TRIANGULATIONS AND HEEGAARD SPLITTINGS

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Introduction

Any 3-manifold can be triangulated. A triangulation of a 3-manifold consists of two parts, a collection of tetrahedra and the manner in which faces of the tetrahedra are identified by face–pairings. However, different triangulations will tell us different aspects of the story of the manifold. To gain more information about the manifolds requires us to have a deeper understanding of triangulations. My advisor, Dr. William Jaco, and Dr. Hyam Rubinstein together discovered some really nice triangulations, which are called efficient triangulations [12]. These triangulations, in general, have only one vertex and some well behaved embedded normal surfaces. For example, in a 0-efficient triangulation of a closed 3-manifold the only normal 2-sphere is vertex linking. They started a program to extend these ideas to restrictions on normal tori in the triangulation, yielding 1-efficient triangulations. This work is ongoing and is of interest to me. Their work has given rise to the study of layered triangulations [13]. There remain a number of unsolved problems on layered triangulations of higher genus handlebodies and their use for giving new combinatorial structures for the study of Heegaard splittings.

Irreducible 3-manifolds consists of Haken manifolds and non-Haken manifolds. We can study Haken manifolds by using the fact that them contains incompressible surfaces. For non-Haken manifolds, we don't have these surfaces. However, we can explore another tool, Heegaard splitting surfaces. Here the underlying philosophy is to embed a surface into a 3-manifold so that the components of its complement are as "simple" as possible.

Heegaard splittings were first introduced by Poul Heegaard [9] in his Ph.D thesis. Now it has become a classical way to study the topology of 3-manifolds. A Heegaard splitting surface is a surface that splits a 3-manifold into two handlebodies necessarily having the same genus. A handlebody is a 3-manifold topologically equivalent to a 3-manifold obtained by thickening of a finite connected graph in R^3. Similar to triangulations, any 3-manifold has a Heegaard splitting. Roughly speaking, given a 3-manifold with a triangulation, the boundary of a regular neighborhood of the 1-skeleton is a Heegaard splitting surface. Furthermore, every 3-manifold admits a Heegaard splitting of arbitrary high genus. However, not every one of them will say much about the topology of the manifold it lies in. In order to gain useful information from Heegaard splittings, we need to add some nontrivial conditions on it. For example, Casson and Gordon in their paper [5] gave the definition of a strongly irreducible Heegaard splitting. They also showed that in a non-Haken manifold, an irreducible Heegaard splitting is strongly irreducible. Hyam Rubinstein [27] proved that for any triangulation of a closed, irreducible 3-manifold, a strongly irreducible Heegaard splitting surface is isotopic to an almost normal surface. This gives us a good connection between Heegaard splittings and almost normal surface theory.

Heegaard splittings are intruduced to construct and classify 3-manifolds. Here arises the classification problem for Heegaard splittings. Nowadays, Heegaard splitting as a tool is also used to study homeomorphisms of 3-manifolds and to compute the mapping class group of some special 3-manifolds. It is also the main tool to show that every homeomorphism of the Poincaré sphere is isotopic to the identity [2].

Recently G. Perelman proved Thurston's Geometrization Conjecture, which says that every 3-manifold can be decomposed into submanifolds, each of which admits one of eight homogeneous geometries including the familiar Euclidean, hyperbolic, and elliptic geometries. The solution of the Poincaré Conjecture is a direct application of this theorem. Perelman proved the Geometrization Conjecture by using Ricci flow with surgery. One may ask whether there is a topological/combinatorial way to prove it. So far, we have a combinatorial approach to the prime decomposition step of the decomposition, a surgery decomposition based on normal 2-spheres. Every compact orientable 3-manifold decomposes uniquely as a connected sum of prime manifolds. Prime orientable manifolds are irreducible except for $S^2 \times S^1$. In the mid 1970's, Jaco-Shalen and Johannson gave a further canonical decomposition of irreducible compact orientable 3-manifolds, splitting along tori, which is called a JSJ decomposition. Each component after the JSJ decomposition is either atoroidal or a Seifert fibered manifold. People realized that the JSJ decomposition is a finer decomposition than the geometric one conjectured by Thurston. One can get a geometric decomposition from the JSJ decomposition by making some identifications along the boundaries of some of the JSJ pieces. Therefore, this may give us a way, by using triangulations, to realize geometric decompositions.

Jaco and Rubinstein present an algorithm [12] that one can modify any triangulation of a compact 3-manifold to arrive at a decomposition of the 3-manifold into a connected sum with the interesting component having 0-efficient triangulation. It is really interesting that this algorithm seems to model the first stage of the Ricci flow in the work of Perelman *et al* [6, 17, 18, 22, 23, 24]. The algorithm starts by searching for a normal 2-sphere or a normal disk obstruction to the triangulation being 0-efficient. Crushing the triangulation along such a normal surface can reduce the complexity of the triangulation or chop off a connected sum factor $S^2 \times S^1$, RP^3, or $D^2 \times S^1$ from the manifold. By repeating this procedure a number of times, we can finally present the original 3-manifold as a connected sum of copies of $S^2 \times S^1$, copies of RP^3, copies of $D^2 \times S^1$, and manifolds with 0-efficient triangulations. Since we crush along all the special normal 2-spheres, at the final stage, the only normal 2-spheres left in the latter case of manifolds are vertex-linking. Therefore, we can get 0-efficient triangulations.

Once we finish the spherical decomposition and have a 0-efficient triangulation of a manifold, we can start to look for certain kinds of normal tori or annuli to crush. The procedure is finite and stops by enabling one to construct the JSJ decomposition. The goal is to arrive at a 0-efficient triangulation along with strong restrictions on embedded normal tori for the factors that are not Seifert fibered. At the final stage in the crushing [14] we have some components that are open 3-manifolds, which are atoroidal with ideal triangulations that are "1-efficient". The problem is in keeping these conditions upon reconstructing the factors in the JSJ decomposition.

Jaco and Rubinstein's algorithm to construct either the JSJ or the geometric decomposition of a 3–manifold starts with a 0-efficient triangulation of a 3-manifold and proceeds to find the JSJ/geometric decomposition by modifying the triangulation via the crushing of certain interesting normal tori (if no such normal tori exists, then the triangulation is 1–efficient). Crushing a given triangulation of an irreducible 3-manifold along normal tori encounters obstructions similar to what happens in the case of crushing along normal 2-spheres. However, here these obstructions are resolved by showing that they give rise to Seifert fibered components in the JSJ or geometric decomposition.

Recently in the papers [15, 16], it is proved that the generalized quaternion spaces S^3/Q_{4k}, which are small Seifert fibered spaces $(S^2 : (2,1),(2,1),(k,-k+1))$, have complexity k, $k \geq 2$. The complexity of a 3-manifold M is the minimal number of tetrahedra in a triangulation of M. The techniques used can be expanded to other infinite families, including showing that the layered chain pair triangulations of the Seifert fibered spaces $(S^2 : (2,-1),(r+1,1),(s+1,1))$ are minimal.

My thesis is to closely study the minimal, 0-efficient triangulations of the above two infinite families of Seifert fiberd spaces. One is called the twisted layered loop triangulation, and the other is called layered chain pair triangulations in paper [4]. We classify all the normal and almost normal surfaces, and identify one-sided incompressible surfaces, and orientable incompressible surfaces if there are any. We also use combinatorial methods to classify Heegaard splitting surfaces. In order to study these two triangulations, we need to first focus on a special family of triangulations, layered chain triangulations, of the solid torus.

In the twisted layered loop triangulations of the generalized quaternion spaces S^3/Q_{4k}, $k \geq 2$. we prove that normal surfaces cannot be Heegaard splitting surfaces in this case. We also prove that a properly embedded surface S is a Heegaard splitting surface if and only if it is an almost normal tubed surface with the almost normal tube at the same level of a thin edge-linking tube. Any genus two Heegaard splitting surface is proved to be vertical. Furthermore, a combinatorial proof is given that all these vertical Heegaard splitting surfaces are the same up to isotopy. Since there are no normal and almost normal octagonal Heegaard splitting surfaces, thus, we classify all the irreducible genus 2 Heegaard splittings, up to isotopy, and get a conclusion that there is a unique irreducible genus 2 Heegaard splitting, up to isotopy, in each of the twisted layered loop triangulations of the generalized quaternion spaces S^3/Q_{4k}, $k \geq 2$.

In the layered chain pair triangulation of Seifert fibered spaces $M_{r,s}$=$(S^2 : (2, -1), (r+1, 1), (s + 1, 1))$, $r, s \geq 1$, we notice that there are some normal surfaces which can be Heegaard splitting surfaces in this case. Furthermore, we prove that the genus 2 almost normal octagonal surface in $M_{3,4}$ and $M_{2,6}$ are Heegaard splitting surfaces. We also prove that an almost normal tubed surface with the almost normal tube at the same level of a thin edge-linking tube is a Heegaard splitting surface. Moreover, if the genus of it is 2, then it is not only an irreducible Heegaard splitting but also a vertical one. We give a combinatorial proof that up to isotopy, there is a unique irreducible vertical Heegaard splitting surface in each of the layered chain pair triangulations of this infinite family of Seifert fibered spaces.

For the octagonal almost normal surface in the layered chain pair triangulation of Seifert fibered spaces $M_{r,s}$, $r, s \geq 1$, we can prove that the genus 2 octagonal almost normal surface in $M_{3,4}$ and $M_{2,6}$ are Heegaard splitting surfaces. We are still working on classify these genus 2 octagonal Heegaard splitting surfaces, up to isotopy. In [1, 19], they showed that in Seifert fibered space $W(2, 4, b)$, with $2 \nmid b$, $b \geq 5$ and $V(2, 3, a)$ with $3 \nmid a$, $a \geq 7$, there are two Heegaard splittings up to isotopy, one vertical and one is horizontal. Here, in our two infinite family triangulations of Seifert fibered spaces with 3 exceptional fibers, only $M_{3,4}$ and $M_{2,6}$ belongs to these two special families of 3-manifolds, and $M_{3,4} = W(2, 4, 5)$ and $M_{2,6} = V(2, 3, 7)$.

Our work follows the methods used by Jaco and Rubinstein in studying layered-triangulations of the solid torus and their classification of normal surfaces and almost normal surfaces in these triangulations [13]. We introduce some basic definitions and properties about triangulation, normal surface theory, Heegaard splittings, and Seifert fibered spaces in the next section.

1.1 Triangulations, normal surfaces and almost normal surfaces

The results presented in this section are based on [10, 12, 13].

Definition 1.1 *A triangulation T of a compact 3-manifold M consists of a finite collection of pairwise disjoint tetrahedron $\Delta = \{\Delta_i | 1 \leq i \leq m\}$ and a family of homeomorphisms $\Phi = \{\phi_j | 1 \leq j \leq n\}$, such that each homeomorphism ϕ_i identifies faces of tetrahedra in pairs and $M = \Delta/\Phi$.*

For a compact, orientable 3-manifold with nonempty boundary, a triangulation is 0-efficient [12] if and only if the only properly embedded, normal disks are vertex-linking. A triangulation of a closed, orientable 3-manifold is 0-efficient if and only if the only embedded, normal 2-spheres are vertex-linking. A 0-efficient triangulation of a closed manifold has only one vertex or the manifold is S^3 and in this case, the triangulation has precisely two vertices.

Hellmuth Kneser originated the concept normal surface in his proof of the prime decomposition theorem for 3-manifolds. In 1961, Wolfgang Haken [7] developed normal surface theory, which is at the basis of many of the algorithms in 3-manifold theory. The notion of almost normal surfaces is due to Hyam Rubinstein.

Definition 1.2 *A normal arc is a simple arc lying in a triangle(often in the face of a tetrahedron) such that its two end points meet two different edges of this triangle. A normal curve in a triangulated surface is a simple closed curve such that it intersects each triangle in the triangulation only in normal arcs.*

The boundary of a tetrahedron is a triangulated 3-sphere. Each normal curve bounds a properly embededed disk in the tetrahedron.

If a normal curve in the boundary of a tetrahedron meets the edges at most once, then it consists of either three or four normal arcs.

Definition 1.3 *If a normal curve in the boundary of a tetrahedron consists of three normal arcs, then the properly embedded disk it bounds in the tetrahedron is called a normal triangle. If it consists of four normal arcs, then the disk is called a normal quadrilateral (quad).*

Definition 1.4 *An embedded surface S is a normal surface with respect to T, if S meets each tetrahedra from the triangulation T only in normal triangles and/or normal quads. See figure 1.1*

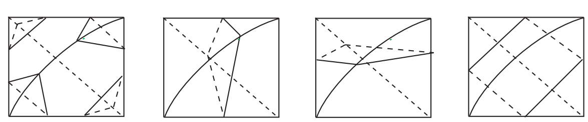

Figure 1.1: Normal triangles and normal quads.

Notice any two different types of quads must intersect with each other inside one tetrahedron. Therefore, in order to make sure that S is an embedded surface, we have to put extra constaints on it.

Definition 1.5 *Quadrilateral condition: the intersection of S with every tetrahedra in T must have no more than one quad type.*

Every embedded normal surface should satisfy the quadrilateral condition.

Definition 1.6 *S is an almost normal surface if S meets all the tetrahedra of T the same way as a normal surface does, except for one tetrahedron where S has either an almost normal tube or an almost normal octagonal disk. Furthermore, S satisfies the quadrilateral condition.*

There are twenty five almost normal tube types for each tetrahedron. Every tube type is one possible type of connection by adding a tube between two different normal quads, normal triangles, or between a normal quad and a normal triangle. There are three different connections between two quads, ten different connection between a triangle and a triangle, twelve between a triangle and a quad.

There are three almost normal octagonal disk types for each tetrahedron. See Figure 1.2.

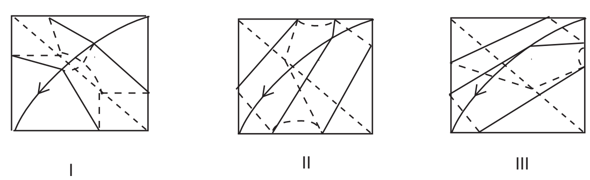

Figure 1.2: Three octagonal disk types.

Note that if there is an almost normal octagonal disk in a tetrahedron, there will be no normal quads in this tetrahedron.

Definition 1.7 *A normal surface in a triangulation T of a 3-manifold is called a splitting surface if it consists of precisely one quadrilateral disc within each tetrahedron of T and no other normal disc.*

1.2 Heegaard splittings and incompressible surfaces

Most of the results in this section are based on [28].

Definition 1.8 *H is a handlebody, if H is topologically equivalent to a regular neighborhood of a graph in R_3.*

H is a 3-manifold with boundary.

Definition 1.9 *A Heegaard splitting for a closed 3-manifold is a decomposition of M into two handlebodies so that $M = H_1 \cup_S H_2$, and $S = H_1 \cap H_2 = \partial H_1 = \partial H_2$. The surface S is called a Heegaard splitting surface.*

Definition 1.10 *The genus of a Heegaard splitting of 3-manifold is the genus of its Heegaard splitting surface.*

Definition 1.11 *The genus of M, $g(M)$, is the least genus of all Heegaard splittings of M.*

Definition 1.12 *Two Heegaard splittings are isotopic if their splitting surfaces are isotopic in M.*

Definition 1.13 *Two Heegaard splittings are homeomorphic if there is a homeomorphism of M carrying the splitting surface of one to the splitting surface of the other.*

Definition 1.14 *A Heegaard splitting is stabilized if there are properly embedded, essential disks $D_1 \subset H_1$ and $D_2 \subset H_2$ such that $|\partial D_1 \cap \partial D_2| = 1$.*

Definition 1.15 *A Heegaard splitting is reducible if there is a 2-sphere which intersects S in a single essential cirle. Otherwise, it is irreducible.*

A Heegaard splitting is reducible iff there are essential disks $D_1 \subset H_1$ and $D_2 \subset H_2$ such that $\partial D_1 = \partial D_2$.

Theorem 1.1 *([33]). Every positive genus Heegaard splitting of S^3 is stabilized.*

Theorem 1.2 *([3]). In a lens space, M, every Heegaard splitting of M with genus $g \geq 2$ is stabilized.*

Theorem 1.3 *Suppose M is an irreducible 3-manifold and $H_1 \cup_S H_2$ is a reducible Heegaard splitting of M. Then $H_1 \cup_S H_2$ is stabilized.*

Definition 1.16 *A Heegaard splitting is weakly reducible if there are essential disks $D_1 \subset H_1$ and $D_2 \subset H_2$ such that $\partial D_1 \cap \partial D_2 = \phi$. Otherwise, it is stongly irreducible.*

Theorem 1.4 *([5]). If $M = H_1 \cup_S H_2$ is a weakly reducible splitting, then either $H_1 \cup_S H_2$ is reducible or M contains an incompressible surface.*

Here are some definitions of surfaces based on [8] and [11].

Definition 1.17 *A surface S in M^3 is incompressible if for each disk $D \subset M$ with $D \cap S = \partial D$, there is a disk $D' \subset S$ with $\partial D' = \partial D$. Otherwise, S is compressible.*

Definition 1.18 *If M is a 3-manifold with boundary and S is a properly embedded surface in M, we say S is ∂-incompressible if for each disk $D \subset M$, such that ∂D is the union of two arcs α and β meeting only at their common endpoints, with $D \cap S = \alpha$ and $D \cap \partial M = \beta$, there is a disk $D' \subset S$, such that $\partial D'$ is the union of two arcs α and γ meeting only at their common endpoints and $\gamma \subset \partial S$. Otherwise, S is ∂-compressible.*

1.3 Seifert fibered spaces

The definitions and theorems in this section are mainly based on [1, 11, 20, 28, 30, 31, 32].

Definition 1.19 *A fibered solid torus is a decomposition of $S^1 \times D^2$ into disjoint circles, called fibers, constructed as follows: Start with $[0,1] \times D^2$ decomposed into the segments $[0,1] \times \{x\}$, identify the disks $0 \times D^2$ and $1 \times D^2$ via a $2\pi\gamma/\alpha$ rotation,* *for $\gamma/\alpha \in Q$ with γ and α relatively prime. The segment $[0,1] \times \{0\}$ becomes a fiber $S^1 \times \{0\}$, where every other fiber in $S^1 \times D^2$ is made from α segments $[0,1] \times \{x\}$.*

Definition 1.20 *A Seifert fibered 3-manifold M is a 3-manifold that can be decomposed into pairwise disjoint circles, the fibers, such that each fiber has a neighborhood homeomorphic, preserving fibers, to a fibered solid torus.*

Since each fiber circle f in a Seifert fibered space M has a neighborhood a fibered solid torus, it has a well-defined multiplicity or index, the number of times a small disk transverse to f meets each nearby fiber. Fibers of multiplicity 1 are called regular fibers and other fibers are singular or exceptional. For a compact Seifert fibered space there are only finitely many exceptional fibers.

The quotient space obtained by identifying each fiber to a point is a surface B, called the base surface of the Seifert manifold. The projection $\pi : M \to B$ in general does not define a fiber bundle, but the restriction does when we exclude the finite number of points $x_1, ..., x_m$ of B that correspond to exceptional fibers $f_1, ..., f_m$ of M.

In this article, we only consider a closed Seifert fibered space over an orientable base surface.

For each exceptional fiber f_i, $i = 1, \cdots, m$, choose β_i, δ_i, such that $\alpha_i\delta_i - \beta_i\gamma_i = 1$. The f_i is called an exceptional fiber of type $\beta_i/\alpha_i(\text{mod } 1)$. We always use β_i/α_i, such that $0 < \beta_i < \alpha_i$ to represent this type of exceptional fiber f_i.

Let the interger e be the usual Euler class representing the obstruction to extend a section given on the boundary components of regular neighborhoods of the exceptional fibers to the complement. Then, the rational Euler number is defined to be

$$e_0 = e - \beta_1/\alpha_1 - ... - \beta_m/\alpha_m$$

.

Definition 1.21 *Let M be an orientable Seifert fibered space with an orientable based space B of genus g_0, m exceptional fibers, and rational Euler number e_0. It will be* *denoted by*

$$M = \{g_0, e_0|(\alpha_1, \beta_1), ..., (\alpha_m, \beta_m)\},$$

where g.c.d.$(\alpha_j, \beta_j) = 1$ *and* β_j *is normalized so that* $0 < \beta_j < \alpha_j$*. The pairs of numbers* (α_j, β_j) *are Seifert invariants of the* j^{th} *exceptional fiber.*

$$\pi_1 M =< f, s_1, ..., s_m|[s_i, f] = 1, s_1 s_2 s_3 f^e = 1, {s_i}^{\alpha_i} f^{\beta_i} = 1, i = 1, ..., m >$$

Definition 1.22 *An orientable Seifert fibered space* M *is called a small Seifert fibered space if it doesn't contain any orientable incompressible surface.*

Since we only consider closed Seifert fibered spaces, we will give the definition of a vertical Heegaard splitting of closed Seifert fibered space. (c.f. [20, 28, 30]).

Definition 1.23 *Suppose that* M *is a closed orientable Seifert fibered manifold with an orientable base* B*, projection* $p : M \to F$*, and singular fibers* $f_1, ..., f_m$ *the inverse images of* $x_1, ..., x_m \in B$*. Let* Γ *be a connected graph in* B *such that some nonempty subsets of the* x_i*,* $1 \leq i \leq m$*, are vertices of* Γ *and each component of* $B - \Gamma$ *is a disk containing a single* x_i*. Let* $H_1 \subset M$ *is a handlebody whose spine is the union of the lift of* Γ *and the exceptional fiber(s) lying over each* $X_i \subset \Gamma$*. The complement of* H_1 *in* M *is also a handlebody, whose spine is the union of exceptional fibers not lying over* Γ *and the lift of a "dual" complex to* Γ*. Therefore,* $M = H_1 \cup H_2$*. This Heegaard splitting is called a vertical Heegaard splitting.*

Now we will give the definition of a horizontal Heegaard splitting.

Let M be a Seifert fibered space and let f_i be a fiber(regular or exceptional) in M. Then $M_0 = M - N(f_i)$ fibers over S^1 with a surface fiber S. Suppose that M is obtained from M_0 by $1/n$-Dehn filling with respect to the framing determined by ∂S. Then the Heegaard splitting for M constructed as following (using M_0 and S) is called a horizontal Heegaard splitting corresponding to the fiber f_i with mutiplicity $\alpha_i = n$.

The construction of horizontal Heegaard splittings in the Seifert fibered spaces M which admitted them is as follows:

Consider a Seifert fibered space M_0, where $M_0 = M - N(f_i)$, an orientable manifold over an orientable base surface $B_0 = B - N(pt)$ with one torus boundary component. Now M_0 has n_0 exceptional fibers, where $n_0 = m$ or $m - 1$. Such manifold fibers as a periodic surface bundle over the circle, $M_0 = S\widetilde{\times}S^1$, where the fiber S is a connected and orientable surface and the orbit of any point under the S^1 action is a fiber in the Seifert fibering. We can write

$$M_0 = S \times I/x \times \{0\} \sim h(x) \times \{1\},$$

where $h : S \to S$ is the periodic homeomorphism associated with the bundle $M_0 = S\widetilde{\times}S^1$. h will have degree $d = lcm\{\alpha_1, ..., \alpha_{n_0}\}$.

Since S is a once punctured surface and hence a regular neighborhood of S is a handlebody H_1 whose genus is $2\times$(genus S). The manifold $M_0 - N(S)$ is homeomorphic to $S \times I$ and is also a handlebody H_2. The two handlebodies H_1,H_2 are glued to each other along their boundaries except for two annuli $A_1 \subset H_1$, $A_2 \subset H_2$. The two annuli are glued to each other along their boundaries to form the boundary torus. Choose two disjoint copies of the surface fiber, S_1 and S_2, and cut along these surfaces to decompose $M_0 = S\widetilde{\times}S_1$ into two pieces, $S \times I_1$ and $S \times I_2$. Label the surfaces S_1 and S_2 and orient I_1 and I_2 so that $S \times {I_1}^- = S_1$; $S \times {I_1}^+ = S_2$; $S \times {I_2}^- = S_2$, and $S \times {I_2}^+ = S_1$.

We obtain M by gluing the solid torus neighborhood of f, $\overline{N(f)}$, to the boundary of M_0, such that the meridian m of the solid torus $\overline{N(f)}$ must intersect ∂S exactly once. Then $m_1 = m \cap A_1$ and $m_2 = m \cap A_2$ will each be a single arc and the manifold M maybe thought of as the quotient $M_0/(A_1 = A_2)$, where the gluing of A_1 and A_2 is defined by identifying the arcs m_1 and m_2. See figure 1.3.

Definition 1.24 *The Heegaard splitting* $M = (S \times I_1) \cup_F (S \times I_2)$, *where* $F =$

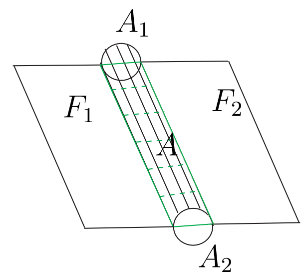

Figure 1.3: $M = M_0/(A_1 = A_2)$.

$S_1 \cup A \cup S_2 = S_2 \cup A \cup S_1$ *is called a horizontal splitting of* M *at* f*.*

We can construct three vertical Heegaard splitting in small Seifert fibered space M with three exceptional fibers f_i, $1 \leq i \leq 3$. Take any two exceptional fibers f_i and f_j, $1 \leq i \neq j \leq 3$. Let's connect them by an arc projected to a simple arc on the base S^2, which gives us a graph in M. Now let H be the regular neighborhood of this graph, we get a handlebody. Notice the complement of H in M is the handlebody which is described as H_1 in the above definition, where Γ is a loop based on one vertex x_k, $i \neq k \neq j$, which separates x_i and x_j on the base B. This gives us a vertical Heegaard splitting of genus 2 in the small Seifert fibered space.

Waldhausen in [33] shows that S^3 has a unique Horizontal irreducible Heegaard splitting. Bonahon and Otal in [3] show that lens spaces have a unique vertical splitting. The main results of [20]and [29] imply that irreducible Heegaard splittings of Seifert fibered spaces are vertical or horizontal. Eric Sedgewick in [30] Shows that if M is a Seifert fibered space which admits a horizontal splitting at the fiber f. If the genus of the horizontal splitting at f is less than the genus of the vertical splittings, its genus will be minimal and the splitting irreducible. Otherwise, this splitting will be irreducible if and only if the multiplicity of the fiber f is strictly greater than the least common multiple of the multiplicities of the other fibers. In particular, each Seifert fibered space possesses at most one irreducible horizontal splitting. The vertical splittings will be reducible if and only if M has a horizontal splitting with genus strictly less than the genus of the vertical splittings.

CHAPTER 2

Layered chain triangulations

In this chapter we will give the definition of a layered chain triangulation of the solid torus, as well as some other important definitions. We will provide detailed proofs for the classification of normal surfaces in layered chain triangulations of the solid torus. A partial classification appears in work of [16] without much detail. The methods are similar to those of [13], where the normal surfaces in a minimal layered triangulation of the solid torus are classified. These results will be applied to the study of twisted layered loop triangulations and layered chain pair triangulations of some infinite families of Seifert fibered spaces.

2.1 Layered chain triangulations of the Solid torus

Define layered chain triangulations of the solid torus. Notice the boundary of the solid torus is a torus that can be obtained by gluing two annuli along their corresponding boundary components t and b. The layered chain triangulation of the solid torus starts from a triangulation of the bottom annulus, denoted by A_0, labelling as in the figure 2.1. Notice that there are two vertices v_1 and v_2 on A_0. The edge t is a loop based at vertex v_1, and the edge b is a loop based at vertex v_2. The edge e_1 and e_2 are oriented from vertex v_1 to vertex v_2.

Given a tetrahedron σ_1, it has four faces. Let any two of these four faces, which share a common edge, glue to the two faces on the A_0, such that the common edge is identified with edge e_1. This operation is called layering the tetrahedron σ_1 on top of A_0 along the edge e_1. See figure 2.2.

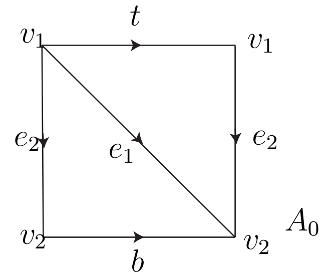

Figure 2.1: A_0 is the bottom annulus of the boundary of a solid torus.

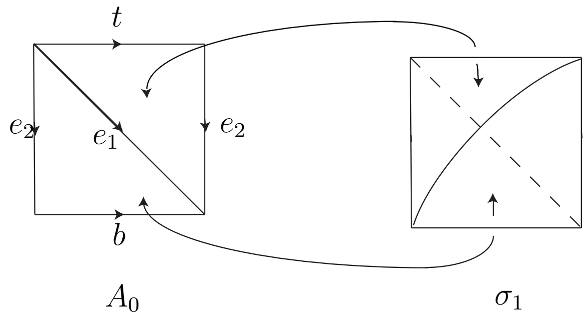

Figure 2.2: Layering the tetrahedron σ_1 on top of A_0 along the edge e_1.

After layering the first tetrahedron σ_1 on top of A_0 along the edge e_1, we get a one tetrahedron triangulation C_1 of a creased solid torus. See figure 2.3.

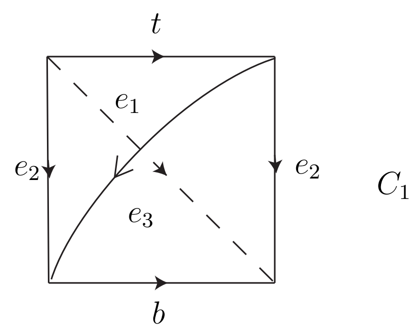

Figure 2.3: C_1, a triangulation of the creased solid torus.

In σ_1, after identification, the top two triangles give us the top annulus A_1 with an edge e_3 oriented from t to b.

Now let's layer the second tetrahedron σ_2 on top of A_1 along the edge e_2. we get a triangulation of the solid torus with two vertices, v_1 and v_2. See figure 2.4. σ_1 and σ_2 together give us a triangulation of 2-tetrahedron of the solid torus, denoted by C_2.

The top two triangles give us the top annulus A_2 with an edge e_4 oriented from v_1 to v_2.

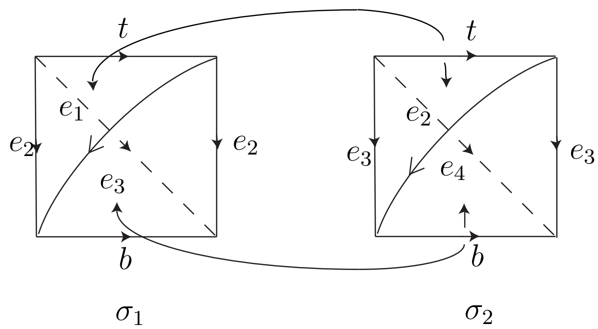

Figure 2.4: C_2, a triangulation of the solid torus.

Keep doing the same procedure, after layering the k^{th} tetrahedron σ_k on top of A_{k-1} along the edge e_k, $k \geq 2$, we get a triangulation of k-tetrahedron of the solid torus, C_k. See figure 2.5. This special way of construction k-tetrahedron triangulation of the solid torus is called *the layered chain triangulation of length k.*

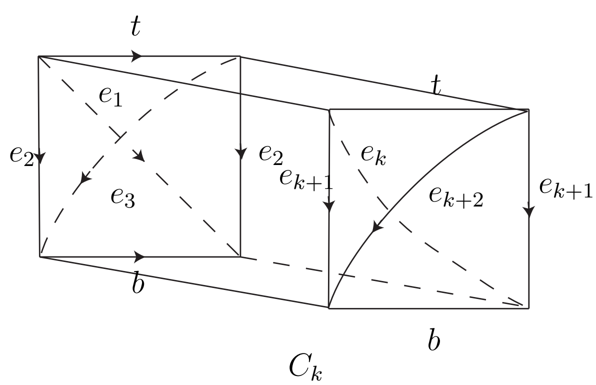

Figure 2.5: C_k, a layered chain triangulation of a solid torus of length k.

This triangulation has 2 vetices, v_1 and v_2. The edge t is a loop based on vertex v_1, and edge b is a loop based on vertex v_2. All the other edge e_i, $1 \leq i \leq k+2$, is oriented from vertex v_1 to vertex v_2.

The boundary of the layered chain triangulation of a solid torus of length k consists of two annuli, the bottom one A_0 and the top one A_k. We will call the annulus A_i, $1 \leq i \leq k-1$ obtained during the layering *the level annulus.* In the triangulation C_k, edge t and b are only edges of degree k, e_1 and e_{k+2} are univalent edge of degree 1. e_2 and e_{k+1} are edges of degree 2, and all the other edges are of degree 3.

2.2 Normal surfaces in the Layered chain triangulations

In this section we will study and give a classification of the normal surfaces in the layered chain triangulations of the solid torus. The results will be applied to the study the minimal, 0-efficient triangulations of two infinite families of small Seifert fiberd spaces.

Definition 2.1 *The edge-weight of a normal arc in the bottom or top annulus of a layered chain triangulation of the solid torus is an ordered 4-tuple* $(wt_t, wt_b, wt_{e_1}, wt_{e_2})$ *or* $(wt_t, wt_b, wt_{e_{k+1}}, wt_{e_{k+2}})$*, respectively, where* wt_x *is the number of intersections of a normal arc with the edge* x*. We call it the bottom or top edge-weight of a normal arc in the layered chain triangulation of the solid torus.*

For a normal curve in a layered triangulation of the solid torus, it will intersects with the bottom and top annulus of the boundary of this triangulation. Therefore, we will use $(wt_t, wt_b, wt_{e_1}, wt_{e_2}); (wt_t, wt_b, wt_{e_{k+1}}, wt_{e_{k+2}})$ to represent the edge-weight of a normal curve.

For a normal surface, we consider the edge-weight of its boundary to be the edge-weight of the surface.

Suppose we have a layered chain triangulation C_k of the solid torus. Now if we layer a new tetrahedron σ_{k+1} on top of A_k along the edge e_{k+1}, we will get a new layered chain triangulation C_{k+1} of the solid torus with $k+1$ tetrahedra. The difference between C_{k+1} and C_k is that we just add a product structure of the top annulus of the triangulation C_k. Now suppose S_{k+1} is a normal surface in C_{k+1}, then $S_{k+1} \cap C_k = S_k$ is a normal surface in C_k. The only difference between S_{k+1} and S_k is a collection of normal triangles and quadrilaterals in the tetrahedron σ_{k+1}. Now we pay close attention to how we get these extra normal pieces from the normal surface S_k to form the new normal surface S_{k+1}. We notice there are two possible ways to add normal disks.

1. **Push-through**. We extend every normal arc in the intersection of S_k with the two faces of the top annulus A_k of the triangulation C_k in σ_k, by adding some of 4 types of normal triangles and/or possibly one of the 3 types of normal quadrilaterals. These are completely determined by the arc types of the intersection of S_k with the top annulus A_k. Obviously, Push-through just adds the product structure to the normal surface. We get a new normal surface which is homeomorphic to the old surface.

2. **Banding**. Instead of pushing through every normal arc, the intersections of S_k with A_k in C_k, we allows to add a band connecting two parallel arcs, if the new band will not intersect with any other new adding normal disks in the tetrahedron σ_{k+1}. Sometime, we can add more than more bands in σ_{k+1}. These bands are the same type of quadrilaterals out of total 3 possible types of quadrilaterals, according to the Quadrilateral condition. Every times we add a band on the surface, the Euler character will be decreased by 1, i.e. χ(old surface)$=\chi$(new surface)-1.

In the layered chain triangulation of the solid torus, if we push-through the normal arc with edge-weight $(2,0,1,1)$ or $(0,2,1,1)$ on the top annulus A_k of C_k by adding the tetrahedron σ_{k+1}, we'll get the same edge-weight $(2,0,1,1)$ or $(0,2,1,1)$, respectively, on the top annulus A_{k+1} in the layered chain triangulation C_{k+1}. Notice the sum of edge-weight $(2,0,1,1)$ and $(0,2,1,1)$ is $(2,2,2,2)$. From now on, we use $(2,2,2,2)$ to represent a pair of normal arcs $(2,0,1,1)$ and $(0,2,1,1)$ on the bottom annulus in a layered chain triangulation of the solid torus. According to the above discussion, if we push-through the normal arcs with bottom edge-weight $(2,2,2,2)$ on the top annulus

A_k of C_k in the σ_{k+1} which lies on top of A_k in C_{k+1}, we'll get the same edge-weight for normal arcs on the top annulus A_{k+1} in the layered chain triangulation C_{k+1}.

If we push-through the normal arc with edge-weight $(0, 0, 1, 1)$ on the top annulus A_k of C_k by adding the tetrahedron σ_{k+1}, we get the same edge-weight $(0, 0, 1, 1)$ on the top annulus A_{k+1} in a layered chain triangulation C_{k+1}.

If we push-through the normal arc with edge-weight $(1, 1, p, p+1)$ on the top annulus A_k of C_k by adding the tetrahedron σ_{k+1}, we will get the edge-weight $(1, 1, p+1, p+2)$ on the top annulus A_{k+1} in a layered chain triangulation C_{k+1}. For the edge-weight $(1, 1, p+1, p)$, we will get the edge-weight $(1, 1, p, |p-1|)$, for any $p \geq 0$.

There are three possible types of quadrilateral disks in the $k + 1^th$ tetrahedron σ_{k+1} in a layered chain triangulation. From the figure 2.6, we can see that a quad of type I is obtained by push-through the arc with edge-weight $(1, 1, 1, 0)$ on the bottom annulus A_k in C_{k+1}. The new surface will have an edge-weight $(1, 1, 0, 1)$ on the top annulus A_{k+1} in C_{k+1}. The quad of type II is obtained by push-through the arc with edge-weight $(0, 0, 1, 1)$ on the bottom annulus A_k in C_{k+1}, and have the same edge-weight on the top annulus A_{k+1} in C_{k+1}. The quad of type III is the only quad type, that is obtained by banding, instead of push-through. It has an edge-weight $(1, 1, 0, 1)$ on A_k and an edge-weight $(1, 1, 1, 0)$ on A_{k+1} in a layered chain triangulation C_{k+1} of the solid torus.

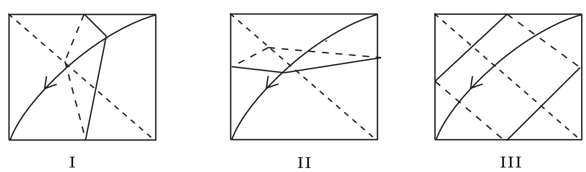

Figure 2.6: Three types of quadrilaterals.

Here are some examples that we can add a band or two bands in σ_{k+1}. If a normal surface intersects with the bottom annulus of σ_{k+1} with two normal arcs with edge-weights $(2, 2, 2, 2)$, then we can add a band between the two arcs parallel to

e_{k+1} and push through the other two arcs, the new surface will have the edge-weight $2 \times (1, 1, 1, 0)$ on the top annulus A_{k+1} in C_{k+1}. If a normal surface intersects with the bottom annulus of σ_{k+1} with an edge-weight $(1, 1, 0, 1)$, then we can add one band between the two parallel arcs, the new surface will have the edge-weight $(1, 1, 1, 0)$. If a normal surface intersects with the bottom annulus of σ_{k+1} with an edge-weight $2 \times (1, 1, 0, 1)$, then we can add a band or two bands between the two parallel arcs, the new surface will have the edge-weight $(2, 2, 2, 2)$ or $2 \times (1, 1, 1, 0)$. See figure 2.7. In fact, $(2, 2, 2, 2)$, $(1, 1, 1, 0)$, $2 \times (1, 1, 1, 0)$ are the only possible edge-weights on the top annulus A_k of C_k that we can add a band by adding the tetrahedron σ_{k+1}.

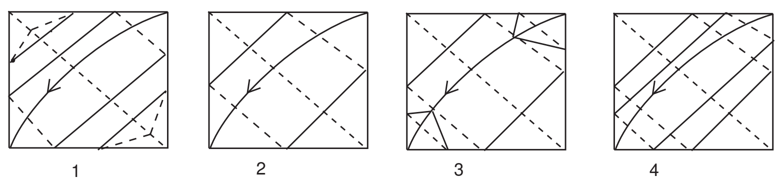

Figure 2.7: Four examples of bandings.

For any connected normal surface S_k in the layered chain triangulation C_k of a solid torus, then ∂S_k meets the top and bottom annuli of C_k in a collection of normal curves. Now we need to find all the possible edge-weights of a normal curve intersecting with the bottom or top annulus in the layered chain triangulation of a solid torus.

Lemma 2.1 *All the possible bottom(top) edge-weights of a connected normal surface in a layered chain triangulation of the solid torus are* $(2, 0, 1, 1)$, $(0, 2, 1, 1)$, $(2, 2, 2, 2)$, $(0, 0, 1, 1)$, $(1, 1, p + 1, p)$, $(1, 1, p, p + 1)$, $p \geq 0$ *or at most 2 copies of the last two cases.*

Proof. Any normal closed curve on the boundary of a solid torus will intersect the bottom annulus A_0 in trivial arcs, essential arcs, essential simple closed curve. Notice a trivial closed curve can not be normally isotopic to a normal curve on the bottom annulus, hence we will not pay attention to this case.

An essential curve of the annulus is a closed curve parallel to the boundary of the annulus. It has the edge-weight $(0,0,1,1)$ on the bottom annulus A_0. The only way to get a new normal surfaces from this arc is by pushing through. The normal surface obtained by this way intersects each tetrahedron with a quad of type II in the figure 2.6 in the triangulation C_k of a solid torus. By calculate the Euler characteristic of this surface, it is a normal annulus separating edge t and edge b. The edge-weight of this normal surface is $(0,0,1,1);(0,0,1,1)$.

The trivial arc on the bottom annulus can only be an arc with end points on the same boundary component of the annulus. Therefore, the only possible trivial normal arc has an edge-weight $(2,0,1,1)$ or $(0,2,1,1)$. The only way to get a new surface from either of them is by pushing through. Hence, we get a normal disk whose boundary with the edge-weight $(2,0,1,1)$; $(2,0,1,1)$ or $(0,2,1,1)$; $(0,2,1,1)$ respectively.

An essential arc of the annulus is a simple arc with two end points on the two different boundary components of the annulus. On the bottom annulus, the normal essential arc can only have one of the following two types of edge-weights, $(1,1,p+1,p)$ or $(1,1,p,p+1)$, $p \geq 0$. See figure 2.8.

If we push through the normal arc with edge-weight $(1,1,p,p+1)$ on the top annulus A_i of C_i by adding the tetrahedron σ_{i+1}, we will get the edge-weight $(1,1,p+1,p+2)$ on the top annulus A_{i+1} in the new layered chain triangulation C_{i+1}. It means by pushing through this normal arc once in a tetrahedron, its last two coordinates of the edge-weight on the top annulus in the new layered chain triangulation will be increased by 1 at the same time. Therefore, an normal arc with the edge-weight $(1,1,p,p+1)$ on the bottom annulus in C_k, will has the edge-weight $(1,1,p+k,p+k+1)$ on the top annulus A_k in the layered chain triangulation C_k. Furthermore, the last coordinate of the edge-weight is still greater than the third coordinate by 1. Therefore,

$(1, 1, p, p+1)$, for $p \geq 1$ is one of the possible top edge-weights of a connected normal surface.

