ARITHMETIC PROPERTIES OF L-FUNCTIONS ATTACHED TO HILBERT MODULAR FORMS

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CHAPTER 1

INTRODUCTION

In Number Theory, analyzing the special values of L-functions have been one of the significant targets of research since the eighteenth century when the Riemann-zeta function was introduced. It has an important role particularly on study of the distribution of prime numbers, and also has applications in various fields including physics and statistics. The Riemann-zeta function, which was originally introduced and studied by Leonhard Euler, is a function of a complex variable defined as

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

It can be checked that this infinite series converges only when the real part of s is larger than 1, and has an Euler product:

$$\zeta(s) = \prod_{p} (1 - p^{-s})^{-1},$$

where p runs through all the prime numbers. This observation first made by Euler says that the Riemann-zeta function relates deeply to prime numbers. The series also admits an analytic continuation, which allows us to extend our definition of $\zeta(s)$ as a meromorphic function on the whole complex plane except for a pole at $s = 1$.

In the nineteenth century, Dirichlet constructed the Dirichlet L-functions which are generalizations of the Riemann-zeta function. A Dirichlet L-series is obtained by "twisting" the Riemann-zeta function by a certain function χ called a Dirichlet character that maps integers to complex numbers, i.e., Dirichlet L-series attached to χ is defined to be

$$L(s, \chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

He investigated this series to prove the theorem that an arithmetic progression of the form $\{a + nd\}_{n=0}^{\infty}$ contains infinitely many primes if a and d are relatively prime. Similar to the Riemann-zeta function, the Dirichlet L-series have Euler products, and can be analytically continued to the complex plane. The continued meromorphic functions are called Dirichlet L-functions. Such analytic properties of the Riemann-zeta function and Dirichlet L-functions are found in many books in Number Theory. See, for example, [34, Section 7.1–7.2].

One of the interesting topics in the area is to analyze the “special values” of L-functions. Some expamples of special values are:

$$\zeta(2) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6},$$

which is due to Euler, or if χ is a Dirichlet character modulo 4 such that $\chi(x) = 1$ for $x \equiv 1 \pmod 4$ and $\chi(x) = -1$ for $x \equiv 3 \pmod 4$, Leibniz observed that

$$L(1, \chi) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}.$$

One can aim to generalize some arithmetic properties of such values. This is a very deep and still highly conjectural topic.

In early twentieth century, Erich Hecke generalized Dirichlet characters to obtain characters of the idèle group of a number field F, and associated them with L-functions. His work was reformulated by John Tate in his Ph.D. thesis in which he showed meromorphic continuation and the functional equations of L-functions attached to a Hecke character by using Fourier analysis on adèle group over number field F and on its local fields. Moving to the adèlic setting removed many technical issues faced in the classical setting. His method is now called the modern GL_1-theory of automorphic forms, and contributed to the development of GL_n-theory. In particular, we deal with the GL_2-theory in the thesis.

While the Riemann-zeta function and Dirichlet L-functions are classical examples of L-functions, there are much broader generalizations in the area nowadays. For example, Hecke contributed deeply to the theory of L-functions attached to modular forms, which leads us to the classical GL_2-theory. A modular form is a complex-valued analytic function defined on the upper-half plane $\mathfrak{h}$ that satisfies a certain functional equation and a growth condition. We say a modular form f is of weight k and level N if, for any element $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $\mathrm{SL}_2(\mathbb{Z})$ with $c \equiv 0 \pmod{N}$, it satisfies

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z).$$

In particular, such a function satisfies $f(z+1) = f(z)$ and therefore has a Fourier expansion;

$$f(z) = \sum_{n=-m}^{\infty} a_n(f) e^{2\pi i n z}.$$

A modular form is called a cusp form if it vanishes at all cusps, or equivalently, the Fourier coefficients $a_n(f)$ are 0 for all $n \leq 0$. The L-function attached a cusp form f is defined as the infinite series given by

$$L(s,f) = \sum_{n=1}^{\infty} \frac{a_n(f)}{n^s}.$$

It has analytic properties of the expected kind, which are thoroughly studied in [33]. Furthermore, the concepts of modular forms were extended to obtain modular forms in several variables. These forms are called Hilbert modular forms. Despite the fact that Hilbert modular forms were introduced in the early stage, the development of their theory was not seen until much later. This thesis focuses on such functions and some of the arithmetic properties of their L-functions.

We now direct our attention to the contents of this thesis. We begin Chapter 2 by introducing some notations and basic facts that we will be using throughout the thesis. Chapter 3 is to give precise definitions regarding Hilbert modular forms. A Hilbert modular form can be lifted to the adèlic setting. The adèlized form is denoted as $\mathbf{f}$ which is an h-tuple of Hilbert modular forms defined on $\mathfrak{h}^n$. See (3.1.2). Here, h represents the narrow class number (described in Section 2.3). It should be noted that we do not have any restriction on the narrow class number anywhere in the thesis, which is often assumed to be one in literature. Section 3.3 focuses on primitive forms, which are the forms with "nice" properties and the objects of our interests.

We would like to pass to the setting of the modern GL_2-theory which is to use a representation theoretical point of view. As mentioned earlier, it simplifies some technical difficulties that arise in the classical setting. In particular, a large narrow class number does not cause any trouble in a study of the representations. A bridge that takes us from Hilbert modular forms to certain automorphic representations of GL_2 is described in detail in Chapter 5. Keeping the correspondence between these objects in our mind, our interest is to compare the (finite) L-functions of Hilbert modular form and of automorphic representation of GL_2. An L-function of Hilbert modular form $\mathbf{f}$ is defined as an infinite sum

$$L_f(s, \mathbf{f}) = \sum_{\mathfrak{m}} \frac{\mathrm{C}(\mathfrak{m}, \mathbf{f})}{\mathrm{N}(\mathfrak{m})^s},$$

where $\mathfrak{m}$ runs through all the integral ideals of the base field F, and $\mathrm{C}(\mathfrak{m}, \mathbf{f})$ is a suitably normalized Fourier coefficient at $\mathfrak{m}$ given in (3.1.3). On the other hand, an L-function of a representation Π is defined to be the product of the local L-factors

$$L_f(s, \Pi) = \prod_{\mathfrak{p}} L_{\mathfrak{p}}(s, \Pi_{\mathfrak{p}}).$$

These two L-functions constructed in completely different manners give the following relation:

$$L(s, \Pi(\mathbf{f})) = L\left(s + \frac{k_0 - 1}{2}, \mathbf{f}\right),$$

where k_0 is determined by the weight of $\mathbf{f}$. Note that, by denoting $\Pi(\mathbf{f})$, we mean the representation corresponding to $\mathbf{f}$. Section 5.3 should be referred to for the construction of the L-functions and their relations.

What is also worth mentioning is the $\mathrm{Aut}(\mathbb{C})$-equivariance of the correspondence. The $\mathrm{Aut}(\mathbb{C})$-actions and the equivariance are described in Section 5.4, and arise as one of the key facts in the proof of algebraicity theorem. See Chapter 6.

We study representations of GL_2 in Chapter 4. We start Section 4.1 with the study of the local representations Π_∞ at infinity with a special focus on the *Langlands correspondence* for $\mathrm{GL}_2(\mathbb{R})$ in Section 4.1.2. This study has an important role to determine whether a global representation Π is *regular and algebraic*, which is the type of representations appearing in the correspondence described in Chapter 5. The regular algebraic cuspidal automorphic representations are discussed in Section 4.2. The rest of the chapter is to study *cohomological* representations, which contains several technical details which are needed in Chapter 6.

Finally, we discuss our main theorems in Chapters 6, 7, and 8. These main theorems are to concern the following:

- Algebraicity theorem discussed in Chapter 6;
- Congruence property in Chapter 7; and
- Non-vanishing property in Chapter 8.

Note that Chapter 6, Chapter 7, as well as their supporting materials are joint work with the author's thesis advisor, A. Raghuram.

Chapter 6 deals with an algebraicity theorem. There is a fundamental result of Shimura in [41] about critical L-values of Hilbert modular form. The result is stated as Theorem 6.0.1, which roughly says that, if $\mathbf{f}$ is primitive, then there are nonzero complex numbers $u(r, \mathbf{f})$ called *periods* so that the critical L-values of $\mathbf{f}$ equal $(2\pi i)^{**}u(r, \mathbf{f})$ (with some integer $**$), up to an element of the rationality field $\mathbb{Q}(\mathbf{f})$. In other words, the L-values divided by $(2\pi i)^{**}u(r, \mathbf{f})$ are algebraic (and belong to $\mathbb{Q}(\mathbf{f})$). More generally, similar statements hold when the L-function is twisted by a finite order Hecke character χ. The aim of this chapter is to give another proof to his theorem. As mentioned earlier, our approach is to study the L-functions attached to the representation $\Pi(\mathbf{f})$. One of the merits of working on the L-function attached to the representation $\Pi(\mathbf{f})$ instead of the one attached to $\mathbf{f}$ is that it is sufficient to prove an algebraicity theorem for only one critical value, namely the central critical value. Algebraicity results for the other critical values follow from the *period relations* proved by Raghuram and Shahidi in [38]. Furthermore, the theorem can be applied to any representation $\Pi(\mathbf{f}) \otimes \chi$ twisted by a finite order Hecke character χ, which gives Shimura's results for $L(s, \mathbf{f}, \chi)$. See Theorem 6.0.2 and Corollary 6.0.3. In Section 6.1, we define the periods and give a brief summary of the period relations for our context of GL_2. Note that our period is not the same as what Shimura used, and it arises out of a comparison of rational structures on two different realizations of Π. The relation between these periods is observed in (6.2.18). The proof for Theorem 6.0.2 is completed in Section 6.2. The idea of the proof is to interpret the Mellin transform as a cohomological map. The summary of the proof is compressed into the diagram (6.2.1).

Chapter 7 is to discuss a congruence property of the central critical L-values. Vatsal proved in [44] that if two elliptic cusp forms are congruent modulo ℓ, for a prime ℓ, then the congruence also holds for their critical L-values. Our goal is to generalize this result to Hilbert modular forms by using the approach described in Chapter 6, and analyzing it integrally. The key ingredient for solving the problem is a refinement of the definition of periods, which follows from studying integral structures on inner cohomology. Definitions of integral structures are provided in Section 7.2. The periods defined in Chapter 6 are canonical up to an element of the rationality field $\mathbb{Q}(\Pi)$ of the representation. However, for a congruence property, it needs to be treated more sensitively. More precisely, it needs to be defined so that it is canonical up to a unit in an ℓ-adic completion of a field containing all the necessary number fields. The proof is completed in Section 7.3 with a special focus on the refinement of the periods in Section 7.3.3.

In Chapter 8, we study a nonvanishing property of the derivatives of L-functions. Theorem 8.0.1 says that, for a primitive holomorphic Hilbert cusp form $\mathbf{f}$ of even weight, if the central critical L-value does not vanish, then neither does the derivative at the center of symmetry. This is a generalization of the result by Gun, Murty, and Rath in [18]. The proof is completed by studying the derivative of the functional equation for a completed L-function. Some properties of Hilbert modular forms stated in Section 3.4 will be useful to know.

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CHAPTER 2

Preliminaries

In this chapter, we will introduce some notations and basic definitions that will be needed throughout the thesis.

2.1 The base field

Let F denote a totally real number field of degree n, $\mathcal{O}_F$ the ring of integers in F, and $\mathfrak{n}$ a fixed integral ideal in F. The real embeddings of F are denoted η_j with $j = 1, \cdots, n$, and we put $\eta = (\eta_1, \cdots, \eta_n)$ with a fixed order of $\{\eta_j\}$. With respect to this η, F naturally sits inside $\mathbb{R}^n$, and an element α in F will be expressed as $(\alpha_1, \cdots, \alpha_n)$ for $(\eta_1(\alpha), \cdots, \eta_n(\alpha))$ to be considered as an element of $\mathbb{R}^n$. We write F_+ for the set of all the totally positive elements in F. (A totally positive element means an element α in F such that $\eta_j(\alpha) > 0$ for all $j = 1, \ldots, n$.)

Let $\mathbb{A}_F$ denote the adèle ring of F, and $\mathbb{A}_{F,f}$ the finite adèles; we will drop the subscript F for the field $\mathbb{Q}$. Hence $\mathbb{A}_F = \mathbb{A} \otimes_{\mathbb{Q}} F$, etc. The infinite part of the adèle F_∞ can be also denoted as $\prod_{v \in S_\infty} F_v = \prod_{j=1}^{n} F_{\eta_j} \simeq \prod_{j=1}^{n} \mathbb{R}$ where S_∞ is the set of all real places $\{\eta_j\}$. We write F_{∞^+} for the subset of all $(x_1, \ldots, x_n)$ in F_∞ such that $x_j > 0$ for all j.

Let $\mathfrak{p}$ denote a prime ideal of $\mathcal{O}_F$, $F_\mathfrak{p}$ the completion of F at $\mathfrak{p}$, and $\mathcal{O}_\mathfrak{p}$ the ring of integers of $F_\mathfrak{p}$. The unique maximal ideal of $\mathcal{O}_\mathfrak{p}$ is $\mathfrak{p}\mathcal{O}_\mathfrak{p}$ and is generated by a uniformizer $\varpi_\mathfrak{p}$. We let $\mathfrak{D}_F$ denote the absolute different of F, i.e., $\mathfrak{D}_F^{-1} = \{x \in F : T_{F/\mathbb{Q}}(x\mathcal{O}_F) \subset \mathbb{Z}\}$. It can be also described as $\mathfrak{D}_F = \prod_\mathfrak{p} \mathfrak{p}^{r_\mathfrak{p}}$ where $r_\mathfrak{p} = \mathrm{ord}_\mathfrak{p}(\mathfrak{D}_F)$.

2.2 The groups $G \supset B \supset T \supset Z$

Let $G = \mathrm{Res}_{F/\mathbb{Q}}(\mathrm{GL}_2)$ which is the Weil restriction of scalars from F to $\mathbb{Q}$ of the algebraic group GL_2 over F. Hence $G(\mathbb{Q}) = \mathrm{GL}_2(F)$, and more generally, for any $\mathbb{Q}$-algebra A, we have $G(A) = \mathrm{GL}_2(A \otimes_{\mathbb{Q}} F)$. For any finite prime p, $G(\mathbb{Q}_p) = \prod_{\mathfrak{p}|p} \mathrm{GL}_2(F_{\mathfrak{p}})$; similarly, $G_\infty := G(\mathbb{R}) = \prod_{j=1}^n \mathrm{GL}_2(F_{\eta_j}) = \prod_{j=1}^n \mathrm{GL}_2(\mathbb{R})$. We write G_∞^+ to mean the set of elements $(g_1, \ldots, g_n)$ in G_∞ such that $\det(g_j) > 0$ for all j. Let $\mathfrak{g}_\infty$ be the complexified Lie algebra of G_∞.

Fix the standard Borel subgroup $B = \mathrm{Res}_{F/\mathbb{Q}}(B_2)$, with B_2 being the standard Borel subgroup of GL_2 of all upper triangular matrices. Let $T = \mathrm{Res}_{F/\mathbb{Q}}(T_2)$, where T_2 stands for the diagonal torus in GL_2. Let $Z = \mathrm{Res}_{F/\mathbb{Q}}(Z_2)$, where Z_2 is the center of GL_2 consisting of scalar matrices. For any $\mathbb{Q}$-algebra A, we can talk of $B(A)$, $T(A)$, and $Z(A)$ as we did for G.

2.3 The narrow class group of F

By the narrow class group, we mean the group $F^\times \backslash \mathbb{A}_F^\times / F_{\infty+}^\times \prod \mathcal{O}_{\mathfrak{p}}^\times$, and the cardinality of this group, which is denoted as $h = h_F$, is called the narrow class number. The narrow class group can be also viewed as the group J_F/P_{F_+} of all fractional ideals of F modulo principal ideals generated by the elements in F_+. The narrow class group is, in general, bigger than the class group J_F/P_F, and one has the following exact sequence. (See, for instance, [34, Section VI. 1].)

$$1 \longrightarrow \mathcal{O}_F^\times/\mathcal{O}_{F,+}^\times \longrightarrow F_\infty^\times/F_{\infty+}^\times \longrightarrow J_F/P_{F_+} \longrightarrow J_F/P_F \longrightarrow 1.$$

We write $\{t_\nu\}_{\nu=1}^h$ for the elements of $\mathbb{A}_F$ whose archimedean part $t_{\nu,\infty}$ is 1 and that form a complete set of representatives of the narrow class group. Put $x_\nu = \left(\begin{smallmatrix} 1 & \\ & t_\nu \end{smallmatrix}\right)$ and $x_\nu^\iota = \left(\begin{smallmatrix} t_\nu & \\ & 1 \end{smallmatrix}\right)$. Here, ι denotes the involution defined as ${}^\iota\mathrm{A} = w_0\, {}^t\mathrm{A} w_0^{-1}$, where t is transpose and $w_0 = \left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right)$.

2.4 Various Subgroups

Let K_∞ stand for the maximal compact subgroup of G_∞ thickened by its center. Hence

$$\mathrm{K}_\infty = \prod_{j=1}^{n} (\mathrm{O}_2(\mathbb{R}) Z_2(\mathbb{R})),$$

where $\mathrm{O}_2(\mathbb{R})$ is the usual maximal compact subgroup of $\mathrm{GL}_2(\mathbb{R})$. For any Lie group $\mathcal{G}$ we will denote $\mathcal{G}^\circ$ the connected component of the identity, and $\pi_0(\mathcal{G}) := \mathcal{G}/\mathcal{G}^\circ$ denotes the group of connected components. Observe then that

$$\mathrm{K}_\infty^\circ = \prod_{j=1}^{n} (\mathrm{SO}_2(\mathbb{R}) Z_2(\mathbb{R})^\circ)$$

and that $\pi_0(G_\infty) = \pi_0(\mathrm{K}_\infty) = \mathrm{K}_\infty/\mathrm{K}_\infty^\circ \simeq (\mathbb{Z}/2\mathbb{Z})^n$. We will identify the dual group $(\mathrm{K}_\infty/\mathrm{K}_\infty^\circ)\widehat{\ }$ with $(\mathbb{Z}/2\mathbb{Z})^n = \{\pm\}^n$, with the $+$ (resp., $-$) denoting the trivial (resp., nontrivial) character of $\mathrm{O}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$.

Let $\mathfrak{k}_\infty$ be the complexified Lie algebra of K_∞ or K_∞°; we will use similar 'standard' notation for the complexified Lie algebras of other Lie groups.

For a non-archimedean place $\mathfrak{p}$, define a subgroup $\mathrm{K}_\mathfrak{p}(\mathfrak{n})$ of $\mathrm{GL}_2(F_\mathfrak{p})$ as

$$\mathrm{K}_\mathfrak{p}(\mathfrak{n}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(F_\mathfrak{p}) \ : \ \begin{array}{c} a\mathcal{O}_\mathfrak{p} + \mathfrak{n}_\mathfrak{p} = \mathcal{O}_\mathfrak{p}, \quad b \in \mathfrak{D}_\mathfrak{p}^{-1}, \\ c \in \mathfrak{n}_\mathfrak{p}\mathfrak{D}_\mathfrak{p}, \qquad d \in \mathcal{O}_\mathfrak{p}, \quad ad - bc \in \mathcal{O}_\mathfrak{p}^\times \end{array} \right\}, \tag{2.4.1}$$

where $\mathfrak{n}_\mathfrak{p}$ and $\mathfrak{D}_\mathfrak{p}$ are $\mathfrak{p}$-parts of $\mathfrak{n}$ and the different $\mathfrak{D}_F$ of $\mathcal{O}_F$, respectively, and put

$$\mathrm{K}_0(\mathfrak{n}) := \prod_{\mathfrak{p}<\infty} \mathrm{K}_\mathfrak{p}(\mathfrak{n}). \tag{2.4.2}$$

We note that $G(\mathbb{A})$ affords a decomposition given as a disjoint union,

$$G(\mathbb{A}) = \cup_{\nu=1}^{h} G(\mathbb{Q}) x_\nu^{-\iota} \left(G_\infty^+ \mathrm{K}_0(\mathfrak{n}) \right). \tag{2.4.3}$$

Let us also define a congruence subgroup $\Gamma_\nu(\mathfrak{n})$ of $G(\mathbb{Q})$ for each ν as

$$\Gamma_\nu(\mathfrak{n}) = \left\{ \begin{pmatrix} a & t_\nu^{-1} b \\ t_\nu c d & \end{pmatrix} \ : \ \begin{array}{c} a \in \mathcal{O}_F, \quad b \in \mathfrak{D}^{-1}, \\ c \in \mathfrak{n}\mathfrak{D}, \quad d \in \mathcal{O}_F, \quad ad - bc \in \mathcal{O}_F^\times \end{array} \right\}.$$

We note that $\Gamma_\nu(\mathfrak{n}) = G_\infty^+ (x_\nu \mathrm{K}_0(\mathfrak{n}) x_\nu^{-1}) \cap G(\mathbb{Q})$.

2.5 Measures and absolute values

The normalized absolute value for any local field L is denoted $|\ |_L$, or simply $|\ |$ when no confusion arises. The product of all the local absolute values gives the adèlic norm $|\ |$ on $\mathbb{A}_F^\times$. All the measures used will be Haar measures, or measures on quotient spaces derived from Haar measures. We will simply denote the underlying measure by dx or dg; the measures are normalized in the usual or 'obvious' way. For example, locally $\mathcal{O}_\mathfrak{p}^\times$ has volume 1, and similarly, so does $\mathrm{GL}_2(\mathcal{O}_\mathfrak{p})$. The global measures on $\mathbb{A}_F^\times$ and $\mathrm{GL}_2(\mathbb{A}_F)$ are the product measures of local measures, etc.

2.6 Hecke algebra

For each place $\mathfrak{p}$ of F, let $\mathcal{H}_\mathfrak{p}(\mathrm{K}_\mathfrak{p})$ be the space of $\mathbb{C}$-valued functions on $\mathrm{GL}_2(F_\mathfrak{p})$ that are smooth, compactly supported, and $\mathrm{K}_\mathfrak{p}$-biinvariant where $\mathrm{K}_\mathfrak{p}$ is an open compact subgroup of $\mathrm{GL}_2(F_\mathfrak{p})$. This space is an algebra under convolution:

$$(\mathbf{f} * \mathbf{g})(x) := \int_{\mathrm{GL}_2(\mathbb{A}_F)} \mathbf{f}(xy^{-1})\mathbf{g}(y)\,dy,$$

where dy is the normalized Haar measure so that the volume of $\mathrm{K}_\mathfrak{p}$ is one. This is the local Hecke algebra at $\mathfrak{p}$, and we denote $\mathcal{H}(\mathrm{K})$ for the global Hecke algebra, which is the restricted tensor product of $\mathcal{H}_\mathfrak{p}(\mathrm{K}_\mathfrak{p})$ with respect to the identity element, namely the characteristic function on $\mathrm{K}_\mathfrak{p}$. Recall that the local Hecke algebra $\mathcal{H}_\mathfrak{p}(\mathrm{K}_\mathfrak{p})$ can be viewed as the space of Hecke operators which is generated by the following two elements:

$$\mathbb{T}_\mathfrak{p} = \mathrm{K}_\mathfrak{p} \begin{pmatrix} \varpi_\mathfrak{p} & \\ & 1 \end{pmatrix} \mathrm{K}_\mathfrak{p}, \text{ and } \mathbb{S}_\mathfrak{p} = \mathrm{K}_\mathfrak{p} \begin{pmatrix} \varpi_\mathfrak{p} & \\ & \varpi_\mathfrak{p} \end{pmatrix} \mathrm{K}_\mathfrak{p}.$$

See Section 3.2 for the Hecke operators.

2.7 Some notes on the various characters

Fix a character ω of $(\mathcal{O}_F/\mathfrak{n})^\times$. We "lift" it to a character $\tilde{\omega}$ of $\mathbb{A}_F^\times/F^\times$ as follows. Write $\mathbb{A}_F^\times/F^\times$ as a disjoint union

$$\mathbb{A}_F^\times/F^\times = \bigcup_{\nu=1}^{h} t_\nu F_{\infty+}^\times \prod_{\mathfrak{p}<\infty} \mathcal{O}_\mathfrak{p}^\times,$$

where $\{t_\nu\}$ are taken to be a set of representatives of the narrow class group, and consider the following diagram where the row is exact:

$$\begin{tikzcd}
1 \arrow[r] & \frac{F^\times(F_{\infty+}^\times \prod \mathcal{O}_\mathfrak{p}^\times)}{F^\times} \arrow[r, hook] \arrow[d] & \frac{\mathbb{A}_F^\times}{F^\times} \arrow[r] & \frac{\mathbb{A}_F^\times}{F^\times(F_{\infty+}^\times \prod \mathcal{O}_\mathfrak{p}^\times)} \arrow[r] & 1 \\
& \frac{F^\times(F_{\infty+}^\times \prod \mathcal{O}_\mathfrak{p}^\times)}{F^\times\left(F_{\infty+}^\times \prod (1+\mathfrak{p}^{f_\mathfrak{p}}\mathcal{O}_\mathfrak{p})\right)} \arrow[d] & & & \\
& (\mathcal{O}_F/\mathfrak{n})^\times \arrow[d, "\omega"] & & & \\
& \mathbb{C}^* & & &
\end{tikzcd}$$

Here $f_\mathfrak{p}$ is the highest power of $\mathfrak{p}$ dividing $\mathfrak{n}$. Using the column, a character ω of $(\mathcal{O}_F/\mathfrak{n})^\times$ can be inflated up to a character, also denoted ω, of $F^\times(F_{\infty+}^\times \prod \mathcal{O}_\mathfrak{p}^\times)/F^\times$. Denote this latter group tentatively by H, and observe that it is a subgroup of finite index inside the abelian group $G := \mathbb{A}_F^\times/F^\times$; the index is the narrow class number h. The representation $\mathrm{Ind}_H^G(\omega)$ is a direct sum of h characters, and we can take $\tilde{\omega}$ to be any such character. We will say that $\tilde{\omega}$ is a character of $\mathbb{A}_F^\times/F^\times$ which restricts to the character ω of $(\mathcal{O}_F/\mathfrak{n})^\times$.

2.8 Whittaker models

We will often be working with Whittaker models, and without any ado we will freely use these standard results. (See, for example, Bump [6, Chapters 3,4].)

We fix, once and for all, an additive character $\psi_{\mathbb{Q}}$ of $\mathbb{Q}\backslash\mathbb{A}$, as in Tate's thesis, namely, $\psi_{\mathbb{Q}}(x) = e^{2\pi i\lambda(x)}$ with the λ as defined in [42, Section 2.2]. In particular, $\lambda = \sum_{p\leq\infty}\lambda_p$; $\lambda_\infty(t) = -t$ for any $t \in \mathbb{R}$; $\lambda_p(x)$ for any $x \in \mathbb{Q}_p$ is that rational number with only p-power denominator such that $x - \lambda_p(x) \in \mathbb{Z}_p$. If we write $\psi_{\mathbb{Q}} = \psi_{\mathbb{R}} \otimes \otimes_p \psi_{\mathbb{Q}_p}$, then $\psi_{\mathbb{R}}(t) = e^{-2\pi i t}$ and $\psi_{\mathbb{Q}_p}$ is trivial on $\mathbb{Z}_p$ and nontrivial on $p^{-1}\mathbb{Z}_p$.

Next, we define a character ψ of $F\backslash\mathbb{A}_F$ by composing $\psi_{\mathbb{Q}}$ with the trace map from F to $\mathbb{Q}$: $\psi = \psi_{\mathbb{Q}} \circ T_{F/\mathbb{Q}}$. If $\psi = \otimes_v \psi_v$, then the local characters are determined analogously. In particular, for all prime ideals $\mathfrak{p}$, suppose $r_{\mathfrak{p}}$ is the highest power of $\mathfrak{p}$ dividing the different $\mathfrak{D}_F$, then the conductor of the local character $\psi_{\mathfrak{p}}$ is $\mathfrak{p}^{-r_{\mathfrak{p}}}$, i.e., $\psi_{\mathfrak{p}}$ is trivial on $\mathfrak{p}^{-r_{\mathfrak{p}}}$ and nontrivial on $\mathfrak{p}^{-r_{\mathfrak{p}}-1}$.

Theorem 2.8.1 (Local Whittaker Models) *For any place v of F, let Π_v be an irreducible admissible infinite-dimensional representation of $\mathrm{GL}_2(F_v)$. Then there exists a unique space $\mathcal{W}(\Pi_v, \psi_v)$, with respect to a fixed additive charater ψ_v on F_v, of smooth functions that satisfy the following conditions. For any function $W \in \mathcal{W}(\Pi_v, \psi_v)$,*

$$W\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) = \psi_v(x)W(g) \qquad \text{for } x \in F_v \text{ and } g \in \mathrm{GL}_2(F_v),$$

and W has the growth condition. The space is invariant under the right translation of $\mathrm{GL}_2(F_v)$, and equivalent to the representation Π_v. This space $\mathcal{W}(\Pi_v, \psi_v)$ is called a (local) Whittaker model for Π_v.

Theorem 2.8.2 (Global Whittaker Models) *Let $\mathbb{A} := \mathbb{A}_F$ be the adèle ring of a number field F, and (Π, V_Π) a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$. Then there exists a unique Whittaker model $\mathcal{W}(\Pi, \psi)$ for Π with respect to a non-trivial* *additive character ψ, that consists of finite linear combinations of functions given by*

$$W_\phi(g) := \int_{\mathbb{A}/F} \phi\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) \overline{\psi(x)}\, dx,$$

where $\phi \in V_\Pi$ and $g \in \mathrm{GL}_2(\mathbb{A})$. This space decomposes as a restricted tensor product of local Whittaker models.

2.9 Gauss sums

For a Hecke character ξ of F, by which we mean a continuous homomorphism $\xi : F^\times\backslash\mathbb{A}_F^\times \to \mathbb{C}^\times$, following Weil [47, Chapter VII, Section 7], we define the Gauss sum of ξ as follows: We let $\mathfrak{c}$ stand for the conductor ideal of ξ_f. Let $y = (y_\mathfrak{p})_\mathfrak{p} \in \mathbb{A}_{F,f}^\times$ be such that $\mathrm{ord}_\mathfrak{p}(y_\mathfrak{p}) = -\mathrm{ord}_\mathfrak{p}(\mathfrak{c}) - r_\mathfrak{p}$. The Gauss sum of ξ is defined as $\mathcal{G}(\xi_f, \psi_f, y) = \prod_\mathfrak{p} \mathcal{G}(\xi_\mathfrak{p}, \psi_\mathfrak{p}, y_\mathfrak{p})$ where the local Gauss sum $\mathcal{G}(\xi_\mathfrak{p}, \psi_\mathfrak{p}, y_\mathfrak{p})$ is defined as

$$\mathcal{G}(\xi_\mathfrak{p}, \psi_\mathfrak{p}, y_\mathfrak{p}) = \int_{\mathcal{O}_\mathfrak{p}^\times} \xi_\mathfrak{p}(u_\mathfrak{p})^{-1} \psi_\mathfrak{p}(y_\mathfrak{p} u_\mathfrak{p})\, du_\mathfrak{p}.$$

For almost all $\mathfrak{p}$, where everything in sight is unramified, we have $\mathcal{G}(\xi_\mathfrak{p}, \psi_\mathfrak{p}, y_\mathfrak{p}) = 1$, and for all $\mathfrak{p}$ we have $\mathcal{G}(\xi_\mathfrak{p}, \psi_\mathfrak{p}, y_\mathfrak{p}) \neq 0$. Note that, unlike Weil, we do not normalize the Gauss sum to make it have absolute value one and we do not have any factor at infinity. Suppressing the dependence on ψ and y, we denote $\mathcal{G}(\xi_f, \psi_f, y)$ simply by $\mathcal{G}(\xi_f)$ or even $\mathcal{G}(\xi)$.

CHAPTER 3

Hilbert modular forms

3.1 Hilbert automorphic forms of holomorphic type

Let $k = (k_1, \cdots, k_n) \in \mathbb{Z}^n$, and $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{R}^n$. We write α^k to mean $\prod_{k=1}^n \alpha_j^{k_j}$.

Let $\gamma = (\gamma_1, \cdots, \gamma_n)$ be an element of G_∞^+, and write $\gamma_j = \left(\begin{smallmatrix} a_j & b_j \\ c_j & d_j \end{smallmatrix}\right)$ for each $j = 1, \cdots, n$. Then γ acts on $\mathfrak{h}^n$ by

$$\gamma.z = \left(\frac{a_1 z_1 + b_1}{c_1 z_1 + d_1}, \cdots, \frac{a_n z_n + b_n}{c_n z_n + d_n}\right),$$

with $z = (z_1, \cdots, z_n) \in \mathfrak{h}^n$. For a holomorphic function f on $\mathfrak{h}^n$, an element $\gamma \in G_\infty^+$, and $k = (k_1, \cdots, k_n) \in \mathbb{Z}^n$, define

$$f||_k\gamma(z) \quad = \quad \det \gamma^{k/2} \mathrm{j}(\gamma, z)^{-k} f(\gamma z)$$

where $\mathrm{j}(\gamma, z) = cz + d = (c_1 z_1 + d_1, \ldots, c_n z_n + d_n)$.

Fix a character ω of $(\mathcal{O}_F/\mathfrak{n})^\times$, and let $\tilde{\omega}$ be a character of $\mathbb{A}_F^\times/F^\times$ induced from ω. (See Section 2.7 for induced characters.) Then, we define a character of $\mathrm{K}_0(\mathfrak{n})$ by $\tilde{\omega}\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right) = \tilde{\omega}(a)$. We put $\mathcal{M}_k(\Gamma_\nu(\mathfrak{n}), \tilde{\omega})$ to be the space of Hilbert modular forms of weight $k = (k_1, \cdots, k_n)$ with respect to $\Gamma_\nu(\mathfrak{n})$, with a character $\tilde{\omega}$, by which we mean a space of $\mathbb{C}$-valued functions f_ν that are holomorphic on $\mathfrak{h}^n$ and at all cusps, and that satisfy $f||_k\gamma = \tilde{\omega}(\gamma) f$ for all $\gamma \in \Gamma_\nu(\mathfrak{n})$ considered as elements of G_∞ on the left hand side. Let us note that it makes sense to apply $\tilde{\omega}$ to $\Gamma_\nu(\mathfrak{n})$. A function f_ν in $\mathcal{M}_k(\Gamma_\nu(\mathfrak{n}), \tilde{\omega})$ has a Fourier expansion of the form:

$$f_\nu(z) = \sum_\xi a_\nu(\xi) e^{2\pi i \xi z}, \tag{3.1.1}$$

where $e^{2\pi i \xi z} = \exp\left(2\pi i \sum_{j=1}^{n} \xi_j z_j\right)$, and ξ runs through all the totally positive elements in $t_\nu \mathcal{O}_F$ and $\xi = 0$. A Hilbert modular form is called a cusp form if, for all $\gamma \in \mathrm{GL}_2^+(F)$, the constant term of $f_\nu||_k\gamma$ in its Fourier expansion is 0, and the space of cusp forms with respect to $\Gamma_\nu(\mathfrak{n})$ is denoted as $\mathcal{S}_k(\Gamma_\nu(\mathfrak{n}), \tilde{\omega})$. To have nonempty spaces of cusp forms $\mathcal{S}_k(\Gamma_\nu(\mathfrak{n}), \tilde{\omega})$, assume henceforth that $k_j \geq 1$. (See, for example, Garrett [12, Theorem 1.7].)

Choose a function $f_\nu \in \mathcal{M}_k(\Gamma_\nu(\mathfrak{n}), \tilde{\omega})$ for each ν, and put $\mathbf{f} = (f_1, \cdots, f_h)$ to be a function of $G(\mathbb{A})$ defined as follows: Using the decomposition given in (2.4.3), any element g in $G(\mathbb{A})$ can be written as $g = \gamma x_\nu g_\infty k_0$ where $\gamma \in G(\mathbb{Q})$, $g_\infty \in G_\infty^+$, and $k_f \in \mathrm{K}_0(\mathfrak{n})$. Then the values of $\mathbf{f}(g)$ are taken as

$$\mathbf{f}(\gamma x_\nu g_\infty k_f) = (f_\nu||_k g_\infty)(\mathbf{i})\tilde{\omega}_f(k_f^\iota), \tag{3.1.2}$$

where $\mathbf{i} = (i, \cdots, i)$ and $\tilde{\omega}_f$ is the finite part of $\tilde{\omega}$. The space of such functions $\mathbf{f}$ will be denoted as $\mathcal{M}_k(\mathfrak{n}, \tilde{\omega})$. In particular, if $f_\nu \in \mathcal{S}_k(\Gamma_\nu(\mathfrak{n}), \tilde{\omega})$ for all ν, then $\mathbf{f}$ is called a cusp form as well, and we write as $\mathcal{S}_k(\mathfrak{n}, \tilde{\omega})$ for the space of cusp forms in the adèlic setting, i.e.,

$$\mathcal{S}_k(\mathfrak{n}, \tilde{\omega}) \;=\; \bigoplus_{\nu=1}^{h} \mathcal{S}_k(\Gamma_\nu(\mathfrak{n}), \tilde{\omega}).$$

For any integral ideal $\mathfrak{m}$ in F, there exist a unique $\nu \in \{1, \cdots, h\}$ and a totally positive element ξ in F so that $\mathfrak{m} = \xi t_\nu^{-1} \mathcal{O}_F$. Put

$$\mathrm{C}(\mathfrak{m}, \mathbf{f}) = a_\nu(\xi)\xi^{-k/2}\mathrm{N}(\mathfrak{m})^{k_0/2} \tag{3.1.3}$$

with $a_\nu(\xi)$ being a Fourier coefficient of f_ν given in (3.1.1) and $k_0 = \max_j\{k_j\}$. This is well-defined because the right hand side of the expression is invariant under the totally positive elements in $\mathcal{O}_F^\times$. For our convenience, set $\mathrm{C}(\mathfrak{m}, \mathbf{f}) = 0$ if $\mathfrak{m}$ is not integral.

