A WEAK CONVERGENCE FOR APPROXIMATION OF AMERICAN OPTION PRICES

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CHAPTER 1

INTRODUCTION

1.1 History of the problem

We study a weak convergence for a sequence of discretized American option price processes arising from the tree-based scheme proposed by Maller, Solomon and Szimayer (2006) for all time. The tree-based method (or lattice method) is more tractable to price American options in practice. Cox, Ross and Rubinstein (1979) presented a binomial model to approximate the Black-Scholes model and gave the option price correspondingly. The approach by Cox, Ross and Rubinstein was extended to the finite activity case of the jump diffusion by Amin (1993) and Mulinacci (1996), and to the infinitely activity case by Këllezi and Webber (2004). Këllezi and Webber (2004) can price the Bermudan options via a lattice method based on transition probabilities.

Maller, Solomon and Szimayer (2006) proposed a multinomial tree for the Lévy process. The approximation is constructed by interpolating a sequence of finite time, finite state space and processes for computational convenience and practical need. But Maller, Solomon and Szimayer (2006) can only show that the discrete American option price processes converge in distribution under Meyer-Zheng (MZ) topology (see Meyer and Zheng (1984)), which implies the convergence only holds for t in a subset of full Lebesque measure in $[0, T]$ but not every $t \in [0, T]$. Clearly the weak convergence in distribution under the Skorokohod J_1-topology is stronger than that under the Meyer-Zhang topology. Maller, Solomon and Szimayer (2006) predicted that their method does not lead convergence for **all** t **in** $[0, T]$, though "it plausibly holds" under their conditions. The main purpose of the present paper is to offer an

affirmative answer to their claim. We prove the convergence for **all** t **in** $[0, T]$ **in distribution**.

More recently, Szimayer and Maller (2007) proposed another path-by-path defined approximation scheme, $L_t(n)$, for a pure jump Lévy process, L_t. The sequence of discrete processes converges to the Lévy process in probability or almost surely under J_1-topology under different conditions. The proof in the last paragraph on page 1446 of Szimayer and Maller (2007) makes use of Skorokhod representation theorem that requires $L_t(n)$ converge to L_t in distribution for each $t \in [0, T]$. However, the law of $X_j(n)$ in Szimayer and Maller (2007) is not given explicitly and the law of $X_j(n)$ must be consistent with (A.2)-(A.5) of Maller, Solomon and Szimayer (2006) in order to achieve the necessary and sufficient conditions for $L_t(n) \to L_t$ in distribution.

Under the multinomial tree scheme in Maller, Solomon and Szimayer (2006), we prove that the discretized American put option prices converge to the continuous time counterpart for **all** t **in** $[0, T]$ **in distribution**. We make use of the Skorokhod representation theorem, some results in Maller, Solomon and Szimayer (2006) and the results of Conquet and Toldo (2007).

In Chapter 2, we first give two specific examples: Brownian motion and Poisson process. Then we are using the Skorokhod Representation Theorem to obtain representatives of the approximation scheme proposed by Maller, Solomon and Szimayer (2006) and the pure jump Lévy process, respectively. Then, we work on the new obtained set up, i.e., the representatives. By a result of Conquet and Toldo (2007), the Snell envelopes of the payoff processes under the representative of the approximation scheme converge to that under the representative of the original Lévy process. Since the original processes and their representatives are equal in distribution, we get the same convergence result for the Snell envelopes under the original set up. In Chapter 3, we get the desired result. That is, the discretized American option price processes, $\pi_t(n)$, converge to the continuous time American option price process, π_t, weakly at

every time $t \in [0, T]$. In Chapter 4, we will give a path by path defined approximation scheme that is different from the one proposed by Szimayer and Maller (2007). However, the new approximation scheme shares almost the same law with the one proposed by Maller, Solomon and Szimayer (2006). At last, we will summarize the results as well as discuss some open questions in Chapter 5.

1.2 preliminary staff

The aim of this section is to give a brief introduction of the definitions, propositions and theorems that we will use in later chapters.

Definition 1.2.1 *Let $(X(n), n \in \mathbb{N})$ be a sequence of $\mathbb{R}^d$-valued random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and X be an $\mathbb{R}^d$-valued random variable also defined on $(\Omega, \mathcal{F}, \mathbb{P})$. We say that:*

$X(n)$ converges to X almost surely (a.s), denoted by " $\overset{a.s.}{\to}$ ", if $\lim_{n\to\infty} X(n)(\omega) = X(\omega)$ for all $\omega \in \Omega \setminus \mathcal{N}$, where $\mathcal{N} \in \mathcal{F}$ satisfies $\mathbb{P}(\mathcal{N}) = 0$;

$X(n)$ converges to X in $L^p (1 \leq p < \infty)$, denoted by " $\overset{L^p}{\to}$ ", if $\lim_{n\to\infty} \mathbb{E}(|X(n) - X|^p) = 0$;

$X(n)$ converges to X in probability, denoted by " $\overset{\mathbb{P}}{\to}$ ", if, for all $a > 0$, $\lim_{n\to\infty} \mathbb{P}(|X(n) - X| > a) = 0$;

$X(n)$ converges to X in distribution or weakly, denoted by " $\overset{D}{\to}$ ", if

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} f(x) p_{X(n)}(dx) = \int_{\mathbb{R}^d} f(x) p_X(dx) \text{ for all } f \in C_b(\mathbb{R}^d),$$

where $C_b(\mathbb{R}^d)$ is the set of continuous bounded functions on $\mathbb{R}^d$ and $p_{X(n)} = \mathbb{P} \circ X(n)^{-1}$, $p_X = \mathbb{P} \circ X^{-1}$ are the probability laws of $X(n)$ and X, respectively.

We have the following relations between the modes of convergence:

almost surely converge $\Rightarrow$ converge in probability $\Rightarrow$ converge in distribution;

L^p-converge $\Rightarrow$ converge in probability $\Rightarrow$ converge in distribution.

Notation 1.2.1. The equivalence in distribution of two $\mathbb{R}^d-$valued random variables, X and Y, is denoted by $X \overset{D}{=} Y$.

Definition 1.2.2 *A function is continuous on the right and always has limit on the left is called a càdlàg function. The space of càdlàg functions from* $[0,T]$ *to* $\mathbb{R}$ *is denoted by* $\mathbb{D}[0,T]$.

Definition 1.2.3 *(Skorokhod* J_1-*topology) For any two càdlàg functions* $X(t),\ Y(t) \in \mathbb{D}[0,T]$*, the Skorokhod distance between them is defined as*

$$\rho(X,Y) \triangleq \inf_{\lambda\in\Lambda}\left\{\sup_{0\le t\le T}|X(t)-Y(\lambda(t))| + \sup_{0\le t\le T}|t-\lambda(t)|\right\},$$

where $\Lambda = \{\lambda :\ [0,T] \to [0,T] |\ \lambda$ *is strictly increasing and continuous satisfying* $\lambda(0)=0,\ \lambda(T)=T\}$*. The topology generated by the Skorokhod distance,* ρ*, is called* J_1-*topology. If a sequence of càdlàg processes* $X(n) = (X_t(n),\ 0\le t\le T)$ *converges to a càdlàg process* $X = (X_t,\ 0\le t\le T)$ *weakly under the* J_1-*topology in* $\mathbb{D}[0,T]$ *as* $n\to\infty$*, write* $X(n) \xrightarrow{\mathfrak{L}} X$*. The equivalence of two càdlàg processes,* X *and* Y*, under the* J_1-*topology in* $\mathbb{D}[0,T]$ *is denoted by* $X \overset{\mathfrak{L}}{=} Y$.

The following proposition is part of Theorem VI.1.14 of Jacod and Shiryaev (2003):

Proposition 1.2.1 $\mathbb{D}[0,T]$ *equipped with the Skorokhod* J_1*-topology is a Polish space, i.e., a separable topology space that is metrisable by a complete metric.*

For more detail results about Skorokhod topology or space $\mathbb{D}[0,T]$, see Billingsley (1968) or Jacod and Shiryaev (2003).

Definition 1.2.4 *Let* $\mathcal{F}$ *be a* $\sigma-$*algebra of subsets of a given set* Ω*. A family* $(\mathcal{F}_t, t\ge 0)$ *of sub* $\sigma-$*algebras of* $\mathcal{F}$ *is called a filtration if*

$$\mathcal{F}_s \subseteq \mathcal{F}_t \quad \textit{whenever } s\le t.$$

Definition 1.2.5 *A probability space* $(\Omega,\mathcal{F},\mathbb{P})$ *that comes equipped with such a family,* $(\mathcal{F}_t, t\ge 0)$*, is said to be filtered. Let* $\mathcal{N}$ *denote the collection of all sets of* $\mathbb{P}$*-measure zero in* $\mathcal{F}$ *and define* $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{N}$ *for each* $t\ge 0$*; then* $(\mathcal{G}_t,\ t\ge 0)$ *is a another filtration of* $\mathcal{F}$ *called the augmented filtration.*

Definition 1.2.6 *Let $X = (X_t, t \geq 0)$ be a stochastic process defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We say it is adapted to the filtration (or $\mathcal{F}_t$-adapted) if*

$$X_t \text{ is } \mathcal{F}_t - \text{measurable for each } t \geq 0.$$

Any process $Y = (Y_t, t \geq 0)$ is adapted to its own filtration which is denoted by $\mathcal{F}^Y = (\mathcal{F}_t^Y,\ t \geq 0)$, where $\mathcal{F}_t^Y = \sigma\{Y_s; 0 \leq s \leq t\}$. This filtration is called the natural filtration. And the filtration $\mathbb{F}^Y = (\bigcap_{s>t} \mathcal{F}_s^Y,\ t \geq 0)$ is called the right continuous filtration generated by process Y.

Definition 1.2.7 *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space filtered with a filtration $(\mathcal{F}_t, t \geq 0)$. A stopping time is a random variable $\tau : \Omega \to [0, \infty]$ for which the event $(\tau \leq t) \in \mathcal{F}_t$ for each $t \geq 0$.*

Definition 1.2.8 *(Definition 2 of Conquet, Mémin, and Slominski(2001)) A sequence of filtrations $\mathbb{F}^n = (\mathcal{F}_t^n,\ 0 \leq t \leq T)_{n\in\mathbb{N}}$ converges weakly to a filtration $\mathbb{F} = (\mathcal{F}_t,,\ 0 \leq t \leq T)$, denoted by $\mathbb{F}^n \xrightarrow{\omega} \mathbb{F}$, if and only if, for all $\mathbf{B} \in \mathcal{F}_T$, the sequence of càdlàg martingales $(\mathbb{E}[1_B|\mathcal{F}_\cdot^n])_{n\in\mathbb{N}}$ converges in probability under the Skorokhod J_1-topology in $\mathbb{D}[0,T]$ to the martingale $(\mathbb{E}[1_B|\mathcal{F}_\cdot])$.*

Proposition 1.2.2 *(Proposition 2 of Conquet, Mémin, and Slominski(2001)) Let $(X^n)_{n\in\mathbb{N}}$ be a sequence of càdlàg processes with independent increments and $X \in \mathbb{D}[0,T]$. If $\rho(X^n, X) \xrightarrow{\mathbb{P}} 0$, then $\mathbb{F}^{X^n} \xrightarrow{\omega} \mathbb{F}^X$.*

Proposition 1.2.3 *(Chebyshev-Markov inequality) Let X be a random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The mean of X is denoted by μ. Then,*

$$\mathbb{P}(|X - \alpha\mu| \geq C) \leq \frac{\mathbb{E}(|X - \alpha\mu|^n)}{C^n},$$

where $C > 0$, $\alpha \in \mathbb{R}$, $n \in \mathbb{N}$.

Definition 1.2.9 *A sequence of processes $(X_t(n),\ 0 \leq t \leq T)$, $n \in \mathbb{N}$, satisfies the Aldous' criterion for tightness if, for any $\varepsilon > 0$,*

$$\lim_{\delta\downarrow 0} \limsup_{n\to\infty} \sup_{\tau,\sigma\in\mathcal{S}_{0,T}^{X(n)}, \sigma\leq\tau\leq\sigma+\delta} \mathbb{P}(|X_\tau(n) - X_\sigma(n)| \geq \epsilon) = 0.$$

Here and later, we denote $\mathcal{S}^Z_{0,T}$ the set of $\mathbb{F}^Z$-stopping times taking values in $[0,T]$ for any process $(Z_t,\ 0 \le t \le T)$.

Theorem 1.2.1 *(Corollary 6 of Conquet and Toldo(2007)) Let $(\gamma^n)_{n\in\mathbb{N}}$ be a sequence of continuous bounded functions on $[0,T]\times\mathbb{R}$ which uniformly converges to a continuous bounded function γ. Let X be a càdlàg process and $(X^n)_{n\in\mathbb{N}}$ be a sequence of càdlàg processes. Suppose that $\rho(X^n, X) \xrightarrow{\mathbb{P}} 0$, that Aldous' Criterion for tightness is filled and that one of the following assertions holds:*
- for every n, $\mathcal{F}^{X^n} \subset \mathbb{F}^X$,
- $\mathcal{F}^{X^n} \xrightarrow{\omega} \mathbb{F}^X$.
Then

$$\operatorname*{ess\,sup}_{\tau^n\in\mathcal{T}^{X^n}_{0,T}} \mathbb{E}(\gamma^n(\tau^n, X^n_{\tau^n})) \to \operatorname*{ess\,sup}_{\tau\in\mathcal{S}^X_{0,T}} \mathbb{E}(\gamma(\tau, X_\tau))$$

as $n\to\infty$. Here and later, we denote $\mathcal{T}^Z_{0,T}$ the set of $\mathcal{F}^Z$-stopping times taking values in $[0,T]$ for any process $(Z_t,\ 0 \le t \le T)$.

Remark 1.2.1 *Let $(X_t,\ 0 \le t \le T)$ be a càdlàg process. Assume that the sample paths of X is of step function style. Then $\mathcal{F}^X = \mathbb{F}^X$ and $\mathcal{T}^X_{0,T} = \mathcal{S}^X_{0,T}$.*

Definition 1.2.10 *For process $(Y_t,\ 0 \le t \le T)$, $\sup\limits_{\tau\in\mathcal{S}^Y_{0,T}} \mathbb{E}(Y_\tau)$ is called the Snell envelope of the process Y.*

Proposition 1.2.4 *(Jensen's inequality) If $f:\ \mathbb{R}\to\mathbb{R}$ is a convex function and random variables X and $f(X)$ are both integrable, then*

$$f(\mathbb{E}(X)) \le \mathbb{E}(f(X)).$$

Proposition 1.2.5 *(Proposition 6.3.14 of Jacod and Shiyavue(2003) page 349) If $X(n) \xrightarrow{\mathcal{L}} X$, then $(X^n_{t_1},\cdots,X^n_{t_k}) \to (X_{t_1},\cdots,X_{t_k})$ in distribution, for any $t_i \in D = \mathbb{R}^+ \setminus J(X) = \mathbb{R}^+ \setminus \{t\ge 0,\ \mathbb{P}(\Delta X_t \ne 0) > 0\}$, $i = 1,\cdots,k$, $k\in\mathbb{N}$, where ΔX_t is the jump of X at t. This is denoted by $X(n) \xrightarrow{\mathcal{L}(D)} X$.*

Theorem 1.2.2 *(Skorkhod Representation Theorem, Theorem 6.7 of Billingsley (1968)) Suppose that $X(n) \to X$ in distribution and the law of X has a separable support. Then there exist $Y(n)$ and Y, defined on a common probability space $(\Omega^Y, \mathcal{F}^Y, \mathbb{P}^Y)$, such that*

$$X(n) = Y(n) \text{ in distribution }, \ X = Y \text{ in distribution}$$

and

$$Y(n)(\omega) \to Y(\omega) \text{ for every } \omega \in \Omega^Y, \text{ as } n \to \infty.$$

Proposition 1.2.6 *((6.3.8) of Jacod and Shiyavue(2003)) Let E be a Polosh space. Assume that $\{X^n, \ n \in \mathbb{N}\}$ and X are E-valued random variables defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Assume that $X^n \to X$ in distribution and that $\mathbb{P}(X \in C) = 1$, where C is the continuity set of the function $h : E \to E'$. Then*
(i) if $E' = \mathbb{R}$ and h is bounded, then $\mathbb{E}(h(X^n)) \to \mathbb{E}(h(X))$;
(ii) if E' is polish, i.e., separable and metrisable by a complete metric, then $h(X^n) \to h(X)$ in distribution.

Theorem 1.2.3 *(Theorem 3.2 of Lamberton and Pagès (1990)) Let $X^n = (X^n_t, \ 0 \leq t \leq T)$, $n \in \mathbb{N}$, and $X = (X_t, \ 0 \leq t \leq T)$ be càdlàg processes defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $C = (C_t, \ 0 \leq t \leq T)$ be the canonical process on $\mathbb{D}[0, T]$ and $\mathcal{T}$ be the set of the $\mathbb{F}^C$-stopping times. Assume there exists a dense $D \subseteq [0, T]$ such that $T \in D$. Under the following three Hypotheses:*

(1) $(X, \ \mathbb{F}^X)$ and $(X^n, \ \mathcal{F}^{X^n})$, $n \in \mathbb{N}$ are all of class D,

(2) for any $\tau \in \mathcal{T}$, $\{X^n_{\tau \circ X^n}, \ n \in \mathbb{N}\}$ is uniformly integrable,

(3) $X^n \overset{\mathcal{L}(D)}{\to} X$,

we have that $\operatorname*{ess\,sup}_{\tau \in \mathcal{S}^X_{0,T}} \mathbb{E}(X_\tau) \leq \liminf_{n \to \infty} \operatorname*{ess\,sup}_{\tau^n \in \mathcal{T}^{X^n}_{0,T}} \mathbb{E}(X^n_{\tau^n})$.

CHAPTER 2

Convergence of The Snell Envelopes of The Discounted Payoff Processes Under An Approximation Scheme for A Lévy Process to The Counterpart Under The Lévy Process

In this chapter, we will give two examples to show convergence results of tree-based approximations for Brownian motion and Poisson process at first. Secondly, some preliminary definitions as well as results about Lévy process are recalled. At last, we will show the first main result of this paper. That is, the sequence of Snell envelopes of the discounted payoff processes under the approximation scheme for a pure jump Lévy process proposed by Maller, Solomon and Szimayer (2006) converges to the Snell envelope of the discounted payoff process under the Lévy process. To show this, we are proving three things: (1) There are representatives of the approximation scheme and the Lévy process, denoted by $\widehat{L}(n)$, $n \in \mathbb{N}$ and $\widehat{L}$, such that $\widehat{L}(n) \to \widehat{L}$ everywhere; (2) The Aldous's criterion of tightness is satisfied by $(\widehat{L}(n))_{n\in\mathbb{N}}$; (3) The sequence of natural filtrations generated by $\widehat{L}(n)$ converges weakly to the right continuous filtration generated by $\widehat{L}$. Thus the first main result will be obtained by a result of Conquet and Toldo (2007).

2.1 Examples

Example 2.1.1: (Brownian motion) A binomial tree based approximation of a Brownian motion $B = (B_t,\ 0 \le t \le T)$ is built by Itô and McKean (1974) or see Knight (1962) also. Let us recall that equally spaced approximation scheme:

Let $\mathcal{P}_n :\quad 0 = t_0^n < t_1^n < ... < t_{N(n)}^n = T$ be a sequence of equally spaced partitions

of a time interval $[0,T]$, where $\lim_{n\to\infty} N(n)=\infty$. Denote $\Delta t(n)=\frac{T}{N(n)}$, $n\in\mathbb{N}$. Define stopping times by $e_0(n)=0$, and for any $n\in\mathbb{N}$,

$$e_j(n)=\inf\{t>e_{j-1}(n):|B_t-B_{e_{j-1}(n)}|>\sqrt{\Delta t(n)}\},\quad j=1,2,\cdots.$$

Let

$$B_t(n)=B_{e_{j-1}(n)},\ \text{when } t_{j-1}(n)\le t<t_j(n).$$

From the proof of Theorem 3.3 in Szimayer and Maller(2007), it follows that

$$\sup_{0\le t\le T}|B_t(n)-B_t|\xrightarrow{\mathbb{P}}0,\ \text{as } n\to\infty. \tag{2.1.1}$$

Moreover, from the construction, it is easy to see that $(B_t(n),\ 0\le t\le T)_{n\in\mathbb{N}}$ are cádlág processes with independent increments. Hence, $\mathbb{F}^{B(n)}\xrightarrow{\omega}\mathbb{F}^B$ by Proposition 2 of Conquet, Mémin and Słominski (2001).

At last, $B(n)$ satisfies Aldous' criterion for tightness.

