1})/r(\mathscr{E}_i/\mathscr{E}_{i-1})$ are strictly decreasing for large n. This filtration is called the *Harder-Narasimhan filtration*. If $\mathscr{E}$ is a p-semistable sheaf of pure dimension d then there is a filtration

$$0 \subset \mathscr{E}_0 \subset \mathscr{E}_1 \subset \cdots \subset \mathscr{E}_k = \mathscr{E}$$

such that the quotients $\mathscr{E}_i/\mathscr{E}_{i-1}$ are p-stable of pure dimension d, with the same normalized Hilbert polynomials.

Lemma 4.1. *Let $\mathcal{C}$ denote the category of p-semistable sheaves of pure dimension d with normalized Hilbert polynomial p_0. Consider an exact sequence of coherent sheaves,*

$$0 \to \mathscr{E} \to \mathscr{F} \to \mathscr{G} \to 0$$

If $\mathscr{E}$ and $\mathscr{G}$ are objects of $\mathcal{C}$ then so is $\mathscr{F}$.

Proof. Let $\mathscr{F}'$ be a coherent subsheaf of $\mathscr{F}$. Consider the intersection $\mathscr{E}' = \mathscr{E} \cap \mathscr{F}'$ and the image $\mathscr{G}'$ of $\mathscr{F}'$ in $\mathscr{G}$. If these sheaves are nonzero then we can write $p(\mathscr{E}') \leq p(\mathscr{E})$ and $p(\mathscr{F}') \leq p(\mathscr{F})$. Since we have the exact sequence

$$0 \to \mathscr{E}' \to \mathscr{F}' \to \mathscr{G}' \to 0$$

we obtain

$$\begin{aligned} P(\mathscr{F}') &= P(\mathscr{E}') + P(\mathscr{G}') \\ &\leq \mathrm{rk}(\mathscr{E}')p_0 + \mathrm{rk}(\mathscr{G}')p_0 \\ &= \mathrm{rk}(\mathscr{F}')p_0 \end{aligned}$$

It follows that $p(\mathscr{F}') \leq p_0$. If $\mathscr{E}' = 0$, then $\mathscr{F}' \cong \mathscr{G}'$. Hence $p(\mathscr{F}') = p(\mathscr{G}') \leq p_0$. If $\mathscr{G}' = 0$, then $\mathscr{F}' = \mathscr{E}'$. Hence $p(\mathscr{F}') = p(\mathscr{E}') \leq p_0$. □

4.2 A_1 case

Let X be an analytic 3-fold, nonsingular along a curve C. Let $\mathscr{I}$ be the ideal sheaf of C in X. Reid gave the following Definition 4.3,

Definition 4.3. [19]

1. A curve $C \subset X$ is a (-2)-curve if $C \cong \mathbf{P^1}$, and $N_{X/C} \cong \mathcal{O}_{\mathbf{P^1}}(a) \oplus \mathcal{O}_{\mathbf{P^1}}(b)$, with $(a, b) = (-1, -1)$ or $(0, -2)$.

2. The *width* of a (-2)-curve $C \subset X$ is given by

$$\begin{aligned} n &= \mathrm{width}(C \subset X) \\ &= \sup\{n \mid \text{ there exists a scheme } C_n \text{ with } C \subset C_n \subset X \text{ and } C_n \cong C \times \mathrm{Spec}\, k[\epsilon]/\epsilon^n\} \end{aligned}$$

If $n < \infty$, C is *isolated.*

Let C be a (-2)-curve, Reid in [19] got the following sequence of ideal sheaves,

$$\mathscr{I}_k \subset \mathscr{I}_{k-1} \subset \cdots \subset \mathscr{I}_2 \subset \mathscr{I} \subset \mathcal{O}_X \quad (1_k)$$

satisfying

$$\mathscr{I}\mathscr{I}_i \subset \mathscr{I}_{i+1} \subset \mathscr{I}_i, \quad \mathscr{I}_i/\mathscr{I}_{i+1} \cong \mathcal{O}_C \quad \text{and} \quad \mathscr{I}_{i+1}/\mathscr{I}\mathscr{I}_i \cong \mathcal{O}_C(2)$$

for all $i \leq k-1$.

For $\mathscr{I}_k/\mathscr{I}\mathscr{I}_k$ there is the exact sequence

$$0 \to \mathscr{I}\mathscr{I}_{k-1}/\mathscr{I}\mathscr{I}_k \to \mathscr{I}_k/\mathscr{I}\mathscr{I}_k \to \mathscr{I}_k/\mathscr{I}\mathscr{I}_{k-1} \to 0 \quad (2_k)$$

satisfying $\mathscr{I}\mathscr{I}_{k-1}/\mathscr{I}\mathscr{I}_k \cong \mathcal{O}_C$ and $\mathscr{I}_k/\mathscr{I}\mathscr{I}_{k-1} \cong \mathcal{O}_C(2)$.

The chain (1_k) can be extended to a chain (1_{k+1}) if and only if (2_k) splits.

Proposition 4.1. *[19] C has width n if and only if there exists a chain (1_n) such that $\mathscr{I}_n/\mathscr{I}\mathscr{I}_n \cong \mathcal{O}(1) \oplus \mathcal{O}(1)$.*

Remark 4.3. Let's first consider a 3-fold Y with a cA_1 singularity which is defined by equation (4.25),

$$xy + z^2 + t^{2n} = 0 \tag{4.25}$$

Let X be a small resolution of Y. Let $p : X \to Y$ be the blow up map. Let C be the exceptional set. Then C is a (-2)-curve, and the width of C is n.

Lemma 4.2. $\mu(\mathcal{O}_X/\mathscr{I}_k) = 1$.

Proof. We know $\mu(\mathcal{O}_{\mathbf{P}^1}) = 1$. Notice in the sequence

$$0 \to \mathscr{I}/\mathscr{I}_2 \to \mathcal{O}_X/\mathscr{I}_2 \to \mathcal{O}_X/\mathscr{I} \to 0$$

$\mathscr{I}/\mathscr{I}_2 \cong \mathcal{O}_{\mathbf{P}^1}$ and $\mathcal{O}_X/\mathscr{I} \cong \mathcal{O}_{\mathbf{P}^1}$. Hence we get $P(\mathcal{O}_X/\mathscr{I}_2) = 2n+2$. For $1 \leq j \leq k-1$, we have

$$0 \to \mathscr{I}_j/\mathscr{I}_{j+1} \to \mathcal{O}_X/\mathscr{I}_{j+1} \to \mathcal{O}_X/\mathscr{I}_j \to 0$$

Notice $\mathscr{I}_j/\mathscr{I}_{j+1} \cong \mathcal{O}_{\mathbf{P}^1}$, and $P(\mathcal{O}_X/\mathscr{I}_j) = jn+j$, (by induction.) We get $\mu(\mathcal{O}_X/\mathscr{I}_{j+1}) = (j+1)n+j+1$. Inductively, we get $P(\mathcal{O}_X/\mathscr{I}_k) = kn+k$. Hence we have $\mu(\mathcal{O}_X/\mathscr{I}_k) = 1$. □

For a finitely generated module M, we have

$$\operatorname{rad}(\operatorname{ann}(M)) = \bigcap_{\mathfrak{p}\in \operatorname{Supp}(M)} \mathfrak{p} = \bigcap_{\mathfrak{p}\in \operatorname{Ass}(M)} \mathfrak{p}.$$

Claim 4.1. *Let $I = (x^k, y^l)$ and $R = \mathbb{C}[x,y,z]$. Then (x^k, y^l) is primary.*

Proof. $\sqrt{(x^k, y^l)} = (x,y)$ implies that $P = (x,y)$ is a minimal prime over $I = (x^k, y^l)$. If $\exists Q \supsetneq P$ and $Q \in Ass_R(R/I)$, then $Q \subseteq \mathfrak{m}$, where $\mathfrak{m}$ is some maximal ideal in R. Then

$$I_{\mathfrak{m}} \subset P_{\mathfrak{m}} \subsetneq Q_{\mathfrak{m}} \in Ass_{R_{\mathfrak{m}}}(R_{\mathfrak{m}}/I_{\mathfrak{m}}).$$

Since $\dfrac{\mathbb{C}[x,y,z]_{\mathfrak{m}}}{(x^k, y^l)}$ is a Cohen-Macaulay $\mathbb{C}[x,y,z]_{\mathfrak{m}}$-module, $Ass_{R_{\mathfrak{m}}}(R_{\mathfrak{m}}/I_{\mathfrak{m}})$ has no embedded prime, hence $Q_{\mathfrak{m}} = P_{\mathfrak{m}}$. This is a contradiction! It follows that $Ass_R(R/I) = \{P\}$, whence I is P-primary. □

Proposition 4.2. *$\mathcal{O}_X/\mathscr{I}_k$ is pure.*

Proof. Let $\mathscr{M}$ be a nontrivial subsheaf of $\mathcal{O}_X/\mathscr{I}_k$. Then locally $\mathscr{M} = \widetilde{M/I} \subset \widetilde{R/I}$, where I is a primary ideal (see Claim 4.1).

$$\begin{aligned}\operatorname{Supp} M/I &= V(\operatorname{ann} \overline{M}) \\ &= V(\bigcap_{m_i \notin I} \operatorname{ann} \overline{m_i}) \\ &= \bigcup_{m_i \notin I} V(\operatorname{ann} \overline{m_i})\end{aligned} \tag{4.26}$$

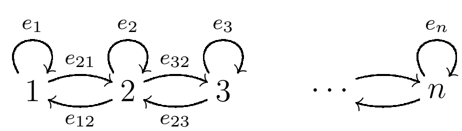
We want to show that $\operatorname{Supp} M/I = V(I)$. But this follows form the following fact: for any $V(\operatorname{ann}\overline{m_i})$ in (4.26), one has $V(\operatorname{ann}\overline{m_i}) = V(I)$. Since $\operatorname{Supp} M/I \subset V(I)$, for any $V(\operatorname{ann}\overline{m_i})$ in (4.26), one has $V(\operatorname{ann}\overline{m_i}) \subset V(I)$. It follows that to show $V(\operatorname{ann}(\overline{m_i})) = V(I)$, one needs only to prove $V(\operatorname{ann}(\overline{m_i}) \supset V(I)$, or equivalently, to prove $\sqrt{\operatorname{ann}\overline{m_i}} \subseteq \sqrt{I}$. If $x \in \sqrt{\operatorname{ann}\bar{m}_i}$, then $\overline{m_i x^n} = 0$. It follows that $m_i x^n \in I$, $m_i \notin I \Rightarrow x^n \in \sqrt{I} \Rightarrow x \in \sqrt{I}$. □

Proposition 4.3. *$\mathcal{O}_X/\mathscr{I}_k$ is μ-semistable.*

Proof. This is a consequence of Lemma 4.1 and Lemma 4.2. □

In [4], the following relations are given for an $N = 1$ A_n quiver,

A_n :



$$(A_n) \qquad Q_{12}Q_{21} + p'(\Phi_1) = 0 \quad - Q_{21}Q_{12} + Q_{23}Q_{32} + p'(\Phi_2) = 0$$

$$\vdots$$

$$-Q_{r-1,r-2}Q_{r-2,r-1} + Q_{r-1,r}Q_{r,r-1} + p'(\Phi_{r-1}) = 0 \qquad - Q_{r,r-1}Q_{r-1,r} + p'(\Phi_r) = 0,$$

and

$$Q_{i,i+1}\Phi_{i+1} = \Phi_i Q_{i,i+1} \qquad \Phi_{i+1}Q_{i+1,i} = Q_{i+1,i}\Phi_i \qquad \text{for} \qquad i = 1, \ldots, r-1.$$

We get the following Theorem 4.1.