3.2 Hecke operators

Let $\mathbf{f}$ be a cuspform of weight $k = (k_1, \cdots, k_n)$, level $\mathfrak{n}$, with a character $\tilde{\omega}$. For each finite place $\mathfrak{p}$, let $\varpi_\mathfrak{p}$ be a uniformizer for $\mathcal{O}_\mathfrak{p}$. The Hecke operator $\mathbb{T}_\mathfrak{p}$ at $\mathfrak{p}$ is defined by

$$(\mathbb{T}_\mathfrak{p}\mathbf{f})(g) = \int_{\mathrm{K}_\mathfrak{p}(\mathfrak{n})} \mathbf{f}\left(gk_p \begin{pmatrix} \varpi_\mathfrak{p} & \\ & 1 \end{pmatrix}\right) \tilde{\omega}^{-1}(k_\mathfrak{p})\, dk_\mathfrak{p}.$$

Suppose that $\mathfrak{p}$ does not divide neither $\mathfrak{n}$ nor $\mathfrak{D}$, then observe that $\mathrm{K}_\mathfrak{p}(\mathfrak{n}) = \mathrm{GL}_2(\mathcal{O}_\mathfrak{p})$, $\tilde{\omega}|_{\mathcal{O}_\mathfrak{p}^\times} \equiv \mathbb{1}$, and that $\mathbf{f}$ is right $\mathrm{K}_\mathfrak{p}(\mathfrak{n})$-fixed. Therefore, it follows that

$$(\mathbb{T}_\mathfrak{p}\mathbf{f})(g) = \int_{\mathrm{K}_\mathfrak{p}(\mathfrak{n})\left(\begin{smallmatrix} \varpi_\mathfrak{p} & \\ & 1 \end{smallmatrix}\right)\mathrm{K}_\mathfrak{p}(\mathfrak{n})} \mathbf{f}(gh)\, dh.$$

Furthermore, it can be rewritten as the finite sum

$$(\mathbb{T}_\mathfrak{p}\mathbf{f})(g) = \mathbf{f}\left(g\begin{pmatrix} 1 & \\ & \varpi_\mathfrak{p} \end{pmatrix}\right) + \sum_{u \in \mathcal{O}_F/\mathfrak{p}} \mathbf{f}\left(g\begin{pmatrix} \varpi_\mathfrak{p} & u \\ & 1 \end{pmatrix}\right). \tag{3.2.1}$$

This is obtained by decomposing the double coset $\mathrm{K}_\mathfrak{p}(\mathfrak{n})\left(\begin{smallmatrix} \varpi_\mathfrak{p} & \\ & 1 \end{smallmatrix}\right)\mathrm{K}_\mathfrak{p}(\mathfrak{n})$ as a disjoint union of right cosets,

$$\mathrm{K}_\mathfrak{p}(\mathfrak{n})\begin{pmatrix} \varpi_\mathfrak{p} & \\ & 1 \end{pmatrix}\mathrm{K}_\mathfrak{p}(\mathfrak{n}) = \begin{pmatrix} 1 & \\ & \varpi_\mathfrak{p} \end{pmatrix}\mathrm{K}_\mathfrak{p}(\mathfrak{n}) \cup \left(\cup_{u \in \mathcal{O}_F/\mathfrak{p}} \begin{pmatrix} \varpi_\mathfrak{p} & u \\ & 1 \end{pmatrix}\mathrm{K}_\mathfrak{p}(\mathfrak{n})\right). \tag{3.2.2}$$

Now, we wish to define, more generally, the Hecke operators $\mathbb{T}_\mathfrak{m}$ for any integral ideal $\mathfrak{m}$. Let $\mathrm{K} = G_\infty^+ \cdot \mathrm{K}_0(\mathfrak{n})$, where $\mathrm{K}_0(\mathfrak{n})$ is as defined earlier. We also let

$$\mathrm{Y}_\mathfrak{p} := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(F_\mathfrak{p}) \ : \ \begin{matrix} a\mathcal{O}_\mathfrak{p} + \mathfrak{n}_\mathfrak{p} = \mathcal{O}_\mathfrak{p}, & b \in \mathfrak{D}_\mathfrak{p}^{-1} \\ c \in \mathfrak{n}_\mathfrak{p}\mathfrak{D}_\mathfrak{p}, & d \in \mathcal{O}_\mathfrak{p} \end{matrix} \right\},$$

and $\mathrm{Y} = \left(G(\mathbb{R}) \cdot \prod_\mathfrak{p} \mathrm{Y}_\mathfrak{p}\right) \cap G(\mathbb{A})$.

The Hecke operator $\mathbb{T}_\mathfrak{m}$ is given by $\mathbb{T}_\mathfrak{m} = \sum_y \mathrm{K}y\mathrm{K}$ where the sum is taken over all the representatives y of the double cosets $\mathrm{K}y\mathrm{K}$ with $y \in \mathrm{Y}$ satisfying $(\det y)\mathcal{O}_F = \mathfrak{m}$.

Noting that each summand $\mathrm{K}y\mathrm{K}$ can be written as a disjoint union $\mathrm{K}y\mathrm{K} = \cup_j \mathrm{K}y_j$ with the archimedean part of y_j being 1, we define

$$(\mathbf{f}|\mathrm{K}y\mathrm{K})(g) = \sum_j \omega'(y_j)^{-1}\mathbf{f}(gy_j^{\iota}),$$

where $\omega'\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right) = \omega(a_{\mathfrak{n}} \mod \mathfrak{n})$. This definition coincides with the integral definition for all the Hecke operators $\mathbb{T}_{\mathfrak{p}}$ with respect to prime ideals $\mathfrak{p}$.

3.3 Primitive form

We first recall the definition of new forms from Shimura [41]. Let $\mathfrak{m}$ be an integral ideal that divides $\mathfrak{n}$ and is divisible by the conductor of $\tilde{\omega}$, and $\mathbf{g} \in \mathcal{S}_k(\mathfrak{m}, \tilde{\omega})$. Let $\mathfrak{a}$ be an integral ideal dividing $\mathfrak{m}^{-1}\mathfrak{n}$ that is generated by an element $\alpha \in \mathbb{A}_F^{\times}$ with $\alpha_\infty = 1$. Define $\mathbf{g}_{\mathfrak{a}}$ by the right translation of $\mathrm{N}(\mathfrak{a})^{-k_0/2}\mathbf{g}$ by $\left(\begin{smallmatrix} \alpha^{-1} & \\ & 1 \end{smallmatrix}\right)$. Such $\mathbf{g}_{\mathfrak{a}}$ is an element in $\mathcal{S}_k(\mathfrak{q}\mathfrak{m}, \tilde{\omega})$. The space $\mathcal{S}_k^{\mathrm{old}}(\mathfrak{n}, \tilde{\omega})$ generated by all such $\mathbf{g}_{\mathfrak{a}}$ is called the space of old forms. The space $\mathcal{S}_k^{\mathrm{new}}(\mathfrak{n}, \tilde{\omega})$ of new forms is defined to be the orthogonal complement of $\mathcal{S}_k^{\mathrm{old}}(\mathfrak{n}, \tilde{\omega})$ with respect to an inner product:

$$\langle \mathbf{f},\, \mathbf{g} \rangle := \sum_{\nu=1}^{h} \frac{1}{\mu\left(\Gamma_\nu \backslash \mathfrak{h}^n\right)} \int_{\Gamma_\nu \backslash \mathfrak{h}^n} \overline{f_\nu(z)} g_\nu(z) y^k \, d\mu(z),$$

where $d\mu(z) = \prod_{j=1}^{n} \frac{dx_j dy_j}{y_j^2}$.

A Hilbert cusp form $\mathbf{f}$ in $\mathcal{S}_k(\mathfrak{n}, \tilde{\omega})$ is said to be primitive if it is a newform, a common eigenfunction of all the Hecke operators $\mathbb{T}_{\mathfrak{p}}$, and normalized so that $\mathrm{C}(\mathcal{O}_F, \mathbf{f}) = 1$.

Miyake proved that if two newforms $\mathbf{f}$ and $\mathbf{g}$ are common eigenfunctions for $\mathbb{T}_{\mathfrak{p}}$ and share the same eigenvalues for almost all primes $\mathfrak{p}$, then $\mathbf{f}$ and $\mathbf{g}$ are a constant multiple of each other. In particular, if they are normalized, we have $\mathbf{f} = \mathbf{g}$. Furthermore, if a newform $\mathbf{f}$ is normalized and a common eigenfunction of $\mathbb{T}_{\mathfrak{p}}$ for all $\mathfrak{p}$ not dividing $\mathfrak{n}$, then it is an eigenfunction for all $\mathbb{T}_{\mathfrak{m}}$ and its eigenvalues are $\mathrm{N}(\mathfrak{m})^{1-k_0/2}\mathrm{C}(\mathfrak{m}, \mathbf{f})$. (See [32] and [41].) For our convenience, we normalize the Hecke operators $\mathbb{T}_{\mathfrak{m}}$ as

$$\mathbb{T}'_{\mathfrak{m}} := \mathrm{N}(\mathfrak{m})^{k_0/2-1}\mathbb{T}_{\mathfrak{m}}$$

so that the eigenvalues of $\mathbf{f}$ for $\mathbb{T}'_{\mathfrak{m}}$ are $\mathrm{C}(\mathfrak{m}, \mathbf{f})$.

We make one more observation regarding a primitive form. Suppose $\mathbf{f} = (f_1, \ldots, f_h)$ is a primitive form. One may ask whether $\mathbf{f}$ is determined by any one of its components f_ν. In general this is not true. For example, take χ to be a non-trivial character of the narrow class group, and put $\mathbf{g} = \mathbf{f} \otimes \chi$, i.e., for any $x \in G(\mathbb{A})$, $\mathbf{g}(x) = \mathbf{f}(x)\chi(\det(x))$. Using (3.1.2) it is trivial to check that $g_1 = f_1$, however, $\mathbf{f} \neq \mathbf{g}$, at least not in general. After we prove the correspondence $\mathbf{f} \leftrightarrow \Pi(\mathbf{f})$, it will follow that $\Pi(\mathbf{f} \otimes \chi) = \Pi(\mathbf{f}) \otimes \chi$, and so if $\Pi(\mathbf{f})$ admits a self-twist, then the twisting character must be quadratic, and $\Pi(\mathbf{f})$ has to come via automorphic induction from the corresponding quadratic extension of F, and in general this would not be the case for a given $\mathbf{f}$. On a related note, one can make an interesting observation based on a refined strong multiplicity one theorem due to Ramakrishnan [39]: suppose, $\mathbf{f}$ and $\mathbf{g}$ are primitive forms, and suppose $f_\nu = g_\nu$ for all ν except, say, $\nu = \nu_0$. This means that $C(\mathfrak{p}, \mathbf{f}) = C(\mathfrak{p}, \mathbf{g})$ for all prime ideals $\mathfrak{p}$ whose class in the narrow class group is not represented by $t_{\nu_0}^{-1}$, or in other words, $C(\mathfrak{p}, \mathbf{f}) = C(\mathfrak{p}, \mathbf{g})$ for all prime ideals $\mathfrak{p}$ outside a set S of finite places with Dirichlet density $1/h$. (See, for example, Koch [28, Theorem 1.111].) It follows from Ramakrishnan's theorem that if the narrow class number is sufficiently large ($h > 8$ will do) then necessarily $\mathbf{f} = \mathbf{g}$.

3.4 Some properties

Some further properties of Hilbert modular forms are discussed in this section, that are needed in Chapter 8.

Proposition 3.4.1 (Shimura, [41]) *Let $\mathbf{f}$ be a holomorphic Hilbert modular form of weight k, level $\mathfrak{n}$, with a character $\tilde{\omega}$. If f is an eigenfunction of $\mathbb{T}_{\mathfrak{m}}$, for an ideal $\mathfrak{m}$ prime to $\mathfrak{n}$, with its eigenvalue $\lambda(\mathfrak{m})$, then $\lambda(\mathfrak{m}) = \omega^*(\mathfrak{m})\overline{\lambda(\mathfrak{m})}$.*

In particular, if the character $\tilde{\omega}$ is trivial, then $\lambda(\mathfrak{m})$ is real.

We also define $\mathbf{f}|\mathrm{J}_{\mathfrak{n}}$ as follows: For each ν, pick a totally positive element q_ν. There exists a unique index λ so that $t_\nu t_\lambda \mathcal{O}\mathfrak{n}\mathfrak{D}_F^2 = q_\nu \mathcal{O}$. Put $\beta_\nu = \left(\begin{smallmatrix} & 1 \\ -q_\nu & \end{smallmatrix}\right)$, and $f'_\lambda = (-1)^k f_\nu||_k \beta_\nu$. Then $\mathbf{f}|\mathrm{J}_{\mathfrak{n}}$ is defined to be

$$\mathbf{f}|\mathrm{J}_{\mathfrak{n}} = (f'_1, \cdots, f'_h). \tag{3.4.2}$$

Then $\mathbf{f}|\mathrm{J}_{\mathfrak{n}}$ has the same weight and level as $\mathbf{f}$. Furthermore, we have the following:

Proposition 3.4.3 (Shimura, [41]) *Let* $\mathbf{f}$ *be a primitive form with conductor* $\mathfrak{n}$. *Then* $\mathbf{f}|\mathrm{J}_n$ *is a nonzero constant times the complex conjugation of* $\mathbf{f}$.

CHAPTER 4

Representation Theory on GL(2)

4.1 Representations of $\mathrm{GL}_2(\mathbb{R})$

4.1.1 The Weil group of $\mathbb{R}$

Let $W_{\mathbb{R}}$ be the Weil group of $\mathbb{R}$. Recall that as a set it is defined as $W_{\mathbb{R}} = \mathbb{C}^* \cup j\mathbb{C}^*$. The group structure is induced from that of $\mathbb{C}^*$ and the relations $jzj^{-1} = \overline{z}$ and $j^2 = -1$. There is a homomorphism $W_{\mathbb{R}} \to \mathbb{R}^*$ which sends $z \in \mathbb{C}^*$ to $|z|_{\mathbb{C}} = z\bar{z}$ and sends j to -1. This homomorphism induces an isomorphism of the abelianization $W_{\mathbb{R}}^{\mathrm{ab}} \to \mathbb{R}^*$.

Let us recall the classification of two-dimensional semi-simple representations of $W_{\mathbb{R}}$. To begin, any (quasi-)character ξ of $\mathbb{C}^*$ looks like $\xi_{(s,w)} : \mathbb{C}^* \to \mathbb{C}^*$ with

$$\xi_{(s,w)}(z) = z^s \bar{z}^w, \quad \text{or} \quad \xi_{(s,w)}(re^{i\theta}) = r^{s+w}e^{i(s-w)\theta},$$

where $s, w \in \mathbb{C}$ and $s - w \in \mathbb{Z}$. As alluded to above, the complex absolute value is $|z|_{\mathbb{C}} := z\bar{z} = \xi_{(1,1)}(z)$. A character $\xi_{(s,w)}$ is unitary, i.e., takes values in $\mathbb{C}^1 = \{z \in \mathbb{C}^* : |z|_{\mathbb{C}} = 1\}$, if and only if $w = -s$ in which case $s \in \frac{1}{2}\mathbb{Z}$. In other words, any unitary character of $\mathbb{C}^*$ is of the form ξ_l for $l \in \mathbb{Z}$, where

$$\xi_l(z) = \left(\frac{z}{\bar{z}}\right)^{l/2} = \left(\frac{z}{\sqrt{|z|_{\mathbb{C}}}}\right)^l, \quad \text{or} \quad \xi_l(re^{i\theta}) = e^{il\theta}.$$

Next, any character χ of $\mathbb{R}^*$ looks like $\chi_{(s,\epsilon)} : \mathbb{R}^* \to \mathbb{C}^*$ with

$$\chi_{(s,\epsilon)}(t) = |t|^s \mathrm{sgn}(t)^\epsilon$$

where $s \in \mathbb{C}$ and ϵ is in $\{0, 1\}$. Via the isomorphism $W_{\mathbb{R}}^{\mathrm{ab}} \to \mathbb{R}^*$ any character θ of $W_{\mathbb{R}}$ also looks like $\theta_{(s,\epsilon)} : W_{\mathbb{R}} \to \mathbb{C}^*$ with

$$\theta_{(s,\epsilon)}(z) = (z\bar{z})^s \quad \text{and} \quad \theta_{(s,\epsilon)}(j) = (-1)^\epsilon.$$

Henceforth, we identify the character $\theta_{(s,\epsilon)}$ of $W_{\mathbb{R}}$ with the character $\chi_{(s,\epsilon)}$ of $\mathbb{R}^*$. Let $\epsilon_{\mathbb{R}} : W_{\mathbb{R}} \to \{\pm 1\}$ denote the sign homomorphism, defined as $\epsilon_{\mathbb{R}}(z) = 1$ and $\epsilon_{\mathbb{R}}(j) = -1$, i.e., $\epsilon_{\mathbb{R}} = \chi_{(0,1)}$. The usual absolute value of a real number t is denoted $|t|$ and this gives an absolute value $|\ |_{\mathbb{R}}$ on $W_{\mathbb{R}}$, defined as $\chi_{(1,0)}$. The restriction of $|\ |_{\mathbb{R}}$ to $\mathbb{C}^*$ via $\mathbb{C}^* \hookrightarrow W_{\mathbb{R}}$ gives $|\ |_{\mathbb{C}}$ on $\mathbb{C}^*$. Since $W_{\mathbb{R}}$ contains $\mathbb{C}^*$ as an abelian subgroup of index two, it is an easy exercise to see that any two-dimensional semi-simple representation τ is one of these two-kinds:

1. an irreducible 2-dimensional representation; $\tau = \tau(l, t)$ parametrized by pairs (l, t) with $l \geq 1$ an integer and $t \in \mathbb{C}$ where

$$\tau(l, t) = \mathrm{Ind}_{\mathbb{C}^*}^{W_{\mathbb{R}}}(\xi_l) \otimes |\ |_{\mathbb{R}}^t = \mathrm{Ind}_{\mathbb{C}^*}^{W_{\mathbb{R}}}(\xi_l \otimes |\ |_{\mathbb{C}}^t).$$

2. a reducible 2-dimensional semi-simple representation; $\tau = \tau(\chi_1, \chi_2)$ with characters $\chi_i = \chi_{(s_i,\epsilon_i)}$ of $W_{\mathbb{R}}$, where

$$\tau(\chi_1, \chi_2) = \chi_1 \oplus \chi_2.$$

4.1.2 The local Langlands correspondence

Let us recall the Langlands classification for $\mathrm{GL}_2(\mathbb{R})$. Let χ_1, χ_2 be characters of $\mathbb{R}^*$ such that $\chi_i = \chi_{(s_i,\epsilon_i)}$. Let $I(\chi_1, \chi_2)$ be the normalized parabolic induction of the character $\chi_1 \otimes \chi_2$ of the standard Borel subgroup to all of $\mathrm{GL}_2(\mathbb{R})$. Suppose that $\Re(s_1) \geq \Re(s_2)$ then $I(\chi_1, \chi_2)$ has a unique irreducible quotient, called the Langlands quotient, which we denote as $J(\chi_1, \chi_2)$. The induced representation $I(\chi_1, \chi_2)$ is reducible if and only if $s_1 - s_2 = l \in \mathbb{Z}_{\geq 1}$; in this case the Langlands quotient is, up to a twist, the irreducible finite-dimensional sub-quotient of dimension l, and the other piece is a twist of the discrete series representation D_l which we now define. (Later we will give this exact sequence precisely.) For any integer $l \geq 1$, let D_l stand for the discrete series representation with lowest non-negative K-type being the character $\left(\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right) \mapsto e^{-i(l+1)\theta}$, and central character $a \mapsto \mathrm{sgn}(a)^{l+1}$. Note the shift from l to $l+1$. The representation at infinity for a holomorphic elliptic modular cusp form of weight k is D_{k-1}. The Langlands classification states that any irreducible admissible representation of $\mathrm{GL}_2(\mathbb{R})$ is, up to equivalence, one of these:

1. $D_l \otimes |\,|_{\mathbb{R}}^t$, for an integer $l \geq 1$ and $t \in \mathbb{C}$; or
2. $J(\chi_1, \chi_2)$, for characters $\chi_i = \chi_{(s_i,\epsilon_i)}$ of $\mathbb{R}^*$ with $\Re(s_1) \geq \Re(s_2)$.

There is a canonical bijection $\pi \leftrightarrow \tau$ between equivalence classes of irreducible admissible representations π of $\mathrm{GL}_2(\mathbb{R})$ and equivalence classes of two dimensional semi-simple representations $\tau = \tau(\pi)$ of $W_{\mathbb{R}}$. We call τ the *Langlands parameter* of π. From the above classifications it is clear that under this correspondence, we have

1. $\pi = D_l \otimes |\,|_{\mathbb{R}}^t \quad \leftrightarrow \quad \tau = \tau(l,t)$; for an integer $l \geq 1$ and $t \in \mathbb{C}$; and
2. $\pi = J(\chi_1, \chi_2) \quad \leftrightarrow \quad \tau = \tau(\chi_1, \chi_2)$; for characters $\chi_i = \chi_{(s_i,\epsilon_i)}$ of $\mathbb{R}^*$.

In the second case, given χ_1 and χ_2, if necessary we reorder them such that $\Re(s_1) \geq \Re(s_2)$ which ensures that $J(\chi_1, \chi_2)$ is defined, while noting that reordering them does not change the equivalence class of $\tau(\chi_1, \chi_2)$. This bijection is canonical in that it preserves local factors and is equivariant under twisting. The local L-factor of an irreducible representation τ of $W_{\mathbb{R}}$ is as follows. (See Knapp [27].)

$$L(s,\tau) = \begin{cases} \pi^{-(s+t)/2}\,\Gamma\left(\frac{s+t}{2}\right) & \text{, if } \tau = |\,|_{\mathbb{R}}^t \\ \pi^{-(s+t+1)/2}\,\Gamma\left(\frac{s+t+1}{2}\right) & \text{, if } \tau = \epsilon_{\mathbb{R}} \otimes |\,|_{\mathbb{R}}^t \\ 2(2\pi)^{-(s+t+l/2)}\;\Gamma(s+t+l/2) & \text{, if } \tau = \mathrm{Ind}_{\mathbb{C}^*}^{W_{\mathbb{R}}}(\xi_l) \otimes |\,|_{\mathbb{R}}^t \text{ with } l \geq 1. \end{cases}$$

4.2 Automorphic representations of $\mathrm{GL}_2(\mathbb{A}_F)$

Following Borel–Jacquet [4, Section 4.6], we say an irreducible representation of $G(\mathbb{A})$ is automorphic if it is isomorphic to an irreducible subquotient of the representation of $G(\mathbb{A})$ on its space of automorphic forms. We say an automorphic representation is cuspidal if it is a subrepresentation of the representation of $G(\mathbb{A})$ on the space of cusp forms $\mathcal{A}_{\text{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A}))$; let V_Π denote the representation space of Π. (In particular, a cuspidal representation need not be unitary.) For an automorphic representation Π of $G(\mathbb{A})$, we have $\Pi = \Pi_\infty \otimes \Pi_f$, where $\Pi_\infty = \otimes_{v \in S_\infty} \Pi_v$ is an irreducible representation of G_∞, and $\Pi_f = \otimes_{v \notin S_\infty} \Pi_v$, which is a restricted tensor product, is an irreducible representation of $G(\mathbb{A}_f)$.

4.2.1 Algebraic automorphic representation

(See Clozel [7, p.89].) Let Π be an irreducible automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$. We will work over a totally real number field F. The representation at infinity Π_∞ is a tensor product

$$\Pi_\infty = \otimes_{\eta \in S_\infty} \Pi_\eta = \Pi_{\eta_1} \otimes \cdots \otimes \Pi_{\eta_n},$$

where Π_η is an irreducible admissible representation of $\mathrm{GL}_2(F_\eta) = \mathrm{GL}_2(\mathbb{R})$. For $1 \leq j \leq n$, let τ_j be the Langlands parameter of Π_{η_j}. The restriction of τ_j to $\mathbb{C}^*$ is a direct sum of characters:

$$\tau_j|_{\mathbb{C}^*} = \xi_{j_1} \oplus \xi_{j_2},$$

with $\xi_{j_i} = \xi_{(s_{j_i}, w_{j_i})}$. We say that an irreducible automorphic representation Π is *algebraic* if

$$s_{j_i} = \frac{1}{2} + p_{j_i}, \quad w_{j_i} = \frac{1}{2} + q_{j_i}, \text{ with } p_{j_i}, q_{j_i} \in \mathbb{Z}.$$

(In other words, a global representation Π is algebraic if all the exponents appearing in the characters of $\mathbb{C}^*$ coming from the representations Π_η at infinity are half plus an integer.)

Note that the data $(s_{j_1}, w_{j_1}, s_{j_2}, w_{j_2})$ depends only on two of these numbers:

1. the restriction of $\tau = \tau(l,t)$ to $\mathbb{C}^*$ is given by

$$\mathrm{Ind}_{\mathbb{C}^*}^{W_R}(\xi_l \otimes |\ |_{\mathbb{C}}^t)|_{\mathbb{C}^*} = \xi_l \otimes |\ |_{\mathbb{C}}^t \oplus \xi_{-l} \otimes |\ |_{\mathbb{C}}^t,$$

   which looks like $(z^s \bar{z}^w, z^w \bar{z}^s)$ with $s = l/2 + t$ and $w = -l/2 + t$.

2. Or, if $\tau = \tau(\chi_1, \chi_2)$ with $\chi_i = \chi_{(s_i, \epsilon_i)}$, then the restriction of τ to $\mathbb{C}^*$ is $((z\bar{z})^{s_1}, (z\bar{z})^{s_2})$.

4.2.2 Regular algebraic cuspidal automorphic representation

Let us first note that this section is borrowed heavily from Clozel [7].

Let Π be an irreducible algebraic automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$. The infinity type of Π is an element of $\prod_{j=1}^n (\mathbb{Z}^2)^{\mathbb{Z}/2}$, i.e., it is an n-tuple of unordered pairs of integers, and is defined as follows: Consider $\Pi' = \Pi \otimes |\ |^{-1/2}$. Since Π is algebraic, all the exponents of the characters of $\mathbb{C}^*$ coming from the infinite components of Π' are integers. For each real place η_j for $1 \leq j \leq n$, the restriction to $\mathbb{C}^*$ of the Langlands parameter of the representation Π'_{η_j}, as described above, looks either like $(z^{p_j} \bar{z}^{q_j}, z^{q_j} \bar{z}^{p_j})$ or like $((z\bar{z})^{p_j}, (z\bar{z})^{q_j})$ for integers p_j and q_j. The infinity type of Π is then defined as:

$$\infty(\Pi) := (\{p_1, q_1\}, \{p_2, q_2\}, \ldots, \{p_n, q_n\}).$$

(See Clozel [7, p.106] for details.)

If an algebraic cuspidal automorphic representation Π has the infinity type of $(\{p_1, q_1\}, \{p_2, q_2\}, \ldots, \{p_n, q_n\})$ with $p_j \neq q_j$ for all $1 \leq j \leq n$, we say that Π is *regular*.

4.2.3 Infinite components of a regular algebraic cuspidal automorphic representation

Let us suppose that Π is such a representation, and let us look closely at the possible exponents of the characters of $\mathbb{C}^*$ for the representations at infinity. Suppose one of the representations at infinity looks like $\Pi_\eta = J(\chi_1, \chi_2)$. Then its Langlands parameter is $\tau = \tau(\chi_1, \chi_2)$ with $\chi_i = \chi_{(s_i, \epsilon_i)}$; the restriction of τ to $\mathbb{C}^*$ as mentioned above looks like $((z\bar{z})^{s_1}, (z\bar{z})^{s_2})$. Since Π is algebraic we have $s_i = \frac{1}{2} + p_i$ with $p_i \in \mathbb{Z}$. Then $s_1 - s_2 \in \mathbb{Z}$. Since the inducing data is of Langlands type, we have $s_1 - s_2 \geq 0$. Since Π is regular, $s_1 - s_2 \geq 1$. But then the full induced representation $I(\chi_1, \chi_2)$ is reducible; hence the Langlands quotient $J(\chi_1, \chi_2)$ is a finite-dimensional representation. But a cuspidal automorphic representation is globally generic (i.e., has a global Whittaker model) and so locally generic everywhere, and so every local component has to be an infinite-dimensional representation. Hence Π_η cannot be equivalent to $J(\chi_1, \chi_2)$, and has to be of the form $D_l \otimes |\ |_{\mathbb{R}}^t$. In this case the exponents of the characters are $l/2 + t$ and $-l/2 + t$; hence if l is even then $t \in \frac{1}{2}\mathbb{Z}$, and if l is odd then $t \in \mathbb{Z}$. We have just proved that *the infinite components of a regular algebraic cuspidal automorphic representation of* $\mathrm{GL}_2(\mathbb{A}_F)$ *are all discrete series representations twisted by integral or half-integral powers of absolute value.* Further, there is a compatibility with all these twists afforded by the fact that there is a twist of the global representation which makes it unitary; see 4.4.4.

4.3 Finite-dimensional representations

Any $t \in T_\infty$ looks like $t = (t_j)_j \in \prod_{j=1}^n T_2(F_{\eta_j}) = \prod_{j=1}^n T_2(\mathbb{R})$. We will also write $t \in T_\infty$ as:

$$t = \left(\begin{pmatrix} x_1 & 0 \\ 0 & y_1 \end{pmatrix}, \begin{pmatrix} x_2 & 0 \\ 0 & y_2 \end{pmatrix}, \ldots, \begin{pmatrix} x_n & 0 \\ 0 & y_n \end{pmatrix} \right).$$

Let $\mu = (\mu_1, \ldots, \mu_n)$ be an integral weight for T_∞, i.e., each $\mu_j = (a_j, b_j) \in \mathbb{Z}^2$ and we have

$$\mu(t) = \prod_j \mu_j(t_j) = \prod_j x_j^{a_j} y_j^{b_j}.$$

Let $X(T_\infty)$ stand for set of all integral weights. Let $X^+(T_\infty)$ be the subset of dominant integral weights; dominant for the choice of Borel subgroup being B. A weight $\mu \in X(T_\infty)$ as above is dominant if and only if $a_j \geq b_j$ for all $1 \leq j \leq n$.

For $\mu \in X^+(T_\infty)$, we let M_μ stand for the irreducible finite-dimensional representation of $G(\mathbb{C})$ of highest weight μ. Since $G(\mathbb{C}) = \prod_{j=1}^n \mathrm{GL}_2(\mathbb{C})$, it is clear that $M_\mu = \otimes_j M_{\mu_j}$ with M_{μ_j} being the irreducible finite-dimensional representation of $\mathrm{GL}_2(\mathbb{C})$ of highest weight μ_j. Since $\mu_j = (a_j, b_j)$ it is well-known that

$$M_{\mu_j} = \mathrm{Sym}^{a_j - b_j}(\mathbb{C}^2) \otimes \det{}^{b_j}$$

where $\mathbb{C}^2$ is the standard representation of $\mathrm{GL}_2(\mathbb{C})$. We let M_μ^{v} stand for the contragredient representation; $M_\mu^{\mathsf{v}} = M_{\mu^{\mathsf{v}}}$ where $\mu^{\mathsf{v}} = (\mu_1^{\mathsf{v}}, \ldots, \mu_n^{\mathsf{v}})$ with $\mu_j^{\mathsf{v}} = (-b_j, -a_j)$. Hence,

$$M_{\mu_j}^{\mathsf{v}} = \mathrm{Sym}^{a_j - b_j}(\mathbb{C}^2) \otimes \det{}^{-a_j}.$$

4.4 Cohomological automorphic representations

4.4.1 Cuspidal cohomology

For any open-compact subgroup $\mathrm{K}_f \subset G(\mathbb{A}_f)$ define the space

$$S^G_{\mathrm{K}_f} := G(\mathbb{Q})\backslash G(\mathbb{A})/\mathrm{K}^\circ_\infty \mathrm{K}_f = \mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A}_F)/\mathrm{K}^\circ_\infty \mathrm{K}_f.$$

This is an example of a locally symmetric space, because such a space is a finite disjoint union of its connected components which are all of the form $\Gamma\backslash G(\mathbb{R})^\circ/\mathrm{K}^\circ_\infty$ for an arithmetic subgroup Γ of $G(\mathbb{R})^\circ$; locally it looks like the symmetric space $G(\mathbb{R})^\circ/\mathrm{K}^\circ_\infty$. In the literature on Hilbert modular forms, these spaces also go by the appellation Hilbert–Blumenthal varieties. (See, for example, Ghate [15, Section 2.2].)

Let $\mu = (\mu_1, \ldots, \mu_n) \in X^+(T)$. The representation $M_\mu^\vee$ defines a local system $\widetilde{M_\mu^\vee}$ on $S^G_{\mathrm{K}_f}$. The precise definition is as follows.

Definition 4.4.1 *Let $P_1 : G(\mathbb{A}_F)/\mathrm{K}^\circ_\infty \mathrm{K}_f \longrightarrow S^G_{\mathrm{K}_f}$ be the projection map. For any open set U in $S^G_{\mathrm{K}_f}$, the section $\widetilde{M_{\mu^\vee}}(U)$ over U is defined as:*

$$\widetilde{M_{\mu^\vee}}(U) := \left\{ \begin{array}{lr} s : P_1^{-1}(U) \to M_{\mu^\vee} : & s \text{ is locally constant, and} \\ & s(\gamma x) = \gamma.s(x)\ \forall \gamma \in \mathrm{GL}_2(F) \end{array} \right\}.$$

(Working with the dual $M_{\mu^\vee}$ instead of just M_μ is for convenience which will become clear later on.) We should be cautious in the existence of such a non-zero sheaf. To obtain a non-zero section, the condition $s(\gamma x) = s(x)$ must be satisfied for any element γ in $Z(F) \cap \mathrm{K}^\circ_\infty \mathrm{K}_f$. For the simplicity, let us assume $\mathrm{K}_f = \prod_{\mathfrak{p}} \mathrm{GL}_2(\mathcal{O}_{\mathfrak{p}})$. Any element in $Z(F) \cap \mathrm{K}^\circ_\infty \mathrm{K}_f$ is described as $\left(\begin{smallmatrix} u & \\ & u \end{smallmatrix} \right)$ where u is a totally positive element in $F^\times$ and sits inside $\mathcal{O}_{\mathfrak{p}}^\times$ for all non-archimedean places $\mathfrak{p}$, i.e., u is a totally positive element in $\mathcal{O}_F^\times$. Hence the condition we need is equivalent to say that $\prod_j \eta_j(u)^{a_j+b_j} = 1$ for all totally positive elements u in $\mathcal{O}_F^\times$ where $\mu = (\mu_j)$ with $\mu_j = (a_j, b_j)$ is a dominant integral weight. It is fulfilled when $a_j + b_j$ is a constant for all j. Indeed, if $a_j + b_j = c$ for all j, then $\prod_j \eta_j(u)^{a_j+b_j} = \left(\prod_j \eta_j(u)\right)^c = 1$. We state this fact as a lemma below.

Lemma 4.1 *Let F be a totally real number field. Let $\mu = (\mu_j)$ with $\mu_j = (a_j, b_j)$ be a dominant integral weight, and M_μ the irreducible finite-dimensional representation of highest weight μ. Then a local system $\widetilde{M_{\mu^\vee}}$ on $S^G_{\mathrm{K}_f}$ defined by $M_{\mu^\vee}$ is a nonzero sheaf if $a_j + b_j$ is independent of j.*

We note that the hypothesis of this lemma is always satisfied under our setting in the latter chapters. See (5.4.5). The open-compact subgroup K_f will be taken as $\mathrm{K}_0(\mathfrak{n})$ defined in (2.4.2) which does not contradict the lemma as $\mathrm{K}_0(\mathfrak{n})$ is a subgroup of $\prod_{\mathfrak{p}} \mathrm{GL}_2(\mathcal{O}_{\mathfrak{p}})$ up to a conjugation by an element of $\mathfrak{D}$.

We are interested in the sheaf cohomology

$$H^\bullet(S^G_{\mathrm{K}_f}, \widetilde{M_{\mu^\vee}}).$$

It is convenient to pass to the limit over all open-compact subgroups K_f and let

$$H^\bullet(S^G, \widetilde{M_{\mu^\vee}}) := \varinjlim_{\mathrm{K}_f} H^\bullet(S^G_{\mathrm{K}_f}, \widetilde{M_{\mu^\vee}}).$$

There is an action of $\pi_0(G_\infty)\times G(\mathbb{A}_f)$ on $H^\bullet(S^G, \widetilde{M_{\mu^\vee}})$, which is usually called a Hecke-action, and one can always recover the cohomology of $S^G_{\mathrm{K}_f}$ by taking invariants:

$$H^\bullet(S^G_{\mathrm{K}_f}, \widetilde{M_{\mu^\vee}}) = H^\bullet(S^G, \widetilde{M_{\mu^\vee}})^{\mathrm{K}_f}.$$

We can compute the above sheaf cohomology via the de Rham complex, and then reinterpreting the de Rham complex in terms of the complex computing relative Lie algebra cohomology, we get the isomorphism:

$$H^\bullet(S^G, \widetilde{M_{\mu^\vee}}) \simeq H^\bullet(\mathfrak{g}_\infty, \mathrm{K}_\infty^\circ;\ C^\infty(G(\mathbb{Q})\backslash G(\mathbb{A})) \otimes M_{\mu^\vee}).$$

With level structure K_f this takes the form:

$$H^\bullet(S^G_{\mathrm{K}_f}, \widetilde{M_{\mu^\vee}}) \simeq H^\bullet(\mathfrak{g}_\infty, \mathrm{K}_\infty^\circ;\ C^\infty(G(\mathbb{Q})\backslash G(\mathbb{A}))^{\mathrm{K}_f} \otimes M_{\mu^\vee}).$$

The inclusion $C^\infty_{\mathrm{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A})) \hookrightarrow C^\infty(G(\mathbb{Q})\backslash G(\mathbb{A}))$ of the space of smooth cusp forms in the space of all smooth functions induces, via results of Borel [3], an injection in cohomology; this defines cuspidal cohomology:

$$\begin{tikzcd}
H^\bullet(S^G, \widetilde{M_{\mu^\vee}}) \arrow[r] & H^\bullet(\mathfrak{g}_\infty, \mathrm{K}_\infty^\circ; C^\infty(G(\mathbb{Q})\backslash G(\mathbb{A})) \otimes M_{\mu^\vee}) \\
H^\bullet_{\mathrm{cusp}}(S^G, \widetilde{M_{\mu^\vee}}) \arrow[r] \arrow[u, hook] & H^\bullet(\mathfrak{g}_\infty, \mathrm{K}_\infty^\circ; C^\infty_{\mathrm{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A})) \otimes M_{\mu^\vee}) \arrow[u, hook]
\end{tikzcd}$$

Using the usual decomposition of the space of cusp forms into a direct sum of cuspidal automorphic representations, we get the following fundamental decomposition of $\pi_0(G_\infty) \times G(\mathbb{A}_f)$-modules:

$$H^\bullet_{\mathrm{cusp}}(S^G, \widetilde{M_{\mu^\vee}}) = \bigoplus_{\Pi} H^\bullet(\mathfrak{g}_\infty, \mathrm{K}_\infty^\circ; \Pi_\infty \otimes M_{\mu^\vee}) \otimes \Pi_f \tag{4.4.2}$$

We say that Π *contributes to the cuspidal cohomology of G with coefficients in $M_{\mu^\vee}$* if Π has a nonzero contribution to the above decomposition. Equivalently, if Π

is a cuspidal automorphic representation whose representation at infinity Π_∞ after twisting by $M_{\mu^\vee}$ has nontrivial relative Lie algebra cohomology. In this situation, we write $\Pi \in \mathrm{Coh}(G, \mu^\vee)$. A fundamental observation of Clozel is that a cuspidal automorphic representation Π is regular algebraic if and only if Π is of cohomological type, i.e., contributes to the cohomology –possibly with nontrivial coefficients–of a locally symmetric space attached to GL_2 over F. (See Clozel [7, p.111].)

Whether Π contributes to cuspidal cohomology or not is determined entirely by its infinite component Π_∞. This is a very well-known and somewhat surprising fact; surprising because local representations at infinity seem to have a such a strong control over a global phenomenon. Further, it was observed by Clozel that this property is in fact captured purely in terms of certain exponents of characters of $\mathbb{C}^*$ appearing in the Langlands parameter of Π_∞. See the following sections. We also refer the reader to Knapp [27].