In fact, for any $\delta>0$, $n\in\mathbb{N}$ and $\tau,\sigma\in\mathcal{S}_{0,T}^{B(n)}$ satisfying $\sigma\le\tau\le\sigma+\delta$, we have that

$$\begin{aligned}
&\mathbb{P}(|B_\tau(n)-B_\sigma(n)|\ge\epsilon)\\
\le&\mathbb{P}(|B_\tau(n)-B_\tau|+|B_\tau-B_\sigma|+|B_\sigma(n)-B_\sigma|\ge\epsilon)\\
\le&\mathbb{P}(|B_\tau(n)-B_\tau|\ge\epsilon/3)+\mathbb{P}(|B_\sigma(n)-B_\sigma|\ge\epsilon/3)+\mathbb{P}(|B_\tau-B_\sigma|\ge\epsilon/3).
\end{aligned}$$

By (2.1.1), we get

$$\begin{aligned}
&\limsup_{n\to\infty}\sup_{\sigma\le\tau\le\sigma+\delta,\tau,\sigma\in\mathcal{S}_{0,T}^{B(n)}}\mathbb{P}(|B_\tau(n)-B_\tau|\ge\epsilon/3)\\
\le&\limsup_{n\to\infty}\mathbb{P}(\sup_{\tau\in\mathcal{S}_{0,T}^{B(n)}(n)}|B_\tau(n)-B_\tau|\ge\epsilon/3)\\
=&0.
\end{aligned}$$

Similarly,

$$\limsup_{n\to\infty}\sup_{\sigma\le\tau\le\sigma+\delta,\tau,\sigma\in\mathcal{S}_{0,T}^{B(n)}}\mathbb{P}(|B_\sigma(n)-B_\sigma|\ge\epsilon/3)=0.$$

Consider that

$$
\begin{aligned}
&\lim_{\delta\downarrow 0} \sup_{\sigma\le\tau\le\sigma+\delta,\tau,\sigma\in\mathcal{S}_{0,T}^{B(n)}} \mathbb{P}(|B_\tau - B_\sigma| \ge \epsilon/3) \\
=&\lim_{\delta\downarrow 0} \sup_{\sigma\le\tau\le\sigma+\delta,\tau,\sigma\in\mathcal{S}_{0,T}^{B(n)}} \mathbb{P}(|B_{\tau-\sigma}| \ge \epsilon/3) \\
\le&\lim_{\delta\downarrow 0} \mathbb{P}(\sup_{\sigma\le\tau\le\sigma+\delta,\tau,\sigma\in\mathcal{S}_{0,T}^{B(n)}} |B_{\tau-\sigma}| \ge \epsilon/3) \\
\le&\lim_{\delta\downarrow 0} \mathbb{P}(\sup_{0\le t\le\delta} |B_t| \ge \epsilon/3) \\
\le&\lim_{\delta\downarrow 0} \frac{\mathbb{E}(\sup_{0\le t\le\delta} |B_t|^2)}{(\varepsilon/3)^2} \\
\le&\lim_{\delta\downarrow 0} \frac{\mathbb{E}(B_\delta^2)}{(\varepsilon/3)^2} \\
=&\lim_{\delta\downarrow 0} \frac{\delta}{(\varepsilon/3)^2} \\
=&0,
\end{aligned}
$$

where the first equality follows from the Strong Markov property, the fourth inequality from the Chebyshev inequality and the fifth from the Doob's inequality that could be used because $(|B_t|, t \ge 0)$ is a positive submartingale. Therefore,

$$
\lim_{\delta\downarrow 0} \limsup_{n\to\infty} \sup_{\sigma\le\tau\le\sigma+\delta,\tau,\sigma\in\mathcal{S}_{0,T}^{B(n)}} \mathbb{P}(|B_\tau(n) - B_\sigma(n)| \ge \epsilon) = 0.
$$

Above all, $\operatorname{ess\,sup}_{\tau\in\mathcal{S}_{0,T}^{B(n)}} \mathbb{E}(\gamma^n(\tau, B_\tau(n))) \to \operatorname{ess\,sup}_{\tau\in\mathcal{S}_{0,T}^{B}} \mathbb{E}(\gamma(\tau, B_\tau))$ as $n \to \infty$, whenever γ^n is a sequence of continuous and bounded function on $[0,T] \times \mathbb{R}$ which uniformly converges to a continuous bounded function, γ, on $[0,T] \times \mathbb{R}$, by Corollary 6 of Conquet and Toldo (2007).

A sample path of a Brownian motion is continuous almost surely. However, a Lévy process usually has jumps. In the following example, we will discuss a Lévy process with finitely many jumps in each finite time interval, almost surely.

Definition 2.1.1 *The Poisson process with intensity $\lambda > 0$ is a Lévy process $(N_t,\ 0 \le t \le T)$ taking values in $\mathbb{N} \cup \{0\}$, where each N_t follows a Poisson distribution denoted*

by $N_t \sim \pi(\lambda t)$, for any $t \in [0,T]$. That is, we have that

$$\mathbb{P}(N_t = n) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}$$

for each $n = 0, 1, 2, ...$

Example 2.1.2: (Poisson process) Let $N = (N_t,\ t \in [0,T])$ be a Poisson process with intensity λ. We wish to construct an approximation of N, denoted by $(N'_t(n),\ 0 \leq t \leq T)$, such that $\rho(N'(n), N) \xrightarrow{\mathbb{P}} 0$, as $n \to \infty$, where ρ is the Skorokhod distance. Suppose that there is a sequence of equally spaced partitions of the time interval $[0,T]$, $\mathcal{P}_n$, which is the same as those in Example 2.1.1. Let $\{m_+(n), n \in \mathbb{N}\}$ be a sequence of positive integer satisfying $\lim_{n\to\infty} m_+(n) = \infty$. Let

$$X_j(n) = (N_{t_j^n} - N_{t_{j-1}^n}) \wedge m_+(n), \text{ for } j = 1, 2, \cdots, N(n).$$

Clearly, $X_j(n),\ j = 1, 2, \cdots, N(n)$, are identically independent distributed (i.i.d.) random variables with common law:

$$\mathbb{P}(X_j(n) = i) = \frac{(\lambda \Delta t(n))^i}{i!} e^{-\lambda \triangle t(n)}$$

for $i = 0, 1, 2, \cdots, m_+(n) - 1$ and

$$\mathbb{P}(X_j(n) = m_+(n)) = 1 - \sum_{i=0}^{m_+(n)-1} \mathbb{P}(X_j(n) = i).$$

Let

$$\overline{N_t}(n) = N_{t_j^n}, \text{ for } t_j^n \leq t < t_{j+1}^n,\ j = 0, 1, 2, \cdots, N(n) - 1, \quad \overline{N_T}(n) = N_T$$

and

$$N'_t(n) = \sum_{j=1}^{\lfloor \frac{N(n)t}{T} \rfloor} X_j(n), \quad \text{for any } t \in [0,T].$$

Lemma 2.1.1 *(1)$\rho(\overline{N}(n),\ N) \xrightarrow{\mathbb{P}} 0$, as $n \to \infty$;*

(2) $\sup_{0 \leq t \leq T} |N'_t(n) - \overline{N}_t(n)| \xrightarrow{\mathbb{P}} 0$, *as $n \to \infty$.*

Proof. (1) W use the idea for the first part of the proof of Theorem 3.2 in Szimayer and Maller (2007). We know that the Poisson process has finitely many jumps in each finite time interval almost surely. That is, for all ω in Ω except on a zero mass set Θ, Poisson process has finitely many jumps in each finite time interval. Fix $\omega \in \Omega \setminus \Theta$, we can take n large enough such that each subinterval $(t^n_{j-1}, t^n_j]$, $j = 1, 2, \cdots, N(n) - 1$, contains at most one jump and the last interval, $(t^n_{N(n)-1},\ T]$ has no jump. Let $\overline{t^n_j}$, $\ j \in \{1, ..., N(n) - 1\}$ be the time when the unique jump occurs in the subinterval $(t^n_{j-1}, t^n_j]$ if there exists such a time. Otherwise, let $\overline{t^n_j} = t^n_j$. Hence $\max\limits_{1\le j\le N(n)-1} |\overline{t^n_j} - t^n_j| = \Delta t(n)$. Define a continuous and strictly increasing function on $[0, T]$, denoted by λ_0, such that $\lambda_0(t^n_j) = \overline{t^n_j}$, $j = 1, 2, \cdots, N(n) - 1$, $\lambda_0(0) = 0$ and $\lambda_0(T) = T$. Thus, we get $\sup\limits_{0\le t\le T} |\overline{N_t}(n) - N_{\lambda_0(t)}| = |X_{N(n)}(n)|$ and so

$$\begin{aligned}
\rho(\overline{N}(n),\ N) =& \inf\{\sup_{0\le t\le T} |\overline{N_t}(n) - N_{\lambda(t)}| + \sup_{0\le t\le T} |t - \lambda(t)|\} \\
\le& \sup_{0\le t\le T} |\overline{N_t}(n) - N_{\lambda_0(t)}| + \sup_{0\le t\le T} |t - \lambda_0(t)|, \\
\le& \Delta t(n) + |X_{N(n)}(n)|.
\end{aligned}$$

Hence, for any $a > 0$, $\mathbb{P}(\rho(\overline{N}(n),\ N) > a) \le \mathbb{P}(|X_{N(n)}(n)| > a/2) \to 0$.

(2) Consider that $\sup\limits_{t\in[0,T]} |N'_t(n) - \overline{N_t}(n)| > 0$ if and only if there exists $j \in \{1, 2, \cdots, N(n)\}$ such that the number of jumps of N_t in $(t^n_{j-1}, t^n_j]$ is greater than $m_+(n)$. Hence,

$$\begin{aligned}
\mathbb{P}\left(\sup_{t\in[0,T]} |N'_t(n) - \overline{N_t}(n)| > 0\right) \le& N(n) \sum_{i=1}^{\infty} \frac{(\lambda\Delta t(n))^{m_+(n)+i}}{(m_+(n) + i)!} e^{-\lambda\Delta t(n)} \\
=& \lambda T e^{-\lambda\Delta t(n)} \sum_{i=1}^{\infty} \frac{(\lambda\Delta t(n))^{m_+(n)+i-1}}{(m_+(n) + i)!} \\
\le& \lambda T e^{-\lambda\Delta t(n)} (\lambda\Delta t(n))^{m_+(n)} \left(\sum_{i=0}^{\infty} \frac{(\lambda\Delta t(n))^i}{i!}\right) \\
=& \lambda T e^{-\lambda\Delta t(n)} (\lambda\Delta t(n))^{m_+(n)} e^{\lambda\Delta t(n)} \\
=& \lambda T (\lambda\Delta t(n))^{m_+(n)}.
\end{aligned}$$

And notice that $(\lambda\Delta t(n))^{m_+(n)} \to 0$, as $n \to \infty$. Hence,

$$\sup_{t\in[0,T]} |N'_t(n) - \overline{N_t}(n)| \xrightarrow{\mathbb{P}} 0 \text{ as } n \to \infty,$$

as required. ■

Lemma 2.1.2 *(Triangular inequality for the Skorokhod distance) The Triangular inequality holds for the Skorokhod distance ρ. That is, for càdlàg functions $X(t),\ Y(t),\ Z(t) \in \mathbb{D}[0,T]$,*

$$\rho(X,Z) \le \rho(X,Y) + \rho(Y,Z).$$

Proof. write $X(t) = X_t,\ Y(t) = Y_t,\ Z(t) = Z_t$, for all $t \in [0,T]$. By definition of the Skorokhod distance ρ, for any $\varepsilon > 0$, there exists a λ_1 in Λ such that

$$\sup_{0\le t\le T} |X_t - Y_{\lambda_1(t)}| + \sup_{0\le t\le T} |t - \lambda_1(t)| \le \rho(X,Y) + \varepsilon. \tag{2.1.2}$$

Consider that

$$\begin{aligned}
\rho(Y,Z) &= \inf_{\lambda\in\Lambda}\{\sup_{0\le t\le T} |Y_t - Z_{\lambda(t)}| + \sup_{0\le t\le T} |t - \lambda(t)|\} \\
&= \inf_{\lambda\in\Lambda}\{\sup_{0\le \lambda_1(t)\le T} |Y_{\lambda_1(t)} - Z_{\lambda(\lambda_1(t))}| + \sup_{0\le \lambda_1(t)\le T} |\lambda_1(t) - \lambda(\lambda_1(t))|\} \\
&= \inf_{\lambda\in\Lambda}\{\sup_{0\le t\le T} |Y_{\lambda_1(t)} - Z_{\lambda(\lambda_1(t))}| + \sup_{0\le t\le T} |\lambda_1(t) - \lambda(\lambda_1(t))|\},
\end{aligned}$$

where the second equality follows from the property $\lambda_1 \in \Lambda$ and the third from $\lambda_1^{-1} \in \Lambda$. By the same argument, there exists $\lambda_2 \in \Lambda$ such that

$$\sup_{0\le t\le T} |Y_{\lambda_1(t)} - Z_{\lambda_2(\lambda_1(t))}| + \sup_{0\le t\le T} |\lambda_1(t) - \lambda_2(\lambda_1(t))| \le \rho(Y,Z) + \varepsilon. \tag{2.1.3}$$

By (2.1.2) and (2.1.3), we get

$$\begin{aligned}
&\rho(X,Y) + \rho(Y,Z) \\
\ge &\sup_{0\le t\le T} |X_t - Y_{\lambda_1(t)}| + \sup_{0\le t\le T} |t - \lambda_1(t)| \\
&+ \sup_{0\le t\le T} |Y_{\lambda_1(t)} - Z_{\lambda_2(\lambda_1(t))}| + \sup_{0\le t\le T} |\lambda_1(t) - \lambda_2(\lambda_1(t))| - 2\varepsilon
\end{aligned}$$

$$\geq \sup_{0\leq t\leq T} |X_t - Z_{\lambda_2(\lambda_1(t))}| + \sup_{0\leq t\leq T} |t - \lambda_2(\lambda_1(t))| - 2\varepsilon$$

$$\geq \rho(X, Z) - 2\varepsilon,$$

where the last inequality follows from $\lambda_2 \circ \lambda_1 \in \Lambda$ and definition of the Skorokhod distance ρ. The result follows from taking $\varepsilon \to 0$. ■

Proposition 2.1.1 $\rho(N'(n),\ N) \xrightarrow{\mathbb{P}} 0$ *as* $n \to \infty$.

Proof. By the definition of $N'_t(n)$, we know that $N'_t(n)$, $\overline{N}_t(n)$ and N_t are all cádlág functions, then

$$\rho(N'(n),\ N) \leq \rho(N'(n),\ \overline{N}(n)) + \rho(\overline{N}(n),\ N),$$

by Lemma 2.1.2. For any $\delta > 0$, the following estimate holds.

$$\begin{aligned}\mathbb{P}(\rho(N'(n),\ N) > \delta) \leq & \mathbb{P}(\rho(N'(n),\ \overline{N}(n)) > \delta/2) + \mathbb{P}(\rho(\overline{N}(n),\ N) > \delta/2) \\ \to & 0,\ \text{as } n \to \infty,\end{aligned}$$

by Lemma 2.1.1. ■

The next proposition verifies that a second required condition of Corollary 6 in Conquet and Toldo(2007) is satisfied by the constructed approximation scheme $\{N'(n)\}_{n\in\mathbb{N}}$.

Proposition 2.1.2 $\{N'(n)\}_{n\in\mathbb{N}}$ *satisfies the Aldous's criterion for tightness.*

Proof. For any $\varepsilon > 0$, $\delta > 0$, $n \in \mathbb{N}$, and $\sigma,\ \tau \in \mathcal{S}_{0,T}^{N'(n)}$ satisying $\sigma \leq \tau \leq \sigma + \delta$, we have that

$$\begin{aligned}&\mathbb{P}(|N'_\tau(n) - N'_\sigma(n)| > \varepsilon) \\ \leq & \mathbb{P}(|N'_\tau(n) - \overline{N}_\tau(n)| > \varepsilon/3) + \mathbb{P}(|N'_\sigma(n) - \overline{N}_\sigma(n)| > \varepsilon/3) + \mathbb{P}(|\overline{N}_\tau(n) - \overline{N}_\sigma(n)| > \varepsilon/3)\end{aligned}$$

By Lemma 2.1.1 (2), it follows that the first two terms on the right hand side of last inequality converge to 0 as $n \to \infty$.

Let $t_\tau(n)$ be $\max\{t_j(n) : \ t_j(n) \leq \tau, \ j = 0, 1, 2, ..., N(n)\}$ and let $t_\sigma(n)$ be $\max\{t_j(n) : \ t_j(n) \leq \sigma, \ j = 0, 1, 2, ..., N(n)\}$, for $n \in \mathbb{N}$. Hence, $|t_\tau(n) - t_\sigma(n)| \leq |\tau - \sigma| + \Delta t(n)$. Since $\sigma \leq \tau \leq \sigma + \delta$, $|t_\tau(n) - t_\sigma(n)| \leq \delta + \Delta t(n)$. By the definition of $\overline{N}_t(n)$, we obtain $\overline{N}_\tau(n) = N_{t_\tau(n)}$ and $\overline{N}_\sigma(n) = N_{t_\sigma(n)}$. Thus,

$$\begin{aligned}
\mathbb{P}(|\overline{N}_\tau(n) - \overline{N}_\sigma(n)| > \varepsilon/3) =& \mathbb{P}(|N_{t_\tau(n)} - N_{t_\sigma(n)}| > \varepsilon/3) \\
=& \mathbb{P}(|N_{t_\tau(n) - t_\sigma(n)}| > \varepsilon/3) \\
\leq& \mathbb{P}(\sup_{0 \leq s \leq \delta + \Delta t(n)} |N_s| > \varepsilon/3) \\
\leq& \mathbb{P}(N_{\delta + \Delta t(n)} \neq 0) \\
=& 1 - e^{-\lambda(\delta + \Delta t(n))} \\
\to& 0,
\end{aligned}$$

as $\delta \downarrow 0$ and $n \to \infty$, where λ is the intensity of the Poisson process N and the second equality follows from the strong Markov property.

Therefore,

$$\lim_{\delta \downarrow 0} \limsup_{n \to \infty} \sup_{\tau, \sigma \in \mathcal{S}_{0,T}^{N'(n)}, \sigma \leq \tau \leq \sigma + \delta} \mathbb{P}(|N'_\tau(n) - N'_\sigma(n)| \geq \epsilon) = 0,$$

as required. ■

Proposition 2.1.3 *If $(\gamma^n)_{n \in \mathbb{N}}$ is a sequence of continuous bounded functions on $[0, T] \times \mathbb{R}$ which uniformly converges to the continuous bounded function γ on $[0, T] \times \mathbb{R}$, then,*

$$\operatorname*{ess\,sup}_{\tau \in \mathcal{S}_{0,T}^{N'(n)}} \mathbb{E}(\gamma^n(\tau, N'_\tau(n))) \to \operatorname*{ess\,sup}_{\tau \in \mathcal{S}_{0,T}^{N}} \mathbb{E}(\gamma(\tau, N_\tau)), \ \textit{as } n \to \infty.$$

Proof. Proposition 2.1.1 and 2.1.2 give two conditions required by Corollary 6 of Conquet and Toldo (2007). The last condition we have to get is that $\mathbb{F}^{N'(n)} \subset \mathbb{F}^N$, for all $n \in \mathbb{N}$, which is true for our construction. Hence, the convergence result follows directly by Corollary 6 of Conquet and Toldo (2007). ■

2.2 Preliminary Results For Lévy Processes

This section is a review of basic theory about Lévy process. Firstly, we introduce the Lévy-Khintchine form and definition of Lévy process.

Definition 2.2.1 *Let $\mathcal{M}_1(\mathbb{R}^d)$ denote the set of all Borel probability measures on $\mathbb{R}^d$. The convolution of two probability measures is as follows:*

$$(\mu_1 * \mu_2)(A) = \int_{\mathbb{R}^d} \mu_1(A - x)\mu_2(dx)$$

for each $\mu_i \in \mathcal{M}_1(\mathbb{R}^d)$, $i = 1, 2$, and each $A \in \mathcal{B}(R)$, where $A - x = \{y - x, y \in A\}$.

Definition 2.2.2 *Let $\mu \in \mathcal{M}_1(\mathbb{R}^d)$. If there exists a measure $\nu \in \mathcal{M}_1(\mathbb{R}^d)$ such that $\mu = \nu * \cdots * \nu$ (n times), we say μ has a convolution nth root.*

Definition 2.2.3 *A random variable X is infinitely divisible if, for all $n \in \mathbb{N}$, there exist i.i.d. random variables $Y_1^{(n)}, \cdots, Y_n^{(n)}$ such that $X = Y_1^{(n)} + \cdots + Y_n^{(n)}$ in distribution.*

Definition 2.2.4 *Let X be a random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and taking values in $\mathbb{R}^d$ with probability law p_X. Its characteristic function is defined as*

$$\phi_X(u) = \mathbb{E}(e^{i(u,X)}) = \int_\Omega e^{i(u,X(\omega))}\mathbb{P}(d\omega) = \int_{\mathbb{R}^d} e^{i(u,y)}p_X(dy),$$

for each $u \in \mathbb{R}^d$. More generally, if $\mu \in \mathcal{M}_1(\mathbb{R}^d)$, then $\phi_\mu(u) = \int_{\mathbb{R}^d} e^{i(u,y)}\mu(dy)$.

Proposition 2.2.1 *Let X be a random variable taking values in $\mathbb{R}^d$ with law μ_X. The following are equivalent:*

(1) X is infinitely divisible;

(2) μ_X has a convolution nth root that is itself the law of a random variable, for each $n \in \mathbb{N}$;

(3) ϕ_X has an nth root that is itself the characteristic function of a random variable, for each $n \in \mathbb{N}$.