Theorem 4.1. *Let X and C be defined as in Remark 4.3. Then: (a) there is a ring isomorphism $\mathcal{O}_X/\mathscr{I}_k \cong \mathcal{O}_C[\epsilon]/\epsilon^k$; (b) there exists a natural one-to-one correspondence between semi-stable sheaves $\{\mathcal{O}_X/\mathscr{I}_k\}_{1\leq k\leq n}$ and indecomposable representations of $N = 1$ A_1 with relation defined in (A_n).*

Proof. (a) From [19], we know that

$$C_k = C \times \operatorname{Spec} \mathbb{C}[\epsilon]/\epsilon^k$$

for $1 \leq k \leq n$. Notice that in this case we have $p'(\Phi) = \Phi^n$. We know all the indecomposable representations are $\{J_i\}_{1 \leq i \leq n}$, where J_i is a standard $i \times i$ Jordan-block with eigenvalue 0 defined by

$$J_i = \begin{pmatrix} 0 & \cdots & & & \\ 1 & 0 & \cdots & & \\ \multicolumn{5}{c}{\cdots\cdots\cdots\cdots\cdots\cdots} \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

clearly $J_i^n = 0$ for $1 \leq i \leq n$. Therefore we get the following one-to-one correspondence between coherent sheaves and indecomposable representations,

$$\mathscr{O}_X/\mathscr{I}_i \leftrightarrow J_i$$

□

Definition 4.4. Let $\pi : X \to Y$ contract C to the point $q \in Y$. The *length* of the the component C_i of C is the length of the scheme with structure sheaf $\mathscr{O}_Y/\pi^{-1}(\mathfrak{m}_{q,Y})$ at a generic point of C_i.

4.3 A_n case

In [25], Thomas Zerger studied the A_n case. He got the following theorem.

Theorem 4.2 (Thomas Zerger). *(c.f. [25]) If $f : X \to Y$ is a contraction map with $f(C) = q$ and $C = \cup_{i=1}^n C_i$, with all components having length 1, then a general hyperplane section of q has an A_n type singularity at q.*

Let $\mathscr{K}$ be the ideal sheaf of $C_{ab} = C_a + \cdots + C_b \subset C$. Zerger got a family of ideal sheaves $\{\mathscr{K}_i\}$ (see page 380 of [25].) This sequence satisfies $\mathscr{K}\mathscr{K}_{i-1} \subset \mathscr{K}_i \subset \mathscr{K}_{i-1}$,

$\mathscr{K}_{i-1}/\mathscr{K}_i \cong \mathcal{O}_{C_{ab}}$ and $\mathscr{K}_i/\mathscr{K}\mathscr{K}_{i-1} \cong \omega^*_{C_{ab}}$, where $\omega^*_{C_{ab}}$ is the dual of the dualizing sheaf of C_{ab}. In local coordinates at p on C_{ab}, $\mathscr{K}_i = (xy + \lambda_1 z + \cdots + \lambda_{i-1} z^{i-1}, z^i)$ or $\mathscr{K}_i = (x^i y^i, z)$.

Lemma 4.3. *$\mathcal{O}_X/\mathscr{K}_{i+1}$ is pure for each i.*

Proof. If $0 \neq \mathscr{F} \subseteq \mathcal{O}_X/\mathscr{K}_{i+1}$, then there exists $C_j \subset C$, such that $\operatorname{Supp} \mathscr{F} \cap C_j \neq 0$. Let $\mathscr{F}'$ be the image of $\mathscr{F}$ in $\mathcal{O}_{C_j}$, then $\mathscr{F}' \neq 0$ since $\operatorname{Supp} \mathscr{F} \cap C_j \neq 0$. We have the following commutative diagram,

$$\begin{array}{ccc} \mathscr{F} & \longrightarrow & \mathscr{F}' \\ \downarrow & & \downarrow \\ \mathcal{O}_X/\mathscr{K}_{i+1} & \longrightarrow & \mathcal{O}_{C_j} \end{array}$$

Since $\mathcal{O}_{C_j} \cong \mathcal{O}_{\mathbf{P^1}}$ is pure, we get

$$1 = \dim \mathcal{O}_X/\mathscr{K}_{i+1} \geq \dim \mathscr{F} \geq \dim \mathscr{F}' = \dim C_j = 1$$

□

Definition 4.5. (c.f. [8]) Generalizing the notion of a subcomplex is that of a *filtered complex* $(F^p K^*, d)$, defined as a decreasing sequence of subcomplexes

$$K^* = F^0 K^* \supset F^1 K^* \supset F^1 K^* \supset \cdots \supset F^n K^* = \{0\}.$$

The spectral sequence of a filtered complex will generalize the long exact cohomology sequence. Before coming to this, we need a few more definitions.

The *associated graded complex* to the filtered complex $(F^p K^*, d)$ is the complex

$$\operatorname{Gr} K^* = \oplus_{p \geq 0} \operatorname{Gr}^p K^*$$

where

$$\operatorname{Gr}^p K^* = \frac{F^p K^*}{F^{p+1} K^*}$$

and the differential is the obvious one. The filtration F^pK^* on K^* also induces a filtration $F^pH^*(K^*)$ on the cohomology by

$$F^pH^q(K^*) = \frac{F^pZ^q}{F^pB^q}.$$

The *associated graded cohomology* is

$$\operatorname{Gr} H^*(K^*) = \oplus_{p,q} \operatorname{Gr}^p H^q(K^*),$$

where

$$\operatorname{Gr}^p H^q(K^*) = \frac{F^pH^q(K^*)}{F^{p+1}H^q(K^*)}.$$

Definition 4.6. (c.f. [8]) A *spectral sequence* is a sequence $\{E_r, d_r\}$ $(r \geq 0)$ of bigraded groups

$$E_r = \oplus_{p,q\geq 0} E_r^{p,q}$$

together with differentials

$$d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}, \quad d_r^2 = 0,$$

such that

$$H^*(E_r) = E_{r+1}.$$

Lemma 4.4. *$P(\mathcal{O}_{C_{ab}}, n) = (\delta+1)n+1$, where $C_{ab} = C_a + \cdots + C_b \subset C$ and $\delta = b - a$.*

Proof. Let's abuse notation by identifying the sheaf

$$\mathcal{O}_{\widetilde{C_{ab}}} = \sum_{i=a}^{b} \mathcal{O}_{\widetilde{C_i}}$$

on the normalization $\widetilde{C_{ab}} = \bigcup \widetilde{C}_i$ of C_{ab} with its direct image on C_{ab}. This is harmless, since the Leray spectral sequence identifies all cohomology of sheaves. We have an exact sequence

$$0 \to \mathcal{O}_{C_{ab}} \to \mathcal{O}_{\widetilde{C_{ab}}} \to \sum_{j=1}^{\delta} \mathbb{C}_{p_j} \to 0 \tag{4.27}$$

Tensoring each term of (4.27) with $\mathcal{O}_X(n)$ and calculating the Hilbert polynomial, we get our desired results. $\square$

Lemma 4.5. *$p(\mathcal{O}_X/\mathscr{K}_i, n)$ is independent of i.*

Proof. We have the following exact sequences,

$$0 \to \mathscr{K}/\mathscr{K}_2 \to \mathcal{O}_X/\mathscr{K}_2 \to \mathcal{O}_X/\mathscr{K} \to 0 \qquad (1)$$

$$\vdots$$

$$0 \to \mathscr{K}_i/\mathscr{K}_{i+1} \to \mathcal{O}_X/\mathscr{K}_{i+1} \to \mathcal{O}_X/\mathscr{K}_i \to 0 \qquad (i)$$

From (1), we obtain that $p(\mathcal{O}_X/\mathscr{K}_2) = p(\mathcal{O}_X/\mathscr{K})$. Inductively, we get that $p(\mathcal{O}_X/\mathscr{K}_i, n)$ is independent of i. $\square$

Lemma 4.6. *$\mathcal{O}_X/\mathscr{K}$ is semistable.*

Proof. Let $\mathscr{F}$ be a proper subsheaf of $\mathcal{O}_X/\mathscr{K}$. For any rational curve $C_i \subset C$, we have following commutative diagram of sheaves,

$$\begin{array}{ccc} \mathscr{F} & \longrightarrow & \mathscr{F}_i \\ \downarrow & & \downarrow \\ \mathcal{O}_X/\mathscr{K} & \longrightarrow & \mathcal{O}_{C_i} \end{array}$$

where $\mathscr{F}_i$ is the image of $\mathscr{F}$ in $\mathcal{O}_{C_i}$. Since $\mathcal{O}_{C_i}$ is pure and 1-dimensional, we get that $\mathscr{F}_i$ is either the 0-sheaf or a 1-dimensional subsheaf of $\mathcal{O}_{C_i}$. Let $I = \{i \mid \mathscr{F}_i \neq 0\}$ and let $C_I = \cup_{i \in I} C_i$. By an argument on page 14 of [24], we see that $\mathscr{F}|_{C_I}$ is an invertible sheaf on C_I and $\mathscr{F}|_{C_{[1,n]-I}} = 0$. Let $\mathscr{H}$ be the kernel of $\mathcal{O}_C \to \mathcal{O}_{C_I}$, we get $\operatorname{Supp} \mathscr{H} \subset \cup_{j \in [1,n]-I} C_j$. Hence $\mathscr{F} \cap \mathscr{H} = 0$. It follows that $\mathscr{F} \cong \mathscr{F}|_{C_I}$. Decomposing C_I into connected components, we get that $C_I = \cup C_{J_k}$, where C_{J_k} is a connected component of C_I and $J_k = [l, l+m]$ for some l and m which depend on J_k. It follows that $\mathscr{F}|_{C_{J_k}} = \mathcal{O}_{C_{J_k}}(a_l, \cdots, a_{l+m})$ for some $a_t \leq 0$, $l \leq t \leq l+m$. We claim that there exists an $a_{t_0} < 0$ for some $l \leq t_0 \leq l+m$. Suppose all $a_t = 0$, then $\mathcal{O}_{C_{J_k}}$ is a subsheaf of $\mathcal{O}_C$. I give two methods to prove that this can't happen.