If we push through the normal arc with edge-weight $(1, 1, p+1, p)$, when $p > 0$, on the top annulus A_i of C_i by adding the tetrahedron σ_{i+1}, we will get the edge-weight $(1, 1, p, p-1)$ on the top annulus A_{i+1} in the new layered chain triangulation C_{i+1}. It means by pushing through this normal arc once in a tetrahedron, its last two coordinates of the edge-weight on the top annulus in the new layered chain triangulation will be decreased by 1 at the same time. If we push through the normal arc with edge-weight $(1, 1, p+1, p)$, when $p = 0$, on the top annulus A_i of C_i by adding the tetrahedron σ_{i+1}, we will get the edge-weight $(1, 1, 0, 1)$=$(1, 1, p, |p-1|)$ on the top annulus A_{i+1} in the new layered chain triangulation C_{i+1}. If we push throught the normal arc with edge-weight $(1, 1, 0, 1)$=$(1, 1, p, p+1)$, $p = 0$ in the layered chain triangulation, then the last two coordinates of the new normal arc on the top of annulus of the new layered chain triangulation will be increased by 1 for each more tetrahedron layering after the i^{th} tetrahedron in C_k. All in all, the normal arc with edge-weight $(1, 1, p+1, p)$ can be pushed through in the layered chain triangulation C_k and has the top edge-weight $(1, 1, |p-k|, |p-k-1|)$ in C_k. Therefore, this case can give us two of the possible top edge-weights $(1, 1, p+1, p)$ and $(1, 1, p, p+1)$ with $p \geq 0$.

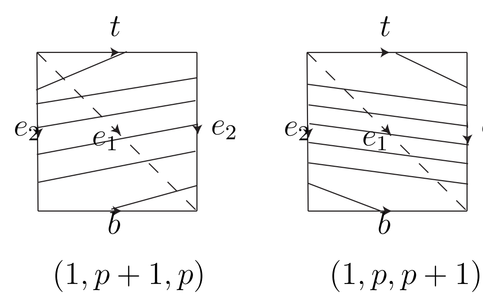

Figure 2.8: Two possible types of essential arcs in the bottom annulus.

Now we want to show that $(2, 2, 2, 2)$, $2\times(1, 1, p+1, p)$, $2\times(1, 1, p, p+1)$, $p \geq 0$ are possible top/bottom edge-weights for a connected normal surface in a layered chain triangulation.

For two trivial normal arcs $(2,0,1,1)$ and $(0,2,1,1)$ together on the bottom annulus of C_k, respectively. Notice they together hit the edge t and edge b same times, the sum of the edge-weights is $(2,2,2,2)$. We can always push through these two arcs in the tetrahedron and get two disjoint surfaces that give us two same nontrivial arcs with edge-weight $(2,2,2,2)$ on the top annulus of that tetrahedron. Suppose we keep pushing through these two arcs in the first i tetrahedra, $0 \leq i < k$, we get two disjoint arcs with same edge-weight $(2,2,2,2)$ on the bottom annulus of σ_{i+1}. From figure 2.7, we can see that we can also add a band on them instead of push-through. Thus we get a connected normal surface with edge-weight $2 \times (1,1,1,0)$ on the top annulus A_{i+1} of σ_{i+1}. If there is a tetrahedron σ_{i+2} layering on top of A_{i+1}, we can only push through the arcs with edge-weight $2 \times (1,1,1,0)$ and get two arcs with edge-weight $2 \times (1,1,0,1)$ on the top annulus A_{i+2} of σ_{i+2}. If there is a tetrahedron σ_{i+3} layering on top of A_{i+2}, we can push through, add a band or add two bands on the arcs with edge-weight $2 \times (1,1,0,1)$ and get the arcs with edge-weight $2 \times (1,1,1,2)$, $2,2,2,2$, or $2 \times (1,1,1,0)$ on on the top annulus A_{i+3} of σ_{i+3}. We realize that for any arc with edge-weight of type $2 \times (1,1,p,p+1)$, $p \geq 1$, we can only push through them and the new arcs will with edge-weight of same type $2 \times (1,1,p',p'+1)$. As for the arcs with edge-weight $2,2,2,2$, we can have the whole argument about its edge-weight which starts from the very beginning of this paragraph again. For the arcs with edge-weight $2 \times (1,1,1,0)$, our argument about its edge-weight next repeated from a connected normal surface with edge-weight $2 \times (1,1,1,0)$ on the top annulus A_{i+1} of σ_{i+1} in this paragraph. Since any sub-layered-chain from σ_i and end at any σ_j,$j \geq i+2$, is still a layered chain triangulation of a solid tous. Therefore, $(2,2,2,2)$, $2 \times (1,1,1,0)$, $2 \times (1,1,0,1)$, are possible bottom edge-weights of a connected surface in the layered chain triangulation C_k of the solid torus. $(2,2,2,2)$, $2 \times (1,1,1,0)$, $2 \times (1,1,p,p+1)$, $p \geq 0$, are the possible top edge-weights of a connected surface in C_k. In particular,

$2 \times (1, 1, p, p+1)$, $p \geq 1$ can not be the edge-weight of a connected surface in C_k. this is because that the last two coordinates of the edge-weight of these two arcs can only be increased by pushing through and can never have a chance to add a band to get a connected surface.

From now on if we mention an arc with edge-weight $(2, 0, 1, 1)$ or $(0, 2, 1, 1)$, it means this arc cannot be part of a pair $(2, 2, 2, 2)$ in a layered chain triangulation of the solid torus, i.e. there is no arc with edge-weight $(0, 2, 1, 1)$ or $(2, 0, 1, 1)$ in the chain that they together make a pair arcs with edge-weight $(2, 2, 2, 2)$.

Now we need to check an edge-weight $2 \times (1, 1, p+1, p)$,$p \geq 1$, is a possible bottom edge-weight of a connect surface in C_k. If we have two arcs with bottom edge-weight $2 \times (1, 1, p+1, p)$ in C_k, we can push only push through them. After we push through them in the first $k+1$ tetrahedra, we will have two disjoint normal surfaces with edge-weight $2 \times (1, 1, |p+1-(p+1)|, |p-(p+1)|)$=$2 \times (1, 1, 0, 1)$. If we keep push through these two disjoint surfaces, we will never have chance to add a band later. In order to get a connected surface, we need to add at least a band here. See the last two cases in figure 2.7. Therefore, $2 \times (1, 1, p+1, p)$,$p \geq 1$, is a possible bottom edge-weight of a connect surface in C_k.

Now the last thing we need to check is that there is no other possible edge-weights for a connected surface in the layered chain triangulation C_k of a solid torus.

First we observes that if there are several arcs on the bottom annulus A_0 of C_k with one of them with bottom edge-weights $(2, 0, 1, 1)$, or $(0, 2, 1, 1)$, then we will not get a connected surface unless all the arcs together give us a edge-weight $(2, 2, 2, 2)$. It means as long as they don't show up in pairs, it can give us a connected surface by itself and will not be added a band to connect with other surfaces.

Another observation is that if there are several arcs on the bottom annulus A_0 of C_k with one of them with bottom edge-weights $(0, 0, 1, 1)$ or $(2, 2, 2, 2)$, then we will not get a connected surface, because $(0, 0, 1, 1)$ and $(2, 2, 2, 2)$ can give us a connected surface by itself and can not be added a band to connected with other surfaces.

Till now, all the possible bottom edge-weights left for a connect surface is from a collection of compatible essential arcs. Furthermore, they need to have same edge-weights $(1, 1, p+1, p)$. If not, no band can be added to them to form a connected surface.

Suppose that there are n copies of an essential arc with bottom edge-weights $n \times (1, 1, p+1, p)$ on the bottom annulus A_0 of C_k. In order to add bands, we have to push through them in the first $p+1$ tetrahedra to get an edge-weight $n \times (1, 1, 0, 1)$ on the top annulus of A_{p+1}, therefore, $k > p+1$. Now we can add at most n bands in σ_{p+2}, which give us at most $n/2$ connected surface if n is even or at most $(n+1)/2$ connected surfaces if n is odd. Furthermore, they will give us $n \times (1, 1, 1, 0)$ on the top annulus A_{p+2} of σ_{p+2}. If there is σ_{p+3} on top of A_{p+2}, we can only push through these surfaces and get an edge-weight $n \times (1, 1, 0, 1)$ again on the top annulus A_{p+3} of σ_{p+3}, we can add at most n bands again, however, all these bands will only add to the original connected surfaces instead of connecting two disjoint surfaces. Hence, although we have chance to add more bands from now on, but banding will not decrease the number of surfaces that are disconnected any more. The smallest number of disconnected surfaces is $n/2$ if n is even or $(n+1)/2$ if n is odd. Therefore, we will get a connected surface only if n is 1 or 2. All these cases we already discussed.

Therefore, we proved the lemma. ■

2.2.1 Some families of normal surfaces in the layered chain triangulations of the solid torus

All the normal surfaces in the layered chain triangulation of the solid torus will be classified in this section. However, before we do that, we give some examples of normal surfaces in layered chain triangulations of a solid torus and develop some terminology for the various families of such normal surfaces. If $\mathcal{T}$ is a triangulation of a 3–manifold and S is a normal surface such that for some edge e the surface S contains the collection of quadrilaterals linking e, we say S has a *thin edge-linking tube* (about e). Other terms we use are either identical with or analogous to those in [13].

0. *Vertex-linking disk,*$(0,2,1,1);(0,2,1,1)$ or $(2,0,1,1);(2,0,1,1)$.

   If S is a normal surface such that for one of the vertice v the surface S contains the collection of triangles linking v, we say S has a *vertex linking disk* (about v). There are two vertices in the layered chain triangulation C_k of the solid torus. Vertex-linking disks are obtained by starting with vertex-linking arcs, $(0,2,1,1)$ or $(2,0,1,1)$, in C_0 and pushing through at each layer. Obviously these two disks have edge-weights $(0,2,1,1);(0,2,1,1)$ or $(2,0,1,1);(2,0,1,1)$.

1. *Vertex-linking disks with thin edge-linking tubes*, $(2,2,2,2);(2,2,2,2)$.

   There are two vertex-linking disks $(2,0,1,1);(2,0,1,1)$ and $(0,2,1,1);(0,2,1,1)$. They together give us vertex-linking disks with edge-weight $(2,2,2,2);(2,2,2,2)$. It is also possible to add a band about an edge e to connect these two disks. If we continue to add all the quads that link the thin edge e as the band does, then we create a thin edge-linking tube about e between the two vertex-linking disks. We call it the vertex-linking disks with a thin edge-linking tube, $(2,2,2,2);(2,2,2,2)$. Notice we can keep adding quads linking other thin edges in this way, hence we get a family of normal surfaces, $(2,2,2,2);(2,2,2,2)$. We call them the vertex-linking disks with thin edge-linking tubes.

2. *Vertical annulus*, $(0,0,1,1);(0,0,1,1)$.

   This is a quadrilateral, splitting surface, splitting the edge t from the edge b. It starts with the essential simple closed curve in C_0 which is pushed through at every stage of the layering. It also is a thin edge-linking annulus about the edge t as well as about the edge b.

3. *Meridian disk.*

   (a) *Meridian disk*, $(1,1,p,p+1);(1,1,p+k,(p+1)+k)$, or

   (b) *Meridian disk*, $(1,1,p+1,p);(1,1,|(p+1)-k|,|p-k|)$.

   Each starts with an essential arc in C_0 having edge weights $(1,1,p,p+1)$ or $(1,1,p+1,p)$, respectively, and at every layer, the new surface is obtained by pushing through. This gives two infinite families of normal meridional disks.

4. *Upper edge-linking disk (possibly) with thin edge-linking tubes*, $(2,2,2,2);2\times(1,1,1,0)$ or $(2,2,2,2);2\times(1,1,q,q+1)$, with $q\geq 0$. In C_{i-1}, the surface is the vertex-linking disks (possibly) with thin edge-linking tubes; in layer i a band is added. This is an upper edge-linking disk (possibly) with thin edge-linking tubes. At all subsequent steps of the layering, push through. If $i=k$, then the edge weights on the top annulus of C_k are $2\times(1,1,0)$ and the surface is the thin edge-linking disk about the edge e_{k+2},(possibly) with thin edge-linking tubes. These normal surfaces are analogous to the edge-linking annuli in [**?**]. It is "upper" since it meets the bottom annulus of C_k only in vertex-linking arcs. Each embedded edge in C_k determines a finite family of these surfaces, the members differing only by the placement of thin edge-linking tubes.

5. *Lower edge-linking disk (possibly) with thin edge-linking tubes*, $2\times(1,1,p+1,p);(2,2,2,2)$ or $2\times(1,1,0,1);(2,2,2,2)$.

   There are two essential normal arcs in C_0 with bottom edge-weights $2\times(1,1,p+1,p)$. By pushing through at each stage, then the edge weights in the top annulus of C_p are $2\times(1,1,1,0)$. The only possibility in C_{p+1} is to push through; however, in C_{p+2} one or two bands can be added. Adding one band gives a lower edge-linking disk. In this case, $k\geq p+2$ and if $k>p+2$, we only add thin edge-linking tubes in subsequent layers, giving a lower edge-linking disk (possibly) with thin edge-linking tubes. In the case of the two arcs in C_0 having edge weights $2 \times (1,1,0,1)$, the lower edge-linking disk is the thin edge-linking disk about the edge e_1. These edge-linking disks are "lower" since they meets the top annulus of C_k only in vertex-linking arcs. Each embedded edge in C_k determines a finite family of these surfaces, the members differing only by the placement of thin edge-linking tubes.

We notice that a lower edge-linking disk (possibly) with tubes is just an inverted upper edge-linking disk (possibly) with tubes and vice-versa.

6. *One-sided (nonorientable) surface.*

   (a) *One-sided surface of genus* c, $(1,1,0,1);(1,1,|k-2c|,(k+1)-2c), k \geq 2c-1$.

   These one-sided surfaces can be obtained by banding immediately in C_1 and then alternately pushing through and banding, possibly eventually just pushing through ; banding adds nonorientable genus while pushing through adds edge-weight to the intersection numbers of the boundary of the surface with the edges in the top annulus.

   It is possible to ∂–compress these surfaces into the bottom annulus of C_k, giving a surface with $c-1$ crosscaps and edge-weights $(1,1,2,1);(1,1,|k-2c|,(k+1)-2c), k \geq 2c-1$. See Example 6(b) with $p=1$ and $c-1$ crosscaps.

   If $k=2c$, the surface is a quadrilateral, one-sided, splitting surface, splitting the odd index edges. It has edge-weights $(1,1,0,1);(1,1,0,1)$.

   If $k=2c+1$, it is possible to ∂–compress the surface into the top annulus of C_k, giving a surface with c crosscaps and edge-weights $(1,1,0,1);(1,1,1,2)$;

∂–compressing into both the top and bottom annulus gives a surface with $c-1$ crosscaps and edge-weights $(1,1,2,1);(1,1,1,2)$. The latter surface appears as a surface in 6(b) with $p=1$.

(b) *One-sided surface of genus* c, $(1,1,p+1,p);(1,1,|(k-1)-(p+2c)|,k-(p+2c)), k \geq (p+2c)$.

For $i=p$, there is a meridian disk having edge weights on the top annulus $(1,1,1,0)$; then at C_{p+2} it is possible to band, giving a Möbius band. The various one-sided surfaces are obtained either by continuing alternatively pushing through or banding, the latter of which adds nonorientable genus, possibly eventually ended by pushing through, which adds edge-weight to the intersection numbers of the boundary of the surface with the edges in the top annulus.

If $p=0$ and $k=2c$, the surface is a quadrilateral, one-sided, splitting surface, splitting the even index edges. It has edge-weights $(1,1,1,0);(1,1,1,0)$.

7. *Annulus (possibly) with thin edge-linking tubes.*

(a) *Annulus (possibly) with thin edge-linking tubes,* which is the double of surface 6(a).

The boundary of the surface with the following two possible boundary edge-weights

(a.1) $2\times(1,1,0,1);2\times(1,1,1,0)$, for $k=2c-1$, $c\geq 1$;

(a.2) $2\times(1,1,0,1);2\times(1,1,q,q+1)$, for $k=2c+q$, $q\geq 0$;

(b) *Annulus (possibly) with thin edge-linking tubes,* which is the double of surface 6(b).

The boundary of the surface with the following two possible weights

(b.1) $2 \times (1, 1, p + 1, p); 2 \times (1, 1, 1, 0)$, for $k = p + 2c$, $p \geq 0$; or

(b.2) $2 \times (1, 1, p + 1, p); 2 \times (1, 1, q, q + 1)$, for $k = p + 2c + q + 1$, p, $q \geq 0$.

(c) *Annulus (possibly) with thin edge-linking tubes,* with the following possible weights.

(c.1) $2 \times (1, 1, 0, 1); 2 \times (1, 1, 1, 0)$,

(c.2) $2 \times (1, 1, 0, 1); 2 \times (1, 1, q, q + 1)$, which includes $2 \times (1, 1, 0, 1); 2 \times (1, 1, 0, 1)$,

(c.3) $2 \times (1, 1, p + 1, p); 2 \times (1, 1, 1, 0)$, which includes $2 \times (1, 1, 1, 0); 2 \times (1, 1, 1, 0)$,

(c.4) $2 \times (1, 1, p + 1, p); 2 \times (1, 1, q, q + 1)$, which includes $2 \times (1, 1, 1, 0); 2 \times (1, 1, 0, 1)$,

The surface above in $7(c)$ is obtained from a lower edge-linking disk (possibly) with thin edge-linking tubes attached along two vertex-linking arcs with an upper edge-linking disk (possibly) with thin edge-linking tubes.

2.2.2 Normal surfaces in C_2

Before we give the classification of the normal surfaces in C_k, let's first studies the normal surfaces in C_2.

Theorem 2.1 *In the layered chain triangulation C_2 of a solid torus, all the connected normal surfaces are one of the following:*

1. *vertex-linking disks,* $(2, 0, 1, 1); (2, 0, 1, 1)$ *and* $(0, 2, 1, 1); (0, 2, 1, 1)$

2. *vertical annulus,* $(0, 0, 1, 1); (0, 0, 1, 1)$.

3. *Meridian disk.*

   (a) *Meridian disk,* $(1, 1, p, p+1); (1, 1, p+2, p+3), p \geq 0,$ *or*

   (b) *Meridian disk,* $(1, 1, p+1, p); (1, 1, |p-1|, |p-2|), p \geq 0.$

   For $p = 0$*, we have the meridian disk* $(1, 1, 1, 0); (1, 1, 1, 2)$ *which is the boundary compression of the the Möbius band with edge weights* $(1, 1, 1, 0); (1, 1, 1, 0)$ *in 5(a) into the top annulus* A_2 *of the boundary of the solid torus.*

   For $p = 1$*, we have the meridian disk* $(1, 1, 2, 1); (1, 1, 0, 1)$ *which is the boundary compression of the the Möbius band with edge weights* $(1, 1, 0, 1); (1, 1, 0, 1)$ *in 5(b) into the bottom annulus* A_0 *of the boundary of the solid torus.*

4. *Upper edge-linking disk,* $(2, 2, 2, 2); 2 \times (1, 1, 0, 1)$ *or* $(2, 2, 2, 2); 2 \times (1, 1, 1, 0)$.

5. *Lower edge-linking disk,* $2 \times (1, 1, 1, 0); (2, 2, 2, 2)$ *or* $2 \times (1, 1, 0, 1); (2, 2, 2, 2)$.

6. *One-sided (nonorientable) surface.*

   (a) *Möbius band,* $(1, 1, 0, 1); (1, 1, 0, 1)$.

   (b) *Möbius band,* $(1, 1, 1, 0); (1, 1, 1, 0)$.

7. *Annulus,*

   (a) $2 \times (1, 1, 1, 0); 2 \times (1, 1, 1, 0)$, *which is the double of the Möbius band in 6(a), or*

   (b) $2 \times (1, 1, 0, 1); 2 \times (1, 1, 0, 1)$, *which is the double of the Möbius band in 6(b), or*

   (c) $2 \times (1, 1, 0, 1); 2 \times (1, 1, 1, 0)$

Proof. According to 2.1, all the possible edge-weights of normal curves from the boundaries of a connect normal surface in the bottom annulus A_0 are $(2, 0, 1, 1)$,

$(0,2,1,1)$, $(2,2,2,2)$, $(0,0,1,1)$, $(1,1,p,p+1)$, $(1,1,p+1,p)$, $2\times(1,1,0,1)$ or $2\times(1,1,p+1,p)$ with $p\geq 0$.

1. The bottom edge-weight of a normal surface is $(2,0,1,1)$ or $(0,2,1,1)$. Then all we can do to these two vertex linking arcs is push-through. Hence we get two vertex-linking disks, $(2,0,1,1);(2,0,1,1)$ and $(0,2,1,1);(0,2,1,1)$. This gives us case 1 in the theorem.

2. The bottom edge-weight of a normal surface is $(2,2,2,2)$. Then we can either push through or add a band in the first tetrahedron.If we push through in both tetrahedra, we will get 2 disjoint vertex-linking disks $(2,2,2,2)$. Since this is not connected, we ignore this case. If we first push through this normal surface in the first tetrahedron and then add a band connecting the two vertex linking disks, by calculating the Euler character, we get an orientable normal disk in this triangulation. Hence we get the first part of case 4, an upper edge-linking disk, $(2,2,2,2);2\times(1,1,1,0)$. If we add a band at the beginning, then we will have edge-weight $(2,2,2,2);2\times(1,1,1,0)$, in C_1. By the above discussion of the change of edge-weight, we know in the next tetrahedron we can only push through this normal surface, hence we find another upper edge-linking disk $(2,2,2,2);2\times(1,1,0,1)$. This gives us the second part of case 4.

3. The bottom edge-weight of a normal surface is $(0,0,1,1)$. We can only push though in the two tetrahedra. We notice that this surface is an annulus in case 2, a vertical annulus, $(0,0,1,1);(0,0,1,1)$.

4. The bottom edge-weight of a normal surface is $(1,1,p,p{+}1)$, where $p\geq 0$. There are two possibilities for this case. If $p\geq 1$, all we can do is to push through the normal arc all the way. Thus we get a meridian disk,$(1,1,p,p{+}1);(1,p{+}2,p{+}3)$, in case $3(a)$ when $p>0$. If p=0, then we have a normal surface with bottom edge-weight $(1,1,0,1)$. In this situation, we can either push through or add

a bandin σ_1. If we just push through, then we just get a meridian disk with edge-weights $(1,1,0,1);(1,1,2,3)$, which is from case $3(a)$ when $p=0$ in C_2. If we add a band in σ_1, it means we add a crosscap to the normal surface with top edge-weight $(1,1,1,0)$. We can only push through in the second tetrahedron. Therefore we get a Möbius band with edge weights $(1,1,0,1);(1,1,0,1)$ giving case $6(a)$.

5. The bottom edge-weight of a normal surface is $(1,1,p+1,p)$, where $p \geq 0$. If $p \geq 1$, then we can only push through in C_2, we get a meridian disk of case $3(b)$ for $p > 0$. If $p = 0$, we can only push through $(1,1,1,0)$ to $(1,1,0,1)$ in σ_1. In σ_2, we can either push through or add a band. If we push through, we get a meridian disk, $(1,1,1,0);(1,1,1,2)$, in case $3(b)$ for $p=0$. If we add a band, then we get a Möbius band with edge-weight $(1,1,1,0),(1,1,1,0)$ giving case $6(b)$. Notice we have the meridian disk $(1,1,1,0);(1,1,1,2)$, which is the boundary compression of the the Möbius band with edge weights $(1,1,1,0);(1,1,1,0)$ into the top annulus A_2 of the boundary of the solid torus. Furthermore, for $p=1$, we get a meridian disk,$(1,1,2,1);(1,1,0,1)$, which is the boundary compression of the the Möbius band with edge weights $(1,1,0,1);(1,1,0,1)$ into the bottom annulus of ∂C_2 See figure 2.9

6. The bottom edge-weight of a normal surface is $2\times(1,1,0,1)$. In this case we have 3 choices in the first tetrahedron. The first choice we can do is to push through. However, if we do this, then we can never add a band in the second tetrahedron to make this two copies of normal surfaces connected. Hence we ignore it. The second choice is to add one band. Then we have the top edge-weight $(2,2,2,2)$ in the first tetrahedron. What we can do next is either to push through or add one band in the σ_2. If we push through, we will have a lower edge-linking disk, $2\times(1,1,0,1);(2,2,2,2)$, which is the second possiblility in case 5. If we add a

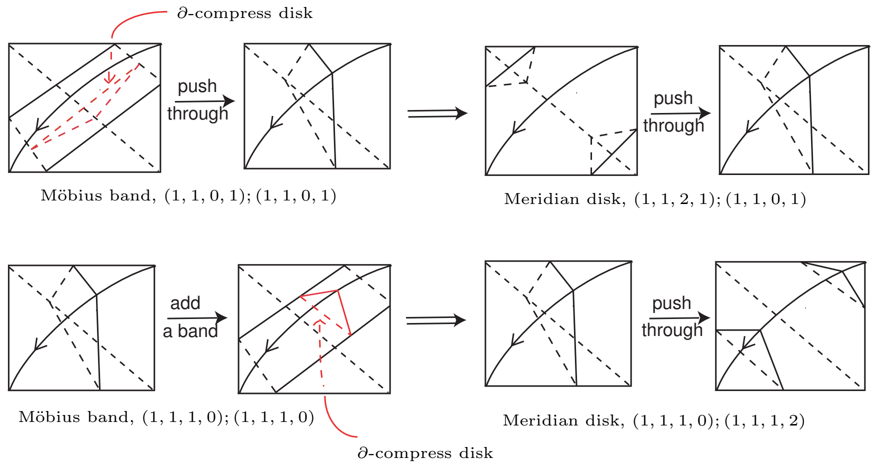

Figure 2.9: ∂-compress in C_2.

band in the σ_2, then we will get a normal annulus, $2 \times (1,1,0,1); 2 \times (1,1,1,0)$, which is the case $7(c)$. The third choice we can do in the σ_1 is to add two bands. The edge-weights changed from $2 \times (1,1,0,1)$ to $2 \times (1,1,1,0)$ by adding two bands. Therefore, we get a normal annulus, $2 \times (1,1,0,1); 2 \times (1,1,0,1)$, which is the case $7(b)$

7. The bottom edge-weight of a normal surface is $2 \times (1,1,p+1,p)$. If $p \geq 1$, The only thing we can do in C_2 is push through. Then we will get two disjoint meridian disks, $2 \times [(1,1,p+1,p); (1,1,p-1,p-2)]$. We ignore this case. If $p = 0$, we have the bottom edge-weight $2 \times (1,1,1,0)$. We can only push through in σ_1 and get the top edge-weight $2 \times (1,1,0,1)$ on A_1. Hence in σ_2, we can either add one band or two bands to get a connected surface. If we add one band, we will get a lower edge-linking disk, $2 \times (1,1,1,0); (2,2,2,2)$, which is the first possiblility of case 5. If we add two bands on it, we will get a normal annulus, $2 \times (1,1,1,0); 2 \times (1,1,1,0)$, which is the case $7(a)$.

■

2.2.3 Classification of normal surfaces in the layered chain triangulations of the solid torus

We show that the examples we studied in the section 2.2.1 represent *all* possible types of connected, embedded, normal surfaces in layered chain triangulations of the solid torus.

Theorem 2.2 *A connected, embedded, normal surface in a layered chain triangulation, C_k, of the solid torus, $k \geq 2$, is normally isotopic to one of the model surfaces listed in the following,*

0. *Vertex-linking disk,*$(0,2,1,1);(0,2,1,1)$ *or* $(2,0,1,1);(2,0,1,1)$

1. *Vertex-linking disks (possibly) with thin edge-linking tubes,* $(2,2,2,2);(2,2,2,2)$.

2. *Vertical annulus,* $(0,0,1,1);(0,0,1,1)$.

3. *Meridian disk.*

   (a) *Meridian disk,* $(1,1,p,p+1);(1,1,p+k,(p+1)+k)$, *or*

   (b) *Meridian disk,* $(1,1,p+1,p);(1,1,|(p+1)-k|,|p-k|)$.

4. *Upper edge-linking disk (possibly) with thin edge-linking tubes,* $(2,2,2,2);2\times(1,1,1,0)$ *or* $(2,2,2,2);2\times(1,1,q,q+1)$, *with* $q \geq 0$.

5. *Lower edge-linking disk (possibly) with thin edge-linking tubes,* $2\times(1,1,p+1,p);(2,2,2,2)$ *or* $2\times(1,1,0,1);(2,2,2,2)$.

6. *One-sided (nonorientable) surface.*

   (a) *One-sided surface of genus* c, $(1,1,0,1);(1,1,|k-2c|,(k+1)-2c), k \geq 2c-1$.

   (b) *One-sided surface of genus* c, $(1,1,p+1,p);(1,1,|(k-1)-(p+2c)|,k-(p+2c)), k \geq (p+2c)$.

7. *Annulus (possibly) with thin edge-linking tubes.*

   (a) *Annulus (possibly) with thin edge-linking tubes, which is the double of surface 6(a).*

   The boundary of the surface with the following two possible boundary edge-weights,

   (a.1) $2 \times (1, 1, 0, 1); 2 \times (1, 1, 1, 0)$, *for* $k = 2c - 1$, $c \geq 1$;

   (a.2) $2 \times (1, 1, 0, 1); 2 \times (1, 1, q, q + 1)$, *for* $k = 2c + q$, $q \geq 0$;

   (b) *Annulus (possibly) with thin edge-linking tubes, which is the double of surface 6(b).*

   The boundary of the surface with the following two possible weights,

   (b.1) $2 \times (1, 1, p + 1, p); 2 \times (1, 1, 1, 0)$, *for* $k = p + 2c$, $p \geq 0$; *or*

   (b.2) $2 \times (1, 1, p + 1, p); 2 \times (1, 1, q, q + 1)$, *for* $k = p + 2c + q + 1$, p, $q \geq 0$.

   (c) *Annulus (possibly) with thin edge-linking tubes, with the following possible weights.*

   (c.1) $2 \times (1, 1, 0, 1); 2 \times (1, 1, 1, 0)$,

   (c.2) $2 \times (1, 1, 0, 1); 2 \times (1, 1, q, q + 1)$, *which includes* $2 \times (1, 1, 0, 1); 2 \times (1, 1, 0, 1)$,

   (c.3) $2 \times (1, 1, p + 1, p); 2 \times (1, 1, 1, 0)$, *which includes* $2 \times (1, 1, 1, 0); 2 \times (1, 1, 1, 0)$,

   (c.4) $2 \times (1, 1, p + 1, p); 2 \times (1, 1, q, q + 1)$, *which includes* $2 \times (1, 1, 1, 0); 2 \times (1, 1, 0, 1)$,

Proof. Our proof is by induction on k, the number of layers in the layered chain triangulation of the solid torus.

We begin with C_2, the first level for which we have a solid torus.

By theorem 2.1, all of the possible normal surfaces in C_2 are listed among those in the above cases. We do not have any thin edge-linking tubes for surfaces in C_2, since there are no interior edges in the triangulation.

Now let's assume that all the normal surfaces in $C_k, k \geq 2$ are among those listed in the theorem. We consider C_{k+1}.

A connected normal surface S_{k+1} in C_{k+1} meets C_k in a normal surface S_k and meets the tetrahedron σ_{k+1} in a collection of normal triangles and normal quads. There are two possibilities that determine the collection of triangles and quads in σ_{k+1}: pushing S_k through or adding band(s) on S_k.

Case A. The surface S_{k+1} is obtained from S_k by *pushing* S_k through σ_{k+1}. In this case the surface S_{k+1} is homeomorphic to S_k; hence, we only need to check the intersection numbers for components of S_{k+1} meeting the top annulus of C_{k+1}.

We have same top edge weight for $(2,0,1,1)$, $(0,2,1,1)$, $(2,2,2,2)$ and $(0,0,1,1)$. For the general case $(1,1,p,p+1)$, with $p \geq 0$ we have $(1,1,p+1,p+2)$ and for $(1,1,p+1,p)$, we have $(1,1,p,|p-1|), p \geq 0$. This satisfies our induction hypothesis.

Case B. The surface S_{k+1} is obtained from S_k by banding in σ_{k+1}. Recall that for a band to be added, the surface S_k must met the top annulus of C_k in slopes $(2,2,2,2)$ or $(1,1,0,1)$ or the double the latter case. Furthermore, since we are only interested in the case the surface S_{k+1} is connected, then either S_k is connected or S_k consists of two copies of a normal surface, each meeting the top annulus of C_k in one essential arc with edge weights $(1,1,0,1)$, according to the proof of Lemma [**?**].

Our induction hypothesis is that we have classified the connected normal surfaces in C_k and they appear in the above list of examples. Hence, by running through the different type of surfaces 1-7 in the list, we can distinguish all possibilities having the edge-weights in the top annulus A_k of C_k either $(2,2,2,2)$ or $(1,1,0,1)$.

We have:

0. S_k is the *Vertex-linking disk*,$(0,2,1,1);(0,2,1,1)$ or $(2,0,1,1);(2,0,1,1)$.

   In this case, We can not add any band on either of the vertex-linking disk. However, if S_k is disconnected normal surface consisting of two different types of vertex-linking disks, $(0,2,1,1);(0,2,1,1)$ and $(2,0,1,1);(2,0,1,1)$, we can add a band between them and get an upper edge-linking disk, $(2,2,2,2); 2\times(1,1,0,1)$. This gives us the surface in case 4.

1. S_k is the *vertex-linking disks (possibly) with thin edge-linking tubes*, $(2,2,2,2);(2,2,2,2)$.

   In this case, we can add a band in σ_{k+1} and S_{k+1} is an upper edge-linking disk (possibly) with thin edge-linking tubes and intersection numbers $(2,2,2,2);(1,1,1,0)$, which appears in case 4.

$3(b_1)$. S_k is a *meridian disk*, $(1,1,p+1,p);(1,1,|(p+1)-k|,|p-k|), p>0$.

   Only when $k=p+1$, the meridian disk has edge-weights $(1,1,p+1,p);(1,1,0,1)$ on C_k. In this case, we can add a band in C_{k+1}, and S_{k+1} is a Möbius band with intersection numbers $(1,1,p+1,p);(1,1,1,0)$, which appears in case $6(b)$ with $k=p+2=p+2c, c=1$.

$3(b_2)$. S_k is two copies of a *meridian disk*, $(1,1,p+1,p);(1,1,|(p+1)-k|,|p-k|), k=p+1, p>0$.

   In this case, we can add two bands and S_{k+1} is an annulus with intersection numbers $2\times(1,1,p+1,p); 2\times(1,1,1,0)$; it is the double the Möbius band in $3(b_1)$, which appears in case 7.

   Or, we can add just one band and two triangles and S_{k+1} is a lower edge-linking disk with intersection numbers $2\times(1,1,p+1,p);(2,2,2,2)$; it is a ∂–compression of the previous annulus, which appears in case 5.

4. S_k is an *upper edge-linking disk (possibly) with thin edge-linking tubes*, $(2,2,2,2); 2\times(1,1,0,1)$.

In this case, we can add two bands and S_{k+1} is new upper edge-linking disk (possibly) with thin edge-linking tubes and with edge weights $2\times(2,2,2,2); 2\times(1,1,1,0)$, which appears in case 4.

Or, we can add just one band and two triangles and S_{k+1} is the vertex-linking disks with thin edge-linking tubes and intersection numbers $(2,2,2,2);(2,2,2,2)$, which appears in case 1.

5. S_k is a *lower edge-linking disk (possibly) with thin edge-linking tubes*, $2\times(1,1,p+1,p);(2,2,2,2)$ or $2\times(1,1,0,1);(2,2,2,2)$.

   In this case, we can add just one band and S_{k+1} is an annulus (possibly) with thin edge-linking tubes and edge weights $2\times(1,1,p+1,p); 2\times(1,1,1,0)$ or $2\times(1,1,0,1); 2\times(1,1,1,0)$, both of which appear in case 7.

6(a). S_k is a *one-sided surface of genus* $c\geq 1$, $(1,1,0,1);(1,1,|k-2c|,(k+1)-2c), k\geq 2c-1$.

For $k=2c$ the intersection numbers for S_k are $(1,1,0,1);(1,1,0,1)$. Hence, we can add one band in σ_{k+1}. In this case F_{k+1} is a one-sided surface of genus $c+1$ and has intersection numbers $(1,1,0,1);(1,1,1,0)$. Note that $k+1=2c+1$ and the general form for the intersection numbers on the top annulus of C_{k+1} is $(1,1,1,0)=(1,1,|(2c+1)-2(c+1)|,2c+2-(2c+2))=(1,1,|((k+1)-2(c+1))|,((k+1)+1)-2(c+1))$, which is the general form and appears in Case $6(a)$.

6(b). S_k is a *one-sided surface of genus* $c\geq 1$, $(1,1,p+1,p);(1,1,|(k-1)-(p+2c)|,k-(p+2c)), k=(p+2c+1)$.

For $k=p+2c+1$ the edge weights for S_k are $(1,1,p+1,p);(1,1,0,1)$. Hence, we can add one band in σ_{k+1}. In this case S_{k+1} is a one-sided surface of genus $c+1$ and has edge weights $(1,1,p+1,p);(1,1,1,0)$. Note that $k+1=p+$

$2c+2$ and the general form for the edge weights on the top annulus of C_{k+1} is $(1,1,1,0) = (1,1,p+2c+1-(p+2c+2), p+2c+2-(p+2c+2)) = (1,1,|((k+1)-1)-(p+2(c+1))|, (k+1)-(p+2(c+1)))$, which is the general form and appears in Case $6(b)$.

Finally, the only remaining possibility for banding in σ_{k+1} is

7. S_k is an *annulus (possibly) with thin edge-linking tubes* and intersection numbers $2\times(1,1,0,1); 2\times(1,1,0,1)$ or $2\times(1,1,p+1,p); 2\times(1,1,0,1)$.

   In either case, we can add two bands and S_{k+1} is an annulus with thin edge-linking tubes and with intersection numbers $2\times(1,1,0,1); 2\times(1,1,1,0)$ or $2\times(1,1,p+1,p); 2\times(1,1,1,0)$, respectively; it can be the the double of the one-sided surface in $6(a)$ or $6(b)$, or just add a thin edge-linking tube to the original surface from $7(c)$. Hence all the surfaces appears in Case 7.

   Or, we can add just one band and two triangles and F_{k+1} is a lower edge-linking disk with thin edge-linking tubes and intersection numbers $2\times(1,1,0,1); (2,2,2,2)$ or $2\times(1,1,p+1,p); (2,2,2,2)$, which appears in Example 5. Furthermore, it is a ∂–compression of the previous annulus.

Hence, all normal surfaces in C_{k+1} are included in the list.

■

CHAPTER 3

Twisted layered loop triangulations

In a layered chain triangulation, C_k, of the solid torus there are two free faces in the tetrahedron σ_k making up the top annulus A_k in ∂C_k and two free faces in the tetrahedron σ_1 making up the bottom annulus A_0 in ∂C_k. If we try to identify the free faces on the boundary of the solid torus in pairs, we may get a new triangulation of some 3-manifolds. Burton in [4] discussed several different cases. In this chapter we will study one of them which is called twisted layered loop triangulation.

3.1 Twisted layered loop triangulations of M_k

. First let's see how Burton constructed this triangulation. We identify the four free faces on the top/bottom annuli of a layered chain triangulation of the solid torus by layering σ_1 onto σ_k along e_k with $e_1 \leftrightarrow -e_{k+1}$, $e_2 \leftrightarrow -e_{k+2}$, and $t \leftrightarrow -b$. See figure 3.1. The result is a closed 3–manifold, denoted M_k, and the triangulation, denoted $\widehat{C}_k$. In [4], he also shows that for each $k \geq 1$, the twisted layered loop $\widehat{C}_k$ is a one-vertex triangulation of the space S^3/Q_{4k}, or equivalently of the Seifert fibered space $SFS(S:(2,1),(2,1),(k,-k+1))$. Recently in the papers [15, 16], it is proved that the generalized quaternion spaces S^3/Q_{4k} have complexity k, $k \geq 2$. The complexity of a 3-manifold M is the minimal number of tetrahedra in a triangulation of M. Therefore, a twisted layered loop triangulation $\widehat{C}_k$ of M_k is a minimal triangulation of M_k.

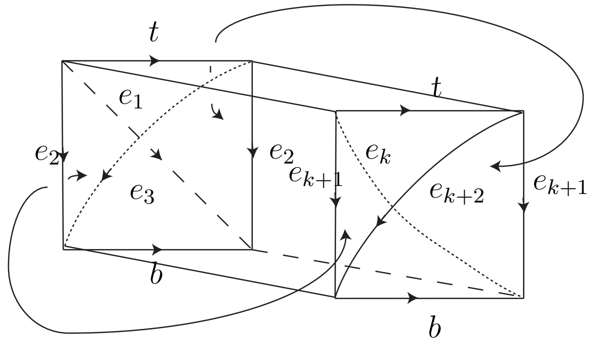

Figure 3.1: Twisted layered loop triangulation.

3.2 Normal surfaces in twisted layered loop triangulations

In this section we will discuss and classify the twisted layered loop triangulations $\widehat{C}_k$ of M_k, where $M_k = S^3/Q_{4k} = S^2((2,1),(2,1)(k,1-k))$.

If $\widehat{S}$ is a normal surface in $\widehat{C}_k$, then $\widehat{S}$ determines a unique normal surface S in the layered chain triangulation C_k of the solid torus and $\widehat{S}$ is obtained from S by identifications along ∂S. Hence, it is necessary that the surface S has the same edge-weights on the identified edges e_{k+1} and e_1, edges e_{k+2} and e_2, and edges t and b; that is, S must have the ordered edge weights $(wt_t, wt_b, wt_{e_1}, wt_{e_2}) = (wt_t, wt_b, wt_{e_{k+1}}, wt_{e_{k+2}})$, i.e. the corresponding coordinates are the same. Furthermore, we realize that $wt_t = wt_b$, since the edge t is identified with $-b$.