4.4.2 Cohomology of a discrete series representation

We will digress for a moment to observe that discrete series representations of $\mathrm{GL}_2(\mathbb{R})$, possibly twisted by a half-integral power of absolute value, have nontrivial cohomology. For brevity, let $(\mathfrak{g}_2, \mathrm{K}_2^\circ) := (\mathfrak{gl}_2, \mathrm{SO}(2)Z_2(\mathbb{R})^\circ)$. For a dominant integral weight $\nu = (a, b)$, with integers $a \geq b$, the basic fact here is that there is a non-split exact sequence of $(\mathfrak{g}_2, \mathrm{K}_2^\circ)$-modules:

$$0 \to D_{a-b+1} \otimes |\ |_{\mathbb{R}}^{(a+b)/2} \to \mathrm{Ind}_{B_2(\mathbb{R})}^{\mathrm{GL}_2(\mathbb{R})}(\chi_{(a,a)}|\ |^{1/2} \otimes \chi_{(b,b)}|\ |^{-1/2}) \to M_\nu \to 0. \quad (4.4.3)$$

(Recall from our earlier notation that $\chi_{(a,a)}(t) = |t|^a \mathrm{sgn}(t)^a = t^a$ for any integer a.) In other words, in the category $\mathcal{C}(\mathfrak{g}_2, \mathrm{K}_2^\circ)$ of admissible $(\mathfrak{g}_2, \mathrm{K}_2^\circ)$-modules, one has

$$\mathrm{Ext}^1_{\mathcal{C}(\mathfrak{g}_2, \mathrm{K}_2^\circ)}(M_\nu, D_{a-b+1} \otimes |\ |_{\mathbb{R}}^{(a+b)/2}) \neq 0.$$

But

$$\begin{aligned} H^1(\mathfrak{g}_2, \mathrm{K}_2^\circ; (D_{a-b+1} \otimes |\ |_{\mathbb{R}}^{(a+b)/2}) \otimes M_{\nu^\vee}) &= \mathrm{Ext}^1_{\mathcal{C}(\mathfrak{g}_2,\mathrm{K}_2^\circ)}(\mathbb{1}, (D_{a-b+1} \otimes |\ |_{\mathbb{R}}^{(a+b)/2}) \otimes M_{\nu^\vee}) \\ &= \mathrm{Ext}^1_{\mathcal{C}(\mathfrak{g}_2,\mathrm{K}_2^\circ)}(M_\nu, D_{a-b+1} \otimes |\ |_{\mathbb{R}}^{(a+b)/2}) \neq 0. \end{aligned}$$

Further, it is well-known that $H^\bullet(\mathfrak{g}_2, \mathrm{K}_2^\circ; (D_{a-b+1} \otimes |\ |_{\mathbb{R}}^{(a+b)/2}) \otimes M_{\nu^\vee}) \neq 0$ if and only if $\bullet = 1$, and that dimension of $H^1(\mathfrak{g}_2, \mathrm{K}_2^\circ; (D_{a-b+1} \otimes |\ |_{\mathbb{R}}^{(a+b)/2}) \otimes M_{\nu^\vee})$ is two, with both the characters of O(2)/SO(2) appearing exactly once. (See, for example, Waldspurger [46, Proposition I.4].) This detail will be useful below; see 4.4.5. Finally, suppose $H^q(\mathfrak{g}_2, \mathrm{K}_2^\circ; \Xi \otimes M_{\nu^\vee}) \neq 0$ for some irreducible admissible infinite-dimensional representation Ξ of $\mathrm{GL}_2(\mathbb{R})$, then the central character restricted to $\mathbb{R}_{>0}$ and the infinitesimal character of Ξ are the same as that of M_ν which can be seen from Wigner's Lemma (Borel-Wallach [5, Theorem I.4.1]). It follows from Langlands classification that $\Xi \simeq D_{a-b+1} \otimes |\ |_{\mathbb{R}}^{(a+b)/2}$.

4.4.3 'Regular algebraic' = 'Cohomological'

Let Π be a cuspidal automorphic representation of $G(\mathbb{A}_F)$. A point of view afforded by Clozel [7] is that

$$\Pi \text{ is regular and algebraic} \iff \Pi \in \mathrm{Coh}(G, \mu^\vee) \text{ for some } \mu \in X^+(T).$$

Let $\Pi \in \mathrm{Coh}(G, \mu^\vee)$. Say, $\mu = (\mu_1, \ldots, \mu_n)$, and each $\mu_j = (a_j, b_j)$. Apply the Künneth theorem (see, for example, Borel-Wallach [5, I.1.3]) to see that

$$H^\bullet(\mathfrak{g}_\infty, \mathrm{K}_\infty^\circ; \Pi_\infty \otimes M_{\mu^\vee}) = \bigoplus_{d_1+\cdots+d_n=\bullet} \otimes_{j=1}^n H^{d_j}(\mathfrak{gl}_2, \mathrm{SO}(2)Z_2(\mathbb{R})^\circ; \Pi_j \otimes M_{\mu_j^\vee}).$$

From 4.4.2 the right hand side is nonzero only for $d_j = 1$ and $\Pi_{\eta_j} = D_{a_j-b_j+1} \otimes |\ |_{\mathbb{R}}^{(a_j+b_j)/2}$. The exponents in the Langlands parameter of Π_{η_j} are therefore given by:

$$\tau(\Pi_{\eta_j})(z) = z^{\frac{1}{2}+a_j}\bar{z}^{-\frac{1}{2}+b_j} + z^{-\frac{1}{2}+b_j}\bar{z}^{\frac{1}{2}+a_j}, \quad \forall z \in \mathbb{C}^* \subset W_{\mathbb{R}}.$$

Hence Π is algebraic. Next, working with $\Pi' = \Pi \otimes |\ |^{-1/2}$ we see that the infinity type of Π is:

$$\infty(\Pi) := (\{a_1, b_1 - 1\}, \{a_2, b_2 - 1\}, \ldots, \{a_n, b_n - 1\}).$$

Since μ is dominant, $a_j \geq b_j$; whence $a_j > b_j - 1$, i.e., Π is regular and algebraic.

Conversely, let Π be a regular algebraic cuspidal automorphic representation of $G(\mathbb{A}_F)$. As in 4.2.2 the infinity type of π is given by

$$\infty(\Pi) := (\{p_1, q_1\}, \{p_2, q_2\}, \ldots, \{p_n, q_n\})$$

for integers p_j, q_j and regularity says that $p_j \neq q_j$. Without loss of generality assume that $p_j > q_j$. Put $a_j = p_j$ and $b_j = q_j + 1$. Now let $\mu_j = (a_j, b_j)$ and $\mu = (\mu_1, \ldots, \mu_n)$. Then $\mu \in X^+(T)$, and it follows from 4.2.3 that $\Pi \in \mathrm{Coh}(G, \mu^{\mathsf{v}})$.

4.4.4 Clozel's purity lemma

(See Clozel [7, Lemme 4.9].) Let $\mu \in X^+(T)$ be a dominant integral weight as above; say, $\mu = (\mu_1, \ldots, \mu_n)$, and each $\mu_j = (a_j, b_j)$ with integers $a_j \geq b_j$. The purity lemma says that if the weight μ supports nontrivial cuspidal cohomology, i.e., if $\mathrm{Coh}(G, \mu^{\mathsf{v}})$ is nonempty, then μ satisfies the 'purity' condition: there exists $\mathsf{w} = \mathsf{w}(\mu) \in \mathbb{Z}$ such that $a_j + b_j = \mathsf{w}$ for all j. This integer w is called the purity weight of μ, and if $\Pi \in \mathrm{Coh}(G, \mu^{\mathsf{v}})$, then we will call w the purity weight of Π as well. (Proof: Given a cuspidal representation Π, there is a complex number w such that the twisted representation $\Pi \otimes |\ |^{\mathsf{w}}$ is unitary; if further Π is algebraic it follows that w must be an integer.) Let us denote the set of all pure dominant integral weights by $X_0^+(T)$. If we start with a primitive holomorphic Hilbert modular form, as will be the case in the latter part of the article, then this condition is automatically fulfilled; however, from the perspective of cohomological automorphic representations, the purity of the weight μ is an important condition to keep in mind.

4.4.5 Pinning down generators for the cohomology class at infinity

Let $\Pi \in \mathrm{Coh}(G, \mu^{\mathsf{v}})$. Say, $\mu = (\mu_1, \ldots, \mu_n)$, and each $\mu_j = (a_j, b_j)$. The space $H^n(\mathfrak{g}_\infty, \mathrm{K}_\infty^\circ; \Pi_\infty \otimes M_{\mu^{\mathsf{v}}})$ is acted upon by $\mathrm{K}_\infty/\mathrm{K}_\infty^\circ$. It follows from the Künneth rule (Borel-Wallach [5, I.1.3]) and 4.4.2 that every character of $\mathrm{K}_\infty/\mathrm{K}_\infty^\circ$ appears with multiplicity one in $H^n(\mathfrak{g}_\infty, \mathrm{K}_\infty^\circ; \Pi_\infty \otimes M_{\mu^{\mathsf{v}}})$. Fix such a character $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ of $\mathrm{K}_\infty/\mathrm{K}_\infty^\circ$. The purpose of this (somewhat tedious) paragraph is to fix a basis $[\Pi_\infty]^\epsilon$ for the one-dimensional vector space $H^n(\mathfrak{g}_\infty, \mathrm{K}_\infty^\circ; \Pi_\infty \otimes M_{\mu^{\mathsf{v}}})(\epsilon)$. (See (4.4.8) below, especially when $\epsilon = (+, \ldots, +)$.) Since Künneth gives:

$$H^n(\mathfrak{g}_\infty, \mathrm{K}_\infty^\circ; \Pi_\infty \otimes M_{\mu^{\mathsf{v}}})(\epsilon) = \bigotimes_{j=1}^{n} H^1(\mathfrak{gl}_2, \mathrm{SO}(2)Z_2(\mathbb{R})^\circ; \Pi_j \otimes M_{\mu_j^{\mathsf{v}}})(\epsilon_j),$$

it suffices to fix a basis $[\Pi_j]^{\epsilon_j}$ for the one-dimensional $H^1(\mathfrak{gl}_2, \mathrm{SO}(2)Z_2(\mathbb{R})^\circ; \Pi_j \otimes M_{\mu_j^{\mathsf{v}}})(\epsilon_j)$ and let

$$[\Pi_\infty]^\epsilon = \bigotimes_{j=1}^{n} [\Pi_j]^{\epsilon_j}.$$

We now proceed to fix $[\Pi_j]^{\epsilon_j}$. Since we are working with only one copy of $\mathrm{GL}_2(\mathbb{R})$, let us omit the subscript j and slightly change our notations: Let $\nu = (\nu_1, \nu_2) \in X^+(T_2)$ be a dominant integral weight for the diagonal torus $T_2(\mathbb{R})$ in $\mathrm{GL}_2(\mathbb{R})$, and M_ν the corresponding finite-dimensional irreducible representation of $\mathrm{GL}_2(\mathbb{C})$ of highest weight ν. Let $\Xi \simeq D_{\nu_1 - \nu_2 + 1} \otimes |\ |^{(\nu_1+\nu_2)/2}$. For any choice of sign in $\{\pm\} := (\mathrm{O}(2)/\mathrm{SO}(2))\widehat{\ }$, with $+$ or $-$ being the trivial or nontrivial character of $\mathrm{O}(2)/\mathrm{SO}(2)$ respectively, we will fix a 1-cocycle $[\Xi]^\pm$ so that

$$H^1(\mathfrak{gl}_2, \mathrm{SO}(2)Z_2(\mathbb{R})^\circ; \Xi \otimes M_{\nu^{\mathsf{v}}})(\pm) \ = \ \mathbb{C}\,[\Xi]^\pm.$$

For any integer $m \geq 1$, let M_m be the $(m-1)^{\text{th}}$ symmetric power of the standard (two-dimensional) representation $\mathbb{C}^2$ of $\mathrm{GL}_2(\mathbb{C})$, i.e., $M_m = \mathrm{Sym}^{m-1}(\mathbb{C}^2)$. The representation M_m is irreducible and of dimension m. Denote the standard basis of $\mathbb{C}^2$ by $\{e_1, e_2\}$, which gives the 'standard' basis $\{e_2^{m-1}, e_2^{m-2}e_1, \ldots, e_1^{m-1}\}$ for M_m. This basis will be denoted as $\{\mathsf{s}_0, \mathsf{s}_1, \ldots, \mathsf{s}_{m-1}\}$, i.e., $\mathsf{s}_j = e_1^j e_2^{m-1-j}$. The finite-dimensional irreducible representation M_ν of $\mathrm{GL}_2(\mathbb{C})$ with highest weight ν is $M_\nu = M_{\nu_1-\nu_2+1} \otimes \det^{\nu_2} = \mathrm{Sym}^{\nu_1-\nu_2}(\mathbb{C}^2) \otimes \det^{\nu_2}$. By restriction, M_ν is also a representation of $\mathrm{GL}_2(\mathbb{R})$. The central character of M_ν is given by $a \mapsto \omega_\nu(a) = a^{\nu_1+\nu_2}$ for all $a \in \mathbb{R}^*$. The contragredient representation of M_ν is denoted M_ν^{v}; one has $M_\nu^{\mathsf{v}} = M_{\nu^{\mathsf{v}}}$, where $\nu^{\mathsf{v}} = (-\nu_2, -\nu_1)$ is the dual weight of ν. Explicitly, $M_{\nu^{\mathsf{v}}} = M_{\nu_1-\nu_2+1} \otimes \det^{-\nu_1}$. We will need information on the restriction of $M_{\nu^{\mathsf{v}}}$ to various subgroups.

In either of the representations M_ν or $M_{\nu^{\mathsf{v}}}$, the action of the diagonal torus in SL_2 on the basis vectors is given by $\left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right) \mathsf{s}_j = t^{-\nu_1+\nu_2+2j}\mathsf{s}_j$. Hence, the standard basis realizes the weights:

$$\{-(\nu_1 - \nu_2), -(\nu_1 - \nu_2) + 2, \ldots, \nu_1 - \nu_2\}.$$

In particular, the highest weight vector of $M_{\nu^{\mathsf{v}}}$ is given by $e^+_{\nu^{\mathsf{v}}} := \mathsf{s}_{\nu_1-\nu_2} = e_1^{\nu_1-\nu_2}$. Observe that the standard basis gives a $\mathbb{Q}$-structure on $M_{\nu^{\mathsf{v}}}$.

The restriction of $M_{\nu^{\mathsf{v}}}$ to $\mathrm{GL}_1(\mathbb{R}) \hookrightarrow \mathrm{GL}_2(\mathbb{R})$ is described by $\left(\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right) \mathsf{s}_j = t^{j-\nu_1}\mathsf{s}_j$. From this we easily deduce the following lemma which will be of use later on; see 6.2.5 below.

Lemma 4.2 *Let $\mathbb{1}$ denote the trivial representation of* $\mathrm{GL}_1(\mathbb{R})$*. Then*

$$\mathrm{Hom}_{\mathrm{GL}_1(\mathbb{R})}(M_{\nu^{\mathsf{v}}}, \mathbb{1}) \neq 0 \iff \nu_1 \geq 0 \geq \nu_2.$$

In this situation, $\mathrm{Hom}_{\mathrm{GL}_1(\mathbb{R})}(M_{\nu^{\mathsf{v}}}, \mathbb{1})$ *is one-dimensional and a nonzero map $\mathcal{T}$ in* $\mathrm{Hom}_{\mathrm{GL}_1(\mathbb{R})}(M_{\nu^{\mathsf{v}}}, \mathbb{1})$ *is given by projecting to the coordinate corresponding to s_{ν_1}, i.e.,*

$$\mathcal{T}(\sum_{j=0}^{\nu_1-\nu_2} c_j \mathsf{s}_j) = c_{\nu_1}.$$

Proof. This is easy to verify and we omit the proof. Let us mention that this is a special case of well-known classical branching laws from $\mathrm{GL}_n(\mathbb{C})$ to $\mathrm{GL}_{n-1}(\mathbb{C})$; see Goodman-Wallach [17]. ∎

The SO(2)-types of M_ν, as well as $M_{\nu^\vee}$, are given by:

$$M_{\nu^\vee}|_{\mathrm{K}_2^1} = M_\nu|_{\mathrm{K}_2^1} = \theta_{-(\nu_1-\nu_2)} \oplus \theta_{-(\nu_1-\nu_2)+2} \oplus \cdots \oplus \theta_{\nu_1-\nu_2-2} \oplus \theta_{\nu_1-\nu_2},$$

where, for any integer n, θ_n is the character of SO(2) given by $\theta_n(r(t)) = e^{-int}$, for all $r(t) = \left(\begin{smallmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{smallmatrix}\right)$ in SO(2). It is necessary to fix an ordered basis giving the above decomposition. Let $\{\mathsf{w}_1, \mathsf{w}_2\}$ denote the basis for $\mathbb{C}^2$ which diagonalizes the SO(2)-action:

$$\mathsf{w}_1 = e_1 + ie_2, \ \mathsf{w}_2 = ie_1 + e_2;$$

it is trivially verified that

$$r(t)\mathsf{w}_1 = e^{-it}\mathsf{w}_1 = \theta_1(r(t))\mathsf{w}_1, \quad r(t)\mathsf{w}_2 = e^{it}\mathsf{w}_2 = \theta_{-1}(r(t))\mathsf{w}_2.$$

The ordered basis $\{\mathsf{w}_2^{\nu_1-\nu_2}, \mathsf{w}_2^{\nu_1-\nu_2-1}\mathsf{w}_1, \ldots, \mathsf{w}_1^{\nu_1-\nu_2}\}$ of $M_{\nu^\vee}$ realizes the above decomposition of $M_{\mu^\vee}$ into its K-types. Let $\mathsf{w}_{\nu^\vee}^+ = \mathsf{w}_1^{\nu_1-\nu_2}$ be the basis vector realizing the highest non-negative K-type in $M_{\nu^\vee}$, i.e., $\theta_{\nu_1-\nu_2}$; similarly, the lowest K-type is realized by $\mathsf{w}_{\nu^\vee}^- = \mathsf{w}_2^{\nu_1-\nu_2}$. In terms of the standard basis:

$$\begin{aligned} \mathsf{w}_{\nu^\vee}^+ = (e_1 + ie_2)^{\nu_1-\nu_2} &= \sum_{\alpha=0}^{\nu_1-\nu_2} \binom{\nu_1-\nu_2}{\alpha} i^{\nu_1-\nu_2-\alpha}\, \mathsf{s}_\alpha \\ \mathsf{w}_{\nu^\vee}^- = (ie_1 + e_2)^{\nu_1-\nu_2} &= \sum_{\alpha=0}^{\nu_1-\nu_2} \binom{\nu_1-\nu_2}{\alpha} i^{\alpha}\, \mathsf{s}_\alpha. \end{aligned} \tag{4.4.4}$$

For a dominant integral weight $\nu = (\nu_1, \nu_2)$, and for $\Xi \simeq D_{\nu_1-\nu_2+1} \otimes |\ |^{(\nu_1+\nu_2)/2}$, using the exact sequence in (4.4.3) we deduce that the SO(2)-types of Ξ are

$$\cdots \oplus \theta_{-(\nu_1-\nu_2+4)} \oplus \theta_{-(\nu_1-\nu_2+2)} \oplus (\text{nothing here}) \oplus \theta_{\nu_1-\nu_2+2} \oplus \theta_{\nu_1-\nu_2+4} \oplus \cdots$$

The missing K-types in *(nothing here)* correspond exactly to the K-types of M_ν. Let $\phi_{\pm(\nu_1-\nu_2+2)}$ be vectors in Ξ with K-types $\theta_{\pm(\nu_1-\nu_2+2)}$, respectively. The vectors in $\Xi = D_{\nu_1-\nu_2+1} \otimes |\ |^{(\nu_1+\nu_2)/2}$ may be identified with vectors in the induced representation $\mathrm{Ind}_{B_2}^{G_2}(\chi_{(\nu_1,\nu_1)}|\ |^{1/2} \otimes \chi_{(\nu_2,\nu_2)}|\ |^{-1/2})$ which is the middle term in the exact sequence (4.4.3). In particular, we may and shall normalize them as

$$\phi_{\pm(\nu_1-\nu_2+2)}\left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\right) = 1.$$

Recall that $(\mathfrak{g}_2, \mathrm{K}_2^\circ) := (\mathfrak{gl}_2(\mathbb{C}), \mathrm{SO}(2)Z_2(\mathbb{R})^\circ)$ and let $\mathrm{K}_2^1 = \mathrm{SO}(2)$. The cochain complex

$$C^\bullet := \mathrm{Hom}_{\mathrm{K}_2^\circ}(\wedge^\bullet \mathfrak{g}_2/\mathfrak{k}_2, \Xi \otimes M_{\nu^\mathsf{v}})$$

computes $(\mathfrak{g}_2, \mathrm{K}_2^\circ)$-cohomology of $\Xi \otimes M_{\nu^\mathsf{v}}$. Since the central characters of Ξ and M_ν are equal, this complex is same as $\mathrm{Hom}_{\mathrm{K}_2^1}(\wedge^\bullet \mathfrak{g}_2/\mathfrak{k}_2, \Xi \otimes M_{\nu^\mathsf{v}})$. It is easy to see that $\mathfrak{g}_2/\mathfrak{k}_2 = \theta_2 \oplus \theta_{-2}$ as a K_2^1-module: let $\{\mathbf{z}_1, \mathbf{z}_2\}$ be the basis for $\mathfrak{g}_2/\mathfrak{k}_2$ given by:

$$\mathbf{z}_1 = i \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \quad \mathbf{z}_2 = i \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix};$$

it is easily checked that

$$\mathrm{Ad}(r(t))(\mathbf{z}_1) = e^{-2it}\mathbf{z}_1 = \theta_2(r(t))\mathbf{z}_1 \text{ and } \mathrm{Ad}(r(t))(\mathbf{z}_2) = e^{2it}\mathbf{z}_2 = \theta_{-2}(r(t))\mathbf{z}_2.$$

From the description of K-types of Ξ and M_ν we see that $C^q = 0$ for all $q \neq 1$, and

$$C^1 = \mathrm{Hom}_{\mathrm{K}_2^1}(\theta_{-2} \oplus \theta_2, \Xi \otimes M_{\nu^\mathsf{v}}) \simeq \mathbb{C}^2.$$

Fix a basis $\{f_{-2}, f_2\}$ for this two dimensional space C^1 as follows: f_{-2} picks up the vector $\phi_{-\nu_1+\nu_2-2} \otimes \mathsf{w}^+_{\nu^\mathsf{v}}$ realizing the character θ_{-2}; similarly, f_2 picks up the vector $\phi_{\nu_1-\nu_2+2} \otimes \mathsf{w}^-_{\nu^\mathsf{v}}$ realizing the character θ_2. More precisely,

$$\begin{aligned} f_{-2}(\mathbf{z}_1) &= 0, & f_{-2}(\mathbf{z}_2) &= \phi_{-\nu_1+\nu_2-2} \otimes \mathsf{w}^+_{\nu^\mathsf{v}}, \\ f_2(\mathbf{z}_1) &= \phi_{\nu_1-\nu_2+2} \otimes \mathsf{w}^-_{\nu^\mathsf{v}}, & f_2(\mathbf{z}_2) &= 0. \end{aligned}$$

Since $C^1 = \mathrm{Hom}_{\mathrm{K}_2^\circ}(\mathfrak{g}_2/\mathfrak{k}_2, \Xi \otimes M_{\nu^\mathsf{v}}) \simeq ((\mathfrak{g}_2/\mathfrak{k}_2)^* \otimes \Xi \otimes M_{\nu^\mathsf{v}})^{\mathrm{K}_2^\circ}$ we can transcribe these expressions for $f_{\pm 2}$ as follows: Let $\{\mathbf{z}_1^*, \mathbf{z}_2^*\}$ be the basis for $(\mathfrak{g}_2/\mathfrak{k}_2)^*$ that is dual to the basis $\{\mathbf{z}_1, \mathbf{z}_2\}$ for $\mathfrak{g}_2/\mathfrak{k}_2$. Then

$$f_{-2} = \mathbf{z}_2^* \otimes \phi_{-\nu_1+\nu_2-2} \otimes \mathsf{w}^+_{\nu^\mathsf{v}}, \text{ and } f_2 = \mathbf{z}_1^* \otimes \phi_{\nu_1-\nu_2+2} \otimes \mathsf{w}^-_{\nu^\mathsf{v}}.$$

To summarize we have:

$$H^1(\mathfrak{g}_2, \mathrm{K}_2^\circ; \Xi \otimes M_{\nu^\mathsf{v}}) = \mathrm{Hom}_{\mathrm{K}_2^\circ}(\wedge^1 \mathfrak{g}_2/\mathfrak{k}_2, \Xi \otimes M_{\nu^\mathsf{v}}) = ((\mathfrak{g}_2/\mathfrak{k}_2)^* \otimes \Xi \otimes M_{\nu^\mathsf{v}})^{\mathrm{K}_2^\circ} = \mathbb{C}f_{-2} \oplus \mathbb{C}f_2$$

with explicit expressions for $f_{\pm 2}$ as relative Lie algebra cocycles.

To identify the class $[\Xi]^{\pm}$, a generator for the one-dimensional space $H^1(\mathfrak{g}_2, \mathrm{K}_2^\circ; \Xi \otimes M_{\nu^{\mathsf{v}}})(\pm)$, we need to know the action of the element $\delta = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ which represents the nontrivial element in $\mathrm{K}_2/\mathrm{K}_2^\circ$. Recall that the action of δ on any $f \in \mathrm{Hom}_{\mathrm{K}_2^\circ}(\wedge^1 \mathfrak{g}_2/\mathfrak{k}_2, \Xi \otimes M_{\nu^{\mathsf{v}}})$ is given by

$$(\delta f)(\mathbf{z}) = (\Xi \otimes M_{\nu^{\mathsf{v}}})(\delta)(f(\mathrm{Ad}(\delta^{-1})\mathbf{z})).$$

Lemma 4.3 *The action of δ on $f_{\pm 2}$ is given by*

$$\delta f_{-2} = i^{\nu_1 - \nu_2} f_2, \quad \text{and} \quad \delta f_2 = i^{-\nu_1+\nu_2} f_{-2}.$$

In particular, δ acts by ± 1 on the cocycle $[\Xi]^{\pm} \;:=\; f_2 \pm i^{-\nu_1+\nu_2} f_{-2}$.

Proof. The proof is routine; here are some useful relations:

$$\begin{aligned} \mathrm{Ad}(\delta^{-1})(\mathbf{z}_1) &= \mathbf{z}_2, \\ \Xi(\delta)(\phi_{\nu_1-\nu_2+2}) &= i^{2\nu_1} \phi_{-(\nu_1-\nu_2+2)}, \\ M_{\nu^{\mathsf{v}}}(\delta)(\mathsf{w}^+_{\nu^{\mathsf{v}}}) &= i^{-(\nu_1+\nu_2)} \mathsf{w}^-_{\nu^{\mathsf{v}}}. \end{aligned}$$

■

Let $\mathcal{W}(\Xi)$ denote the Whittaker model of Ξ with respect to a nontrivial additive character $\psi_{\mathbb{R}}$ of $\mathbb{R}$; which we recall from 2.8, is taken to be $x \mapsto \psi_{\mathbb{R}}(x) = e^{-2\pi i x}$. For any $\phi \in \Xi$, let $\lambda = w(\phi)$ denote the corresponding Whittaker vector. The cohomology class $[\Xi]^{\pm}$ which generates $H^1(\mathfrak{g}_2, \mathrm{K}_2^\circ; \mathcal{W}(\Xi) \otimes M_{\nu^{\mathsf{v}}})(\pm)$ is explicitly given by

$$[\Xi]^{\pm} \;=\; \mathbf{z}_1^* \otimes \lambda_{\nu_1-\nu_2+2} \otimes \mathsf{w}^-_{\nu^{\mathsf{v}}} \;\pm\; i^{-\nu_1+\nu_2} \mathbf{z}_2^* \otimes \lambda_{-(\nu_1-\nu_2+2)} \otimes \mathsf{w}^+_{\nu^{\mathsf{v}}}$$

Using (4.4.4) we can also express this class as:

$$[\Xi]^{\pm} = \sum_{l=1}^{2} \sum_{\alpha=0}^{\nu_1-\nu_2} \mathbf{z}_l^* \otimes \lambda^{\pm}_{l,\alpha} \otimes \mathsf{s}_\alpha \tag{4.4.5}$$

where

$$\lambda^{\pm}_{1,\alpha} = \binom{\nu_1-\nu_2}{\alpha} i^{\alpha} \lambda_{\nu_1-\nu_2+2}, \quad \lambda^{\pm}_{2,\alpha} = \pm \binom{\nu_1-\nu_2}{\alpha} i^{-\alpha} \lambda_{-(\nu_1-\nu_2+2)} \tag{4.4.6}$$

Let us now go back to $\Pi \in \mathrm{Coh}(G, \mu^{\mathsf{v}})$ and write down $[\Pi_\infty]^{++}$ explicitly, where $++$ is short for $(+, \dots, +)$. Since

$$[\Pi_\infty]^{++} = \bigotimes_{j=1}^{n} [\Pi_j]^+$$

we will tensor over j the class $[\Pi_j]^+$.

Let $\{\mathsf{s}_{j,0}, \mathsf{s}_{j,1}, \dots, \mathsf{s}_{j,a_j-b_j}\}$ denote the standard basis for the representation $M_{\mu_j^{\mathsf{v}}}$. Let $\alpha = (\alpha_1, \dots, \alpha_n)$ be an n-tuple of integers such that $0 \le \alpha_j \le a_j - b_j$. Let

$$\mathsf{s}_\alpha = \otimes_{j=1}^n \mathsf{s}_{j,\alpha_j}.$$

Then the set $\{\mathsf{s}_\alpha\}_\alpha$, as α runs through all n-tuples as above gives a basis for $M_{\mu^{\mathsf{v}}}$. Next, let $l = (l_1, \dots, l_n)$ be an n-tuple of integers such that $l_j \in \{1, 2\}$. For each such l, put

$$\mathbf{z}_l^* = \otimes_{j=1}^n \mathbf{z}_{j,l_j}^*,$$

where for each $1 \le j \le n$ we let $\mathbf{z}_{j,1} = \mathbf{z}_1$ and $\mathbf{z}_{j,2} = \mathbf{z}_2$ as elements of $\mathfrak{gl}_2 = \mathrm{Lie}(\mathrm{GL}_2(F_{\eta_j}))$; and as before $\mathbf{z}^*$ is the corresponding element in the dual basis. For each $1 \le j \le n$, and α_j as above, let

$$\lambda_{j,1,\alpha_j} = \binom{\nu_1 - \nu_2}{\alpha_j} i^{\alpha_j} \lambda_{\nu_1-\nu_2+2}, \quad \lambda_{j,2,\alpha_j} = \binom{\nu_1 - \nu_2}{\alpha_j} i^{-\alpha_j} \lambda_{-(\nu_1-\nu_2+2)}$$

and for any l and α put

$$W_{l,\alpha,\infty} = \otimes_{j=1}^n \lambda_{j,l_j,\alpha_j} \in \mathcal{W}(\Pi_\infty, \psi_\infty).$$

We have the following expression

$$[\Pi_\infty]^{++} = \bigotimes_{j=1}^{n} [\Pi_j]^+ = \bigotimes_{j=1}^{n} \left(\sum_{l_j=1}^{2} \sum_{\alpha_j=0}^{a_j-b_j} \mathbf{z}_{j,l_j}^* \otimes \lambda_{j,l_j,\alpha_j} \otimes \mathsf{s}_{j,\alpha_j} \right). \tag{4.4.7}$$

Interchanging the tensor and the summations and regrouping we get:

$$\boxed{[\Pi_\infty]^{++} = \sum_{l=(l_1,\dots,l_n)} \sum_{\alpha=(\alpha_1,\dots,\alpha_n)} \mathbf{z}_l^* \otimes W_{l,\alpha,\infty} \otimes \mathsf{s}_\alpha} \tag{4.4.8}$$

which is our chosen generator of the one-dimensional space $H^n(\mathfrak{g}_\infty, \mathrm{K}_\infty^\circ; \Pi_\infty \otimes M_{\mu^\vee})(+, \dots, +)$ and is expressed as a K_∞-fixed element of

$$(\mathfrak{g}_\infty/\mathfrak{k}_\infty)^* \otimes \mathcal{W}(\Pi_\infty, \psi_\infty) \otimes M_{\mu^\vee}.$$

CHAPTER 5

Dictionary

It is well-known to experts that there is a dictionary between holomorphic Hilbert modular forms and automorphic representations of GL_2 over a totally real number field F. However, it is difficult to find all the details in the literature. We do not want any restriction on the base field F while most treatments assume at some point that the narrow class number of F is one. Besides, we could not find anywhere the answer to the question: is the dictionary $\mathrm{Aut}(\mathbb{C})$-equivariant? Some of the standard books on Hilbert modular forms like Freitag [11], Garrett [12] or van der Geer [43] do not have what we want; although Garrett's book has a definitive treatment of the action of $\mathrm{Aut}(\mathbb{C})$ on spaces of Hilbert modular forms–which is called the 'arithmetic structure theorem' in his book. In this chapter, we write down such a dictionary, give enough details to make the presentation self-contained, and also analyze its arithmetic properties. (We refer the reader to Blasius-Rogawski [1] and Harris [22] for some intimately related arithmetic issues about Hilbert modular forms.) The correspondence is summarized in the following theorem.

Theorem 5.0.1 (The dictionary) *There is a bijection* $\mathbf{f} \leftrightarrow \Pi$ *between*

- $\mathbf{f} \in S_k(\mathfrak{n}, \tilde{\omega})_{\mathrm{prim}}$, *that is* $\mathbf{f}$ *is a primitive holomorphic Hilbert modular form of weight* $k = (k_1, \ldots, k_n)$, *of level* $\mathfrak{n}$ *and nebentypus character* $\tilde{\omega}$ *which is a character of* $\mathbb{A}_F/F$ *induced from a character* $(\mathcal{O}_F/\mathfrak{n})^\times$*; by primitive we mean it is an eigenform for all Hecke operators* $\mathbb{T}'_{\mathfrak{p}}$*, a newform, and it is normalized as* $\mathrm{C}(\mathcal{O}_F, \mathbf{f}) = 1$.

- Π *is a cuspidal automorphic representation of* $\mathrm{GL}_2(\mathbb{A}_F)$ *whose representation at infinity* $\Pi_\infty = \otimes_j D_{k_j-1}$*, of conductor* $\mathfrak{n}$*, and central character* $\omega_\Pi = \tilde{\omega}$*–the adelization of* ω*; here* D_l *is the discrete series representation of* $\mathrm{GL}_2(\mathbb{R})$ *with lowest non-negative* K*-type being the character* $\left(\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right) \mapsto e^{-i(l+1)\theta}$*, and central character* $a \mapsto \mathrm{sgn}(a)^{l+1}$.

We devote this chapter to give a self-contained proof of this theorem and to describe the relation of their L-functions.

5.1 Attaching a cuspidal automorphic representation

Let $L_0^2(G(\mathbb{Q})\backslash G(\mathbb{A}), \tilde{\omega})$ be the space of functions on $G(\mathbb{A})$ such that

$$\begin{aligned} \phi(\gamma g) &= \phi(g) \qquad \text{for all}\, \gamma \in G(\mathbb{Q}), \\ \phi(zg) &= \tilde{\omega}(z)\phi(g) \qquad \text{for all}\, z \in \mathbb{A}_F^\times, \end{aligned}$$

ϕ is square integrable modulo the center, and ϕ satisfies the cuspidality condition

$$\int_{F^\times\backslash\mathbb{A}_F^\times} \phi\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) dx = 0$$

for almost all g in $G(\mathbb{A})$. The regular representation of $G(\mathbb{A})$ on this space will be denoted $\rho_0^{\tilde{\omega}}$.

Let $\mathbf{f}$ be a primitive holomorphic Hilbert cusp form of weight $k = (k_1, \cdots, k_n)$, level $\mathfrak{n}$, with a Hecke character $\tilde{\omega}$. Let $\mathcal{H}(\mathbf{f})$ be a space spanned by right translations of $\mathbf{f}$ under $G(\mathbb{A})$. Then the resulting representation $\Pi(\mathbf{f})$ on this space $\mathcal{H}(\mathbf{f})$ occurs in the regular representation $\rho_0^{\tilde{\omega}}$ on the cusp forms. The goal of this section is to prove the following theorem.

Theorem 5.1.1 *With the notions above, the representation* $\Pi(\mathbf{f})$ *on the space* $\mathcal{H}(\mathbf{f})$ *is irreducible. Furthermore, the local representation* Π_{η_j} *at each archimedean place* η_j *is the discrete series representation* D_{k_j-1} *of lowest weight* k_j*.*

To prove the first part of this theorem, let us recall some important theorems regarding automorphic representations. (See, for example, Cogdell [8].)

Theorem 5.1.2 (Multiplicity One Theorem) *The representation $\rho_0^{\tilde{\omega}}$ decomposes as the direct sum of irreducible representations, each of which appear with multiplicity one.*

Theorem 5.1.3 (Tensor Product Theorem) *Let (Π, V_Π) be an automorphic representation of $G(\mathbb{A})$. Then Π is the restricted tensor product of the local representations Π_v, where v runs through all the places of F, and each Π_v is irreducible admissible representation of $\mathrm{GL}_2(F_v)$.*

Theorem 5.1.4 (Strong Multiplicity One Theorem) *Let (Π, V_Π) and (Π', V'_Π) be irreducible admissible constituents of the regular representation of GL_2 on the cusp forms. If Π_v is equivalent to Π'_v for almost all non-archimedean places v, then $\Pi \approx \Pi'$.*

Theorem 5.1.2 and Theorem 5.1.3 guarantee that the representation $\Pi(\mathbf{f})$ can be written as $\Pi(\mathbf{f}) = \oplus_i \Pi^i$, with each irreducible constituent Π^i being a restricted tensor product of local representations Π^i_v. Therefore, in order to show that $\Pi(\mathbf{f})$ is irreducible, it is now enough to show that $\Pi^i_v \approx \Pi^j_v$ for almost all non-archimedean places v and for all i and j by Theorem 5.1.4. Write $\mathbf{f} = \oplus_i \mathbf{f}^i$ with each $\mathbf{f}^i$ in the space of Π^i. Now consider an irreducible constituent Π^i.

Let $\mathfrak{p}$ be a prime ideal of F not dividing either $\mathfrak{n}$ or the different $\mathfrak{D}$. For such an ideal $\mathfrak{p}$, $\Pi^i_{\mathfrak{p}}$ is a spherical representation $\pi(\chi_{1,\mathfrak{p}}, \chi_{2,\mathfrak{p}})$ induced from some unramified characters $\chi_{1,\mathfrak{p}}$ and $\chi_{2,\mathfrak{p}}$. (We will work with only normalized parabolic induction.) Since $\mathbf{f}$ is an eigenfunction of $\mathbb{T}_{\mathfrak{p}}$, so is $\mathbf{f}^i$, since the projection from $\Pi(\mathbf{f})$ to the i-th coordinate in $\oplus_i \Pi^i$ is a Hecke-equivariant map. Furthermore, it can be seen that the eigenvalue is $q_{\mathfrak{p}}^{1/2}\left(\chi_{1,\mathfrak{p}}(\varpi_{\mathfrak{p}}) + \chi_{2,\mathfrak{p}}(\varpi_{\mathfrak{p}})\right)$, where $q_{\mathfrak{p}}$ is the cardinality of the residue field $\mathcal{O}_{\mathfrak{p}}/\mathfrak{p}\mathcal{O}_{\mathfrak{p}}$, and $\varpi_{\mathfrak{p}}$ is a uniformizer. Indeed, applying $g = 1$ in (3.2.1), we obtain that

$$
\begin{aligned}
(\mathbb{T}_{\mathfrak{p}}\mathbf{f}^i)(1) &= \mathbf{f}^i\begin{pmatrix} 1 & \\ & \varpi_{\mathfrak{p}} \end{pmatrix} + \sum_{u\in\mathcal{O}_{\mathfrak{p}}^{\times}} \mathbf{f}^i\begin{pmatrix} \varpi_{\mathfrak{p}} & u \\ & 1 \end{pmatrix} \qquad (5.1.5)\\
&= q_{\mathfrak{p}}^{1/2}\left(\chi_{1,\mathfrak{p}}(\varpi_{\mathfrak{p}}) + \chi_{2,\mathfrak{p}}(\varpi_{\mathfrak{p}})\right)\mathbf{f}^i(1).
\end{aligned}
$$

This shows that $\chi_{1,\mathfrak{p}}(\varpi_{\mathfrak{p}}) + \chi_{2,\mathfrak{p}}(\varpi_{\mathfrak{p}}) = q_{\mathfrak{p}}^{(1-k_0)/2}\mathrm{C}(\mathfrak{p},\mathbf{f})$ where $k_0 = \max_j\{k_j\}$, and that, together with $\chi_{1,\mathfrak{p}}\chi_{2,\mathfrak{p}}$ being the central character of $\Pi^i_{\mathfrak{p}}$, the characters $\chi_{1,\mathfrak{p}}$ and $\chi_{2,\mathfrak{p}}$ are uniquely determined by $\mathbf{f}$ and they are independent of i. Hence $\Pi^i_{\mathfrak{p}} = \Pi^j_{\mathfrak{p}}$ for almost all $\mathfrak{p}$. Strong multiplicity one implies that $\Pi^i \approx \Pi^j$, and multiplicity one will imply that $\Pi(\mathbf{f})$ is irreducible, This completes the proof for the first part of Theorem 5.1.1.

