# THE APPLICATION OF STOCHASTIC CONTROL THEORY TO HEDGE RATIO OPTIMIZATION IN RISK MANAGEMENT

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THE APPLICATION OF STOCHASTIC CONTROL THEORY TO HEDGE RATIO OPTIMIZATION IN RISK MANAGEMENT

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# CHAPTER 1

# INTRODUCTION

## 1.1 Definitions and Notations

For convenience, we list some notations that are used in this thesis.

$\mathbb{R}^n$: $n$-dimensional Euclidean space, $n \geq 1$, $\mathbb{R} = \mathbb{R}^1$

$U$: An open subset of $\mathbb{R}^n$

$\Omega$: A bounded subset of $\mathbb{R}^n$

$T,\ T_0$: A bounded time variable, $0 \leq T,\ T_0 < \infty$

$Q_0 = [0, T_0) \times \mathbb{R}$: An unbounded subset in $\mathbb{R}^2$

$\bar{Q}_0$: The closure of $Q_0$

$\partial Q$: The boundary of $Q_0$

$Q = [0, T_0) \times \Omega$: A bounded subset of $\mathbb{R}^n$

$B(x, r)$: A closed ball with center $x$ and radius $r > 0$

$C^k(U) = \{u : U \to \mathbb{R} |\ u$ is $k$-times continuously differentiable$\}$

$C^{1,k}(U) = \{u : [0, T) \times U \to \mathbb{R} |\ u$ is continuous in $t$ and $k$-times continuously differentiable in $x \in U\}$

$C^\infty(U) = \{u : U \to \mathbb{R} |\ u$ is infinitely differentiable$\} = \cap_{k=0}^{\infty} C^k(U)$ (We say $u$ is smooth provided $u$ is infinitely differentiable.)

$C_c^\infty(U) = \{u : U \to \mathbb{R} |\ u \in C^\infty(U)$ with compact support. The support of a function is denoted by supp $u\}$

$C_0^\infty(U) = \{u : U \to \mathbb{R} |\ u$ is infinitely differentiable and $u$ vanishes at infinity$\}$

$\dot{C}^1(U) = \{u : \bar{U} \to \mathbb{R} |\ u$ is continuous and vanish on the boundary of $U\}$

$L^p(U) = \{u : U \to \mathbb{R}|\ u$ is Lebesgue measurable, $\|u\|_{L^p(U)} < \infty\}$, where

$$\|u\|_{L^p(U)} = \Big(\int_U |u|^p dx\Big)^{\frac{1}{p}} \quad (1 \leq p < \infty)$$

$L^p_{loc}(U) = \{u : U \to \mathbb{R}|\ u \in L^p(V)$ for each $V \subsetneq U\}$

$S^{2,2}(U)$: Sobolev Space with $k = 2$ and $p = 2$

$S^{1,2}_2(U)$: Sobolev Space with time $t$ involved, $t = 1$, $k = 2$, and $p = 2$

$S^{1,2}_0(U)$: The closure of $C^\infty_0(U)$ in $S^{1,2}(U)$

$\dot{S}^{1,2}_2(U)$: The closure of $\dot{C}^1(U)$ in $S^{1,2}_2(U)$

$\mathbf{e}_i = (0, \ldots, 0, 1, 0, \ldots, 0) = i^{th}$ standard coordinate vector

$D^\alpha u(x) := \dfrac{\partial^{|\alpha|} u(x)}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} = \partial^{\alpha_1}_{x_1} \cdots \partial^{\alpha_n}_{x_n} u$, where $|\alpha| = \alpha_1 + \cdots + \alpha_n$

$\mathcal{F}(U)$ : $\sigma$-algebra

$\mathbb{P}^y$: Probability measure with respect to $y = (s, w)$, the initial point of a process

$\mathbf{E}^y$: The expectation with respect to the probability measure $\mathbb{P}^y$, where $y = (s, w)$ is the initial point of a stochastic process

$Z_{it}$: One-dimensional Brownian motion, $i = 1, 2, ...$

$X$: A stochastic process $X(t, w)$, with $t \in [0, T_0]$ and $w \in \Omega$

Constants: We use the letter $C$ and $K$ to denote any constant that can be explicitly computed in terms of known quantities.

There are two types of definitions in this thesis, mathematics definitions and finance definitions. We present these definitions in the order they appear respectively. All of these definitions are from [2], [4], [9] [13], and [16].

**Definition 1.1** *If $U$ is a given set, then a* $\sigma$-algebra $\mathcal{F}$ *on $U$ is a family of subsets of $U$ with the following properties:*

*(i)* $\emptyset \in \mathcal{F}$

*(ii)* $F \in \mathcal{F} \Rightarrow F^C \in \mathcal{F}$, *where* $F^C = U \backslash F$ *is the complement of $F$ in $U$*

*(iii)* $A_1, A_2, ... \in \mathcal{F} \Rightarrow A := \cup_{i=1}^{\infty} A_i \in \mathcal{F}$

*The $\sigma$-algebra obtained by beginning with closed intervals and adding everything else necessary in order to have a $\sigma$-algebra is called* Borel $\sigma$-algebra *of subsets of* $[0,1]$ *and the sets in this $\sigma$-algebra are called* Borel sets.

**Definition 1.2** *A* probability measure $\mathbb{P}$ *on a measurable space* $(\Omega, \mathcal{F})$ *is a function* $\mathbb{P} : \mathcal{F} \to [0,1]$ *such that*

*(a)* $\mathbb{P}(\emptyset) = 0, \ \mathbb{P}(\Omega) = 1$

*(b) If* $A_1, A_2, ... \in \mathcal{F}$ *and* $\{A_i\}_{i=1}^{\infty}$ *is* disjoint *(i.e.* $A_i \cap A_j = \emptyset$ *if* $i \neq j$*) then*

$$\mathbb{P}\Big(\bigcup_{i=1}^{\infty} A_i\Big) = \sum_{i=1}^{\infty} \mathbb{P}(A_i) \ .$$

*The triple* $(\Omega, \mathcal{F}, \mathbb{P})$ *is called a* probability space.

**Definition 1.3** *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space. A* random variable *is a real-valued function $X$ defined on $\Omega$ with the property that for every Borel subset $B$ of* $\mathbb{R}$*, the subset of $\Omega$ given by*

$$\{X \in B\} = \{\omega \in \Omega;\ X(\omega) \in B\}$$

*is in the $\sigma$-algebra $\mathcal{F}$.*

*Let $X$ be a random variable on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*. The* distribution measure *of $X$ is the probability measure $\mu_X$ that assigns to each Borel subset $B$ of* $\mathbb{R}$ *the mass* $\mu_X(B) = \mathbb{P}\{X \in B\}$.

**Definition 1.4** *Let $X$ be a random variable on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*. The* the expectation *(or* expected value*) of $X$ is defined to be*

$$\mathbf{E}[X] := \int_{\Omega} X(\omega)\, d\mathbb{P}(\omega)$$

*This definition makes sense if $X$ is* integrable*, i.e., if*

$$\mathbf{E}[|X|] = \iint_{\Omega} |X(\omega)|\, d\mathbb{P}(\omega) < \infty$$

**Definition 1.5** *Let* $\Omega$ *be a nonempty set. Let* $T$ *be a fixed positive number, and assume that for each* $t \in [0,T]$ *there is a* $\sigma$*-algebra* $\mathcal{F}_t$*. Assume further that if* $s \leq t$*, then every set in* $\mathcal{F}_s$ *is also in* $\mathcal{F}_t$*. Then we call the collection of* $\sigma$*-algebras* $\mathcal{F}_t$, $0 \leq t \leq T$, *a* filtration*.*

**Definition 1.6** *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space, let* $T$ *be a fixed positive number. A* stochastic process *is a parametrized collection of random variables*

$$\{X_t\}_{t \in T}$$

*assuming values in* $\mathbb{R}^n$*.*

**Definition 1.7** *Let* $\Omega$ *be a nonempty sample space equipped with a filtration* $\mathcal{F}_t$, $0 \leq t \leq T$*. Let* $X_t$ *be a collection of random variables indexed by* $t \in [0,T]$*. We say this collection of random variables is an* adapted stochastic process *if, for each* $t$*, the random variable* $X_t$ *is* $\mathcal{F}_t$*-measurable.*

**Definition 1.8** *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space, let* $T$ *be a fixed positive number, and let* $\mathcal{F}_t$, $0 \leq t \leq T$, *be a filtration of sub-*$\sigma$*-algebras of* $\mathcal{F}$*. Consider an adapted stochastic process* $X_t$, $0 \leq t \leq T$*. Assume that for* $0 \leq s \leq t \leq T$ *and for every nonnegative, Borel-measurable function* $f$*, there is another Borel-measurable function* $g$ *such that*

$$\mathbf{E}[f(X_t)|\mathcal{F}_s] = g(X_s)$$

*Then we say that the* $X_t$ *is a* Markov process*.*

**Definition 1.9** *A* stopping time $\tau$ *is a random variable taking values in* $[0, \infty]$ *and satisfying*

$$\{\tau \leq t\} \in \mathcal{F}_t, \quad \textit{for all} \;\; t \geq 0\,.$$

*Let* $U \subset \mathbb{R}^n$ *be open. Then the* first exit time

$$\tau_U := \inf\{t > 0;\, X_t \notin U\}$$

*is a stopping time in* $\mathcal{F}_t$*.*

**Definition 1.10** *A (time-homogeneous)* Itô diffusion *is a stochastic process* $X_t(\omega) = X(t,\omega) : [0,\infty) \times \Omega \to \mathbb{R}^n$ *satisfying a stochastic differential equation of the form*

$$dX_t = b(X_t)\,dt + \sigma(X_t)\,dZ_t, \quad t \geq s;\ X_s = x$$

*where* $Z_t$ *is* $m$*-dimensional Brownian motion and* $b : \mathbb{R}^n \to \mathbb{R}$, $\sigma : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ *satisfy the condition:*

$$|b(x) - b(y)| + |\sigma(x) - \sigma(y)| \leq D|x-y|; \quad x, y \in \mathbb{R}^n$$

*where* $|\sigma|^2 = \sum |\sigma_{ij}|^2$.

**Definition 1.11** *For a bounded Borel function* $f \in \mathbb{R}^n$, *the Itô diffusion satisfies the* strong Markov property *if*

$$\mathbf{E}^x[f(X_{\tau+h})|\mathcal{F}_\tau] = \mathbf{E}^x[f(X_h)] \quad \text{for all} \quad h \geq 0\,,$$

*where* $\tau$ *is a stopping time with respect to* $\mathcal{F}$, *and* $\tau < \infty$.

**Definition 1.12** *Let* $\{X_t\}$ *be a (time-homogeneous) Itô diffusion in* $\mathbb{R}^n$. The (infinitesimal) generator $A$ *of* $X_t$ *is defined by*

$$Af(x) = \lim_{t\downarrow 0} \frac{\mathbf{E}^x[f(X_t)] - f(x)}{t}; \quad x \in \mathbb{R}^n$$

*The set of functions* $f : \mathbb{R}^n \to \mathbb{R}$ *such that the limit exists at* $x$ *is denoted by* $\mathcal{D}_A(x)$, *while* $\mathcal{D}_A$ *denotes the set of functions for which the limit exists for all* $x \in \mathbb{R}^n$.

*If* $f \in C_0^2(\mathbb{R}^n)$, *then* $f \in \mathcal{D}_A$ *and*

$$Af(x) = \sum_i b_i(x)\frac{\partial f}{\partial x_i} + \frac{1}{2}\sum_{i,j}(\sigma\sigma^T)_{i,j}(x)\frac{\partial^2 f}{\partial x_i \partial x_j}$$

**Definition 1.13** *Define the* performance function $J^u(y)$ *to be*

$$J^u(y) = \mathbf{E}^y\Big[\int_0^T f^u(Y_t)dt + g(Y_T \cdot \chi_{\{T<\infty\}})\Big]$$

*where* $f : \mathbb{R} \times \mathbb{R}^n \times U \to \mathbb{R}$ *and* $g : \mathbb{R} \times \mathbb{R}^b \to \mathbb{R}$ *are given continuous functions.*

**Definition 1.14**

$$\Phi(y) := \sup_{u(t,\omega)} \{J^u(y)\} = J^{u^*}(y)$$

*A family* $\mathcal{A}$ *of* admissible controls *are controls contained in the set of all* $\mathcal{F}^{(m)}$*-adapted process* $\{u(t)\}$ *with values in* $U$*. If such a control* $u^*$ *exists, it is called an* optimal control *and* $\Phi$ *is called the* optimal performance or the value function.

**Definition 1.15** *Functions* $u(t,\omega)$ *of the form* $u(t,\omega) = u_0(t, X_t(\omega))$ *for some function* $u_0 : \mathbb{R}^n \to U \subset \mathbb{R}^m$*. If* $u$ *does not depend on the starting point* $y = (s,x)$*, and the value at time* $t$ *only depends on the state of the system at this time. Then,* $u(t,\omega)$ *are called* Markov controls, *because with such* $u$*, the corresponding process* $X_t$ *becomes an Itô diffusion, in particular a Markov process, denoted by* $u(Y_t) = u(t, X_t)$.

**Definition 1.16** *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space. A family* $\{f_j\}_{j\in J}$ *of real measurable functions* $f_j$ *on* $\Omega$ *is called* uniformly $\mathbb{P}^y$-integrable *if*

$$\lim_{M\to\infty} \left( \sup_{j\in J} \left\{ \int_{\{|g_j|>M\}} |f_j| d\mathbb{P} \right\} \right) = 0\,.$$

**Definition 1.17** *We say that a function* $u(t,x)$ *satisfies a* polynomial growth condition *on* $Q$ *if, for some constants* $C$*,* $k$*,* $|u(t,x)| \leq C(1+|x|)^k$ *when* $(t,x) \in Q$*. The class of* $u$ *in* $C^{1,2}$ *which satisfies a polynomial growth condition on* $Q$ *is denoted by* $C_p^{1,2}$.

**Definition 1.18** *In the domain* $Q_0$*, if the boundary data are imposed at the final time* $T$*:*

$$u(T,x) = g(x), \quad x \in \mathbb{R}$$

*such data at a fixed time* $T$ *are called* Cauchy data.

**Definition 1.19** *Define the* standard mollifier $\xi \in C^\infty(\mathbb{R}^n)$ *by*

$$\xi(x) := \begin{cases} C \exp\left(\frac{1}{|x|^2-1}\right), & \text{if } |x| < 1; \\ 0, & \text{if } |x| \geq 1, \end{cases}$$

*the constant* $C > 0$ *selected so that* $\displaystyle\int_{\mathbb{R}^n} \xi \, dx = 1$.

**Definition 1.20** *Suppose* $u$, $v \in L^1_{loc}(U)$, *and* $\alpha$ *is a multi-index. We say that* $v$ *is* the weak $\alpha$-derivative of $u$, *written* $D^\alpha u = v$ *if*

$$\int_U \zeta v \, dx = (-1)^{|\alpha|} \int_U u \mathrm{D}^\alpha \zeta \, dx$$

*for all test functions* $\zeta \in C_c^\infty$.

**Definition 1.21** *The* Sobolev space

$$S^{k,p}(U)$$

*consists of all locally summable functions* $u : U \to \mathbb{R}$ *such that for each multiplier* $\alpha$ *with* $|\alpha| < k$, $D^\alpha u$ *exists in the weak sense and belongs to* $L^p(U)$.

*If time* $t$ *involves, denote it by* $S_p^{t,k}(U)$.

**Definition 1.22** *The* Steklov average $\upsilon_\delta$ *of a function* $\upsilon$ *for a nonzero constant* $\delta$ *is defined by*

$$\upsilon_\delta(X) = \frac{1}{\delta} \int_t^{t+\delta} \upsilon(s, x) \, ds$$

*For* $\delta > 0$, $\upsilon_\delta$ *gives an average of* $\upsilon$ *over later times, and for* $\delta < 0$, *it gives an average over earlier times.*

**Definition 1.23** *The* payoff *is the cash realized by the holder of an option or other derivative at the end of its life.*

**Definition 1.24** Deadweight cost/loss *is the extent to which the value and impact of a tax, tax relief or SUBSIDY is reduced because of its side-effects.*

**Definition 1.25** *In finance,* hedge *is a trade designed to reduce risk.* Hedge ratio *is the ratio of the size of a position in a hedging instrument to the size of the position being hedged.*

**Definition 1.26** Forward contract *is a contract that obligates the holder to buy or sell an asset for a predetermined delivery price at a predetermined future time.*

Futures contract *is a contract that obligates the holder to buy or sell an asset for a predetermined delivery price during a specified future time period. The contract is marked to market daily.*

## 1.2 Problems and Results

There are a lot of academic research papers in finance, most of which study Corporation Risk Management problems. The purpose of risk management nowadays is not just reducing any risk that any corporation has. In 1990's, some people raised the idea of coordinating risk, for example, Schrand and Unal(1998), [15]. At that time, finance analysts discussed much details about the importance of hedging in risk management, but not a few of them questioned the mathematical part.

Froot, Scharfstein, and Stein (1993), [6] presented a paper about solving for the optimal hedging strategy of risk management, in which they not only introduced the model of hedging the wealth of a firm but also illustrated that a corporation can hedge a ratio of its' total wealth. The authors answered the question logically and deduced the optimal hedge ratio $h^*$ of one variable and two variables, if linear hedging strategies are considered. It also introduced the idea of non-linear hedging strategy and gave the corresponding optimal hedge ratio result. We start from their important research outcomes and develop the content of Chapter 2 in this thesis.

We notice that there was one important assumption in Froot, Scharfstein, and Stein (1993), [6]: all the processes are non-stochastic processes. However, this raise a question to us: What if the process is stochastic? Can we find any solution of maximizing the expected profit function if there is an optimal stochastic hedge process? We studied one paper of stochastic optimal control problems written by Huang and Liu (2007), [8], in which the dynamic programming (the HJB Equation) method was applied. Based on this paper, we present the existence of a weak solution of the HJB Equation on an unbounded domain with a free boundary condition problem and develop the required theoretical proof in Chapter 3.

This thesis is mainly about the mathematical theory and computation about deriving the hedge ratio for the non-stochastic $n$-dimension case in single period and showing the existence of a weak solution to the HJB Equation if a stochastic process is considered. In Chapter 2, we derive the multinational risk management coordination, mainly when a company changes the investment opportunities, we solve for the optimal hedge ratio in two periods. Then, in Chapter3 we develop a stochastic optimization model with a controlled Markov process, and apply the dynamic programming (the Hamilton-Jacobi-Bellman (HJB) Equation), a generally used stochastic optimization method in stochastic control theory to prove that if there exists an optimal hedge process $h^*$, the corresponding HJB Equation with a free boundary condition has a weak solution on an unbounded domain. The last part of this thesis is the conclusion, in which we summarize the thesis and give conclusions.

### 1.2.1 Background Introduction

Froot, Scharfstein, and Stein (1993), [6] presented the steps of finding the optimal hedging model with changing investment and financing opportunities of linear hedging strategies (i.e. forward sales or purchases) situation for one variable case. They introduced the hedging decision model:

$$w = w_0(h + (1 - h)\epsilon) \tag{1.1}$$

where $w_0$ is the initial wealth of a company, $h$ is the hedge ratio, and $\epsilon$ is the return rate of the investment.

The expected profit function $P(w)$ is defined as:

$$P(w) = \max_{I} \{\theta\, f(I) - I - C(e)\} \tag{1.2}$$

where $I$ is the total investment, $\theta = \alpha(\epsilon - \bar{\epsilon}) + 1$, $\alpha$ is a measure of the correlation between investment opportunities and the risk to be hedged, $f(I)$ is the product function, and $C(e)$ is the dead weight cost.

The question in the paper was finding an optimal hedging policy $h^*$ so that the expected profits $P$ would be maximized:

$$\max_{h} \mathbf{E}[P(w)] \tag{1.3}$$

The following result was derived:

$$h^* = 1 + \alpha \frac{\mathbf{E}[f_I P_{ww}/\theta f_{II}]}{w_0 \bar{P}_{ww}}, \tag{1.4}$$

where $\bar{P}_{ww} = \mathbf{E}[P_{ww}]$.

Then, the coordinating investment opportunities for multinational companies' risk management strategy of two variables were introduced. There were two investments in the model, home investment $I^H$ and abroad investment $I^A$. The expected profits $P(w)$ was given by

$$P(w) = f^H(I^H) + \theta\, f^A(I^A) - I^H - \gamma\, I^A - C(e) \tag{1.5}$$

where $\theta = \alpha(\epsilon - \bar{\epsilon}) + 1,\ \gamma = \beta(\epsilon - \bar{\epsilon}) + 1$. Here $\epsilon$ is the home currency price of the foreign currency, and $0 \leq \alpha,\ \beta \leq 1$ are parameters indexing the sensitivity of foreign revenues and foreign investment costs to the exchange rate.

Using similar arguments to develop $h^*$ of one variable, the optimal hedge ratio $h^*$ was solved as follows:

$$h^* = 1 + \frac{\mathbf{E}[(\alpha\gamma - \beta\theta) f_I^A P_{ww}/\theta\, f_{II}^A]}{w_0 \bar{P}_{ww}} - \beta\, \frac{\mathbf{E}[I^A P_{ww}]}{w_0 \bar{P}_{ww}}, \tag{1.6}$$

where

$$P_{ww} = \frac{\theta\, f_{II}^{H} f_{II}^{A} C_{ee}}{C_{ee}(\gamma^2 f_{II}^{H} - \theta\, f_{II}^{A}) - \theta\, f_{II}^{H} f_{II}^{A}} \; < \; 0 \tag{1.7}$$

After studied this paper, we are very interested in two problems:

- Extending the idea of finding the optimal hedge ratio $\mathbf{h^{0*}}$ to $n$ dimension, where $n \geq 2$ in period zero and period one;
- Changing the non-stochastic processes to stochastic processes and solving for the corresponding optimal hedge control.

Assume that in Chapter 2, only linear hedging strategies and non-stochastic processes are considered. We first develop the multinational risk management model to $n$ variables of single period in Chapter 2. We also find that the hedge ratio $\mathbf{h^{1*}}$ of $n$ variables can be calculated in period one if we treat random variables as functions.

Two $n = 2$ cases are presented to check that our model can be calculated, one of which was given in Froot, Scharfstein, and Stein (1993), [6] with slight change.

### 1.2.2 Stochastic Optimal Control Model

Since we consider changing some non-stochastic processes in [6] to stochastic processes, we are thinking of the stochastic optimal control theory. We found that Huang and Liu (2007), [8] applied the HJB Equation method in stochastic control theory to one finance problem:

Given the initial wealth $W_{0^-} > 0$ and the prior $(M_{0^-}, V(0^-))$, choose the number $N \in \mathcal{F}_0$ of news updates, the news accuracies $\alpha_\varepsilon, \alpha_\nu \in \mathcal{F}_0$, and an optimal trading strategy to maximize the expected utility function at the terminal wealth $W_T$,

$$\max_{N,\alpha_\varepsilon,\alpha_\nu,\theta} \mathbf{E}[u(W_T)],$$

subject to the stochastic process of $W_t$ with initial condition $W_0$ and $u(W)$ is a power function, increasing and concave.

The value function is

$$J(W, M, t; \alpha_\nu) = \max_\theta \mathbf{E}[u(W_T)|W_t = W, M_t = M].$$

and the corresponding HJB Equation is:

$$J_t + \max_\theta\{\frac{1}{2}\theta^2\sigma_S J_{WW} + \theta(\mu_0 + \mu_1 M - r)J_W + \theta\sigma_S\sigma_{M1}(t)J_{WM}\}$$
$$+ rWJ_W + \frac{1}{2}\sigma_M(t)^2 J_{MM} + (g_0 + g_1 M)J_M = 0$$

with the terminal condition

$$J(W, M, T; \alpha_\nu) = u(W).$$

The authors found a solution of this HJB Equation, that is the value function $J(W, M, t; \alpha_\nu)$ with the stochastic optimal trading strategy.

The difference of our model is that, instead of maximizing the expected utility function $u(W)$, our problem is maximizing the expected profit function $P(W)$ at the terminal time $T_0$. Then, the boundary condition is not a fixed boundary but a free boundary. To solve a free boundary problem, we studied another paper, Muthuraman and Kumar (2008), [12], which is about solving the free-boundary (stopping time) problems in finance.

With these two papers, we set up our model as follows:

Given the initial wealth $w_0 > 0$ of a company, choose an optimal hedging strategy $h(t) \in U$ to maximize the expected profit function $P(W_t)$ at a terminal time $T_0$,

$$\Phi(y) = \sup_{h\in U}\{\mathbf{E}^{0,w_0}[P(W^h_{T_0})\}$$

subject to

$$dW_t = w_0\big[((1 - h_t)r_W + h_t\mu_W)\,dt + h_t\sigma_W\,dZ_{1t}\big]$$

and initial wealth $W(0) = w_0 > 0$.

If $\Phi \in C_0^2(Q_0)$, the HJB Equation will be

$$\sup_{h \in U}\{\mathcal{L}^h \Phi(y)\} = 0$$

with the terminal condition $\Phi(T_0, w) = P(w)$ on $\{T_0\} \times [m, M]$, $m$, $M > 0$.

In Chapter 3, we prove that there exists a weak solution of the HJB Equation on an unbounded domain $Q_0 = [0, T_0) \times \mathbb{R}$ with a free boundary condition $P(w)$ in three sections.

First, prove the existence of a solution to the HJB Equation with a smooth and bounded boundary condition $g(w)$ on $Q_0$.

Second, there exists a sequence of the solutions $\Phi_n$ of the HJB Equation convergent in some vector space $V$ with proper norm. Then, the limit of the convergent sequence can be defined as a weak solution of the HJB Equation on a bounded set $Q$.

Finally, extend the weak solution from $Q$ to $Q_0$.

# CHAPTER 2

# MULTINATIONAL RISK MANAGEMENT - COORDINATING CORPORATION INVESTMENT

In the Financial Risk Management field, when a multinational corporation has sales and production opportunities in a number of different countries, there are many factors involved in the product function, which also affect the expected profit. If the total investments $\mathbf{I}$ of the corporation contains not only the internal funds $w$, but also some external funds $e$, the corporation needs to apply some derivative tools, such as linear hedging strategies forward and futures contract to hedge the total wealth $w$ so that it could coordinate the risk. There are two sections in this chapter, Section 2.1 is about the single period case and Section 2.2 is the multi period case. In addition, we present $n = 2$ to illustrate that the optimal hedge ratio $\mathbf{h}^*$ can be calculated in each section. When $n = 2$, the single period example was given in [6].

We assume that all the processes in this chapter are non-stochastic processes.

## 2.1 The Financial Environment - Single Period

In the first section of Chapter 2, we establish a model of solving the optimal hedge ratio of the total wealth $w_0$ to maximize the expected profit for $n$ variables, which is applied to some multinational corporations around the world.

Suppose that a multinational corporation has sales and production opportunities in a number of different countries. More than one factor complicate the hedging problem for multinational corporations, for example, the random exchange rate between countries, the random stock market price in different countries, the random price of goods in different countries, and so on. As a consequence, it is meaningful to build and solve an $n$ variable mathematical model to find an optimal hedging strategy, with which the expected profit of a multinational company can be maximized.

### 2.1.1 Construction of the $n$ Variables Model

To set up the model, we need to make some necessary assumptions. First, we assume that the financial market is complete, and we use the right superscription 0 to represent period zero in this section.

Assume that the multinational company can invest at $n$ different locations in period zero with the investments $(\mathbf{I^0})^T = (I_1^0,\, I_2^0, ..., \, I_n^0)$, and $\mathrm{cov}(I_i^0,\, I_j^0) = 0$ for each $i \neq j$.

$$\mathbf{f}(\mathbf{I^0}) : \mathbb{R}^n \rightarrow \mathbb{R}^n$$

are the product functions from $n$ locations in period zero, that is

$$\mathbf{I^0} \mapsto (f_1(\mathbf{I^0}), ..., f_n(\mathbf{I^0})).$$

Define the net present value of *investment expenditures*

$$\mathbf{F} : \mathbb{R}^n \rightarrow \mathbb{R},$$

given by

$$\mathbf{F}(\mathbf{I^0}) = (\theta^0)^T \cdot \mathbf{f}(\mathbf{I^0}) - (\gamma^0)^T \cdot \mathbf{I^0} \tag{2.1}$$

For all $1 \leq j \leq n$, let

$$(\theta^0)^T = \{\theta_1^0, ..., \theta_n^0\}, \qquad \theta_j^0 = \alpha_j^0(\epsilon_j^0 - \bar{\epsilon_j}^0) + 1,$$

$$(\gamma^0)^T = \{\gamma_1^0, ..., \gamma_n^0\}, \qquad \gamma_j^0 = \beta_j^0(\epsilon_j^0 - \bar{\epsilon_j}^0) + 1$$

where $\epsilon_j^0$, representing the home currency price of the foreign currency, is a random variable, with the mean $\bar{\epsilon_j}^0$, the variance $(\sigma_j^0)^2$ for all $1 \leq j \leq n$, and $\mathrm{cov}(\epsilon_i,\, \epsilon_j) = 0$ for $i \neq j$.

$0 \leq \alpha_j^0$, $\beta_j^0 \leq 1$ are parameters indexing the sensitivity of foreign revenues and foreign investment costs to the exchange rate in period zero.

Also assume that for each $1 \leq j \leq n$, the $j$th product function $f_j : \mathbb{R}^n \to \mathbb{R}$ have the first order derivative function:

$$\mathrm{D}f_j = \Big(\frac{\partial f_j}{\partial I_1}, \frac{\partial f_j}{\partial I_2}, ..., \frac{\partial f_j}{\partial I_n}\Big)$$

where $\dfrac{\partial f_j}{\partial I_i} > 0$ for all $1 \leq i, j \leq n$.

The corresponding first order derivative function in a matrix form is

$$\mathrm{D}\mathbf{f}(\mathbf{I^0}) = \begin{pmatrix} \frac{\partial f_1(\mathbf{I^0})}{\partial I_1^0} & \frac{\partial f_1(\mathbf{I^0})}{\partial I_2^0} & \cdots & \frac{\partial f_1(\mathbf{I^0})}{\partial I_n^0} \\ \frac{\partial f_2(\mathbf{I^0})}{\partial I_1^0} & \frac{\partial f_2(\mathbf{I^0})}{\partial I_2^0} & \cdots & \frac{\partial f_2(\mathbf{I^0})}{\partial I_n^0} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n(\mathbf{I^0})}{\partial I_1^0} & \frac{\partial f_n(\mathbf{I^0})}{\partial I_2^0} & \cdots & \frac{\partial f_n(\mathbf{I^0})}{\partial I_n^0} \end{pmatrix}$$

For all $j = 1, ..., n$, each component of the second derivative functions is a Hessian matrix given by,

$$\mathrm{D}^2 f_j(\mathbf{I^0}) = \begin{pmatrix} \frac{\partial^2 f_j(\mathbf{I^0})}{\partial (I_1^0)^2} & \frac{\partial^2 f_j(\mathbf{I^0})}{\partial I_2^0 \partial I_1^0} & \cdots & \frac{\partial^2 f_j(\mathbf{I^0})}{\partial I_n^0 \partial I_1^0} \\ \frac{\partial^2 f_j(\mathbf{I^0})}{\partial I_1^0 \partial I_2^0} & \frac{\partial^2 f_j(\mathbf{I^0})}{\partial (I_2^0)^2} & \cdots & \frac{\partial^2 f_j(\mathbf{I^0})}{\partial I_n^0 \partial I_2^0} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f_j(\mathbf{I^0})}{\partial I_1^0 \partial I_n^0} & \frac{\partial^2 f_j(\mathbf{I^0})}{\partial I_2^0 \partial I_n^0} & \cdots & \frac{\partial^2 f_j(\mathbf{I^0})}{\partial (I_n^0)^2} \end{pmatrix}$$

with $\dfrac{\partial^2 f_j}{\partial I_i \partial I_j} < 0$ for all $i, j$.

Let the external funds $e_1$ in period zero be

$$e_1 = (\gamma^0)^T \cdot \mathbf{I^0} - w_1,$$

then all the deadweight costs $C(e_1)$ are defined as:

$$C(e_1) : \mathbb{R} \to \mathbb{R}$$

with $C_{e_1} > 0$ and $C_{e_1 e_1} > 0$.

Using the product function $\mathbf{f}(\mathbf{I^0})$, the investments $\mathbf{I^0}$ at $n$ locations, and the dead weight cost function $C(e_1)$, we define the profits function $P(w_1)$ to be

$$P(w_1) = \max_{\mathbf{I^0}}\{\mathbf{F}(\mathbf{I}) - C(e_1)\} = \max_{\mathbf{I^0}}\{(\theta^0)^T \cdot \mathbf{f}(\mathbf{I^0}) - (\gamma^0)^T \cdot \mathbf{I^0} - C(e_1)\} \qquad (2.2)$$

In $P(w_1)$, $w_1$ is the amount of liquid assets in period zero, where $w_0$ is the initial total wealth of the corporation, and $w_1$ is defined to be a function of the random variable $\epsilon_j^0$. Thus, $w_1$ is also a random variable, and the issue of hedging the total wealth arises. If we consider the linear hedging strategies such as forward and futures contract, the hedge ratio $\mathbf{h}^0 = \{h_1^0, ..., h_n^0\}$ of $w_1$ appears. And we define

$$w_1 = w_0(h_1^0\epsilon_1^0 + h_2^0\epsilon_2^0 + \cdots + h_n^0\epsilon_n^0)$$

with

$$h_1^0 + \cdots + h_n^0 = 1$$

Once we give all the necessary definitions and assumptions, the problem now becomes: A corporation needs to adjust the hedging strategy to determine the hedge ratio $\mathbf{h}^0$, which will get the expected profit $\mathbf{E}[P(w_1)]$ maximized, and the corresponding hedge ratio will be the optimal hedge ratio and denoted by $\mathbf{h^0}^*$. As noted above, if $P(w_1)$ is a concave function, the random fluctuation in $\epsilon_j^0$ will reduce expected profits. In addition, only when $P_{w_1w_1} < 0$ for all $w_1$, hedging part of the total wealth could raise average the profits.

The goal of this section is to solve for the optimal hedge ratio $\mathbf{h^0}^*$ at period zero. To fulfill this purpose, we will find the first order condition of $P(w_1)$. After we obtain the first order condition of $P(w_1)$, compute the second order derivative $P_{w_1w_1}$, and show that $P_{w_1w_1} < 0$, which is a concave function. Then, use the Implicit Theorem to solve for $\dfrac{\partial \mathbf{I}^*}{\partial \epsilon_j}$, $1 \leq j \leq n$ as a vector. In addition, we apply the method of covariance in probability to gain $\text{cov}(P, \epsilon) = 0$, which gives us linear equation systems. Finally, we can solve for the optimal hedge ratio $\mathbf{h^0}^*$ in period zero.