Theorem 3.1 *A connected, embedded normal surface in the twisted layered loop triangulation $\widehat{C}_k$ is normally isotopic to one of the following surfaces:*

(i) vertex-linking 2–sphere (possibly) with thin edge-linking tubes; or

(ii) a Klein bottle, which is a quadrilateral splitting surface, splitting the opposite edges $t = -b$ in each tetrahedron; or

(iii) k is even, and there is

- *a nonorientable surface of genus $\frac{k}{2}+1$, which is a quadrilateral splitting surface, splitting the odd index edges, and*

- *a nonorientable surface of genus $\frac{k}{2}+1$, which is a quadrilateral splitting surface, splitting the even index edges.*

Remark 3.1 *The double of the surface in (ii) is a thin edge-linking torus about the edge $t = -b$. The doubles of the surfaces in (iii) are vertex-linking 2–spheres with thin edge-linking tubes; in the first case of (iii), the thin edge-linking tubes are about the odd index edges and in the second case of (iii), the thin edge-linking tubes are about the even index edges. In all these doubles, we have quadrilateral surfaces that fall into (i).*

Proof. We only need to list the embedded normal surfaces in the layered chain triangulation C_k of the solid torus as in theorem 2.2 and consider those normal surfaces where we have ordered edge weights $(wt_t, wt_b, wt_{e_1}, wt_{e_2}) = (wt_t, wt_b, wt_{e_{k+1}}, wt_{e_{k+2}})$, and $wt_t = wt_b$.

0. No surfaces result from Vertex-linking disk. For any Vertex-linking disk,$(0,2,1,1)$;$(0,2,1,1)$ or $(2,0,1,1); (2,0,1,1)$, we can not get any normal surfaces in the twisted layered loop triangulation, since $wt_t \neq wt_b$

1. From the vertex-linking disks (possibly) with thin edge-linking tubes, $(2,2,2,2); (2,2,2,2)$, we get the *vertex-linking 2–sphere (possibly) with thin edge-linking tubes.* The two vertex-linking disks are identified along their boundaries to give the single vertex-linking 2–sphere; if there are thin edge-linking tubes, then the resulting surface has the same thin edge-linking tubes.

2. From the vertical annulus, $(0,0,1,1); (0,0,1,1)$, we have *a Klein bottle, which is a quadrilateral splitting surface, splitting the opposite edges $t = -b$ in each tetrahedron.*

   In this case the two boundaries of the vertical annulus have an orientation reversing identification, giving a Klein bottle.

3. No surfaces result from the meridian disks.

   For the meridian disks from $3(a)$ with boundary edge-weights, $(1, 1, p, p+1); (1, 1, p+k, (p+1)+k)$, we have $k > 0$ and thus $p \neq p+k$.

   For the meridian disks from $3(b)$ with boundary edge-weights, $(1, 1, p+1, p); (1, 1, |(p+1)-k|, |p-k|)$, we have $k > 0$; hence, for $0 < k \leq p$, we have $0 \leq p - k \neq p$ and for $k > p$, we have $|(p+1)-k| < |p-k|$, whereas $p+1 > p$. It follows that in both cases the boundary edge-weights do not match upon identification.

4. No surfaces result from an upper edge-linking disk (possibly) with thin edge-linking tubes, $(2, 2, 2, 2); 2 \times (1, 1, q, q+1)$ or $(2, 2, 2, 2); 2 \times (1, 1, 1, 0)$, because the boundary edge-weights do not match upon identification.

5. Similarly, no surfaces result from a lower edge-linking disk (possibly) with thin edge-linking tubes, $2 \times (1, 1, p+1, p); (2, 2, 2, 2)$ or $2 \times (1, 1, 0, 1); (2, 2, 2, 2)$.

6. If k is even we get two one-sided (nonorientable) surfaces each having genus $\frac{k}{2} + 1$.

   (a) From the one-sided surfaces of genus c, $(1, 1, 0, 1); (1, 1, 0, 1)$, we get a nonorientable surface of genus $\frac{k}{2} + 1$, $\widehat{S}_1$, which is a quadrilateral splitting surface, splitting the odd index edges.

   In case 6(a) of theorem 2.2 we have a family of one-sided surface of genus c, $(1, 1, 0, 1); (1, 1, |k-2c|, (k+1)-2c), k = 2c$. For a surface from this family to give a surface in $\widehat{C}_k$, we must have the edge-weight $|k-2c| = 0$, which happens if and only if $k = 2c$. We noted above in Example 6(a) that this surface is a quadrilateral splitting surface, splitting the odd index edges. This surface has connected boundary in C_k and upon identification of its boundary, we add another crosscap, giving a one-sided surface with genus $c + 1 = \frac{k}{2} + 1$ in $\widehat{C}_k$. Notice there is no other nonorientable surface exist such that this surface can be compress to, therefore, it is incompressible.

(b) From the one-sided surface of genus $c = \frac{k}{2}$, $(1,1,1,0);(1,1,1,0)$, we get a nonorientable surface of genus $\frac{k}{2}+1$, $\widehat{S}_2$, which is a quadrilateral splitting surface, splitting the even index edges.

In case 6(b) of theorem 2.2 we have a family of one-sided surface of genus c, $(1,1,p+1,p);(1,1,|(k-1)-(p+2c)|,k-(p+2c)), k \geq (p+2c)$. For a surface from this family to give a surface in $\widehat{C}_k$, we must have the edge-weight $k-(p+2c) = p$, which happens if and only if $k = 2p+2c$. However, we also must have the edge-weight $|(k-1)-(p+2c)| = p+1$ for $k = 2p+2c$, which happens if and only if $p = 0$. So, from Example 6(b) above, we must have the one-sided surface of genus $c = \frac{k}{2}$, $(1,1,1,0);(1,1,1,0)$. We noted that this surface is a quadrilateral splitting surface, splitting the even index edges; it has connected boundary in C_k and upon identification of its boundary, we add another crosscap, giving a one-sided surface with genus $c+1 = \frac{k}{2}+1$ in $\widehat{C}_k$, and it is incompressible.

7. Finally, we have annuli (possibly) with thin edge-linking tubes and edge-weights in ∂C_k, $2\times(1,1,0,1);2\times(1,1,1,0)$, or $2\times(1,1,0,1);2\times(1,1,q,q+1)$, or $2\times(1,1,p+1,p);2\times(1,1,1,0)$, or $2\times(1,1,p+1,p);2\times(1,1,q,q+1)$, where p, $q \geq 0$.

   Obviously, we can not get a match from $2\times(1,1,0,1);2\times(1,1,1,0)$. From $2\times(1,1,0,1);2\times(1,1,q,q+1)$, we must have $q = 0$ and we have the double of the one-sided surface in 6(a),or an annulus in 7(c). From $2\times(1,1,p+1,p);2\times(1,1,1,0)$, we must have $p = 0$ and it is the double of the one-sided surface in 6(b) or annulus in 7(c). In the last case where the edge-weights are $2\times(1,1,p+1,p);2\times(1,1,q,q+1)$, we can never get a match, since the maximum edge-weight is on e_1 in the bottom annulus of ∂C_k but is on e_{k+2}, not e_{k+1}, in the top annulus. These edge-weights are not match under the identification.

Note that the only surfaces we obtained from 7 are the vertex-linking 2–sphere with thin edge-linking tubes.

This completes the proof. ■

Corollary 3.1 *Let $M_k = S^3/Q_{4k} = S^2((2,1),(2,1),(k,1-k))$. If k is even, then M_k has precisely three distinct (up to isotopy) embedded, connected, one-sided, incompressible surfaces; one is a Klein bottle, each of the other two have genus $\frac{k}{2}+1$.*

Proof. It is well-known (c.f. [11, 21]) that the fundamental group of M_k is

$$\pi_1 M_k =< f, s_1, s_2, s_3 | [s_i, f], {s_1}^2 f, {s_2}^2 f, {s_3}^k f, s_1 s_2 s_3 f^e = 1, i = 1, 2, 3 >$$

where the interger e is the usual Euler class representing the obstruction to extend a section given on the boundary components of regular neighborhoods of the exceptional fibers to the complement. Here $e = 1$.

Therefore, the first homology group $H_1(M_k, \mathbb{Z}_2)$ is $\mathbb{Z}_2 + \mathbb{Z}_2$ for k is even, and is $\mathbb{Z}_4$ for k is odd. There is an one-sided incompressible nonorientable surface associated with any nonzero class in $H_1(M_k, \mathbb{Z}_2) \cong H^1(M_k, \mathbb{Z}_2) \cong H_2(M_k, \mathbb{Z}_2)$ (See details in [26]). Therefore, for k even, there are 3 nonzero class in $H_1(M_k, \mathbb{Z}_2)$ is $\mathbb{Z}_2 + \mathbb{Z}_2$. Hence, M_k has at least three distinct (up to isotopy), embedded, one-sided incompressible surfaces. It follows that for any triangulation of M_k there must then be at least three distinct, embedded, one-sided, incompressible surfaces. In particular, this is true for the twisted layered loop triangulation $\widehat{C}_k$. Since the triangulation $\widehat{C}_k$ has precisely three one-sided normal surfaces: a Klein bottle and two surfaces each with genus $\frac{k}{2} + 1$, it follows that these surfaces are not isotopic and are incompressible. ■

CHAPTER 4

Layered chain pair triangulations

In this chapter we will discuss another family of triangulations, layered chain pair triangulations, based on layered chain triangulations of the solid torus, constructed by Ben Burton in [4].

4.1 Layered chain pair triangulations of $M_{r,s}$

The starting point for a layered chain pair triangulation is two layered chain triangulations C_r and C_s of the solid torus, of length r and s, respectively. The boundary of the solid torus in a layered chain triangulation is made up of four triangles. Two of them make up what we are calling the top annulus and the other two make up the bottom annulus. In the layered chain C_r we label the boundary edges $\tau, \beta, f_1, f_2, f_{r+1}$, and f_{r+2}, where the labels are analogous with those in Figure 3.24 in [4], and direct them as in Figure 3.24 in [4]. In the layered chain C_s we label the boundary edges t, b, e_1, e_2, e_{s+1} and e_{s+2} and direct them as in Figure 3.24 in [4]. A manifold with triangulation is then obtained by identifying the four boundary faces of C_r with the four boundary faces of C_s, using the following edge identifications. See Figure 4.1.

$$b \leftrightarrow f_{r+1} \qquad\qquad t \leftrightarrow -f_2$$
$$e_2 \leftrightarrow -\tau \qquad\qquad e_{s+1} \leftrightarrow \beta$$
$$e_1 \leftrightarrow -f_1 \leftrightarrow -e_{s+2} \leftrightarrow f_{r+2}.$$

The result is a closed 3–manifold, denoted $M_{r,s}$, and the triangulation, denoted $C_{r,s}$, is called an (r, s) *layered chain pair* after [4].

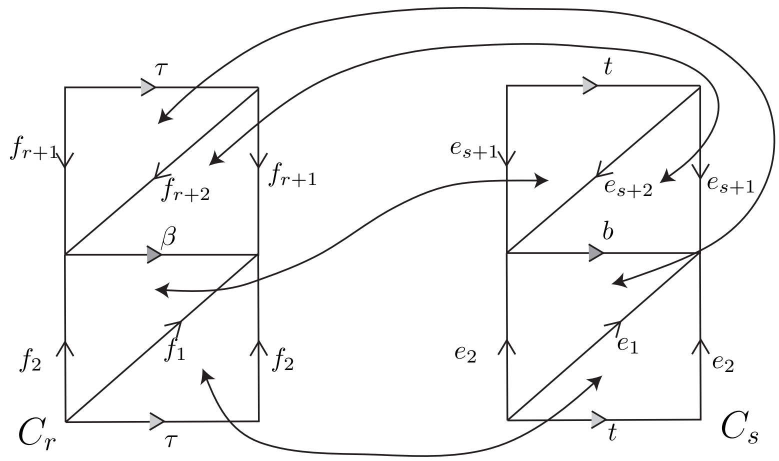

Figure 4.1: Face identifications of two layered chain triangulations, C_r and C_s, giving a layered chain pair triangulation, $C_{r,s}$.

Ben Burton proved the following theorem in [4].

Theorem 4.1 *(Burton) For each $r, s \geq 1$, the layered chain pair $C_{r,s}$ is a triangulation of the Seifert fibred space $(S^2 : (2, -1), (r + 1, 1), (s + 1, 1))$.*

Lemma 4.1 *(Burton) The layered chain pairs $C_{r,s}$ and $C_{s,r}$ are isomorphic triangulations. Furthermore, the layered chain pair $C_{r,1}$ is in fact simply the twisted layered loop $\widehat{C}_{r+1}$.*

4.2 Normal surfaces in layered chain pair triangulations

If S is a normal surface in $C_{r,s}$, then S determines a unique normal surface S_r in C_r and a unique normal surface S_s in C_s. Notice, the normal surface S_r and S_s are not necessary connected. Furthermore, S is obtained from S_r and S_s by identifications along their boundaries. Hence, it is necessary that the boundary of the surface S_r has the same edge-weights in C_r as the boundary of the surface S_s has in C_s on edges matching under the above face identifications to obtain $C_{r,s}$. In particular, we must have the edge-weights $wt_{f_2}(S_r) = wt_t(S_s)$, $wt_{f_{r+1}}(S_r) = wt_b(S_s)$, $wt_\tau(S_r) = wt_{e_2}(S_s)$, and $wt_\beta(S_r) = wt_{e_{s+1}}(S_s)$, respectively, and edge-weights $wt_{f_1}(S_r) = wt_{e_1}(S_s) =$

$wt_{f_{r+2}}(S_r) = wt_{e_{s+2}}(S_s)$.

In our notation, the edge-weights for the boundary of the surface S_r are given as $(wt_\tau, wt_\beta, wt_{f_1}, wt_{f_2}); (wt_\tau, wt_\beta, wt_{f_{r+1}}, wt_{f_{r+2}})$; whereas, those for the surface S_s are given as $(wt_t, wt_b, wt_{e_1}, wt_{e_2}); (wt_t, wt_b, wt_{e_{s+1}}, wt_{e_{s+2}})$. Hence, in order for S_r and S_s to match to give a normal surface in $C_{r,s}$, we must have pairs of 4-tuple:

$$(x, y, z, u); (x, y, v, z) \leftrightarrow (u, v, z, x); (u, v, y, z), \tag{4.1}$$

where the first pair $(x, y, z, u); (x, y, v, z)$ are the parameterizations for edge-weights of the boundary of S_r in the bottom annulus and the top annulus of C_r,respectively, and the second pair $(u, v, z, x); (u, v, y, z)$ are the parametrizations for the edge-weights of the boundary of S_s in the bottom annulus and the top annulus of C_s, respectively. From now on, we will identify the unique normal surface S obtained from S_r and S_s by the edge-weight matching equation $(x, y, z, u); (x, y, v, z) \leftrightarrow (u, v, z, x); (u, v, y, z)$, determined from S_r and S_s.

Theorem 4.2 *A connected, embedded, normal surface S in the triangulated chain pair, $C_{r,s}$=$C_{s,r}$, r, $s > 1$ is isotopic to one of the following surfaces:*

For the orientable case,

I. S is a vertex-linking s^2 (possibly) with thin edge-linking tubes in all triangulation $C_{r,s}$, $r, s > 1$.

It has one of the following possible edge-weight matching equations,

1. $(2, 2, 2, 2); (2, 2, 2, 2) \leftrightarrow (2, 2, 2, 2); (2, 2, 2, 2)$
2. $2 \times (1, 1, 0, 1); 2 \times (1, 1, 1, 0) \leftrightarrow 2 \times (1, 1, 0, 1); 2 \times (1, 1, 1, 0)$
3. $[(0, 2, 1, 1); (0, 2, 1, 1)] + [(0, 0, 1, 1); (0, 0, 1, 1)] \leftrightarrow 2 \times (1, 1, 1, 0); (2, 2, 2, 2)$
4. $[(2, 0, 1, 1); (2, 0, 1, 1)]+[(0, 0, 1, 1); (0, 0, 1, 1)] \leftrightarrow (2, 2, 2, 2) \leftrightarrow 2\times(1, 1, 0, 1)$
5. $2 \times (0, 0, 1, 1); 2 \times (0, 0, 1, 1) \leftrightarrow 2 \times (1, 1, 1, 0); 2 \times (1, 1, 0, 1)$

In fact, for very edge in the layered chain pair triangulation, there is a vertex-linking S^2 with a thin edge-linking tube around it. Furthermore, For any proper compatible subsets of all edges in the layered chain pair triangulation, there is a vertex linking S^2 with thin edge-linking tubes around each edge in this subset.

II. *S is an orientable normal surface, which is not a thin edge-linking tube surface. Assume S_r in C_r has genus g', and S_s in C_s has genus g.*

1. *S is a nonseparating torus in $C_{2,5} = C_{5,2}$ with edge-weights matching equation $(2,2,3,1);(2,2,1,3) \leftrightarrow (1,1,3,2);(1,1,2,3)$.*

   It is a torus fiber in the fibration of $M_{2,5} = M_{5,2}$ over S_1.

2. *S is an orientable surface with genus $g+2$ in $C_{2,n} = C_{n,2}$, $n \geq 7$, with edge-weights matching equation $2\times(2,2,3,1);2\times(2,2,1,3) \leftrightarrow 2\times(1,1,3,2);2\times(1,1,2,3)$.*

3. *S is a nonseparating torus in $C_{3,3}$ with edge-weights matching equation $(1,1,2,1);(1,1,1,2) \leftrightarrow (1,1,2,1);(1,1,1,2)$.*

   It is a torus fiber in the fibration of $M_{3,3}$ over S^1.

4. *S is an orientable surface with genus $g'+g+2$ in $C_{3,5} = C_{5,3}$ or $C_{r,s}$, $r,s \geq 5$, with edge-weights matching equation $2\times(1,1,2,1);2\times(1,1,1,2) \leftrightarrow 2\times(1,1,2,1);2\times(1,1,1,2)$.*

For the nonorientable case,

I. *r even and s odd, $s = 2c+1, c \geq 1$ (or r odd, $r = 2c'+1, c' \geq 1$ and s even):*

1. *a nonorientable incompressible surface of genus c (or c') if $r = 2$, $s = 2c+1$,$s \geq 7$, i.e. $c \geq 3$.*

   It has an edge-weights matching equation $(2,2,3,1);(2,2,1,3) \leftrightarrow (1,1,3,2);(1,1,2,3)$.

   The double of it is an orientable surface with genus $c-1$ in the orientable case II.2. Also, if $s = 5$ this surface is a nonseparating torus in $C_{2,5}$.

2. *a nonorientable, quadrilateral, splitting surface of genus* $c+2$ *(or* $c'+2$*), if* $r=2, s=2c+1, s \geq 3$. *The edge-matching equation is* $(0,0,1,1);(0,0,1,1) \leftrightarrow (1,1,1,0)(1,1,0,1)$.

   In $C_{2,2c+1}$, $c \geq 2$ *i.e.* $s \geq 5$, *it is a compressible surface and can be compressed to a surface mentioned in the above case 1. Otherwise, it is incompressible. The double of this surface is a vertex-linking* S^2 *with* $c+1$ *thin edge-linking tubes. It always has at least two thin edge-linking tubes around edge* $\tau = -e_2$ *and* $\beta = e_{s+1}$, *respectively.*

II. r, s *both odd,* $r = 2c'+1$, $s = 2c+1$, $c', c \geq 1$

1. *a nonorientable, incompressible, quadrilateral splitting surface of genus* $c+2$, *with edge-weights matching equation* $(0,0,1,1);(0,0,1,1) \leftrightarrow (1,1,1,0)(1,1,0,1)$.

2. *a nonorientable, incompressible, quadrilateral splitting surface of genus* $c'+2$, $(1,1,1,0)(1,1,0,1) \leftrightarrow (0,0,1,1);(0,0,1,1)$.

3. *a nonorientable, compressible, quadrilateral splitting surface of genus* $c'+c+2$, *with edge-weights matching equation* $(1,1,0,1);(1,1,1,0) \leftrightarrow (1,1,0,1);(1,1,1,0)$.

   This surface can be compressed to get a surface in the following case.

4. *a nonorientable, incompressible surface of genus* $c'+c$, *except for* $c' = c = 1$. *It has the edge-weights matching equation* $(1,1,2,1);(1,1,1,2) \leftrightarrow (1,1,2,1);(1,1,1,2)$.

   If we double this surface we will get a normal surface with genus $c'+c-1$ *in the orientable case II.4. When* $c' = c = 1$, *this surface is a nonseperating torus in* $C_{3,3}$ *in the orientable case II.3.*

Proof. As noted above, a normal surface S in $C_{r,s}$ determines unique normal surfaces S_r in C_r and S_s in C_s and is formed by identification of the boundary of S_r with the boundary of S_s. Furthermore, in order for the boundaries of S_r and S_s to match to give a normal surface in $C_{r,s}$, we must have the pairs of 4-tuples of edge-weights match as in equation 4.1, including possible multiplicities.

For the proof, we analyze the possible matches of these 4-tuples, using the classification of normal surfaces in a layered chain triangulation of the solid torus given in Theorem 2.2.

Notice all the families of normal surfaces in the Theorem 2.2 have edge-weight 4-tuples satisfy the equation $wt_t = wt_b$, except for the vertex linking disk, $(0, 2, 1, 1); (0, 2, 1, 1)$ and $(2, 0, 1, 1); (2, 0, 1, 1)$ in case 0. Therefore, $|x - y| = 2m$ holds for the edge-weights of the normal surface S_r, $(x, y, z, u); (x, z, v, y)$, in C_r, where m is the number of extra copies of $(0, 2, 1, 1); (0, 2, 1, 1)$ or $(2, 0, 1, 1); (2, 0, 1, 1)$ in case 0. If $x \geq y$, then m is where m is the number of extra copies of $(0, 2, 1, 1); (0, 2, 1, 1)$, and vice versa. Similarly, $|u - v| = 2n$ holds for the the edge-weights of the boundary of S_s, $(u, v, z, x); (u, v, y, z)$, in C_s, where n is the the number of extra copies of $(0, 2, 1, 1); (0, 2, 1, 1)$ or $(2, 0, 1, 1); (2, 0, 1, 1)$ in case 0.

Since $C_{r,s}$ and $C_{s,r}$ are isomorphic, we only need to consider the following 3 cases.

- if $x = y$ and $u = v$,
- if $x = y$ and $u \neq v$,
- if $x \neq y$ and $u \neq v$.

Let's start from the first case.

- if $x = y$ and $u = v$, In order for the S_r and S_s to match to give a normal surface in $C_{r,s}$, we must have pairs of 4-tuples:

$$(x, x, z, u); (x, x, u, z) \leftrightarrow (u, u, z, x); (u, u, x, z), \tag{4.2}$$

we analyze the possible matches of these 4-tuples, using the classification of normal surfaces in a layered chain triangulation of the solid torus given in

Theorem 2.2. Obviously that the normal surface S_r or S_s cannot be vertex linking disk, $(2,0,1,1);(2,0,1,1)$ or $(0,2,1,1);(0,2,1,1)$, Since its edge-weight does't satisfy $x = y$. Hence we ignore this case. We start from S_r is from case 1 in Theorem 2.2.

Case 1. The surface S_r is the vertex-linking disks (possibly) with thin edge-linking tubes, $(2,2,2,2);(2,2,2,2)$.

The match is

$$(2,2,2,2);(2,2,2,2) \leftrightarrow (2,2,2,2);(2,2,2,2),$$

giving that S_s must also be the vertex-linking disks (possibly) with thin edge-linking tubes. The surface S in $C_{r,s}$ is *the vertex-linking* 2*–sphere (possibly) with thin edge-linking tubes.*

Case 2.S_r is the vertical annulus, $(0,0,1,1);(0,0,1,1)$.

It follows from the necessary matching of edge weights in equation 4.2 that the edge-weights for the surface S_s must be $(1,1,1,0);(1,1,0,1)$ and the match is:

$$(0,0,1,1);(0,0,1,1) \leftrightarrow (1,1,1,0);(1,1,0,1).$$

Hence, we consider the possibilities for S_s.

Possibility 2.1. S_s is from $3(b)$ of Theorem 2.2, a meridian disk, $(1,1,p+1,p);(1,1,|(p+1)-s|,|p-s|)$. Hence, necessarily $p = 0$ and $s = 1$ and S is the Klein bottle in $\widehat{C}_{r+1}$. Since here we are assuming $s > 1$, we temporarily ignore this case, and address it again in the corollary following this proof.

Possibility 2.2. S_s is from $6(b)$ of Theorem 2.2, a one-sided surface of genus $c \geq 1$, $(1,1,p+1,p);(1,1,|(s-1)-(p+2c)|,s-(p+2c)), s \geq (p+2c)$. So, necessarily $p = 0$ and $(s-1)-2c = 0$ giving $s = 2c+1, c \geq 1$. The surface S in $C_{r,2c+1}$ is *a nonorientable, quadrilateral, splitting surface of genus* $c+2$. In general, this surface is is **compressible**, if $r = 2, s = 2c+1, s \geq 7$. Otherwise, it is incompressible.

If r is odd, $r = 2c' + 1$, then we can reverse the roles of r and s and we have the surface S in $C_{2c'+1,s}$ a nonorientable, quadrilateral, splitting surface of genus $c' + 2$ (See Possibility 6.2.1 below).

If $r = 2c' + 1$ and $s = 2c + 1$ are both odd, then we have both of these in $C_{2c'+1,2c+1}$, giving *distinct, nonorientable, quadrilateral, splitting surfaces of genus* $c + 2$ *and* $c' + 2$.

Possibility 2.3. If we double the surface S_r, we have the match

$$2 \times (0,0,1,1); 2 \times (0,0,1,1) \leftrightarrow 2 \times (1,1,1,0); 2 \times (1,1,0,1).$$

.

S_s is from $7(a.1)$ or $7(c.1)$ of Theorem 2.2, an annulus with thin edge-linking tubes, $2 \times (1,1,1,0); 2 \times (1,1,0,1)$.

Notice if S_s is double of a meridian disk, then S is not connected which consists of two disjoint surfaces. we ignore this case. Hence, S_s is from $7(a.1)$ or $7(c.1)$ of Theorem 2.2, an annulus with thin edge-linking tubes, $2 \times (1,1,1,0); 2 \times (1,1,0,1)$.

In this situation, S is a vertex-linking 2–sphere that has thin edge-linking tubes. Furthermore, S obtained from $(7(a.1)$ is a quadrilateral vertex-linking 2–sphere with thin edge-linking tubes which is the double of *Possibility 2.2*.

Case 3. S_r is from $3(b)$ of Theorem 2.2, a meridian disk, $(1,1,p+1,p); (1,1,|(p+1)-r|, |p-r|)$.

We have that $3(a)$ of Theorem 2.2 can not occur. For in this case, the edge-weights for the boundary of the meridian disk meeting the edges in the bottom annulus and the top annulus of C_r are $(1,1,p,p+1); (1,1,p+r,(p+1)+r)$, where

$r > 1$. However, the matching from equation 4.2 would require $p = p + 1 + r$ i.e. $r = -1$, and $p + 1 = p + r$, which is impossible.

If we have $3(b)$, the edge-weights for the boundary of the meridian disk S_r is $(1, 1, p+1, p); (1, 1, |(p+1)-r|, |p-r|)$. Since the edge-weights for the boundary of the meridian disk meeting the edges in the bottom annulus is $(1, 1, p+1, p)$, it follows from equation 4.2 that the match is

$$(1, 1, p + 1, p); (1, 1, p, p + 1) \leftrightarrow (p, p, p + 1, 1); (p, p, 1, p + 1),$$

By comparing the coordinates of the edge-weights in the above equation, we have $r = 2p + 1$. we also notice S_r is a meridian disk from $3(b)$, $(1, 1, p + 1, p); (1, 1, p, p + 1)$, $p \geq 0$.

Now, we consider the possibilities for the surface S_s. Its edge-weights must be $(p, p, p + 1, 1); (p, p, 1, p + 1)$.

notice the edge-weights for the boundary of S_s meeting the edges in the bottom annulus is

$$(p, p, p + 1, 1) = \begin{cases} (1, 1, 2, 1) + (p - 1) \times (1, 1, 1, 0), p \geq 2 \\ (1, 1, 2, 1), p = 1 \\ (0, 0, 1, 1), p = 0 \end{cases} \tag{4.3}$$

Let's consider the case $p \geq 2$, the edge-weights for the boundary of S_s meeting the edges in the bottom annulus is

$$\begin{aligned}
&(p, p, p + 1, 1) = (1, 1, 2, 1) + (p - 1) \times (1, 1, 1, 0) \\
&\xrightarrow{push} (1, 1, 1, 0) + (p - 1) \times (1, 1, 0, 1) \\
&\xrightarrow{push} (1, 1, 0, 1) + (p - 1) \times (1, 1, 1, 2) \\
&\xrightarrow{push} (1, 1, 1, 2) + (p - 1) \times (1, 1, 2, 3) \\
&\xrightarrow{push} \cdots \\
&\xrightarrow{push} (1, 1, p', p' + 1) + (p - 1) \times (1, 1, p' + 1, p' + 2),\ p' \geq 0
\end{aligned}$$

Hence, the edge-weights for the boundary of S_s meeting the edges in the top annulus with bottom edge-weights (p,p,p+1,1)can only possible be

$$(1,1,1,0) + (p-1) \times (1,1,0,1), or$$

$$(1,1,p',p'+1) + (p-1) \times (1,1,p'+1,p'+2),\ \ p' \geq 0$$

While, according to the match $(1,1,p{+}1,p);(1,1,p,p{+}1) \leftrightarrow (p,p,p{+}1,1);(p,p,1,p+1)$, the edge-weights for the boundary of S_s meeting the edges in the top annulus need to be

$$(p,p,1,p+1) = \begin{cases} (1,1,1,2) + (p-1) \times (1,1,0,1), p \geq 2 \\ (1,1,1,2), p = 1 \\ (0,0,1,1), p = 0 \end{cases} \tag{4.4}$$

Notice, for $p \geq 2$ case, if $(1,1,1,2) + (p-1) \times (1,1,0,1) = (1,1,1,0) + (p-1) \times (1,1,0,1)$, or $(1,1,p',p'+1) + (p-1) \times (1,1,p'+1,p'+2)$, $p' \geq 0$, we get p can only be 2, and the edge-weights of S_s are $(2,2,3,1);(2,2,1,3)$ i.e. $(1,1,2,1) + (1,1,1,0);(1,1,0,1) + (1,1,1,2)$

Therefore, if the surfaces S_r with S_s together satisfying

$$(1,1,p+1,p);(1,1,p,p+1) \leftrightarrow (p,p,p+1,1);(p,p,1,p+1);$$

Therefore, $p = 0, p = 1$, or $p = 2$.

Possibility 3.1. If $p = 0$, the match is $(0,0,1,1);(0,0,1,1) \leftrightarrow (1,1,1,0);(1,1,0,1)$. Notice that no meridian disk is of the boundary edge-weights $(0,0,1,1),(0,0,1,1)$. Hence this case is impossible.

Possibility 3.2. If $p = 1$, then $r = 3$ and the match is $(1,1,2,1);(1,1,1,2) \leftrightarrow (1,1,2,1);(1,1,1,2)$. For this case, there are two possibilities for S_s.

Possibility 3.2.1. S_s is from $3(b)$ of Theorem 2.2, a meridian disk, $(1,1,2,1);(1,1,s-2,s-1), s>1$.

From this we have $s-2=1$, $s=3$; S_3 is a meridian disk in $C_s=C_3$ with boundary edge-weights $(1,1,2,1);(1,1,1,2)$ and joining it with the meridian disk S_3 in $C_r=C_3$ we have *the triangulation is* $C_{3,3}$ *and* S *is the torus fiber in the fibration of* $M_{3,3}$ *over* S^1.

Possibility 3.2.2. S_s is from $6(b)$ of Theorem 2.2, a one-sided surface of genus c, $(1,1,2,1);(1,1,|s-2-2c|,s-1-2c), s\geq 2c+1$.

From this we have $s-1-2c=2$ i.e. $s=2c+3, c\geq 1$. When we calculate the Euler characteristics of the normal surface S in this case, we have a nonorientable surface of genus $c+2$; thus in C_{2c+1} the nonorientable genus is $c+1$. In $C_{3,2c+1}$ we have *a nonorientable, incompressible surface of genus* $1+c$. (Note this is related to Possibility 6.2.2 below and for $3=2c'+1, c'=1$ and we have genus $c'+c=1+c$ as in that possibility.)

Possibility 3.2.3. S_s is from 7 of Theorem 2.2, an annulus with thin edge-linking tubes, $2\times(1,1,2,1);2\times(1,1,1,2)$.

This is case that we double the edge-weights of S_r and S_s, we have the match $2\times(1,1,2,1);2\times(1,1,1,2)\leftrightarrow 2\times(1,1,2,1);2\times(1,1,1,2)$. Thus in $C_r=C_3$ we have two copies of a meridian disk, $(1,1,2,1);(1,1,1,2)$ and in C_s we have 2 possibilities. One is two copies of meridian disk, $2\times(1,2,1);2\times(1,1,2)$. we will get two copies of S in $C_{3,3}$. We ignore this case. The other one is from 7 of Theorem 2.2. Furthermore, it is from $7(b.2)$ or $7(c.4)$. Both of them give us the vertex-linking 2–sphere with thin edge-linking tubes. Notice the one from $7(b.2)$ is the double of the one-sided surface in *Possibility 3.2.2*.

Possibility 3.3. If $p=2$, then $r=5$ and the match is $(1,1,3,2);(1,1,2,3)\leftrightarrow(2,2,3,1);(2,2,1,3)$. For this case, S_s can only be two disjoint meridian disks with edge-weight $(1,1,2,1)+(1,1,1,0);(1,1,0,1)+(1,1,1,2)$.

Notice $(1,1,2,1)+(1,1,1,0) \xrightarrow{push} (1,1,1,0)+(1,1,0,1) \xrightarrow{push} (1,1,0,1)+(1,1,1,2)$

Obviously, S_s can only be two disjoint meridian disks. In $C_{5,2}$, we calculate the Euler characteristics of the normal surface S, we get S is a torus.

If we double the edge-weights of S, we get two disjoint surfaces, so we ignore this case.

Now we need to consider the case that we have any combination of meridian disks, $n_1m_1 + n_2m_2 + ... + n_km_k$, where $n_i \in \{1,2,...\}$, and m_i is any meridian disk from case 3.

Notice,

- If C_r consists of n copies of a meridian disks, by the above argument, then we may have n copies of disjoint surfaces S.
- If C_r consists of different copies of different type of meridian disks, some of which are from $3(a)$, others are from $3(b)$, there are two different quadrilateral normal disks in the same tetrahedron. This contradicts to the quadrilateral condition that it can not contain more than one type of quadrilateral normal disk in one tetrahedron. Hence all of the meridian disks, summands of C_r, are either all from $3(a)$, or all from $3(b)$.
- If C_r is only from the combination of meridian disks from $3(a)$, $n_1m_1 + n_2m_2 + ... + n_km_k$, then the bottom edge-weights of it is $n_1(1,1,p_1,p_1+1) + n_2(1,1,p_2,p_2+1) + ... + n_k(1,1,p_k,p_k+1) = (t,t,u,t+u)$, where t, u are positive intergers. However,the matching from equation 4.2 would require the top edge-weights of it is $(t,t,t+u,u)$, which is impossible since the weights $(t,t,u,t+u)$ can only be $(t,t,u+k,t+u+k)$ obtained by push.

Hence C_r can only be any combination of meridian disks, $n_1 m_1 + n_2 m_2 + ... + n_k m_k$, where $n_i \in \{1, 2, ...\}$, where m_i is any meridian disk from $3(b)$ with bottom edge-weights $(t, t, t+v, v)$. Therefore, C_r is the disconnected surface with edge-weights $(t, t, t+v, v); (t, t, v, t+v)$ Now consider C_r's the bottom edge-weights $(t, t, t+v, v)$. It's not difficult to show that:

$$(t, t, t+v, v) = \begin{cases} v(1,1,2,1) + (t-v)(1,1,1,0), t > v \\ t(1,1,2,1), t = v \\ r(1,1,n+2,n+1) + (t-r)(1,1,n+1,n), t < v, v = nt + r \end{cases} \tag{4.5}$$

If the bottom edge-weights of the surface is $(t, t, v, t+v)$, by same argument as above we get

$$(t, t, v, t+v) = \begin{cases} v(1,1,1,2) + (t-v)(1,1,0,1), t > v \\ t(1,1,1,2), t = v \\ r(1,1,n,n+2) + (t-r)(1,1,n,n+1), t < v, v = nt + r \end{cases} \tag{4.6}$$

For C_r, the surface of any combination of meridian disks, where all meridian disks are from $3(b)$ with edge-weights $(t, t, t+v, v); (t, t, v, t+v)$, there are 3 possibilities by considering its bottom edge-weights:

- The first case $t > v$.

  Possibility 3.4. C_r is two disjoint meridian disks $(1,1,2,1); (1,1,0,1)$ and $(1,1,1,0); (1,1,1,2)$, where $r = 2$

  For $t > v$, C_r the surface of any combination of meridian disks, where all meridian disks are from $3(b)$ with edge-weights $(t, t, t+v, v); (t, t, v, t+v)$ with $t > v$.

$$\begin{aligned}(t, t, t+v, v) =& v(1,1,2,1) + (t-v)(1,1,1,0) \\ &\xrightarrow{push} v(1,1,1,0) + (t-v)(1,1,0,1)\end{aligned}$$

$$\xrightarrow{push} v(1,1,0,1) + (t-v)(1,1,1,2) \xrightarrow{push} \cdots$$
$$\xrightarrow{push} v(1,1,p,p+1) + (t-v)(1,1,p+1,p+2),\ p \geq 0.$$

From the above, we notice that in order to have the top edge-weights $(t,t,v,t+v)$, we need to require $v = t - v$, i.e. $t = 2v$. Hence, C_r with edge-weights

$$(2v,2v,3v,v);(2v,2v,v,3v) = v[(2,2,3,1);(2,2,1,3)]$$

$$= v[(1,1,2,1)+(1,1,1,0);(1,1,0,1)+(1,1,1,2)].$$

. This is v copies of the sum of two meridian disks $(1,1,2,1);(1,1,0,1)$ and $(1,1,1,0);(1,1,1,2)$.

Consider one copy of the sum of two meridian disks $(1,1,2,1);(1,1,0,1)$ and $(1,1,1,0);(1,1,1,2)$. Look at the matching equation 4.2. We have the match:

$$(2,2,3,1);(2,2,1,3) \leftrightarrow (1,1,3,2);(1,1,2,3),\ i.e.$$

$$(1,1,2,1)+(1,1,1,0);(1,1,0,1)+(1,1,1,2) \leftrightarrow (1,1,3,2);(1,1,2,3).$$

There are two possibilities for c_s with edge-weights $(1,1,3,2);(1,1,2,3)$.

Possibility 3.4.1. C_r is two disjoint meridian disks $(1,1,2,1);(1,1,0,1)$ and $(1,1,1,0);(1,1,1,2)$, where $r = 2$. C_s is a meridian disk, $(1,1,3,2);(1,1,2,3)$, with $s = 5$. This gives us that S is a torus.

Possibility 3.4.2. C_r is two disjoint meridian disks $(1,1,2,1);(1,1,0,1)$ and $(1,1,1,0);(1,1,1,2)$, where $r = 2$. C_s is a a one-sided nonorientable surface of genus c from $6(b)$, $(1,1,3,2);(1,1,2,3)$. Since $p = 2$, then $r = 2c+5$. so S_{2c+5} is a one-sided nonorientable surface of genus c, $c \geq 1$. This is equivalently to say S_{2c+1} is a one-sided nonorientable surface of genus

$c-2$, $c \geq 3$. In this case $s \geq 7$. S is an incompressible non-orientable surface in $C_{2,2c+1}$ with genus $c-2+2=c$, where $c \geq 7$, i.e $s \geq 7$.

Now consider two copies of the sum of two meridian disks $(1,1,2,1);(1,1,0,1)$ and $(1,1,1,0);(1,1,1,2)$. Look at the matching equation 4.2. We have the match:

$$2 \times (2,2,3,1); 2 \times (2,2,1,3) \leftrightarrow 2 \times (1,1,3,2); (1,1,2,3), \ i.e.$$

$$2 \times [(1,1,2,1)+(1,1,1,0)]; 2 \times [(1,1,0,1)+(1,1,1,2)]$$

$$\leftrightarrow 2 \times (1,1,3,2); 2 \times (1,1,2,3).$$

Possibility 3.4.3. C_r is two copies of two disjoint meridian disks $(1,1,2,1);(1,1,0,1)$ and $(1,1,1,0);(1,1,1,2)$, where $r=2$ and c_s is from $7(b.2)$ or $7(c.4)$ of Theorem 2.2. Any of the resulting surfaces is a vertex-linking 2–sphere with thin edge-linking tubes. The one with S_s from $7(b.2)$ is the double of the one-sided surface from *Possibility 3.4.2.*.

Notice, if $v \geq 3$, we will get disjoint surface S. We ignore this case.

– The second case $t=v$. We have C_r is any combination of meridian disks, where all meridian disks are from $3(b)$ with edge-weights $(t,t,t+v,v);(t,t,v,t+v)$, s.t. $t=v$. Therefore, C_r's t copies of meridian disks $(1,1,2,1);(1,1,1,2)$. Which is the same case as *Possibility 3.2.*

– The third case $t<v$. We have C_r is any combination of meridian disks, where all meridian disks are from $3(b)$ with edge-weights $(t,t,t{+}v,v);(t,t,v,t+v)$, s.t. $t<v$. Then C_r is some combination of meridian disks $r(1,1,n+2,n+1)+(t-r)(1,1,n+1,n);r(1,1,n+1,n+2)+(t-r)(1,1,n,n+1)$, where $v=nt+r$. Let's consider the possible top edge-weights of the surfaces with bottom edge-weights $(1,1,n+2,n+1)+(t-r)(1,1,n+1,n)$.By the same argument as above, we have $r=t-r$, i.e. $t=2r$.

If $r = 0$, C_r is t copies of meridian surface, $(1,1,n+1,n);(1,1,n,n+1)$. We discuss this case before. If $r > 0$, C_r is the sum of r copies of 2 meridian surfaces, $(1,1,n+2,n+1)+(1,1,n+1,n);(1,1,n+1,n+2)+(1,1,n,n+1)$. Look at the matching equation 4.2. We have the match:

$$(1,1,n+2,n+1)+(1,1,n+1,n);(1,1,n+1,n+2)+(1,1,n,n+1)$$

$$\leftrightarrow (1,1,n+1,n+2)+(1,1,n,n+1);(1,1,n+2,n+1)+(1,1,n+1,n+2)$$

Notice if this is the case, then C_s is of edge-weights $(1,1,n+1,n+2)+(1,1,n,n+1);(1,1,n+2,n+1)+(1,1,n+1,n+2)$, which is impossible.

Case 4. S_r is from 4 of Theorem 2.2, an upper edge-linking disk, $(2,2,2,2);2\times(1,1,q,q+1)$. From the matching in equation 4.2, we would necessarily have $2 = 2q$ and $2 = 2+2q$, which is impossible.

Case 5. S_r is from 5 of Theorem 2.2, a lower edge-linking disk, $2\times(1,1,p+1,p);(2,2,2,2)$. Similar to Case 4, this situation is impossible.

Case 6. S_r is from 6 of Theorem 2.2, either $6(a)$ or $6(b)$.