Method 1. On the one hand, $1 \in H^0(\mathcal{O}_{C_{J_k}}) = \mathbb{C}$; on the other hand, 0 is the only section outside of C_{J_k}. Hence we obtain a contradiction!

Method 2. Let $\mathscr{H}'$ be the kernel of $\mathcal{O}_C \to \mathcal{O}_{C_{J_k}}$. Thus we get a splitting exact sequence of sheaves,

$$0 \to \mathscr{H}' \to \mathcal{O}_C \to \mathcal{O}_{C_{J_k}} \to 0$$

This can't happen.

Now let's calculate $\mu(\mathscr{F}, n)$. We have

$$P(\mathscr{F}, n) = P(\mathscr{F}|C_I, n) = \sum P(\mathscr{F}|C_{J_k}, n)$$

and

$$\begin{aligned}
& P(\mathscr{F}|C_{J_k}, n) \\
= \; & P\left(\mathcal{O}_{C_{J_k}}(a_l, \cdots, a_{l+m}), n\right) \\
= \; & P\left(\mathcal{O}_{C_{J_k}-C_{l+m}}(a_l, \cdots, a_{l+m-2}, a_{l+m-1}-1), n\right) + (n + a_{l+m} + 1) \\
= \; & (m+1)n + \sum_{l}^{m+l} a_t + 1
\end{aligned}$$

Since for $l \leq t \leq l+m$ all $a_t \leq 0$ and there exists one $a_{t_0} < 0$, we conclude that $\mu(\mathcal{O}_{C_{J_k}}(a_l, \cdots, a_{l+m})) \leq 0$. It follows that $\mu(\mathscr{F}, n) \leq 0$. ☐

Proposition 4.4. *$\mathcal{O}_X/\mathscr{K}_i$ is semistable.*

Proof. This is a consequence of Lemma 4.1, Lemma 4.5 and Lemma 4.6. ☐

Theorem 4.3. *Let X be a Calabi-Yau 3-fold with a rational curve $C = \cup_{i=1}^{n} C_i \subset X$ which contracts to a cA_n singularity. Let*

$$\mathcal{A} = \{p_{ab}(x) = p'_a(x) + \cdots + p'_b(x) \mid 1 \leq a \leq b \leq n, p'_i(x) \textit{ as in relation } (1.1)\}$$

Suppose no two elements in $\mathcal{A}$ have a common root, then there exists a one-to-one natural correspondence between the semistable sheaves which have a support on rational curves and indecomposable representations as in Theorem 2.1. Explicitly, if $p_{ab}(x) = (x-\lambda)^m g(x)$ where $(x-\lambda)$ is not a factor of $g(x)$, then one has the following natural correspondence between sheaves and indecomposable representations:

$$\mathcal{O}/\mathscr{K}_m \leftrightarrow (V_{ab}^m, f)$$

$$\mathcal{O}/\mathscr{K}_{m-1} \leftrightarrow (V_{ab}^{m-1}, f)$$

$$\vdots$$

$$\mathcal{O}/\mathscr{K}_1 \leftrightarrow (V_{ab}^1, f)$$

where (V_{ab}^l, f), $1 \leq l \leq m$, is defined by

$$V(i) = \begin{cases} \mathbb{C}[x]/(x-\lambda)^l & a \leq i \leq b \\ 0 & \textit{otherwise} \end{cases}$$

Proof. Suppose $p_{ab}(x) = (x-\lambda)^m g(x)$ where $(x-\lambda)$ is not a factor of $g(x)$. Let $C_{ab} = C_a + \cdots + C_b \subset C$. Let $\mathscr{K}$ be the ideal sheaf of C_{ab}. Let $\{\mathscr{K}_i\}$ be the family of ideal sheaves in Lemma 4.3 (page 388) of [25]. We know that in local coordinates, $K_i = (x^i y^i, z)$ or $K_i = (xy + g_1 z + g_2 z^2 + \cdots + g_{i-1} z^{i-1}, z^i)$, (see Lemma 4.3 (page 388) of [25].) It follows that $\mathcal{O}_X/\mathscr{K}_i$ and $(\mathcal{O}_X/\mathscr{K})[\epsilon]/\epsilon^i$ have the same multiplicity i. Since there is a natural correspondence $(\mathcal{O}_X/\mathscr{K})[\epsilon]/\epsilon^i \leftrightarrow (V_{ab}^i, f)$, the desired conclusion follows.

□

Proposition 4.5. *(c.f. [19]) Let $P \in X$ be a Gorenstein 3-fold singularity having a small resolution $f: Y \to X$; then $P \in X$ is cDV.*

Let's consider an example:

Example 4.1. Let $\widetilde{X}$ be defined by

$$xy + z^2 + t^4 = 0$$

This is a Gorenstein 3-fold with a cA_1singularity. There exists a small resolution $p: X \to \widetilde{X}$, such that $p^{-1}((0,0,0)) = C$ is a rational curve. Let $\mathscr{I}$ be the ideal sheaf of the curve C. We know that

$$\mathscr{I}/\mathscr{I}^2 \cong \mathcal{O}_C \oplus \mathcal{O}_C(2)$$

In this case, we have two indecomposable representations corresponding to the relation (1.1) (see [4]) $i)\mathbb{C}$, $ii)\mathbb{C}^2$, which correspond to a stable sheaf $\mathcal{O}_C$ and a semistable sheaf $\mathcal{O}_X/\mathscr{J}$, where $\mathscr{J} = \ker(\mathscr{I} \to \mathscr{I}/\mathscr{I}^2 \to \mathcal{O}_C)$. It's easy to see that $\mathcal{O}_X/\mathscr{J}$ has support C. To prove stability (semi-stability,) we have to know $d(\mathcal{F})$ and $r(\mathcal{F})$ for a coherent sheaf $\mathcal{F}$. This is related to the "width" in Reid's paper. See Proposition 4.3 (page 76) for the proof of the semistability of the sheaves $\mathcal{O}_C$ and $\mathcal{O}_X/\mathscr{J}$.

Remark 4.4. Laufer [15] defined the above X in the following way,

$$\begin{cases} z_1 = y_1 + f(x, y_1, y_2) \\ z_2 = x^2 y_2 + g(x, y_1, y_2) \\ w_1 = x^{-1} \end{cases}$$

where $f(x, y_1, y_2)$ and $g(x, y_1, y_2)$ are sections of $\mathscr{I}^2$.

X contains a rational curve C and there is a contraction map

$$p : X \to \widetilde{X}$$

satisfying $p(C) = o \in \widetilde{X}$ and $\widetilde{X}$ is defined by $xy + z^2 + t^{2n} = 0$.

The equation for a surface with an A_n type singularity is

$$xy + z^{n+1} = 0$$

The deformation of the A_n surface is

$$xy + z^{n+1} + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n = 0$$

where $a_i \in \mathbb{C}[V]^{\mathfrak{W}}$ and $\mathfrak{W}$ is the Weyl group which is generated by reflections. (See **Theorem 1** of [14].) The 3-fold which has a cA_n singularity is a one dimensional deformation of a surface with an A_n singularity. The generic hyperplane section depends on the the length at the singular point. We can also write the equations for surfaces with D_n or E_n singularities. The equation for a surface with a D_n singularity is

$$x^2 + y^2 z - z^{n-1} = 0$$

The deformation for a surface with a D_n singularity is

$$x^2 + y^2 z - z^{n-1} - \sum_{i=1}^{n-1} \delta_{2i} z^{n-i-1} + 2\gamma_n y = 0.$$

The equation for a surface with an E_6 singularity is

$$x^2 + xz^2 - y^3 = 0$$

The equation for a surface with an E_7 singularity is

$$x^2 + y^3 + 16yz^3 = 0$$

The equation for a surface with an E_8 singularity is

$$x^2 - y^3 + z^5 = 0.$$

The deformation equations for the E_n case are very complicated. (See [14]).

5 Field equations and the deformation theory of rational curves

In this chapter, we observe that the $N = 1$ ADE physical field equations can have geometrical consequences. Namely, they provide constraints on deformations of A-D-E singularities.

5.1 Deformations of ADE rational curves and field equations

We need the following famous Reid's Lemma.

Lemma 5.1. *[19, (1.1),(1.14)] Let $\pi : Y \to X$ be a resolution of an isolated Gorenstein threefold singularity $P \in X$. Suppose that the exceptional set of π has pure dimension 1. Let X_0 be a generic hyperplane section of X that passes through P. Then X_0 has a rational double point at P.*

Moreover, if X_0 is any hyperplane section through P with a rational double point, and Y_0 is its proper transform, then Y_0 is normal, and the minimal resolution $Z_0 \to X_0$ factors through the induced map $\pi\mid_{Y_0}: Y_0 \to X_0$.

Follwing Wahl [23], a map $Y_0 \to X_0$ through which the minimal resolution $Z_0 \to X_0$ factors is called a *partial resolution* of X_0 (provided that Y_0 is normal). There is a natural graph associated to such a map. Start with the dual graph Γ of the components of the exceptional divisor of the minimal resolution $Z_0 \to X_0$. The curves contracted by $Y_0 \to X_0$ correspond to vertices in the graph that span a subgraph Γ_0; we call $\Gamma_0 \subset \Gamma$ the *partial resolution graph* of π. The vertices corresponding to Γ_0 are shown with open circle ($\circ$), while those corresponding to $\Gamma - \Gamma_0$ are shown with a closed circle ($\bullet$).