### 2.1.2 Risk Management for Multinationals in Period Zero, $n \geq 2$

We start from the first order condition of the profit function $P(w_1)$, and notice that we can write the first order condition of $P(w_1)$ directly with respect to $I_j^0$ as follows:

$$\frac{\partial}{\partial I_j^0}\big(\mathbf{F}(\mathbf{I^0}) - C(e_1)\big) = 0, \quad 1 \leq j \leq n,$$

that is,

$$\begin{aligned}
\frac{\partial}{\partial I_j^0}\big(\mathbf{F}(\mathbf{I^0}) - C(e_1)\big) &= \frac{\partial}{\partial I_j^0}\big((\theta^0)^T \cdot \mathbf{f}(\mathbf{I^0}) - (\gamma^0)^T \cdot \mathbf{I^0} - C(e_1)\big) \\
&= (\theta^0)^T \cdot \frac{\partial}{\partial I_j^0}\mathbf{f}(\mathbf{I^0}) - (\gamma^0)^T \cdot \frac{\partial}{\partial I_j^0}\mathbf{I^0} - C_{e_1}\big((\gamma^0)^T \cdot \frac{\partial}{\partial I_j^0}\mathbf{I^0} - 0\big) \\
&= (\theta^0)^T \cdot \big(\mathrm{D}\mathbf{f}(\mathbf{I^0}) \cdot \mathbf{e}_j\big) - (1 + C_{e_1})(\gamma^0)^T \cdot \mathbf{e}_j \\
&= 0
\end{aligned}$$

In general, the first order condition of $P(w_1)$ is, for all $1 \leq j \leq n$,

$$(\theta^0)^T \cdot \mathrm{D}\mathbf{f}(\mathbf{I^0}) \cdot \mathbf{e}_j = \gamma_j^0(1 + C_{e_1}) \tag{2.3}$$

and the matrix form is given by:

$$(\mathrm{D}\mathbf{f}(\mathbf{I^0}))^T \cdot \theta^0 = (1 + C_{e_1})\,\gamma^0\,. \tag{2.4}$$

We can also obtain the following expression from 2.4,

$$1 + C_{e_1} = \frac{1}{\gamma_j^0}\big((\theta^0)^T \cdot \mathrm{D}\mathbf{f}(\mathbf{I^0}) \cdot \mathbf{e}_j\big) \tag{2.5}$$

which will be used later in this section.

Denote the first order condition of $P(w_1)$ as

$$(\mathbf{I^{0*}})^T = (I_1^{0^*},\, I_2^{0^*}, ...,\, I_n^{0^*})$$

Compute the Hessian matrices $\mathrm{D}^2 f_j(\mathbf{I^{0*}})$ for each component $f_j$ $(1 \leq j \leq n)$, and by assumption of $\mathbf{I^0}$, we have the following lemma:

**Lemma 2.1** $\mathrm{D}^2 f_j(\mathbf{I^{0*}})$ *is symmetric for all* $1 \le j \le n$.

*Proof.* Since all the $n$ investments are independent to each other, i.e. $\mathrm{cov}(I_i^0,\, I_j^0) = 0$ for $i \neq j$, the Hessian $\mathrm{D}^2 f_j(\mathbf{I^{0*}})$ are symmetric for all $1 \le j \le n$. ■

As a consequent, the tensor defined by

$$\mathrm{D}^2\mathbf{f}(\mathbf{I^{0*}})\Big(\frac{\mathrm{d}\mathbf{I^{0*}}}{\mathrm{d}w_1}\Big) \tag{2.6}$$

is symmetric, and assume it is negative definite.

**Proposition 2.1** *(i)* $P_{w_1} = C_{e_1}$.

*(ii)* $P_{w_1w_1} = (\theta^0)^T \cdot \mathrm{D}^2\mathbf{f}(\mathbf{I^{0*}})\Big(\frac{\mathrm{d}\mathbf{I^{0*}}}{\mathrm{d}w_1}\Big) \cdot \frac{\mathrm{d}\mathbf{I^{0*}}}{\mathrm{d}w_1} - C_{e_1e_1}\Big((\gamma^0)^T \cdot \frac{\mathrm{d}\mathbf{I^{0*}}}{\mathrm{d}w_1} - 1\Big)^2$

*Proof.* At the first order condition $(\mathbf{I^{0*}})^T$, compute the first and second order derivative functions of $P(w_1)$, where

$$P(w_1) = (\theta^0)^T \cdot \mathbf{f}(\mathbf{I^{0*}}) - (\gamma^0)^T \cdot \mathbf{I^{0*}} - C(e_1)$$

The first order derivative is:

$$\begin{aligned}
P_{w_1} &= (\theta^0)^T \cdot \mathrm{D}\mathbf{f}(\mathbf{I^{0*}}) \cdot \frac{\mathrm{d}\mathbf{I^{0*}}}{\mathrm{d}w_1} - (\gamma^0)^T \cdot \frac{\mathrm{d}\mathbf{I^{0*}}}{\mathrm{d}w_1} - C_{e_1}\big((\gamma^0)^T \cdot \frac{\mathrm{d}\mathbf{I^{0*}}}{\mathrm{d}w_1} - 1\big) \\
&= (\mathrm{D}\mathbf{f}(\mathbf{I^{0*}})^T \cdot \theta^0)^T \cdot \frac{\mathrm{d}\mathbf{I^{0*}}}{\mathrm{d}w_1} - (1 + C_{e_1})(\gamma^0)^T \cdot \frac{\mathrm{d}\mathbf{I^{0*}}}{\mathrm{d}w_1} + C_{e_1} \\
&= \big(\mathrm{D}\mathbf{f}(\mathbf{I^{0*}})^T \cdot \theta^0 - (1 + C_{e_1})\gamma^0\big)^T \cdot \frac{\mathrm{d}\mathbf{I^{0*}}}{\mathrm{d}w_1} + C_{e_1} \\
&= C_{e_1}
\end{aligned} \tag{2.7}$$

where the first order condition gives

$$\mathrm{D}\mathbf{f}(\mathbf{I^{0*}})^T \cdot \theta^0 = (1 + C_{e_1})\,\gamma^0$$

Apply the product rule to

$$P_{w_1} = (\theta^0)^T \cdot \mathrm{D}\mathbf{f}(\mathbf{I^{0*}}) \cdot \frac{\mathrm{d}\mathbf{I^{0*}}}{\mathrm{d}w_1} - (\gamma^0)^T \cdot \frac{\mathrm{d}\mathbf{I^{0*}}}{\mathrm{d}w_1} - C_{e_1}\big((\gamma^0)^T \cdot \frac{\mathrm{d}\mathbf{I^{0*}}}{\mathrm{d}w_1} - 1\big) \tag{2.8}$$

Then, the second order derivative function of $P(w_1)$ follows:

$$\begin{aligned} P_{w_1w_1} &= (\theta^0)^T \cdot \Big(\frac{\mathrm{d}\mathbf{I}^{\mathbf{0}*}}{\mathrm{d}w_1}\Big)^T \mathrm{D}^2\mathbf{f}(\mathbf{I}^{\mathbf{0}*}) \cdot \frac{\mathrm{d}\mathbf{I}^{\mathbf{0}*}}{\mathrm{d}w_1} + (\theta^0)^T \cdot \mathrm{D}\mathbf{f}(\mathbf{I}^{\mathbf{0}*}) \cdot \frac{\mathrm{d}^2\mathbf{I}^{\mathbf{0}*}}{\mathrm{d}w_1^2} - (\gamma^0)^T \cdot \frac{\mathrm{d}^2\mathbf{I}^{\mathbf{0}*}}{\mathrm{d}w_1^2} \\ &\quad - C_{e_1e_1}\big((\gamma^0)^T \cdot \frac{\mathrm{d}\mathbf{I}^{\mathbf{0}*}}{\mathrm{d}w_1} - 1\big)^2 - C_{e_1}\Big((\gamma^0)^T \cdot \frac{\mathrm{d}^2\mathbf{I}^{\mathbf{0}*}}{\mathrm{d}w_1^2}\Big) \\ &= (\theta^0)^T \cdot \mathrm{D}^2\mathbf{f}(\mathbf{I}^{\mathbf{0}*})\Big(\frac{\mathrm{d}\mathbf{I}^{\mathbf{0}*}}{\mathrm{d}w_1}\Big) \cdot \frac{\mathrm{d}\mathbf{I}^{\mathbf{0}*}}{\mathrm{d}w_1} - C_{e_1e_1}\big((\gamma^0)^T \cdot \frac{\mathrm{d}\mathbf{I}^{\mathbf{0}*}}{\mathrm{d}w_1} - 1\big)^2 \end{aligned} \tag{2.9}$$

■

**Proposition 2.2** *Suppose that the tensor* $\mathrm{D}^2\mathbf{f}(\mathbf{I}^{\mathbf{0}*})\Big(\frac{\mathrm{d}\mathbf{I}^{\mathbf{0}*}}{\mathrm{d}w_1}\Big)$ *is negative definite, then the second order derivative function* $P_{w_1w_1} < 0$ *and* $P(w_1)$ *is a concave function.*

*Proof.* Since the tensor $\mathrm{D}^2\mathbf{f}(\mathbf{I}^{\mathbf{0}*})\Big(\frac{\mathrm{d}\mathbf{I}^{\mathbf{0}*}}{\mathrm{d}w_1}\Big)$ is negative definite, it is clear that the inner product

$$\Big\langle \mathrm{D}^2\mathbf{f}(\mathbf{I}^{\mathbf{0}*})\Big(\frac{\mathrm{d}\mathbf{I}^{\mathbf{0}*}}{\mathrm{d}w_1}\Big), \frac{\mathrm{d}\mathbf{I}^{\mathbf{0}*}}{\mathrm{d}w_1}\Big\rangle < 0.$$

Then the first term of $P_{w_1w_1}$ in 2.9 is negative, and $C_{e_1e_1} > 0$, then

$$P_{w_1w_1} = (\theta^0)^T \cdot \mathrm{D}^2\mathbf{f}(\mathbf{I}^{\mathbf{0}*})\Big(\frac{\mathrm{d}\mathbf{I}^{\mathbf{0}*}}{\mathrm{d}w_1}\Big) \cdot \frac{\mathrm{d}\mathbf{I}^{\mathbf{0}*}}{\mathrm{d}w_1} - C_{e_1e_1}\big((\gamma^0)^T \cdot \frac{\mathrm{d}\mathbf{I}^{\mathbf{0}*}}{\mathrm{d}w_1} - 1\big)^2 < 0$$

Thus, $P(w_1)$ is a concave function.

■

There is a vector $\frac{\mathrm{d}\mathbf{I}^{\mathbf{0}*}}{\mathrm{d}w_1}$ in $P_{w_1w_1}$, and we can solve for this vector in order to simplify $P_{w_1w_1}$.

**Theorem 2.3** *Suppose that* $C_{e_1e_1} > 0$*, and* $\gamma_j^0 > 0$ *for all* $1 \leq j \leq n$*. Then*

$$\frac{\mathrm{d}\mathbf{I}^{\mathbf{0}*}}{\mathrm{d}w_1} = -C_{e_1e_1}\big((\theta^0)^T \cdot \mathrm{D}^2\mathbf{f}(\mathbf{I}^{\mathbf{0}*}) - C_{e_1e_1}(\gamma^0 \cdot (\gamma^0)^T)\big)^{-1} \cdot \gamma^0 \tag{2.10}$$

*Proof.* To solve for the vector $\frac{\mathrm{d}\mathbf{I}^{\mathbf{0}*}}{\mathrm{d}w_1}$, we apply the Implicit Theorem to the first order condition of $P(w_1)$ with respect to $w_1$, then

$$(\theta^0)^T \cdot \mathrm{D}^2\mathbf{f}(\mathbf{I}^{\mathbf{0}*})\Big(\frac{\mathrm{d}\mathbf{I}^{\mathbf{0}*}}{\mathrm{d}w_1}\Big) = C_{e_1e_1} \cdot \gamma^0 \cdot \big((\gamma^0)^T \cdot \frac{\mathrm{d}\mathbf{I}^{\mathbf{0}*}}{\mathrm{d}w_1} - 1\big)$$

Simplify and group the vector,

$$(\theta^0)^T \cdot \mathrm{D}^2\mathbf{f}(\mathbf{I^{0*}})\Big(\frac{\mathrm{d}\mathbf{I^{0*}}}{\mathrm{d}w_1}\Big) - C_{e_1e_1} \cdot (\gamma^0 \cdot (\gamma^0)^T) \cdot \frac{\mathrm{d}\mathbf{I^{0*}}}{\mathrm{d}w_1} = -C_{e_1e_1} \cdot \gamma^0$$

Since $C_{e_1e_1} > 0$ and $\gamma_j^0 > 0$ for all $1 \leq j \leq n$, the matrix $C_{e_1e_1}(\gamma^0 \cdot (\gamma^0)^T)$ is symmetric and positive definite, then $-C_{e_1e_1}(\gamma^0 \cdot (\gamma^0)^T)$ is negative definite.

Define a matrix

$$\mathbf{A}_0 := (\theta^0)^T \cdot \mathrm{D}^2\mathbf{f}(\mathbf{I^{0*}}) - C_{e_1e_1}(\gamma^0 \cdot (\gamma^0)^T) \tag{2.11}$$

which is symmetric and negative definite.

Since $\theta^0$ and $\gamma^0$ are random variables with probability, $\mathbf{A}_0$ is almost invertible. If $\mathbf{A}_0$ is symmetric and negative definite, $\mathbf{A}_0^{-1}$ is also symmetric and negative definite.

Therefore, the vector is solved as

$$\begin{aligned}\frac{\mathrm{d}\mathbf{I^{0*}}}{\mathrm{d}w_1} &= -C_{e_1e_1}\big((\theta^0)^T \cdot \mathrm{D}^2\mathbf{f}(\mathbf{I^{0*}}) - C_{e_1e_1}(\gamma^0 \cdot (\gamma^0)^T)\big)^{-1} \cdot \gamma^0 \\ &= -C_{e_1e_1}\,\mathbf{A}_0^{-1} \cdot \gamma^0\end{aligned}$$

■

**Proposition 2.4** *Let $\mathbf{A}_0$ be the matrix 2.11 defined in Theorem 2.3. The vector $\dfrac{\mathrm{d}\mathbf{I^{0*}}}{\mathrm{d}w_1}$ is also given by the Equation 2.10 in Theorem 2.3, then the second derivative function $P_{w_1w_1}$ is*

$$P_{w_1w_1} = -C_{e_1e_1} - C_{e_1e_1}^2(\gamma^0)^T \cdot \mathbf{A}_0^{-1} \cdot \gamma^0 \tag{2.12}$$

*Proof.* The vector $\dfrac{\mathrm{d}\mathbf{I^{0*}}}{\mathrm{d}w_1} = -C_{e_1e_1}\mathbf{A}_0^{-1} \cdot \gamma^0$ in Theorem 2.3, then:

$$
\begin{aligned}
P_{w_1w_1} &= (\theta^0)^T \cdot \mathrm{D}^2\mathbf{f}(\mathbf{I^{0*}})\Big(\frac{\mathrm{d}\mathbf{I^{0*}}}{\mathrm{d}w_1}\Big) \cdot \frac{\mathrm{d}\mathbf{I^{0*}}}{\mathrm{d}w_1} - C_{e_1e_1}\big((\gamma^0)^T \cdot \frac{\mathrm{d}\mathbf{I^{0*}}}{\mathrm{d}w_1} - 1\big)^2 \\
&= \Big(\frac{\mathrm{d}\mathbf{I^{0*}}}{\mathrm{d}w_1}\Big)^T \big((\theta^0)^T \cdot \mathrm{D}^2\mathbf{f}(\mathbf{I^{0*}})\big) \cdot \frac{\mathrm{d}\mathbf{I^{0*}}}{\mathrm{d}w_1} - C_{e_1e_1}\big((\gamma^0)^T \cdot \frac{\mathrm{d}\mathbf{I^{0*}}}{\mathrm{d}w_1} \cdot (\gamma^0)^T \cdot \frac{\mathrm{d}\mathbf{I^{0*}}}{\mathrm{d}w_1} - 2(\gamma^0)^T \cdot \frac{\mathrm{d}\mathbf{I^{0*}}}{\mathrm{d}w_1} + 1\big) \\
&= \Big(\frac{\mathrm{d}\mathbf{I^{0*}}}{\mathrm{d}w_1}\Big)^T \cdot \big((\theta^0)^T \cdot \mathrm{D}^2\mathbf{f}(\mathbf{I^{0*}}) - C_{e_1e_1}(\gamma^0 \cdot (\gamma^0)^T)\big) \cdot \frac{\mathrm{d}\mathbf{I^{0*}}}{\mathrm{d}w_1} - C_{e_1e_1}\big(1 - 2(\gamma^0)^T \cdot \frac{\mathrm{d}\mathbf{I^{0*}}}{\mathrm{d}w_1}\big) \\
&= (-C_{e_1e_1}\mathbf{A}_0^{-1} \cdot \gamma^0)^T \cdot \mathbf{A}_0 \cdot (-C_{e_1e_1}\mathbf{A}_0^{-1} \cdot \gamma^0) - C_{e_1e_1}\big(1 - 2(\gamma^0)^T \cdot (-C_{e_1e_1}\mathbf{A}_0^{-1} \cdot \gamma^0)\big) \\
&= C_{e_1e_1}^2(\gamma^0)^T \cdot (\mathbf{A}_0^{-1})^T \cdot \mathbf{A}_0 \cdot \mathbf{A}_0^{-1}\gamma^0 - C_{e_1e_1} - 2C_{e_1e_1}^2(\gamma^0)^T \cdot \mathbf{A}_0^{-1} \cdot \gamma^0 \\
&= -C_{e_1e_1} - C_{e_1e_1}^2(\gamma^0)^T \cdot \mathbf{A}_0^{-1} \cdot \gamma^0
\end{aligned}
$$

Thus, $P_{w_1w_1}$ is simplified as Equation 2.9. ■

**Theorem 2.5** *Suppose that $C_{e_1e_1} > 0$ and $\gamma_j^0 > 0$ for all $1 \le j \le n$. Then*

$$
\frac{\partial \mathbf{I^{0*}}}{\partial \epsilon_j^0} = \mathbf{A}_0^{-1} \cdot \Big\{C_{e_1e_1}(\beta_j^0 I_j^{0^*} - w_0h_j^0) \cdot \gamma^0 + \Big(\frac{\beta_j^0}{\gamma_j^0}((\theta^0)^T \cdot \mathrm{D}\mathbf{f}(\mathbf{I^{0*}}) \cdot \mathbf{e}_j) - \alpha_j^0 \mathrm{D}\mathbf{f}(\mathbf{I^{0*}})\Big) \cdot \mathbf{e}_j\Big\} \tag{2.13}
$$

*Proof.* Similar to Theorem 2.3, we apply the Implicit Theorem to the first order condition of $P(w_1)$:

$$
(\theta^0)^T \cdot \mathrm{D}\mathbf{f}(\mathbf{I^{0*}}) \cdot \mathbf{e}_j = \gamma_j^0(1 + C_{e_1})
$$

with respect to $\epsilon_j^0$, for all $1 \le j \le n$.

That is

$$
\begin{aligned}
&(\theta^0)^T \cdot \big(\mathrm{D}^2\mathbf{f}(\mathbf{I^{0*}}) \cdot \mathbf{e}_j\big) \cdot \frac{\partial \mathbf{I^{0*}}}{\partial \epsilon_j^0} + \alpha_j^0 \mathbf{e}_j^T \cdot \mathrm{D}\mathbf{f}(\mathbf{I^{0*}}) \cdot \mathbf{e}_j = \beta_j^0(1 + C_{e_1}) \\
&\quad + \gamma_j^0 C_{e_1e_1}\big((\gamma^0)^T \cdot \frac{\partial \mathbf{I^{0*}}}{\partial \epsilon_j^0} + \beta_j^0 I_j^{0^*} - w_0h_j^0\big) \ \Rightarrow \\
&(\theta^0)^T \mathrm{D}^2\mathbf{f}(\mathbf{I^{0*}}) \cdot \frac{\partial \mathbf{I^{0*}}}{\partial \epsilon_j^0} - C_{e_1e_1}(\gamma^0 \cdot (\gamma^0)^T) \cdot \frac{\partial \mathbf{I^{0*}}}{\partial \epsilon_j^0} = C_{e_1e_1}(\beta_j^0 I_j^{0^*} - w_0h_j^0) \cdot \gamma^0 \\
&\quad + (1 + C_{e_1})\beta_j^0 \cdot \mathbf{e}_j - \alpha_j^0 \mathrm{D}\mathbf{f}(\mathbf{I^{0*}}) \cdot \mathbf{e}_j \ \Rightarrow
\end{aligned}
$$

$$\Big((\theta^0)^T \cdot \mathrm{D}^2\mathbf{f}(\mathbf{I^{0*}}) - C_{e_1e_1}(\gamma^0 \cdot (\gamma^0)^T)\Big) \cdot \frac{\partial \mathbf{I^{0*}}}{\partial \epsilon_j^0} = C_{e_1e_1}(\beta_j^0 I_j^{0^*} - w_0 h_j^0) \cdot \gamma^0$$
$$+ \Big(\frac{\beta_j^0}{\gamma_j^0}((\theta^0)^T \cdot \mathrm{D}\mathbf{f}(\mathbf{I^{0*}}) \cdot \mathbf{e}_j) - \alpha_j^0 \mathrm{D}\mathbf{f}(\mathbf{I^{0*}})\Big) \cdot \mathbf{e}_j$$

Then, the vector $\frac{\partial \mathbf{I^{0*}}}{\partial \epsilon_j^0}$ is as follows:

$$\frac{\partial \mathbf{I^{0*}}}{\partial \epsilon_j^0} = \mathbf{A}_0^{-1} \cdot \Big\{C_{e_1e_1}(\beta_j^0 I_j^{0^*} - w_0 h_j^0) \cdot \gamma^0 + \Big(\frac{\beta_j^0}{\gamma_j^0}((\theta^0)^T \cdot \mathrm{D}\mathbf{f}(\mathbf{I^{0*}}) \cdot \mathbf{e}_j) - \alpha_j^0 \mathrm{D}\mathbf{f}(\mathbf{I^{0*}})\Big) \cdot \mathbf{e}_j\Big\}$$

■

We have gained the vectors and matrices that are needed to maximize the expected profit function $\mathbf{E}[P(w_1)]$ in period zero with the hedge ratio $\mathbf{h^0}^T = (h_1^0, ..., h_n^0)$ at the vector $(\mathbf{I^{0*}})^T = (I_1^{0^*}, ..., I_n^{0^*})$. The process of maximizing $\mathbf{E}[P(w_1)]$ will give the solution of the optimal hedge ratio $\mathbf{h^{0*}}$.

Remember in Section 2.1, we assume that the covariance between distinct $\epsilon_i$ and $\epsilon_j$ is 0, i.e. $\mathrm{cov}(\epsilon_i^0, \epsilon_j^0) = 0, i \neq j$.

**Lemma 2.2** *[14][Appendix] If $x$ and $y$ are normally distributed, and $a(x)$ and $b(y)$ are differentiable functions, then*

$$cov(a(x), b(y)) = \mathbf{E}_x[a_x]\mathbf{E}_y[b_y]cov(x, y) \tag{2.14}$$

We also need

$$P_{w_1} = C_{e_1}$$

and

$$P_{w_1w_1} = -C_{e_1e_1} - C_{e_1e_1}^2(\gamma^0)^T \cdot \mathrm{A}_0^{-1} \cdot \gamma^0$$

to solve for the optimal hedge ratio $\mathbf{h^{0*}}$.

**Theorem 2.6** *Suppose that all the assumptions in this section are satisfied. Also assume that at* $\mathbf{I^{0*}}$*,* $P(w_1)$ *is maximized and* $\mathrm{cov}(\epsilon_i^0, \epsilon_j^0) = 0$ *for* $i \neq j$*. Then, for all* $1 \leq j \leq n$*,* $h_j^{0^*}$*, the* $j^{th}$ *component of the optimal hedge ratio*

$$\mathbf{h^{0^*}} = \{h_1^{0^*}, h_2^{0^*}, ..., h_n^{0^*}\}$$

*is given by*

$$\begin{aligned} h_j^{0^*} &= \frac{\mathbf{E}[P_{w_1}]}{(\sigma_j^0)^2 w_0 \mathbf{E}_{\epsilon_j^0}[P_{w_1 w_1}]} + \beta_j^0 \frac{\mathbf{E}_{\epsilon_j^0}[I_j^{0^*} P_{w_1 w_1}]}{w_0 \mathbf{E}_{\epsilon_j^0}[P_{w_1 w_1}]} \\ &- \frac{\mathbf{E}_{\epsilon_j^0}\big[C_{e_1 e_1}(\gamma^0)^T \cdot \mathbf{A}_0^{-1} \cdot \big(\frac{\beta_j^0}{\gamma_j^0}((\theta^0)^T \cdot \mathrm{D}\mathbf{f}(\mathbf{I^{0^*}}) \cdot \mathbf{e}_j) - \alpha_j^0 \mathrm{D}\mathbf{f}(\mathbf{I^{0^*}})\big) \cdot \mathbf{e}_j\big]}{w_0 \mathbf{E}_{\epsilon_j^0}[P_{w_1 w_1}]} \end{aligned} \tag{2.15}$$

*Proof.* Since $w_1 = w_0(h_1^0 \epsilon_1^0 + \cdots + h_n^0 \epsilon_n^0)$, then for all $1 \leq j \leq n$,

$$\frac{\partial\, w_1}{\partial\, h_j^0} = w_0 \epsilon_j^0$$

If order to maximize $\mathbf{E}[P(w_1)]$, we require the corresponding first order condition:

$$\mathbf{E}\big[P_{w_1} \cdot \frac{\partial w_1}{\partial h_j^0}\big] = 0 \tag{2.16}$$

for all $1 \leq j \leq n$.

By the definition of covariance between two random variables and Equation 2.14, we derive the first order condition 2.16, and obtain:

$$\mathbf{E}\big[P_{w_1} \cdot \frac{\partial w_1}{\partial h_j^0}\big] = \mathbf{E}[P_{w_1} \cdot w_0 \epsilon_j^0] = w_0 \mathbf{E}[P_{w_1} \cdot \epsilon_j^0] = 0 \Leftrightarrow$$

$$\mathbf{E}[P_{w_1} \cdot \epsilon_j^0] = \mathbf{E}[P_{w_1}] \cdot \mathbf{E}[\epsilon_j^0] - \mathrm{cov}(P_{w_1}, \epsilon_j^0) = 0 \text{ (since } \mathbf{E}[\epsilon_j^0] = 1) \Leftrightarrow$$

$$\mathbf{E}[P_{w_1}] - \mathbf{E}_{\epsilon_j^0}[P_{w_1 \epsilon_j^0}] \cdot \mathbf{E}_{\epsilon_j^0}[\epsilon_j^0] \cdot \mathrm{cov}(\epsilon_j^0, \epsilon_j^0) = 0 \Leftrightarrow$$

$$\mathbf{E}[P_{w_1} \cdot \epsilon_j^0] = \mathbf{E}[P_{w_1}] - \mathbf{E}_{\epsilon_j^0}[P_{w_1 \epsilon_j^0}] \cdot (\sigma_j^0)^2 = 0.$$

Since $P_{w_1} = C_{e_1}$,

$$\mathbf{E}_{\epsilon_j^0}[P_{w_1 \epsilon_j^0}] = \mathbf{E}_{\epsilon_j^0}[(P_{w_1})_{\epsilon_j^0}] = \mathbf{E}_{\epsilon_j^0}[(C_{e_1})_{\epsilon_j^0}] = \mathbf{E}_{\epsilon_j^0}\big[C_{e_1 e_1} \cdot \frac{\partial e_1}{\partial \epsilon_j^0}\big]$$

that is,

$$\mathbf{E}[P_{w_1}] - (\sigma_j^0)^2 \mathbf{E}_{\epsilon_j^0}\big[C_{e_1e_1} \cdot \frac{\partial e_1}{\partial \epsilon_j^0}\big] = 0 \tag{2.17}$$

By the definition of the external funds $e_1$, we have the derivative of $e_1$ with respect to $\epsilon_j^0$ as following:

$$\frac{\partial e_1}{\partial \epsilon_j^0} = (\gamma^0)^T \cdot \frac{\partial \mathbf{I^{0*}}}{\partial \epsilon_j^0} + \beta_j^0 I_j^{0^*} - w_0 h_j^0$$

for all $1 \leq j \leq n$.

Put it into Equation 2.17,

$$\mathbf{E}[P_{w_1}] - (\sigma_j^0)^2 \mathbf{E}_{\epsilon_j^0}\big[C_{e_1e_1}((\gamma^0)^T \cdot \frac{\partial \mathbf{I^{0^*}}}{\partial \epsilon_j^0} + \beta_j^0 I_j^{0^*} - w_0 h_j^0)\big] = 0$$

Simplify and rearrange it,

$$\mathbf{E}[P_{w_1}] - (\sigma_j^0)^2 \mathbf{E}_{\epsilon_j^0}\big[C_{e_1e_1}((\gamma^0)^T \cdot \frac{\partial \mathbf{I^{0^*}}}{\partial \epsilon_j^0})\big] - (\sigma_j^0)^2 \mathbf{E}_{\epsilon_j^0}[C_{e_1e_1}\beta_j^0 I_j^{0^*}] + (\sigma_j^0)^2 \mathbf{E}_{\epsilon_j^0}[C_{e_1e_1} w_0 h_j^0] = 0$$

for all $1 \leq j \leq n$.

Replace the vector $\dfrac{\partial \mathbf{I^{0*}}}{\partial \epsilon_j^0}$ in Equation 2.13, Equation 2.17 becomes

$$\begin{aligned} &\mathbf{E}[P_{w_1}] - (\sigma_j^0)^2 \mathbf{E}_{\epsilon_j^0}\big[C_{e_1e_1}(\gamma^0)^T \cdot \mathbf{A}_0^{-1} \cdot (C_{e_1e_1}(\beta_j^0 I_j^{0^*} - w_0 h_j^0) \cdot \gamma^0 \\ &+ \big(\frac{\beta_j^0}{\gamma_j^0}((\theta^0)^T \cdot \mathrm{D}\mathbf{f}(\mathbf{I^{0*}}) \cdot \mathbf{e}_j) - \alpha_j^0 \mathrm{D}\mathbf{f}(\mathbf{I^{0*}})\big) \cdot \mathbf{e}_j)\big] \\ &- (\sigma_j^0)^2 \beta_j^0 \mathbf{E}_{\epsilon_j^0}[C_{e_1e_1} I_j^{0^*}] + (\sigma_j^0)^2 w_0 h_j^0 \mathbf{E}_{\epsilon_j^0}[C_{e_1e_1}] = 0 \end{aligned}$$

Expand every term of the left hand side of this equation,

$$\begin{aligned} &\mathbf{E}[P_{w_1}] - (\sigma_j^0)^2\, \mathbf{E}_{\epsilon_j^0}\big[C_{e_1e_1}^2 (\gamma^0)^T \cdot \mathbf{A}_0^{-1} \cdot (\beta_j^0 I_j^{0^*}) \cdot \gamma^0\big] + (\sigma_j^0)^2\, w_0\, h_j^0\, \mathbf{E}_{\epsilon_j^0}\big[C_{e_1e_1}^2 (\gamma^0)^T \cdot \mathbf{A}_0^{-1} \cdot \gamma^0\big] \\ &- (\sigma_j^0)^2\, \frac{\beta_j^0}{\gamma_j^0}\, \mathbf{E}_{\epsilon_j^0}\big[C_{e_1e_1}(\gamma^0)^T \cdot \mathbf{A}_0^{-1} \cdot ((\theta^0)^T \cdot \mathrm{D}\mathbf{f}(\mathbf{I^{0*}}) \cdot \mathbf{e}_j) \cdot \mathbf{e}_j\big] \\ &+ (\sigma_j^0)^2\, \alpha_j^0\, \mathbf{E}_{\epsilon_j^0}\big[C_{e_1e_1}(\gamma^0)^T \cdot \mathbf{A}_0^{-1} \cdot \mathrm{D}\mathbf{f}(\mathbf{I^{0*}}) \cdot \mathbf{e}_j\big] \\ &- (\sigma_j^0)^2\, \beta_j^0\, \mathbf{E}_{\epsilon_j^0}\big[C_{e_1e_1} I_j^{0^*}\big] + (\sigma_j^0)^2 w_0 h_j^0 \mathbf{E}_{\epsilon_j^0}\big[C_{e_1e_1}\big] = 0 \end{aligned}$$

Put all the coefficients out of the expectations, and group those terms involved in $h_j^0$ together,

$$\begin{aligned}
&\mathbf{E}[P_{w_1}] - (\sigma_j^0)^2\, \beta_j^0\, \mathbf{E}_{\epsilon_j^0}\big[I_j^{0^*} C_{e_1e_1}^2 ((\gamma^0)^T \cdot \mathbf{A}_0^{-1} \cdot \gamma^0)\big] \\
&- (\sigma_j^0)^2\, \frac{\beta_j^0}{\gamma_j^0}\, \mathbf{E}_{\epsilon_j^0}\big[C_{e_1e_1}((\theta^0)^T \cdot \mathrm{D}\mathbf{f}(\mathbf{I^{0*}}) \cdot \mathbf{e}_j)((\gamma^0)^T \cdot \mathbf{A}_0^{-1} \cdot \mathbf{e}_j)\big] \\
&+ (\sigma_j^0)^2\, \alpha_j^0\, \mathbf{E}_{\epsilon_j^0}\big[C_{e_1e_1}((\gamma^0)^T \cdot \mathbf{A}_0^{-1} \cdot \mathrm{D}\mathbf{f}(\mathbf{I^{0*}}) \cdot \mathbf{e}_j)\big] - (\sigma_j^0)^2 \beta_j^0 \mathbf{E}_{\epsilon_j^0}[C_{e_1e_1} I_j^{0^*}] \\
&= -(\sigma_j^0)^2\, w_0\, h_j^0\, \mathbf{E}_{\epsilon_j^0}\big[C_{e_1e_1}^2 (\gamma^0)^T \cdot \mathbf{A}_0^{-1} \cdot \gamma^0\big] - (\sigma_j^0)^2\, w_0\, h_j^0\, \mathbf{E}_{\epsilon_j^0}\big[C_{e_1e_1}\big]
\end{aligned}$$

Rearrange and group some of the terms of both sides,

$$\begin{aligned}
&\mathbf{E}[P_{w_1}] + (\sigma_j^0)^2\, \beta_j^0\, \mathbf{E}_{\epsilon_j^0}\big[I_j^{0^*}(-C_{e_1e_1} - C_{e_1e_1}^2 (\gamma^0)^T \cdot \mathbf{A}_0^{-1} \cdot \gamma^0)\big] \\
&- (\sigma_j^0)^2\, \frac{\beta_j^0}{\gamma_j^0}\, \mathbf{E}_{\epsilon_j^0}\big[C_{e_1e_1}(\gamma^0)^T \cdot \mathbf{A}_0^{-1} \cdot ((\theta^0)^T \cdot \mathrm{D}\mathbf{f}(\mathbf{I^{0*}}) \cdot \mathbf{e}_j) \cdot \mathbf{e}_j\big] \\
&+ (\sigma_j^0)^2\, \alpha_j^0\, \mathbf{E}_{\epsilon_j^0}\big[C_{e_1e_1}(\gamma^0)^T \cdot \mathbf{A}_0^{-1} \cdot \mathrm{D}\mathbf{f}(\mathbf{I^{0*}}) \cdot \mathbf{e}_j\big] \\
&= (\sigma_j^0)^2\, w_0\, \mathbf{E}_{\epsilon_j^0}\big[-C_{e_1e_1} - C_{e_1e_1}^2 (\gamma^0)^T \cdot \mathbf{A}_0^{-1} \cdot \gamma^0\big]\, h_j^0
\end{aligned}$$

Since

$$P_{w_1w_1} = -C_{e_1e_1} - C_{e_1e_1}^2 (\gamma^0)^T \cdot \mathbf{A}_0^{-1} \cdot \gamma^0$$

then, the above equation is,

$$\begin{aligned}
&\mathbf{E}[P_{w_1}] + (\sigma_j^0)^2\, \beta_j^0\, \mathbf{E}_{\epsilon_j^0}\big[I_j^{0^*} P_{w_1w_1}\big] \\
&- (\sigma_j^0)^2\, \mathbf{E}_{\epsilon_j^0}\big[C_{e_1e_1}(\gamma^0)^T \cdot \mathbf{A}_0^{-1} \cdot (\frac{\beta_j^0}{\gamma_j^0}((\theta^0)^T \cdot \mathrm{D}\mathbf{f}(\mathbf{I^{0*}}) \cdot \mathbf{e}_j) - \alpha_j^0 \mathrm{D}\mathbf{f}(\mathbf{I^{0*}})) \cdot \mathbf{e}_j\big] \\
&= (\sigma_j^0)^2\, w_0\, \mathbf{E}_{\epsilon_j^0}\big[P_{w_1w_1}\big]\, h_j^0
\end{aligned}$$

When the first order condition of $\mathbf{E}\big[P(w_1)\big]$ is zero, it will be maximized. We denote the corresponding hedge ratio $\mathbf{h^0}$ by

$$\mathbf{h^{0*}} = \{h_1^{0^*},\, h_2^{0^*}, ...,\, h_n^{0^*}\}$$

Finally, we solve $h_j^{0^*}$ in the above equation for all $1 \le j \le n$, and we gain

$$h_j^{0^*} = \frac{\mathbf{E}[P_{w_1}]}{(\sigma_j^0)^2\, w_0\, \mathbf{E}_{\epsilon_j^0}\big[P_{w_1w_1}\big]} + \beta_j^0\, \frac{\mathbf{E}_{\epsilon_j^0}\big[I_j^{0^*} P_{w_1w_1}\big]}{w_0\, \mathbf{E}_{\epsilon_j^0}\big[P_{w_1w_1}\big]}$$
$$- \frac{\mathbf{E}_{\epsilon_j^0}\big[C_{e_1e_1}(\gamma^0)^T \cdot \mathbf{A}_0^{-1} \cdot \big(\frac{\beta_j^0}{\gamma_j^0}((\theta^0)^T \cdot \mathrm{D}\mathbf{f}(\mathbf{I^{0^*}}) \cdot \mathbf{e}_j) - \alpha_j^0\, \mathrm{D}\mathbf{f}(\mathbf{I^{0^*}})\big) \cdot \mathbf{e}_j\big]}{w_0\, \mathbf{E}_{\epsilon_j^0}\big[P_{w_1w_1}\big]}$$

■

### 2.1.3 When $n = 2$, The Optimal Hedge Ratio $\mathbf{h^{0^*}}$

In this section, we calculate $n = 2$ case. In [6], the authors presented a similar example for $n = 2$.