For archimedean places, note that the local representation Π_{η_j} at each place η_j is a $(\mathfrak{gl}(2), \mathrm{O}(2))$-module, so it is enough to consider the eigenvalue λ_j for the Casimir operator $\Delta_j = -y_j^2\left(\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2}\right) - y_j\frac{\partial^2}{\partial x_j \partial\theta_j}$. Since Δ_j acts on $\mathbf{f}$ as a function on $\mathrm{GL}_2(F_{\eta_j})$, we only need to see the action on $(f_\nu||_k g_j)(\mathbf{i})$ for each ν. Writing $g_j = \begin{pmatrix} y_j^{1/2} & x_j y_j^{-1/2} \\ & y_j^{-1/2}\end{pmatrix}\begin{pmatrix} \cos\theta_j & -\sin\theta_j \\ \sin\theta_j & \cos\theta_j\end{pmatrix}$, a direct computation shows that $\lambda_j = \frac{k_j}{2}\left(1 - \frac{k_j}{2}\right)$ for any ν. An irreducible admissible infinite-dimensional representation of $\mathrm{GL}_2(\mathbb{R})$, with infinitesimal character determined by $\frac{k_j}{2}\left(1-\frac{k_j}{2}\right)$ and central character trivial on $\mathbb{R}_{>0}$ has to be the discrete series representation D_{k_j-1}. This says that $\Pi_{\eta_j} = D_{k_j-1}$. (Infinite-dimensionality of Π_{η_j} is guaranteed by existence of Whittaker models.)

5.2 Retrieving a Hilbert modular form from a representation

Let (Π, V_Π) be a cuspidal automorphic representation with the central character $\tilde{\omega}$ that is trivial on $F^{\times}_{\infty+}$, and such that the representation at infinity is equivalent to $\otimes_{j=1}^n D_{k_j-1}$, where D_{k_j-1} is a discrete series representation of the lowest weight k_j. Let the conductor of Π be $\mathfrak{n}$. We note that, for any non-archimedean place v not dividing $\mathfrak{n}$, the local representation Π_v is equivalent to a spherical representation induced from some unramified character $\chi_{1,v}\otimes\chi_{2,v}$. In order to retrieve a primitive holomorphic

Hilbert cusp form from this representation, it is quite useful to consider a Whittaker model of Π.

Recall from Section 2.8 our non-trivial additive character ψ of $\mathbb{A}/F$, and write ψ_v for v-component of this character. The isomorphism between the representation space V_Π and the Whittaker space $\mathcal{W}(\Pi, \psi)$ allows us to determine a unique holomorphic Hilbert cusp form that corresponds to Π by choosing a suitable element from each local Whittaker model $\mathcal{W}(\Pi_v, \psi_v)$. For almost all v, $W_v \in \mathcal{W}(\Pi_v, \psi_v)$ is a spherical element, and is normalized so that $W_v(k_v) = 1$ for all $k_v \in \mathrm{GL}_2(\mathcal{O}_v)$. The choices for the local vectors should be made in the following manner.

For each archimedean place $v = \eta_j$, let $W^\circ_{\eta_j}$ be the lowest weight vector in $\mathcal{W}(\Pi_{\eta_j}, \psi_{\eta_j})$. By the lowest weight vector, we shall mean the element given as follows:

$$W^\circ_{\eta_j}\left(\begin{pmatrix} a & \\ & a \end{pmatrix}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} y & \\ & 1 \end{pmatrix}\right) = \omega_{\eta_j}(a)\psi_{\eta_j}(x)e^{-ik_j\theta}e^{-2\pi y}.$$

For a non-archimedean place $\mathfrak{p}$, a suitably normalized $\mathrm{K}_\mathfrak{p}(\mathfrak{n})$-fixed vector needs to be chosen, where $K_\mathfrak{p}(\mathfrak{n})$ is an open compact subgroup of $\mathrm{GL}_2(F_\mathfrak{p})$ defined in (2.4.1). For this purpose, let f and r be the highest powers of $\mathfrak{p}$, which are possibly zeros, that divide $\mathfrak{n}$ and the different $\mathfrak{D}_F$, respectively. It is clear that $\mathrm{K}_\mathfrak{p}(\mathfrak{n})$ can be written as

$$\mathrm{K}_\mathfrak{p}(\mathfrak{n}) = \begin{pmatrix} \varpi_\mathfrak{p}^{-r} & \\ & 1 \end{pmatrix}\Gamma_0(\mathfrak{p}^f)\begin{pmatrix} \varpi_\mathfrak{p}^{r} & \\ & 1 \end{pmatrix},$$

where $\Gamma_0(\mathfrak{p}^f) = \left\{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathcal{O}_\mathfrak{p}) : c \equiv 0 \bmod \mathfrak{p}^f\right\}$. Let $W^{\mathrm{new}}_\mathfrak{p}$ be the new vector in $\mathcal{W}(\Pi_\mathfrak{p}, \psi_\mathfrak{p})$, i.e., $W^{\mathrm{new}}_\mathfrak{p}$ is an element such that $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot W^{\mathrm{new}}_\mathfrak{p} = \omega_\mathfrak{p}(d)W^{\mathrm{new}}_\mathfrak{p}$ for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(\mathfrak{p}^f)$, and normalized in a way specified below. Define $W^\circ_\mathfrak{p} = \left(\begin{smallmatrix} \varpi_\mathfrak{p}^{-r} & \\ & 1 \end{smallmatrix}\right) \cdot W^{\mathrm{new}}_\mathfrak{p}$. Then $W^\circ_\mathfrak{p}$ is an "almost" $\mathrm{K}_\mathfrak{p}(\mathfrak{n})$-fixed vector. (Note: this $W^\circ_\mathfrak{p}$ is slightly different from the $W^\circ_\mathfrak{p}$ of 6.2.1.) In particular,

$$W^\circ_\mathfrak{p}\begin{pmatrix} 1 & \\ & 1 \end{pmatrix} = W^{\mathrm{new}}_\mathfrak{p}\begin{pmatrix} \varpi_\mathfrak{p}^{-r} & \\ & 1 \end{pmatrix}. \tag{5.2.1}$$

We claim that the right hand side of the above expression is not zero for any $\mathfrak{p}$, and hence $W_{\mathfrak{p}}^{\circ}$ can be normalized so that $W_{\mathfrak{p}}^{\circ}(1) = 1$. In order to show that our claim is true, we first assume that $r = 0$, and hence the conductor of $\psi_{\mathfrak{p}}$ is $\mathcal{O}_{\mathfrak{p}}$. Pass $W_{\mathfrak{p}}^{\circ} = W_{\mathfrak{p}}$ to the new vector $\kappa_{\mathfrak{p}}$ in the Kirillov model $\mathcal{K}(\Pi_{\mathfrak{p}}, \psi_{\mathfrak{p}})$ with respect to the same additive character $\psi_{\mathfrak{p}}$, by defining $\kappa_{\mathfrak{p}}(x) = W_{\mathfrak{p}}^{\circ}\left(\begin{smallmatrix} x & \\ & 1 \end{smallmatrix}\right)$, and observe $\kappa_{\mathfrak{p}}(1) \neq 0$. ([40, Section 2.4].) The isomorphism between the Whittaker model $\mathcal{W}(\Pi_{\mathfrak{p}}, \psi_{\mathfrak{p}})$ and the Kirillov model $\mathcal{K}(\Pi_{\mathfrak{p}}, \psi_{\mathfrak{p}})$ guarantees that $W_{\mathfrak{p}}^{\circ}\left(\begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix}\right) \neq 0$. Normalize this vector, (and call it $W_{\mathfrak{p}}^{\circ}$ again), so that $W_{\mathfrak{p}}^{\circ}(1) = 1$.

Next, let $r > 0$. Let $\psi_{\mathfrak{p}, \varpi_{\mathfrak{p}}^{-r}}$ be an additive character defined by $\psi_{\mathfrak{p}, \varpi_{\mathfrak{p}}^{-r}}(x) := \psi_{\mathfrak{p}}(\varpi_{\mathfrak{p}}^{-r} x)$. Then since the conductor of $\psi_{\mathfrak{p}, \varpi_{\mathfrak{p}}^{-r}}$ is $\mathcal{O}_{\mathfrak{p}}$, the same argument as above applies to show that $W_{\mathfrak{p}, \varpi_{\mathfrak{p}}^{-r}}(1) \neq 0$ where $W_{\mathfrak{p}, \varpi_{\mathfrak{p}}^{-r}}$ is the new vector in a Whittaker model with respect to $\psi_{\mathfrak{p}, \varpi_{\mathfrak{p}}^{-r}}$. Observing that

$$W_{\mathfrak{p}, \varpi_{\mathfrak{p}}^{-r}} \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} = W_{\mathfrak{p}} \begin{pmatrix} \varpi_{\mathfrak{p}}^{-r} & \\ & 1 \end{pmatrix},$$

the same normalization can be done in this case as well.

Let $W^{\circ} = \otimes_v W_v^{\circ}$, which is an element of $\mathcal{W}(\Pi, \psi)$. Then there is a corresponding element $\mathbf{f}$ in V_{Π} by the usual isomorphism $V_{\Pi} \to \mathcal{W}(\Pi, \psi)$. The vectors $\mathbf{f}$ and W° are related by

$$\mathbf{f}(g) = \sum_{\alpha \in F^{\times}} W^{\circ} \left(\begin{pmatrix} \alpha & \\ & 1 \end{pmatrix} g \right). \tag{5.2.2}$$

Furthermore, we claim that, in the above expression, α only runs through totally positive elements in F. To see this, put $g = \left(\begin{smallmatrix} y_{\infty} & \\ & 1 \end{smallmatrix}\right)$ where y_{∞} is an element of $\mathbb{A}$ whose finite part is 1. Then it is easy to see that $W^{\circ}\left(\begin{smallmatrix} y_{\infty}\alpha & \\ & 1 \end{smallmatrix}\right) = e^{-2\pi y_{\infty}\alpha} \prod_{v<\infty} W_v^{\circ}\left(\begin{smallmatrix} \alpha & \\ & 1 \end{smallmatrix}\right)$ must be zero unless α is totally positive for the summation to be bounded. Hence the Fourier expansion of $\mathbf{f}$ simplifies to:

$$\mathbf{f}(g) = \sum_{\alpha \in F_+^\times} W^\circ\left(\begin{pmatrix} \alpha & \\ & 1 \end{pmatrix} g\right). \tag{5.2.3}$$

The rest of the section will be devoted to show that $\mathbf{f}$ is the desired Hilbert modular form.

Theorem 5.2.4 *Let $\mathcal{A}_0(k, \mathfrak{n}, \tilde{\omega})$ be a subspace of $\mathcal{A}_{\text{cusp}}(G(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}), \tilde{\omega})$ that consists of elements satisfying the following properties.*

1. *$\phi(gr(\theta)) = e^{-ik\theta}\phi(g)$ where $r(\theta) := \left\{\begin{pmatrix} \cos\theta_j & -\sin\theta_j \\ \sin\theta_j & \cos\theta_j \end{pmatrix}\right\}_j \in \mathrm{SO}(2)^n$,*

2. *$\phi(gk_0) = \tilde{\omega}_f(k_0^\iota)\phi(g)$, where $\tilde{\omega}_f$ is the finite part of $\tilde{\omega}$ and $k_0 \in \mathrm{K}_0(\mathfrak{n})$, and*

3. *ϕ is an eigenfunction of the Casimir element $\Delta := (\Delta_1, \cdots, \Delta_n)$ as a function of $\mathrm{GL}_2(\mathbb{R})^n$, with its eigenvalue $\lambda = \prod_{j=1}^n \frac{k_j}{2}\left(1 - \frac{k_j}{2}\right)$.*

Then $\mathcal{A}_0(k, \mathfrak{n}, \tilde{\omega})$ is isomorphic to $\mathcal{S}_k(\mathfrak{n}, \tilde{\omega})$.

Proof. These are essentially the same spaces defined from different points of view. To see this, observe first that any holomorphic Hilbert cusp form $\mathbf{f}$ is in $\mathcal{A}_0(k, \mathfrak{n}, \tilde{\omega})$. So it remains to recover a holomorphic Hilbert cusp form from any element ϕ in $\mathcal{A}_0(k, \mathfrak{n}, \tilde{\omega})$. For each element $g = \gamma x_\nu^{-\iota} g_\infty k_0$ of $G(\mathbb{A})$, put

$$f_\nu(z) = \phi(x_\nu^{-\iota} g_\infty) \det g_\infty^{-k/2} \mathrm{j}(g_\infty, \mathbf{i})^k,$$

where $g_\infty(\mathbf{i}) = z$. Holomorphy of f_ν can be shown by checking that it is annihilated by the first order differential opeator $\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}$. (As mentioned in Gelbart [13, Proof of Proposition 2.1], details of this argument appear in Gelfand–Graev–Piatetski-Shapiro[14, Chapter 1, Section 4].) A direct computation shows that $\mathbf{f} := (f_1, \cdots, f_h) \in \mathcal{S}_k(\mathfrak{n}, \tilde{\omega})$ and $\mathbf{f} = \phi$. ∎

Going back to the $\mathbf{f}$ that corresponds to the global Whittaker vector W°, it is clear that $\mathbf{f}$ belongs to $\mathcal{A}_0(k, \mathfrak{n}, \tilde{\omega})$. Indeed the first two conditions follow from (5.2.3) immediately, and the third condition holds because $\Pi_\infty = \otimes D_{k_j-1}$ and it follows that $\Delta W_\infty^\circ = \lambda W_\infty^\circ$ with λ given in the theorem. Therefore, it now only remains to show that $\mathbf{f}$ is primitive.

To prove that $\mathbf{f}$ is a newform, suppose that there exists an integral ideal $\mathfrak{m}$ that divides $\mathfrak{n}$ and such that $\mathbf{f} \in \mathcal{S}_k(\mathfrak{m}, \tilde{\omega})$. Writing $\mathbf{f} = (f_1, \cdots, f_h)$ with $f_\nu \in \mathcal{S}_k(\Gamma_\nu(\mathfrak{m}), \omega)$ for $\nu = 1, \cdots, h$, it shows that $f_\nu||_k\gamma = \tilde{\omega}(\gamma) f_\nu$ for all $\gamma \in \Gamma_\nu(\mathfrak{m})$ which contradicts the fact that the conductor of Π is $\mathfrak{n}$.

Next, it needs to be proven that $\mathbf{f}$ is a common eigenfunction of the Hecke operators $\mathbb{T}_\mathfrak{p}$ for almost all prime ideals $\mathfrak{p}$, namely $\mathfrak{p}$ not dividing neither $\mathfrak{n}$ nor the different $\mathfrak{D}_F$. Recall that for such an ideal $\mathfrak{p}$, $\mathrm{K}_\mathfrak{p}(\mathfrak{n}) = \mathrm{GL}_2(\mathcal{O}_\mathfrak{p})$, $\mathbf{f}$ is right $\mathrm{K}_\mathfrak{p}(\mathfrak{n})$-fixed, and the local representatoin $\Pi_\mathfrak{p}$ is equivalent to a spherical representation, $\pi(\chi_{1,\mathfrak{p}}, \chi_{2,\mathfrak{p}})$, induced from some unramified characters $\chi_{1,\mathfrak{p}}$ and $\chi_{2,\mathfrak{p}}$. Let $\mathbf{f}_\mathfrak{p}^\circ$ be the normalized spherical vector in the induced model. Then $\mathbf{f}_\mathfrak{p}^\circ$ is an eigenfunction of $\mathbb{T}_\mathfrak{p}$ with eigenvalue is $q_\mathfrak{p}^{1/2}(\chi_{1,\mathfrak{p}}(\varpi_\mathfrak{p}) + \chi_{2,\mathfrak{p}}(\varpi_\mathfrak{p}))$. (See (5.1.5).) Hence $W_\mathfrak{p}^\circ$ is an eigenfunction for $\mathbb{T}_\mathfrak{p}$ with same eigenvalue. It follows from (5.2.2) that $\mathbf{f}$ is also an eigenfunction for $\mathbb{T}_\mathfrak{p}$ with same eigenvalue.

Finally, we will prove that $\mathbf{f}$ is normalized, i.e., $\mathrm{C}(\mathcal{O}_F, \mathbf{f}) = 1$. Put

$$\mathrm{c}(\mathfrak{m}, \mathbf{f}) = a_\nu(\xi)\xi^{-k/2}$$

if $\mathfrak{m} = \xi t_\nu \mathcal{O}_F$ is an integral ideal in F, and 0 otherwise. This means that

$$\mathrm{c}(\mathfrak{m}, \mathbf{f}) = \mathrm{C}(\mathfrak{m}, \mathbf{f}) \mathrm{N}(\mathfrak{m})^{-k_0/2}$$

for any ideal $\mathfrak{m}$. Then $\mathbf{f}$ affords an expansion,

$$\mathbf{f}\left(\begin{pmatrix} y & x \\ & 1 \end{pmatrix}\right) = \sum_{\xi \in F_+^\times} \mathrm{c}(\xi y \mathcal{O}_F, \mathbf{f})(\xi y_\infty)^{k/2} e^{-2\pi \xi y_\infty} \mu(\xi x),$$

where $y \in \mathbb{A}_F^\times$ with $y_\infty \in F_{\infty^+}$, $x \in \mathbb{A}_F$, and μ is some additive character of $\mathbb{A}_F/F$. This expression is obtained by observing that $\mathrm{c}(\xi \mathcal{O}_F, \mathbf{f}) = \mathrm{c}(\xi y \mathcal{O}_F, \mathbf{f})$ for any totally positive element y in $\mathcal{O}_F^\times$. (See, for example, Garrett [12] or Shimura [41].) In particular,

$$\mathbf{f}\left(\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}\right) = \sum_{\xi \in F_+^\times} c(\xi\mathcal{O}_F, \mathbf{f})\xi^{k/2}e^{-2\pi\xi}.$$

On the other hand, by (5.2.3),

$$\begin{aligned} \mathbf{f}\left(\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}\right) &= \sum_{\alpha \in F_+^\times} W^\circ \begin{pmatrix} \alpha & \\ & 1 \end{pmatrix} \\ &= \sum_{0 \ll \alpha \in F^\times} e^{-2\pi\alpha} \prod_{v<\infty} W_v^\circ \begin{pmatrix} \alpha & \\ & 1 \end{pmatrix}. \end{aligned}$$

Comparing these expressions, we obtain $\mathrm{C}(\mathcal{O}_F, \mathbf{f}) = \mathrm{c}(\mathcal{O}_F, \mathbf{f}) = \prod_{v<\infty} W_v^\circ \left(\begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix}\right) = 1$ as desired.

This result, together with the argument in the previous section, completes the proof of the correspondence between primitive holomorphic Hilbert modular forms in $\mathcal{S}_k(\mathfrak{n}, \tilde{\omega})$ and cuspidal automorphic representations of $G(\mathbb{A})$ over a totally real number field F satisfying the following conditions: the local representations at infinite places are the discrete series representations D_{k_j-1} of lowest weight k_j for each $j = 1, \ldots, n$, the conductor is $\mathfrak{n}$, and the central character is trivial on the totally positive elements F_{∞^+} in $\mathbb{R}^n$.

5.3 L-functions

5.3.1 L-function attached to f

Let $\mathbf{f} = (f_1, \cdots, f_h)$ be a primitive holomorphic Hilbert modular form of weight k, level $\mathfrak{n}$, and with a character $\tilde{\omega}$. Recall that $\mathrm{C}(\mathfrak{m}, \mathbf{f})$ is defined to be $a_\nu(\xi)\xi^{-k/2}\mathrm{N}(\mathfrak{m})^{k_0/2}$ for any integral $\mathfrak{m} = \xi t_\nu \mathcal{O}_F$, and it is 0 for $\mathfrak{m}$ not integral. The (finite) L-function attached to $\mathbf{f}$ is defined to be

$$L_f(s, \mathbf{f}) = \sum_{\mathfrak{m}} \frac{\mathrm{C}(\mathfrak{m}, \mathbf{f})}{\mathrm{N}(\mathfrak{m})^s},$$

where $\mathfrak{m}$ runs through all the integral ideals of F. Let ω^* be a character of the group of ideals prime to $\mathfrak{n}$ defined by $\omega^*(\mathfrak{p}) = \tilde{\omega}(\varpi_{\mathfrak{p}})$ for all prime ideals $\mathfrak{p}$ that do not divide $\mathfrak{n}$; and let $\omega^*(\mathfrak{p}) = 0$ if $\mathfrak{p}$ divides $\mathfrak{n}$. Then, the L-function has an Euler product,

$$L_f(s, \mathbf{f}) = \prod_{\mathfrak{p}} \left(1 - \mathrm{C}(\mathfrak{p}, \mathbf{f})\mathrm{N}(\mathfrak{p})^{-s} + \omega^*(\mathfrak{p})\mathrm{N}(\mathfrak{p})^{k_0-1-2s}\right)^{-1},$$

with $k_0 = \max_j\{k_j\}$. The product is taken over all the prime ideals $\mathfrak{p}$. Define the local factors at archimedean places by

$$L_{\eta_j}(s, \mathbf{f}) = (2\pi)^{-\left(s - \frac{k_0 - k_j}{2}\right)}\Gamma\left(s - \frac{k_0 - k_j}{2}\right),$$

and for convenience write

$$L_\infty(s, \mathbf{f}) = \prod_{j=1}^{n} L_{\eta_j}(s, \mathbf{f}) = (2\pi)^{-\left(s - \frac{k_0 - k}{2}\right)}\Gamma\left(s - \frac{k_0 - k}{2}\right).$$

Define the completed L-function by

$$L(s, \mathbf{f}) = L_f(s, \mathbf{f})L_\infty(s, \mathbf{f}).$$

The above definitions are all for $\Re(s) \gg 0$. It is part of standard 'Hecke Theory' for Hilbert modular forms that $L(s, \mathbf{f})$ has an analytic continuation to all of $\mathbb{C}$ and satisfies a functional equation of the expected kind.

5.3.2 L-functions attached to Π

In this section, we recall the definition of the L-function attached to a cuspidal automorphic representation Π. First recall the GL_1-theory. For a Hecke character $\chi = \otimes_v \chi_v$ of finite order, the local L-factors at the finite places are given by

$$\begin{aligned} L_v(s, \chi_v) &= (1 - \chi_v(\varpi_v)q_v^{-s})^{-1} && \text{if } \chi_v \text{ is unramified, and} \\ L_v(s, \chi_v) &= 1 && \text{if } \chi_v \text{ is ramified.} \end{aligned}$$

Define the local L-factors for GL_2 as follows: if the local representation $\Pi_{\mathfrak{p}}$ at a place $\mathfrak{p}$ is equivalent to a principal series representation $\pi(\chi_{1,\mathfrak{p}}, \chi_{2,\mathfrak{p}})$, then put

$$L_{\mathfrak{p}}(s, \Pi_{\mathfrak{p}}) = L_{\mathfrak{p}}(s, \chi_{1,\mathfrak{p}})L_{\mathfrak{p}}(s, \chi_{2,\mathfrak{p}}).$$

Note that both factors are non-trivial if and only if $\Pi_{\mathfrak{p}}$ is spherical. For the other representations, define $L_{\mathfrak{p}}(s, \Pi_{\mathfrak{p}}) = 1$ for a supercuspidal representation $\Pi_{\mathfrak{p}}$, and

$$L_{\mathfrak{p}}(s, \Pi_{\mathfrak{p}}) = L_{\mathfrak{p}}(s + 1/2, \chi_{\mathfrak{p}})$$

for $\Pi_{\mathfrak{p}} = \mathrm{St}_{\mathrm{GL}_2(F_{\mathfrak{p}})} \otimes \chi_{\mathfrak{p}}$, the twist of the Steinberg representation $\mathrm{St}_{\mathrm{GL}_2(F_{\mathfrak{p}})}$ by $\chi_{\mathfrak{p}}$. (See, for example, Kudla [29, Section 3].) At the infinite places, the factors are

$$L_{\eta_j}(s, \Pi_{\eta_j}) = (2\pi)^{-\left(s + \frac{k_j - 1}{2}\right)} \Gamma\left(s + \frac{k_j - 1}{2}\right).$$

Again, we use a multi-index notation

$$L_{\infty}(s, \Pi_{\infty}) = (2\pi)^{-\left(s + \frac{k-1}{2}\right)} \Gamma\left(s + \frac{k-1}{2}\right)$$

to mean the product of all $L_{\eta_j}(s, \Pi_{\eta_j})$. The global L-function attached to Π is

$$L(s, \Pi) = \otimes_v L_v(s, \Pi_v).$$

The above definitions are all for $\Re(s) \gg 0$. It is part of standard 'Hecke Theory', due to Jacquet and Langlands [25], for cuspidal representations of $\mathrm{GL}_2(\mathbb{A})$ that $L(s, \Pi)$ has an analytic continuation to all of $\mathbb{C}$ and satisfies a functional equation of the expected kind.

5.3.3 Relation between $L(s, \mathbf{f})$ and $L(s, \Pi)$

Having L-functions attached to a Hilbert cusp form $\mathbf{f}$ and to a cuspidal automorphic representation Π, a natural question to ask is how $L(s, \mathbf{f})$ and $L(s, \Pi(\mathbf{f}))$ relate to each other where $\Pi(\mathbf{f})$ is a representation attached to a primitive cusp form $\mathbf{f}$. The main theorem of this section is:

Theorem 5.3.1 *Let $\mathbf{f} \in \mathcal{S}_k(\mathfrak{n}, \tilde{\omega})$ be primitive, and $\Pi(\mathbf{f})$ a cuspidal automorphic representation attached to $\mathbf{f}$. Then the completed L-functions attached to $\mathbf{f}$ and attached to $\Pi(\mathbf{f})$ satisfy the following relation:*

$$L\left(s, \Pi(\mathbf{f})\right) = L\left(s + \frac{k_0 - 1}{2}, \mathbf{f}\right),$$

where $k_0 = \mathrm{Max}\{k_1, \cdots, k_n\}$. *The same relation holds between the finite and infinite parts of the two L-functions.*

Proof. Let $\Re(s) \gg 0$. For any place v of F, and any vector W_v in a local Whittaker model $\mathcal{W}(\Pi(\mathbf{f})_v, \psi_v)$, define a local ζ-integral by

$$\zeta_v(s, W_v) = \int_{F_v^\times} W_v \begin{pmatrix} \alpha & \\ & 1 \end{pmatrix} |\alpha|^{s-1/2} \, d^\times \alpha.$$

A global ζ-integral is similarly defined for $W \in \mathcal{W}(\Pi, \psi)$ as

$$\zeta(s, W) = \int_{\mathbb{A}_F^\times} W \begin{pmatrix} \alpha & \\ & 1 \end{pmatrix} |\alpha|^{s-1/2} \, d^\times \alpha.$$

This integral is eulerian, i.e., if the global Whittaker vector W factorizes as $W = \otimes W_v$ into local Whittaker vectors then

$$\zeta(s, W) = \prod_{v \leq \infty} \zeta_v(s, W_v).$$

In particular, take ψ to be the additive character that has been fixed in Section 2.8, and W_v the normalized new vector W_v° as in Section 5.2. Then, one can show that $L_v(s, \Pi(\mathbf{f})_v) = \zeta_v(s, W_v^\circ)$. (See, for example, Gelbart [13, Proposition 6.17].) Therefore $L(s, \Pi(\mathbf{f})) = \prod_v L_v(s, \Pi(\mathbf{f})_v) = \prod_v \zeta_v(s, W_v^\circ) = \zeta(s, W^\circ)$. On the other hand, we have

$$\begin{aligned} \int_{\mathbb{A}_F^\times / F^\times} \mathbf{f} \begin{pmatrix} y & \\ & 1 \end{pmatrix} |y|^{s-1/2} \, d^\times y &= \int_{\mathbb{A}_F^\times / F^\times} \sum_{\alpha \in F^\times} W^\circ \begin{pmatrix} \alpha y & \\ & 1 \end{pmatrix} |\alpha y|^{s-1/2} \, d^\times y \\ &= \int_{\mathbb{A}_F^\times} W^\circ \begin{pmatrix} \alpha & \\ & 1 \end{pmatrix} |\alpha|^{s-1/2} \, d^\times \alpha. \end{aligned}$$

We recall that $\mathbb{A}_F^\times / F^\times = \cup_{\nu=1}^h t_\nu F_{\infty^+}^\times \prod \mathcal{O}_\mathfrak{p}^\times$ (a disjoint union), and it follows that, for any $y \in \mathbb{A}_F^\times$,

$$\mathbf{f} \begin{pmatrix} y & \\ & 1 \end{pmatrix} |y|^{s-1/2} = y_\infty^{k/2} f_\nu(\mathbf{i} y_\infty) \left(|t_\nu| |y_\infty| \right)^{s-1/2},$$

with a unique ν where y_∞ is the infinite part of y and $\mathbf{i} = (i, \cdots, i)$. Hence

$$\begin{aligned} L(s, \Pi(\mathbf{f})) &= \int_{\mathbb{A}_F^\times/F^\times} \mathbf{f}\begin{pmatrix} y & \\ & 1 \end{pmatrix} |y|^{s-1/2}\, d^\times y \\ &= \sum_{\nu=1}^{h} \int_{F_{\infty+}} f_\nu(\mathbf{i}y) y^{s+\frac{k-1}{2}} |t_\nu|^{s-1/2} \frac{dy}{y}. \end{aligned}$$

Applying the Fourier expansion $f_\nu(z) = \sum_\xi a_\nu(\xi) \exp(2\pi i \xi z)$, the proof can be completed as follows.

$$\begin{aligned} L(s, \Pi(\mathbf{f})) &= \sum_{\nu,\xi} a_\nu(\xi) |t_\nu|^{s-1/2} \int_{F_{\infty+}} e^{-2\pi\xi y} y^{s+\frac{k-1}{2}} \frac{dy}{y} \\ &= \sum_{\nu,\xi} \frac{a_\nu(\xi)}{(2\pi\xi)^{s+\frac{k-1}{2}} |t_\nu|^{-(s-1/2)}} \int_{F_{\infty+}} e^{-y} y^{s+\frac{k-1}{2}} \frac{dy}{y} \\ &= (2\pi)^{-\left(s+\frac{k-1}{2}\right)} \Gamma\left(s + \frac{k-1}{2}\right) \sum_{\mathfrak{m}} \frac{\mathrm{C}(\mathfrak{m}, \mathbf{f})}{\mathrm{N}(\mathfrak{m})^{s+\frac{k_0-1}{2}}} \\ &= L\left(s + \frac{k_0 - 1}{2}, \mathbf{f}\right). \end{aligned}$$

The equality is for $\Re(s) \gg 0$. Both sides have analytic continuation to all of $\mathbb{C}$ and hence we have equality everywhere.

From the definitions of the infinite parts of the two L-functions, we see that

$$L_\infty(s, \Pi(\mathbf{f})_\infty) = L_\infty\left(s + \frac{k_0 - 1}{2}, \mathbf{f}\right).$$

It follows that the same relations hold for the finite part since we have $L_f(s, \Pi_f) = L(s, \Pi)/L_\infty(s, \Pi_\infty)$, and similarly, $L_f(s, \mathbf{f}) = L(s, \mathbf{f})/L_\infty(s, \mathbf{f})$. ■

5.4 The Action of $\mathrm{Aut}(\mathbb{C})$

5.4.1 The action of $\mathrm{Aut}(\mathbb{C})$ on Hilbert modular forms

Let σ be an automorphism of $\mathbb{C}$, and let it act on $\mathbb{R}^n = \prod_{j=1}^n F_{\eta_j}$ by permuting the coordinates. Then $\sigma \circ \eta$ gives another embedding of F into $\mathbb{R}^n$. Considering $\sigma(\xi^k) = \prod_{j=1}^n \sigma(\eta_j(\xi))^{k_j}$ for $\xi \in F$ and $k \in \mathbb{Z}^n$, we can view σ as a permutation of $\{k_j\}$. We will use this identification from now on, and denote it as k^σ.

Let f be a Hilbert modular form of weight k, level $\mathfrak{n}$, with a character $\tilde{\omega}$, and write its Fourier expansion as $f(z) = \sum_{\xi} a_\nu(\xi) \exp(2\pi i \xi z)$. We define f^σ to be

$$f^\sigma(z) = \sum_{\xi} a_\nu^\sigma(\xi) \exp(2\pi i \xi z),$$

with $a_\nu^\sigma(\xi) = \sigma(a_\nu(\xi))$. We have the following

Proposition 5.4.1 (Shimura, [41]) *Let* $\sigma \in \mathrm{Aut}(\mathbb{C})$. *If* $f \in \mathcal{M}_k(\Gamma, \omega)$, *then* $f^\sigma \in \mathcal{M}_{k^\sigma}(\Gamma, \omega^\sigma)$, *where* $\omega^\sigma = \sigma \circ \omega$.

In order to attain a similar result in the adèlic setting, we normalize f^σ as follows: For $f_\nu \in \mathcal{M}_k(\Gamma_\nu, \omega)$ with Γ_ν defined in Section 3.1, put

$$f_\nu^{[\sigma]} = f_\nu^\sigma \cdot \left(\mathrm{N}(t_\nu)^{k_0/2}\right)^\sigma \mathrm{N}(t_\nu)^{-k_0/2},$$

where $k_0 = \mathrm{Max}\{k_1, \cdots, k_n\}$. If $\mathbf{f}$ is a holomorphic Hilbert modular form given as $\mathbf{f} = (f_1, \cdots, f_h)$, we define $\mathbf{f}^\sigma$ to be $\mathbf{f}^\sigma = (f_1^{[\sigma]}, \cdots, f_h^{[\sigma]})$.

Proposition 5.4.2 (Shimura, [41]) *Let* $\mathbf{f} = (f_1, \cdots, f_h)$ *be in* $\mathcal{M}_k(\mathfrak{n}, \tilde{\omega})$, *and* $\sigma \in \mathrm{Aut}(\mathbb{C})$. *Then* $\mathbf{f}^\sigma \in \mathcal{M}_{k^\sigma}(\mathfrak{n}, \tilde{\omega}^\sigma)$, *and* $\mathrm{C}(\mathfrak{m}, \mathbf{f}^\sigma) = \mathrm{C}(m, \mathbf{f})^\sigma$. *Furthermore,* $\mathbf{f}^\sigma$ *is primitive whenever* $\mathbf{f}$ *is.*

5.4.2 The action of $\mathrm{Aut}(\mathbb{C})$ on global representations

The following theorem is due to Harder [19] and Waldspurger [46] for GL_2 over any number field (although we state it only for our totally real base field F). It was generalized to GL_n over any number field by Clozel [7]. We have adapted the statement from Clozel's and Waldspurger's articles. In a classical context of Hilbert modular forms it is due to Shimura [41]; see also Garrett [12, Theorem 6.1].

Theorem 5.4.3 *Let* Π *be a regular algebraic cuspidal automorphic representation. For any* $\sigma \in \mathrm{Aut}(\mathbb{C})$, *define an abstract irreducible representation* ${}^\sigma\Pi = \otimes_v {}^\sigma\Pi_v$ *of* $\mathrm{GL}_2(\mathbb{A}_F)$ *as follows:*

- *For any finite place v, suppose the representation space of Π_v is V_v, then pick any σ-linear isomorphism $A_v : V_v \to V_v'$, and define ${}^\sigma\Pi_v$ as the representation of $\mathrm{GL}_2(F_v)$ acting on V_v' by ${}^\sigma\Pi_v(g) = A_v \circ \Pi_v(g) \circ A_v^{-1}$. The definition of ${}^\sigma\Pi_v$ is, up to equivalence, independent of all the choices made.*

- *For $v \in S_\infty$, define ${}^\sigma\Pi_v := \Pi_{\sigma^{-1}v}$, i.e., $({}^\sigma\Pi)_\infty = \otimes_\eta \Pi_{\sigma^{-1}\circ\eta}$ where η runs through the set $\mathrm{Hom}(F, \mathbb{C})$ of all infinite places of the totally real field F.*

Then ${}^\sigma\Pi$ is also a regular algebraic cuspidal automorphic representation. The rationality field $\mathbb{Q}(\Pi_f)$, which is defined as the subfield of $\mathbb{C}$ fixed by $\{\sigma : {}^\sigma(\Pi_f) \simeq \Pi_f\}$, is a number field. For any field E containing $\mathbb{Q}(\Pi_f)$, the representation Π_f of $\mathrm{GL}_2(\mathbb{A}_{F,f})$ has an E-structure that is unique up to homotheties.

(Note that the above action is a left-action, i.e., ${}^{\sigma\tau}\Pi = {}^\sigma({}^\tau\Pi)$.) Suppose σ fixes Π_f for a representation Π as in the theorem above, then by the strong multiplicity one theorem, σ fixes Π; which justifies a change in notation: $\mathbb{Q}(\Pi)$ instead of $\mathbb{Q}(\Pi_f)$.

5.4.3 $\mathrm{Aut}(\mathbb{C})$-equivariance of the dictionary

Proposition 5.4.2 guarantees that $\mathbf{f}^\sigma$ is a primitive holomorphic Hilbert modular form if $\mathbf{f}$ is. Therefore, by the bijection discussed in Section 5.1 and Section 5.2, there exists a cuspidal automorphic representation $\Pi(\mathbf{f}^\sigma)$ of a certain type that corresponds to $\mathbf{f}^\sigma$. Now, the question is: how one can compare the $\mathrm{Aut}(\mathbb{C})$-action on the space of Hilbert modular forms with the $\mathrm{Aut}(\mathbb{C})$-action on the space of cuspidal automorphic representations? The obvious guess that $\Pi(\mathbf{f}^\sigma) = {}^\sigma\Pi(\mathbf{f})$ is not quite correct; indeed, ${}^\sigma\Pi(\mathbf{f})$ may not even be an automorphic representation. The following theorem answers our question.