Proposition 5.1. *For a $N = 1$ A_n quiver,suppose the underlying Dynkin diagram A_n is the dual graph of $C_{A_n} = \cup C_i \subset Y_0$. For any rational curve $C \subset C_{A_n}$ defined by $C = \cup_k^l C_j \cap C_{A_n}$, where $\cup_k^l C_j \subset Z_0$, let $\mathscr{A}_C = \{[k', l'] \mid C = \cup_{k'}^{l'} C_j \cap C_{A_n}, \cup_{k'}^{l'} C_j \subset Z_0\}$.*

Then the deformation of C can be described by the field equation

$$\prod_{[k',l']\in\mathscr{A}_C}\sum_{i\in[k',l']} p_i'(\Phi_e)=0$$

where e is a vertex corresponding to curve $C_e \subset C$.

Proof. For the vertex e such that $C_e \subset C$, let $A = Q_{e,e-1}Q_{e-1,e}$ and $B = -Q_{e,e+1}Q_{e+1,e}$, by [4], we have field equations

$$A\prod_{1\leq j\leq e-1}(A+p_{e-1}'(\Phi_e)+\cdots+p_j'(\Phi_e))=0$$

and

$$B\prod_{e+1\leq l\leq n}(B+p_{e+1}'(\Phi_e)+\cdots+p_l'(\Phi_e))=0$$

we also have

$$A+B=p_l'(\Phi_l)$$

Let $p_i'(\Phi)=t_i(\Phi)-t_{i+1}(\Phi)$. Then the resultant of the eigenvalue equations of these two field equations at vertex e is

$$\prod_{1\leq i\leq e}\prod_{e+1\leq j\leq n+1}(t_i-t_j)=0$$

The locus where C lifts is

$$\prod_{[i,j-1]\in\mathscr{A}_C}(t_i-t_j)=0$$

Hence the corresponding field equation is

$$\prod_{[k',l']\in\mathscr{A}_C}\sum_{i\in[k',l']} p_i'(\Phi_e)=0$$

□

Proposition 5.2. *(A) Let Γ be the underlying Dynkin diagram of the $N = 1$ D_n quiver. Let $\Gamma_0 \subset \Gamma$ be the set of $(\circ)$ vertices which contains vertex $n-2$. For any I such that $n-2 \in I \subset \Gamma_0$, setting $\mathscr{A}_I = \{J \subset \Gamma - \Gamma_0 \mid I \cup J$ is a connected subset of $\Gamma\}$, the deformation of $\cup_{i\in I} C_i$ can be described by the field equation*

$$\prod_{J\in\mathscr{A}_I}\left(\sum_{i\in I} p_i'(\Phi_{n-2}) + \sum_{j\in J} p_j'(\Phi_{n-2})\right) = 0$$

(B) Let $\cup_I \cup_{i\in I} C_i + 2\cup_{I'}\cup_{i'\in I'} C_{i'}$ be a curve. Let $a = \min \cup I$, $b = \max \cup I$, and $a' = \min \cup I'$. Then the deformation of the curve $\cup_I \cup_{i\in I} C_i + 2\cup_{I'}\cup_{i'\in I'} C_{i'}$ can be described by field equation

$$\prod_{i\le a,\, b\le k\le a'}\left(\sum_{i\le j\le k-1} p_j'(\Phi_{n-2}) + 2\sum_{k\le j\le n-2} p_j'(\Phi_{n-2}) + p_{n-1}'(\Phi_{n-2}) + p_n'(\Phi_{n-2})\right) = 0$$

Proof. (A) Following page 19 of [4], for the one dimensional representation case, we have following equation

$$\prod_{i=1}^{n-2}({t_{n-1}}^2 - {t_i}^2)({t_n}^2 - {t_i}^2) = 0$$

By [14], we have the commutative diagram

$$\begin{array}{ccc} \mathcal{Z} & \longrightarrow & \mathcal{Y} \\ \downarrow & & \downarrow \\ V & \longrightarrow & V/\mathfrak{W}_0 \end{array}$$

All the connected curves with the form of $\sum_{i\in I} C_i + \sum_{j\in J} C_j$ in $\mathcal{Z}$ contract to curve $\sum_{i\in I} C_i$ in $\mathcal{Y}$. The curve $\sum_{i\in I} C_i + \sum_{j\in J} C_j$ corresponds to $\sum_{i\in I} p_i'(\Phi_{n-2}) + \sum_{j\in J} p_j'(\Phi_{n-2})$, where $p_a'(\Phi_{n-2}) = t_a(\Phi_{n-2}) - t_{a+1}(\Phi_{n-2})$ for $a < n$ and $p_n'(\Phi_{n-2}) = t_{n-1}(\Phi_{n-2}) + t_n(\Phi_{n-2})$. Then the desired conclusion follows.

(B) Following page 19 of [4], for the two dimensional representation case, we have the following equation

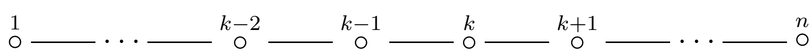
$$t_i + t_j = 0$$

where $i, j \in \{1, \cdots, n-2\}$. As for part (A), we know all curves which contract to curve $C = \cup_I \cup_{i \in I} C_i + 2 \cup_{I'} \cup_{i' \in I'} C_{i'}$ contribute to the deformation of C. Then the desired result follows. □

Remark 5.1. I believe that the results of Proposition 5.1 and Proposition 5.2 can be generalized to E_n case.

5.2 Examples

Suppose the underlying A_n Dynkin diagram is



The field equations are given by,

$$Q_{12}Q_{21} + p_1'(\Phi_1) = 0$$

$$-Q_{21}Q_{12} + Q_{23}Q_{32} + p_2'(\Phi_2) = 0$$

$$\vdots$$

$$-Q_{n-1,n-2}Q_{n-2,n-1} + Q_{n-1,n}Q_{n,n-1} + p_{n-1}'(\Phi_{n-1}) = 0$$

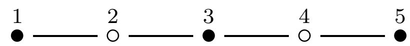
$$-Q_{n,n-1}Q_{n-1,n} + p_n'(\Phi_n) = 0,$$

and

$$Q_{i,i+1}\Phi_{i+1} = \Phi_i Q_{i,i+1} \quad \Phi_{i+1}Q_{i+1,i} = Q_{i+1,i}\Phi_i \qquad i = 1, \ldots, n-1.$$

The solution of this system field equation should correspond to the deformation of curve $C = C_1 \cup ... \cup C_n$.

Example 5.1. Suppose the underlying Dynkin diagram is



For the second node, let $A = Q_{21}Q_{12}$ and $B = -Q_{23}Q_{32}$, we have field equations

$$A(A + p_1'(\Phi_2)) = 0$$

$$B(B + p_3'(\Phi_2))(B + p_3'(\Phi_2) + p_4'(\Phi_2))(B + p_3'(\Phi_2) + p_4'(\Phi_2) + p_5'(\Phi_2)) = 0$$

$$A + B = p_2'(\Phi_2)$$

So the resultant of the eigenvalue equations of these two field equations at node 2 is

$$\prod_{i=1}^{2}\prod_{j=3}^{6}(t_i - t_j) = 0$$

The locus where C_2 lifts is given by

$$\prod_{i=1}^{2}\prod_{j=3}^{4}(t_i - t_j) = 0$$

Let $p_i'(\Phi) = t_i(\Phi) - t_{i+1}(\Phi)$, then we get the corresponding field equation

$$\prod_{1 \le i \le 2 \le j \le 3} (p_i'(\Phi_2) + \cdots + p_j'(\Phi_2)) = 0$$

Let C_i contracts to q_i, then this field equations gives the deformation of C_2, $q_1 \cup C_2$, $C_2 \cup q_3$, $q_1 \cup C_2 \cup q_3$.

The locus where $C_2 + C_4$ lifts is given by

$$\prod_{i=1}^{2}\prod_{j=5}^{6}(t_i - t_j) = 0$$

It corresponds to field equation

$$\prod_{1\le i\le 2,\, 4\le j\le 5}(p'_i(\Phi_2) + \cdots + p'_j(\Phi_2)) = 0$$

This field equation gives the deformation of $C_2 \cup q_3 \cup C_4$, $C_2 \cup q_3 \cup C_4 \cup q_5$, $q_1 \cup C_2 \cup q_3 \cup C_4$, $q_1 \cup C_2 \cup q_3 \cup C_4 \cup q_5$.

Similarly, at the 4-th node, we have field equations

$$C(C + p'_5) = 0$$

$$D(D + p'_3)(B + p'_3 + p'_2)(D + p'_3 + p'_2 + p'_1) = 0$$

$$C + D = p'_4$$

So the resultant is

$$\prod_{i=1}^{4}\prod_{j=5}^{6}(t_i - t_j) = 0$$

The deformation of C_4 is given by

$$\prod_{i=3}^{4}\prod_{j=5}^{6}(t_i - t_j) = 0$$

The corresponding field equation is

$$\prod_{3\le i\le 4\le j\le 5}(p'_i(\Phi_4) + \cdots + p'_j(\Phi_4)) = 0$$

The deformation of $C_2 + C_4$ is given by

$$\prod_{i=1}^{2}\prod_{j=5}^{6}(t_i - t_j) = 0$$

The corresponding field equation is

$$\prod_{1\le i\le 2,\, 4\le j\le 5}(p_i'(\Phi_4) + \cdots + p_j'(\Phi_4)) = 0$$

Example 5.2. For A_n Dynkin diagram, at the $k-th$ node, we have the following fields equation.

$$X_k + Y_k = p_k'(\Phi_k)$$

$$X_k(X_k + p_{k-1}'(\Phi_k))...(X_k + p_{k-1}'(\Phi_k) + ... + p_1'(\Phi_k)) = 0 \tag{5.28}$$

$$Y_k(Y_k + p_{k+1}'(\Phi_k))...(Y_k + p_{k+1}'(\Phi_k) + ... + p_n'(\Phi_k)) = 0 \tag{5.29}$$

Where $X_k = Q_{k,k-1}Q_{k-1,k}$ and $Y_k = -Q_{k,k+1}Q_{k+1,k}$. So we get

$$Y_k = p_k'(\Phi_k) - X_k = p_k'(\Phi_k) + p_{k-1}'(\Phi_k) + ... + p_j'(\Phi_k)$$

for $1 \le j \le k$. Substitute them back to (5.29), we get a system equations which have geometric explanations. Multiply all these equation together, we get the field equation for the deformation of curve C_k, where $i \le k \le j \le n$.

