CONFORMALLY INVARIANT SYSTEMS OF DIFFERENTIAL OPERATORS ASSOCIATED TO TWO-STEP NILPOTENT MAXIMAL PARABOLICS OF NON-HEISENBERG TYPE

By

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CONFORMALLY INVARIANT SYSTEMS OF DIFFERENTIAL OPERATORS ASSOCIATED TO TWO-STEP NILPOTENT MAXIMAL PARABOLICS OF NON-HEISENBERG TYPE

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CHAPTER 1

Introduction

The main work of this thesis concerns systems of differential operators that are equivariant under an action of a Lie algebra. We call such systems *conformally invariant.* To explain the meaning of the equivariance condition, suppose that $\mathcal{V} \to M$ is a vector bundle over a smooth manifold M and $\mathfrak{g}$ is a Lie algebra of first-order differential operators that act on sections of $\mathcal{V}$. A linearly independent list $D_1, \ldots, D_n$ of linear differential operators on sections of $\mathcal{V}$ is called a *conformally invariant system* if, for each $X \in \mathfrak{g}$, there are smooth functions $C_{ij}^X(m)$ on M so that, for all $1 \leq i \leq n$, and sections f of $\mathcal{V}$, we have

$$\big([X, D_i]\bullet f\big)(m) = \sum_{j=1}^{n} C_{ji}^X(m)(D_j\bullet f)(m), \tag{1.0.1}$$

where $[X, D_j] = XD_j - D_jX$, and the dot $\bullet$ denotes the action of differential operators on smooth functions. (See Definition 2.1.4 for the precise definition.)

A typical example for a conformally invariant system of one differential operator is the wave operator $\square = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} - \frac{\partial^2}{\partial x_4^2}$ on the Minkowski space $\mathbb{R}^{3,1}$. If X is an element of $\mathfrak{g} = \mathfrak{so}(4,2)$ acting as a first-order differential operators on sections of an appropriate line bundle over $\mathbb{R}^{3,1}$ then there is a smooth function C^X on $\mathbb{R}^{3,1}$ so that

$$[X, \square] = C^X \square.$$

An important consequence of the definition (1.0.1) is that the common kernel of the operators in the conformally invariant system $D_1, \ldots, D_n$ is invariant under a Lie algebra action. The representation theoretic question of understanding the common kernel as a $\mathfrak{g}$-module is an open question (except for a small number of very special examples).

The notion of conformally invariant systems generalizes that of quasi-invariant differential operators introduced by Kostant in [19] and is related to a work of Huang ([8]). It is also compatible with the definition given by Ehrenpreis in [6]. Conformally invariant systems are explicitly or implicitly presented in the work of Davidson-Enright-Stanke ([5]), Kable ([12], [13]), Kobayashi-Ørsted ([16], [17], [18]), Wallach ([25]), among others. Much of the published work is for the case that $M = G/Q$ with $Q = LN$, N abelian. The systematic study of conformally invariant systems started with the work of Barchini-Kable-Zierau in [1] and [2]

Although the theory of conformally invariant systems can be viewed as a geometric-analytic theory, it is closely related to algebraic objects such as generalized Verma modules. It has been shown in [2] that a conformally invariant system yields a homomorphism between certain generalized Verma modules. The classification of non-standard homomorphisms between generalized Verma modules is an open problem.

The main goal of this thesis is to build systems of differential operators that satisfy the condition (1.0.1), when M is a homogeneous manifold G/Q with Q a maximal two-step nilpotent parabolic subgroup. This is to construct systems $D_1, \ldots, D_n$ acting on sections of bundles $\mathcal{V}_s \to G/Q$ over G/Q in a systematic manner and to determine the bundles $\mathcal{V}_s$ on which the systems are conformally invariant. The method that we use is different from one used by Barchini-Kable-Zierau in [1]. The systems that we build yield explicit homomorphisms between appropriate generalized Verma modules. We show that the most of those homomorphisms are non-standard.

To describe our work more precisely, let G be a complex, simple, connected, simply-connected Lie group with Lie algebra $\mathfrak{g}$. It is known that $\mathfrak{g}$ has a $\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{j=-r}^{r} \mathfrak{g}(j)$ so that $\mathfrak{q} = \mathfrak{g}(0) \oplus \bigoplus_{j>0} \mathfrak{g}(j) = \mathfrak{l} \oplus \mathfrak{n}$ is a parabolic subalgebra of $\mathfrak{g}$. Let $Q = N_G(\mathfrak{q}) = LN$. For a real form $\mathfrak{g}_0$ of $\mathfrak{g}$, define G_0 to be an analytic subgroup of G with Lie algebra $\mathfrak{g}$. Set $Q_0 = N_{G_0}(\mathfrak{q})$. Our manifold is $M = G_0/Q_0$ and we consider a line bundle $\mathcal{L}_{-s} \to G_0/Q_0$ for each $s \in \mathbb{C}$.

It is known, by the Bruhat theory, that G_0/Q_0 admits an open dense submanifold $\bar{N}_0 Q_0/Q_0$. We restrict our bundle to this submanifold. The systems that we study act on sections of the restricted bundle.

To build systems of differential operators we observe that L acts by the adjoint representation on $\mathfrak{g}(1)$ with a unique open orbit. This makes $\mathfrak{g}(1)$ a prehomogeneous vector space. Our construction is based on the invariant theory of a prehomogeneous vector space. It is natural to associate L-equivariant polynomial maps called *covariant maps* to the prehomogeneous vector space $(L, \mathrm{Ad}, \mathfrak{g}(1))$. To define our systems of differential operators, we use covariant maps that are associated to $\mathfrak{g}(1)$. We denote the covariant maps by τ_k. Each τ_k can be thought of as giving the symbols of the differential operators that we study. For $0 \leq k \leq 2r$, the maps τ_k are defined by

$$\begin{aligned} \tau_k : \mathfrak{g}(1) &\to \mathfrak{g}(-r+k) \otimes \mathfrak{g}(r) \\ X &\mapsto \frac{1}{k!}\mathrm{ad}(X)^k \omega_0, \end{aligned} \tag{1.0.2}$$

where ω_0 is a certain element in $\mathfrak{g}(-r+k) \otimes \mathfrak{g}(r)$. (See Definition 2.5.1.)

Let

$$\mathfrak{g}(-r+k) \otimes \mathfrak{g}(r) = V_1 \oplus \cdots \oplus V_m \tag{1.0.3}$$

be the irreducible decomposition of $\mathfrak{g}(-r+k) \otimes \mathfrak{g}(r)$ as an L-module. Covariant map τ_k induces an L-equivariant linear map $\tilde{\tau}_k|_{V_j^*} : V_j^* \to \mathcal{P}^k(\mathfrak{g}(1)))$ with V_j^* the dual of an irreducible constituent V_j of $\mathfrak{g}(-r+k) \otimes \mathfrak{g}(r)$ and $\mathcal{P}^k(\mathfrak{g}(1))$ the space of polynomials on $\mathfrak{g}(1)$ of degree k. We define differential operators from $\tilde{\tau}_k|_{V_j^*}(Y^*)$. For $Y^* \in V_j^*$, let $\Omega_k(Y^*)$ denote the k-th order differential operators that are constructed from $\tilde{\tau}_k|_{V_j^*}(Y^*)$.

We say that a list of differential operators $D_1, \ldots, D_n$ is the $\Omega_k|_{V_j^*}$ *system* if it is equivalent (in the sense of Definition 2.1.5) to a list of differential operators

$$\Omega_k(Y_1^*), \ldots, \Omega_k(Y_n^*), \tag{1.0.4}$$

where $\{Y_1^*, \ldots, Y_n^*\}$ is a basis for V_j^* over $\mathbb{C}$. By construction the $\Omega_k|_{V_j^*}$ system consists of $\dim_{\mathbb{C}}(V_j)$ operators.

It is not necessary for the $\Omega_k|_{V_j^*}$ system to be conformally invariant; the conformal invariance of the operators (1.0.4) strongly depends on the complex parameter s for the line bundle $\mathcal{L}_{-s}$. Then we say that the $\Omega_k|_{V_j^*}$ system has *special value* s_0 if the system is conformally invariant on the line bundle $\mathcal{L}_{s_0}$. The special values for the case that $\dim([\mathfrak{n}, \mathfrak{n}]) = 1$ for $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{n}$ are studied by Barchini-Kable-Zierau in [1] and [2], and myself in [20].

In this thesis we consider a more general case; namely, $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{n}$ is a maximal parabolic subalgebra and $\mathfrak{n}$ satisfies the condition that $[\mathfrak{n}, [\mathfrak{n}, \mathfrak{n}]] = 0$ and $\dim_{\mathbb{C}}([\mathfrak{n}, \mathfrak{n}]) > 1$. We call such parabolic subalgebras maximal two-step nilpotent parabolic subalgebras of non-Heisenberg type. In this case we have $r = 2$ in (2.5.6). Therefore the Ω_k systems for $k \geq 5$ are zero. The main results of this thesis are Theorem 4.2.5 and Theorem 7.3.6, where the special values of the Ω_1 system and Ω_2 systems for the parabolic subalgebras are determined. We also classify the non-standard homomorphisms between the generalized Verma modules that arise from our systems of differential operators.

We may want to remark that, although the special value of s for the Ω_1 system is easily found by computing the bracket $[X, \Omega_1(Y_i^*)]$, it is in general not easy to find the special values for the Ω_2 systems by a direct computation. (See Section 5 of [1].) In this thesis, to find the special value for the $\Omega_2|_{V_j^*}$ system, we use two reduction techniques to compute the special values. First, in order to show the equivariance condition (1.0.1) for $D_i = \Omega_2(Y_i^*)$ with $Y_i^* \in V_j^*$, it is enough to compute $[X, \Omega_2(Y_i^*)]$ at the identity e. Furthermore, we show that it is even sufficient to compute only $[X_h, \Omega_2(Y_l^*)]$ at e, where X_h and Y_l^* are a highest weight vector of $\mathfrak{g}(1) \subset \mathfrak{g}$ and a lowest weight vector of V_j^*, respectively. These two techniques significantly reduce the amount of computations.

We now outline the contents of this thesis. In Chapter 2 we study conformally invariant systems of differential operators. We recapitulate Section 2 of [2] in Section 2.1. In Sections 2.2 and 2.3 we specialize the theory of conformally invariant systems to the situation that we are interested in. Two useful formulas on differential operators will be shown in Section 2.4. In Section 2.5, the general construction of the Ω_k systems is given. Section 2.6 discusses two technical lemmas on the Ω_k systems, and in Section 2.7, we describe a relationship between the Ω_k systems and generalized Verma modules.

The aim of Chapter 3 is to study the $\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{j=-r}^{r} \mathfrak{g}(j)$ on $\mathfrak{g}$ and a maximal two-step nilpotent parabolic subalgebra $\mathfrak{q}$ of non-Heisenberg type. We begin this chapter by classifying the k-step nilpotent parabolic subalgebras in Section 3.1. In Section 3.2 and Section 3.3, we study a maximal two-step nilpotent parabolic subalgebra $\mathfrak{q}$ of non-Heisenberg type and the associated 2-grading $\mathfrak{g} = \bigoplus_{j=-2}^{2} \mathfrak{g}(j) = \mathfrak{z}(\bar{\mathfrak{n}}) \oplus \mathfrak{g}(-1) \oplus \mathfrak{l} \oplus \mathfrak{g}(1) \oplus \mathfrak{z}(\mathfrak{n})$ of $\mathfrak{g}$.

In Chapter 4, we construct the Ω_1 system and find the special value of the system. In Section 4.1, we fix normalizations for root vectors. The normalizations play an important role to construct the system. In Section 4.2 we show that the special value s_1 of s for the Ω_1 system is $s_1 = 0$. This is done in Theorem 4.2.5.

To build the Ω_2 systems, we need to find the irreducible constituents V^* of $\mathfrak{l}^* \otimes \mathfrak{z}(\mathfrak{n})^*$ so that $\tilde{\tau}_2|_{V^*} \neq 0$. In Chapters 5 and 6, we show preliminary results to find such irreducible constituents. In Chapter 5 we decompose $\mathfrak{l} \otimes \mathfrak{z}(\mathfrak{n})$ into the direct sum of the irreducible constituents. We first summarize our main decomposition results, Theorem 5.1.3, in Section 5.1. Section 5.2 contains preliminary results and technical lemmas that are used to prove the theorem. The proof for Theorem 5.1.3 is given in Section 5.3. In Chapter 6, by using the decomposition results, we determine the candidates of the irreducible constituents V^* so that $\tilde{\tau}_2|_{V^*} \neq 0$. We call such constituents *special*. In Section 6.1 we define the special constituents. We then classify such constituents in Section 6.2. In Section 6.3 we collect the technical results on the special constituents, which are used to find the special values for the Ω_2 systems.

In Chapter 7, we build the Ω_2 systems and find their special values. First, it is shown in Section 7.1 that the covariant maps τ_2 and the induced linear maps $\tilde{\tau}_2|_{V^*}$ for certain special constituents V^* are non-zero. We then construct the Ω_2 systems in Section 7.2, and in Section 7.3, we find their special values. This is done in Theorem 7.3.6.

In Chapter 8, we determine whether or not the homomorphisms φ_{Ω_k} that are induced by the Ω_k systems between appropriate generalized Verma modules are standard for $k = 1, 2$. In Section 8.1 we review the well-known results on the standard map between generalized Verma modules. Technical results to determine the standardness of the maps φ_{Ω_k} are also shown in this section. We then determine the standardness of φ_{Ω_1} and φ_{Ω_2} in Section 8.2 and Section 8.3, respectively.

In this thesis we also have the appendices. In Appendix A, as an Ω_k system that is conformally invariant on the line bundle $\mathcal{L}_{s_0}$ induces the reducibility of a scalar generalized Verma module $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{q})} \mathbb{C}_{-s_0}$, to support the results for the special values for the Ω_2 systems, we show the reducibility points for the scalar generalized Verma modules for $\mathfrak{g}$ exceptional algebras. To determine the reducibility we use a criterion due to Jantzen. (See Section A.2.)

In Appendices B, C, and D, we collect miscellaneous useful data. Namely, Appendix B contains the Dynkin diagrams with the multiplicities of the simple roots in the highest root of $\mathfrak{g}$ and extended Dynkin diagrams. Appendix C summarizes the useful data for the parabolic subalgebras under consideration such as the roots for $\mathfrak{l}$, $\mathfrak{g}(1)$, and $\mathfrak{z}(\mathfrak{n})$. In Appendix D we include the lists of the positive roots for the exceptional algebras.

Finally, I would like to thank my advisor, Dr. Leticia Barchini, for introducing this topic for me and for her generous help. I would also like to thank Dr. Anthony Kable and Dr. Roger Zierau for their valuable comments on this work.

CHAPTER 2

Conformally Invariant Systems and the Ω_k Systems

The purpose of this chapter is to study conformally invariant systems of differential operators, that are the main objects of this thesis. In particular, we define systems of differential operators of order k, which we call the Ω_k systems.

2.1 Conformally Invariant Systems

The aim of this section is to introduce the definition of conformally invariant systems. Suppose that V and W are finite dimensional complex vector spaces and $C^\infty(\mathbb{R}^n, V)$ is the space of smooth V-valued functions on $\mathbb{R}^n$. A linear map $D : C^\infty(\mathbb{R}^n, V) \to C^\infty(\mathbb{R}^n, W)$ is called a **differential operator** if it is of the form

$$D\bullet h = \sum_{|\alpha|\leq k} T_\alpha \left(\frac{\partial^\alpha}{\partial x^\alpha} \bullet h \right) \tag{2.1.1}$$

for some $k \in \mathbb{Z}_{\geq 0}$ and all $h \in C^\infty(\mathbb{R}^n, V)$, where T_α are smooth functions from $\mathbb{R}^n$ to $\mathrm{Hom}_{\mathbb{C}}(V, W)$, and multi-index notation is being used. Here, the dot $\bullet$ denotes the action of differential operators on smooth functions.

Now let M be a smooth manifold, and let $\mathrm{pr}_V : \mathcal{V} \to M$ and $\mathrm{pr}_W : \mathcal{W} \to M$ be smooth vector bundles over M of finite rank with pr_V and pr_W the bundle projections. For each $p \in M$, there exists an open neighborhood U of p so that the local trivializations $\mathrm{pr}_V^{-1}(U) \cong U \times V$ and $\mathrm{pr}_W^{-1}(U) \cong U \times W$ hold. Then a linear map D from smooth sections of $\mathcal{V}$ to smooth sections of $\mathcal{W}$ is called a **differential operator** if in each local trivialization D is of the form of (2.1.1). The smallest integer k with $|\alpha| \leq k$ in (2.1.1), for which $T_\alpha \neq 0$, is called the **order** of D. We denote by $\mathbb{D}(\mathcal{V})$ the space of differential operators on the smooth sections of $\mathcal{V}$. Note that we regard smooth functions f on M as elements in $\mathbb{D}(\mathcal{V})$ by identifying them with the multiplication operator they induce.

Let $\mathfrak{g}_0$ be a real Lie algebra and $\mathfrak{X}(M)$ be the space of smooth vector fields on M.

Definition 2.1.2 [2, page 790] *A smooth manifold M is called a $\mathfrak{g}_0$-**manifold** if there is an $\mathbb{R}$-linear map $\pi_M : \mathfrak{g}_0 \to C^\infty(M) \oplus \mathfrak{X}(M)$ so that*

$$\pi_M([X, Y]) = [\pi_M(X), \pi_M(Y)]$$

for all $X, Y \in \mathfrak{g}_0$.

For each $X \in \mathfrak{g}_0$, we write $\pi_M(X) = \pi_0(X) + \pi_1(X)$ with $\pi_0(X) \in C^\infty(M)$ and $\pi_1(X) \in \mathfrak{X}(M)$.

Definition 2.1.3 [2, page 791] *Let M be a $\mathfrak{g}_0$-manifold. A vector bundle $\mathcal{V} \to M$ is called a $\mathfrak{g}_0$-**bundle** if there is an $\mathbb{R}$-linear map $\pi_\mathcal{V} : \mathfrak{g}_0 \to \mathbb{D}(\mathcal{V})$ that satisfies the following properties:*

(B1) *We have $\pi_\mathcal{V}([X, Y]) = [\pi_\mathcal{V}(X), \pi_\mathcal{V}(Y)]$ for all $X, Y \in \mathfrak{g}_0$.*

(B2) *In $\mathbb{D}(\mathcal{V})$, $[\pi_\mathcal{V}(X), f] = \pi_1(X) \bullet f$ for all $X \in \mathfrak{g}_0$ and $f \in C^\infty(M)$.*

Now we introduce the definition of conformally invariant systems.

Definition 2.1.4 [2, page 791] *Let $\mathcal{V} \to M$ be a $\mathfrak{g}_0$-bundle. A **conformally invariant system** on $\mathcal{V}$ with respect to $\pi_\mathcal{V}$ is a list of differential operators $D_1, \ldots, D_m \in \mathbb{D}(\mathcal{V})$ so that the following two conditions hold:*

(S1) *At each point $p \in M$, the list $D_1, \ldots, D_m$ is linearly independent over $\mathbb{C}$.*

(S2) *For each $X \in \mathfrak{g}_0$, there is a matrix $C(X)$ in $M_{m\times m}(C^\infty(M))$ so that*

$$[\pi_\mathcal{V}(X), D_i] = \sum_{j=1}^{m} C_{ji}(X) D_j$$

in $\mathbb{D}(\mathcal{V})$.

*The map $C : \mathfrak{g}_0 \to M_{m\times m}(C^\infty(M))$ is called the **structure operator** of the conformally invariant system.*

If $\mathfrak{g}$ is the complexification of $\mathfrak{g}_0$ then $\mathfrak{g}$-manifolds and $\mathfrak{g}$-bundles are defined by extending the $\mathfrak{g}_0$-action $\mathbb{C}$-linearly.

Definition 2.1.5 [2, page 792] *Two conformally invariant systems $D_1, \ldots, D_n$ and $D'_1, \ldots, D'_n$ are said to be **equivalent** if there is a matrix $A \in GL(n, \mathbb{C}^\infty(M))$ so that*

$$D'_i = \sum_{j=1}^{n} A_{ji} D_j$$

for $1 \le i \le n$.

Definition 2.1.6 [2, page 793] *A conformally invariant system $D_1, \ldots, D_n$ is called **reducible** if there is an equivalent system $D'_1, \ldots, D'_n$ and an $m < n$ such that the system $D'_1, \ldots, D'_m$ is conformally invariant. Otherwise we say that $D_1, \ldots, D_n$ is **irreducible**.*

The case that M is a homogeneous manifold is of our particular interest. In Section 2.2 and Section 2.3, we will specify the $\mathfrak{g}$-manifold and $\mathfrak{g}$-bundle that we will work with.

2.2 A Specialization on a $\mathfrak{g}$-manifold and $\mathfrak{g}$-bundle

In this section we shall introduce the specializations on a smooth manifold M and a vector bundle $\mathcal{V} \to M$, as in Section 5 of [2].

Let G be a complex, simple, connected, simply-connected Lie group with Lie algebra $\mathfrak{g}$. Such G contains a maximal connected solvable subgroup B. Write $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{u}$ for its Lie algebra with $\mathfrak{h}$ the Cartan subalgebra and $\mathfrak{u}$ the nilpotent subalgebra. Let $\mathfrak{q} \supset \mathfrak{b}$ be a parabolic subalgebra of $\mathfrak{g}$. We define $Q = N_G(\mathfrak{q})$, a parabolic subgroup of G. It follows from Section 8.4 of [24] that Q is connected. Write $Q = LN$ for the Levi decomposition of Q with L the Levi subgroup and N the nilpotent subgroup.

It is known, see Corollary 7.11 of [15], that the Levi subgroup L is the commuting product $L = Z(L)^\circ L_{ss}$, where $Z(L)^\circ$ is the identity component of the center of L and L_{ss} is the semisimple part of L.

Let $\mathfrak{g}_0$ be a real form of $\mathfrak{g}$ and let G_0 be the analytic subgroup of G with Lie algebra $\mathfrak{g}_0$. Define $Q_0 = N_{G_0}(\mathfrak{q}) \subset Q$, and write $Q_0 = L_0 N_0$. We will work on $M = G_0/Q_0$ for a class of maximal parabolic Q_0 that will be specified in Chapter 3.

Next, we need to specify a vector bundle $\mathcal{V}$ on M. To this end we recall the bijection between the standard parabolic subalgebras and the subsets of simple roots. Let $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ be the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$. We denote by Δ^+ the positive system so that $\mathfrak{u} = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$ with $\mathfrak{g}_\alpha$ the root spaces for α. We write Π for the set of simple roots.

Observe that the parabolic $\mathfrak{q}$ contains the fixed Borel subalgebra $\mathfrak{b}$. Therefore, it is of the form

$$\mathfrak{q} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Xi} \mathfrak{g}_\alpha$$

with $\Delta^+ \subset \Xi \subset \Delta$. Each subset Ξ can be described in terms of a subset $S \subset \Pi$ of simple roots. Indeed,

$$\Xi = \Delta^+ \cup \{\alpha \in \Delta \mid \alpha \in \operatorname{span}(\Pi \backslash S)\},$$

where $\Pi \backslash S$ is the complementary subset of S in Π . If $\Delta_S = \{\alpha \in \Delta \mid \alpha \in \operatorname{span}(\Pi \backslash S)\}$ then $\Xi = \Delta_S \cup (\Delta^+ \backslash \Delta_S)$. Then $\mathfrak{q}$ may be written as

$$\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{n} \tag{2.2.1}$$

with

$$\mathfrak{l} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta_S} \mathfrak{g}_\alpha \quad \text{and} \quad \mathfrak{n} = \bigoplus_{\alpha \in \Delta^+ \backslash \Delta_S} \mathfrak{g}_\alpha. \tag{2.2.2}$$

The subalgebras $\mathfrak{l}$ and $\mathfrak{n}$ are called the Levi factor and the nilpotent radical, respectively. The Lie algebra $\mathfrak{l}$ is reductive and $\mathfrak{n}$ is a nilpotent ideal in $\mathfrak{q}$.

Now we state the well-known fact that there exists a one-to-one correspondence between the standard parabolic subalgebras $\mathfrak{q}$ and subsets of Π.

Theorem 2.2.3 *There exists a one-to-one correspondence between parabolic subalgebras* $\mathfrak{q}$ *containing* $\mathfrak{b}$ *and the subsets* S *of the set of simple roots* Π*. The parabolic subalgebra* $\mathfrak{q}_S$ *corresponding to the subset* S *is of the form (2.2.1) with (2.2.2).*

Since our parabolic Q_0 will be maximal, by Theorem 2.2.3, there exists the corresponding simple root $\alpha_{\mathfrak{q}} \in \Pi$ so that $\mathfrak{q} = \mathfrak{q}_{\{\alpha_{\mathfrak{q}}\}}$. Call $\lambda_{\mathfrak{q}}$ the fundamental weight of $\alpha_{\mathfrak{q}}$. The weight $\lambda_{\mathfrak{q}}$ is orthogonal to any roots α with $\mathfrak{g}_\alpha \subset [\mathfrak{l}, \mathfrak{l}]$. Hence it exponentiates to a character $\chi_{\mathfrak{q}}$ of L. As $\chi_{\mathfrak{q}}$ takes real values on L_0, for $s \in \mathbb{C}$, character $\chi^{-s} = |\chi_{\mathfrak{q}}|^{-s}$ is well-defined on L_0. Let $\mathbb{C}_{\chi^{-s}}$ be the one-dimensional representation of L_0 with character χ^{-s}. The representation χ^{-s} is extended to a representation of Q_0 by making it trivial on N_0. Then it deduces a line bundle $\mathcal{L}_{-s}$ on $M = G_0/Q_0$ with fiber $\mathbb{C}_{\chi^{-s}}$.

The group G_0 acts on the space

$$\begin{aligned} &C^\infty_\chi(G_0/Q_0, \mathbb{C}_{\chi^{-s}}) \\ &= \{F \in C^\infty(G_0, \mathbb{C}_{\chi^{-s}}) \mid F(gq) = \chi^{-s}(q^{-1})F(g) \text{ for all } q \in Q_0 \text{ and } g \in G_0\} \end{aligned}$$

by left translation. The action π_s of $\mathfrak{g}_0$ on $C^\infty_\chi(G_0/Q_0, \mathbb{C}_{\chi^{-s}})$ arising from this action is given by

$$(\pi_s(Y)\bullet F)(g) = \frac{d}{dt}F(\exp(-tY)g)\big|_{t=0}$$

for $Y \in \mathfrak{g}_0$. This action is extended $\mathbb{C}$-linearly to $\mathfrak{g}$ and then naturally to the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. We use the same symbols for the extended actions.

Let $\bar{N}_0$ be the nilpotent subgroup opposite to N_0. By the Bruhat theory, the subset $\bar{N}_0 Q_0$ is open and dense in G_0. Then the restriction map $C^\infty_\chi(G_0/Q_0, \mathbb{C}_{\chi^{-s}}) \to C^\infty(\bar{N}_0, \mathbb{C}_{\chi^{-s}})$ is an injection, where $C^\infty(\bar{N}_0, \mathbb{C}_{\chi^{-s}})$ is the space of the smooth functions from $\bar{N}_0$ to $\mathbb{C}_{\chi^{-s}}$. Then, for $u \in \mathcal{U}(\mathfrak{g})$ and $F \in C^\infty_\chi(G_0/Q_0, \mathbb{C}_{\chi^{-s}})$, we let $f = F|_{\bar{N}_0}$ and define the action of $\mathcal{U}(\mathfrak{g})$ on the image of the restriction map by

$$\pi_s(u)\bullet f = \big(\pi_s(u)\bullet F\big)|_{\bar{N}_0}. \tag{2.2.4}$$

The line bundle $\mathcal{L}_{-s} \to G_0/Q_0$ restricted to $\bar{N}_0$ is the trivial bundle $\bar{N}_0 \times \mathbb{C}_{\chi^{-s}} \to \bar{N}_0$. By slight abuse of notation, we refer to the trivial bundle over $\bar{N}_0$ as $\mathcal{L}_{-s}$. Then in practice our manifold M will be $M = \bar{N}_0$ and our vector bundle will be the trivial bundle. In the next section we shall show that $\bar{N}_0$ and the trivial bundle $\mathcal{L}_{-s}$ are a $\mathfrak{g}$-manifold and $\mathfrak{g}$-bundle with the action π_s, respectively.

2.3 A $\mathfrak{g}$-manifold $\bar{N}_0$ and $\mathfrak{g}$-bundle $\mathcal{L}_{-s}$

Here we prove that with the linear map π_s defined in (2.2.4),

(1) the manifold $\bar{N}_0$ is a $\mathfrak{g}$-manifold, and

(2) the trivial bundle $\mathcal{L}_{-s}$ is a $\mathfrak{g}$-bundle.

Let $\bar{\mathfrak{n}}$ and $\mathfrak{q}$ be the complexifications of the Lie algebras of $\bar{N}_0$ and Q_0, respectively; we have the direct sum $\mathfrak{g} = \bar{\mathfrak{n}} \oplus \mathfrak{q}$. For $Y \in \mathfrak{g}$, write $Y = Y_{\bar{\mathfrak{n}}} + Y_{\mathfrak{q}}$ for the decomposition of Y in this direct sum. Similarly, write the Bruhat decomposition of $g \in \bar{N}_0 Q_0$ as $g = \bar{\mathbf{n}}(g)\mathbf{q}(g)$ with $\bar{\mathbf{n}}(g) \in \bar{N}_0$ and $\mathbf{q}(g) \in Q_0$. For $Y \in \mathfrak{g}_0$, we have

$$Y_{\bar{\mathfrak{n}}} = \frac{d}{dt}\bar{\mathbf{n}}(\exp(tY))\big|_{t=0}, \tag{2.3.1}$$

and a similar equality holds for $Y_{\mathfrak{q}}$. Define a right action R of $\mathcal{U}(\bar{\mathfrak{n}})$ on $C^\infty(\bar{N}_0, \mathbb{C}_{\chi^{-s}})$ by

$$\big(R(X)\bullet f\big)(\bar{n}) = \frac{d}{dt} f\big(\bar{n}\exp(tX)\big)\big|_{t=0} \tag{2.3.2}$$

for $X \in \bar{\mathfrak{n}}_0$ and $f \in C^\infty(\bar{N}_0, \mathbb{C}_{\chi^{-s}})$. Observe that, by definition, the differential $d\chi$ of χ is $d\chi = \lambda_{\mathfrak{q}}$.

Proposition 2.3.3 *We have*

$$\big(\pi_s(Y)\bullet f\big)(\bar{n}) = -s\lambda_{\mathfrak{q}}\big((\mathrm{Ad}(\bar{n}^{-1})Y)_{\mathfrak{q}}\big)f(\bar{n}) - \Big(R\big((\mathrm{Ad}(\bar{n}^{-1})Y)_{\bar{\mathfrak{n}}}\big)\bullet f\Big)(\bar{n}) \tag{2.3.4}$$

for $Y \in \mathfrak{g}$ and f in the image of the restriction map $C^\infty_\chi(G_0/Q_0, \mathbb{C}_{\chi^{-s}}) \to C^\infty(\bar{N}_0, \mathbb{C}_{\chi^{-s}})$.