When $n = 2$ in period zero, $(\mathbf{I^0})^T = (I_1^0,\, I_2^0)$ and

$$P(w_1) = \max_{\mathbf{I^0}}\{\theta_1^0 f_1(I_1^0) + \theta_2^0 f_2(I_2^0) - \gamma_1^0 I_1^0 - \gamma_2^0 I_2^0 - C(e_1)\}$$

with the following settings,

$w_1 = w_0(h_1^0\epsilon_1^0 + h_2^0\epsilon_2^0)$, $h_1^0 + h_2^0 = 1$, $e_1 = \gamma_1^0 I_1^0 + \gamma_2^0 I_2^0 - w_1$

$\theta_j^0 = \alpha_j^0(\epsilon_j^0 - \bar{\epsilon_j}^0) + 1$, $\gamma_j^0 = \beta_j^0(\epsilon_j^0 - \bar{\epsilon_j}^0) + 1$, $\epsilon_j^0 \sim \mathbf{N}\big(\bar{\epsilon_j}^0, (\sigma_j^0)^2\big)$, $j = 1, 2$

$\mathrm{D}f_1(I_1) = \dfrac{\partial f_1(I_1^0)}{\partial I_1^0}$, and $\mathrm{D}f_2(I_2) = \dfrac{\partial f_2(I_2^0)}{\partial I_2^0}$,

$\mathrm{D}^2 f_1(I_1^0) = \dfrac{\partial^2 f_1(I_1^0)}{\partial (I_1^0)^2}$ and $\mathrm{D}^2 f_2(I_2^0) = \dfrac{\partial^2 f_2(I_2^0)}{\partial (I_2^0)^2}$

We will solve for the optimal hedge ratio $\mathbf{h^{0^*}}$ in the following steps:

STEP 1:

Obtain the first order condition of $P(w_1)$ with respect to $\mathbf{I^0}$

$$\theta_1^0 \frac{\partial f_1(I_1^0)}{\partial I_1^0} = (1 + C_{e_1})\gamma_1^0\,, \qquad \theta_2^0 \frac{\partial f_2(I_2^0)}{\partial I_2^0} = (1 + C_{e_1})\gamma_2^0$$

STEP 2:

Compute $P_{w_1}$ and $P_{w_1w_1}$ at $\mathbf{I^0}^*$. Notice that from Equation 2.7, we have $P_{w_1} = C_{e_1}$. Differentiate $P_{w_1}$ in 2.8, $P_{w_1w_1}$ follows:

$$P_{w_1w_1} = \theta_1^0\Big(\frac{\partial^2 f_1(I_1^{0^*})}{\partial(I_1^0)^2}(\frac{\mathrm{d}I_1^{0^*}}{\mathrm{d}w_1})^2\Big)+\theta_2^0\Big(\frac{\partial^2 f_2(I_2^{0^*})}{\partial(I_2^0)^2}(\frac{\mathrm{d}I_2^{0^*}}{\mathrm{d}w_1})^2\Big)-C_{e_1e_1}\Big(\gamma_1^0\frac{\mathrm{d}I_1^{0^*}}{\mathrm{d}w_1}+\gamma_2^0\frac{\mathrm{d}I_2^{0^*}}{\mathrm{d}w_1}-1\Big)^2$$

Step 3:

Solve for the vector $\frac{\mathrm{d}\mathbf{I^0}^*}{\mathrm{d}w_1}$ and simplify $P_{w_1w_1}$.

Apply the Implicit Theorems to the first order condition in Step 1 and get,

$$\theta_1^0\Big(\frac{\partial^2 f_1(I_1^{0^*})}{\partial(I_1^{0^*})^2}\frac{\mathrm{d}I_1^{0^*}}{\mathrm{d}w_1}\Big) - \gamma_1^0 C_{e_1e_1}\Big(\gamma_1^0\frac{\mathrm{d}I_1^{0^*}}{\mathrm{d}w_1} + \gamma_2^0\frac{\mathrm{d}I_2^{0^*}}{\mathrm{d}w_1} - 1\Big) = 0$$

$$\theta_2^0\Big(\frac{\partial^2 f_2(I_2^{0^*})}{\partial(I_2^{0^*})^2}\frac{\mathrm{d}I_2^{0^*}}{\mathrm{d}w_1}\Big) - \gamma_2^0 C_{e_1e_1}\Big(\gamma_1^0\frac{\mathrm{d}I_1^{0^*}}{\mathrm{d}w_1} + \gamma_2^0\frac{\mathrm{d}I_2^{0^*}}{\mathrm{d}w_1} - 1\Big) = 0$$

Similarly, the matrix $\mathbf{A}_0$ is:

$$\mathbf{A}_0 = \begin{pmatrix} \theta_1^0\mathrm{D}^2 f_1(I_1^{0^*}) & 0 \\ 0 & \theta_2^0\mathrm{D}^2 f_2(I_2^{0^*}) \end{pmatrix} - C_{e_1e_1}\begin{pmatrix} (\gamma_1^0)^2 & \gamma_1^0\gamma_2^0 \\ \gamma_2^0\gamma_1^0 & (\gamma_2^0)^2 \end{pmatrix}$$

The vector $\frac{\mathrm{d}\mathbf{I^0}^*}{\mathrm{d}w_1}$ is given by

$$\begin{pmatrix} \frac{\mathrm{d}I_1^{0^*}}{\mathrm{d}w_1} \\ \frac{\mathrm{d}I_2^{0^*}}{\mathrm{d}w_1} \end{pmatrix} = -C_{e_1e_1}\mathbf{A}_0^{-1}\cdot\begin{pmatrix} \gamma_1^0 \\ \gamma_2^0 \end{pmatrix}$$

Simplify $P_{w_1w_1}$ as

$$P_{w_1w_1} = -C_{e_1e_1} - C_{e_1e_1}^2\begin{pmatrix} \gamma_1^0 & \gamma_2^0 \end{pmatrix}\cdot\mathbf{A}_0^{-1}\cdot\begin{pmatrix} \gamma_1^0 \\ \gamma_2^0 \end{pmatrix}$$

Step 4:

Solve for the vector $\frac{\partial\mathbf{I^0}^*}{\partial\epsilon_j}$ for $j = 1, 2$.

Again, we differentiate the first order condition in Step 1 with respect to $\epsilon_1$,

$$\theta_1^0\Big(\frac{\partial^2 f_1(I_1^{0^*})}{\partial (I_1^{0^*})^2}\cdot\frac{\partial I_1^{0^*}}{\partial \epsilon_1^0}\Big)+\alpha_1^0\frac{\partial f_1(I_1^{0^*})}{\partial I_1^{0^*}} = \beta_1^0(1+C_{e_1})+\gamma_1^0 C_{e_1e_1}\Big(\gamma_1^0\frac{\partial I_1^{0^*}}{\partial \epsilon_1^0}+\beta_1^0 I_1^{0^*}+\gamma_2^0\frac{\partial I_2^{0^*}}{\partial \epsilon_1^0}-w_0h_1^0\Big)$$

$$\theta_2^0\Big(\frac{\partial^2 f_2(I_2^{0^*})}{\partial (I_2^{0^*})^2}\cdot\frac{\partial I_2^{0^*}}{\partial \epsilon_1^0}\Big) = 0(1+C_{e_1})+\gamma_2^0 C_{e_1e_1}\Big(\gamma_1^0\frac{\partial I_1^{0^*}}{\partial \epsilon_1^0}+\beta_1^0 I_1^{0^*}+\gamma_2^0\frac{\partial I_2^{0^*}}{\partial \epsilon_1^0}-w_0h_1^0\Big)$$

Write this in the vector and matrix form:

$$\Big[\theta_1^0\begin{pmatrix}\frac{\partial^2 f_1(\mathbf{I^{0*}})}{\partial (I_1^{0^*})^2} & 0\end{pmatrix}+\theta_2^0\begin{pmatrix}0 & 0\end{pmatrix}\Big]\begin{pmatrix}\frac{\partial I_1^{0^*}}{\partial \epsilon_1^0}\\ \frac{\partial I_2^{0^*}}{\partial \epsilon_1^0}\end{pmatrix}+\alpha_1^0\frac{\partial f_1(\mathbf{I^{0*}})}{\partial I_1^{0^*}}$$

$$= \beta_1^0(1+C_{e_1})+\gamma_1^0 C_{e_1e_1}\Big(\begin{pmatrix}\gamma_1^0 & \gamma_2^0\end{pmatrix}\begin{pmatrix}\frac{\partial I_1^{0^*}}{\partial \epsilon_1^0}\\ \frac{\partial I_2^{0^*}}{\partial \epsilon_1^0}\end{pmatrix}+\beta_1^0 I_1^{0^*}-w_0h_1^0\Big)$$

$$\Big[\theta_1^0\begin{pmatrix}0 & 0\end{pmatrix}+\theta_2^0\begin{pmatrix}0 & \frac{\partial^2 f_2(\mathbf{I^{0*}})}{\partial (I_2^{0^*})^2}\end{pmatrix}\Big]\begin{pmatrix}\frac{\partial I_1^{0^*}}{\partial \epsilon_1^0}\\ \frac{\partial I_2^{0^*}}{\partial \epsilon_1^0}\end{pmatrix}$$

$$= 0(1+C_{e_1})+\gamma_2^0 C_{e_1e_1}\Big(\begin{pmatrix}\gamma_1^0 & \gamma_2^0\end{pmatrix}\begin{pmatrix}\frac{\partial I_1^{0^*}}{\partial \epsilon_1^0}\\ \frac{\partial I_2^{0^*}}{\partial \epsilon_1^0}\end{pmatrix}+\beta_1^0 I_1^{0^*}-w_0h_1^0\Big)$$

Then, the vector is

$$\begin{pmatrix}\frac{\partial I_1^{0^*}}{\partial \epsilon_1^0}\\ \frac{\partial I_2^{0^*}}{\partial \epsilon_1^0}\end{pmatrix} = \mathbf{A}_0^{-1}\cdot C_{e_1e_1}\Big\{(\beta_1^0 I_1^{0^*}-w_0h_1^0)\begin{pmatrix}\gamma_1^0\\ \gamma_2^0\end{pmatrix}+(1+C_{e_1})\begin{pmatrix}\beta_1^0\\ 0\end{pmatrix}-\alpha_1^0\begin{pmatrix}\frac{\partial f_1(I_1^{0^*})}{\partial I_1^{0^*}}\\ 0\end{pmatrix}\Big\}$$

$$= \mathbf{A}_0^{-1}\cdot\Big\{C_{e_1e_1}(\beta_1^0 I_1^{0^*}-w_0h_1^0)\begin{pmatrix}\gamma_1^0\\ \gamma_2^0\end{pmatrix}$$

$$+\Big[\frac{\beta_1^0}{\gamma_1^0}\big(\theta_1^0\mathrm{D}f_1(I_1^{0^*})+\theta_2^0\mathrm{D}f_2(I_2^{0^*})\big)\cdot\begin{pmatrix}1\\ 0\end{pmatrix}\Big]\cdot\begin{pmatrix}1\\ 0\end{pmatrix}-\alpha_1^0\begin{pmatrix}\frac{\partial f_1(I_1^{0^*})}{\partial I_1^{0^*}}\\ 0\end{pmatrix}\Big\}$$

For $j=2$, differentiate the first order condition with respect to $\epsilon_2$,

$$\theta_1^0\Big(\frac{\partial^2 f_1(I_1^{0^*})}{\partial (I_1^{0^*})^2}\cdot\frac{\partial I_1^{0^*}}{\partial \epsilon_2^0}\Big) = 0(1+C_{e_1})+\gamma_2^0 C_{e_1e_1}\Big(\gamma_1^0\frac{\partial I_1^{0^*}}{\partial \epsilon_2^0}+\beta_2^0 I_2^{0^*}+\gamma_2^0\frac{\partial I_2^{0^*}}{\partial \epsilon_2^0}-w_0h_2^0\Big)$$

$$\theta_2^0\Big(\frac{\partial^2 f_2(I_2^{0^*})}{\partial(I_2^{0^*})^2}\cdot\frac{\partial I_2^{0^*}}{\partial\epsilon_2^0}\Big)+\alpha_2^0\frac{\partial f_2(I_2^{0^*})}{\partial I_2^{0^*}} = \beta_2^0(1+C_{e_1})+\gamma_2^0 C_{e_1e_1}\Big(\gamma_1^0\frac{\partial I_1^{0^*}}{\partial\epsilon_2^0}+\beta_2^0 I_2^{0^*}+\gamma_2^0\frac{\partial I_2^{0^*}}{\partial\epsilon_2^0}-w_0h_2^0\Big)$$

The vector and matrix form is,

$$\Big[\theta_1^0\begin{pmatrix}\frac{\partial^2 f_1(I_1^{0^*})}{\partial(I_1^{0^*})^2} & 0\end{pmatrix}+\theta_2^0\begin{pmatrix}0 & 0\end{pmatrix}\Big]\begin{pmatrix}\frac{\partial I_1^{0^*}}{\partial\epsilon_2^0}\\ \frac{\partial I_2^{0^*}}{\partial\epsilon_2^0}\end{pmatrix}$$

$$= 0(1+C_{e_1})+\gamma_1^0 C_{e_1e_1}\Big(\begin{pmatrix}\gamma_1^0 & \gamma_2^0\end{pmatrix}\begin{pmatrix}\frac{\partial I_1^{0^*}}{\partial\epsilon_2^0}\\ \frac{\partial I_2^{0^*}}{\partial\epsilon_2^0}\end{pmatrix}+\beta_2^0 I_2^{0^*}-w_0h_2^0\Big)$$

$$\Big[\theta_1^0\begin{pmatrix}0 & 0\end{pmatrix}+\theta_2^0\begin{pmatrix}0 & \frac{\partial^2 f_2(I_2^{0^*})}{\partial(I_2^{0^*})^2}\end{pmatrix}\Big]\begin{pmatrix}\frac{\partial I_1^{0^*}}{\partial\epsilon_2^0}\\ \frac{\partial I_2^{0^*}}{\partial\epsilon_2^0}\end{pmatrix}+\alpha_2^0\frac{\partial f_2(I_2^{0^*})}{\partial I_2^{0^*}}$$

$$= \beta_2^0(1+C_{e_1})+\gamma_2^0 C_{e_1e_1}\Big(\begin{pmatrix}\gamma_1^0 & \gamma_2^0\end{pmatrix}\begin{pmatrix}\frac{\partial I_1^{0^*}}{\partial\epsilon_2^0}\\ \frac{\partial I_2^{0^*}}{\partial\epsilon_2^0}\end{pmatrix}+\beta_2^0 I_2^{0^*}-w_0h_2^0\Big)$$

The corresponding vector when $j=2$ is given by

$$\begin{pmatrix}\frac{\partial I_1^{0^*}}{\partial\epsilon_2^0}\\ \frac{\partial I_2^{0^*}}{\partial\epsilon_2^0}\end{pmatrix} = \mathbf{A}_0^{-1}\cdot\Big\{(\beta_2^0 I_2^{0^*}-w_0h_2^0)C_{e_1e_1}\begin{pmatrix}\gamma_1^0\\ \gamma_2^0\end{pmatrix}+(1+C_{e_1})\begin{pmatrix}0\\ \beta_2^0\end{pmatrix}-\alpha_2^0\begin{pmatrix}0\\ \frac{\partial f_2(I_2^{0^*})}{\partial I_2^{0^*}}\end{pmatrix}\Big\}$$

$$= \mathbf{A}_0^{-1}\cdot\Big\{C_{e_1e_1}(\beta_2^0 I_2^{0^*}-w_0h_2^0)\begin{pmatrix}\gamma_1^0\\ \gamma_2^0\end{pmatrix}$$

$$+\Big[\frac{\beta_2^0}{\gamma_2^0}\big(\theta_1^0\mathrm{D}f_1(I_1^{0^*})+\theta_2^0\mathrm{D}f_2(I_2^{0^*})\big)\cdot\begin{pmatrix}0\\ 1\end{pmatrix}\Big]\cdot\begin{pmatrix}0\\ 1\end{pmatrix}-\alpha_2^0\begin{pmatrix}0\\ \frac{\partial f_2(I_2^{0^*})}{\partial I_2^{0^*}}\end{pmatrix}\Big\}$$

Step 5:

Finally, solve for the optimal hedge ratio $\mathbf{h}^{\mathbf{0}^*}=\{h_1^{0^*},h_2^{0^*}\}$.

$P_{w_1}=C_{e_1}$ and $P_{w_1w_1}=-C_{e_1e_1}-C_{e_1e_1}^2(\gamma^0)^T\cdot\mathbf{A}_0^{-1}\cdot\gamma$.

$w_1=w_0(h_1^0\epsilon_1^0+h_2^0\epsilon_2^0)$, then for $j=1,2$,

$$\frac{\partial w_1}{\partial h_j^0}=w_0\epsilon_j^0$$

$\max_{\mathbf{h}} \mathbf{E}[P(w_1)]$ requires that the first order condition equations

$$\mathbf{E}\big[P_{w_1} \cdot \frac{\partial w_1}{\partial h_1^0}\big] = 0$$

$$\mathbf{E}\big[P_{w_1} \cdot \frac{\partial w_1}{\partial h_2^0}\big] = 0$$

Apply Lemma 2.14 and the definition of covariance with respect to $h_1^0$, we obtain

$$\mathbf{E}\big[P_{w_1} \cdot \frac{\partial w_1}{\partial h_1^0}\big] = \mathbf{E}[P_{w_1} \cdot w_0 \epsilon_1^0] = w_0 \mathbf{E}[P_{w_1} \cdot \epsilon_1^0] = 0 \Leftrightarrow$$

$$\mathbf{E}[P_{w_1} \cdot \epsilon_1^0] = \mathbf{E}[P_{w_1}] \cdot \mathbf{E}[\epsilon_1^0] - \text{cov}(P_{w_1}, \epsilon_1^0) = 0 \text{ (since } \mathbf{E}[\epsilon_1^0] = 1) \Leftrightarrow$$

$$\mathbf{E}[P_{w_1}] - \mathbf{E}_{\epsilon_1^0}[P_{w_1 \epsilon_1^0}] \cdot \mathbf{E}_{\epsilon_1^0}[\epsilon_1^0] \cdot \text{cov}(\epsilon_1^0, \epsilon_1^0) = 0 \Leftrightarrow$$

$$\mathbf{E}[P_{w_1} \cdot \epsilon_1^0] = \mathbf{E}[P_{w_1}] - \mathbf{E}_{\epsilon_1^0}[P_{w_1 \epsilon_1^0}] \cdot (\sigma_1^0)^2 = 0$$

Notice that

$$\mathbf{E}_{\epsilon_1^0}[P_{w_1 \epsilon_1^0}] = \mathbf{E}_{\epsilon_1^0}[(P_{w_1})_{\epsilon_1^0}] = \mathbf{E}_{\epsilon_1^0}[(C_{e_1})_{\epsilon_1^0}] = \mathbf{E}_{\epsilon_1^0}\big[C_{e_1 e_1} \cdot \frac{\partial e_1}{\partial \epsilon_1^0}\big]$$

then,

$$\mathbf{E}[P_{w_1}] - (\sigma_1^0)^2 \mathbf{E}_{\epsilon_1^0}\big[C_{e_1 e_1} \cdot \frac{\partial e_1}{\partial \epsilon_1^0}\big] = 0 \quad (*)$$

By the definition of $e_1$, the derivative of $e_1$ with respect to $\epsilon_1$ is

$$\frac{\partial e_1}{\partial \epsilon_1^0} = \gamma_1^0 \frac{\partial I_1^{0^*}}{\partial \epsilon_1^0} + \gamma_2^0 \frac{\partial I_2^{0^*}}{\partial \epsilon_1^0} + \beta_1^0 I_1^{0^*} - w_0 h_1^0$$

Plug it into the equation $(*)$, then

$$\mathbf{E}[P_{w_1}] - (\sigma_1^0)^2 \mathbf{E}_{\epsilon_1^0}\Big[C_{e_1 e_1}\big(\gamma_1^0 \frac{\partial I_1^{0^*}}{\partial \epsilon_1^0} + \gamma_2^0 \frac{\partial I_2^{0^*}}{\partial \epsilon_1^0} + \beta_1^0 I_1^{0^*} - w_0 h_1^0\big)\Big] = 0$$

That is,

$$\mathbf{E}[P_{w_1}] - (\sigma_1^0)^2 \mathbf{E}_{\epsilon_1^0}\Big[C_{e_1 e_1}\big(\gamma_1^0 \frac{\partial I_1^{0^*}}{\partial \epsilon_1^0} + \gamma_2^0 \frac{\partial I_2^{0^*}}{\partial \epsilon_1^0}\big)\Big] - (\sigma_1^0)^2 \mathbf{E}_{\epsilon_1^0}[C_{e_1 e_1} \beta_1^0 I_1^{0^*}] + (\sigma_1^0)^2 \mathbf{E}_{\epsilon_1^0}[C_{e_1 e_1} w_0 h_1^0] = 0$$

Use the vectors $\dfrac{\partial \mathbf{I^{0*}}}{\partial \epsilon_1^0}$ from Step 4, then

$$\mathbf{E}[P_{w_1}] - (\sigma_1^0)^2 \mathbf{E}_{\epsilon_1^0}\Big[C_{e_1e_1}\begin{pmatrix} \gamma_1^0 & \gamma_2^0 \end{pmatrix} \cdot \mathbf{A}_0^{-1}\big(C_{e_1e_1}(\beta_1^0 I_1^{0^*} - w_0 h_1^0)\begin{pmatrix} \gamma_1^0 \\ \gamma_2^0 \end{pmatrix}$$

$$+ \frac{\beta_1^0}{\gamma_1^0}\big(\theta_1^0 \mathrm{D}^2 f_1(I_1^{0^*}) + \theta_2^0 \mathrm{D}^2 f_2(I_2^{0^*})\big)\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \alpha_1 \frac{\partial f_1(I_1^{0^*})}{\partial I_1^{0^*}}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\big) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}\Big]$$

$$- (\sigma_1^0)^2 \beta_1^0 \mathbf{E}_{\epsilon_1^0}[C_{e_1e_1} I_1^{0^*}] + (\sigma_1^0)^2 w_0 h_1^0 \mathbf{E}_{\epsilon_1^0}[C_{e_1e_1}] = 0$$

Expand and simplify,

$$\mathbf{E}[P_{w_1}] - (\sigma_1^0)^2 \beta_1^0 \mathbf{E}_{\epsilon_1^0}\Big[I_1^{0^*} C_{e_1e_1}\begin{pmatrix} \gamma_1^0 & \gamma_1^0 \end{pmatrix} \cdot \mathbf{A}_0^{-1} C_{e_1e_1}\begin{pmatrix} \gamma_1^0 \\ \gamma_2^0 \end{pmatrix}\Big]$$

$$+ (\sigma_1^0)^2 w_0 h_1^0 \mathbf{E}_{\epsilon_1^0}\Big[C_{e_1e_1}\begin{pmatrix} \gamma_1^0 & \gamma_2^0 \end{pmatrix} \cdot \mathbf{A}_0^{-1} \cdot \begin{pmatrix} \gamma_1^0 \\ \gamma_2^0 \end{pmatrix}\Big]$$

$$- (\sigma_1^0)^2 \frac{\beta_1^0}{\gamma_1^0} \mathbf{E}_{\epsilon_1^0}\Big[C_{e_1e_1}\begin{pmatrix} \gamma_1^0 & \gamma_2^0 \end{pmatrix} \cdot \mathbf{A}_0^{-1} \cdot \Big(\big(\theta_1^0 \mathrm{D}^2 f_1(I_1^{0^*}) + \theta_2^0 \mathrm{D}^2 f_2(I_2^{0^*})\big)\begin{pmatrix} 1 \\ 0 \end{pmatrix}\Big) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$+ (\sigma_1^0)^2 \alpha_1^0 \frac{\partial f_1(I_1^{0^*})}{\partial I_1^{0^*}}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\Big] - (\sigma_1^0)^2 \beta_1^0 \mathbf{E}_{\epsilon_1^0}[C_{e_1e_1} I_1^{0^*}] + (\sigma_1^0)^2 w_0 h_1^0 \mathbf{E}_{\epsilon_1^0}[C_{e_1e_1}] = 0$$

$$\mathbf{E}[P_{w_1}] - (\sigma_1^0)^2 \beta_1^0 \mathbf{E}_{\epsilon_1^0}\Big[I_1^{0^*} C_{e_1e_1}\begin{pmatrix} \gamma_1^0 & \gamma_1^0 \end{pmatrix} \cdot \mathbf{A}_0^{-1} C_{e_1e_1}\begin{pmatrix} \gamma_1^0 \\ \gamma_2^0 \end{pmatrix}\Big]$$

$$- (\sigma_1^0)^2 \frac{\beta_1^0}{\gamma_1^0} \mathbf{E}_{\epsilon_1^0}\Big[C_{e_1e_1}\begin{pmatrix} \gamma_1^0 & \gamma_2^0 \end{pmatrix} \cdot \mathbf{A}_0^{-1} \cdot \Big(\big(\theta_1^0 \mathrm{D}^2 f_1(I_1^{0^*}) + \theta_2^0 \mathrm{D}^2 f_2(I_2^{0^*})\big)\begin{pmatrix} 1 \\ 0 \end{pmatrix}\Big) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$+ (\sigma_1^0)^2 \alpha_1^0 \frac{\partial f_1(I_1^{0^*})}{\partial I_1^{0^*}}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\Big] - (\sigma_1^0)^2 \beta_1^0 \mathbf{E}_{\epsilon_1^0}[C_{e_1e_1} I_1^{0^*}]$$

$$= -(\sigma_1^0)^2 w_0 h_1^0 \mathbf{E}_{\epsilon_1^0}\Big[C_{e_1e_1}\begin{pmatrix} \gamma_1^0 & \gamma_2^0 \end{pmatrix} \cdot \mathbf{A}_0^{-1} \cdot \begin{pmatrix} \gamma_1^0 \\ \gamma_2^0 \end{pmatrix}\Big] + (\sigma_1^0)^2 w_0 h_1^0 \mathbf{E}_{\epsilon_1^0}[C_{e_1e_1}]$$

$$\mathbf{E}[P_{w_1}] + (\sigma_1^0)^2\beta_1^0\mathbf{E}_{\epsilon_1^0}\Big[I_1^{0^*}\big( - C_{e_1e_1} - C_{e_1e_1}\begin{pmatrix} \gamma_1^0 & \gamma_1^0 \end{pmatrix}\cdot \mathbf{A}_0^{-1}C_{e_1e_1}\begin{pmatrix} \gamma_1^0 \\ \gamma_2^0 \end{pmatrix}\big)\Big]$$

$$- (\sigma_1^0)^2\frac{\beta_1^0}{\gamma_1^0}\mathbf{E}_{\epsilon_1^0}\Big[C_{e_1e_1}\begin{pmatrix} \gamma_1^0 & \gamma_2^0 \end{pmatrix}\cdot \mathbf{A}_0^{-1}\cdot\Big(\big(\theta_1^0\mathrm{D}^2f_1(I_1^{0^*}) + \theta_2^0\mathrm{D}^2f_2(I_2^{0^*})\big)\begin{pmatrix} 1 \\ 0 \end{pmatrix}\Big)\cdot\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$+ (\sigma_1^0)^2\alpha_1^0\frac{\partial f_1(I_1^{0^*})}{\partial I_1^{0^*}}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\Big] = (\sigma_1^0)^2 w_0 h_1^0\mathbf{E}_{\epsilon_1^0}\Big[ - C_{e_1e_1} - C_{e_1e_1}\begin{pmatrix} \gamma_1^0 & \gamma_2^0 \end{pmatrix}\cdot \mathbf{A}_0^{-1}\cdot\begin{pmatrix} \gamma_1^0 \\ \gamma_2^0 \end{pmatrix}\Big]$$

$$\mathbf{E}[P_{w_1}] + (\sigma_1^0)^2\beta_1^0\mathbf{E}_{\epsilon_1^0}\Big[I_1^{0^*}P_{w_1w_1}\Big] - (\sigma_1^0)^2\mathbf{E}_{\epsilon_1^0}\Big[C_{e_1e_1}\begin{pmatrix} \gamma_1^0 & \gamma_2^0 \end{pmatrix}\cdot \mathbf{A}_0^{-1}\cdot$$

$$\Big(\frac{\beta_1^0}{\gamma_1^0}\big(\theta_1^0\mathrm{D}^2f_1(I_1^{0^*}) + \theta_2^0\mathrm{D}^2f_2(I_2^{0^*})\big)\begin{pmatrix} 1 \\ 0 \end{pmatrix}\Big)\cdot\begin{pmatrix} 1 \\ 0 \end{pmatrix} + (\sigma_1^0)^2\alpha_1^0\frac{\partial f_1(I_1^{0^*})}{\partial I_1^{0^*}}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\Big]$$

$$- (\sigma_1^0)^2\beta_1^0\mathbf{E}_{\epsilon_1^0}[C_{e_1e_1}I_1^{0^*}] = (\sigma_1^0)^2 w_0 h_1^0\mathbf{E}_{\epsilon_1^0}\big[P_{w_1w_1}\big]$$

Similarly, we need $\dfrac{\partial e_1}{\partial \epsilon_2^0} = \gamma_1^0\dfrac{\partial I_1^{0^*}}{\partial \epsilon_2^0} + \gamma_2^0\dfrac{\partial I_2^{0^*}}{\partial \epsilon_2^0} + \beta_2^0 I_2^{0^*} - w_0h_2^0$ to solve $h_2^0$.

$$\mathbf{E}[P_{w_1}] - (\sigma_2^0)^2\mathbf{E}_{\epsilon_2^0}[C_{e_1e_1}(\gamma_1^0\frac{\partial I_1^{0^*}}{\partial \epsilon_2^0} + \gamma_2^0\frac{\partial I_2^{0^*}}{\partial \epsilon_2^0} + \beta_2^0 I_2^{0^*} - w_0h_2^0)] = 0$$

Then,

$$\mathbf{E}[P_{w_1}] - (\sigma_2^0)^2\mathbf{E}_{\epsilon_2^0}[C_{e_1e_1}(\gamma_1^0\frac{\partial I_1^{0^*}}{\partial \epsilon_2^0} + \gamma_2^0\frac{\partial I_2^{0^*}}{\partial \epsilon_2^0})] - (\sigma_2^0)^2\mathbf{E}_{\epsilon_2^0}[C_{e_1e_1}\beta_2^0 I_2^0] + (\sigma_2^0)^2\mathbf{E}_{\epsilon_2^0}[C_{e_1e_1}w_0h_2^0] = 0$$