Example 5.3. For a D_4 singularity, we have following field equation

$$Q_{1,2}Q_{2,1} + p_1'(\Phi_1) = 0$$

$$-Q_{2,1}Q_{1,2} + Q_{2,3}Q_{3,2} + Q_{2,4}Q_{4,2} + p_2'(\Phi_2) = 0$$

$$-Q_{3,2}Q_{2,3} + p_3'(\Phi_3) = 0$$

$$-Q_{4,2}Q_{2,4} + p_4'(\Phi_4) = 0$$

$$Q_{ij}\Phi_j = \Phi_i \mathbf{Q}_{ij}$$

Let $X_2 = Q_{2,1}Q_{1,2}$,$Y_2 = -Q_{2,3}Q_{3,2}$, and $Z_2 = -Q_{2,4}Q_{4,2}$

Conjugate, we get

$$X_2(X_2 + p_1'(\Phi_2)) = 0$$

$$Y_2(Y_2 + p_3'(\Phi_2)) = 0$$

$$Z_2(Z_2 + p_4'(\Phi_2)) = 0$$

and

$$X_2 + Y_2 + Z_2 = p_2'$$

For D_4, we know $p_1' = t_1 - t_2$, $p_2' = t_2 - t_3$, $p_3' = t_3 - t_4$, and $p_4' = t_3 + t_4$

Make a shift, $X_2 \longrightarrow X_2 - t_2(\Phi_2)$,$Y_2 \longrightarrow Y_2 - \frac{1}{2}(t_3(\Phi_2) - t_4(\Phi_2))$ and $Z_2 \longrightarrow Z_2 - \frac{1}{2}(t_3(\Phi_2) + t_4(\Phi_2))$, then we get

$$(X_2 + t_2(\Phi_2))(X_2 + t_1(\Phi_2)) = 0$$

$$Y_2^2 = \frac{1}{4}(t_3(\Phi_2) - t_4(\Phi_2)^2 = 0$$

$$Z_2^2 = \frac{1}{4}(t_3(\Phi_2) + t_4(\Phi_2)^2 = 0$$

$$X_2 + Y_2 + Z_2 = 0$$

Following page 19 of [4], for one dimensional representation, we have following equation

$$\prod_{i=1}^{2}({t_3}^2 - {t_i}^2)({t_4}^2 - {t_i}^2) = 0$$

Mathematically, the deformation of Y_0 is

$$x^2 + y^2 z + \frac{(z+t_1{}^2)(z+t_2{}^2)(z+t_3{}^2)(z+t_4{}^2) - t_1{}^2 t_2{}^2 t_3{}^2 t_4{}^2}{z} + 2t_1 t_2 t_3 t_4 y = 0$$

Suppose $\Gamma_0 = \{2\} \subset \{1,2,3,4\} = \Gamma$, then the deformation of C_2 is given by

$$\prod_{i=1}^{2} (t_3{}^2 - t_i{}^2)(t_4{}^2 - t_i{}^2) = 0$$

and the corresponding field equation is

$$\prod_{J \subset \Gamma - \{2\}} \left(p_2'(\Phi_2) + \sum_{j \in J} p_j'(\Phi_2) \right) = 0$$

Suppose $\Gamma_0 = \{1,2\} \subset \{1,2,3,4\} = \Gamma$, then the deformation of C_2 is given by

$$(t_3{}^2 - t_2{}^2)(t_4{}^2 - t_2{}^2) = 0$$

and the corresponding field equation is

$$\prod_{J \subset \Gamma - \Gamma_0} \left(p_2'(\Phi_2) + \sum_{j \in J} p_j'(\Phi_2) \right) = 0$$

The deformation of $C_1 + C_2$ is given by

$$(t_3{}^2 - t_1{}^2)(t_4{}^2 - t_1{}^2) = 0$$

and the corresponding field equation is

$$\prod_{J \subset \Gamma - \Gamma_0} \left(p_1'(\Phi_2) + p_2'(\Phi_2) + \sum_{j \in J} p_j'(\Phi_2) \right) = 0$$

The field equations for the following cases are easy to written out by Proposition 5.2, we omit them. We only write out the deformation equations.

Suppose $\Gamma_0 = \{2, 3\} \subset \{1, 2, 3, 4\} = \Gamma$, then the deformation of C_2 is given by

$$\prod_{i=1}^{2}(t_i - t_3)\prod_{1}^{2}(t_i + t_4) = 0$$

The deformation of $C_2 + C_3$ is given by

$$\prod_{i=1}^{2}(t_i - t_4)\prod_{1}^{2}(t_i + t_3) = 0$$

Suppose $\Gamma_0 = \{2, 4\} \subset \{1, 2, 3, 4\} = \Gamma$, then the deformation of C_2 is given by

$$\prod_{i=1}^{2}\prod_{j=3}^{4}(t_i - t_j) = 0$$

The deformation of $C_2 + C_4$ is given by

$$\prod_{i=1}^{2}\prod_{j=3}^{4}(t_i + t_j) = 0$$

Suppose $\Gamma_0 = \{1, 2, 3\} \subset \{1, 2, 3, 4\} = \Gamma$, then the deformation of C_2 is given by

$$(t_2 - t_3)(t_2 + t_4) = 0$$

The deformation of $C_1 + C_2$ is given by

$$(t_1 - t_3)(t_1 + t_4) = 0$$

The deformation of $C_2 + C_3$ is given by

$$(t_2 - t_4)(t_2 + t_3) = 0$$

The deformation of $C_1 + C_2 + C_3$ is given by

$$(t_1 - t_4)(t_1 + t_3) = 0$$

Suppose $\Gamma_0 = \{1, 2, 4\} \subset \{1, 2, 3, 4\} = \Gamma$, then the deformation of C_2 is given by

$$(t_2 - t_3)(t_2 - t_4) = 0$$

The deformation of $C_1 + C_2$ is given by

$$(t_1 - t_3)(t_1 - t_4) = 0$$

The deformation of $C_2 + C_4$ is given by

$$(t_2 + t_4)(t_2 + t_3) = 0$$

The deformation of $C_1 + C_2 + C_4$ is given by

$$(t_1 + t_4)(t_1 + t_3) = 0$$

Suppose $\Gamma_0 = \{2, 3, 4\} \subset \{1, 2, 3, 4\} = \Gamma$, then the deformation of C_2 is given by

$$(t_1 - t_3)(t_2 - t_3) = 0$$

The deformation of $C_2 + C_3$ is given by

$$(t_1 - t_4)(t_2 - t_4) = 0$$

The deformation of $C_2 + C_4$ is given by

$$(t_2 + t_4)(t_1 + t_4) = 0$$

The deformation of $C_2 + C_3 + C_4$ is given by

$$(t_1 + t_3)(t_2 + t_3) = 0$$

Suppose $\Gamma_0 = \{1, 2, 3, 4\} \subset \{1, 2, 3, 4\} = \Gamma$, then we have the following: The deformation of C_2 is $t_2 - t_3 = 0$. The deformation of $C_1 + C_2$ is $t_1 - t_3 = 0$. The deformation of $C_2 + C_3$ is $t_2 - t_4 = 0$. The deformation of $C_2 + C_4$ is $t_2 + t_4 = 0$. The deformation of $C_1 + C_2 + C_3$ is $t_1 - t_4 = 0$. The deformation of $C_1 + C_2 + C_4$ is $t_1 + t_4 = 0$. The deformation of $C_2 + C_3 + C_4$ is $t_2 + t_3 = 0$. The deformation of $C_1 + C_2 + C_3 + C_4$ is $t_1 + t_3 = 0$.

For 2-dimensional representation, again following page 19 of [4], we have equation

$$t_1 + t_2 = 0$$

Which gives the deformation of $C_1 + 2C_2 + C_3 + C_4$.

So for D_4, we get that the one dimensional representation and two dimensional representation provides us all deformation information about curves C which contains C_2.

For general D_n, at $n - 2$ node, we have the field equation

$$\prod_{i=1}^{n-2}(X + t_i) = 0$$

$$Y^2 = \frac{1}{4}(t_{n-1} + t_n)^2$$

$$Z^2 = \frac{1}{4}(t_{n-1} - t_n)^2$$

$$X + Y + Z = 0$$

For one dimensional representation, we have

$$\prod_{i=1}^{n-2}({t_{n-1}}^2 - {t_i}^2)({t_n}^2 - {t_i}^2) = 0$$

For two dimensional representation, we have

$$t_i + t_j = 0$$

with $i, j = 1, ..., n-2$ and $i \neq j$.

As for D_4 case, we can consider the deformation of curves.

The following Example 5.4 says that we can deform a A_n curve to A_1 curve.

Example 5.4. For a rational curve C_k with A_{k-1} and A_{n-k} singularity, we get

$$xy + (z^k + a_1 z^{k-1} + ... + a_{k-1}z + a_k)(z^{n+1-k} + b_1 z^{n-k} + ... + b_{n+1-k}) = 0$$

where $a_1 + b_1 = 0$. Let a_i ,b_j be constants for $1 \leq i \leq k-1$ and $1 \leq j \leq n+1-k$. For generically a_k and b_{n+1-k} which vanishes at $t = 0$, we can let $a_k = at + \sum a_d t^d$ where $d > 1$ and $a \neq 0$ and $b_{n+1-k} = bt + \sum b_l t^l$ where $l > 1$ and $b \neq 0$. Then at $z = 0$, we get a A_1 singularity.

6 Generalization of Reid's Pagoda Technique

6.1 Introduction

Let X be a Calabi-Yau 3-fold, $C \subseteq X$, $C \cong \mathbf{P}^1$, C contracts to a cD_4 point. Let $\mathscr{I}$ be the ideal sheaf of C. Then

$$\mathscr{I}/\mathscr{I}^2 \cong \mathcal{O}_C(-1) \oplus \mathcal{O}_C(3).$$

Let $\mathscr{J} = \ker(\mathscr{I} \to \mathscr{I}/\mathscr{I}^2 \to \mathcal{O}_C(-1))$, then $\mathscr{I}/\mathscr{J} = \mathcal{O}_C(-1)$, and $\mathscr{J}/\mathscr{I}^2 = \mathcal{O}_C(3)$. We get the following exact sequence

$$0 \to \mathscr{I}^2/\mathscr{I}\mathscr{J} \to \mathscr{J}/\mathscr{I}\mathscr{J} \to \mathscr{J}/\mathscr{I}^2 \to 0.$$

It's easy to see that $\mathscr{I}^2/\mathscr{I}\mathscr{J} = S^2(\mathscr{I}/\mathscr{J}) = \mathcal{O}_C(-2)$.