Proof. Suppose that $f = F|_{\bar{N}_0}$ for some $F \in C^\infty_\chi(G_0/Q_0, \mathbb{C}_{\chi^{-s}})$. If $g^{-1}\bar{n} \in \bar{N}_0 Q_0$ then we have

$$(g \cdot f)(\bar{n}) = F(g^{-1}\bar{n}) = \chi^{-s}(\mathbf{q}(g^{-1}\bar{n})^{-1}) f(\bar{\mathbf{n}}(g^{-1}\bar{n})). \tag{2.3.5}$$

Observe that if g is close enough to the identity then $g^{-1}\bar{n} \in \bar{N}_0 Q_0$ by the openness of $\bar{N}_0 Q_0$. By replacing g by $\exp(tY)$ in (2.3.5) with $Y \in \mathfrak{g}_0$ and differentiating at $t = 0$, we have

$$\begin{aligned}
&(\pi_s(Y) \bullet f)(\bar{n}) \\
&= \frac{d}{dt}\chi^{-s}\big(\mathbf{q}(\exp(-tY)\bar{n})^{-1}\big) f(\bar{\mathbf{n}}(\exp(-tY)\bar{n}))|_{t=0} \\
&= \frac{d}{dt}\chi^{-s}\big(\mathbf{q}(\exp(-tY)\bar{n})^{-1}\big)|_{t=0}\ f(\bar{n}) + \frac{d}{dt} f(\bar{\mathbf{n}}(\exp(-tY)\bar{n}))|_{t=0} \\
&= \frac{d}{dt}\chi^{-s}\big(\mathbf{q}(\exp(-t\mathrm{Ad}(\bar{n}^{-1})Y))^{-1}\big)|_{t=0}\ f(\bar{n}) + \frac{d}{dt} f(\bar{n}\bar{\mathbf{n}}(\exp(-t\mathrm{Ad}(\bar{n}^{-1})Y)))|_{t=0} \\
&= -s\lambda_{\mathfrak{q}}\big((\mathrm{Ad}(\bar{n}^{-1})Y)_{\mathfrak{q}}\big) f(\bar{n}) - \big(R\big((\mathrm{Ad}(\bar{n}^{-1})Y)_{\bar{\mathfrak{n}}}\big) \bullet f\big)(\bar{n}).
\end{aligned}$$

Note that the equality (2.3.1) is used from line three to line four. Now the proposed formula is obtained by extending the action $\mathbb{C}$-linearly. ■

Equation (2.3.4) implies that the representation π_s extends to a representation of $\mathcal{U}(\mathfrak{g})$ on the whole space $C^\infty(\bar{N}_0, \mathbb{C}_{\chi^{-s}})$. Moreover, it also shows that for all $Y \in \mathfrak{g}$, the linear map $\pi_s(Y)$ is in $C^\infty(\bar{N}_0) \oplus \mathfrak{X}(\bar{N}_0)$. Therefore, with this linear map π_s, $\bar{N}_0$ is a $\mathfrak{g}$-manifold.

Next, we show that the linear map π_s gives $\mathcal{L}_{-s}$ the structure of a $\mathfrak{g}$-bundle. As π_s is a representation of $\mathfrak{g}$, the condition (B1) of Definition 2.1.3 is trivial. Thus it suffices to show that the condition (B2) holds. Since $\mathcal{L}_{-s}$ is the trivial bundle of $\bar{N}_0$ with fiber $\mathbb{C}_{\chi^{-s}}$, the space of smooth sections of $\mathcal{L}_{-s}$ is identified with $C^\infty(\bar{N}_0, \mathbb{C}_{\chi^{-s}})$.

Proposition 2.3.6 *In $\mathbb{D}(\mathcal{L}_{-s})$ we have*

$$\big([\pi_s(Y), f]\big)(\bar{n}) = -\big(R\big((\mathrm{Ad}(\bar{n}^{-1})Y)_{\bar{\mathfrak{n}}}\big) \bullet f\big)(\bar{n})$$

for $Y \in \mathfrak{g}$ and $f \in C^\infty(\bar{N}_0)$. In particular, the trivial bundle $\mathcal{L}_{-s}$ with π_s is a $\mathfrak{g}$-bundle.

Proof. Take $h \in C^\infty(\bar{N}_0, \mathbb{C}_{\chi^{-s}})$. Since $[\pi_s(Y), f] = \pi_s(Y)f - f\pi_s(Y)$ in $\mathbb{D}(\mathcal{L}_{-s})$, the operator $[\pi_s(Y), f]$ acts on h by

$$\big([\pi_s(Y), f]\bullet h\big)(\bar{n}) = \big(\pi_s(Y)\bullet(fh)\big)(\bar{n}) - f(\bar{n})\big(\pi_s(Y)\bullet h\big)(\bar{n}). \tag{2.3.7}$$

It follows from Proposition 2.3.3 that the first term evaluates to

$$\big(\pi_s(Y)\bullet(fh)\big)(\bar{n}) = -s\lambda_{\mathfrak{q}}\big((\mathrm{Ad}(\bar{n}^{-1})Y)_{\mathfrak{q}}\big)f(\bar{n})h(\bar{n}) - \big(R\big((\mathrm{Ad}(\bar{n}^{-1})Y)_{\bar{\mathfrak{n}}}\big)\bullet(fh)\big)(\bar{n}) \tag{2.3.8}$$

with

$$\begin{aligned}&\big(R\big((\mathrm{Ad}(\bar{n}^{-1})Y)_{\bar{\mathfrak{n}}}\big)\bullet(fh)\big)(\bar{n})\\ &= \big(R\big((\mathrm{Ad}(\bar{n}^{-1})Y)_{\bar{\mathfrak{n}}}\big)\bullet f\big)(\bar{n})h(\bar{n}) + f(\bar{n})\big(R\big((\mathrm{Ad}(\bar{n}^{-1})Y)_{\bar{\mathfrak{n}}}\big)\bullet h\big)(\bar{n}).\end{aligned}$$

Similarly, the second term evaluates to

$$f(\bar{n})\big(\pi_s(Y)\bullet h\big)(\bar{n}) = -s\lambda_{\mathfrak{q}}\big((\mathrm{Ad}(\bar{n}^{-1})Y)_{\mathfrak{q}}\big)f(\bar{n})h(\bar{n}) - f(\bar{n})\big(R\big((\mathrm{Ad}(\bar{n}^{-1})Y)_{\bar{\mathfrak{n}}}\big)\bullet h\big)(\bar{n}). \tag{2.3.9}$$

Now the proposed equality is obtained by substituting (2.3.8) and (2.3.9) into (2.3.7).

■

In the next section we are going to construct systems of differential operators on $\mathcal{L}_{-s}$. The systems of operators will satisfy several properties of conformally invariant systems. To end this section we collect those properties here.

Definition 2.3.10 [2, page 806] *A conformally invariant system $D_1, \ldots, D_m$ on the line bundle $\mathcal{L}_{-s}$ is called L_0-**stable** if there is a map $c : L_0 \to GL(n, C^\infty(\bar{N}_0))$ such that*

$$l \cdot D_i = \sum_{j=1}^{m} c(l)_{ji} D_j,$$

where the action $l \cdot D_i$ is given by (2.5.10).

It is known that there exists a semisimple element $H_0 \in \mathfrak{l}$, so that $\mathrm{ad}(H_0)$ has only integer eigenvalues on $\mathfrak{g}$ with $\mathfrak{g}(1) \neq \{0\}$, $\mathfrak{l} = \mathfrak{g}(0)$, $\mathfrak{n} = \bigoplus_{j>0} \mathfrak{g}(j)$, and $\bar{\mathfrak{n}} = \bigoplus_{j>0} \mathfrak{g}(-j)$, where $\mathfrak{g}(j)$ is the j-eigenspace of $\mathrm{ad}(H_0)$ (see for example [15, Section X.3]).

Definition 2.3.11 [2, page 804] *A conformally invariant system $D_1, \ldots, D_m$ is called **homogeneous** if $C(H_0)$ is a scalar matrix, where C is the structure operator of the conformally invariant system (see Definition 2.1.4).*

Proposition 2.3.12 [2, Proposition 17] *Any irreducible conformally invariant system is homogeneous.*

Define

$$\mathbb{D}(\mathcal{L}_{-s})^{\bar{\mathfrak{n}}} = \{D \in \mathbb{D}(\mathcal{L}_{-s}) \mid [\pi_s(X), D] = 0 \text{ for all } X \in \bar{\mathfrak{n}}\}.$$

Observe that in the sense of [2, page 796], the $\mathfrak{g}$-manifold $\bar{N}_0$ is *straight* with respect to the subalgebra $\bar{\mathfrak{n}}$ of $\mathfrak{g}$ ([2, page 799]). Then we state the definition of *straight* conformally invariant systems specialized to the present situation. For the general definition see p.797 of [2].

Definition 2.3.13 *We say that a conformally invariant system $D_1, \ldots, D_m$ is **straight** if $D_j \in \mathbb{D}(\mathcal{L}_{-s})^{\bar{\mathfrak{n}}}$ for $j = 1, \ldots, m$.*

In general, to show that a given list $D_1, \ldots, D_m$ of differential operators on $\bar{N}_0$ is a conformally invariant system, we need check (S2) of Definition 2.1.4 at each point of $\bar{N}_0$. Proposition 2.3.14 below shows that in the case $D_1, \ldots, D_m$ in $\mathbb{D}(\mathcal{L}_{-s})^{\bar{\mathfrak{n}}}$, it suffices to check the condition only at the identity e.

Proposition 2.3.14 [2, Proposition 13] *Let $D_1, \ldots, D_m$ be a list of operators in $\mathbb{D}(\mathcal{L}_{-s})^{\bar{\mathfrak{n}}}$. Suppose that the list is linearly independent at e and that there is a map $b : \mathfrak{g} \to \mathfrak{gl}(m, \mathbb{C})$ such that*

$$\big([\pi_s(Y), D_i] \bullet f\big)(e) = \sum_{j=1}^{m} b(Y)_{ji} (D_j \bullet f)(e)$$

for all $Y \in \mathfrak{g}$, $f \in C^\infty(\bar{N}_0, \mathbb{C}_{\chi^{-s}})$, *and* $1 \leq i \leq m$. *Then* $D_1, \ldots, D_m$ *is a conformally invariant system on* $\mathcal{L}_{-s}$. *The structure operator of the system is given by* $C(Y)(\bar{n}) = b(\mathrm{Ad}(\bar{n}^{-1})Y)$ *for all* $\bar{n} \in \bar{N}_0$ *and* $Y \in \mathfrak{g}$.

2.4 Useful Formulas

In this section we are going to show two formulas that will be helpful, when we study the conformal invariance of certain systems of differential operators on $\bar{N}_0$ in Chapter 4 and Chapter 7.

Proposition 2.4.1 *For* $Y \in \mathfrak{g}$, $X \in \bar{\mathfrak{n}}$, *and* $f \in C^\infty(\bar{N}_0, \mathbb{C}_{\chi^{-s}})$, *we have*

$$\big([\pi_s(Y), R(X)]\bullet f\big)(\bar{n}) = \big(R([(\mathrm{Ad}(\bar{n}^{-1})Y)_{\mathfrak{q}}, X]_{\bar{\mathfrak{n}}})\bullet f\big)(\bar{n}) + s\lambda_{\mathfrak{q}}\big([\mathrm{Ad}(\bar{n}^{-1})Y, X]_{\mathfrak{q}}\big) f(\bar{n}).$$

Proof. Since $[\pi_s(Y), R(X)] = \pi_s(Y)R(X) - R(X)\pi_s(Y)$, it suffices to consider the contributions from each term. By Proposition 2.3.3, the contribution from $\pi_s(Y)R(X)$ is

$$\begin{aligned} &\big((\pi_s(Y)R(X))\bullet f\big)(\bar{n}) \\ &= -s\lambda_{\mathfrak{q}}\big((\mathrm{Ad}(\bar{n}^{-1})Y)_{\mathfrak{q}}\big)(R(X)\bullet f)(\bar{n}) - \big((R((\mathrm{Ad}(\bar{n}^{-1})Y)_{\bar{\mathfrak{n}}})R(X))\bullet f)\big)(\bar{n}). \end{aligned} \tag{2.4.2}$$

To obtain the contribution from $R(X)\pi_s(Y)$, observe that

$$\big(R(X)\pi_s(Y)\bullet f\big)(\bar{n}) = \frac{d}{dt}\big(\pi_s(Y)\bullet f\big)(\bar{n}\exp(tX))|_{t=0}.$$

By applying Proposition 2.3.3, differentiating with respect to t, and setting $t = 0$, the contribution from this term is

$$\begin{aligned} \big(R(X)\pi_s(Y)\bullet f\big)(\bar{n}) &= s\lambda_{\mathfrak{q}}\big([X, \mathrm{Ad}(\bar{n}^{-1})Y]_{\mathfrak{q}}\big) f(\bar{n}) - s\lambda_{\mathfrak{q}}\big((\mathrm{Ad}(\bar{n}^{-1})Y)\mathfrak{q}\big)(R(X)\bullet f)(\bar{n}) \\ &\quad + \big(R([X, \mathrm{Ad}(\bar{n}^{-1})Y]_{\bar{\mathfrak{n}}})\bullet f\big)(\bar{n}) - \big((R(X)R((\mathrm{Ad}(\bar{n}^{-1})Y)_{\bar{\mathfrak{n}}}))\bullet f\big)(\bar{n}). \end{aligned} \tag{2.4.3}$$

Since $R([X, (\mathrm{Ad}(\bar{n}^{-1})Y)_{\bar{\mathfrak{n}}}]) = R(X)R((\mathrm{Ad}(\bar{n}^{-1})Y)_{\bar{\mathfrak{n}}}) - R((\mathrm{Ad}(\bar{n}^{-1})Y)_{\bar{\mathfrak{n}}})R(X)$, it follows from (2.4.2) and (2.4.3) that $\big([\pi_s(Y), R(X)]\bullet f\big)(\bar{n})$ evaluates to

$$\big([\pi_s(Y), R(X)]\bullet f\big)(\bar{n}) = \tag{2.4.4}$$
$$\big(R([X, (\mathrm{Ad}(\bar{n}^{-1})Y)_{\bar{\mathfrak{n}}}])\bullet f\big)(\bar{n}) - \big(R([X, \mathrm{Ad}(\bar{n}^{-1})Y]_{\bar{\mathfrak{n}}})\bullet f\big)(\bar{n}) + s\lambda_{\mathfrak{q}}([\mathrm{Ad}(\bar{n}^{-1})Y, X]_{\mathfrak{q}}\big) f(\bar{n}).$$

As $\mathrm{Ad}(\bar{n}^{-1})Y = (\mathrm{Ad}(\bar{n}^{-1})Y)_{\bar{\mathfrak{n}}} + (\mathrm{Ad}(\bar{n}^{-1})Y)_{\mathfrak{q}}$ and $X \in \bar{\mathfrak{n}}$, we have

$$[X, \mathrm{Ad}(\bar{n}^{-1})Y]_{\bar{\mathfrak{n}}} = [X, (\mathrm{Ad}(\bar{n}^{-1})Y)_{\bar{\mathfrak{n}}}] + [X, (\mathrm{Ad}(\bar{n}^{-1})Y)_{\mathfrak{q}}]_{\bar{\mathfrak{n}}}.$$

Now the proposed formula follows from substituting this into the second term of the right hand side of (2.4.4). ■

Proposition 2.4.5 *For $Y \in \mathfrak{g}$, $X_1, X_2 \in \bar{\mathfrak{n}}$, and $f \in C^\infty(\bar{N}_0, \mathbb{C}_{\chi^{-s}})$, we have*

$$\begin{aligned}
&\big([\pi_s(Y), R(X_1)R(X_2)]\bullet f\big)(\bar{n})\\
&= \big(R([(\mathrm{Ad}(\bar{n}^{-1})Y)_{\mathfrak{q}}, X_1]_{\bar{\mathfrak{n}}})R(X_2)\bullet f\big)(\bar{n}) + \big(R(X_1)R([(\mathrm{Ad}(\bar{n}^{-1})Y)_{\mathfrak{q}}, X_2]_{\bar{\mathfrak{n}}})\bullet f\big)(\bar{n})\\
&\quad + \big(R([[\mathrm{Ad}(\bar{n}^{-1})Y, X_1]_{\mathfrak{q}}, X_2]_{\bar{\mathfrak{n}}})\bullet f\big)(\bar{n}) + s\lambda_{\mathfrak{q}}([\mathrm{Ad}(\bar{n}^{-1})Y, X_1]_{\mathfrak{q}})(R(X_2)\bullet f)(\bar{n})\\
&\quad + s\lambda_{\mathfrak{q}}([\mathrm{Ad}(\bar{n}^{-1})Y, X_2]_{\mathfrak{q}})(R(X_1)\bullet f)(\bar{n}) + s\lambda_{\mathfrak{q}}([[\mathrm{Ad}(\bar{n}^{-1})Y, X_1], X_2]_{\mathfrak{q}}) f(\bar{n}).
\end{aligned}$$

Proof. Observe that $[\pi_s(Y), R(X_1)R(X_2)]$ is the sum of two terms

$$[\pi_s(Y), R(X_1)R(X_2)] = [\pi_s(Y), R(X_1)]R(X_2) + R(X_1)[\pi_s(Y), R(X_2)].$$

The contribution from the first term is

$$\begin{aligned}
&\big([\pi_s(Y), R(X_1)]\bullet(R(X_2)\bullet f)\big)(\bar{n})\\
&= \big(R([(\mathrm{Ad}(\bar{n}^{-1})Y)_{\mathfrak{q}}, X_1]_{\bar{\mathfrak{n}}})\bullet(R(X_2)\bullet f)\big)(\bar{n}) + s\lambda_{\mathfrak{q}}\big([\mathrm{Ad}(\bar{n}^{-1})Y, X_1]_{\mathfrak{q}}\big)(R(X_2)\bullet f)(\bar{n}).
\end{aligned} \tag{2.4.6}$$

The second term evaluates to

$$\begin{aligned}
&\big(R(X_1)[\pi_s(Y), R(X_2)]\bullet f\big)(\bar{n}) \\
&= \frac{d}{dt}\big([\pi_s(Y), R(X_2)]\bullet f\big)(\bar{n}\exp(tX_1))|_{t=0} \\
&= -\big(R([X_1, \mathrm{Ad}(\bar{n}^{-1})Y]_{\mathfrak{q}}, X_2]_{\bar{\mathfrak{n}}})\bullet f\big)(\bar{n}) + \big(R(X_1)R([(\mathrm{Ad}(\bar{n}^{-1})Y)_{\mathfrak{q}}, X_2]_{\bar{\mathfrak{n}}})\bullet f\big)(\bar{n}) \\
&\quad - s\lambda_{\mathfrak{q}}\big([[X_1, \mathrm{Ad}(\bar{n}^{-1})Y], X_2]_{\mathfrak{q}}\big)f(\bar{n}) + s\lambda_{\mathfrak{q}}\big([\mathrm{Ad}(\bar{n}^{-1})Y, X_2]_{\mathfrak{q}}\big)(R(X_1)\bullet f)(\bar{n}).
\end{aligned}$$

Now the proposed formula follows from adding this to (2.4.6). ∎

2.5 The Ω_k Systems

The purpose of this section is to construct systems of differential operators in $\mathbb{D}(\mathcal{L}_{-s})^{\bar{\mathfrak{n}}}$ in a systematic manner.

We start with a $\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{j=-r}^{r} \mathfrak{g}(j)$ on $\mathfrak{g}$ with $\mathfrak{g}(1) \neq 0$. It is known that $\mathfrak{g}(0)$ is reductive (see for instance [15, Corollary 10.17]). By construction, $\mathfrak{q} = \mathfrak{g}(0) \oplus \bigoplus_{j>0} \mathfrak{g}(j)$ is a parabolic subalgebra. Take L to be the analytic subgroup of G with Lie algebra $\mathfrak{g}(0)$. Vinberg's Theorem ([15, Theorem 10.19]) shows that the adjoint action of L on $\mathfrak{g}(1)$ has only finitely many orbits; in particular, L has an open orbit in $\mathfrak{g}(1)$. Such a space is called prehomogeneous. In the theory of prehomogeneous vector spaces, it is natural to associate certain maps called *covariant maps* to a prehomogeneous vector space. To define our systems of differential operators, we use covariant maps that are associated to prehomogeneous vector space $(L, \mathrm{Ad}, \mathfrak{g}(1))$. We denote the covariant maps by τ_k and define them below. These maps can be thought to give symbols of a class of differential operators that we will study. We would like to acknowledge that the construction of τ_k as in this thesis was suggested by Anthony Kable.

Definition 2.5.1 *Let* $\mathfrak{g} = \bigoplus_{j=-r}^{r} \mathfrak{g}(j)$ *be a graded complex simple Lie algebra with*

$\mathfrak{g}(1) \neq 0$. *Then, for* $0 \leq k \leq 2r$, *the map* τ_k *on* $\mathfrak{g}(1)$ *is defined by*

$$\tau_k : \mathfrak{g}(1) \to \mathfrak{g}(-r+k) \otimes \mathfrak{g}(r)$$
$$X \mapsto \frac{1}{k!}\mathrm{ad}(X)^k \omega_0$$

with $\omega_0 = \sum_{\gamma_j \in \Delta(\mathfrak{g}(r))} X_{-\gamma_j} \otimes X_{\gamma_j}$, *where* X_{γ_j} *are root vectors for* γ_j *and* $\Delta(\mathfrak{g}(r))$ *is the set of roots* α *so that* $\mathfrak{g}_\alpha \subset \mathfrak{g}(r)$.

Here, we mean by $\mathrm{ad}(X)^k\omega_0$ that X acts on the tensor product diagonally via the action $\mathrm{ad}(\cdot)^k$. Observe that since $X \in \mathfrak{g}(1)$ and $[\mathfrak{g}(1), \mathfrak{g}(r)] = 0$, we have $\mathrm{ad}(X)^k X_{\gamma_j} = 0$ for all $\gamma_j \in \Delta(\mathfrak{g}(r))$. Therefore, $\mathrm{ad}(X)^k\omega_0 = \sum_{\gamma_j} \mathrm{ad}(X)^k(X_{-\gamma_j}) \otimes X_{\gamma_j}$.

When $\mathfrak{g}(1)$ and $\mathfrak{g}(-r+k) \otimes \mathfrak{g}(r)$ are viewed as affine varieties, the maps τ_k are indeed morphisms of varieties. We shall check in Lemma 2.5.4 that these maps are L-equivariant. This will show that τ_k satisfy the definition of covariant maps.

To simplify a proof for Lemma 2.5.4, we first show that ω_0 in Definition 2.5.1 is independent of a choice of a basis for $\mathfrak{g}(r)$.

Lemma 2.5.2 *If* $Y_1, \ldots, Y_m$ *is a basis for* $\mathfrak{g}(r)$ *and* $Y_1^*, \ldots, Y_m^*$ *is the dual basis for* $\mathfrak{g}(-r)$ *with respect to the Killing form* κ *then* $\omega_0 = \sum_{i=1}^m (Y_i \otimes Y_i^*)$.

Proof. If $\Delta(\mathfrak{g}(r)) = \{\gamma_1, \ldots, \gamma_m\}$ then each Y_i may be expressed by $Y_i = \sum_{r=1}^m a_{ir} X_{\gamma_r}$ for $a_{ir} \in \mathbb{C}$. Let $[a_{ir}]$ be the change of basis matrix and set $[b_{ir}] = [a_{ir}]^{-1}$. Define $Y_i^* = \sum_{s=1}^m b_{si} X_{-\gamma_s}$ for $i = 1, \ldots, m$. Since $\sum_{s=1}^m a_{is} b_{sj} = \delta_{ij}$ and $\kappa(X_{\gamma_i}, X_{-\gamma_j}) = \delta_{ij}$ with δ_{ij} the Kronecker delta, it follows that $\kappa(Y_i, Y_j^*) = \delta_{ij}$. Thus $\{Y_1^*, \ldots, Y_m^*\}$ is the dual basis of $\{Y_1, \ldots, Y_m\}$. Hence,

$$\sum_{i=1}^m (Y_i^* \otimes Y_i) = \sum_{r,s=1}^m \Big(\sum_{i=1}^m b_{si} a_{ir}\Big)(X_{-\gamma_s} \otimes X_{\gamma_r}) = \sum_{s=1}^m (X_{-\gamma_s} \otimes X_{\gamma_s}).$$

■

Corollary 2.5.3 *Let* $\mathfrak{g} = \bigoplus_{j=-r}^r \mathfrak{g}(j)$ *be a graded complex simple Lie algebra with* $\mathfrak{g}(1) \neq 0$ *and* G *be a complex analytic group with Lie algebra* $\mathfrak{g}$. *If* L *is the analytic* *subgroup of G with Lie algebra $\mathfrak{g}(0)$ and ω_0 is as in Definition 2.5.1 then, for all $l \in L$,*

$$(\mathrm{Ad}(l) \otimes \mathrm{Ad}(l))\omega_0 = \omega_0.$$

Proof. If $g \in L$ then $\{\mathrm{Ad}(l)X_{\gamma_j} \mid \gamma_j \in \Delta(\mathfrak{g}(r))\}$ forms a basis for $\mathfrak{g}(r)$. It also holds that $\{\mathrm{Ad}(l)X_{-\gamma_j} \mid \gamma_j \in \Delta(\mathfrak{g}(r))\}$ is the dual basis for $\mathfrak{g}(-r)$ with respect to the Killing form. Now the assertion follows from Lemma 2.5.2 ■

Now we show that τ_k are L-equivariant.

Lemma 2.5.4 *Let $\mathfrak{g} = \bigoplus_{j=-r}^{r} \mathfrak{g}(j)$ be a graded complex simple Lie algebra with $\mathfrak{g}(1) \neq 0$ and G be a complex analytic group with Lie algebra $\mathfrak{g}$. If L is the analytic subgroup of G with Lie algebra $\mathfrak{g}(0)$ then, for all $l \in L$, $X \in \mathfrak{g}(1)$, and for $0 \leq k \leq 2r$, we have*

$$\tau_k(\mathrm{Ad}(l)X) = (\mathrm{Ad}(l) \otimes \mathrm{Ad}(l))\tau_k(X). \tag{2.5.5}$$

Proof. For $l \in L$, we have

$$\begin{aligned}
\tau_k(\mathrm{Ad}(l)X) &= \frac{1}{k!}\mathrm{ad}(\mathrm{Ad}(l)(X))^k\omega_0 \\
&= \frac{1}{k!}\sum_{\gamma_j \in \Delta(\mathfrak{z}(\mathfrak{n}))} \mathrm{ad}(\mathrm{Ad}(l)(X))^k(X_{-\gamma_j}) \otimes X_{\gamma_j} \\
&= \frac{1}{k!}\sum_{\gamma_j \in \Delta(\mathfrak{z}(\mathfrak{n}))} \mathrm{Ad}(l)\big(\mathrm{ad}(X)^k(\mathrm{Ad}(l^{-1})X_{-\gamma_j})\big) \otimes X_{\gamma_j} \\
&= (\mathrm{Ad}(l) \otimes \mathrm{Ad}(l))\left(\frac{1}{k!}\sum_{\gamma_j \in \Delta(\mathfrak{z}(\mathfrak{n}))} \mathrm{ad}(X)^k(\mathrm{Ad}(l^{-1})X_{-\gamma_j}) \otimes \mathrm{Ad}(l^{-1})(X_{\gamma_j})\right) \\
&= (\mathrm{Ad}(l) \otimes \mathrm{Ad}(l))\left(\frac{1}{k!}\mathrm{ad}(X)^k\omega_0\right) \\
&= (\mathrm{Ad}(l) \otimes \mathrm{Ad}(l))\tau_k(X).
\end{aligned}$$

Note that Corollary 2.5.3 is applied from line four to line five. ■

Now we are going to build the systems of differential operators in $\mathbb{D}(\mathcal{L}_{-s})^{\bar{\mathfrak{n}}}$ that we study. It is useful to observe that $\tau_k : \mathfrak{g}(1) \to \mathfrak{g}(-r+k) \otimes \mathfrak{g}(r) = W$ are L-equivariant polynomial maps of degree k. Here, by a polynomial map we mean a map for which each coordinate is a polynomial in $\mathfrak{g}(1)$. Therefore the maps τ_k can be thought of as elements in $(\mathcal{P}^k(\mathfrak{g}(1)) \otimes W)^L$, where $\mathcal{P}^k(\mathfrak{g}(1))$ denotes the space of homogeneous polynomials on $\mathfrak{g}(1)$ of degree k. Then the isomorphism $(\mathcal{P}^k(\mathfrak{g}(1)) \otimes W)^L \cong \mathrm{Hom}_L(W^*, \mathcal{P}^k(\mathfrak{g}(1)))$ yields the L-intertwining operators $\tilde{\tau}_k$, that are given by

$$\tilde{\tau}_k(Y^*)(X) = Y^*(\tau_k(X)), \tag{2.5.6}$$

where W^* is the dual module of W with respect to the Killing form. For each $Y^* \in W^*$, we have $\tilde{\tau}_k(Y^*) \in \mathcal{P}^k(\mathfrak{g}(1)) \cong \mathrm{Sym}^k(\mathfrak{g}(-1))$. We define differential operators in $\mathbb{D}(\mathcal{L}_{-s})^{\bar{\mathfrak{n}}}$ from $\tilde{\tau}_k(Y^*)$. This is done as follows. Let $\sigma : \mathrm{Sym}^k(\mathfrak{g}(-1)) \to \mathcal{U}(\bar{\mathfrak{n}})$ be the symmetrization operator. Identify $\mathcal{U}(\bar{\mathfrak{n}})$ with $\mathbb{D}(\mathcal{L}_{-s})^{\bar{\mathfrak{n}}}$ by making $\bar{\mathfrak{n}}$ act on $C^\infty(\bar{N}_0, \mathbb{C}_{\chi^{-s}})$ via right differentiation R. Then we have a composition of linear maps

$$W^* \xrightarrow{\tilde{\tau}_k} \mathcal{P}^k(\mathfrak{g}(1)) \cong \mathrm{Sym}^k(\mathfrak{g}(-1)) \xhookrightarrow{\sigma} \mathcal{U}(\bar{\mathfrak{n}}) \xrightarrow{R} \mathbb{D}(\mathcal{L}_{-s})^{\bar{\mathfrak{n}}}.$$

For $Y^* \in W^*$, we define a differential operator $\Omega_k(Y^*) \in \mathbb{D}(\mathcal{L}_{-s})^{\bar{\mathfrak{n}}}$ by

$$\Omega_k(Y^*) = R \circ \sigma \circ \tilde{\tau}_k(Y^*).$$

As we will work with irreducible systems we need to be a little more careful with our construction; in particular, we need to take an irreducible constituent of $\mathfrak{g}(-r+k)^* \otimes \mathfrak{g}(r)^*$. Let

$$\mathfrak{g}(-r+k) \otimes \mathfrak{g}(r) = V_1 \oplus \cdots \oplus V_m$$

be the irreducible decomposition of $\mathfrak{g}(-r+k) \otimes \mathfrak{g}(r)$ as an L-module, and let

$$\mathfrak{g}(-r+k)^* \otimes \mathfrak{g}(r)^* = V_1^* \oplus \cdots \oplus V_m^*$$

be the corresponding irreducible decomposition of $\mathfrak{g}(-r+k)^* \otimes \mathfrak{g}(r)^*$, where $\mathfrak{g}(j)^*$ are the dual spaces of $\mathfrak{g}(j)$ with respect to the Killing form. For each irreducible constituent V_j^* of $\mathfrak{g}(-r+k)^* \otimes \mathfrak{g}(r)^*$, there exists an L-intertwining operator $\tilde{\tau}_k|_{V_j^*} \in \mathrm{Hom}_L(V_j^*, \mathcal{P}^k(\mathfrak{g}(1)))$ given as in (2.5.6). Then we define a linear operator $\Omega_k|_{V_j^*}$: $V_j^* \to \mathbb{D}(\mathcal{L}_{-s})^{\bar{\mathfrak{n}}}$ by

$$\Omega_k|_{V_j^*} = R \circ \sigma \circ \tilde{\tau}_k|_{V_j^*}.$$

Since, for $Y^* \in V_j^*$, we have $\Omega_k|_{V_j^*}(Y^*) = \Omega_k(Y^*)$ as a differential operator, we simply write $\Omega_k(Y^*)$ for the differential operator arising from $Y^* \in V_j^*$.