Possibility 6.1. S_r is from $6(a)$, a one-sided surface of genus c', $(1,1,0,1);(1,1,|r-2c'|,(r+1)-2c')$, $r \geq 2c'-1$. In this case, By matching equation 4.2, we have the match: $(1,1,0,1);(1,1,1,0) \leftrightarrow (1,1,0,1);(1,1,1,0)$. Hence, the boundary edge-weights for the surface S_r are $(1,1,0,1);(1,1,1,0)$, hence $r+1-2c' = 0$, i.e. $r = 2c'-1$.

The only possibility for the surface S_s in this case is a one-sided surface of genus c from $6(a)$ with $s = 2c-1$. Now, changing the index to match that above, we have for $r = 2c'+1$ with $c' \geq 0$ and $s = 2c+1$ with $c \geq 0$, nonorientable surfaces, S_r and S_s, of genus $c'+1$ and $c+1$, respectively. Considering how these surfaces are attached along their boundaries, we have in $C_{2c+1,2c'+1}$ *a nonorientable,* *compressible, quadrilateral splitting surface of genus* $c' + c + 2$. Notice that this surface compresses about the valence 4 edge $e_1 \leftrightarrow -f_1 \leftrightarrow -e_{s+2} \leftrightarrow f_{r+2}$.

Possibility 6.2. S_r is from $6(b)$, a one-sided surface of genus c', $(1, 1, p + 1, p); (1, 1, |(r-1) - (p + 2c')|, r - (p + 2c')$, $r \geq p + 2c'$. Again, we look at the matching equation 4.2. We have

$$(1, 1, p+1, p); (1, 1, p, p+1) \leftrightarrow (p, p, p+1, 1); (p, p, 1, p+1)$$

Hence, $p + 1 = r - (p + 2c')$ and therefore, $r = 2p + 2c' + 1$, $c' \geq 1$. By the same argument in case 3, we get $p = 0, p = 1$, or $p = 2$.

Possibility 6.2.1. Suppose $p = 0$, we have S_r is a one-sided surface of genus c', $r = 2c' + 1$.

Look at the matching equation 4.2. We have the match:

$$(1, 1, 1, 0); (1, 1, 0, 1) \leftrightarrow (0, 0, 1, 1); (0, 0, 1, 1).$$

This is the reversal of the roles of r and s from *Possibility 2.2* above and we have S_r is a one-sided surface of genus c', $r = 2c' + 1$, and S_s is the vertical annulus. Hence, The surface S in $C_{2c'+1,s}$ is *a nonorientable, quadrilateral, splitting surface of genus* $c' + 2$. In general, this surface is compressible, if $r = 2c' + 1$, $s = 2$, $r \geq 7$. Otherwise, it is incompressible.

Possibility 6.2.2. Suppose $p = 1$. we have S_r is a one-sided surface of genus c', $r = 2c' + 3$

we look at the matching equation 4.2. We have the match:

$$(1, 1, 2, 1); (1, 1, 1, 2) \leftrightarrow (1, 1, 2, 1); (1, 1, 1, 2).$$

Hence, $r = 2c' + 3$ and s is odd, say $s = 2c + 3$. If $c = 0$ and $s = 3$, then we have $S_s = S_3$ is a meridian disk and we have the reverse roles of r and s from *Possibility 3.2* above.

If $c \neq 0 \neq c'$, then upon identification, we have a nonorientable, incompressible surface of genus $c' + c + 2$. Hence, for $r = 2c' + 1, s = 2c + 1$, the nonorientable genera of S_r and S_s are $c' - 1$ and $c - 1$, respectively. Combining these results with those of Case 3 above which handles these surfaces when either $r = 3$ or $s = 3$, we have in $C_{2c'+1,2c+1}$, except for $c' = 1 = c$, *a nonorientable, incompressible surface of genus* $c' + c$; for $c' = 1 = c$ and in $C_{3,3}$, this surface is the torus fiber in the fibration of $M_{3,3}$ over S^1. In all cases, these surfaces are compressions of a surface in *Possibility 6.2.1.*

Notice that for either $r = 3$ or $s = 3$, the surfaces found in Case 3 are special cases of this family of surfaces where S_3 in C_r, $r = 3$, is a meridian disk rather than a one-sided surface. The genus of the surface S in $C_{2c'+1,2c+1}$ is $c' + c$ and agrees with the special cases $c' = 1, c > 1$ or $c = 1$, $c' > 1$.

Possibility 6.3. Suppose $p = 2$. we have S_r is a one-sided surface of genus c', and $r = 2c' + 5$, where $c' \geq 1$. Hence $r \geq 7$

Look at the matching equation 4.2. We have the match:

$$(1, 1, 3, 2); (1, 1, 2, 3) \leftrightarrow (2, 2, 3, 1); (2, 2, 1, 3), \ \textit{i.e.}$$

$$(1, 1, 3, 2); (1, 1, 2, 3) \leftrightarrow [(1, 1, 2, 1) + (1, 1, 1, 0)]; [(1, 1, 0, 1) + (1, 1, 1, 2)]$$

Since $p = 2$, then $r = 2c' + 5$. so $S_{2c'+5}$ is a one-sided nonorientable surface of genus c', $c' \geq 1$. This is equivalently to say $S_{2c'+1}$ is a one-sided nonorientable surface of genus $c' - 2$, $c' \geq 3$. In this case $r \geq 7$. S_s can only be two disjointed meridian disks $(1, 1, 2, 1); (1, 1, 1, 0)$ and $(1, 1, 1, 0); (1, 1, 1, 2)$. By the same argument as *Possibility 3.3.*, $s = 2$. Therefore, S is a non-orientable surface in $C_2c' + 1, 2$ with genus $c' - 2 + 2 = c'$. This is an incompressible surface, a compression of *Possibility 6.2.1.*, when $c' \geq 3$ i.e. $r \geq 7$.

Case 7. S_r is from 7 of Theorem 2.2, an annulus possible with thin edge-linking tubes. We notice that the edge-weights for the boundary of S_r meeting the edges in the bottom annulus is either $2 \times (1,1,0,1)$ or $2 \times (1,1,p+1,p)$.

Possibility 7.1. S_r is an annulus (possibly) with thin edge-linking tubes with bottom edge-weights $2 \times (1,1,0,1)$.we get the matching of the boundary edge-weights for S_r with those of S_s as

$$2 \times (1,1,0,1); 2 \times (1,1,1,0) \leftrightarrow 2 \times (1,1,0,1); 2 \times (1,1,1,0).$$

The only possibilities for S_r are from $7(a.1)$ and $7(c.1)$ of Theorem 2.2. The possibility for S_s is from 3, $7(a.1)$ or $7(c.1)$. In any case, the resulting surface is a vertex-linking 2–sphere with thin edge-linking tubes, and the one with S_r and S_s both from $7(a.1)$ is the double of the one-sided surface from 6.1 above,the resulting surfaces is a quadrilateral vertex-linking 2–sphere with thin edge-linking tubes.

Possibility 7.2. S_r is an annulus (possibly) with thin edge-linking tubes with bottom edge-weights $2\times(1,1,p+1,p)$, then it is from $7(b.2)$ or $7(c.4)$ of Theorem 2.2. Since the bottom edge-weight is $2 \times (1,1,p+1,p)$, we get the matching of the boundray edge-weights for S_r with those of S_s as

$$2 \times (1,1,p+1,p); 2 \times (1,1,p,p+1) \leftrightarrow 2 \times (p,p,p+1,1); 2 \times (p,p,1,p+1).$$

Hence $p = 0$, $p = 1$ or $p = 2$.

Possibility 7.2.1. Suppose $p = 0$. S_r is from $7(b.2)$ or $7(c.4)$ of Theorem 2.2, an annulus possible with thin edge-linking tubes, $2 \times (1,1,1,0); 2 \times (1,1,0,1)$. By the matching equation 4.2, we have the match:

$$2 \times (1,1,1,0); 2 \times (1,1,0,1) \leftrightarrow 2 \times (0,0,1,1); 2 \times (0,0,1,1).$$

Hence, S_s is two copies of the vertical annulus. The resulting surfaces is a vertex-linking 2–sphere with thin edge-linking tubes. The one with S_r from $7(b.2)$ is the double of the one-sided surface from *Possibility 6.2.1* above and we have S_r is the double of a one-sided surface of genus c', $r = 2c' + 1$, This is the reversal of the roles of r and s in *Possibility 2.3* above.

Possibility 7.2.2. Suppose $p = 1$. S_r is from $7(b.2)$ or $7(c.4)$ of Theorem 2.2, an annulus possible with thin edge-linking tubes, $2 \times (1,1,2,1); 2 \times (1,1,1,2)$. By the matching equation 4.2, we have the match:

$$2 \times (1,1,2,1); 2 \times (1,1,1,2) \leftrightarrow 2 \times (1,1,2,1); 2 \times (1,1,1,2).$$

Hence, S_s is also from two copies of 3, $7(b.2)$ or $7(c.4)$. Any of the resulting surfaces is a vertex-linking 2–sphere with thin edge-linking tubes. The one with S_r and S_s both from $7(b.2)$ is the double of the one-sided surface from *Possibility 6.2.2* above and we have S_r is the double of a one-sided surface of genus c', $r = 2c' + 1$, This is the reversal of the roles of r and s in *Possibility 6.2.2.* above.

Possibility 7.2.3 Suppose $p = 2$. S_r is from $7(b.2)$ or $7(c.4)$ of Theorem 2.2, an annulus possible with thin edge-linking tubes, $2 \times (1,1,3,2); 2 \times (1,1,2,3)$. By the matching equation 4.2, we have the match:

$$2 \times (1,1,3,2); 2 \times (1,1,2,3) \leftrightarrow 2 \times (2,2,3,1); 2 \times (2,2,1,3),$$

$$i.e. (1,1,3,2); (1,1,2,3) \leftrightarrow [(1,1,2,1) + (1,1,1,0)]; [(1,1,0,1) + (1,1,1,2)]$$

We have S_s is the two copies of two disjoint meridian disks (1,1,2,1);(1,1,1,0) and (1,1,1,0);(1,1,1,2). Notice $s = 2$. Any of the resulting surfaces is a vertex-linking 2–sphere with thin edge-linking tubes. The one with S_r from $7(b.2)$ is the double of the one-sided surface from *Possibility 6.3*.

2. if $x = y$ and $u \neq v$,

We know S_r can be any family of normal surfaces including the vertex-linking pair except copies of only one type of vertex-linking disk $(2,0,1,1); (2,0,1,1)$ or

$(0,2,1,1);(0,2,1,1)$. The possibilities of S_r is the same as the discussion in the above case. However, $u \neq v$, it means that S_s can only consist of copies of only one type vertex-linking disks $(2,0,1,1);(2,0,1,1)$ or $(0,2,1,1);(0,2,1,1)$ or the combination of them with other normal surfaces. It is also easy to see that If we have more than one copy of vertex-linking disk in the above cases, we can never have a connected surface S in $C_{r,s}$ obtained by identifying the boundaries of the S_r and S_s.

we start to analyze the possible matches of these 4-tuples, using the classification of normal surfaces in a layered chain triangulation of the solid torus given in Theorem 2.2. we must have the pairs of 4-tuples of edge-weights match as in equation 4.1, including possible multiplicities.

case 1. The surface S_r is the vertex-linking disks (possibly) with thin edge-linking tubes, $(2,2,2,2);(2,2,2,2)$. The match is

$$(2,2,2,2);(2,2,2,2) \leftrightarrow (2,2,2,2);(2,2,2,2),$$

giving that S_s must also be the vertex-linking disks (possibly) with thin edge-linking tubes. Hence $U = V = 2$. This contradicts the fact that $u \neq v$.

Case 2. S_r is the vertical annulus, $(0,0,1,1);(0,0,1,1)$ It follows from the necessary matching of edge weights in equation 4.1 that the edge-weights for the surface S_s must be $(1,1,1,0);(1,1,0,1)$ and the match is:

$$(0,0,1,1);(0,0,1,1) \leftrightarrow (1,1,1,0);(1,1,0,1).$$

Hence, Hence $U = V = 1$. This contradicts the fact that $u \neq v$.

Case 3. S_r is a meridian disk from $3(a)$ or $3(b)$ of Theorem 2.2.

Possibility 3.1 S_r is a meridian disk from $3(a)$,$(1,1,p,p+1);(1,1,p+r,p+1+r)$. It follows from the necessary matching of edge weights in equation 4.1 that $z = p$ and $z = p+1+r$, which is impossible, since $r \neq -1$.

Possibility 3.2 S_r is a meridian disk from $3(b)$, $(1,1,p+1,p);(1,1,|(p+1)-r|,|p-r|)$. It follows from the necessary matching of edge weights in equation 4.1 that $u = p$,

$v = |(p+1) - r|$,$z = p+1$ and $z = |p-r|$.

From $z = p+1 = |p-r|$, we get

$$p+1 = \begin{cases} p-r, p > r \\ 0, p = r \\ r-p, p < r \end{cases} \tag{4.7}$$

. Obviously, if $p > r$, then $r = -1$. If $p = r$, $p = -1$. They are all impossible. If $p < r$, we get $p+1 = r-p$, hence $r = 2p+1$, where p is a nonnegative integer. Hence we have $u = p$, $v = |(p+1) - r| = r - (p+1) = p$. This contradicts the fact $u \neq v$.

Case 4. S_r is from 4 of Theorem 2.2, an upper edge-linking disk, $(2,2,2,2); 2 \times (1,1,1,0)$ or $(2,2,2,2); 2 \times (1,1,q,q+1)$.

Possibility 4.1 If S_r is from 4 of Theorem 2.2, an upper edge-linking disk, $(2,2,2,2); 2 \times (1,1,1,0)$. We would necessarily have $u = v = 2$, from the matching in equation 4.1. This is impossible.

Possibility 4.2 If S_r is from 4 of Theorem 2.2, an upper edge-linking disk, $(2,2,2,2); 2 \times (1,1,q,q+1)$.

It follows from the necessary matching of edge weights in equation 4.1 that $u = 2$, $v = 2q$, $z = 2$ and $z = 2(q+1)$. From $z = 2 = 2(q+1)$, we have $q = 0$, hence $v = 0$. Then the edge-weight of the boundary of normal surface S_s is $(2,0,2,2); (2,0,2,2)$, which is the sum of edge-weights $(2,0,1,1); (2,0,1,1)$ and $(0,0,1,1); (0,0,1,1)$. Hence the matching equation is

$$(2,2,2,2); 2 \times (1,1,0,1) \leftrightarrow (2,0,2,2); (2,0,2,2),$$

i.e.

$$(2,2,2,2); 2 \times (1,1,0,1) \leftrightarrow [(2,0,1,1); (2,0,1,1) + (0,0,1,1); (0,0,1,1)]$$

This gives us the vertex-linking S^2 with a thin edge-linking tube around the edge $f_{r+1} = b$ in $C_{r,s}$ and possibly with other thin edge-linking tubes.

Since we can reverse the role of r and s, so we can get a vertex-linking S^2 with a thin edge-linking tube around the edge $e_{s+1} = \beta$ in $C_{r,s}$ and possibly with other thin edge-linking tubes, with the matching equation

$$[(2,0,1,1);(2,0,1,1)+(0,0,1,1);(0,0,1,1)] \leftrightarrow (2,2,2,2);2\times(1,1,0,1)$$

Case 5. S_r is from 5 of Theorem 2.2, a lower edge-linking disk, $2\times(1,1,0,1);(2,2,2,2)$ or $2\times(1,1,p+1,p);(2,2,2,2)$.

Possibility 5.1 If S_r is from 5 of Theorem 2.2, an lower edge-linking disk, $2\times(1,1,0,1);(2,2,2,2)$. We would necessarily have $u = v = 2$, from the matching in equation 4.1. This is impossible.

Possibility 5.2 If S_r is from 5 of Theorem 2.2, an Lower edge-linking disk, $2\times(1,1,p+1,p);(2,2,2,2)$.

It follows from the necessary matching of edge weights in equation 4.1 that $u = p$, $v = 2$, $z = 2$ and $z = 2(p+1)$.From $z = 2 = 2(p+1)$, we have $p = 0$, hence $u = 0$. Then the edge-weight of the boundary of normal surface S_s is $(0,2,2,2);(0,2,2,2)$, which is the sum of edge-weights $(0,2,1,1);(0,2,1,1)$ and $(0,0,1,1);(0,0,1,1)$. Hence the matching equation is

$$2\times(1,1,1,0);(2,2,2,2) \leftrightarrow (0,2,2,2);(0,2,2,2),$$

i.e.

$$2\times(1,1,1,0);(2,2,2,2) \leftrightarrow [(0,2,1,1);(0,2,1,1)+(0,0,1,1);(0,0,1,1)]$$

This gives us the vertex-linking S^2 with a thin edge-linking tube around the edge $f_2 = -t$ in $C_{r,s}$ and possibly with other thin edge-linking tubes.

Since we can reverse the role of r and s, so we can get a vertex-linking S^2 with a thin edge-linking tube around the edge $e_2 = -\tau$ in $C_{r,s}$ and possibly with other thin edge-linking tubes, with the matching equation

$$[(0,2,1,1);(0,2,1,1)+(0,0,1,1);(0,0,1,1)] \leftrightarrow 2\times(1,1,1,0);(2,2,2,2)$$

Case 6. S_r is from 6 of Theorem 2.2, either $6(a)$ or $6(b)$.

Possibility 6.1. S_r is from $6(a)$, a one-sided surface of genus c', $(1,1,0,1);(1,1,|r-2c'|,(r+1)-2c')$, $r \geq 2c'-1$. In this case, By matching equation 4.1, we get $z = 0 = k+2c-1$, which means$k = 2c-1$, $U = 1$ and $v = |r-2c'|$. Since $k = 2c-1$, so $v = |r-2c'| = |2c-1-2c| = 1$. Hence,$u = v = 1$, which is a contradiction.

Possibility 6.2. S_r is from $6(b)$, a one-sided surface of genus c', $(1,1,p+1,p);(1,1,|(r-1)-(p+2c')|,r-(p+2c')$, $r \geq p+2c'$. Again, we look at the matching equation 4.1. We get $u = p$,$v = |r-1)-(p+2c')|$ and $z = p+1 = r-(p+2c')$. The last relation gives us $r = 2p+1+2c'$. Hence $v = |2p+1+2c'-1-p-2c'| = 0$.Hence $z = r-(p+2c') = 1 = p+1$,which means $p = 0$. Therefore, $u = p = 0 = v$, which is a contradiction.

Case 7. S_r is from 7 of Theorem 2.2, an annulus possible with thin edge-linking tubes with edge weight $2\times(1,1,0,1);2\times(1,1,1,0)$, $2\times(1,1,0,1);2\times(1,1,q,q+1)$, $2\times(1,1,0,1);2\times(1,1,1,0)$, $2\times(1,1,p+1,p);2\times(1,1,1,0)$ or $2\times(1,1,p+1,p);2\times(1,1,q,q+1)$.

Possibility 7.1. S_r is an annulus (possibly) with thin edge-linkin tubes with edge-weights $2\times(1,1,0,1);2\times(1,1,1,0)$. Obviously it implies $u = v = 2$, which is a contradiction.

Possibility 7.2. S_r is an annulus (possibly) with thinedge-linking tubes with edge-weights $2\times(1,1,0,1);2\times(1,1,q,q+1)$. It implies that $z = 0 = 2(q+1)$.Hence q is negative, which is impossible.

Possibility 7.3. S_r is an annulus (possibly) with thin edge-linking tubes with edge-weights $2\times(1,1,p+1,p);2\times(1,1,1,0)$. This implies that$z = 2(p+1) = 0$. Hence p is negative, which is impossible.

Possibility 7.4. S_r is an annulus (possibly) with thin edge-linking tubes with edge-weights $2\times(1,1,p+1,p);2\times(1,1,q,q+1)$. This implies that$u = 2p$,$v = 2q$,$z = 2(p+1) = 2(q+1)$. From the last relation, we get $p = q$. Hence $u = v$, which is a contradiction.

3. if $x \neq y$ and $u \neq v$.

In this case, S_r can only consist of copies of only one type vertex-linking disks $(2,0,1,1);(2,0,1,1)$ or $(0,2,1,1);(0,2,1,1)$ or the combination of them with other normal surfaces, which include vertex-linking disk pairs.

Case 1. If S_r is a vertex-linking disk $(2,0,1,1);(2,0,1,1)$ or $(0,2,1,1);(0,2,1,1)$.
Possibility 1.1. If S_r is a vertex-linking disk $(2,0,1,1);(2,0,1,1)$.By the matching equation, we get

$$(2,0,1,1);(2,0,1,1) \leftrightarrow (1,1,1,2);(1,1,0,1)$$

Notice no normal surface can has the edge weight $(1,1,1,2);(1,1,0,1)$. Obviously it is impossible for S_r to be more than one copy of this type of disk. Also notice $u = v = 1$, which is a contradiction.

Possibility 1.2. If S_r is a vertex-linking disk $(0,2,1,1);(0,2,1,1)$. By the matching equation, we get

$$(0,2,1,1);(0,2,1,1) \leftrightarrow (1,1,1,0);(1,1,2,1)$$

Notice no normal surface can has the edge weight $(1,1,1,0);(1,1,2,1)$. Obviously it is impossible for S_r to be more than one copy of this type of disk. Also notice that $u = v = 1$, which is a contradiction.

Case 2. If S_r is the combination of one type of vertex linking disk with other normal surfaces, which include vertex-linking disk pairs.

Now let's consider S_r is the combination of one type of vertex linking disk with one connected normal surfaces. According to the discussion in **Case 1.**, we find out that if S_r is vertex-linking disk, then by matching equation, it will give us $u = v$, no matter the corresponding S_s exists or not.

Moreover,by the discussion in the above two cases $x = y, u = v$ and $x = y, u \neq v$.

We notice for any normal surface S_r in the $1-3, 6, 7$ of the theorem 2.2,we will get a matching equation s.t. $u = v$,no matter the corresponding S_s exists or not.

Therefore, S_r in this case can not be the combination of vertex-linking disk with the normal surfaces in the $1-3, 6, 7$ of the theorem 2.2.

In order to have $u \neq v$, we only need to consider the normal surface in $4, 5$ of the theorem 2.2, we will have $u \neq v$ for S_s.

1. If S_r is vertex-linking disk $(2, 0, 1, 1); (2, 0, 1, 1)$ or $(0, 2, 1, 1); (0, 2, 1, 1)$ together with an upper edge-linking disk, $(2, 2, 2, 2); 2\times(1, 1, 1, 0)$ or $(2, 2, 2, 2); 2\times(1, 1, q, q+1)$, where $q \geq 0$.

Since in the *Possibility 4.1* of case $x = y, u \neq v$, we find out if S_r is from 4 of Theorem 2.2, an upper edge-linking disk, $(2, 2, 2, 2); 2 \times (1, 1, 1, 0)$. We would necessarily have $u = v = 2$, from the matching in equation 4.1. This is a contradiction. Hence we only need to consider the combination of vertex-linking disk with the other case in 4, which is an upper edge-linking disk, $(2, 2, 2, 2); 2 \times (1, 1, q, q + 1)$, where $q \geq 1$.

Possibility 1.1 If S_r is vertex-linking disk $(2, 0, 1, 1); (2, 0, 1, 1)$ with an upper edge-linking disk, $(2, 2, 2, 2); 2 \times (1, 1, q, q + 1)$, where $q \geq 1$.

It follows from the necessary matching of edge weights in equation 4.1 that S_r has the edge-weight$(2, 0, 1, 1)+(2, 2, 2, 2); (2, 0, 1, 1)+2\times(1, 1, q, q+1)$, i.e. $(4, 2, 3, 3); (4, 2, 1+2q, 1+2(q+1))$ Hence $x = 4$, $y = 2$, $u = 3$, $v = 2q+1$, $z = 3$ and $z = 1+2(q+1)$.From $z = 3 = 1 + 2(q + 1)$, we have $q = 0$, hence $v = 1$. Therefore we get the matching equation

$$[(2, 0, 1, 1) + (2, 2, 2, 2) = (4, 2, 3, 3); (2, 0, 1, 1) + 2 \times (1, 1, 0, 1) = (4, 2, 1, 3)]$$

$$\leftrightarrow (3, 1, 3, 4); (3, 1, 2, 3)$$

which equals

$$[(2, 0, 1, 1) + (2, 2, 2, 2) = (4, 2, 3, 3); (2, 0, 1, 1) + 2 \times (1, 1, 0, 1) = (4, 2, 1, 3)]$$

$$\leftrightarrow [(2,0,1,1)+(1,1,2,3);(2,0,1,1)+(1,1,1,2)]$$

The only normal surface with edge-weights satisfies $u \geq v$ is vertex-linking disk $(2,0,1,1);(2,0,1,1)$. Hence, we can decompose the edge-weight of later part to the following combination.

$$[(2,0,1,1);(2,0,1,1)+(2,2,2,2);2\times(1,1,0,1)]$$

$$\leftrightarrow [(2,0,1,1);(2,0,1,1)+(1,1,2,3);(1,1,1,2)]$$

Notice there is no normal surface with the edge-weights $(1,1,2,3);(1,1,1,2)$. This is impossible.

Possibility 1.2 If S_r is vertex-linking disk $(0,2,1,1);(0,2,1,1)$ with an upper edge-linking disk, $(2,2,2,2);2\times(1,1,q,q+1)$, where $q \geq 1$. It follows from the necessary matching of edge weights in equation 4.1 that S_r has the edge-weight$(0,2,1,1)+(2,2,2,2);(0,2,1,1)+2\times(1,1,q,q+1)$, i.e. $(2,4,3,3);(2,4,1+2q,1+2(q+1))$ Hence $x=2$, $y=4$, $u=3$, $v=2q+1$, $z=3$ and $z=1+2(q+1)$.From $z=3=1+2(q+1)$, we have $q=0$, hence $v=1$. Therefore we get the matching equation

$$[(2,0,1,1)+(2,2,2,2)=(2,4,3,3);(2,0,1,1)+2\times(1,1,0,1)=(2,4,1,3)]$$

$$\leftrightarrow [(3,1,3,2);(3,1,4,3)]$$

The only normal surface with edge-weights satisfies $u \geq v$ is vertex-linking disk $(2,0,1,1);(2,0,1,1)$. Hence, we can decompose the edge-weight of later part to the following combination.

$$[(0,2,1,1);(0,2,1,1)+(2,2,2,2);2\times(1,1,0,1)]$$

$$\leftrightarrow [(2,0,1,1);(2,0,1,1)+(1,1,2,1);(1,1,3,2)]$$

Notice that there is no normal surface with edge-weights $(1,1,2,1);(1,1,3,2)$, hence it is impossible.

2.If S_r is vertex-linking disk $(2,0,1,1);(2,0,1,1)$ or $(0,2,1,1);(0,2,1,1)$ together with an lower edge-linking disk, $2\times(1,1,0,1);(2,2,2,2)$ or $2\times(1,1,p+1,p);(2,2,2,2)$, where $p \geq 0$.

If S_r is vertex-linking disk $(2,0,1,1);(2,0,1,1)$ with an lower edge-linking disk,$2\times(1,1,0,1);(2,2,2,2)$. We would necessarily have $u = v = 3$, from the matching in equation 4.1. This is impossible. Hence, S_r is vertex-linking disk $(2,0,1,1);(2,0,1,1)$ or $(0,2,1,1);(0,2,1,1)$ together with an lower edge-linking disk, $2\times(1,1,p+1,p);(2,2,2,2)$, where $p \geq 0$

Possibility 2.1 If S_r is vertex-linking disk $(2,0,1,1);(2,0,1,1)$ with an upper edge-linking disk, $2\times(1,1,p+1,p);(2,2,2,2)$, where $q \geq 1$.

It follows from the necessary matching of edge weights in equation 4.1 that S_r has the edge-weight $(2,0,1,1)+2\times(1,1,p+1,p);(2,0,1,1)+(2,2,2,2)$, i.e. $(4,2,1+2(p+1),1+2p);(4,2,3,3)$ Hence $x=4$, $y=2$, $u=1+2p$, $v=3$, $z=3$ and $z=1+2(p+1)$.From $z=3=1+2(p+1)$, we have $p=0$, hence $u=1$. Therefore we get the matching equation

$$[(2,0,1,1)+2\times(1,1,1,0)=(4,2,3,1);(2,0,1,1)+(2,2,2,2)=(4,2,3,3)]$$

$$\leftrightarrow (1,3,3,4);(1,3,2,3)$$

The only normal surface with edge-weights satisfies $v \geq u$ is vertex-linking disk $(0,2,1,1);(0,2,1,1)$. Hence, we can decompose the edge-weight of later part to the following combination.

$$[(2,0,1,1);(2,0,1,1)+2\times(1,1,1,0);(2,2,2,2))]$$

$$\leftrightarrow [(0,2,1,1);(0,2,1,1)+(1,1,2,3);(1,1,1,2)]$$

Notice there is no normal surface with the edge-weights $(1,1,2,3);(1,1,1,2)$. This is impossible. *Possibility 2.2* If S_r is vertex-linking disk $(0,2,1,1);(0,2,1,1)$ with an upper edge-linking disk, $2\times(1,1,p+1,p);(2,2,2,2)$, where $q \geq 1$.

It follows from the necessary matching of edge weights in equation 4.1 that S_r has the edge-weight $(0,2,1,1) + 2 \times (1,1,p+1,p); (0,2,1,1) + (2,2,2,2)$, i.e. $(2,4,1+2(p+1), 1+2p); (2,4,3,3)$ Hence $x = 2$, $y = 4$, $u = 1+2p$, $v = 3$, $z = 3$ and $z = 1+2(p+1)$.From $z = 3 = 1+2(p+1)$, we have $p = 0$, hence $u = 1$. Therefore we get the matching equation

$$[(0,2,1,1) + 2 \times (1,1,1,0) = (2,4,3,1); (0,2,1,1) + (2,2,2,2) = (2,4,3,3)]$$

$$\leftrightarrow (1,3,3,2); (1,3,4,3)$$

The only normal surface with edge-weights satisfies $v \geq u$ is vertex-linking disk $(0,2,1,1); (0,2,1,1)$. Hence, we can decompose the edge-weight of later part to the following combination.

$$[(2,0,1,1); (2,0,1,1) + 2 \times (1,1,1,0); (2,2,2,2))]$$

$$\leftrightarrow [(0,2,1,1); (0,2,1,1) + (1,1,2,1); (1,1,3,2)]$$

Notice there is no normal surface with the edge-weights $(1,1,2,1); (1,1,3,2)$. This is impossible.

Therefore we can not get any surface from the case $x \neq y$ and $u \neq v$.

All in all, any normal surface is isotopic to one of the surfaces in the list.

■

From the observation above that $\widehat{C}_k = C_{k-1,1}$, we can carry the analysis of the normal surfaces in a layered chain pair triangulation to obtain an alternate proof of Theorem 3.1.

Corollary 4.1 *A connected, embedded, normal surface in the triangulated chain pair, $C_{k-1,1} = \widehat{C}_k$ is normally isotopic to one of the following surfaces:*

(i) A vertex-linking $2-$sphere (possibly) with thin edge-linking tubes; or

(ii) When $k-1$ is even, a Klein bottle, which is a nonorientable, incompressible, quadrilateral splitting surface; or

(iii) When $k-1$ is odd, three distinct (up to isotopy) nonorientable surfaces:

(a) A nonorientable, incompressible, quadrilateral splitting surface of genus 2 (genus $c+2, c=0$), a Klein bottle,

(b) A nonorientable, incompressible, quadrilateral splitting surface of genus $\frac{k}{2}+1$ (genus $c'+2, c'=\frac{k}{2}-1$,

(c) A nonorientable, incompressible, quadrilateral splitting surface of genus $\frac{k}{2}+1$ (genus $c+c'+2, c=0, c'=\frac{k}{2}-1$).

From Possibility 2.1 of theorem, we note that for $s=1$, then $C_{r,1}=\widehat{C}_{r+1}$ and S is a Klein bottle. The similar situation occurs reversing the roles of r and s. From Possibility 3.1 and $(1,1,1,0);(1,1,0,1)\leftrightarrow(0,0,1,1);(0,0,1,1)$; in which case, S_s can only be the vertical annulus in item 2 of Theorem 2.2. This is the reversal of the roles of r and s from Possibility 2.1 above in Case 2, which gives the Klein bottle in $C_{1,s}=\widehat{C}_{s+1}$.

CHAPTER 5

Almost Normal Octagonal Surfaces

In this chapter we will provide detailed proofs for the classification of almost normal octagonal surfaces in layered chain triangulations of the solid torus.

5.1 Almost normal octagonal surfaces in the layered chain triangulations

We can list all the almost normal octagonal surfaces by studying their possibilities of edge-weights on the bottom annulus of the tetrahedron in C_k.

Theorem 5.1 *The connected orientable almost normal octagonal surfaces in the layered chain triangulation C_k of the solid torus is an octagonal disk (possibly) with thin edge-linking tubes or an octagonal annulus (possibly) with thin edge-linking tubes.*

1. *An octagonal disk (possibly) with thin edge-linking tubes, which has one of the following edge-weights,*

   (a) $(2,2,2,2); 2\times(1,1,p',p'+1)$, $p' \geq 1$;

   (b) $2\times(1,1,p+1,p); (2,2,2,2)$, $p \geq 1$;

   (c) $(2,2,2,2)+(1,1,p+1,p); (1,1,0,1)+2\times(1,1,1,0)$, $p \geq 0$

   (d) $(2,2,2,2)+(1,1,p+1,p); (1,1,p'+1,p'+2)+2\times(1,1,p',p'+1)$, $p,\ p' \geq 0$.

   (e) $2\times(1,1,0,1)+(1,1,1,0); (2,2,2,2)+(1,1,p',p'+1)$, $p' \geq 0$;

   (f) $2\times(1,1,p+1,p)+(1,1,p+2,p+1); (2,2,2,2)+(1,1,p',p'+1)$, $p,\ p' \geq 0$;

   (g) $(2,2,2,2)+(1,1,p+1,p); 2\times(1,1,1,0)+(1,1,2,1)$, $p \geq 2$;

(h) $(2,2,2,2)+(1,1,p+1,p); 2\times(1,1,0,1)+(1,1,1,0)$, $p\geq 2$;

(i) $(2,2,2,2)+(1,1,p+1,p); 2\times(1,1,p'+1,p'+2)+(1,1,p',p'+1)$, $p\geq 2$, $p'\geq 0$.

(j) $2\times(1,1,0,1)+(1,1,1,2); (2,2,2,2)+(1,1,p',p'+1)$, $p'\geq 2$;

(k) $2\times(1,1,1,0)+(1,1,0,1); (2,2,2,2)+(1,1,p',p'+1)$, $p'\geq 2$;

(l) $2\times(1,1,p+2,p+1)+(1,1,p+1,p); (2,2,2,2)+(1,1,p',p'+1)$, $p\geq 0$, $p'\geq 2$.

2. *An octagonal annulus (possibly) with thin edge-linking tubes, which has one of the following edge-weights*

(a) $2\times(1,1,p+1,p); 2\times(1,1,p',p'+1)$, $p\geq 0$,$p'\geq 1$;

(b) $2\times(1,1,0,1); 2\times(1,1,p',p'+1)$, $p'\geq 1$;

(c) $2\times(1,1,p+1,p); 2\times(1,1,p',p'+1)$, $p\geq 1$,$p'\geq 0$

(d) $2\times(1,1,p+1,p); 2\times(1,1,1,0)$, $p\geq 1$;

(e) $(1,1,1,0)+(1,1,0,1); (1,1,p',p'+1)+(1,1,p'+1,p'+2)$, $p'\geq 0$;

(f) $(1,1,p+2,p+1)+(1,1,p+1,p); (1,1,1,0)+(1,1,0,1)$, $p\geq 0$;

(g) $(1,1,p+2,p+1)+(1,1,p+1,p); (1,1,p',p'+1)+(1,1,p'+1,p'+2)$, p, $p'\geq 0$;

(h) $(2,2,2,2)+(0,0,1,1); (1,1,2,1)+(1,1,1,0)$;

(i) $(2,2,2,2)+(0,0,1,1); (1,1,1,0)+(1,1,0,1)$;

(j) $(2,2,2,2)+(0,0,1,1); (1,1,p',p'+1)+(1,1,p'+1,p'+2)$, $p'\geq 0$;

(k) $(1,1,0,1)+(1,1,1,2); (2,2,2,2)+(0,0,1,1)$;

(l) $(1,1,1,0)+(1,1,0,1); (2,2,2,2)+(0,0,1,1)$;

(m) $(1,1,p+2,p+1)+(1,1,p+1,p); (2,2,2,2)+(0,0,1,1)$, $p\geq 0$.

Proof. There are three normal octagonal disk types for each tetrahedron. We can investigate all the possible edge-weights on the bottom annulus of the tetrahedron in C_k where we can add an octagonal disk in it, and furthermore which can give us a connected surface in C_k after the identification. Let's denote the tetrahedron where we add the octagonal disk σ_i, with $1 \leq i \leq k$.

Case 1. An almost normal octagonal surfaces with an octagonal disk of type I.

It's not hard to verify that there are three possible edge-weights on the bottom annulus of one tetrahedron which will allow us to add an octagonal disk of type I. See figure 5.1.

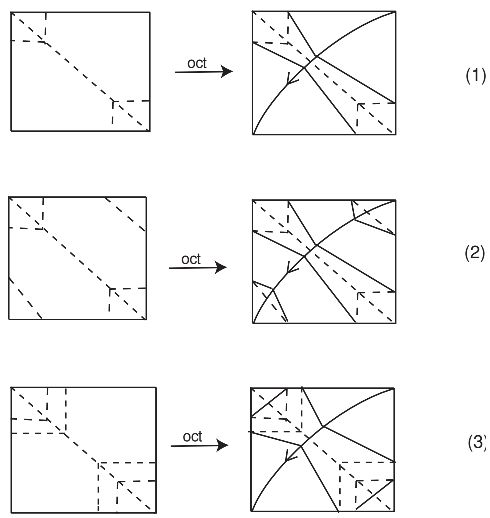

Figure 5.1: Three possible octagonal disk of type I.

1. An octagonal Möbius band,$(1, 1, p+1, p); (1, 1, p', p'+1)$, with $p \geq 2$, $p' \geq 1$.

   In the case (1) of type I, after we add an octagonal disk, the edge-weights is changed from $(1, 1, 2, 1)$ to $(1, 1, 1, 2)$. Note the almost normal surface intersect other tetrahedron with only triangles and quads as normal surface does. Hence we can use the same discussion about the change in the edge-weights in the chapter two here.

   Notice by theorem 2.2 the only normal surface with the top-edge weight $(1, 1, 2, 1)$ is a normal surface with the bottom edge-weight $(1, 1, p+1, p)$, for $p \geq 2$, and obtained by pushing through $p-2$ times. Furthermore, the only normal surface with the bottom edge-weight $(1,1,1,2)$ can only be obtained by push through.

Hence we get this relationship of the edge-weight of this almost normal surface.

$$\begin{aligned}(1,1,p+1,p) &\xrightarrow{push} (1,1,p,p-1) \xrightarrow{push} \cdots \\ &\xrightarrow{push} (1,1,2,1) \xrightarrow{oct} (1,1,1,2) \xrightarrow{push} \cdots \\ &\xrightarrow{push} (1,1,p',p'+1),\ p \geq 2,\ p' \geq 1.\end{aligned}$$

After we identify the corresponding edges and calculate the Euler characteristic, we find out this surface is an octagonal Möbius band.

Since $k \geq 2$, we have if $p = 1$, then $p' \geq 2$. Moreover, if $p' = 1$, then $p \geq 3$

2. (a) An octagonal disk (possibly) with thin edge-linking tubes, $(2,2,2,2); 2 \times (1,1,p',p'+1)$, with $p' \geq 1$.

   (b) An octagonal annulus (possibly) with thin edge-linking tubes, with edge-weights $2 \times (1,1,p+1,p); 2 \times (1,1,p',p'+1)$, with $p \geq 0$ and $p' \geq 1$ or $2 \times (1,1,0,1); 2 \times (1,1,p',p'+1)$, where $p' \geq 1$.

In the case (2), after we add an octagonal disk, the edge-weights is changed from $(2,2,2,2)$ to $2 \times (1,1,1,2)$.

By theorem 2.2, the possible normal surfaces with top edge-weight $(2,2,2,2)$ are (a) vertex-linking disks (possibly) with thin edge-linking tubes, $(2,2,2,2); (2,2,2,2)$, and (b) lower edge-linking disk (possibly) with thin edge-linking tubes, $2 \times (1,1,p+1,p); (2,2,2,2)$ or $2 \times (1,1,0,1); (2,2,2,2)$

The only normal surfaces with bottom edge-weight $2 \times (1,1,1,2)$ are two disjoint copies of normal meridian disks, $2 \times (1,1,1,2); 2 \times (1,1,p',p'+1)$, with $p' \geq 1$.

After we identify the corresponding edges and calculate the Euler characteristic, we find out this surface is either (a) an octagonal disk (possibly) with thin edge-linking tubes, $(2,2,2,2); 2\times(1,1,p',p'+1)$ or (b) an octagonal annulus (possibly) with thin edge-linking tubes;$2\times(1,1,p+1,p); 2\times(1,1,p',p'+1)$ or $2\times(1,1,1,2); 2\times(1,1,p',p'+1)$, where $p'\geq 1$.

3. (a) An octagonal disk (possibly) with thin edge-linking tubes, $2\times(1,1,p+1,p);(2,2,2,2)$, with $p\geq 1$.

   (b) An octagonal annulus (possibly) with thin edge-linking tubes, with edge-weights $2\times(1,1,p+1,p); 2\times(1,1,p',p'+1)$, with $p\geq 1$ and $p'\geq 0$ or $2\times(1,1,0,1); 2\times(1,1,1,0)$.

In the case (3), after we add an octagonal disk in σ_i, the edge-weight is changed from $2\times(1,1,2,1)$ to $(2,2,2,2)$.

By theorem 2.2, the possible normal surfaces with top edge-weight $2\times(1,1,2,1)$ are two disjoint copies of meridian disks, $2\times(1,1,p+1,p); 2\times(1,1,2,1)$, with $p\geq 1$.

The possible normal surfaces with bottom edge-weight $(2,2,2,2)$ are either vertex-linking disks (possibly) with thin edge-linking tubes, $(2,2,2,2);(2,2,2,2)$ or lower edge-linking disk (possibly) with thin edge-linking tubes, $(2,2,2,2); 2\times(1,1,1,0)$ or $(2,2,2,2); 2\times(1,1,p',p'+1)$, with $p'\geq 0$.

After we identify the corresponding edges and calculate the Euler characteristic, we find out this surface is either (a) an octagonal disk (possibly) with thin edge-linking tubes, $2\times(1,1,p+1,p);(2,2,2,2)$ or (b) an octagonal annulus (possibly) with thin edge-linking tubes; $2\times(1,1,p+1,p); 2\times(1,1,1,0)$ or $2\times(1,1,p+1,p); 2\times(1,1,p',p'+1)$, where $p\geq 1$ and $p'\geq 0$.

Case 2. An almost normal octagonal surfaces with an octagonal disk of type II.

It's not hard to verify that there are three possible edge-weights on the bottom annulus of one tetrahedron which will allow us to add an octagonal disk of type I. See figure 5.2.