Therefore, we get

$$\mathscr{J}/\mathscr{I}\mathscr{J} = \begin{cases} \mathcal{O}_C(3) \oplus \mathcal{O}_C(-2) & (A) \\ \mathcal{O}_C(2) \oplus \mathcal{O}_C(-1) & (B) \\ \mathcal{O}_C \oplus \mathcal{O}_C(1) & (C) \end{cases}$$

(A) can't happen, since $H^1(\mathcal{O}_C(3) \oplus \mathcal{O}_C(-2)) \neq 0$. Now we will prove that (B) can't happen.

Let $f : X \to Y$ be the contraction map, $f(C) = p$. Let $g \in \mathfrak{m}_p$, then $g \circ f \in f^*(\mathfrak{m}_p) \subset \mathscr{I}$.

$$0 \to \mathscr{J} \to \mathscr{I} \to \mathcal{O}_C(-1) \to 0$$

Since $H^0(\mathcal{O}_C(-1)) = 0$, then $g \circ f \in H^0(\mathscr{J})$.

If $\mathscr{J}/\mathscr{I}\mathscr{J} = \mathcal{O}_C(2) \oplus \mathcal{O}_C(-1)$, then we can find $\mathscr{I}_3 \subset \mathscr{J}$, such that

$$\mathscr{I}_3 = \ker(\mathscr{J} \to \mathscr{J}/\mathscr{I}\mathscr{J} \to \mathcal{O}_C(-1)).$$

It follows that $\mathscr{J}/\mathscr{I}_3 \cong \mathcal{O}_C(-1)$ and $\mathscr{I}_3/\mathscr{I}\mathscr{J} \cong \mathcal{O}_C(2)$.

Again, since $H^0(\mathcal{O}_C(-1)) = 0$, then $g \circ f \in H^0(\mathscr{I}_3)$. Therefore, we have $f^*(\mathfrak{m}_p) \subset \mathscr{I}_3$. It follows that

$$\mathcal{O}_X/f^*(\mathfrak{m}_p) \supsetneq \mathcal{O}_X/\mathscr{I}_3 \supsetneq \mathcal{O}_X/\mathscr{J} \supsetneq \mathcal{O}_X/\mathscr{I}.$$

Therefore, $\text{length}(\mathcal{O}_X/f^*(\mathfrak{m}_p)) \geq 3$. But for cD_4, $\text{length}(\mathcal{O}_X/f^*(\mathfrak{m}_p)) = 2$. So case (B) can't happen.

Then we get $\mathscr{J}/\mathscr{I}\mathscr{J} \cong \mathcal{O}_C \oplus \mathcal{O}_C(1)$.

Locally, let $\mathscr{I} = (y, z)$ and $\mathscr{J} = (y, z^2)$. Let

$$\mathcal{O}_{C_2} = \mathcal{O}_X/\mathscr{J}$$

and

$$C_2 = \operatorname{Spec} \mathcal{O}_{C_2}$$

In this chapter, I will study some properties of C_2.

6.2 Some sheaf properties of C_2

It's easy to see that $\mathscr{I}^2 \subset \mathscr{J} \subsetneq \mathscr{I}$, so $(C_2)_{red} = C$. Since C_2 is non-reduced everywhere, it's singular everywhere. We want to prove that $\mathscr{J}/\mathscr{J}^2$ is a locally free sheaf of module of $\mathcal{O}_{C_2}$ of rank 2. (If C is nonsingular curve inside X, by [9], we know $\mathscr{I}/\mathscr{I}^2$ is locally free of rank 2. But C_2 is singular, so we can't apply the result in [9].) We had to find another way to prove it.

To prove this, I use some result from Matsumura.

Theorem 6.1. *[16, Theorem 19.9] Let A be a Noetherian local ring, and I a proper ideal of A; assume that $Projdim I < \infty$. Then I is generated by an A-sequence $\iff I/I^2$ is a free module over A/I.*

Definition 6.1. Let $a_1, ..., a_n \in A$, set $I = \sum_1^n a_i A$, and let M be an A-module with $IM \neq M$. We say that $a_1, \cdots, a_n$ is an M-quasi-regular sequence if the following condition holds for each r: $F(X_1,...,X_n) \in M[X_1,...,X_n]$ is homogeneous of degree r and $F(a) \in I^{r+1}M$ implies that all the coefficients of F are in IM.

Facts 6.1. *If $a_1,...,a_n$ is an M-sequence, then it is a M-quasi-regular.*

Lemma 6.1. *$\mathscr{J}/\mathscr{J}^2$ is a locally free sheaf of rank 2 over $\mathcal{O}_X/\mathscr{J}$.*

Proof. $\mathscr{I}$ is generated by a regular sequence since C is smooth hence a local complete intersection. Hence $\mathscr{J}$ is generated by a regular sequence (since (a,b) is a regular sequence iff (a,b^2) is a regular sequence.) Hence $\mathscr{J}/\mathscr{J}^2$ is locally free over $O_X/\mathscr{J}$ by Theorem 6.1. □

Next, I prove C_2 is a rational curve.

Lemma 6.2. *C_2 is a rational curve.*

Proof. Let $\mathscr{I} = (y,z)$ be the ideal sheaf of C, $\mathscr{J} = (y,z^2)$ be the ideal sheaf of C_2, then $\mathscr{I}/\mathscr{J}$ be the ideal sheaf of C in C_2. We have the following short exact sequence of sheaves,

$$0 \longrightarrow \mathscr{I}/\mathscr{J} \longrightarrow \mathcal{O}_{C_2} \longrightarrow \mathcal{O}_C \longrightarrow 0$$

Then we have the corresponding long exact sequence of cohomology groups. Notice $\mathscr{I}/\mathscr{J} = \mathcal{O}_C(-1) = \mathcal{O}_{P^1}(-1)$, so $H^1(\mathscr{I}/\mathscr{J}) = 0$, and $H^1(\mathcal{O}_C) = 0$, then $H^1(\mathcal{O}_{C_2}) = 0$. Since $H^0(\mathscr{I}/\mathscr{J}) = 0$, then we get $H^0(\mathcal{O}_{C_2}) = H^0(\mathcal{O}_C) = C$. Hence $P_a(C_2) = 1 - \chi(\mathcal{O}_{C_2}) = 0$. Therefore, C_2 is a rational curve.

□

Remark 6.1. Because C_2 is not reduced, C_2 is not a variety.

Let $\mathscr{I} = (y,z)$, $\mathscr{J} = (y,z^2)$. Then $\mathscr{I}^2 \subset \mathscr{J} \subset \mathscr{I}$. Let $\mathcal{O}_C = \mathcal{O}_X/\mathscr{I}$. We have the following exact sequence of sheaves

$$0 \longrightarrow \mathscr{I}\mathscr{J}/\mathscr{J}^2 \longrightarrow \mathscr{J}/\mathscr{J}^2 \longrightarrow \mathscr{J}/\mathscr{I}\mathscr{J} \longrightarrow 0$$

Lemma 6.3. $\mathscr{I}\mathscr{J}/\mathscr{J}^2 = \mathscr{I}/\mathscr{J} \otimes \mathscr{J}/\mathscr{I}\mathscr{J}$

Proof. Actually, we have a natural map, $\mathscr{I} \otimes \mathscr{J} \longrightarrow \mathscr{I}\mathscr{J}/\mathscr{J}^2$, it kills $\mathscr{J} \otimes \mathscr{J}$ and $\mathscr{I} \otimes \mathscr{I}\mathscr{J}$, because their images are $\mathscr{J}^2$ and $\mathscr{I}^2\mathscr{J}$, both are contained in $\mathscr{J}^2$.

Because $\mathscr{J}/\mathscr{I}\mathscr{J}$ is generated by z^2 and y as an $\mathcal{O}_C$ module, we get that $\mathscr{I}/\mathscr{J} \otimes \mathscr{J}/\mathscr{I}\mathscr{J}$ is generated by z^3 and yz as an $\mathcal{O}_C$ module. We know that $\mathscr{I}\mathscr{J}/\mathscr{J}^2$ is also generated by z^3 and yz as an $\mathcal{O}_C$ module. Hence $\mathscr{I}\mathscr{J}/\mathscr{J}^2$ and $\mathscr{I}/\mathscr{J} \otimes \mathscr{J}/\mathscr{I}\mathscr{J}$ are generated by the same elements.

Now I will prove $\mathscr{I}\mathscr{J}/\mathscr{J}^2$ is a locally free sheaf of rank 2 over $\mathcal{O}_C$

Define $\mathcal{O}_C \oplus \mathcal{O}_C \longrightarrow \mathscr{I}\mathscr{J}/\mathscr{J}^2$ by $g : (f, h) \longrightarrow fz^3 + hyz$. This map is surjective since it sends the generators of $\mathcal{O}_C \oplus \mathcal{O}_C$ to the generator of $\mathscr{I}\mathscr{J}/\mathscr{J}^2$. This map is also injective since the image element $fz^3 + hyz$ is in $\mathscr{J}^2$ only if f, h is divisible by y or z (Since $\mathscr{J}^2 = (z^4, z^2y, y^2)$). That is, $(f, h) \longrightarrow 0$ implies $f, h \in (y, z) = \mathscr{I}$. Therefore, g is an isomorphism, and $\mathscr{I}\mathscr{J}/\mathscr{J}^2$ is locally free of rank 2 over $\mathcal{O}_C$.

Therefore, $\mathscr{I}\mathscr{J}/\mathscr{J}^2 = \mathscr{I}/\mathscr{J} \otimes \mathscr{J}/\mathscr{I}\mathscr{J} = \mathcal{O}_C(-1) \otimes (\mathcal{O}_C \oplus \mathcal{O}_C(1)) = \mathcal{O}_C(-1) \oplus \mathcal{O}_C$.

□

Lemma 6.4. $H^0(\mathscr{J}/\mathscr{J}^2) \longrightarrow H^0(\mathcal{O}_C \oplus \mathcal{O}_C(1))$ *is surjective.*

Proof. Notice that

$$H^1(\mathscr{I}\mathscr{J}/\mathscr{J}^2) = H^1(\mathcal{O}_C(-1) \oplus \mathcal{O}_C) = 0$$

Therefore,

$$H^0(\mathscr{J}/\mathscr{J}^2) \longrightarrow H^0(\mathcal{O}_C \oplus \mathcal{O}_C(1))$$

is surjective. □

Lemma 6.5. $Pic(C_2) = Z$

Proof. We have an exact sequence of sheaves.

$$0 \longrightarrow Z \longrightarrow \mathcal{O}_{C_2} \longrightarrow \mathcal{O}^*_{C_2} \longrightarrow 0$$

Since $H^1(\mathcal{O}_{C_2}) = 0$ and $H^2(C_2, Z) = Z$.(because C_2 is a rational curve).Therefore, $H^1(\mathcal{O}^*_{C_2}) = Z$. Hence, $Pic(C_2) = Z$. □

Since every section of $\mathcal{O}_C \oplus \mathcal{O}_C(1)$ can be extended to a section of $\mathscr{I}/\mathscr{I}^2$, then for a nowhere vanishing section s of $\mathcal{O}_C \oplus \mathcal{O}_C(1)$, it can be extended to a nowhere vanishing section $\tilde{s}$ of $\mathscr{I}/\mathscr{I}^2$.