Definition 2.2.5 *Let ν be a Borel measure defined on $\mathbb{R}^d - \{0\} = \{x \in \mathbb{R}^d, x \neq 0\}$. We say that it is a Lévy measure if*

$$\int_{\mathbb{R}^d - \{0\}} (|y|^2 \wedge 1)\nu(dy) < \infty.$$

Theorem 2.2.1 *(Lévy-Khintchine form) Let μ be a Borel probability measure on $\mathbb{R}^d$. μ is infinitely divisible if there exists a vector $b \in \mathbb{R}^d$, a positive definite symmetric $d \times d$ matrix A and a Lévy measure ν on $\mathbb{R}^d - \{0\}$ such that, for all $u \in \mathbb{R}^d$,*

$$\phi_\mu(u) = exp\{i(b,u) - \frac{1}{2}(u, Au) + \int_{\mathbb{R}^d - \{0\}} [e^{i(u,y)} - 1 - i(u,y)\chi_{B_1(0)}(y)]\nu(dy)\}.$$

Conversely, any mapping of the above form is the characteristic function of an infinitely divisible probability measure on $\mathbb{R}^d$.

Definition 2.2.6 *Let $X = (X(t), t \geq 0)$ be a stochastic process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We say that X is a Lévy process if:*

(L1) $X(0) = 0$ (a.s);

(L2) X has independent and stationary increments, i.e., $X(t) - X(s) = X(t-s)$ in distribution and $X(t) - X(s)$ is independent of $X(s)$ for any $0 \leq s < t$;

(L3) X is stochastocally continuous, i.e. for all $a > 0$ and for all $s \geq 0$

$$\lim_{t \to s} \mathbb{P}(|X(t) - X(s)| > a) = 0.$$

Theorem 2.2.2 *(Proposition 1.3.1 of Applebaum (2004)) If X is a Lévy process, then $X(t)$ is infinitely divisible for each $t \geq 0$.*

Theorem 2.2.3 *((1.18) of Applebaum (2004) page 42) If X is a Lévy process, then there exists a vector $\gamma \in \mathbb{R}^d$, a positive definite symmetric $d \times d$ matrix A and a Lévy measure μ on $\mathbb{R}^d - \{0\}$ such that, for each $u \in \mathbb{R}^d$, $t \geq 0$,*

$$\mathbb{E}(e^{i(u,X(t))}) = \exp(t\{i(\gamma, u) - \frac{1}{2}(u, Au) + \int_{\mathbb{R}^d - \{0\}} [e^{i(u,y)} - 1 - i(u,y)\chi_{B_1(0)}(y)]\mu(dy)\}),$$

where $i(b,u) - \frac{1}{2}(u, Au) + \int_{\mathbb{R}^d - 0} [e^{i(u,y)} - 1 - i(u,y)\chi_{B_1(0)}(y)]\mu(dy)$ is called the Lévy symbol of the Lévy process X. (γ, A, μ) is called the Lévy triplet.

Next, let us recall the strong Markove property.

Definition 2.2.7 *Let $X = (X_t,\ t \geq 0)$ be an adapted process defined on a filtered probability space that also satisfies the integrability requirement $\mathbb{E}(|X_t|) < \infty$ for all $t \geq 0$. We say that it is a martingale if, for all $0 \leq s < t < \infty$,*

$$\mathbb{E}(X_t|\mathcal{F}_s) = X_s,\ \ a.s..$$

Proposition 2.2.2 *(Doob's martingale inequality) If $(X(t), t \geq 0)$ is a positive martingale, then, for any $p > 1$,*

$$\mathbb{E}(\sup_{0\leq s\leq t} X(s)^p) \leq q^p \mathbb{E}(X(t)^p),$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 2.2.4 *If X is a Lévy process with càdlàg paths, then its augmented natural filtration is right continuous.*

Theorem 2.2.5 *(Strong Markov property, i.e., theorem 2.2.11 of Applebaum(2004)) Let $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ be a filtration. Assume that $X = (X_t,\ t \geq 0)$ is a $\mathcal{F}_t-$adapted Lévy process and that τ is a $\mathcal{F}_t-$stopping time. Let $X_\tau(t) = X_{\tau+t} - X_\tau$. Then, on $(\tau < \infty)$:*

(1) X_τ is a Lévy process that is independent of $\mathcal{F}_\tau$;

(2) for each $t \geq 0$, $X_\tau(t)$ has the same law as X_t;

(3) X_τ has càdlàg paths and is $\mathcal{F}_{\tau+t}-$adapted.

At last, some preliminary results about the jumps of Lévy process are listed as follows.

Theorem 2.2.6 *(Lemma 2.3.2 of Applebaum(2004)) If X is a Lévy process, then for any fixed $t > 0$, $\Delta X(t) = X(t) - X(t^-) = 0$ (a.s.), where $X(t^-)$ is the left limit of X at t.*

Definition 2.2.8 *Let $\mathcal{B}(\mathbb{R}^d - \{0\})$ be the Borel $\sigma-$algebra on $\mathbb{R}^d - \{0\}$. We say that $A \in \mathcal{B}(\mathbb{R}^d - \{0\})$ is bounded below if $0 \notin \overline{A}$.*

Definition 2.2.9 *Let $L = (L_t,\ t \geq 0)$ be a Lévy process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For any $A \in \mathcal{B}(\mathbb{R}^d - \{0\})$, we define random variables on Ω: for any $t \geq 0$ and $0 \leq s_1 < s_2$,*

$$N(t,\ A)(\omega) = \sharp\{0 \leq s \leq t;\ \Delta L_s(\omega) \in A\},\ \textit{for any } \omega \in \Omega,$$

$$N((s_1, s_2],\ A)(\omega) = \sharp\{s_1 < s \leq s_2;\ \Delta L_s(\omega) \in A\},\ \textit{for any } \omega \in \Omega.$$

Theorem 2.2.7 *(Theorem 2.3.5 of Applebaum (2004))*

(1) If A is bounded below, then $(N(t, A), t \geq 0)$ is a Poisson process with intensity $\mu(A) = \mathbb{E}(N(1, A))$.

(2) If $A_1, \cdots, A_m \in \mathbb{R}^d - \{0\}$ are disjoint, then the random variables $N(t, A_1), \cdots, N(t, A_m)$ are independent.

Definition 2.2.10 *Let f be a Borel measurable function from $\mathbb{R}^d$ to $\mathbb{R}^d$ and let A be bounded below. Then for each $t > 0$, $\omega \in \Omega$, we define the Poisson integral of f as a random finite sum by*

$$\int_A f(x)N(t,\ dx)(\omega) = \sum_{x \in A} f(x)N(t,\ \{x\})(\omega) = \sum_{0 \leq u \leq t} f(\Delta L_u(\omega))\chi_A(\Delta L_u(\omega)).$$

Theorem 2.2.8 *(The Lévy-Itô Decomposition) If X is a Lévy process, then there exists $b \in \mathbb{R}^d$, a Brownian motion B_A with covariance matrix A and an independent Poisson random measure N on $\mathbb{R}^+ \times (\mathbb{R}^d - \{0\})$ such that, for each $t \geq 0$,*

$$X(t) = bt + B_A(t) + \int_{|x|<1} x\widetilde{N}(t, dx) + \int_{|x| \geq 1} xN(t, dx),$$

Where $\int_{|x|<1} x\widetilde{N}(t, dx) = \int_{|x|<1} xN(t, dx) - t\int_{|x|<1} x\mu(dx),\ \mu(\cdot) = \mathbb{E}(N(1, \cdot))$.

2.3 An Approximation Scheme For A Lévy Process and Their Representatives

Let $L = (L_t,\ t \geq 0)$ be a Lévy process with càdlàg paths defined on a completed probability space $(\Omega,\ \mathcal{F},\ \mathbb{P})$, and let $\mathbb{F}^L = (\mathcal{F}_t^L)_{t \geq 0}$ be the right continuous filtration

generated by $(L_t,\ t \geq 0)$. Suppose that $\mathcal{F}_0^L$ contains all $\mathbb{P}-$null sets and that $\mathcal{F}_\infty^L = \mathcal{F}$. We assume that the Lévy triplet of $(L_t,\ t \geq 0)$ is $(\gamma,\ 0,\ \Pi)$, where $\gamma \in \mathbb{R}$ and $\Pi(\cdot)$ is a Lévy measure. This is a pure jump Lévy process. We also assume that $\int\limits_{|x|>1} |x|\Pi(dx) < \infty$.

Assume that the approximation of the Lévy process is only on the finite time interval $[0, T]$. In this present section, the tree-based approximation scheme, $L(n) = (L_t(n),\ 0 \leq t \leq T),\ n \in \mathbb{N}$, is exactly the one proposed by Maller and Szimayer (2006). The scheme is set up so similar as the binomial tree for the Black-Sholes model that the corresponding option price could be computed straightforward by the backward induction technique as in J. Neveu(1975).

Let us recall the construction of $L(n)$ in Maller, Solomon and Szimayer (2006). The number of time steps per unit time is denoted by $N(n)$, and each time period is $\Delta t(n) = 1/N(n)$ for $n \in \mathbb{N}$. The increments of $L_t(n)$ take values of integer multiples of $\Delta(n)$. The range of the increments is determined by the number of possible steps up: $m_+(n)$, and down: $m_-(n)$.

Let us choose sequences $\{\Delta(n)\} \downarrow 0$ and $\{N(n)\} \uparrow \infty$, as $n \to \infty$, satisfying

$$\liminf_{n\to\infty} \sqrt{N(n)}\Delta(n) > 0. \tag{2.3.1}$$

Suppose that the sequences $m_\pm(n),\ n = 1, 2, \cdots$, satisfy

$$\lim_{n\to\infty} \Delta(n) m_\pm(n) = \infty.$$

Denote, for all $n \in \mathbb{N}$,

$$\mathcal{M}(n) = \{-m_-(n),\ \cdots,\ -1,\ 1,\ \cdots,\ m_+(n)\},$$

$$I_k(n) = ((k - \frac{1}{2})\Delta(n), (k + \frac{1}{2})\Delta(n)], \qquad k \in \mathcal{M}(n).$$

Note that there is no 0 in $\mathcal{M}(n)$, and the union of nonoverlapping intervals $I_k(n)$ is

$$\mathcal{I}(n) = \bigcup_{k\in\mathcal{M}(n)} I_k = (-(m_-(n) + \frac{1}{2})\Delta(n),\ (m_+(n) + \frac{1}{2})\Delta(n)] \setminus (-\frac{\Delta(n)}{2},\ \frac{\Delta(n)}{2}].$$

Definition 2.3.1 *For each $n \in \mathbb{N}$, let $X(n)$ be a random variable taking values in $\{k\Delta(n),\ k \in \mathcal{M}(n) \cup 0\}$. The law of $X(n)$ is given by*

$$\mathbb{P}(X(n) = k\Delta(n)) = \frac{1}{N(n)}\Pi(I_k(n)),\ k \in \mathcal{M}(n),$$

and

$$\mathbb{P}(X(n) = 0) = 1 - \sum_{k \in \mathcal{M}(n)} \mathbb{P}(X(n) = k\Delta(n)).$$

Let $X_j(n),\ 1 \leq j \leq \lfloor N(n)T \rfloor$ be i.i.d. copies of $X(n)$. Define

$$L_t(n) = \sum_{j=1}^{\lfloor N(n)t \rfloor} (X_j(n) - a(n)),$$

where $a(n) = -\frac{\gamma}{N(n)} + \mathbb{E}(X(n)1_{\{|X(n)| \leq 1\}}) + b(n)$, $b(n) = o(1/N(n))$, $n \in \mathbb{N}$.

Proposition 2.3.1 *For the processes $L(n) = (L_t(n),\ 0 \leq t \leq T)$, $n \in \mathbb{N}$ and $L = (L_t,\ 0 \leq t \leq T)$ defined above, there exist $\mathbb{D}[0,T]-$valued random variables $\widehat{L}(n),\ n \in \mathbb{N}$ and $\widehat{L}$ defined on a common (complete) probability space $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathbb{P}})$ such that*

$$\widehat{L} \stackrel{\mathfrak{L}}{=} L, \quad \widehat{L}(n) \stackrel{\mathfrak{L}}{=} L(n),\ n \in \mathbb{N},$$

and

$$\widehat{L}(n)(\omega) \to \widehat{L}(\omega) \text{ under the } J_1 - \text{topology, for every } \omega \in \widehat{\Omega}.$$

Proof. By Theorem 3.1 of Maller, Solomon and Szimayer (2006), $L(n) \stackrel{\mathfrak{L}}{\to} L$ in $\mathbb{D}[0,T]$. Note that $\mathbb{D}[0,T]$ is a Polish space under the J_1-topology and is separable. By the Skorokhod representation theorem, there exist $\mathbb{D}[0,T]-$valued random variables $\widehat{L}(n),\ n \in \mathbb{N}$ and $\widehat{L}$ defined on a common (complete) probability space $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathbb{P}})$ such that

$$\widehat{L}(n) \stackrel{\mathfrak{L}}{=} L(n),\ n \in \mathbb{N}, \quad \widehat{L} \stackrel{\mathfrak{L}}{=} L$$

and

$$\widehat{L}(n)(\omega) \to \widehat{L}(\omega) \text{ under the } J_1 - \text{topology, for each } \omega \in \widehat{\Omega}.$$

■

Before we give the next theorem, we need to prove two lemmas first.

Lemma 2.3.1 *There exist random variables, $Y_j(n)$, $j = 1, 2, \cdots, \lfloor N(n)T \rfloor$, defined on $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathbb{P}})$ such that, for any $t \in [0, T]$,*

$$\widehat{L}_t(n) = \sum_{j=1}^{\lfloor N(n)t \rfloor} (Y_j(n) - a(n)),$$

where $a(n)$ is as in Definition 2.3.1.

Proof. Let $M(n) = (m_+(n) + m_-(n) + 1) \times \lfloor N(n)T \rfloor$ for each $n \in \mathbb{N}$. By the definition of $L_t(n)$, we get $L_\cdot(n)$ has $M(n)$ step function style paths, denoted by $f_1(t), f_2(t), \cdots, f_{M(n)}(t)$, $t \in [0, T]$. Then, $\sum_{l=1}^{M(n)} \mathbb{P}(L_\cdot(n) = f_l) = 1$. By Proposition 2.3.1 with $\widehat{L}(n) \overset{\mathfrak{L}}{=} L(n)$ and $\widehat{\mathbb{P}}(\widehat{L}_\cdot(n) = f_l) = \mathbb{P}(L_\cdot(n) = f_l)$ for any $l = 1, 2, \cdots, M(n)$, we have

$$\sum_{l=1}^{M(n)} \widehat{\mathbb{P}}(\widehat{L}_\cdot(n) = f_l) = \sum_{l=1}^{M(n)} \mathbb{P}(L_\cdot(n) = f_l) = 1.$$

That is, $\widehat{\mathbb{P}}(\widehat{L}_\cdot(n) \in \{f_1, f_2, \cdots, f_{M(n)}\}) = 1$. Hence, the paths of $\widehat{L}(n)$ are of step function style with jumps occurring only at the grid points $j\Delta t(n)$, $j = 1, 2, \cdots, \lfloor N(n)T \rfloor$ with probability 1. Therefore, we have

$$\widehat{L}_t(n) = \sum_{j=1}^{\lfloor N(n)t \rfloor} Z_j(n),$$

where $Z_j(n)$ are random variables defined on $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathbb{P}})$ representing the jumps of $\widehat{L}_\cdot(n)$ occurring at the grid point $j\Delta t(n)$, $j = 1, 2, \cdots, \lfloor N(n)T \rfloor$. Let $Y_j(n) = Z_j(n) + a(n)$. Hence, the required identity is obtained. ■

Lemma 2.3.2 *For each $n \in \mathbb{N}$, $j = 1, 2, \cdots, \lfloor N(n)T \rfloor$,*

$$X_j(n) \overset{D}{=} Y_j(n).$$

Proof. Let $\Delta f_l(j\Delta t(n))$ be the jump of function f_l occurring at $j\Delta t(n)$, for any $j = 1, 2, \cdots, \lfloor N(n)T \rfloor$ and $l = 1, 2, \cdots, M(n)$. By the definitions of $X_j(n)$ and $Y_j(n)$ and

arguments in Lemma 2.3.1, we get

$$\begin{aligned}
&\mathbb{P}(X_j(n) = k\Delta(n)) \\
=&\sum_{l=1}^{M(n)} \mathbb{P}(L_\cdot(n) = f_l) 1_{\{\Delta f_l(j\Delta t(n)) = k\Delta(n) - a(n)\}} \\
=&\sum_{l=1}^{M(n)} \widehat{\mathbb{P}}(\widehat{L}_\cdot(n) = f_l) 1_{\{\Delta f_l(j\Delta t(n)) = k\Delta(n) - a(n)\}} \\
=&\widehat{\mathbb{P}}(Y_j(n) = k\Delta(n)),
\end{aligned}$$

for any $k \in \mathcal{M}(n) \cup \{0\}$. Thus the lemma follows. ■

Proposition 2.3.2 $\mathbb{F}^{\widehat{L}(n)} \underset{\rightarrow}{\omega} \mathbb{F}^{\widehat{L}}$ *as* $n \to \infty$.

Proof. Proposition 2 of Conquet, Mémin and Słominski (2001) states that if the sequence of càdlàg processes, $(\widehat{L}(n), n \in \mathbb{N})$, converges to a càdlàg process, $\widehat{L}$, in probability under the J_1-topology and $\widehat{L}(n)$ has independent increments for each $n \in \mathbb{N}$, then $\mathbb{F}^{\widehat{L}(n)} \xrightarrow{w} \mathbb{F}^{\widehat{L}}$. By Proposition 2.3.1, $\widehat{L}(n)$, for all $n \in \mathbb{N}$, and $\widehat{L}$ are all càdlàg processes and $\widehat{L}(n)(\omega) \to \widehat{L}(\omega)$ under the $J_1 -$ topology, for each $\omega \in \widehat{\Omega}$. Therefore, in order to prove $\mathbb{F}^{\widehat{L}(n)} \xrightarrow{w} \mathbb{F}^{\widehat{L}}$, we only need to show that $\widehat{L}(n)$ has independent increments for each $n \in \mathbb{N}$.

By Lemma 2.3.2, $Y_j(n) \overset{D}{=} X_j(n)$, for all $j = 1, 2, \cdots, \lfloor N(n)T \rfloor$. Hence, $(Y_j(n))_{j=1,2,\cdots,\lfloor N(n)T \rfloor}$ are identically distributed and

$$\begin{aligned}
&\widehat{\mathbb{P}}(Y_i(n) = k_1\Delta(n), Y_j(n) = k_2\Delta(n)) \\
=&\sum_{l=1}^{M(n)} \widehat{\mathbb{P}}(\widehat{L}_\cdot(n) = f_l) 1_{\{\Delta f_l(i\Delta t(n)) = k_1\Delta(n) - a(n)\}} 1_{\{\Delta f_l(j\Delta t(n)) = k_2\Delta(n) - a(n)\}} \\
=&\sum_{l=1}^{M(n)} \mathbb{P}(L_\cdot(n) = f_l) 1_{\{\Delta f_l(i\Delta t(n)) = k_1\Delta(n) - a(n)\}} 1_{\{\Delta f_l(j\Delta t(n)) = k_2\Delta(n) - a(n)\}} \\
=&\mathbb{P}(X_i(n) = k_1\Delta(n), X_j(n) = k_2\Delta(n)) \\
=&\mathbb{P}(X_i(n) = k_1\Delta(n))\mathbb{P}(X_j(n) = k_2\Delta(n)) \\
=&\widehat{\mathbb{P}}(Y_i(n) = k_1\Delta(n))\widehat{\mathbb{P}}(Y_j(n) = k_2\Delta(n)),
\end{aligned}$$

for any $i \neq j,\ 1 \leq i, j \leq \lfloor N(n)T \rfloor$ and $k_1,\ k_2 \in \mathcal{M}(n) \cup \{0\}$, where the first and the third equality follows from the definition, the second from Proposition 2.3.1, the fourth from the i.i.d. property of $(X_j(n))_{j=1,2,\cdots,\lfloor N(n)T \rfloor}$ and the last from Lemma 2.3.2. Hence, $Y_j(n),\ j = 1, 2, \cdots, \lfloor N(n)T \rfloor$ are mutually independent. Therefore, $(Y_j(n))_{j=1,2,\cdots,\lfloor N(n)T \rfloor}$ are i.i.d..

Note that $\widehat{L}_t(n) - \widehat{L}_s(n) = \sum_{j=\lfloor N(n)s \rfloor+1}^{\lfloor N(n)t \rfloor} Y_j(n)$ and $\widehat{L}_s(n) = \sum_{j=1}^{\lfloor N(n)s \rfloor} (Y_j(n) - a(n))$.

$$
\begin{aligned}
&\widehat{\mathbb{P}}((\widehat{L}_t(n) - \widehat{L}_s(n)) \cdot \widehat{L}_s(n)) \\
=&\widehat{\mathbb{P}}\left(\sum_{j=\lfloor N(n)s \rfloor+1}^{\lfloor N(n)t \rfloor} Y_j(n) \cdot \sum_{j=1}^{\lfloor N(n)s \rfloor} (Y_j(n) - a(n))\right) \\
=&\widehat{\mathbb{P}}\left(\sum_{j=\lfloor N(n)s \rfloor+1}^{\lfloor N(n)t \rfloor} Y_j(n)\right) \cdot \widehat{\mathbb{P}}\left(\sum_{j=1}^{\lfloor N(n)s \rfloor} (Y_j(n) - a(n))\right) \\
=&\widehat{\mathbb{P}}(\widehat{L}_t(n) - \widehat{L}_s(n)) \cdot \widehat{\mathbb{P}}(\widehat{L}_s(n)),
\end{aligned}
$$

where the second equality is by the mutually independence of $Y_j(n),\ j = 1, 2, \cdots, \lfloor N(n)T \rfloor$. Hence, $(\widehat{L}_t(n),\ t \in [0, T])$ has independent increments for all $n \in \mathbb{N}$. Therefore, our result, $\mathbb{F}^{\widehat{L}(n)} \xrightarrow{w} \mathbb{F}^{\widehat{L}}$, follows from Proposition 2 of Conquet, Mémin and Słominski (2001). ■

Next, I start to prove that the new approximation scheme, $\widehat{L}(n)$, satisfies the Aldous' criterion for tightness.