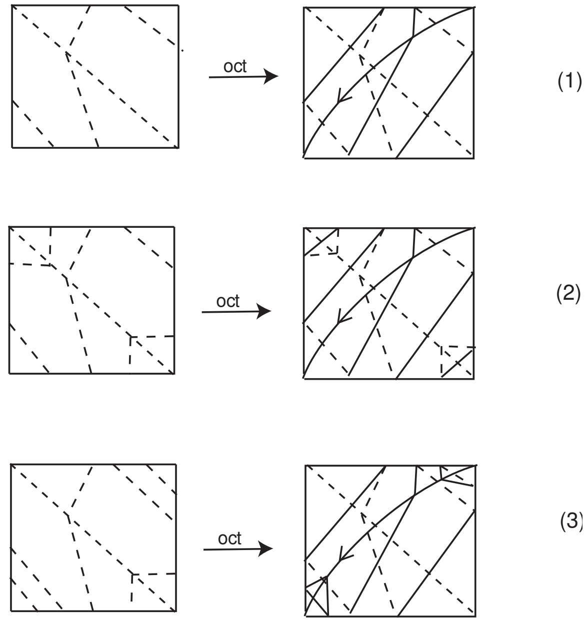

Figure 5.2: Three possible octagonal disk of type II.

1. An octagonal annulus with one of the following possible edge-weight,

$$(1,1,1,0)+(1,1,0,1);(1,1,p',p'+1)+(1,1,p'+1,p'+2),\ p'\geq 0$$

$$(1,1,p+2,p+1)+(1,1,p+1,p);(1,1,1,0)+(1,1,0,1),\ p\geq 0$$

$$(1,1,p+2,p+1)+(1,1,p+1,p);(1,1,p',p'+1)+(1,1,p'+1,p'+2),\ p'\geq 0$$

In the case (1) of type II, after we add an octagonal disk, the edge-weights is changed from (1,1,0,1)+(1,1,1,0) to $(1,1,1,0)+(1,1,0,1)$.

By theorem 2.2 and the discussion on chapter four, the only normal surfaces with the top edge-weight $(1,1,0,1)+(1,1,1,0)$ is two disjoint meridian disks with the bottom edge-weight $(1,1,p+2,p+1)+(1,1,p+1,p)$, for $p\geq 0$, and obtained by pushing through p times. Furthermore, the only normal surface with the bottom edge-weight (1,1,1,0)+(1,1,0,1) can only be obtained by pushing through.

Hence we get this relationship of the edge-weight of this almost normal surface.

$$(1,1,p+2,p+1)+(1,1,p+1,p)\xrightarrow{push}\cdots\xrightarrow{Oct}(1,1,1,0)+(1,1,0,1)\xrightarrow{push}\cdots$$

$\xrightarrow{push} (1,1,p',p'+1)+(1,1,p'+1,p'+2),\ p\geq 0,\ p'\geq 0.$

After we identify the corresponding edges and calculate the Euler characteristic, we find out this surface is an octagonal annulus.

Since $k\geq 2$, we can not have an octagonal annulus with edge-weight $(1,1,1,0)+(1,1,0,1); (1,1,1,0)+(1,1,0,1)$.

2. (a) An octagonal disk, with edge-weight $(2,2,2,2)+(1,1,p+1,p); (1,1,0,1)+2\times(1,1,1,0)$ or $(2,2,2,2)+(1,1,p+1,p); (1,1,p'+1,p'+2)+2\times(1,1,p',p'+1)$, where p, $p'\geq 0$.

   (b) A nonorientable octagonal surface.

In the case (2), after we add an octagonal disk, the edge-weights is changed from $(2,2,2,2)+(1,1,1,0)$ to $(1,1,0,1)+2\times(1,1,1,0)$.

By theorem 2.2 and the discussion on chapter four, the possible normal surfaces with top edge-weight $(2,2,2,2)+(1,1,1,0)$ are (a) the disjoint union of vertex-linking disks and a meridian disk, $(2,2,2,2)+(1,1,p+1,p); (2,2,2,2)+(1,1,1,0)$, (b) the disjoint union of vertex-linking disks with no tubes and a nonorientable surface, with bottom edge-weight $(2,2,2,2)+(1,1,0,1)$, $(2,2,2,2)+(1,1,p+1,p)$, $3\times(1,1,0,1)$ or $3\times(1,1,p+1,p)$, where $p\geq 0$. It is not hard to check that the disjoint union of lower edge-linking disk and a orientable surface can not give us the top edge-weight $(2,2,2,2)+(1,1,1,0)$. We can investigate a surface with bottom edge-weight $3\times(1,1,0,1)$ to see why this is true.

The only normal surfaces with bottom edge-weight $(1,1,0,1)+2\times(1,1,1,0)$ are three disjoint surfaces obtained by pushing through.

Hence after identify all the pieces together, we get two possible octagonal surfaces as mentioned above.

3. (a) An octagonal disk, $2 \times (1,1,0,1) + (1,1,1,0); (2,2,2,2) + (1,1,p',p'+1)$ or $2 \times (1,1,p+1,p) + (1,p+2,p+1); (2,2,2,2) + (1,1,p',p'+1)$, where p, $p' \geq 0$.

   (b) A nonorientable octagonal surface, with the same possible edge-weights as above (a) and two more, $2 \times (1,1,0,1) + (1,1,1,0); (2,2,2,2) + (1,1,1,0)$ and $2 \times (1,1,p+1,p) + (1,p+2,p+1); (2,2,2,2) + (1,1,1,0)$, where $p \geq 0$.

In the case (3), after we add an octagonal disk, the edge-weights is changed from $2 \times (1,1,0,1) + (1,1,1,0)$ to $(2,2,2,2) + (1,1,0,1)$.

By theorem 2.2 and the discussion on chapter four, the possible normal surfaces with top edge-weight $2 \times (1,1,0,1) + (1,1,1,0)$ are (a) the disjoint union of two copies of one meridian disk and one copy of another meridian disk, with bottom edge-weight $2 \times (1,1,0,1) + (1,1,1,0)$ or $2 \times (1,1,p+1,p) + (1,p+2,p+1)$, $p \geq 0$.

The possible normal surfaces with bottom edge-weight, $(2,2,2,2) + (1,1,0,1)$, are either the disjoint union of vertex-linking disks and a meridian disk, $(2,2,2,2) + (1,1,0,1); (2,2,2,2) + (1,1,p',p'+1)$ with $p' \geq 0$ or the disjoint union of vertex-linking disks and a nonorientable disk, $2 \times (1,1,0,1) + (1,1,1,0); (2,2,2,2) + (1,1,1,0)$ or $(2,2,2,2) + (1,1,0,1); (2,2,2,2) + (1,1,p',p'+1)$, with $p' \geq 0$

Hence after identify all the pieces together, we get two possible octagonal surfaces as mentioned above.

Case 3. An almost normal octagonal surfaces with an octagonal disk of type III.

It's not hard to verify that there are five possible edge-weights on the bottom annulus of one tetrahedron which will allow us to add an octagonal disk of type III. See figure 5.3.

1. A nonorientable octagonal surface with one of the following possible edge-weights,

$$(1,1,p,p+1); (1,1,p',p'+1); \text{ or } (1,1,p,p+1); (1,1,p'+1,p'), 0 \leq p,\ p' \leq 1$$

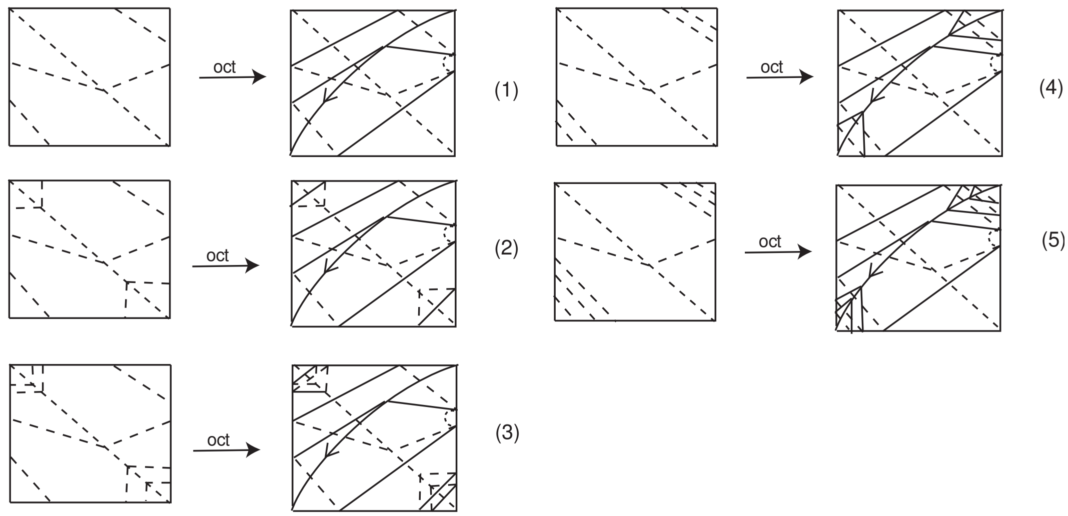

Figure 5.3: Three possible octagonal disk of type III.

$(1,1,p+1,p); (1,1,p'+1,p')$;or $(1,1,p,p+1); (1,1,p',p'+1)$, $p \geq 0$, $0 \leq p' \leq 1$

In the case (1) of type III, after we add an octagonal disk, the edge-weights is changed from $(1,1,1,2)$ to $(1,1,2,1)$.

By theorem 2.2, the possibly normal surfaces with the top edge-weight $(1,1,1,2)$ is a meridian disk or a nonorientable surface with the bottom edge-weight $(1,1,p,p+1)$, with $0 \leq p \leq 1$ or $(1,1,p+1,p)$, with $p \geq 0$. Furthermore, the possible normal surface with the bottom edge-weight $(1,1,2,1)$ can also be either a meridian disk or a nonorientable surface with the top edge-weight $(1,1,p'+1,p')$, with $0 \leq p' \leq 1$ or $(1,1,p,p+1)$, with $p' \geq 0$.

After we identify the corresponding edges and calculate the Euler characteristic, we find out all the possible surfaces are nonorientable.

2. An octagonal annulus, $(2,2,2,2)+(0,0,1,1); (1,1,2,1)+(1,1,1,0)$, $(2,2,2,2)+(0,0,1,1); (1,1,1,0)+(1,1,0,1)$, or $(2,2,2,2)+(0,0,1,1); (1,1,p',p'+1)+(1,1,p'+1,p'+2)$, with $p' \geq 0$.

In the case (2) of type III, after we add an octagonal disk, the edge-weights is changed from $(2,2,2,2)+(0,0,1,1)$ to $(1,1,2,1)+(1,1,1,0)$.

By theorem 2.2, the possibly normal surfaces with the top edge-weight $(2,2,2,2)+(0,0,1,1)$ is the disjoint union of vertex-linking disks $(2,2,2,2);(2,2,2,2)$ and vertical annulus $(0,0,1,1);(0,0,1,1)$. Furthermore, we know any normal surface with bottom edge-weight $(1,1,2,1)+(1,1,1,0)$ can only be obtained by pushing through two disjoint meridian disks. Therefore, the possible top edge-weights are $(1,1,2,1)+(1,1,1,0)$, $(1,1,1,0)+(1,1,0,1)$ and $(1,1,p',p'+1)+(1,1,p'+1,p'+2)$, with $p'\geq 0$.

After we identify the corresponding edges and calculate the Euler characteristic, we find out all the possible result surfaces are octagonal annulus.

3. An octagonal disk, $(2,2,2,2)+(1,1,p+1,p);2\times(1,1,1,0)+(1,1,2,1)$, $(2,2,2,2)+(1,1,p+1,p);2\times(1,1,0,1)+(1,1,1,0)$, or $(2,2,2,2)+(1,1,p+1,p);2\times(1,1,p'+1,p'+2)+(1,1,p',p'+1)$, with $p\geq 2$, $p'\geq 0$.

   In the case (3) of type III, after we add an octagonal disk, the edge-weights is changed from $(2,2,2,2)+(1,1,3,2)$ to $2\times(1,1,1,0)+(1,1,2,1)$.

   By theorem 2.2, the possibly normal surfaces with the top edge-weight $(2,2,2,2)+(1,1,3,2)$ is the disjoint union of vertex-linking disks $(2,2,2,2);(2,2,2,2)$ and a meridian disk $(1,1,p+1,p);(1,1,3,2)$, with $p\geq 2$. Furthermore, we know any normal surface with bottom edge-weight $2\times(1,1,0,1)+(1,1,1,0)$ can only be obtained by pushing through three disjoint meridian disks.

   Hence we get this relationship of the edge-weight of this almost normal surface.

$$(2,2,2,2)+(1,1,p+1,p)\xrightarrow{push}\cdots\xrightarrow{push}(2,2,2,2)+(1,1,3,2)$$
$$\xrightarrow{oct}2\times(1,1,1,0)+(1,1,2,1)\xrightarrow{push}2\times(1,1,0,1)+(1,1,1,0)$$
$$\xrightarrow{push}\cdots\xrightarrow{push}2\times(1,1,p'+1,p'+2)+(1,1,p',p'+1),\ p\geq 2,\ p'\geq 0.$$

After we identify the corresponding edges and calculate the Euler characteristic, we find out this surface is an octagonal disk with the possible edge-weight listed as above.

4. An octagonal annulus, $(1,1,0,1)+(1,1,1,2);(2,2,2,2)+(0,0,1,1)$, $(1,1,1,0)+(1,1,0,1);(2,2,2,2)+(0,0,1,1)$, or $(1,1,p+2,p+1)+(1,1,p+1,p);(2,2,2,2)+(0,0,1,1)$, with $p \geq 0$.

   In the case (4) of type III, after we add an octagonal disk, the edge-weights is changed from $(1,1,1,2)+(1,1,0,1)$ to $(2,2,2,2)+(0,0,1,1)$.

   By theorem 2.2, we know any normal surface with top edge-weight $(1,1,0,1)+(1,1,1,2)$ can only be obtained by pushing through two disjoint meridian disks.Furthermore, the possibly normal surfaces with the top edge-weight $(2,2,2,2)+(0,0,1,1)$ is the disjoint union of vertex-linking disks $(2,2,2,2);(2,2,2,2)$ and vertical annulus $(0,0,1,1);(0,0,1,1)$.

   Hence we get this relationship of the edge-weight of this almost normal surface.

   $$(1,1,p+2,p+1)+(1,1,p+1,p) \xrightarrow{push} \cdots \xrightarrow{push} (1,1,1,0)+(1,1,0,1)$$
   $$\xrightarrow{push} (1,1,0,1)+(1,1,1,2) \xrightarrow{oct} (2,2,2,2)+(0,0,1,1)$$
   $$\xrightarrow{push} \cdots \xrightarrow{push} (2,2,2,2)+(0,0,1,1),\ p \geq 0.$$

   After we identify the corresponding edges and calculate the Euler characteristic, we find out all the possible result surfaces are octagonal annulus with the edge-weight listed as above.

5. An octagonal disk, $2\times(1,1,0,1)+(1,1,1,2);(2,2,2,2)+(1,1,p',p'+1)$, $2\times(1,1,1,0)+(1,1,0,1);(2,2,2,2)+(1,1,p',p'+1)$, or $2\times(1,1,p+2,p+1)+(1,1,p+1,p);(2,2,2,2)+(1,1,p',p'+1)$, with $p \geq 0$, $p' \geq 2$.

In the case (5) of type III, after we add an octagonal disk, the edge-weights is changed from $2 \times (1,1,1,2) + (1,1,0,1)$ to $(2,2,2,2) + (1,1,2,3)$.

By theorem 2.2, we know any normal surface with top edge-weight $2\times(1,1,1,2)+(1,1,0,1)$ can only be obtained by pushing through three disjoint meridian disks. Furthermore, the possibly normal surfaces with the bottom edge-weight $(2,2,2,2)+(1,1,2,3)$ is the disjoint union of vertex-linking disks $(2,2,2,2)$; $(2,2,2,2)$ and a meridian disk $(1,1,2,3)$; $(1,1,p',p'+1)$, with $p' \geq 2$.

Hence we get this relationship of the edge-weight of this almost normal surface.

$$2 \times (1,1,p+2,p'+1) + (1,1,p+1,p) \xrightarrow{push} \cdots \xrightarrow{push} 2 \times (1,1,1,0) + (1,1,0,1)$$
$$\xrightarrow{push} 2 \times (1,1,0,1) + (1,1,1,2) \xrightarrow{oct} (2,2,2,2) + (1,1,2,3)$$
$$\xrightarrow{push} \cdots \xrightarrow{push} (2,2,2,2) + (1,1,p',p'+1),\ p \geq 0,\ p' \geq 2.$$

After we identify the corresponding edges and calculate the Euler characteristic, we find out this surface is an octagonal disk with the possible edge-weight listed as above.

According to the above discussion, we showed that all the possible orientable surfaces are included in the list of the theorem.

■

Now by theorem 5.1, we can give the list of almost normal octagonal surfaces in the triangulation C_2 of the solid torus. Furthermore, this can give us a clear ideal how the Euler characteristic is changed by adding an octagonal disk.

5.2 Almost normal octagonal surfaces in the twisted layered loop triangulations

In chapter 3 we showed that if $\widehat{F}$ is a normal surface in the twisted layered loop triangulation $\widehat{C}_k$, then $\widehat{F}$ determines a unique normal surface F in the layered chain triangulation C_k and $\widehat{F}$ is obtained from F by identifications along ∂F, and its edge weights satisfies

$$(wt_t, wt_t, wt_{e_1}, wt_{e_2}) = (wt_t, wt_t, wt_{e_{k+1}}, wt_{e_{k+2}}), \tag{5.1}$$

,

Similarly, if $\widehat{S}$ is an almost normal octagonal surface in the twisted layered loop triangulation $\widehat{C}_k$, then S intersects all the tetrahedron with triangle and/or quads except for one, which is an octagonal disk. Hence, $\widehat{S}$ determines a unique almost normal octagonal surface S in the layered chain triangulation C_k and $\widehat{S}$ is obtained from S by identifications along ∂S, and its edge weights satisfies the matching equation 5.1. The correspondence between the set of almost normal surfaces in $\widehat{C}_k$ and the set of almost normal surfaces in C_k is one-to-one and onto. Therefore, all the possible orientable almost normal octagonal surfaces comes from those listed in the theorem 5.1, which also satisfies the matching equation 5.1.

Theorem 5.2 *There is no connected orientable almost normal octagonal surface in the twisted layered loop triangulation $\widehat{C}_k$ of the Seifert fibered space M_k, $k \geq 2$.*

Proof. There are two types of almost normal octagonal surfaces in the layered chain triangulation of the solid torus. One is an octagonal disk (possibly) with thin edge-linking tubes, and the other is an octagonal annulus (possibly) with thin edge-linking tubes. Let's check all edge-weights of octagonal surfaces in the theorem 5.1 to see which one satisfies the condition 5.1.

For the first case, an octagonal disk (possibly) with thin edge-linking tubes,

$(2,2,2,2); 2\times(1,1,p',p'+1)$, with $p'\geq 1$. Obviously, $wt_t = wt_b = 2$. However, if we let $2 = wt_{e_1} = wt_{e_{k+1}} = 2p'$, then $p' = 1$. Hence $wt_{e_{k+2}} = 2(p'+1) = 4$ which is not equal to $wt_{e_2} = 2$. Hence this almost normal surface in the layered chain triangulation of a solid torus can not give us an almost normal surface in the twisted layered loop triangulation $\widehat{C}_k$ of the small Seifert fibered space M_k.

By the same argument, it is not hard to check that no edge-weights listed in the theorem 5.1 satisfies the condition $(wt_t, wt_b, wt_{e_1}, wt_{e_2}) = (wt_t, wt_b, wt_{e_{k+1}}, wt_{e_{k+2}})$, where $wt_t = wt_b$. Therefore, there is no connected orientable almost normal surface here is no connected orientable almost normal octagonal surface in the twisted layered loop triangulation $\widehat{C}_k$ of M_k. ■

5.3 Almost normal octagonal surfaces in the Layered chain pair triangulations

In the closed 3-manifold $M_{r,s}$, if S is a normal surface in $C_{r,s}$, then S determines a unique normal surface S_r in C_r and a unique normal surface S_s in C_s. Similarly, if S is an almost normal octagonal surface in $C_{r,s}$, then S intersects all the tetrahedron with triangle and/or quads except for one, which is an octagonal disk. This means that S will determines a normal surface in one of the layered chain triangulation of the solid torus and determines an almost normal octagonal surface in the other one. The surface it determines in C_r is S_r and in C_s is S_s, respectively. Notice that S is obtained from S_r and S_s by identifications along their boundaries. It is necessary that the boundary of the surface S_r has the same edge-weights as the boundary of the surface S_s on matching edges under the face identifications in chapter 4. Hence, the edge-weights of S_r and S_s satisfies the matching equation 4.1, which is $(x,y,z,u);(x,y,v,z) \leftrightarrow (u,v,z,x);(u,v,y,z)$, where the first pair $(x,y,z,u);(x,y,v,z)$ are the parameterizations for edge-weights of the boundary of S_r in the bottom annulus and the top annulus of C_r, and the second pair

$(u, v, z, x); (u, v, y, z)$ gives the edge-weights of the boundary of S_s in the bottom annulus and the top annulus of C_s, respectively.

Notice the almost normal octagonal surface in the layered chain triangulation of the solid torus is either an octagonal disk (possibly) with thin edge-linking tubes or an octagonal annulus (possibly) with thin edge-linking tubes.If S_r is an almost normal octagonal surface, we will assume that S_r is an octagonal disk/annulus with c thin edge-linking tubes. If S_s is orientable normal surface, then it is a disk/annulus with c' thin edge-linking tubes.

Theorem 5.3 *The connected orientable almost normal octagonal surface in the layered chain pair triangulation $C_{r,s}$ of the Seifert fibered space $M_{r,s}$, $r, s \geq 2$ are isomorphic to one of the following:*

1. *An almost octagonal surfaces S with genus $c + 2$ in $C_{r,3}$, $r \geq 4$, with an edge-weight matching equation,* $2 \times (1, 1, 2, 1); 2 \times (1, 1, 1, 2) \leftrightarrow 2 \times (1, 1, 2, 1); 2 \times (1, 1, 1, 2)$. *Here S_r is an octagonal annulus with c thin edge-linking tubes,* $0 \leq c \leq \lfloor \frac{r-4}{2} \rfloor$ *and S_s is two copies of meridian disks,* $(1, 1, 2, 1); (1, 1, 1, 2)$. *In particular, in $C_{4,3}$, we will have two orientable octagonal surfaces with genus two. See figure 5.4.*

2. *An almost normal octagonal surfaces S with genus $c + c' + 3$ in $C_{r,s}$, $r \geq 4$ and $s \geq 5$.It has an edge-weight matching equation of ∂S_r and ∂S_s,* $2 \times (1, 1, 2, 1); 2 \times (1, 1, 1, 2) \leftrightarrow 2 \times (1, 1, 2, 1); 2 \times (1, 1, 1, 2)$ *Here S_r is an octagonal annulus with c thin edge-linking tubes and S_s is an annulus with c' tubes, where* $0 \leq c \leq \lfloor \frac{r-4}{2} \rfloor$ *and* $0 \leq c' \leq \lfloor \frac{s-3}{2} \rfloor$.

3. *An almost normal octagonal surfaces S with genus $2 + c$ in $C_{r,2}$, $r \geq 6$. It has an edge-weight matching equation* $2 \times (1, 1, 3, 2); 2 \times (1, 1, 2, 3) \leftrightarrow 2 \times (2, 2, 3, 1); 2 \times (2, 2, 1, 3)$ *i.e.* $2 \times (1, 1, 3, 2); 2 \times (1, 1, 2, 3) \leftrightarrow 2 \times (1, 1, 2, 1) + 2 \times (1, 1, 1, 0); 2 \times (1, 1, 1, 2) + 2 \times (1, 1, 0, 1)$. *Here S_r is an octagonal annulus*

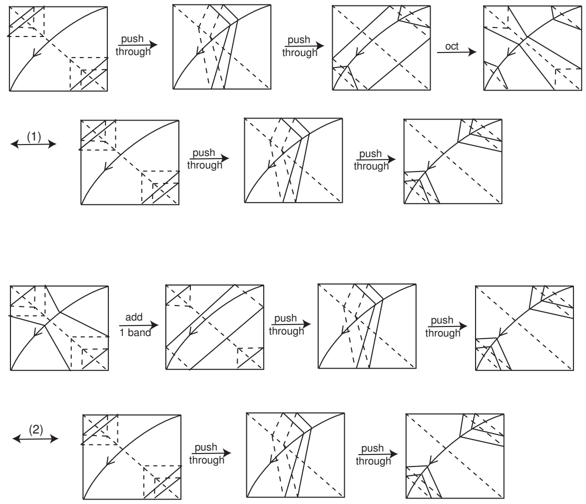

Figure 5.4: Two genus two octagonal surfaces in $C_{4,3}$.

with c thin edge-linking tubes, $0 \leq c \leq \lfloor \frac{r-6}{2} \rfloor$, and S_s is two copies of meridian disks, $(1,1,2,1);(1,1,0,1)$ and $(1,1,1,0);(1,1,1,2)$. In particular, when $r=6$, then $c=0$, we get two genus two octagonal surfaces S in $C_{6,2}$, see figure 5.5.

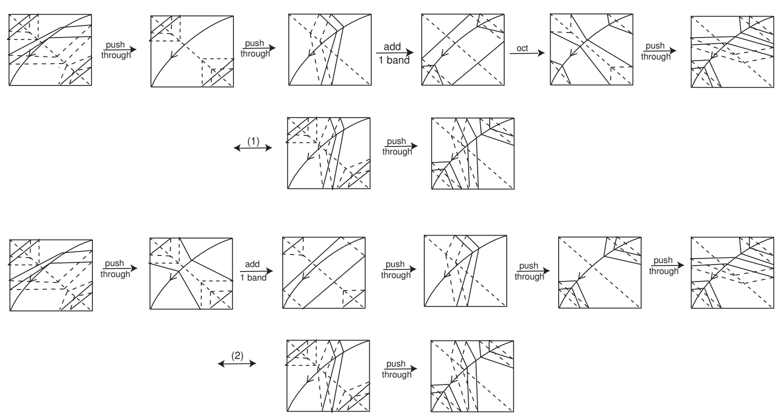

Figure 5.5: Two genus two octagonal surfaces in $C_{6,2}$.

Proof. Let S be an almost normal octagonal surface in $C_{r,s}$. S determines a unique surface S_r in C_r and a unique surface S_s in C_s.

For each $r, s \geq 1$, the layered chain pair $C_{r,s}$ is a triangulation of the Seifert fibred space $(S^2 : (2,-1),(r+1,1),(s+1,1)$. Furthermore, $C_{r,s}$ and $C_{s,r}$ are isomorphic. WLOG, we can assume that S_r in C_r is an almost normal octagonal surface and S_s

in C_s is a normal surface, which is possibly disconnected.

Since S_r is an almost normal surface in the layered chain triangulation C_r, we can list all the possible cases according to the theorem 5.1.

Case 1. S_r is an almost normal octagonal disk (possibly) with thin edge-linking tubes.

1. S_r has edge-weight $(2, 2, 2, 2); 2 \times (1, 1, p', p' + 1)$, with $p' \geq 1$. This case is impossible because according to the edge-weight matching equation 4.1,

$$(x, y, z, u); (x, y, v, z) \leftrightarrow (u, v, z, x); (u, v, y, z)$$

. we will have $x = y = 2$, $u = 2$, $v = 2p'$ and $z = 2 = 2(p' + 1)$. Hence, $p' = 0$, which contradicts to the fact that $p' \geq 1$.

2. S_r has edge-weight $2 \times (1, 1, p + 1, p); (2, 2, 2, 2)$, with $p \geq 1$. This case is impossible because according to the edge-weight matching equation 4.1, we will have $x = y = 2$, $u = p$, $v = 2$ and $z = 2 = 2(p + 1)$. Hence, $p = 0$, which contradicts to the fact that $p' \geq 1$.

3. S_r has edge-weight $(2, 2, 2, 2) + (1, 1, p + 1, p); (1, 1, 0, 1) + 2 \times (1, 1, 1, 0)$, $p \geq 0$.

   This case is impossible because according to the edge-weight matching equation 4.1, we will have $x = y = 3$, $u = 2 + p$, $v = 2$ and $z = 1 = 2 + p + 1$. Hence, $p = -1$,which contradicts to the fact that $p' \geq 0$.

4. S_r has edge-weight $(2, 2, 2, 2)+(1, 1, p+1, p); (1, 1, p'+1, p'+2)+2\times(1, 1, p', p'+1)$, p, $p' \geq 0$.

   According to the edge-weight matching equation 4.1, we will have $x = y = 3$, $u = 2 + p$, $v = 3p' + 1$ and $z = 2 + p + 1 = 3p' + 4$. Hence, $p = 3p' + 1$. We get $v = p$. Now the possible edge-weight for the normal surface S_s in C_s is $(u, v, z, x); (u, v, y, z) = (p + 2, p, p + 3, 3); (p + 2, p, 3, p + 3)$, which is

$(2,0,1,1)+(p,p,p+2,2);(2,0,1,1)+(p,p,2,p+2)$. Notice for $(p,p,p+2,2)$, we have

$$(p,p,p+2,2) = \begin{cases} 2\times(0,0,1,1), p=0 \\ (1,1,3,2), p=1 \\ 2\times(1,1,2,1), p=2 \\ 2\times(1,1,2,1)+(p-2)\times(1,1,1,0), p\geq 2 \end{cases} \tag{5.2}$$

Notice for $(p,p,2,p+2)$, we have

$$(p,p,2,p+2) = \begin{cases} 2\times(0,0,1,1), p=0 \\ (1,1,2,3), p=1 \\ 2\times(1,1,1,2), p=2 \\ 2\times(1,1,1,2)+(p-2)\times(1,1,0,1), p\geq 2 \end{cases} \tag{5.3}$$

Possibility 1. If $p=0$, by $p=3p'+1$ we get p' is not integer, which is a contradiction.

Possibility 2. If $p=1$, $P'=0$, since $p=3p'+1$. Hence

S_r has edge-weight $(2,2,2,2)+(1,1,2,1);(1,1,1,2)+2\times(1,1,0,1)$. According to the ***Case 2.*** *2(a) of the theorem 5.1. This is an octagonal disk and $r=3$. By the edge-weight matching equation 4.1, we have S_s in C_s has edge-weight $(2,0,1,1)+(1,1,3,2);(2,0,1,1)+(1,1,2,3)$. By theorem 2.2, we know that there are two possibilities for S_s. First, it is a disjoint union of vertex-linking disk $(2,0,1,1);(2,0,1,1)$ and a meridian disk $(1,1,3,2);(1,1,2,3)$. Hence we get $s=5$. After identifying the corresponding edges on ∂S_r and ∂S_s together, we get an orientable almost normal octagonal surface S in $C_{3,5}$. $(2,2,2,2)+(1,1,2,1);(1,1,1,2)+2\times(1,1,0,1)\leftrightarrow(2,0,1,1)+(1,1,3,2);(2,0,1,1)+(1,1,2,3)$. However, after identification, we notice this is a nonorientable surface. We ignore it.*

The other possibility for S_s *is nonorientable surface from* $6(b)$. *Then the almost normal* S *is obtained from this* S_s *is a nonorientable surface. Hence we ignore it.*

Possibility 3. If $p = 2$, *by* $p = 3p' + 1$ *we get* p' *is not integer, which is a contradiction.*

Possibility 4. If $p \geq 2$, *we need to have* $3|(p-1)$, *since* $p = 3p' + 1$ *and* p' *is a nonnegative integer. Notice* S_s *has the edge-weight* $2 \times (1,1,2,1) + (p-2) \times (1,1,1,0); 2 \times (1,1,1,2) + (p-2) \times (1,1,0,1)$, *when* $p \geq 2$.

Notice for any normal surfaces with bottom edge-weight $2 \times (1,1,2,1) + (p-2) \times (1,1,1,0)$ *can only be obtained by pushing through. The change of edge-weight after pushing through is showed as follows*

$$2 \times (1,1,2,1) + (p-2) \times (1,1,1,0) \xrightarrow{push} 2 \times (1,1,1,0) + (p-2) \times (1,1,0,1)$$
$$\xrightarrow{push} 2 \times (1,1,0,1) + (p-2) \times (1,1,1,2)$$
$$\xrightarrow{push} 2 \times (1,1,1,2) + (p-2) \times (1,1,2,3) \xrightarrow{push} \cdots$$

Since S_s *has the top edge-weight* $2 \times (1,1,1,2) + (p-2) \times (1,1,0,1)$, *when* $p \geq 2$. *We realize the only possibly case is when* $p-2 = 2$, *i.e.* $p = 4$. *In this case* $s = 2$ *and* S_s *is the disjoint union of two copies of meridian disk,* $(1,1,2,1); (1,1,0,1)$ *and two copies of meridian disk* $(1,1,1,0); (1,1,1,2)$ *and vertex linking disk* $(2,0,1,1); (2,0,1,1)$. *Furthermore, when* $p = 4$, $p' = 1$. *Hence* S_r *is octagonal disk with edge-weight* $(2,2,2,2)+(1,1,5,4); (1,1,2,3)+2\times(1,1,1,2)$, *with* $r = 6$. *Therefore, we get a almost octagonal normal surface* S *in* $C_{6,2}$. *The matching equation for* S_r *and* S_s *is* $(2,2,2,2) + (1,1,5,4); (1,1,2,3) + 2 \times (1,1,1,2) \leftrightarrow 2 \times [(1,1,1,2) + (1,1,0,1)]$. *However, after identification, we notice this is a nonorientable surface. We ignore it.*

5. *S_r has edge-weight* $2\times(1,1,0,1)+(1,1,1,0);(2,2,2,2)+(1,1,p',p'+1)$, $p'\geq 0$. *It is impossible.*

According to the edge-weight matching equation 4.1, we will have $x=y=3$, $u=2$, $v=p'+2$ *and* $z=1=p'+3$. *Hence,* $p'=-2$, *which contradicts to the fact that* $p'\geq 0$.

6. *S_r has edge-weight* $2\times(1,1,p+1,p)+(1,1,p+2,p+1);(2,2,2,2)+(1,1,p',p'+1)$, p, $p'\geq 0$.

According to the edge-weight matching equation 4.1, we will have $x=y=3$, $u=3p+1$, $v=p'+2$ *and* $z=3p+4=p'+3$. *Hence,* $p'=3p+1=u$. *Now the possible edge-weight for the normal surface* S_s *in* C_s *is* $(u,v,z,x);(u,v,y,z)=(p',p'+2,p'+3,3);(p',p'+2,3,p'+3)$, *which is* $(0,2,1,1)+(p',p',p'+2,2);(0,2,1,1)+(p',p',2,p'+2)$. *According to the equations 4.5 and 4.6. we have the following argument.*

For $(p',p',p'+2,2)$, *we have*

$$(p',p',p'+2,2)=\begin{cases} 2\times(0,0,1,1),p'=0 \\ (1,1,3,2),p'=1 \\ 2\times(1,1,2,1),p'=2 \\ 2\times(1,1,2,1)+(p'-2)\times(1,1,1,0),p'\geq 2 \end{cases} \tag{5.4}$$

Notice for $(p',p',2,p'+2)$, *we have*

$$(p',p',2,p'+2)=\begin{cases} 2\times(0,0,1,1),p'=0 \\ (1,1,2,3),p'=1 \\ 2\times(1,1,1,2),p'=2 \\ 2\times(1,1,1,2)+(p'-2)\times(1,1,0,1),p'\geq 2 \end{cases} \tag{5.5}$$

Possibility 1. If $p'=0$, *by* $p'=3p+1$ *we get* p *is not integer, which is a contradiction.*

Possibility 2. If $p' = 1,\ P = 0,$ *since* $p' = 3p + 1$. *Hence*

S_r *has edge-weight* $2 \times (1,1,1,0) + (1,1,2,1); (2,2,2,2) + (1,1,1,2)$. *According to the* ***Case 2.*** $3(a)$ *of the theorem 5.1. This is an octagonal disk and* $r = 3$. *By the edge-weight matching equation 4.1, we have* S_s *in* C_s *has edge-weight* $(0,2,1,1) + (p', p', p' + 2, 2); (0,2,1,1) + (p', p', 2, p' + 2) = (0,2,1,1) + (1,1,3,2); (0,2,1,1) + (1,1,2,3)$. *By theorem 2.2, we know that there are two possibilities for* S_s. *First, it is a disjoint union of vertex-linking disk* $(0,2,1,1)$;$(0,2,1,1)$ *and a meridian disk* $(1,1,3,2); (1,1,2,3)$. *Hence we get* $s = 5$.

After identifying the corresponding edges on ∂S_r *and* ∂S_s *together, we get an orientable almost normal octagonal surface* S *in* $C_{3,5}$. *The matching equation for* S_r *and* S_s *is* $2 \times (1,1,1,0) + (1,1,2,1); (2,2,2,2) + (1,1,1,2) \leftrightarrow (0,2,1,1) + (1,1,3,2); (0,2,1,1) + (1,1,2,3)$ *However, after identification, we notice this is a nonorientable surface. We ignore it.*

The other possibility for S_s *is nonorientable surface from* $6(b)$. *Then the almost normal* S *is obtained from this* S_s *is a nonorientable surface. Hence we ignore it.*

Possibility 3. If $p' = 2$, *by* $p' = 3p + 1$ *we get* p *is not integer, which is a contradiction.*

Possibility 4. If $p' \geq 2$, *we need to have* $3|(p' - 1)$, *since* $p' = 3p + 1$ *and* p *is a nonnegative integer. Notice* S_s *has the edge-weight* $2 \times (1,1,2,1) + (p' - 2) \times (1,1,1,0); 2 \times (1,1,1,2) + (p' - 2) \times (1,1,0,1)$, *when* $p \geq 2$.

Notice for any normal surfaces with bottom edge-weight $2 \times (1,1,2,1) + (p' - 2) \times (1,1,1,0)$ *can only be obtained by pushing through. The change of edge-weight after pushing through is showed as follows*

$$2 \times (1,1,2,1) + (p' - 2) \times (1,1,1,0) \xrightarrow{push} 2 \times (1,1,1,0) + (p' - 2) \times (1,1,0,1)$$

$\xrightarrow{push} 2 \times (1,1,0,1) + (p'-2) \times (1,1,1,2)$

$\xrightarrow{push} 2 \times (1,1,1,2) + (p'-2) \times (1,1,2,3) \xrightarrow{push} \cdots$

Since S_s *has the top edge-weight* $2 \times (1,1,1,2) + (p'-2) \times (1,1,0,1)$*, when* $p \geq 2$*. We realize the only possibly case is when* $p'-2 = 2$*, i.e.* $p' = 4$*. In this case* $s = 2$ *and* S_s *is the disjoint union of two copies of meridian disk,* $(1,1,2,1);(1,1,0,1)$*, two copies of meridian disk* $(1,1,1,0);(1,1,1,2)$ *and two copies of* $(0,2,1,1);(0,2,1,1)$*. Furthermore, when* $p' = 4$*,* $p = 1$*. Hence* S_r *is octagonal disk with edge-weight* $2\times(1,1,2,1)+(1,1,3,2);(2,2,2,2)+(1,1,4,5)$*, with* $r = 6$*.*

After identifying the corresponding edges on ∂S_r *and* ∂S_s *together, we get an orientable almost normal octagonal surface* S *in* $C_{6,2}$*. The matching equation for* S_r *and* S_s *is* $2\times(1,1,2,1)+(1,1,3,2);(2,2,2,2)+(1,1,4,5) \leftrightarrow 2\times[(1,1,2,1)+(1,1,1,0)];2\times[(1,1,0,1)+(1,1,1,2)]$*. However, after identification, we notice this is a nonorientable surface. We ignore it.*

7. S_r *has edge-weight* $(2,2,2,2)+(1,1,p+1,p);2\times(1,1,1,0)+(1,1,2,1)$, $p \geq 2$*. It is impossible to get a* S *from this surface. According to the edge-weight matching equation 4.1, we will have* $x = y = 3$*,* $u = 2+p$*,* $v = 4$ *and* $z = p+3 = 1$*. Hence,* $p = -2$*, which contradicts to the fact that* $p \geq 2$*.*

8. S_r *has edge-weight* $(2,2,2,2)+(1,1,p+1,p);2\times(1,1,0,1)+(1,1,1,0)$, $p \geq 2$*. It is impossible to get a* S *from this surface. According to the edge-weight matching equation 4.1, we will have* $x = y = 3$*,* $u = 2+p$*,* $v = 1$ *and* $z = p+3 = 2$*. Hence,* $p = -1$*, which contradicts to the fact that* $p \geq 2$*.*

9. S_r *has edge-weight* $(2,2,2,2)+(1,1,p+1,p);2\times(1,1,p'+1,p'+2)+(1,1,p',p'+1)$*,* $p \geq 2$*,* $p' \geq 0$*. According to the edge-weight matching equation 4.1, we will have*

$x = y = 3$, $u = 2 + p$, $v = 3p' + 2$ and $z = p + 3 = 3p' + 5$. Hence, $p = 3p' + 2 = v$. Hence S_s has edge-weight $(p+2, p, p+3, 3); (p+2, p, 3, p+3) = (2, 0, 1, 1) + (p, p, p+2, 2); (2, 0, 1, 1) + (p, p, 2, p+2)$, $p \geq 2$. According to equation 5.2, $(p, p, p+2, 2) = 2 \times (1, 1, 2, 1) + (p-2) \times (1, 1, 1, 0)$ and $(p, p, 2, p+2) = 2 \times (1, 1, 1, 2) + (p-2) \times (1, 1, 0, 1)$, with $p \geq 2$.

Possibility 1. $p = 2$. Since $p = 3p' + 2$, we get $p' = 0$. Hence, S_r is an octagonal disk with edge-weight $(2, 2, 2, 2) + (1, 1, 3, 2); 2 \times (1, 1, 1, 2) + (1, 1, 0, 1)$, with $r = 3$. S_s has edge-weight $(2, 0, 1, 1) + 2 \times (1, 1, 2, 1); (2, 0, 1, 1) + 2 \times (1, 1, 1, 2)$. Hence S_s is a disjoint union of vertex-linking disk $(2, 0, 1, 1); (2, 0, 1, 1)$ and a normal surface with edge-weight $2 \times (1, 1, 2, 1); 2 \times (1, 1, 1, 2)$. There are 3 possibilities for this normal surface.