Again,

$$\mathbf{E}[P_{w_1}] - (\sigma_2^0)^2\mathbf{E}_{\epsilon_2^0}\Big[C_{e_1e_1}\begin{pmatrix} \gamma_1^0 & \gamma_2^0 \end{pmatrix}\cdot \mathbf{A}_0^{-1}\big(C_{e_1e_1}(\beta_2^0 I_2^{0^*} - w_0h_2^0)\begin{pmatrix} \gamma_1^0 \\ \gamma_2^0 \end{pmatrix}$$

$$+ \frac{\beta_2^0}{\gamma_2^0}\big(\theta_1^0\mathrm{D}^2f_1(I_1^{0^*}) + \theta_2^0\mathrm{D}^2f_2(I_2^{0^*})\big)\begin{pmatrix} 0 \\ 1 \end{pmatrix} - \alpha_2\frac{\partial f_2(I_2^{0^*})}{\partial I_2^{0^*}}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\big)\cdot\begin{pmatrix} 0 \\ 1 \end{pmatrix}\Big]$$

$$- (\sigma_2^0)^2\beta_2^0\mathbf{E}_{\epsilon_2^0}[C_{e_1e_1}I_2^{0^*}] + (\sigma_2^0)^2 w_0 h_2^0\mathbf{E}_{\epsilon_2^0}[C_{e_1e_1}] = 0$$

$$\mathbf{E}[P_{w_1}] - (\sigma_2^0)^2\beta_2^0\mathbf{E}_{\epsilon_2^0}\Big[I_2^{0^*}C_{e_1e_1}\begin{pmatrix} \gamma_1^0 & \gamma_1^0 \end{pmatrix}\cdot \mathbf{A}_0^{-1}C_{e_1e_1}\begin{pmatrix} \gamma_1^0 \\ \gamma_2^0 \end{pmatrix}\Big]$$

$$+ (\sigma_2^0)^2 w_0 h_2^0\mathbf{E}_{\epsilon_2^0}\Big[C_{e_1e_1}\begin{pmatrix} \gamma_1^0 & \gamma_2^0 \end{pmatrix}\cdot \mathbf{A}_0^{-1}\cdot\begin{pmatrix} \gamma_1^0 \\ \gamma_2^0 \end{pmatrix}\Big]$$

$$- (\sigma_2^0)^2\frac{\beta_2^0}{\gamma_2^0}\mathbf{E}_{\epsilon_2^0}\Big[C_{e_1e_1}\begin{pmatrix} \gamma_1^0 & \gamma_2^0 \end{pmatrix}\cdot \mathbf{A}_0^{-1}\cdot\Big(\big(\theta_1^0\mathrm{D}^2 f_1(I_1^{0^*}) + \theta_2^0\mathrm{D}^2 f_2(I_2^{0^*})\big)\begin{pmatrix} 0 \\ 1 \end{pmatrix}\Big)\cdot\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$+ (\sigma_2^0)^2\alpha_2^0\frac{\partial f_2(I_2^{0^*})}{\partial I_2^{0^*}}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\Big] - (\sigma_2^0)^2\beta_2^0\mathbf{E}_{\epsilon_2^0}[C_{e_1e_1}I_2^{0^*}] + (\sigma_2^0)^2 w_0 h_2^0\mathbf{E}_{\epsilon_2^0}[C_{e_1e_1}] = 0$$

$$\mathbf{E}[P_{w_1}] - (\sigma_2^0)^2\beta_2^0\mathbf{E}_{\epsilon_2^0}\Big[I_2^{0^*}C_{e_1e_1}\begin{pmatrix} \gamma_1^0 & \gamma_1^0 \end{pmatrix}\cdot \mathbf{A}_0^{-1}C_{e_1e_1}\begin{pmatrix} \gamma_1^0 \\ \gamma_2^0 \end{pmatrix}\Big]$$

$$- (\sigma_2^0)^2\frac{\beta_2^0}{\gamma_2^0}\mathbf{E}_{\epsilon_2^0}\Big[C_{e_1e_1}\begin{pmatrix} \gamma_1^0 & \gamma_2^0 \end{pmatrix}\cdot \mathbf{A}_0^{-1}\cdot\Big(\big(\theta_1^0\mathrm{D}^2 f_1(I_1^{0^*}) + \theta_2^0\mathrm{D}^2 f_2(I_2^{0^*})\big)\begin{pmatrix} 0 \\ 1 \end{pmatrix}\Big)\cdot\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$+ (\sigma_2^0)^2\alpha_2^0\frac{\partial f_2(I_2^{0^*})}{\partial I_2^{0^*}}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\Big] - (\sigma_2^0)^2\beta_2^0\mathbf{E}_{\epsilon_2^0}[C_{e_1e_1}I_1^{0^*}]$$

$$= -(\sigma_2^0)^2 w_0 h_2^0\mathbf{E}_{\epsilon_2^0}\Big[C_{e_1e_1}\begin{pmatrix} \gamma_1^0 & \gamma_2^0 \end{pmatrix}\cdot \mathbf{A}_0^{-1}\cdot\begin{pmatrix} \gamma_1^0 \\ \gamma_2^0 \end{pmatrix}\Big] + (\sigma_2^0)^2 w_0 h_2^0\mathbf{E}_{\epsilon_2^0}[C_{e_1e_1}]$$

$$\mathbf{E}[P_{w_1}] + (\sigma_2^0)^2\beta_2^0\mathbf{E}_{\epsilon_2^0}\Big[I_2^{0^*}\big( - C_{e_1e_1} - C_{e_1e_1}\begin{pmatrix} \gamma_1^0 & \gamma_1^0 \end{pmatrix}\cdot \mathbf{A}_0^{-1}C_{e_1e_1}\begin{pmatrix} \gamma_1^0 \\ \gamma_2^0 \end{pmatrix}\big)\Big]$$

$$- (\sigma_2^0)^2\frac{\beta_2^0}{\gamma_2^0}\mathbf{E}_{\epsilon_2^0}\Big[C_{e_1e_1}\begin{pmatrix} \gamma_1^0 & \gamma_2^0 \end{pmatrix}\cdot \mathbf{A}_0^{-1}\cdot\Big(\big(\theta_1^0\mathrm{D}^2 f_1(I_1^{0^*}) + \theta_2^0\mathrm{D}^2 f_2(I_2^{0^*})\big)\begin{pmatrix} 0 \\ 1 \end{pmatrix}\Big)\cdot\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$+ (\sigma_2^0)^2\alpha_2^0\frac{\partial f_2(I_2^{0^*})}{\partial I_2^{0^*}}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\Big] = (\sigma_2^0)^2 w_0 h_2^0\mathbf{E}_{\epsilon_2^0}\Big[ - C_{e_1e_1} - C_{e_1e_1}\begin{pmatrix} \gamma_1^0 & \gamma_2^0 \end{pmatrix}\cdot \mathbf{A}_0^{-1}\cdot\begin{pmatrix} \gamma_1^0 \\ \gamma_2^0 \end{pmatrix}\Big]$$

$$\mathbf{E}[P_{w_1}] + (\sigma_2^0)^2\beta_2^0\mathbf{E}_{\epsilon_2^0}\Big[I_2^{0^*}P_{w_1w_1}\Big] - (\sigma_2^0)^2\mathbf{E}_{\epsilon_2^0}\Big[C_{e_1e_1}\begin{pmatrix}\gamma_1^0 & \gamma_2^0\end{pmatrix}\cdot\mathbf{A}_0^{-1}\cdot$$

$$\Big(\frac{\beta_2^0}{\gamma_2^0}\big(\theta_1^0\mathrm{D}^2f_1(I_1^{0^*}) + \theta_2^0\mathrm{D}^2f_2(I_2^{0^*})\big)\begin{pmatrix}0\\1\end{pmatrix}\Big)\cdot\begin{pmatrix}0\\1\end{pmatrix} + (\sigma_2^0)^2\alpha_2^0\frac{\partial f_2(I_2^{0^*})}{\partial I_2^{0^*}}\begin{pmatrix}0\\1\end{pmatrix}\Big]$$

$$-(\sigma_2^0)^2\beta_2^0\mathbf{E}_{\epsilon_2^0}[C_{e_1e_1}I_1^{0^*}] = (\sigma_2^0)^2w_0h_2^0\mathbf{E}_{\epsilon_2^0}\Big[P_{w_1w_1}\Big]$$

As a consequence, we solve the optimal hedge ratio $\mathbf{h^{0^*}}$ for $n = 2$, which is

$$h_1^{0^*} = \frac{\mathbf{E}[P_{w_1}]}{(\sigma_1^0)^2w_0\mathbf{E}_{\epsilon_1^0}[P_{w_1w_1}]} + \beta_1^0\frac{\mathbf{E}_{\epsilon_1^0}[I_1^{0^*}P_{w_1w_1}]}{w_0\mathbf{E}_{\epsilon_1^0}[P_{w_1w_1}]}$$
$$-\frac{\mathbf{E}_{\epsilon_1^0}\Big[C_{e_1e_1}\begin{pmatrix}\gamma_1^0 & \gamma_2^0\end{pmatrix}\cdot\mathbf{A}_0^{-1}\cdot\Big(\frac{\beta_1^0}{\gamma_1^0}\big((\theta_1^0\mathrm{D}f_1(I_1^{0^*}) + \theta_2^0\mathrm{D}f_2(I_2^{0^*}))\cdot\begin{pmatrix}1\\0\end{pmatrix}\big)\Big)\cdot\begin{pmatrix}1\\0\end{pmatrix}\Big]}{w_0\mathbf{E}_{\epsilon_1^0}[P_{w_1w_1}]}$$
$$+\alpha_1^0\frac{\mathbf{E}_{\epsilon_1^0}\Big[C_{e_1e_1}\frac{\partial f_1(I_1^{0^*})}{\partial I_1^{0^*}}\begin{pmatrix}\gamma_1^0 & \gamma_2^0\end{pmatrix}\cdot\mathbf{A}_0^{-1}\cdot\begin{pmatrix}1\\0\end{pmatrix}\Big]}{w_0\mathbf{E}_{\epsilon_1^0}[P_{w_1w_1}]}$$

$$h_2^{0^*} = \frac{\mathbf{E}[P_{w_1}]}{(\sigma_2^0)^2w_0\mathbf{E}_{\epsilon_2^0}[P_{w_1w_1}]} + \beta_2^0\frac{\mathbf{E}_{\epsilon_2^0}[I_2^{0^*}P_{w_1w_1}]}{w_0\mathbf{E}_{\epsilon_2^0}[P_{w_1w_1}]}$$
$$-\frac{\mathbf{E}_{\epsilon_2^0}\Big[C_{e_1e_1}\begin{pmatrix}\gamma_1^0 & \gamma_2^0\end{pmatrix}\cdot\mathbf{A}_0^{-1}\cdot\Big(\frac{\beta_2^0}{\gamma_2^0}\big((\theta_1^0\mathrm{D}f_1(I_1^{0^*}) + \theta_2^0\mathrm{D}f_2(I_2^{0^*}))\cdot\begin{pmatrix}0\\1\end{pmatrix}\big)\Big)\cdot\begin{pmatrix}0\\1\end{pmatrix}\Big]}{w_0\mathbf{E}_{\epsilon_2^0}[P_{w_1w_1}]}$$
$$+\alpha_2^0\frac{\mathbf{E}_{\epsilon_2^0}\Big[C_{e_1e_1}\frac{\partial f_2(I_2^{0^*})}{\partial I_2^{0^*}}\begin{pmatrix}\gamma_1^0 & \gamma_2^0\end{pmatrix}\cdot\mathbf{A}_0^{-1}\cdot\begin{pmatrix}0\\1\end{pmatrix}\Big]}{w_0\mathbf{E}_{\epsilon_2^0}[P_{w_1w_1}]}$$

In [6], $h_1^0 = h$, $h_2^0 = 1-h$, $\epsilon_1^0 = 1$, $\epsilon_2^0 = \epsilon$, $\theta_1^0 = 1$ and $\theta_2^0 = \theta$, $\gamma_1^0 = 1$ and $\gamma_2^0 = \gamma$. Replace the corresponding $\theta$ and $\gamma$ in the optimal hedge ratio, we achieve the result in [6].

The financial meaning of this optimal hedge ratio $\mathbf{h^0}^*$ is that the multinational corporation can use some linear hedge strategy, such as forward and futures to hedge partial of its total wealth $w_1$ in period zero so that it could maximize the expected profit $\mathbf{E}[P(w_1)]$. For detailed financial explanations, please refer to the paper [6].

## 2.2 The Financial Environment - Period One to Period Two

### 2.2.1 Introduction of the Model in Period One

In Section 2.1, we derive the optimal hedge ratio $\mathbf{h^0}^*$ in period zero, which is single period. We will solve a second period formula for the optimal hedge ratio from period one to period two.

When a firm invests from period zero to period one and has an optimal hedge ratio $\mathbf{h^0}^*$ in period zero. Then, the total wealth $w_1$ is a random variable depending on $\mathbf{h^0}^*$. Similarly, the total investments $\mathbf{I^1}^*$ is a random variable too. Assume that all the conditions in previous section are satisfied, then all the models in period zero are the same as in period one, for example, the product function $\mathbf{f}$ and deadweight cost function $C$. Please notice that in period one, all these functions are random variables.

The net present value of investment expenditures is:

$$\mathbf{F}(\mathbf{I^1}) = (\theta^1)^T \cdot \mathbf{f}(\mathbf{I^1}) - (\gamma^1)^T \cdot \mathbf{I^1} \tag{2.18}$$

where $(\mathbf{I^1})^T = (I_1^1, I_2^1, ..., I_n^1)$ is the total investment, a random variable in period one, and $\mathbf{f}(\mathbf{I^1}) : \mathbb{R}^n \to \mathbb{R}^n$ are the product functions from $n$ locations in period one with

$$\mathbf{I^1} \mapsto (f_1(\mathbf{I^1}), ..., f_n(\mathbf{I^1}))$$

We use the right superscription 0 to represent period zero in Section 2.1. All the superscriptions are updated to 1 for period one in this section.

Similarly, we have

$$(\theta^1)^T = \{\theta_1^1, ..., \theta_n^1\}, \qquad \theta_j^1 = \alpha_j^1(\epsilon_j^1 - \bar{\epsilon_j}^1) + 1,$$

$$(\gamma^1)^T = \{\gamma_1^1, ..., \gamma_n^1\}, \qquad \gamma_j^1 = \beta_j^1(\epsilon_j^1 - \bar{\epsilon_j}^1) + 1$$

for all $1 \leq j \leq n$, and $\epsilon_j^1$, which is defined in Section 2.1.

For each $1 \leq j \leq n$, the $j$th product $f_j : \mathbb{R}^n \to \mathbb{R}$ satisfies

$$\mathrm{D}f_j = \Big(\frac{\partial f_j}{\partial I_1}, \frac{\partial f_j}{\partial I_2}, ..., \frac{\partial f_j}{\partial I_n}\Big)$$

and $\mathrm{D}^2 f_j$ is weakly negative definite.

Similarly, the corresponding first and second order derivative matrices are

$$\mathrm{D}\mathbf{f}(\mathbf{I^1}) = \begin{pmatrix} \frac{\partial f_1(\mathbf{I^1})}{\partial I_1^1} & \frac{\partial f_1(\mathbf{I^1})}{\partial I_2^1} & \cdots & \frac{\partial f_1(\mathbf{I^1})}{\partial I_n^1} \\ \frac{\partial f_2(\mathbf{I^1})}{\partial I_1^1} & \frac{\partial f_2(\mathbf{I^1})}{\partial I_2^1} & \cdots & \frac{\partial f_2(\mathbf{I^1})}{\partial I_n^1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n(\mathbf{I^1})}{\partial I_1^1} & \frac{\partial f_n(\mathbf{I^1})}{\partial I_2^1} & \cdots & \frac{\partial f_n(\mathbf{I^1})}{\partial I_n^1} \end{pmatrix}$$

and

$$\mathrm{D}^2 f_j(\mathbf{I^1}) = \begin{pmatrix} \frac{\partial^2 f_j(\mathbf{I^1})}{\partial (I_1^1)^2} & \frac{\partial^2 f_j(\mathbf{I^1})}{\partial I_2^1 \partial I_1^1} & \cdots & \frac{\partial^2 f_j(\mathbf{I^1})}{\partial I_n^1 \partial I_1^1} \\ \frac{\partial^2 f_j(\mathbf{I^1})}{\partial I_1^1 \partial I_2^1} & \frac{\partial^2 f_j(\mathbf{I^1})}{\partial (I_2^1)^2} & \cdots & \frac{\partial^2 f_j(\mathbf{I^1})}{\partial I_n^1 \partial I_2^1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f_j(\mathbf{I^1})}{\partial I_1^1 \partial I_n^1} & \frac{\partial^2 f_j(\mathbf{I^1})}{\partial I_2^1 \partial I_n^1} & \cdots & \frac{\partial^2 f_j(\mathbf{I^1})}{\partial (I_n^1)^2} \end{pmatrix}, \quad j = 1, .., n$$

The external funds $e_2$ in period one is also a random variable, given by

$$e_2 = (\gamma^1)^T \cdot \mathbf{I^1} - w_2,$$

All the deadweight loss function in period one is

$$C(e_2) : \mathbb{R} \to \mathbb{R}, \quad \text{with } C_{e_2} > 0, \ C_{e_2 e_2} > 0$$

In period one, the profit function $P$ is the same as in period zero,

$$P(w_2) = \max_{\mathbf{I^1}} \{\mathbf{F}(\mathbf{I^1}) - C(e_2)\} = \max_{\mathbf{I^1}} \{(\theta^1)^T \cdot \mathbf{f}(\mathbf{I^1}) - (\gamma^1)^T \cdot \mathbf{I^1} - C(e_2)\} \qquad (2.19)$$

where $w_2 = w_1(h_1^1\epsilon_1^1 + h_2^1\epsilon_2^1 + \cdots + h_n^1\epsilon_n^1)$, and $w_1$ is a random variable in period zero. Then, $P(w_2)$ has random values in period one.

Similarly to the period zero and as noted above, if $P(w_2)$ is a concave function, the random fluctuation in $\epsilon_j^1$, $1 \le j \le n$, will reduce expected profits. And only when $P_{w_2w_2} < 0$ for all $w_2$, the hedging could raise average profits.

We apply all the results in Section 2.1 to all the lemmas, propositions, and theorems in this section.

### 2.2.2 $n$ Variables Hedge Ratio Optimization in Period One

$P(w_2)$ are random variables, when we compute the first order condition of $P(w_2)$, we treat it as a function, then for all $1 \le j \le n$,

$$\frac{\partial}{\partial I_j^1}\big(\mathbf{F}(\mathbf{I^1}) - C(e_2)\big) = 0,$$

that is,

$$(\theta^1)^T \cdot \big(\mathrm{D}\mathbf{f}(\mathbf{I^1}) \cdot \mathbf{e}_j\big) - (1 + C_{e_2})(\gamma^1)^T \cdot \mathbf{e}_j = 0$$

We write it as

$$(\theta^1)^T \cdot \mathrm{D}\mathbf{f}(\mathbf{I^1}) \cdot \mathbf{e}_j = \gamma_j^1(1 + C_{e_2}). \tag{2.20}$$

And the matrix form is: $(\mathrm{D}\mathbf{f}(\mathbf{I^1}))^T \cdot \theta^1 = (1 + C_{e_2})\gamma^1$

Denote the first order condition as $(\mathbf{I^1}^*)^T = (I_1^{1^*}, I_2^{1^*}, ..., I_n^{1^*})$, then $\mathbf{I^1} = \mathbf{I^1}^*$, $P(w_2)$ is maximized.

The tensor $\mathrm{D}^2\mathbf{f}(\mathbf{I^1}^*)\Big(\dfrac{\mathrm{d}\mathbf{I^1}^*}{\mathrm{d}w_2}\Big)$ is weakly negative definite. This is because when we treat each entry of the Hessian matrix $\mathrm{D}^2 f_j(\mathbf{I^1}^*)$ as a function, the Hessian matrices are symmetric and $D^2 f_j < 0$ for all $1 \le j \le n$.

**Proposition 2.7** *If we treat $P(w_2)$ as a function, then*

*(i)* $P_{w_2} = C_{e_2}$

*(ii)* $P_{w_2w_2} = (\theta^1)^T \cdot \mathrm{D}^2\mathbf{f}(\mathbf{I^1}^*)\Big(\frac{\mathrm{d}\mathbf{I^1}^*}{\mathrm{d}w_2}\Big) \cdot \frac{\mathrm{d}\mathbf{I^1}^*}{\mathrm{d}w_2} - C_{e_2e_2}\Big((\gamma^1)^T \cdot \frac{\mathrm{d}\mathbf{I^1}^*}{\mathrm{d}w_2} - 1\Big)^2$

*Moreover, $P_{w_2w_2} < 0$, and $P(w_2)$ is a concave function.*

*Proof.* Since $P(w_2) = (\theta^1)^T \cdot \mathbf{f}(\mathbf{I^1}^*) - (\gamma^1)^T \cdot \mathbf{I^1}^* - C(e_2)$ at $(\mathbf{I^1}^*)^T$, treat it as a function so that we can apply the product rule and the first order condition of $P(w_2)$, then we get

$$\begin{aligned}
P_{w_2} &= (\theta^1)^T \cdot \mathrm{D}\mathbf{f}(\mathbf{I^1}^*) \cdot \frac{\mathrm{d}\mathbf{I^1}^*}{\mathrm{d}w_2} - (\gamma^1)^T \cdot \frac{\mathrm{d}\mathbf{I^1}^*}{\mathrm{d}w_2} - C_{e_2}\big((\gamma^1)^T \cdot \frac{\mathrm{d}\mathbf{I^1}^*}{\mathrm{d}w_2} - 1\big) \\
&= (\mathrm{D}\mathbf{f}(\mathbf{I^1}^*)^T \cdot \theta^1)^T \cdot \frac{\mathrm{d}\mathbf{I^1}^*}{\mathrm{d}w_2} - (1 + C_{e_2})(\gamma^1)^T \cdot \frac{\mathrm{d}\mathbf{I^1}^*}{\mathrm{d}w_2} + C_{e_2} \\
&= \big(\mathrm{D}\mathbf{f}(\mathbf{I^1}^*)^T \cdot \theta^1 - (1 + C_{e_2})\gamma^1\big)^T \cdot \frac{\mathrm{d}\mathbf{I^1}^*}{\mathrm{d}w_2} + C_{e_2} \\
&= C_{e_2}
\end{aligned}$$

Similarly, treat the first order derivative function $P_{w_2}$ as a function, and derive the second derivative of $P(w_2)$, where

$$P_{w_2} = (\theta^1)^T \cdot \mathrm{D}\mathbf{f}(\mathbf{I^1}^*) \cdot \frac{\mathrm{d}\mathbf{I^1}^*}{\mathrm{d}w_2} - (\gamma^1)^T \cdot \frac{\mathrm{d}\mathbf{I^1}^*}{\mathrm{d}w_2} - C_{e_2}\big((\gamma^1)^T \cdot \frac{\mathrm{d}\mathbf{I^1}^*}{\mathrm{d}w_2} - 1\big)$$

then,

$$\begin{aligned}
P_{w_2w_2} &= (\theta^1)^T \cdot \Big(\frac{\mathrm{d}\mathbf{I^1}^*}{\mathrm{d}w_2}\Big)^T \mathrm{D}^2\mathbf{f}(\mathbf{I^1}^*) \cdot \frac{\mathrm{d}\mathbf{I^1}^*}{\mathrm{d}w_2} + (\theta^1)^T \cdot \mathrm{D}\mathbf{f}(\mathbf{I^1}^*) \cdot \frac{\mathrm{d}^2\mathbf{I^1}^*}{\mathrm{d}w_2^2} - (\gamma^1)^T \cdot \frac{\mathrm{d}^2\mathbf{I^1}^*}{\mathrm{d}w_2^2} \\
&\quad - C_{e_2e_2}\big((\gamma^1)^T \cdot \frac{\mathrm{d}\mathbf{I^1}^*}{\mathrm{d}w_2} - 1\big)^2 - C_{e_2}\Big((\gamma^1)^T \cdot \frac{\mathrm{d}^2\mathbf{I^1}^*}{\mathrm{d}w_2^2}\Big) \\
&= (\theta^1)^T \cdot \mathrm{D}^2\mathbf{f}(\mathbf{I^1}^*)\Big(\frac{\mathrm{d}\mathbf{I^1}^*}{\mathrm{d}w_2}\Big) \cdot \frac{\mathrm{d}\mathbf{I^1}^*}{\mathrm{d}w_2} - C_{e_2e_2}\big((\gamma^1)^T \cdot \frac{\mathrm{d}\mathbf{I^1}^*}{\mathrm{d}w_2} - 1\big)^2
\end{aligned}$$

By the assumption, we have $\Big\langle \mathrm{D}^2\mathbf{f}(\mathbf{I^1}^*)\Big(\frac{\mathrm{d}\mathbf{I^1}^*}{\mathrm{d}w_2}\Big), \frac{\mathrm{d}\mathbf{I^1}^*}{\mathrm{d}w_2} \Big\rangle < 0$, and $C_{e_1e_1} > 0$, which means that $P_{w_2w_2} < 0$.

■

Similarly to Section 2.1.2, we need to solve for the vector $\frac{\mathrm{d}\mathbf{I^1}^*}{\mathrm{d}w_2}$. Again, we treat $C(e_2)$, $C_{e_2}$, $C_{e_2e_2}$ as functions.

**Theorem 2.8** *Treat $C(e_2)$ as a function, $C_{e_2e_2} > 0$, and $\gamma^1_j > 0$ for all $1 \leq j \leq n$, then*

$$\frac{\mathrm{d}\mathbf{I^1}^*}{\mathrm{d}w_2} = -C_{e_2e_2}\big((\theta^1)^T \cdot \mathrm{D}^2\mathbf{f}(\mathbf{I^1}^*) - C_{e_2e_2}(\gamma^1 \cdot (\gamma^1)^T)\big)^{-1} \cdot \gamma^1$$

*Proof.* The proof is the same as Theorem 2.3, and all the random variables in period one are treated as functions so that we can get the derivative functions. First, we have

$$(\theta^1)^T \cdot \mathrm{D}^2\mathbf{f}(\mathbf{I^{1}}^*)\Big(\frac{\mathrm{d}\mathbf{I^{1}}^*}{\mathrm{d}w_2}\Big) = C_{e_2e_2} \cdot \gamma^1 \cdot \Big((\gamma^1)^T \cdot \frac{\mathrm{d}\mathbf{I^{1}}^*}{\mathrm{d}w_2} - 1\Big)$$

$$\Rightarrow$$

$$(\theta^1)^T \cdot \mathrm{D}^2\mathbf{f}(\mathbf{I^{1}}^*)\Big(\frac{\mathrm{d}\mathbf{I^{1}}^*}{\mathrm{d}w_2}\Big) - C_{e_2e_2} \cdot (\gamma^1 \cdot (\gamma^1)^T) \cdot \frac{\mathrm{d}\mathbf{I^{1}}^*}{\mathrm{d}w_2} = -C_{e_2e_2} \cdot \gamma^1$$

Let

$$\mathbf{A}_1 := (\theta^1)^T \cdot \mathrm{D}^2\mathbf{f}(\mathbf{I^{1}}^*) - C_{e_2e_2}(\gamma^1 \cdot (\gamma^1)^T) \tag{2.21}$$

Since $C_{e_2e_2} > 0$ and $\gamma^1_j > 0$ for all $1 \leq j \leq n$, the matrix $C_{e_2e_2}(\gamma^1 \cdot (\gamma^1)^T)$ is symmetric and weakly positive definite, then $-C_{e_2e_2}(\gamma^1 \cdot (\gamma^1)^T)$ is weakly negative definite, so as $\mathbf{A}_1$.

Similarly, $\theta$ is a random variable with probability, so $\mathbf{A}_1$ is almost invertible. We have,

$$\begin{aligned}\frac{\mathrm{d}\mathbf{I^{1}}^*}{\mathrm{d}w_2} &= -C_{e_2e_2}\big((\theta^1)^T \cdot \mathrm{D}^2\mathbf{f}(\mathbf{I^{1}}^*) - C_{e_2e_2}(\gamma^1 \cdot (\gamma^1)^T)\big)^{-1} \cdot \gamma^1 \\ &= -C_{e_2e_2} \cdot \mathbf{A}_1^{-1} \cdot \gamma^1\end{aligned} \tag{2.22}$$

If $\mathbf{A}_1$ is symmetric and weakly negative definite, $\mathbf{A}_1^{-1}$ is also symmetric and weakly negative definite.

■

**Proposition 2.9** *Let $\mathbf{A}_1$ be the matrix 2.21 defined in Theorem 2.8. The vector $\frac{\mathrm{d}\mathbf{I^{0}}^*}{\mathrm{d}w_2}$ is also given by the Equation 2.22, then the second derivative function $P_{w_2w_2}$ is*

$$P_{w_2w_2} = -C_{e_2e_2} - C^2_{e_2e_2}(\gamma^1)^T \cdot \mathbf{A}_1^{-1} \cdot \gamma^1 \tag{2.23}$$

*Proof.* $P(w_2)$ can be viewed as a function, then we have

$$\begin{aligned}
P_{w_2w_2} &= (\theta^1)^T \cdot \mathrm{D}^2\mathbf{f}(\mathbf{I^{1*}})\Big(\frac{\mathrm{d}\mathbf{I^{1*}}}{\mathrm{d}w_2}\Big) \cdot \frac{\mathrm{d}\mathbf{I^{1*}}}{\mathrm{d}w_2} - C_{e_2e_2}\big((\gamma^1)^T \cdot \frac{\mathrm{d}\mathbf{I^{1*}}}{\mathrm{d}w_2} - 1\big)^2 \\
&= \Big(\frac{\mathrm{d}\mathbf{I^{1*}}}{\mathrm{d}w_2}\Big)^T \big((\theta^1)^T \cdot \mathrm{D}^2\mathbf{f}(\mathbf{I^{1*}})\big) \cdot \frac{\mathrm{d}\mathbf{I^{1*}}}{\mathrm{d}w_2} - C_{e_2e_2}\big((\gamma^1)^T \cdot \frac{\mathrm{d}\mathbf{I^{1*}}}{\mathrm{d}w_2} \cdot (\gamma^1)^T \cdot \frac{\mathrm{d}\mathbf{I^{1*}}}{\mathrm{d}w_2} - 2(\gamma^1)^T \cdot \frac{\mathrm{d}\mathbf{I^{1*}}}{\mathrm{d}w_2} + 1\big) \\
&= \Big(\frac{\mathrm{d}\mathbf{I^{1*}}}{\mathrm{d}w_2}\Big)^T \cdot \big((\theta^1)^T \cdot \mathrm{D}^2\mathbf{f}(\mathbf{I^{1*}}) - C_{e_2e_2}(\gamma^1 \cdot (\gamma^1)^T)\big) \cdot \frac{\mathrm{d}\mathbf{I^{1*}}}{\mathrm{d}w_2} - C_{e_2e_2}\big(1 - 2(\gamma^1)^T \cdot \frac{\mathrm{d}\mathbf{I^{1*}}}{\mathrm{d}w_2}\big) \\
&= (-C_{e_2e_2}\mathbf{A}_1^{-1} \cdot \gamma^1)^T \cdot \mathbf{A}_1 \cdot (-C_{e_2e_2}\mathbf{A}_1^{-1} \cdot \gamma^1) - C_{e_2e_2}\big(1 - 2(\gamma^1)^T \cdot (-C_{e_2e_2}\mathbf{A}_1^{-1} \cdot \gamma^1)\big) \\
&= C_{e_2e_2}^2(\gamma^1)^T \cdot \big(\mathbf{A}_1^{-1}\big)^T \cdot \mathbf{A}_1 \cdot \mathbf{A}_1^{-1}\gamma^1 - C_{e_2e_2} - 2C_{e_2e_2}^2(\gamma^1)^T \cdot \mathbf{A}_1^{-1} \cdot \gamma^1 \\
&= -C_{e_2e_2} - C_{e_2e_2}^2(\gamma^1)^T \cdot \mathbf{A}_1^{-1} \cdot \gamma^1
\end{aligned}$$

which is as asserted.

■

Similar to the vector $\frac{\mathrm{d}\mathbf{I^{1*}}}{\mathrm{d}w_2}$, we can calculate the vectors $\frac{\partial\mathbf{I^{1*}}}{\partial\epsilon_j^1}$ for all $1 \le j \le n$ by viewing all the random variables in period one as functions.

**Theorem 2.10** *If $C_{e_2e_2} > 0$ and $\gamma_j^1 > 0$ for all $1 \le j \le n$. Then,*

$$\frac{\partial\mathbf{I^{1*}}}{\partial\epsilon_j^1} = \mathbf{A}_1^{-1} \cdot \Big\{C_{e_2e_2}(\beta_j^1 I_j^{1^*} - w_1h_j^1) \cdot \gamma^1 + \big(\frac{\beta_j^1}{\gamma_j^1}((\theta^1)^T \cdot \mathrm{D}\mathbf{f}(\mathbf{I^{1*}}) \cdot \mathbf{e}_j) - \alpha_j^1\mathrm{D}\mathbf{f}(\mathbf{I^{1*}})\big) \cdot \mathbf{e}_j\Big\} \quad (2.24)$$

*Proof.* Apply the method of the proof of Theorem 2.8 and view every random variables as functions, we have the vector

$$\frac{\partial\mathbf{I^{1*}}}{\partial\epsilon_j^1} = \mathbf{A}_1^{-1} \cdot \Big\{C_{e_2e_2}(\beta_j^1 I_j^{1^*} - w_1h_j^1) \cdot \gamma^1 + \big(\frac{\beta_j^1}{\gamma_j^1}((\theta^1)^T \cdot \mathrm{D}\mathbf{f}(\mathbf{I^{1*}}) \cdot \mathbf{e}_j) - \alpha_j^1\mathrm{D}\mathbf{f}(\mathbf{I^{1*}})\big) \cdot \mathbf{e}_j\Big\}$$

■

Next, we will solve for the optimal hedge ratio $\mathbf{h^{1*}} = \{h_1^{1^*}, ..., h_n^{1^*}\}$ in period one to maximize $\mathbf{E}[P(w_2)]$ at $\mathbf{I^{1*}} = \{I_1^{1^*}, ..., I_n^{1^*}\}$ if $\mathrm{cov}(\epsilon_i^1, \epsilon_j^1) = 0, i \ne j$. Notice that since we take the expectation of $P(w_2)$, it becomes a function, then we can find the derivative of $\mathbf{E}[P(w_2)]$.

**Theorem 2.11** *Suppose that all the assumptions in this section are satisfied. Also assume that at* $\mathbf{I^{1^*}}$, $P(w_2)$ *is maximized and* $\mathrm{cov}(\epsilon_i^1, \epsilon_j^1) = 0$ *for* $i \neq j$. *Then, the optimal hedge ratio*

$$\mathbf{h^{1^*}} = \{h_1^{1^*}, h_2^{1^*}, ..., h_n^{1^*}\}$$

*where* $h_j^{1^*}$ *are given by*

$$\begin{aligned} h_j^{1^*} &= \frac{\mathbf{E}[P_{w_2}]}{(\sigma_j^1)^2 w_1 \mathbf{E}_{\epsilon_j^1}[P_{w_2w_2}]} + \beta_j^1 \frac{\mathbf{E}_{\epsilon_j^1}[I_j^{1^*} P_{w_2w_2}]}{w_1 \mathbf{E}_{\epsilon_j^1}[P_{w_2w_2}]} \\ &- \frac{\mathbf{E}_{\epsilon_j^1}\big[C_{e_2e_2}(\gamma^1)^T \cdot \mathbf{A}_1^{-1} \cdot \big(\frac{\beta_j^1}{\gamma_j^1}((\theta^1)^T \cdot \mathrm{D}\mathbf{f}(\mathbf{I^{1^*}}) \cdot \mathbf{e}_j) - \alpha_j^1 \mathrm{D}\mathbf{f}(\mathbf{I^{1^*}})\big) \cdot \mathbf{e}_j\big]}{w_1 \mathbf{E}_{\epsilon_j^1}[P_{w_1w_1}]} \end{aligned} \tag{2.25}$$

*for all* $1 \leq j \leq n$.

*Proof.* The proof is also similar to the proof of Theorem 2.6. Notice that $w_1$ here is not a constant but a random variable depending on $\epsilon_j^0$.