Theorem 5.4.4 *Let $\mathbf{f} \in \mathcal{S}_k(\mathfrak{n}, \tilde{\omega})$ be primitive, with $k = (k_1, \cdots, k_n)$. Assume that $k_1 \equiv \cdots \equiv k_n \mod 2$. Then the map $\mathbf{f} \mapsto \Pi(\mathbf{f}) \otimes |\ |^{k_0/2}$ is $\mathrm{Aut}(\mathbb{C})$-equivariant, where $k_0 = \mathrm{Max}\{k_1, \cdots, k_n\}$.*

Proof. First, let us note that $\Pi(\mathbf{f})$ is algebraic if $k_0 \equiv 0 \mod 2$, and $\Pi(\mathbf{f}) \otimes |\ |^{1/2}$ is algebraic when $k_0 \equiv 1 \mod 2$; these follow easily from 4.2.1. These cases may be uniformized by considering the twist $\Pi(\mathbf{f})$ by $|\ |^{k_0/2}$ to say that $\Pi(\mathbf{f}) \otimes |\ |^{k_0/2}$ is an algebraic cuspidal automorphic representation for all k that satisfy the parity condition in the hypothesis. Further, if $k_j \geq 2$ for all j, then $\Pi(\mathbf{f}) \otimes |\ |^{k_0/2}$ is a regular algebraic cuspidal automorphic representation; this can be seen immediately from 4.4.3 after one notes that $\Pi(\mathbf{f}) \otimes |\ |^{k_0/2} \in \mathrm{Coh}(G, \mu^{\vee})$, where the weight $\mu = (\mu_1, \ldots, \mu_n)$ is given by:

$$\mu_j = \left(\frac{k_0 + k_j - 2}{2}, \frac{k_0 - k_j + 2}{2} \right). \tag{5.4.5}$$

(Let us add a comment about $k_j \geq 2$. Even in the elliptic modular case, a weight 1 form is not of motivic type; another facet of the same phenomenon is that the associated representation after twisting by $|\ |^{1/2}$ is algebraic but not regular; or that the associated L-function has no critical points.)

By Theorem 5.4.3, the representation ${}^{\sigma}(\Pi(\mathbf{f}) \otimes |\ |^{k_0/2})$ is also a regular algebraic cuspidal automorphic representation. Let us note that this representation, however, does not have an appropriate central character to apply the "dictionary." In order to modify the central character, twist it by $|\ |^{-k_0/2}$ and consider the representation $\Pi' := {}^{\sigma}(\Pi(\mathbf{f}) \otimes |\ |^{k_0/2}) \otimes |\ |^{-k_0/2}$. This representation is cuspidal and automorphic, whose local representations at infinity places are $\otimes_{\eta} \Pi(\mathbf{f})_{\sigma^{-1}\eta}$, i.e., the permutation of the discrete series representations $\{D_{k_j - 1}\}$, and such that the conductor is $\mathfrak{n}$, and that the central character is trivial on F_{∞^+}. Therefore, by Section 5.2, there is a primitive holomorphic Hilbert modular form of weight k^{σ} and level $\mathfrak{n}$. It remains to show that this cusp form is actually $\mathbf{f}^{\sigma}$, and that the central character of Π' is $\tilde{\omega}^{\sigma}$.

By Theorem 5.1.4, it is enough to show that $\Pi'_{\mathfrak{p}}$ coincides with $\Pi(\mathbf{f}^{\sigma})_{\mathfrak{p}}$ for almost all finite places $\mathfrak{p}$. In particular, let $\mathfrak{p}$ be a place of F that does not divide $\mathfrak{n}$. Then, the local representation $\Pi'_{\mathfrak{p}}$ at $\mathfrak{p}$ is a spherical representation, say induced from $\chi'_{1,\mathfrak{p}}$ and $\chi'_{2,\mathfrak{p}}$, and write it as $\Pi'_{\mathfrak{p}} = \pi\left(\chi'_{1,\mathfrak{p}}, \chi'_{2,\mathfrak{p}}\right)$. We use the following lemma to see these characters more carefully.

Lemma 5.1 (Waldspurger, [46]) *Let $\Pi = \pi(\chi_1, \chi_2)$ be a spherical representation induced from χ_1 and χ_2, then ${}^\sigma\Pi$ is also spherical, and it is induced from characters defined as $|\ |^{-1/2}\sigma(\chi_i \cdot |\ |^{1/2})$, where $i = 1,\, 2$.*

Let $\Pi(\mathbf{f})_\mathfrak{p} = \pi(\chi_{1,\mathfrak{p}},\, \chi_{2,\mathfrak{p}})$. By the lemma above, the characters $\chi'_{i,\mathfrak{p}}$ can be described as

$$\chi'_{i,\mathfrak{p}} = |\ |_\mathfrak{p}^{-\frac{k_0+1}{2}}\sigma\left(\chi_{i,\mathfrak{p}} \cdot |\ |_\mathfrak{p}^{\frac{k_0+1}{2}}\right).$$

Therefore, a direct computation shows that

$$\begin{aligned}
\left(\chi'_{1,\mathfrak{p}} + \chi'_{2,\mathfrak{p}}\right)(\varpi_\mathfrak{p}) &= q_\mathfrak{p}^{-\frac{k_0+1}{2}}\left(q_\mathfrak{p}^{\frac{k_0+1}{2}}\right)^\sigma \left(q_\mathfrak{p}^{\frac{1-k_0}{2}}\mathrm{C}(\mathfrak{p}, \mathbf{f})\right)^\sigma \\
&= q_\mathfrak{p}^{\frac{1-k_0}{2}}\mathrm{C}(\mathfrak{p}, \mathbf{f})^\sigma \\
&= q_\mathfrak{p}^{\frac{1-k_0}{2}}\mathrm{C}(\mathfrak{p}, \mathbf{f}^\sigma) \qquad \text{by Proposition 5.4.2,}
\end{aligned}$$

and

$$\chi'_{1,\mathfrak{p}} \cdot \chi'_{2,\mathfrak{p}} = \sigma(\tilde{\omega}).$$

This says that $q_\mathfrak{p}^{1/2}\left(\chi'_{1,\mathfrak{p}} + \chi'_{2,\mathfrak{p}}\right)(\varpi_\mathfrak{p})$ gives the eigenvalue for the Hecke operator $\mathbb{T}_\mathfrak{p}$ applied to $\mathbf{f}^\sigma$, which can be seen by the same computation done in (5.1.5), and that $\chi'_{1,\mathfrak{p}} \cdot \chi'_{2,\mathfrak{p}}$ is the central character $\tilde{\omega}^\sigma$ of $\Pi(\mathbf{f}^\sigma)$. This completes the proof of Theorem 5.4.4. ■

5.4.4 Rationality fields

Let $\mathbf{f}$ be a primitive form in $\mathcal{S}_k(\mathfrak{n}, \tilde{\omega})$. Define the rationality field of $\mathbf{f}$ as

$$\mathbb{Q}(\mathbf{f}) := \mathbb{Q}(\mathrm{C}(\mathfrak{m}, \mathbf{f}) : \text{for all integral ideals } \mathfrak{m}\}). \tag{5.4.6}$$

This is the field generated over $\mathbb{Q}$ by the eigenvalues of $\mathbf{f}$ for all the normalized Hecke operators $\mathbb{T}'_\mathfrak{m}$. Shimura [41, Proposition 2.8] proves that this field is a number field which is in fact generated by $\mathrm{C}(\mathfrak{p}, \mathbf{f})$ for almost all prime ideals $\mathfrak{p}$. Further, this number field is either totally real, or a totally imaginary quadratic extension of a totally real number field.

Let $\Pi = \Pi(\mathbf{f})$, and we have checked that $\Pi(\mathbf{f}) \otimes |\ |^{k_0/2}$ is a regular cuspidal automorphic representation. Define the rationality field $\mathbb{Q}(\Pi)$ to be

$$\mathbb{Q}(\Pi) := \mathbb{C}^{\{\sigma\in \mathrm{Aut}(\mathbb{C}) : {}^{\sigma}\Pi_f = \Pi_f\}}. \tag{5.4.7}$$

That is the subfield of $\mathbb{C}$ fixed by the group $\{\sigma \in \mathrm{Aut}(\mathbb{C}) : {}^{\sigma}\Pi_f = \Pi_f\}$ of all $\mathbb{C}$-automorphisms which fix Π_f. By strong multiplicity one, ${}^{\sigma}\Pi_f = \Pi_f$ if and only if ${}^{\sigma}\Pi_{\mathfrak{p}} = \Pi_{\mathfrak{p}}$ for almost all prime ideals $\mathfrak{p}$.

Using Proposition 5.4.2 it is clear that $\mathbb{Q}(\mathbf{f})$ is the subfield of $\mathbb{C}$ fixed by the group $\{\sigma \in \mathrm{Aut}(\mathbb{C}) : \sigma(\mathrm{C}(\mathfrak{m}, \mathbf{f})) = \mathrm{C}(\mathfrak{m}, \mathbf{f}))\}$. It follows that $\mathbb{Q}(\mathbf{f}) = \mathbb{Q}(\Pi(\mathbf{f}))$.

CHAPTER 6

Algebraicity Theorem

Shimura proved the following fundamental result (see [41, Theorem 4.3]) on the critical values of the standard L-function attached to a holomorphic Hilbert modular form.

Theorem 6.0.1 (Shimura) *Let* $\mathbf{f}$ *be a primitive holomorphic Hilbert modular cusp form of type* $(k, \tilde{\omega})$ *over a totally real number field* F *of degree* n*. Assume that the weight* $k = (k_1, \ldots, k_n)$ *satisfies the parity condition*

$$k_1 \equiv \cdots \equiv k_n \pmod 2.$$

Let $k^0 = \min(k_1, \ldots, k_n)$ *and* $k_0 = \max(k_1, \ldots, k_n)$*. There exist nonzero complex numbers* $u(r, \mathbf{f}^\sigma)$ *defined for* $r \in \mathbb{Z}^n/2\mathbb{Z}^n$ *and* $\sigma \in \mathrm{Aut}(\mathbb{C})$ *such that for any Hecke character* χ *of* $\mathbb{A}_F^\times$ *of finite order, for any integer* m *with*

$$(k_0 - k^0)/2 < m < (k_0 + k^0)/2,$$

and for any $\sigma \in \mathrm{Aut}(\mathbb{C})$*, we have*

$$\sigma\left(\frac{L_f(m, \mathbf{f}, \chi)}{(2\pi i)^{mn}\, \tau(\chi)\, u(\epsilon, \mathbf{f})}\right) \;=\; \frac{L_f(m, \mathbf{f}^\sigma, \chi^\sigma)}{(2\pi i)^{mn}\, \tau(\chi^\sigma)\, u(\epsilon, \mathbf{f}^\sigma)},$$

where ϵ *is prescribed by:* $\chi(a) = \mathrm{sgn}[a^\epsilon N(a)^m]$*; the quantity* $\tau(\chi)$ *is the Gauss sum attached to* χ*, and* $L_f(s, \mathbf{f}, \chi)$ *is the (finite part of the) standard* L*-function attached to* $\mathbf{f}$*, twisted by* χ*.*

The purpose of this chapter is to give another proof of the above theorem, which is rather different from Shimura's proof. However, before proceeding any further, let us mention that our proof is contained in the union of these papers: Harder [19], Hida [24]; see also Dou [10]. What is different from these papers is an organizational principle based on the period relations proved in Raghuram-Shahidi [38] while working in the context of regular algebraic cuspidal automorphic representations.

The first main theorem proved in this chapter is:

Theorem 6.0.2 (Central critical value) *Let Π be a regular algebraic cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$, for a totally real number field F of degree $n = n_F = [F : \mathbb{Q}]$. For every $\epsilon \in \{\pm\}^n$, we define $p^\epsilon(\Pi) \in \mathbb{C}^*$ which have the following property: Assume that $s = 1/2$ is critical for the standard L-function $L(s, \Pi)$ attached to Π. Then for any $\sigma \in \mathrm{Aut}(\mathbb{C})$ we have*

$$\sigma\left(\frac{L_f(\frac{1}{2}, \Pi)}{(2\pi i)^{d_\infty} p^{(+,\dots,+)}(\Pi)}\right) = \frac{L_f(\frac{1}{2}, {}^\sigma\Pi)}{(2\pi i)^{d_\infty} p^{(+,\dots,+)}({}^\sigma\Pi)},$$

where $d_\infty = d(\Pi_\infty) = d({}^\sigma\Pi_\infty)$ is an integer determined by the representation at infinity; see Proposition 6.2.16.

In particular,

$$L_f(1/2, \Pi) \sim_{\mathbb{Q}(\Pi)} (2\pi i)^{d_\infty} p^{(+,\dots,+)}(\Pi),$$

where, by $\sim_{\mathbb{Q}(\Pi)}$, we mean up to an element of the number field $\mathbb{Q}(\Pi)$.

The following result on all critical values for twisted L-functions follows from the period relations proved in [38].

Corollary 6.0.3 (All critical values) *Let Π be a regular algebraic cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$, for a totally real number field F of degree $n = n_F = [F : \mathbb{Q}]$. Let $\eta_1, \dots, \eta_n$ be all the infinite places of F. Assume that $s = \frac{1}{2} + m \in \frac{1}{2} + \mathbb{Z}$ is critical for the standard L-function $L(s, \Pi)$ attached to Π. Then, for any finite order character χ of $F^\times \backslash \mathbb{A}_F^\times$, and for any $\sigma \in \mathrm{Aut}(\mathbb{C})$ we have*

$$\sigma\left(\frac{L_f(\frac{1}{2}+m, \Pi \otimes \chi)}{(2\pi i)^{d_\infty + nm}\, p^{((-1)^m \epsilon_\chi)}(\Pi)\, \mathcal{G}(\chi)}\right) = \frac{L_f(\frac{1}{2}+m, {}^\sigma\Pi \otimes {}^\sigma\chi)}{(2\pi i)^{d_\infty + nm}\, p^{((-1)^m \epsilon_{\sigma_\chi})}({}^\sigma\Pi)\, \mathcal{G}({}^\sigma\chi)},$$

where $\epsilon_\chi = (\chi_{\eta_1}(-1), \ldots, \chi_{\eta_n}(-1))$ is the 'parity' of χ determined completely by $\chi_\infty = \otimes_{j=1}^n \chi_{\eta_j}$; and $\mathcal{G}(\chi)$ is the Gauss sum of χ.

In particular,

$$L_f(1/2 + m, \Pi \otimes \chi) \ \sim_{\mathbb{Q}(\Pi,\chi)} \ (2\pi i)^{d_\infty + nm} p^{((-1)^m \epsilon_\chi)}(\Pi)\mathcal{G}(\chi),$$

where, by $\sim_{\mathbb{Q}(\Pi,\chi)}$, we mean up to an element of the compositum of $\mathbb{Q}(\Pi)$ and $\mathbb{Q}(\chi)$.

To see that the corollary exactly corresponds to Shimura's Theorem 6.0.1 above, we need to know some arithmetic properties of the dictionary between primitive holomorphic Hilbert modular forms for F and regular algebraic cuspidal automorphic representations of GL_2 over F. The main statements are summarized in the theorem below.

Theorem 6.0.4 *Let $\mathbf{f}$ and $\Pi(\mathbf{f})$ be such as described in Theorem 5.0.1. Then it has the following properties.*

1. *(L-functions) For any finite order character χ of $F^\times \backslash \mathbb{A}_F^\times$ we have an equality of (completed) L-functions:*

$$L(s, \Pi(\mathbf{f}) \otimes \chi) = L(s + \frac{k_0 - 1}{2}, \mathbf{f}, \chi),$$

where the left hand side is the standard L-function defined as in Jacquet and Langlands [25], and the right hand side is defined via a Dirichlet series as in Shimura [41].

2. *(Algebraicity)*

   (a) *if $k_1 \equiv \cdots \equiv k_n \equiv 0 \pmod 2$ then $\Pi(\mathbf{f})$ is algebraic;*

   (b) *if $k_1 \equiv \cdots \equiv k_n \equiv 1 \pmod 2$ then $\Pi(\mathbf{f}) \otimes |\ |^{1/2}$ is algebraic;*

   (c) *if $k_i \not\equiv k_j \pmod 2$ for some i and j then no twist of $\Pi(\mathbf{f})$ is algebraic.*

Note that (a), (b) and (c) can all be put-together as

$$k_1 \equiv \cdots \equiv k_n \pmod 2 \iff \Pi(\mathbf{f}) \otimes |\ |^{k_0/2} \textit{is algebraic.}$$

3. *(Regularity) Suppose now that* $k_1 \equiv \cdots \equiv k_n \pmod 2$. *Then* $\Pi(\mathbf{f}) \otimes |\ |^{k_0/2}$ *is regular exactly when each* $k_j \geq 2$.

4. *(Galois equivariance) Let* $k_1 \equiv \cdots \equiv k_n \pmod 2$ *with* $k_j \geq 2$ *for all* j. *Then, for any* $\sigma \in \mathrm{Aut}(\mathbb{C})$ *we have:*

$$^{\sigma}(\Pi(\mathbf{f}) \otimes |\ |^{k_0/2}) = \Pi(\mathbf{f}^{\sigma}) \otimes |\ |^{k_0/2},$$

where the action of σ *on representations is as in Clozel [7] or Waldspurger [46], and on Hilbert modular forms is as in Shimura [41].*

5. *(Rationality field) Let* $\mathbb{Q}(\mathbf{f})$ *be the field generated by the Fourier coefficients of* $\mathbf{f}$, *and let* $\mathbb{Q}(\Pi(\mathbf{f}))$ *be the subfield of complex numbers fixed by the set of all* $\sigma \in \mathrm{Aut}(\mathbb{C})$ *such that* $^{\sigma}\Pi(\mathbf{f})_v = \Pi(\mathbf{f})_v$ *for all finite places* v. *Then* $\mathbb{Q}(\mathbf{f}) = \mathbb{Q}(\Pi(\mathbf{f}))$.

The first property was introduced in Section 5.3.3 for the case that χ is trivial, and the properties 2, 3, and 4 are contained in Theorem 5.4.4. Recalling arithmetic issues of regular algebraic cuspidal automorphic representations discussed in 4.4, we start Section 6.1 by providing a summary of the definition of certain periods which arise via a comparison of a rational structure on a Whittaker model of Π with a rational structure on a cohomological realization of Π. We also record certain relations amongst these periods as in Raghuram-Shahidi [38]. Section 6.2 is devoted to complete the proofs of Theorem 6.0.2 and Corollary 6.0.3.

6.1 Periods and period relations

6.1.1 Periods

We now look closely at the assertion that Π_f has an E-structure. On the one hand, a cuspidal automorphic representation Π of $\mathrm{GL}_n(\mathbb{A}_F)$ admits a Whittaker model, and these models carry a natural rational structure. On the other hand, if Π is regular and algebraic, then it contributes to cuspidal cohomology and from this arises a rational structure on a cohomological realization of Π. One defines periods by playing off these rational structures against each other. (Another word for these 'periods' might be 'regulators', as the definition our periods is very close in spirit to Borel's regulators [2].) The rest of 6.1 is a very brief summary of Raghuram-Shahidi [38].

As a matter of definition/notation, given a $\mathbb{C}$-vector space V, and given a subfield $E \subset \mathbb{C}$, by an E-structure on V we mean an E-subspace V_E such that the canonical map $V_E \otimes_E \mathbb{C} \to V$ is an isomorphism. Further, if V is a representation space for the action of a group G, then we will need V_E to be G-stable. Fixing an E-structure gives an action of $\mathrm{Aut}(\mathbb{C}/E)$ on V, by making it act on the second factor in $V = V_E \otimes_E \mathbb{C}$. Having fixed an E-structure, for any extension E'/E, we have a canonical E'-structure by letting $V_{E'} = V_E \otimes_E E'$.

6.1.2 Rational structures on Whittaker models

Recall from 2.8 that we have fixed a nontrivial character $\psi = \psi_\infty \otimes \psi_f$ of $F\backslash\mathbb{A}_F$. Let $\mathcal{W}(\Pi, \psi)$ be the Whittaker model of Π, and this factors as $\mathcal{W}(\Pi, \psi) = \mathcal{W}(\Pi_\infty, \psi_\infty) \otimes \mathcal{W}(\Pi_f, \psi_f)$. There is a semilinear action of $\mathrm{Aut}(\mathbb{C})$ on $\mathcal{W}(\Pi_f, \psi_f)$ which is defined as follows. (See Harder [19, pp.79-80].) Consider:

$$\begin{array}{ccccccccc}
\mathrm{Aut}(\mathbb{C}/\mathbb{Q}) & \to & \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) & \to & \mathrm{Gal}(\mathbb{Q}(\mu_\infty)/\mathbb{Q}) & \to & \widehat{\mathbb{Z}}^\times \simeq \prod_p \mathbb{Z}_p^\times & \subset & \prod_p \prod_{\mathfrak{p}|p} \mathcal{O}_\mathfrak{p}^\times \\
\sigma & \mapsto & \sigma|_{\overline{\mathbb{Q}}} & \mapsto & \sigma|_{\mathbb{Q}(\mu_\infty)} & \mapsto & t_\sigma & \mapsto & t_\sigma = (t_{\sigma,\mathfrak{p}})_\mathfrak{p}
\end{array}$$

where the last inclusion is the one induced by the diagonal embedding of $\mathbb{Z}_p^\times$ into $\prod_{\mathfrak{p}|p} \mathcal{O}_\mathfrak{p}^\times$. The element t_σ at the end can be thought of as an element of $\mathbb{A}_{F,f}^\times$. Let $[t_\sigma^{-1}]$ denote the diagonal matrix $\mathrm{diag}(t_\sigma^{-1}, 1)$ regarded as an element of $\mathrm{GL}_2(\mathbb{A}_{F,f})$. For $\sigma \in \mathrm{Aut}(\mathbb{C})$ and $W \in \mathcal{W}(\Pi_f, \psi_f)$, define the function ${}^\sigma W$ by

$${}^\sigma W(g_f) = \sigma(W([t_\sigma^{-1}]g_f))$$

for all $g_f \in \mathrm{GL}_2(\mathbb{A}_{F,f})$. Note that this action makes sense locally, by replacing t_σ by $t_{\sigma,\mathfrak{p}}$. Further, if $\Pi_{\mathfrak{p}}$ is unramified, then a spherical vector is mapped to a spherical vector under σ. If we normalize the spherical vector to take the value 1 on the identity, then σ fixes this vector. This makes the local and global actions of σ compatible.

Lemma 6.1 *With notation as above, $W \mapsto {}^\sigma W$ is a σ-linear $\mathrm{GL}_2(\mathbb{A}_{F,f})$-equivariant isomorphism from $\mathcal{W}(\Pi_f, \psi_f)$ onto $W({}^\sigma\Pi_f, \psi_f)$. For any finite extension $E/\mathbb{Q}(\Pi_f)$ we have an E-structure on $\mathcal{W}(\Pi_f, \psi_f)$ by taking invariants:*

$$\mathcal{W}(\Pi_f, \psi_f)_E = \mathcal{W}(\Pi_f, \psi_f)^{\mathrm{Aut}(\mathbb{C}/E)}.$$

Proof. See Raghuram-Shahidi [38, Lemma 3.2]; it amounts to saying that a normalized new-vector generates the E-structure obtained by taking invariants under $\mathrm{Aut}(\mathbb{C}/E)$. (It helps to keep Waldspruger's [46, Lemme I.1] in mind.) Later, we will work with some carefully normalized new-vectors; see 6.2.1 below. ∎

As a notational convenience, when we talk of Whittaker models, we will henceforth suppress the additive character ψ, since that has been fixed once and for all; for example, $\mathcal{W}(\Pi_f)$ will denote $\mathcal{W}(\Pi_f, \psi_f)$. Next, $\mathcal{W}(\Pi_f)_0$ will denote the $\mathbb{Q}(\Pi)$-rational structure on $\mathcal{W}(\Pi_f)$.

6.1.3 Rational structures on cohomological representations

Let $\mu \in X_0^+(T)$ and $\Pi \in \mathrm{Coh}(G, \mu^{\mathsf{v}})$. For any character ϵ of $\pi_0(G_\infty)$, the cohomology space $H^n(\mathfrak{g}_\infty, \mathrm{K}_\infty^\circ; V_\Pi \otimes M_{\mu^{\mathsf{v}}})(\epsilon)$, which as a representation of the group $\pi_0(G_\infty) \times G(\mathbb{A}_{F,f})$ is isomorphic to $\epsilon \otimes \Pi_f$, has a natural $\mathbb{Q}(\Pi)$-structure which may be seen as follows. Consider the following diagram:

$$\begin{array}{ccccc}
H^n(\mathfrak{g}_\infty, \mathrm{K}_\infty^\circ; V_\Pi \otimes M_{\mu^\vee})(\epsilon) & \simeq & \epsilon \otimes \Pi_f & & \\
\downarrow & & \downarrow & & \\
H^n(\mathfrak{g}_\infty, \mathrm{K}_\infty^\circ; \mathcal{A}_{\mathrm{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A})) \otimes M_{\mu^\vee}) & \simeq & H^n_{\mathrm{cusp}}(S^G, \widetilde{M_{\mu^\vee}}) & & \\
\downarrow & & \downarrow & & \\
H^n(\mathfrak{g}_\infty, \mathrm{K}_\infty^\circ; C^\infty(G(\mathbb{Q})\backslash G(\mathbb{A})) \otimes M_{\mu^\vee}) & \simeq & H^n_{dR}(S^G, \widetilde{M_{\mu^\vee}}) & \simeq & H^n_B(S^G, \widetilde{M_{\mu^\vee}})
\end{array}$$

where all the vertical arrows are injections induced by inclusions. Indeed, the rational structures on all the above spaces come from a rational structure on the Betti cohomology space on which it is easy to describe an action of $\mathrm{Aut}(\mathbb{C})$–see Clozel [7]. The point is that cuspidal cohomology admits a $\mathbb{Q}(\mu)$-structure which it inherits from 'the' canonical $\mathbb{Q}(\mu)$-structure on Betti cohomology $H^n_B(S^G, \widetilde{M_{\mu^\vee}})$. (By $\mathbb{Q}(\mu)$ we mean the subfield of $\mathbb{C}$ fixed by $\{\sigma : {}^\sigma\mu = \mu\}$, where the action of σ on μ, or any quantity indexed by the infinite places S_∞, is via permuting these places, exactly as the action of $\mathrm{Aut}(\mathbb{C})$ on Π_∞ described in Theorem 5.4.3.) Since $\epsilon \otimes \Pi_f \simeq H^n(\mathfrak{g}_\infty, \mathrm{K}_\infty^\circ; V_\Pi \otimes M_{\mu^\vee})(\epsilon)$ is a Hecke eigenspace (i.e., is an irreducible subspace for the action of $\pi_0(G_\infty) \times G(\mathbb{A}_{F,f})$) of cuspidal cohomology, it follows that this eigenspace admits a $\mathbb{Q}(\Pi)$-rational structure.

6.1.4 Comparing Whittaker models and cohomological representations

We have the following comparison isomorphism $\mathcal{F}_\Pi^\epsilon$, which is the composition of three isomorphisms:

$$\begin{aligned}
\mathcal{W}(\Pi_f) &\longrightarrow \mathcal{W}(\Pi_f) \otimes H^n(\mathfrak{g}_\infty, \mathrm{K}_\infty^\circ; \mathcal{W}(\Pi_\infty) \otimes M_{\mu^\vee})(\epsilon) \\
&\longrightarrow H^n(\mathfrak{g}_\infty, \mathrm{K}_\infty^\circ; \mathcal{W}(\Pi) \otimes M_{\mu^\vee})(\epsilon) \\
&\longrightarrow H^n(\mathfrak{g}_\infty, \mathrm{K}_\infty^\circ; V_\Pi \otimes M_{\mu^\vee})(\epsilon),
\end{aligned}$$

where the first map is $W_f \mapsto W_f \otimes [\Pi_\infty]^\epsilon$ for all $W_f \in \mathcal{W}(\Pi_f)$ with $[\Pi_\infty]^\epsilon$ being the generator (as in 4.4.5) of the one-dimensional space $H^n(\mathfrak{g}_\infty, \mathrm{K}_\infty^\circ; \mathcal{W}(\Pi_\infty) \otimes M_{\mu^\vee})(\epsilon)$;

the second map is the obvious one; and the third map is the map induced in cohomology by the inverse of the map which gives the Fourier coefficient of a cusp form in V_Π–the space of functions in $\mathcal{A}_{\rm cusp}(G(\mathbb{Q})\backslash G(\mathbb{A}))$ which realizes Π.

6.1.5 Definition of the periods

The isomorphism $\mathcal{F}_\Pi^\epsilon$ need not preserve rational structures on either side. Each side is an irreducible representation space for the action of $\pi_0(G_\infty) \times G(\mathbb{A}_{F,f})$ and rational structures being unique up to homotheties (by Waldspurger [46, Lemme I.1]), we see that we can adjust the isomorphism $\mathcal{F}_\Pi^\epsilon$ by a scalar–which is the period–such that the adjusted map preserves rational structures. Let us state this more precisely:

Let $\Pi = \Pi_f \otimes \Pi_\infty$ be a regular algebraic cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$. Let $\mu \in X_0^+(T)$ be such that $\Pi \in \mathrm{Coh}(G, \mu^\vee)$. Let ϵ be a character of $\mathrm{K}_\infty/\mathrm{K}_\infty^\circ$. Let $[\Pi_\infty]^\epsilon$ be a generator of the one dimensional vector space $H^n(\mathfrak{g}_\infty, \mathrm{K}_\infty^\circ, \Pi_\infty \otimes M_{\mu^\vee})(\epsilon)$. To such a datum $(\Pi_f, \epsilon, [\Pi_\infty]^\epsilon)$, there is a nonzero complex number $p^\epsilon(\Pi)$, such that the normalized map

$$\mathcal{F}_{\Pi,0}^\epsilon := p^\epsilon(\Pi)^{-1}\mathcal{F}_\Pi^\epsilon$$

is $\mathrm{Aut}(\mathbb{C})$-equivariant, i.e., the following diagram commutes:

$$\begin{tikzcd}
\mathcal{W}(\Pi_f) \arrow[r, "\mathcal{F}_{\Pi,0}^\epsilon"] \arrow[d, "\sigma"'] & H^n(\mathfrak{g}_\infty, \mathrm{K}_\infty^\circ; V_\Pi \otimes M_{\mu^\vee})(\epsilon) \arrow[d, "\sigma"] \\
\mathcal{W}({}^\sigma\Pi_f) \arrow[r, "\mathcal{F}_{{}^\sigma\Pi,0}^{{}^\sigma\epsilon}"] & H^n(\mathfrak{g}_\infty, \mathrm{K}_\infty^\circ; V_{{}^\sigma\Pi} \otimes E_{{}^\sigma\mu^\vee})({}^\sigma\epsilon)
\end{tikzcd}$$

The complex number $p^\epsilon(\Pi)$, called a period, is well-defined only up to multiplication by elements of $\mathbb{Q}(\Pi)^*$. if we change $p^\epsilon(\Pi)$ to $\alpha p^\epsilon(\Pi)$ with a $\alpha \in \mathbb{Q}(\Pi)^*$ then the period $p^{{}^\sigma\epsilon}({}^\sigma\Pi)$ changes to $\sigma(\alpha)p^{{}^\sigma\epsilon}({}^\sigma\Pi)$.

In terms of the un-normalized maps, we can describe the above commutative diagram by

$$\sigma \circ \mathcal{F}_\Pi^\epsilon = \left(\frac{\sigma(p^\epsilon(\Pi))}{p^{{}^\sigma\epsilon}({}^\sigma\Pi)}\right) \mathcal{F}_{{}^\sigma\Pi}^{{}^\sigma\epsilon} \circ \sigma. \tag{6.1.1}$$

6.1.6 Period relations

The following is the main result proved in Raghuram-Shahidi [38], but stated below for our context of GL_2 over a totally real F. Let $\mu \in X_0^+(T)$ be such that $\Pi \in \mathrm{Coh}(G, \mu^\vee)$. Let ϵ be a character of $\mathrm{K}_\infty/\mathrm{K}_\infty^\circ$. Let ξ be an algebraic Hecke character of F with signature ϵ_ξ which is defined as follows: any such ξ is of the form $\xi = |\ |^m \otimes \xi^0$ for an integer m, and a finite order character ξ^0, then

$$\epsilon_\xi = (-1)^m(\xi^0_{\eta_1}(-1), \ldots, \xi^0_{\eta_n}(-1)).$$

For any $\sigma \in \mathrm{Aut}(\mathbb{C})$ we have

$$\sigma\left(\frac{p^{\epsilon\cdot\epsilon_\xi}(\Pi\otimes\xi)}{\mathcal{G}(\xi)\,p^\epsilon(\Pi)}\right) = \left(\frac{p^{{}^\sigma\epsilon\cdot\epsilon_{{}^\sigma\xi}}({}^\sigma\Pi\otimes{}^\sigma\xi)}{\mathcal{G}({}^\sigma\xi)\,p^{{}^\sigma\epsilon}({}^\sigma\Pi)}\right).$$

The action of $\mathrm{Aut}(\mathbb{C})$ on ϵ is via permuting the infinite places. Define $\mathbb{Q}(\xi)$ as the field obtained by adjoining the values of ξ^0, and let $\mathbb{Q}(\Pi, \xi)$ be the compositum of the number fields $\mathbb{Q}(\Pi)$ and $\mathbb{Q}(\xi)$. We have

$$p^{\epsilon\cdot\epsilon_\xi}(\Pi\otimes\xi) \ \sim_{\mathbb{Q}(\Pi,\xi)}\ \mathcal{G}(\xi)\,p^\epsilon(\Pi).$$

By $\sim_{\mathbb{Q}(\Pi,\xi)}$ we mean up to an element of $\mathbb{Q}(\Pi, \xi)$.

6.2 Proof of Theorem 6.0.2

We finally complete the proofs of Theorem 6.0.2 and Corollary 6.0.3. The proof is rather technical, but the idea is based on a cohomological interpretation of the classical Mellin transform. We start the section by providing a diagram that summarizes the proof.

$$\begin{tikzcd}
\mathrm{H}^n(\mathfrak{g}_\infty, \mathrm{K}_\infty^\circ : V_\Pi \otimes M_{\mu^\vee}) \arrow[r, hook] & \mathrm{H}_c^n(\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A}_F)/\mathrm{K}_\infty^\circ \mathrm{K}_f, \widetilde{M_{\mu^\vee}}) \arrow[d, "\iota^*"] \\
\mathcal{W}(\Pi_f) \arrow[u, "\mathcal{F}_\Pi^{++}"] \arrow[dd] & \mathrm{H}_c^n(F^\times \backslash \mathbb{A}_F^\times/\iota^*\mathrm{K}_f, \iota^*\widetilde{M_{\mu^\vee}}) \arrow[d, "\mathcal{T}^*"] \\
 & \mathrm{H}_c^n(F^\times \backslash \mathbb{A}_F^\times/\iota^*\mathrm{K}_f, \mathbb{C}) \arrow[d, "\int_M"] \\
\mathbb{C} & \oplus_{\nu=1}^h \mathbb{C} \arrow[l]
\end{tikzcd} \tag{6.2.1}$$

A normalized new vector W_Π° in the finite part of Whittaker model $\mathcal{W}(\Pi_f)$ is mapped to a cohomology class ϑ_Π in the relative Lie algebra cohomology $\mathrm{H}^n(\mathfrak{g}_\infty, \mathrm{K}_\infty^\circ : V_\Pi \otimes M_{\mu^\vee})$ via a map $\mathcal{F}_\Pi^{++}$. This map is normalized by the period $p^{(++)}(\Pi)$ so that it preserves rational structures on each side. (See Section 6.1.) The cohomology class ϑ_Π, viewed as an element in the compactly supported cohomology space $\mathrm{H}_c^n(\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A}_F)/\mathrm{K}_\infty^\circ \mathrm{K}_f, \widetilde{M_{\mu^\vee}})$, can be mapped to a top-degree compactly supported cohomology class for GL_1 with the constant coefficients by the map $\mathcal{T}^*\iota^*$ shown in the diagram above. Applying the Poincaré duality, which maps $\mathcal{T}^*\iota^*\vartheta_\Pi$ to a complex number by integrating it over the entire manifold $\mathcal{M} = F^\times \backslash \mathbb{A}_F^\times/\iota^*\mathrm{K}_f$, we have

$$\int_{\mathcal{M}} \mathcal{T}^*\iota^*\vartheta_\Pi = \frac{c \cdot L(1/2, \Pi)}{(2\pi i)^{d_\infty} p^{(++)}(\Pi)}, \tag{6.2.2}$$

where c is an integer. Theorem 6.0.2 follows from this equation and the rationality of each map used in the diagram. Each step is discussed in detail.

6.2.1 Normalized new vectors

We now show that local new-vectors when normalized appropriately give a very explicit element in the rational structure $\mathcal{W}(\Pi_f)_0$ of the (finite part of the) global Whittaker model. The new-vectors will be denoted as $W_\mathfrak{p}^\circ$ after a suitable normalization. We should be aware that the normalization is slightly different from what is done in Section 5.2.

Recall from 2.8 our choice of additive character ψ. Pick an element $\mathsf{d}_F \in \mathcal{O}_F$ such that $\mathrm{ord}_{\mathfrak{p}}(\mathsf{d}_F) = r_{\mathfrak{p}} = \mathrm{ord}(\mathfrak{D}_F)$; this is possible by strong approximation. Now define a character ψ' by $\psi'(x) = \psi(\mathsf{d}_F^{-1}x)$; then it is trivially checked that $\psi'_{\mathfrak{p}}$ has conductor $\mathcal{O}_{\mathfrak{p}}$ for all prime ideals $\mathfrak{p}$. We have a map $\mathcal{W}(\Pi_f, \psi'_f) \to \mathcal{W}(\Pi_f, \psi_f)$, given by $W'_f \mapsto W_f$ where

$$W_f(g) = W'_f\left(\begin{pmatrix} \mathsf{d}_F & 0 \\ 0 & 1 \end{pmatrix} g\right).$$

This also makes sense locally: $W_{\mathfrak{p}}(g_{\mathfrak{p}}) = W'_{\mathfrak{p}}\left(\left(\begin{smallmatrix} \mathsf{d}_{\mathfrak{p}} & 0 \\ 0 & 1 \end{smallmatrix}\right) g_{\mathfrak{p}}\right)$, where, by $\mathsf{d}_{\mathfrak{p}}$, we mean d_F as an element of $F_{\mathfrak{p}}$.

A Whittaker vector $W'_{\mathfrak{p}}$ is completely determined by the function on $F_{\mathfrak{p}}^*$

$$x_{\mathfrak{p}} \mapsto \phi'_{\mathfrak{p}}(x_{\mathfrak{p}}) := W'_{\mathfrak{p}}\left(\left(\begin{pmatrix} x_{\mathfrak{p}} & 0 \\ 0 & 1 \end{pmatrix}\right)\right),$$

i.e., the map $W'_{\mathfrak{p}} \mapsto \phi'_{\mathfrak{p}}$ is injective. (See, for example, Godement [16, Lemma 3 on p.1.5].) The set of all such functions $\kappa'_{\mathfrak{p}}$ is the Kirillov model $\mathcal{K}(\Pi_{\mathfrak{p}}, \psi'_{\mathfrak{p}})$. Working in the Kirillov model $\mathcal{K}(\Pi_{\mathfrak{p}}, \psi'_{\mathfrak{p}})$, we have the following explicit formulae for new-vectors taken from Schmidt [40, p.141]. For each representation $\Pi_{\mathfrak{p}}$ we have a very special vector $\kappa_{\mathfrak{p}}^{\mathrm{new}} \in \mathcal{K}(\Pi_{\mathfrak{p}}, \psi'_{\mathfrak{p}})$ that is the local new-vector in that model. Since the table consists of purely local information, we will abuse our notation by dropping the subscript $\mathfrak{p}$.