(a) *If this normal surface is 2 copies of meridian disks, $(1, 1, 2, 1); (1, 1, 1, 2)$, with $s = 3$. After identifying the corresponding edges on ∂S_r and ∂S_s together, we get an orientable almost normal octagonal surface S in $C_{3,3}$. The matching equation for S_r and S_s is $(2, 2, 2, 2) + (1, 1, 3, 2); 2 \times (1, 1, 1, 2) + (1, 1, 0, 1) \leftrightarrow (2, 0, 1, 1) + 2 \times (1, 1, 2, 1); (2, 0, 1, 1) + 2 \times (1, 1, 1, 2)$. However, after identification, we notice this is a nonorientable surface. We ignore it.*

(b) *If this normal surface is an annulus (possibly) with tubes, which is double of nonorientable suface from $7(b.2)$ of theorem 2.2, $(1, 1, 2, 1); (1, 1, 1, 2)$, with $s = 3 + 2c$, $c \geq 0$, which is the number of tetrahedron we add bands in.*

After identifying the corresponding edges on ∂S_r and ∂S_s together, we get an orientable almost normal octagonal surface S in $C_{3,3+2c}$. The matching equation for S_r and S_s is $(2, 2, 2, 2) + (1, 1, 3, 2); 2 \times (1, 1, 1, 2) + (1, 1, 0, 1) \leftrightarrow (2, 0, 1, 1) + 2 \times (1, 1, 2, 1); (2, 0, 1, 1) + 2 \times (1, 1, 1, 2)$. However, after iden-

tification, we notice this is a nonorientable surface. We ignore it.

(c) *If this normal surface is an annulus (possibly) with tubes from* 7(*c*.4). *Then* $s \geq 6$. *After identifying the corresponding edges on* ∂S_r *and* ∂S_s *together, we get an orientable almost normal octagonal surface* S *in* $C_{3,3+2c}$. *The matching equation for* S_r *and* S_s *is* $(2,2,2,2)+(1,1,3,2); 2\times(1,1,1,2)+(1,1,0,1) \leftrightarrow (2,0,1,1)+2\times(1,1,2,1); (2,0,1,1)+2\times(1,1,1,2)$. *However, after identification, we notice this is a nonorientable surface. We ignore it.*

Possibility 2. For $p \geq 2$, *by the similar argument as in above case* 4 *possibility 4. we have* $p-2=2$, *i.e.* $p=4$ *to have normal surface* S_s. *However,* $p=3p'+2$, *hence,* $p'=2/3$, *which is a contradiction.*

10. S_r *has edge-weight* $2\times(1,1,0,1)+(1,1,1,2); (2,2,2,2)+(1,1,p',p'+1)$, $p' \geq 2$. *It is impossible to get an octagonal surface* S. *According to the edge-weight matching equation 4.1, we will have* $x=y=3$, $u=4$, $v=p'+2$ *and* $z=1=p'+3$. *Hence,* $p'=-2$, *which is a contradiction to the fact* $p' \geq 2$.

11. S_r *has edge-weight* $2\times(1,1,1,0)+(1,1,0,1); (2,2,2,2)+(1,1,p',p'+1)$, $p' \geq 2$.*It is impossible to get an octagonal surface* S. *According to the edge-weight matching equation 4.1, we will have* $x=y=3$, $u=1$, $v=p'+2$ *and* $z=2=p'+3$. *Hence,* $p'=-1$, *which is a contradiction to the fact* $p' \geq 2$.

12. S_r *has edge-weight* $2\times(1,1,p+2,p+1)+(1,1,p+1,p); (2,2,2,2)+(1,1,p',p'+1)$, $p \geq 0$, $p' \geq 2$. *According to the edge-weight matching equation 4.1, we will have* $x=y=3$, $u=3p+2$, $v=p'+2$ *and* $z=3p+5=p'+3$. *Hence,* $p'=3p+2=u$, *Now the possible edge-weight for the normal surface* S_s *in* C_s *is* $(u,v,z,x); (u,v,y,z) = (p',p'+2,p'+3,3); (p',p'+2,3,p'+3)$, *which is* $(0,2,1,1)+(p',p',p'+2,2); (0,2,1,1)+(p',p',2,p'+2)$. *The argument is similar to the case [6.].*

Possibility 1. *If* $p' = 0$*, by* $p' = 3p + 2$ *we get* p *is not integer, which is a contradiction.*

Possibility 2. *If* $p' = 1$*,by* $p' = 3p + 2$ *we get* p *is not integer, which is a contradiction.*

Possibility 3. *If* $p' = 2$*, by* $p' = 3p + 2$ *we get* $p = 0$*. Now* S_r *is an octagonal disk,* $2 \times (1, 1, 2, 1) + (1, 1, 1, 0); (2, 2, 2, 2) + (1, 1, 2, 3)$*, with* $r = 3$*.* S_s *is a normal surface with edge-weight* $(u, v, z, x); (u, v, y, z) = (2, 4, 5, 3); (2, 4, 3, 5)$*, i.e.* $(0, 2, 1, 1) + 2 \times (1, 1, 2, 1); (0, 2, 1, 1) + 2 \times (1, 1, 1, 2)$*. There are three possibilities for* S_s*.*

(a) *If this normal surface is a disjoint union of vertex-linking disk,* $(0, 2, 1, 1)$*;*$(0, 2, 1, 1)$*, and 2 copies of meridian disks,* $(1, 1, 2, 1); (1, 1, 1, 2)$*, with* $s = 3$*.*

    After identifying the corresponding edges on ∂S_r *and* ∂S_s *together, we get an orientable almost normal octagonal surface* S *in* $C_{3,3}$*. . The matching equation for* S_r *and* S_s *is* $2 \times (1, 1, 2, 1) + (1, 1, 1, 0); (2, 2, 2, 2) + (1, 1, 2, 3) \leftrightarrow (0, 2, 1, 1) + 2 \times (1, 1, 2, 1); (0, 2, 1, 1) + 2 \times (1, 1, 1, 2)$*. However, after identification, we notice this is a nonorientable surface. We ignore it.*

(b) *If this normal surface is a disjoint union of vertex-linking disk,* $(0, 2, 1, 1)$*;*$(0, 2, 1, 1)$*, and an annulus (possibly) with tubes, which is double of nonorientable suface from 7(b.2) of theorem 2.2,* $(1, 1, 2, 1); (1, 1, 1, 2)$*, with* $s = 3 + 2c$*,* $c \geq 0$*, with* $c \geq 1$*.*

    After identifying the corresponding edges on ∂S_r *and* ∂S_s *together, we get an orientable almost normal octagonal surface* S *in* $C_{3,3+2c}$*, with* $c \geq 1$*. The matching equation for* S_r *and* S_s *is* $2 \times (1, 1, 2, 1) + (1, 1, 1, 0); (2, 2, 2, 2) + (1, 1, 2, 3) \leftrightarrow (0, 2, 1, 1) + 2 \times (1, 1, 2, 1); (0, 2, 1, 1) + 2 \times (1, 1, 1, 2)$*. However, after identification, we notice this is a nonorientable surface. We ignore it.*

(c) If this normal surface is an annulus (possibly) with tubes from 7(*c*.4)*. Then* $s \geq 6$*. After identifying the corresponding edges on* ∂S_r *and* ∂S_s *together, we get an orientable almost normal octagonal surface* S *in* $C_{3,s}$*, with* $s \geq 6$*. The matching equation for* S_r *and* S_s *is* $2 \times (1,1,2,1) + (1,1,1,0); (2,2,2,2) + (1,1,2,3) \leftrightarrow (0,2,1,1) + 2 \times (1,1,2,1); (0,2,1,1) + 2 \times (1,1,1,2)$ *However, after identification, we notice this is a nonorientable surface. We ignore it.*

Possibility 4. If $p' \geq 2$*, we need to have* $3|(p'-2)$*, since* $p' = 3p+2$ *and* p *is a nonnegative integer. By the similar argument we have in case [6.] We must have* $p' - 2 = 2$ *in order to have a corresponding normal surface* S_s*. Then this contradicts to the fact* $3|(p'-2)$*. Therefore, this is impossible.*

Case 2. S_r is an almost normal octagonal annulus (possibly) with thin edge-linking tubes.

1. S_r has edge-weight $2 \times (1,1,p+1,p); 2 \times (1,1,p',p'+1)$, $p \geq 0$,$p' \geq 1$. According to the edge-weight matching equation 4.1,

$$(x,y,z,u); (x,y,v,z) \leftrightarrow (u,v,z,x); (u,v,y,z)$$

we will have $x = y = 2$, $u = 2p$, $v = 2p'$ and $z = 2(p+1) = 2(p'+1)$. Hence, $p = p'$ and $p \geq 1$.

Now the possible edge-weight for the normal surface S_s in C_s is $(u,v,z,x); (u,v,y,z) = (2p,2p,2p+2,2); (2p,2p,2,2p+2)$.

For the bottom edge-weight $(2p,2p,2p+2,2)$, according to the equation 4.5 we have

$$(2p,2p,2p+2,2) = \begin{cases} 2 \times (0,0,1,1), p = 0 \\ 2 \times (1,1,2,1), p = 1 \\ 2 \times (1,1,2,1) + (2p-2)(1,1,1,0), p \geq 2 \end{cases} \quad (5.6)$$

For the top edge-weight $(2p, 2p, 2, 2p+2)$,according to the equation 4.6 we have

$$(2p, 2p, 2, 2p+2) = \begin{cases} 2 \times (0,0,1,1), p = 0 \\ 2 \times (1,1,1,2), p = 1 \\ 2 \times (1,1,1,2) + (2p-2)(1,1,0,1), p \geq 2 \end{cases} \tag{5.7}$$

Since $p = p' \geq 1$ *Possibility 1. If* $p = 1$*, then* $p' = p = 1$*. we have* S_r *is an octagonal annulus (possibly) with c thin edge-linking tubes,*$2 \times (1,1,2,1); 2 \times (1,1,1,2)$*. Then* $r \geq 4$*, since it needs at least 3 steps from the edge-weight* $2 \times (1,1,2,1)$ *to* $(2,2,2,2)$*, and one more step to add an octagonal disk to get a surface with top edge-weight* $2 \times (1,1,1,2)$*. Moreover, we will have* S_s *has edge-weight* $2 \times (1,1,2,1); 2 \times (1,1,1,2)$*. There are three possible choices for* S_s*.*

(a) If S_s *is a normal surface of two copies of meridian disks,* $2 \times (1,1,2,1); 2 \times (1,1,1,2)$*. We notice* $s = 3$*. After identifying the corresponding edges on* ∂S_r *and* ∂S_s *together, we get an orientable almost normal octagonal surface* S *with genus* $c+2$ *in* $C_{r,3}$*, where* $r \geq 4$*. In particular, if* $r = 4$*, we get a genus 2 orientable octagonal surface* S *in* $C_{4,3}$*. Later, we will show that it is a Heegaard splitting. The matching equation for* S_r *and* S_s *is*

$$2 \times (1,1,2,1); 2 \times (1,1,1,2) \leftrightarrow 2 \times (1,1,2,1); 2 \times (1,1,1,2)$$

(b) If S_s *is an annulus (possibly) with* c' *thin edge-linking tubes, which is double of nonorientable suface from* $7(b.2)$ *of theorem 2.2,* $2 \times (1,1,2,1); 2 \times (1,1,1,2)$*, with* $s = 3 + 2c'$*.*

After identifying the corresponding edges on ∂S_r *and* ∂S_s *together, we get an orientable almost normal octagonal surface* S *with genus* $c + c' + 3$ *in* $C_{r,3+2c'}$*,* $r \geq 4$*. The matching equation for* S_r *and* S_s *is*

$$2 \times (1,1,2,1); 2 \times (1,1,1,2) \leftrightarrow 2 \times (1,1,2,1); 2 \times (1,1,1,2)$$

(c) *If this normal surface is an annulus (possibly) with tubes from* $7(c.4)$*. Then* $s \geq 6$*. After identifying the corresponding edges on* ∂S_r *and* ∂S_s *together, we get an orientable almost normal octagonal surface* S *with genus* $3+c+c'$ *in* $C_{r,s}$*, where* $r \geq 4$ *and* $s \geq 6$*. The matching equation for* S_r *and* S_s *is*

$$2 \times (1,1,2,1); 2 \times (1,1,1,2) \leftrightarrow 2 \times (1,1,2,1); 2 \times (1,1,1,2)$$

Possibility 2. If $p \geq 2$*.* S_s *has edge-weight* $2\times(1,1,2,1)+(2p-2)(1,1,1,0); 2\times(1,1,1,2)+(2p-2)(1,1,0,1)$*. Notice*

$$2 \times (1,1,2,1) + (2p-2)(1,1,1,0) \xrightarrow{push} 2 \times (1,1,1,0) + (2p-2) \times (1,1,0,1)$$
$$\xrightarrow{push} 2 \times (1,1,0,1) + (2p-2) \times (1,1,1,2)$$
$$\xrightarrow{push} 2 \times (1,1,1,2) + (2p-2) \times (1,1,2,3) \xrightarrow{push} \cdots$$

Compare this with the top edge-weight $2 \times (1,1,1,2) + (2p-2)(1,1,0,1)$ *of* S_s*. The only possible choice for p is* $2p-2=2$*, which* $p=2$*.*

When $p = 2$*,* $p' = p = 2$*. we have* S_r *is an octagonal annulus (possibly) with c thin edge-linking tubes,* $2\times(1,1,3,2); 2\times(1,1,2,3)$*.Then* $r \geq 6$*, since it needs at least 4 steps from the edge-weight* $2 \times (1,1,2,1)$ *to* $(2,2,2,2)$*, a step to add an octagonal disk and one more step to push through the surface to get top edge-weight* $2 \times (1,1,2,3)$*. Moreover, we will have* S_s *has edge-weight* $2\times(1,1,2,1)+2\times(1,1,1,0); 2\times(1,1,1,2)+2\times(1,1,0,1)$*. Hence,* S_s *can only obtained two copies of disconnected normal meridian disks,* $(1,1,2,1); (1,1,0,1)$ *and* $(1,1,1,0); (1,1,1,2)$*. Therefore,* $s = 2$

After identifying the corresponding edges on ∂S_r *and* ∂S_s *together, we get an orientable almost normal octagonal surface* S *with genus* $2+c$ *in* $C_{r,2}$*,* $r \geq 6$*. The matching equation for* S_r *and* S_s *is*

$$2\times(1,1,3,2); 2\times(1,1,2,3) \leftrightarrow 2\times(1,1,2,1)+2\times(1,1,1,0); 2\times(1,1,1,2)+2\times(1,1,0,1)$$

In particular, when in $C_{6,2}$, we will have an orientable octagonal surface with genus two, this is the Heegaard splitting surface.

2. *S_r has edge-weight $2 \times (1,1,0,1); 2 \times (1,1,p',p'+1)$, $p' \geq 1$. It is impossible, because according to the edge-weight matching equation 4.1, we will have $x = y = 2$, $u = 2$, $v = 2p'$ and $z = 0 = 2(p'+1)$. Hence, $p' = -1$, which is a contradiction to the fact that $p' \geq 1$.*

3. *S_r has edge-weight $2 \times (1,1,p+1,p); 2 \times (1,1,p',p'+1)$, $p \geq 1$,$p' \geq 0$. This edge-weight is the same as the case [1.], except the domain of p and p'. According to the edge-weight matching equation 4.1, we will have $x = y = 2$, $u = 2p$, $v = 2p'$ and $z = 2(p+1) = 2(p'+1)$. Hence, $p = p'$ and $p \geq 1$. Therefore, the discussion is exactly follow the first case in* ***Case 2.***

   Possibility 1. If $p = 1$, then $p' = p = 1$. we have S_r is an octagonal annulus (possibly) with c thin edge-linking tubes,$2 \times (1,1,2,1); 2 \times (1,1,1,2)$, with $r \geq 4$. There are three possible choices for S_s.

   (a) *If S_s is a normal surface of two copies of meridian disks, $2 \times (1,1,2,1); 2 \times (1,1,1,2)$, with $s = 3$. After identifying the corresponding edges on ∂S_r and ∂S_s together, we get an orientable almost normal octagonal surface S with genus $c + 2$ in $C_{r,3}$, where $r \geq 4$. In particular, if $r = 4$, we get a genus two orientable octagonal surface S in $C_{4,3}$. Later, we will show that it is a Heegaard splitting. The matching equation for S_r and S_s is*

   $$2 \times (1,1,2,1); 2 \times (1,1,1,2) \leftrightarrow 2 \times (1,1,2,1); 2 \times (1,1,1,2)$$

   (b) *If S_s is an annulus (possibly) with c' thin edge-linking tubes, which is double of nonorientable suface from 7(b.2) of theorem 2.2, $2 \times (1,1,2,1); 2 \times (1,1,1,2)$, with $s = 3 + 2c'$.*

After identifying the corresponding edges on ∂S_r and ∂S_s together, we get an orientable almost normal octagonal surface S with genus $c + c' + 3$ in $C_{r,3+2c'}$, $r \geq 4$. The matching equation for S_r and S_s is

$$2 \times (1,1,2,1); 2 \times (1,1,1,2) \leftrightarrow 2 \times (1,1,2,1); 2 \times (1,1,1,2)$$

(c) *If this normal surface is an annulus (possibly) with tubes from* $7(c.4)$. *Then* $s \geq 6$. *After identifying the corresponding edges on ∂S_r and ∂S_s together, we get an orientable almost normal octagonal surface S with genus* $3+c+c'$ *in $C_{r,s}$, where $r \geq 4$ and $s \geq 6$. The matching equation for S_r and S_s is*

$$2 \times (1,1,2,1); 2 \times (1,1,1,2) \leftrightarrow 2 \times (1,1,2,1); 2 \times (1,1,1,2)$$

Possibility 2. If $p = 2$, $p' = p = 2$. we have S_r is an octagonal annulus (possibly) with c thin edge-linking tubes, $2\times(1,1,3,2); 2\times(1,1,2,3)$, *with $r \geq 6$. Moreover, we will have S_s has edge-weight* $2 \times (1,1,2,1) + 2 \times (1,1,1,0); 2 \times (1,1,1,2) + 2 \times (1,1,0,1)$. *Hence, S_s can only obtained two copies of disconnected normal meridian disks,* $(1,1,2,1); (1,1,0,1)$ *and* $(1,1,1,0); (1,1,1,2)$. *Therefore, $s = 2$*

After identifying the corresponding edges on ∂S_r and ∂S_s together, we get an orientable almost normal octagonal surface S with genus $2 + c$ in $C_{r,2}$, $r \geq 6$. The matching equation for S_r and S_s is $2 \times (1,1,3,2); 2 \times (1,1,2,3) \leftrightarrow 2 \times (1,1,2,1) + 2 \times (1,1,1,0); 2 \times (1,1,1,2) + 2 \times (1,1,0,1)$ *In particular, when in $C_{6,2}$, we will have an orientable octagonal surface with genus two, this is the Heegaard splitting surface.*

4. *S_r has edge-weight* $2 \times (1,1,p+1,p); 2 \times (1,1,1,0)$, $p \geq 1$. *It is impossible. According to the edge-weight matching equation 4.1, we will have $x = y = 2$, $u = 2p$, $v = 2$ and $z = 2(p+1) = 0$. Hence $p = -1$, which contradicts to the fact that $p \geq 1$.*

5. *S_r has edge-weight* $(1,1,1,0)+(1,1,0,1);(1,1,p',p'+1)+(1,1,p'+1,p'+2)$, $p'\geq 0$. *It is impossible. According to the edge-weight matching equation 4.1, we will have* $x=y=2$, $u=1$, $v=2P'+1$ *and* $z=1=2P'+3$. *Hence* $p'=-1$,*which contradicts to the fact that* $p'\geq 0$.

6. *S_r has edge-weight* $(1,1,p+2,p+1)+(1,1,p+1,p);(1,1,1,0)+(1,1,0,1)$, $p\geq 0$. *It is impossible. According to the edge-weight matching equation 4.1, we will have* $x=y=2$, $u=2p+1$, $v=1$ *and* $z=2P+3=1$. *Hence* $p=-1$,*which contradicts to the fact that* $p\geq 0$.

7. *S_r has edge-weight* $(1,1,p+2,p+1)+(1,1,p+1,p);(1,1,p',p'+1)+(1,1,p'+1,p'+2)$, p ,$p'\geq 0$. *According to the edge-weight matching equation 4.1, we will have* $x=y=2$, $u=2p+1$, $v=2p'+1$ *and* $z=2P+3=2p'+3$. *Hence* $p=p'$. *Now the possible edge-weight for the normal surface S_s in C_s is* $(u,v,z,x);(u,v,y,z)=(2p+1,2p+1,2p+3,2);(2p+1,2p+1,2,2p+3)$.

   For the bottom edge-weight $(2p+1,2p+1,2p+3,2)$, *according to the equation 4.5 we have*

$$(2p+1,2p+1,2p+3,2)=\begin{cases} ((1,1,3,2),p=0 \\ 2\times(1,1,2,1)+(2p-1)(1,1,1,0),p\geq 1\end{cases} \tag{5.8}$$

   For the top edge-weight $(2p+1,2p+1,2,2p+3)$,*according to the equation 4.6 we have*

$$(2p,2p,2,2p+2)=\begin{cases} (1,1,3,2),p=0 \\ 2\times(1,1,1,2)+(2p-1)(1,1,0,1),p\geq 1\end{cases} \tag{5.9}$$

   Possibility 1. If $p=0$, *then* $p'=p=0$. *we have S_r is an octagonal annulus (with no tubes),* $(1,1,2,1)+(1,1,1,0);(1,1,0,1)+(1,1,1,2)$. *Notice* $r=3$, *because*

$$(1,1,2,1)+(1,1,1,0)\xrightarrow{push}(1,1,1,0)+(1,1,0,1)$$

$$\xrightarrow{oct} (1,1,1,0) + (1,1,0,1)$$
$$\xrightarrow{push} (1,1,0,0) + (1,1,1,2)$$

Furthermore, the possible edge-weight for the normal surface S_s in C_s is $(u,v,z,x);(u,v,y,z)=$ $(1,1,3,2);(1,1,2,3)$. *There are two possible choices for S_s.*

(a) *S_s is a copy of meridian disk,* $(1,1,3,2);(1,1,2,3)$. *Then we get $s=5$. After identifying the corresponding edges on ∂S_r and ∂S_s together, we get an orientable almost normal octagonal surface S in $C_{3,5}$.*

   The matching equation for S_r and S_s is

$$(1,1,2,1)+(1,1,1,0);(1,1,0,1)+(1,1,1,2) \leftrightarrow (1,1,3,2);(1,1,2,3)$$

   However, after identification, we notice this is a nonorientable surface. We ignore it.

(b) *S_s is a nonorientable surface,*$(1,1,3,2);(1,1,2,3)$. *This will give us a nonorientable octagonal S after identifying the corresponding edges on ∂S_r and ∂S_s together. Therefore, we will ignore this case.*

Possibility 2. If $p \geq 1$, since

$$2\times(1,1,2,1)+(2p-1)(1,1,1,0) \xrightarrow{push} 2\times(1,1,1,0)+(2p-1)\times(1,1,0,1)$$
$$\xrightarrow{push} 2\times(1,1,0,1)+(2p-1)\times(1,1,1,2)$$
$$\xrightarrow{push} 2\times(1,1,1,2)+(2p-2)\times(1,1,2,3) \xrightarrow{push} \cdots$$

Therefore, if we need to find a surface S_s with edge-weight $(2p+1,2p+1,2p+3,2);(2p+1,2p+1,2,2p+3)$ *for $p \geq 1$, it requires that $2p-1=2$, hence p is not an integer. Therefore it is impossible.*

8. S_r *has edge-weight*$(2,2,2,2)+(0,0,1,1);(1,1,2,1)+(1,1,1,0)$. *It is impossible. According to the edge-weight matching equation 4.1, we will have* $x=y=2$, $u=3$, $v=3$ *and* $z=3=1$, *which is impossible.*

9. S_r *has edge-weight*$(2,2,2,2)+(0,0,1,1);(1,1,1,0)+(1,1,0,1)$. *It is impossible. According to the edge-weight matching equation 4.1, we will have* $x=y=2$, $u=3$, $v=1$ *and* $z=3=1$, *which is impossible.*

10. S_r *has edge-weight*$(2,2,2,2)+(0,0,1,1);(1,1,p',p'+1)+(1,1,p'+1,p'+2)$, $p'\geq 0$. *According to the edge-weight matching equation 4.1, we will have* $x=y=2$, $u=3$, $v=2p'+1$ *and* $z=3=2p'+3$. *Hence* $p'=0$. S_r *has edge-weight*$(2,2,2,2)+(0,0,1,1);(1,1,0,1)+(1,1,1,2)$, *with* $r\geq 3$, *since*

$$\begin{aligned}&(2,2,2,2)+(0,0,1,1)\xrightarrow{push}\cdots\xrightarrow{push}(2,2,2,2)+(0,0,1,1)\\&\xrightarrow{oct}(1,1,2,1)+(1,1,1,0)\\&\xrightarrow{push}(1,1,1,0)+(1,1,0,1)\xrightarrow{push}(1,1,0,1)+(1,1,1,2)\end{aligned}$$

Now the possible edge-weight for the normal surface S_s *in* C_s *is* $(u,v,z,x);(u,v,y,z)=(3,1,3,2);(3,1,2,3)=(2,0,1,1)+(1,1,2,1);(2,0,1,1)+(1,1,1,2)$. *Therefore, there are two possible choices for* S_s.

(a) S_s *is a disjoint union of vertex-linking disk,* $(2,0,1,1);(2,0,1,1)$ *and a meridian disk* $(1,1,2,1);(1,1,1,2)$. *Notice here* $s=3$. *After identifying the corresponding edges on* ∂S_r *and* ∂S_s *together, we get an orientable almost normal octagonal surface* S *in* $C_{r,3}$, $r\geq 3$.

    The matching equation for S_r *and* S_s *is* $(2,2,2,2)+(0,0,1,1);(1,1,0,1)+(1,1,1,2)\leftrightarrow(2,0,1,1)+(1,1,2,1);(2,0,1,1)+(1,1,1,2)$. *However, this is a nonorientable surface. We ignore it.*

(b) S_s *is a disjoint union of vertex-linking disk,* $(2,0,1,1);(2,0,1,1)$ *and a nonorientable surface* $(1,1,2,1);(1,1,1,2)$. *Then after identifying the corresponding edges on* ∂S_r *and* ∂S_s *together, we get a nonorientable octagonal surface. We ignore this case.*

11. S_r *has edge-weight*$(1,1,0,1)+(1,1,1,2);(2,2,2,2)+(0,0,1,1)$. *This case is impossible. According to the edge-weight matching equation 4.1, we will have* $x=y=2$, $u=3$, $v=3$ *and* $z=1=3$, *which is impossible.*

12. S_r *has edge-weight*$(1,1,1,0)+(1,1,0,1);(2,2,2,2)+(0,0,1,1)$. *This case is impossible. According to the edge-weight matching equation 4.1, we will have* $x=y=2$, $u=1$, $v=3$ *and* $z=1=3$, *which is impossible.*

13. S_r *has edge-weight*$(1,1,p+2,p+1)+(1,1,p+1,p);(2,2,2,2)+(0,0,1,1)$, $p\geq 0$. *According to the edge-weight matching equation 4.1, we will have* $x=y=2$, $u=2p+1$, $v=3$ *and* $z=2p+3=3$. *Hence we get* $p=0$. *Hence* S_r *has edge-weight* $(1,1,2,1)+(1,1,1,0);(2,2,2,2)+(0,0,1,1)$. *Notice here* $r\geq 3$

$$(1,1,2,1)+(1,1,1,0)\xrightarrow{push}(1,1,1,0)+(1,1,0,1)\xrightarrow{push}(1,1,0,1)+(1,1,1,2)$$
$$\xrightarrow{oct}(2,2,2,2)+(0,0,1,1)$$
$$\xrightarrow{push} cdots \xrightarrow{push}(2,2,2,2)+(0,0,1,1)$$

Now the possible edge-weight for the normal surface S_s *in* C_s *is* $(u,v,z,x);(u,v,y,z)=(1,3,3,2);(1,3,2,3)=(0,2,1,1)+(1,1,2,1);(0,2,1,1)+(1,1,1,2)$. *Therefore, there are two possible choices for* S_s.

(a) S_s *is a disjoint union of vertex-linking disk,* $(0,2,1,1);(0,2,1,1)$ *and a meridian disk* $(1,1,2,1);(1,1,1,2)$. *Notice here* $s=3$. *After identifying the corresponding edges on* ∂S_r *and* ∂S_s *together, we get an orientable* *almost normal octagonal surface S in* $C_{r,3}$, $r \geq 3$. *The matching equation for* S_r *and* S_s *is* $(1,1,2,1)+(1,1,1,0); (2,2,2,2)+(0,0,1,1) \leftrightarrow (0,2,1,1)+(1,1,2,1); (0,2,1,1)+(1,1,1,2)$. *However, this is a nonorientable surface. We ignore it.*

(b) S_s *is a disjoint union of vertex-linking disk,* $(0,2,1,1); (0,2,1,1)$ *and a nonorientable surface* $(1,1,2,1); (1,1,1,2)$. *Then after identifying the corresponding edges on* ∂S_r *and* ∂S_s *together, we get a nonorientable octagonal surface. We ignore this case.*

■

Let's consider the smallest genus octagonal surfaces in each layered chain pair triangulation of $M_{r,s}$. We notice according to theorem 5.3, this happens if and only if $c = c' = 0$.

Corollary 5.1 *The octagonal almost normal surface with the smallest genus in the layered chain triangulation of* $M_{r,s}$,$r, s \geq 2$ *are isotopic to one of the followings,*

1. *In* $C_{n,3} = C_{3,n}$, $n \geq 5$, *there are only two almost normal octagonal surfaces of genus 2.*

2. *In* $C_{r,2} = C_{2,r}$, $r \geq 6$, *there are only two almost normal octagonal surfaces with genus* 2.

3. *In* $C_{r,s} = C_{s,r}$, $r, s \geq 5$ *or* $C_{4,5} = C_{5,4}$, *all the almost normal octagonal surfaces has genus at least 3.*

4. *There is no octagonal surface in other* $C_(r, s)$. *The list of them are* $C_{3,3}$, $C_{4,4}$, $C_{2,s} = C_{s,2}$, $2 \leq s \leq 5$.

All in all, there are almost normal octagonal surfaces in any layered chain pair triangulation $C_{r,s}$ of $M_{r,s}$, except for $M_{2,2}$, $M_{2,3}$, $M_{2,4}$, $M_{2,5}$, $M_{3,3}$ and $M_{4,4}$. Notice the first 5 Seifert fibered manifolds satisfies the condition $\sum 1/\alpha_i \geq 1$, with $i = 1, 2, 3$.

We also notice that $M_{4,3} = M_{3,4}$=$W(2,4,b)$, where $b = 5$, and $M_{2,6} = M_{6,2}$=$V(2,3,a)$, where $a = 7$.

1. $V(2,3,a)$ is a Brieskon manifold.

   $V(2,3,a)$ =$\{z \in \mathbb{C}^{\Vdash} | {z_1}^2 + {z_2}^3 + {z_3}^a = 0, \|z\| = 1\}$, with $3 \nmid a$, $a \geq 7$.

   If a is even, $V(2,3,a) = S(0; -\frac{1}{6a}; \frac{1}{2}, \frac{(-a)^{-1}}{3}, \frac{6^{-1}}{a})$, otherwise

   $V(2,3,2a')$=$S(0; -\frac{1}{3a}; \frac{-a'^{-1}}{3}, \frac{-a'^{-1}}{3}, \frac{3^{-1}}{a'})$,

2. $W(2,4,b)$ is the link of the singularity.

   $W(2,4,b)$=$\{z \in \mathbb{C}^{\Vdash} | {z_1}^2 + ({z_2}^2 + {z_3}^b)z_2 = 0, \|z\| = 1\}$, with $2 \nmid b$, $b \geq 5$
   $W(2,4,b) = S(0; -\frac{1}{4b}; \frac{1}{2}, \frac{(-b)^{-1}}{4}, \frac{4^{-1}}{b})$.

In fact, by comparing the seifert invariant notations of $M_{r,s}$ with $V(2,3,a)$ and $W(2,4,b)$. We notice $M_{4,3} = M_{3,4}$ and $M_{2,6} = M_{6,2}$ are the only two manifolds in $M_{r,s}$ belongs to these two special families of manifolds.

CHAPTER 6

Heegaard splitting surfaces

In this chapter, we will discuss Heegaard splitting surfaces in twisted layered loop triangulations of Seifert fibred spaces $M_k = S^3/Q_{4k} = S^2((2,1),(2,1)(k,1-k))$, and layered chain pairs triangulation of Seifert fibred space $M_{r,s}$=$(S^2 : (2,-1),(r+1,1),(s+1,1))$, respectively.

Definition 6.1 *In the twisted layered loop triangulation $\widehat{C}_k$ of the manifold $\widehat{M}$, if we cut the triangulation along a level annulus that meets a thin edge-linking tube, which is not $t = b$, then $\widehat{M}$ turns into a solid torus, and the twisted layered loop triangulation turns into the layered chain triangulation. Furthermore, the boundaries of this thin edge-linking tube in $\widehat{M}$ will separate the torus boundary into two annulus. The annulus which contains the vertices is called the companion annulus for this tube. The other annulus is called the complementary annulus.*

There are two possible companion/complementary annulus for the same thin edge-linking tube. It depends where we cut the level annulus. See figure 6.1

Definition 6.2 *If an almost normal tubed surface is obtained by adding an almost normal tube to a normal surface along an edge, which intersects with some quad(s) of a thin edge-linking tube from this normal surface, it is called an almost normal tubed surface with the almost normal tube at the same level of a thin edge-linking tube.*

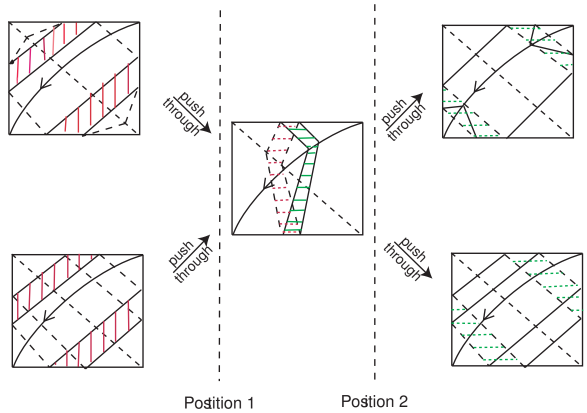

Figure 6.1: The complementary annuli w.r.t. a thin edge-linking tube.

6.1 Heegaard splitting surfaces in the twisted layered loop triangulations

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In this section we will show that Heegaard splitting surface can not be normal in the twisted layered loop triangulation of $\widehat{M}$. Moreover, we will discuss what surfaces are Heegaard splitting surfaces and answer the classification problem for Heegaard splitting surfaces.

Theorem 6.1 *Any orientable normal surface in the twisted layered loop triangulation of small Seifert fibred space $\widehat{M}$ is not Heegaard splitting surface.*

Proof. First notice that any orientable normal surface in twisted layered loop triangulation is a vertex-linking 2-sphere (possibly) with thin edge-linking tubes by theorem 3.1.

The manifold $\widehat{M}$ is a small Seifert fibered space, not S^3 or lens space. According to the papers [33], $\widehat{M}$ can not have genus 0 or 1 Heegaard splittings. Therefore, we only consider the normal surface S^2 with at least two thin edge-linking tubes.

Let S be a normal surface, a vertex-linking disk with at least two thin edge-linking tubes. We assume that S has genus g, then $g \geq 2$. We will prove this theorem by contradiction.

Suppose S is a Heegaard splitting surface of $\widehat{M}$. S separates $\widehat{M}$ into 2 handlebodies H, H'.

Let H be the handlebody containing the unique vertex of the twisted layered loop triangulation $\widehat{C}_k$. Then H is a handlebody. If we cut $\widehat{M}$ along a level annulus that meets a thin edge-linking tube, then S become a normal surface S', an annulus with at least one tube in the layered chain triangulation of the solid torus M. S' has a companion annulus A and a complementary annulus B on the boundary of the solid torus M.

Let's first make an observation. Notice B is an embedded Möbius band in the handlebody H' before cutting. This is because $t = -b$ and B is doesn't contains vertice and $B \cap H{=}B \cap \partial H{=}B \cap S{=}\partial B(=\partial A)$. Since any embedded surface in a handlebody is compressible or ∂-compressible in H'.

First we will show that Möbius band is not compressible. Let γ be a noncontractable simple closed curve on the Möbius band. This curve is orientation reversing curve. Suppose that we can find a compression disk for this Möbius and ∂D is γ. Let's take the small regular neighborhood of D, we get a ball with curve γ on its boundary. Since γ is orientation reversing curve, the small regular neighborhood of it on the 2-sphere is a Möbius band. Furthermore, any simple closed curve on the 2-sphere bounds a disk. Hence, we can add a disk on the boundary of this Möbius, therefore, we get a $\mathbb{RP}^2$ on the surface 2-sphere, which is impossible because the first homology of $\mathbb{RP}^2$ is $\mathbb{Z}_2$, which can not be a subgroup of $\mathbb{Z}$. Möbius band is therefore not compressible.

Möbius band is ∂-compressible in a handlebody. Now let's consider all ∂-compressing disks for all possible Möbius bands Bs in the handlebody H'.

Let D be a ∂-compressing disk for the some Möbius band B with respect to thin edge-linking tube e, where $-b \neq e \neq t$, so that it has minimal intersection with all the Bs in H', where $\partial D \cap S = \alpha$, $\partial D \cap B = \beta$, and $\partial D = \alpha \cup \beta$.Notice $D \subset H'$.

Let's first consider the case if there is an innermost simple closed intersection curve in D.

Notice this simple closed curve can only be a trivial curve. Otherwise, it is an orientation reversing curve on the Möbius band, we know it cannot bounds a disk here. If it is a trivial curve in some B', then we can use standard techniques to modify the intersection to reduce the number of intersection components in D, Since every S^2 bounds a ball in an irreducible manifold. Therefore, we get a new ∂-compressing disk for the annulus B which has less intersection with the collection of the complementary annuli. This is a contradiction to the fact that D has the minimal intersection with the collection of complementary annuli. It is impossible.

Now Let's consider the outermost intersection arc in D with the collection of complementary annuli .

If it is a trivial arc in some B', it will cobound a disk with an arc γ on the boundary of B'. Moreover, this trivial arc will separate ∂D into two parts, and obviously that the two endpoints of the arc are on the boundary of $B' \subset S$, hence they are in the arc α. Now we can construct a new ∂-compressing disk D' for B, where $\partial D' = \beta \cup \gamma \cup$ the two segments on the ∂D between the endpoints of γ and β.

If it is an essential arc, denoted by γ in some B'. we will get a new ∂-compressing disk D' for B', with $\partial D'$ is the union of γ and the arc which is the segment in $\alpha \subset S$ obtained by separation from the endpoints of γ on the ∂D. Obviously $D' \subset D$ has less intersection with the collection of complementary annuli for S. This is a contradiction.

Therefore, the ∂-compressing disk D for the Möbius band B in H' doesn't intersect with any other Möbius band Bs with respect to other thin edge-linking tubes.

Furthermore, we get that D is also a ∂-compressing disk for the annulus with respect to edge e of the normal surface S' into its complementary annulus B on the boundary of the layered chain triangulation of the solid torus. First we notice that $\beta \subset B$ has two endpoints on two different boundaries of B in the layered chain triangulation. Therefore, it is essential arc on the annulus B. Moreover, $\alpha \subset S$ is an arc on the annulus part of S' without hitting other tubes and it have same endpoints as β. Therefore, α is an arc with endpoints on the different boundaries of the annulus of S with respect to the thin edge-linking tube e. Therefore, We get that α is essential with respect to the thin edge-linking tube e. Therefore, We get the annulus of S' with respect to thin edge-linking tube e is ∂-compressed to its complementary annulus B, and D is a ∂-compressing disk in the solid torus. Therefore, the companion annulus of the annulus in S' is ∂-compressed into its the complementary annulus. Hence, the thin edge that the annulus is around should be longitudinal. However, this edge e is not t or b. This is a contradiction.

Therefore, no normal surface in the twisted layered loop triangulation is a Heegaard splitting surface.

■

Lemma 6.1 *If S is an almost normal tubed surface which is obtained by adding an almost normal tube between two connected normal surfaces in the twisted layered loop triangulation, then S can not be a Heegaard splitting surface.*

Proof. Let S be an almost normal surface which is obtained by adding an almost normal tube between two compatible connected normal surfaces S_1 and S_2 in the twisted layered loop triangulation. Furthermore, S_1 separates S_2 from the vertex. Obviously, S_1 and S_2 are both vertex-linking S^2 (possibly) with thin edge-linking tubes by theorem 3.1. They all bounds a handlebody on the side which contains the unique vertex in the twisted layered loop triangulation of $\widehat{M}$. We will prove the theroem by contradiction.

Let's assume that S is a Heegaard Splitting surface. S separates M into two handlebodies H and H', where H is the one that contains the vertex. Then S_2 is a disk connected summand of S and it bounds a handlebody that is a connected summand of H on the side that not only doesn't contain the vertex, but also is disjoint from S_1. Hence, S_2 boundsa handlebody on both sides. Therefore, it is a Heegaard splitting surface. However, According to the theorem 6.1, no normal surface can be a Heegaard splitting surface. This is a contradiction. Therefore, S can not be a Heegaard splitting surface. ■

Now we will come to answwer the question what kind of surface in the twisted layered triangulation can be a Heegaard splitting surface. Before that, let's first make some important observation.

By theorem 3.1, the only orientable normal surface in the twisted layered loop triangulation is vertex-linking S^2 (possibly) with thin edge-linking tubes. These surfaces come from the following orientable surfaces in the layered chain triangulation after identification their boundary according to the match equation $(wt_t, wt_b, wt_{e_1}, wt_{e_2}) = (wt_t, wt_b, wt_{e_{k+1}}, wt_{e_{k+2}})$, where $wt_t = wt_b$.

1. Vertex-linking disks (possibly) with thin edge-linking tubes, $(2, 2, 2, 2); (2, 2, 2, 2)$

2. The double cover of a Klein bottle, which has the edge-weights $2 \times (0, 0, 1, 1); 2 \times (0, 0, 1, 1)$ and is a quadrilateral splitting surface, splitting the opposite edges $t = -b$ in each tetrahedron of the layered chain triangulation. This give us a vertex-linking S^2 with a thin edge-linking tube around the edge $t = -b$ in the twisted layered loop.

3. If k is even,

    - the double cover of a nonorientable surface of genus $\frac{k}{2} + 1$, which has the edge-weights $2 \times (1, 1, 0, 1); 2 \times (1, 1, 0, 1)$ and is a quadrilateral splitting surface, splitting the odd index edges in the layered chain triangulation.

This can give us a vertex-linking S^2 with thin edge-tubes around all the odd edges with genus $\frac{k}{2}$.

- the double cover of a nonorientable surface of genus $\frac{k}{2}+1$, which has the edge-weights $2 \times (1,1,1,0); 2 \times (1,1,1,0)$ and is a quadrilateral splitting surface, splitting the even index edges in the layered chain triangulation. This can give us a vertex-linking S^2 with thin edge-tubes around all the even edges with genus $\frac{k}{2}$.