Since $w_2 = w_1(h_1^1\epsilon_1^1 + \cdots + h_n^1\epsilon_n^1)$,then for all $1 \leq j \leq n$,

$$\frac{\partial w_2}{\partial h_j^1} = w_1\epsilon_j^1$$

For all $1 \leq j \leq n$, we need

$$\mathbf{E}\Big[P_{w_2} \cdot \frac{\partial w_2}{\partial h_j^1}\Big] = 0$$

Apply Lemma 2.14, and notice that $w_1$ is a random variable, not like $w_0$, a constant number, so we can not put it out of the expectation, then

$$\mathbf{E}\big[P_{w_2} \cdot \frac{\partial w_2}{\partial h_j^1}\big] = \mathbf{E}[P_{w_2} \cdot w_1\epsilon_j^1] = \mathbf{E}[w_1P_{w_2} \cdot \epsilon_j^1] = 0 \Leftrightarrow$$

$$\mathbf{E}[w_1P_{w_2} \cdot \epsilon_j^1] = \mathbf{E}[w_1P_{w_2}] \cdot \mathbf{E}[\epsilon_j^1] - \mathrm{cov}(w_1P_{w_2}, \epsilon_j^1) = 0 \text{ (since } \mathbf{E}[\epsilon_j^1] = 1) \Leftrightarrow$$

$$\mathbf{E}[w_1P_{w_2} \cdot \epsilon_j^1] = \mathbf{E}[w_1P_{w_2}] - \mathbf{E}_{\epsilon_j^1}[w_1P_{w_2\epsilon_j^1}] \cdot \mathbf{E}_{\epsilon_j^1}[\epsilon_j^1] \cdot \mathrm{cov}(\epsilon_j^1, \epsilon_j^1) = 0 \Leftrightarrow$$

$$\mathbf{E}[w_1P_{w_2} \cdot \epsilon_j^1] = \mathbf{E}[w_1P_{w_2}] - \mathbf{E}_{\epsilon_j^1}[w_1P_{w_2\epsilon_j^1}] \cdot (\sigma_j^1)^2 = 0$$

Since $P_{w_2} = C_{e_2}$,

$$\mathbf{E}_{\epsilon_j^1}[w_1P_{w_2\epsilon_j^1}] = \mathbf{E}_{\epsilon_j^1}[w_1(P_{w_2})_{\epsilon_j^1}] = \mathbf{E}_{\epsilon_j^1}[w_1(C_{e_2})_{\epsilon_j^1}] = \mathbf{E}_{\epsilon_j^1}\Big[w_1C_{e_2e_2} \cdot \frac{\partial e_2}{\partial \epsilon_j^1}\Big]$$

By the definition of $e_2$, for all $1 \leq j \leq n$, we have

$$\frac{\partial e_2}{\partial \epsilon_j^1} = (\gamma^1)^T \cdot \frac{\partial \mathbf{I^1}^*}{\partial \epsilon_j^1} + \beta_j^1 I_j^{1^*} - w_1 h_j^1$$

Then,

$$\mathbf{E}[w_1 P_{w_2}] - (\sigma_j^1)^2 \mathbf{E}_{\epsilon_j^1}\big[w_1 C_{e_2e_2}((\gamma^1)^T \cdot \frac{\partial \mathbf{I^1}^*}{\partial \epsilon_j^1} + \beta_j^1 I_j^{1^*} - w_1 h_j^1)\big] = 0$$

which can be expanded as:

$$\begin{aligned}\mathbf{E}[w_1 P_{w_2}] - (\sigma_j^1)^2 \mathbf{E}_{\epsilon_j^1}\big[w_1 C_{e_2e_2}((\gamma^1)^T \cdot \frac{\partial \mathbf{I^1}^*}{\partial \epsilon_j^1})\big] \\ - (\sigma_j^1)^2 \mathbf{E}_{\epsilon_j^1}[w_1 C_{e_2e_2} \beta_j^1 I_j^{1^*}] + (\sigma_j^1)^2 \mathbf{E}_{\epsilon_j^1}[(w_1)^2 C_{e_2e_2} h_j^1] = 0\end{aligned} \tag{2.26}$$

The vector $\frac{\partial \mathbf{I^1}^*}{\partial \epsilon_j^1}$ has been solved in Theorem 2.10, which is given in Equation 2.24, then plug this vector into the above equation to get

$$\begin{aligned}&\mathbf{E}[w_1 P_{w_2}] - (\sigma_j^1)^2 \mathbf{E}_{\epsilon_j^1}\big[w_1 C_{e_2e_2}(\gamma^1)^T \cdot \mathbf{A}_1^{-1} \cdot (C_{e_2e_2}(\beta_j^1 I_j^{1^*} - w_1 h_j^1) \cdot \gamma^1 \\ &+ \big(\frac{\beta_j^1}{\gamma_j^1}((\theta^1)^T \cdot \mathrm{D}\mathbf{f}(\mathbf{I^1}^*) \cdot \mathbf{e}_j) - \alpha_j^1 \mathrm{D}\mathbf{f}(\mathbf{I^1}^*)\big) \cdot \mathbf{e}_j)\big] \\ &- (\sigma_j^1)^2 \beta_j^1 \mathbf{E}_{\epsilon_j^1}[w_1 C_{e_2e_2} I_j^{1^*}] + (\sigma_j^1)^2 h_j^1 \mathbf{E}_{\epsilon_j^1}[(w_1)^2 C_{e_2e_2}] = 0\end{aligned}$$

Expand each term of left hand side of the equation:

$$\begin{aligned}&\mathbf{E}[w_1 P_{w_2}] - (\sigma_j^1)^2 \mathbf{E}_{\epsilon_j^1}[w_1 C_{e_2e_2}^2 (\gamma^1)^T \cdot \mathbf{A}_1^{-1} \cdot (\beta_j^1 I_j^{1^*}) \cdot \gamma^1] + (\sigma_j^1)^2 h_j^1 \mathbf{E}_{\epsilon_j^1}[(w_1)^2 C_{e_2e_2}^2 (\gamma^1)^T \cdot \mathbf{A}_1^{-1} \cdot \gamma^1] \\ &- (\sigma_j^1)^2 \frac{\beta_j^1}{\gamma_j^1} \mathbf{E}_{\epsilon_j^1}[w_1 C_{e_2e_2} (\gamma^1)^T \cdot \mathbf{A}_1^{-1} \cdot ((\theta^1)^T \cdot \mathrm{D}\mathbf{f}(\mathbf{I^1}^*) \cdot \mathbf{e}_j) \cdot \mathbf{e}_j] \\ &+ (\sigma_j^1)^2 \alpha_j^1 \mathbf{E}_{\epsilon_j^1}[w_1 C_{e_2e_2} (\gamma^1)^T \cdot \mathbf{A}_1^{-1} \cdot \mathrm{D}\mathbf{f}(\mathbf{I^1}^*) \cdot \mathbf{e}_j] \\ &- (\sigma_j^1)^2 \beta_j^1 \mathbf{E}_{\epsilon_j^1}[w_1 C_{e_2e_2} I_j^{1^*}] + (\sigma_j^1)^2 h_j^1 \mathbf{E}_{\epsilon_j^1}[(w_1)^2 C_{e_2e_2}] = 0\end{aligned}$$

Rewrite the equation and group all terms having the optimal hedge ratio $h_j^1$ together,

$$\begin{aligned}&\mathbf{E}[w_1 P_{w_2}] - (\sigma_j^1)^2 \beta_j^1 \mathbf{E}_{\epsilon_j^1}[w_1 I_j^{1^*} C_{e_2e_2}^2 ((\gamma^1)^T \cdot \mathbf{A}_1^{-1} \cdot \gamma^1)] \\ &- (\sigma_j^1)^2 \frac{\beta_j^1}{\gamma_j^1} \mathbf{E}_{\epsilon_j^1}[w_1 C_{e_2e_2} ((\theta^1)^T \cdot \mathrm{D}\mathbf{f}(\mathbf{I^1}^*) \cdot \mathbf{e}_j)((\gamma^1)^T \cdot \mathbf{A}_1^{-1} \cdot \mathbf{e}_j)] \\ &+ (\sigma_j^1)^2 \alpha_j^1 \mathbf{E}_{\epsilon_j^1}[w_1 C_{e_2e_2} ((\gamma^1)^T \cdot \mathbf{A}_1^{-1} \cdot \mathrm{D}\mathbf{f}(\mathbf{I^1}^*) \cdot \mathbf{e}_j)] - (\sigma_j^1)^2 \beta_j^1 \mathbf{E}_{\epsilon_j^1}[w_1 C_{e_2e_2} I_j^{1^*}] \\ &= -(\sigma_j^1)^2 h_j^1 \mathbf{E}_{\epsilon_j^1}[(w_1)^2 C_{e_2e_2}^2 (\gamma^1)^T \cdot \mathbf{A}_1^{-1} \cdot \gamma^1] - (\sigma_j^1)^2 h_j^1 \mathbf{E}_{\epsilon_j^1}[(w_1)^2 C_{e_2e_2}]\end{aligned}$$

Rearrange and group some terms of the equation,

$$\begin{aligned}
&\mathbf{E}[w_1 P_{w_2}] + (\sigma_j^1)^2 \beta_j^1 \mathbf{E}_{\epsilon_j^1}[w_1 I_j^{1^*}(-C_{e_2e_2} - C_{e_2e_2}^2 (\gamma^1)^T \cdot \mathbf{A}_1^{-1} \cdot \gamma^1)] \\
&\quad - (\sigma_j^1)^2 \frac{\beta_j^1}{\gamma_j^1} \mathbf{E}_{\epsilon_j^1}[w_1 C_{e_2e_2} (\gamma^1)^T \cdot \mathbf{A}_1^{-1} \cdot ((\theta^1)^T \cdot \mathrm{D}\mathbf{f}(\mathbf{I^{1^*}}) \cdot \mathbf{e}_j) \cdot \mathbf{e}_j] \\
&\quad + (\sigma_j^1)^2 \alpha_j^1 \mathbf{E}_{\epsilon_j^1}[w_1 C_{e_2e_2} (\gamma^1)^T \cdot \mathbf{A}_1^{-1} \cdot \mathrm{D}\mathbf{f}(\mathbf{I^{1^*}}) \cdot \mathbf{e}_j] \\
&\quad = (\sigma_j^1)^2 \mathbf{E}_{\epsilon_j^1}[(w_1)^2 (-C_{e_2e_2} - C_{e_2e_2}^2 (\gamma^1)^T \cdot \mathbf{A}_1^{-1} \cdot \gamma^1)] h_j^1
\end{aligned}$$

Since $P_{w_2w_2} = -C_{e_2e_2} - C_{e_2e_2}^2 (\gamma^1)^T \cdot \mathbf{A}_1^{-1} \cdot \gamma^1$, the above equation becomes

$$\begin{aligned}
&\mathbf{E}[w_1 P_{w_2}] + (\sigma_j^1)^2 \beta_j^1 \mathbf{E}_{\epsilon_j^1}[w_1 I_j^{1^*} P_{w_2w_2}] \\
&\quad - (\sigma_j^1)^2 \mathbf{E}_{\epsilon_j^1}[w_1 C_{e_2e_2} (\gamma^1)^T \cdot \mathbf{A}_1^{-1} \cdot (\frac{\beta_j^1}{\gamma_j^1}((\theta^1)^T \cdot \mathrm{D}\mathbf{f}(\mathbf{I^{1^*}}) \cdot \mathbf{e}_j) - \alpha_j^1 \mathrm{D}\mathbf{f}(\mathbf{I^{1^*}})) \cdot \mathbf{e}_j] \\
&\quad = (\sigma_j^1)^2 \mathbf{E}_{\epsilon_j^1}[(w_1)^2 P_{w_2w_2}] h_j^1
\end{aligned}$$

Similarly, when the first order condition of $\mathbf{E}\big[P(w_2)\big]$ under the hedge ratio $\mathbf{h^1}$ is zero, it will be maximized, and the corresponding hedge ratio is the optimal hedge ratio $\mathbf{h^1}^*$.

Solve $h_j^1$ for all $1 \le j \le n$, and we obtain,

$$\begin{aligned}
h_j^{1^*} &= \frac{\mathbf{E}[w_1 P_{w_2}]}{(\sigma_j^1)^2 \mathbf{E}_{\epsilon_j^1}[(w_1)^2 P_{w_2w_2}]} + \beta_j^1 \frac{\mathbf{E}_{\epsilon_j^1}[w_1 I_j^{1^*} P_{w_2w_2}]}{\mathbf{E}_{\epsilon_j^1}[(w_1)^2 P_{w_2w_2}]} \\
&\quad - \frac{\mathbf{E}_{\epsilon_j^1}\big[w_1 C_{e_2e_2} (\gamma^1)^T \cdot \mathbf{A}_1^{-1} \cdot \big(\frac{\beta_j^1}{\gamma_j^1}((\theta^1)^T \cdot \mathrm{D}\mathbf{f}(\mathbf{I^{1^*}}) \cdot \mathbf{e}_j) - \alpha_j^1 \mathrm{D}\mathbf{f}(\mathbf{I^{1^*}})\big) \cdot \mathbf{e}_j\big]}{\mathbf{E}_{\epsilon_j^1}[(w_1)^2 P_{w_2w_2}]}
\end{aligned} \tag{2.27}$$

■

We conclude that if the product function, total wealth, total investment, and the profit function keep the same for period one, and these random variables can be treated as functions, then the optimal hedge ratio can be calculated. In the next section, we will present $n = 2$ case similar to Section 2.1.3.

### 2.2.3 The Optimal Hedge Ratio of $n = 2$ in Period One

In period one with the case $n = 2$ , we can calculate the optimal hedge ratio $\mathbf{h^{1^*}} = \{h_1^{1^*}, h_2^{1^*}\}$ with some assumptions.

Let $(\mathbf{I^1})^T = (I_1^1, I_2^1)$, and $\mathbf{F}(\mathbf{I^0}) = \theta_1^1 f_1(I_1^1, I_2^1) + \theta_2^1 f_2(I_1^1, I_2^1) - \gamma_1^1 I_1^1 - \gamma_2^1 I_2^1 - C(e_2)$.

$$w_2 = w_1(h_1^1\epsilon_1^1 + h_2^1\epsilon_2^1) = w_0(h_1^0\epsilon_1^0 + h_2^0\epsilon_2^0)(h_1^1\epsilon_1^1 + h_2^1\epsilon_2^1), \quad h_1^1 + h_2^1 = 1,$$

$$e_2 = \gamma_1^1 I_1^1 + \gamma_2^1 I_2^1 - w_2$$

$$\theta_1^1 = \alpha_1^1(\epsilon_1^1 - \bar{\epsilon_1}^1) + 1, \quad \theta_2^1 = \alpha_2^1(\epsilon_2^1 - \bar{\epsilon_2}^1) + 1$$

$$\gamma_1^1 = \beta_1^1(\epsilon_1^1 - \bar{\epsilon_1}^1) + 1, \quad \gamma_2^1 = \beta_2^1(\epsilon_2^1 - \bar{\epsilon_2}^1) + 1$$

$$\epsilon_j^1 \sim \mathbf{N}(\bar{\epsilon}_j^1, (\sigma_j^1)^2), \ j = 1, 2$$

$$\mathrm{D}\mathbf{f}(\mathbf{I^1}) = \begin{pmatrix} \frac{\partial f_1(\mathbf{I^1})}{\partial I_1^1} & \frac{\partial f_1(\mathbf{I^1})}{\partial I_2^1} \\ \frac{\partial f_2(\mathbf{I^1})}{\partial I_1^1} & \frac{\partial f_2(\mathbf{I^1})}{\partial I_2^1} \end{pmatrix},$$

$$\mathrm{D}^2 f_1(\mathbf{I^1}) = \begin{pmatrix} \frac{\partial^2 f_1(\mathbf{I^1})}{\partial (I_1^1)^2} & \frac{\partial^2 f_1(\mathbf{I^1})}{\partial I_2^1 \partial I_1^1} \\ \frac{\partial^2 f_1(\mathbf{I^1})}{\partial I_1^1 \partial I_2^1} & \frac{\partial^2 f_1(\mathbf{I^1})}{\partial (I_2^1)^2} \end{pmatrix} \text{ and } \mathrm{D}^2 f_2(\mathbf{I^1}) = \begin{pmatrix} \frac{\partial^2 f_2(\mathbf{I^1})}{\partial (I_1^1)^2} & \frac{\partial^2 f_2(\mathbf{I^1})}{\partial I_2^1 \partial I_1^1} \\ \frac{\partial^2 f_2(\mathbf{I^1})}{\partial I_1^1 \partial I_2^1} & \frac{\partial^2 f_2(\mathbf{I^1})}{\partial (I_2^1)^2} \end{pmatrix}$$

Step 1:

The first order condition of $P(w_2)$ with respect to $\mathbf{I^1}$ is given by

$$\theta_1^1 \frac{\partial f_1(\mathbf{I^1})}{\partial I_1^1} + \theta_2^1 \frac{\partial f_2(\mathbf{I^1})}{\partial I_1^1} = (1 + C_{e_2})\gamma_1^1 \qquad \theta_1^1 \frac{\partial f_1(\mathbf{I^1})}{\partial I_2^1} + \theta_2^1 \frac{\partial f_2(\mathbf{I^1})}{\partial I_2^1} = (1 + C_{e_2})\gamma_2^1$$

Step 2:

Same as the example in Section 2.1.3, we have $P_{w_2} = C_{e_2}$.

$$P_{w_2 w_2} = \theta_1^1 \Big( \frac{\partial^2 f_1(\mathbf{I^{1^*}})}{\partial (I_1^1)^2} \big( \frac{\mathrm{d} I_1^{1^*}}{\mathrm{d} w_2} \big)^2 + \frac{\partial^2 f_1(\mathbf{I^{1^*}})}{\partial (I_2^1)^2} \big( \frac{\mathrm{d} I_2^{1^*}}{\mathrm{d} w_2} \big)^2 \Big) + \theta_2^1 \Big( \frac{\partial^2 f_2(\mathbf{I^{1^*}})}{\partial (I_1^1)^2} \big( \frac{\mathrm{d} I_1^{1^*}}{\mathrm{d} w_2} \big)^2 + \frac{\partial^2 f_2(\mathbf{I^{1^*}})}{\partial (I_2^1)^2} \big( \frac{\mathrm{d} I_2^{1^*}}{\mathrm{d} w_2} \big)^2 \Big) - C_{e_2 e_2} \Big( \gamma_1^1 \frac{\mathrm{d} I_1^{1^*}}{\mathrm{d} w_2} + \gamma_2^1 \frac{\mathrm{d} I_2^{1^*}}{\mathrm{d} w_2} - 1 \Big)^2$$

Step 3:

Solve for $\dfrac{\mathrm{d}\mathbf{I^1}^*}{\mathrm{d}w_2}$ and simplify $P_{w_2w_2}$.

$$\theta_1^1\Big(\frac{\partial^2 f_1(\mathbf{I^1}^*)}{\partial (I_1^{1^*})^2}\frac{\mathrm{d}I_1^{1^*}}{\mathrm{d}w_2}+\frac{\partial^2 f_1(\mathbf{I^1}^*)}{\partial I_1^{1^*}\partial I_2^{1^*}}\frac{\mathrm{d}I_2^{1^*}}{\mathrm{d}w_2}\Big)+\theta_2^1\Big(\frac{\partial^2 f_2(\mathbf{I^1}^*)}{\partial (I_1^{1^*})^2}\frac{\mathrm{d}I_1^{1^*}}{\mathrm{d}w_2}+\frac{\partial^2 f_2(\mathbf{I^1}^*)}{\partial I_1^{1^*}\partial I_2^{1^*}}\frac{\mathrm{d}I_2^{1^*}}{\mathrm{d}w_2}\Big)$$
$$-\gamma_1^1 C_{e_2e_2}\Big(\gamma_1^1\frac{\mathrm{d}I_1^{1^*}}{\mathrm{d}w_2}+\gamma_2^1\frac{\mathrm{d}I_2^{1^*}}{\mathrm{d}w_2}-1\Big)=0$$

$$\theta_1^1\Big(\frac{\partial^2 f_1(\mathbf{I^1}^*)}{\partial I_2^{1^*}\partial I_1^{1^*}}\frac{\mathrm{d}I_1^{1^*}}{\mathrm{d}w_2}+\frac{\partial^2 f_1(\mathbf{I^1}^*)}{\partial (I_2^{1^*})^2}\frac{\mathrm{d}I_2^{1^*}}{\mathrm{d}w_2}\Big)+\theta_2^1\Big(\frac{\partial^2 f_2(\mathbf{I^1}^*)}{\partial I_2^{1^*}\partial I_1^{1^*}}\frac{\mathrm{d}I_1^{1^*}}{\mathrm{d}w_2}+\frac{\partial^2 f_2(\mathbf{I^1}^*)}{\partial (I_2^{1^*})^2}\frac{\mathrm{d}I_2^{1^*}}{\mathrm{d}w_2}\Big)$$
$$-\gamma_2^1 C_{e_2e_2}\Big(\gamma_1^1\frac{\mathrm{d}I_1^{1^*}}{\mathrm{d}w_2}+\gamma_2^1\frac{\mathrm{d}I_2^{1^*}}{\mathrm{d}w_2}-1\Big)=0$$

Let

$$\mathbf{A}_1=\theta_1^1\mathrm{D}^2 f_1(\mathbf{I^1}^*)+\theta_2^1\mathrm{D}^2 f_2(\mathbf{I^1}^*)-C_{e_2e_2}\begin{pmatrix}(\gamma_1^1)^2 & \gamma_1^1\gamma_2^1\\ \gamma_2^1\gamma_1^1 & (\gamma_2^1)^2\end{pmatrix}$$

then,

$$\begin{pmatrix}\frac{\mathrm{d}I_1^{1^*}}{\mathrm{d}w_2}\\ \frac{\mathrm{d}I_2^{1^*}}{\mathrm{d}w_2}\end{pmatrix}=-C_{e_2e_2}\mathbf{A}_1^{-1}\cdot\begin{pmatrix}\gamma_1^1\\ \gamma_2^1\end{pmatrix}$$

and so

$$P_{w_2w_2}=-C_{e_2e_2}-C_{e_2e_2}^2\begin{pmatrix}\gamma_1^1 & \gamma_2^1\end{pmatrix}\cdot\mathbf{A}_1^{-1}\cdot\begin{pmatrix}\gamma_1^1\\ \gamma_2^1\end{pmatrix}$$

Step 4:

Solve for $\dfrac{\partial\mathbf{I^1}^*}{\partial\epsilon_j}$, $j=1,2$.

For $j=1$, we have,

$$\theta_1^1\Big(\frac{\partial^2 f_1(\mathbf{I^1}^*)}{\partial (I_1^{1^*})^2}\cdot\frac{\partial I_1^{1^*}}{\partial\epsilon_1^1}+\frac{\partial^2 f_1(\mathbf{I^1}^*)}{\partial I_1^{1^*}\partial I_2^{1^*}}\cdot\frac{\partial I_2^{1^*}}{\partial\epsilon_1^1}\Big)+\theta_2^1\Big(\frac{\partial^2 f_2(\mathbf{I^1}^*)}{\partial (I_1^{1^*})^2}\cdot\frac{\partial I_1^{1^*}}{\partial\epsilon_1^1}+\frac{\partial^2 f_2(\mathbf{I^1}^*)}{\partial I_1^{1^*}\partial I_2^{1^*}}\cdot\frac{\partial I_2^{1^*}}{\partial\epsilon_1^1}\Big)$$
$$+\alpha_1^1\frac{\partial f_1(\mathbf{I^1}^*)}{\partial I_1^{1^*}}=\beta_1^1(1+C_{e_2})+\gamma_1^1 C_{e_2e_2}\Big(\gamma_1^1\frac{\partial I_1^{1^*}}{\partial\epsilon_1^1}+\beta_1^1 I_1^{1^*}+\gamma_2^1\frac{\partial I_2^{1^*}}{\partial\epsilon_1^1}-w_0(h_1^0\epsilon_1^0+h_2^0\epsilon_2^0)h_1^1\Big)$$

$$\theta_1^1\Big(\frac{\partial^2 f_1(\mathbf{I^1}^*)}{\partial I_2^{1^*}\partial I_1^{1^*}}\cdot\frac{\partial I_1^{1^*}}{\partial\epsilon_1^1}+\frac{\partial^2 f_1(\mathbf{I^1}^*)}{\partial (I_2^{1^*})^2}\cdot\frac{\partial I_2^{1^*}}{\partial\epsilon_1^1}\Big)+\theta_2^1\Big(\frac{\partial^2 f_2(\mathbf{I^1}^*)}{\partial I_2^{1^*}\partial I_1^{1^*}}\cdot\frac{\partial I_1^{1^*}}{\partial\epsilon_1^1}+\frac{\partial^2 f_2(\mathbf{I^1}^*)}{\partial (I_2^{1^*})^2}\cdot\frac{\partial I_2^{1^*}}{\partial\epsilon_1^1}\Big)$$
$$+\alpha_1^1\frac{\partial f_1(\mathbf{I^1}^*)}{\partial I_2^{1^*}}=0(1+C_{e_2})+\gamma_2^1 C_{e_2e_2}\Big(\gamma_1^1\frac{\partial I_1^{1^*}}{\partial\epsilon_1^1}+\beta_1^1 I_1^{1^*}+\gamma_2^1\frac{\partial I_2^{1^*}}{\partial\epsilon_1^1}-w_0(h_1^0\epsilon_1^0+h_2^0\epsilon_2^0)h_1^1\Big)$$

Write it in a vector form:

$$\Big[\theta_1^1\begin{pmatrix}\frac{\partial^2 f_1(\mathbf{I^{1*}})}{\partial(I_1^{1^*})^2} & \frac{\partial^2 f_1(\mathbf{I^{1*}})}{\partial I_1^{1^*}\partial I_2^{1^*}}\end{pmatrix}+\theta_2^1\begin{pmatrix}\frac{\partial^2 f_2(\mathbf{I^{1*}})}{\partial(I_1^{1^*})^2} & \frac{\partial^2 f_2(\mathbf{I^{1*}})}{\partial I_1^{1^*}\partial I_2^{1^*}}\end{pmatrix}\Big]\begin{pmatrix}\frac{\partial I_1^{1^*}}{\partial\epsilon_1^1}\\ \frac{\partial I_2^{1^*}}{\partial\epsilon_1^1}\end{pmatrix}+\alpha_1^1\frac{\partial f_1(\mathbf{I^{1*}})}{\partial I_1^{1^*}}$$

$$=\beta_1^1(1+C_{e_2})+\gamma_1^1 C_{e_2e_2}\Big(\begin{pmatrix}\gamma_1^1 & \gamma_2^1\end{pmatrix}\begin{pmatrix}\frac{\partial I_1^{1^*}}{\partial\epsilon_1^1}\\ \frac{\partial I_2^{1^*}}{\partial\epsilon_1^1}\end{pmatrix}+\beta_1^1 I_1^{1^*}-w_0(h_1^0\epsilon_1^0+h_2^0\epsilon_2^0)h_1^1\Big)$$

$$\Big[\theta_1^1\begin{pmatrix}\frac{\partial^2 f_1(\mathbf{I^{1*}})}{\partial I_2^{1^*}\partial I_1^{1^*}} & \frac{\partial^2 f_1(\mathbf{I^{1*}})}{\partial(I_2^{1^*})^2}\end{pmatrix}+\theta_2^1\begin{pmatrix}\frac{\partial^2 f_2(\mathbf{I^{1*}})}{\partial I_2^{1^*}\partial I_1^{1^*}} & \frac{\partial^2 f_2(\mathbf{I^{1*}})}{\partial(I_2^{1^*})^2}\end{pmatrix}\Big]\begin{pmatrix}\frac{\partial I_1^{1^*}}{\partial\epsilon_1^1}\\ \frac{\partial I_2^{1^*}}{\partial\epsilon_1^1}\end{pmatrix}+\alpha_1^1\frac{\partial f_1(\mathbf{I^{1*}})}{\partial I_2^{1^*}}$$

$$=0(1+C_{e_2})+\gamma_2^1 C_{e_2e_2}\Big(\begin{pmatrix}\gamma_1^1 & \gamma_2^1\end{pmatrix}\begin{pmatrix}\frac{\partial I_1^{1^*}}{\partial\epsilon_1^1}\\ \frac{\partial I_2^{1^*}}{\partial\epsilon_1^1}\end{pmatrix}+\beta_1^1 I_1^{1^*}-w_0(h_1^0\epsilon_1^0+h_2^0\epsilon_2^0)h_1^1\Big)$$

Then, the vector is

$$\begin{pmatrix}\frac{\partial I_1^{1^*}}{\partial\epsilon_1^1}\\ \frac{\partial I_2^{1^*}}{\partial\epsilon_1^1}\end{pmatrix}=\mathbf{A}_1^{-1}\cdot C_{e_2e_2}\Big\{(\beta_1^1 I_1^{1^*}-w_1h_1^1)\begin{pmatrix}\gamma_1^1\\ \gamma_2^1\end{pmatrix}+(1+C_{e_2})\begin{pmatrix}\beta_1^1\\ 0\end{pmatrix}-\alpha_1^1\begin{pmatrix}\frac{\partial f_1(\mathbf{I^{1*}})}{\partial I_1^{1^*}}\\ \frac{\partial f_1(\mathbf{I^{1*}})}{\partial I_2^{1^*}}\end{pmatrix}\Big\}$$

$$=\mathbf{A}_1^{-1}\cdot\Big\{C_{e_2e_2}(\beta_1^1 I_1^{1^*}-w_1h_1^1)\begin{pmatrix}\gamma_1^1\\ \gamma_2^1\end{pmatrix}$$

$$+\Big[\frac{\beta_1^1}{\gamma_1^1}\big(\theta_1^1\mathrm{D}f_1(\mathbf{I^{1*}})+\theta_2^1\mathrm{D}f_2(\mathbf{I^{1*}})\big)\cdot\begin{pmatrix}1\\ 0\end{pmatrix}\Big]\cdot\begin{pmatrix}1\\ 0\end{pmatrix}-\alpha_1^1\begin{pmatrix}\frac{\partial f_1(\mathbf{I^{1*}})}{\partial I_1^{1^*}}\\ \frac{\partial f_1(\mathbf{I^{1*}})}{\partial I_2^{1^*}}\end{pmatrix}\Big\}$$

For $j=2$, again, we have,

$$\theta_1^1\Big(\frac{\partial^2 f_1(\mathbf{I^{1*}})}{\partial(I_1^{1^*})^2}\cdot\frac{\partial I_1^{1^*}}{\partial\epsilon_2^1}+\frac{\partial^2 f_1(\mathbf{I^{1*}})}{\partial I_1^{1^*}\partial I_2^{1^*}}\cdot\frac{\partial I_2^{1^*}}{\partial\epsilon_2^1}\Big)+\theta_2^1\Big(\frac{\partial^2 f_2(\mathbf{I^{1*}})}{\partial(I_1^{1^*})^2}\cdot\frac{\partial I_1^{1^*}}{\partial\epsilon_2^1}+\frac{\partial^2 f_2(\mathbf{I^{1*}})}{\partial I_1^{1^*}\partial I_2^{1^*}}\cdot\frac{\partial I_2^{1^*}}{\partial\epsilon_2^1}\Big)$$
$$+\alpha_2^1\frac{\partial f_2(\mathbf{I^{1*}})}{\partial I_1^{1^*}}=0(1+C_{e_2})+\gamma_2^1 C_{e_2e_2}\Big(\gamma_1^1\frac{\partial I_1^{1^*}}{\partial\epsilon_2^1}+\beta_2^1 I_2^{1^*}+\gamma_2^1\frac{\partial I_2^{1^*}}{\partial\epsilon_2^1}-w_0(h_1^0\epsilon_1^0+h_2^0\epsilon_2^0)h_2^1\Big)$$

$$\theta_1^1\Big(\frac{\partial^2 f_1(\mathbf{I^{1*}})}{\partial I_2^{1^*}\partial I_1^{1^*}}\cdot\frac{\partial I_1^{1^*}}{\partial\epsilon_2^1}+\frac{\partial^2 f_1(\mathbf{I^{1*}})}{\partial(I_2^{1^*})^2}\cdot\frac{\partial I_2^{1^*}}{\partial\epsilon_2^1}\Big)+\theta_2^1\Big(\frac{\partial^2 f_2(\mathbf{I^{1*}})}{\partial I_2^{1^*}\partial I_1^{1^*}}\cdot\frac{\partial I_1^{1^*}}{\partial\epsilon_2^1}+\frac{\partial^2 f_2(\mathbf{I^{1*}})}{\partial(I_2^{1^*})^2}\cdot\frac{\partial I_2^{1^*}}{\partial\epsilon_2^1}\Big)$$
$$+\alpha_2^1\frac{\partial f_2(\mathbf{I^{1*}})}{\partial I_2^{1^*}}=\beta_2^1(1+C_{e_2})+\gamma_2^1 C_{e_2e_2}\Big(\gamma_1^1\frac{\partial I_1^{1^*}}{\partial\epsilon_2^1}+\beta_2^1 I_2^{1^*}+\gamma_2^1\frac{\partial I_2^{1^*}}{\partial\epsilon_2^1}-w_0(h_1^0\epsilon_1^0+h_2^0\epsilon_2^0)h_2^1\Big)$$

$$\Big[\theta_1^1 \begin{pmatrix} \frac{\partial^2 f_1(\mathbf{I^{1*}})}{\partial (I_1^{1^*})^2} & \frac{\partial^2 f_1(\mathbf{I^{1*}})}{\partial I_1^{1^*} \partial I_2^{1^*}} \end{pmatrix} + \theta_2^1 \begin{pmatrix} \frac{\partial^2 f_2(\mathbf{I^{1*}})}{\partial (I_1^{1^*})^2} & \frac{\partial^2 f_2(\mathbf{I^{1*}})}{\partial I_1^{1^*} \partial I_2^{1^*}} \end{pmatrix}\Big] \begin{pmatrix} \frac{\partial I_1^{1^*}}{\partial \epsilon_2^1} \\ \frac{\partial I_2^{1^*}}{\partial \epsilon_2^1} \end{pmatrix} + \alpha_2^1 \frac{\partial f_2(\mathbf{I^{1*}})}{\partial I_1^{1^*}}$$

$$= 0(1 + C_{e_2}) + \gamma_1^1 C_{e_2e_2} \Big( \begin{pmatrix} \gamma_1^1 & \gamma_2^1 \end{pmatrix} \begin{pmatrix} \frac{\partial I_1^{1^*}}{\partial \epsilon_2^1} \\ \frac{\partial I_2^{1^*}}{\partial \epsilon_2^1} \end{pmatrix} + \beta_2^1 I_2^{1^*} - w_0(h_1^0\epsilon_1^0 + h_2^0\epsilon_2^0)h_2^1 \Big)$$

$$\Big[\theta_1^1 \begin{pmatrix} \frac{\partial^2 f_1(\mathbf{I^{1*}})}{\partial I_2^{1^*} \partial I_1^{1^*}} & \frac{\partial^2 f_1(\mathbf{I^{1*}})}{\partial (I_2^{1^*})^2} \end{pmatrix} + \theta_2^1 \begin{pmatrix} \frac{\partial^2 f_2(\mathbf{I^{1*}})}{\partial I_2^{1^*} \partial I_1^{1^*}} & \frac{\partial^2 f_2(\mathbf{I^{1*}})}{\partial (I_2^{1^*})^2} \end{pmatrix}\Big] \begin{pmatrix} \frac{\partial I_1^{1^*}}{\partial \epsilon_2^1} \\ \frac{\partial I_2^{1^*}}{\partial \epsilon_2^1} \end{pmatrix} + \alpha_2^1 \frac{\partial f_2(\mathbf{I^{1*}})}{\partial I_2^{1^*}}$$

$$= \beta_2^1(1 + C_{e_2}) + \gamma_2^1 C_{e_2e_2} \Big( \begin{pmatrix} \gamma_1^1 & \gamma_2^1 \end{pmatrix} \begin{pmatrix} \frac{\partial I_1^{1^*}}{\partial \epsilon_2^1} \\ \frac{\partial I_2^{1^*}}{\partial \epsilon_2^1} \end{pmatrix} + \beta_2^1 I_2^{1^*} - w_0(h_1^0\epsilon_1^0 + h_2^0\epsilon_2^0)h_2^1 \Big)$$

Finally, the vector with respect to $\epsilon_2^1$ is given by

$$\begin{pmatrix} \frac{\partial I_1^{1^*}}{\partial \epsilon_2^1} \\ \frac{\partial I_2^{1^*}}{\partial \epsilon_2^1} \end{pmatrix} = \mathbf{A}_1^{-1} \cdot \Big\{ (\beta_2^1 I_2^{1^*} - w_1 h_2^1) C_{e_2e_2} \begin{pmatrix} \gamma_1^1 \\ \gamma_2^1 \end{pmatrix} + (1 + C_{e_2}) \begin{pmatrix} 0 \\ \beta_2^1 \end{pmatrix} - \alpha_2^1 \begin{pmatrix} \frac{\partial f_2(\mathbf{I^{1*}})}{\partial I_1^{1^*}} \\ \frac{\partial f_2(\mathbf{I^{1*}})}{\partial I_2^{1^*}} \end{pmatrix} \Big\}$$

$$= \mathbf{A}_1^{-1} \cdot \Big\{ C_{e_2e_2} (\beta_2^1 I_2^{1^*} - w_1 h_2^1) \begin{pmatrix} \gamma_1^1 \\ \gamma_2^1 \end{pmatrix}$$

$$+ \Big[ \frac{\beta_2^1}{\gamma_2^1} \big( \theta_1^1 \mathrm{D} f_1(\mathbf{I^{1*}}) + \theta_2^1 \mathrm{D} f_2(\mathbf{I^{1*}}) \big) \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Big] \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \alpha_2^1 \begin{pmatrix} \frac{\partial f_2(\mathbf{I^{1*}})}{\partial I_1^{1^*}} \\ \frac{\partial f_2(\mathbf{I^{1*}})}{\partial I_2^{1^*}} \end{pmatrix} \Big\}$$

Step 5:

Solve for the optimal hedge ratio $\mathbf{h^{1*}} = \{h_1^{1^*}, h_2^{1^*}\}$.