Let $\delta > 0$ and $\sigma,\ \tau \in \mathcal{S}_{0,T}^{\widehat{L}(n)}$ satisfying $\sigma \leq \tau \leq \sigma + \delta$. By the construction of $\widehat{L}_t(n)$ and a similar argument as in (A.7) of Maller, Solomon and Szimayer (2006),

$$
\begin{aligned}
&\mathbb{E}|\widehat{L}_\tau(n) - \widehat{L}_\sigma(n)| \\
=&\mathbb{E}\left|\sum_{j=\lfloor N(n)\sigma \rfloor+1}^{\lfloor N(n)\tau \rfloor} (Y_j(n) - a(n))\right| \\
=&\mathbb{E}\left|\sum_{j=\lfloor N(n)\sigma \rfloor+1}^{\lfloor N(n)\tau \rfloor} [Y_j(n) - \mathbb{E}(Y_1(n)1_{\{|Y_1(n)| \leq 1\}}) + \frac{\gamma}{N(n)} - b(n)]\right|
\end{aligned}
$$

$$\leq \mathbb{E}\left|\sum_{j=\lfloor N(n)\sigma\rfloor+1}^{\lfloor N(n)\tau\rfloor}[Y_j(n)-\mathbb{E}(Y_1(n)1_{\{|Y_1(n)|\leq 1\}})]\right| + \mathbb{E}\left|\sum_{j=\lfloor N(n)\sigma\rfloor+1}^{\lfloor N(n)\tau\rfloor}[\frac{\gamma}{N(n)}-b(n)]\right|.$$

Define $I = \mathbb{E}\left|\sum_{j=\lfloor N(n)\sigma\rfloor+1}^{\lfloor N(n)\tau\rfloor}[Y_j(n)-\mathbb{E}(Y_1(n)1_{\{|Y_1(n)|\leq 1\}})]\right|$ and $II = \mathbb{E}\left|\sum_{j=\lfloor N(n)\sigma\rfloor+1}^{\lfloor N(n)\tau\rfloor}[\frac{\gamma}{N(n)}-b(n)]\right|$. Then we have $\mathbb{E}|\widehat{L}_\tau(n)-\widehat{L}_\sigma(n)| \leq I + II$.

Lemma 2.3.3 *Let $\delta > 0$ and $\sigma,\ \tau \in \mathcal{S}_{0,T}^{\widehat{L}(n)}$ satisfying $\sigma \leq \tau \leq \sigma+\delta$. Then:*

(1) $\lfloor N(n)\tau\rfloor - \lfloor N(n)\sigma\rfloor \leq N(n)\delta + 1$;

(2) $II \leq C_0\delta + \frac{C_0}{N(n)}$ for some constant $C_0 > 0$.

Proof. (1) Let $0 \leq \varepsilon_1, \varepsilon_2 < 1$ be the numbers such that $\lfloor N(n)\tau\rfloor = N(n)\tau - \varepsilon_1$ and $\lfloor N(n)\sigma\rfloor = N(n)\sigma - \varepsilon_2$. Then we have $0 \leq |\varepsilon_1 - \varepsilon_2| < 1$ and

$$\begin{aligned}
\lfloor N(n)\tau\rfloor - \lfloor N(n)\sigma\rfloor =& (N(n)\tau-\varepsilon_1)-(N(n)\sigma-\varepsilon_2)\\
=& N(n)(\tau-\sigma)-(\varepsilon_1-\varepsilon_2)\\
\leq& N(n)(\tau-\sigma)+1\\
\leq& N(n)\delta+1.
\end{aligned}$$

$$\begin{aligned}
(2)\ II = \mathbb{E}\left|\sum_{j=\lfloor N(n)\sigma\rfloor+1}^{\lfloor N(n)\tau\rfloor}[\frac{\gamma}{N(n)}-b(n)]\right| \leq& (\frac{|\gamma|}{N(n)}+|b(n)|)\mathbb{E}(\lfloor N(n)\tau\rfloor-\lfloor N(n)\sigma\rfloor)\\
\leq& (\frac{|\gamma|}{N(n)}+|b(n)|)(N(n)\delta+1)\\
=& \delta|\gamma|+\frac{|\gamma|}{N(n)}+|b(n)|N(n)\delta+|b(n)|\\
=& \delta|\gamma|+\frac{|\gamma|}{N(n)}+o(\delta)+o\left(\frac{1}{N(n)}\right)\\
\leq& \delta C_0+\frac{C_0}{N(n)},
\end{aligned}$$

for some constant $C_0 > |\gamma| > 0$, where the second inequality follows from part (1) and the second equality from $b(n) = o\left(\frac{1}{N(n)}\right)$. ■

Lemma 2.3.4 *Let* $I_1 = \mathbb{E}\left|\sum_{j=\lfloor N(n)\sigma\rfloor+1}^{\lfloor N(n)\tau\rfloor}[Y_j(n)1_{\{|Y_j(n)|\le 1\}} - \mathbb{E}(Y_1(n)1_{\{|Y_1(n)|\le 1\}})]\right|$, $I_2 = \mathbb{E}\left|\sum_{j=\lfloor N(n)\sigma\rfloor+1}^{\lfloor N(n)\tau\rfloor} Y_j(n)1_{\{|Y_j(n)|>1\}}\right|$. *Then* $I \le I_1 + I_2$.

Proof. By the definition of I, we get

$$\begin{aligned} I =& \mathbb{E}\left|\sum_{j=\lfloor N(n)\sigma\rfloor+1}^{\lfloor N(n)\tau\rfloor}[Y_j(n) - \mathbb{E}(Y_1(n)1_{\{|Y_1(n)|\le 1\}})]\right| \\ \le& \mathbb{E}\left|\sum_{j=\lfloor N(n)\sigma\rfloor+1}^{\lfloor N(n)\tau\rfloor}[Y_j(n)1_{\{|Y_j(n)|\le 1\}} - \mathbb{E}(Y_1(n)1_{\{|Y_1(n)|\le 1\}})]\right| + \mathbb{E}\left|\sum_{j=\lfloor N(n)\sigma\rfloor+1}^{\lfloor N(n)\tau\rfloor} Y_j(n)1_{\{|Y_j(n)|>1\}}\right| \\ =& I_1 + I_2. \end{aligned}$$

■

Lemma 2.3.5 *The following estimates hold.*

(1) $\mathbb{E}\left|Y_1(n)1_{\{|Y_1(n)|>1\}}\right| \le \frac{3}{N(n)}\left\{\int_{|x|>1}|x|\Pi(dx) + \Pi(1-\frac{\Delta(n)}{2},1]\right\}$;

(2) $\mathbb{E}(Y_1(n)1_{\{|Y_1(n)|\le 1\}})^2 \le \frac{9}{N(n)}\left[\int_{|x|\le 1}x^2\Pi(dx) + \int_{1<|x|\le 1+\frac{\Delta(n)}{2}}x^2\Pi(dx)\right]$;

(3) for all $s \in \mathbb{Z}^+$ *satisfying* $\lfloor N(n)\sigma\rfloor + s \le \lfloor N(n)T\rfloor$,

$$\mathbb{E}\left|Y_{\lfloor N(n)\sigma\rfloor+s}(n)1_{\{|Y_{\lfloor N(n)\sigma\rfloor+s}(n)|>1\}}\right| = \mathbb{E}|Y_1(n)1_{\{|Y_1(n)|>1\}}|;$$

(4) for all $s \in \mathbb{Z}^+$ *satisfying* $\lfloor N(n)\sigma\rfloor + s \le \lfloor N(n)T\rfloor$,

$$\begin{aligned} &\mathbb{E}\left[Y_{\lfloor N(n)\sigma\rfloor+s}(n)1_{\{|Y_{\lfloor N(n)\sigma\rfloor+s}(n)|\le 1\}} - \mathbb{E}(Y_1(n)1_{\{|Y_1(n)|\le 1\}})\right]^2 \\ &= \mathbb{E}[Y_1(n)1_{\{|Y_1(n)|\le 1\}} - \mathbb{E}(Y_1(n)1_{\{|Y_1(n)|\le 1\}})]^2. \end{aligned}$$

Proof. (1) By the proof of Lemma B.1 in Appendix B of Maller, Solomon and Szimayer (2006), for any $x \in I_k(n),\ k \in \mathcal{M}(n)$, we have that $|k\Delta(n) - x| \le \Delta(n)$ and $|x| \ge \frac{\Delta(n)}{2}$. By the triangular inequality, $|k\Delta(n)| - |x| \le |k\Delta(n) - x|$. Thus, $|k\Delta(n)| - |x| \le \Delta(n)$

and so $|k\Delta(n)| \leq |x| + \Delta(n) \leq |x| + 2|x| = 3|x|$. If $x \in I_k(n)$, then $k\Delta(n) - \frac{\Delta(n)}{2} < x \leq k\Delta(n) + \frac{\Delta(n)}{2}$. If $|k\Delta(n)| > 1$ i.e., $k\Delta(n) > 1$ or $k\Delta(n) < -1$, then $x > k\Delta(n) - \frac{\Delta(n)}{2} > 1 - \frac{\Delta(n)}{2}$ or $x \leq k\Delta(n) + \frac{\Delta(n)}{2} < -(1 - \frac{\Delta(n)}{2})$ correspondingly. Hence $|x| > 1 - \frac{\Delta(n)}{2}$. Therefore,

$$\begin{aligned}
\mathbb{E}|Y_1(n)1_{\{|Y_1(n)|>1\}}| =& \sum_{k\in\mathcal{M}(n),\ |k\Delta(n)|>1} |k\Delta(n)|\frac{1}{N(n)}\Pi(I_k(n)) \\
\leq& \frac{3}{N(n)} \sum_{k\in\mathcal{M}(n),\ |k\Delta(n)|>1} \int_{I_k(n)} |x|\Pi(dx) \\
\leq& \frac{3}{N(n)} \int_{|x|>1-\frac{\Delta(n)}{2}} |x|\Pi(dx) \\
=& \frac{3}{N(n)} \left[\int_{|x|>1} |x|\Pi(dx) + \int_{1-\frac{\Delta(n)}{2}<|x|\leq 1} |x|\Pi(dx)\right] \\
\leq& \frac{3}{N(n)} \left[\int_{|x|>1} |x|\Pi(dx) + \Pi(1 - \frac{\Delta(n)}{2}, 1]\right].
\end{aligned}$$

(2) This follows from (A.6) of Maller, Solomon and Szimayer (2006) and the equality after (A.6).

(3) For all $s \in \mathbb{Z}^+$ with $\lfloor N(n)\sigma\rfloor + s \leq \lfloor N(n)T\rfloor$, we have

$$\begin{aligned}
&\mathbb{E}|Y_{\lfloor N(n)\sigma\rfloor+s}(n)1_{\{|Y_{\lfloor N(n)\sigma\rfloor+s}(n)|>1\}}| \\
=& \sum_{j=0}^{\lfloor N(n)T\rfloor-s} \mathbb{E}(|Y_{j+s}(n)1_{\{|Y_{j+s}(n)|>1\}}|\,|\lfloor N(n)\sigma\rfloor = j)\mathbb{P}(\lfloor N(n)\sigma\rfloor = j) \\
=& \sum_{j=0}^{\lfloor N(n)T\rfloor-s} \mathbb{E}(|Y_{j+s}(n)1_{\{|Y_{j+s}(n)|>1\}}|)\mathbb{P}(\lfloor N(n)\sigma\rfloor = j) \\
=& \mathbb{E}(|Y_1(n)1_{\{|Y_1(n)|>1\}}|) \sum_{j=0}^{\lfloor N(n)T\rfloor-s} \mathbb{P}(\lfloor N(n)\sigma\rfloor = j) \\
=& \mathbb{E}|Y_1(n)1_{\{|Y_1(n)|>1\}}|,
\end{aligned}$$

where the second equality follows from the independence of $(\lfloor N(n)\sigma\rfloor = j)$ and $(Y_{j+s}(n)1_{\{|Y_{j+s}(n)|>1\}} = k\Delta(n))$, $k \in \mathcal{M}(n) \cup \{0\}$.

(4) For all $s \in \mathbb{Z}^+$ with $\lfloor N(n)\sigma \rfloor + s \leq \lfloor N(n)T \rfloor$, we have

$$
\begin{aligned}
&\mathbb{E}\left[Y_{\lfloor N(n)\sigma\rfloor+s}(n)1_{\{|Y_{\lfloor N(n)\sigma\rfloor+s}(n)|\leq 1\}} - \mathbb{E}(Y_1(n)1_{\{|Y_1(n)|\leq 1\}})\right]^2 \\
=&\sum_{j=0}^{\lfloor N(n)T\rfloor - s} \mathbb{E}\left([Y_{j+s}(n)1_{\{|Y_{j+s}(n)|\leq 1\}} - \mathbb{E}(Y_1(n)1_{\{|Y_1(n)|\leq 1\}})]^2 | \lfloor N(n)\sigma\rfloor = j\right)\mathbb{P}(\lfloor N(n)\sigma\rfloor = j) \\
=&\sum_{j=0}^{\lfloor N(n)T\rfloor - s} \mathbb{E}([Y_{j+s}(n)1_{\{|Y_{j+s}(n)|\leq 1\}} - \mathbb{E}(Y_1(n)1_{\{|Y_1(n)|\leq 1\}})]^2)\mathbb{P}(\lfloor N(n)\sigma\rfloor = j) \\
=&\sum_{j=0}^{\lfloor N(n)T\rfloor - s} \{\mathbb{E}[Y_{j+s}(n)1_{\{|Y_{j+s}(n)|\leq 1\}}]^2 - \mathbb{E}^2(Y_1(n)1_{\{|Y_1(n)|\leq 1\}})\}\mathbb{P}(\lfloor N(n)\sigma\rfloor = j) \\
=&\{\mathbb{E}[Y_1(n)1_{\{|Y_1(n)|\leq 1\}}]^2 - \mathbb{E}^2(Y_1(n)1_{\{|Y_1(n)|\leq 1\}})\}\sum_{j=0}^{\lfloor N(n)T\rfloor - s} \mathbb{P}(\lfloor N(n)\sigma\rfloor = j) \\
=&\mathbb{E}[Y_1(n)1_{\{|Y_1(n)|\leq 1\}} - \mathbb{E}(Y_1(n)1_{\{|Y_1(n)|\leq 1\}})]^2.
\end{aligned}
$$

■

Lemma 2.3.6 *There exist $\overline{n}_1 \in \mathbb{N}$ and a positive constant C_1 such that for $n > \overline{n}_1$,*

$$I_2 \leq 3C_1\delta + \frac{3C_1}{N(n)}.$$

Proof. Since $\Pi(1-\frac{\Delta(n)}{2}, 1] \to 0$ as $n \to \infty$, there exists $\overline{n}_1 \in \mathbb{N}$ such that $\Pi(1-\frac{\Delta(n)}{2}, 1]$ is bounded for $n > \overline{n}_1$. By our earlier assumption, $\int_{|x|>1} |x|\Pi(dx) < \infty$. Let $C_1 > 0$ be a constant such that $\int_{|x|>1} |x|\Pi(dx) + \Pi(1-\frac{\Delta(n)}{2}, 1] \leq C_1$ for $n > \overline{n}_1$. For $n > \overline{n}_1$,

$$
\begin{aligned}
I_2 =&\mathbb{E}\left|\sum_{j=\lfloor N(n)\sigma\rfloor+1}^{\lfloor N(n)\tau\rfloor} [Y_j(n)1_{\{|Y_j(n)|>1\}}]\right| \\
\leq&\mathbb{E}\left(\sum_{j=\lfloor N(n)\sigma\rfloor+1}^{\lfloor N(n)\tau\rfloor} \left|Y_j(n)1_{\{|Y_j(n)|>1\}}\right|\right) \\
\leq&\mathbb{E}\left(\sum_{j=\lfloor N(n)\sigma\rfloor+1}^{(\lfloor N(n)\sigma\rfloor+N(n)\delta+1)\wedge\lfloor N(n)T\rfloor} |Y_j(n)1_{\{|Y_j(n)|>1\}}|\right) \\
=&(N(n)\delta+1)\cdot\mathbb{E}|Y_1(n)1_{\{|Y_1(n)|>1\}}| \\
\leq&(N(n)\delta+1)\frac{3}{N(n)}\left\{\int_{|x|>1} |x|\Pi(dx) + \Pi(1-\frac{\Delta(n)}{2}, 1]\right\}
\end{aligned}
$$

$$\leq 3C_1\delta + \frac{3C_1}{N(n)},$$

where the second inequality follows from triangle inequality, the third from Lemma 2.3.3 (1), the fourth identity from Lemma 2.3.5 (3), and the fifth from Lemma 2.3.5 (1). Thus the result follows. ■

Lemma 2.3.7 *There exists a positive constant C_2 such that*

$$I_1 \leq \left\{18C_2(\delta + \frac{1}{N(n)})\right\}^{\frac{1}{2}}.$$

Proof. Note that $\int\limits_{|x|\leq 1} x^2\Pi(dx) + \int\limits_{1<|x|\leq 1+\frac{\Delta(n)}{2}} x^2\Pi(dx)$ is bounded because of the definition of Lévy measure. Let $C_2 > 0$ be a constant such that $\int\limits_{|x|\leq 1} x^2\Pi(dx) + \int\limits_{1<|x|\leq 1+\frac{\Delta(n)}{2}} x^2\Pi(dx) \leq C_2$. We have the following estimates.

$$\begin{aligned}
I_1^2 =& \left(\mathbb{E}|\sum_{j=\lfloor N(n)\sigma\rfloor+1}^{\lfloor N(n)\tau\rfloor} [Y_j(n)1_{\{|Y_j(n)|\leq 1\}} - \mathbb{E}(Y_1(n)1_{\{|Y_1(n)|\leq 1\}})]|\right)^2 \\
\leq& \mathbb{E}\left(\sum_{j=\lfloor N(n)\sigma\rfloor+1}^{\lfloor N(n)\tau\rfloor} [Y_j(n)1_{\{|Y_j(n)|\leq 1\}} - \mathbb{E}(Y_1(n)1_{\{|Y_1(n)|\leq 1\}})]\right)^2 \\
=& (N(n)\delta+1)\mathbb{E}[Y_1(n)1_{\{|Y_1(n)|\leq 1\}} - \mathbb{E}(Y_1(n)1_{\{|Y_1(n)|\leq 1\}})]^2 \\
=& (N(n)\delta+1)\{\mathbb{E}(Y_1(n)1_{\{|Y_1(n)|\leq 1\}})^2 - (\mathbb{E}(Y_1(n)1_{\{|Y_1(n)|\leq 1\}}))^2\} \\
\leq& (N(n)\delta+1)\{\mathbb{E}(Y_1(n)1_{\{|Y_1(n)|\leq 1\}})^2 + (\mathbb{E}(Y_1(n)1_{\{|Y_1(n)|\leq 1\}}))^2\} \\
\leq& 2(N(n)\delta+1)\mathbb{E}(Y_1(n)1_{\{|Y_1(n)|\leq 1\}})^2 \\
\leq& 2(N(n)\delta+1)\cdot\frac{9}{N(n)}\left[\int_{|x|\leq 1} x^2\Pi(dx) + \int_{1<|x|\leq 1+\frac{\Delta(n)}{2}} x^2\Pi(dx)\right] \\
\leq& 18C_2(\delta + \frac{1}{N(n)}),
\end{aligned}$$

where the second inequality follows from Jensen's inequality, the third identity from Lemma 2.3.5 (4) and the seventh from Lemma 2.3.5 (2). ■

Proposition 2.3.3 *The sequence of processes $(\widehat{L}_t(n),\ 0 \leq t \leq T)_{n\in\mathbb{N}}$ satisfies the Aldous' criterion for tightness.*

Proof. By Lemma 2.3.4 and the definition of I and II, we have

$$\mathbb{E}|\widehat{L}_\tau(n) - \widehat{L}_\sigma(n)| \leq I + II \leq I_1 + I_2 + II.$$

By lemma 2.3.3 (2), 2.3.6 and 2.3.7, we have

$$\mathbb{E}|\widehat{L}_\tau(n) - \widehat{L}_\sigma(n)| \leq (C_0\delta + \frac{C_0}{N(n)}) + (3C_1\delta + \frac{3C_1}{N(n)}) + \left\{18C_2(\delta + \frac{1}{N(n)})\right\}^{\frac{1}{2}}.$$

By taking limit for $n \to \infty$, and $\delta \to 0^+$, therefore we obtain

$$\lim_{\delta\downarrow 0}\limsup_{n\to\infty} \sup_{\sigma,\ \tau\in\mathcal{S}_{0,T}^{\widehat{L}(n)},\sigma\leq\tau\leq\sigma+\delta} \mathbb{E}|\widehat{L}_\tau(n) - \widehat{L}_\sigma(n)| = 0.$$

■

Theorem 2.3.1 *Assume that* $(\gamma^n(s,x), n \in \mathbb{N})$ *is a sequence of continuous bounded functions on* $[0,T]\times\mathbb{R}$ *which uniformly converges to the continuous bounded function* $\gamma(s,x)$ *on* $[0,T]\times\mathbb{R}$. *Then*

$$\operatorname*{ess\,sup}_{\tau\in\mathcal{S}_{0,T}^{\widehat{L}(n)}} \mathbb{E}(\gamma^n(\tau,\widehat{L}_\tau(n))) \to \operatorname*{ess\,sup}_{\tau\in\mathcal{S}_{0,T}^{\widehat{L}}} \mathbb{E}(\gamma(\tau,\widehat{L}_\tau)) \text{ as } n\to\infty.$$

Proof. It is easy to see that Proposition 2.3.1, 2.3.2, 2.3.3 give the three required conditions of Theorem 1.2.1 for $\widehat{L}(n)$ and $\widehat{L}$. Hence, we obtain that

$$\operatorname*{ess\,sup}_{\tau\in\mathcal{S}_{0,T}^{\widehat{L}(n)}} \mathbb{E}(\gamma^n(\tau,\widehat{L}_\tau(n))) \to \operatorname*{ess\,sup}_{\tau\in\mathcal{S}_{0,T}^{\widehat{L}}} \mathbb{E}(\gamma(\tau,\widehat{L}_\tau)) \text{ as } n\to\infty$$

when $(\gamma^n)_{n\in\mathbb{N}}$ is a sequence of continuous bounded functions on $[0,T]\times\mathbb{R}$ which uniformly converges to a continuous bounded function γ on $[0,T]\times\mathbb{R}$. ■

Having obtained the representative $\widehat{L}_t(n)$ of the approximation $L_t(n)$, we show that the snell envelope of the discounted payoff process achieves the same value. For this goal, we use results in Jacod and Shiryaev (2003) and some technique Theorem of Lamberton and Pagès (1990).