4. If $k \geq 3$, the annulus with c thin-edge linking tubes, $2\times(1,1,0,1); 2\times(1,1,0,1)$, or $2 \times (1,1,1,0); 2 \times (1,1,1,0)$. The gunus is $c+1$.

Lemma 6.2 *If we add an almost normal tube to a connected orientable normal surface with at least one thin edge-linking tube in the twisted layered loop triangulation, the new surface is either isotopic to a connected normal surface with one more thin edge-linking tube, or an almost normal surface such that the almost normal tube is along the edge $t = -b$, or an almost normal surface such that the almost normal tube is along any edge except for edge $t = -b$.*

Proof. Suppose we have an orientable normal surface S, a vertex-linking S^2 possibly with thin edge-linking tubes. For the case that S has no thin edge-linking tube around edge $t = -b$. If we add an almost tube along an edge, except for edge $t = -b$, which has no thin edge-linking tube around it, then this almost normal tube can be normally isotopic to a normal tube around this edge. If we add an almost normal tube along other edges, which has a thin-edge linking tube around it, then it always can be normally isotopic to an almost normal tube along the edge $t = -b$. Notice there is no thin edge-linking tube around edge $t = -b$.

For the case that S has a thin edge-linking tube around edge $t = -b$. We notice from the above observation, S is obtained by identify the boundaries of the double cover of of a Klein bottle, $2\times(0,0,1,1); 2\times(0,0,1,1)$, in the layered chain triangulation. This is a quadrilateral splitting surface, splitting the opposite edges $t = -b$ in each tetrahedron. Hence, the almost normal tube can be normally isotopic to along any edge, except for $t = -b$, in the twisted layered loop triangulation of 3-manifold $\widehat{M}$.

■

By the proof of the above lemma, we notice that the almost normal surface S with a thin edge-linking tube around edge $t = -b$, is a surface of genus 2. We will prove that it is a Heegaard splitting surface.

Theorem 6.2 *If S is an almost normal tubed surface with a thin edge-linking tube around the edge $t = -b$, then S is an irreducible Heegaard splitting of $\widehat{M}$. Furthermore, it is a vertical Heegaard splitting.*

Proof. Since S is a vertex-linking S^2 with an almost normal tube and a thin edge-linking tube around the edge $t = -b$, it separates $\widehat{M}$ into two parts, H and H'. If H is the part contains the vertex, then H is a handlebody of genus two. Now we want to show that H' is also a handlebody.

Since S is obtained from adding an almost normal tube to an orientalbe normal surface S' in the twist layered loop triangulation of $\widehat{M}$, which is obtained by identifying the boundaries of 2 copies of vertical annuli, $2 \times (0, 0, 1, 1); 2 \times (0, 0, 1, 1)$ in the layered chain triangulation of a solid torus. This normal surface S' is a torus which is a double cover of a Klein bottle K, i.e the boundary of $K \tilde{\times} I$. After we add an almost normal tube on it, it is equivalent to say that we drill a tunnel along the direction of the I-boundle. Hence H' is a manifold obtained by the I-bounble of $K - intD$, which is same as the regular neighborhood of $S^1 \vee S^1$. Hence H' is a handlebody of genus 2.

Therefore, S is a Heegaard splitting surface of genus 2 in $\widehat{M}$.

Since $\widehat{M}$ is a Seifert manifold, not a lens space or S^3, the smallest genus of its Heegaard splitting is 2. Hence, it is irreducible Heegaard splitting.

Now we will show that S is a vertical Heegaard splitting. By the definition given in the paper [1], we know any vertical Heegaard splitting surface is the boundary of the neighborhood of two exceptional seifert fibers and an arc, which are projected to be an arc connecting two singular points projected by these two exceptional fibers. In our twist layered loop triangulation of $\widehat{M}$, Since the Klein bottle K is an embedded imcompressible surface in $\widehat{M}$, by Proposition 3 in [25], $\widehat{M} - \text{int}H$ is the Handlebody H', which can be fibered by circles, with two exceptional fibers of multiplicity 2 at the centers of the Möbius bands on K. Moreover, the edge $t = -b$ is at the core of the solid torus H bounded by S. Therefore, $\widehat{M} = H \cup H'$ is a vertical Heegaard splitting. ■

Now let's consider an almost normal tubed surface with exactly one thin edge-linking tube around edge e, which is not the edge $t = -b$. We will show that it is a Heegaard splitting surface of $\widehat{M}$ and it is isotopic to a vertical Heegaard splitting. In fact, we find two methods to prove that it is a genus 2 Heegaard splitting surface.

Theorem 6.3 *If S is an almost normal tubed surface with exactly one thin edge-linking tube around an edge e, where e is not $t = -b$, in the twisted layered loop triangulation of $\widehat{M}$, S is a Heegaard splitting surface of $\widehat{M}$. Moreover, S has genus 2 and it gives us an irreducible Heegaard splitting, which is isotopic to a vertical Heegaard splitting of $\widehat{M}$ in the theorem 6.2.*

Proof. Since S is an almost normal tubed surface with exactly one thin edge-linking tube around edge e, which is not $t = -b$, it separates $\widehat{M}$ in to two parts H and H', where H is the part that contains the unique vertex of the triangulation, hence H is a handlebody of genus 2. Now we need to prove that H' is also a handlebody of genus 2.

There are two ways to prove this part. Method 1 is the first method we found, and later we realize there is a much easier proof by using barrier surface theory.

Method 1: By Lemma 6.2, we know that the almost tube of S can always be isotopic to along the edge $t = -b$. Furthermore, we can always push the almost tube to be in the position at the same level of the tube around the thin edge e indicated as figure 6.2.

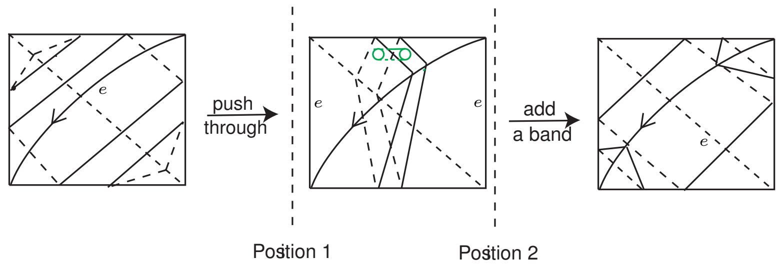


Figure 6.2: An almost normal tube along the edge $t = -b$ at the same level of the tube around the thin edge e.

If we cut the triangulation open along the level annulus at the position 1 or 2, we will get a layered chain triangulation of the solid torus. WLOG, we can cut it at the position 2. Before we cut it open, we can push the almost normal tube until it meet with the level annulus indicated as figure 6.3.

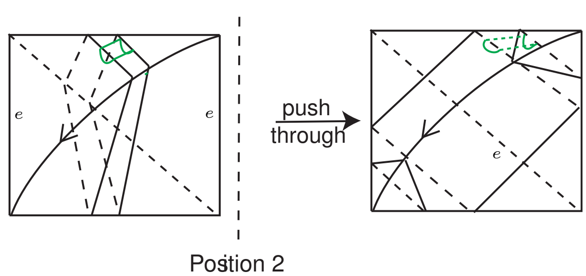


Figure 6.3: Push the almost normal tube to the level annulus.

Once we cut the triangulation open at the position 2, we get a solid torus and the surface S becomes a ∂-parallel surface, see figure 6.4. Hence, Let $\widehat{H}$ be the manifold of $\widehat{M} - intH$. Notice $\widehat{H}$ is a solid torus.

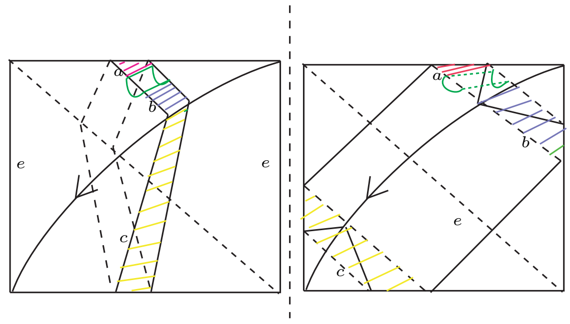


Figure 6.4: Push the tube up to the level annulus.

In figure 6.5, We notice on the bottom and top annuli, we have two disks, acb, where a,b,c indicate the order of pieces connected together to form a big disk when we try to identify the boundaries of the solid torus to get the twist layered loop triangulation. Notice, H' is obtained by gluing these two disks together on the boundary of the solid torus $\widehat{H}$. Therefore, H' is a handlebody of genus 2.

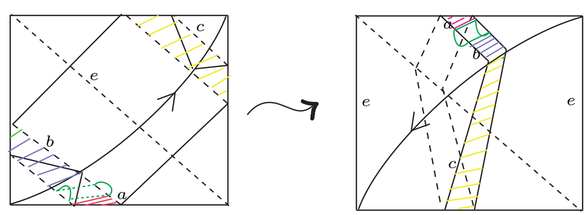


Figure 6.5: Cut along the level annulus.

Method 2: Since S has only one thin edge-linking tube and the almost normal tube can always be isotopic to at the same level of this tube. See Figure 6.6. Now we'll isotopy S by pushing the almost normal tube through the edge $t = -b$ in the H', see figure 6.7, which is same as the surface in figure 6.8. Realize the piece of the normal surface in the triangulation where two tubes meet is isotopic to the one indicated in figure 6.9, which can never isotopic it to a normal surface. In this case, we say S has no normal surface as a barrier surface on this side, then it bounds a handlebody (c.f.[12]). Therefore, we proved that H' is a handlebody.

Therefore, S separates $\widehat{M}$ into two genus 2 handlebodies. It is a genus 2 Heegaard splitting. Since $\widehat{M}$ is not a lens space or S^3, then this is a Heegaard splitting with minimal genus, therefore, it is an irreducible Heegaard splitting.

Now we will show that S is isotopic to the vertical Heegaard splitting surface, denote it S' in the theorem 6.2.

In the Lemma 6.2, we showed that the almost normal tube S' can be moved to along any edge which is not $t = -b$. Therefore, we can move it to same tetrahedron where S has two tubes meet together and along the edge e. See figure 6.10. Now we isotopic S' in certain ways shown in figure 6.11, we get the exactly same surface in figure 6.8inside this triangulation. This shows that this two surface are isotopic to each other.

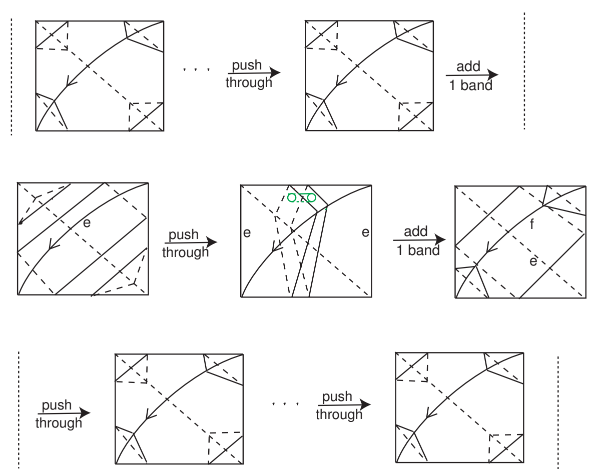


Figure 6.6: An almost tube at the same level of the thin edge-linking tube.

After further isotopy the surface in 6.8, we realize that they are all isotopic to a handlebody $\widehat{S}$ indicated as figure 6.12. It's not hard to realize that this give us a genus 2 handlebody (see figure 6.13) in the twisted layered loop triangulation of small Seifert fibered space $\widehat{M}$.

Hence, S is isotopic to a vertical Heegaard splitting S'. Since this is true for any thin edge-linking tube e, therefore, all the almost normal tubed surface with genus

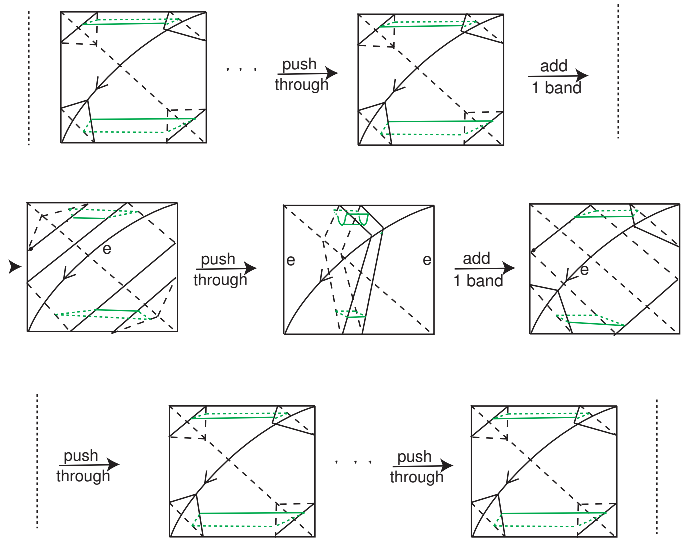


Figure 6.7: Push the almost tube through the edge $t = -b$.

2, obtained by adding a tube on the a connected normal surface, are isotopic to the vertical Heegaard splitting S'. ■

Theorem 6.4 *Any embedded orientable surface in the twisted layered loop triangulation of the small Seifert manifold $\widehat{M}$ is a Heegaard splitting surface, if and only if it is an almost normal tubed surface such that it is obtained by adding an almost normal tube on any orientable normal surface with at least one thin edge-linking tube in $\widehat{M}$.*

Proof. $\Longrightarrow$

If S is an embedded orientable Heegaard splitting surface in the twisted layered loop triangulation of a Seifert manifold $\widehat{M}$, it is not normal by theorem 6.1. Hence, S can only be an almost normal surface in $\widehat{M}$. By theorem 5.2, there is no connected orientable almost normal octagonal surface. Therefore, S can only be an almost normal tubed surface.

Since S is a Heegaard splitting, it separates $\widehat{M}$ into two handlebodies H and H', where H is the one that contains the unique vertex in the triangulation. By lemma 6.1, we know that S can not be a surface obtained by adding an almost normal tube to two disconnected normal surfaces. Hence, it can only be obtained by adding a tube

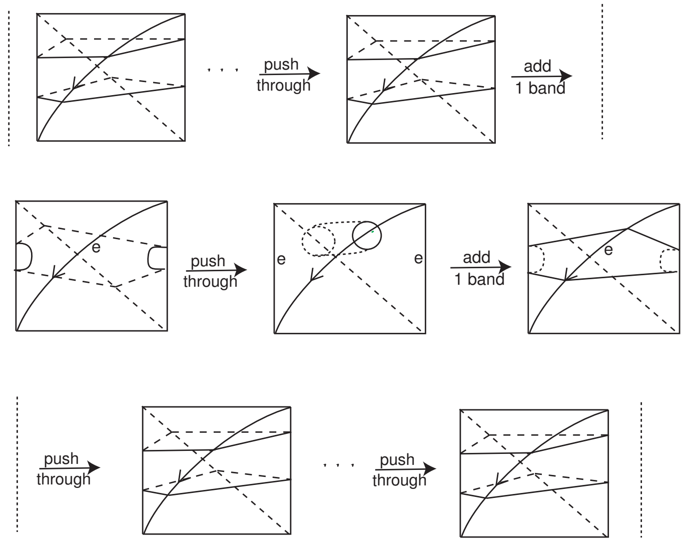


Figure 6.8: The isotopy of the surface S.

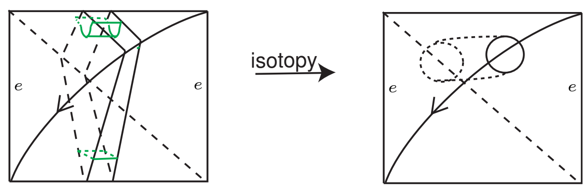


Figure 6.9:

to a connected normal surface, denoted by S'. since S' is orientable normal surface, so S' is a vertex-linking S^2 possibly with thin edge-linking tubes. Therefore, S is a vertex-linking S^2 with an almost normal tube and possibly with thin edge-linking tubes. Since $\widehat{M}$ is a small Seifert fibered surface,not a lens space or S^3, the Heegaard splitting surface of it can not have genus 0 or genus 1. Therefore, S should have at least 2 tubes. Since one of them should be an almost normal tube, S should have at least one thin edge-linking tube.

$\Longleftarrow$

If S is an almost normal tubed surface obtained by adding an almost normal tube on any orientable normal surface with at least one thin edge-linking tube in $\widehat{M}$. By the Lemma 6.2, we realize that there are 2 possibilities for S in the twisted layered chain triangulation, an almost normal tubed surface such that the thin edge-linking

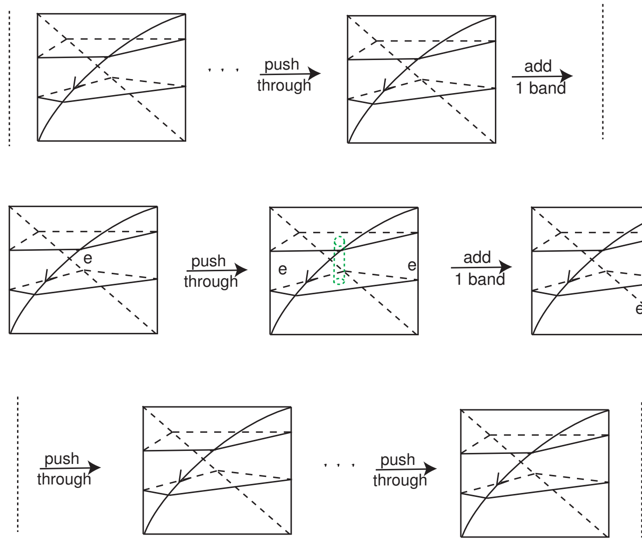


Figure 6.10: $\widehat{S}$ is the genus 2 handlebody.

tube is around the edge $t = -b$, and the almost normal tube is along the edge $t = -b$ and at the same level of the thin edge-linking tube.

By theorem 6.2, If we add the almost tube to the vertex linking 2-sphere with a thin edge-linking tube around the edge $t = -b$, it is a Heegaard splitting surface.

For the latter case, S has at least one thin edge-linking tube and the almost normal tube is along the edge $t = -b$. Notice this surface always bounds a handlebody on the side which contains the vertex. Because any vertex-linking S^2 with thin edge-linking tube(s) will bounds a handlebody on the side containing the vertex. If we add one handle along an edge of the triangulation to this surface, it is still bounds a handlebody on this side.

Now we need to show that S bounds a handlebody on the other side. Recall that the almost tube can always be put at the same level with a thin edge-linking tube around edge $e \neq t$. If we isotopy S by pushing the almost tube through the edge $t = -b$ in the direction away from the vertex, we realize that the piece of surface (figure 6.9) inside this tetrahedron can not be normalized later. Therefore, S has no normal surface as a barrier surface in this direction. It means it bounds a handlebody.

Therefore, S bounds a handlebody on both sides, so it is a Heegaard splitting

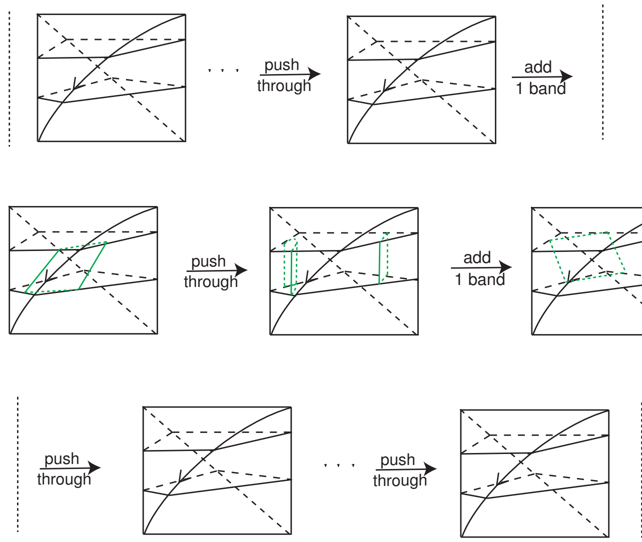


Figure 6.11: The isotopy surface of S.

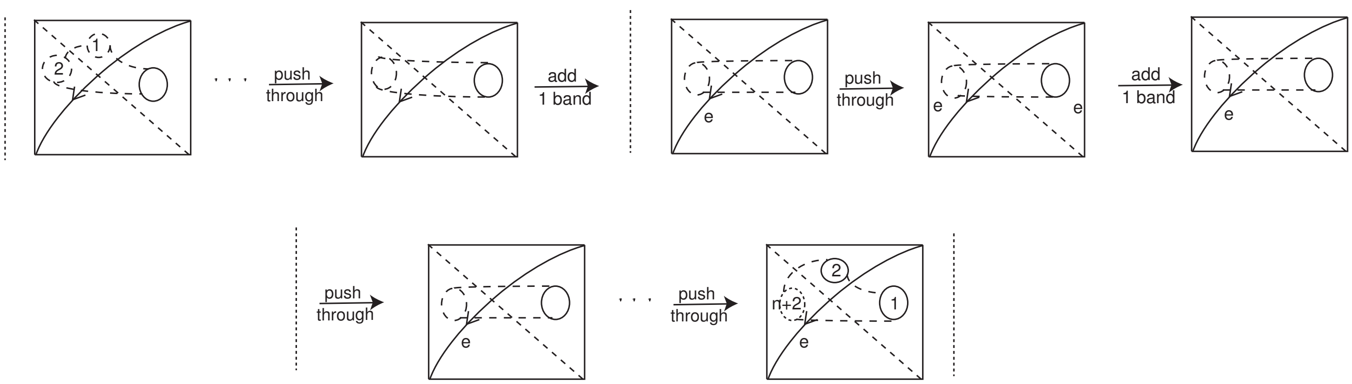


Figure 6.12: The isotopy surface of S.

surface of $\widehat{M}$.

■

According to this theorem 6.4, theorem 6.2 and 6.3 we can directly get the following conclusion.

Theorem 6.5 *(Isotopy theorem) There is a unique irreducible Heegaard splitting in the twisted layered loop triangulation of Seifert fibered space* $\widehat{M} = M_k = S^3/Q_{4k} = S^2((2,1),(2,1)(k,1-k))$*, up to isotopy. Furthermore, it is a vertical Heegaard split-*

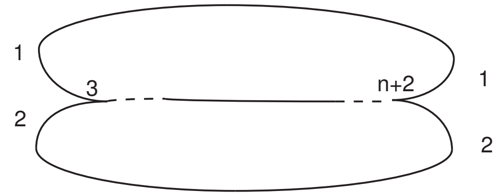


Figure 6.13: The isotopy surface of S.

ting.

Proof. From the theorem 3.1, we know there is no orientable incompressible surface in M_k, therefore, M_k is a small Seifert fibered manifold, which is non-Haken. Because any triangulation can catch each strongly irreducible Heegaard splitting class up to isotopy. In the non-Haken manifold, every irreducible Heegaard splitting surface is strongly irreducible. All the strongly irreducible Heegaard splittings will be isotopic to an almost normal surface. Since there is no octagonal almost normal surface in the twisted layered loop triangulation of M_k, all the irreducible Heegaard splitting surface are almost normal tubed surfaces.

We showed that all the Heegaard splitting surfaces of genus 2 in $\widehat{M}$ are almost normal tubed surfaces which obtained by adding an almost normal tube to a vertex-linking S^2 with a thin edge-linking tube in theorem 6.4.

We also showed in theorem 6.3that they are all isotopic to the vertical Heegaard splitting surface, S, discussed in theorem 6.2.

Therefore, all the genus 2 almost normal surfaces which are Heegaard splitting surfaces are isotopic to a vertical Heegaard splitting S. And there is no other Heegaard splittings with less genus than S. Therefore, S is irreducible (c.f.[30]). Therefore, there is a unique irreducible Heegaard splitting surface of genus 2 in $\widehat{M}$. ∎

6.2 Heegaard splitting surfaces in the layered chain pairs triangulations

In this section, we will study more about the surfaces in the layered chain pair triangulations of Seifert fibered spaces $M_{r,s}$=$(S^2 : (2, -1), (r + 1, 1), (s + 1, 1))$, and try to tell which surface is a Heegaard splitting surface.

6.2.1 Almost normal octaognal Heegaard splitting surfaces

Theorem 6.6 *The orientable almost normal octagonal surfaces with genus 2 in the layered chain pair triangulation of the Seifert fibred spaces $M_{4,3} = M_{3,4}$ and $M_{2,6} = M_{6,2}$ are irreducible Heegaard splitting surfaces.*

Proof. According to theorem 5.3, there are two almost normal octagonal surfaces with genus 2 in $M_{4,3} = M_{3,4}$ and $M_{2,6} = M_{6,2}$, respectively.

1 .The almost octagonal surfaces S and S' of genus 2 in $C_{4,3}$ (or $C_{3,4}$), with an edge-weights matching equation of ∂S_r and ∂S_s, $2 \times (1, 1, 2, 1); 2 \times (1, 1, 1, 2) \leftrightarrow 2 \times (1, 1, 2, 1); 2 \times (1, 1, 1, 2)$. Here S_r is an octagonal annulus and S_s is two copies of meridian disks. Notice S is an octagonal surface of genus two. See figure 5.4. In fact, they are homeomorphic to each other.

On one hand, we can isotopy S and S' towards the vertex, and from the figure 6.14, and figure 6.15, we can tell the S is isotopic to a normal surface with two thin edge-linking tubes, which bounds a handlebody of genus 2 on the side containing the vertex. On the other hand, we isotopy S and s' away from the vertex, and from figure 6.16, it's not hard to check that S can not fall on any normal surface. By the barrier theory in the paper [12], S bounds a handlebody on the side away from vertex. It can be a fun exercise to further isotopy it and see how it finally looks like a genus two handlebody in $C_{4,3}$ (or $C_{3,4}$. Therefore, S and S' are genus two Heegaard splitting surfaces in $C_{4,3}$ (or $C_{3,4}$).

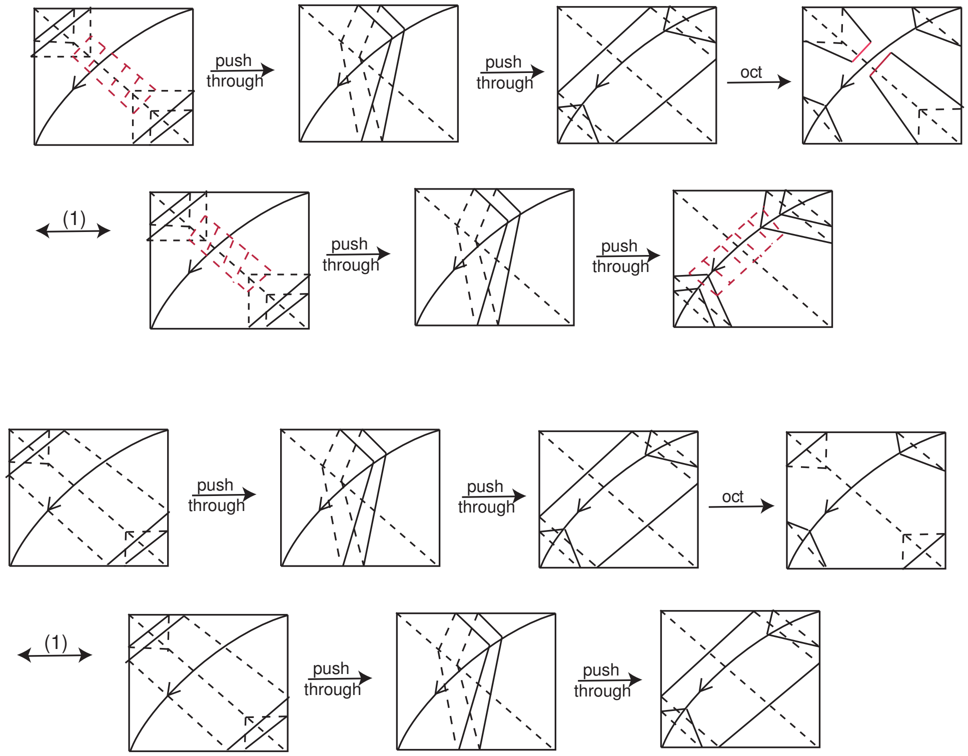


Figure 6.14: Isotopy one possible S towards the vertex.

2 . Two almost normal octagonal surfaces S, S' of genus 2 in $C_{6,2}$ (or $C_{2,6}$), with an edge-matching equation of ∂S_r and ∂S_s, $2\times(1,1,3,2); 2\times(1,1,2,3) \leftrightarrow 2\times(1,1,2,1)+2\times(1,1,1,0); 2\times(1,1,1,2)+2\times(1,1,0,1)$. Here S_r is an octagonal annulus and S_s is two copies of meridian disks, $(1,1,2,1); (1,1,0,1)$ and $(1,1,1,0); (1,1,1,2)$ then the genus of S is 2. See figure 5.5. In fact, they are homeomorphic to each other.

In these case, we can try to isotopy S, S' to both side. See figure 6.17.S can not isotopy to any normal surface. By the barrier theory in the paper [12], S bounds a handlebody on each side. Therefore, S is a genus two Heegaard splitting surface in $C_{6,2}$ (or $C_{2,6}$). Since the Heegaard genus of these two manifolds are 2, so these octagonal surfaces are irreducible Heegaard splittings.

■

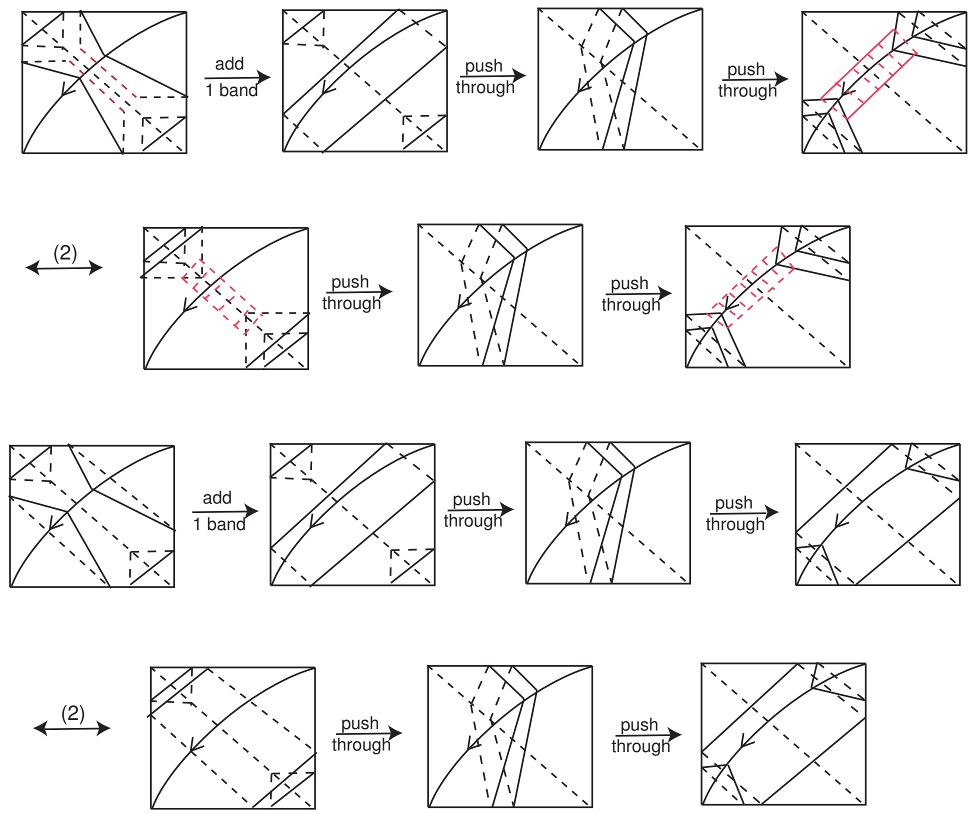


Figure 6.15: Isotopy S' towards the vertex.

Corollary 6.1 *The normal surface in the layered chain pair triangulation $C_{4,3}$ (or $C_{3,4}$) that is isotopic to an octagonal almost normal surface is a genus two irreducible Heegaard splitting.*

Proof. Since two almost octagonal surfaces S of genus 2 in $C_{4,3}$ (or $C_{3,4}$) are isotopic to two normal surfaces, repectively. See figure 5.4. Furthermore, these two octagonal surfaces are Heegaard splitting surfaces. So does these two normal surfaces. ■

Remark: These two normal Heegaard splitting surfaces in $C_{4,3}$ are a vertex-linking S^2 with thin edge-linking tubes around edge f_3 and edge e_3, and a vertex-linking S^2 with thin edge-linking tubes around edge f_4 and edge e_3, respectively. see figure 6.14 and 6.15.

Theorem 6.7 *Any genus 2 octagonal almost normal surface is isotopic to a normal*

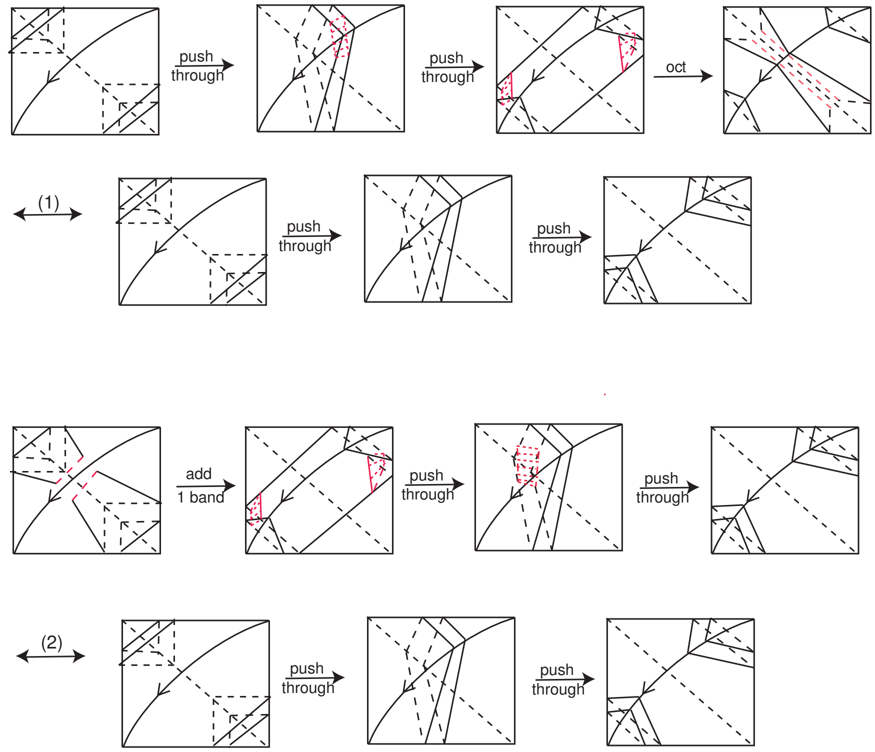


Figure 6.16: Isotopy S away from the vertex.

surface in $C_{r,s}$, $r, s \geq 1$, except for the ones in $C_{2,6}$. Furthermore, each genus 2 octagonal almost normal surface isotopes to a normal surface if we isotopic it in the direction away from the vertex in $C_{r,s}$, $r, s \geq 2$, except for the ones in $C_{3,4}$ and $C_{2,6}$. Furthermore, the genus 2 octagonal almost normal surfaces in $C_{3,5}$ and $C_{2,7}$ are not Heegaard splitting surfaces.

Proof. By the corollary 5.1, we notice genus 2 octagonal almost surface only exist in case 1 and case 2 of the corollary.

1. In $C_{n,3} = C_{3,n}$, $n \geq 5$, there are only one almost normal octagonal surfaces of genus 2.

   It has edge-weight matching equation $2 \times (1,1,2,1); 2 \times (1,1,1,2) \leftrightarrow 2 \times (1,1,2,1); 2 \times (1,1,1,2)$.

   Notice here S_r is an annulus with no tubes. Therefore, we realize all the pieces

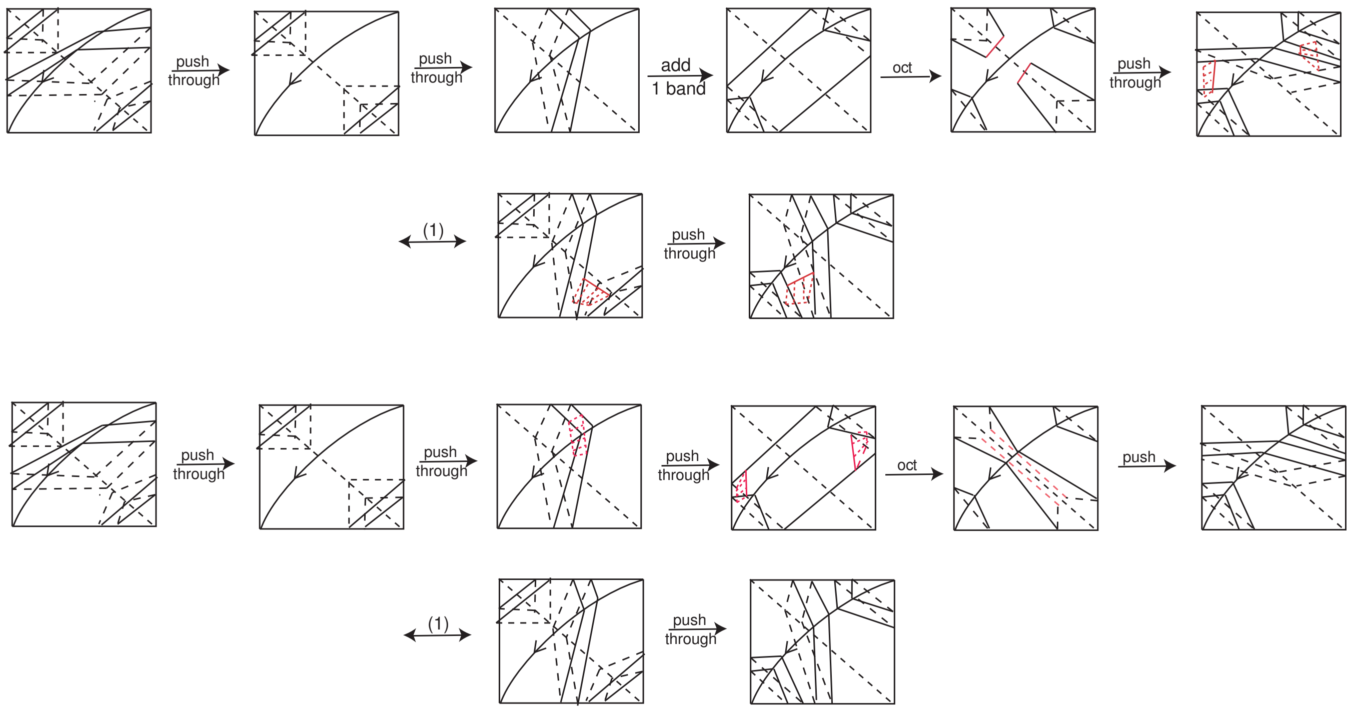


Figure 6.17: Isotopy S towards/away from the vertex.

in the layered chain triangulation in C_r are vertex-linking disks except for the remaining 4 tetrahedra at the beginning and end of layered chain triangulation which were restricted by the edge-weights of the annulus S_r. See figure 6.18.

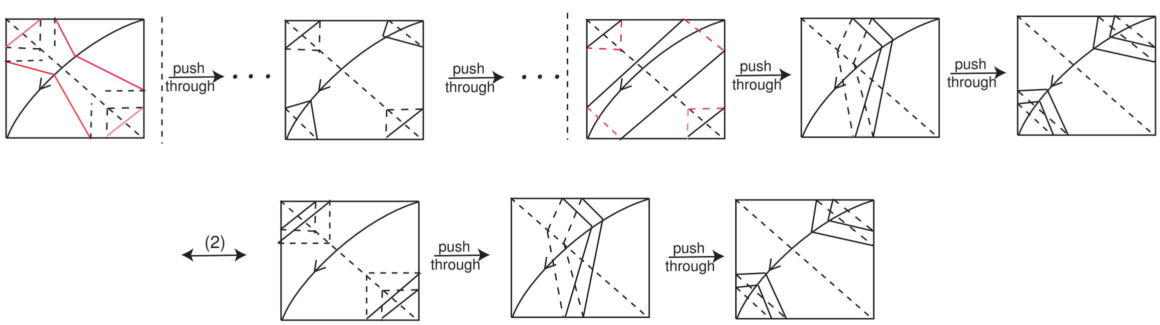


Figure 6.18: A genus 2 octagonal almost normal surface in $C_{n,3}$.

2. In $C_{r,2} = C_{2,r}$, $r \geq 7$, there are only one almost normal octagonal surface of genus 2.

   It has edge-weight matching equation $2 \times (1, 1, 3, 2); 2 \times (1, 1, 2, 3) \leftrightarrow 2 \times$

$(2,2,3,1); 2 \times (2,2,1,3)$ i.e. $2 \times (1,1,3,2); 2 \times (1,1,2,3) \leftrightarrow 2 \times (1,1,2,1) + 2 \times (1,1,1,0); 2 \times (1,1,1,2) + 2 \times (1,1,0,1)$.

Notice here S_r is an annulus with no tubes. Therefore, we realize all the pieces in the layered chain triangulation in C_r are vertex-linking disks except for the remaining 6 tetrahedra at the beginning and end of layered chain triangulation which were restricted by the edge-weights of the annulus S_r. See figure 6.19.

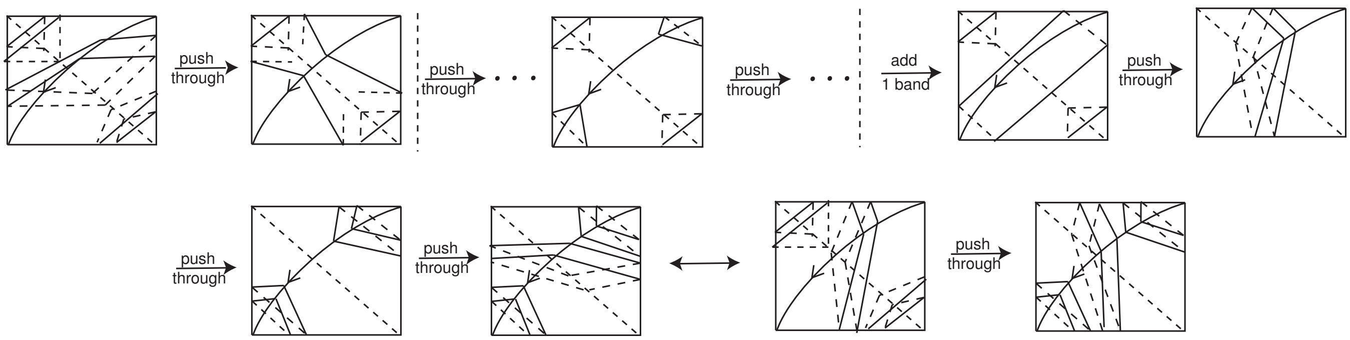


Figure 6.19: A genus 2 octagonal almost normal surface in $C_{n,2}$.