We have an exact sequence of sheaves.

$$0 \longrightarrow \mathcal{O}_{C_2} \xrightarrow{\tilde{s}} \mathscr{I}/\mathscr{I}^2 \longrightarrow L \longrightarrow 0$$

where L is the cokernel.

Now I want to prove $L = \mathcal{O}_{C_2}(1)$

Lemma 6.6. $L = \mathcal{O}_{C_2}(1)$

Proof. By adjunction formula,$\omega(C_2) = \omega_X \otimes \wedge^2(\mathscr{I}/\mathscr{I}^2)^* = \wedge^2(\mathscr{I}/\mathscr{I}^2)^*$, (since X is Calabi-Yau,then we know $\omega_X = O_X$.) Because $c_1(\mathcal{O}_{C_2}) = 0$, so $c_1(L) = c_1(\mathscr{I}/\mathscr{I}^2) = c_1(\omega(\mathcal{O}_{C_2})) = 1$. Therefore,$L = \mathcal{O}_{C_2}(1)$. □

6.3 The sequence of sheaves

In this section, I mimic the proof of Theorem 5.4 in Reid's paper [19]. I get a family of exact sequence of sheaves. Consequently, I will prove $\mathscr{I}_k = (y, z^{2k})$ or $\mathscr{I}_k = (y^k, z^2)$ in the sequence. But first, let me prove a lemma.

Lemma 6.7.

$$0 \longrightarrow \mathcal{O}_{C_2} \longrightarrow \mathscr{I}/\mathscr{I}^2 \longrightarrow \mathcal{O}_{C_2}(1) \longrightarrow 0 \quad (**)$$

does not always split.

Proof. We calculate

$$\begin{aligned}
& \mathrm{Ext}^1_{\mathcal{O}_{C_2}}(\mathcal{O}_{C_2}(1), \mathcal{O}_{C_2}) \\
\cong\ & \mathrm{Ext}^1_{\mathcal{O}_{C_2}}(\mathcal{O}_{C_2}, \mathcal{O}_{C_2}(-1)) \\
\cong\ & H^1(\mathcal{O}_{C_2}(-1)) \\
\cong\ & H^0(\mathcal{O}_{C_2})^* \\
\cong\ & \mathbb{C} \\
\neq\ & 0
\end{aligned}$$

So $(**)$ does not always split. □

We assume $\mathscr{I}/\mathscr{I}^2 = \mathcal{O}_{C_2} \oplus \mathcal{O}_{C_2}(1)$. Then we can have a sequence

$$0 \longrightarrow \mathscr{I}_2 \longrightarrow \mathscr{I} \longrightarrow \mathscr{I}/\mathscr{I}^2 \longrightarrow \mathcal{O}_{C_2} \longrightarrow 0$$

We get $\mathscr{I}_2 = (ay + bz^2, \mathscr{I}^2)$. So a or b must be a unit.(Because $\mathscr{I}_2/\mathscr{I}^2$ is an invertible subsheaf of $\mathscr{I}/\mathscr{I}^2$, so we can't have a and b both vanishing locally at the same time.)

Case 1. If a is a unit, we let $Y = y + bz^2$. Then we have

$$\mathscr{I}_2 = (Y, (Y - bz^2, z^2)(Y - bz^2, z^2)) = (Y, z^4)$$

We have $\mathscr{I}/\mathscr{I}_2 = \mathcal{O}_{C_2}$, and $\mathscr{I}_2/\mathscr{I}^2 = \mathcal{O}_{C_2}(1)$.

$$0 \longrightarrow \mathscr{I}/\mathscr{I}_2 \longrightarrow \mathscr{I}/\mathscr{I}^2 \longrightarrow \mathscr{I}_2/\mathscr{I}^2 \longrightarrow 0$$

$$0 \to \mathscr{I}^2/\mathscr{I}\mathscr{I}_2 \underset{\alpha}{\to} \mathscr{I}_2/\mathscr{I}\mathscr{I}_2 \underset{\beta}{\to} \mathscr{I}_2/\mathscr{I}^2 \to 0 \qquad (2)$$

Then we have

$$\mathscr{I}^2/\mathscr{I}\,\mathscr{I}_2 = S^2(\mathscr{I}/\mathscr{I}_2) = \mathscr{I}/\mathscr{I}_2 \otimes \mathscr{I}/\mathscr{I}_2 = \mathcal{O}_{C_2}$$

Suppose (2) splits. Then we have

$$\mathscr{I}^2/\mathscr{I}\,\mathscr{I}_2 \underset{\rho}{\longleftarrow} \mathscr{I}_2/\mathscr{I}\,\mathscr{I}_2$$

and

$$\mathscr{I}_2/\mathscr{I}\,\mathscr{I}_2 \underset{\tau}{\longleftarrow} \mathscr{I}_2/\mathscr{I}^2$$

such that $\rho\alpha = id$ and $\beta\tau = id$.

Then we can define $\mathscr{I}_3$. Let $\mathscr{I}_1 = \mathscr{I}$.

$$0 \longrightarrow \mathscr{I}_3 \longrightarrow \mathscr{I}_2 \longrightarrow \mathscr{I}_2/\mathscr{I}\,\mathscr{I}_2 \underset{\rho}{\rightarrow} \mathcal{O}_{C_2} \longrightarrow 0$$

Then

$$\mathscr{I}_2/\mathscr{I}_3 = \mathcal{O}_{C_2}$$

and we have

$$0 \longrightarrow \mathscr{I}_3/\mathscr{I}\,\mathscr{I}_2 \longrightarrow \mathscr{I}_2/\mathscr{I}\,\mathscr{I}_2 \longrightarrow \mathscr{I}_2/\mathscr{I}_3 \longrightarrow 0 \quad (2')$$

Since $(2')$ splits, we get

$$\mathscr{I}_3/\mathscr{I}\,\mathscr{I}_2 = \mathcal{O}_{C_2}(1)$$

Now $\mathscr{I}_3 = (ay + bz^4, \mathscr{I}\,\mathscr{I}_2)$.

Lemma 6.8. *Let $s = ay + bz^4$ in the definition of $\mathscr{I}_3$,then we can view a as a unit.*

Proof. let $\tilde{s}$ be the image of s in $\mathscr{I}_2/\mathscr{I}\,\mathscr{I}_2$. Then $\rho(\tilde{s}) = 0$. Therefore, there exists $f \in \mathscr{I}_2/\mathscr{I}^2$, such that $\tau(f) = \tilde{s}$. We can define $x' = \beta(\tilde{s}) = f$. Notice that

$\mathscr{I}^2/\mathscr{I}\mathscr{I}_2$ is generated by z^4 locally, therefore we get $\tilde{s} = x' + cz^4$ locally. Hence we can view a as a unit in s. □

Corollary 6.1.

$$\mathscr{I}_3 = (y, z^6)$$

Proof. Since a is a unit, we let $Y = y + bz^4$. Then $y = Y - bz^4$. Notice

$$\mathscr{I}\mathscr{I}_2 = (y, z^2)(y, z^4) = (Y - bz^4, z^2)(Y - bz^4, z^4) = (Y, z^2)(Y, z^4)$$

Then

$$\mathscr{I}_3 = (Y, z^6)$$

□

$$0 \longrightarrow \mathscr{I}\mathscr{I}_2/\mathscr{I}\mathscr{I}_3 \longrightarrow \mathscr{I}_3/\mathscr{I}\mathscr{I}_3 \longrightarrow \mathscr{I}_3/\mathscr{I}\mathscr{I}_2 \longrightarrow 0 \quad (3)$$

Lemma 6.9. *We have*

$$\mathscr{I}\mathscr{I}_2/\mathscr{I}\mathscr{I}_3 = \mathcal{O}_{C_2}$$

Proof. We have a natural map

$$\mathscr{I} \times \mathscr{I}_2 \longrightarrow \mathscr{I}\mathscr{I}_2/\mathscr{I}\mathscr{I}_3$$

by multiplication. Notice $\mathscr{I}_2^2 = (y, z^4)^2 = (y^2, yz^4, z^8)$, and $\mathscr{I}\mathscr{I}_3 = (y, z^2)(y, z^6) = (y^2, yz^2, z^8)$. Therefore, $\mathscr{I}_2^2 \subset \mathscr{I}\mathscr{I}_3$. Since $\mathscr{I}\mathscr{I}_2/\mathscr{I}\mathscr{I}_3$ and $\mathscr{I}/\mathscr{I}_2 \otimes \mathscr{I}_2/\mathscr{I}_3$ are both generated by z^6 locally. So we conclude

$$\mathscr{I}\mathscr{I}_2/\mathscr{I}\mathscr{I}_3 = \mathscr{I}/\mathscr{I}_2 \otimes \mathscr{I}_2/\mathscr{I}_3 = \mathcal{O}_{C_2} \otimes \mathcal{O}_{C_2} = \mathcal{O}_{C_2}$$

□

Now suppose by induction that there is a sequence of ideals

$$\mathscr{I}_k \subset \mathscr{I}_{k-1} \subset ... \subset \mathscr{I}_2 \subset \mathscr{I}_1 \subset \mathcal{O}_X,$$

satisfying

$$\mathscr{I}\mathscr{I}_i \subset \mathscr{I}_{i+1} \subset \mathscr{I}_i, \quad \mathscr{I}_i/\mathscr{I}_{i+1} = \mathcal{O}_{C_2} \quad \mathscr{I}_{i+1}/\mathscr{I}\mathscr{I}_i = \mathcal{O}_{C_2}(1)$$

and $\mathscr{I}_i = (y, z^{2i})$ for all $i \leqq k-1$

Then we have exact sequence

$$0 \longrightarrow \mathscr{I}\mathscr{I}_{k-1}/\mathscr{I}\mathscr{I}_k \longrightarrow \mathscr{I}_k/\mathscr{I}\mathscr{I}_k \longrightarrow \mathscr{I}_k/\mathscr{I}\mathscr{I}_{k-1} \longrightarrow 0 \quad (k)$$

Where

$$\mathscr{I}\mathscr{I}_{k-1}/\mathscr{I}\mathscr{I}_k = \mathscr{I}/\mathscr{I}_2 \otimes \mathscr{I}_{k-1}/\mathscr{I}_k = \mathcal{O}_{C_2}$$

and

$$\mathscr{I}_k/\mathscr{I}\mathscr{I}_{k-1} = \mathcal{O}_{C_2}(1)$$

We can define $\mathscr{I}_{k+1}$.

$$0 \longrightarrow \mathscr{I}_{k+1} \longrightarrow \mathscr{I}_k \longrightarrow \mathscr{I}_k/\mathscr{I}\mathscr{I}_k \longrightarrow \mathcal{O}_{C_2} \longrightarrow 0$$

If the (k) splits, using the same argument as in Lemma 6.8, Corollary 6.1 and Lemma 6.9, we get $\mathscr{I}_{k+1} = (y, z^{2(k+1)})$.