Lemma 2.3.8 *Let $(X_t,\ t \in [0,T])$, $(Y_t,\ t \in [0,T])$ be two cádlág processes defined on probability spaces $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathbb{P}})$, respectively. Assume that X is a process satisfying $\Delta X_t = 0$ almost surely for any $t \in [0,T]$ and that $X \overset{\mathfrak{L}}{=} Y$ in $\mathbb{D}[0,T]$. Then*

$$\operatorname*{ess\,sup}_{\tau \in \mathcal{S}^X_{0,T}} \mathbb{E}(\gamma(\tau, X_\tau)) = \operatorname*{ess\,sup}_{\tau \in \mathcal{S}^Y_{0,T}} \mathbb{E}(\gamma(\tau, Y_\tau)),$$

where $\gamma(s,x)$ is a continuous bounded function on $[0,T] \times \mathbb{R}$.

Proof. Since γ is continuous, we have $\mathbb{F}^X = \mathbb{F}^{\gamma(\cdot, X_\cdot)}$ and $\mathbb{F}^Y = \mathbb{F}^{\gamma(\cdot, Y_\cdot)}$. Correspondingly, $\mathcal{S}^X_{0,T} = \mathcal{S}^{\gamma(\cdot, X_\cdot)}_{0,T}$ and $\mathcal{S}^Y_{0,T} = \mathcal{S}^{\gamma(\cdot, Y_\cdot)}_{0,T}$. Since γ is bounded,

$$\sup_{\tau \in \mathcal{S}^X_{0,T}} \mathbb{E}(\gamma(\tau, X_\tau)) < \infty; \quad \sup_{\tau \in \mathcal{S}^Y_{0,T}} \mathbb{E}(\gamma(\tau, Y_\tau)) < \infty.$$

Now we have that $(\gamma(\cdot, X_\cdot),\ \mathbb{F}^{\gamma(\cdot, X_\cdot)})$ and $(\gamma(\cdot, Y_\cdot),\ \mathbb{F}^{\gamma(\cdot, Y_\cdot)})$ are both of class D. See term (7) on page 345 of Lamberton and Pagès (1990) for the definition of class D. Let $C = (C_t,\ 0 \le t \le T)$ be the canonical process on $\mathbb{D}[0,T]$ and $\mathcal{T}$ be the set of the $\mathbb{F}^C$-stopping times. Let $Z^n = \gamma(\cdot,\ Y_\cdot)$, $n \in \mathbb{N}$, $Z = \gamma(\cdot,\ X_\cdot)$. Thus, for any $\tau \in \mathcal{T}$, $\{Z^n_{\tau \circ Z^n},\ n \in \mathbb{N}\}$ is uniformly integrable by the boundedness of γ (see Section 3.1 of Lamberton and Pagès (1990) for $\tau \circ Z^n$ in detail). Since $Y \overset{\mathfrak{L}}{=} X$ and $\Delta X_t = 0$ almost surely for any $t \in [0,T]$, which implies $J(X) = \emptyset$, $Y \overset{\mathcal{L}([0,T])}{=} X$ by Proposition 1.2.6. Since γ is continuous, we have

$$\gamma(\cdot, Y_\cdot) \overset{\mathcal{L}([0,T])}{=} \gamma(\cdot, X_\cdot), \text{i.e.,} \quad Z^n \overset{\mathcal{L}([0,T])}{\longrightarrow} Z. \tag{2.3.2}$$

By Theorem 1.2.3, we obtain that

$$\sup_{\tau \in \mathcal{S}^Z_{0,T}} \mathbb{E}(Z_\tau) \le \sup_{\tau \in \mathcal{S}^{Z^n}_{0,T}} \mathbb{E}(Z^n_\tau).$$

That is,

$$\sup_{\tau \in \mathcal{S}^X_{0,T}} \mathbb{E}(\gamma(\tau, X_\tau)) \le \sup_{\tau \in \mathcal{S}^Y_{0,T}} \mathbb{E}(\gamma(\tau, Y_\tau)).$$

By switching X and Y, let $Z^n = \gamma(\cdot,\ X_\cdot)$, $n \in \mathbb{N}$, $Z = \gamma(\cdot,\ Y_\cdot)$. Thus, we have $Z^n \overset{\mathcal{L}([0,T])}{\longrightarrow} Z$ since they are equal in finite dimensional distribution, see (2.3.3). By

Theorem 1.2.3 again, we have

$$\sup_{\tau\in\mathcal{S}^Y_{0,T}} \mathbb{E}(\gamma(\tau, Y_\tau)) \leq \sup_{\tau\in\mathcal{S}^X_{0,T}} \mathbb{E}(\gamma(\tau, X_\tau)).$$

Therefore $\sup_{\tau\in\mathcal{S}^X_{0,T}} \mathbb{E}(\gamma(\tau, X_\tau)) = \sup_{\tau\in\mathcal{S}^Y_{0,T}} \mathbb{E}(\gamma(\tau, Y_\tau))$. ■

Theorem 2.3.2 *Let $(\gamma^n(s,x), n \in \mathbb{N})$ be a sequence of continuous bounded functions on $[0,T]\times\mathbb{R}$ which uniformly converges to the continuous bounded function $\gamma(s,x)$ defined on $[0,T]\times\mathbb{R}$. Then,*

$$\operatorname*{ess\,sup}_{\tau\in\mathcal{S}^{L(n)}_{0,T}} \mathbb{E}(\gamma^n(\tau, L_\tau(n))) \to \operatorname*{ess\,sup}_{\tau\in\mathcal{S}^{L}_{0,T}} \mathbb{E}(\gamma(\tau, L_\tau)) \text{ as } n\to\infty.$$

Proof. Since L is a Lévy process, $J(L)=\emptyset$ by Theorem 2.2.6. Since $L \overset{\mathfrak{L}}{=} \widehat{L}$ and both of $\widehat{L}$ and L are cádlág processes, by Lemma 2.3.8 we obtain

$$\sup_{\tau\in\mathcal{S}^{L}_{0,T}} \mathbb{E}(\gamma(\tau, L_\tau)) = \sup_{\tau\in\mathcal{S}^{\widehat{L}}_{0,T}} \mathbb{E}(\gamma(\tau, \widehat{L}_\tau)).$$

By the arguments as those in the proof of lemma 2.3.1, both $L_t(n)$ and $\widehat{L}_t(n)$, $t\in[0,T]$, take only finitely many values,

$$k_1\Delta(n) - \lfloor N(n)t\rfloor a(n),\ k_2\Delta(n) - \lfloor N(n)t\rfloor a(n), \cdots, k_{m_n(t)}\Delta(n) - \lfloor N(n)t\rfloor a(n),$$

where $m_n(t) = (m_+(n)+m_-(n))\lfloor N(n)t\rfloor + 1$. We know that $L(n) \overset{\mathfrak{L}}{=} \widehat{L}(n)$. Hence, for each $i = 1, 2, \cdots, m_n(t)$,

$$\begin{aligned}
\mathbb{P}(L_t(n) = k_i(n)\Delta(n) - \lfloor N(n)t\rfloor a(n)) &= \sum_{l=1}^{M(n)} \mathbb{P}(L_\cdot(n) = f_l) 1_{\{f_l(t) = k_i(n)\Delta(n) - \lfloor N(n)t\rfloor a(n)\}} \\
&= \sum_{l=1}^{M(n)} \widehat{\mathbb{P}}(\widehat{L}_\cdot(n) = f_l) 1_{\{f_l(t) = k_i(n)\Delta(n) - \lfloor N(n)t\rfloor a(n)\}} \\
&= \widehat{\mathbb{P}}(\widehat{L}_t(n) = k_i(n)\Delta(n) - \lfloor N(n)t\rfloor a(n)),
\end{aligned}$$

where $M(n)$ is as in Lemma 2.3.1. Thus $L_t(n) \overset{D}{=} \widehat{L}_t(n)$ and $\gamma^n(t, L_t(n)) \overset{D}{=} \gamma^n(t, \widehat{L}_t(n))$ for every $t\in[0,T]$ by the definition of convergence in distribution. Similarly, by

taking $t = t_1, t_2, \cdots, t_m$, $m \in \mathbb{N}$, we obtain

$$\mathbb{P}(L_{t_1}(n) = k_{i_1}(n)\Delta(n) - \lfloor N(n)t_1 \rfloor a(n), \ \cdots, \ L_{t_m}(n) = k_{i_m}(n)\Delta(n) - \lfloor N(n)t_m \rfloor a(n))$$
$$= \widehat{\mathbb{P}}(\widehat{L}_{t_1}(n) = k_{i_1}(n)\Delta(n) - \lfloor N(n)t_1 \rfloor a(n), \ \cdots, \ \widehat{L}_{t_m}(n) = k_{i_m}(n)\Delta(n) - \lfloor N(n)t_m \rfloor a(n)),$$

where $k_{i_l}(n)\Delta(n) - \lfloor N(n)t_l \rfloor a(n)$ is an possible value of $L_{t_l}(n)$. Hence $\gamma^n(\cdot, L(n)) \overset{\mathcal{L}([0,T])}{=} \gamma^n(\cdot, \widehat{L}(n))$. By Theorem 1.2.3 and the same arguments with Lemma 2.3.8,

$$\operatorname*{ess\,sup}_{\tau \in \mathcal{S}_{0,T}^{L(n)}} \mathbb{E}(\gamma^n(\tau, L_\tau(n))) = \operatorname*{ess\,sup}_{\tau \in \mathcal{S}_{0,T}^{\widehat{L}(n)}} \mathbb{E}(\gamma^n(\tau, \widehat{L}_\tau(n))).$$

Therefore our result follows by Theorem 2.3.1. ■

CHAPTER 3

Convergence of American Option price processes

In this chapter, we discuss the American option price process in a Lévy process model. We wish to show that the American put option price process could be approximated in distribution by the sequence of price processes under the discrete model proposed by Maller, Solomon and Szimayer (2006). In order to show this, we introduce value function and prove that the value function defined under the Lévy process can be approximated by its discrete counterpart by using the result of chapter 2. And hence, we will show that the American style put option price processes under the approximation converge to that under the continuous time Lévy process weakly for all time $t \in [0, T]$.

3.1 An Approximation of The American Option Price Process and The Corresponding Value Functions

Assume the stock price process is given by

$$S_t = S_0 e^{L_t},\ 0 \leq t \leq T, \tag{3.1.1}$$

where L_t is the Lévy process defined in Section 2.3 with triplet $(\gamma, 0, \Pi)$ and $S_0 \in \mathbb{R}^+$ is an initial stock price, which is a random variable independent of $(L_t, 0 \leq t \leq T)$. Assume that $\mathbb{E}(S_0) < \infty$, $\mathbb{E}(e^{L_t}) < \infty$ and a discount bond with maturity $T > 0$ and unit face value is traded. Assume the instantaneous interest rate $r > 0$ is constant for all maturities. Let $g(x)$ be the payoff function. Suppose that the option is not exercised before time t. Let $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}$ be the right continuous filtration generated

by $(S_t,\ t \in [0,T])$. Let $\mathcal{S}_{s_1,s_2}$ be the set of $\mathbb{F}-$stopping times taking values in $[s_1, s_2]$. The American option price can be given as the solution to the optimal stopping problem (see Myneni(1992)): For $0 \le t \le T$,

$$\pi_t = \operatorname*{ess\,sup}_{\tau \in \mathcal{S}_{t,T}} \mathbb{E}(e^{-r(\tau-t)} g(S_\tau)|\mathcal{F}_t).$$

Using the discretization $L(n)$ illustrated in Section 2.3, a discrete approximation of the American option price process could be achieved. Similar to (3.1.1), let

$$S_t(n) = S_0(n) e^{L_t(n)}, \quad \text{for} \quad 0 \le t \le T,$$

where $S_0(n) > 0$ is the starting value of the discrete stock price process independent of $(L_t(n))_{0 \le t \le T}$, for each $n \in \mathbb{N}$. Assume that $S_0(n) \xrightarrow{D} S_0$, as $n \to \infty$. For computational convenience, we assume that $S_0(n)$ takes only finitely many values for each $n \in \mathbb{N}$. One example is that $S_0(n) = \{\overline{m}(n) \wedge \lfloor \frac{S_0}{\Delta(n)} \rfloor\}\Delta(n)$, where $\{\overline{m}(n)\Delta(n)\} \uparrow \infty$. In fact, as mentioned in the Remark 4.5 of Maller, Solomon and Szimayer (2006), $S_0(n) = S_0$, a constant, is often taken in most cases. See the VG and NIG examples in Maller, Solomon and Szimayer (2006) and the set up of of Szimayer and Maller (2007).

Let $\mathbb{F}^n = (\mathcal{F}_t^n)_{t \in [0,T]}$ be the natural filtration generated by $(S_t(n),\ 0 \le t \le T)$ and $\mathcal{S}_{s_1,s_2}(n)$ be the set of $\mathbb{F}^n-$stopping times taking values in $[s_1, s_2]$. Notice that $\mathbb{F}^n$ is right continuous. The discounted price process of the not-exercised option under the approximation, $L(n)$, is given by the Snell envelop

$$\pi_t'(n) = \operatorname*{ess\,sup}_{\tau \in \mathcal{S}_{t,T}(n)} \mathbb{E}(e^{-r(\tau-t)} g(S_\tau(n))|\mathcal{F}_t^n).$$

Here, $\pi_t'(n)$ is exactly the same as $\pi_t(n)$ defined in (4.4) of Maller, Solomon and Szimayer (2006).

We define another discrete price process, $\pi_t(n)$, which equals $\pi_t'(n)$ eventually. Let

$$\pi_t(n) = \begin{cases} \operatorname*{ess\,sup}_{\tau \in \mathcal{S}_{t,T}(n)} \mathbb{E}(e^{-r(\tau-t)} g(S_\tau(n))|\mathcal{F}_t^n), \ t = j\Delta t(n), \ j = 0, 1, \cdots, \lfloor N(n)T \rfloor \\ \pi_{j\Delta t(n)}(n), \quad j\Delta t(n) \le t < (j+1)\Delta t(n) \wedge T, \ j = 0, 1, \cdots, \lfloor N(n)T \rfloor. \end{cases}$$

The term $\pi_t(n)$ is an interim value between $\pi'_t(n)$ and π_t. It is for the convenience of our later proof.

As in Lamberton (1998) and Szimayer and Maller (2007), the option prices can be expressed by their value functions.

Definition 3.1.1 *For any $(t,x) \in [0,T] \times \mathbb{R}^+$, the value function of π_t is defined by*

$$v(t,x) = \operatorname*{ess\,sup}_{\tau \in \mathcal{S}_{0,T-t}} \mathbb{E}(e^{-r\tau} g(xS_0 e^{L_\tau})),$$

and the value function of $\pi_t(n)$ is defined by

$$v_n(t,x) = \operatorname*{ess\,sup}_{\tau \in \mathcal{S}_{0,T-t}(n)} \mathbb{E}(e^{-r\tau} g(xS_0(n) e^{L_\tau(n)})),$$

for $t = j\Delta t(n)$, $j = 0,1,2,\cdots,\lfloor N(n)T \rfloor$ and

$$v_n(t,x) = v_n(j\Delta t(n), x), \text{ for } j\Delta t(n) \le t < (j+1)\Delta t(n) \wedge T.$$

Remark 3.1.1. Notice that $\pi_t(n) = v_n(t, e^{L_t(n)})$ and $\pi_t = v(t, e^{L_t})$. By Remark 5 of Szimayer and Maller (2007), for any $t = j\Delta t(n)$, $j = 0,1,2,\cdots,\lfloor N(n)T \rfloor$, it is easy to see that the stopping time in $\mathcal{S}_{t,T}(n)$ that maximize $v_n(t,x)$ must take values on the discrete grid $[t,T] \cap \{j\Delta t(n) : j = 0,1,\cdots,\lfloor N(n)T \rfloor\}$.

3.2 Weak Convergence of The American Option Price Processes Under The Multinomial Tree Scheme

Maller, Solomon and Szimayer (2006) proved that the sequence of American put option price processes under the multinomial tree scheme, $\pi(n)$, converges to the American put option price process under the Lévy process, π, in $\mathbb{D}[0,T]$, under the Meyer-Zheng topology[see Meyer and Zheng(1984) or Mulinacci and Pratelli(1998)]. That result can not satisfy practical needs since the convergence in Mayer-Zheng topology only implies that $\pi_t(n)$ converge to π_t in distribution for t in a full Lebesque measure subset of $[0,T]$ but not every $t \in [0,T]$. In this section, we will show that

the discrete value functions converge to the continuous time value function defined in section 3.1 pointwisely. And, we will give the weak convergence result of $\pi_t(n)$ to π_t, for each $t \in [0,T]$, which complete the problem in Maller, Solomon and Szimayer (2006).

From the proofs of Proposition 2.3.1-2.3.3, Theorem 2.3.1, Lemma 2.3.8 and Theorem 2.3.2, we can see the conditions we need therein are as follows:

(1) L_t and $L_t(n)$, $n \in \mathbb{N}$ are all càdlàg processes;

(2) $L_t(n)$ has only finitely many step function style paths;

(3) $L(n) \xrightarrow{\mathfrak{L}} L$;

(4) $\Delta L_t(n) = 0$ almost surely for each $t \in [0,T]$;

(5) Jumps of $L_t(n)$, $X_j(n)$, $j = 1, 2, \cdots, \lfloor N(n)T \rfloor$, are i.i.d. with law $\mathbb{P}(X_j(n) = k\Delta(n)) = \frac{1}{N(n)}\Pi(I_k(n))$, $j = 1, 2, \cdots, \lfloor N(n)T \rfloor$, $k \in \mathcal{M}(n)$ and $\mathbb{P}(X_j(n) = 0) = 1 - \sum_{k \in \mathcal{M}(n)} \mathbb{P}(X_j(n) = k\Delta(n))$.

Let process, $R = (R_t,\ 0 \le t \le T)$, where $R_t = \ln S_0 + L_t$ for each $t \in [0,T]$. Let $R(n) = (R_t(n),\ 0 \le t \le T)$ where $R_t(n) = \ln S_0(n) + L_t(n)$ for each $t \in [0,T]$, $n \in \mathbb{N}$. By these definitions, the difference of L and R is that the initial value is changed from 0 to $\ln S_0$ and that of $L(n)$ and $R(n)$ is that the initial value is changed from 0 to $\ln S_0(n)$. Hence, R and $R(n)$ satisfy the above conditions (1), (4) and (5). We know that $L(n) \xrightarrow{\mathfrak{L}} L$, $S_0(n) \xrightarrow{D} S_0$ and S_0 is independent of L_t, $t \in [0,T]$, $S_0(n)$ is independent of $L_t(n)$, $t \in [0,T]$ for any $n \in \mathbb{N}$. Hence, $R(n) \xrightarrow{\mathfrak{L}} R$ in $\mathbb{D}[0,T]$ as $n \to \infty$. Both S_0 and $S_0(n)$, $n \in \mathbb{N}$, take only finitely many values. Thus, conditions (2) and (3) still hold for R and $R(n)$. Therefore, Proposition 2.3.1-2.3.3, Theorem 2.3.1, Lemma 2.3.8 and Theorem 2.3.2 are true for R and $R(n)$. Let us restate Theorem 2.3.2 here for R and $R(n)$. Notice that $\mathcal{S}_{0,T}^{R(n)} = \mathcal{S}_{0,T}(n)$ and that $\mathcal{S}_{0,T}^{R} = \mathcal{S}_{0,T}$.