Definition 2.5.7 *Let $\mathfrak{g} = \bigoplus_{j=-r}^{r} \mathfrak{g}(j)$ be an r-graded complex simple Lie algebra with $\mathfrak{g}(1) \neq 0$, and $\mathfrak{q} = \bigoplus_{j=0}^{r} \mathfrak{g}(j)$ be the parabolic subalgebra of $\mathfrak{g}$ associated with the r-grading. If V^* is an irreducible constituent of $\mathfrak{g}(-r+k)^* \otimes \mathfrak{g}(r)^*$ so that $\tilde{\tau}_k|_{V^*}$ is not identically zero then a list of differential operators $D_1, \ldots, D_n \in \mathbb{D}(\mathcal{L}_{-s})^{\bar{\mathfrak{n}}}$ is called the $\Omega_k|_{V^*}$ **system** if it is equivalent to a list of differential operators*

$$\Omega_k(Y_1^*), \ldots, \Omega_k(Y_n^*), \tag{2.5.8}$$

where $\{Y_1^*, \ldots, Y_n^*\}$ is a basis for V^* over $\mathbb{C}$.

Each $\Omega_k|_{W^*}$ system is also simply referred to as an Ω_k system. We want to remark that the construction of the Ω_k systems might require additional modification to secure the conformal invariance. See Section 6 in [1] and Section 3 in [20] for the modification for the Ω_3 systems of the Heisenberg parabolic subalgebra.

It is important to notice that it is not necessary for the Ω_k systems to be conformally invariant; their conformal invariance strongly depends on the complex parameter s for the line bundle $\mathcal{L}_{-s}$. So, we give the following definition.

Definition 2.5.9 *Let V^* be an irreducible constituent of $\mathfrak{g}(-r+k)^* \otimes \mathfrak{g}(r)^*$. Then we say that the $\Omega_k|_{V^*}$ system has **special value** s_0 if the system is conformally invariant on the line bundle $\mathcal{L}_{s_0}$.*

The goal of this thesis is to find the special values of the Ω_1 system and the Ω_2 systems of a maximal two-step nilpotent parabolic subalgebra of non-Heisenberg type. This is done in Chapter 4 and Chapter 7.

To finish this section we define an action of L_0 on $\mathbb{D}(\mathcal{L}_{-s})^{\bar{\mathfrak{n}}}$ so that the linear operator $\Omega_k|_{V^*} : V^* \to \mathbb{D}(\mathcal{L}_{-s})^{\bar{\mathfrak{n}}}$ is an L_0-intertwining operator. This will allow that the $\Omega_k|_{V^*}$ system is L_0-stable (see Definition 2.3.10). As on p.805 of [2], we first define an action of L_0 on $C^\infty(\bar{N}_0, \mathbb{C}_{\chi^{-s}})$ by

$$(l \cdot f)(\bar{n}) = \chi^{-s}(l) f(l^{-1}\bar{n}l).$$

This action agrees with the action of L_0 by the left translation on the image of the restriction map $C^\infty_\chi(G_0/Q_0, \mathbb{C}_{\chi^{-s}}) \to C^\infty(\bar{N}_0, \mathbb{C}_{\chi^{-s}})$. In terms of this action we define an action of L_0 on $\mathbb{D}(\mathcal{L}_{-s})$ by

$$(l \cdot D)\bullet f = l \cdot (D\bullet(l^{-1} \cdot f)). \tag{2.5.10}$$

One can check that we have $l \cdot R(u) = R(\mathrm{Ad}(l)u)$ for $l \in L_0$ and $u \in \mathcal{U}(\bar{\mathfrak{n}})$; in particular, this action stabilizes the subspace $\mathbb{D}(\mathcal{L}_{-s})^{\bar{\mathfrak{n}}}$. With the adjoint action of L_0 on $\mathcal{U}(\bar{\mathfrak{n}})$, the linear isomorphism $\mathcal{U}(\bar{\mathfrak{n}}) \stackrel{R}{\to} \mathbb{D}(\mathcal{L}_{-s})^{\bar{\mathfrak{n}}}$ is L_0-equvariant. It is clear that each map in $V^* \stackrel{\tilde{\tau}_k|_{V^*}}{\to} \mathcal{P}^k(\mathfrak{g}(1)) \cong \mathrm{Sym}^k(\mathfrak{g}(-1)) \stackrel{\sigma}{\hookrightarrow} \mathcal{U}(\bar{\mathfrak{n}})$ is L_0-equivariant with respect to the natural actions of L_0 on each space, which are induced by the adjoint action of L_0 on $\mathfrak{g}$. Therefore, with the L_0-action (2.5.10), the operator $\Omega_k|_{V^*} : V^* \to \mathbb{D}(\mathcal{L}_{-s})^{\bar{\mathfrak{n}}}$ is an L_0-intertwining operator.

Now we summarize the properties of the $\Omega_k|_{V^*}$ system.

Remark 2.5.11 *It follows from the definition and the observation above that the $\Omega_k|_{V^*}$ system satisfies the following properties:*

1. *The $\Omega_k|_{V^*}$ system satisfies the condition (S1) of Definition 2.1.4.*

2. *When the* $\Omega_k|_{V^*}$ *system is conformally invariant then it is an irreducible, straight, and* L_0*-stable system. By Proposition 2.3.12, it is also a homogeneous system.*

2.6 Technical Lemmas

The aim of this section is to show two technical lemmas that will be used in Section 7.3. For $D \in \mathbb{D}(\mathcal{L}_{-s})$, we denote by $D_{\bar{n}}$ the linear functional $f \mapsto (D \bullet f)(\bar{n})$ for $f \in C^\infty(\bar{N}_0, \mathbb{C}_{\chi^{-s}})$. A simple observation shows that $(D_1 D_2)_{\bar{n}} = (D_1)_{\bar{n}} D_2$ for $D_1, D_2 \in \mathbb{D}(\mathcal{L}_{-s})$; in particular, if $(D_1)_{\bar{n}} = 0$ then $[D_1, D_2]_{\bar{n}} = -(D_2)_{\bar{n}} D_1$.

Lemma 2.6.1 *Suppose that* V^* *is an irreducible constituent of* $\mathfrak{g}(-r+k)^* \otimes \mathfrak{g}(r)^*$*. Let* $X_1, X_2 \in \mathfrak{g}$ *and* $Y_1^*, \ldots, Y_n^* \in V^*$*. If* $\pi_s(X_1)_e = 0$ *and if* $[\pi_s(X_i), \Omega_k(Y_t^*)]_e \in \operatorname{span}_{\mathbb{C}}\{\Omega_k(Y_j^*)_e \mid j = 1, \ldots n\}$ *for all* $i = 1, 2$ *then*

$$\big[\pi_s(X_1), [\pi_s(X_2), \Omega_k(Y_t^*)]\big]_e \in \operatorname{span}_{\mathbb{C}}\{\Omega_k(Y_1^*)_e, \ldots, \Omega_k(Y_n^*)_e\}. \qquad (2.6.2)$$

Proof. Observe that $[\pi_s(X_1), [\pi_s(X_2), \Omega_k(Y_t^*)]]$ is

$$\pi_s(X_1)[\pi_s(X_2), \Omega_k(Y_t^*)] - [\pi_s(X_2), \Omega_k(Y_t^*)]\pi_s(X_1). \qquad (2.6.3)$$

Since, by assumption, we have $\pi_s(X_1)_e = 0$, the first term is zero at e. By assumption, $[\pi_s(X_2), \Omega_k(Y_t^*)]_e$ is a linear combination of $\Omega_k(Y_1^*)_e, \ldots, \Omega_k(Y_n^*)_e$ over $\mathbb{C}$. So it may be written as

$$[\pi_s(X_2), \Omega_k(Y_t^*)]_e = \sum_{j=1}^{n} a_{jt} \Omega_k(Y_j^*)_e$$

with $a_{jt} \in \mathbb{C}$. Then, at the identity e, the second term in (2.6.3) evaluates to

$$-\sum_{j=1}^{n} a_{jt} \Omega_k(Y_j^*)_e \pi_s(X_1).$$

Since $(\pi_s(X_1)\Omega_k(Y_j^*))_e = \pi_s(X_1)_e \Omega_k(Y_j^*) = 0$, we obtain

$$\begin{aligned}
[\pi_s(X_1), [\pi_s(X_2), \Omega_k(Y_t^*)]]_e &= -\sum_{j=1}^{n} a_{jt} \Omega_k(Y_j^*)_e \pi_s(X_1) \\
&= -\sum_{j=1}^{n} a_{jt} [\pi_s(X_1) \Omega_k(Y_j^*)]_e.
\end{aligned}$$

Now the proposed result follows from the assumption that $[\pi_s(X_1), \Omega_k(Y_t^*)]_e$ is a linear combination of $\Omega_k(Y_j^*)_e$ over $\mathbb{C}$. ■

We call

$$\mathfrak{u}_{\mathfrak{l}} = \bigoplus_{\Delta^+(\mathfrak{l})} \mathfrak{g}_\alpha \text{ and } \bar{\mathfrak{u}}_{\mathfrak{l}} = \bigoplus_{\Delta^+(\mathfrak{l})} \mathfrak{g}_{-\alpha},$$

where $\Delta^+(\mathfrak{l})$ is the set of positive roots in $\mathfrak{l}$.

Lemma 2.6.4 *Suppose that $\mathfrak{g}(1)$ is irreducible and that V^* is an irreducible constituent of $\mathfrak{g}(-r+k)^* \otimes \mathfrak{g}(r)^*$. Let X_h be a highest weight vector for $\mathfrak{g}(1)$ and Y_l^* be a lowest weight vector for V^*. If*

$$[\pi_s(X_h), \Omega_k(Y_l^*)]_e = \text{span}_{\mathbb{C}}\{\Omega_k(Y_1^*)_e, \ldots, \Omega_k(Y_n^*)_e\} \tag{2.6.5}$$

with $\{Y_1^*, \ldots, Y_n^*\}$ a basis for V^* then, for any $X \in \mathfrak{g}(1)$ and $Y^* \in V^*$,

$$[\pi_s(X), \Omega_k(Y^*)]_e \in \text{span}_{\mathbb{C}}\{\Omega_k(Y_1^*)_e, \ldots, \Omega_k(Y_n^*)_e\}.$$

Proof. Set $E = \text{span}_{\mathbb{C}}\{\Omega_k(Y_1^*)_e, \ldots, \Omega_k(Y_n^*)_e\}$. We first show that for each $X \in \mathfrak{g}(1)$,

$$[\pi_s(X), \Omega_k(Y_l^*)]_e \in E. \tag{2.6.6}$$

Observe that since $(L, \mathfrak{g}(1))$ is assumed to be irreducible, the L-module $\mathfrak{g}(1)$ is given by $\mathfrak{g}(1) = \mathcal{U}(\bar{\mathfrak{u}}_{\mathfrak{l}})X_h$. Then, as π_s is linear on $\mathfrak{g}(1)$, it suffices to show that (2.6.6) holds when $X = \bar{u}_k \cdot X_h$ with $\bar{u}_k$ a monomial in $\mathcal{U}(\bar{\mathfrak{u}}_{\mathfrak{l}})$. This is done by induction on the order of $\bar{u}_k$. Indeed, the proof is clear once we show that (2.6.6) holds for $X = \bar{Z} \cdot X_h = [\bar{Z}, X_h]$ with $\bar{Z} \in \bar{\mathfrak{u}}_{\mathfrak{l}}$.

By the Jocobi identity, the commutator $[\pi_s([\bar{Z}, X_h]), \Omega_k(Y_l^*)]$ is

$$[\pi_s([\bar{Z}, X_h]), \Omega_k(Y_l^*)] = [\pi_s(\bar{Z}), [\pi_s(X_h), \Omega_k(Y_l^*)]] - [\pi_s(X_h), [\pi_s(\bar{Z}), \Omega_k(Y_l^*)]]. \tag{2.6.7}$$

By the $\mathfrak{l}$-equivariance of the operator $\Omega_k : V^* \to \mathbb{D}(\mathcal{L}_{-s})^{\bar{\mathfrak{n}}}$, it follows that

$$[\pi_s(\bar{Z}), \Omega_k(Y_l^*)] = \Omega_k([\bar{Z}, Y_l^*]).$$

Since $\bar{Z} \in \bar{\mathfrak{u}}_{\mathfrak{l}}$ and Y_l^* is a lowest weight vector, we have $\Omega_k([\bar{Z}, Y_l^*]) = 0$, and so is the second term of the right hand side of (2.6.7). Thus we have

$$[\pi_s([\bar{Z}, X_h]), \Omega_k(Y_l^*)]_e = [\pi_s(\bar{Z}), [\pi_s(X_h), \Omega_k(Y_l^*)]]_e. \tag{2.6.8}$$

Now, by hypotheses and the $\mathfrak{l}$-equivariance of Ω_k, it follows that

$$[\pi_s(X_h), \Omega_k(Y_l^*)]_e, [\pi_s(\bar{Z}), \Omega_k(Y_l^*)]_e \in E.$$

As $\bar{Z} \in \bar{\mathfrak{u}}_{\mathfrak{l}}$, by Proposition 2.3.3, we have $\pi_s(\bar{Z})_e = 0$. Thus, by Lemma 2.6.1, we obtain $[\pi_s(\bar{Z}), [\pi_s(X_h), \Omega_k(Y_l^*)]]_e \in E$, and so, by (2.6.8), $[\pi_s([\bar{Z}, X_h]), \Omega_k(Y_l^*)]_e \in E$.

Next we show that for any $X \in \mathfrak{g}(1)$ and $Y^* \in V^*$,

$$[\pi_s(X), \Omega_k(Y^*)]_e \in E. \tag{2.6.9}$$

Once again since V^* is irreducible, it is given by $V^* = \mathcal{U}(\mathfrak{u}_{\mathfrak{l}})Y_l^*$. As before, it is enough to show that (2.6.9) holds for $Y^* = Z \cdot Y_l^*$ with $Z \in \mathfrak{u}_{\mathfrak{l}}$. Since $\Omega_k(Z \cdot Y_l^*) = [\pi_s(Z), \Omega_k(Y_l^*)]$, by the Jacobi identity, the commutator $[\pi_s(X), \Omega_k(Z \cdot Y_l^*)]$ is

$$[\pi_s(X), \Omega_k(Z \cdot Y_l^*)] = [\pi_s(Z), [\pi_s(X), \Omega_k(Y_l^*)]] - [[\pi_s(Z), \pi_s(X)], \Omega_k(Y_l^*)]. \tag{2.6.10}$$

We showed above that $[\pi_s(X), \Omega_k(Y_l^*)]_e \in E$. Since $\pi_s(Z)_e = 0$ and $[\pi_s(Z), \Omega_k(Y_l^*)]_e \in E$, by Lemma 2.6.1, the first term of the right hand side of (2.6.10) satisfies

$$[\pi_s(Z), [\pi_s(X), \Omega_k(Y_l^*)]]_e \in E.$$

Moreover, as $[\pi_s(Z), \pi_s(X)] = \pi_s([Z, X])$ with $[Z, X] \in \mathfrak{g}(1)$, by what we have shown above, the second term satisfies

$$[[\pi_s(Z), \pi_s(X)], \Omega_k(Y_l^*)]_e \in E.$$

Hence, $[\pi_s(X), \Omega_k(Z \cdot Y_l^*)]_e \in E$. ∎

2.7 The Ω_k Systems and Generalized Verma Modules

To conclude this chapter, we show that conformally invariant Ω_k systems induce non-zero $\mathcal{U}(\mathfrak{g})$-homomorphisms between certain generalized Verma modules. The main idea is that conformally invariant Ω_k systems yield finite dimensional simple $\mathfrak{l}$-submodules of generalized Verma modules, on which $\mathfrak{n}$ acts trivially.

In general, to describe the relationship between conformally invariant systems on a $\mathfrak{g}_0$-bundle $\mathcal{V} \to M$ and generalized Verma modules, we realize generalized Verma modules as the space of smooth distributions on M supported at the identity. However, in our setting that the vector bundle $\mathcal{V}$ is a line bundle $\mathcal{L}_{-s}$, it is not necessary to use the realization. Thus, in this section, we are going to describe the relationship without using the realization. For more general theory on the relationship between conformally invariant systems and generalized Verma modules, see Sections 3, 5, and 6 of [2].

A **generalized Verma module** $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{q})} W$ is a $\mathcal{U}(\mathfrak{g})$-module that is induced from a finite dimensional simple $\mathfrak{l}$-module W on which $\mathfrak{n}$ acts trivially. See Section A.1 for more details on generalized Verma modules. In this section we parametrize those modules as

$$M_{\mathfrak{q}}[W] = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{q})} W.$$

We first observe that the differential operators in $\mathbb{D}(\mathcal{L}_{-s})^{\bar{\mathfrak{n}}}$ can be described in terms of elements of $M_{\mathfrak{q}}[\mathbb{C}_{s\lambda_{\mathfrak{q}}}]$, where $\mathbb{C}_{s\lambda_{\mathfrak{q}}}$ is the $\mathfrak{q}$-module derived from the Q_0-representation $(\chi^s, \mathbb{C})$. By identifying $M_{\mathfrak{q}}[\mathbb{C}_{s\lambda_{\mathfrak{q}}}]$ as $\mathcal{U}(\bar{\mathfrak{n}}) \otimes \mathbb{C}_{s\lambda_{\mathfrak{q}}}$, the map $M_{\mathfrak{q}}[\mathbb{C}_{s\lambda_{\mathfrak{q}}}] \to \mathcal{U}(\bar{\mathfrak{n}})$ given by $u \otimes 1 \mapsto u$ is an isomorphism of vector spaces. The composition

$$M_{\mathfrak{q}}[\mathbb{C}_{s\lambda_{\mathfrak{q}}}] \to \mathcal{U}(\bar{\mathfrak{n}}) \xrightarrow{R} \mathbb{D}(\mathcal{L}_{-s})^{\bar{\mathfrak{n}}} \tag{2.7.1}$$

is then a vector-space isomorphism.

Let W^* be an irreducible constituent of $\mathfrak{g}(-r+k)^* \otimes \mathfrak{g}(k)^*$ so that the L_0-intertwining operator $\Omega_k|_{W^*} : W^* \to \mathbb{D}(\mathcal{L}_{-s})^{\bar{\mathfrak{n}}}$ is not identically zero. For $Y^* \in W^*$, if $\omega_k(Y^*) = \omega_k|_{W^*}(Y^*)$ denotes the element in $\mathcal{U}(\bar{\mathfrak{n}})$ that corresponds to $\Omega_k(Y^*) = \Omega_k|_{W^*}(Y^*)$ in $\mathbb{D}(\mathcal{L}_{-s})^{\bar{\mathfrak{n}}}$ via right differentiation R in (2.7.1) then the linear operator $\omega_k|_{W^*} : W^* \to \mathcal{U}(\bar{\mathfrak{n}})$ is L-equivariant. Indeed, for $l \in L$ and $Y^* \in W^*$, we have

$$\omega_k(l \cdot Y^*) = \mathrm{Ad}(l)\omega_k(Y^*),$$

where the action $l \cdot Y^*$ is the standard action of L on W^*, which is induced from the adjoint action of L on W.

Define

$$M_{\mathfrak{q}}[W]^{\mathfrak{n}} = \{v \in M_{\mathfrak{q}}[W] \mid X \cdot v = 0 \text{ for all } X \in \mathfrak{n}\}.$$

The following result is the specialization of Theorem 19 in [2] to the present situation.

Theorem 2.7.2 *If $D = D_1, \ldots, D_m$ is a straight L_0-stable homogeneous conformally invariant system on the line bundle $\mathcal{L}_{-s}$, and if ω_j denotes the element in $\mathcal{U}(\bar{\mathfrak{n}})$ that corresponds to D_j for $j = 1, \ldots, m$ via right differentiation R then the space*

$$F(D) = \mathrm{span}_{\mathbb{C}}\{\omega_j \otimes 1 \mid j = 1, \ldots, m\}$$

is an L-invariant subspace of $M_{\mathfrak{q}}[\mathbb{C}_{s\lambda_{\mathfrak{q}}}]^{\mathfrak{n}}$.

If the $\Omega_k|_{W^*}$ system is

$$\Omega_k|_{W^*} = \Omega_k(Y_1^*), \ldots, \Omega_k(Y_m^*),$$

where $\{Y_1^*, \ldots, Y_m^*\}$ is a basis of W^*, then the space $F(\Omega_k|_{W^*})$ is given by

$$F(\Omega_k|_{W^*}) = \mathrm{span}_{\mathbb{C}}\{\omega_k(Y_j^*) \otimes 1 \mid j = 1, \ldots, m\} \subset M_{\mathfrak{q}}[\mathbb{C}_{s\lambda_{\mathfrak{q}}}].$$

Corollary 2.7.3 *If the $\Omega_k|_{W^*}$ system is conformally invariant on the line bundle $\mathcal{L}_{s_0}$ then $F(\Omega_k|_{W^*})$ is an L-invariant subspace of $M_{\mathfrak{q}}[\mathbb{C}_{-s_0\lambda_{\mathfrak{q}}}]^{\mathfrak{n}}$.*

Proof. By Remark 2.5.11, if the $\Omega_k|_{W^*}$ system is conformally invariant then it is a straight, L_0-stable, and homogeneous system. Now this corollary follows from Theorem 2.7.2. ■

Now suppose that the $\Omega_k|_{W^*}$ system is conformally invariant over $\mathcal{L}_{s_0}$. Then, by Corollary 2.7.3, it follows that $F(\Omega_k|_{W^*})$ is an L-invariant subspace of $M_{\mathfrak{q}}[\mathbb{C}_{-s_0\lambda_{\mathfrak{q}}}]^{\mathfrak{n}}$. On the other hand, there exists a vector space isomorphism

$$F(\Omega_k|_{W^*}) \to W^* \otimes \mathbb{C}_{-s\lambda_{\mathfrak{q}}}, \tag{2.7.4}$$

that is given by $\omega_k(Y_j^*) \otimes 1 \mapsto Y_j^* \otimes 1$. It is clear that the vector space isomorphism is L-equivariant with respect to the standard action of L on the tensor products $F(\Omega_k|_{W^*}) \subset \mathcal{U}(\bar{\mathfrak{n}}) \otimes \mathbb{C}_{-s\lambda_{\mathfrak{q}}}$ and $W^* \otimes \mathbb{C}_{-s\lambda_{\mathfrak{q}}}$. In particular, since W^* is an irreducible L-module, if W^* has highest weight ν then $F(\Omega_k|_{W^*})$ is the irreducible L-module with highest weight $\nu - s_0\lambda_{\mathfrak{q}}$. [1] Moreover, as $F(\Omega_k|_{W^*}) \subset M_{\mathfrak{q}}[\mathbb{C}_{-s_0\lambda_{\mathfrak{q}}}]^{\mathfrak{n}}$, the nilradical $\mathfrak{n}$ acts on $F(\Omega_k|_{W^*})$ trivially. Therefore the inclusion map $\iota \in \mathrm{Hom}_L\big(F(\Omega_k|_{W^*}), M_{\mathfrak{q}}[\mathbb{C}_{-s_0\lambda_{\mathfrak{q}}}]\big)$ induces a non-zero $\mathcal{U}(\mathfrak{g})$-homomorphism $\varphi_{\Omega_k} \in \mathrm{Hom}_{\mathcal{U}(\mathfrak{g}),L}\big(M_{\mathfrak{q}}[F(\Omega_k|_{W^*})], M_{\mathfrak{q}}[\mathbb{C}_{-s_0\lambda_{\mathfrak{q}}}]\big)$ of generalized Verma modules, that is given by

$$\begin{gathered} M_{\mathfrak{q}}[F(\Omega_k|_{W^*})] \stackrel{\varphi_{\Omega_k}}{\to} M_{\mathfrak{q}}[\mathbb{C}_{-s_0\lambda_{\mathfrak{q}}}] \\ u \otimes \big(\omega_k(Y) \otimes 1\big) \mapsto u \cdot \iota\big(\omega_k(Y) \otimes 1\big). \end{gathered} \tag{2.7.5}$$

If $F(\Omega_k|_{W^*}) = \mathbb{C}_{-s_0\lambda_{\mathfrak{q}}}$ then the map in (2.7.5) is just the identity map. However, Proposition 2.7.6 below shows that it does not happen.

Proposition 2.7.6 *Let W^* be an irreducible constituent of $\mathfrak{g}(-r+k)^* \otimes \mathfrak{g}(r)^*$ with $k = 1, \ldots, 2r$, so that $\Omega_k|_{W^*} : W^* \to \mathbb{D}(\mathcal{L}_{-s})^{\bar{\mathfrak{n}}}$ is not identically zero. If the $\Omega_k|_{W^*}$ system is conformally invariant on the line bundle $\mathcal{L}_{s_0}$ then $F(\Omega_k|_{W^*}) \neq \mathbb{C}_{-s_0\lambda_{\mathfrak{q}}}$*

Proof. Observe that if ν is the highest weight for W^* then $F(\Omega_k|_{W^*})$ has highest weight $\nu - s_0\lambda_{\mathfrak{q}}$. If $F(\Omega_k|_{W^*}) = \mathbb{C}_{-s_0\lambda_{\mathfrak{q}}}$ then $\nu = 0$, and so the irreducible constituent $W \subset \mathfrak{g}(-r+k) \otimes \mathfrak{g}(r)$ would have highest weight 0. It is known that if γ is the highest weight for $\mathfrak{g}(r)$ then the highest weight of any irreducible constituent of $\mathfrak{g}(-r+k) \otimes \mathfrak{g}(r)$

[1]See Section 3.2 for the details of what we mean by a highest weight of a finite dimensional representation of reductive group L.

is of the form $\gamma+\eta$ with η some weight for $\mathfrak{g}(-r+k)$ (see for instance [21, Proposition 3.2]). Thus, the highest weight 0 for W must be of the form $0 = \gamma + (-\gamma)$. However, $-\gamma$ cannot be a weight for $\mathfrak{g}(-r + k)$ for any $k = 1, \ldots, 2r$, since only $\mathfrak{g}(-r)$ has weight $-\gamma$. Therefore $F(\Omega_k|_{W^*}) \neq \mathbb{C}_{-s_0\lambda_\mathfrak{q}}$. ■

Corollary 2.7.7 *Under the same hypotheses for Proposition 2.7.6, the generalized Verma module* $M_\mathfrak{q}[\mathbb{C}_{-s_0\lambda_\mathfrak{q}}]$ *is reducible.*

Proof. If ν is the highest weight for W^* then, by the proof for Proposition 2.7.6, it follows that $F(\Omega_k|_{W^*}) \neq \mathbb{C}_{-s_0\lambda_\mathfrak{q}}$. Now this corollary follows from (2.7.5). ■

CHAPTER 3

Parabolic Subalgebras and $\mathbb{Z}$-gradings

It has been observed in Section 2.5 that the $\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{j=-r}^{r} \mathfrak{g}(j)$ on $\mathfrak{g}$ and the parabolic subalgebra $\mathfrak{q}$ play a role to construct the Ω_k systems. In this chapter we study those in detail for $\mathfrak{q}$ a maximal two-step nilpotent parabolic subalgebra of non-Heisenberg type. The Ω_1 system and the Ω_2 systems of those parabolics will be studied in Chapter 4 and Chapter 7, respectively.

3.1 k-step Nilpotent Parabolic Subalgebras

We shall later construct the Ω_1 system and the Ω_2 systems of a maximal two-step nilpotent parabolic $\mathfrak{q}$. To do so, in this section we classify the k-step nilpotent parabolic subalgebras $\mathfrak{q}$ by the subsets of simple roots. This is done in Proposition 3.1.4.

Let $\mathfrak{r}$ be any nonzero Lie algebra. Put $\mathfrak{r}_0 = \mathfrak{r}$, $\mathfrak{r}_1 = [\mathfrak{r}, \mathfrak{r}]$, and $\mathfrak{r}_k = [\mathfrak{r}, \mathfrak{r}_{k-1}]$ for $k \in \mathbb{Z}_{>0}$. We call $\mathfrak{r}_k$ the **k-th step** of $\mathfrak{r}$ for $k \in \mathbb{Z}_{\geq 0}$. The Lie algebra $\mathfrak{r}$ is called **nilpotent** if $\mathfrak{r}_k = 0$ for some k, and it is called **k-step nilpotent** if $\mathfrak{r}_{k-1} \neq 0$ and $\mathfrak{r}_k = 0$. In particular, if $[\mathfrak{r}, \mathfrak{r}] = 0$ then $\mathfrak{r}$ is called **abelian**, and if $\dim([\mathfrak{r}, \mathfrak{r}]) = 1$ then $\mathfrak{r}$ is called **Heisenberg**. Note that $\mathfrak{r}$ is Heisenberg if and only if its center $\mathfrak{z}(\mathfrak{r})$ is one-dimensional. If the nilpotent radical $\mathfrak{n}$ of a parabolic subalgebra $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{n}$ is k-step nilpotent (resp. abelian or Heisenberg) then we say that $\mathfrak{q}$ is a **k-step nilpotent** (resp. **abelian** or **Heisenberg)** parabolic.

If $\beta = \sum_{\alpha \in \Pi} m_\alpha \alpha \in \sum_{\alpha \in \Pi} \mathbb{Z}\alpha$ then we say that $|m_\alpha|$ are the **multiplicities** of α in β. Proposition 3.1.4 below determines k-step nilpotent parabolic subalgebras

$\mathfrak{q}_S$ by the sum of the multiplicities of the simple roots of $S \subset \Pi$ in the highest root. Although it is a well-known fact, we include a proof in this thesis, since we couldn't find one in the literature. To prove the proposition it is convenient to show two technical lemmas, namely, Lemma 3.1.2 and Lemma 3.1.3. In Lemma 3.1.2 and Lemma 3.1.3, the subalgebras $\mathfrak{l}$ and $\mathfrak{n}$ are assumed to be the Levi factor and the nilpotent radical of $\mathfrak{q}_S$ with $S = \{\alpha_{i_1}, \dots, \alpha_{i_r}\}$, respectively.

Remark 3.1.1 *It is easily shown by the Jacobi identity and the induction on k that we have $[\mathfrak{l}, \mathfrak{n}_k] \subset \mathfrak{n}_k$ for each k. In particular, if $\alpha + \beta \in \Delta$ with $\alpha \in \Delta(\mathfrak{l})$ and $\beta \in \Delta(\mathfrak{n}_k)$ then $\alpha + \beta \in \Delta(\mathfrak{n}_k)$, where $\Delta(\mathfrak{l})$ and $\Delta(\mathfrak{n}_k)$ are the subsets of roots that contribute to $\mathfrak{l}$ and $\mathfrak{n}_k$, respectively.*

Lemma 3.1.2 *Suppose that β is a root in Δ and let m_{i_j} be the multiplicity of α_{i_j} in β. If $\sum_{j=1}^{r} m_{i_j} = k$ then $\beta \in \Delta(\mathfrak{n}_{k-1})$.*

Proof. For $\beta \in \Delta$, it is well known that there exists an ordered set $O_\beta = \{\alpha_1, \dots, \alpha_s\}$ of simple roots so that $\beta = \sum_{t=1}^{s} \alpha_t$ having the property that each ordered partial sum is a root (see for instance [9, Corollary 10.2A]). Note that some of the roots in O_β belong to S and others are in $\Pi(\mathfrak{l}) = \Delta(\mathfrak{l}) \cap \Pi$.