Again, $P_{w_2} = C_{e_2}$ and $P_{w_2w_2} = -C_{e_2e_2} - C_{e_2e_2}^2 (\gamma^0)^T \cdot \mathbf{A}_1^{-1} \cdot \gamma$.

Since $w_2 = w_0(h_1^0\epsilon_1^0 + h_2^0\epsilon_2^0)(h_1^1\epsilon_1^1 + h_2^1\epsilon_2^1) = w_1(h_1^1\epsilon_1^1 + h_2^1\epsilon_2^1$, for $j = 1, 2$

$$\frac{\partial w_2}{\partial h_j^1} = w_0(h_1^0\epsilon_1^0 + h_2^0\epsilon_2^0)\epsilon_j^1 = w_1\epsilon_j^1$$

It requires that the first order conditions of $\mathbf{E}[P(w_2)]$ are zero,which are

$$\mathbf{E}\big[P_{w_2} \cdot \frac{\partial w_2}{\partial h_1^1}\big] = 0$$

$$\mathbf{E}\big[P_{w_2}\cdot\frac{\partial w_2}{\partial h_2^1}\big]=0$$

Similarly to the example in Section 2.1.3, we have

$$\mathbf{E}\big[P_{w_2}\cdot\frac{\partial w_2}{\partial h_1^1}\big]=\mathbf{E}[P_{w_2}\cdot w_1\epsilon_1^1]=\mathbf{E}[w_1P_{w_2}\cdot\epsilon_1^1]=0\Leftrightarrow$$

$$\mathbf{E}[w_1P_{w_2}\cdot\epsilon_1^1]=\mathbf{E}[w_1P_{w_2}]\cdot\mathbf{E}[\epsilon_1^1]-\mathrm{cov}(w_1P_{w_2},\epsilon_1^1)=0\ (\text{since }\mathbf{E}[\epsilon_1^1]=1)\Leftrightarrow$$

$$\mathbf{E}[w_1P_{w_2}]-\mathbf{E}_{\epsilon_1^1}[w_1P_{w_2\epsilon_1^1}]\cdot\mathbf{E}_{\epsilon_1^1}[\epsilon_1^1]\cdot\mathrm{cov}(\epsilon_1^1,\epsilon_1^1)=0\Leftrightarrow$$

$$\mathbf{E}[w_1P_{w_2}\cdot\epsilon_1^1]=\mathbf{E}[w_1P_{w_2}]-\mathbf{E}_{\epsilon_1^1}[w_1P_{w_2\epsilon_1^1}]\cdot(\sigma_1^1)^2=0.$$

Since

$$\mathbf{E}_{\epsilon_1^1}[w_1P_{w_2\epsilon_1^1}]=\mathbf{E}_{\epsilon_1^1}[w_1(P_{w_2})_{\epsilon_1^1}]=\mathbf{E}_{\epsilon_1^1}[w_1(C_{e_2})_{\epsilon_1^1}]=\mathbf{E}_{\epsilon_1^1}[w_1C_{e_2e_2}\cdot\frac{\partial e_2}{\partial\epsilon_1^1}]$$

then

$$\mathbf{E}[w_1P_{w_2}]-(\sigma_1^1)^2\mathbf{E}_{\epsilon_1^1}[w_1C_{e_2e_2}\cdot\frac{\partial e_2}{\partial\epsilon_1^1}]=0\quad(**)$$

Also by definition, we have

$$\frac{\partial e_2}{\partial\epsilon_1^1}=\gamma_1^1\frac{\partial I_1^{1^*}}{\partial\epsilon_1^1}+\gamma_2^1\frac{\partial I_2^{1^*}}{\partial\epsilon_1^1}+\beta_1^1I_1^{1^*}-w_1h_1^1$$

Then, $(**)$ is

$$\mathbf{E}[w_1P_{w_2}]-(\sigma_1^1)^2\mathbf{E}_{\epsilon_1^1}[w_1C_{e_2e_2}(\gamma_1^1\frac{\partial I_1^{1^*}}{\partial\epsilon_1^1}+\gamma_2^1\frac{\partial I_2^{1^*}}{\partial\epsilon_1^1}+\beta_1^1I_1^{1^*}-w_1h_1^1)]=0$$

Rewrite it as:

$$\mathbf{E}[w_1P_{w_2}]-(\sigma_1^1)^2\mathbf{E}_{\epsilon_1^1}\Big[w_1C_{e_2e_2}(\gamma_1^1\frac{\partial I_1^{1^*}}{\partial\epsilon_1^1}+\gamma_2^1\frac{\partial I_2^{1^*}}{\partial\epsilon_1^1})\Big]$$
$$-(\sigma_1^1)^2\mathbf{E}_{\epsilon_1^1}[w_1C_{e_2e_2}\beta_1^1I_1^{1^*}]+(\sigma_1^1)^2\mathbf{E}_{\epsilon_1^1}[w_1C_{e_2e_2}w_1h_1^1]=0$$

From Step 4, we have $\dfrac{\partial\mathbf{I^1}^*}{\partial\epsilon_1^1}$, then,

$$\mathbf{E}[w_1P_{w_2}]-(\sigma_1^1)^2\mathbf{E}_{\epsilon_1^1}\Big[w_1C_{e_2e_2}\begin{pmatrix}\gamma_1^1 & \gamma_2^1\end{pmatrix}\cdot\mathbf{A}_1^{-1}\big(C_{e_2e_2}(\beta_1^1I_1^{1^*}-w_1h_1^1)\begin{pmatrix}\gamma_1^1\\ \gamma_2^1\end{pmatrix}$$
$$+\frac{\beta_1^1}{\gamma_1^1}\big(\theta_1^1\mathrm{D}^2f_1(\mathbf{I^1}^*)+\theta_2^1\mathrm{D}^2f_2(\mathbf{I^1}^*)\big)\begin{pmatrix}1\\0\end{pmatrix}-\alpha_1\mathrm{D}\mathbf{f}(\mathbf{I^1}^*)\begin{pmatrix}1\\0\end{pmatrix}\big)\cdot\begin{pmatrix}1\\0\end{pmatrix}\Big]$$
$$-(\sigma_1^1)^2\beta_1^1\mathbf{E}_{\epsilon_1^1}[w_1C_{e_2e_2}I_1^{1^*}]+(\sigma_1^1)^2h_1^1\mathbf{E}_{\epsilon_1^1}[(w_1)^2C_{e_2e_2}]=0$$

$$
\begin{aligned}
&\mathbf{E}[w_1 P_{w_2}] - (\sigma_1^1)^2 \beta_1^1 \mathbf{E}_{\epsilon_1^1}\Big[w_1 I_1^{1^*} C_{e_2e_2} \begin{pmatrix} \gamma_1^1 & \gamma_1^1 \end{pmatrix} \cdot \mathbf{A}_1^{-1} C_{e_2e_2} \begin{pmatrix} \gamma_1^1 \\ \gamma_2^1 \end{pmatrix} \Big] \\
&+ (\sigma_1^1)^2 h_1^1 \mathbf{E}_{\epsilon_1^1}\Big[(w_1)^2 C_{e_2e_2} \begin{pmatrix} \gamma_1^1 & \gamma_2^1 \end{pmatrix} \cdot \mathbf{A}_1^{-1} \cdot \begin{pmatrix} \gamma_1^1 \\ \gamma_2^1 \end{pmatrix} \Big] \\
&- (\sigma_1^1)^2 \frac{\beta_1^1}{\gamma_1^1} \mathbf{E}_{\epsilon_1^1}\Big[w_1 C_{e_2e_2} \begin{pmatrix} \gamma_1^1 & \gamma_2^1 \end{pmatrix} \cdot \mathbf{A}_1^{-1} \cdot \Big(\big(\theta_1^1 \mathrm{D}^2 f_1(\mathbf{I}^{\mathbf{1}^*}) + \theta_2^1 \mathrm{D}^2 f_2(\mathbf{I}^{\mathbf{1}^*})\big) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Big) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&+ (\sigma_1^1)^2 \alpha_1^1 \mathrm{D}\mathbf{f}(\mathbf{I}^{\mathbf{1}^*}) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Big] - (\sigma_1^1)^2 \beta_1^1 \mathbf{E}_{\epsilon_1^1}[w_1 C_{e_2e_2} I_1^{1^*}] + (\sigma_1^1)^2 h_1^1 \mathbf{E}_{\epsilon_1^1}[(w_1)^2 C_{e_2e_2}] = 0
\end{aligned}
$$

$$
\begin{aligned}
&\mathbf{E}[w_1 P_{w_2}] - (\sigma_1^1)^2 \beta_1^1 \mathbf{E}_{\epsilon_1^1}\Big[w_1 I_1^{1^*} C_{e_2e_2} \begin{pmatrix} \gamma_1^1 & \gamma_1^1 \end{pmatrix} \cdot \mathbf{A}_1^{-1} C_{e_2e_2} \begin{pmatrix} \gamma_1^1 \\ \gamma_2^1 \end{pmatrix} \Big] \\
&- (\sigma_1^1)^2 \frac{\beta_1^1}{\gamma_1^1} \mathbf{E}_{\epsilon_1^1}\Big[w_1 C_{e_2e_2} \begin{pmatrix} \gamma_1^1 & \gamma_2^1 \end{pmatrix} \cdot \mathbf{A}_1^{-1} \cdot \Big(\big(\theta_1^1 \mathrm{D}^2 f_1(\mathbf{I}^{\mathbf{1}^*}) + \theta_2^1 \mathrm{D}^2 f_2(\mathbf{I}^{\mathbf{1}^*})\big) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Big) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&+ (\sigma_1^1)^2 \alpha_1^1 \mathrm{D}\mathbf{f}(\mathbf{I}^{\mathbf{1}^*}) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Big] - (\sigma_1^1)^2 \beta_1^1 \mathbf{E}_{\epsilon_1^1}[w_1 C_{e_2e_2} I_1^{1^*}] \\
&= -(\sigma_1^1)^2 h_1^1 \mathbf{E}_{\epsilon_1^1}\Big[(w_1)^2 C_{e_2e_2} \begin{pmatrix} \gamma_1^1 & \gamma_2^1 \end{pmatrix} \cdot \mathbf{A}_1^{-1} \cdot \begin{pmatrix} \gamma_1^1 \\ \gamma_2^1 \end{pmatrix} \Big] + (\sigma_1^1)^2 h_1^1 \mathbf{E}_{\epsilon_1^1}[(w_1)^2 C_{e_2e_2}]
\end{aligned}
$$

$$
\begin{aligned}
&\mathbf{E}[w_1 P_{w_2}] + (\sigma_1^1)^2 \beta_1^1 \mathbf{E}_{\epsilon_1^1}\Big[w_1 I_1^{1^*} \big( - C_{e_2e_2} - C_{e_2e_2} \begin{pmatrix} \gamma_1^1 & \gamma_1^1 \end{pmatrix} \cdot \mathbf{A}_1^{-1} C_{e_2e_2} \begin{pmatrix} \gamma_1^1 \\ \gamma_2^1 \end{pmatrix} )\Big] \\
&- (\sigma_1^1)^2 \frac{\beta_1^1}{\gamma_1^1} \mathbf{E}_{\epsilon_1^1}\Big[w_1 C_{e_2e_2} \begin{pmatrix} \gamma_1^1 & \gamma_2^1 \end{pmatrix} \cdot \mathbf{A}_1^{-1} \cdot \Big(\big(\theta_1^1 \mathrm{D}^2 f_1(\mathbf{I}^{\mathbf{1}^*}) + \theta_2^1 \mathrm{D}^2 f_2(\mathbf{I}^{\mathbf{1}^*})\big) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Big) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&+ (\sigma_1^1)^2 \alpha_1^1 \mathrm{D}\mathbf{f}(\mathbf{I}^{\mathbf{1}^*}) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Big] = (\sigma_1^1)^2 h_1^1 \mathbf{E}_{\epsilon_1^1}\Big[(w_1)^2 (-C_{e_2e_2} - C_{e_2e_2} \begin{pmatrix} \gamma_1^1 & \gamma_2^1 \end{pmatrix} \cdot \mathbf{A}_1^{-1} \cdot \begin{pmatrix} \gamma_1^1 \\ \gamma_2^1 \end{pmatrix} )\Big]
\end{aligned}
$$

$$\mathbf{E}[w_1 P_{w_2}] + (\sigma_1^1)^2\beta_1^1\mathbf{E}_{\epsilon_1^1}\Big[w_1 I_1^{1^*} P_{w_2w_2}\Big] - (\sigma_1^1)^2\mathbf{E}_{\epsilon_1^1}\Big[w_1 C_{e_2e_2}\begin{pmatrix}\gamma_1^1 & \gamma_2^1\end{pmatrix}\cdot\mathbf{A}_1^{-1}\cdot$$

$$\Big(\frac{\beta_1^1}{\gamma_1^1}\big(\theta_1^1\mathrm{D}^2 f_1(\mathbf{I^1}^*) + \theta_2^1\mathrm{D}^2 f_2(\mathbf{I^1}^*)\big)\begin{pmatrix}1\\0\end{pmatrix}\Big)\cdot\begin{pmatrix}1\\0\end{pmatrix} + (\sigma_1^1)^2\alpha_1^1\mathrm{D}\mathbf{f}(\mathbf{I^1}^*)\begin{pmatrix}1\\0\end{pmatrix}\Big]$$

$$-(\sigma_1^1)^2\beta_1^1\mathbf{E}_{\epsilon_1^1}[w_1 C_{e_2e_2} I_1^{1^*}] = (\sigma_1^1)^2 h_1^1\mathbf{E}_{\epsilon_1^1}\Big[(w_1)^2 P_{w_2w_2}\Big]$$

Similarly, for $\epsilon_2^1$, the derivative function is

$$\frac{\partial e_2}{\partial\epsilon_2^1} = \gamma_1^1\frac{\partial I_1^{1^*}}{\partial\epsilon_2^1} + \gamma_2^1\frac{\partial I_2^{1^*}}{\partial\epsilon_2^1} + \beta_2^1 I_2^{1^*} - w_1 h_2^1$$

which is need to solve for $h_2^1$.

$$\mathbf{E}[w_1 P_{w_2}] - (\sigma_2^1)^2\mathbf{E}_{\epsilon_2^1}[w_1 C_{e_2e_2}(\gamma_1^1\frac{\partial I_1^{1^*}}{\partial\epsilon_2^1} + \gamma_2^1\frac{\partial I_2^{1^*}}{\partial\epsilon_2^1} + \beta_2^1 I_2^{1^*} - w_1 h_2^1)] = 0$$

Then,

$$\mathbf{E}[w_1 P_{w_2}] - (\sigma_2^1)^2\mathbf{E}_{\epsilon_2^1}\Big[w_1 C_{e_2e_2}(\gamma_1^1\frac{\partial I_1^{1^*}}{\partial\epsilon_2^1} + \gamma_2^1\frac{\partial I_2^{1^*}}{\partial\epsilon_2^1})\Big]$$
$$- (\sigma_2^1)^2\mathbf{E}_{\epsilon_2^1}[w_1 C_{e_2e_2}\beta_2^1 I_2^1] + (\sigma_2^1)^2\mathbf{E}_{\epsilon_2^1}[C_{e_2e_2}(w_1)^2 h_2^1] = 0$$

Replace the vector $\dfrac{\partial\mathbf{I^1}^*}{\partial\epsilon_2^1}$, we have

$$\mathbf{E}[w_1 P_{w_2}] - (\sigma_2^1)^2\mathbf{E}_{\epsilon_2^1}\Big[w_1 C_{e_2e_2}\begin{pmatrix}\gamma_1^1 & \gamma_2^1\end{pmatrix}\cdot\mathbf{A}_1^{-1}\big(C_{e_2e_2}(\beta_2^1 I_2^{1^*} - w_1 h_2^1)\begin{pmatrix}\gamma_1^1\\\gamma_2^1\end{pmatrix}$$

$$+\frac{\beta_2^1}{\gamma_2^1}\big(\theta_1^1\mathrm{D}^2 f_1(\mathbf{I^1}^*) + \theta_2^1\mathrm{D}^2 f_2(\mathbf{I^1}^*)\big)\begin{pmatrix}0\\1\end{pmatrix} - \alpha_2\mathrm{D}\mathbf{f}(\mathbf{I^1}^*)\begin{pmatrix}0\\1\end{pmatrix})\cdot\begin{pmatrix}0\\1\end{pmatrix}\Big]$$

$$-(\sigma_2^1)^2\beta_2^1\mathbf{E}_{\epsilon_2^1}[w_1 C_{e_2e_2} I_2^{1^*}] + (\sigma_2^1)^2 h_2^1\mathbf{E}_{\epsilon_2^1}[(w_1)^2 C_{e_2e_2}] = 0$$

$$\mathbf{E}[w_1 P_{w_2}] - (\sigma_2^1)^2 \beta_2^1 \mathbf{E}_{\epsilon_2^1}\Big[ w_1 I_2^{1^*} C_{e_2e_2} \begin{pmatrix} \gamma_1^1 & \gamma_1^1 \end{pmatrix} \cdot \mathbf{A}_1^{-1} C_{e_2e_2} \begin{pmatrix} \gamma_1^1 \\ \gamma_2^1 \end{pmatrix} \Big]$$

$$+ (\sigma_2^1)^2 h_2^1 \mathbf{E}_{\epsilon_2^1}\Big[ (w_1)^2 C_{e_2e_2} \begin{pmatrix} \gamma_1^1 & \gamma_2^1 \end{pmatrix} \cdot \mathbf{A}_1^{-1} \cdot \begin{pmatrix} \gamma_1^1 \\ \gamma_2^1 \end{pmatrix} \Big]$$

$$- (\sigma_2^1)^2 \frac{\beta_2^1}{\gamma_2^1} \mathbf{E}_{\epsilon_2^1}\Big[ w_1 C_{e_2e_2} \begin{pmatrix} \gamma_1^1 & \gamma_2^1 \end{pmatrix} \cdot \mathbf{A}_1^{-1} \cdot \Big( \big(\theta_1^1 \mathrm{D}^2 f_1(\mathbf{I}^{\mathbf{1}^*}) + \theta_2^1 \mathrm{D}^2 f_2(\mathbf{I}^{\mathbf{1}^*})\big) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Big) \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$+ (\sigma_2^1)^2 \alpha_2^1 \mathrm{D}\mathbf{f}(\mathbf{I}^{\mathbf{1}^*}) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Big] - (\sigma_2^1)^2 \beta_2^1 \mathbf{E}_{\epsilon_2^1}[w_1 C_{e_2e_2} I_2^{1^*}] + (\sigma_2^1)^2 h_2^1 \mathbf{E}_{\epsilon_2^1}[(w_1)^2 C_{e_2e_2}] = 0$$

$$\mathbf{E}[w_1 P_{w_2}] - (\sigma_2^1)^2 \beta_1^1 \mathbf{E}_{\epsilon_2^1}\Big[ w_1 I_2^{1^*} C_{e_2e_2} \begin{pmatrix} \gamma_1^1 & \gamma_1^1 \end{pmatrix} \cdot \mathbf{A}_1^{-1} C_{e_2e_2} \begin{pmatrix} \gamma_1^1 \\ \gamma_2^1 \end{pmatrix} \Big]$$

$$- (\sigma_2^1)^2 \frac{\beta_2^1}{\gamma_2^1} \mathbf{E}_{\epsilon_2^1}\Big[ w_1 C_{e_2e_2} \begin{pmatrix} \gamma_1^1 & \gamma_2^1 \end{pmatrix} \cdot \mathbf{A}_1^{-1} \cdot \Big( \big(\theta_1^1 \mathrm{D}^2 f_1(\mathbf{I}^{\mathbf{1}^*}) + \theta_2^1 \mathrm{D}^2 f_2(\mathbf{I}^{\mathbf{1}^*})\big) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Big) \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$+ (\sigma_2^1)^2 \alpha_2^1 \mathrm{D}\mathbf{f}(\mathbf{I}^{\mathbf{1}^*}) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Big] - (\sigma_2^1)^2 \beta_2^1 \mathbf{E}_{\epsilon_2^1}[w_1 C_{e_2e_2} I_1^{1^*}]$$

$$= -(\sigma_2^1)^2 h_2^1 \mathbf{E}_{\epsilon_2^1}\Big[ (w_1)^2 C_{e_2e_2} \begin{pmatrix} \gamma_1^1 & \gamma_2^1 \end{pmatrix} \cdot \mathbf{A}_1^{-1} \cdot \begin{pmatrix} \gamma_1^1 \\ \gamma_2^1 \end{pmatrix} \Big] + (\sigma_2^1)^2 h_2^1 \mathbf{E}_{\epsilon_2^1}[(w_1)^2 C_{e_2e_2}]$$

$$\mathbf{E}[w_1 P_{w_2}] + (\sigma_2^1)^2 \beta_2^1 \mathbf{E}_{\epsilon_2^1}\Big[ w_1 I_2^{1^*} \big( - C_{e_2e_2} - C_{e_2e_2} \begin{pmatrix} \gamma_1^1 & \gamma_1^1 \end{pmatrix} \cdot \mathbf{A}_1^{-1} C_{e_2e_2} \begin{pmatrix} \gamma_1^1 \\ \gamma_2^1 \end{pmatrix} ) \Big]$$

$$- (\sigma_2^1)^2 \frac{\beta_2^1}{\gamma_2^1} \mathbf{E}_{\epsilon_2^1}\Big[ w_1 C_{e_2e_2} \begin{pmatrix} \gamma_1^1 & \gamma_2^1 \end{pmatrix} \cdot \mathbf{A}_1^{-1} \cdot \Big( \big(\theta_1^1 \mathrm{D}^2 f_1(\mathbf{I}^{\mathbf{1}^*}) + \theta_2^1 \mathrm{D}^2 f_2(\mathbf{I}^{\mathbf{1}^*})\big) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Big) \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$+ (\sigma_2^1)^2 \alpha_2^1 \mathrm{D}\mathbf{f}(\mathbf{I}^{\mathbf{1}^*}) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Big] = (\sigma_2^1)^2 h_2^1 \mathbf{E}_{\epsilon_2^1}\Big[ (w_1)^2 (-C_{e_2e_2} - C_{e_2e_2} \begin{pmatrix} \gamma_1^1 & \gamma_2^1 \end{pmatrix} \cdot \mathbf{A}_1^{-1} \cdot \begin{pmatrix} \gamma_1^1 \\ \gamma_2^1 \end{pmatrix} ) \Big]$$

$$\mathbf{E}[w_1 P_{w_2}] + (\sigma_2^1)^2 \beta_2^1 \mathbf{E}_{\epsilon_2^1}\Big[w_1 I_2^{1^*} P_{w_2 w_2}\Big] - (\sigma_2^1)^2 \mathbf{E}_{\epsilon_2^1}\Big[w_1 C_{e_2 e_2} \begin{pmatrix} \gamma_1^1 & \gamma_2^1 \end{pmatrix} \cdot \mathbf{A}_1^{-1} \cdot$$

$$\Big(\frac{\beta_2^1}{\gamma_2^1}\big(\theta_1^1 \mathrm{D}^2 f_1(\mathbf{I^{1^*}}) + \theta_2^1 \mathrm{D}^2 f_2(\mathbf{I^{1^*}})\big) \begin{pmatrix} 0 \\ 1 \end{pmatrix}\Big) \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} + (\sigma_2^1)^2 \alpha_2^1 \mathrm{D}\mathbf{f}(\mathbf{I^{1^*}}) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Big]$$

$$- (\sigma_2^1)^2 \beta_2^1 \mathbf{E}_{\epsilon_2^1}[w_1 C_{e_2 e_2} I_1^{1^*}] = (\sigma_2^1)^2 h_2^1 \mathbf{E}_{\epsilon_2^1}\Big[(w_1)^2 P_{w_2 w_2}\Big]$$

Finally, we achieve the optimal hedge ratio $\mathbf{h^{1^*}} = \{h_1^{1^*}, h_2^{1^*}\}$, where

$$h_1^1 = \frac{\mathbf{E}[w_1 P_{w_2}]}{(\sigma_1^1)^2 \mathbf{E}_{\epsilon_1^1}[(w_1)^2 P_{w_2 w_2}]} + \beta_1^1 \frac{\mathbf{E}_{\epsilon_1^1}[w_1 I_1^{1^*} P_{w_2 w_2}]}{\mathbf{E}_{\epsilon_1^1}[(w_1)^2 P_{w_2 w_2}]}$$

$$- \frac{\mathbf{E}_{\epsilon_1^1}\Big[w_1 C_{e_2 e_2} \begin{pmatrix} \gamma_1^1 & \gamma_2^1 \end{pmatrix} \cdot \mathbf{A}_1^{-1} \cdot \Big(\frac{\beta_1^1}{\gamma_1^1}\big((\theta_1^1 \mathrm{D} f_1(\mathbf{I^{1^*}}) + \theta_2^1 \mathrm{D} f_2(\mathbf{I^{1^*}})) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}\big)\Big) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}\Big]}{\mathbf{E}_{\epsilon_1^1}[(w_1)^2 P_{w_2 w_2}]}$$

$$+ \alpha_1^1 \frac{\mathbf{E}_{\epsilon_1^1}\Big[w_1 C_{e_2 e_2} \begin{pmatrix} \gamma_1^1 & \gamma_2^1 \end{pmatrix} \cdot \mathbf{A}_1^{-1} \cdot \mathrm{D}\mathbf{f}(\mathbf{I^{1^*}}) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}\Big]}{\mathbf{E}_{\epsilon_1^1}[(w_1)^2 P_{w_2 w_2}]}$$

$$h_2^1 = \frac{\mathbf{E}[w_1 P_{w_2}]}{(\sigma_2^1)^2 \mathbf{E}_{\epsilon_2^1}[(w_1)^2 P_{w_2 w_2}]} + \beta_2^1 \frac{\mathbf{E}_{\epsilon_2^1}[w_1 I_2^{1^*} P_{w_2 w_2}]}{\mathbf{E}_{\epsilon_2^1}[(w_1)^2 P_{w_2 w_2}]}$$

$$- \frac{\mathbf{E}_{\epsilon_2^1}\Big[w_1 C_{e_2 e_2} \begin{pmatrix} \gamma_1^1 & \gamma_2^1 \end{pmatrix} \cdot \mathbf{A}_1^{-1} \cdot \Big(\frac{\beta_2^1}{\gamma_2^1}\big((\theta_1^1 \mathrm{D} f_1(\mathbf{I^{1^*}}) + \theta_2^1 \mathrm{D} f_2(\mathbf{I^{1^*}})) \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}\big)\Big) \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}\Big]}{\mathbf{E}_{\epsilon_2^1}[(w_1)^2 P_{w_2 w_2}]}$$

$$+ \alpha_2^1 \frac{\mathbf{E}_{\epsilon_2^1}\Big[w_1 C_{e_2 e_2} \begin{pmatrix} \gamma_1^1 & \gamma_2^1 \end{pmatrix} \cdot \mathbf{A}_1^{-1} \cdot \mathrm{D}\mathbf{f}(\mathbf{I^{1^*}}) \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}\Big]}{\mathbf{E}_{\epsilon_2^1}[(w_1)^2 P_{w_2 w_2}]}$$

In this section, we compute the optimal hedge ratio $\mathbf{h^{1^*}}$ for $n = 2$ in period one, if the concrete model is given, and any linear hedging strategy is considered, it is feasible to calculate the optimal hedge ratio $\mathbf{h^{1^*}}$ in period one.

# CHAPTER 3

## Stochastic Optimization Application of Risk Coordination

In Chapter 2, we discuss the situation that when a company has $n$ different locations, it is possible to find the optimal hedge ratio $\mathbf{h^{0}}^*$ and $\mathbf{h^{1}}^*$ with linear hedging strategies so that the expected profit function is maximized. However, all the processes in Chapter 2 are non-stochastic processes. If we consider all the stochastic processes, whether there exists any (weak) solution of the stochastic optimal control problem. To study this problem, we construct a stochastic model and apply the method of dynamic programming (the HJB Equation).

### 3.1 Introduction of Stochastic Optimal Control Problem

#### 3.1.1 Construction of The Stochastic Model

First, suppose that a company has a similar investment pattern in Chapter 2, that is, the company has initial wealth $w_0$, a positive constant, and it acquires external funds $L_t$ from some risk-free asset with interest rate $r_L$, a constant. Then, the company invests the total liquid asset, which is the sum of $W_t$ and $L_t$ to some project. Our problem is if there exists an optimal stochastic control $h^*$, the expected profit $\mathbf{E}[P(W_t)]$ can be maximized under $h^*$. We also need the following assumptions and definitions.

Denote the domain

$$Q_0 = [0, T_0) \times \mathbb{R}$$

and

$$Q = [0, T_0) \times \Omega \subsetneq Q_0$$

where $\Omega$ is a bounded set in $\mathbb{R}$ with compact support.

Denote $T_0$ as a bounded stopping time on $[0, \infty)$ such that when $t = T_0$, the profit function $P$ is maximized.

Let the initial wealth $w_0 > 0$ be a constant, and denote $y = (0, w_0)$ as the initial point when $t = 0$.

Let $X_t$ be the predictable stock price and follow a stochastic process

$$dX_t = (\mu_0 + \mu_1 X_t)\, dt + \rho\, \sigma_X\, dZ_{1t} + \sqrt{1 - \rho^2}\, \sigma_X\, dZ_{2t} \tag{3.1}$$

Let the external funds $L_t$ follow the process

$$dL_t = r_L dt \tag{3.2}$$

Let $h = h(t, w) = h_t$ be a Markov control process and $h \in U$, a compact Borel set in $\mathbb{R}$ and $|h_t| \leq 1$. This is equivalent to say that at time $t$, $h$ is the corresponding hedge ratio of the total wealth of the company.

Let the total wealth of a company be $W_t$ and follow

$$dW_t = w_0 \big[((1 - h_t) r_W + h_t\, \mu_W)\, dt + h_t\, \sigma_W\, dZ_{1t}\big] \tag{3.3}$$

Then, the total investments $K_t$ is given by

$$dK_t = dW_t + dL_t = \big(w_0 (1 - h_t) r_W + w_0\, h_t\, \mu_W + r_L\big)\, dt + w_0\, h_t\, \sigma_W\, dZ_{1t} \tag{3.4}$$

Here $\mu_0$, $\mu_1$, $\rho$, $\sigma_X$, $r_L$, $w_0$, $\mu_W$, and $\sigma_W$ are all constants.

Let $p_t$ be the unit price and follow the process

$$dp_t = (p_0 + p_1\, X_t)\, dt + \sigma_p\, dZ_{1t} \tag{3.5}$$

Let the payoff function be

$$f : [0, T_0] \times \mathbb{R} \to \mathbb{R}$$

with $f \in C^2(Q_0), f' > 0, f'' < 0$, and define the deadweight loss to be

$$D : [0, T_0] \times \mathbb{R} \to \mathbb{R}$$

with $D \in C^2(Q_0), D' > 0, D'' > 0$.

Finally, we define the profit function $P(W_t)$ to be

$$P(W_t) = \max_{K_t}\{p_t \cdot f(K_t) - K_t - D(L_t)\} \tag{3.6}$$

Define the performance function $J^h(y)$ to be

$$\begin{aligned} J^h(0, w_0) &= \mathbf{E}^{0,w_0}\big[P(W^h_{T_0})\big] \\ J^h(y) &= \mathbf{E}^{y}\big[P(W^h_{T_0})\big] \end{aligned} \tag{3.7}$$

Then, the value function $\Phi(y)$ with the optimal control $h^*$ can be defined as

$$\Phi(y) = \sup_h\{J^h(y),\ h(y) \text{ Markov control}\} = J^{h^*}(y), \tag{3.8}$$

subject to

$$\begin{cases} dW_t = (w_0(1 - h_t)\, r_W + w_0\, h_t\, \mu_W + r_L)\, dt + w_0\, h_t\, \sigma_W\, dZ_{1t} \\ W(0) = w_0 > 0 \end{cases}$$

with a boundary condition

$$\Phi(T_0, w) = P(w) = \begin{cases} w, & [m, M] \\ 0, & \text{Otherwise} \end{cases} \tag{3.9}$$

The problem is that for each initial point $y = (0, w_0) \in Q_0$, find a number $\Phi(y)$ and a Markov control $h^* = h^*(t, w) \in \mathcal{A}$ such that

$$\Phi(y) = \sup_{h \in U}\{J^h(y)\} = J^{h^*}(y)$$

where the supremum is taken over a given family of *admissible* controls, contained in the set of a Markov process $\{h_t\}$ with values in $U$.

**Definition 3.1** *For the Markov control* $\bar{h}(t,w) \in U$ *and* $\Phi \in C_0^2(Q_0)$*, denote the* linear parabolic differential operator $\mathcal{L}^{\bar{h}}$ *by*

$$(\mathcal{L}^{\bar{h}}\Phi)(t,w) := \Phi_t(t,w) + \big(w_0(1-\bar{h})\, r_W + w_0\, \bar{h}\, \mu_W + r_L\big)\Phi_w(t,w) + \frac{1}{2}(w_0\, \bar{h}\, \sigma_W)^2\Phi_{ww}(t,w) \tag{3.10}$$

For notation convenience, let

$$\sigma^{\bar{h}} = w_0\, \bar{h}\, \sigma_W, \qquad a^{\bar{h}} = \frac{1}{2}(\sigma^{\bar{h}})^2,$$

and

$$b^{\bar{h}} = w_0(1-\bar{h})\, r_W + w_0\, \bar{h}\, \mu_W + r_L.$$

Then, Equation 3.10 can be written as

$$(\mathcal{L}^{\bar{h}}\Phi)(t,w) := \Phi_t(t,w) + b^{\bar{h}}\Phi_w(t,w) + a^{\bar{h}}\Phi_{ww}(t,w) \tag{3.11}$$

and similarly, the HJB Equation in our model is

$$\sup_{h\in U}\{(\mathcal{L}^{h}\Phi)(t,w)\} = 0 \tag{3.12}$$

### 3.1.2 Methodology Background: Hamilton-Jacobi-Bellman Equation

Our model is in fact a stochastic control problem. The general used method in stochastic control theory is the dynamic programming method, also called the Hamilton-Jacobi-Bellman Equation method. We have studied Huang and Liu (2007), [8] about the HJB Equation in financial mathematics. In that paper, the authors applied the HJB Equation method to solve a stochastic optimization problem:

Given the initial wealth, choose the number of news updates, the news accuracies, and an optimal trading strategy to maximize the expected utility function at the terminal wealth subject to the stochastic process of the total wealth with initial condition and the utility is a power function, increasing and concave.