Theorem 3.2.1 *Let $(\gamma^n(s,x), n \in \mathbb{N})$ be a sequence of continuous bounded functions on $[0,T] \times \mathbb{R}$ which uniformly converges to the continuous bounded function $\gamma(s,x)$*

defined on $[0,T]\times\mathbb{R}$. *Then*

$$\operatorname*{ess\,sup}_{\tau\in\mathcal{S}_{0,T}(n)} \mathbb{E}(\gamma^n(\tau,R_\tau(n))) \to \operatorname*{ess\,sup}_{\tau\in\mathcal{S}_{0,T}} \mathbb{E}(\gamma(\tau,R_\tau)) \text{ as } n\to\infty.$$

Theorem 3.2.2 *Suppose that the option is an American put option, i.e., the payoff function* $g(x)=(K-x)^+$, *where* K *is the strike price and* x *is the stock price when the option is exercised. Then, whenever* $\{x_n\}\to x$ *as* $n\to\infty$, *we have*

$$\lim_{n\to\infty} v_n(t,x_n)=v(t,x), \quad \text{for any } (t,x)\in[0,T]\times\mathbb{R}^+, \tag{3.2.1}$$

$$\pi_t(n)\xrightarrow{D}\pi_t, \text{ as } n\to\infty, \text{ for each } t\in[0,T], \tag{3.2.2}$$

$$\pi'_t(n)\xrightarrow{D}\pi_t, \text{ as } n\to\infty, \text{ for each } t\in[0,T]. \tag{3.2.3}$$

Remark 3.2.1: The idea we use to show (3.2.1) is similar to that used in the proof of Theorem 5.1 of Szimayer and Maller (2007). First of all, we define a sequence of functions, $\widetilde{v}_n(t,x)$ on $[0,T]\times\mathbb{R}^+$. For any $(t,x)\in[0,T]\times\mathbb{R}^+$, let $\widetilde{v}_n(t,x)=\operatorname*{ess\,sup}_{\tau\in\mathcal{S}_{0,T-t}(n)} \mathbb{E}(e^{-r\tau}g(xe^{R_\tau(n)}))$. Hence $\widetilde{v}_n(t,x_n)=\operatorname*{ess\,sup}_{\tau\in\mathcal{S}_{0,T-t}(n)} \mathbb{E}(e^{-r\tau}g(x_ne^{R_\tau(n)}))=\operatorname*{ess\,sup}_{\tau\in\mathcal{S}_{0,T-t}(n)} \mathbb{E}(\gamma^n(\tau,R_\tau(n))$ where $\gamma^n(\tau,y)=e^{-r\tau}g(x_ne^y)$. Then γ^n is continuous and bounded for $(\tau,y)\in[0,T]\times\mathbb{R}$ since g is continuous and bounded. Also write $v(t,x)=\operatorname*{ess\,sup}_{\tau\in\mathcal{S}_{0,T-t}} \mathbb{E}(\gamma(\tau,R_\tau))$, where $\gamma(\tau,y)=e^{-r\tau}g(xe^y)$. So γ is also continuous and bounded for $(\tau,y)\in[0,T]\times\mathbb{R}$.

Before we give the proof of Theorem 3.2.2, we first show a lemma and two propositions.

Lemma 3.2.1 *The sequence of continuous bounded functions* $\gamma^n(\tau,y)$ *converges to* $\gamma(\tau,y)$ *uniformly on* $(\tau,y)\in[0,T]\times\mathbb{R}$.

Proof. We give a proof for the sake of completeness. Let K be a fixed positive number. If $xe^y<K$, then there exists $\delta>0$ such that $xe^y\le K-\delta$. Since $\lim_{n\to\infty}x_n=x$, a positive number, there exists $n_1\in\mathbb{N}$ such that $|x_n-x|<\frac{\delta x}{K}$ when $n\ge n_1$. Then, $|x_ne^y-xe^y|=e^y|x_n-x|<\delta$. And so, $x_ne^y<K$ when $n\ge n_1$.

Similarly, if $xe^y>K$, there exists $n_2\in\mathbb{N}$ such that $x_ne^y>K$ for $n\ge n_2$.

If $xe^y = K$, $e^y = \frac{K}{x}$. Hence for $n \geq \max\{n_1,\ n_2\}$, we have that

$$|(K - x_n e^y)^+ - (K - xe^y)^+| = \begin{cases} |x - x_n|e^y, & xe^y < K \\ 0, & xe^y > K \\ \frac{K}{x}(x - x_n)^+, & xe^y = K. \end{cases}$$

Since $\lim_{n\to\infty} x_n = x$, for any $\varepsilon > 0$, there exists $n_3 \in \mathbb{N}$ such that $|x - x_n| < \frac{x\varepsilon}{K}$, for $n \geq n_3$. For $n \geq \max\{n_1, n_2, n_3\}$, $|(K - x_n e^y)^+ - (K - xe^y)^+| \leq \varepsilon$ uniformly for $y \in \mathbb{R}$. Therefore, $\gamma^n(\tau, y) \to \gamma(\tau, y)$ uniformly for $(\tau, y) \in [0, T] \times \mathbb{R}$ as $n \to \infty$. ■

Proposition 3.2.1 *For any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for $n > N$, $\widetilde{v}_n(t, x_n) \leq v_n(t, x_n) \leq \widetilde{v}_n(t, x_n) + \varepsilon$.*

Proof. For any $(t, x) \in [0, T] \times \mathbb{R}^+$, by the definition of $v_n(t, x)$, we could write

$$v_n(t, x) = \operatorname*{ess\,sup}_{\tau \in \mathcal{S}_{0, T-t+\rho_n(t)}(n)} \mathbb{E}(e^{-r\tau} g(xe^{R_\tau(n)})),$$

where $\rho_n(t) = t - \lfloor N(n)t \rfloor \Delta t(n)$, for any $n \in \mathbb{N}$. Clearly, $0 \leq \rho_n(t) < \Delta t(n)$.

Let $\tau_0 \in \mathcal{S}_{0, T-t+\rho_n(t)}(n)$ be the optimal stopping time of $v_n(t, x_n)$. By Remark 3.1.1,

$$\tau_0 \in [0, T - t] \cap \{j\Delta t(n),\ j = 0, 1, \cdots, \lfloor N(n)(T - t) \rfloor\}$$

and

$$v_n(t, x_n) = \mathbb{E}(e^{-r\tau_0} g(x_n e^{R_{\tau_0}(n)})).$$

Taking $\tau_1 = \tau_0 \wedge (T - t)$, then $\tau_1 \in \mathcal{S}_{0, T-t}(n)$ and $0 \leq \tau_0 - \tau_1 \leq \rho_n(t) < \Delta t(n)$. Consider that

$$\begin{aligned}
&|\mathbb{E}(e^{-r\tau_0} g(x_n e^{R_{\tau_0}(n)})) - \mathbb{E}(e^{-r\tau_1} g(x_n e^{R_{\tau_1}(n)}))| \\
\leq &|\mathbb{E}(e^{-r\tau_0} g(x_n e^{R_{\tau_0}(n)})) - \mathbb{E}(e^{-r\tau_0} g(x_n e^{R_{\tau_1}(n)}))| + |\mathbb{E}(e^{-r\tau_0} g(x_n e^{R_{\tau_1}(n)})) - \mathbb{E}(e^{-r\tau_1} g(x_n e^{R_{\tau_1}(n)}))| \\
\leq &\mathbb{E}|e^{-r\tau_0} g(x_n e^{R_{\tau_0}(n)}) - e^{-r\tau_0} g(x_n e^{R_{\tau_1}(n)})| + \mathbb{E}|e^{-r\tau_0} g(x_n e^{R_{\tau_1}(n)}) - e^{-r\tau_1} g(x_n e^{R_{\tau_1}(n)})| \\
\leq &\mathbb{E}|g(x_n e^{R_{\tau_0}(n)}) - g(x_n e^{R_{\tau_1}(n)})| + \mathbb{E}|[e^{-r\tau_0} - e^{-r\tau_1}] g(x_n e^{R_{\tau_1}(n)})|.
\end{aligned}$$

By (2.3.1), there exists $C_{inf} > 0$ and $\widehat{n}_0 \in \mathbb{N}$ such that $0 < C_{inf} \leq \Delta(n)\sqrt{N(n)}$ for $n > \widehat{n}_0$. Therefore, $N(n) \geq \frac{C_{inf}^2}{\Delta^2(n)}$ for $n > \widehat{n}_0$. By $g(x) = (K-x)^+$, $0 \leq g(\cdot) \leq K$ and the definition of $L_t(n)$, τ_0 and τ_1, we have for $n > \widehat{n}_0$,

$$\begin{aligned}
&\mathbb{E}|g(x_n e^{R_{\tau_0}(n)}) - g(x_n e^{R_{\tau_1}(n)})| \\
\leq& K\mathbb{P}(L_{\tau_0}(n) \neq L_{\tau_1}(n)) \\
\leq& K\mathbb{P}(X_{\lfloor N(n)(T-t)\rfloor}(n) 1_{\{\tau_0 \neq \tau_1\}} \neq 0) \\
\leq& K\mathbb{P}(X_{\lfloor N(n)(T-t)\rfloor}(n) \neq 0) \\
=& K \sum_{k \in \mathcal{M}(n)} \frac{1}{N(n)} \Pi(I_k(n)) \\
\leq& \frac{K}{C_{inf}^2} \sum_{k \in \mathcal{M}(n)} (\Delta(n))^2 \Pi(I_k(n)) \\
\leq& \frac{K}{C_{inf}^2} (\Delta(n))^2 \overline{\Pi}(\frac{\Delta(n)}{2}),
\end{aligned}$$

where the second inequality follows from $L_{\tau_0}(n) \neq L_{\tau_1}(n)$, which implies $\tau_0 \neq \tau_1$ and so $\tau_0 > T - t$. Hence, $\tau_0 = \lfloor N(n)(T-t)\rfloor \Delta t(n) > T - t$ and so $L_{\tau_0}(n) - L_{\tau_1}(n) = X_{\lfloor N(n)(T-t)\rfloor}(n) 1_{\{\tau_0 \neq \tau_1\}}$. Since $\Pi(\cdot)$ is a Lévy measure, $(\Delta(n))^2 \overline{\Pi}(\frac{\Delta(n)}{2}) \to 0$ as $n \to \infty$. For any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $(\Delta(n))^2 \overline{\Pi}(\frac{\Delta(n)}{2}) < \frac{\varepsilon C_{inf}^2}{2K}$ for $n \geq n_0$. Hence,

$$\mathbb{E}|g(x_n e^{R_{\tau_0}(n)}) - g(x_n e^{R_{\tau_1}(n)})| \leq \frac{\varepsilon}{2}$$

for $n \geq \max\{n_0, \widehat{n}_0\}$. Note that

$$\begin{aligned}
\mathbb{E}|[e^{-r\tau_0} - e^{-r\tau_1}] g(x_n e^{R_{\tau_1}(n)})| \leq& K\mathbb{E}|e^{-r\tau_0} - e^{-r\tau_1}| \\
=& K\mathbb{E}|e^{-r\tau_0}(1 - e^{r(\tau_0 - \tau_1)})| \\
=& K\mathbb{E}(e^{-r\tau_0}(e^{r(\tau_0-\tau_1)} - 1)) \\
\leq& K\mathbb{E}(e^{r\rho_n(t)} - 1) \\
\to& 0 \text{ as } n \to \infty.
\end{aligned}$$

Hence, there exists $n_0' \in \mathbb{N}$ such that $K\mathbb{E}(e^{r\rho_n(t)} - 1) < \frac{\varepsilon}{2}$ for $n \geq n_0'$, and so

$$\mathbb{E}|(e^{-r\tau_0} - e^{-r\tau_1}) g(x_n e^{R_{\tau_1}(n)})| \leq \frac{\varepsilon}{2}$$

for $n \geq n_0'$. Therefore, for $n > \max\{n_0, n_0', \widehat{n}_0\}$, we get

$$v_n(t, x_n) \leq \mathbb{E}(e^{-r\tau_1} g(x_n e^{R_{\tau_1}(n)})) + \varepsilon.$$

Since $\tau_1 \in \mathcal{S}_{0,T-t}(n)$, $\mathbb{E}(e^{-r\tau_1} g(x_n e^{R_{\tau_1}(n)})) \leq \widetilde{v}_n(t, x_n)$ and so,

$$v_n(t, x_n) \leq \widetilde{v}_n(t, x_n) + \varepsilon.$$

On the other hand, by the construction of $\widetilde{v}_n(t, x_n)$ and $v_n(t, x_n)$, it is easy to see that, for any $n \in \mathbb{N}$,

$$\widetilde{v}_n(t, x_n) \leq v_n(t, x_n).$$

Taking $N = \max\{n_0, n_0', \widehat{n}_0\}$, the result follows. ■

Proposition 3.2.2 *For each $t \in [0, T]$, $\lim_{n\to\infty} \pi_t(n) = \lim_{n\to\infty} \pi_t'(n)$.*

Proof. Fix $t \in [0, T]$. For each $n \in \mathbb{N}$, let $t \in [j\Delta t(n), (j+1)\Delta t(n))$. Consider that

$$\pi_t(n) = \pi_{j\Delta t(n)}(n) = \operatorname*{ess\,sup}_{\tau \in \mathcal{S}_{j\Delta t(n),T}(n)} \mathbb{E}(e^{-r(\tau - j\Delta t(n))} g(S_\tau(n)) | \mathcal{F}^n_{j\Delta t(n)}).$$

Since $\mathcal{F}^n_{j\Delta t(n)} = \mathcal{F}^n_t$,

$$\pi_t'(n) e^{-r(\Delta t(n))} \leq \pi_t'(n) e^{-r(t - j\Delta t(n))} \leq \pi_t(n).$$

Let $\pi_t(n) = \mathbb{E}(e^{-r(\sigma_0 - j\Delta t(n))} g(S_{\sigma_0}(n)) | \mathcal{F}^n_t)$, where $\sigma_0 \in \mathcal{S}_{j\Delta t(n),T}(n)$ is the optimal stopping time of $\pi_t(n)$. By Remark 3.1.1, σ_0 only takes values in $\{k\Delta t(n) | k = j, j+1, \cdots, \lfloor N(n)T \rfloor\}$. By taking $\sigma_1 = \sigma_0 \vee t$, we obtain that $\sigma_1 \in \mathcal{S}_{t,T}(n)$, $\sigma_0 \leq \sigma_1 < \sigma_0 + \Delta t(n)$ and $S_{\sigma_1}(n) = S_{\sigma_0}(n)$. Hence,

$$\begin{aligned}
\pi_t(n) e^{-r(\Delta t(n))} =& \mathbb{E}(e^{-r(\sigma_0 - j\Delta t(n) + \Delta t(n))} g(S_{\sigma_0}(n)) | \mathcal{F}^n_t) \\
\leq& \mathbb{E}(e^{-r(\sigma_1 - j\Delta t(n))} g(S_{\sigma_1}(n)) | \mathcal{F}^n_t) \\
\leq& \mathbb{E}(e^{-r(\sigma_1 - t)} g(S_{\sigma_1}(n)) | \mathcal{F}^n_t)
\end{aligned}$$

$$\leq \pi_t'(n).$$

By the Squeeze law, the result follows. ■

Proof of Theorem 3.2.2. By Lemma 3.2.1 and Theorem 3.2.1,

$$\widetilde{v}_n(t, x_n) \to v(t, x) \text{ whenever } x_n \to x \text{ as } n \to \infty. \tag{3.2.4}$$

By Proposition 3.2.1, $|v_n(t, x_n) - \widetilde{v}_n(t, x_n)| \leq \varepsilon$ when $n > N$. Hence, $v_n(t, x_n) \to v(t, x)$ whenever $x_n \to x$ as $n \to \infty$. Thus (3.2.1) is proved.

Since $L(n) \xrightarrow{\mathfrak{L}} L$ in $\mathbb{D}[0, T]$, $L_t(n) \xrightarrow{D} L_t$ as $n \to \infty$ by Proposition 1.2.5. So, $e^{L_t(n)} \xrightarrow{D} e^{L_t}$ as $n \to \infty$, for any given $t \in [0, T]$, by Proposition 1.2.6. From the Skorokhod representation theorem, it follows that there exist random variables $Z_t(n)$, $n \in \mathbb{N}$ and Z_t defined on a common probability space $(\Omega^{Z_t}, \mathcal{F}^{Z_t}, \mathbb{P}^{Z_t})$, such that $Z_t(n) \stackrel{D}{=} e^{L_t(n)}$, $Z_t \stackrel{D}{=} e^{L_t}$ and $Z_t(n) \to Z_t$ for every $\omega \in \Omega^{Z_t}$, as $n \to \infty$. By (3.2.1), we get that

$$v_n(t, Z_t(n)) \to v(t, Z_t) \text{ for every } \omega \in \Omega^{Z_t}, \text{ as } n \to \infty. \tag{3.2.5}$$

Consider that, for any fixed $t \in [0, T]$ and fixed $n \in \mathbb{N}$, $Z_t(n) \stackrel{D}{=} e^{L_t(n)}$ take only finitely many values. So, $v_n(t, Z_t(n)) \stackrel{D}{=} \pi_t(n)$. On the other hand, if $L(n) = L$ for each $n \in \mathbb{N}$, then $\widetilde{v}_n(t, x_n) = v(t, x_n)$. (3.2.4) implies that, for any $(t, x) \in [0, T] \times \mathbb{R}^+$, $v(t, x_n) \to v(t, x)$, whenever $\{x_n\} \to x$. So, the value function $v(t, x)$ is bounded and continuous with respect to x. Hence, $v(t, e^{L_t}) \stackrel{D}{=} v(t, Z_t)$, i.e., $v(t, Z_t) \stackrel{D}{=} \pi_t$, by the definition of convergence in distribution. Therefore, $\pi_t(n) \xrightarrow{D} \pi_t$, as $n \to \infty$, for any $t \in [0, T]$ by (3.2.5).

By Proposition 3.2.2, $\lim_{n\to\infty} \mathbb{E}(f(\pi_t'(n))) = \lim_{n\to\infty} \mathbb{E}(f(\pi_t(n))) = \lim_{n\to\infty} \mathbb{E}(f(\pi_t))$ for any continuous bounded function $f : \mathbb{R} \to \mathbb{R}$. Therefore, $\pi_t'(n) \xrightarrow{D} \pi_t$, as $n \to \infty$, for any $t \in [0, T]$.

Remark 3.2.1 *In the proof of Theorem 3.2.2, the continuity and boundedness of the payoff function are required. Although the payoff function of a call option is not bounded, we can modify it to be a bounded one. Let the payoff function of a modified call option is of the form*

$$g(x) = (x - K)^+ \wedge M,$$

where M is a large positive number. Then g is continuous bounded. By a similar proof as that of Theorem 3.2.2, we could get the same convergence results as those of Theorem 3.2.2 for the modified American call option.

Corollary 3.2.1 *Suppose that the option is an modified American call option with the payoff function $g(x) = (x - K)^+ \wedge M$ for a (sufficiently) large positive number M. Then we have,*

$$\lim_{n\to\infty} v_n(t, x_n) = v(t, x), \quad \textit{for any } (t, x) \in [0, T] \times \mathbb{R}^+,$$

provided $\{x_n\} \to x$ as $n \to \infty$ and

$$\pi_t(n) \xrightarrow{D} \pi_t, \ \textit{as } n \to \infty, \ \textit{for each } t \in [0, T],$$

$$\pi'_t(n) \xrightarrow{D} \pi_t, \ \textit{as } n \to \infty, \ \textit{for each } t \in [0, T].$$

Proof. From the proofs of Lemma 3.2.1, Proposition 3.2.1, 3.2.2 and Theorem 3.2.2, we only need to show $\gamma^n(\tau, y) \to \gamma(\tau, y)$ uniformly on $(\tau, y) \in [0, T] \times \mathbb{R}$.

By a very similar argument as that in Lemma 3.2.1, we have there exists $\widehat{n}_1 \in \mathbb{N}$ such that, when $n \geq \widehat{n}_1$,

$$|(x_n e^y - K)^+ \wedge M - (xe^y - K)^+ \wedge M| = \begin{cases} 0, & xe^y < K \\ \frac{K}{x}(x_n - x)^+, & xe^y = K \\ |x_n - x|e^y, & M + K > xe^y > K \\ |(x_n e^y - K)^+ \wedge M - M|, & xe^y = M + K \\ 0, & xe^y > M + K. \end{cases}$$

Therefore, $\gamma^n(\tau, y) \to \gamma(\tau, y)$ uniformly for $(\tau, y) \in [0, T] \times \mathbb{R}$ as $n \to \infty$.

$\blacksquare$

CHAPTER 4

Another Approximation Scheme

Let $L = (L_t,\ t \in [0,T])$ be the pure jump Lévy process defined in Chapter 2. In this chapter, we are setting up an approximation scheme, $(L(n), t \in [0,T])_{n\in\mathbb{N}}$, which is defined pathwisely and explicitly so that we could show that this setting also satisfies the conditions of Corollary 6 of Conquet and Toldo (2007). Therefore, we can obtain the same convergence results for Snell envelopes of the payoff processes and American option price processes under the new scheme as those stated in Theorem 3.2.1 and Theorem 3.2.2. Moreover, the convergence rate is discussed in section 4.2.