We prove this lemma by induction on the sum, $\sum_{j=1}^{r} m_{i_j}$, of the multiplicities of α_{i_j} in S. When $\sum_{j=1}^{r} m_{i_j} = 1$, we have $O_\beta \cap S = \{\alpha_h\}$ for some $\alpha_h \in S \subset \Delta(\mathfrak{n})$. Write $\delta = \sum_{t=1}^{h} \alpha_t$. If $\alpha_h = \alpha_1$ then $\delta = \alpha_h \in \Delta(\mathfrak{n}) = \Delta(\mathfrak{n}_0)$. If $\alpha_h \neq \alpha_1$ then since each partial sum is a root, we have $\sum_{t=1}^{h-1} \alpha_t \in \Delta(\mathfrak{l})$. Since $[\mathfrak{l}, \mathfrak{n}_0] \subset \mathfrak{n}_0$, it follows that

$$\delta = \sum_{t=1}^{h} \alpha_t = \sum_{t=1}^{h-1} \alpha_t + \alpha_h \in \Delta(\mathfrak{n}_0).$$

Since each sum $\sum_{t=1}^{d} \alpha_t$ for $d \geq h$ is a root and all α_t for $t > h$ are in $\Delta(\mathfrak{l})$, by Remark 3.1.1, we conclude that

$$\beta = \delta + \alpha_{h+1} + \cdots + \alpha_s \in \Delta(\mathfrak{n}_0).$$

Now we assume that the proposed statement holds for $k-1 \geq \sum_{j=1}^{r} m_{i_j} \geq 1$. Let $\sum_{j=1}^{r} m_{i_j} = k$. There are two cases, $\alpha_s \in S$ or $\alpha_s \in \Pi(\mathfrak{l})$. If $\alpha_s \in S$ then the sum of the multiplicities of the simple roots in S contributing to $\beta - \alpha_s$ is equal to $k-1$. By induction hypothesis, we have $\beta - \alpha_s \in \Delta(\mathfrak{n}_{k-2})$. Therefore, $\beta = (\beta - \alpha_s) + \alpha_s \in \Delta([\mathfrak{n}, \mathfrak{n}_{k-2}]) = \Delta(\mathfrak{n}_{k-1})$.

When $\alpha_s \in \Pi(\mathfrak{l})$, let α_l be the largest root in the order of O_β so that $\alpha_l \in S$. Then the sum of the multiplicities of the simple roots from S in the root $\sum_{t=1}^{l} \alpha_t$ is equal to k. Assuming as before, we conclude that $\sum_{t=1}^{l} \alpha_t \in \Delta(\mathfrak{n}_{k-1})$. Now, once again, since each sum $\sum_{t=1}^{d} \alpha_t$ for $d \geq l$ is a root and all α_t for $t > l$ are in $\Delta(\mathfrak{l})$, by Remark 3.1.1, we conclude that

$$\beta = \sum_{t=1}^{l} \alpha_t + \alpha_{l+1} + \cdots + \alpha_s \in \Delta(\mathfrak{n}_{k-1}).$$

■

Lemma 3.1.3 *If $\beta \in \Delta(\mathfrak{n}_k)$ and m_{i_j} are the multiplicities of α_{i_j} in β then $\sum_{j=1}^{r} m_{i_j} \geq k+1$.*

Proof. We prove it by induction on k. Observe that if $\beta \in \Delta(\mathfrak{n}) = \Delta^+ \backslash \Delta(\mathfrak{l})$ then there exists $\alpha_{i_j} \in S$ so that the multiplicity of α_{i_j} in β is non-zero, because we would have $\beta \in \Delta(\mathfrak{l})$, otherwise. Thus the case $k = 0$ is clear. We then assume that this holds for $k = l$. Let $\beta \in \Delta(\mathfrak{n}_{l+1})$. Since $\mathfrak{n}_{l+1} = [\mathfrak{n}_l, \mathfrak{n}]$, the root β may be written as $\beta = \beta' + \beta''$ with $\beta' \in \Delta(\mathfrak{n}_l)$ and $\beta'' \in \Delta(\mathfrak{n})$. Denoting by $m_{i_j}(\beta)$ the multiplicities of α_{i_j} in β, we have

$$\sum_{j=1}^{r} m_{i_j}(\beta) = \sum_{j=1}^{r} m_{i_j}(\beta' + \beta'') = \sum_{j=1}^{r} m_{i_j}(\beta') + \sum_{j=1}^{r} m_{i_j}(\beta'') \geq (l+1) + 1 = l+2.$$

By induction the lemma follows. ■

We remark that if the highest root γ is $\gamma = \sum_{\alpha \in \Pi} m_\alpha \alpha$ then for any root $\beta = \sum_{\alpha \in \Pi} n_\alpha \alpha$, it follows that $n_\alpha \leq m_\alpha$ for all $\alpha \in \Pi$.

Proposition 3.1.4 *Let $\mathfrak{g}$ be a complex simple Lie algebra with highest root γ, and $\mathfrak{q}_S = \mathfrak{l} \oplus \mathfrak{n}$ be the parabolic subalgebra of $\mathfrak{g}$ that is parametrized by S with $S = \{\alpha_{i_1}, \ldots, \alpha_{i_r}\} \subset \Pi$. Then $\mathfrak{n}$ is k-step nilpotent if and only if $k = m_{i_1} + m_{i_2} + \cdots + m_{i_r}$, where m_{i_j} are the multiplicities of α_{i_j} in γ.*

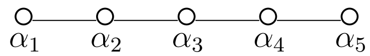
Proof. First we show that if $k = \sum_{j=1}^{r} m_{i_j}$ then $\mathfrak{n}$ is k-step nilpotent. If $k = \sum_{j=1}^{r} m_{i_j}$ then, by Lemma 3.1.2, we have $\gamma \in \Delta(\mathfrak{n}_{k-1})$; in particular, $\mathfrak{n}_{k-1} \neq 0$. If $\mathfrak{n}_k \neq 0$ then there would exist $\beta \in \Delta(\mathfrak{n}_k)$. If n_{i_j} are the multiplicities of α_{i_j} in β then, by Lemma 3.1.3, it follows that

$$\sum_{j=1}^{r} n_{i_j} \geq k + 1 > k.$$

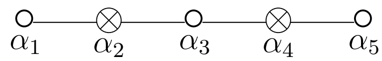
This contradicts the remark above. Therefore $\mathfrak{n}_k = 0$, and so $\mathfrak{n}$ is k-step nilpotent. Conversely, suppose that $\mathfrak{n}$ is k-step nilpotent. If $\sum_{j=1}^{r} m_{i_j} = l$ then, as we showed above, $\mathfrak{n}$ is l-step nilpotent. Hence, $l = k$. ■

To finish this section we introduce subdiagrams of Dynkin diagrams that associate to parabolics $\mathfrak{q}_S$ and classification types of them. First, Theorem 2.2.3 shows that there exists a bijection between the standard parabolics $\mathfrak{q}_S$ and the subsets S of simple roots. This allows us to associate $\mathfrak{q}_S$ to subdiagrams of Dynkin diagrams. The subdiagrams that associates to $\mathfrak{q}_S$ are obtained by deleting the nodes of the Dynkin diagram of $\mathfrak{g}$ that correspond to the simple roots in S, and the edges in incident on them. We call such subdiagrams **deleted Dynkin diagrams**. With the multiplicities of simple roots in the highest root of $\mathfrak{g}$ in hand, by Proposition 3.1.4, we can also see the number of steps of nilradical $\mathfrak{n}$ of $\mathfrak{q}_S$ from the deleted Dynkin diagram. Example 3.1.5 below describes the deleted Dynkin diagram of a given parabolic $\mathfrak{q}_S$ and how we read the diagram. For simplicity, we depict deleted Dynkin diagrams by crossing out the deleted nodes.

Example 3.1.5 *Take* $\mathfrak{g} = \mathfrak{sl}(6, \mathbb{C})$*. The set of simple roots* Π *is* $\Pi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$ *with Dynkin diagram*



Choose $S = \{\alpha_2, \alpha_4\}$*. Then the deleted Dynkin diagram of parabolic subalgebra* $\mathfrak{q}_S$ *corresponding to the subset* S *is*



Moreover, Figure B.2 in Appendix B shows that the multiplicity of each simple root in the highest root of $\mathfrak{g}$ *of type* A_n *is 1, so this parabolic* $\mathfrak{q}_S$ *is a two-step nilpotent parabolic.*

In later sections we often refer to parabolic subalgebras $\mathfrak{q}_S$ by their corresponding subset S of simple roots. To this end, we are going to define classification types of parabolics $\mathfrak{q}_S$. In Definition 3.1.6 below, we mean by classification type $\mathcal{T}$ of $\mathfrak{g}$ type A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, or G_2.

Definition 3.1.6 *If* $\mathfrak{g}$ *is a complex simple Lie algebra of classification type* $\mathcal{T}$ *and* S *is a subset of* Π *of simple roots then we say that a parabolic subalgebra* $\mathfrak{q}_S$ *of* $\mathfrak{g}$ *is of* ***type*** $\mathcal{T}(S)$*, or* ***type*** $\mathcal{T}(i_1, \ldots, i_k)$ *if* $S = \{\alpha_{i_1}, \ldots, \alpha_{i_k}\}$*.*

For example, the parabolic subalgebra $\mathfrak{q}_S$ in Example 3.1.5 is of type $A_5(2, 4)$. Any maximal parabolic subalgebra is of type $\mathcal{T}(i)$ for some $\alpha_i \in \Pi$. In this thesis we use the Bourbaki conventions [4] for the labels of the simple roots (see Figure B.1 in Appendix B for the labels).

3.2 Maximal Two-Step Nilpotent Parabolic $\mathfrak{q}$ of Non-Heisenberg type

The aim of this section is to study the 2-grading $\mathfrak{g} = \bigoplus_{j=-2}^{2} \mathfrak{g}(j)$ on $\mathfrak{g}$, that is induced from a maximal two-step nilpotent parabolic subalgebra $\mathfrak{q}$ of non-Heisenberg type.

Assume that $\mathfrak{g}$ has rank greater than one and that $\alpha_{\mathfrak{q}}$ is a simple root, so that the parabolic subalgebra $\mathfrak{q} = \mathfrak{q}_{\{\alpha_{\mathfrak{q}}\}} = \mathfrak{l} \oplus \mathfrak{n}$ parameterized by $\alpha_{\mathfrak{q}}$ is a maximal two-step nilpotent parabolic with $\dim([\mathfrak{n}, \mathfrak{n}]) > 1$. Let $\langle \cdot, \cdot \rangle$ be the inner product induced on $\mathfrak{h}^*$ corresponding to the Killing form κ. Write $||\alpha||^2 = \langle \alpha, \alpha \rangle$ for $\alpha \in \Delta$. The coroot of α is $\alpha^\vee = 2\alpha/\langle \alpha, \alpha \rangle$.

Recall from Section 2.2 that $\lambda_{\mathfrak{q}}$ denotes the fundamental weight for $\alpha_{\mathfrak{q}}$. As $\Delta(\mathfrak{l}) = \{\alpha \in \Delta \mid \alpha \in \text{span}(\Pi \backslash \{\alpha_{\mathfrak{q}}\})\}$ and $\Delta(\mathfrak{n}) = \Delta^+ \backslash \Delta(\mathfrak{l})$, we have

$$\langle \lambda_{\mathfrak{q}}, \beta \rangle \begin{cases} = 0 & \text{if } \beta \in \Delta(\mathfrak{l}) \\ > 0 & \text{if } \beta \in \Delta(\mathfrak{n}) \ . \end{cases}$$

Observe that if $H_{\lambda_{\mathfrak{q}}} \in \mathfrak{h}$ is defined by $\kappa(H, H_{\lambda_{\mathfrak{q}}}) = \lambda_{\mathfrak{q}}(H)$ for all $H \in \mathfrak{h}$ and if

$$H_{\mathfrak{q}} = \frac{2}{||\alpha_{\mathfrak{q}}||^2} H_{\lambda_{\mathfrak{q}}} \tag{3.2.1}$$

then $\beta(H_{\mathfrak{q}})$ is the multiplicity of $\alpha_{\mathfrak{q}}$ in β. In particular, it follows from Proposition 3.1.4 that for $\beta \in \Delta^+$, $\beta(H_{\mathfrak{q}})$ only can assume the values of 0, 1, or 2. Therefore, if $\mathfrak{g}(j)$ denotes the j-eigenspace of $\text{ad}(H_{\mathfrak{q}})$ then the action of $\text{ad}(H_{\mathfrak{q}})$ on $\mathfrak{g}$ induces a 2-grading

$$\mathfrak{g} = \mathfrak{g}(-2) \oplus \mathfrak{g}(-1) \oplus \mathfrak{g}(0) \oplus \mathfrak{g}(1) \oplus \mathfrak{g}(2)$$

with parabolic subalgebra

$$\mathfrak{q} = \mathfrak{g}(0) \oplus \mathfrak{g}(1) \oplus \mathfrak{g}(2).$$

Here we have $\mathfrak{l} = \mathfrak{g}(0)$ and $\mathfrak{n} = \mathfrak{g}(1) \oplus \mathfrak{g}(2)$. The subalgebra $\bar{\mathfrak{n}}$, the opposite of $\mathfrak{n}$, is given by

$$\bar{\mathfrak{n}} = \mathfrak{g}(-1) \oplus \mathfrak{g}(-2).$$

Observe that L acts on each of the subspaces $\mathfrak{g}(j)$ via the adjoint representation. The goal of this section is to show that $\mathfrak{g}(j)$ are irreducible L-modules for $j \neq 0$.

Via the Killing form, $\mathfrak{g}(-1)$ and $\mathfrak{g}(-2)$ are dual to $\mathfrak{g}(1)$ and $\mathfrak{g}(2)$, respectively. Thus, we will show that $\mathfrak{g}(1)$ and $\mathfrak{g}(2)$ are L-irreducible; hence, so are true for $\mathfrak{g}(-1)$ and $\mathfrak{g}(-2)$.

The following proposition is well known. However, since the argument used in the proof will be referred in the proof for Corollary 3.2.3 below, we give a proof.

Proposition 3.2.2 *Assume that $\mathfrak{g}$ is a graded complex semisimple Lie algebra with $\mathfrak{g} = \bigoplus_j \mathfrak{g}(j)$, and let $\mathfrak{q} = \mathfrak{g}(0) \oplus \bigoplus_{j>0} \mathfrak{g}(j)$ with $\mathfrak{g}(1) \neq 0$. Then $\mathfrak{g}(1)$ is $\mathfrak{g}(0)$-irreducible if and only if $\mathfrak{q}$ is a maximal parabolic.*

Proof. We first show that if $\mathfrak{q}$ is not maximal then $\mathfrak{g}(1)$ is not $\mathfrak{g}(0)$-irreducible. Under this assumption there are at least two distinct simple roots in $\Pi \backslash \Delta(\mathfrak{g}(0))$, say β_1 and β_2. Let X_{β_1} and X_{β_2} be root vectors for β_1 and β_2, respectively. If $\mathcal{U}(\mathfrak{g}(0))$ denotes the universal enveloping algebra of $\mathfrak{g}(0)$ then $\mathcal{U}(\mathfrak{g}(0))X_{\beta_1}$ and $\mathcal{U}(\mathfrak{g}(0))X_{\beta_2}$ are two $\mathfrak{g}(0)$-submodules of $\mathfrak{g}(1)$. Since β_1 and β_2 are simple, $\mathcal{U}(\mathfrak{g}(0))X_{\beta_1} \neq \mathcal{U}(\mathfrak{g}(0))X_{\beta_2}$. Hence $\mathfrak{g}(1)$ is reducible.

To prove the converse, as $\mathfrak{g}(0) = \mathfrak{z}(\mathfrak{g}(0)) \oplus \mathfrak{g}(0)_{ss}$ and the center $\mathfrak{z}(\mathfrak{g}(0))$ acts by scalars on $\mathfrak{g}(1)$, it suffices to show that $\mathfrak{g}(1)$ is an irreducible $\mathfrak{g}(0)_{ss}$-module. As in [9, Corollary 10.2A] we write $\delta \in \Delta^+$ as

$$\delta = \alpha_{i_1} + \cdots + \alpha_{i_n}$$

with $\alpha_{i_j} \in \Pi$ (not necessarily distinct) in such a way that each partial sum $\alpha_{i_1} + \cdots + \alpha_{i_j}$ is a root. If $\mathfrak{q}$ is maximal then there exists unique simple root $\beta \in \Pi \backslash \Delta(\mathfrak{g}(0))$. Each root $\delta \in \Delta(\mathfrak{g}(1))$ is of the form

$$\delta = \alpha_{i_1} + \cdots + \alpha_{i_k} + \beta + \alpha_{i_m} + \cdots + \alpha_{i_n},$$

where the sum $\alpha_{i_1} + \cdots + \alpha_{i_k} \equiv \alpha_{\mathfrak{q}}$ is a root with $\alpha_{i_j} \in \Delta(\mathfrak{g}(0))$. Let $X_{\alpha_{\mathfrak{q}}}$ and X_β be root vectors for $\alpha_{\mathfrak{q}}$ and β, respectively. If X_j is a root vector for α_{i_j} then

$$0 \neq \mathrm{ad}(X_n)\mathrm{ad}(X_{n-1}) \cdots \mathrm{ad}(X_{m+1})\mathrm{ad}(X_m)\mathrm{ad}(X_{\alpha_{\mathfrak{q}}})X_\beta$$

is a non-zero element in $(\mathcal{U}(\mathfrak{g}(0)_{ss})X_\beta) \cap \mathfrak{g}_\delta$. Since $\delta \in \Delta(\mathfrak{g}(1))$ is arbitrary, it is followed that $\mathfrak{g}(1) = \mathcal{U}(\mathfrak{g}(0)_{ss})X_\beta$. We quote the Theorem of the Highest Weight to conclude that $\mathfrak{g}(1)$ is $\mathfrak{g}(0)_{ss}$-irreducible with lowest weight β. ∎

Let $\mathfrak{l} = \mathfrak{z}(\mathfrak{l}) \oplus \mathfrak{l}_{ss}$ be the decomposition of $\mathfrak{l}$, that corresponds to $L = Z(L)^\circ L_{ss}$ with $Z(L)^\circ$ the identity component of the center of L and L_{ss} the semisimple part of L. We say that a weight $\nu \in \mathfrak{h}^*$ is a highest weight of a finite dimensional L-module V if $\nu|_{\mathfrak{h}_{ss}}$ is a highest weight of V as an L_{ss}-module, where $\mathfrak{h}_{ss} = \mathfrak{h} \cap \mathfrak{l}_{ss}$. A lowest weight of a finite dimensional L-module is similarly defined.

Corollary 3.2.3 *If $\mathfrak{q} = \mathfrak{g}(0) \oplus \mathfrak{g}(1) \oplus \mathfrak{g}(2)$ is the maximal two-step nilpotent parabolic of non-Heisenberg type determined by $\alpha_\mathfrak{q}$ then $\mathfrak{g}(1)$ is the irreducible L-module with lowest weight $\alpha_\mathfrak{q}$.*

Proof. Observe that since a root vector for $\alpha_\mathfrak{q}$ is an element of $\mathfrak{g}(1)$, we have $\mathfrak{g}(1) \neq \emptyset$. As $\mathrm{Ad}(L)$ preserves $\mathfrak{g}(1)$, Proposition 3.2.2 implies that $\mathfrak{g}(1)$ is L-irreducible. ∎

Next we show that $\mathfrak{g}(2)$ is the irreducible L-module with highest weight γ. Since the argument of the proof works for general r-grading $\mathfrak{g} = \bigoplus_{j=-r}^{r} \mathfrak{g}(j)$, we give the proof in the general setting.

Proposition 3.2.4 *Assume that $\mathfrak{g} = \bigoplus_{j=-r}^{r} \mathfrak{g}(j)$ is a graded complex simple Lie algebra with $\mathfrak{n} = \bigoplus_{j=1}^{r} \mathfrak{g}(j)$. If the positive system Δ^+ is chosen so that $\Delta^+ = \Delta^+(\mathfrak{g}(0)) \cup \Delta(\mathfrak{n})$ and γ is the highest root of $\mathfrak{g}$ with respect to Δ^+ then $\mathfrak{g}(r)$ is the irreducible $\mathfrak{g}(0)$-module with highest weight γ.*

Proof. As $\mathfrak{g}$ is simple and γ is the highest root with respect to Δ^+,

$$\mathfrak{g} = \mathcal{U}(\mathfrak{g})X_\gamma = \mathcal{U}(\bar{\mathfrak{n}})(\mathcal{U}(\mathfrak{g}(0))X_\gamma).$$

Observe that since $X_\gamma \in \mathfrak{g}(r)$ and $\mathfrak{g}(r)$ is $\mathfrak{g}(0)$-stable, we have $\mathcal{U}(\mathfrak{g}(0))X_\gamma \subset \mathfrak{g}(r)$. On the other hand, as $\bar{\mathfrak{n}} = \bigoplus_{j=-r}^{-1} \mathfrak{g}(j)$, it follows that

$$\mathcal{U}(\bar{\mathfrak{n}})\mathfrak{g}(r) \subset \bigoplus_{j=-r}^{r-1} \mathfrak{g}(j).$$

As $\mathfrak{g} = \bigoplus_{j=-r}^{r} \mathfrak{g}(j)$, this shows that $\mathcal{U}(\mathfrak{g}(0))X_\gamma \supset \mathfrak{g}(r)$. ■

Corollary 3.2.5 *If $\mathfrak{q} = \mathfrak{g}(0) \oplus \mathfrak{g}(1) \oplus \mathfrak{g}(2)$ is the maximal two-step nilpotent parabolic of non-Heisenberg type determined by $\alpha_\mathfrak{q}$ then $\mathfrak{g}(2)$ is the irreducible L-module with highest weight γ.*

Proof. Observe that γ is the highest root of $\mathfrak{g}$ for $\Delta^+ = \Delta^+(\mathfrak{l}) \cup \Delta(\mathfrak{n})$. Now, as $\mathrm{Ad}(L)$ preserves $\mathfrak{g}(2)$, Proposition 3.2.4 implies that $\mathfrak{g}(2)$ is L-irreducible. ■

To conclude this section we show that $\mathfrak{z}(\mathfrak{n}) = \mathfrak{g}(2)$ and $\mathfrak{z}(\bar{\mathfrak{n}}) = \mathfrak{g}(-2)$, where $\mathfrak{z}(\mathfrak{n})$ and $\mathfrak{z}(\bar{\mathfrak{n}})$ are the centers of $\mathfrak{n}$ and $\bar{\mathfrak{n}}$, respectively. Because of the identification of $\mathfrak{g}(-j)$ with $\mathfrak{g}(j)^*$ via the Killing form, it suffices to show that $\mathfrak{z}(\mathfrak{n}) = \mathfrak{g}(2)$. The following technical lemma will simplify the expositions.

Lemma 3.2.6 *If $\mathfrak{q} = \mathfrak{g}(0) \oplus \mathfrak{g}(1) \oplus \mathfrak{g}(2)$ is a maximal two-step nilpotent parabolic of non-Heisenberg type with $\mathfrak{n} = \mathfrak{g}(1) \oplus \mathfrak{g}(2)$ then $\mathfrak{z}(\mathfrak{n}) \cap \mathfrak{g}(1) = \{0\}$.*

Proof. One can easily check that $\mathfrak{z}(\mathfrak{n})$ is an $\mathfrak{l}$-module by using the Jacobi identity and the fact that $\mathfrak{n}$ is an $\mathfrak{l}$-module. Therefore the intersection $\mathfrak{z}(\mathfrak{n}) \cap \mathfrak{g}(1)$ is an $\mathfrak{l}$-submodule of $\mathfrak{g}(1)$. The irreducibility of $\mathfrak{g}(1)$ from Corollary 3.2.3 then forces that $\mathfrak{z}(\mathfrak{n}) \cap \mathfrak{g}(1) = \{0\}$ or $\mathfrak{g}(1)$. However, the second is impossible; otherwise, we would have

$$[\mathfrak{n}, \mathfrak{n}] = [\mathfrak{g}(1), \mathfrak{g}(1)] = 0,$$

contrary to $[\mathfrak{n}, \mathfrak{n}] \neq 0$. Therefore, $\mathfrak{z}(\mathfrak{n}) \cap \mathfrak{g}(1) = \{0\}$. ■

Lemma 3.2.7 *If* $\mathfrak{q} = \mathfrak{g}(0) \oplus \mathfrak{g}(1) \oplus \mathfrak{g}(2)$ *is a maximal two-step nilpotent parabolic of non-Heisenberg type with* $\mathfrak{n} = \mathfrak{g}(1) \oplus \mathfrak{g}(2)$ *then* $\mathfrak{z}(\mathfrak{n}) = \mathfrak{g}(2)$.

Proof. Since $\mathfrak{g}(2) \subset \mathfrak{z}(\mathfrak{n})$, it suffices to show the other inclusion. Take $X \in \mathfrak{z}(\mathfrak{n})$. Since $\mathfrak{n} = \mathfrak{g}(1) \oplus \mathfrak{g}(2)$, there exist $X_j \in \mathfrak{g}(j)$ for $j = 1, 2$ so that $X = X_1 + X_2$. Since $X, X_2 \in \mathfrak{z}(\mathfrak{n})$, we have for any $Y \in \mathfrak{n}$,

$$[Y, X_1] = [Y, X_1] + [Y, X_2] = [Y, X] = 0.$$

Thus $X_1 \in \mathfrak{z}(\mathfrak{n}) \cap \mathfrak{g}(1)$. Lemma 3.2.6 then concludes that $X_1 = 0$, and so we have $X = X_2 \in \mathfrak{g}(2)$. Since $X \in \mathfrak{z}(\mathfrak{n})$ is arbitrary, this yields that $\mathfrak{z}(\mathfrak{n}) \subset \mathfrak{g}(2)$. ■

Now, since $\mathfrak{l} = \mathfrak{g}(0)$, $\mathfrak{g}(2) = \mathfrak{z}(\mathfrak{n})$ and $\mathfrak{g}(-2) = \mathfrak{z}(\bar{\mathfrak{n}})$, we write the 2-grading $\mathfrak{g} = \bigoplus_{j=-2}^{2} \mathfrak{g}(j)$ as

$$\mathfrak{g} = \mathfrak{z}(\bar{\mathfrak{n}}) \oplus \mathfrak{g}(-1) \oplus \mathfrak{l} \oplus \mathfrak{g}(1) \oplus \mathfrak{z}(\mathfrak{n}) \tag{3.2.8}$$

with parabolic subalgebra

$$\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{g}(1) \oplus \mathfrak{z}(\mathfrak{n}). \tag{3.2.9}$$

3.3 The Simple Subalgebras $\mathfrak{l}_\gamma$ and $\mathfrak{l}_{n\gamma}$

The purpose of this section is to study the structure of the Levi subalgebra $\mathfrak{l} = \mathfrak{z}(\mathfrak{l}) \oplus \mathfrak{l}_{ss}$. The material of this section will play a role in Chapter 5 and Chapter 6 when we decompose $\mathfrak{l} \otimes \mathfrak{z}(\mathfrak{n})$ into L-irreducible subspaces.

The center $\mathfrak{z}(\mathfrak{l})$ is of the form $\mathfrak{z}(\mathfrak{l}) = \bigcap_{\alpha \in \Pi(\mathfrak{l})} \ker(\alpha)$. Since $\mathfrak{g}$ has rank greater than one and $\Pi(\mathfrak{l}) = \Pi \backslash \{\alpha_{\mathfrak{q}}\}$, $\mathfrak{z}(\mathfrak{l})$ is non-zero and one-dimensional. It is clear from (3.2.1) that $H_{\mathfrak{q}}$ is an element of $\mathfrak{z}(\mathfrak{l})$. Therefore we have $\mathfrak{z}(\mathfrak{l}) = \mathbb{C}H_{\mathfrak{q}}$.

Next we consider the structure of $\mathfrak{l}_{ss}$. Observe that the Dynkin diagram of $\mathfrak{g}$ can be extended by attaching the lowest root $-\gamma$ to the diagram. If $\mathfrak{g}$ is not of type A_n then there is exactly one simple root, that is connected to $-\gamma$ in the extended diagram (see Figure B.3 in Appendix B). Let α_γ denote such a unique simple root. It is easy to see that $\mathfrak{q}_{\{\alpha_\gamma\}}$ is the Heisenberg parabolic of $\mathfrak{g}$; that is, the two-step nilpotent parabolic with $\dim([\mathfrak{n},\mathfrak{n}]) = 1$. Hence, if $\mathfrak{q}_{\{\alpha_\mathfrak{q}\}}$ is a maximal two-step nilpotent parabolic with $\dim([\mathfrak{n},\mathfrak{n}]) > 1$ then $\alpha_\gamma \in \Pi(\mathfrak{l}) = \Pi \backslash \{\alpha_\mathfrak{q}\}$. If we delete the node corresponding to $\alpha_\mathfrak{q}$ then we obtain one, two, or three subgraphs with one subgraph containing α_γ. This implies that the subalgebra $\mathfrak{l}_{ss}$ is either simple or the direct sum of two or three simple subalgebras with only one simple subalgebra containing the root space $\mathfrak{g}_{\alpha_\gamma}$ for α_γ. The three subgraphs occur only when $\mathfrak{q}$ is of type $D_n(n-2)$. So, if $\mathfrak{q}$ is not of type $D_n(n-2)$ then there are at most two subgraphs. In this case we denote by $\mathfrak{l}_\gamma$ (resp. $\mathfrak{l}_{n\gamma}$) the simple subalgebra of $\mathfrak{l}$ whose subgraph in the deleted Dynkin diagram contains (resp. does not contain) the node for α_γ. Thus the Levi subalgebra $\mathfrak{l}$ may decompose into

$$\mathfrak{l} = \mathbb{C}H_\mathfrak{q} \oplus \mathfrak{l}_\gamma \oplus \mathfrak{l}_{n\gamma}. \tag{3.3.1}$$

Then, for the rest of this chapter, we assume that $\mathfrak{q}$ is not of type $D_n(n-2)$, so that the Levi subalgebra $\mathfrak{l}$ can be expressed as (3.3.1). Recall from Definition 3.1.6 that if $\mathfrak{g}$ is of type $\mathcal{T}$ then we say that the parabolic subalgebra $\mathfrak{q}$ determined by $\alpha_i \in \Pi$ is of type $\mathcal{T}(i)$. Then the parabolic subalgebras $\mathfrak{q}$ under consideration are given as follows:

$$B_n(i)\ (3 \le i \le n), \quad C_n(i)\ (2 \le i \le n-1), \quad D_n(i)\ (3 \le i \le n-3), \tag{3.3.2}$$

and

$$E_6(3),\ E_6(5),\ E_7(2),\ E_7(6),\ E_8(1),\ F_4(4). \tag{3.3.3}$$

Note that in type A_n the nilradical $\mathfrak{n}$ of any maximal parabolic subalgebra is abelian.

Write $\Pi(\mathfrak{l}_\gamma) = \{\alpha \in \Pi \mid \alpha \in \Delta(\mathfrak{l}_\gamma)\}$ and $\Pi(\mathfrak{l}_{n\gamma}) = \{\alpha \in \Pi \mid \alpha \in \Delta(\mathfrak{l}_{n\gamma})\}$. Example 3.3.4 below exhibits the subgraphs for $\mathfrak{l}_\gamma$ and $\mathfrak{l}_{n\gamma}$ of $\mathfrak{q}$ of type $B_5(3)$ with $\Pi(\mathfrak{l}_\gamma)$ and

$\Pi(\mathfrak{l}_{n\gamma})$. One can find those data in Appendix C for each maximal parabolic subalgebra in (3.3.2) or (3.3.3).