Based on this idea, we develop our stochastic model and try to apply the HJB Equation method. In Stochastic Differential Equations (SDE) theory, it is necessary to know if there exists a solution of the given SDE with a boundary condition on a domain. In addition, Stochastic Control Theory requires that the value function is a solution of the HJB Equation. That is, we need to prove that we can apply the HJB Equation to our stochastic control problem, then prove the existence of a weak solution to the corresponding HJB Equation if an optimal hedge control exists.

In this section, we will introduce the Hamilton-Jacobi-Bellman Equation method in Stochastic Control Theory. There are some basic theorems in the Stochastic Control Theory from Øksendal, 2005, [13], Chapter 11:

Notice that $G$ is a fixed domain in $\mathbb{R} \times \mathbb{R}^n$.

**Theorem 3.1** *[13][11.2.1 (The Hamilton-Jacobi-Bellman (HJB) equation (I))] Define*

$$\Phi(y) = \sup\{J^u(y);\ u = u(Y) = u(t, X_t) \ \textit{Markov control}\}\ .$$

*Suppose that* $\Phi \in C^2(G) \cap C(\bar{G})$ *satisfies*

$$\mathbf{E}^y\Big[|\Phi(Y_\tau)| + \int_0^\alpha |\mathcal{L}^v\Phi(Y_t)|\, dt\Big] < \infty$$

*for all bounded stopping times* $\tau \leq \tau_G$, *all* $y \in G$, *and all* $v \in U$. *Moreover, suppose that an optimal Markov control* $u^*$ *exists and that* $\partial G$ *is regular for* $Y_t^{u^*}$. *Then*

$$\sup_{v\in U}\{f^v(y) + (\mathcal{L}^v\Phi)(y)\} = 0 \quad \textit{for all}\ \ y \in G \tag{3.13}$$

*and*

$$\Phi(y) = g(y) \quad \textit{for all}\ \ y \in \partial G\ . \tag{3.14}$$

*The supremum in ( 3.13) is obtained if* $v = u^*$ *where* $h^*(y)$ *is optimal. In other words,*

$$f(y, u^*(y)) + (\mathcal{L}^{u^*(y)}\Phi)(y) = 0 \quad \textit{for all}\ \ y \in G\ . \tag{3.15}$$

**Theorem 3.2** *[13][11.2.2 (The HJB (II) equation - a verification theorem)] Let $\phi$ be a function in $C^2(G) \cap C(\bar{G})$ such that, for all $v \in U$,*

$$f^v(y) + (\mathcal{L}^v \phi)(y) \leq 0; \qquad y \in G \tag{3.16}$$

*with boundary values*

$$\lim_{t \to \tau_G} \phi(Y_t) = g(Y_{\tau_G}) \cdot \chi_{\{\tau_G < \infty\}} \quad a.s. \quad \mathbb{P}^y \tag{3.17}$$

*and such that $\{\phi^-(Y_\tau);\ \tau$ stopping time, $\tau \leq \tau_G\}$ is uniformly $\mathbb{P}^y$-integrable for all Markov controls $u$ and all $y \in G$.*

*Then*

$$\phi(y) \geq J^u(y) \quad \text{for all Markov controls } u \text{ and all } y \in G\ . \tag{3.18}$$

*Moreover, if for each $y \in G$ we have found $u_0(y)$ such that*

$$f^{u_0(y)}(y) + (\mathcal{L}^{u_0(y)} \phi)(y) = 0 \tag{3.19}$$

*and $\{\phi(Y_\tau^{u_0});\ \tau$ stopping time, $\tau \leq \tau_G\}$ is uniformly $\mathbb{P}^y$-integrable for all $y \in G$ then $u_0 = u_0(y)$ is a Markov control such that*

$$\phi(y) = J^{u_0}(y)$$

*and hence if $u_0$ is admissible, then $u_0$ must be an optimal control and $\phi(y) = \Phi(y)$.*

**Theorem 3.3** *[13][11.2.3] Let*

$$\Phi_M(y) = \sup\{J^u(y);\ u = u(Y) \text{ Markov control}\}$$

*and*

$$\Phi_a(y) = \sup\{J^u(y);\ u = u(t, \omega)\ \mathcal{F}_t^{(m)} - \text{adapted control}\}.$$

*Suppose there exists an optimal Markov control $u_0 = u_0(Y)$ for the Markov control problem (i.e. $\Phi_M(y) = J^{u_0}(y)$ for all $y \in G$) such that all the boundary points of $G$*

*are regular with respect to* $Y_t^{u_0}$ *and that* $\Phi_{_M}$ *is a bounded function in* $C^2(G) \cap C(\bar{G})$ *satisfying*

$$\mathbf{E}^y\Big[|\Phi_{_M}(Y_\alpha)| + \int_0^\alpha |\mathcal{L}^u \Phi_{_M}(Y_t)|\, dt\Big] < \infty \tag{3.20}$$

*for all bounded stopping times* $\tau \leq \tau_{_G}$*, all adapted controls* $h$ *and all* $y \in G$*. Then*

$$\Phi_{_M}(y) = \Phi_a(y) \quad \textit{for all} \;\; y \in G\ .$$

Notice that all the three basic theorems of the HJB Equation method require that the domain $G$ be a bounded set in $[0, T) \times \mathbb{R}^n$, and the boundary conditions are smooth and bounded on the boundary of $G$. Theorem 3.1 shows that if there is any optimal Markov control, the solution $\Phi(y)$ of the HJB Equation 3.13 reach the supremum zero with the optimal Markov control. Also, the supremum of the performance function $J^u(y)$ can be achieved, which is $\Phi(y)$. Theorem 3.2 describes that if the HJB Equation 3.13 has a solution $\phi(y)$ with the supremum is zero, the corresponding Markov control is the optimal control and the solution is the value function $\Phi(y)$. Theorem 3.3 extends the Markov control to any adapted control for the HJB Equation.

In our model, the domain is not a bounded domain, or the boundary condition is not a fixed boundary. Therefore, we will show that the HJB Equation can be applied to our model, which is, if there exists an optimal Markov control $h^*$, there is a weak solution of the HJB Equation 3.12 on $Q_0 = [0, T_0) \times \mathbb{R}$ with the given boundary condition 3.9.

## 3.2 Existence of A Weak Solution to The Stochastic HJB Equation

### 3.2.1 The Necessary Condition of The HJB Equation with Domain $Q_0 = [0, T_0) \times \mathbb{R}$

In the following sections, all the notations and definitions are from Section 3.1.

Let $g(w) \in C_0^\infty(\{T_0\} \times \mathbb{R}) \cap L^2(\mathbb{R})$ be a smooth and bounded function vanishing at infinity, then we have a lemma:

**Lemma 3.1** *[7][Lemma 4.1] Suppose that $g(w) \in C_0^\infty(\mathbb{R})$ and satisfies for some $C > 0$ and $\beta > 0$, $|g(w)| + |g'(w)| + |g''(w)| \leq C(1 + |w|^\beta)$, and $b^{\bar{h}}(t, w)$, $\sigma^{\bar{h}}(t, w)$ are continuous in both $t$ and $w$. Also suppose that*

$$|b^{\bar{h}}(t, w)|^2 + |\sigma^{\bar{h}}(t, w)|^2 \leq \bar{C}(1 + |w|^2) \ ,$$

*then*

$$\lim_{\delta \to 0} \frac{1}{\delta}\big[\mathbf{E}^{t-\delta, w_0}[P(W(t + \delta))] - g(w)\big] = b^{\bar{h}}(t, w)g'(w) + \frac{1}{2}(\sigma^{\bar{h}}(t, w))^2 g''(w) \ .$$

The proof is given in [7], Lemma 4.1, using Itô Lemma.

**Theorem 3.4** *[5][Theorem 6.1] Assume that:*

*(i) $b^{\bar{h}}(t, w), \sigma^{\bar{h}}(t, w)$ are continuous and satisfy that there exists a constant $C$ such that for all $(t, w) \in \bar{Q}_0$, $|b^{\bar{h}}(t, w)| + |\sigma^{\bar{h}}(t, w)| \leq C(1 + |w|)$.*

*(ii) $b^{\bar{h}}(t, \cdot), \sigma^{\bar{h}}(t, \cdot)$ are $C^2$ for each $t \in [0, T_0]$, moreover, $b_w^{\bar{h}}, \sigma_w^{\bar{h}}$ are bounded on $Q_0 = [0, T_0) \times \mathbb{R}$ and the partial derivatives $b_w^{\bar{h}}$, $b_{ww}^{\bar{h}}$, $\sigma_w^{\bar{h}}$, $\sigma_{ww}^{\bar{h}}$ satisfy for some constant $C$, $\beta$,*

$$|b_w^{\bar{h}}(t, w)| + |b_{ww}^{\bar{h}}(t, w)| + |\sigma_w^{\bar{h}}(t, w)| + |\sigma_{ww}^{\bar{h}}(t, w)| \leq C(1 + |w|^\beta)$$

*with $(t, w) \in Q_0$*

*(iii) $g(w) \in C_0^\infty(\mathbb{R})$ is bounded.*

*Then $\Phi(y) = \mathbf{E}^y\big[P(W_{T_0})\big]$, $y = (0, w_0)$ is a solution in $C_p^{1,2}(\bar{Q}_0)$ of the homogeneous backward equation*

$$\mathcal{L}^{\bar{h}}\Phi(t, w) := \Phi_t + b^{\bar{h}}\Phi_w + a^{\bar{h}}\Phi_{ww} = 0, \qquad (3.21)$$

*where $b^{\bar{h}}$, $a^{\bar{h}}$ are the same in Equation 3.11 and with the Cauchy data*

$$\lim_{t \uparrow T_0} \Phi(t, w) = g(w), \quad w \in \mathbb{R}.$$

The proof is given in [5], Theorem 6.1 or [7], Chapter 3, §11. Theorem 1.

**Remark 3.1** *This is the existence theorem of the homogeneous backward equation with the boundary of Cauchy data in [3]. With this theorem, we can state and prove the theorem for the necessary condition of the stochastic HJB Equation 3.12 with the boundary of Cauchy data $g(w)$, which is smooth and bounded on $Q_0$.*

The following lemma and Dynkin's Formula in[13] are used in our proof.

**Lemma 3.2** *[13][Equation 7.2.6] Let $H \subset \mathbb{R}$ be a measurable set and let $\tau_H$ be the first exit time from $H$ for an Itô diffusion $X_t$. Let $\alpha$ be another stopping time, $f$ be a bounded continuous function on $\mathbb{R}$ and put*

$$\eta = f(X_{\tau_H})\chi_{\{\tau_H<\infty\}} \ , \quad \tau_H^\alpha = \inf\{t > \alpha;\, X_t \notin H\}$$

*then,*

$$\theta_\alpha \eta \cdot \chi_{\{\alpha<\infty\}} = f(X_{\tau_H^\alpha})\chi_{\tau_H^\alpha<\infty} \tag{3.22}$$

**Theorem 3.5** *[13][Theorem 7.4.1 (Dynkin's Formula)] Let $f \in C_0^2(\mathbb{R})$. Suppose $\tau$ is a stopping time, $\mathbf{E}^x[\tau] < \infty$. Then*

$$\mathbf{E}^x[f(X_\tau)] = f(x) + \mathbf{E}^x\Big[\int_0^\tau Af(X_s)\,ds\Big] \tag{3.23}$$

**Theorem 3.6** *Let $\Phi(y)$ be the value function defined in Equation 3.8.*

*Suppose that $\Phi \in C_0^2(Q_0)$ and*

$$\mathbf{E}^y\Big[\int_0^\tau |\mathcal{L}^h\Phi(W_t)|dt + |\Phi(W_\tau)|\Big] < \infty$$

*for all bounded stopping time $\tau \le T_0$, all $y \in Q_0$ and all $h \in U$.*

*Moreover, suppose that there exists an optimal Markov control $h^*$, then the HJB Equation 3.12*

$$\sup_{\bar{h}\in U}\{(\mathcal{L}^{\bar{h}}\Phi)(y)\} = 0, \qquad \textit{for all} \ \ y \in Q_0 \ ,$$

*and*

$$\Phi(y) = g(w), \forall\, y \in \partial Q_0 = \{T_0\} \times \mathbb{R}\ . \tag{3.24}$$

*The supremum is obtained if* $\bar{h} = h^*(y)$*, where* $h^*(y)$ *is optimal, that is*

$$(\mathcal{L}^{h^*(y)}\Phi)(y) = 0, \qquad \textit{for all}\ \ y \in Q_0\ .$$

*Proof.* We first show that when $h^*$ exists, the supremum of 3.12 is obtained and the boundary condition 3.24 is satisfied.

When $h^* = h^*(y)$ is optimal,

$$\Phi(y) = J^{h^*}(y) = \mathbf{E}^y\big[P(W_{T_0}^{h^*})\big]$$

Then, by Theorem 3.4, we have

$$(\mathcal{L}^{h^*(y)}\Phi)(y) = 0, \quad \text{for all}\ \ y \in Q_0\ .$$

If $y \in \partial Q_0 = \{T_0\} \times \mathbb{R}$, then

$$\Phi(y) = \Phi(T_0, w) = \mathbf{E}^y\big[P(W_{T_0})\big] = \mathbf{E}^{T_0,w}\big[P(W_{T_0})\big] = P(W_{T_0}) = g(w)\ .$$

Next, we will show that

$$(\mathcal{L}^{\bar{h}}\Phi)(y) \leq 0, \quad \text{for all}\ \ y \in Q_0\ .$$

Fix $y = (s, w) \in Q_0$ and choose a Markov control $h \in U$.

Choose $Q_{t_0} \subset Q_0$ with the form $Q_{t_0} = \{(r, z) \in Q_0;\ r < t_0\}$, where $s < t_0 < T_0$. In our model, $t_0$ is the first exit time of $Q_{t_0}$, and $0 < T_0 < \infty$.

By the definition of performance function,

$$J^h(y) = \mathbf{E}^y\big[P(W_{T_0}^h)\big].$$

Then, apply the strong Markov property and formula 3.22 to get

$$\begin{aligned}
\mathbf{E}^y[J^h(W_{t_0})] &= \mathbf{E}^y\big[\mathbf{E}^{W_{t_0}}[P(W^h_{T_0})]\big] \\
&= \mathbf{E}^y\big[\mathbf{E}^y[\theta_{t_0}\big(P(W^h_{T_0})\big)\big|\mathcal{F}_{t_0}]\big] \\
&= \mathbf{E}^y\big[\mathbf{E}^y[P(W^h_{T_0})\big|\mathcal{F}_{t_0}]\big] \\
&= \mathbf{E}^y\big[P(W^h_{T_0})\big] \\
&= J^h(y)
\end{aligned}$$

That is, $J^h(y) = \mathbf{E}^y\big[J^h(W_{t_0})\big]$.

Assume an optimal control $h^*(y) = h^*(r,z)$ exists and let

$$h(r,z) = \begin{cases} \bar{h}, & \text{if } (r,z) \in Q_{t_0} \\ h^*(r,z), & \text{if } (r,z) \notin Q_{t_0} \end{cases}$$

where $\bar{h} \in U$ is arbitrary. Thus, we have

$$\Phi(W_{t_0}) = J^{h^*}(W_{t_0}) = J^h(W_{t_0}).$$

Since $\Phi(y)$ is the supremum of $J^h(y)$, then

$$\Phi(y) \geq J^h(y) = \mathbf{E}^y\big[J^h(W_{t_0})\big] = \mathbf{E}^y\big[\Phi(W_{t_0})\big].$$

Also because $\Phi \in C_0^2(Q_0)$, by Dynkin's formula,

$$\mathbf{E}^y\big[\Phi(W_{t_0})\big] = \Phi(y) + \mathbf{E}^y\Big[\int_0^{t_0} (\mathcal{L}^{\bar{h}}\Phi)(W_r)dr\Big]$$

Therefore,

$$\Phi(y) \geq \mathbf{E}^y\big[\Phi(W_{t_0})\big] = \Phi(y) + \mathbf{E}^y\Big[\int_0^{t_0} (\mathcal{L}^{\bar{h}}\Phi)(W_r)dr\Big]$$

which is

$$\mathbf{E}^y\Big[\int_0^{t_0} (\mathcal{L}^{\bar{h}}\Phi)(W_r)dr\Big] \leq 0 \ .$$

Finally, we get

$$\frac{\mathbf{E}^y\big[(\mathcal{L}^{\bar{h}}\Phi)(W_r)dr\big]}{\mathbf{E}^y[t_0]} \leq 0, \qquad \text{for all such } Q_{t_0} \ .$$

Since $(\mathcal{L}^{\bar{h}}\Phi)(\cdot)$ is continuous at $y$, and let $t_0 \downarrow s$, we get $(\mathcal{L}^{\bar{h}}\Phi)(y) \leq 0$. Done. ■

Theorem 3.6 states the necessary condition of the HJB Equation 3.21, similar to Theorem 3.1 from [13], we need to prove a verification theorem of the HJB Equation 3.12, which is also the sufficient condition.

**Theorem 3.7** *Let $\phi(y) \in C_0^2(Q_0)$ be a function such that, for all $\bar{h} \in U$,*

$$\sup_{\bar{h}\in U}\{(\mathcal{L}^{\bar{h}}\phi)(y)\} = 0, \qquad \text{for all } \; y \in Q_0$$

*with boundary values*

$$\lim_{t\to T_0} \phi(W_t) = g(W_{T_0}) \cdot \chi_{\{T_0<\infty\}}, \quad a.s. \; \mathbb{P}^y$$

*and such that $\{\phi^-(W_{t_0});\, t_0$ stopping time, $t_0 \le T_0\}$ is uniformly $\mathbb{P}^y$-integrable for all Markov controls $h$ and all $y \in Q_0$.*

*Then $\phi(y) \ge J^h(y)$ for all Markov controls $h$ and all $y \in Q_0$.*

*Moreover, if for each $y \in Q_0$, we have found $\hat{h}$ such that*

$$(\mathcal{L}^{\hat{h}}\phi)(y) = 0$$

*and $\{\phi(W_{t_0}^{\hat{h}});\; t_0$ stopping time, $t_0 \le T_0\}$ is uniformly $\mathbb{P}^y$-integrable for all $y \in Q_0$, then $\hat{h} = \hat{h}(y)$ is a Markov control such that*

$$\phi(y) = J^{\hat{h}}(y)$$

*and if $\hat{h}$ is admissible, then $\hat{h}$ is an optimal control and $\phi(y) = \Phi(y)$.*

*Proof.* First, show that $\phi(y) \ge J^h(y)$ for all Markov controls $h$ and all $y \in Q_0$.

For each $\bar{h} \in U,\; y \in Q_0$, we have $(\mathcal{L}^{\bar{h}})\Phi(y) \le 0$ in $Q_0$.

Let $h$ be a Markov control in $U$, and apply Dynkin's formula to get

$$\begin{aligned}\mathbf{E}^y\big[\phi(W_{T_R})\big] &= \phi(y) + \mathbf{E}^y\Big[\int_0^{T_R} (\mathcal{L}^{\bar{h}}\phi)(W_r)dr\Big] \\ &\le \phi(y)\end{aligned}$$

where

$$T_R = \min\big\{T_0,\; \inf\{t > 0; |W_t| \ge R\}\big\}$$

for all $R < \infty$.

Since $\lim_{t \to T_0} \phi(W_{t_0}) = g(W_{T_0}) \cdot \chi_{\{T_0 < \infty\}} < \infty$, and $\phi^-(W_{T_0})$ is uniformly $\mathbb{P}^y$-integrable for all Markov controls $h$ and all $y \in Q_0$, Fatou's lemma gives

$$\begin{aligned}
\phi(y) &\geq \liminf_{R\to\infty} \mathbf{E}^y\big[\phi(W_{T_R})\big] \\
&\geq \mathbf{E}^y\big[\phi(W_{T_0})\big] \\
&= J^h(y).
\end{aligned}$$

Now, if $(\mathcal{L}^{\hat{h}}\phi)(y) = 0$ and $\{\phi(W_{t_0}^{\hat{h}});\ t_0 \text{ stopping time}, t_0 \leq T_0\}$ is uniformly $\mathbb{P}^y$-integrable for all $y \in Q_0$, we obtain the equality part of the statement.

Again, Dynkin's formula gives

$$\begin{aligned}
\mathbf{E}^y\big[\phi(W_{T_R})\big] &= \phi(y) + \mathbf{E}^y\Big[\int_0^{T_R} (\mathcal{L}^{\hat{h}}\phi)(W_r)dr\Big] \\
&= \phi(y)
\end{aligned}$$

Since

$$\mathbf{E}^y\Big[\int_0^{\tau} |\mathcal{L}^h\Phi(W_t)|dt + |\Phi(W_\tau)|\Big] < \infty$$

for all $\tau < T_0$ and all $y \in Q_0$, by Dominate Convergence Theorem,

$$\begin{aligned}
\phi(y) &= \mathbf{E}^y\big[\phi(W_{T_R}^{\hat{h}})\big] \\
&= \lim_{R\to\infty} \mathbf{E}^y\big[\phi(W_{T_R}^{\hat{h}})\big] \\
&= \mathbf{E}^y\big[\lim_{R\to\infty} \phi(W_{T_R}^{\hat{h}})\big] \\
&= \mathbf{E}^y\big[\phi(W_{T_0}^{\hat{h}})\big] \\
&= J^{\hat{y}}(y)
\end{aligned}$$

By the definition of admissible and value function, $\hat{h}$ is an optimal control and $\phi(y) = \Phi(y)$ is the value function.

■

### 3.2.2 Existence of A Weak Solution to The HJB Equation with $L^2$ Boundary Condition on $Q = [0, T_0) \times \Omega$

**Remark 3.2** *The boundary conditions in Theorem 3.6 and Theorem 3.7 require that $g(w) \in C_0^\infty(\mathbb{R})$, but in our model, the boundary condition is $\Phi(T_0, w) = P(w)$, where $P(w)$ is not continuous on $\{T_0\} \times \mathbb{R}$. Thus, we want to find a sequence of functions $g_n(w) \in C_0^\infty(\mathbb{R})$ such that $g_n(w)$ converges to $P(w)$ in $L^2(\Omega)$ as $n \to \infty$. And this can be done by using mollifiers. For detail definitions and proofs of mollifiers, please see [1], Chapter 2.*

Consider $Q = [0, T_0) \times \Omega$, a bounded set defined at the beginning of Section 3.1, and let $S_2^{1,2}(\Omega)$ be the Sobolev space with time $t = 1$ and $k = 2$, $p = 2$.

Given constants $0 < m$, $M < \infty$, and let $\Omega = (m-1, M+1)$. $P(w)$ is given by Equation 3.9:

$$P(w) = \begin{cases} w, & w \in [m, M] \\ 0, & w \notin [m, M], \end{cases}$$

then $P(w) \in L^2(\Omega)$.

Let $\xi$ be the standard mollifier, and for all $n = 1, 2, ...$, set

$$\xi_n(w) := n^l \xi(nw)$$

then $\int_{\mathbb{R}} \xi_n \, dx = 1$ and supp $(\xi_n) \subset B(0, \frac{1}{n})$, where $B(0, \frac{1}{n})$ is a closed ball with center 0 and radius $1/n$.

**Definition 3.2** *[1] If $\tilde{g} \in L^1_{loc}(\Omega)$, define its* mollification

$$g_n := \xi_n * \tilde{g} \ \in \Omega_n,$$

*which is*

$$g_n(x) = \int_\Omega \xi_n(x-y)\, \tilde{g}(y)\, dy = \int_{B(0,\frac{1}{n})} \xi_n(y)\, \tilde{g}(x-y)\, dy$$

*for $x \in \Omega_n$.*

**Theorem 3.8** *[1][Theorem 2.18 (Properties of mollifiers)]*

*(i)* $g_n = \xi_n * \tilde{g} \in C^\infty(\Omega_n)$.

*(ii)* *If supp* $g \subsetneq \Omega$, *and if* $\frac{1}{n} <$ *dist(supp* $g$, $\partial\Omega)$, *then* $g_n \in C_0^\infty(\Omega)$.

*(iii)* *If* $\tilde{g} \in C(\Omega)$, *then* $g_n \to \tilde{g}$ *uniformly on compact subsets of* $\Omega$.

*(iv)* *If* $1 \leq p < \infty$ *and* $\tilde{g} \in L^p(\Omega)$, *then* $g_n \in L^p(\Omega)$, *and*

$$\|g_n\|_{L^p} \leq \|\tilde{g}\|_{L^p} \quad and \quad \lim_{\frac{1}{n} \to 0^+} \|g_n - \tilde{g}\|_{L^p} = 0.$$

**Lemma 3.3** *[17][§2.1] If* $\tilde{g}$ *is a function from* $\Omega$ *to* $\mathbb{R}$, *and* $\tilde{g} \in L^p(\Omega)$ , *where* $\Omega$ *is an open subset of* $\mathbb{R}$, *then* $\tilde{g} \in L^1_{loc}(\Omega)$, *that is,* $\tilde{g}$ *is integrable on each compact subset of* $\Omega$.

**Remark 3.3** *Let* $\tilde{g}(w) = P(w)$, *by the definition of* $P(w)$ *in our model,* $P(w) \in L^2(\bar{\Omega})$. *Then,* $P(w) \in L^1_{loc}(\Omega)$. *Suppose that there is a sequence of functions* $g_n(w) \in C_0^\infty(\Omega)$, *and all the properties of mollifiers follow. Then,*

$$g_n(w) \to P(w) \quad in \quad L^2(\Omega)$$

Next, we show that the weak derivative of $P(w)$ exists on $\Omega$ and denote it by $D^\alpha P$.

**Proposition 3.9** $\Omega$ *is a bounded set of* $\mathbb{R}$. *Suppose that* $P$, $\tilde{P} \in L^2(\Omega)$, *and* $\alpha$ *is a multi-index number. If there is a sequence* $\{g_n\} \subset C_0^\infty(\Omega)$ *such that* $g_n \to P$ *and* $D^\alpha g_n \to \tilde{P}$ *in* $L^2(\Omega)$, *then* $\tilde{P} = D^\alpha P$.

*Proof.* Since $P$ and $\tilde{P}$ are in $L^2(\Omega)$, by Lemma 3.3, $P$ and $\tilde{P}$ are locally integrable, that is $P$, $\tilde{P} \in L^1_{loc}(\Omega)$.

Given a sequence $\{g_n\} \subset C_0^\infty(\Omega)$, and $g_n \to P$ and $D^\alpha g_n \to \tilde{P}$ in $L^2(\Omega)$, they also converge in $L^1_{loc}(\Omega)$. Using integration by parts, we have

$$(-1)^{|\alpha|} \int_\Omega g_n D^\alpha \zeta \, dx = \int_\Omega D^\alpha g_n \zeta \, dx$$

Let $n \to \infty$, then

$$(-1)^{|\alpha|} \int_\Omega P D^\alpha \zeta \, dx = \int_\Omega \tilde{P} \zeta \, dx$$

Therefore, $\tilde{P} = D^\alpha P$.

■

If $g_n \in C_0^\infty(\Omega)$, $g_n \in L^2(\Omega)$ for all $n = 1, 2, ...$ by Theorem 3.8, $D^\alpha g_n$, $P$ and $D^\alpha P$ are in $L^2(\Omega)$. Then, $D^\alpha P \in L^2(\Omega)$, $\alpha = 1,\, 2$, which gives $P \in S^{1,2}(\Omega)$.

By Theorem 3.6, and Theorem 3.7, we know that for any boundary function $g(w) \in C_0^\infty(\mathbb{R})$, there exists an optimal Markov control $h^*$ such that $\Phi(y) = J^{h^*}(y)$ on $Q_0$. However, the boundary condition $P(w)$ in Equation 3.9 is not continuous on $\mathbb{R}$. To show the existence of a weak solution to the HJB Equation on $Q_0$, we consider a bounded domain $Q$ with the boundary condition $P(w)$.

The idea is to find a sequence of solutions $\{\Phi_n\}_{n=1}^\infty$ of the HJB Equation 3.12 with the corresponding boundary conditions $g_n(w)$ and the optimal control $h_n^*$ such that when $g_n(w)$ converges to $P(w)$ in $L^2(\Omega)$, the sequence of solutions $\{\Phi_n\}_{n=1}^\infty$ is a Cauchy sequence, which converges weakly to some function in some vector space $V$ with proper norm on it. Thus, the limit of $\{\Phi_n\}_{n=1}^\infty$ in $V$ can be defined as a weak solution of the HJB Equation 3.12 with the boundary condition $P(w)$ on $Q$.

Define a vector space $V$, the set of all $\phi \in L^2(Q)$, $D\phi \in L^2(Q)$, $\phi(t, \cdot) \in L^2(\omega(t))$ for all $t \in [0, T_0]$, and the norm on it is defined by

$$\|\phi\|_V^2 = \iint_Q |D\phi|^2 \, dx \, dt + \sup_{t \in [0, T_0]} \int_{\omega(t)} \phi^2 \, dx \tag{3.25}$$

which is finite. We will use this vector space and the norm 3.25 from Lieberman, [11] for the following statements and proofs, and we will show that the solutions $\Phi_n$ converges weakly in $V(Q)$ when $g_n(w) \to P(w)$ in $L^2(Q)$.

Write $\dot{C}^1$ for the set of all functions in $C^1(\bar{Q})$ which vanish on $\partial Q$ and $V_0$ for the closure of $\dot{C}^1$ in the norm of $V$.

We also define a weak solution of the backward parabolic equation 3.21 if $\phi \in V_0$ and

$$\iint_{Q(\tau)} (\phi_t + (a^h D^2\phi + b^h D\phi))\varphi \, dx \, dt = \iint_{Q(\tau)} f\varphi \, dx \, dt \Longrightarrow$$
$$\int_{\omega(\tau)} \phi\varphi dx - \iint_{Q(\tau)} \phi\varphi_t \, dx \, dt - \iint_{Q(\tau)} (a^h D\phi + b^h\phi) D\varphi \, dx \, dt \tag{3.26}$$
$$= \iint_{Q(\tau)} f\varphi \, dx \, dt - \int_{\partial Q} g\varphi \, dx$$

for all $\varphi \in \dot{C}^1$ and almost $\tau \in [0, T_0]$. We may set $\varphi \in \dot{S}_2^{1,2}(Q)$, which means $\varphi$ is the limit in $S_2^{1,2}(Q)$ of $\dot{C}^1$ functions.

Claim that all the solutions $\Phi_n$ are in the vector space $V$. To show this claim, we need the Maximum Principle for Linear Parabolic Equations.

**Theorem 3.10 (Maximum Principle)** *[18][Theorem 8.1.4] Let $Q = [0, T_0) \times \Omega$ be bounded in $Q_0$, $\phi \in C^2(Q) \cap C(\bar{Q})$ satisfy $\mathcal{L}\phi = f \leq 0$ in $Q$, then*

$$\sup_Q \phi(t, x) \leq \sup_{\partial Q} \phi_+(t, x)$$

*where $\phi_+ = \max\{\phi, 0\}$.*

**Theorem 3.11** *[18][Theorem 8.1.7] Let $b^{\bar{h}}$ be bounded in $Q = [0, T_0) \times \Omega$, $\phi \in C^2(Q) \cap C(\bar{Q})$ satisfy $\mathcal{L}\phi = f$ in $Q$, then*

$$\sup_Q |\phi| \leq \sup_{\partial Q} |\phi| + C \sup_Q |f|$$

**Lemma 3.4** *$Q = [0, T_0) \times \Omega$ is a bounded subset of $Q_0$ and $\omega(t) \subset \Omega$ when $t \in [0, T_0]$.*

*If $\Phi_n$ is a solution of Equation 3.21*

$$\mathcal{L}^{\bar{h}}\Phi(t, w) := \Phi_t + b^{\bar{h}}\Phi_w + a^{\bar{h}}\Phi_{ww} = 0$$

*then $\Phi_n$ are in $V(Q)$ for all $n = 1, 2, ....$*

*Proof.* To show that $\Phi_n \in V(Q)$ for all $n = 1, 2, ...,$ we check that $\Phi_n \in L^2(Q)$, $D\Phi_n \in L^2(Q)$, as well as $\Phi_n(t, \cdot) \in L^2(\omega(t))$ for all $t \in [0, T_0]$ and for all $n$. In addition, the norm 3.25 on $Q$ is finite.

Since $\Phi_n$ is a solution of Equation 3.21 for each $n$ in the classical sense, for each multi-index $\alpha \leq 2$ and $t \leq 1$, the weak derivatives $(\Phi_n)_w$, $(\Phi_n)_{ww}$, and $(\Phi_n)_t$ exist for each $n$.

Next, show that for each $n$, $\Phi_n \in L^2(Q)$, $(\Phi_n)_w \in L^2(Q)$, and $\Phi_n(t, \cdot) \in L^2(\omega(t))$.

Since $\Phi_n$ satisfy Equation 3.21 for each $n$, by Maximum Principle 3.10 and Theorem 3.11, we have

$$\sup_Q |\Phi_n| \leq \sup_{\partial Q} |\Phi_n|,$$

with $f = 0$ in Equation 3.21.

Notice that $\Phi_n(T_0, w) = g_n(w)$, where $g_n(w) \in L^2(\Omega)$ for each $n$, then

$$\begin{aligned}
\iint_Q |\Phi_n|^2 \, dw \, dt &= \int_0^{T_0} \int_\Omega |\Phi_n|^2 \, dw \, dt \\
&\leq \int_0^{T_0} \int_\Omega |\Phi_n(T_0, w)|^2 \, dw \, dt \\
&\leq \int_0^{T_0} \int_\Omega |g_n(w)|^2 \, dw \, dt \\
&< \infty,
\end{aligned}$$

that is $\Phi_n \in L^2(Q)$ for each $n$, and similarly, $(\Phi_n)_w \in L^2(Q)$.

For $\Phi_n(t, \cdot)$, fix $t \in [0, T_0)$. Since $\Phi_n$ are solutions of Equation 3.21, $\Phi_n$ are continuous and differentiable in the variable $t$. $\Phi_n$ are bounded in the variable $t$ and $\omega(t) \subset \Omega$ is compact, then $\omega(t)$ is a bounded subset of $\mathbb{R}$. Thus, we have

$$\int_{\omega(t)} (\Phi_n)^2 \, dw < \infty$$

for all $t \in [0, T_0)$. Thus, $\Phi_n(t, \cdot) \in L^2(\omega(t))$.