1. Principal series representation $\pi(\chi_1, \chi_2)$, with χ_1, χ_2 unramified, and $\chi_1\chi_2^{-1} \neq |\ |^{\pm 1}$. Then

$$\kappa_{\mathfrak{p}}^{\mathrm{new}}(x) = |x|^{1/2}\left(\sum_{k+l=v(x)} \chi_1(\varpi^k)\chi_2(\varpi^l)\right)\mathbf{1}_{\mathcal{O}}(x).$$

2. Principal series representation $\pi(\chi_1, \chi_2)$ with exactly one of the characters being unramified; say χ_1 unramified and χ_2 ramified. Then

$$\kappa_{\mathfrak{p}}^{\mathrm{new}}(x) = |x|^{1/2}\chi_1(x)\mathbf{1}_{\mathcal{O}}(x).$$

3. Unramified twist of the Steinberg representation: St $\otimes \chi$ with χ unramified. Then

$$\kappa_{\mathfrak{p}}^{\mathrm{new}}(x) = |x|\chi(x)\mathbf{1}_{\mathcal{O}}(x).$$

4. In all other cases (principal series $\pi(\chi_1, \chi_2)$ with both χ_1, χ_2 ramified; ramified twist of the Steinberg representation; any supercuspidal representation) we have

$$\kappa_{\mathfrak{p}}^{\mathrm{new}}(x) = \mathbf{1}_{\mathcal{O}^\times}(x)$$

Let $W_{\mathfrak{p}}^{\mathrm{new}} \in \mathcal{W}(\Pi_{\mathfrak{p}}, \psi'_{\mathfrak{p}})$ correspond to $\kappa_{\mathfrak{p}}^{\mathrm{new}}$, and finally we let $W_{\mathfrak{p}}^{\circ} \in \mathcal{W}(\Pi_{\mathfrak{p}}, \psi_{\mathfrak{p}})$ correspond to $W_{\mathfrak{p}}^{\mathrm{new}}$. That is we have:

$$W_{\mathfrak{p}}^{\circ} \ \leftrightarrow \ W_{\mathfrak{p}}^{\mathrm{new}} \ \leftrightarrow \ \kappa_{\mathfrak{p}}^{\mathrm{new}}. \tag{6.2.3}$$

We will also denote $W_{\mathfrak{p}}^{\circ}$ by $W_{\Pi_{\mathfrak{p}}}^{\circ}$, and observe that

$$W_{\Pi_{\mathfrak{p}}}^{\circ}\left(\begin{pmatrix} x_{\mathfrak{p}} & 0 \\ 0 & 1 \end{pmatrix}\right) = W_{\Pi_{\mathfrak{p}}}^{\mathrm{new}}\left(\begin{pmatrix} \mathsf{d}_{\mathfrak{p}} x_{\mathfrak{p}} & 0 \\ 0 & 1 \end{pmatrix}\right) = \kappa_{\mathfrak{p}}^{\mathrm{new}}(\mathsf{d}_{\mathfrak{p}} x_{\mathfrak{p}}).$$

Proposition 6.2.4 *Let* Π *be a cuspidal automorphic representation of* $\mathrm{GL}_2(\mathbb{A}_F)$*. For each prime ideal* $\mathfrak{p}$*, let* $W_{\Pi_{\mathfrak{p}}}^{\circ}$ *be the normalized new-vector as defined in (6.2.3) of the representation* $\Pi_{\mathfrak{p}}$ *which is realized in its Whittaker model* $\mathcal{W}(\Pi_{\mathfrak{p}}, \psi_{\mathfrak{p}})$*. For any* $\sigma \in \mathrm{Aut}(\mathbb{C})$ *we have*

$${}^{\sigma}W_{\Pi_{\mathfrak{p}}}^{\circ} = W_{{}^{\sigma}\Pi_{\mathfrak{p}}}^{\circ}.$$

Let $W^{\circ} = \otimes_{\mathfrak{p}} W_{\Pi_{\mathfrak{p}}}^{\circ} \in \mathcal{W}(\Pi_f, \psi_f)$*. Then* W° *is fixed by* $\mathrm{Aut}(\mathbb{C}/\mathbb{Q}(\Pi))$*, and hence* $W^{\circ} \in \mathcal{W}(\Pi_f, \psi_f)_0$*.*

Proof. To see ${}^{\sigma}W_{\Pi_{\mathfrak{p}}}^{\circ} = W_{{}^{\sigma}\Pi_{\mathfrak{p}}}^{\circ}$, it suffices to check that both Whittaker vectors give the same vector in the Kirillov model; which is then verified using a case-by-case analysis using the above table.

Suppose $\Pi_{\mathfrak{p}}$ is an unramified principal series representation, and let us write $\Pi_{\mathfrak{p}} = \pi(\chi_{1,\mathfrak{p}}, \chi_{2,\mathfrak{p}})$ for characters $\chi_{j,\mathfrak{p}} : F_{\mathfrak{p}}^* \to \mathbb{C}^*$. Let us describe ${}^{\sigma}\Pi_{\mathfrak{p}}$. For this, given any character χ of F^*, and any $\sigma \in \mathrm{Aut}(\mathbb{C})$, define ${}^\sigma\chi$ as $\sigma \circ \chi$, i.e., ${}^\sigma\chi(x) = \sigma(\chi(x))$. Define a twisted action of $\sigma \in \mathrm{Aut}(\mathbb{C})$ on characters by:

$$ {}^{\sigma'}\chi(x) = |x|^{-1/2}\sigma(\chi(x)|x|^{1/2}). $$

As is checked in Waldspurger [46, I.2], we have

$$ {}^{\sigma}\pi(\chi_{1,\mathfrak{p}}, \chi_{2,\mathfrak{p}}) = \pi({}^{\sigma'}\chi_{1,\mathfrak{p}}, {}^{\sigma'}\chi_{2,\mathfrak{p}}). $$

On the one hand, using the formula for $\kappa_{\mathfrak{p}}^{\mathrm{new}}$ for $\Pi_{\mathfrak{p}} = \pi(\chi_{1,\mathfrak{p}}, \chi_{2,\mathfrak{p}})$ we have:

$$
\begin{aligned}
{}^{\sigma}W^{\circ}_{\Pi_{\mathfrak{p}}}\left(\begin{pmatrix} x_{\mathfrak{p}} & 0 \\ 0 & 1 \end{pmatrix}\right) &= \sigma\left(W^{\circ}_{\Pi_{\mathfrak{p}}}\left(\begin{pmatrix} t_{\sigma,\mathfrak{p}}^{-1}x_{\mathfrak{p}} & 0 \\ 0 & 1 \end{pmatrix}\right)\right) \\
&= \sigma\left(W^{\mathrm{new}}_{\Pi_{\mathfrak{p}}}\left(\begin{pmatrix} \mathsf{d}_{\mathfrak{p}}t_{\sigma,\mathfrak{p}}^{-1}x_{\mathfrak{p}} & 0 \\ 0 & 1 \end{pmatrix}\right)\right) \\
&= \sigma\left(|\mathsf{d}_{\mathfrak{p}}t_{\sigma,\mathfrak{p}}^{-1}x_{\mathfrak{p}}|^{1/2}\left(\sum_{k+l=r_{\mathfrak{p}}+v(x)} \chi_{1,\mathfrak{p}}(\varpi^k)\chi_{2,\mathfrak{p}}(\varpi^l)\right)\mathbf{1}_{\mathcal{O}_{\mathfrak{p}}}(\mathsf{d}_{\mathfrak{p}}t_{\sigma,\mathfrak{p}}^{-1}x_{\mathfrak{p}})\right) \\
&= \sigma(|\mathsf{d}_{\mathfrak{p}}x_{\mathfrak{p}}|^{1/2})\left(\sum_{k+l=r_{\mathfrak{p}}+v(x)} \sigma\left(\chi_{1,\mathfrak{p}}(\varpi^k)\chi_{2,\mathfrak{p}}(\varpi^l)\right)\right)\mathbf{1}_{\mathcal{O}_{\mathfrak{p}}}(\mathsf{d}_{\mathfrak{p}}x_{\mathfrak{p}})
\end{aligned}
$$

since $t_{\sigma,\mathfrak{p}} \in \mathcal{O}_{\mathfrak{p}}^{\times}$. On the other hand, using the same formula for $\kappa_{\mathfrak{p}}^{\mathrm{new}}$, but now for the representation ${}^{\sigma}\Pi_{\mathfrak{p}} = \pi({}^{\sigma'}\chi_{1,\mathfrak{p}}, {}^{\sigma'}\chi_{2,\mathfrak{p}})$ we have:

$$
\begin{aligned}
W^{\circ}_{{}^{\sigma}\Pi_{\mathfrak{p}}}\left(\begin{pmatrix} x_{\mathfrak{p}} & 0 \\ 0 & 1 \end{pmatrix}\right) &= W^{\mathrm{new}}_{{}^{\sigma}\Pi_{\mathfrak{p}}}\left(\begin{pmatrix} \mathsf{d}_{\mathfrak{p}}x_{\mathfrak{p}} & 0 \\ 0 & 1 \end{pmatrix}\right) \\
&= |\mathsf{d}_{\mathfrak{p}}x_{\mathfrak{p}}|^{1/2}\left(\sum_{k+l=r_{\mathfrak{p}}+v(x)} {}^{\sigma'}\chi_{1,\mathfrak{p}}(\varpi^k){}^{\sigma'}\chi_{2,\mathfrak{p}}(\varpi^l)\right)\mathbf{1}_{\mathcal{O}_{\mathfrak{p}}}(\mathsf{d}_{\mathfrak{p}}x_{\mathfrak{p}}) \\
&= |\mathsf{d}_{\mathfrak{p}}x_{\mathfrak{p}}|^{1/2}|\varpi_{\mathfrak{p}}|^{-\frac{r+v(x)}{2}} \\
&\quad \left(\sum_{k+l=r_{\mathfrak{p}}+v(x)} \sigma\left(\chi_{1,\mathfrak{p}}(\varpi^k)\chi_{2,\mathfrak{p}}(\varpi^l)|\varpi_{\mathfrak{p}}|^{\frac{r+v(x)}{2}}\right)\right)\mathbf{1}_{\mathcal{O}_{\mathfrak{p}}}(\mathsf{d}_{\mathfrak{p}}x_{\mathfrak{p}})
\end{aligned}
$$

Using $|\varpi_{\mathfrak{p}}|^{\frac{r+v(x)}{2}} = |\mathsf{d}_{\mathfrak{p}}x_{\mathfrak{p}}|^{1/2}$ the final expression also simplifies to

$$ \sigma(|\mathsf{d}_{\mathfrak{p}}x_{\mathfrak{p}}|^{1/2})\left(\sum_{k+l=r_{\mathfrak{p}}+v(x)} \sigma\left(\chi_{1,\mathfrak{p}}(\varpi^k)\chi_{2,\mathfrak{p}}(\varpi^l)\right)\right)\mathbf{1}_{\mathcal{O}_{\mathfrak{p}}}(\mathsf{d}_{\mathfrak{p}}x_{\mathfrak{p}}). $$

This concludes the proof in the case of an unramified principal series representation. In all the other cases, the above calculation is much simpler. Let us note that in the case of the Steinberg representation one has ${}^{\sigma}(\mathrm{St} \otimes \chi) = \mathrm{St} \otimes {}^{\sigma}\chi$. We omit further details. ■

6.2.2 The global integral

Let Π be a cuspidal automorphic representation as in Theorem 6.0.2. Piece together all the normalized Whittaker vectors $W_{\mathfrak{p}}^{\circ}$ in 6.2.1 and let $W^{\circ} = \otimes_{\mathfrak{p}} W_{\mathfrak{p}}^{\circ}$. For each infinite place η_j pick any Whittaker vector $W_j \in \mathcal{W}(\Pi_{\eta_j}, \psi_{\eta_j})$, and put $W_\infty = \otimes_{j=1}^{n} W_j$. Now put,

$$W = W_\infty \otimes W^{\circ} \in \mathcal{W}(\Pi_\infty) \otimes \mathcal{W}(\Pi_f) = \mathcal{W}(\Pi).$$

Let $\phi \in V_\Pi$ be the cusp form that corresponds to W under the isomorphism $V_\Pi \to \mathcal{W}(\Pi)$ of taking the ψ-Fourier coefficient. For any place v, and any $W_v \in \mathcal{W}(\Pi_v)$, define the zeta-integral

$$\zeta_v(s, W_v) = \int_{x \in F_v^*} W_v\left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}\right) |x|^{s-\frac{1}{2}} dx, \quad \Re(s) \gg 0.$$

Hecke theory for GL_2 (see Gelbart [13, Section 6]) says that these integrals have a meromorphic continuation to all of $\mathbb{C}$. The assumption that $s = 1/2$ is critical for $L(s, \Pi)$ says that $\zeta_\eta(\frac{1}{2}, W_\eta)$ is finite for every infinite place η. Lastly, let $\zeta_\infty(s, W_\infty) = \prod_{\eta \in S_\infty} \zeta_\eta(s, W_\eta)$.

Proposition 6.2.5 *With the notations as above,*

$$\int_{F^\times \backslash \mathbb{A}_F^\times} \phi\left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}\right) dx = \zeta_\infty(1/2, W_\infty) L_f(1/2, \Pi).$$

Proof. The usual unfolding argument gives

$$\int_{F^\times \backslash \mathbb{A}_F^\times} \phi\left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}\right) |x|^{s-\frac{1}{2}} dx = \int_{\mathbb{A}_F^\times} W\left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}\right) |x|^{s-\frac{1}{2}} dx.$$

The integral on the left converges absolutely everywhere (since ϕ has rapid decay). The integral on the right converges for $\Re(s) \gg 0$, and there it is eulerian, and so factorizes as $\prod_v \zeta_v(s, W_v)$. For every prime ideal $\mathfrak{p}$, we know that the zeta-integral of the local new vector gives the local L-function; more precisely, we have

$$\zeta_{\mathfrak{p}}(s, W_{\mathfrak{p}}^{\circ}) = |\mathsf{d}_{\mathfrak{p}}|^{s-1/2}\zeta_{\mathfrak{p}}(s, W_{\mathfrak{p}}^{\text{new}}) = |\mathsf{d}_{\mathfrak{p}}|^{s-1/2} L_{\mathfrak{p}}(s, \Pi_{\mathfrak{p}}).$$

We deduce that for $\Re(s) \gg 0$ we have

$$\int_{F^\times \backslash \mathbb{A}_F^\times} \phi\left(\left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}\right)\right) |x|^{s-\frac{1}{2}} dx = \zeta_\infty(s, W_\infty) L_f(s, \Pi) |\mathfrak{d}_F|^{s-1/2}$$

where $\mathfrak{d}_F$ is the absolute discriminant of F. However, the left hand side converges for all s, and the right hand side has a meromorphic continuation for all s, and so we can evaluate at $s = 1/2$ to finish the proof of the proposition. ■

6.2.3 The cohomology class ϑ_Π attached to W_Π°

Consider the map

$$\mathcal{F}_\Pi^{++} : \mathcal{W}(\Pi_f) \to H^n(\mathfrak{g}_\infty, \mathrm{K}_\infty^\circ; V_\Pi \otimes M_{\mu^{\mathsf{v}}})(++)$$

as in 6.1.4, and let ϑ_Π be the image of W_Π° under this map, i.e.,

$$\vartheta_\Pi = \mathcal{F}_\Pi^{++}(W_\Pi^\circ).$$

Fix an open compact subgroup K_f which leaves W_Π° invariant; an optimal one is related to the conductor of Π, but this will not play a role here. From 4.4.1, we have

$$\vartheta_\Pi \in H^n_{\text{cusp}}(S^G_{\mathrm{K}_f}, \widetilde{M_{\mu^{\mathsf{v}}}}) \subset H^n(S^G_{\mathrm{K}_f}, \widetilde{M_{\mu^{\mathsf{v}}}}).$$

It is a fundamental fact (see Clozel [7]) that cuspidal cohomology injects into cohomology with compact supports, i.e.,

$$H^n_{\text{cusp}}(S^G_{\mathrm{K}_f}, \widetilde{M_{\mu^{\mathsf{v}}}}) \hookrightarrow H^n_c(S^G_{\mathrm{K}_f}, \widetilde{M_{\mu^{\mathsf{v}}}}).$$

Therefore

$$\vartheta_\Pi \in H^n_c(S^G_{\mathrm{K}_f}, \widetilde{M_{\mu^\vee}}).$$

Recall that map $\mathcal{F}^{++}_\Pi$ is a composition of three isomorphisms, and the first one maps W°_Π to $W^\circ_\Pi \otimes [\Pi_\infty]^{++}$, where the class $[\Pi_\infty]^{++}$ is given in (4.4.8). Using an analogous notation, we may write the class ϑ_Π in terms of Lie algebra cocycles as

$$\vartheta_\Pi = \sum_{l=(l_1,\ldots,l_n)} \sum_{\alpha=(\alpha_1,\ldots,\alpha_n)} \mathbf{z}^*_l \otimes \phi_{l,\alpha} \otimes \mathsf{s}_\alpha \tag{6.2.6}$$

where $\phi_{l,\alpha} \in V_\Pi$ are cuspforms whose corresponding Whittaker functions in $\mathcal{W}(\Pi) = \mathcal{W}(\Pi_\infty) \otimes \mathcal{W}(\Pi_f)$ are

$$\phi_{l,\alpha} \leftrightarrow W_{l,\alpha} = W_{l,\alpha,\infty} \otimes W^\circ_\Pi.$$

For later use, let us record the action of $\mathrm{Aut}(\mathbb{C})$ on ϑ_Π which is given by the following

Proposition 6.2.7

$$^\sigma\vartheta_\Pi = \frac{\sigma(p^{++}(\Pi))}{p^{++}(^\sigma\Pi)}\vartheta_{^\sigma\Pi}.$$

Proof. This follows from Equation (6.1.1) and Proposition 6.2.4:

$$\begin{aligned} ^\sigma\vartheta_\Pi = \sigma(\mathcal{F}^{++}_\Pi(W^\circ_\Pi)) &= \left(\frac{\sigma(p^{++}(\Pi))}{p^{++}(^\sigma\Pi)}\right)\mathcal{F}^{++}_{^\sigma\Pi}(^\sigma W^\circ_\Pi) \\ &= \left(\frac{\sigma(p^{++}(\Pi))}{p^{++}(^\sigma\Pi)}\right)\mathcal{F}^{++}_{^\sigma\Pi}(W^\circ_{^\sigma\Pi}) = \frac{\sigma(p^{++}(\Pi))}{p^{++}(^\sigma\Pi)}\vartheta_{^\sigma\Pi}. \end{aligned}$$

■

6.2.4 Pulling back to get a GL_1-class $\iota^*\vartheta_\Pi$

Let $\iota : \mathrm{GL}_1 \to \mathrm{GL}_2$ be the map $x \mapsto \left(\begin{smallmatrix} x & \\ & 1 \end{smallmatrix}\right)$. Then ι induces a map at the level of local and global groups, and between appropriate symmetric spaces of GL_1 and GL_2, all of which will also be denoted by ι again; this should cause no confusion. The pullback (of a subset, a function, a differential form, or a cohomology class) via ι will be denoted ι^*. A little more precisely, ι induces an injection:

$$\iota : \mathrm{GL}_1(F)\backslash\mathrm{GL}_1(\mathbb{A}_F)/\iota^*\mathrm{K}_\infty^\circ\iota^*\mathrm{K}_f \hookrightarrow \mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}_F)/\mathrm{K}_\infty^\circ\mathrm{K}_f.$$

Note that $\iota^*\mathrm{K}_\infty^\circ = \{1\}$, and let us denote $\mathrm{R}_f := \iota^*\mathrm{K}_f$ which is an open compact subgroup of $\mathbb{A}_{F,f}^\times$. The above injection will be denoted $\iota : \bar{S}^{G_1}_{\mathrm{R}_f} \hookrightarrow S^G_{\mathrm{K}_f}$, where

$$\bar{S}^{G_1}_{\mathrm{R}_f} = F^\times\backslash\mathbb{A}_F^\times/\mathrm{R}_f.$$

As a manifold $\bar{S}^{G_1}_{\mathrm{R}_f}$ is an oriented n-dimensional manifold all of whose connected components are isomorphic to $\prod_{j=1}^n \mathbb{R}_{>0}$. (Choose the obvious orientation on each connected component.) It is a standard fact that this inclusion $\iota : \bar{S}^{G_1}_{\mathrm{R}_f} \hookrightarrow S^G_{\mathrm{K}_f}$ is a proper map, and hence we can pull back $\vartheta_\Pi \in H^n_c(S^G_{\mathrm{K}_f}, \widetilde{M_{\mu^\vee}})$ by ι, to get

$$\iota^*\vartheta_\Pi \in H^n_c(\bar{S}^{G_1}_{\mathrm{R}_f}, \iota^*\widetilde{M_{\mu^\vee}})$$

where $\iota^*\widetilde{M_{\mu^\vee}}$ is the sheaf on $\bar{S}^{G_1}_{\mathrm{R}_f}$ given by the pullback of the sheaf $\widetilde{M_{\mu^\vee}}$.

We note that the construction of the pullback sheaf is somehow complicated in general: We first define the presheaf ${}^p(\iota^*\widetilde{M^\vee_\mu})$ on $\bar{S}^{G_1}_{\mathrm{R}_f}$ by putting

$${}^p(\iota^*\widetilde{M^\vee_\mu})(U) := \varinjlim_{V\supset\iota(U)} \widetilde{M^\vee_\mu}(V)$$

for any open set U in $\bar{S}^{G_1}_{\mathrm{R}_f}$. The sheafification of this presheaf gives us the pullback sheaf $\iota^*\widetilde{M_{\mu^\vee}}$ of $\widetilde{M_{\mu^\vee}}$. We now claim that this pullback sheaf is the restriction of the representation $M_{\mu^\vee}$ to GL_1 via ι, i.e., $\widetilde{\iota^*M_{\mu^\vee}}$. To see this, we recall some basic facts from sheaf theory. (See [21, Chapter 3] for details.)

Proposition 6.2.8 *Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves on X. A morphism $\phi : \mathcal{F} \longrightarrow \mathcal{G}$ between two sheaves is an isomorphism if and only if, for all $x \in X$, the homomorphism at the level of stalks is an isomorphism, i.e., $\phi_x : \mathcal{F}_x \xrightarrow{\sim} \mathcal{G}_x$.*

For our case, a morphism from $\iota^*\widetilde{M_{\mu^\vee}}$ to $\widetilde{\iota^*M_{\mu^\vee}}$ is given by the restriction. Consider the following diagram.

$$\begin{tikzcd}
\mathbb{A}^\times/\iota^*\mathrm{K}_f \arrow[r, hook, "\iota"] \arrow[d, "p_1"] & G(\mathbb{A})/\mathrm{K}_\infty^\circ\mathrm{K}_f \arrow[d, "p_2"] \\
\bar{S}^{\mathrm{GL}_1}_{\iota^*\mathrm{K}_f} \arrow[r, hook, "\iota"] & S^{\mathrm{GL}_2}_{\mathrm{K}_f}
\end{tikzcd}$$

Here, the vertical arrows are projections. The stalks at x in $\bar{S}^{\mathrm{GL}_1}_{\iota^*\mathrm{K}_f}$ for each sheaf is described as the following.

$$\begin{aligned}
\left(\iota^*\widetilde{M_{\mu^\vee}}\right)_x &\stackrel{\text{def}}{=} \left(\widetilde{M_{\mu^\vee}}\right)_{\iota(x)} \\
&= \varinjlim_{\left(\begin{smallmatrix} x & \\ & 1\end{smallmatrix}\right)\in V} \left\{ s_V : p_2^{-1}(V) \longrightarrow M_{\mu^\vee} : \begin{array}{c} s_V(\gamma g) = \gamma . s_V(g) \text{ for } g \in S^{\mathrm{GL}_2}_{\mathrm{K}_f}, \\ s_V \text{ is locally constant} \end{array} \right\}
\end{aligned}$$

$$\left(\widetilde{\iota^* M_{\mu^\vee}}\right)_x = \varinjlim_{x\in U} \left\{ t_U : p_1^{-1}(U) \longrightarrow M_{\mu^\vee} : \begin{array}{c} t_U(\gamma g) = \gamma . t_U(g) \text{ for } g \in \bar{S}^{\mathrm{GL}_1}_{\iota^*\mathrm{K}_f}, \\ t_U \text{ is locally constant} \end{array} \right\}$$

The isomorphisms of these stalks given by the restriction need to be shown. Let us recall the definition of germ:

Definition 6.2.9 *An element s_p in the stalk $\mathcal{F}_p$ at p is called a germ of a section. A germ s_p can be represented by a section $s_U \in \mathcal{F}(U)$ for an open set U containing the point p: $s_p = s_U|_p$.*

In particular, a germ of a continuous function at p is a continuous function $\phi : U_p \longrightarrow \mathbb{C}$ defined in an open neighborhood U_p of p modulo the following equivalence relation:

$$(\phi : U_p \longrightarrow \mathbb{C}) \sim (\psi : V_p \longrightarrow \mathbb{C})$$

if and only if there is a neighborhood $W_p \subset U_p \cap V_p$ of p such that and $\phi|_{W_p} = \psi|_{W_p}$.

For any section $s_V \in \iota^*\widetilde{M_{\mu^\vee}}(V)$, the restriction to GL_1, i.e., $s_V|_{\iota^{-1}}$ gives a section in $\widetilde{\iota^* M_{\mu^\vee}}(U)$ where $U = \iota^{-1}(V)$. To see that this map is surjective, let us take any element x in $\bar{S}^{\mathrm{GL}_1}_{\mathrm{R}_f}$ and a connected open neighborhood U_x of x. Then any section t_{U_x}

in $\widetilde{\iota^* M_{\mu^\vee}}(U_x)$ is a constant function, say $t_{U_x}(g) = m$. Choose a connected open set V in $S^{\mathrm{GL}_2}_{\mathrm{K}_f}$ that contains $\iota(U_x)$. A constant map $s_V : p_2^{-1}(V) \longrightarrow \{m\}$ is an element in $\iota^* \widetilde{M_{\mu^\vee}}(V)$ and its restriction to U_x equals t_{U_x}. This proves the surjectivity. The injectivity follows from the surjectivity because $M_{\mu^\vee}$ is finite dimensional. This proves that

$$\iota^* \widetilde{M_{\mu^\vee}} = \widetilde{\iota^* M_{\mu^\vee}}.$$

6.2.5 Criticality of $s = 1/2$ and the coefficient μ

We now appeal to the hypothesis that $s = 1/2$ is critical to deduce that we can work with cohomology with trivial coefficients, i.e., in $H^n(\bar{S}^{G_1}_{\mathrm{R}_f}, \mathbb{C})$. For this, let us first record all the critical points for the L-function at hand:

Proposition 6.2.10 *Let* $\Pi \in \mathrm{Coh}(G, \mu^\vee)$*, with* $\mu \in X_0^+(T)$*. Suppose* $\mu = (\mu_1, \ldots, \mu_n)$ *where* $\mu_j = (a_j, b_j)$ *and* $a_j \geq b_j$*. Then*

$$s = \frac{1}{2} + m \in \frac{1}{2} + \mathbb{Z} \textit{ is critical for } L(s, \Pi) \iff -a_j \leq m \leq -b_j, \ \forall j.$$

Proof. Recall the definition (as stated, for example, in Deligne [9]) for a point to be critical. If we are working with an L-function for GL_n then the so-called motivic normalization says that critical points are in the set $\frac{n-1}{2} + \mathbb{Z}$. In our situation, we would say $s = s_0 \in \frac{1}{2} + \mathbb{Z}$ is critical for $L(s, \Pi)$ if and only if both $L_\infty(s, \Pi_\infty)$ and $L_\infty(1-s, \Pi_\infty^\vee)$ are regular at $s = s_0$, i.e., the L-factors at infinity on both sides of the functional equation do not have poles at $s = s_0$. (Automorphic L-functions are always normalized so that the functional equation looks like $L(s, \Pi) = \varepsilon(s, \Pi) L(1-s, \Pi^\vee)$.)

Given $\Pi \in \mathrm{Coh}(G, \mu^\vee)$, we know from 4.2.3 and 4.4.2 that

$$\Pi_\infty = \otimes_j \Pi_{\eta_j} = \otimes_j (D_{a_j - b_j + 1} \otimes |\ |^{(a_j + b_j)/2}).$$

Since D_l is self-dual, we also have

$$\Pi_\infty^\vee = \otimes_j (D_{a_j - b_j + 1} \otimes |\ |^{-(a_j + b_j)/2}).$$

Using the information in 4.1.2 on the local factors for $\mathrm{GL}_2(\mathbb{R})$, and ignoring nonzero constants and exponentials (which are irrelevant to compute critical points) we have:

$$L_\infty(s,\Pi_\infty) \sim \prod_j \Gamma\left(s+\frac{1}{2}+a_j\right), \quad L_\infty(1-s,\Pi_\infty^\vee) \sim \prod_j \Gamma\left(1-s+\frac{1}{2}-b_j\right).$$

Hence, $L_\infty(s,\Pi_\infty)$ is regular at $\frac{1}{2}+m$ if and only if $m+a_j \geq 0$; similarly, $L_\infty(1-s,\Pi_\infty^\vee)$ is regular at $\frac{1}{2}+m$ if and only if $-m-b_j \geq 0$. ■

Corollary 6.2.11 *Let $\Pi \in \mathrm{Coh}(G,\mu^\vee)$, with $\mu \in X_0^+(T)$. Suppose $\mu = (\mu_1,\ldots,\mu_n)$ where $\mu_j = (a_j,b_j)$ and $a_j \geq b_j$. The center of symmetry $s = 1/2$ is critical if and only if* $\mathrm{Hom}_{\mathrm{GL}_1(F_\infty)}(M_{\mu^\vee},\mathbb{1}) \neq 0$.

Proof. Follows from the above proposition and Lemma 4.2. ■

6.2.6 The cohomology class $\mathcal{T}^*\iota^*\vartheta_\Pi$ with trivial coefficients

When $s = 1/2$ is critical, let us let

$$\mathcal{T} \in \mathrm{Hom}_{\mathrm{GL}_1(F_\infty)}(M_{\mu^\vee},\mathbb{1})$$

be the nonzero element as prescribed by Lemma 4.2. Since everything factors over infinite places, we can let $\mathcal{T} = \otimes_{j=1}^n \mathcal{T}_j$, with $\mathcal{T}_j \in \mathrm{Hom}_{\mathrm{GL}_1(F_{\eta_j})}(M_{\mu_j{}^\vee},\mathbb{1})$. The map $\mathcal{T}$ induces a morphism of sheaves on the space $\bar{S}^{G_1}_{\mathrm{R}_f}$, and by functoriality, a homomorphism

$$\mathcal{T}^* : H_c^n(\bar{S}^{G_1}_{\mathrm{R}_f}, \widetilde{\iota^* M_{\mu^\vee}}) \to H_c^n(\bar{S}^{G_1}_{\mathrm{R}_f}, \mathbb{C}).$$

The image of the class $\iota^*\vartheta_\Pi$ under $\mathcal{T}^*$, expressed in terms of relative Lie algebra cocycles, is given by:

$$\mathcal{T}^*\iota^*\vartheta_\Pi = \sum_{l=(l_1,\ldots,l_n)} \sum_{\alpha=(\alpha_1,\ldots,\alpha_n)} \iota^*\mathbf{z}_l^* \otimes \iota^*\phi_{l,\alpha} \otimes \mathcal{T}(\mathsf{s}_\alpha) = \sum_{l=(l_1,\ldots,l_n)} \iota^*\mathbf{z}_l^* \otimes \iota^*\phi_{l,\underline{a}} \quad (6.2.12)$$

where $\underline{a} = (a_1,\ldots,a_n)$. This follows from (6.2.6) and Lemma 4.2.

Since the map $\mathcal{T}$ is defined over $\mathbb{Q}$, as after all the standard basis for $M_{\mu_j{}^\vee}$ gives it a $\mathbb{Q}$-structure, the morphism $\mathcal{T}^*$ is rational, i.e, for all $\sigma \in \mathrm{Aut}(\mathbb{C})$ we have

$$\sigma \circ \mathcal{T}^* = \mathcal{T}^* \circ \sigma \tag{6.2.13}$$

Observe that $\mathcal{T}^*\iota^*\vartheta_\Pi$ is a top-degree compactly supported cohomology class for $\bar{S}^{\mathrm{GL}_1}_{\mathrm{R}_f}$.

6.2.7 Top-degree cohomology with compact supports

Let us recall some basic topological facts here. Let $\mathcal{M}$ be an oriented n-dimensional manifold with h connected components, indexed by ν with $1 \le \nu \le h$. Then Poincaré duality implies that

$$H^n_c(\mathcal{M}, \mathbb{C}) \simeq \oplus_{\nu=1}^h \mathbb{C}.$$

(See, for example, Harder [21, 4.8.5].) The map is integration over the entire manifold with some chosen orientation; for each connected component you get a complex number. Now let us add these numbers to get a map $\vartheta \mapsto \int_\mathcal{M} \vartheta$

$$\int_\mathcal{M} : H^n_c(\mathcal{M}, \mathbb{C}) \to \mathbb{C}.$$

As explained in Raghuram [36, 3.2.3], such a map given by Poincaré duality is rational, i.e.,

$$\sigma\left(\int_\mathcal{M} \vartheta\right) = \int_\mathcal{M} {}^\sigma\vartheta. \tag{6.2.14}$$

6.2.8 The main identity

Recall that $\mathcal{T}^*\iota^*\vartheta_\Pi \in H^n_c(\bar{S}^{G_1}_{\mathrm{R}_f}, \mathbb{C})$ is a top-degree compactly supported cohomology class. We can integrate it over all of $\bar{S}^{G_1}_{\mathrm{R}_f}$. The main technical theorem needed to analyze the arithmetic properties of the special value $L(1/2, \Pi)$ is

Theorem 6.2.15

$$\int_{\bar{S}^{G_1}_{\mathrm{R}_f}} \mathcal{T}^*\iota^*\vartheta_\Pi \;=\; \frac{\langle[\Pi_\infty]^{++}\rangle}{(4i)^n \mathrm{vol}(\mathrm{R}_f)}\, L_f(1/2, \Pi)$$

where

$$\langle [\Pi_\infty]^{++} \rangle = \sum_{l=(l_1,\dots,l_n)} \zeta_\infty(1/2, W_{l,\underline{a},\infty}).$$

Proof. Recall that (6.2.12) gives

$$\mathcal{T}^*\iota^*\vartheta_\Pi = \sum_{l=(l_1,\dots,l_n)} \sum_{\alpha=(\alpha_1,\dots,\alpha_n)} \iota^*\mathbf{z}_l^* \otimes \iota^*\phi_{l,\alpha} \otimes \mathcal{T}(\mathsf{s}_\alpha) = \sum_{l=(l_1,\dots,l_n)} \iota^*\mathbf{z}_l^* \otimes \iota^*\phi_{l,\underline{a}}.$$

We will identify the terms $\iota^*\mathbf{z}_l^*$. Consider just one copy of $\mathrm{GL}_1(\mathbb{R})$ sitting inside $\mathrm{GL}_2(\mathbb{R})$ via ι. Let $\mathbf{t}_1 := 1$ be a basis for $\mathfrak{g}_1 = \mathbb{C}$. (Fixing $\mathbf{t}_1$ is tantamount to fixing an orientation on $\mathbb{R}_{>0} = \mathrm{GL}_1(\mathbb{R})^\circ$. Taking all the infinite places together, this will be fixing an orientation on each connected component of $\bar{S}^{G_1}_{\mathrm{R}_f}$.) Note that

$$\begin{aligned} \iota(\mathbf{t}_1) &= \frac{1}{4i}\left(\mathbf{z}_1 + \mathbf{z}_2 + \left(\begin{smallmatrix} 2i & 0 \\ 0 & 2i \end{smallmatrix}\right)\right), \text{ in } \mathfrak{g}_2, \\ &= \frac{1}{4i}\left(\mathbf{z}_1 + \mathbf{z}_2\right), \text{ in } \mathfrak{g}_2/\mathfrak{k}_2, \end{aligned}$$

hence $\iota^*\mathbf{z}_1^* = \iota^*\mathbf{z}_2^* = \frac{1}{4i}\mathbf{t}_1^*$. Applying this to each infinite place, we see

$$\iota^*\mathbf{z}_l^* = \otimes \iota^*\mathbf{z}_{j,l_j}^* = \frac{1}{(4i)^n} \otimes_{j=1}^n \mathbf{t}_{j,1}^*,$$

where $\mathbf{t}_{j,1}$ is the element $\mathbf{t}_1$ for the infinite place η_j.