Example 3.3.4 *Let $\mathfrak{q}$ be the parabolic subalgebra of type $B_5(3)$ with deleted Dynkin diagram*

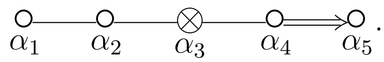
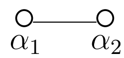
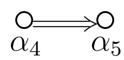


Figure B.3 in Appendix B shows that $\alpha_\gamma = \alpha_2$. Therefore, the subgraph for $\mathfrak{l}_\gamma$ is



and that for $\mathfrak{l}_{n\gamma}$ is



with $\Pi(\mathfrak{l}_\gamma) = \{\alpha_1, \alpha_2\}$ and $\Pi(\mathfrak{l}_{n\gamma}) = \{\alpha_4, \alpha_5\}$.

Remark 3.3.5 *It is clear from the extended Dynkin diagrams that $\langle \gamma, \alpha_\gamma \rangle > 0$ and $\langle \gamma, \alpha \rangle = 0$ for any other simple roots α. In particular, $\langle \alpha, \gamma \rangle = 0$ for all $\alpha \in \Pi(\mathfrak{l}_{n\gamma})$.*

3.4 Technical Facts on the Highest Weights for $\mathfrak{l}_\gamma$, $\mathfrak{l}_{n\gamma}$, $\mathfrak{g}(1)$, and $\mathfrak{z}(\mathfrak{n})$

In this section we summarize technical lemmas on the L-highest weights for $\mathfrak{l}_\gamma$, $\mathfrak{l}_{n\gamma}$, $\mathfrak{g}(1)$, and $\mathfrak{z}(\mathfrak{n})$. These technical facts will be used in later computations.

Proposition 3.2.4 shows that $\mathfrak{z}(\mathfrak{n})$ has highest weight γ, which is the highest root of $\mathfrak{g}$. We denote by ξ_γ, $\xi_{n\gamma}$, and μ the highest weights for $\mathfrak{l}_\gamma$, $\mathfrak{l}_{n\gamma}$, and $\mathfrak{g}(1)$, respectively. In Appendix C we give the explicit values for these highest weights for each of the parabolic subalgebras under consideration. We remark that all these highest weights are indeed roots in Δ^+. Observe that the highest weights ξ_γ and $\xi_{n\gamma}$ of $\mathfrak{l}_\gamma$ and $\mathfrak{l}_{n\gamma}$, respectively, are also the highest roots of $\mathfrak{l}_\gamma$ and $\mathfrak{l}_{n\gamma}$ as simple algebras; in particular, the multiplicities of $\alpha \in \Pi(\mathfrak{l}_\gamma)$ (resp. $\alpha \in \Pi(\mathfrak{l}_{n\gamma})$) in ξ_γ (resp. $\xi_{n\gamma}$) are all strictly positive.

Lemma 3.4.1 *If $\alpha_{\mathfrak{q}}$ is the simple root that determines $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{g}(1) \oplus \mathfrak{z}(\mathfrak{n})$ then $\xi_\gamma + \alpha_{\mathfrak{q}}$ and $\xi_{n\gamma} + \alpha_{\mathfrak{q}}$ are roots.*

Proof. We only prove that $\xi_\gamma + \alpha_{\mathfrak{q}} \in \Delta$; the other assertion that $\xi_{n\gamma} + \alpha_{\mathfrak{q}} \in \Delta$ can be proven similarly. It suffices to show that $\langle \xi_\gamma, \alpha_{\mathfrak{q}} \rangle < 0$, since both ξ_γ and $\alpha_{\mathfrak{q}}$ are roots. For $\alpha \in \Pi$ we observe that $\langle \alpha, \alpha_{\mathfrak{q}} \rangle < 0$ if α is adjacent to $\alpha_{\mathfrak{q}}$ in the Dynkin diagram and $\langle \alpha, \alpha_{\mathfrak{q}} \rangle = 0$ otherwise. An observation on the deleted Dynkin diagrams shows that there exists a unique simple root α_k in $\Pi(\mathfrak{l}_\gamma)$ that is adjacent to $\alpha_{\mathfrak{q}}$. Since ξ_γ is the highest root for $\mathfrak{l}_\gamma$ as a simple algebra, the multiplicity of α_k in ξ_γ is strictly positive. Thus $\langle \xi_\gamma, \alpha_{\mathfrak{q}} \rangle < 0$. ∎

Lemma 3.4.2 *If ξ_γ, $\xi_{n\gamma}$, μ, and γ are the highest weights of $\mathfrak{l}_\gamma$, $\mathfrak{l}_{n\gamma}$, $\mathfrak{g}(1)$, and $\mathfrak{z}(\mathfrak{n})$, respectively, then the following hold:*

(1) $\gamma - \xi_\gamma \in \Delta$, but $\gamma - \xi_{n\gamma} \notin \Delta$.

(2) $\gamma - \mu \in \Delta$.

(3) $\mu - \xi_\gamma, \mu - \xi_{n\gamma} \in \Delta$.

Proof. To prove $\gamma - \xi_{n\gamma} \notin \Delta$, we recall a well-known fact that if n and m are the largest non-negative integers so that $\gamma - n\xi_{n\gamma} \in \Delta$ and $\gamma + m\xi_{n\gamma} \in \Delta$, respectively, then $\langle \gamma, \xi_{n\gamma}^\vee \rangle$ is given by $\langle \gamma, \xi_{n\gamma}^\vee \rangle = n - m$ (see for instance [9, Section 9.4]). Observe that the roots in $\Delta(\mathfrak{l}_{n\gamma})$ are orthogonal to γ; in particular, $\langle \gamma, \xi_{n\gamma}^\vee \rangle = 0$. Thus, we have $n = m$. As $\xi_{n\gamma} \in \Delta^+$ and γ is the highest root, $\gamma + \xi_{n\gamma} \notin \Delta$. Therefore, $n = m = 0$, which concludes that $\gamma - \xi_{n\gamma}$ is not a root. To prove $\gamma - \xi_\gamma \in \Delta$, it suffices to show that $\langle \gamma, \xi_\gamma \rangle > 0$, since both γ and ξ_γ are roots. Write ξ_γ in terms of simple roots in $\Pi(\mathfrak{l}_\gamma)$. Observe that each $\alpha \in \Pi(\mathfrak{l}_\gamma)$ has positive multiplicity m_α in ξ_γ. As γ is orthogonal to α for any $\alpha \in \Pi(\mathfrak{l}_\gamma) \backslash \{\alpha_\gamma\}$, we have $\langle \gamma, \xi_\gamma \rangle = m_{\alpha_\gamma} \langle \gamma, \alpha_\gamma \rangle > 0$.

To prove the assertion (2), we show that $\langle \mu, \gamma \rangle > 0$. Since, for α simple and $\alpha \neq \alpha_\gamma$, we have $\langle \alpha, \gamma \rangle = 0$ and $\langle \alpha_\gamma, \gamma \rangle > 0$, it suffices to show that the multiplicity

n_{α_γ} of α_γ in μ is $n_{\alpha_\gamma} > 0$. Observe that the root $\theta = \sum_{\alpha\in\Pi} \alpha$ belongs to $\Delta(\mathfrak{g}(1))$. The multiplicity of α_γ in θ is one. As $\mathfrak{g}(1)$ is an irreducible L-module with highest weight μ, the root θ is of the form $\theta = \mu - \sum_{\alpha\in\Pi(\mathfrak{l})} c_\alpha \alpha$ with c_α non-negative integers. Therefore $\mu = \theta + \sum_{\alpha\in\Pi(\mathfrak{l})} c_\alpha\alpha$, and so $n_{\alpha_\gamma} = 1 + c_{\alpha_\gamma} > 0$.

Next we show that $\mu - \xi_{n\gamma} \in \Delta$. The other assertion in (3) is proven in a similar manner. It suffices to show that $\langle \mu, \xi_{n\gamma}\rangle > 0$. We write μ as

$$\mu = \sum_{\alpha\in\Pi(\mathfrak{l}_\gamma)} m_\alpha \varpi_\alpha + \sum_{\beta\in\Pi(\mathfrak{l}_{n\gamma})} n_\beta \widetilde{\varpi}_\beta \quad \text{with } m_\alpha, n_\beta \in \mathbb{Z}_{\geq 0}, \tag{3.4.3}$$

where ϖ_α and $\widetilde{\varpi}_\beta$ are the fundamental weights of $\alpha \in \Pi(\mathfrak{l}_\gamma)$ and $\beta \in \Pi(\mathfrak{l}_{n\gamma})$, respectively. The root $\xi_{n\gamma}$ is an integer combination of simple roots in $\Pi(\mathfrak{l}_{n\gamma})$ of the form

$$\xi_{n\gamma} = \sum_{\beta\in\Pi(\mathfrak{l}_{n\gamma})} m_\beta \beta \quad \text{with } m_\beta \in \mathbb{Z}_{>0}.$$

Then $\langle \varpi_\alpha, \xi_{n\gamma}\rangle = 0$ for all $\alpha \in \Pi(\mathfrak{l}_\gamma)$, and $\langle \widetilde{\varpi}_\beta, \xi_{n\gamma}\rangle > 0$ for all $\beta \in \Pi(\mathfrak{l}_{n\gamma})$. It follows from Lemma 3.4.1 that $\mathfrak{l}_{n\gamma}$ acts on $\mathfrak{g}(1)$ nontrivially. Thus, there exists $\beta' \in \Pi(\mathfrak{l}_{n\gamma})$ so that $n_{\beta'} \neq 0$ in (3.4.3), and so we obtain $\langle \mu, \xi_{n\gamma}\rangle \geq n_{\beta'} m_{\beta'} > 0$.

■

When $\mathfrak{g}$ is not simply laced then there are two root lengths in Δ. A root α is called long or short accordingly. The following technical lemma will simplify arguments concerning the long roots later. We regard any root as a long root, when $\mathfrak{g}$ is simply laced.

Lemma 3.4.4 *Suppose that $\alpha \in \Delta$ is a long root. For any $\beta \in \Delta$, the following hold.*

(1) If $\beta - \alpha \in \Delta$ then $\langle \beta, \alpha^\vee\rangle = 1$.

(2) If $\beta + \alpha \in \Delta$ then $\langle \beta, \alpha^\vee\rangle = -1$.

(3) If $\beta \pm \alpha \in \Delta$ then $\beta \mp \alpha \notin \Delta$.

(4) $\beta \pm 2\alpha \notin \Delta$.

Proof. Assume that $\beta-\alpha \in \Delta$. Since α is a long root, we have $1 \geq ||\beta-\alpha||^2/||\alpha||^2 > 0$. Thus,

$$1 \geq \frac{||\beta||^2}{||\alpha||^2} - \langle \beta, \alpha^\vee \rangle + 1 > 0,$$

which implies that

$$0 < \frac{||\beta||^2}{||\alpha||^2} \leq \langle \beta, \alpha^\vee \rangle < 1 + \frac{||\beta||^2}{||\alpha||^2} \leq 2.$$

Therefore $\langle \beta, \alpha^\vee \rangle = 1$. Part (2) may be shown similarly, and (3) and (4) follow from (1) and (2) with the fact that $\langle \beta, \alpha^\vee \rangle = p_{\alpha,\beta} - q_{\alpha,\beta}$, where $p_{\alpha,\beta} = \max\{j \in \mathbb{Z}_{\geq 0} \mid \beta - j\alpha \in \Delta\}$ and $q_{\alpha,\beta} = \max\{j \in \mathbb{Z}_{\geq 0} \mid \beta + j\alpha \in \Delta\}$. ■

Lemma 3.4.5 *If ξ_γ, $\xi_{n\gamma}$, μ, and γ are the highest weights of $\mathfrak{l}_\gamma$, $\mathfrak{l}_{n\gamma}$, $\mathfrak{g}(1)$, and $\mathfrak{z}(\mathfrak{n})$, respectively, then the following hold:*

(1) $\gamma - \mu + \xi_{n\gamma} \in \Delta$.

(2) $\gamma - \mu - \xi_{n\gamma} \notin \Delta$.

(3) If ξ_γ is a long root then $\gamma - \mu \pm \xi_\gamma \notin \Delta$.

Proof. Lemma 3.4.2 shows that $\gamma - \mu \in \Delta$. Then in order to prove (1), it is enough to show that $\langle \xi_{n\gamma}, \gamma - \mu \rangle < 0$. It follows from Remark 3.3.5 that $\langle \xi_{n\gamma}, \gamma \rangle = 0$. On the other hand, we have $\langle \xi_{n\gamma}, \mu \rangle > 0$ by the proof for (3) of Lemma 3.4.2. Therefore,

$$\langle \xi_{n\gamma}, \gamma - \mu \rangle = \langle \xi_{n\gamma}, \gamma \rangle - \langle \xi_{n\gamma}, \mu \rangle < 0.$$

When $\xi_{n\gamma}$ is a long root of $\mathfrak{g}$, the assertion (2) follows from (1) and Lemma 3.4.4. The data in Appendix C shows that $\xi_{n\gamma}$ is a long root unless $\mathfrak{q}$ is of type $B_n(n-1)$. If $\mathfrak{q}$ is of type $B_n(n-1)$ then we have $\gamma = \varepsilon_1 + \varepsilon_2$, $\mu = \varepsilon_1 + \varepsilon_n$, and $\xi_{n\gamma} = \varepsilon_n$. Thus $\gamma - \mu - \xi_{n\gamma} \notin \Delta$.

To show (3), observe that, by Lemma 3.4.2, we have $\gamma - \xi_\gamma,\ \mu - \xi_\gamma \in \Delta$. Since ξ_γ is assumed to be a long root, it follows from Lemma 3.4.4 that $\langle \gamma, \xi_\gamma^\vee \rangle = \langle \mu, \xi_\gamma^\vee \rangle = 1$.

Therefore $\langle \gamma - \mu, \xi_\gamma^\vee \rangle = 0$, which forces that

$$||\gamma - \mu \pm \xi_\gamma||^2 = ||\gamma - \mu||^2 + ||\xi_\gamma||^2. \tag{3.4.6}$$

Since $\gamma - \mu$ is a root, we have $||\gamma - \mu|| \neq 0$. As ξ_γ is assumed to be a long root, (3.4.6) implies that $(\gamma - \mu) \pm \xi_\gamma \notin \Delta$. ■

Remark 3.4.7 *Direct observation shows that ξ_γ is a long root, unless $\mathfrak{q}$ is of type $C_n(i)$. If $\mathfrak{q}$ is of type $C_n(i)$ then the data in Appendix C shows $\gamma = 2\varepsilon_1$, $\mu = \varepsilon_1 + \varepsilon_{i+1}$, and $\xi_\gamma = \varepsilon_1 - \varepsilon_i$. Thus $\gamma - \mu + \xi_\gamma \notin \Delta$, but $\gamma - \mu - \xi_\gamma \in \Delta$.*

CHAPTER 4

The Ω_1 System

The aim of this chapter is to determine the complex parameter $s_1 \in \mathbb{C}$ for the line bundle $\mathcal{L}_{-s}$ so that the Ω_1 system of a maximal two-step nilpotent parabolic $\mathfrak{q}$ of non-Heisenberg type is conformally invariant on $\mathcal{L}_{s_1}$. The special value is given in Theorem 4.2.5.

4.1 Normalizations

The purpose of this section is to fix normalizations for root vectors. In the next section we are going to construct the Ω_1 system and determine its special value of s. To do so, it is essential to set up convenient normalizations.

If $\alpha, \beta \in \Delta$ then define

$$\begin{aligned} p_{\alpha,\beta} &= \max\{j \in \mathbb{Z}_{\geq 0} \mid \beta - j\alpha \in \Delta\} \text{ and} \\ q_{\alpha,\beta} &= \max\{j \in \mathbb{Z}_{\geq 0} \mid \beta + j\alpha \in \Delta\}. \end{aligned} \tag{4.1.1}$$

In particular, we have

$$\langle \beta, \alpha^\vee \rangle = p_{\alpha,\beta} - q_{\alpha,\beta}. \tag{4.1.2}$$

It is known that we can choose $X_\alpha \in \mathfrak{g}_\alpha$ and $H_\alpha \in \mathfrak{h}$ for each $\alpha \in \Delta$ in such a way that the following conditions hold (see for instance [7, Sections III.4 and III.5]). The reader may want to notice that our normalizations are different from those used in [1].

(H1) For each $\alpha \in \Delta^+$, $\{X_\alpha, X_{-\alpha}, H_\alpha\}$ is an $\mathfrak{sl}(2,\mathbb{C})$ triple; in particular,

$$[X_\alpha, X_{-\alpha}] = H_\alpha.$$

(H2) For each $\alpha, \beta \in \Delta^+$, $[H_\alpha, X_\beta] = \beta(H_\alpha)X_\beta$.

(H3) For $\alpha \in \Delta$ we have $\kappa(X_\alpha, X_{-\alpha}) = 1$.

(H4) For $\alpha, \beta \in \Delta$ we have $\beta(H_\alpha) = \langle \alpha, \beta \rangle$.

(H5) For $\alpha, \beta \in \Delta$ with $\alpha + \beta \neq 0$, there is a constant $N_{\alpha,\beta}$ so that

$$[X_\alpha, X_\beta] = N_{\alpha,\beta} X_{\alpha+\beta} \quad \text{if } \alpha + \beta \in \Delta,$$
$$N_{\alpha,\beta} = 0 \quad \text{if } \alpha + \beta \notin \Delta.$$

(H6) If $\alpha_1, \alpha_2, \alpha_3 \in \Delta^+$ with $\alpha_1 + \alpha_2 + \alpha_3 = 0$ then

$$N_{\alpha_1,\alpha_2} = N_{\alpha_2,\alpha_3} = N_{\alpha_3,\alpha_1}.$$

(H7) If $\alpha, \beta \in \Delta$ and $\alpha + \beta \in \Delta$ then

$$N_{\alpha,\beta} N_{-\alpha,-\beta} = -\frac{q_{\alpha,\beta}(1 + p_{\alpha,\beta})}{2} \alpha(H_\alpha).$$

In particular, $N_{\alpha,\beta}$ is non-zero if $\alpha + \beta \in \Delta$.

We call the constants $N_{\alpha,\beta}$ structure constants.

4.2 The Ω_1 System

In this section we shall build the Ω_1 system and determine its special value. As we have observed in Section 2.5, we use the covariant map τ_1 and the associated L-intertwining operators $\tilde{\tau}_1|_{V^*}$, where V^* are irreducible constituents of $\mathfrak{g}(-1)^* \otimes \mathfrak{g}(2)^*$. By Definition 2.5.1, the covariant map τ_1 is given by

$$\tau_1 : \mathfrak{g}(1) \to \mathfrak{g}(-1) \otimes \mathfrak{z}(\mathfrak{n})$$
$$X \mapsto \mathrm{ad}(X)\omega_0$$

with $\omega_0 = \sum_{\gamma_j \in \Delta(\mathfrak{z}(\mathfrak{n}))} X_{-\gamma_j} \otimes X_{\gamma_j}$. It is clear that τ_1 is not identically zero. Indeed, if $X = X_\mu$ with μ the highest weight for $\mathfrak{g}(1)$ then

$$\begin{aligned}\tau_1(X_\mu) &= \mathrm{ad}(X_\mu)\omega_0 \\ &= \sum_{\Delta_\mu(\mathfrak{z}(\mathfrak{n}))} N_{\mu,-\gamma_j} X_{\mu-\gamma_j} \otimes X_{\gamma_j}\end{aligned}$$

with $\Delta_\mu(\mathfrak{z}(\mathfrak{n})) = \{\gamma_j \in \Delta(\mathfrak{z}(\mathfrak{n})) \mid \mu - \gamma_j \in \Delta\}$. By Lemma 3.4.2, we have $\mu - \gamma \in \Delta$ with γ the highest weight for $\mathfrak{z}(\mathfrak{n})$, so $\Delta_\mu(\mathfrak{z}(\mathfrak{n})) \neq \emptyset$. Since the vectors $X_{\mu-\gamma_j} \otimes X_{\gamma_j}$ for $\gamma_j \in \Delta_\mu(\mathfrak{z}(\mathfrak{n}))$ are linearly independent, we have $\tau_1(X_\mu) \neq 0$.

For each irreducible constituent V^* of $\mathfrak{g}(-1)^* \otimes \mathfrak{z}(\mathfrak{n})^*$, there exists an associated L-intertwining operator $\tilde{\tau}_1|_{V^*} \in \mathrm{Hom}_L(V^*, \mathcal{P}^1(\mathfrak{g}(1)))$ so that, for all $Y^* \in V^*$,

$$\tilde{\tau}_1|_{V^*}(Y^*)(X) = Y^*(\tau_1(X)).$$

Observe that the duality for V^* is defined with respect to the Killing form κ. Moreover, via the Killing form κ, we have $\mathfrak{g}(-1)^* \otimes \mathfrak{z}(\mathfrak{n})^* \cong \mathfrak{g}(1) \otimes \mathfrak{z}(\bar{\mathfrak{n}})$. Thus, if $Y^* = X_\alpha \otimes X_{-\gamma_t}$ with $\alpha \in \Delta(\mathfrak{g}(1))$ and $\gamma_t \in \Delta(\mathfrak{z}(\mathfrak{n}))$ then $Y^*(\tau_1(X))$ is given by

$$Y^*(\tau_1(X)) = \sum_{\gamma_j \in \Delta(\mathfrak{z}(\mathfrak{n}))} \kappa(X_\alpha, \mathrm{ad}(X)X_{-\gamma_j})\kappa(X_{-\gamma_t}, X_{\gamma_j}), \tag{4.2.1}$$

as $\tau_1(X) = \sum_{\gamma_j \in \Delta(\mathfrak{z}(\mathfrak{n}))} \mathrm{ad}(X)X_{-\gamma_j} \otimes X_{\gamma_j}$.

Now we wish to determine all the irreducible constituents V^* of $\mathfrak{g}(1) \otimes \mathfrak{z}(\bar{\mathfrak{n}})$, so that $\tilde{\tau}_1|_{V^*}$ are not identically zero. Observe that $\mathcal{P}^1(\mathfrak{g}(1)) \cong \mathrm{Sym}^1(\mathfrak{g}(-1)) = \mathfrak{g}(-1)$ and that $\mathfrak{g}(-1)$ is an irreducible L-module, as $\mathfrak{q}$ is a maximal parabolic. Thus, if $\tilde{\tau}_1|_{V^*}$ is not identically zero then $V^* \cong \mathfrak{g}(-1)$. Proposition 4.2.2 below shows that the converse also holds.

Proposition 4.2.2 *Let V^* be an irreducible constituent of $\mathfrak{g}(1) \otimes \mathfrak{z}(\bar{\mathfrak{n}})$. Then $\tilde{\tau}_1|_{V^*}$ is not identically zero if and only if $V^* \cong \mathfrak{g}(-1)$.*

Proof. First observe that $\mathfrak{g}(-1)$ is an irreducible constituent of $\mathfrak{g}(1) \otimes \mathfrak{z}(\bar{\mathfrak{n}})$. Indeed, since τ_1 is linear, we have $\tau_1(\mathfrak{g}(1)) \cong \mathfrak{g}(1)$ as an L-module; in particular, $\mathfrak{g}(1)$ is an irreducible constituent of $\mathfrak{g}(-1) \otimes \mathfrak{z}(\mathfrak{n})$. Therefore $\mathfrak{g}(-1) \cong \mathfrak{g}(1)^*$ is an irreducible constituent of $\mathfrak{g}(1) \otimes \mathfrak{z}(\bar{\mathfrak{n}}) \cong (\mathfrak{g}(-1) \otimes \mathfrak{z}(\mathfrak{n}))^*$.

To prove $\tilde{\tau}_1|_{\mathfrak{g}(-1)}$ is a non-zero map, it suffices to show that $\tilde{\tau}_1|_{\mathfrak{g}(-1)}(Y^*) \neq 0$ for some $Y^* \in \mathfrak{g}(-1) \subset \mathfrak{g}(1) \otimes \mathfrak{z}(\bar{\mathfrak{n}})$. To do so, consider a map

$$\begin{aligned} \bar{\tau}_1 : \mathfrak{g}(-1) &\to \mathfrak{g}(1) \otimes \mathfrak{z}(\bar{\mathfrak{n}}) \\ \bar{X} &\mapsto \mathrm{ad}(\bar{X})\bar{\omega}_0 \end{aligned}$$

with $\bar{\omega}_0 = \sum_{\gamma_t \in \Delta(\mathfrak{z}(\mathfrak{n}))} X_{\gamma_t} \otimes X_{-\gamma_t}$. This is a non-zero L-intertwining operator. Thus $\bar{\tau}_1(\mathfrak{g}(-1)) \cong \mathfrak{g}(-1)$ as an L-module, and $\bar{\tau}_1(X_{-\alpha})$ is a weight vector with weight $-\alpha$ for all $\alpha \in \Delta(\mathfrak{g}(1))$. As $\mathfrak{g}(1)$ has highest weight μ, the lowest weight for $\mathfrak{g}(-1)$ is $-\mu$.

Now we set

$$c_\mu = \sum_{\gamma_t \in \Delta_\mu(\mathfrak{z}(\mathfrak{n}))} N_{-\mu,\gamma_t} N_{\mu,-\gamma_t}$$

with $\Delta_\mu(\mathfrak{z}(\mathfrak{n})) = \{\gamma_t \in \Delta(\mathfrak{z}(\mathfrak{n})) \mid \gamma_t - \mu \in \Delta\}$. By Lemma 3.4.2, it follows that $\gamma - \mu \in \Delta$; in particular, $\Delta_\mu(\mathfrak{z}(\mathfrak{n})) \neq \emptyset$. The normalization (H7) in Section 4.1 shows that $N_{-\mu,\gamma_t} N_{\mu,-\gamma_t} < 0$ for all $\gamma_t \in \Delta_\mu(\mathfrak{z}(\mathfrak{n}))$. Therefore $c_\mu \neq 0$. Then define $Y_l^* \in \mathfrak{g}(-1)$ by means of

$$Y_l^* = \frac{1}{c_\mu} \bar{\tau}_1(X_{-\mu}) = \frac{1}{c_\mu} \sum_{\gamma_t \in \Delta_\mu(\mathfrak{z}(\mathfrak{n}))} N_{-\mu,\gamma_t} X_{\gamma_t - \mu} \otimes X_{-\gamma_t}.$$

We claim that $\tilde{\tau}_1|_{\mathfrak{g}(-1)}(Y_l^*)(X) \neq 0$. By (4.2.1), the polynomial $\tilde{\tau}_1|_{\mathfrak{g}(-1)}(Y_l^*)(X)$ is

$$\begin{aligned} \tilde{\tau}_1|_{\mathfrak{g}(-1)}(Y_l^*)(X) &= Y_l^*(\tau_1(X)) \\ &= \frac{1}{c_\mu} \sum_{\substack{\gamma_t \in \Delta_\mu(\mathfrak{z}(\mathfrak{n})) \\ \gamma_j \in \Delta(\mathfrak{z}(\mathfrak{n}))}} N_{-\mu,\gamma_t} \kappa(X_{\gamma_t - \mu}, \mathrm{ad}(X) X_{-\gamma_j}) \kappa(X_{-\gamma_t}, X_{\gamma_j}) \\ &= \frac{1}{c_\mu} \sum_{\gamma_t \in \Delta_\mu(\mathfrak{z}(\mathfrak{n}))} N_{-\mu,\gamma_t} \kappa(X_{\gamma_t - \mu}, \mathrm{ad}(X) X_{-\gamma_t}). \end{aligned}$$

Write $X = \sum_{\alpha \in \Delta(\mathfrak{g}(1))} \eta_\alpha X_\alpha$, where $\eta_\alpha \in \mathfrak{n}^*$ is the coordinate dual to X_α with respect to the Killing form κ. Then,

$$\begin{aligned}
\tilde{\tau}_1|_{\mathfrak{g}(-1)}(Y_l^*)(X) &= \frac{1}{c_\mu} \sum_{\gamma_t \in \Delta_\mu(\mathfrak{z}(\mathfrak{n}))} N_{-\mu,\gamma_t} \kappa(X_{\gamma_t - \mu}, \mathrm{ad}(X) X_{-\gamma_t}) \\
&= \frac{1}{c_\mu} \sum_{\substack{\alpha \in \Delta(\mathfrak{g}(1)) \\ \gamma_t \in \Delta_\mu(\mathfrak{z}(\mathfrak{n}))}} N_{-\mu,\gamma_t} \eta_\alpha \kappa(X_{\gamma_t - \mu}, \mathrm{ad}(X_\alpha) X_{-\gamma_t}) \\
&= \frac{1}{c_\mu} \sum_{\gamma_t \in \Delta_\mu(\mathfrak{z}(\mathfrak{n}))} N_{-\mu,\gamma_t} N_{\mu,-\gamma_t} \eta_\mu \\
&= \eta_\mu \\
&= \kappa(X, X_{-\mu}). \qquad (4.2.3)
\end{aligned}$$

Hence $\tilde{\tau}_1|_{\mathfrak{g}(-1)}(Y_l^*)(X) \neq 0$. ■

Since only $\mathfrak{g}(-1)$ contributes to the construction of the Ω_1 systems, we simply refer to the Ω_1 system as the $\Omega_1|_{\mathfrak{g}(-1)}$ system.

As we observed in Section 2.5, the operator $\Omega_1|_{\mathfrak{g}(-1)} : \mathfrak{g}(-1) \to \mathbb{D}(\mathcal{L}_{-s})^{\bar{\mathfrak{n}}}$ is obtained via the composition of maps

$$\mathfrak{g}(-1) \stackrel{\tilde{\tau}_1|_{\mathfrak{g}(-1)}}{\to} \mathcal{P}^1(\mathfrak{g}(1)) \to \mathfrak{g}(-1) \stackrel{\sigma}{\hookrightarrow} \mathcal{U}(\bar{\mathfrak{n}}) \stackrel{R}{\to} \mathbb{D}(\mathcal{L}_{-s})^{\bar{\mathfrak{n}}}.$$

By (4.2.3), we have $\tilde{\tau}_1|_{\mathfrak{g}(-1)}(Y_l^*)(X) = \kappa(X, X_{-\mu})$. Therefore,

$$\Omega_1(Y_l^*) = R(X_{-\mu}).$$

Now, for all $\alpha \in \Delta(\mathfrak{g}(1))$, set

$$Y_{-\alpha} = \bar{\tau}_1(X_{-\alpha}).$$

Then, as $Y_l^* = (1/c_\mu)\bar{\tau}_1(X_{-\mu})$, we have

$$\Omega_1(Y_{-\mu}) = c_\mu R(X_{-\mu}).$$

Since both $\Omega_1|_{\mathfrak{g}(-1)}$ and $\bar{\tau}_1$ are L_0-intertwining operators and $\mathfrak{g}(-1) = \mathcal{U}(\mathfrak{l})X_{-\mu}$, for any $\alpha \in \Delta(\mathfrak{g}(1))$, we obtain

$$\Omega_1(Y_{-\alpha}) = c_\alpha R(X_{-\alpha}) \tag{4.2.4}$$

with some constant c_α. Then, for $\Delta(\mathfrak{g}(1)) = \{\alpha_1, \ldots, \alpha_m\}$, the Ω_1 system is given by

$$R(X_{-\alpha_1}), \ldots, R(X_{-\alpha_m}).$$

The following theorem shows that the Ω_1 system is conformally invariant on $\mathcal{L}_0$.