The norm of $\Phi_n$ on $V(Q)$ is given by Equation 3.25:

$$\|\Phi_n\|_V^2 = \iint_Q |(\Phi_n)_w|^2 \, dw \, dt + \sup_{t \in [0, T_0]} \int_{\omega(t)} (\Phi_n)^2 \, dw$$

Since $(\Phi_n)_w \in L^2(Q)$, the first part of the norm is finite, and we need to show that the second part of the norm is finite too.

We have already gained that $\Phi_n(t, \cdot) \in L^2(\omega(t))$ for all $t \in [0, T_0]$, if we can show that, $\int_{\omega(t)} (\Phi_n)^2 \, dw$ is increasing, then its supremum appears when $t$ is close to $T_0$. On the other hand, when $t \in [0, T_0)$, we apply the fact that $\Phi_n$ are solutions of Equation 3.21 in the classical sense, which can induce that the second part of the norm is finite.

Write $(\Phi_n)_t + b^{\bar{h}}(\Phi_n)_w + a^{\bar{h}}(\Phi_n)_{ww} = 0$ as

$$(\Phi_n)_t = -b^{\bar{h}}(\Phi_n)_w - a^{\bar{h}}(\Phi_n)_{ww}$$

Then, take the inner product with $\Phi_n$ and apply integration by parts:

$$\begin{aligned}
\frac{\partial}{\partial t}(\Phi_n, \Phi_n) &= -b^{\bar{h}}((\Phi_n)_w, \Phi_n) - a^{\bar{h}}((\Phi_n)_{ww}, \Phi_n) \\
&= -b^{\bar{h}} \int (\Phi_n)_w \Phi_n - a^{\bar{h}} \int (\Phi_n)_{ww} \Phi_n \qquad (3.27) \\
&= -b^{\bar{h}}(\Phi_n)^2\Big|_{\partial Q} + b^{\bar{h}} \int (\Phi_n)(\Phi_n)_w - a^{\bar{h}}(\Phi_n)_w \Phi_n\Big|_{\partial Q} + a^{\bar{h}} \int (\Phi_n)_w^2
\end{aligned}$$

Notice that

$$a^{\bar{h}} = \frac{1}{2}(w_0 \bar{h} \sigma_W)^2 > 0$$

When $t = T_0$, since $\Phi_n \in C_0^2(Q)$, $\Phi_n = 0$ on the boundary of $Q$, then

$$\begin{aligned}
b^{\bar{h}} \int (\Phi_n)_w \Phi_n &= b^{\bar{h}}(\Phi_n)^2\Big|_{\partial Q} - b^{\bar{h}} \int (\Phi_n)(\Phi_n)_w \\
2b^{\bar{h}} \int (\Phi_n)_w \Phi_n &= b^{\bar{h}}(\Phi_n)^2\Big|_{\partial Q} = 0
\end{aligned}$$

Then, the inner product 3.27 at $t = T_0$ will be

$$\frac{\partial}{\partial t} \int_{\omega(t)} (\Phi_n)^2 \, dw\Big|_{t=T_0} = a^{\bar{h}} \int (\Phi_n)_w^2 > 0$$

which gives

$$\frac{\partial}{\partial t} \int_{\omega(t)} (\Phi_n)^2 \, dw\Big|_{t=T_0} > 0$$

Let $\varepsilon > 0$, for $t \in (T_0 - \varepsilon, T_0]$, $\int_{\omega(t)} (\Phi_n)^2 \, dw$ is increasing, which means that

$$\sup_{(T_0-\varepsilon,T_0]} \int_{\omega(t)} (\Phi_n)^2 \, dw \leq \sup_{t=T_0} \int_{\omega(t)} (\Phi_n)^2 \, dw < \infty$$

Second, show that when $t \in [0, T_0 - \frac{\varepsilon}{2})$, the supremum of the above integral is finite too.

For each $n = 1, 2, ...$, since $\Phi_n$ are classical solutions of Equation 3.21 with the boundary condition $g_n(w) \in C_0^\infty$, by regularity, $\Phi_n \in C^{1,2}(Q)$ for all $n$. That is, $\Phi_n$ is bounded on the interval $t \in [0, T_0 - \frac{\varepsilon}{2})$ for each $n$.

Thus, $\int_{\omega(t)} (\Phi_n)^2 \, dw$ is bounded on $t \in [0, T_0 - \frac{\varepsilon}{2})$, and so

$$\sup_{t\in[0,T_0-\frac{\varepsilon}{2})} \int_{\omega(t)} (\Phi_n)^2 \, dw < \infty$$

Consequently, for $t \in [0, T_0]$, the second part of the norm on $V$ is finite and we have

$$\Phi_n \in V(Q)$$

for all $n = 1, 2, ...$

■

The next step is to prove that $\{\Phi_n\}_{n=1}^\infty$ in Lemma 3.4 is a Cauchy sequence in $V(Q)$, which is equivalent to show that, for any $n$, $m > 0$,

$$\|\Phi_n - \Phi_m\|_V \to 0, \text{ as } n, m \to \infty.$$

To show $\{\Phi_n\}_{n=1}^\infty$ is Cauchy, we need the following estimate:

**Theorem 3.12** *Let $Q$ be a bounded subset of $Q_0$ and let $\phi$ be a solution of the Backward Parabolic Equation 3.21 with a boundary function $g(w) \in C_0^\infty(\Omega) \cap L^2(\Omega)$. $a^{\bar{h}}$, $b^{\bar{h}}$ are constants and if there are positive constants $\lambda$, $\Lambda$, and $\kappa$ such that*

$$\lambda \leq |a^{\bar{h}}| \leq \Lambda\lambda,$$

$$|b^{\bar{h}}| \leq \kappa\lambda,$$

*then there is a constant $C$ depending only on $\lambda$ and $\kappa$, such that*

$$\|\phi\|_V \leq Ce^{CT_0}(\|\mathcal{L}^{\bar{h}}\phi\|_{L^2} + \|g\|_{L^2}) \tag{3.28}$$

*Proof.* Choose $\tau \in (0, T_0)$ and $\delta \in (\tau, T_0)$. Let $X = (t, w)$ denote a point in $Q$.

Set $\eta \in \dot{S}_2^{1,2}((\delta, T_0) \times \Omega)$ with $\eta(X) = 0$ if $t = \delta$ or $t = T_0$, and set $\varphi = \eta_\delta$, $\upsilon = \eta_t$. Then by calculation, we get

$$\varphi_t = (\eta_\delta)_t = (\eta_t)_\delta = \upsilon_\delta$$

and so,

$$\begin{aligned}
\int_Q \phi\,\varphi_t\,dX &= \int_Q \phi\,\upsilon_\delta\,dX \\
&= \int_\Omega \int_0^{T_0} \phi(X) \frac{1}{\delta} \int_t^{t+\delta} \upsilon(s, w)\,ds\,dt\,dw \\
&= \int_\Omega \int_\delta^{T_0+\delta} \frac{1}{\delta} \int_{s-\delta}^{s} \phi(X)\,\upsilon(s, w)\,dt\,ds\,dw \\
&= \int_\Omega \int_\delta^{T_0+\delta} \phi_{-\delta}(X)\,\upsilon(s, w)\,ds\,dw \\
&= \int_Q \phi_{-\delta}(X)\upsilon(X)\,dX
\end{aligned}$$

For any $\eta \in \dot{S}_2^{1,2}$, it vanishes for $t < \delta$, and $\upsilon(s, w) = 0$ for $s < \delta$ and $s > T_0$. Apply the integration by parts, the above result becomes

$$\int_Q \phi\,\varphi_t\,dX = \int_Q \phi_{-\delta}\,\upsilon\,dX = \int_Q \phi_{-\delta}\,\eta_t\,dX = -\int_Q \phi_{-\delta t}\,\eta\,dX$$

where $\phi_{-\delta t}$ is the derivative with respect to $t$ of $\phi_{-\delta}$.

Since $\eta \in \dot{S}_2^{1,2}((\delta, T_0) \times \Omega)$, and $\eta(X) = 0$ if $t = \delta,\ t = T_0$, by the definition of the weak solution 3.26 in $V$ with $\phi_{-\delta}$, we have

$$\begin{aligned}
\int_{\omega(\tau)} \phi_{-\delta}\,\eta\,dw - \iint_Q \phi_{-\delta}\,\eta_t\,dw\,dt - \iint_Q (a^{\bar{h}} D\phi + b^{\bar{h}}\phi)_{-\delta} D\eta\,dw\,dt \\
= \iint_Q f_{-\delta}\,\eta\,dw\,dt - \int_{\partial Q} g\,\eta\,dw
\end{aligned}$$

$$-\iint_Q (a^{\bar{h}} D\phi + b^{\bar{h}}\phi)_{-\delta} D\eta \, dw\, dt = \iint_Q f_{-\delta}\, \eta\, dw\, dt + \iint_Q \phi_{-\delta}\, \eta_t\, dw\, dt$$

$$\iint_Q (a^{\bar{h}} D\phi + b^{\bar{h}}\phi)_{-\delta}\, D\eta \, dw\, dt = -\iint_Q f_{-\delta}\, \eta\, dw\, dt + \iint_Q \phi_{-\delta t}\, \eta\, dw\, dt$$

$$\int_Q (a^{\bar{h}} D\phi + b^{\bar{h}}\phi)_{-\delta}\, D\eta \, dX = -\int_Q (\mathcal{L}^{\bar{h}}\phi)_{-\delta}\, \eta\, dX + \int_Q \phi_{-\delta t}\, \eta\, dX$$

Next, we will show the above result is still true if we replace $\eta$ by $\phi_{-\delta}\chi(t)$, where

$$\chi(t) = 1,\ t > T_0 - \tau$$

$$\chi(t) = 0,\ t < T_0 - \tau,$$

with the idea of cutting function.

Fix $n$, a sufficiently large integer, and define a continuous function $z_n$, which is linear on $(T_0 - \tau, T_0 - \tau + \frac{1}{n}) \cup (T_0 - \frac{1}{n}, T_0)$, is 0 on $(-\infty, T_0 - \tau) \cup (T_0, \infty)$, and is 1 on $(T_0 - \tau + \frac{1}{n}, T_0 - \frac{1}{n})$.

Let $\eta_n = \phi_{-\delta}\, z_n$, which is an admissible test function, we can take the limit as $n \to \infty$, then $\eta \to \phi_{-\delta}$ and we infer that

$$\int_{Q(\tau)} (a^{\bar{h}} D\phi + b^{\bar{h}}\phi)_{-\delta}\, D\phi_{-\delta} \, dX = -\int_{Q(\tau)} (\mathcal{L}^{\bar{h}}\phi)_{-\delta}\, \phi_{-\delta}\, dX + \int_{Q(\tau)} \phi_{-\delta t}\, \phi_{-\delta}\, dX$$

where $Q(\tau) = (T_0 - \tau, T_0) \times \Omega$.

Denote $\omega(\tau) = \{T_0 - \tau\} \times \Omega$, and $\omega(T_0) = \{T_0\} \times \Omega$.

Integrate the above expression with respect to $t$, and the second term of the right hand side of the expression is

$$\int_{Q(\tau)} \phi_{-\delta t}\, \phi_{-\delta}\, dX = \frac{1}{2}\int_{\omega(T_0)} \phi_{-\delta}(X)^2\, dw - \frac{1}{2}\int_{\omega(\tau)} \phi_{-\delta}(X)^2\, dw$$

Let $\delta \to 0$, then

$$\int_{Q(\tau)} (a^{\bar{h}} D\phi + b^{\bar{h}}\phi)\, D\phi \, dX = -\int_{Q(\tau)} (\mathcal{L}^{\bar{h}}\phi)\, \phi\, dX + \frac{1}{2}\int_{\omega(T_0)} \phi(X)^2\, dw - \frac{1}{2}\int_{\omega(\tau)} \phi(X)^2\, dw$$

$$\frac{1}{2}\int_{\omega(\tau)} \phi^2\, dw + \int_{Q(\tau)} (a^{\bar{h}} D\phi + b^{\bar{h}}\phi) D\phi\, dX = -\int_{Q(\tau)} (\mathcal{L}^{\bar{h}}\phi)\phi\, dX + \frac{1}{2}\int_{\omega(T_0)} g^2\, dw$$

$$\frac{1}{2}\int_{\omega(\tau)} \phi^2\, dw + \int_{Q(\tau)} a^{\bar{h}}(D\phi)^2\, dX + \int_{Q(\tau)} b^{\bar{h}}\phi D\phi\, dX = -\int_{Q(\tau)} (\mathcal{L}^{\bar{h}}\phi)\phi\, dX + \frac{1}{2}\int_{\omega(T_0)} g^2\, dw$$

Since $\lambda \le a^{\bar{h}} \le \Lambda\lambda$ and $|b^{\bar{h}}| \le \kappa\lambda$, both imply that

$$\int_{\omega(\tau)} \phi^2\, dw + \int_{Q(\tau)} |D\phi|^2\, dX \le C\Big(\int_Q (\mathcal{L}^{\bar{h}}\phi)^2\, dX + \iint_{Q(\tau)} \phi^2\, dw\, dt + \int_{\omega(T_0)} g^2\, dw\Big).$$

Set $H(\tau) = \displaystyle\int_{\omega(\tau)} \phi^2\, dw$, and $K = C(\|\mathcal{L}^{\bar{h}}\phi\|_{L^2}^2 + \|g\|_{L^2}^2)$, then consider the inequality

$$H(\tau) \le C\int_{T_0-\tau}^{T_0} H(t)dt + K,$$

Apply Gronwall's inequality to get

$$\int_{T_0-\tau}^{T_0} H(t)dt \le \frac{K}{C}(e^{C\tau} - 1)$$

thus,

$$\|\phi\|_V^2 \le Ce^{CT_0}(\|\mathcal{L}^{\bar{h}}\phi\|_{L^2}^2 + \|g\|_{L^2}^2)$$

■

Using Estimate 3.28, we have the following theorem:

**Theorem 3.13** *Suppose that $g_n(w) \to P(w)$ in $L^2(Q)$ as $n \to \infty$, and $P(w)$ is defined in Equation 3.9. Also suppose that with each boundary function $g_n(w)$, $n = 1, 2, ...$, Equation 3.21 has a solution $\Phi_n(y)$ for all $n = 1, 2, ...$, then we have*

*(i) $\{\Phi_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $V(Q)$.*

*(ii) $\Phi_n$ converges to some function in $V(Q)$.*

*Proof.* Apply Estimate 3.28 in Theorem 3.12. Since $\mathcal{L}^{\bar{h}}$ is a linear operator, for $n, m > 0$, we have

$$\|\Phi_n - \Phi_m\|_{V(Q)} \le Ce^{CT_0}\big(\|\mathcal{L}^{\bar{h}}(\Phi_n - \Phi_m)\|_{L^2} + \|g_n - g_m\|_{L^2}\big)$$

Also because $\mathcal{L}^{\bar{h}}\Phi_n = 0$ for all $n = 1, 2, ...,$

$$\|\Phi_n - \Phi_m\|_{V(Q)} \leq C\, e^{CT_0}(\|g_n - g_m\|_{L^2})$$

It is clear that the boundary conditions $\{g_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $L^2(Q)$, thus $\{\Phi_n\}_{n=1}^{\infty}$ is Cauchy in $V(Q)$.

Since $V$ is a Banach space, it is complete, and $\Phi_n$ is a Cauchy sequence in $V(Q)$, $\Phi_n$ converges to some function in $V(Q)$.

■

We define a weak solution to our HJB Equation 3.12 in $V(Q)$ as follows:

**Definition 3.3** *If there exists an optimal Markov control process $h^*$, and if the HJB Equation 3.12 with the boundary condition $P(w)$ has a weak solution in $V(Q)$, then we define a* weak solution *of the HJB Equation to be the limit function of $\{\Phi_n\}_{n=1}^{\infty}$ in $V(Q)$ in Theorem 3.13. And denote it by $\Phi$ on $Q$.*

**Remark 3.4** *For each smooth and bounded function $g_n$, $\Phi_n$ is the corresponding solution of the HJB Equation 3.12 on $Q$. If the boundary function is $P(w) \in L^2(Q)$, the corresponding weak solution of the HJB Equation on $Q$ is $\Phi$.*

### 3.2.3 Extension of A Weak Solution to The HJB Equation from $Q$ to $Q_0$

The domain $Q$ of the HJB Equation in Theorem 3.13 is a bounded subset in an unbounded domain $Q_0$. Thus, we need to extend the weak solution $\Phi$ of the HJB Equation 3.12 from $Q$ to $Q_0 = [0, T_0) \times \mathbb{R}$. In order to achieve this goal, we will prove one extension theorem in the vector space $V$.

We need Poincarè's Inequality in the proof of the extension theorem:

**Lemma 3.5 (Poincaré's Inequality, [18], 1.3.4)** *Let $p = 2$ and $\Omega \subset \mathbb{R}$ be a bounded domain. If $\phi \in S_0^{1,2}(\Omega)$, then*

$$\int_\Omega |\phi|^2\, dx \leq C \int_\Omega |\mathrm{D}\phi|^2\, dx \tag{3.29}$$

Consider a compact subset $Q_R = [0, T_0) \times [-R, R]$, where $Q \subset Q_R$. That is $\Omega = [m, M] \subsetneq [-R, R]$. Then, the following theorem will extend a function in $V(Q)$ to $V(Q_0)$.

**Theorem 3.14** *Suppose that $Q_R$ is a bounded set of $Q_0 = [0, T_0) \times \mathbb{R}$, and $\Phi_R \in V(Q_R)$ is a weak solution of the HJB Equation 3.12 with $P_R(w)$. Then there exists a bounded linear operator*

$$E : V(Q_R) \to V(Q_0)$$

*such that for each $\Phi_R \in V(Q_R)$:*

*(i)* $E\Phi_R = \Phi_R$ *a.e. in* $Q_R$,

*(ii)* $E\Phi_R$ *has support within* $Q_R$,

*(iii)* $\|E\Phi_R\|_{V(Q_0)} \leq C\|\Phi_R\|_{V(Q_R)}$, *where the constant $C$ is independent of $R$.*

*Proof.* Fix $k > 0$ be a sufficiently large constant, and suppose that $\Phi_R \in V(Q_R)$. Define a function $\tilde{\Phi}_R$ as

$$\tilde{\Phi}_{R_k} = \begin{cases} \Phi_R(t, w), & \text{if } w \in [-R,\ R]; \\ 2\Phi_R(t, R) - \Phi_R(t, 2R - w), & \text{if } w \in (-R - k, -R) \cup (R, R + k). \end{cases}$$

which extends $\Phi_R$ from $Q_R = [0, T_0) \times [-R, R]$ to $Q_{R_k} = [0, T_0) \times (-R-k, R+k)$ and keeps it continuous and differentiable at two endpoints $R$ and $-R$.

Fix a time variable $t \in [0, T_0)$, and when $w \in (R, R + k)$, we get

$$\lim_{w \to R^+} \tilde{\Phi}_{R_k} = 2\Phi_R(t, R) - \Phi_R(t, 2R - R) = \Phi_R(t, R),$$

then the function $\tilde{\Phi}_{R_k}$ is continuous at $w = R$.

Similarly, check $(\tilde{\Phi}^-_{R_k})_w|_{w\to R^-} = (\tilde{\Phi}^+_{R_k})_w|_{w\to R^+}$, which is

$$(\tilde{\Phi}^-_{R_k})_w|_{w\to R^-} = (\Phi_R)_w(t,R), \qquad (\tilde{\Phi}^+_{R_k})_w|_{w\to R^+} = -(\Phi_R)_w(t,2R-R)\cdot(-1) = (\Phi_R)_w(t,R)$$

When $w = -R$, the same results follow.

Define a smooth cutting function $\chi_k$ on $\mathbb{R}$, where $\chi_k(w) = 0$ if $w > R+k$ or $w < -R-k$, $\chi_k(w) = 1$ if $w \in [-R,R]$, and $\chi_k(w)$ is a smooth function, if $w \in (-R-k,-R) \cup (R,R+k)$. Then, $\|(\chi_k)_w\|_{L^2} \le C$.

Let $\tilde{\tilde{\Phi}}_{R_k} = \chi_k \tilde{\Phi}_{R_k}$ on $Q_0$.

Next, we will show that the norm of $\tilde{\tilde{\Phi}}_{R_k}$ in $V(Q_0))$ is bounded by some constant times the norm of $\Phi_R$ in $V(Q_R)$.

Claim:

$$\|\tilde{\Phi}_{R_k}\|_{V(Q_{R_k})} \le C_1 \|\Phi_R\|_{V(Q_R)}$$

Consider a bounded set $Q_{R_k^+} = [0,T_0) \times (R-k, R+k)$, then

$$\begin{aligned}
\|\tilde{\Phi}_{R_k}\|_{V(Q_{R_k^+})} &\le \|\tilde{\Phi}_R\|_{V([0,T_0)\times(R-k,R))} + \|\tilde{\Phi}_R\|_{V([0,T_0)\times(R,R+k))} \\
&= \|\Phi_R(t,w)\|_{V([0,T_0)\times(R-k,R))} + \|2\Phi_R(t,R) - \Phi_R(t,2R-w)\|_{V([0,T_0)\times(R,R+k))} \\
&\le \|\Phi_R\|_{V([0,T_0)\times(R-k,R))} + \|\Phi_R\|_{V([0,T_0)\times(R-k,R))} \\
&\le 2\|\Phi_R\|_{V([0,T_0)\times(R-k,R))} \\
&\le C_2 \|\Phi_R\|_{V(Q_R)}
\end{aligned}$$

where $C_2$ is independent of $R$.

Similarly, we can show that on the bounded set $Q_{R_k^-} = [0,T_0) \times (-R-k, -R+k)$, the same inequality follows,

$$\|\tilde{\Phi}_{R_k}\|_{V(Q_{R_k^-})} \le C_2 \|\Phi_R\|_{V(Q_R)}$$

Therefore,

$$\|\tilde{\Phi}_{R_k}\|_{V(Q_{R_k})} \le C_1 \|\Phi_R\|_{V(Q_R)}$$

Next, prove that the norm of $\tilde{\tilde{\Phi}}_{R_k}$ in $V(Q_0)$ is bounded by some constant multiplies the norm of $\Phi_R$ in $V(Q_R)$, and the constant only depends on $k$, but is independent of $R$, which is,

$$\|\tilde{\tilde{\Phi}}_{R_k}\|_{V(Q_0)} \leq C\|\Phi_R\|_{V(Q_R)}$$

By the definition of $\tilde{\tilde{\Phi}}$ on $Q_0$, the norm in $V(Q_0)$ is:

$$\begin{aligned}
\|\tilde{\tilde{\Phi}}_{R_k}\|_{V(Q_0)} &= \|\chi_k \tilde{\Phi}_{R_k}\|_{V(Q_{R_k})} \\
&\leq \|\chi_k \tilde{\Phi}_{R_k}\|_{V([0,T_0)\times(-R-k,-R))} + \|\Phi_R\|_{V(Q_R)} + \|\chi_k \tilde{\Phi}_{R_k}\|_{V([0,T_0)\times(R,R+k))} \\
&\leq \|\Phi_R\|_{V(Q_R)} + \iint_{[0,T_0)\times(-R-k,-R)} |(\chi_k \tilde{\Phi}_{R_k})_w|^2 \, dw \, dt + \sup_{t\in[0,T_0)} \int_{\omega(t)} (\chi_k \tilde{\Phi}_R)^2 \, dw \\
&\quad + \iint_{[0,T_0)\times(R,R+k)} |(\chi_k \tilde{\Phi}_{R_k})_w|^2 \, dw \, dt + \sup_{t\in[0,T_0)} \int_{\omega(t)} (\chi_k \tilde{\Phi}_R)^2 \, dw \\
&\leq \|\Phi_R\|_{V(Q_R)} \\
&\quad + \iint_{[0,T_0)\times(-R-k,-R)} |(\chi_k)_w \tilde{\Phi}_{R_k}|^2 \, dw \, dt + \iint_{[0,T_0)\times(-R-k,-R)} |\chi_k (\tilde{\Phi}_{R_k})_w|^2 \, dw \, dt \\
&\quad + \sup_{t\in[0,T_0)} \int_{\omega(t)} (\tilde{\Phi}_{R_k})^2 \, dw \\
&\quad + \iint_{[0,T_0)\times(R,R+k)} |(\chi_k)_w \tilde{\Phi}_{R_k}|^2 \, dw \, dt + \iint_{[0,T_0)\times(R,R+k)} |\chi_k (\tilde{\Phi}_{R_k})_w|^2 \, dw \, dt \\
&\quad + \sup_{t\in[0,T_0)} \int_{\omega(t)} (\tilde{\Phi}_{R_k})^2 \, dw \\
&\leq \|\Phi_R\|_{V(Q_R)} \\
&\quad + C_3 \iint_{[0,T_0)\times(-R-k,-R)} |\tilde{\Phi}_{R_k}|^2 \, dw \, dt + \iint_{[0,T_0)\times(-R-k,-R)} |(\tilde{\Phi}_{R_k})_w|^2 \, dw \, dt \\
&\quad + C_3 \iint_{[0,T_0)\times(R,R+k)} |\tilde{\Phi}_{R_k}|^2 \, dw \, dt + \iint_{[0,T_0)\times(R,R+k)} |(\tilde{\Phi}_{R_k})_w|^2 \, dw \, dt \\
&\quad + 2 \sup_{t\in[0,T_0)} \int_{\omega(t)} (\tilde{\Phi}_{R_k})^2 \, dw
\end{aligned}$$

Since $\Phi_R$ is a weak solution of the HJB Equation 3.12 on a bounded set $Q_R$, then the weak derivatives of $\Phi_R$ exist and it is clear that $(\Phi_R)_w$ belongs to $L^2(\Omega)$. Then, $\Phi_R \in S^{1,2}(\Omega)$.

Apply Poincaré's inequality 3.29 to

$$\iint_{[0,T_0)\times(-R-k,-R)} |\tilde{\Phi}_{R_k}|^2\,dw\,dt \text{ and } \iint_{[0,T_0)\times(R,R+k)} |\tilde{\Phi}_{R_k}|^2\,dw\,dt$$

Since $n = 1$ and $S_0^{1,2}(\Omega) = S^{1,2}(\Omega)$ in our model, and $\Omega \subset \mathbb{R}$. Then, on the set $[0,T_0) \times (-R-k,-R)$, we have

$$\begin{aligned}\iint_{[0,T_0)\times(-R-k,-R)} |\tilde{\Phi}_{R_k}|^2\,dw\,dt &= \int_0^{T_0}\int_{(-R-k,-R)} |\tilde{\Phi}_{R_k}|^2\,dw\,dt\\ &\le C_4\int_0^{T_0}\int_{(-R-k,-R)} |(\tilde{\Phi}_{R_k})_w|^2\,dw\,dt\\ &\le C_4\iint_{[0,T_0)\times(-R-k,-R)} |(\tilde{\Phi}_{R_k})_w|^2\,dw\,dt\end{aligned}$$

Similarly,

$$\iint_{[0,T_0)\times(R,R+k)} |\tilde{\Phi}_{R_k}|^2\,dw\,dt \le C_4\iint_{[0,T_0)\times(R,R+k)} |(\tilde{\Phi}_{R_k})_w|^2\,dw\,dt$$

on the set $[0,T_0) \times (R,R+k)$.

Then,

$$\begin{aligned}\|\tilde{\tilde{\Phi}}_{R_k}\|_{V(Q_0)} \le{}& \|\Phi_R\|_{V(Q_R)}\\ &+ C_3C_4\iint_{[0,T_0)\times(-R-k,-R)} |(\tilde{\Phi}_{R_k})_w|^2\,dw\,dt + \iint_{[0,T_0)\times(-R-k,-R)} |(\tilde{\Phi}_{R_k})_w|^2\,dw\,dt\\ &+ C_3C_4\iint_{[0,T_0)\times(R,R+k)} |(\tilde{\Phi}_{R_k})_w|^2\,dw\,dt + \iint_{[0,T_0)\times(R,R+k)} |(\tilde{\Phi}_{R_k})_w|^2\,dw\,dt\\ &+ 2\sup_{t\in[0,T_0)}\int_{\omega(t)} (\tilde{\Phi}_{R_k})^2\,dw\\ \le{}& C_5\|\Phi_R\|_{V(Q_R)} + C_5\Big(\iint_{[0,T_0)\times(-R-k,-R)} |(\tilde{\Phi}_{R_k})_w|^2\,dw\,dt + \sup_{t\in[0,T_0)}\int_{\omega(t)} (\tilde{\Phi}_{R_k})^2\,dw\Big)\\ &+ C_5\Big(\iint_{[0,T_0)\times(R,R+k)} |(\tilde{\Phi}_{R_k})_w|^2\,dw\,dt + \sup_{t\in[0,T_0)}\int_{\omega(t)} (\tilde{\Phi}_{R_k})^2\,dw\Big)\\ ={}& C_5\|\Phi_R\|_{V(Q_R)} + C_5\|\tilde{\Phi}_{R_k}\|_{V(Q_{R_k^-})} + C_5\|\tilde{\Phi}_{R_k}\|_{V(Q_{R_k^+})}\\ \le{}& C_5\|\Phi_R\|_{V(Q_R)} + C_1\|\Phi_R\|_{V(Q_R)}\\ \le{}& C\|\Phi_R\|_{V(Q_R)}\end{aligned}$$

where $C$ depends on $k$, but is independent of $R$.

Define the linear operator $E := \tilde{\tilde{\Phi}}_{R_k}$, and by the definition of $\tilde{\tilde{\Phi}}_{R_k}$, we get $E\Phi_R = \Phi_R$ a.e. in $Q_R$ and has support within $Q_R$.

■

By this extension theorem, we achieve a similar extension of the weak solution to the HJB Equation 3.12 from $Q_R$ to $Q_0$.

**Theorem 3.15** *Suppose that all the assumptions of Theorem 3.14 are satisfied, and if $\{\Phi_R\}$ are weak solutions of the HJB Equation 3.12 with the boundary conditions $P_R(w)$, then $\{E\Phi_R\}$ is uniformly bounded and there exists a subsequence of $\{E\Phi_R\}$ that converges in $V(Q_0)$.*

*Proof.* Consider the linear operator $E\Phi_R$ on $Q_0$, and from Theorem 3.14, we have

$$\begin{aligned}
\|E\Phi_R\|_{V(Q_0)} &\leq C\|\Phi_R\|_{V(Q_R)} \\
&\leq C(C'e^{C'T_0}(\|\mathcal{L}^{\bar{h}}\Phi_R\|_{L^2} + \|P_R(w)\|_{L^2})) \\
&\leq CC'e^{C'T_0}\|P_R(w)\|_{L^2}
\end{aligned}$$

By the definition of the boundary function $P_R(w)$, it is an $L^2(Q_R)$ function, which means it is finite. Let $\|P_R(w)\|_{L^2} \leq C''$, we have

$$\|E\Phi_R\|_{V(Q_0)} \leq CC'e^{C'T_0}C'' \leq K$$

where $K$ is some constant, a uniform bound.

Since $V$ is a Banach space, it is complete, when $R \to \infty$, there exist a subsequence of $\{E\Phi_R\}$ such that the subsequence converges in $V(Q_0)$.

■

We have proved that the linear operator $E$ on $Q_0$ is bounded by some constant, as $R \to \infty$, then we extend a weak solution $\Phi_R$ of the HJB Equation 3.12 from $Q_R$ to $Q_0$. Consequently, we define a weak solution of the HJB Equation 3.12 on $Q_0$:

**Definition 3.4** *If an optimal Markov control $h^*$ exists and the boundary function at time $T_0$ is given by Equation 3.9 on $Q_0$, we define the limit of the subsequence of $\{E\Phi_R\}$ in $V(Q_0)$ in Theorem 3.15 to be a* weak solution *of the HJB Equation 3.12, and denote it by $\Psi$.*

**Remark 3.5**

1. *If an optimal Markov control $h^*$ exists, there could be several weak solutions of the HJB Equation 3.12.*
2. *Assume that an optimal Markov control $h^*$ exists, then we can show that there exists a weak solution $\Psi$ in Definition 3.4. It is interesting to know if a weak solution in Definition 3.4 can give an optimal hedge control $h^*$.*

# CHAPTER 4

# CONCLUSIONS

Froot, Scharfstein, and Stein (1993), [6] introduced and presented finding the optimal hedge ratio to maximize the expected profit of a corporation for $n = 1$ and $n = 2$ in period zero if hedging strategies are linear. In [6], the authors not only gave an optimal hedge ratio, but also discussed the detailed financial meaning of each term in the result. With the results [6], we develop the optimal hedge ratio problem to a general $n$ variable case when similar linear hedging strategies (forward or futures contract) are considered. In addition, we also find that the optimal hedge ratio can be calculated in period one if all the elements in the model keep the same, and treat the new random variables as functions. We compute the $n = 2$ case in our model for period zero and period one.

In Chapter 2, we assume that all the processes are not stochastic processes, which is not always the real situation in the world. In fact, we find that if we consider stochastic processes, it is more realistic in finance. However, if the control process, the investment process, and the product process involve any stochastic process, we have to show that the corresponding stochastic differential equations have solutions in weak sense. We studied Huang and Liu (2007), [8], and Øksendal, [13] of the HJB Equation Method. The difference in our model is that instead of utility functions, we study profit functions, which are more complicated and have free boundary conditions. Thus, in Chapter 3, we present the following result: If there exists an optimal Markov control $h^*$, the HJB Equation 3.12 has a weak solution $\Psi$ in the domain $Q_0 = [0, T_0) \times \mathbb{R}$ with the boundary condition $P(w)$, which is a free boundary.

We develop the existence of a weak solution to the HJB Equation in three parts. First, if the boundary condition $g(w)$ is smooth and bounded on $Q_0$, the HJB Equation has a solution on $Q_0$. Second, if $g_n(w) \to P(w)$ in $L^2(\Omega)$, then there exists a weak solution $\Phi$ of the HJB Equation in $V(Q)$. Finally, the weak solution $\Phi$ of the HJB Equation 3.12 can be extended from a bounded domain $Q$ to $Q_0$.

There are some interesting questions we would like to study in the future. For example, we now only solve the existence of a weak solution of the HJB Equation on $Q_0$, but we are interested in the uniqueness of the solutions. Moreover, we also wonder if there is a weak solution of the HJB Equation on $Q_0$, the corresponding hedge control $h^0$ is an optimal hedge control. In addition, we have the question that whether we can extend the hedge control to any adapted control. We hope that we can obtain these conjecture proved in the near future.

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Optimal hedge ratio problem with linear hedging strategy in risk management were introduced by Froot, Scharfstein, and Stein (1993). This thesis extends the idea to find the optimal hedge ratio $\mathbf{h}^*$ in $n$-dimension for two-period situation. It also develops a stochastic model of the optimizing hedge ratio problem and shows that when an optimal Markov control $h^*$ exists, there is a weak solution

$$\Psi(y) = \max_{h \in U}\{\mathbf{E}^y[P(W^h_{T_0})]\} = J^{h^*}(y)$$

in the domain $Q_0 = [0, T_0) \times \mathbb{R}$ of the Hamilton-Jacobi-Bellman Equation

$$\sup\{\mathcal{L}^h\Phi(y)\} = \sup\{\Phi_t(y) + a^h\Phi_{ww}(y) + b^h\Phi_w(y)\} = 0$$

subject to a stochastic process

$$dW_t = w_0\big[((1 - h(t))r_W + h(t)\mu_W)\,dt + h(t)\sigma_W\,dZ_{1t}\big]$$

and a free boundary condition.

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