Case 2.If b is a unit, then we let $Z^2 = ay + z^2$. So now we have

$$\mathscr{I}_2 = (Z^2, (y, Z^2 - ay)(y, Z^2 - ay)) = (y^2, Z^2)$$

We have $\mathscr{I}/\mathscr{I}_2 = \mathcal{O}_{C_2}$, and $\mathscr{I}_2/\mathscr{I}^2 = \mathcal{O}_{C_2}(1)$.

$$0 \longrightarrow \mathscr{I}/\mathscr{I}_2 \longrightarrow \mathscr{I}/\mathscr{I}^2 \longrightarrow \mathscr{I}_2/\mathscr{I}^2 \longrightarrow 0$$

$$0 \to \mathscr{I}^2/\mathscr{I}\,\mathscr{I}_2 \underset{\alpha}{\to} \mathscr{I}_2/\mathscr{I}\,\mathscr{I}_2 \underset{\beta}{\to} \mathscr{I}_2/\mathscr{I}^2 \to 0 \qquad (2)$$

Then we have

$$\mathscr{I}^2/\mathscr{I}\,\mathscr{I}_2 = S^2(\mathscr{I}/\mathscr{I}_2) = \mathscr{I}/\mathscr{I}_2 \otimes \mathscr{I}/\mathscr{I}_2 = \mathcal{O}_{C_2}$$

Suppose (2) splits. Then we have

$$\mathscr{I}^2/\mathscr{I}\,\mathscr{I}_2 \underset{\rho}{\leftarrow} \mathscr{I}_2/\mathscr{I}\,\mathscr{I}_2$$

and

$$\mathscr{I}_2/\mathscr{I}\,\mathscr{I}_2 \underset{\tau}{\leftarrow} \mathscr{I}_2/\mathscr{I}^2$$

such that $\rho\alpha = id$ and $\beta\tau = id$.

Then we can define $\mathscr{I}_3$. Let $\mathscr{I}_1 = \mathscr{I}$.

$$0 \longrightarrow \mathscr{I}_3 \longrightarrow \mathscr{I}_2 \longrightarrow \mathscr{I}_2/\mathscr{I}\,\mathscr{I}_2 \underset{\rho}{\to} \mathcal{O}_{C_2} \longrightarrow 0$$

Then

$$\mathscr{I}_2/\mathscr{I}_3 = \mathcal{O}_{C_2}$$

and we have

$$0 \longrightarrow \mathscr{I}_3/\mathscr{I}\,\mathscr{I}_2 \longrightarrow \mathscr{I}_2/\mathscr{I}\,\mathscr{I}_2 \longrightarrow \mathscr{I}_2/\mathscr{I}_3 \longrightarrow 0 \quad (2')$$

Since $(2')$ splits, we get

$$\mathscr{I}_3/\mathscr{I}\,\mathscr{I}_2 = \mathcal{O}_{C_2}(1)$$

Now $\mathscr{I}_3 = (ay^2 + bz^2, \mathscr{I}\mathscr{I}_2)$.

Lemma 6.10. *Let $s = ay^2 + bz^2$ in the definition of $\mathscr{I}_3$,then we can view b as a unit.*

Proof. let $\tilde{s}$ be the image of s in $\mathscr{I}_2/\mathscr{I}\mathscr{I}_2$. Then $\rho(\tilde{s}) = 0$. Therefore, there exists $f \in \mathscr{I}_2/\mathscr{I}^2$, such that $\tau(f) = \tilde{s}$. We can define $x' = \beta(\tilde{s}) = f$. Notice that $\mathscr{I}^2/\mathscr{I}\mathscr{I}_2$ is generated by y^2 locally, therefore we get $\tilde{s} = x' + cy^2$ locally. Hence we can view b as a unit in s. □

Corollary 6.2.

$$\mathscr{I}_3 = (y^3, z^2)$$

Proof. Since b is a unit, we let $Z^2 = ay^2 + z^2$. Then $z^2 = Z^2 - ay^2$. Notice

$$\mathscr{I}\mathscr{I}_2 = (y, z^2)(y^2, z^2) = (y, Z^2 - ay^2)(y^2, Z^2 - ay^2) = (y, Z^2)(y^2, Z^2)$$

Then

$$\mathscr{I}_3 = (y^3, Z^2)$$

□

$$0 \longrightarrow \mathscr{I}\mathscr{I}_2/\mathscr{I}\mathscr{I}_3 \longrightarrow \mathscr{I}_3/\mathscr{I}\mathscr{I}_3 \longrightarrow \mathscr{I}_3/\mathscr{I}\mathscr{I}_2 \longrightarrow 0 \quad (3)$$

Lemma 6.11. *We have*

$$\mathscr{I}\mathscr{I}_2/\mathscr{I}\mathscr{I}_3 = \mathcal{O}_{C_2}$$

Proof. We have a natural map

$$\mathscr{I} \times \mathscr{I}_2 \longrightarrow \mathscr{I}\mathscr{I}_2/\mathscr{I}\mathscr{I}_3$$

by multiplication. Notice $\mathscr{I}_2^2 = (y^2, z^2)^2 = (y^4, y^2z^2, z^4)$, and $\mathscr{I}\,\mathscr{I}_3 = (y, z^2)(y^3, z^2) = (y^4, yz^2, z^4)$. Therefore, $\mathscr{I}_2^2 \subset \mathscr{I}\,\mathscr{I}_3$. Since $\mathscr{I}\,\mathscr{I}_2/\mathscr{I}\,\mathscr{I}_3$ and $\mathscr{I}/\mathscr{I}_2 \otimes \mathscr{I}_2/\mathscr{I}_3$ are both generated by y^3 locally. So we conclude

$$\mathscr{I}\,\mathscr{I}_2/\mathscr{I}\,\mathscr{I}_3 = \mathscr{I}/\mathscr{I}_2 \otimes \mathscr{I}_2/\mathscr{I}_3 = \mathcal{O}_{C_2} \otimes \mathcal{O}_{C_2} = \mathcal{O}_{C_2}$$

□

Now suppose by induction that there is a sequence of ideals

$$\mathscr{I}_k \subset \mathscr{I}_{k-1} \subset \dots \subset \mathscr{I}_2 \subset \mathscr{I}_1 \subset \mathcal{O}_X,$$

satisfying

$$\mathscr{I}\,\mathscr{I}_i \subset \mathscr{I}_{i+1} \subset \mathscr{I}_i, \quad \mathscr{I}_i/\mathscr{I}_{i+1} = \mathcal{O}_{C_2} \quad \mathscr{I}_{i+1}/\mathscr{I}\,\mathscr{I}_i = \mathcal{O}_{C_2}(1)$$

and $\mathscr{I}_i = (y^i, z^2)$ for all $i \leqq k-1$

Then we have exact sequence

$$0 \longrightarrow \mathscr{I}\,\mathscr{I}_{k-1}/\mathscr{I}\,\mathscr{I}_k \longrightarrow \mathscr{I}_k/\mathscr{I}\,\mathscr{I}_k \longrightarrow \mathscr{I}_k/\mathscr{I}\,\mathscr{I}_{k-1} \longrightarrow 0 \quad (k)$$

Where

$$\mathscr{I}\,\mathscr{I}_{k-1}/\mathscr{I}\,\mathscr{I}_k = \mathscr{I}/\mathscr{I}_2 \otimes \mathscr{I}_{k-1}/\mathscr{I}_k = \mathcal{O}_{C_2}$$

and

$$\mathscr{I}_k/\mathscr{I}\,\mathscr{I}_{k-1} = \mathcal{O}_{C_2}(1)$$

We can define $\mathscr{I}_{k+1}$.

$$0 \longrightarrow \mathscr{I}_{k+1} \longrightarrow \mathscr{I}_k \longrightarrow \mathscr{I}_k/\mathscr{I}\,\mathscr{I}_k \longrightarrow \mathcal{O}_{C_2} \longrightarrow 0$$

If the (k) splits, using the same argument as in Lemma 6.10, Corollary 6.2 and Lemma 6.11 , we get $\mathscr{J}_{k+1} = (y^{k+1}, z^2)$.

Thus we have proved the following proposition.

Proposition 6.1. *Let $C_2 = \operatorname{Spec} \mathcal{O}_X/\mathscr{J}$. If there exists a sequence of ideal sheaves*

$$\mathscr{J}_k \subset \mathscr{J}_{k-1} \subset ... \subset \mathscr{J}_2 \subset \mathscr{J}_1 \subset \mathcal{O}_X,$$

such that, for all $1 \leq i < k$,

$$\mathscr{J}\mathscr{J}_i \subset \mathscr{J}_{i+1} \subset \mathscr{J}_i, \quad \mathscr{J}_i/\mathscr{J}_{i+1} = \mathcal{O}_{C_2}, \quad \mathscr{J}_{i+1}/\mathscr{J}\mathscr{J}_i = \mathcal{O}_{C_2}(1)$$

and

$$0 \longrightarrow \mathscr{J}\mathscr{J}_{i-1}/\mathscr{J}\mathscr{J}_i \longrightarrow \mathscr{J}_i/\mathscr{J}\mathscr{J}_i \longrightarrow \mathscr{J}_i/\mathscr{J}\mathscr{J}_{i-1} \longrightarrow 0 \qquad (i)$$

splits, then there exists an ideal sheaf $\mathscr{J}_{k+1}$ satisfying $\mathscr{J}_{k+1} \subset \mathscr{J}_k$ and $\mathscr{J}_{k+1}/\mathscr{J}\mathscr{J}_k = \mathcal{O}_{C_2}(1)$.

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