Theorem 4.2.5 *Let $\mathfrak{g}$ be a complex simple Lie algebra, and let $\mathfrak{q}$ be a maximal two-step nilpotent parabolic subalgebra of non-Heisenberg type. Then the Ω_1 system is conformally invariant on $\mathcal{L}_{-s}$ if and only if $s = 0$.*

Proof. By Remark 2.5.11, we only need to show that the condition (S2) in Definition 2.1.4 holds if and only if $s = 0$. By Theorem 2.4.1,

$$\begin{aligned}&\big([\pi_s(Y), R(X_{-\alpha_j})]\bullet f\big)(\bar{n})\\&= \big(R([(\mathrm{Ad}(\bar{n}^{-1})Y)_{\mathfrak{q}}, X_{-\alpha_j}]_{\bar{\mathfrak{n}}})\bullet f\big)(\bar{n}) + s\lambda_{\mathfrak{q}}\big([\mathrm{Ad}(\bar{n}^{-1})Y, X_{-\alpha_j}]_{\mathfrak{q}}\big)f(\bar{n})\end{aligned}$$

for any $Y \in \mathfrak{g}$ and any $f \in C^\infty(\bar{N}_0, \mathbb{C}_{\chi^{-s}})$. Hence, the condition (S2) holds if and only if $s = 0$. ∎

CHAPTER 5

Irreducible Decomposition of $\mathfrak{l} \otimes \mathfrak{z}(\mathfrak{n})$

Our next goal is to construct the Ω_2 systems and to find their special values. To do so, we need to detect the irreducible constituents V^* of $\mathfrak{l}^* \otimes \mathfrak{z}(\mathfrak{n})^*$ so that $\tilde{\tau}_2|_{V^*}$ is not identically zero. (see Section 2.5 for the general construction of the Ω_k systems). In this chapter and the next one, we shall show preliminary results to find such irreducible constituents.

5.1 Irreducible Decomposition of $\mathfrak{l}_\gamma \otimes \mathfrak{z}(\mathfrak{n})$

We continue with $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{g}(1) \oplus \mathfrak{z}(\mathfrak{n})$ a maximal two-step nilpotent parabolic subalgebra of non-Heisenberg type listed in (3.3.2) or (3.3.3), and $Q = LN = N_G(\mathfrak{q})$. The Levi subgroup L acts on $\mathfrak{l} \otimes \mathfrak{z}(\mathfrak{n}) \subset \mathfrak{g} \otimes \mathfrak{g}$ via the standard action on the tensor product induced by the adjoint representation on $\mathfrak{l}$ and $\mathfrak{z}(\mathfrak{n})$. As L is complex reductive, this action is completely reducible. Since $\mathfrak{l} = \mathfrak{z}(\mathfrak{l}) \oplus \mathfrak{l}_\gamma \oplus \mathfrak{l}_{n\gamma}$ with $\mathfrak{z}(\mathfrak{l}) = \mathbb{C}H_\mathfrak{q}$, we have

$$\mathfrak{l} \otimes \mathfrak{z}(\mathfrak{n}) = \big(\mathbb{C}H_\mathfrak{q} \otimes \mathfrak{z}(\mathfrak{n})\big) \oplus \big(\mathfrak{l}_\gamma \otimes \mathfrak{z}(\mathfrak{n})\big) \oplus \big(\mathfrak{l}_{n\gamma} \otimes \mathfrak{z}(\mathfrak{n})\big). \tag{5.1.1}$$

It is clear that $\mathbb{C}H_\mathfrak{q} \otimes \mathfrak{z}(\mathfrak{n}) \cong \mathfrak{z}(\mathfrak{n}) = \mathfrak{g}(2)$ as an L-module. Thus, by Corollary 3.2.5, $\mathbb{C}H_\mathfrak{q} \otimes \mathfrak{z}(\mathfrak{n})$ is L-irreducible. It is also easy to show that $\mathfrak{l}_{n\gamma} \otimes \mathfrak{z}(\mathfrak{n})$ is L-irreducible. Let L_γ (resp. $L_{n\gamma}$) be the analytic subgroup of L with Lie algebra $\mathfrak{l}_\gamma$ (resp. $\mathfrak{l}_{n\gamma}$). As in Section 3.2, we call a weight ν for a finite dimensional L-module V a highest weight for V if the restriction $\nu|_{\mathfrak{h}_{ss}}$ onto $\mathfrak{h}_{ss}$ is a highest weight for V as an L_{ss}-module.

Proposition 5.1.2 *Suppose that $\mathfrak{l}_{n\gamma} \neq 0$. If $\xi_{n\gamma}$ and γ are the highest weights of $\mathfrak{l}_{n\gamma}$ and $\mathfrak{z}(\mathfrak{n})$, respectively, then $\mathfrak{l}_{n\gamma} \otimes \mathfrak{z}(\mathfrak{n})$ is the irreducible L-module with highest weight*

$\xi_{n\gamma} + \gamma$.

Proof. First we observe that $L_{n\gamma}$ acts trivially on $\mathfrak{z}(\mathfrak{n})$. By Corollary 3.2.5, we have $\mathfrak{z}(\mathfrak{n}) = \mathfrak{g}(2) = \mathcal{U}(\mathfrak{l}_{ss})X_\gamma$. By the observation made in Remark 3.3.5, it follows that $\alpha \perp \gamma$ for all $\alpha \in \Delta(\mathfrak{l}_{n\gamma})$. Thus $\mathfrak{z}(\mathfrak{n}) = \mathcal{U}(\mathfrak{l}_\gamma)X_\gamma$. Hence $L_{n\gamma}$ acts trivially; in particular, the irreducible L-module $\mathfrak{z}(\mathfrak{n})$ is L_γ-irreducible. On the other hand, it is clear that L_γ acts on $\mathfrak{l}_{n\gamma}$ trivially. Therefore the representation $(L, \mathrm{Ad}\otimes\mathrm{Ad}, \mathfrak{l}_{n\gamma}\otimes\mathfrak{z}(\mathfrak{n}))$ is equivalent to $(L_\gamma \times L_{n\gamma}, \mathrm{Ad}\hat{\otimes}\mathrm{Ad}, \mathfrak{l}_{n\gamma}\otimes\mathfrak{z}(\mathfrak{n}))$, where $\hat{\otimes}$ denotes the outer tensor product. Since $\mathfrak{l}_{n\gamma}$ and $\mathfrak{z}(\mathfrak{n})$ have highest weight $\xi_{n\gamma}$ and γ, respectively, the lemma follows. ■

Now we focus on the decomposition of $\mathfrak{l}_\gamma \otimes \mathfrak{z}(\mathfrak{n})$ into irreducible L-submodules. As noted in the proof for Lemma 5.1.2, the subgroup $L_{n\gamma}$ acts trivially on $\mathfrak{l}_\gamma \otimes \mathfrak{z}(\mathfrak{n})$. Hence we study $\mathfrak{l}_\gamma \otimes \mathfrak{z}(\mathfrak{n})$ as an L_γ-module. For $\lambda \in \mathfrak{h}^*$ with $\langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}_{\geq 0}$ for all $\alpha \in \Pi(\mathfrak{l}_\gamma)$, we will denote by $V(\lambda)$ the irreducible constituent with highest weight $\lambda|_{\mathfrak{h}_\gamma}$, where $\mathfrak{h}_\gamma = \mathfrak{h} \cap \mathfrak{l}_\gamma$. For classical algebra $\mathfrak{g}$, we use the standard realization of the roots ε_i, the dual basis of the standard orthonormal basis for $\mathbb{R}^n$.

Theorem 5.1.3 *The L-module $\mathfrak{l}_\gamma \otimes \mathfrak{z}(\mathfrak{n})$ is reducible. If $V(\lambda)$ denotes the irreducible representation of L with highest weight $\lambda|_{\mathfrak{h}_\gamma}$ then the irreducible decomposition of $\mathfrak{l}_\gamma \otimes \mathfrak{z}(\mathfrak{n})$ is given as follows.*

1. $B_n(i),\ 3 \leq i \leq n:$

$$\mathfrak{l}_\gamma \otimes \mathfrak{z}(\mathfrak{n}) = \begin{cases} V(\xi_\gamma + \gamma) \oplus V(\gamma) \oplus V(\xi_\gamma + (\varepsilon_1 + \varepsilon_3)) & \text{if } i = 3 \\ V(\xi_\gamma + \gamma) \oplus V(\gamma) \oplus V(\xi_\gamma + (\varepsilon_1 + \varepsilon_i)) \oplus V(\xi_\gamma + (\varepsilon_2 + \varepsilon_3)) & \text{if } 4 \leq i \leq n \end{cases}$$

2. $C_n(i),\ 2 \leq i \leq n-1:$

$$\mathfrak{l}_\gamma \otimes \mathfrak{z}(\mathfrak{n})$$

$$= \begin{cases} V(\xi_\gamma+\gamma)\oplus V(\gamma)\oplus V(\xi_\gamma+2\varepsilon_2) & \text{if } i=2 \\ V(\xi_\gamma+\gamma)\oplus V(\gamma)\oplus V(\xi_\gamma+(\varepsilon_2+\varepsilon_i))\oplus V(\xi_\gamma+(\varepsilon_1+\varepsilon_2)) & \text{if } 3\le i\le n-1 \end{cases}$$

3. $D_n(i),\ 3 \le i \le n-3:$

$$\begin{aligned}&\mathfrak{l}_\gamma \otimes \mathfrak{z}(\mathfrak{n}) \\ &= \begin{cases} V(\xi_\gamma+\gamma)\oplus V(\gamma)\oplus V(\xi_\gamma+(\varepsilon_1+\varepsilon_3)) & \text{if } i=3 \\ V(\xi_\gamma+\gamma)\oplus V(\gamma)\oplus V(\xi_\gamma+(\varepsilon_1+\varepsilon_i))\oplus V(\xi_\gamma+(\varepsilon_2+\varepsilon_3)) & \text{if } 4\le i\le n-3 \end{cases}\end{aligned}$$

4. All exceptional cases ($E_6(3)$, $E_6(5)$, $E_7(2)$, $E_7(6)$, $E_8(1)$, $F_4(4)$):

$$\mathfrak{l}_\gamma \otimes \mathfrak{z}(\mathfrak{n}) = V(\xi_\gamma+\gamma)\oplus V(\gamma)\oplus V(\xi_\gamma+\gamma_0),$$

where γ_0 is the following root contributing to $\mathfrak{z}(\mathfrak{n})$:

$$E_6(3) : \gamma_0 = \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$$

$$E_6(5) : \gamma_0 = \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$$

$$E_7(2) : \gamma_0 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$$

$$E_7(6) : \gamma_0 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$$

$$E_8(1) : \gamma_0 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 2\alpha_7 + \alpha_8$$

$$F_4(4) : \gamma_0 = \alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4.$$

5.2 Technical Results on $\mathfrak{l}_\gamma \otimes \mathfrak{z}(\mathfrak{n})$

In general, the study of tensor product decomposition of irreducible finite dimensional representations is complicated. Techniques from representation theory and algebraic geometry have been used to study the problem (See for instance [21]). In our setting $\mathfrak{l}_\gamma = V(\xi_\gamma)$ and $\mathfrak{z}(\mathfrak{n}) = V(\gamma)$, the standard techniques suffice to decompose

$V(\xi_\gamma) \otimes V(\gamma)$ under L_γ-action. We have already observed that this action is completely reducible. The goal is to find all the constituents and their multiplicities. To this end, it is enough to study $V(\xi_\gamma) \otimes V(\gamma)$ as an $\mathfrak{l}_\gamma$-module.

Our main technique is to analyze the character formula for $\mathfrak{l}_\gamma \otimes \mathfrak{z}(\mathfrak{n}) = V(\xi_\gamma) \otimes V(\gamma)$ as an $\mathfrak{l}_\gamma$-module. We will freely use the standard notions of dominant weights and regular weights. When we say that ν is $\Delta(\mathfrak{l}_\gamma)$**-dominant** (resp. $\Delta(\mathfrak{l}_\gamma)$**-regular**), we mean that $\langle \nu, \alpha \rangle \geq 0$ (resp. $\langle \nu, \alpha \rangle \neq 0$) for all $\alpha \in \Delta^+(\mathfrak{l}_\gamma)$. For $V(\lambda)$, the finite dimensional $\mathfrak{l}_\gamma$-module with highest weight $\lambda|_{\mathfrak{h}_\gamma}$, and a weight $\nu \in \mathfrak{h}^*$, we denote by $m_\lambda(\nu)$ the multiplicity $\nu|_{\mathfrak{h}_\gamma}$ in $V(\lambda)$; that is, the dimension of the weight space $V(\lambda)_{\nu|_{\mathfrak{h}_\gamma}}$ in $V(\lambda)$.

A weight ν is either $\Delta(\mathfrak{l}_\gamma)$-regular or not. If ν is $\Delta(\mathfrak{l}_\gamma)$-regular then no nontrivial element w in the Weyl group $W(\mathfrak{l}_\gamma)$ of $\mathfrak{l}_\gamma$ fixes ν. Otherwise, there is $w \neq 1$ in $W(\mathfrak{l}_\gamma)$ so that $w\nu = \nu$. Hence, if ν is a $\Delta(\mathfrak{l}_\gamma)$-regular weight then there is a unique $w_\nu \in W(\mathfrak{l}_\gamma)$ so that $w_\nu \nu$ is $\Delta(\mathfrak{l}_\gamma)$-dominant. We will write $d(\nu) = w_\nu \nu$. Define

$$\operatorname{sgn}(\nu) = \begin{cases} 0 & \text{if some } w \neq 1 \text{ in } W(\mathfrak{l}_\gamma) \text{ fixes } \nu \\ (-1)^{l(w_\nu)} & \text{otherwise, where } w_\nu \in W(\mathfrak{l}_\gamma) \text{ so that } w_\nu \nu = d(\nu), \end{cases}$$

where $l(w_\nu)$ is the length of w_ν. We denote by $\rho(\mathfrak{l}_\gamma)$ half the sum of positive roots in $\Delta^+(\mathfrak{l}_\gamma)$. Then if χ_λ (resp. $\chi_{\lambda'}$) is the character for $V(\lambda)$ (resp. $V(\lambda')$) then the character formula for the character $\chi_\lambda \chi_{\lambda'}$ for the $\mathfrak{l}_\gamma$-module $V(\lambda) \otimes V(\lambda')$ is

$$\chi_\lambda \chi_{\lambda'} = \sum_{\lambda'' \in \Delta(V(\lambda))} m_\lambda(\lambda'') \operatorname{sgn}(\lambda'' + \lambda' + \rho(\mathfrak{l}_\gamma)) \chi_{d(\lambda'' + \lambda' + \rho(\mathfrak{l}_\gamma)) - \rho(\mathfrak{l}_\gamma)}, \tag{5.2.1}$$

where $\Delta(V(\lambda))$ is the set of the weights for $V(\lambda)$. This character formula is due to Klimyk [14, Corollary]. Among the standard facts, we use the following to analyze (5.2.1):

(I) The constituent $V(\lambda + \lambda')$ occurs exactly once in $V(\lambda) \otimes V(\lambda')$. Moreover, if v_λ

and $v_{\lambda'}$ are highest weight vectors of $V(\lambda)$ and $V(\lambda')$, respectively, then $v_\lambda \otimes v_{\lambda'}$ is a highest weight vector of $V(\lambda) \otimes V(\lambda')$.

(II) If λ'' is the highest weight of some irreducible constituent of $V(\lambda) \otimes V(\lambda')$ then λ'' is of the form $\lambda'' = \lambda + \nu$ for some weight ν of $V(\lambda')$.

(III) If all weights of $V(\lambda)$ have multiplicity one then each irreducible constituent of $V(\lambda) \otimes V(\lambda')$ has multiplicity one.

The unique irreducible constituent $V(\lambda + \lambda')$ is called the **Cartan component** of $V(\lambda) \otimes V(\lambda')$ (see for instance [21, page 1230]). In our setting $\mathfrak{l}_\gamma \otimes \mathfrak{z}(\mathfrak{n}) = V(\xi_\gamma) \otimes V(\gamma)$, the weights ξ_γ and γ are roots. By Fact (I) the highest weights of the irreducible constituents of $\mathfrak{l}_\gamma \otimes \mathfrak{z}(\mathfrak{n})$ are of the form $\xi_\gamma + \gamma_j$ with $\gamma_j \in \Delta(\mathfrak{z}(\mathfrak{n}))$.

The character formula (5.2.1) is particularly simple when $\Pi(\mathfrak{l}_\gamma)$ consists solely of long roots. We obtain a couple of results under this assumption.

Lemma 5.2.2 *Suppose that* $\Pi(\mathfrak{l}_\gamma)$ *consists solely of long roots of* $\mathfrak{g}$*. If* $\xi_\gamma + \gamma_j$ *is not* $\Delta(\mathfrak{l}_\gamma)$*-dominant then* $\mathrm{sgn}(\xi_\gamma + \gamma_j + \rho(\mathfrak{l}_\gamma)) = 0$.

Proof. We show that there exists $\alpha \in \Pi(\mathfrak{l}_\gamma)$ so that s_α fixes $\xi_\gamma + \gamma_j + \rho(\mathfrak{l}_\gamma)$. Since $\langle \rho(\mathfrak{l}_\gamma), \alpha^\vee \rangle = 1$ for all $\alpha \in \Pi(\mathfrak{l}_\gamma)$, it suffices to show that $\langle \xi_\gamma + \gamma_j, \alpha^\vee \rangle = -1$ for some $\alpha \in \Pi(\mathfrak{l}_\gamma)$. Under our hypothesis $\xi_\gamma + \gamma_j$ is not $\Delta(\mathfrak{l}_\gamma)$-dominant. Hence there exists $\alpha \in \Pi(\mathfrak{l}_\gamma)$ so that $\langle \xi_\gamma + \gamma_j, \alpha^\vee \rangle < 0$. On the other hand, since ξ_γ is the highest weight of $\mathfrak{l}_\gamma$, it follows that $\langle \xi_\gamma, \alpha^\vee \rangle \geq 0$. We have

$$\langle \gamma_j, \alpha^\vee \rangle < -\langle \xi_\gamma, \alpha^\vee \rangle \leq 0, \tag{5.2.3}$$

and $\gamma_j + \alpha \in \Delta$. Since $\Pi(\mathfrak{l}_\gamma)$ contains only long roots, Lemma 3.4.4 shows that $\langle \gamma_j, \alpha^\vee \rangle = -1$. Then (5.2.3) forces $\langle \xi_\gamma, \alpha^\vee \rangle = 0$, since $\langle \xi_\gamma, \alpha^\vee \rangle$ is an integer. Therefore $\langle \xi_\gamma + \gamma_j, \alpha^\vee \rangle = -1$. ■

Remark 5.2.4 *If $\xi_\gamma + \gamma_j$ is $\Delta(\mathfrak{l}_\gamma)$-dominant then $\xi_\gamma + \gamma_j + \rho(\mathfrak{l}_\gamma)$ is $\Delta(\mathfrak{l}_\gamma)$-dominant and $\Delta(\mathfrak{l}_\gamma)$-regular. Hence, we have $\mathrm{sgn}(\xi_\gamma + \gamma_j + \rho(\mathfrak{l}_\gamma)) = 1$.*

Proposition 5.2.5 *Suppose that $\Pi(\mathfrak{l}_\gamma)$ consists solely of long roots of $\mathfrak{g}$. Then $V(\xi_\gamma + \gamma_j)$ is an irreducible constituent of $\mathfrak{l}_\gamma \otimes \mathfrak{z}(\mathfrak{n})$ if and only if $\xi_\gamma + \gamma_j$ is $\Delta(\mathfrak{l}_\gamma)$-dominant.*

Proof. One of the directions is obvious. We then show that $V(\xi_\gamma+\gamma_j)$ is an irreducible constituent if $\xi_\gamma+\gamma_j$ is $\Delta(\mathfrak{l}_\gamma)$-dominant. By Klimyk's character formula, the character $\chi_{\xi_\gamma}\chi_\gamma$ is of the form

$$\chi_{\xi_\gamma}\chi_\gamma = \sum_{\gamma_j \in \Delta(\mathfrak{z}(\mathfrak{n}))} m_\gamma(\gamma_j)\mathrm{sgn}(\xi_\gamma + \gamma_j + \rho(\mathfrak{l}_\gamma))\chi_{d(\xi_\gamma+\gamma_j+\rho(\mathfrak{l}_\gamma))-\rho(\mathfrak{l}_\gamma)}. \tag{5.2.6}$$

Since the weights of $\mathfrak{z}(\mathfrak{n})$ are roots of $\mathfrak{g}$, they have multiplicity one. Thus $m_\gamma(\gamma_j) = 1$ for all $\gamma_j \in \Delta(\mathfrak{z}(\mathfrak{n}))$. Moreover, Lemma 5.2.2 and Remark 5.2.4 show that

$$\mathrm{sgn}(\xi_\gamma + \gamma_j + \rho(\mathfrak{l}_\gamma)) = \begin{cases} 1 & \text{if } \xi_\gamma + \gamma_j \text{ is } \Delta(\mathfrak{l}_\gamma)\text{-dominant} \\ 0 & \text{otherwise.} \end{cases}$$

Thus (5.2.6) is reduced to

$$\chi_{\xi_\gamma}\chi_\gamma = \sum \chi_{\xi_\gamma+\gamma_j}, \tag{5.2.7}$$

where the sum runs over all $\gamma_j \in \Delta(\mathfrak{z}(\mathfrak{n}))$ so that $\xi_\gamma + \gamma_j$ is $\Delta(\mathfrak{l}_\gamma)$-dominant. Now the proposed assertion follows. ∎

Corollary 5.2.8 *If $\Pi(\mathfrak{l}_\gamma)$ consists solely of long roots of $\mathfrak{g}$ then $V(\gamma)$ occurs in the decomposition of $\mathfrak{l}_\gamma \otimes \mathfrak{z}(\mathfrak{n})$ into irreducibles.*

Proof. By Lemma 3.4.2, we have $\gamma - \xi_\gamma \in \Delta(\mathfrak{z}(\mathfrak{n}))$. Thus there exists $\gamma_j \in \Delta(\mathfrak{z}(\mathfrak{n}))$ so that $\xi_\gamma + \gamma_j = \gamma$. Since γ is $\Delta(\mathfrak{l}_\gamma)$-dominant, the corollary follows from Proposition 5.2.5. ∎

Remark 5.2.9 *Theorem 5.1.3 shows that $V(\gamma)$ in fact occurs in $\mathfrak{l}_\gamma \otimes \mathfrak{z}(\mathfrak{n})$ in every case.*

5.3 Proof of Theorem 5.1.3

In the previous section we have shown that the character formula (5.2.1) is simple, when $\Pi(\mathfrak{l}_\gamma)$ consists solely of long roots. Then in order to prove Theorem 5.1.3, we consider two cases, namely,

Case 1: $\Pi(\mathfrak{l}_\gamma)$ consists solely of long roots.

Case 2: $\Pi(\mathfrak{l}_\gamma)$ contains at least one short root.

When $\mathfrak{g}$ is simply laced, we regard any roots as long roots. Direct observation shows that the parabolic subalgebras $\mathfrak{q}$ in (3.3.2) and (3.3.3) are then classified as follows:

Case 1: $B_n(i)$, $D_n(i)$, $E_6(3)$, $E_6(5)$, $E_7(2)$, $E_7(6)$, $E_8(1)$

Case 2: $C_n(i)$, $F_4(4)$

We start by proving Theorem 5.1.3 for parabolic subalgebras $\mathfrak{q}$ in Case 1.

Proof. [Proof for Theorem 5.1.3 for Case 1] Let Γ be the set of all roots $\gamma_j \in \Delta(\mathfrak{z}(\mathfrak{n}))$ so that $\xi_\gamma + \gamma_j$ is $\Delta(\mathfrak{l}_\gamma)$-dominant. It follows from Fact (III) and Proposition 5.2.5 that the character $\chi_{\xi_\gamma}\chi_\gamma$ is of the form

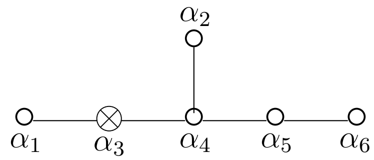
$$\chi_{\xi_\gamma}\chi_\gamma = \sum_{\gamma_j \in \Gamma} \chi_{\xi_\gamma+\gamma_j}. \tag{5.3.1}$$

Moreover, Fact (I) and Corollary 5.2.8 show that $V(\xi_\gamma + \gamma)$ and $V(\gamma)$ occur in the decomposition. Therefore (5.3.1) might be expressed as

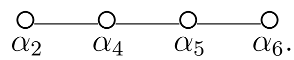
$$\chi_{\xi_\gamma}\chi_\gamma = \chi_{\xi_\gamma+\gamma} + \chi_\gamma + \sum_{\gamma_j \in \Gamma\backslash\{\gamma,\gamma-\xi_\gamma\}} \chi_{\xi_\gamma+\gamma_j}.$$

It remains to identify the roots in $\Gamma\backslash\{\gamma, \gamma-\xi_\gamma\}$. This is done in a case by case fashion. We include the computation for type $E_6(3)$. Other cases may be handled similarly.

The parabolic subalgebra $\mathfrak{q}$ of type $E_6(3)$ corresponds to the deleted Dynkin diagram



The subgraph corresponding to $\mathfrak{l}_\gamma$ is



So the simple subalgebra $\mathfrak{l}_\gamma$ is isomorphic to $\mathfrak{sl}(5,\mathbb{C})$. Write the fundamental weights of $\mathfrak{sl}(5,\mathbb{C})$ corresponding to $\alpha_2,\ \alpha_4,\ \alpha_5,\ \alpha_6$ as $\varpi_1,\ \varpi_2,\ \varpi_3,\ \varpi_4$, respectively. The $\mathfrak{l}_\gamma$-module $\mathfrak{z}(\mathfrak{n})$ has highest weight γ. As $\langle \gamma, \alpha_i \rangle = \delta_{i,2}$ with $\delta_{i,2}$ the Kronecker delta for all $i = 2, 4, 5, 6$, we have $\mathfrak{z}(\mathfrak{n}) = V(\varpi_1)$. Thus, the adjoint representation $\mathfrak{l}_\gamma$ on $\mathfrak{z}(\mathfrak{n})$ is equivalent to the standard representation of $\mathfrak{sl}(5,\mathbb{C})$ on $\mathbb{C}^5$. We then identify the weights of the adjoint action of $\mathfrak{l}_\gamma$ on $\mathfrak{z}(\mathfrak{n})$ with those of the standard action of $\mathfrak{sl}(5,\mathbb{C})$ on $\mathbb{C}^5$; that is,

$$\Delta(\mathfrak{z}(\mathfrak{n})) = \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5\}.$$

In terms of the fundamental weights we have

$$\varepsilon_1 = \varpi_1, \quad \varepsilon_2 = -\varpi_1 + \varpi_2, \quad \varepsilon_3 = -\varpi_2 + \varpi_3, \quad \varepsilon_4 = -\varpi_3 + \varpi_4, \quad \varepsilon_5 = -\varpi_4.$$

The highest weight ξ_γ of $\mathfrak{l}_\gamma$ is $\xi_\gamma = \varpi_1 + \varpi_4$. Therefore, the weights $\gamma_j \in \Delta(\mathfrak{z}(\mathfrak{n}))$ that make $\xi_\gamma + \gamma_j$ $\Delta(\mathfrak{l}_\gamma)$-dominant are $\gamma_j = \varpi_1,\ -\varpi_4$, or $-\varpi_1 + \varpi_2$. Here, we have $\xi_\gamma + \varpi_1 = \xi_\gamma + \gamma$, $\xi_\gamma + (-\varpi_4) = \varpi_1 = \gamma$, and $\xi_\gamma + (-\varpi_1 + \varpi_2) = \xi_\gamma + \gamma_0$ with γ_0 the root in $\Delta(\mathfrak{z}(\mathfrak{n}))$ listed in Theorem 5.1.3. ∎

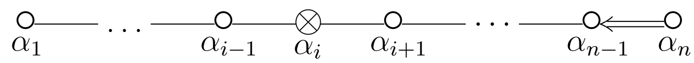
We next show Theorem 5.1.3 for parabolic subalgebras $\mathfrak{q}$ in Case 2, namely, $C_n(i)$ for $2 \leq i \leq n-1$, and $F_4(4)$.

Proof. [Proof for Theorem 5.1.3 for Case 2] The character formula of the tensor product (5.2.6) is of the form

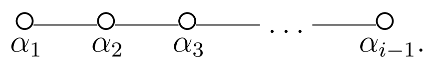
$$\chi_{\xi_\gamma}\chi_\gamma = \sum_{\gamma_j \in \Delta(\mathfrak{z}(\mathfrak{n}))} \operatorname{sgn}(\xi_\gamma + \gamma_j + \rho(\mathfrak{l}_\gamma))\chi_{d(\xi_\gamma+\gamma_j+\rho(\mathfrak{l}_\gamma))-\rho(\mathfrak{l}_\gamma)}. \tag{5.3.2}$$

Here, we use the fact that $m_\gamma(\gamma_j) = 1$ for γ_j roots in $\mathfrak{z}(\mathfrak{n})$. Our strategy is to first find all $\gamma_j \in \Delta(\mathfrak{z}(\mathfrak{n}))$ so that $\xi_\gamma + \gamma_j$ is $\Delta(\mathfrak{l}_\gamma)$-dominant. We then consider the contributions from roots γ_j with $\xi_\gamma + \gamma_j$ not $\Delta(\mathfrak{l}_\gamma)$-dominant. The case $C_n(i)$ for $2 \le i \le n-1$ is demonstrated first. Later, we handle the $F_4(4)$ case.

Let $\mathfrak{q}$ be of type $C_n(i)$ for $2 \le i \le n-1$. The deleted Dynkin diagram is



and the subgraph corresponding to $\mathfrak{l}_\gamma$ is



(5.3.3)

The data in Appendix C shows that

$$\Delta^+(\mathfrak{l}_\gamma) = \{\varepsilon_j - \varepsilon_k \mid 1 \le j < k \le i\}$$

and

$$\Delta(\mathfrak{z}(\mathfrak{n})) = \{\varepsilon_j + \varepsilon_k \mid 1 \le j < k \le i\} \cup \{2\varepsilon_j \mid 1 \le j \le i\}.$$

We have $\xi_\gamma = \varepsilon_1 - \varepsilon_i$ and $\gamma = 2\varepsilon_1$. If Γ is the set of all $\gamma_j \in \Delta(\mathfrak{z}(\mathfrak{n}))$ so that $\xi_\gamma + \gamma_j$ is $\Delta(\mathfrak{l}_\gamma)$-dominant then, by Remark 5.2.4, the character $\chi_{\xi_\gamma}\chi_\gamma$ may be written as

$$\chi_{\xi_\gamma}\chi_\gamma = \sum_{\gamma_j \in \Gamma} \chi_{\xi_\gamma+\gamma_j} + \sum_{\gamma_j \in \Delta(\mathfrak{z}(\mathfrak{n}))\setminus\Gamma} \operatorname{sgn}(\xi_\gamma + \gamma_j + \rho(\mathfrak{l}_\gamma))\chi_{d(\xi_\gamma+\gamma_j+\rho(\mathfrak{l}_\gamma))-\rho(\mathfrak{l}_\gamma)}. \tag{5.3.4}$$

One can see by direct computation that

$$\Gamma = \begin{cases} \{\gamma, \varepsilon_1+\varepsilon_2, 2\varepsilon_2\} & \text{if } i = 2 \\ \{\gamma, \varepsilon_1+\varepsilon_2, \varepsilon_1+\varepsilon_3, \varepsilon_2+\varepsilon_3\} & \text{if } i = 3 \\ \{\gamma, \varepsilon_1+\varepsilon_2, \varepsilon_1+\varepsilon_i, \varepsilon_2+\varepsilon_3, \varepsilon_2+\varepsilon_i\} & \text{if } 4 \le i \le n-1\ . \end{cases}$$

When $i = 2$, we have $\Gamma = \Delta(\mathfrak{z}(\mathfrak{n}))$, and so, $\chi_{\xi_\gamma}\chi_\gamma$ is

$$\chi_{\xi_\gamma}\chi_\gamma = \sum_{\gamma_j \in \Gamma} \chi_{\xi_\gamma+\gamma_j} = \chi_{\xi_\gamma+\gamma} + \chi_{\xi_\gamma+(\varepsilon_1+\varepsilon_2)} + \chi_{\xi_\gamma+(2\varepsilon_2)}.$$

Since $\xi_\gamma = \varepsilon_1 - \varepsilon_2$, we have $\xi_\gamma + (\varepsilon_1 + \varepsilon_2) = 2\varepsilon_1 = \gamma$. When $i = 3$, it follows that $\Delta(\mathfrak{z}(\mathfrak{n}))\backslash\Gamma = \{2\varepsilon_2, 2\varepsilon_3\}$. Since we have $s_{\varepsilon_1-\varepsilon_2}(\xi_\gamma + 2\varepsilon_2 + \rho(\mathfrak{l}_\gamma)) = \xi_\gamma + 2\varepsilon_2 + \rho(\mathfrak{l}_\gamma)$ and $s_{\varepsilon_2-\varepsilon_3}(\xi_\gamma + 2\varepsilon_3 + \rho(\mathfrak{l}_\gamma)) = \xi_\gamma + 2\varepsilon_3 + \rho(\mathfrak{l}_\gamma)$, both weights are not $\Delta(\mathfrak{l}_\gamma)$-regular and do not contribute to the character. Therefore, when $i = 3$,

$$\chi_{\xi_\gamma}\chi_\gamma = \sum_{\gamma_j \in \Gamma} \chi_{\xi_\gamma+\gamma_j} = \chi_{\xi_\gamma+\gamma} + \chi_{\xi_\gamma+(\varepsilon_1+\varepsilon_2)} + \chi_{\xi_\gamma+(\varepsilon_1+\varepsilon_3)} + \chi_{\xi_\gamma+(\varepsilon_2+\varepsilon_3)}.$$

Since $\xi_\gamma = \varepsilon_1 - \varepsilon_3$, we have $\xi_\gamma + (\varepsilon_1 + \varepsilon_3) = 2\varepsilon_1 = \gamma$.