We already shows that genus two octagonal surfaces in $C_{4,3} = C_{3,4}$ and $C_{2,6} = C_{6,2}$ are not isotopic to a normal surface if we push them outwards the vertex. This is how we prove that they are Heegaard splitting surfaces. However, In $C_{4,3} = C_{3,4}$, each octagonal surface is isotopic to a normal surface when we push it towards the vertex. Therefore, Only the genus 2 almost normal octagonal surfaces in $C_{2,6}$ is not isotopic to a normal surfaces.

Now let's consider the case in $C_{n,3} = C_{3,n}$, $n \geq 5$.

First, consider the case in $C_{5,3}$. The genus 2 octagonal almost normal surface will isotopic to a normal surface indicated in figure 6.20.

Notice the normal surface of genus 2 is double cover of a genus 3 nonorientable surface. Hence, it bounds a twist I boundle of a nonorientable surface, which is not a handlebody. Therefore, the genus 2 Octagonal almost normal surface in $C_{5,3}$ is not a Heegaard splitting surface.

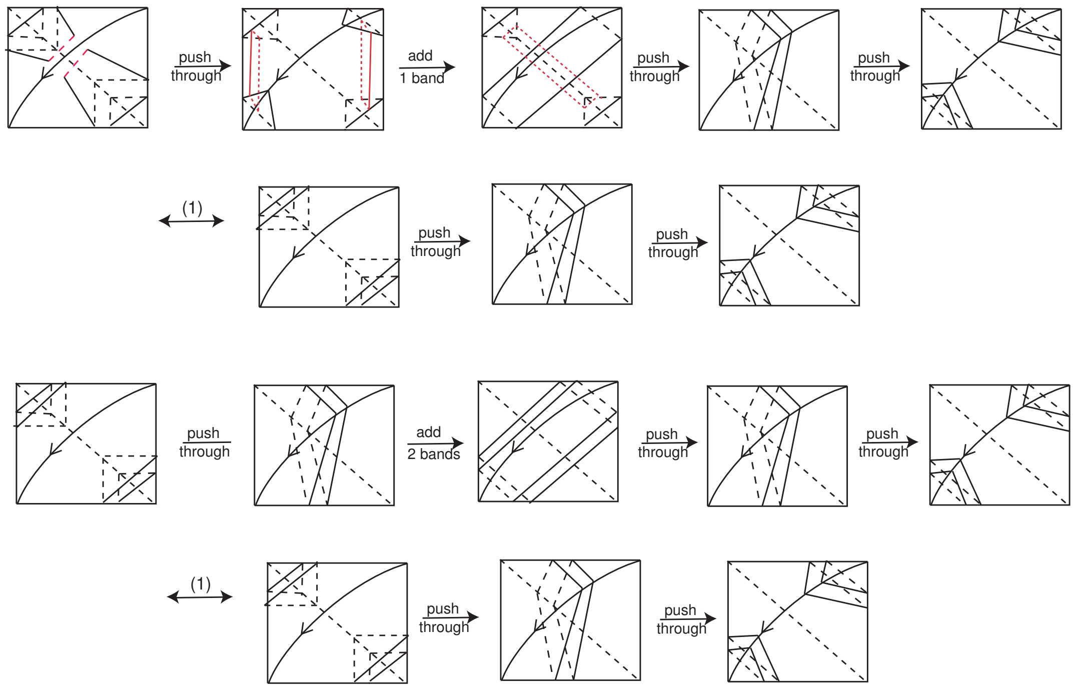


Figure 6.20: The barrier normal surface in $C_{5,3}$.

We can use the same reason to prove that the genus 2 octagonal almost normal surface in $C_{7,2}$ is not a Heegaard splitting surface. It is also isotopic to the double cover of a nonorientable surface. See figure 6.21

It's not hard to discuss the case in $C_{n,3}, n \geq 6$, and $C_{n,2}$, $n \geq 8$. They are all isotopic to a normal surface. See figure 6.22 and 6.23

■

Open question: From Theorem 6.7, we have a good reason to expect that genus 2 octagonal almost normal surfaces in $C_{2,n}$, $n \geq 8$ and $C_{n,3}$, $n \geq 6$, are not Heegaard splitting surfaces. Notice they are all isotopic to a normal surface of genus 2 in $C_{r,s}$ with edge-weight matching equation either $2 \times (1,1,2,1); 2 \times (1,1,1,2) \leftrightarrow 2 \times (1,1,2,1); 2 \times (1,1,1,2)$ or $2 \times (1,1,3,2); 2 \times (1,1,2,3) \leftrightarrow 2 \times (2,2,3,1); 2 \times (2,2,1,3)$.

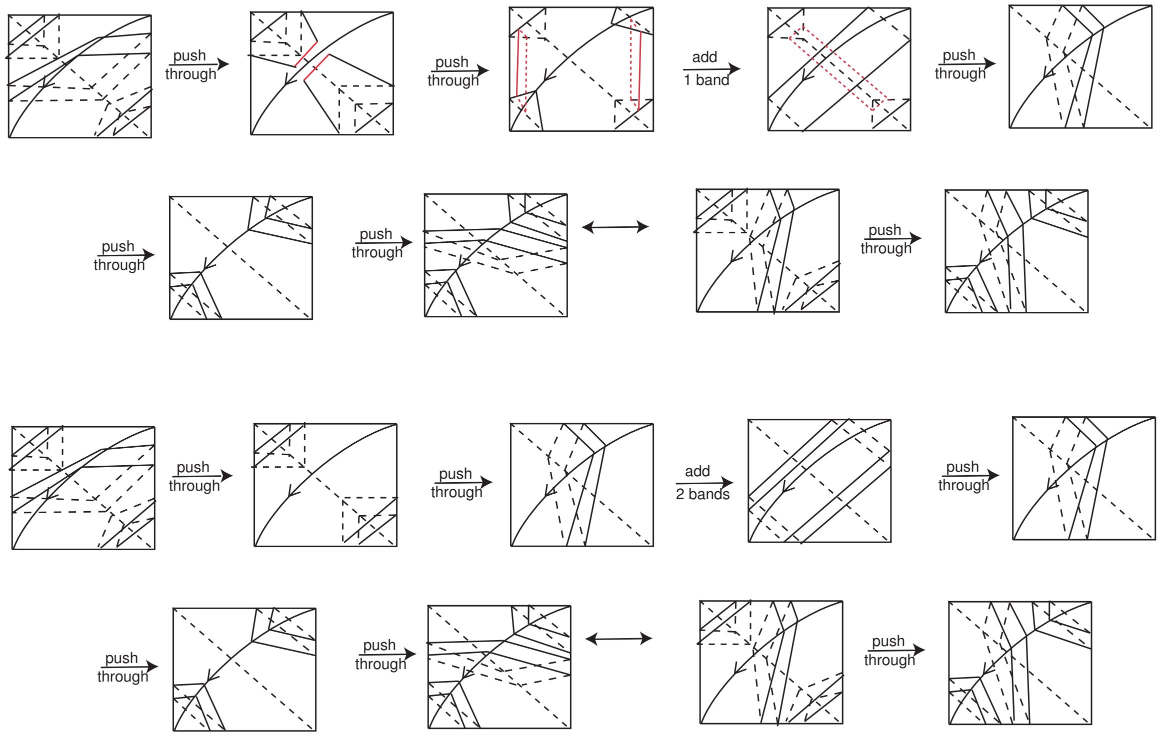


Figure 6.21: The barrier normal surface in $C_{5,3}$.

6.2.2 Almost normal tubed Heegaard splitting surfaces

Now we will consider the almost normal tubed surfaces in the layered chain pair triangulation of $M_{r,s}$. We will give the classification of the irreducible Heegaard splitting.

According to the theorem 4.2, there are three types of orientable normal surfaces in $C_{r,s}$.

1. a vertex-linking S^2 (possibly) with thin edge-linking tubes, denoted by type I surface.

2. a surface of genus n, $n \geq 2$, which has non thin edge-linking tubes, denoted by type II normal surfaces.

   They have edge-weights matching equations either $2\times(1,1,2,1); 2\times(1,1,1,2) \leftrightarrow 2\times(1,1,2,1); 2\times(1,1,1,2)$ or $2\times(1,1,3,2); 2\times(1,1,2,3) \leftrightarrow 2\times(2,2,3,1); 2\times(2,2,1,3)$.

3. a nonseparating torus, which only exist in $C_{3,3}$ and $C_{5,2} = C_{2,5}$. They have edge-weights matching equation $2 \times (1,1,2,1); 2 \times (1,1,1,2) \leftrightarrow 2 \times (1,1,2,1); 2 \times (1,1,1,2)$, $2 \times (1,1,3,2); 2 \times (1,1,2,3) \leftrightarrow 2 \times (2,2,3,1); 2 \times (2,2,1,3)$, respectively.

Now let's consider what kind of surfaces we will get if we add an almost normal tube to a connected orientable normal surface.

Theorem 6.8 *If we add an almost normal tube along an edge to a nonseparating torus in $C_{3,3}$ and $C_{2,5}$, we will get a nonorientable surface.*

Proof. The only way to add the almost normal tube to the nonseparating torus in $C_{3,3}$, up to isotopy, are shown in figure 6.24. It is isotpoic to a genus 2 nonorientable surface, with edge-weight matching equation $(1,1,0,1); (1,1,1,0) \leftrightarrow (1,1,0,1); (1,1,1,0)$.

The same method can also applied to the nonseparating torus in $C_{2,5}$. It is isotopic to a genus 2 nonorientable surface, with edge-weight matching equation $(1,1,1,0); (1,1,0,1) \leftrightarrow (0,0,1,1); (0,0,1,1)$.

See figure 6.25.

■

According to the theorem 4.2, every normal surface with genus 1 is either vertex-linking 2-sphere with one thin edge-linking tube in any $C_{r,s}$, $r, s \geq 2$ or a nonseparating torus in $C_{3,3}$ or $C_{5,2}$. In the theorem 6.8, we just showed that if we add an almost normal tube to a nonseparating torus in $C_{3,3}$ or $C_{5,2}$, we will get a nonorientable surface.If we attached the almost normal tube to other orientable normal surface from type II, which has a non thin-edge linking tube, then it is a surface with genus at least 3. It is also not hard to show that these surfaces are also Heegaard splitting surfaces.

Let S be a vertex-linking 2-sphere with a thin edge-linking tube. If the almost normal tube is attached along an edge which doesn't intersect with any quads of

S, it will normally isotopic to a vertex-linking 2-sphere with two thin edge-linking tubes. From the discussion of last section, we know some of this type of surfaces are Heegaard splitting surfaces and some of them are not. From now on we only consider the surface which is a vertex-linking 2-sphere with 2 tubes such that the almost normal tube is at the same level of the thin edge-linking tube.

Theorem 6.9 *Let S be an orientable genus 2 almost normal tubed surface with an almost normal tube at the same level of a thin edge-linking tube in $M_{r,s}$, $r, s \geq 2$, then it is a Heegaard splitting surface. Furthermore, they are isotopic to each other and are all vertical Heegaard splitting surfaces.*

Proof. Since S is an orientable genus 2 almost normal tubed surface with an almost normal tube at the same level of an thin edge-linking tube. For convenience, we use S_i to denote this type of surface and with one thin edge-lining tube around the edge i in $C_{r,s}$. The complete list of all these S_i is $S_{f_1} = S_{f_{r+2}} = S_{e_1} = S_{e_{s+2}}$, $S_\tau = S_{e_2}$, $S_\beta = S_{e_{s+1}}$, $S_t = S_{f_2}$, $S_b = S_{f_{k+1}}$ and S_e, e is any other edge in $C_{r,s}$. Now we want to show these surfaces are the same, up to isotopy.

For surface S_e, if the edge e is in C_r, and e is any edge except for $\tau = -e_2$, $\beta = -e_{r+1}$ and $f_1 = -f_{r+2} = -e_1 = e_{s+2}$. Now we will show S_e is isotopy to $S_\tau = S_{e_2}$. This is indicated in figure 6.26. S_e is on the top of the figure and S_τ is on the bottom of the figure. These two surfaces looks exactly the same after isotopy, when they were pushed away from the vertex.

Let S_β be a genus 2 almost normal tubed surface with the thin edge-linking tube around the edge β. Since we can isotopy the almost normal tube on S_e to along the edge β. See figure 6.27. S_e is the surface on the top of the figure and can be isotopy to S_β on the bottom of the figure.

Therefore, $S_\tau = S_e = S_\beta$, up to isotopic.

For S_{f_1}, an orientable genus 2 almost normal tubed surface, it is obtained by adding an almost normal tube to a vertex-linking S^2 with one thin edge-linking tube around $f_1 = -f_{k+2} = e_1 = -e_{k+2}$. See figure 6.28. S_{f_1} is the surface on the top of the figure and it can be isotopy to S_τ on the bottom of the figure.

Hence, we have $S_{f_1} = S_\tau = S_{e_2}$, up to the isotopic.

Therefore, we proved $S_\tau = S_\beta$=S_e, for e is every edge in C_r except for $f_2 = -t$ and $f_{r+1} = b$.

Since $f_1 = -f_{r+2} = e_1 = -e_{s+2}$, so S_{e_1} and S_{f_1} is the same surface. Since $C_{r,s} = C_{s,r}$, up to isomorphisim, if we switch the role of r and s, we can show that S_t=S_b=S_e, up to isotopy, where e is every edge in C_s, except for $e_2 = -\tau$ and $e_{s+1} = -\beta$, by using the same figure 6.26, 6.27 and 6.28.

All in all, S_τ=S_β=$_t$=S_b=S_e, e is any other edge in $C_{r,s}$, up to isotopy.

Therefore, all the genus two almost normal tubed surface are isotopy.

Furthermore, we notice they are all vertical Heegaard splitting surfaces, Because they can all be viewed as a boundary surface of a neighborhood of an exceptional fiber with an arc attach to it, which will be projected to be a loop based on the project point of this exceptional fibers. Moreover, S_{f_1} is a verical Heegaard splitting surface with respective to the exceptional fiber with multiplicity 2, S_τ is a verical Heegaard splitting surface with respective to the exceptional fiber with multiplicity $r + 1$, and S_t is a verical Heegaard splitting surface with respective to the exceptional fiber with multiplicity s+1. Notice these are all possible vertical Heegaard splittings in $M_{r,s}$, and they are isotopic to each other, therefore, there is unique vertical Heegaard splitting surface, up to isotopy, in the layered chain triangulation. ■

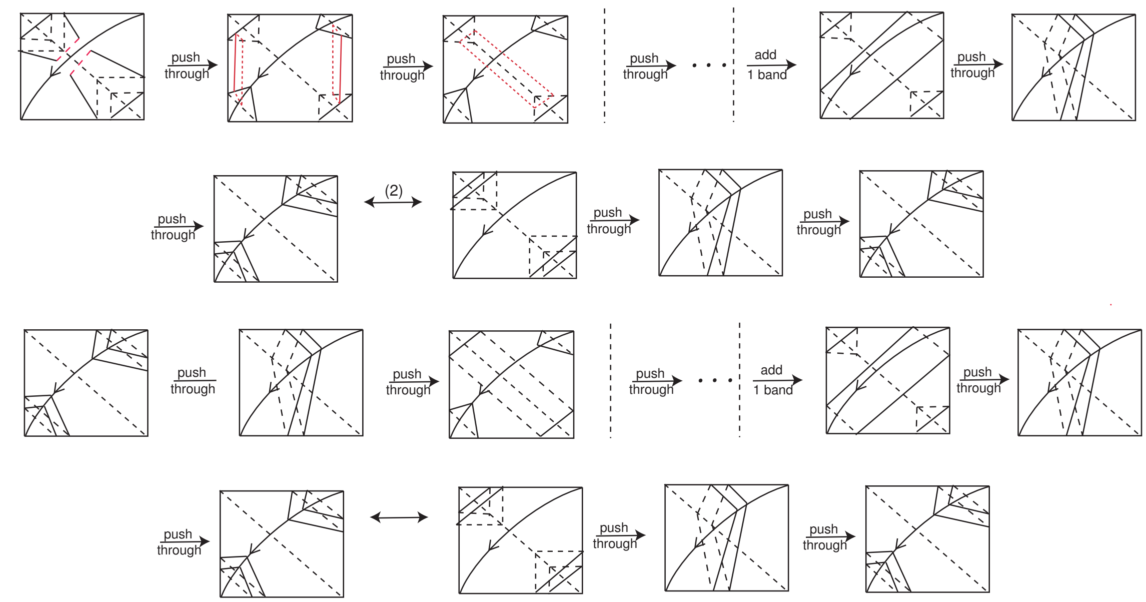
Theorem 6.10 *(Isotopy theorem) In a layered chain pair triangulation of* $M_{r,s}$, $r, s \geq 1$, *there exists a unique vertical Heegaard splitting, up to isotopy.*

Proof. Since in any layered chain triangulation of $C_{r,s}$, the genus 2 almost tubed surfaces with an almost normal tube at the same level of an thin edge-linking tube are Heegaard splitting surfaces. Furthermore, we know all these almost normal tubed surfaces are not only vertical Heegaard splittings, but also includes all three possible vertical Heegaard splitting surfaces, up to isotopy. We proved that they are isotopic to each other for $s, r \geq 2$ according to theorem 6.9. Hence, there is a unique vertical Heegaard splitting surface, up to isotopy. If $s = 1$ or $r = 1$, by theorem 6.5, there is only one vertical Heegaard splitting up to isotopy. Therefore, in a layered chain pair triangulation of Mr, s, $r, s \geq 1$, there exists a unqiue vertical Heegaard splitting, up to isotopy. ∎

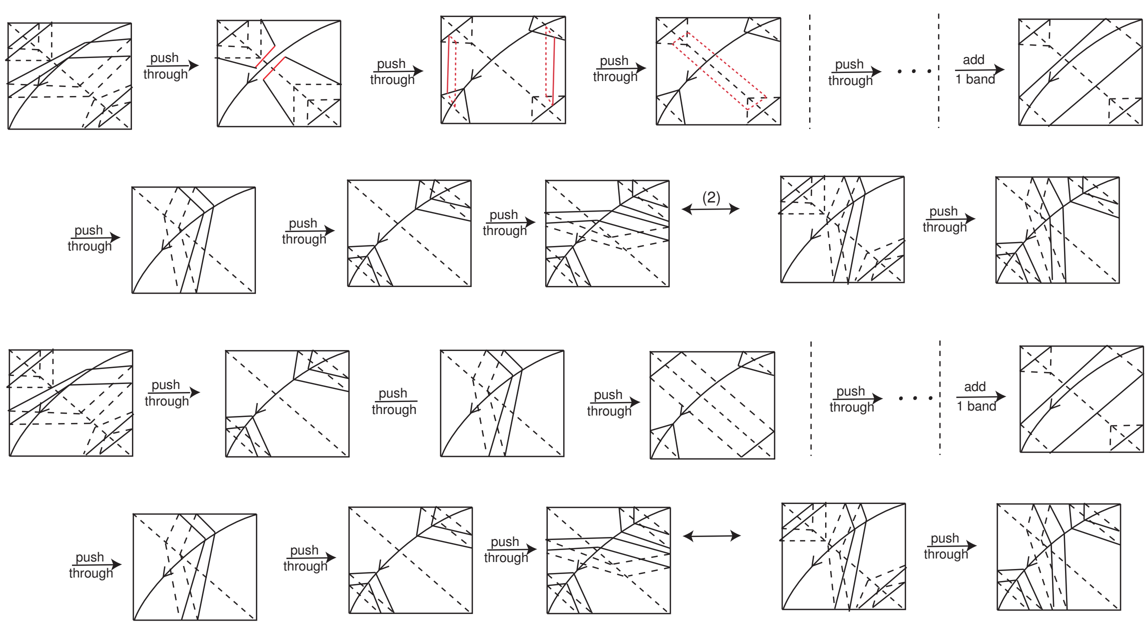
Corollary 6.2 *There is only one genus 2 irreducible Heegaard splitting surface in* $M_{2,7}$ *and* $M_{3,5}$*, up to isotopy.*

Proof. Since we know that in small Seifert fiber spaces $M_{2,7}$ and $M_{3,5}$, every irreducible Heegaard splitting surfaces are strongly irreducible. Furthermore, every strongly irreducible Heegaard splitting surface up to isotopy will be normally isotopy to an almost normal surface. In these two manifolds, we proved in Theorem 6.7 that genus 2 octagonal surfaces are not Heegaard splitting surfaces. Therefore, all the Heegaard splitting surfaces of genus 2 can only be almost normal tubed surfaces. By Theorem 6.10, there is only one genus 2 irreducible Heegaard splitting surface up to isotopy. ∎

Here, we finish our discussion of genus 2 irreducible Heegaard splitting surfaces in $M_{r,s}$, $r, s \geq 1$. There is some questions that we are still working on. For example, we hope to give a proof that all the genus two octagonal almost normal surfaces are not Heegaard splitting surface in $M_{r,s}$, except for $M_{3,4}$ and $M_{2,6}$. Two octagonal Heegaard splitting surfaces are very likely to be horizontal Heegaard splitting surfaces in $M_{3,4}$ and $M_{2,6}$, respectively. This may leads to a complete classification of genus two Heegaard splitting surfaces, up to isotopy.



isotopy in $C_{n,3}$, $n \geq 6$



isotopy in $C_{n,2}$, $n \geq 8$

Figure 6.22: The barrier normal surface in $C_{n,3}$.

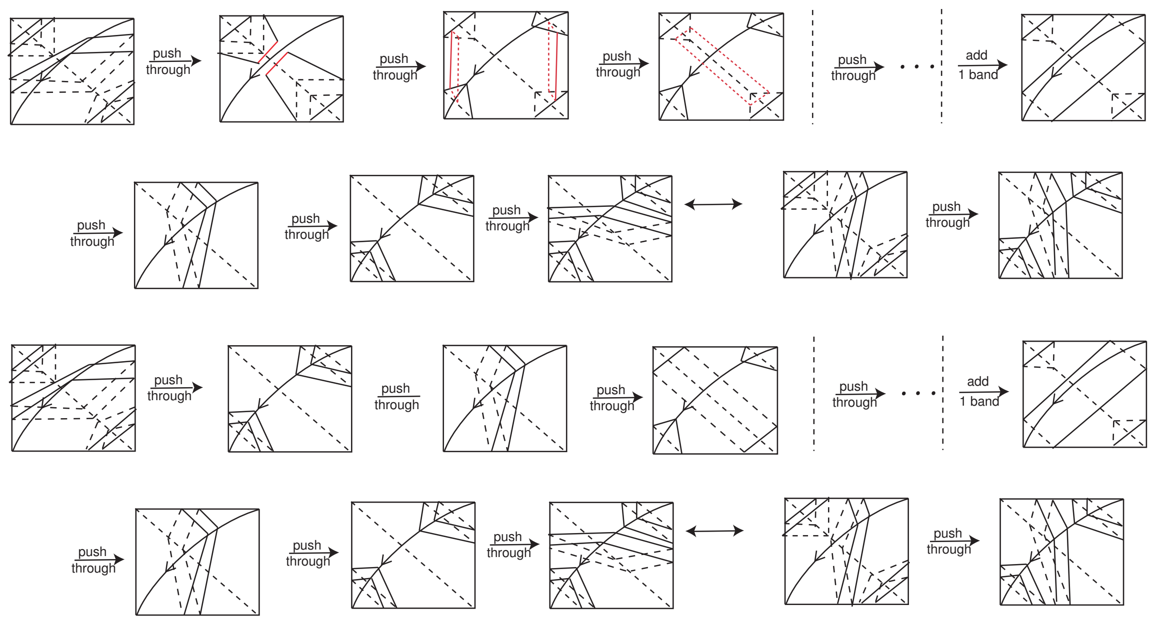


Figure 6.23: The barrier normal surface in $C_{n,2}$

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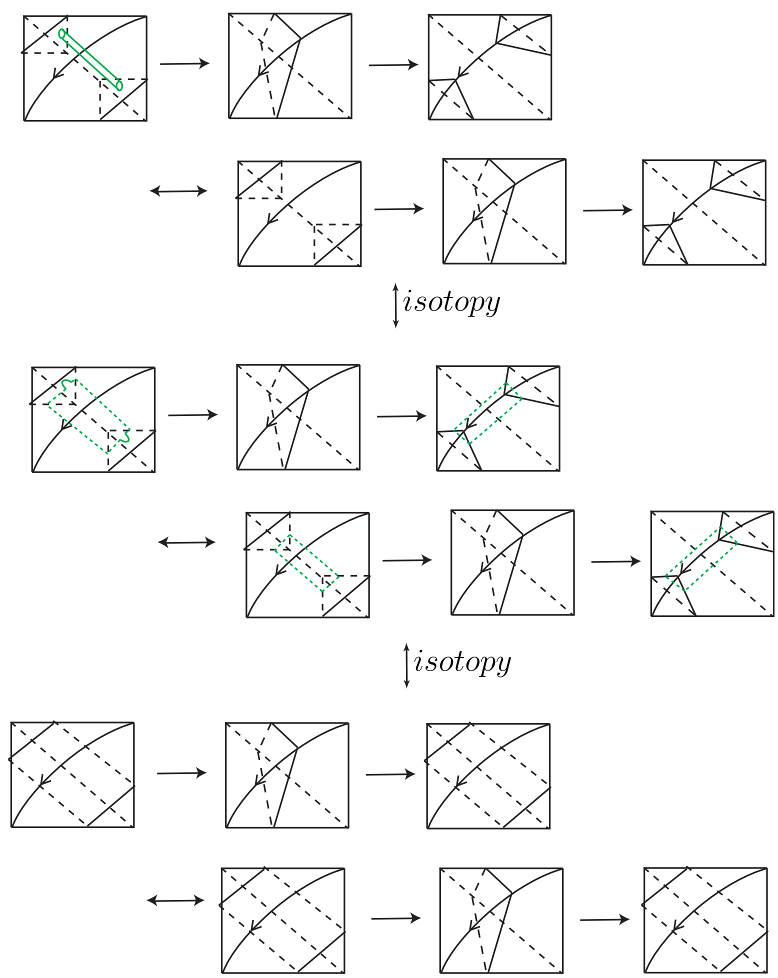

Figure 6.24:

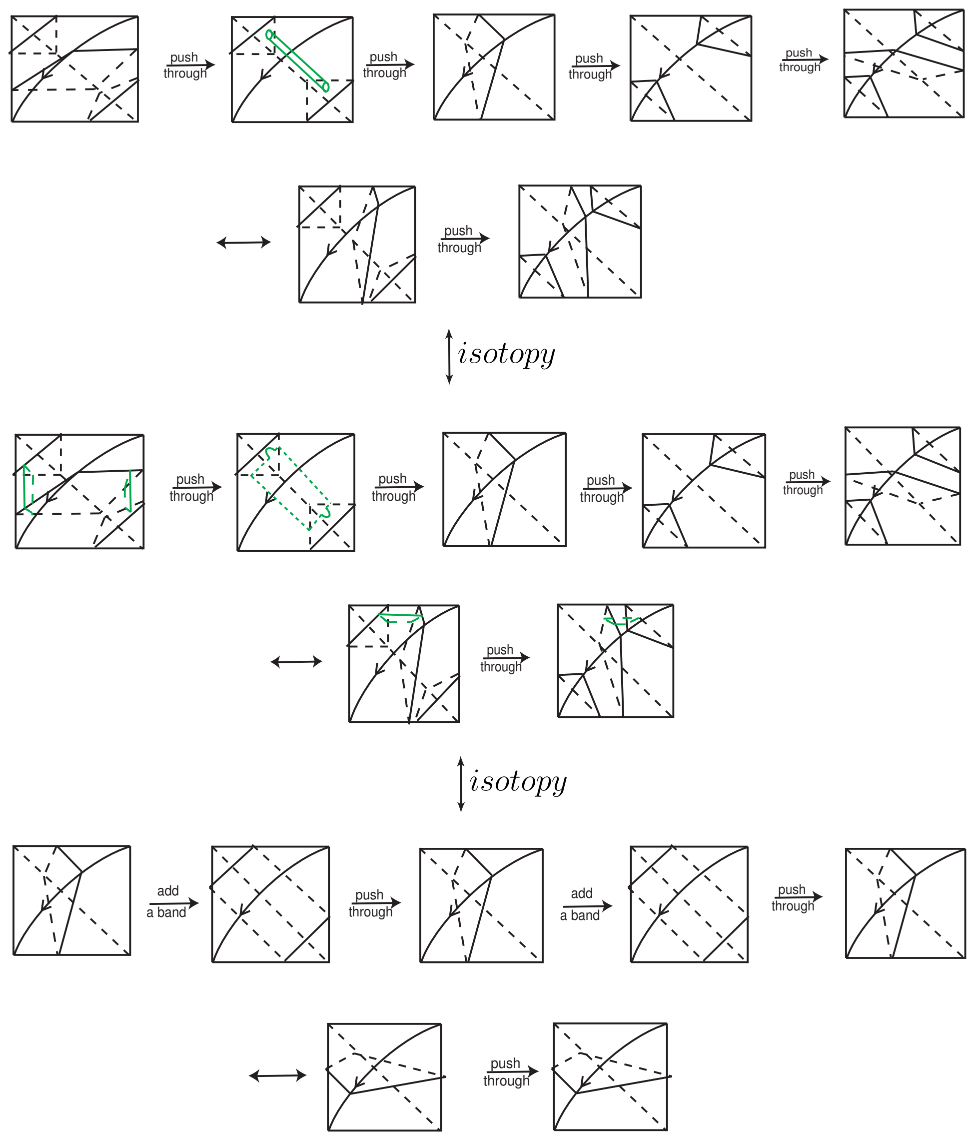

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Figure 6.25:

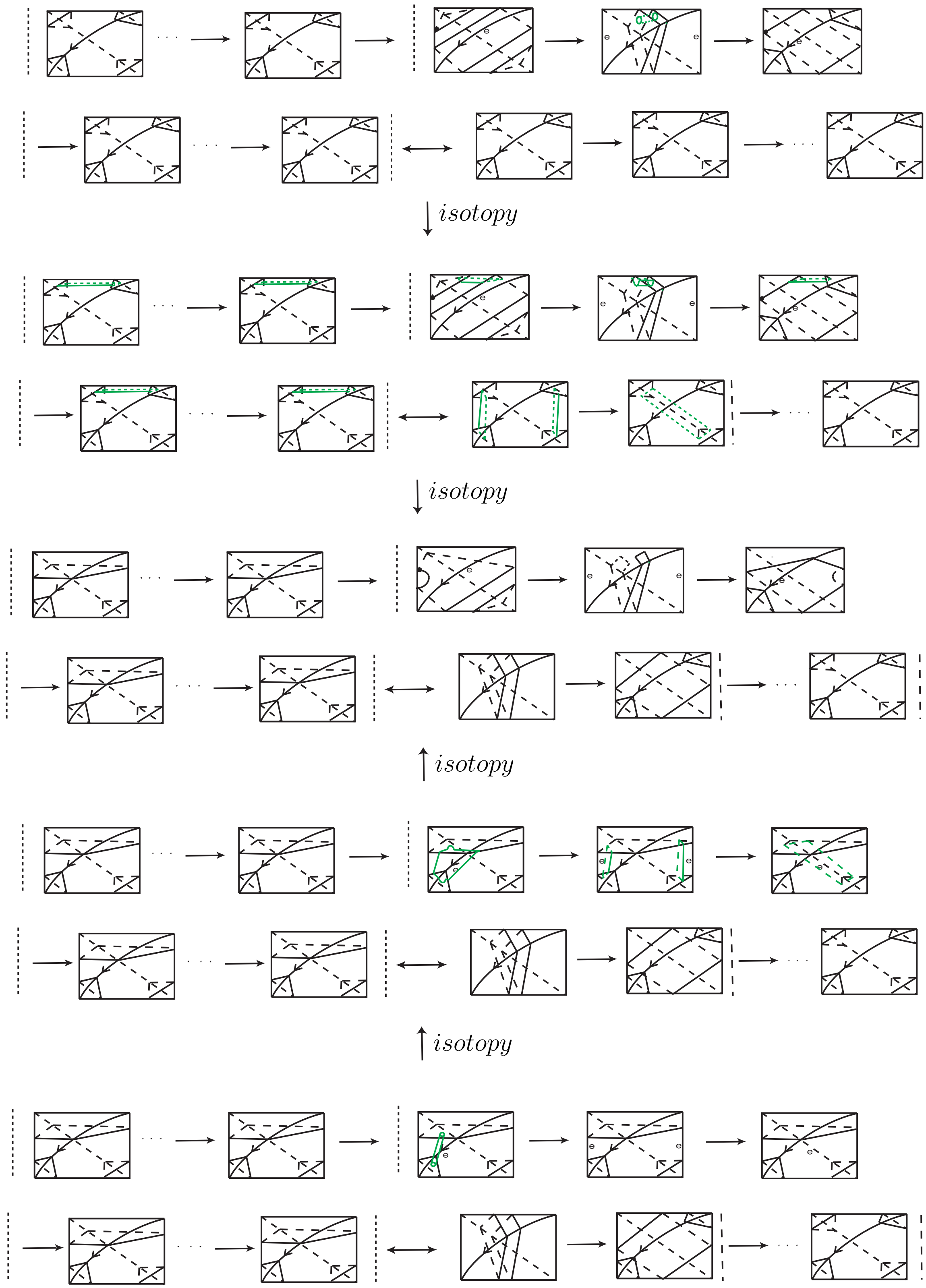

Figure 6.26:

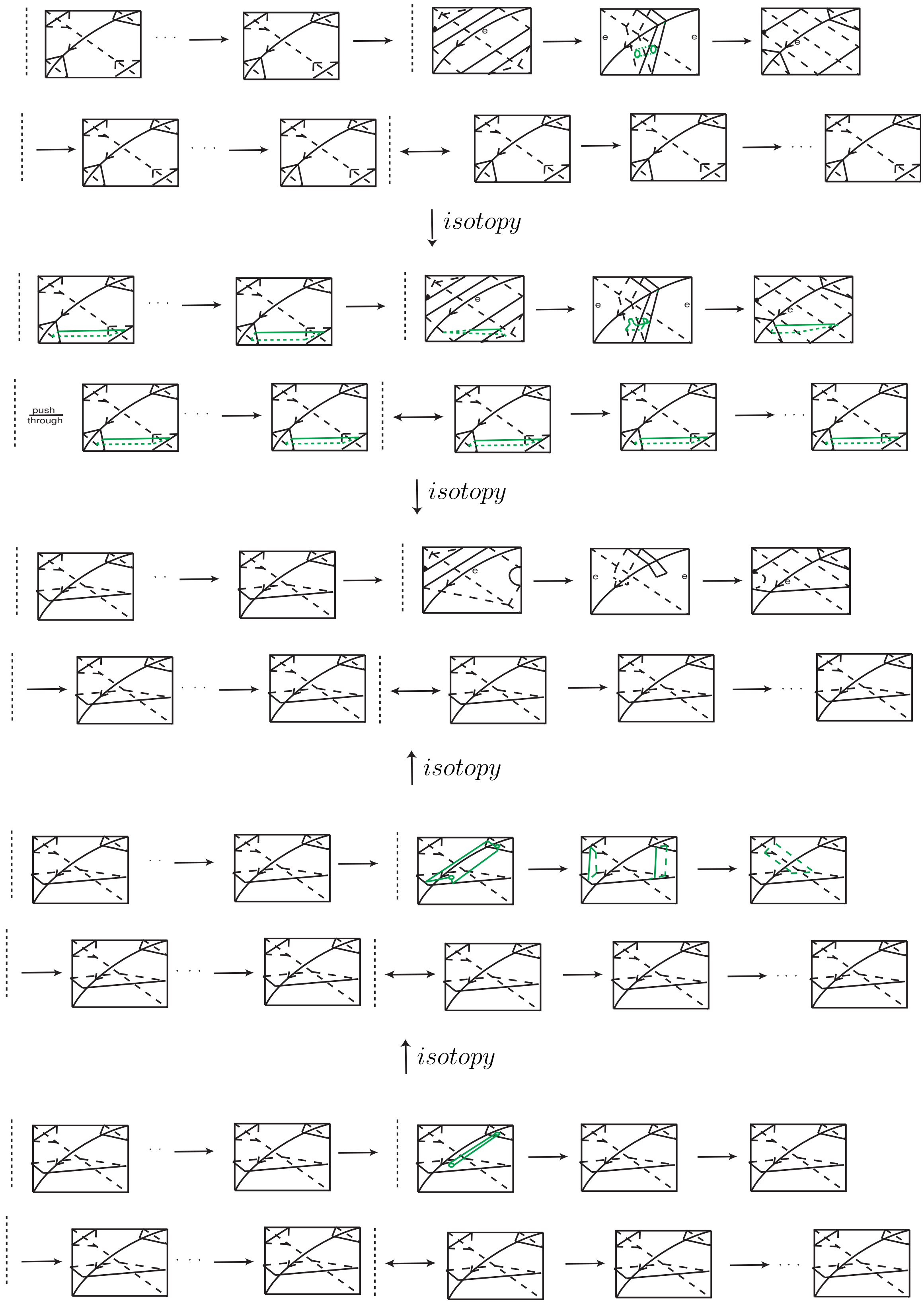

Figure 6.27:

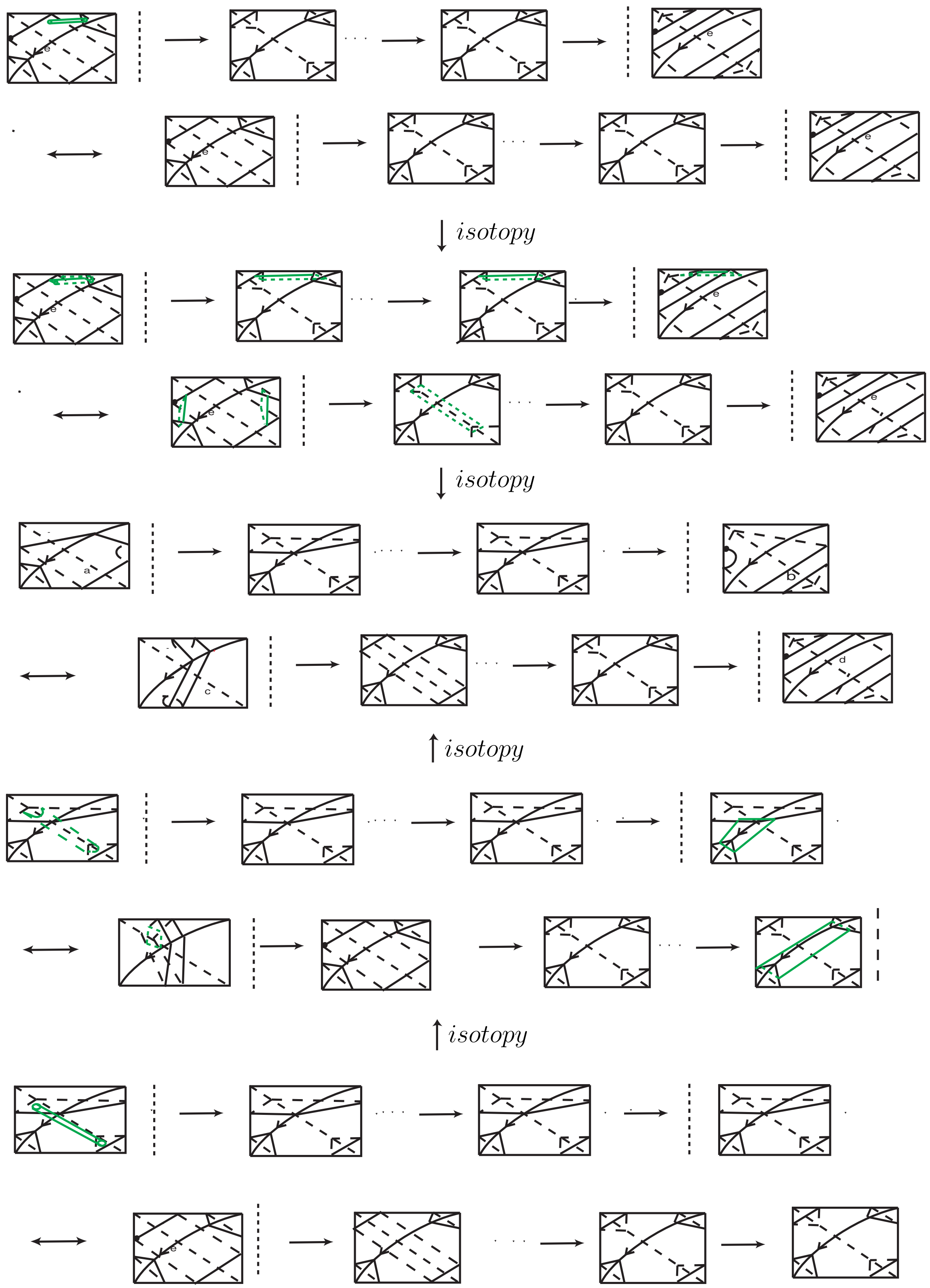

Figure 6.28:

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Recently William Jaco, J. Hyam Rubinstein and Stephan Tillmann together proved that the generalized quaternion spaces S^3/Q_{4k}, $k \geq 2$, which are small Seifert fibered spaces M_k= $(S^2 : (2,1),(2,1),(k,-k+1))$, have complexity k, which is the minimal number of tetrahedra in a triangulation of M_k. The techniques used can be expanded to show that the layered chain pair triangulations of Seifert fibered spaces $(S^2 : (2,-1),(r+1,1),(s+1,1))$, $r, s \geq 1$ are minimal.

My thesis is to closely study the minimal, 0-efficient triangulations of the above two infinite families of Seifert fiberd spaces. One family is called the twisted layered loop triangulation, and the other family is called layered chain pair triangulations. They were named by Ben Burton. We classify all normal and almost normal surfaces by identifying one-sided incompressible surfaces, orientable incompressible surfaces and Heegaard splitting surfaces. We also use combinatorial methods to study and classify irreducible Heegaard splitting surfaces, up to isotopy, in these two infinite families of Seifert fibered manifolds.

In the twisted layered loop triangulations of the Seifert fibered space M_k, $k \geq 2$. We prove that a properly embedded surface S is a Heegaard splitting surface if and only if it is an almost normal tubed surface with the almost normal tube at the same level of a thin edge-linking tube. Furthermore, any genus two Heegaard splitting surface is vertical. A combinatorial proof is given that there is a unique irreducible genus 2 Heegaard splitting surface, up to isotopy, in M_k, $k \geq 2$.

In the layered chain pair triangulation of Seifert fibered spaces $M_{r,s}$=$(S^2 : (2,-1),(r+1,1),(s+1,1))$, $r, s \geq 1$, we prove that an almost normal tubed surface with the almost normal tube at the same level of a thin edge-linking tube is a Heegaard splitting surface. Moreover, if the genus of it is 2, then it is not only an irreducible Heegaard splitting but also a vertical one. We give a combinatorial proof that there is a unique irreducible vertical Heegaard splitting surface, up to isotopy, in $M_{r,s}$, $r, s \geq 1$.

Our work follows the methods used by Jaco and Rubinstein in studying layered-triangulations of the solid torus and their classification of normal surfaces in these triangulations.

ADVISOR'S APPROVAL: ______________________________