Using the fact that $\phi_{l,\alpha}$ is fixed by K_f which implies that $\iota^*\phi_{l,\underline{a}}$ is fixed by $\mathrm{R}_f = \iota^*\mathrm{K}_f$ we get

$$\begin{aligned} \int_{\bar{S}^{G_1}_{\mathrm{R}_f}} \mathcal{T}^*\iota^*\vartheta_\Pi &= \sum_{l=(l_1,\dots,l_n)} \frac{1}{(4i)^n} \int_{\bar{S}^{G_1}_{\mathrm{R}_f}} \iota^*\phi_{l,\underline{a}} \\ &= \frac{1}{(4i)^n} \sum_{l=(l_1,\dots,l_n)} \int_{F^\times\backslash\mathbb{A}_F^\times/\mathrm{R}_f} \phi_{l,\underline{a}}\left(\begin{pmatrix} x & \\ & 1 \end{pmatrix}\right) dx \\ &= \frac{1}{(4i)^n \mathrm{vol}(\mathrm{R}_f)} \sum_{l=(l_1,\dots,l_n)} \int_{F^\times\backslash\mathbb{A}_F^\times} \phi_{l,\underline{a}}\left(\begin{pmatrix} x & \\ & 1 \end{pmatrix}\right) dx \\ &= \frac{1}{(4i)^n \mathrm{vol}(\mathrm{R}_f)} \sum_{l=(l_1,\dots,l_n)} \left(\zeta_\infty(1/2, W_{l,\underline{a},\infty}) L_f(1/2, \Pi)\right) \end{aligned}$$

where the last equality is due to Proposition 6.2.5. ■

6.2.9 Archimedean computations

Proposition 6.2.16 *Given* $\Pi \in \mathrm{Coh}(G, \mu^{\vee})$, *with* $\mu = (\mu_1, \dots, \mu_n)$ *and* $\mu_j = (a_j, b_j)$, *we have*

$$\langle [\Pi_\infty]^{++} \rangle \;=\; c\,(2\pi i)^{-d_\infty} i^n$$

where $d_\infty = \sum_{j=1}^n (a_j + 1)$, *and* c *is a nonzero integer (which is made explicit in the proof).*

Proof. To compute $\langle [\Pi_\infty]^{++} \rangle = \sum_{l=(l_1,\dots,l_n)} \zeta_\infty(1/2, W_{l,\underline{a},\infty})$, let us begin by noting that each summand is a product over infinite places:

$$\zeta_\infty(1/2, W_{l,\underline{a},\infty}) = \prod_{j=1}^n \zeta_{\eta_j}(1/2, \lambda_{j,l_j,a_j})$$

where the λ_{j,l_j,a_j} are as in (4.4.7). We can rewrite the expression for $\langle [\Pi_\infty]^{++} \rangle$ as

$$\sum_{l=(l_1,\dots,l_n)} \prod_{j=1}^n \zeta_{\eta_j}(1/2, \lambda_{j,l_j,a_j}) = \prod_{j=1}^n \left(\zeta_{\eta_j}(1/2, \lambda_{j,1,a_j}) + \zeta_{\eta_j}(1/2, \lambda_{j,2,a_j}) \right).$$

The j-th factor in the right hand side is the value at $s = 1/2$ of the sum of two zeta-integrals:

$$\zeta_{\eta_j}(s, \lambda_{j,1,a_j}) + \zeta_{\eta_j}(s, \lambda_{j,2,a_j}).$$

Using the definitions of ζ_η and λ_{j,l_j,a_j} we get

$$\int_{x\in\mathbb{R}^*} \binom{a_j - b_j}{a_j} i^{a_j} \lambda_{(a_j-b_j+2)} \left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \right) |x|^{s-\frac{1}{2}} dx$$
$$+ \int_{x\in\mathbb{R}^*} \binom{a_j - b_j}{a_j} i^{-a_j} \lambda_{-(a_j-b_j+2)} \left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \right) |x|^{s-\frac{1}{2}} dx.$$

Recall that the integrals converge for $\Re s \gg 0$. In the second integral, using the fact that $\delta(\lambda_{a_j-b_j+2}) = i^{2a_j} \lambda_{-(a_j-b_j+2)}$, and changing variable from x to $-x$, we see that it is the same as the first integral. Hence

$$\zeta_{\eta_j}(s, \lambda_{j,1,a_j}) + \zeta_{\eta_j}(s, \lambda_{j,2,a_j}) = 2 \binom{a_j - b_j}{a_j} i^{a_j} \int_{x\in\mathbb{R}^*} \lambda_{(a_j-b_j+2)} \left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \right) |x|^{s-\frac{1}{2}} dx.$$

It is well-known that the zeta-integral of 'the lowest weight vector' $\lambda_{(a_j-b_j+2)}$ gives the local L-factor $L(s,\Pi_{\eta_j})$, i.e., the local factor for the representation $\Pi_{\eta_j} = D_{a_j-b_j+1} \otimes |\ |^{(a_j+b_j)/2}$. (See, for example, Gelbart [13, Proposition 6.17].) Using the information on local factors in 4.1.2, we get

$$\zeta_{\eta_j}(s,\lambda_{j,1,a_j}) + \zeta_{\eta_j}(s,\lambda_{j,2,a_j}) = 4\binom{a_j-b_j}{a_j} i^{a_j}(2\pi)^{-(s+\frac{1}{2}+a_j)}\Gamma\left(s+\frac{1}{2}+a_j\right).$$

The left hand side converges for $\Re s \gg 0$ and has a meromorphic continuation to all s, and in the right hand side the Γ-function also makes sense, after continuation, to all s; hence we can put $s = 1/2$ to get

$$\begin{aligned}\zeta_{\eta_j}(1/2,\lambda_{j,1,a_j}) + \zeta_{\eta_j}(1/2,\lambda_{j,2,a_j}) &= 4\binom{a_j-b_j}{a_j}\Gamma(a_j+1)i^{a_j}(2\pi)^{-(a_j+1)} \\ &= 4\frac{(a_j-b_j)!}{(-b_j)!}(-1)^{a_j} i(2\pi i)^{-(a_j+1)}.\end{aligned}$$

Hence $\langle[\Pi_\infty]^{++}\rangle = c\,(2\pi i)^{-d_\infty} i^n$ where c is the integer:

$$c = 4^n\prod_{j=1}^{n}(-1)^{a_j}\frac{(a_j-b_j)!}{(-b_j)!}.$$

■

Let us note that $\sigma \in \mathrm{Aut}(\mathbb{C})$ acts on Π_∞ by permuting the infinite places; in particular,

$$\langle[{}^\sigma\Pi_\infty]^{++}\rangle = \langle[\Pi_\infty]^{++}\rangle.$$

This kind of an explicit computation very quickly escalates in complexity when we go from GL_2 to higher GL_n. Indeed, there are many conditional theorems on special values of L-functions that have been proved under the assumption that a quantity analogous to $\langle[\Pi_\infty]^{++}\rangle$ is nonzero. See, for example, Kazhdan-Mazur-Schmidt [26], Mahnkopf [31], or Raghuram [36].

6.2.10 Concluding part of the proof of Theorem 6.0.2

We can now finish the proof as follows. Using Proposition 6.2.16 in the main identity of Theorem 6.2.15 we have

$$\int_{\bar{S}^{G_1}_{\mathrm{R}_f}} \mathcal{T}^*\iota^*\vartheta_\Pi \;=\; \frac{c}{4^n \mathrm{vol}(\mathrm{R}_f)}\frac{L_f(1/2,\Pi)}{(2\pi i)^{d_\infty}}. \tag{6.2.17}$$

Apply $\sigma \in \mathrm{Aut}(\mathbb{C})$ to both sides, while noting that $c/(4^n\mathrm{vol}(\mathrm{R}_f))$ is a nonzero rational number, to get

$$\sigma\left(\int_{\bar{S}^{G_1}_{\mathrm{R}_f}} \mathcal{T}^*\iota^*\vartheta_\Pi\right) \;=\; \frac{c}{4^n \mathrm{vol}(\mathrm{R}_f)}\sigma\left(\frac{L_f(1/2,\Pi)}{(2\pi i)^{d_\infty}}\right).$$

Using (6.2.14), that σ commutes with ι^*–since restriction of a class to a submanifold is a rational operation, (6.2.13), and using Proposition 6.2.7, we get that the left hand side simplifies to

$$\sigma\left(\int_{\bar{S}^{G_1}_{\mathrm{R}_f}} \mathcal{T}^*\iota^*\vartheta_\Pi\right) \;=\; \frac{\sigma(p^{++}(\Pi))}{p^{++}({}^\sigma\Pi)}\int_{\bar{S}^{G_1}_{\mathrm{R}_f}} \mathcal{T}^*\iota^*\vartheta_{{}^\sigma\Pi} = \frac{\sigma(p^{++}(\Pi))}{p^{++}({}^\sigma\Pi)}\frac{c}{4^n \mathrm{vol}(\mathrm{R}_f)}\frac{L_f(1/2,{}^\sigma\Pi)}{(2\pi i)^{d_\infty}},$$

where the last equality follows by applying (6.2.17) for the representation ${}^\sigma\Pi$. Hence, we have

$$\frac{c}{4^n \mathrm{vol}(\mathrm{R}_f)}\sigma\left(\frac{L_f(1/2,\Pi)}{(2\pi i)^{d_\infty}}\right) \;=\; \frac{\sigma(p^{++}(\Pi))}{p^{++}({}^\sigma\Pi)}\frac{c}{4^n \mathrm{vol}(\mathrm{R}_f)}\frac{L_f(1/2,{}^\sigma\Pi)}{(2\pi i)^{d_\infty}}.$$

The proof of Theorem 6.0.2 follows easily from this equation.

6.2.11 Proof of Corollary 6.0.3

Let $s = \frac{1}{2} + m \in \frac{1}{2} + \mathbb{Z}$ be any critical point for $L(s,\Pi)$. Let us note that

$$L(s+m,\Pi) = L(s, \Pi \otimes |\ |^m).$$

Hence $1/2$ is critical for $L(s,\Pi\otimes |\ |^m)$. Now we apply Theorem 6.0.2 to the representation $\Pi \otimes |\ |^m$. There is one nontrivial point to note, i.e., the coefficient system has changed. It is easy to see that

$$\Pi \in \mathrm{Coh}(G,\mu^\vee) \implies \Pi\otimes |\ |^m \in \mathrm{Coh}(G,(\mu+m)^\vee),$$

where, if $\mu = (\mu_1, \dots, \mu_j)$, with $\mu_j = (a_j, b_j)$, then $\mu + m = (\mu_1 + m, \dots, \mu_j + m)$, with $\mu_j = (a_j + m, b_j + m)$. Hence the integer $d_\infty = d(\Pi_\infty)$ also changes:

$$d(\Pi_\infty \otimes |\ |^m) = d(\Pi_\infty) + mn.$$

Further, let us note that the main theorem of Raghuram-Shahidi [38], applied to the special case when the twisting character is $|\ |^m$, gives the period relation:

$$p^{++}(\Pi \otimes |\ |^m) = p^{(-1)^m(++)}(\Pi).$$

Note that twisting by a finite order character χ of $F^\times \backslash \mathbb{A}_F^\times$ does not change the set of critical points. Corollary 6.0.3 follows by the period relations of Raghuram-Shahidi [38] as recalled in 6.1.6.

6.2.12 Proof of Theorem 6.0.1 and period relations

The proof of Theorem 6.0.1 is a totally formal consequence of Theorem 6.0.2 plus Corollary 6.0.3, together with properties of the dictionary as in Theorem 6.0.4; we leave the details to the reader after observing, as mentioned in the proof of Theorem 5.4.4 above, that if $\mathbf{f} \in \mathcal{S}_k(\mathfrak{n}, \tilde{\omega})$ and suppose for convenience that all the weights k_j are even, then $\Pi = \Pi(\mathbf{f}) \in \mathrm{Coh}(G, \mu^\vee)$ with the highest weight $\mu = (\mu_1, \dots, \mu_n)$ being given by:

$$\mu_j = \left(\frac{k_j - 2}{2}, -\frac{k_j - 2}{2} \right) =: (a_j, b_j).$$

Note that Shimura's periods $u(r, \mathbf{f})$ and our periods $p^\epsilon(\Pi)$ have different definitions. With this in mind, it is interesting to see the formal consequences of the fact that these periods appear in the critical values of the 'same' L-function. Using the notation as in Theorem 5.0.1, when all the weights k_j are even integers, then $k_0/2$ is a critical point for $L(s, \mathbf{f})$, which corresponds to the central critical point $1/2$ for $L(s, \Pi(\mathbf{f}))$. We have the following consequence:

$$(2\pi i)^{\frac{nk_0}{2}} u(++, \mathbf{f}) \ \sim\ (2\pi i)^{d_\infty} p^{++}(\Pi(\mathbf{f})).$$

where $\sim$ means up to an element of $\mathbb{Q}(\mathbf{f})^* = \mathbb{Q}(\Pi(\mathbf{f}))^*$. Note that $d_\infty = \sum_j (a_j + 1) = (\sum_j k_j)/2$. Hence

$$p^{++}(\Pi(\mathbf{f})) \;\sim\; (2\pi i)^{\sum_j (k_0 - k_j)/2}\, u(++, \mathbf{f}). \tag{6.2.18}$$

Twisting by a quadratic character of prescribed signature, one can deduce a similar relation between $u(\epsilon, \mathbf{f})$ and $p^\epsilon(\Pi)$ for any $\epsilon \in (\mathbb{Z}/2\mathbb{Z})^n$. Similarly, one can deduce period relations when all the weights k_j are odd.

CHAPTER 7

Congruences of L-functions

7.1 Main theorem

The aim of this chapter is to study congruences of central critical L-values for Hilbert modular froms. The congruences can be discussed in an ℓ-adic setting for a prime ℓ. Throughout the chapter, we fix a prime ℓ and write $\mathbb{Q}_\ell$ for the ℓ-adic completion of $\mathbb{Q}$ as usual. We denote E for a finite extension of $\mathbb{Q}_\ell$ contained in $\bar{\mathbb{Q}}_\ell$, $\mathcal{O}_E$ for the ring of integers in E, and ϖ_E for a generator of the maximal ideal of $\mathcal{O}_E$. Fix an isomorphism $\mathbb{C} \xrightarrow{\sim} \bar{\mathbb{Q}}_\ell$. Vatsal proved in [44] the following theorem in the case of elliptic modular forms.

Theorem 7.1.1 (Vatsal, [44]) *Let $f = \sum a_n q^n$ and $g = \sum b_n q^n$ be normalized Hecke eigenforms on $\Gamma_1(M)$, where M is an integer greater than 4, of weight $k \geq 2$. Write $M = N\ell^s$, $s \geq 0$, and $\ell \nmid N$. Suppose there exists a finite extension E of $\mathbb{Q}_\ell$ contained in $\bar{\mathbb{Q}}_\ell$ such that all a_n and b_n are in $\mathcal{O}_E$ and that*

$$a_n \equiv b_n \mod \varpi_E^r$$

for some integer $r \geq 1$. Then, under a certain hypothesis about isomorphisms of $\mathcal{H}_k$-modules (See [44, Sectoin 1.2].), we have

$$\tau(\bar{\chi})(m-1)!\frac{L(m,f,\chi)}{(-2\pi i)^m \Omega_f^\alpha} \equiv \tau(\bar{\chi})(m-1)!\frac{L(m,g,\chi)}{(-2\pi i)^m \Omega_g^\alpha} \mod \varpi_E^r$$

for each character χ with conductor prime to N, and for each integer m between 1 and $k-1$. Here, τ is the Gauss sum and Ω_*^α are the canonical periods of Vatsal.

We generalize this result to Hilbert modular forms by using the approach described in Chapter 6. Our main theorem in the chapter is the following.

Theorem 7.1.2 *Let $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n$ with k_j being even and at least 4 for all j. Let $\mathbf{f}$ and $\mathbf{f}'$ be primitive forms in $\mathcal{S}_k(\mathfrak{n}, \tilde{\omega})$. Suppose there exists a finite extension E of $\mathbb{Q}_\ell$ inside $\bar{\mathbb{Q}}_\ell$ such that it contains $\mathbb{Q}(\mathbf{f})$, $\mathbb{Q}(\mathbf{f}')$ and the base field F, and that its ring of integers $\mathcal{O}_E$ contains $\mathbb{Z}[\tilde{\omega}]$. Assuming the existence of an isomorphism concerning integral cohomology (see 7.3.3) we define 'canonical periods' $p(\Pi)^\circ$ and $p(\Pi')^\circ$ which are in $\mathbb{C}^* \simeq \bar{\mathbb{Q}}_\ell^*$ and well defined up to $\mathcal{O}_E^\times$. These periods are such that if the Fourier coefficients $\mathrm{C}(\mathfrak{p}, \mathbf{f})$ of $\mathbf{f}$ and $\mathrm{C}(\mathfrak{p}, \mathbf{f}')$ of $\mathbf{f}'$ are contained in $\mathcal{O}_E$ for each prime ideal $\mathfrak{p}$ and satisfy*

$$\mathrm{C}(\mathfrak{p}, \mathbf{f}) \equiv \mathrm{C}(\mathfrak{p}, \mathbf{f}') \mod \varpi_E^s$$

for a positive integer s, then, we have the following:

$$c(k) \cdot \frac{L(k_0/2, \mathbf{f})}{(2\pi i)^{d_\infty} p(\Pi)^\circ} \equiv c(k) \cdot \frac{L(k_0/2, \mathbf{f}')}{(2\pi i)^{d_\infty} p(\Pi')^\circ} \mod \varpi_E^s,$$

where $c(k)$ is an integer depending only on the weight k.

Two Hilbert cusp forms are said to be congruent modulo ϖ_E^s if they fulfill the conditions in the theorem, i.e., Fourier coefficients at primes $\mathfrak{p}$ are in $\mathcal{O}_E$ and congruent modulo ϖ_E^s.

The key ingredient for solving the problem is a refinement of the definition of periods, which follows from studying integral structures for Whittaker models and integral structures on cuspidal cohomology. In the theorem, $\Pi = \Pi(\mathbf{f})$ and $\Pi' = \Pi(\mathbf{f}')$ are regular algebraic cuspidal automorphic representations of $G(\mathbb{A})$ corresponding to $\mathbf{f}$ and $\mathbf{f}'$, respectively. The integer $c(k)$ will be specified in the proof toward the end of the chapter. We first introduce each integral structure in Section 7.2 and complete the proof of Theorem 7.1.2 in Section 7.3.

7.2 Integral structure

7.2.1 $\mathcal{O}_E$-structure for Whittaker model

For each prime $\mathfrak{p}$, let $r_{\mathfrak{p}}$ be the highest power of $\mathfrak{p}$ that divides $\mathfrak{D}_F$. We fix an additive character $\psi = \otimes_v \psi_v$ of $F\backslash\mathbb{A}_F$ such that a local character $\psi_{\mathfrak{p}}$ at non-archimedean place $\mathfrak{p}$ is trivial on $\mathfrak{p}^{-r_{\mathfrak{p}}}$ and non-trivial on $\mathfrak{p}^{-r_{\mathfrak{p}}-1}$. For details, see Section 2.8.

Let (Π, V_Π) be an irreducible infinite dimensional representation of $\mathrm{GL}_2(\mathbb{A}_F)$. With respect to the additive character ψ, there is a unique Whittaker model $\mathcal{W}(\Pi, \psi)$ for Π. We will simply write $\mathcal{W}(\Pi)$ for this model as the additive character ψ is fixed throughout the chapter. We also recall that $\mathcal{W}(\Pi)$ decomposes as a restricted tensor product of local Whittaker model $\mathcal{W}(\Pi_v)$.

Lemma 7.1 *Let* (Π, V_Π) *be a cuspidal automorphic representation of* $\mathrm{GL}_2(\mathbb{A}_F)$*, and* $\mathcal{W}(\Pi)$ *be its Whittaker model with respect to a non-trivial additive character* ψ*. For a prime* ℓ*, if* E *is a finite extension of* $\mathbb{Q}_\ell$ *containing* $\mathbb{Q}(\Pi)$*, and* $\mathcal{O}_E$ *is its ring of integers, then* $\mathcal{W}(\Pi)$ *has* $\mathcal{O}_E$*-structure, i.e., there exists an* $\mathcal{O}_E$*-module* $\mathcal{W}_{\mathcal{O}_E}$ *satisfying:*

1. $\mathcal{W}_{\mathcal{O}_E}$ *is* $\mathrm{GL}_2(\mathcal{O}_F)$*-stable,*

2. *the canonical map* $\mathcal{W}_{\mathcal{O}_E} \otimes_{\mathcal{O}_E} \bar{\mathbb{Q}}_\ell \longrightarrow \mathcal{W}(\Pi)$ *is an isomorphism.*

Proof. The proof is almost identical to [46, Lemma 1.1]. Let W° be a normalized new vector in $\mathcal{W}(\Pi)$ where the normalization is taken as in Section 6.2.1, and let $\mathcal{W}_{\mathcal{O}_E}$ be an $\mathcal{O}_E$-span of $\{g.W^\circ : g \in \mathrm{GL}_2(\mathcal{O}_F)\}$. A new vector means that it satisfies the condition, $k.W^\circ = \tilde{\omega}(k)W^\circ$ where $\tilde{\omega}$ is the central character of Π and k is an element of a compact subgroup in $\mathrm{GL}_2(\mathbb{A}_F)$. We claim that the space $\mathcal{W}_{\mathcal{O}_E}$ satisfies all the conditions. It is clearly $\mathrm{GL}_2(\mathcal{O}_F)$-stable as, for any element $V = \sum_i \alpha_i g_i.W^\circ \in \mathcal{W}_{\mathcal{O}_E}$, we have $g.V = \sum_i \alpha_i (gg_i).W^\circ$. The surjectivity of the map in the second condition follows from the irreducibility of Π. Therefore, it only remains to show injectivity. If it is not, then there exists a nonzero element $\sum_{i=1}^n W_i \otimes \lambda_i$ in $\mathcal{W}_{\mathcal{O}_E} \otimes_{\mathcal{O}_E} \bar{\mathbb{Q}}_\ell$, that is mapped to zero in $\mathcal{W}_\Pi$. Without loss of generality, let us assume that each W_i is of the form $g_i.W^\circ$ and $\lambda_1 = 1$. Let us suppose n is minimal for such elements. We first claim that all λ_i must belong to the rationality field $\mathbb{Q}(\Pi)$, which is defined to be the subfield $\mathbb{C}^{\mathcal{S}(\Pi)}$ of $\mathbb{C}$ that is fixed by all the $\mathbb{C}$-automorphisms in $\mathcal{S}(\Pi) := \{\sigma : {}^\sigma\Pi \sim \Pi\}$. Suppose $\lambda_2 \notin \mathbb{Q}(\Pi)$. There exists a σ in $\mathcal{S}(\Pi)$ that does not fix λ_2. Recall that ${}^\sigma\Pi$ is defined as

$$ {}^\sigma\Pi(g) = s \circ \Pi(g) \circ s^{-1} $$

with some σ-linear isomorphism s from V_Π to $V_{{}^\sigma\Pi}$. This map s, together with an intertwining operator f for Π and ${}^\sigma\Pi$, we have a σ-linear isomorphism t from V_Π to V_Π so that

$$ \Pi(g) = t \circ \Pi(g) \circ t^{-1}. $$

Viewing t as an automorphism of $\mathcal{W}(\Pi)$, it can be easily verified that $t(W^\circ)$ is a new vector. Indeed,

$$ k.t(W^\circ) = t(k.W^\circ) = t(\tilde{\omega}(k)W^\circ) = \tilde{\omega}(k)t(W^\circ), $$

since $\tilde{\omega}(k) \in \mathbb{Q}(\Pi)^\times$ and σ fixes $\mathbb{Q}(\Pi)$. We normalize t so that $t(W^\circ) = W^\circ$. This is possible because the space of new vectors is 1-dimensional. With this normalization, t fixes $g.W^\circ$ for any $g \in \mathrm{GL}_2(\mathcal{O}_F)$. It gives us that

$$ t\left(\sum_{i=1}^n \lambda_i W_i\right) = \sum_{i=1}^n \sigma(\lambda_i) W_i = 0, $$

and therefore

$$ \sum_{i=2}^n (\lambda_i - \sigma(\lambda_i))\, W_i = 0. $$

This contradicts the minimality of n, and our claim is proved.

If $\sum_{i=1}^n W_i \otimes \lambda_i$ is a nonzero element, there is an i so that $\lambda_i \notin \mathcal{O}_E$. Say λ_2 is such an element. Since it is in E by our claim above, we may write $\lambda_2 = \alpha_2/\beta_2$ with α_2,

$\beta_2 \in \mathcal{O}_E$. But then $\beta_2 \sum W_i \otimes \lambda_i$ is mapped to zero in $\mathcal{W}(\Pi)$. On the other hand,

$$\begin{aligned}\beta_2 \sum_{i=1}^{n} W_i \otimes \lambda_i &= (\beta_2 W_1 \otimes 1) + (\alpha_2 W_2 \otimes 1) + \sum_{i=3}^{n} \beta_2 W_i \otimes \lambda_i \\ &= (\beta_2 W_1 + \alpha_2 W_2) \otimes 1 + \sum_{i=3}^{n} \beta_2 W_i \otimes \lambda_i,\end{aligned}$$

which causes a contradiction to the minimality of n. This says that the map in the second condition must be injective, and thus the proof is completed.

■

7.2.2 Integral sheaf

Our aim is to define the integral structure for the cohomology group, but before that we need to define an integral sheaf, i.e., a sheaf of $\mathcal{O}_E$-modules on $S^G_{\mathrm{K}_f}$. We borrow heavily from Günter Harder's book in progress on the cohomology of arithmetic groups. See Chapter III of `http://www.math.uni-bonn.de/people/harder/Manuscripts/buch/`. First, we slightly refine our notation. The finite-dimensional representation of the highest weight μ was denoted simply as M_μ in earlier chapters. From now on, we write it as $M_{\mu,\mathbb{C}}$ and its dual as $M_{\mu^\vee,\mathbb{C}}$ to emphasize its base field. Similarly, an L-structure for $M_{\mu^\vee,\mathbb{C}}$ will be denoted as $M_{\mu^\vee,L}$ where $L = \mathbb{Q}(\mu)$, E, or $\mathcal{O}_E$. Recall from Section 4.4.5 that the standard basis for $M_{\mu^\vee,\mathbb{C}}$ was given by $\{\mathsf{s}_0, \mathsf{s}_1, \ldots, \mathsf{s}_{\alpha-\beta}\}$ with $\mathsf{s}_j = e_1^j e_2^{\alpha-\beta-j}$. This space affords an $\mathcal{O}_E$-integral structure $M_{\mu^\vee,\mathcal{O}_E}$ which is the $\mathcal{O}_E$-span of $\{g.\mathsf{s}_\alpha : g \in \mathrm{GL}_2(\mathcal{O}_F)\}$. By $\mathcal{O}_E$-structure, we mean an $\mathcal{O}_E$-submodule that is stable under then action of $\mathrm{GL}_2(\mathcal{O}_F)$ and such that $M_{\mu^\vee,\mathcal{O}_E} \otimes_{\mathcal{O}_E} \bar{\mathbb{Q}}_\ell \equiv M_{\mu^\vee,\mathbb{C}}$ after the identification of $\mathbb{C} \simeq \bar{\mathbb{Q}}_\ell$. Similarly, E-rational structure of $M_{\mu^\vee,\mathbb{C}}$ is given as E-span of $\{g.\mathsf{s}_\alpha : g \in \mathrm{GL}_2(F)\}$. See Section 4.4.5 for the details of E-structure.

Let us define the integral sheaf $\widetilde{M_{\mu^\vee,\mathcal{O}_E}}$ on $S^G_{\mathrm{K}_f} = G(\mathbb{Q})\backslash G(\mathbb{A})/\mathrm{K}^\circ_\infty \mathrm{K}_f$. We begin this section by recalling the definition of E-rational sheaf on $S^G_{\mathrm{K}_f}$, which is essentially the same as Definition 4.4.1.

Definition 7.2.1 *Let $P_1 : G(\mathbb{R})^\circ/\mathrm{K}_\infty^\circ \times G(\mathbb{A}_f)/\mathrm{K}_f \longrightarrow S^G_{\mathrm{K}_f}$ be the projection map. For any open set U in $S^G_{\mathrm{K}_f}$, the sections $\widetilde{M_{\mu^\vee,E}}(U)$ of the E-rational sheaf over U is defined as:*

$$\widetilde{M_{\mu^\vee,E}}(U) := \left\{ \begin{array}{lr} s : P_1^{-1}(U) \to M_{\mu^\vee,E} \ : & s \text{ is locally constant, and} \\ & s(\gamma(x_\infty, g_f\mathrm{K}_f)) = \gamma.s(x_\infty, g_f\mathrm{K}_f) \end{array} \right\}.$$

It is not obvious how to define the integral sheaf inside the rational sheaf directly. The approach to take here is that we first adèlize the rational sheaf $\widetilde{M_{\mu^\vee,E}}$ and then use the "adèlized" integral sheaf to introduce the sheaf of our interest. See (7.2.2). For adelization, we use the projection map P instead of P_1 in the following diagram.

$$\begin{tikzcd} G(\mathbb{R})^\circ/\mathrm{K}_\infty^\circ \times G(\mathbb{A}_f) \arrow[r] \arrow[dr, "P"'] & G(\mathbb{R})^\circ/\mathrm{K}_\infty^\circ \times G(\mathbb{A}_f)/\mathrm{K}_f \arrow[d, "P_1"] \\ & S^G_{\mathrm{K}_f} \end{tikzcd}$$

The sections $\left(\widetilde{M_{\mu^\vee,E} \otimes_{\mathbb{Q}} \mathbb{A}_f}\right)(U)$ of the adèlized E-rational sheaf on the same space $S^G_{\mathrm{K}_f}$ over U can be defined to be the set of locally constant maps, $\tilde{s} : P^{-1}(U) \longrightarrow M_{\mu^\vee,E} \otimes_{\mathbb{Q}} \mathbb{A}_f$, satisfying

$$\tilde{s}(x_\infty, g_f) = g_f^{-1}.(s(x_\infty, g_f\mathrm{K}_f) \otimes 1).$$

One can see from the definition that $\tilde{s}$ is $G(\mathbb{Q})$-invariant as, for any γ in $G(\mathbb{Q})$, we have

$$\begin{aligned} \tilde{s}(\gamma(x_\infty, g_f)) &= \tilde{s}(\gamma x_\infty, \gamma g_f) = g_f^{-1}\gamma^{-1}.\left(s(\gamma x_\infty, \gamma g_f\mathrm{K}_f) \otimes 1\right) \\ &= g_f^{-1}\gamma^{-1}\gamma.\left(s(x_\infty, g_f\mathrm{K}_f) \otimes 1\right) \\ &= \tilde{s}(x_\infty, g_f), \end{aligned}$$

and also K_f acts from the right as follows.

$$\tilde{s}(x_\infty, g_f k_f) = k_f^{-1}g_f^{-1}.\left(s(x_\infty, g_f\mathrm{K}_f) \otimes 1\right) = k_f^{-1}.\tilde{s}(x_\infty, g_f).$$

Now, let us define the adèlized integral sheaf $\widetilde{M_{\mu^\vee,\mathcal{O}_E} \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}}$ on $S^G_{\mathrm{K}_f}$, that sits inside the adèlized rational sheaf $\widetilde{M_{\mu^\vee,E} \otimes_{\mathbb{Q}} \mathbb{A}_f}$. It must be done component-wise. This is because, if arbitrary open set U is chosen in $S^G_{\mathrm{K}_f}$, the image of $P^{-1}(U)$ under $\tilde{s}$ does not necessarily sit inside $M_{\mu^\vee,\mathcal{O}_E} \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$ where P and $\tilde{s}$ are taken as above. We use the following decomposition of $S^G_{\mathrm{K}_f}$.

Lemma 7.2.2 *Let $S^G_{\mathrm{K}_f} = G(\mathbb{Q})\backslash G(\mathbb{A})/\mathrm{K}^\circ_\infty \mathrm{K}_f$ as before. Then $S^G_{\mathrm{K}_f}$ can be written as a disjoint union of $\Gamma_\nu\backslash G(\mathbb{R})^\circ/\mathrm{K}^\circ_\infty$ where $\nu \in \{1, \ldots, h\}$ with the narrow class number h of F and $\Gamma_\nu = G(\mathbb{Q}) \cap (G(\mathbb{R})^\circ, x_\nu \mathrm{K}_f x_\nu^{-1})$ is a congruence subgroup of $G(\mathbb{Q})$ for each ν.*

Proof. Recall that $G(\mathbb{A})$ has a decomposition;

$$G(\mathbb{A}) = \cup_\nu G(\mathbb{Q}) x_\nu G(\mathbb{R})^\circ \mathrm{K}_f \ \text{ (disjoint)}$$

and so $S^G_{\mathrm{K}_f} = \cup_\nu x_\nu G(\mathbb{R})^\circ/\mathrm{K}^\circ_\infty$. Let us consider a map $G(\mathbb{R})^\circ \longrightarrow x_\nu G(\mathbb{R})^\circ/\mathrm{K}^\circ_\infty$, given by $g_\infty \mapsto [x_\nu g_\infty \mathrm{K}^\circ_\infty]$. This is clearly a surjective map. Suppose that g_∞ and h_∞ are mapped to the same element, say $x_\nu g_\infty K^\circ_\infty$. It means that $x_\nu g_\infty = \gamma x_\nu h_\infty k_\infty k_f$ for some $\gamma \in G(\mathbb{Q})$, $k_\infty \in K^\circ_\infty$, and $k_f \in \mathrm{K}_f$. Comparing these elements for archimedean and non-archimedean parts separately, we see that $g_\infty = \gamma h_\infty k_\infty$ and $x_\nu = \gamma x_\nu k_f$. (Recall that the infinite part of x_ν is one.) It is deduced from the second condition that γ is in $x_\nu \mathrm{K}_f x_\nu^{-1}$, and therefore $\gamma \in G(\mathbb{Q}) \cap (G(\mathbb{R})^\circ, x_\nu \mathrm{K}_f x_\nu^{-1}) =: \Gamma_\nu$. This fact together with the first condition shows that the injectivity of the map holds if the space $G(\mathbb{R})^\circ$ is quotiented by Γ_ν from the left and by K°_∞ from the right. ∎

Throughout the article, each component of the space $S^G_{\mathrm{K}_f}$ is denoted as

$$S^G_{\mathrm{K}_f,\nu} := \Gamma_\nu\backslash G(\mathbb{R})^\circ/\mathrm{K}^\circ_\infty,$$

and assume that $S^G_{\mathrm{K}_f,1}$ is the identity component.

Now, let us consider the integral sheaf on this identity component $S^G_{\mathrm{K}_f,1}$. Write the integral sheaf on the identity component as $M_{\mu^\vee,\mathcal{O}_E,1} \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$. Then, the section over any open set U in $S^G_{\mathrm{K}_f,1}$ can be described as

$$\left(\widetilde{M_{\mu^\vee,\mathcal{O}_E,1} \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}}\right)(U)$$
$$= \left\{ \tilde{s}|_{P^{-1}(U)} \, : \, \tilde{s} \in \left(\widetilde{M_{\mu^\vee,E} \otimes_{\mathbb{Q}} \mathbb{A}_f}\right)(V) \text{ for any open set } V \text{ in } S^G_{\mathrm{K}_f} \text{ containing } U \right\}. \tag{7.2.3}$$

Note that the image of $P^{-1}(U)$ under the map $\tilde{s}$ is inside $\underset{\sim}{M_{\mu^\vee,\mathcal{O}_E}} \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$ as desired. The sheaves on the other components are derived from $M_{\mu^\vee,\mathcal{O}_E,1} \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$.

Lemma 7.2.4 *Let all the notations be as above. If $x_\eta x_\nu = x_\mu$ then*

$$S^G_{x_\nu \mathrm{K}_f x_\nu^{-1},\eta} \simeq S^G_{\mathrm{K}_f,\mu}.$$

In particular, $S^G_{x_\nu \mathrm{K}_f x_\nu^{-1},1} \simeq S^G_{\mathrm{K}_f,\nu}$ for any ν.

Proof. The proof is esentially the same as Lemma 7.2.2. Let $G(\mathbb{R})^\circ \longrightarrow S^G_{x_\nu \mathrm{K}_f x_\nu^{-1},\eta}$ be a map given by $x_\infty \mapsto [x_\infty x_\eta]$. Then any element equivalent to $[x_\infty x_\eta]$ must be of the form $\gamma x_\eta g_\infty k_\infty (x_\nu k_f x_\nu^{-1})$ where $\gamma \in G(\mathbb{Q})$, $g_\infty \in G(\mathbb{R})^\circ$, $k_\infty \in \mathrm{K}^\circ_\infty$, and $k_f \in \mathrm{K}_f$. Comparing the finite and infinite parts separately, we obtain that $x_\eta = \gamma x_\eta x_\nu k_f x_\nu^{-1}$ and $x_\infty = \gamma g_\infty k_\infty$. Therefore, $\gamma = x_\mu k_f^{-1} x_\mu^{-1}$ which concludes that $\gamma \in \Gamma_\mu$. We conclude the proof by observing that quotienting $G(\mathbb{R})^\circ$ by Γ_μ from the left and by K°_∞ from the right, i.e., $S^G_{\mathrm{K}_f,\mu}$, gives the injectivity for the map and so that the two spaces are isomorphic. ■

Let us call the isomorphism from $S^G_{x_\nu \mathrm{K}_f x_\nu^{-1},1}$ to $S^G_{\mathrm{K}_f,\nu}$ as τ_ν, where the isomorphism is given by $[g_\infty] \mapsto [x_\nu g_\infty]$. Since one can define the integral sheaf $\widetilde{M^\nu_{\mu^\vee,\mathcal{O}_E,1} \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}}$ on the identity component $S^G_{x_\nu \mathrm{K}_f x_\nu^{-1},1}$ for $S^G_{x_\nu \mathrm{K}_f x_\nu^{-1}}$, the integral sheaf on $S^G_{\mathrm{K}_f,\nu}$ can be defined by taking the direct image functor $\tau_{\nu,*}$. More precisely,

$$\begin{aligned} \left(\widetilde{M_{\mu^\vee,\mathcal{O}_E,\nu} \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}}\right)(U) &:= \tau_{\nu,*}\left(\widetilde{M^\nu_{\mu^\vee,\mathcal{O}_E,1} \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}}\right)(U) \\ &= \left(\widetilde{M^\nu_{\mu^\vee,\mathcal{O}_E,1} \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}}\right)(\tau_\nu^{-1}(U)). \end{aligned}$$

Finally, we take the adèlized integral sheaf on the whole space $S^G_{\mathrm{K}_f}$ to be the disjoint sum of those local sheaves:

$$\widetilde{M_{\mu^\vee,\mathcal{O}_E}\otimes_{\mathbb{Z}}}\hat{\mathbb{Z}} := \oplus_\nu \widetilde{M_{\mu^\vee,\mathcal{O}_E,\nu}\otimes_{\mathbb{Z}}}\hat{\mathbb{Z}}.$$

The diagram below summarize the definitions. Note that the bottom row describes several spaces used in the definitions while the top row is for the sheaves on each space.

$$\begin{array}{ccccc}
\widetilde{M_{\mu^\vee,\mathcal{O}_E}\otimes_{\mathbb{Z}}}\hat{\mathbb{Z}} & \xleftarrow{\oplus_\nu} & \widetilde{M_{\mu^\vee,\mathcal{O}_E,\nu}\otimes_{\mathbb{Z}}}\hat{\mathbb{Z}} & \xleftarrow{\tau_{\nu,*}} & \widetilde{M^\nu_{\mu^\vee,\mathcal{O}_E,1}\otimes_{\mathbb{Z}}}\hat{\mathbb{Z}} \\
 & & & & \\
S^G_{\mathrm{K}_f} & \simeq & \bigcup_\nu S^G_{\mathrm{K}_f,\nu} & \xleftarrow{\tau_\nu} & \bigcup_\nu S^G_{x_\nu \mathrm{K}_f x_\nu^{-1},1}
\end{array}$$

"The" integral sheaf $\widetilde{M_{\mu^\vee,\mathcal{O}_E}}$ is taken to be the intersection of the E-rational sheaf $\widetilde{M_{\mu^\vee,E}}$ and the adèlized integral sheaf $\widetilde{M_{\mu^\vee,\mathcal{O}_E}\otimes_{\mathbb{Z}}}\hat{\mathbb{Z}}$. Both sheaves sit inside the adèlized sheaf $\widetilde{M_{\mu,E}\otimes_{\mathbb{Q}}}\mathbb{A}_f$. See the following diagram.

$$\begin{tikzcd}
\widetilde{M_{\mu^\vee,E}} \arrow[r, hook] & \widetilde{M_{\mu^\vee,E}\otimes_{\mathbb{Q}}}\mathbb{A}_f \\
\widetilde{M_{\mu^\vee,\mathcal{O}_E}} \arrow[r, hook] \arrow[u, hook] & \widetilde{M_{\mu^\vee,\mathcal{O}_E}\otimes_{\mathbb{Z}}}\hat{Z} \arrow[u, hook]
\end{tikzcd}$$

Note that $\widetilde{M_{\mu^\vee,\mathcal{O}_E}} = \widetilde{M_{\mu^\vee,E}} \cap \widetilde{M_{\mu^\vee,\mathcal{O}_E}\otimes_{\mathbb{Z}}}\hat{Z}$, is actually a sheaf. To prove this, let us recall the definition of the sheaf: A presheaf $\mathcal{F}$ is a sheaf if and only if the following sequence is exact.

$$0 \longrightarrow \mathcal{F}(U) \xrightarrow{p_0} \prod_\alpha \mathcal{F}(U_\alpha) \rightrightarrows \prod_{(\alpha,\beta)} \mathcal{F}(U_\alpha \cap U_\beta),$$

where $\{U_\alpha\}_\alpha$ is an open covering of an open set U, p_0 is the set of restriction morphisms $s \mapsto (s|_{U_\alpha})_\alpha$, and the last map is given by $(s_\alpha) \mapsto (s_\alpha|_{U_\alpha\cap U_\beta} - s_\beta|_{U_\alpha\cap U_\beta})$. For our situation, suppose (s_α) is in $\prod \widetilde{M_{\mu^\vee,\mathcal{O}_E}}(U_\alpha)$ and such that $(s_\alpha|_{U_\alpha\cap U_\beta} - s_\beta|_{U_\alpha\cap U_\beta}) = 0$ in

$\prod \widetilde{M_{\mu^\vee,\mathcal{O}_E}}(U_\alpha \cap U_\beta)$. We need to show that there exists a unique element in $\widetilde{M_{\mu^\vee,\mathcal{O}_E}}(U)$ that is mapped to (s_α) by p_0. Since (s_α) belongs to $\prod \widetilde{M_{\mu^\vee,E}}(U_\alpha)$, there is a unique s in $\widetilde{M_{\mu^\vee,E}}(U)$ such that $p_0(s) = (s_\alpha)$. Similarly, there is a unique s' in $M_{\mu^\vee,\widetilde{\mathcal{O}_E} \otimes} \hat{Z}(U)$ such that $p_0(s') = (s_\alpha)$. But then, since both s and s' are in $M_{\mu^\vee,\widetilde{E \otimes}} \mathbb{A}_f(U)$, they must coincide by the uniqueness of such sections. Hence $s = s'$ is the desired element in $\left(\widetilde{M_{\mu^\vee,E}} \cap M_{\mu^\vee,\widetilde{\mathcal{O}_E} \otimes} \hat{Z}\right)(U) =: \widetilde{M_{\mu^\vee,\mathcal{O}_E}}(U)$.