If $4 \le i \le n-1$ then $\gamma_j \in \Delta(\mathfrak{z}(\mathfrak{n}))\backslash\Gamma$ is

$\varepsilon_1 + \varepsilon_k$ for $3 \le k \le i-1$,

$\varepsilon_2 + \varepsilon_k$ for $4 \le k \le i-1$,

$\varepsilon_r + \varepsilon_k$ for $3 \le r < k \le i$, or

$2\varepsilon_r$ for $2 \le r \le i$.

An observation shows that, for each $\gamma_j \in \Delta(\mathfrak{z}(\mathfrak{n}))\backslash\Gamma$ with $\gamma_j \neq 2\varepsilon_3$, there exists $w \in W(\mathfrak{l}_\gamma)$ with $w \neq 1$ so that w fixes $\xi_\gamma + \gamma_j + \rho(\mathfrak{l}_\gamma)$. Indeed, it is clear from (5.3.3) that $\mathfrak{l}_\gamma$ is of type A_{i-1}. Thus $\rho(\mathfrak{l}_\gamma)$ is given by

$$\rho(\mathfrak{l}_\gamma) = \sum_{s=1}^{i} \left(\frac{i-(2s-1)}{2}\right)\varepsilon_s. \tag{5.3.5}$$

If

$$w = \begin{cases} s_{\varepsilon_{k-1}-\varepsilon_k} & \text{when } \gamma_j = \varepsilon_1+\varepsilon_k, \varepsilon_2+\varepsilon_k \\ s_{\varepsilon_{r-1}-\varepsilon_r} & \text{when } \gamma_j = \varepsilon_r+\varepsilon_k \\ s_{\varepsilon_1-\varepsilon_2} & \text{when } \gamma_j = 2\varepsilon_2 \\ s_{\varepsilon_{r-2}-\varepsilon_r} & \text{when } \gamma_j = 2\varepsilon_r \text{ for } 4 \le r \le i-1 \\ s_{\varepsilon_{i-1}-\varepsilon_i} & \text{when } \gamma_j = 2\varepsilon_i \end{cases}$$

then $w(\xi_\gamma + \gamma_j + \rho(\mathfrak{l}_\gamma)) = \xi_\gamma + \gamma_j + \rho(\mathfrak{l}_\gamma)$. Therefore $\mathrm{sgn}(\xi_\gamma + \gamma_j + \rho(\mathfrak{l}_\gamma)) = 0$ for such γ_j.

Now suppose that $\gamma_j = 2\varepsilon_3$. We first show that $\xi_\gamma + 2\varepsilon_3 + \rho(\mathfrak{l}_\gamma)$ is $\Delta(\mathfrak{l}_\gamma)$-regular. By (5.3.5), we have

$$\begin{aligned}&\xi_\gamma + 2\varepsilon_3 + \rho(\mathfrak{l}_\gamma) = \\ &(\frac{i+1}{2})\varepsilon_1 + (\frac{i-3}{2})\varepsilon_2 + (\frac{i-1}{2})\varepsilon_3 + \sum_{s=4}^{i-1}(\frac{i-(2s-1)}{2})\varepsilon_s + (-\frac{i+1}{2})\varepsilon_i.\end{aligned} \tag{5.3.6}$$

The coefficients of ε_s and ε_t with $s \neq t$ in (5.3.6) are different. Since roots in $\Delta^+(\mathfrak{l}_\gamma)$ are of the form $\varepsilon_s - \varepsilon_t$ with $s < t$, this shows that the weight $\xi_\gamma + 2\varepsilon_3 + \rho(\mathfrak{l}_\gamma)$ is $\Delta(\mathfrak{l}_\gamma)$-regular. The reflection $s_{\varepsilon_2-\varepsilon_3}$ conjugates $\xi_\gamma + 2\varepsilon_3 + \rho(\mathfrak{l}_\gamma)$ to the $\Delta(\mathfrak{l}_\gamma)$-dominant weight

$$s_{\varepsilon_2-\varepsilon_3}(\xi_\gamma + 2\varepsilon_3 + \rho(\mathfrak{l}_\gamma)) = \xi_\gamma + (\varepsilon_2 + \varepsilon_3) + \rho(\mathfrak{l}_\gamma).$$

Thus $\mathrm{sgn}(\xi_\gamma + \gamma_j + \rho(\mathfrak{l}_\gamma)) = -1$ and $d(\xi_\gamma + \gamma_j + \rho(\mathfrak{l}_\gamma)) = \xi_\gamma + (\varepsilon_2 + \varepsilon_3) + \rho(\mathfrak{l}_\gamma)$; we have

$$\mathrm{sgn}(\xi_\gamma + \gamma_j + \rho(\mathfrak{l}_\gamma))\chi_{d(\xi_\gamma+\gamma_j+\rho(\mathfrak{l}_\gamma))-\rho(\mathfrak{l}_\gamma)} = -\chi_{\xi_\gamma+(\varepsilon_2+\varepsilon_3)}.$$

Hence,

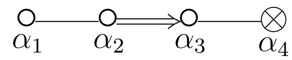
$$\begin{aligned}\chi_{\xi_\gamma}\chi_\gamma &= \sum_{\gamma_j\in\Gamma}\chi_{\xi_\gamma+\gamma_j} + \sum_{\gamma_j\in\Delta(\mathfrak{z}(\mathfrak{n}))\setminus\Gamma}\mathrm{sgn}(\xi_\gamma + \gamma_j + \rho(\mathfrak{l}_\gamma))\chi_{d(\xi_\gamma+\gamma_j+\rho(\mathfrak{l}_\gamma))-\rho(\mathfrak{l}_\gamma)} \\ &= \sum_{\gamma_j\in\Gamma}\chi_{\xi_\gamma+\gamma_j} - \chi_{\xi_\gamma+(\varepsilon_2+\varepsilon_3)}\end{aligned} \tag{5.3.7}$$

with $\Gamma = \{\gamma, \varepsilon_1+\varepsilon_2, \varepsilon_1+\varepsilon_i, \varepsilon_2+\varepsilon_3, \varepsilon_2+\varepsilon_i\}$ for $4 \leq i \leq n-1$. Then we obtain

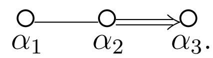
$$\begin{aligned}\chi_{\xi_\gamma}\chi_\gamma &= \sum_{\gamma_j\in\Gamma}\chi_{\xi_\gamma+\gamma_j} - \chi_{\xi_\gamma+(\varepsilon_2+\varepsilon_3)} \\ &= \chi_{\xi_\gamma+\gamma} + \chi_{\xi_\gamma+(\varepsilon_1+\varepsilon_2)} + \chi_{\xi_\gamma+(\varepsilon_1+\varepsilon_i)} + \chi_{\xi_\gamma+(\varepsilon_2+\varepsilon_3)} + \chi_{\xi_\gamma+(\varepsilon_2+\varepsilon_i)} - \chi_{\xi_\gamma+(\varepsilon_2+\varepsilon_3)} \\ &= \chi_{\xi_\gamma+\gamma} + \chi_{\xi_\gamma+(\varepsilon_1+\varepsilon_2)} + \chi_{\xi_\gamma+(\varepsilon_1+\varepsilon_i)} + \chi_{\xi_\gamma+(\varepsilon_2+\varepsilon_i)}.\end{aligned}$$

Since $\xi_\gamma = \varepsilon_1 - \varepsilon_i$, we have $\xi_\gamma + (\varepsilon_1 + \varepsilon_i) = 2\varepsilon_1 = \gamma$.

Next we consider the case that $\mathfrak{q}$ is of type $F_4(4)$. The deleted Dynkin diagram is



and the subgraph corresponding to $\mathfrak{l}_\gamma$ is



The simple subalgebra $\mathfrak{l}_\gamma$ is isomorphic to $\mathfrak{so}(7,\mathbb{C})$. If we write the fundamental weights of $\mathfrak{l}_\gamma \cong \mathfrak{so}(7,\mathbb{C})$ corresponding to α_1, α_2, α_3 as ϖ_1, ϖ_2, ϖ_3, respectively, then the highest weights ξ_γ for $\mathfrak{l}_\gamma$ and γ for $\mathfrak{z}(\mathfrak{n})$ are written in terms of the fundamental weights as $\xi_\gamma = \varpi_2$ and $\gamma = \varpi_1$; we have $\mathfrak{l}_\gamma = V(\varpi_2)$ and $\mathfrak{z}(\mathfrak{n}) = V(\varpi_1)$. Therefore the adjoint action of $\mathfrak{l}_\gamma$ on itself (resp. on $\mathfrak{z}(\mathfrak{n})$) is equivalent to the standard action of $\mathfrak{so}(7,\mathbb{C})$ on $\wedge^2\mathbb{C}^7$ (resp. on $\mathbb{C}^7$). We then identify the $\mathfrak{l}_\gamma$-module $\mathfrak{l}_\gamma \otimes \mathfrak{z}(\mathfrak{n})$ as the $\mathfrak{so}(7,\mathbb{C})$-module $(\wedge^2\mathbb{C}^7) \otimes (\mathbb{C}^7)$, and consider the irreducible decomposition of $(\wedge^2\mathbb{C}^7) \otimes (\mathbb{C}^7)$.

Let Δ^+ be the standard choice of a positive system of $\mathfrak{so}(7,\mathbb{C})$ and ρ be half the sum of the positive roots; that is,

$$\Delta^+ = \{\varepsilon_1 \pm \varepsilon_2, \varepsilon_2 \pm \varepsilon_3, \varepsilon_1 \pm \varepsilon_3\} \cup \{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$$

and

$$\rho = \frac{5}{2}\varepsilon_1 + \frac{3}{2}\varepsilon_2 + \frac{1}{2}\varepsilon_3.$$

If

$$\Gamma = \{\eta \in \Delta(\mathbb{C}^7) \mid \varpi_2 + \eta \text{ is dominant}\}$$

with $\Delta(\mathbb{C}^7)$ the set of weights for $\mathbb{C}^7$ then the character $\chi_{\varpi_2}\chi_{\varpi_1}$ for $(\wedge^2\mathbb{C}^7) \otimes (\mathbb{C}^7) =$

$V(\varpi_2) \otimes V(\varpi_1)$ is

$$\begin{aligned}
\chi_{\varpi_2}\chi_{\varpi_1} &= \sum_{\eta \in \Delta(\mathbb{C}^7)} m_{\varpi_1}(\eta)\mathrm{sgn}(\varpi_2 + \eta + \rho)\chi_{d(\varpi_2+\eta+\rho)-\rho} \\
&= \sum_{\eta \in \Delta(\mathbb{C}^7)} \mathrm{sgn}(\varpi_2 + \eta + \rho)\chi_{d(\varpi_2+\eta+\rho)-\rho} \\
&= \sum_{\eta \in \Gamma} \chi_{\varpi_2+\eta} + \sum_{\eta \in \Delta(\mathbb{C}^7)\backslash\Gamma} \mathrm{sgn}(\varpi_2 + \eta + \rho)\chi_{d(\varpi_2+\eta+\rho)-\rho}.
\end{aligned}$$

We need determine the contributions from $\eta \in \Delta(\mathbb{C}^7)\backslash\Gamma$. The weights for $\mathbb{C}^7$ under the standard action of $\mathfrak{so}(7, \mathbb{C})$ are

$$\Delta(\mathbb{C}^7) = \{\pm\varepsilon_1, \pm\varepsilon_2, \pm\varepsilon_3, 0\}.$$

In terms of the fundamental weights ϖ_1, ϖ_2, and ϖ_3, we have

$$\varepsilon_1 = \varpi_1, \quad \varepsilon_2 = -\varpi_1 + \varpi_2, \quad \varepsilon_3 = -\varpi_2 + 2\varpi_3.$$

Therefore, the weights for $\mathbb{C}^7$ may be written in terms of the fundamental weights as

$$\Delta(\mathbb{C}^7) = \{\pm\varpi_1, \pm(-\varpi_1 + \varpi_2), \pm(-\varpi_2 + 2\varpi_3), 0\}.$$

If η is a weight for $\mathbb{C}^7$ so that $\varpi_2 + \eta$ is $\Delta(\mathfrak{l}_\gamma)$-dominant then η must be

$$\eta = \varpi_1, \varpi_2 - \varpi_2, -\varpi_2 + 2\varpi_3, \text{ or } 0. \tag{5.3.8}$$

Thus,

$$\Delta(\mathbb{C}^7)\backslash\Gamma = \{-\varpi_1, -\varpi_1 + \varpi_2, \varpi_2 - 2\varpi_3\} = \{-\varepsilon_1, \varepsilon_2, -\varepsilon_3\}.$$

Observe that when $\eta = -\varepsilon_1$ or ε_2, there exists a Weyl group element $w \in W$ of $\mathfrak{so}(7, \mathbb{C})$ that fixes $\varpi_2 + \eta + \rho$. Indeed, for either case $\eta = -\varepsilon_1$ or ε_2, the root reflection $s_{\varepsilon_1-\varepsilon_2}$ fixes $\varpi_2 + \eta + \rho$, as $\varpi_2 = \varepsilon_1 + \varepsilon_2$. Thus $\mathrm{sgn}(\varpi_2 + \eta + \rho) = 0$ when $\eta = -\varepsilon_1$ or ε_2. On the other hand, when $\eta = -\varepsilon_3$, we have

$$\varpi_2 - \varepsilon_3 + \rho = \frac{7}{2}\varepsilon_1 + \frac{5}{2}\varepsilon_2 - \frac{1}{2}\varepsilon_3. \tag{5.3.9}$$

The coefficients of ε_s and ε_t with $s \neq t$ in (5.3.9) are different. Since roots in Δ^+ are of the form $\varepsilon_s \pm \varepsilon_t$ with $s < t$ or ε_s, this shows that the weight $\varpi_2 - \varepsilon_3 + \rho$ is $\Delta(\mathfrak{l}_\gamma)$-regular. The reflection s_{ε_3} conjugates $\varpi_2 - \varepsilon_3 + \rho$ to the $\Delta(\mathfrak{l}_\gamma)$-dominant weight

$$s_{\varepsilon_3}(\varpi_2 - \varepsilon_3 + \rho) = \frac{7}{2}\varepsilon_1 + \frac{5}{2}\varepsilon_2 + \frac{1}{2}\varepsilon_3.$$

Thus $\mathrm{sgn}(\varpi_2 - \varepsilon_3 + \rho) = -1$ and $d(\varpi_2 - \varepsilon_3 + \rho) - \rho = \varepsilon_1 + \varepsilon_2 = \varpi_2$; we have

$$\mathrm{sgn}(\varpi_2 - \varepsilon_3 + \rho)\chi_{d(\varpi_2 - \varepsilon_3 + \rho) - \rho} = -\chi_{\varpi_2}.$$

Hence,

$$\begin{aligned} \chi_{\varpi_2}\chi_{\varpi_1} &= \sum_{\eta\in\Gamma} \chi_{\varpi_2+\eta} + \sum_{\eta\in\Delta(\mathbb{C}^7)\setminus\Gamma} \mathrm{sgn}(\varpi_2 + \eta + \rho)\chi_{d(\varpi_2+\eta+\rho)-\rho} \\ &= \sum_{\eta\in\Gamma} \chi_{\varpi_2+\eta} - \chi_{\varpi_2}. \end{aligned}$$

By (5.3.8), we have $\Gamma = \{\varpi_1, \varpi_2 - \varpi_2, -\varpi_2 + 2\varpi_3, 0\}$. Therefore,

$$\begin{aligned} \chi_{\varpi_2}\chi_{\varpi_1} &= \sum_{\eta\in\Gamma} \chi_{\varpi_2+\eta} - \chi_{\varpi_2} \\ &= \chi_{\varpi_2+\varpi_1} + \chi_{\varpi_2+(\varpi_1-\varpi_2)} + \chi_{\varpi_2+(-\varpi_2+2\varpi_3)}. \end{aligned}$$

We have $\varpi_2 + \varpi_1 = \xi_\gamma + \gamma$, $\varpi_2 + (\varpi_1 - \varpi_2) = \varpi_1 = \gamma$, and $\varpi_2 + (-\varpi_2 + 2\varpi_3) = \xi_\gamma + \gamma_0$ with γ_0 the root in $\Delta(\mathfrak{z}(\mathfrak{n}))$ in Theorem 5.1.3. This completes the proof. ■

CHAPTER 6

Special Constituents of $\mathfrak{l} \otimes \mathfrak{z}(\mathfrak{n})$

In this chapter, by using the decomposition results in Chapter 5, we shall determine the candidates of the irreducible constituents of $\mathfrak{l} \otimes \mathfrak{z}(\mathfrak{n})$ that will contribute to the Ω_2 systems; that is, the irreducible constituents $V(\nu)$ so that $\tilde{\tau}_2|_{V(\nu)^*}$ are not identically zero.

6.1 Special Constituents

Given $V(\nu)$, an irreducible constituent in $\mathfrak{l} \otimes \mathfrak{z}(\mathfrak{n})$, we build an L-intertwining map

$$\tilde{\tau}_2|_{V(\nu)^*} \in \mathrm{Hom}_L(V(\nu)^*, \mathcal{P}^2(\mathfrak{g}(1)))$$

with $V(\nu)^*$ the dual of $V(\nu)$ with respect to the Killing form κ. From $\tilde{\tau}_2|_{V(\nu)^*}$, we construct operator $\Omega_2|_{V(\nu)^*} : V(\nu)^* \to \mathbb{D}(\mathcal{L}_{-s})^{\bar{\mathfrak{n}}}$. To do so, it is necessary to determine which irreducible constituents $V(\nu)$ have property that $\tilde{\tau}_2|_{V(\nu)^*} \neq 0$.

We start by observing the vector space isomorphism $\mathcal{P}^2(\mathfrak{g}(1)) \cong \mathrm{Sym}^2(\mathfrak{g}(1))^*$. With the natural L-action on $\mathcal{P}^2(\mathfrak{g}(1))$ and $\mathrm{Sym}^2(\mathfrak{g}(1))^*$, this vector space isomorphism is L-equivariant. Thus, if $\tilde{\tau}_2\big|_{V(\nu)^*}$ is a non-zero map then $V(\nu)$ is an irreducible constituent of $\mathrm{Sym}^2(\mathfrak{g}(1)) \subset \mathfrak{g}(1) \otimes \mathfrak{g}(1)$; in particular, by Fact (II) in Section 5.2, ν is of the form $\nu = \mu + \epsilon$ for some $\epsilon \in \Delta(\mathfrak{g}(1))$, where μ is the highest weight of $\mathfrak{g}(1)$.

One can see from the decompositions in Theorem 5.1.3 that $V(\gamma)$ is an irreducible constituent of $\mathfrak{l} \otimes \mathfrak{z}(\mathfrak{n})$ for any $\mathfrak{q}$ under consideration. By Lemma 3.4.2, we have $\gamma = \mu + \epsilon$ for some $\epsilon \in \Delta(\mathfrak{g}(1))$. Now we claim that $\tilde{\tau}_2|_{V(\gamma)^*}$ is identically zero. It is well-known that

$$\mathfrak{g}(1) \otimes \mathfrak{g}(1) = \mathrm{Sym}^2(\mathfrak{g}(1)) \oplus \wedge^2(\mathfrak{g}(1)) \tag{6.1.1}$$

as an L-module. Since each weight for $\mathfrak{g}(1)$ is a root of $\mathfrak{g}$, by Fact (III) in Section 5.2, the L-module decomposition (6.1.1) is multiplicity free.

Proposition 6.1.2 *The L-module $V(\gamma)$ is an irreducible constituent of $\wedge^2(\mathfrak{g}(1))$.*

Proof. Define a linear map $\varphi : \mathfrak{z}(\mathfrak{n}) \to \wedge^2(\mathfrak{g}(1))$ by means of

$$\varphi(W) = \sum_{\beta \in \Delta(\mathfrak{g}(1))} \mathrm{ad}(W)X_{-\beta} \wedge X_\beta.$$

By using an argument similar to that for Lemma 2.5.4, one can show that φ is L-equivariant. Then, since $\mathfrak{z}(\mathfrak{n}) \cong V(\gamma)$ as an irreducible L-module, it suffices to show that φ is a non-zero map. Write $\Delta_\gamma(\mathfrak{g}(1)) = \{\beta \in \Delta(\mathfrak{g}(1)) \mid \gamma - \beta \in \Delta\}$. By Lemma 3.4.2, we have $\gamma - \mu \in \Delta$. Hence $\Delta_\gamma(\mathfrak{g}(1)) \neq \emptyset$. By writing $\beta' = \gamma - \beta$ for $\beta \in \Delta_\gamma(\mathfrak{g}(1))$, $\varphi(X_\gamma)$ is given by

$$\varphi(X_\gamma) = \sum_{\beta \in \Delta(\mathfrak{g}(1))} \mathrm{ad}(X_\gamma)X_{-\beta} \wedge X_\beta = \sum_{\beta \in \Delta_\gamma(\mathfrak{g}(1))} N_{\gamma,-\beta} X_{\beta'} \wedge X_\beta.$$

Observe that for each $\beta \in \Delta_\gamma(\mathfrak{g}(1))$, we have $\gamma - \beta \in \Delta_\gamma(\mathfrak{g}(1))$. Moreover, by Property (H6) of our normalizations in Section 4.1, it follows that $N_{\gamma,-\beta'} = -N_{\gamma,-\beta}$. Therefore,

$$N_{\gamma,-\beta} X_{\beta'} \wedge X_\beta + N_{\gamma,-\beta'} X_\beta \wedge X_{\beta'} = 2N_{\gamma,-\beta} X_{\beta'} \wedge X_\beta. \tag{6.1.3}$$

Since $N_{\gamma,-\beta} \neq 0$ for $\beta \in \Delta_\gamma(\mathfrak{g}(1))$, equation (6.1.3) is non-zero. On the other hand, if $\beta \in \Delta_\gamma(\mathfrak{g}(1))$ and $\eta \in \Delta_\gamma(\mathfrak{g}(1))$ is so that $\eta \neq \beta, \beta'$ then $X_{\beta'} \wedge X_\beta$ and $X_\eta \wedge X_\beta$ are linearly independent. Hence, $\varphi(X_\gamma) \neq 0$. ∎

Definition 6.1.4 *An irreducible constituent $V(\nu)$ of $\mathfrak{l} \otimes \mathfrak{z}(\mathfrak{n})$ is called **special** if $\nu \neq \gamma$ and there exists $\epsilon \in \Delta(\mathfrak{g}(1))$ so that $\nu = \mu + \epsilon$, where μ and γ are the highest weights for $\mathfrak{g}(1)$ and $\mathfrak{z}(\mathfrak{n})$, respectively.*

Proposition 6.1.5 *Let $V(\nu)$ be an irreducible constituent of $\mathfrak{l} \otimes \mathfrak{z}(\mathfrak{n})$. Then $\tilde{\tau}_2\big|_{V(\nu)^*}$ is not identically zero only if $V(\nu)$ is a special constituent of $\mathfrak{l} \otimes \mathfrak{z}(\mathfrak{n})$.*

Proof. At the beginning of this section we observed that if $\tilde{\tau}_2|_{V(\nu)^*} \neq 0$ then ν must be of the form $\nu = \mu + \epsilon$ for some $\epsilon \in \Delta(\mathfrak{g}(1))$. Then $V(\nu)$ is either a special constituent or $V(\gamma)$ (by Lemma 3.4.2, γ satisfies the form). However, by Proposition 6.1.2, it follows that $\tilde{\tau}_2|_{V(\gamma)^*}$ is identically zero. Therefore, $V(\nu)$ must be a special constituent. ■

We will show in Chapter 7 that the converse of Proposition 6.1.5 also holds for certain special constituents (see Proposition 7.1.6).

6.2 Types of Special Constituents

The aim of this section is to determine and classify all the special constituents of $\mathfrak{l} \otimes \mathfrak{z}(\mathfrak{n})$. Such a classification will play a role in the explicit construction of the Ω_2 systems. We use the decomposition results in Chapter 5 for the rest of this chapter. The parabolic subalgebra $\mathfrak{q}$ under consideration is assumed to be one in (3.3.2) or (3.3.3).

Since $\mathfrak{l} \otimes \mathfrak{z}(\mathfrak{n}) = (\mathbb{C}H_{\mathfrak{q}} \otimes \mathfrak{z}(\mathfrak{n})) \oplus (\mathfrak{l}_{ss} \otimes \mathfrak{z}(\mathfrak{n}))$ and $\mathbb{C}H_{\mathfrak{q}} \otimes \mathfrak{z}(\mathfrak{n}) = V(\gamma)$, it suffices to consider $\mathfrak{l}_{ss} \otimes \mathfrak{z}(\mathfrak{n}) = (\mathfrak{l}_{\gamma} \otimes \mathfrak{z}(\mathfrak{n})) \oplus (\mathfrak{l}_{n\gamma} \otimes \mathfrak{z}(\mathfrak{n}))$. We start by observing that, by Proposition 5.1.2, $\mathfrak{l}_{n\gamma} \otimes \mathfrak{z}(\mathfrak{n}) = V(\xi_{n\gamma} + \gamma)$.

Proposition 6.2.1 *Suppose that $\mathfrak{l}_{n\gamma} \neq 0$. Then the irreducible constituent $V(\xi_{n\gamma}+\gamma)$ is special.*

Proof. We need to show that $\xi_{n\gamma} + \gamma = \mu + \beta$ for some $\beta \in \Delta(\mathfrak{g}(1))$. This is precisely the statement (1) of Lemma 3.4.5. ■

We next investigate the Cartan component $V(\xi_{\gamma} + \gamma)$ of $\mathfrak{l}_{\gamma} \otimes \mathfrak{z}(\mathfrak{n}) = V(\xi_{\gamma}) \otimes V(\gamma)$.

Lemma 6.2.2 *The Cartan component* $V(\xi_\gamma + \gamma)$ *of* $\mathfrak{l}_\gamma \otimes \mathfrak{z}(\mathfrak{n})$ *is not special.*

Proof. Lemma 3.4.5 and Remark 3.4.7 show that $\xi_\gamma + \gamma - \mu \notin \Delta(\mathfrak{g}(1))$, which implies that $\xi_\gamma + \gamma \neq \mu + \beta$ for all $\beta \in \Delta(\mathfrak{g}(1))$. ■

We determine all the special constituents of $\mathfrak{l}_\gamma \otimes \mathfrak{z}(\mathfrak{n})$ in two steps. First we assume that $\mathfrak{g}$ is a classical algebra, and then consider the case that $\mathfrak{g}$ is an exceptional algebra.

For classical cases the parabolic subalgebras $\mathfrak{q}$ under consideration are of type $B_n(i)$ $(3 \leq i \leq n)$, $C_n(i)$ $(2 \leq i \leq n-1)$, or $D_n(i)$ $(3 \leq i \leq n-3)$. It will be convenient to write $\beta \in \Delta(\mathfrak{g}(1))$ in terms of the fundamental weights of $\mathfrak{l}_\gamma$ and $\mathfrak{l}_{n\gamma}$. It is clear from the deleted Dynkin diagrams that, for each of the cases, $\Pi(\mathfrak{l}_\gamma)$ and $\Pi(\mathfrak{l}_{n\gamma})$ are given by

$$\Pi(\mathfrak{l}_\gamma) = \{\alpha_r \mid 1 \leq r \leq i-1\}$$

and

$$\Pi(\mathfrak{l}_{n\gamma}) = \{\alpha_{i+s} \mid 1 \leq s \leq n-i\},$$

where α_j are the simple roots with the standard numbering. By using the standard realizations of roots, we have $\alpha_r = \varepsilon_r - \varepsilon_{r+1}$ for $1 \leq r \leq i-1$, $\alpha_{i+s} = \varepsilon_{i+s} - \varepsilon_{i+s+1}$ for $1 \leq s \leq n-i-1$, and

$$\alpha_n = \begin{cases} \varepsilon_n & \text{if } \mathfrak{g} \text{ is of type } B_n \\ 2\varepsilon_n & \text{if } \mathfrak{g} \text{ is of type } C_n \\ \varepsilon_{n-1} + \varepsilon_n & \text{if } \mathfrak{g} \text{ is of type } D_n. \end{cases}$$

The data in Appendix C shows that $\Delta(\mathfrak{g}(1))$ is

$$\Delta(\mathfrak{g}(1)) =$$
$$\begin{cases} \{\varepsilon_j \pm \varepsilon_k \mid 1 \leq j \leq i \text{ and } i+1 \leq k \leq n\} \cup \{\varepsilon_j \mid 1 \leq j \leq i\} & \text{if } \mathfrak{q} \text{ is of type } B_n(i) \\ \{\varepsilon_j \pm \varepsilon_k \mid 1 \leq j \leq i \text{ and } i+1 \leq k \leq n\} & \text{if } \mathfrak{q} \text{ is of type } C_n(i) \text{ or } D_n(i). \end{cases}$$

Since we have two simple algebras $\mathfrak{l}_\gamma$ and $\mathfrak{l}_{n\gamma}$, we use the notation ϖ_r for the fundamental weights of $\alpha_r \in \Pi(\mathfrak{l}_\gamma)$ and $\tilde{\varpi}_s$ for those of $\alpha_{i+s} \in \Pi(\mathfrak{l}_{n\gamma})$. Direct computation then shows that each $\beta \in \Delta(\mathfrak{g}(1))$ is exactly one of the following form:

$$\beta = \begin{cases} \varpi_1 + \sum_{s=1}^{n-i} \tilde{m}_s \tilde{\varpi}_s, \\ (-\varpi_r + \varpi_{r+1}) + \sum_{s=1}^{n-i} \tilde{m}_s \tilde{\varpi}_s \text{ with } 1 \leq r \leq i-2, \text{ or} \\ -\varpi_{i-1} + \sum_{s=1}^{n-i} \tilde{m}_s \tilde{\varpi}_s \end{cases} \tag{6.2.3}$$

for some $\tilde{m}_s \in \mathbb{Z}$.