4.1 Construction of An Approximation Scheme

The idea of our construction is similar to that in Szimayer and Maller (2007) relying on the Lévy Itô decomposition. However, we give the law of the jump random variable explicitly and the probabilities are almost the same with those given in section 2.3. Hence this scheme can be seen as a model with practical meaning for the approximation scheme proposed in section 2.3 or Maller, Solomon and Szimayer (2006). Before the final approximation $L(n)$ is given, we are giving three interim approximation schemes $\overline{L}(n)$ and $\widetilde{L}(n)$.

Let sequences $N(n)$, $\Delta t(n)$, $\Delta(n)$, $m_{\pm}(n)$, $\mathcal{M}(n)$, $I_k(n),\ k \in \mathcal{M}(n)$ and $\mathcal{I}(n)$ be those in Chapter 2. The sequence of partitions is denoted by $\mathcal{P}(n) : 0 = t_0^n < t_1^n < \cdots < t^n_{\lfloor N(n)T\rfloor} \leq T$, where $t_j^n = j\Delta t(n)$, $j = 0, 1, \cdots, \lfloor N(n)T\rfloor$.

Definition 4.1.1 *For any $n \in \mathbb{N}$, $j = 1, 2, \ \cdots,\ \lfloor N(n)T\rfloor$, define random variable $X_j(n)$ as follows: if there is a unique jump with size in $\mathcal{I}(n)$ of L_t occurs in the*

subinterval $(t_{j-1}^n, t_j^n]$, *let* $X_j(n)$ *be this unique jump. Otherwise, let* $X_j(n) = 0$.

Hence, $X_j(n),\ j = 1, 2, \cdots, \lfloor N(n)T \rfloor$ are i.i.d. and the law of $X_j(n)$ is as follows:

$$\mathbb{P}(X_j(n) = 0) = 1 - \sum_{k \in \mathcal{M}(n)} \mathbb{P}(X_j(n) \in I_k(n)),$$

$$\begin{aligned}
\mathbb{P}(X_j(n) \in I_k(n)) =& \mathbb{P}(N(\Delta t(n),\ I_k(n)) = 1) \cdot \prod_{j \in \mathcal{M}(n), j \neq k} \mathbb{P}(N(\Delta t(n),\ I_j(n)) = 0), \\
=& \Delta t(n) \Pi(I_k(n)) e^{-\Delta t(n) \Pi(I_k(n))} \cdot \prod_{j \in \mathcal{M}(n), j \neq k} e^{-\Delta t(n) \Pi(I_j(n))} \\
=& \Delta t(n) \Pi(I_k(n)) e^{-\Delta t(n) \Pi(\mathcal{I}(n))},
\end{aligned}$$

where the second equality follows from Theorem 2.2.7 and the last equality from

$$\begin{aligned}
\Pi(A_1) + \Pi(A_2) =& \mathbb{E}(N(1, A_1)) + \mathbb{E}(N(1, A_2)) \\
=& \mathbb{E}(N(1, A_1) + N(1, A_2)) \\
=& \mathbb{E}(N(1, A_1 \cup A_2)) \\
=& \Pi(A_1 \cup A_2),
\end{aligned}$$

where $A_1,\ A_2$ are two disjoint Borel sets in $\mathbb{R} - \{0\}$.

Definition 4.1.2 *For any* $t \in [0, T]$, $n \in \mathbb{N}$, *let*

$$\overline{L}_t(n) = \lambda t - t \int_{\frac{\Delta(n)}{2} < |x| < 1} x \Pi(dx) + \sum_{j=1}^{\lfloor N(n)T \rfloor} X_j(n) 1_{\{\tau_j^n \leq t\}},$$

where $\tau_j^n \in (t_{j-1}^n, t_j^n]$ *is the time at which the jump* $X_j(n) \neq 0$ *occurs. If* $X_j(n) = 0$ *for some* $j \in \{1, 2, \ \cdots, \ \lfloor N(n)T \rfloor\}$, *let* $\tau_j^n = t_{j-1}^n$.

Proposition 4.1.1 *Assume that*

$$\lim_{n \to \infty} \Delta t(n) \overline{\Pi}^2 (\frac{\Delta(n)}{2}) = 0, \tag{4.1.1}$$

where $\overline{\Pi}(x) = \Pi(-\infty, -x] \cup \Pi(x, \infty)$, *for* $x > 0$. *Then*

$$\sup_{0 \leq t \leq T} |\overline{L}_t(n) - L_t| \xrightarrow{\mathbb{P}} 0, \ \ as\ n \to \infty.$$

Proof. By the Lévy-Itô decomposition,

$$\begin{aligned} L_t =& \gamma t - t\int_{|x|<1} x\Pi(dx) + \sum_{0\le s\le t} \Delta L_s 1_{\{\Delta L_s\in\mathbb{R}\setminus(\mathcal{I}(n)\cup(-\frac{\Delta(n)}{2},\frac{\Delta(n)}{2}])\}} \\ &+ \sum_{0\le s\le t} \Delta L_s 1_{\{\Delta L_s\in\mathcal{I}(n)\}} + \sum_{0\le s\le t} \Delta L_s 1_{\{\Delta L_s\in(-\frac{\Delta(n)}{2},\frac{\Delta(n)}{2}]\}}. \end{aligned}$$

By the definition of $\overline{L}_t(n)$, for any $t\in[0,T]$, $n\in\mathbb{N}$ and the construction of $X_j(n)$, for any $j=1,2,\cdots,\lfloor N(n)T\rfloor$, we have that

$$\begin{aligned} &|\overline{L}_t(n) - L_t| \\ =&|\sum_{0\le s\le t} \Delta L_s 1_{\{\Delta L_s\in\mathbb{R}\setminus(\mathcal{I}(n)\cup(-\frac{\Delta(n)}{2},\frac{\Delta(n)}{2}])\}} \\ &+ \sum_{j=1}^{\lfloor N(n)T\rfloor} \sum_{t_{j-1}^n<s\le t_j^n\wedge t} \Delta L_s 1_{\{N((t_{j-1}^n,t_j^n],\mathcal{I}(n))\ge 2\}} 1_{\{\Delta L_s\in\mathcal{I}(n)\}} \\ &+ \sum_{t_{\lfloor N(n)T\rfloor}^n<s\le t} \Delta L_s 1_{\{\Delta L_s\in\mathcal{I}(n)\}} \\ &+ \sum_{0\le s\le t} \Delta L_s 1_{\{\Delta L_s\in(-\frac{\Delta(n)}{2},\frac{\Delta(n)}{2}]\}} - t\int_{|x|\le\frac{\Delta(n)}{2}} x\Pi(dx)|. \end{aligned}$$

Hence,

$$\sup_{0\le t\le T} |\overline{L}_t(n) - L_t| \le (I) + (II) + (III) + (IV), \tag{4.1.2}$$

where

$$(I) = \sum_{0\le s\le T} |\Delta L_s| 1_{\{\Delta L_s\in\mathbb{R}\setminus(\mathcal{I}(n)\cup(-\frac{\Delta(n)}{2},\frac{\Delta(n)}{2}])\}}, \tag{4.1.3}$$

$$(II) = \sum_{j=1}^{\lfloor N(n)T\rfloor} \sum_{t_{j-1}^n<s\le t_j^n} |\Delta L_s| 1_{\{N((t_{j-1}^n,t_j^n],\mathcal{I}(n))\ge 2\}} 1_{\{\Delta L_s\in\mathcal{I}(n)\}}, \tag{4.1.4}$$

$$(III) = \sum_{t_{\lfloor N(n)T\rfloor}^n<s\le T} |\Delta L_s| 1_{\{\Delta L_s\in\mathcal{I}(n)\}}, \tag{4.1.5}$$

$$(IV) = \sup_{0\le t\le T} |\sum_{0\le s\le t} \Delta L_s 1_{\{\Delta L_s\in(-\frac{\Delta(n)}{2},\frac{\Delta(n)}{2}]\}} - t\int_{|x|\le\frac{\Delta(n)}{2}} x\Pi(dx)|. \tag{4.1.6}$$

Consider that

$$\mathbb{P}((I)>0) = \mathbb{P}(N(T,\mathbb{R}\setminus(\mathcal{I}(n)\cup(-\frac{\Delta(n)}{2},\frac{\Delta(n)}{2}])) \ge 1) = 1 - e^{-\lambda_1 T},$$

where $\lambda_1 = \mathbb{E}[N(1,\ \mathbb{R}\backslash(\mathcal{I}(n)\cup(-\frac{\Delta(n)}{2},\frac{\Delta(n)}{2}]))] = \Pi(m_+(n)\Delta(n)+\frac{\Delta(n)}{2},\infty)+\Pi(-\infty,-m_-(n)\Delta(n)-\frac{\Delta(n)}{2}]$. Since $\Pi(x,\infty)\to 0$, $\Pi(-\infty,-x]\to 0$ as $x\to\infty$, $\mathbb{P}((I)>0)\to 0$ as $n\to\infty$.

Notice that

$$\begin{aligned}
\mathbb{P}((II)>0) \le & \sum_{j=1}^{\lfloor N(n)T\rfloor} \mathbb{P}(\sum_{t_{j-1}^n<s\le t_j^n} |\Delta L_s| 1_{\{N((t_{j-1}^n,t_j^n],\mathcal{I}(n)\ge 2)\}} 1_{\{\Delta L_s\in\mathcal{I}(n)\}}>0) \\
\le & N(n)T\mathbb{P}(N(\Delta t(n),\ \mathcal{I}(n))\ge 2) \\
= & N(n)Te^{-\Delta t(n)\Pi(\mathcal{I}(n))}\sum_{k=2}^{\infty}\frac{[\Delta t(n)\Pi(\mathcal{I}(n))]^k}{k!} \\
= & N(n)Te^{-\Delta t(n)\Pi(\mathcal{I}(n))}[\Delta t(n)\Pi(\mathcal{I}(n))]^2\sum_{k=2}^{\infty}\frac{[\Delta t(n)\Pi(\mathcal{I}(n))]^{k-2}}{k!} \\
= & N(n)Te^{-\Delta t(n)\Pi(\mathcal{I}(n))}[\Delta t(n)\Pi(\mathcal{I}(n))]^2\sum_{k=0}^{\infty}\frac{[\Delta t(n)\Pi(\mathcal{I}(n))]^{k}}{(k+2)!} \\
\le & N(n)Te^{-\Delta t(n)\Pi(\mathcal{I}(n))}[\Delta t(n)\Pi(\mathcal{I}(n))]^2\sum_{k=0}^{\infty}\frac{[\Delta t(n)\Pi(\mathcal{I}(n))]^{k}}{k!} \\
= & N(n)Te^{-\Delta t(n)\Pi(\mathcal{I}(n))}[\Delta t(n)\Pi(\mathcal{I}(n))]^2 e^{\Delta t(n)\Pi(\mathcal{I}(n))} \\
= & T\Delta t(n)(\Pi(\mathcal{I}(n)))^2 \\
\le & T\Delta t(n)(\overline{\Pi}(\frac{\Delta(n)}{2}))^2.
\end{aligned}$$

Hence,

$$\mathbb{P}((II)>0)\to 0 \text{ as } n\to\infty,$$

by the assumption (4.1.1). Since

$$\mathbb{P}((III)>0) = \mathbb{P}\left(N((\lfloor N(n)T\rfloor,T],\mathcal{I}(n))\ge 1\right) \le \mathbb{P}\left(N(\Delta t(n),\mathcal{I}(n))\ge 1\right) = 1-e^{-\lambda_2\Delta t(n)},$$

where $\lambda_2 = \mathbb{E}N(1,\ \mathcal{I}(n)) = \Pi(\mathcal{I}(n)) \le \overline{\Pi}(\frac{\Delta(n)}{2})$. So, $\mathbb{P}((III)>0)\le 1-e^{-\Delta t(n)\overline{\Pi}(\frac{\Delta(n)}{2})}\to 0$ as $n\to\infty$, by the assumption (4.1.1). At last, by (6.1) of Maller and Szimayer(2007),

$$\mathbb{E}((IV)) \le \sqrt{4T}\int_{|x|\le\frac{\Delta(n)}{2}} x^2\Pi(dx)\to 0,\ \text{as } n\to\infty \tag{4.1.7}$$

since $\Pi(\cdot)$ is a Lévy measure. Hence, for any $\delta>0$, $\mathbb{P}((IV)>\delta)\to 0$ as $n\to\infty$ by Chebyshev's inequality.

Above all, we have proved that

$$\sup_{0\le t\le T} |\overline{L}_t(n) - L_t| \xrightarrow{\mathbb{P}} 0, \text{ as } n \to \infty.$$

■

Definition 4.1.3 *For any $j = 1, 2, \cdots, \lfloor N(n)T \rfloor$, $n \in \mathbb{N}$, let $Y_j(n) = [\frac{X_j(n)}{\Delta(n)}]\Delta(n)$, where $[x] = k$ if $x \in (k - \frac{1}{2}, k + \frac{1}{2}]$, $k \in \mathbb{Z}$. For any $t \in [0, T]$, $n \in \mathbb{N}$, define*

$$\widetilde{L}_t(n) = \gamma t - t \int\limits_{\frac{\Delta(n)}{2}<|x|<1} x\Pi(dx) + \sum_{j=1}^{\lfloor N(n)T \rfloor} Y_j(n) 1_{\{\tau_j^n \le t\}},$$

where τ_j^n is as in Definition 4.1.2.

Remark 4.1.1. By the definition above, we get that the law of $Y_j(n)$ is as follows:

$$\mathbb{P}(Y_j(n) = 0) = 1 - \sum_{k\in\mathcal{M}(n)} \mathbb{P}(Y_j(n) = k\Delta(n)),$$

and

$$\mathbb{P}(Y_j(n) = k\Delta(n)) = \Delta t(n)\Pi(I_k(n))e^{-\Delta t(n)\Pi(\mathcal{I}(n))}.$$

Proposition 4.1.2 *Assume that*

$$\lim_{n\to\infty} \Delta(n)\overline{\Pi}(\frac{\Delta(n)}{2}) = 0. \tag{4.1.8}$$

Then

$$\sup_{0\le t\le T} |\overline{L}_t(n) - \widetilde{L}_t(n)| \xrightarrow{\mathbb{P}} 0, \text{ as } n \to \infty.$$

Proof. For any $\delta > 0$,

$$\mathbb{P}(\sup_{0\le t\le T} |\overline{L}_t(n) - \widetilde{L}_t(n)| > \delta) \le \frac{\mathbb{E}(\sup_{0\le t\le T} |\overline{L}_t(n) - \widetilde{L}_t(n)|)}{\delta}$$

by Chebyshev's inequality. Note that, by definitions of $Y_j(n)$ and $X_j(n)$,

$$\mathbb{E}(\sup_{0\le t\le T} |\overline{L}_t(n) - \widetilde{L}_t(n)|) \le \mathbb{E}(\sum_{j=1}^{\lfloor N(n)T \rfloor} |Y_j(n) - X_j(n)|)$$

$$
\begin{aligned}
&\leq \mathbb{E}(\Delta(n) \sum_{j=1}^{\lfloor N(n)T \rfloor} 1_{\{X_j(n) \neq 0\}}) \\
&= \Delta(n) \sum_{j=1}^{\lfloor N(n)T \rfloor} \mathbb{E}(1_{\{X_j(n) \neq 0\}}) \\
&= \Delta(n) \sum_{j=1}^{\lfloor N(n)T \rfloor} \mathbb{P}(X_j(n) \neq 0) \\
&= \Delta(n) \lfloor N(n)T \rfloor \mathbb{P}(N(\Delta t(n), \mathcal{I}(n)) = 1) \\
&= \Delta(n)(\lfloor N(n)T \rfloor + 1) \Delta t(n) \Pi(\mathcal{I}(n)) e^{-\Delta t(n) \Pi(\mathcal{I}(n))} \\
&\leq 2T \Delta(n) \overline{\Pi}(\frac{\Delta(n)}{2}),
\end{aligned}
$$

where the first inequality follows from the definitions of $\overline{L}_t(n)$ and $\widetilde{L}_t(n)$ and the second from the definition of $Y_j(n)$. Hence the proposition follows by (4.1.8). ■

Proposition 4.1.3 *Let* $L_t(n) = \widetilde{L}_{t_j^n}(n)$ *if* $t \in [t_j^n, t_{j+1}^n)$, $j = 0, 1, \cdots, \lfloor N(n)T \rfloor$. *Then,*

$$\rho(L(n), \widetilde{L}(n)) \xrightarrow{\mathbb{P}} 0 \text{ as } n \to \infty.$$

Proof. In fact, the paths of $L_t(n)$ can be obtained by postponing the jumps of $\widetilde{L}_t(n)$ to the next grid points. Hence, by the same arguments with the proof of Theorem 3.2 in Szimayer and Maller (2007), the required convergence result follows. ■

Theorem 4.1.1 *Assume* (4.1.1) *and* (4.1.8). *Then*

$$\rho(L(n), L) \xrightarrow{\mathbb{P}} 0 \text{ as } n \to \infty.$$

Proof. By the triangular inequality,

$$\rho(L(n), L) \leq \rho(L(n), \widetilde{L}(n)) + \sup_{0 \leq t \leq T} |\widetilde{L}_t(n) - \overline{L}_t(n)| + \sup_{0 \leq t \leq T} |\overline{L}_t(n) - L_t|.$$

Thus, the theorem follows immediately by proposition 4.1.1, 4.1.2 and 4.1.3. ■

4.2 Convergence Rate

In this section, our discussion on the convergence rate of our setting, $\widetilde{L}_t(n)$, to L_t follows the same procedure with that in Szimayer and Maller (2007). That is, we give $\mathbb{E}(\sup_{0\le t\le T}|\widetilde{L}_t(n)-L_t|)$ with respect to $\Delta t(n)$ or the computational complexity of performing the backward induction technique, $C(n)$.

Note that the approximation $(\widetilde{L}_t(n), 0 \le t \le T)$ is a multinomial tree with $\lfloor\frac{T}{\Delta t(n)}\rfloor + \frac{1}{2}(m_+(n)+m_-(n))(\lfloor\frac{T}{\Delta t(n)}\rfloor - 1)\lfloor\frac{T}{\Delta t(n)}\rfloor$ nodes. And, at each node, there are $m_+(n)+m_-(n)+1$ calculations. Hence, $C(n) = (m_+(n)+m_-(n)+1)\cdot\Big[\lfloor\frac{T}{\Delta t(n)}\rfloor + \frac{1}{2}(m_+(n)+m_-(n))(\lfloor\frac{T}{\Delta t(}$ For the convenience of notation, let $m_+(n) = m_-(n) = \mu(n)$. Thus,

$$C(n) \sim \left(\frac{(\mu(n))T}{\sqrt{2}\Delta t(n)}\right)^2. \tag{4.2.1}$$

Recall that the Blumenthal-Getoor index and the Tail-weight index of L_t are defined as follows:

$$\alpha_* = \inf\{\alpha > 0 : \int_{|x|\le 1} |x|^\alpha \Pi(dx) < \infty\} \in [0,2], \tag{4.2.2}$$

$$\beta^* = \sup\{\beta > 0 : \int_{|x|>1} |x|^\beta \Pi(dx) < \infty\} \in [0,+\infty]. \tag{4.2.3}$$

By (4.1.2), $\mathbb{E}(\sup_{0\le t\le T}|\widetilde{L}_t(n)-L_t|)$ is bounded by $\mathbb{E}(I)+\mathbb{E}(II)+\mathbb{E}(III)+\mathbb{E}(IV)$. So we need to give the following lemma first.

Lemma 4.2.1 *Assume (4.1.3), (4.1.4) and (4.1.5). Then:*

(1) $\mathbb{E}(I) \le T \int_{|x|>(\mu(n)+\frac{1}{2})\Delta(n)} |x|\Pi(dx)$;

(2) $\mathbb{E}(II) \le T\Delta t(n)\Pi^2(\mathcal{I}(n))\left\{1 + [\Pi(\mathcal{I}(n))]^{-1}\int_{|x|>1}|x|\Pi(dx)\right\}$;

(3) $\mathbb{E}(III) \le \Delta t(n)\overline{\Pi}(\frac{\Delta(n)}{2}) + \Delta t(n)\int_{|x|>1}|x|\Pi(dx)$.

Proof. (1) By (4.1.3),

$$\mathbb{E}(I) = \mathbb{E}\left(\sum_{0\le s\le T}|\Delta L_s| 1_{\{\Delta L_s\in\mathbb{R}\setminus(\mathcal{I}(n)\cup(-\frac{\Delta(n)}{2},\ \frac{\Delta(n)}{2}]\}}\right)$$

$$\leq T \int\limits_{|x|>(\mu(n)+\frac{1}{2})\Delta(n)} |x|\Pi(dx),$$

where the second inequality follows from (6.3) of Szimayer and Maller (2007).