7.2.3 $\mathcal{O}_E$-structure for cohomology group

In order to define the integral structure for cohomology group, let us introduce some facts from sheaf theory. All the details are found, for example, in [20]. As it was mentioned in Section 6.2.3, cuspidal cohomology injects into cohomology with compact supports: $H^n_{\mathrm{cusp}}(S^G_{\mathrm{K}_f}, \widetilde{M_{\mu^\vee,\mathbb{C}}}) \hookrightarrow H^n_c(S^G_{\mathrm{K}_f}, \widetilde{M_{\mu^\vee,\mathbb{C}}})$. Furthermore, the image of $H^n_c(S^G_{\mathrm{K}_f}, \widetilde{M_{\mu^\vee,\mathbb{C}}})$ in the full cohomology is called the inner cohomology and denoted as

$$H^n_!(S^G_{\mathrm{K}_f}, \widetilde{M_{\mu^\vee,\mathbb{C}}}) := \mathrm{Im}\left(H^n_c(S^G_{\mathrm{K}_f}, \widetilde{M_{\mu^\vee,\mathbb{C}}}) \longrightarrow H^n(S^G_{\mathrm{K}_f}, \widetilde{M_{\mu^\vee,\mathbb{C}}})\right).$$

It is a fact that cuspidal cohomology always sits inside inner cohomology. (See Clozel [7].) In particular, if μ is regular, then one has the following theorem.

Theorem 7.2.5 (Harder, [20]) *If μ is regular, then*

$$H^\bullet_{cusp}(S^G_{\mathrm{K}_f}, \widetilde{M_{\mu^\vee,\mathbb{C}}}) \simeq H^\bullet_!(S^G_{\mathrm{K}_f}, \widetilde{M_{\mu^\vee,\mathbb{C}}}).$$

By saying $\mu = (\mu_1, \ldots, \mu_n)$ is regular, we mean that $a_j > b_j$ for all j where $\mu_j = (a_j, b_j)$. In the case of holomorphic Hilbert cusp forms, μ is regular if the weight k_j is greater than 2 for all j, which follows from the highest weight μ given by

$$\mu_j = (a_j, b_j) = \left(\frac{k_j - 2}{2}, -\frac{k_j - 2}{2}\right)$$

as seen in Section 6.2.12.

It is assumed in Theorem 7.1.2 that all k_j's are at least 4, which allows us to view cuspidal cohomology $H^n_{\mathrm{cusp}}(S^G_{\mathrm{K}_f}, \widetilde{M_{\mu^\vee,\mathbb{C}}})$ as inner cohomology $H^n_!(S^G_{\mathrm{K}_f}, \widetilde{M_{\mu^\vee,\mathbb{C}}})$. The purpose of this identification is to consider integral cohomology. Inner cohomology is sheaf-theoretically defined and admits E-rational structure $H^n_!(S^G_{\mathrm{K}_f}, \widetilde{M_{\mu^\vee,E}})$ and $\mathcal{O}_E$-integral structure $H^n_!(S^G_{\mathrm{K}_f}, \widetilde{M_{\mu^\vee,\mathcal{O}_E}})$. We consider the map

$$H^n_!(S^G_{\mathrm{K}_f}, \widetilde{M_{\mu^\vee,\mathcal{O}_E}}) \longrightarrow H^n_!(S^G_{\mathrm{K}_f}, \widetilde{M_{\mu^\vee,E}}),$$

and write $\bar{H}^n_!(S^G_{\mathrm{K}_f}, \widetilde{M_{\mu^\vee,\mathcal{O}_E}})$ for the image of the inclusion map above. This is done in order to avoid any trouble caused by the torsion elements in the integral cohomology.

7.3 Proof of Theorem 7.1.2

7.3.1 Some notes and assumptions on integral cohomology

Let $\mathcal{S}_k(\mathfrak{n}, \tilde{\omega})_{\mathcal{O}_E}$ be the space of Hilbert cusp forms of weight k, level $\mathfrak{n}$, and with a Hecke character $\tilde{\omega}$, such that all the normalized Fourier coefficients $\mathrm{C}(\mathfrak{m}, *)$ are in $\mathcal{O}_E$. The Hecke algebra $\mathcal{H}_k(\mathfrak{n}, \tilde{\omega})_{\mathcal{O}_E}$ is defined to be an $\mathcal{O}_E$-subalgebra of $\mathrm{End}_{\mathbb{C}}(\mathcal{S}_k(\mathfrak{n}, \tilde{\omega}))$ generated by normalized Hecke operators $\{\mathbb{T}'_{\mathfrak{m}}\}$ defined in Section 3.3 for all integral ideals $\mathfrak{m}$. We note that it is a commutative algebra with the identity where $\mathbb{T}'_{\mathcal{O}_F} = \mathbb{T}_{\mathcal{O}_F}$ is the identity element. Furthermore, it also contains $\mathbb{S}_{\mathfrak{p}} := \mathrm{K}_{\mathfrak{p}} \left(\begin{smallmatrix} \varpi_{\mathfrak{p}} & \\ & \varpi_{\pi} \end{smallmatrix}\right) \mathrm{K}_{\mathfrak{p}}$, for all $\mathfrak{p}$ not dividing $\mathfrak{n}$, which is obtained by $(\mathbb{T}'_{\mathfrak{p}})^2 - \mathbb{T}'_{\mathfrak{p}^2}$ up to a scalar. See [23, Section 3] for details. If $\mathcal{O}_E$ contains $\mathbb{Z}[\tilde{\omega}]_\ell$ then $\mathcal{S}_k(\mathfrak{n}, \tilde{\omega})_{\mathcal{O}_E}$ is stable under the action of the Hecke algebra $\mathcal{H}_k(\mathfrak{n}, \tilde{\omega})_{\mathcal{O}_E}$. The necessity of the condition $\mathbb{Z}[\tilde{\omega}]_\ell \subset \mathcal{O}_E$ is verified by computing

$$\mathrm{C}(\mathfrak{m}, \mathbb{T}'_{\mathfrak{n}}\mathbf{f}) = \sum_{\mathfrak{m}+\mathfrak{n}\subset\mathfrak{a}} \tilde{\omega}(\mathfrak{a}) N(\mathfrak{a})^{k_0-1} \mathrm{C}(\mathfrak{a}^{-2}\mathfrak{m}\mathfrak{n}, \mathbf{f}). \tag{7.3.1}$$

(See [41] for details.)

Theorem 7.3.2 (Hida, [23]) *Let $\tilde{\omega}$ be a Hecke character, and E a finite extension of $\mathbb{Q}_\ell$ such that $\mathcal{O}_E$ contains $\mathbb{Z}[\tilde{\omega}]$. Then one has the following isomorphism.*

$$S_k(\mathfrak{n}, \tilde{\omega})_{\mathcal{O}_E} \simeq \mathcal{H}_k(\mathfrak{n}, \tilde{\omega})^*_{\mathcal{O}_E} = \mathrm{Hom}_{\mathcal{O}_E}(\mathcal{H}_k(\mathfrak{n}, \tilde{\omega})_{\mathcal{O}_E}, \mathcal{O}_E).$$

Proof. The isomorphism is induced from the pairing:

$$<,>:\ \mathcal{H}_k(\mathfrak{n},\tilde{\omega})_{\mathcal{O}_E}\times\mathcal{S}_k(\mathfrak{n},\tilde{\omega})_{\mathcal{O}_E}\longrightarrow\mathcal{O}_E$$

given by $<\mathbb{T},\mathbf{f}>\mapsto \mathrm{C}(\mathcal{O}_F,\mathbb{T}\mathbf{f})$. It can be seen by (7.3.1) that, for any integral ideal $\mathfrak{m}$, $\mathrm{C}(\mathcal{O}_F,\mathbb{T}'_{\mathfrak{m}}\mathbf{f})=\mathrm{C}(\mathfrak{m},\mathbf{f})$.

Let us first prove the result for E-rational structure. Since $\mathcal{H}_k(\mathfrak{n},\tilde{\omega})_E$ and $\mathcal{S}_k(\mathfrak{n},\tilde{\omega})_E$ are finite-dimensional, it is enough to show the non-degeneracy of the pairing. If $<\mathbb{T},\mathbf{f}>=0$ for all $\mathbf{f}\in\mathcal{S}_k(\mathfrak{n},\tilde{\omega})_E$, then one has

$$\mathrm{C}(\mathfrak{m},\mathbb{T}\mathbf{f})=\mathrm{C}(\mathcal{O}_F,\mathbb{T}'_{\mathfrak{m}}\mathbb{T}\mathbf{f})=\mathrm{C}(\mathcal{O}_F,\mathbb{T}\mathbb{T}'_{\mathfrak{m}}\mathbf{f})=<\mathbb{T},\mathbb{T}'_{\mathfrak{m}}\mathbf{f}>=0.$$

Therefore $\mathbb{T}\mathbf{f}=0$ for all $\mathbf{f}$, which means that $\mathbb{T}=0$ as an operator. Now suppose $<\mathbb{T},\mathbf{f}>=0$ for all $\mathbb{T}$. It gives that

$$\mathrm{C}(\mathfrak{m},\mathbf{f})=<\mathbb{T}'_{\mathfrak{m}},\mathbf{f}>=0$$

for all $\mathfrak{m}$ and that $\mathbf{f}=0$. This completes the proof for

$$\mathcal{S}_k(\mathfrak{n},\tilde{\omega})_E\simeq\mathcal{H}_k(\mathfrak{n},\tilde{\omega})^*_E.$$

Now, let ϕ be a homomorphism in $\mathcal{H}_k(\mathfrak{n},\tilde{\omega})^*_{\mathcal{O}_E}$. It can be extended to a E-linear map $\tilde{\phi}$ in $\mathcal{H}_k(\mathfrak{n},\tilde{\omega})^*_E$. This can be done because $\mathcal{H}_k(\mathfrak{n},\tilde{\omega})_{\mathcal{O}_E}\otimes_{\mathcal{O}_E}E=\mathcal{H}_k(\mathfrak{n},\tilde{\omega})_E$. But by the first part of the proof, there is a corresponding Hilbert cusp form $\mathbf{f}$ in $\mathcal{S}_k(\mathfrak{n},\tilde{\omega})_E$ so that $\tilde{\phi}(\mathbb{T})=<\mathbb{T},\mathbf{f}>$ for all $\mathbb{T}$ in $\mathcal{H}_k(\mathfrak{n},\tilde{\omega})_E$ by applying the first case. In particular, $\tilde{\phi}(\mathbb{T}'_{\mathfrak{n}})=<\mathbb{T}'_{\mathfrak{n}},\mathbf{f}>$ for all $\mathbb{T}'_{\mathfrak{n}}$ in $\mathcal{H}_k(\mathfrak{n},\tilde{\omega})_{\mathcal{O}_E}$. Hence,

$$\mathrm{C}(\mathfrak{n},\mathbf{f})=\mathrm{C}(\mathcal{O}_F,\mathbb{T}'_{\mathfrak{n}}\mathbf{f})=<\mathbb{T}'_{\mathfrak{n}},\mathbf{f}>=\phi(\mathbb{T}'_{\mathfrak{n}})\in\mathcal{O}_E.$$

Hence $\mathbf{f}$ is in $\mathcal{S}_k(\mathfrak{n},\tilde{\omega})_{\mathcal{O}_E}$. ■

On the other hand, there is an action of $\mathcal{H}_k(\mathfrak{n},\tilde{\omega})_{\mathcal{O}_E}$ on the integral inner cohomology $\bar{H}^n_!(S^{\mathrm{GL}_2}_{\mathrm{K}_f},\widetilde{M_{\mu^\vee,\mathcal{O}_E}})$. See Harder's book in progress mentioned above. For a Hecke character $\tilde{\omega}$, let us define

$$\bar{H}^n_!(S^{\mathrm{GL}_2}_{\mathrm{K}_f},\widetilde{M_{\mu^\vee,\mathcal{O}_E}})(\tilde{\omega}):=\left\{\xi\in\bar{H}^n_!(S^{\mathrm{GL}_2}_{\mathrm{K}_f},\widetilde{M_{\mu^\vee,\mathcal{O}_E}})\ :\ \mathbb{S}_{\mathfrak{p}}\xi=\tilde{\omega}(\varpi_{\mathfrak{p}})\xi\text{ for }\mathfrak{p}\nmid\mathfrak{n}.\right\}.$$

For each character ϵ of $\mathrm{K}_\infty/\mathrm{K}_\infty^\circ$, we assume the existence of an isomorphism of $\mathcal{O}_E$-modules:

$$\bar{H}_!^n(S^{\mathrm{GL}_2}_{\mathrm{K}_f}, \widetilde{M_{\mu^\vee,\mathcal{O}_E}})(\tilde{\omega}\times\epsilon) \simeq \mathcal{H}_k(\mathfrak{n},\tilde{\omega})^*_{\mathcal{O}_E}. \tag{7.3.3}$$

This isomorphism and Theorem 7.3.2 provide the following proposition.

Proposition 7.3.4 *Assuming the isomorphism in (7.3.3), there is an isomorphism ϑ^ϵ of $\mathcal{H}_k(\mathfrak{n},\tilde{\omega})_{\mathcal{O}_E}$-modules;*

$$\vartheta^\epsilon \,:\, \mathcal{S}_k(\mathfrak{n},\tilde{\omega})_{\mathcal{O}_E} \xrightarrow{\ \sim\ } \bar{H}_!^n(S^{\mathrm{GL}_2}_{\mathrm{K}_f}, \widetilde{M_{\mu^\vee,\mathcal{O}_E}})(\tilde{\omega}\times\epsilon)$$

for each $\epsilon \in \left(\widehat{\mathrm{K}_\infty/\mathrm{K}_\infty^\circ}\right)$.

7.3.2 Integral interpretation of the diagram (6.2.1)

To complete the proof of Theorem 7.1.2, we analyze each step in the diagram 6.2.1 integrally.

Let us let ι be the inclusion map from GL_1 to GL_2 as in Section 6.2.4. Everything discussed in the section works integrally as well, and we obtain a map

$$\bar{H}_!^n(S^{\mathrm{GL}_2}_{\mathrm{K}_f}, \widetilde{M_{\mu^\vee,\mathcal{O}_E}})(\epsilon) \xrightarrow{\iota^*} \bar{H}_!^n(\bar{S}^{\mathrm{GL}_1}_{\iota^*\mathrm{K}_f}, \iota^*\widetilde{M_{\mu^\vee,\mathcal{O}_E}})(\epsilon) \xrightarrow{\ \sim\ } \bar{H}_!^n(\bar{S}^{\mathrm{GL}_1}_{\iota^*\mathrm{K}_f}, \iota^*\widetilde{M_{\mu^\vee,\mathcal{O}_E}})(\epsilon).$$

We now consider the map $\mathcal{T}^*$ to obtain the cohomology class with trivial coefficients as in Section 6.2.6. Recall that, from Lemma 4.2, a nonzero map $\mathcal{T} \in \mathrm{Hom}_{\mathrm{GL}_1(F_\infty)}(M_{\mu^\vee,\mathbb{C}}, \mathbb{1}_{\mathbb{C}})$ is given by $\sum_{j=0}^{\alpha-\beta} c_j \mathsf{s}_j \mapsto c_\alpha$. It is clear from the definition of the integral structure $M_{\mu^\vee,\mathcal{O}_E}$,which is the $\mathcal{O}_E$-span of $\{g.\mathsf{s}_\alpha \,:\, g \in \mathrm{GL}_2(\mathcal{O}_F)\}$, that the exact same $\mathcal{T}$ can be chosen. Noting that the image of the restricted map $\mathcal{T}|_{M_{\mu^\vee,\mathcal{O}_E}}$ lies into $\mathcal{O}_E$, it induces a homomorphism

$$(\mathcal{T}|_{M_{\mu^\vee,\mathcal{O}_E}})^* \,:\, \bar{H}_!^n\left(\bar{S}^{\mathrm{GL}_1}_{\mathrm{R}_f}, \iota^*\widetilde{M_{\mu^\vee,\mathcal{O}_E}}\right)(\tilde{\omega}\times\epsilon) \longrightarrow \bar{H}_!^n\left(\bar{S}^{\mathrm{GL}_1}_{\mathrm{R}_f}, \mathbb{1}_{\mathcal{O}_E}\right)(\tilde{\omega}\times\epsilon).$$

We shall drop the subscript of this map and simply write $\mathcal{T}^*$ as well.

Finally, Poincaré duality can be applied integrally which gives a map

$$\int_{\mathcal{M}} \,:\, \bar{H}_!^n(\mathcal{M}, \mathbb{1}_{\mathcal{O}_E}) \longrightarrow \mathcal{O}_E.$$

(See [21].)

Putting $\mathfrak{S} := \displaystyle\int_{\mathcal{M}} \mathcal{T}^*\iota^*$, we now have an $\mathcal{O}_E$-linear map

$$\mathfrak{S} \,:\, \bar{H}_!^n\left(S^G_{\mathrm{K}_f}, \widetilde{M_{\mu^\vee,\mathcal{O}_E}}\right)(\tilde{\omega}\times\epsilon) \longrightarrow \mathcal{O}_E.$$

Notice that the image of this map is an ideal in $\mathcal{O}_E$, say $\mathfrak{P}_E^r$ for some $r \geq 0$. If $r \geq 1$, let us modify the integral structure $M_{\mu^\vee,\mathcal{O}_E}$ of $M_{\mu^\vee,\mathbb{C}}$ by multiplying by ϖ_E^{-r}, i.e., the integral structure is taken to be $\varpi_E^{-r} M_{\mu^\vee,\mathcal{O}_E} := \left\langle \{g.\varpi_E^{-r}\mathsf{s}_\alpha \,:\, g \in \mathrm{GL}_2(F)\}\right\rangle_{\mathcal{O}_E}$. Any cohomology class ϑ_r in $\bar{H}_!^n\left(S^G_{\mathrm{K}_f}, \widetilde{\varpi_E^{-r} M_{\mu^\vee,\mathcal{O}_E}}\right)(\tilde{\omega}\times\epsilon)$ is described as $\varpi_E^{-r}\vartheta$ with some ϑ in $\bar{H}_!^n\left(S^G_{\mathrm{K}_f}, \widetilde{M_{\mu^\vee,\mathcal{O}_E}}\right)(\tilde{\omega}\times\epsilon)$. It can be easily viewed from considering the element in $H^n\left(\mathfrak{g}_\infty, \mathrm{K}_\infty^\circ; \Pi_\infty \otimes M_{\mu^\vee,\mathbb{C}}\right)$ as in Section 4.4.5. Hence $\mathfrak{S}(\vartheta_r) = \varpi_E^{-r}\mathfrak{S}(\vartheta)$, and it follows that the image of $\mathfrak{S}$ is exactly $\mathcal{O}_E$. This normalization needs to be considered, as otherwise, the algebraic parts of critical L-values will always be in $\mathfrak{P}_E^r$ and therefore the congruence property of our interest becomes a trivial statement. See also Section 7.3.4. In the following sections, it is understood that a suitable normalization is taken on the integral structure and that the normalized integral structure is denoted as $\widetilde{M_{\mu^\vee,\mathcal{O}_E}}$. This shall cause no confusion.

7.3.3 Refinement of the periods

Recall from Section 6.1 that our period $p^\epsilon(\Pi)$ was chosen so that a map from the K_f-fixed Whittaker space to the cohomology group preserves the E-rational structure of each side, i.e.,

$$\mathcal{F}^\epsilon_{\Pi,0} \,:\, \mathcal{W}(\Pi_f)_E^{\mathrm{K}_f} \longrightarrow H_!^n(S^G_{\mathrm{K}_f}, \widetilde{M_{\mu^\vee,E}})(\epsilon),$$

where ϵ stands for a character of $\mathrm{K}_\infty/\mathrm{K}_\infty^\circ = \{\pm\}^n$. For a finer treatment, these periods need to be modified so that it preserves the integral structure on both sides.

Consider the following maps.

$$\begin{tikzcd}
\mathcal{S}_k(\mathfrak{n},\tilde{\omega})_{\mathcal{O}_E} \arrow[r, "\vartheta^\epsilon"] & \bar{H}^n_!\left(S^G_{\mathrm{K}_f}, \widetilde{M_{\mu^\vee,\mathcal{O}_E}}\right)(\tilde{\omega}\times\epsilon) \arrow[r, hook] & H^n_!\left(S^G_{\mathrm{K}_f}, \widetilde{M_{\mu^\vee,\mathbb{C}}}\right)(\tilde{\omega}\times\epsilon) \\
 & & \mathcal{W}(\Pi_f)^{\mathrm{K}_f} \arrow[u, hook, "\mathcal{F}^\epsilon_\Pi"']
\end{tikzcd}$$

As discussed in Section 7.3.2, the integral sheaf $\widetilde{M_{\mu^\vee,\mathcal{O}_E}}$ might have been modified in which case it differs from the one originally defined in Section 7.2.2 by a scalar. The map ϑ^ϵ is an isomorphism given in Proposition 7.3.3, and $\mathcal{F}^\epsilon_\Pi$ is as defined in Section 6.1.4. Let $\mathbf{f}$ be a primitive form in $\mathcal{S}_k(\mathfrak{n},\tilde{\omega})_{\mathcal{O}_E}$. We saw already in Chapter 6 that it is mapped to W° in $\mathcal{W}(\Pi_f)^{\mathrm{K}_f}$ where Π is the representation corresponding to $\mathbf{f}$, and that W° is mapped to ϑ^ϵ_Π via $\mathcal{F}^\epsilon_\Pi$. On the other hand, we have $\vartheta^\epsilon(\mathbf{f})$ in $\bar{H}^n_!\left(S^G_{\mathrm{K}_f}, \widetilde{M_{\mu^\vee,\mathcal{O}_E}}\right)(\tilde{\omega}\times\epsilon)$. So we have two classes $\vartheta^\epsilon(\mathbf{f})$ and ϑ^ϵ_Π in $H^n_!\left(S^G_{\mathrm{K}_f}, \widetilde{M_{\mu^\vee,\mathbb{C}}}\right)(\tilde{\omega}\times\epsilon)$ and if K_f is chosen well, both correspond to the same irreducible one-dimensional representation $\Pi_f^{\mathrm{K}_f}$ which appears with multiplicity one. Hence they differ by a scalar. In other words, there exists $p^\epsilon(\Pi)^\circ$ in $\mathbb{C}^*$ so that

$$\vartheta^\epsilon(\mathbf{f}) = \frac{1}{p^\epsilon(\Pi)^\circ}\vartheta^\epsilon_\Pi,$$

which is the canonical period of our interest. In order to have a uniformity in our notations, we shall call $\vartheta^\epsilon(\mathbf{f}) =: \vartheta^{\epsilon,\circ}_\Pi$.

7.3.4 Concluding part of the proof of Theorem 7.1.2

As in Chapter 6, we are only interested in the case $\epsilon = (+,\dots,+)$, and we suppress the notation ϵ if $\epsilon = (+,\dots,+)$. So $\vartheta^\epsilon = \vartheta$, etc. Now, summarizing all the details discussed in Chapter 6 and this chapter, we obtain the following diagram.

$$\begin{tikzcd}
\mathcal{S}_k(\mathfrak{n},\tilde{\omega})_{\mathcal{O}_E} \arrow[r, "\vartheta"] & \bar{H}^n_!\left(S^G_{\mathrm{K}_f}, \widetilde{M_{\mu^\vee,\mathcal{O}_E}}\right)(\tilde{\omega}\times(++)) \arrow[r, "\mathfrak{S}"] \arrow[d, hook] & \mathcal{O}_E \arrow[d, hook] \\
 & H^n_!\left(S^G_{\mathrm{K}_f}, \widetilde{M_{\mu^\vee,\mathbb{C}}}\right)(\tilde{\omega}\times(++)) \arrow[r, "\mathfrak{S}"] & \mathbb{C}
\end{tikzcd}$$

with $\mathfrak{S} = \int_{\mathcal{M}} \mathcal{T}^* \iota^*$. If $\mathbf{f}$ is a primitive form in $\mathcal{S}_k(\mathfrak{n}, \tilde{\omega})_{\mathcal{O}_E}$, it is mapped to ϑ_Π° in $\bar{H}^n_! \left(S^G_{\mathrm{K}_f}, \widetilde{M_{\mu^\vee, \mathcal{O}_E}} \right) (\tilde{\omega} \times (++))$. Applying $\mathfrak{S}$, we see that

$$\mathfrak{S}(\vartheta_\Pi^\circ) = \frac{1}{p(\Pi)^\circ} \mathfrak{S}(\vartheta_\Pi) = \frac{c(k)}{p(\Pi)^\circ (2\pi i)^{d_\infty}} L\left(\frac{1}{2}, \Pi\right) \in \mathcal{O}_E,$$

where

$$c(k) = \prod_j (-1)^{(k_j-2)/2} \frac{(k_j-2)!}{((k_j-2)/2)!}.$$

The last equality follows from (6.2.17). (Notice that $\mathrm{vol}(\mathrm{R}_f) = 1$.)

Now, let $\mathbf{f}$ and $\mathbf{f}'$ be primitive forms in $\mathcal{S}_k(\mathfrak{n}, \tilde{\omega})_{\mathcal{O}_E}$ such that $\mathbf{f} \equiv \mathbf{f}' \pmod{\varpi_E^s}$. By $\mathcal{O}_E$-linearity of $\vartheta = \vartheta^{(++)}$ and $\mathfrak{S}$, the congruence holds at each level, i.e., $\vartheta_\Pi^\circ \equiv \vartheta_{\Pi'}^\circ \pmod{\varpi_E^s}$. and $\mathfrak{S}(\vartheta_\Pi^\circ) \equiv \mathfrak{S}(\vartheta_{\Pi'}^\circ) \pmod{\varpi_E^s}$. This, together with (6.2.17), we obtain

$$c(k) \frac{L(1/2, \Pi)}{p(\Pi)^\circ (2\pi i)^{d_\infty}} \equiv c(k) \frac{L(1/2, \Pi')}{p(\Pi')^\circ (2\pi i)^{d_\infty}} \mod \varpi_E^s.$$

This completes the proof of Theorem 7.1.2. ■

CHAPTER 8

Non-vanishing of derivatives of L-functions

In this chapter, we study a non-vanishing property of the derivative of the L-function of a Hilbert modular cusp form at the center of symmetry. The property was originally proven by Gun, Murty, and Rath for the case of an elliptic modular cusp form. (See [18, Theorem 4.1].) Our aim is to generalize their result to Hilbert modular forms. A precise statement of our theorem is as follows:

Theorem 8.0.1 *Let* $\mathbf{f}$ *be a holomorphic Hilbert modular cusp form of weight* $k = (k_1, \cdots, k_n)$*, level* $\mathfrak{n}$*, with trivial character, over a totally real number field* F *of degree* n*. Assume that* $\mathbf{f}$ *is primitive, and the weight satisfies the following conditions:* $k_j \geq 4$ *for all* j *and* $k_1 \equiv \cdots \equiv k_n \equiv 0 \mod 2$*. Let* $k_0 = \max(k_1, \ldots, k_n)$*. If* $L_f(k_0/2, \mathbf{f}) \neq 0$*, then*

$$\frac{L_f'(k_0/2, \mathbf{f})}{L_f(k_0/2, \mathbf{f})} = -\frac{\log \mathrm{N}(\mathfrak{n}\mathfrak{D}_F^2)}{2} + n\log(2\pi) - \sum_{j=1}^{n} \psi\left(\frac{k_j}{2}\right),$$

where $\mathfrak{D}_F$ *is the different ideal of* F*, and* ψ *is the logarithmic derivative of the gamma function. Furthermore,* $L_f'(k_0/2, \mathbf{f}) \neq 0$*, i.e., if the central critical value is nonzero then so is the derivative at the center of symmetry.*

Proof. Gun, Murty, and Rath proved the case $n = 1$ in [18, Theorem 4.1]. So we assume that $n \geq 2$. For this proof, the properties discussed in Section 3.4 will be applied. Also, the (finite) L-function needs to be completed in a different way from Section 5.3.1. To distinguish those functions, let us call it $\Lambda(s, \mathbf{f})$ that is defined to be

$$\Lambda(s, \mathbf{f}) := \mathrm{N}(\mathfrak{n}\mathfrak{D}_F^2)^{s/2}(2\pi)^{-ns}\prod_{j=1}^{n}\Gamma\left(s - \frac{k_0 - k_j}{2}\right)L_f(s, \mathbf{f}).$$

It converges for $\Re(s) \gg 0$, and has an analytic continuation to $\mathbb{C}$. The completed L-function $\Lambda(s, \mathbf{f})$ satisfies the functional equation:

$$\Lambda(s, \mathbf{f}) = i^{\sum k_j} \Lambda(k_0 - s, \mathbf{f}|\mathrm{J}_{\mathfrak{n}}), \tag{8.0.2}$$

where $\mathbf{f}|\mathrm{J}_{\mathfrak{n}}$ is as in (3.4.2).

Let $\mathbf{f}$ be a normalized common eigenform for $\mathbb{T}'_{\mathfrak{m}}$. Then, as given in Section 3.3, the eigenvalue for $\mathbb{T}'_{\mathfrak{m}}$ is $C(\mathfrak{m}, \mathbf{f})$. Moreover, it is real by Proposition 3.4.1. It follows by Proposition 3.4.3, that $\mathbf{f}|\mathrm{J}_{\mathfrak{n}} = c \cdot \mathbf{f}$ with some constant c. Therefore, the finite L-function attached to $\mathbf{f}|\mathrm{J}_{\mathfrak{n}}$, i.e., attached to $c \cdot \mathbf{f}$ is:

$$L_f(s, \mathbf{f}|\mathrm{J}_{\mathfrak{n}}) = L_f(s, c\mathbf{f}) = \sum \frac{c \cdot C(\mathfrak{m}, \mathbf{f})}{\mathrm{N}(\mathfrak{m})^s} = cL_f(s, \mathbf{f}).$$

The functional equation given in (8.0.2) can be written as

$$\begin{aligned} &\mathrm{N}(\mathfrak{n}\mathfrak{D}_F^2)^{s/2}(2\pi)^{-ns} \prod_{j=1}^{n} \Gamma\left(s - \frac{k_0 - k_j}{2}\right) L_f(s, \mathbf{f}) \\ &\qquad = c \cdot i^{\sum k_j} \mathrm{N}(\mathfrak{n}\mathfrak{D}_F^2)^{(k_0 - s)/2}(2\pi)^{-n(k_0 - s)} \prod_{j=1}^{n} \Gamma\left(\frac{k_0 + k_j}{2} - s\right) L_f(k_0 - s, \mathbf{f}). \end{aligned}$$

Taking the logarithmic derivative on both sides with respect to s, one has

$$\begin{aligned} &\frac{\log \mathrm{N}(\mathfrak{n}\mathfrak{D}_F^2)}{2} - n\log(2\pi) + \sum_{j=1}^{n} \psi\left(s - \frac{k_0 - k_j}{2}\right) + \frac{L'_f(s, \mathbf{f})}{L_f(s, \mathbf{f})} \\ &\qquad = -\frac{\log \mathrm{N}(\mathfrak{n}\mathfrak{D}_F^2)}{2} + n\log(2\pi) - \sum_{j=1}^{n} \psi\left(\frac{k_0 + k_j}{2} - s\right) - \frac{L'_f(k_0 - s, \mathbf{f})}{L_f(k_0 - s, \mathbf{f})}. \end{aligned}$$

Here, $\psi(k) = \mathrm{H}_{k-1} - \gamma$ with $\mathrm{H}_{k-1} := \sum_{m=1}^{k-1} 1/m$ being the $(k-1)$-th harmonic number, and γ the Euler's constant.

The first part of the theorem is obtained by letting $s = k_0/2$. For the second part, suppose that $L'_f(k_0/2, \mathbf{f}) = 0$. Then one has

$$n\,(\gamma + \log(2\pi)) = \frac{1}{2} \log \mathrm{N}(\mathfrak{n}) + \log(d_F) + \sum_{j=1}^{n} \mathrm{H}_{k_j/2-1}, \tag{8.0.3}$$

where d_F is the discriminant of F. By our assumption on k_j's, $\min\{\sum_j H_{k_j/2-1}\} = n$ that is attained when all the k_j's are 4. Using this and the Minkowski bound:

$$|d_F| \geq \frac{n^{2n}}{(n!)^2},$$

we see that $2n\log(n) - 2\log(n!) + n$ is a lower bound of the right hand side of the equation (8.0.3). But, for $n \geq 7$, this value is larger than $n\,(\gamma + \log(2\pi))$ which is bounded above by $2.4151n$. Hence (8.0.3) cannot be attained.

Now, we only need to check when $n \leq 6$. The table below shows the minimal discriminant of each degree extension; see Voight [45, Table 3].

n	2	3	4	5	6
minimal d_F	5	49	725	14641	300125

Applying each minimal d_F in (8.0.3) for $n \geq 4$, one can check that the right hand side exceeds the left hand side for any weight and level, as long as all the k_j's are at least 4.

If $n = 2$ or 3, one needs to examine several cases. Without loss of generality, let us assume that $k_i \leq k_{i+1}$. If $n = 3$ and the weight is at least $k = (4, 4, 6)$, the right hand side of (8.0.3) exceeds the left hand side for any level and any discriminant. So the only remaining case is $k = (4, 4, 4)$. But it can be easily verified that the equality in (8.0.3) never be satisfied. Checking the case $n = 2$ similarly completes the proof for the second part of the theorem.

■

The necessity of the hypotheses on the weight $k = (k_1, \cdots, k_n)$ are stated in Remark 8.0.6 and 8.0.7. The theorem leads us to some applications in transcendental number theory, as in [18]. See Corollary 8.0.4 and 8.0.5 below.

Corollary 8.0.4 *Suppose that* $\mathbf{f}$ *satisfies all the conditions given in Theorem 8.0.1.*

Then

$$\exp\left(\frac{L_f'(k_0/2,\mathbf{f})}{L_f(k_0/2,\mathbf{f})}+\sum_{j=1}^{n}\psi\left(\frac{k_j}{2}\right)\right)$$

is transcendental.

Proof. This follows from the first part of Theorem 8.0.1:

$$\exp\left(\frac{L_f'(k_0/2,\mathbf{f})}{L_f(k_0/2,\mathbf{f})}+\sum_{j=1}^{n}\psi\left(\frac{k_j}{2}\right)\right)=\exp\left(n\log(2\pi)-\frac{\log \mathrm{N}(\mathfrak{n}\mathfrak{D}_F^2)}{2}\right)=\frac{(2\pi)^n}{\mathrm{N}(\mathfrak{n}\mathfrak{D}_F^2)^{1/2}},$$

which is transcendental. ■

Corollary 8.0.5 *Fix* $k=(k_1,\cdots,k_n)$ *with* $k_j\equiv 0 \mod 2$ *for all* j*, and let* $\mathfrak{S}_k$ *be the set of all primitive Hilbert cusp forms* $\mathbf{f}$ *of weight* k *that satisfy* $L_f(k_0/2,\mathbf{f})\neq 0$*. Then there is at most one algebraic element in the set*

$$\left\{\frac{L_f'(k_0/2,\mathbf{f})}{L_f(k_0/2,\mathbf{f})}\ :\ \mathbf{f}\in\mathfrak{S}_k\right\}.$$

Proof. The first part of Theorem 8.0.1 shows that the logarithmic derivatives of the finite L-functions at $k_0/2$ give the same value if two cusp forms have the same level. Suppose that there are two cusp forms $\mathbf{f}$ and $\mathbf{g}$, with different levels $\mathfrak{n}$ and $\mathfrak{m}$ respectively, and that $L_f'(k_0/2,\mathbf{f})/L_f(k_0/2,\mathbf{f})$ and $L_f'(k_0/2,\mathbf{g})/L_f(k_0/2,\mathbf{g})$ are both algebraic. But then

$$\frac{L_f'(k_0/2,\mathbf{f})}{L_f(k_0/2,\mathbf{f})}-\frac{L_f'(k_0/2,\mathbf{g})}{L_f(k_0/2,\mathbf{g})}=\frac{1}{2}\log\left(\frac{\mathrm{N}(\mathfrak{m}\mathfrak{D}_F^2)}{\mathrm{N}(\mathfrak{n}\mathfrak{D}_F^2)}\right)$$

must be also algebraic, which is a contradiction. ■

Remark 8.0.6 The parity condition of the weight, $k_1\equiv\cdots\equiv k_n \mod 2$, makes $\mathbf{f}$ a Hilbert modular form of algebraic type. Under this condition, any integer m with $(k_0-k^0)/2<m<(k_0+k^0)/2$ is a critical point of the (finite) L-function attached to $\mathbf{f}$, where $k^0=\min(k_1,\dots,k_n)$. In particular, if $k_1\equiv\cdots\equiv k_n\equiv 0 \mod 2$, then $k_0/2$ is a critical point for $L_f(s,\mathbf{f})$. (See Theorem 6.0.4.)

Remark 8.0.7 When the condition $k_j \geq 4$ for all j is not satisfied, the first part of the theorem still holds. However, a difficulty arises to prove the second part, as the right hand side of (8.0.3) does not give a good bound. For example, if $k_1 = \cdots = k_n = 2$, one needs to show that $n(\gamma + \log(2\pi)) = 1/2 \log \mathrm{N}(\mathfrak{n}) + \log(d_F)$ cannot hold. One way to show this is to prove that $e^\gamma \pi$ is transcendental, which to the best of our knowledge seems to be unknown.

It should be also noted that in case the degree n of F is large enough, and $k_j \geq 4$ for enough j's (but not necessarily all of them), the non-vanishing of $L'_f(k_0/2, \mathbf{f})$ can be shown in the same way.

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VITA

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Scope and Method of Study: The goal of this thesis is to study some arithmetic properties of L-functions attached to Hilbert modular forms. We mainly use a representation theoretical point of view for the study, which can be done by associating Hilbert modular forms of our interests with automorphic representations of $\mathrm{GL}_2(\mathbb{A}_F)$. Furthermore, their L-functions are deeply related. We use this realization to analyze the critical L-values for Hilbert modular forms, which reduces some technical difficulties.

Findings and Conclusions: The thesis focuses on three main theorems which concern: Algebraicity theorem; Congruence property; and Non-vanishing property. The first theorem is completed by interpreting the Mellin transform cohomologically, and the second follows from analyzing it integrally. The third theorem is obtained by studying the completed L-functions of Hilbert modular forms.

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