(2) Let $\Delta L_k^j(n)$ be the kth jump in the subinterval $(t_{j-1}^n,\ t_j^n]$ with size in $\mathcal{I}(n)$ of L and $l_j^n = N((t_{j-1}^n,\ t_j^n],\ \mathcal{I}(n))$, $j = 1, \cdots, \lfloor N(n)T \rfloor$. By (4.1.4),

$$\begin{aligned}
\mathbb{E}(II) =& \mathbb{E}\left(\sum_{j=1}^{\lfloor N(n)T\rfloor} \sum_{t_{j-1}^n < s \leq t_j^n} |\Delta L_s| 1_{\{N((t_{j-1}^n, t_j^n], \mathcal{I}(n)) \geq 2\}} 1_{\{\Delta L_s \in \mathcal{I}(n)\}}\right) \\
=& \sum_{j=1}^{\lfloor N(n)T\rfloor} \mathbb{E}\left(\sum_{k=1}^{l_j^n 1_{\{l_j^n \geq 2\}}} |\Delta L_k^j(n)|\right) \\
=& \sum_{j=1}^{\lfloor N(n)T\rfloor} \left(\mathbb{E}|\Delta L_1^j(n)| \mathbb{E}(l_j^n 1_{\{l_j^n \geq 2\}})\right) \\
\leq& \sum_{j=1}^{\lfloor N(n)T\rfloor} \left\{1 + [\Pi(\mathcal{I}(n))]^{-1} \int\limits_{|x|>1} |x|\Pi(dx)\right\} \cdot \left(\Delta t(n)\Pi(\mathcal{I}(n)) - \Pi(\mathcal{I}(n))\Delta t(n) e^{-\Pi(\mathcal{I}(n))\Delta t(n)}\right) \\
\leq& \lfloor N(n)T \rfloor \left\{1 + [\Pi(\mathcal{I}(n))]^{-1} \int\limits_{|x|>1} |x|\Pi(dx)\right\} (\Delta t(n)\Pi(\mathcal{I}(n)))^2 \\
\leq& T\Delta t(n)\Pi^2(\mathcal{I}(n)) \left\{1 + [\Pi(\mathcal{I}(n))]^{-1} \int\limits_{|x|>1} |x|\Pi(dx)\right\},
\end{aligned}$$

where the third equality follows from the i.i.d. property of $\Delta L_k^j(n)$, for all $1 \leq k \leq l_j^n$, and the fourth inequality from (6.6) of Szimayer and Maller (2007).

(3) Let $\Delta L_k(n)$ be the kth jump in the subinterval $(t_{\lfloor N(n)T\rfloor}^n,\ T]$ with size in $\mathcal{I}(n)$ of L and $u_n = N((t_{\lfloor N(n)T\rfloor}^n,\ T],\ \mathcal{I}(n))$. By (4.1.5),

$$\begin{aligned}
\mathbb{E}(III) =& \mathbb{E}\left(\sum_{t_{\lfloor N(n)T\rfloor}^n < s \leq T} |\Delta L_s| 1_{\{\Delta L_s \in \mathcal{I}(n)\}}\right) \\
=& \mathbb{E}\left(\sum_{k=1}^{u_n} |\Delta L_k(n)|\right) \\
=& \mathbb{E}|\Delta L_1(n)| \mathbb{E}(u_n)
\end{aligned}$$

$$
\begin{aligned}
&=\mathbb{E}|\Delta L_1(n)|(T-t^n_{\lfloor N(n)T\rfloor})\Pi(\mathcal{I}(n))\\
&\leq\mathbb{E}|\Delta L_1(n)|\Delta t(n)\overline{\Pi}(\frac{\Delta(n)}{2})\\
&\leq\left[1+(\Pi(\mathcal{I}(n)))^{-1}\int\limits_{|x|>1}|x|\Pi(dx)\right]\cdot\Delta t(n)\overline{\Pi}(\frac{\Delta(n)}{2})\\
&=\Delta t(n)\overline{\Pi}(\frac{\Delta(n)}{2})+\Delta t(n)\int\limits_{|x|>1}|x|\Pi(dx),
\end{aligned}
$$

where the third equality follows from the i.i.d. property of $\Delta L_k(n)$, for all $1\leq k\leq u_n$, and the sixth inequality from (6.6) of Szimayer and Maller (2007). ■

Theorem 4.2.1 *(1) Suppose that $\alpha_*<2$ and $2<\beta^*<\infty$. Assume that $\mu(n)$ and $\Delta(n)$ satisfies*

$$\mu(n)\sim A_1(\Delta t(n))^{-\theta_1},\ \Delta(n)\sim A_2(\Delta t(n))^{\theta_2}. \tag{4.2.4}$$

for some positive constants $A_1,\ A_2,\ \theta_1,\ \theta_2$. The convergence rate with respect to the computational complexity, $C(n)$, is minimized when

$$\theta_1=\frac{\beta^*+1-\alpha_*}{(\beta^*-1)(2+\alpha_*)},\ \theta_2=\frac{1}{2+\alpha_*}. \tag{4.2.5}$$

And, the convergence rate of $\widetilde{L}_t(n)$ to L_t satisfies

$$\mathbb{E}(\sup_{0\leq t\leq T}|\widetilde{L}_t(n)-L_t|)=o(C(n)^{-c}), \tag{4.2.6}$$

$$\mathbb{E}(\sup_{0\leq t\leq T}|\widetilde{L}_t(n)-L_t|)=o(\Delta t(n)^r) \tag{4.2.7}$$

for any $c<c(\alpha_*,\ \beta^*)$ and $r<r(\alpha_*)$, where

$$c(\alpha_*,\ \beta^*)=\frac{(2-\alpha_*)(\beta^*-1)}{2\left[(2+\alpha_*)(\beta^*-1)+1+\beta^*-\alpha_*\right]},\ r(\alpha_*)=\frac{2-\alpha_*}{2+\alpha_*}. \tag{4.2.8}$$

(2) If $\alpha_*<2$, $\beta^*=\infty$, we assume that

$$\lim_{n\to\infty}(\Delta t(n))^n\mu(n)=0 \text{ for all } n\in\mathbb{N}, \tag{4.2.9}$$

instead of the first assumption in (4.2.4). Then, (4.2.6) and (4.2.7) are still both true for $c<c(\alpha_*,\ \infty)=\frac{2-\alpha_*}{2(3+\alpha_*)}$ and $r<r(\alpha_*)=\frac{2-\alpha_*}{2+\alpha_*}$.

Proof. From (4.1.2), it follows that

$$\mathbb{E}(\sup_{0\le t\le T}|\widetilde{L}_t(n)-L_t|)\le \mathbb{E}(I)+\mathbb{E}(II)+\mathbb{E}(III)+\mathbb{E}(IV).$$

By Lemma 4.2.1 and equality (4.1.6), when n is large enough,

$$\begin{aligned}
&\mathbb{E}(\sup_{0\le t\le T}|\widetilde{L}_t(n)-L_t|)\\
\le& T\int\limits_{|x|>(\mu(n)+\frac{1}{2})\Delta(n)}|x|\Pi(dx)+T\Delta t(n)\Pi^2(\mathcal{I}(n))\left\{1+[\Pi(\mathcal{I}(n))]^{-1}\int\limits_{|x|>1}|x|\Pi(dx)\right\}\\
&+\Delta t(n)\overline{\Pi}(\frac{\Delta(n)}{2})+\Delta t(n)\int\limits_{|x|>1}|x|\Pi(dx)+\sqrt{4T}\int_{|x|\le\frac{\Delta(n)}{2}}x^2\Pi(dx)\\
\le& T\int\limits_{|x|>\mu(n)\Delta(n)}|x|\Pi(dx)+C_0\Delta t(n)\Pi^2(\mathcal{I}(n))+\sqrt{4T}\int_{|x|\le\frac{\Delta(n)}{2}}x^2\Pi(dx),
\end{aligned}$$

for some constant $C_0\ge T\left\{1+[\Pi(\mathcal{I}(n))]^{-1}\int\limits_{|x|>1}|x|\Pi(dx)\right\}+\left(\overline{\Pi}(\frac{\Delta(n)}{2})\right)^{-1}+\int\limits_{|x|>1}|x|\Pi(dx)\left(\overline{\Pi}(\frac{\Delta(n)}{2})\right)^{-2}$, as n is large enough.

By the argument in the proof of Theorem 4.1 in Szimayer and Maller (2007) and the definitions of α_* and β^*, for all $\alpha>\alpha_*$ and $\beta<\beta^*$,

$$\int\limits_{|y|\ge x}|y|\Pi(dy)=o(x^{1-\beta}),\ \text{as } x\to\infty,$$

$$\overline{\Pi}(x)=o(x^{-\alpha}),\quad \int\limits_{|y|\le x}y^2\Pi(dy)=o(x^{2-\alpha}),\ \text{as } x\downarrow 0.$$

Thus

$$\begin{aligned}
&\mathbb{E}(\sup_{0\le t\le T}|\widetilde{L}_t(n)-L_t|)\\
=&o\left((\mu(n)\Delta(n))^{1-\beta}\right)+o\left(\Delta t(n)\Delta(n)^{-2\alpha}\right)+o\left(\Delta(n)^{2-\alpha}\right)\\
=&o\left((\Delta t(n))^{(\theta_1-\theta_2)(\beta-1)}\right)+o\left((\Delta t(n))^{1-2\alpha\theta_2}\right)+o\left((\Delta t(n))^{(2-\alpha)\theta_2}\right)\\
=&o\left((\Delta t(n))^{r^*(\theta_1,\theta_2)}\right),
\end{aligned}$$

where $r^*(\theta_1, \theta_2) = \min\{(\theta_1 - \theta_2)(\beta - 1),\ 1 - 2\alpha\theta_2,\ (2-\alpha)\theta_2\}$ and the second equality follows from the assumption (4.2.4). By (4.2.1),

$$\mathbb{E}(\sup_{0\le t\le T} |\widetilde{L}_t(n) - L_t|) = o\left(C(n)^{-\frac{r^*(\theta_1,\theta_2)}{2(1+\theta_1)}}\right).$$

Through an elementary calculation, the maximal value of the function $\frac{r^*(\theta_1,\theta_2)}{2(1+\theta_1)}$ is

$$\frac{(2-\alpha)(\beta-1)}{2\left[(2+\alpha)(\beta-1)+1+\beta-\alpha\right]}$$

which can be obtained by taking $\theta_1 = \frac{\beta+1-\alpha}{(\beta-1)(2+\alpha)},\ \theta_2 = \frac{1}{2+\alpha}$.

Define $r(\alpha) = \frac{2-\alpha}{2+\alpha}$ and $c(\alpha,\beta) = \frac{(2-\alpha)(\beta-1)}{2[(2+\alpha)(\beta-1)+1+\beta-\alpha]}$. Then $r(\alpha)$ is decreasing in α and $c(\alpha,\beta)$ is increasing in β and decreasing in α. Thus, (4.2.6) and (4.2.7) hold. ■

Remark 4.2.1. The convergence rate of $L_t(n)$ to L_t, $\mathbb{E}\rho(L(n),\ L)$, has the same results as those of $\widetilde{L}_t(n)$ to L_t illustrated in Theorem 4.2.1.

4.3 Convergence of The Discrete American Put Option Price Processes Under The New Approximations

In this section, we will show that the sequence of American Put Option Price Processes under the new approximation processes, $(L_t(n), 0 \le t \le T)_{n\in\mathbb{N}}$ proposed in section 4.1 converges to the American Put Option Price Process under the Lévy process.

The following lemma is a result in the proof of Theorem 3.2 of Szimayer and Maller (2007).

Lemma 4.3.1 *For any $\varepsilon > 0,\ \ \delta > 0$, we have*

$$\lim_{\delta\downarrow 0} \mathbb{P}(\sup_{0\le t\le\delta} |L_t| \ge \varepsilon) = 0.$$

Theorem 4.3.1 *Let $(L(n),\ n \in \mathbb{N})$ be the sequence of processes proposed in section 4.1. Assume that* (4.1.1) *and* (4.1.2) *are both satisfied. Then, $(L(n), n \in \mathbb{N})$ satisfies Aldou's criterion for tightness.*

Proof. Fix $\delta > 0$. For any $\sigma,\ \tau \in \mathcal{S}_{0,T}^{L(n)}$, such that $\sigma \le \tau \le \sigma + \delta$, we have that

$$\begin{aligned}
&\mathbb{P}(|L_\sigma(n) - L_\tau(n)| \ge \varepsilon) \\
\le &\mathbb{P}(|L_\sigma(n) - \widetilde{L}_\sigma(n)| \ge \varepsilon/3) + \mathbb{P}(|L_\tau(n) - \widetilde{L}_\tau(n)| \ge \varepsilon/3) + \mathbb{P}(|\widetilde{L}_\sigma(n) - \widetilde{L}_\tau(n)| \ge \varepsilon/3).
\end{aligned}$$

Let $j_\sigma^n = \lfloor \sigma N(n) \rfloor + 1$. By the definitions of $L_t(n)$ and $\widetilde{L}_t(n)$,

$$|L_\sigma(n) - \widetilde{L}_\sigma(n)| \le |Y_{j_\sigma^n}(n)| + \Delta t(n)(\gamma + \int_{\frac{\Delta(n)}{2} < |x| < 1} |x| \Pi(dx)).$$

Hence,

$$\mathbb{P}(|L_\sigma(n) - \widetilde{L}_\sigma(n)| > \frac{\varepsilon}{3}) \le \mathbb{P}(|Y_{j_\sigma^n}(n)| + \Delta t(n)(\gamma + \int_{\frac{\Delta(n)}{2} < |x| < 1} |x| \Pi(dx)) > \frac{\varepsilon}{3}).$$

Since $\Delta t(n) \int_{\frac{\Delta(n)}{2} < |x| < 1} |x| \Pi(dx) \le \Delta t(n) \overline{\Pi}(\frac{\Delta(n)}{2}) \to 0$ as $n \to \infty$ by (4.1.1), there exists $\overline{n} \in \mathbb{N}$ such that $\Delta t(n)(\gamma + \int_{\frac{\Delta(n)}{2} < |x| < 1} |x| \Pi(dx)) \le \varepsilon/6$ when $n > \overline{n}$. Thus,

$$\mathbb{P}(|L_\sigma(n) - \widetilde{L}_\sigma(n)| > \frac{\varepsilon}{3}) \le \mathbb{P}(|Y_{j_\sigma^n}(n)| > \varepsilon/6)$$

when $n > \overline{n}$. Consider that

$$\begin{aligned}
\mathbb{P}(|Y_{j_\sigma^n}(n)| > \varepsilon/6) &= \sum_{k=1}^{\lfloor N(n)T \rfloor} \mathbb{P}(|Y_k(n)| > \varepsilon/6 | j_\sigma^n = k) \\
&= \sum_{k=1}^{\lfloor N(n)T \rfloor} \mathbb{P}(|Y_k(n)| > \varepsilon/6 | t_{k-1}^n \le \sigma < t_k^n) \\
&= \sum_{k=1}^{\lfloor N(n)T \rfloor} \mathbb{P}(|Y_k(n)| > \varepsilon/6) \mathbb{P}(t_{k-1}^n \le \sigma < t_k^n) \\
&= \mathbb{P}(|Y_1(n)| > \varepsilon/6) \sum_{k=1}^{\lfloor N(n)T \rfloor} \mathbb{P}(t_{k-1}^n \le \sigma < t_k^n) \\
&= \mathbb{P}(|Y_1(n)| > \varepsilon/6) \\
&\le \mathbb{P}(|Y_1(n)| \ne 0) \\
&= \Delta t(n) \Pi(\mathcal{I}(n)) e^{-\Delta t(n) \Pi(\mathcal{I}(n))} \\
&\le \Delta t(n) \overline{\Pi}(\frac{\Delta(n)}{2})
\end{aligned}$$

$$\to 0$$

as $n \to \infty$ by (4.1.1), where the third identity follows from the independence of $Y_k(n)$ and $(t_{k-1}^n \le \sigma < t_k^n)$. Therefore,

$$\mathbb{P}(|L_\sigma(n) - \widetilde{L}_\sigma(n)| > \frac{\varepsilon}{3}) \to 0 \text{ as } n \to \infty.$$

Similarly,

$$\mathbb{P}(|L_\tau(n) - \widetilde{L}_\tau(n)| > \frac{\varepsilon}{3}) \to 0 \text{ as } n \to \infty.$$

Note that

$$\begin{aligned}
&\mathbb{P}(|\widetilde{L}_\sigma(n) - \widetilde{L}_\tau(n)| \ge \varepsilon/3) \\
\le &\mathbb{P}(|\widetilde{L}_\sigma(n) - L_\sigma| \ge \varepsilon/9) + \mathbb{P}(|\widetilde{L}_\tau(n) - L_\tau| \ge \varepsilon/9) + \mathbb{P}(|L_\sigma - L_\tau| \ge \varepsilon/9).
\end{aligned}$$

From Propositions 4.1.1 and 4.1.2, it follows that the first two terms on the right hand side of the inequality above converges to 0 as $n \to \infty$. Since $\sigma \le \tau \le \sigma + \delta$, $0 \le \tau - \sigma \le \delta$. Hence

$$\mathbb{P}(|L_\sigma - L_\tau| \ge \varepsilon/9) \le \mathbb{P}(\sup_{0 \le s \le \delta} |L_s| \ge \varepsilon/9) \to 0, \text{ as } \delta \to 0,$$

by the Strong Markov inequality and Lemma 4.3.1.

Above all, the theorem follows. ■

Theorem 4.3.2 *Assume that* $(L(n),\ n \in \mathbb{N})$ *is the sequence of processes defined in section 4.1 and that* (4.1.1), (4.1.2) *are satisfied. Then,*

$$\mathbb{F}^{L(n)} \xrightarrow{\omega} \mathbb{F}^L, \text{ as } n \to \infty.$$

Proof. By the construction of $L_t(n)$, for any $t \in [0, T]$, and $n \in \mathbb{N}$, we know that $(L(n),\ n \in \mathbb{N})$, is a sequence of càdlàg processes with independent increments. From Theorem 4.1.1 and Proposition 2 of Conquet, Mémin, and S*l*ominski(2001), the result follows directly. ■

Remark 4.2.1. Comparing the probabilities of $X_j(n)$ given in definition 4.1.1 with those in definition 2.2.1, it is easy to see that the new approximation scheme we defined in section 4.1 also converges to L in distribution under the J_1-topology in $\mathbb{D}[0,T]$. $L(n) \xrightarrow{\mathfrak{L}} L$ in $\mathbb{D}[0,T]$. Then, by the same arguments as those in chapter 3, we obtain that $\pi_t(n) \xrightarrow{D} \pi_t$, for each $t \in [0,T]$ under this new scheme $(L(n), n \in \mathbb{N})$.

CHAPTER 5

CONCLUSIONS

The approximation scheme proposed by Maller, Solonmon and Szimayer (2006) can be seen as a generalization of the binomial tree for the Black-Sholes model. The tree-based scheme makes it easier to compute American option prices in practice. Just as in Maller, Solonmon and Szimayer (2006), the essential advantage of the tree-based scheme is that the model and the valuation principles are easily implemented and understood without deep knowledge of the underlying financial, mathematical and probabilistic fundamentals. They proved that $\pi_t(n)$ converge to π_t for each t in a full measure set of $[0, T]$ but not every time $t \in [0, T]$.

This convergence result can not satisfy practical need because we need to have a scheme to price an American option at any time.

The approximation scheme proposed by Szimayer and Maller (2007) is defined path-by-path. The idea to achieve the convergence of the sequence of Snell envelopes under the approximation scheme in Szimayer and Maller (2007) is to apply Theorem 5 and Corollary 6 of Conquet and Toldo (2007) by verifying the conditions therein.

In this paper, we have adapted the same principle with Szimayer and Maller (2007) to the approximation scheme given in the multinomial tree of Maller, Solomon and Szimayer (2006). But the directly checking the conditions of Theorem 5 and Corollary 6 of Conquet and Toldo (2007) fails. We have to construct another discrete approximation model which is equal in distribution from the Skorokhod representation theorem. This relies on a basic result proved in Maller, Solomon and Szimayer (2006). The main result of this paper is that the sequence of American (put) option

price processes under the multinomial tree scheme proposed by Maller, Solonmon and Szimayer (2006) converges to the continuous time counterpart in distribution **for all** $t \in [0, T]$**.** Therefore we have overcome the main difficulty in the weak convergence issue in Maller, Solonmon and Szimayer (2006), and *our result is strong enough to fulfill the practical need.* Our proof is not only applicable for American put options but also applicable for any option whose payoff function is continuous bounded and satisfies the statement in Lemma 3.2.1. For call option cases, we only discuss modified call options in Remark 3.2.1 and Corollary 3.2.1.

In chapter 4, a new approximation scheme is given. And we also prove the weak convergence result for American option price processes under this new scheme. The convergence rate is also discussed for this approximation scheme.

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Based on a sequence of discretized American option price processes under the multinomial model proposed by Maller, Solomon and Szimayer (2006), the sequence converges to the counterpart under the original Lévy process in distribution for almost all time. We prove a weak convergence in this case for American put options for all time. By adapting Skorokhod representation theorem, a new sequence of approximating processes with the same laws with the multinomial tree model defined by Maller, Solomon and Szimayer (2006) is obtained. The new sequence of approximating processes satisfies Aldous' criterion for tightness. And, the sequence of filtrations generated by the new approximation converges to the filtration generated by the representative of Lévy process weakly. By using results of Coquet and Toldo (2007), we give a complete proof of the weak convergence for the approximation of American put option prices for all time. Moreover, a path-by-path defined approximation that shares an almost same law with the one in Maller, Solomon and Szimayer (2006) is proposed. The rate of convergence for the path-by-path defined approximation is also discussed.

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