Proposition 6.2.4 *Let $V(\nu)$ be an irreducible constituent of $\mathfrak{l}_\gamma \otimes \mathfrak{z}(\mathfrak{n})$.*

1. *If $\mathfrak{q}$ is of type $B_n(i)$ $(3 \leq i \leq n)$ or $D_n(i)$ $(3 \leq i \leq n-3)$ then $V(\nu)$ is a special constituent if and only if $\nu = 2\varepsilon_1$.*

2. *If $\mathfrak{q}$ is of type $C_n(i)$ $(2 \leq i \leq n-1)$ then $V(\nu)$ is a special constituent if and only if $\nu = \varepsilon_1 + \varepsilon_2$.*

Proof. Suppose that $\mathfrak{q}$ is of type $B_n(i)$, $C_n(i)$, or $D_n(i)$. By Definition 6.1.4, we need to find all ν of the form $\nu = \mu + \beta$ for some $\beta \in \Delta(\mathfrak{g}(1))$. Here μ, the highest weight for $\mathfrak{g}(1)$, is

$$\mu = \begin{cases} \varepsilon_1 + \varepsilon_{i+1} & \text{if } \mathfrak{q} \text{ is of type } B_n(i) \text{ with } i \neq n,\ C_n(i), \text{ or } D_n(i) \\ \varepsilon_1 & \text{if } \mathfrak{q} \text{ is of type } B_n(n). \end{cases}$$

We write μ in terms of the fundamental weights of $\mathfrak{l}_\gamma$ and $\mathfrak{l}_{n\gamma}$; that is,

$$\mu = \begin{cases} \varpi_1 + \tilde{\varpi}_1 & \text{if } \mathfrak{q} \text{ is of type } B_n(i) \text{ with } i \neq n,\ C_n(i), \text{ or } D_n(i) \\ \varpi_1 & \text{if } \mathfrak{q} \text{ is of type } B_n(n), \end{cases} \tag{6.2.5}$$

where ϖ_1 and $\tilde{\varpi}_1$ are the fundamental weights of $\alpha_1 = \varepsilon_1 - \varepsilon_2$ and $\alpha_{i+1} = \varepsilon_{i+1} - \varepsilon_{i+2}$, respectively. As $\mathfrak{l}_{n\gamma}$ acts trivially on both $\mathfrak{l}_\gamma$ and $\mathfrak{z}(\mathfrak{n})$, the highest weight ν for a constituent $V(\nu) \subset \mathfrak{l}_\gamma \otimes \mathfrak{z}(\mathfrak{n})$ is of the form

$$\nu = \sum_{j=1}^{i-1} n_j \varpi_j \quad \text{for } n_j \in \mathbb{Z}_{\geq 0}. \tag{6.2.6}$$

If there exists $\beta \in \Delta(\mathfrak{g}(1))$ so that $\nu = \mu + \beta$ then (6.2.5) and (6.2.6) imply that $\beta = \nu - \mu$ is of the form

$$\beta = \begin{cases} (n_1 - 1)\varpi_1 + \sum_{j=2}^{i-1} n_j \varpi_j - \tilde{\varpi}_1 & \text{if } \mathfrak{q} \text{ is of type } B_n(i) \text{ with } i \neq n,\, C_n(i), \text{ or } D_n(i) \\ (n_1 - 1)\varpi_1 + \sum_{j=2}^{i-1} n_j \varpi_j & \text{if } \mathfrak{q} \text{ is of type } B_n(n) \end{cases} \tag{6.2.7}$$

for $n_j \in \mathbb{Z}_{\geq 0}$. On the other hand, we observed that the root β must be one of the forms in (6.2.3). Then observation shows that if β satisfies both (6.2.3) and (6.2.7) then β must be

$$\beta = \begin{cases} \varpi_1 - \tilde{\varpi}_1 \text{ or } (-\varpi_1 + \varpi_2) - \tilde{\varpi}_1 & \text{if } \mathfrak{q} \text{ is of type } B_n(i) \text{ with } i \neq n,\, C_n(i), \text{ or } D_n(i) \\ \varpi_1 \text{ or } (-\varpi_1 + \varpi_2) & \text{if } \mathfrak{q} \text{ is of type } B_n(n). \end{cases}$$

Therefore $\nu = \mu + \beta$ is $\nu = 2\varpi_1$ or ϖ_2, which shows that $\nu = 2\varepsilon_1$ or $\varepsilon_1 + \varepsilon_2$. As $\xi_\gamma = \varepsilon_1 - \varepsilon_i$ for $\mathfrak{q}$ of type $B_n(i)$, $C_n(i)$, or $D_n(i)$, Theorem 5.1.3 shows that both $V(2\varepsilon_1)$ and $V(\varepsilon_1 + \varepsilon_2)$ occur in $\mathfrak{l}_\gamma \otimes \mathfrak{z}(\mathfrak{n})$. Now the assertions follow from the fact that the highest root γ of $\mathfrak{g}$ is $\gamma = \varepsilon_1 + \varepsilon_2$ if $\mathfrak{g}$ is of type B_n or D_n, and $\gamma = 2\varepsilon_1$ if $\mathfrak{g}$ is of type C_n. ■

If $\mathfrak{g}$ is an exceptional algebra then the parabolic subalgebras $\mathfrak{q}$ under consideration are

$$E_6(3), E_6(5), E_7(2), E_7(6), E_8(1), \text{and } F_4(4). \tag{6.2.8}$$

Lemma 6.2.9 *If $\mathfrak{q}$ is of exceptional type as in (6.2.8) then $V(\xi_\gamma + \gamma_0)$ in Theorem 5.1.3 is a special constituent.*

Proof. This is done by a direct computation. The roots ϵ_γ in $\Delta(\mathfrak{g}(1))$ so that $\xi_\gamma + \gamma_0 = \mu + \epsilon_\gamma$ are given in Table 6.4 below. ■

Proposition 6.2.10 *There exists a unique special constituent in* $\mathfrak{l}_\gamma \otimes \mathfrak{z}(\mathfrak{n})$.

Proof. If $\mathfrak{q}$ is of classical type then this proposition follows from Proposition 6.2.4. For $\mathfrak{q}$ of exceptional type, by Theorem 5.1.3, the tensor product $\mathfrak{l}_\gamma \otimes \mathfrak{z}(\mathfrak{n})$ decomposes into

$$\mathfrak{l}_\gamma \otimes \mathfrak{z}(\mathfrak{n}) = V(\xi_\gamma + \gamma) \oplus V(\gamma) \oplus V(\xi_\gamma + \gamma_0)$$

with $\gamma_0 \in \Delta(\mathfrak{n})$ as in Theorem 5.1.3. Then Lemma 6.2.2 and Lemma 6.2.9 show that $V(\xi_\gamma + \gamma_0)$ is the unique special constituent. ■

Since the weight $\epsilon \in \Delta(\mathfrak{g}(1))$ so that $\mu + \epsilon$ is the highest weight of a special constituent will play a role later, we introduce the notation related to ϵ.

Definition 6.2.11 *We denote by* ϵ_γ *the root contributing to* $\mathfrak{g}(1)$ *so that* $V(\mu + \epsilon_\gamma)$ *is the special constituent of* $\mathfrak{l}_\gamma \otimes \mathfrak{z}(\mathfrak{n})$. *Similarly, we denote by* $\epsilon_{n\gamma}$ *the root for* $\mathfrak{g}(1)$ *so that* $V(\mu + \epsilon_{n\gamma}) = \mathfrak{l}_{n\gamma} \otimes \mathfrak{z}(\mathfrak{n})$.

In Table 6.1, Table 6.2, Table 6.3, and Table 6.4 we summarize the results of this section. Table 6.1 and Table 6.2 contain the highest weight of each special constituent occurring in $\mathfrak{l} \otimes \mathfrak{z}(\mathfrak{n})$ for each parabolic $\mathfrak{q}$ of classical algebras and exceptional algebras. Table 6.3 and Table 6.4 list the roots μ, ϵ_γ, and $\epsilon_{n\gamma}$ for each $\mathfrak{q}$. A dash indicates that no special constituent exists for the case.

Table 6.1: Highest Weights for Special Constituents (Classical Cases)

Type	$V(\mu + \epsilon_\gamma)$	$V(\mu + \epsilon_{n\gamma})$
$B_n(i),\ 3 \le i \le n-2$	$2\varepsilon_1$	$\varepsilon_1 + \varepsilon_2 + \varepsilon_{i+1} + \varepsilon_{i+2}$
$B_n(n-1)$	$2\varepsilon_1$	$\varepsilon_1 + \varepsilon_2 + \varepsilon_n$
$B_n(n)$	$2\varepsilon_1$	$-$
$C_n(i),\ 2 \le i \le n-1$	$\varepsilon_1 + \varepsilon_2$	$2\varepsilon_1 + 2\varepsilon_{i+1}$
$D_n(i),\ 3 \le i \le n-3$	$2\varepsilon_1$	$\varepsilon_1 + \varepsilon_2 + \varepsilon_{i+1} + \varepsilon_{i+2}$

Table 6.2: Highest Weights for Special Constituents (Exceptional Cases)

Type	$V(\mu+\epsilon_\gamma)$	$V(\mu+\epsilon_{n\gamma})$
$E_6(3)$	$\alpha_1+2\alpha_2+2\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6$	$2\alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6$
$E_6(5)$	$2\alpha_1+2\alpha_2+3\alpha_3+4\alpha_4+2\alpha_5+\alpha_6$	$\alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+2\alpha_6$
$E_7(2)$	$2\alpha_1+2\alpha_2+4\alpha_3+5\alpha_4+4\alpha_5+3\alpha_6+2\alpha_7$	–
$E_7(6)$	$2\alpha_1+3\alpha_2+4\alpha_3+6\alpha_4+4\alpha_5+2\alpha_6+\alpha_7$	$2\alpha_1+2\alpha_2+3\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+2\alpha_7$
$E_8(1)$	$2\alpha_1+4\alpha_2+5\alpha_3+8\alpha_4+7\alpha_5+6\alpha_6+4\alpha_7+2\alpha_8$	–
$F_4(4)$	$2\alpha_1+4\alpha_2+6\alpha_3+2\alpha_4$	–

Table 6.3: The Roots μ, ϵ_γ, and $\epsilon_{n\gamma}$ (Classical Cases)

Type	μ	ϵ_γ	$\epsilon_{n\gamma}$
$B_n(i),\ 3\le i\le n-2$	$\varepsilon_1+\varepsilon_{i+1}$	$\varepsilon_1-\varepsilon_{i+1}$	$\varepsilon_2+\varepsilon_{i+2}$
$B_n(n-1)$	$\varepsilon_1+\varepsilon_n$	$\varepsilon_1-\varepsilon_n$	ε_2
$B_n(n)$	ε_1	ε_1	–
$C_n(i),\ 2\le i\le n-1$	$\varepsilon_1+\varepsilon_{i+1}$	$\varepsilon_2-\varepsilon_{i+1}$	$\varepsilon_1+\varepsilon_{i+1}$
$D_n(i),\ 3\le i\le n-3$	$\varepsilon_1+\varepsilon_{i+1}$	$\varepsilon_1-\varepsilon_{i+1}$	$\varepsilon_2+\varepsilon_{i+2}$

Table 6.4: The Roots μ, ϵ_γ, and $\epsilon_{n\gamma}$ (Exceptional Cases)

Type	ϵ_γ	$\epsilon_{n\gamma}$
$E_6(3)$	$\alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6$	$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$
$E_6(5)$	$\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5$	$\alpha_2 + \alpha_4 + \alpha_5 + \alpha_6$
$E_7(2)$	$\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7$	$-$
$E_7(6)$	$\alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$	$\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7$
$E_8(1)$	$\alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8$	$-$
$F_4(4)$	$\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4$	$-$

with

$E_6(3) : \mu = \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6$

$E_6(5) : \mu = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6$

$E_7(2) : \mu = \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$

$E_7(6) : \mu = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7$

$E_8(1) : \mu = \alpha_1 + 3\alpha_2 + 3\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8$

$F_4(4) : \mu = \alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4$

By Proposition 6.1.5, only special constituents could contribute to the construction of the Ω_2 systems. Next we want to show that $\tilde{\tau}_2|_{V^*} \neq 0$ when V is a special constituent. An observation on the highest weights for the special constituents will simplify the argument. We classify them by their highest weights and call them type 1a, type 1b, type 2, and type 3.

Definition 6.2.12 *We say that a special constituent $V(\nu)$ of $\mathfrak{l} \otimes \mathfrak{z}(\mathfrak{n})$ is of*

1. ***type 1a** if $\nu = \mu + \epsilon$ is not a root with $\epsilon \neq \mu$ and both μ and ϵ are long roots,*
2. ***type 1b** if $\nu = \mu + \epsilon$ is not a root with $\epsilon \neq \mu$ and either μ or ϵ is a short root,*
3. ***type 2** if $\nu = \mu + \epsilon = 2\mu$ is not a root, or*
4. ***type 3** if $\nu = \mu + \epsilon$ is a root,*

where μ is the highest weight for $\mathfrak{g}(1)$ and $\epsilon = \epsilon_\gamma$ or $\epsilon_{n\gamma}$ is the root in $\Delta(\mathfrak{g}(1))$ defined in Definition 6.2.11.

Example 6.2.13 *The following are examples of each type of special constituents:*

1. *type 1a: $V(\mu + \epsilon_\gamma)$ for type $B_n(n-1)$ $\big(\mu + \epsilon_\gamma = (\varepsilon_1 + \varepsilon_n) + (\varepsilon_1 - \varepsilon_n)\big)$*
2. *type 1b: $V(\mu + \epsilon_{n\gamma})$ for type $B_n(n-1)$ $\big(\mu + \epsilon_{n\gamma} = (\varepsilon_1 + \varepsilon_n) + (\varepsilon_2)\big)$*
3. *type 2: $V(\mu + \epsilon_{n\gamma})$ for type $C_n(i)$ $\big(\mu + \epsilon_{n\gamma} = 2(\varepsilon_1 + \varepsilon_{i+1}) = 2\mu\big)$*
4. *type 3: $V(\mu + \epsilon_\gamma)$ for type $C_n(i)$ $\big(\mu + \epsilon_\gamma = \varepsilon_1 + \varepsilon_2\big)$*

Table 6.5 summarizes the types of special constituents for each parabolic subaglebra $\mathfrak{q}$. One may want to observe that almost all the special constituents are of type 1a. We regard any roots as long roots, if $\mathfrak{g}$ is simply laced. A dash indicates that no special constituent exists in the case.

Table 6.5: Types of Special Constituents

Type	$V(\mu+\epsilon_\gamma)$	$V(\mu+\epsilon_{n\gamma})$
$B_n(i),\ 3 \le i \le n-2$	Type 1a	Type 1a
$B_n(n-1)$	Type 1a	Type 1b
$B_n(n)$	Type 2	–
$C_n(i),\ 2 \le i \le n-1$	Type 3	Type 2
$D_n(i),\ 3 \le i \le n-3$	Type 1a	Type 1a
$E_6(3)$	Type 1a	Type 1a
$E_6(5)$	Type 1a	Type 1a
$E_7(2)$	Type 1a	–
$E_7(6)$	Type 1a	Type 1a
$E_8(1)$	Type 1a	–
$F_4(4)$	Type 2	–

Remark 6.2.14 *It is observed from Table 6.3 and Table 6.4 that we have $\mu \pm \epsilon \notin \Delta$, unless $V(\mu+\epsilon)$ is of type 3.*

Remark 6.2.15 *Table 6.5 shows that when $V(\mu+\epsilon)$ is a special constituent of type 1a, the parabolic subalgebra $\mathfrak{q}$ is of type $B_n(i)$ $(3 \le i \le n-1)$, $D_n(i)$, $E_6(3)$, $E_6(5)$, $E_7(2)$, $E_7(6)$, or $E_8(1)$. The data in Appendix C shows that when $\mathfrak{q}$ is of type $B_n(i)$ for $3 \le i \le n-1$, the simple root $\alpha_{\mathfrak{q}} = \varepsilon_i - \varepsilon_{i+1}$ that parametrizes $\mathfrak{q}$ is a long root and $\Delta(\mathfrak{z}(\mathfrak{n}))$ contains solely long roots. Since we regard any roots as long roots for $\mathfrak{g}$ simply laced, it follows that when $V(\mu+\epsilon)$ is of type 1a, the simple root $\alpha_{\mathfrak{q}}$ and any root $\gamma_j \in \Delta(\mathfrak{z}(\mathfrak{n}))$ are all long roots.*

6.3 Technical Results

In this section we collect technical results on the special constituents, so that certain arguments will go smoothly in Chapter 7. The weight vectors X_α and the structure constants $N_{\alpha,\beta}$ are normalized as in Section 4.1.

Lemma 6.3.1 *Let $V(\mu+\epsilon)$ be a special constituent $\mathfrak{l}\otimes\mathfrak{z}(\mathfrak{n})$ of type 1a, and $\alpha \in \Delta^+(\mathfrak{l})$. If $\epsilon+\alpha \in \Delta$ then $\mu-\alpha \in \Delta$.*

Proof. We show that $\langle \mu, \alpha\rangle > 0$. Since $\mu+\epsilon$ is the highest weight of an irreducible $\mathfrak{l}$-module, it is $\Delta(\mathfrak{l})$-dominant. Thus,

$$\langle \mu+\epsilon, \alpha\rangle = \langle \mu, \alpha\rangle + \langle \epsilon, \alpha\rangle \geq 0. \tag{6.3.2}$$

Observe that, as $\mu+\epsilon$ is of type 1a, ϵ is a long root of $\mathfrak{g}$. Since $\alpha+\epsilon$ is assumed to be a root, Lemma 3.4.4 implies that $\langle \alpha, \epsilon^\vee\rangle = -1$; in particular, $\langle \epsilon, \alpha\rangle < 0$. Now, by (6.3.2), we have

$$\langle \mu, \alpha\rangle \geq -\langle \epsilon, \alpha\rangle > 0.$$

■

Lemma 6.3.3 *Let $V(\mu+\epsilon)$ be a special constituent of $\mathfrak{l}\otimes\mathfrak{z}(\mathfrak{n})$ of type 1a. Then, for $\alpha \in \Delta^+(\mathfrak{l})$ with $\alpha+\epsilon \in \Delta$, we have*

$$\mathrm{ad}(X_\mu)\mathrm{ad}(X_{\alpha+\epsilon})X_{-\gamma_j} = 0$$

for all $\gamma_j \in \Delta(\mathfrak{z}(\mathfrak{n}))$.

Proof. If $(\alpha+\epsilon)-\gamma_j \notin \Delta$ then there is nothing to prove. So we assume that $(\alpha+\epsilon)-\gamma_j \in \Delta$ and $\mu+(\alpha+\epsilon)-\gamma_j \in \Delta$. Since $\mu+\epsilon$ is assumed to be of type 1a, the root μ is long. Lemma 3.4.4 then implies that

$$\langle (\alpha+\epsilon)-\gamma_j, \mu^\vee\rangle = -1. \tag{6.3.4}$$

By Remark 6.2.14, we have $\langle \epsilon, \mu^\vee \rangle = 0$. Thus (6.3.4) becomes

$$\langle \alpha, \mu^\vee \rangle - \langle \gamma_j, \mu^\vee \rangle = -1. \tag{6.3.5}$$

Since μ is the highest weight for $\mathfrak{g}(1)$, $\gamma_j \in \Delta(\mathfrak{z}(\mathfrak{n}))$, and $\alpha \in \Delta^+(\mathfrak{l})$, neither $\mu + \alpha$ nor $\gamma_j + \mu$ is a root. Then, as μ is a long root, (6.3.5) holds if and only if $\langle \alpha, \mu^\vee \rangle = 0$ and $\langle \gamma_j, \mu^\vee \rangle = 1$. On the other hand, since $\alpha + \epsilon$ is a root by hypothesis and by Lemma 6.3.1, $\mu - \alpha$ is a root. In particular, by Lemma 3.4.4, $\langle \alpha, \mu^\vee \rangle = 1$. Now we have $\langle \alpha, \mu^\vee \rangle = 1$ and $\langle \alpha, \mu^\vee \rangle = 0$, which is a contradiction. ■

For any $\mathrm{ad}(\mathfrak{h})$-invariant subspace $W \subset \mathfrak{g}$ and any weight $\nu \in \mathfrak{h}^*$, we write

$$\Delta_\nu(W) = \{\alpha \in \Delta(W) \mid \nu - \alpha \in \Delta\}.$$

In Chapter 7, we will construct the $\Omega_2|_{V(\mu+\epsilon)^*}$ systems and find their special values, when $V(\mu+\epsilon)$ is of either type 1a or type 2. When we do so, the roots $\beta \in \Delta_{\mu+\epsilon}(\mathfrak{g}(1))$ and $\gamma_j \in \Delta_{\mu+\epsilon}(\mathfrak{z}(\mathfrak{n}))$ will play a role. Therefore, for the rest of this section, we shall show several technical results about those roots, so that certain argument will become simple.

First of all, we need check that $\Delta_{\mu+\epsilon}(\mathfrak{g}(1))$ and $\Delta_{\mu+\epsilon}(\mathfrak{z}(\mathfrak{n}))$ are not empty. It is clear that $\Delta_{\mu+\epsilon}(\mathfrak{g}(1)) \neq \emptyset$, since μ, $\epsilon \in \Delta_{\mu+\epsilon}(\mathfrak{g}(1))$. Moreover, Lemma 6.3.6 below shows that when $V(\mu+\epsilon)$ is of type 2, we have $\Delta_{\mu+\epsilon}(\mathfrak{g}(1)) = \{\mu\}$.

Lemma 6.3.6 *If $V(\mu+\epsilon)$ is a special constituent of $\mathfrak{l}\otimes\mathfrak{z}(\mathfrak{n})$ of type 2 then $\Delta_{\mu+\epsilon}(\mathfrak{g}(1)) = \{\mu\}$.*

Proof. First we claim that μ has the maximum height among the roots $\beta \in \Delta(\mathfrak{g}(1))$. As $\mathfrak{g}(1)$ is the irreducible L-module with highest weight μ, any root $\beta \in \Delta(\mathfrak{g}(1))$ is of the form $\beta = \mu - \sum_{\alpha \in \Pi(\mathfrak{l})} n_\alpha \alpha$ with $n_\alpha \in \mathbb{Z}_{\geq 0}$. Then if $\mathrm{ht}(\mu)$ and $\mathrm{ht}(\beta)$ denote the heights of μ and β, respectively, then

$$\mathrm{ht}(\mu) = \mathrm{ht}(\beta) + \sum_{\alpha \in \Pi(\mathfrak{l})} n_\alpha \geq \mathrm{ht}(\beta).$$

Now as $V(\mu+\epsilon)$ is of type 2, by definition, we have $\mu + \epsilon = 2\mu$. If $\beta \in \Delta_{2\mu}(\mathfrak{g}(1))$ then $2\mu - \beta \in \Delta(\mathfrak{g}(1))$. In particular, the height $\mathrm{ht}(2\mu-\beta)$ satisfies $\mathrm{ht}(\mu) \geq \mathrm{ht}(2\mu-\beta)$. If $\beta = \mu - \sum_{\alpha\in\Pi(\mathfrak{l})} n_\alpha \alpha$ with $n_\alpha \in \mathbb{Z}_{\geq 0}$ then

$$\mathrm{ht}(\mu) \geq \mathrm{ht}(2\mu - \beta) = 2\mathrm{ht}(\mu) - \mathrm{ht}(\beta) = 2\mathrm{ht}(\mu) - \mathrm{ht}(\mu) + \sum_{\alpha\in\Pi(\mathfrak{l})} n_\alpha = \mathrm{ht}(\mu) + \sum_{\alpha\in\Pi(\mathfrak{l})} n_\alpha.$$

This forces that $\sum_{\alpha\in\Pi(\mathfrak{l})} n_\alpha = 0$. Therefore $\beta = \mu$. ■

Lemma 6.3.7 *If $V(\mu+\epsilon)$ is a special constituent of $\mathfrak{l} \otimes \mathfrak{z}(\mathfrak{n})$ then $\Delta_{\mu+\epsilon}(\mathfrak{z}(\mathfrak{n})) \neq \emptyset$.*

Proof. By Fact (II) in Section 5.2, the highest weight $\mu+\epsilon$ of $V(\mu+\epsilon) \subset \mathfrak{l}\otimes\mathfrak{z}(\mathfrak{n})$ is of the form

$$\mu + \epsilon = \begin{cases} \xi_\gamma + \gamma' & \text{if } V(\mu+\epsilon) \subset \mathfrak{l}_\gamma \otimes \mathfrak{z}(\mathfrak{n}) \\ \xi_{n\gamma} + \gamma'' & \text{if } V(\mu+\epsilon) = \mathfrak{l}_{n\gamma} \otimes \mathfrak{z}(\mathfrak{n}) \end{cases}$$

for some $\gamma', \gamma'' \in \Delta(\mathfrak{z}(\mathfrak{n}))$, where ξ_γ and $\xi_{n\gamma}$ are the highest weights for $\mathfrak{l}_\gamma$ and $\mathfrak{l}_{n\gamma}$, respectively. Then we have $\gamma', \gamma'' \in \Delta_{\mu+\epsilon}(\mathfrak{z}(\mathfrak{n}))$. ■

The following simple technical lemma will simplify an argument in later proofs.

Lemma 6.3.8 *Let $\alpha, \beta, \delta \in \Delta$ with $\alpha,\ \beta \neq \delta$. If $\alpha + \beta \notin \Delta$ and $\alpha + \beta - \delta \in \Delta$ then the following hold:*

(1) $\alpha - \delta,\ \beta - \delta \in \Delta$, and

(2) $N_{\beta,\alpha-\delta} N_{\alpha,-\delta} = N_{\alpha,\beta-\delta} N_{\beta,-\delta}$.

Proof. For the first assertion, we show that $\alpha - \delta \in \Delta$. Suppose that $\alpha - \delta \notin \Delta$, so $\langle \alpha, \delta \rangle \leq 0$. By hypothesis, we have $\langle \alpha, \beta \rangle \geq 0$. Thus it follows that

$$\langle \alpha + \beta - \delta, \alpha \rangle = \langle \alpha, \alpha \rangle + \langle \beta, \alpha \rangle - \langle \delta, \alpha \rangle > 0.$$

Therefore, $\beta - \delta = (\alpha + \beta - \delta) - \alpha$ is a root. Now let X_α, X_β, and $X_{-\delta}$ be the root vectors of α, β, and $-\delta$, respectively, normalized as in Section 4.1. Since $\beta - \delta \in \Delta$, we have $N_{\beta,-\delta} \neq 0$ (see Property (H7) in Section 4.1). Moreover, the conditions that $\beta-\delta, \alpha+\beta-\delta \in \Delta$ imply that $N_{\alpha,\beta-\delta} \neq 0$. On the other hand, we have $[X_\alpha, X_{-\delta}] = 0$ by assumption, and $[X_\alpha, X_\beta] = 0$ by hypothesis. So it follows from the Jacobi identity that

$$0 = [X_\beta, [X_\alpha, X_{-\delta}]] = [X_\alpha, [X_\beta, X_{-\delta}]] = N_{\alpha,\beta-\delta}N_{\beta,-\delta}X_{\alpha+\beta-\delta} \neq 0,$$

which is absurd. Therefore $\alpha-\delta \in \Delta$. Since it may be shown similarly that $\beta-\delta \in \Delta$, we omit the proof.

Observe that the condition $\alpha+\beta \notin \Delta$ implies that $\mathrm{ad}(X_\alpha)\mathrm{ad}(X_\beta) = \mathrm{ad}(X_\beta)\mathrm{ad}(X_\alpha)$ by the Jacobi identity. Therefore, $\mathrm{ad}(X_\alpha)\mathrm{ad}(X_\beta)X_{-\delta} = \mathrm{ad}(X_\beta)\mathrm{ad}(X_\alpha)X_{-\delta}$, which implies that

$$N_{\beta,\alpha-\delta}N_{\alpha,-\delta} = N_{\alpha,\beta-\delta}N_{\beta,-\delta}.$$

■

Lemma 6.3.9 *Let W be any ad($\mathfrak{h}$)-invariant subspace of $\mathfrak{g}$ with $\Delta_{\mu+\epsilon}(W)\backslash\{\mu, \epsilon\} \neq \emptyset$. If $V(\mu + \epsilon)$ is a special constituent of $\mathfrak{l} \otimes \mathfrak{z}(\mathfrak{n})$ of type 1a, type 1b, or type 2 then, for any $\delta \in \Delta_{\mu+\epsilon}(W)\backslash\{\mu, \epsilon\}$, we have $\delta - \mu$, $\delta - \epsilon \in \Delta$.*

Proof. If $V(\mu + \epsilon)$ is of type 1a, type 1b, or type 2 then, by definition, $\mu + \epsilon$ is not a root. Then this lemma simply follows from Lemma 6.3.8 ■

Remark 6.3.10 *A direct observation shows that if $V(\mu + \epsilon)$ is a special constituent of type 1a then $\Delta_{\mu+\epsilon}(\mathfrak{g}(1))\backslash\{\mu, \epsilon\} \neq \emptyset$.*

Lemma 6.3.11 *If $V(\mu + \epsilon)$ is a special constituent of $\mathfrak{l} \otimes \mathfrak{z}(\mathfrak{n})$ of type 1a then, for any $\alpha \in \Delta_{\mu+\epsilon}(\mathfrak{g}(1))$ and any $\gamma_j \in \Delta_{\mu+\epsilon}(\mathfrak{z}(\mathfrak{n}))$, we have $\gamma_j - \alpha \in \Delta$.*

Proof. By Lemma 6.3.8, we have $\gamma_j - \mu, \gamma_j - \epsilon \in \Delta$. So, let $\alpha \neq \mu, \epsilon$. We show that $\langle \gamma_j, \alpha \rangle > 0$. Observe that since $\alpha \in \Delta(\mathfrak{g}(1))$ and $\gamma_j \in \Delta(\mathfrak{z}(\mathfrak{n}))$, we have $\gamma_j + \alpha \notin \Delta$.

Thus $\langle \gamma_j, \alpha \rangle \geq 0$. Since $\alpha \in \Delta_{\mu+\epsilon}(\mathfrak{g}(1))\backslash\{\mu, \epsilon\}$ and $\gamma_j \in \Delta_{\mu+\epsilon}(\mathfrak{z}(\mathfrak{n}))$, by Lemma 6.3.9, we have $\mu - \alpha$, $\epsilon - \gamma_j \in \Delta$. Then we first claim that if $\langle \gamma_j, \alpha \rangle = 0$ then $(\mu - \alpha) + (\epsilon - \gamma_j) \in \Delta$. Since $V(\mu + \epsilon)$ is assumed to be of type 1a, both μ and ϵ are long roots. Thus, by Lemma 3.4.4, $\langle \gamma_j, \mu^\vee \rangle = \langle \alpha, \epsilon^\vee \rangle = 1$; in particular, $\langle \gamma_j, \mu \rangle$, $\langle \alpha, \epsilon \rangle > 0$. By Remark 6.2.14, we have $\langle \mu, \epsilon \rangle = 0$. Then,

$$\langle \mu - \alpha, \epsilon - \gamma_j \rangle = -\langle \mu, \gamma_j \rangle - \langle \alpha, \epsilon \rangle < 0.$$

Therefore, as $\mu - \alpha, \epsilon - \gamma_j \in \Delta$, it follows that $(\mu - \alpha) + (\epsilon - \gamma_j) \in \Delta$. On the other hand, since $\langle \mu, \epsilon \rangle = 0$ and $\langle \gamma_j, \alpha \rangle$ is assumed to be 0, we have

$$\begin{aligned} &||(\mu - \alpha) + (\epsilon - \gamma_j)||^2 \\ &= ||\mu||^2 + ||\alpha||^2 + ||\epsilon||^2 + ||\gamma_j||^2 - 2\langle \alpha, \mu \rangle - 2\langle \alpha, \epsilon \rangle - 2\langle \gamma_j, \mu \rangle - 2\langle \gamma_j, \epsilon \rangle. \end{aligned}$$

For $\nu = \alpha, \gamma_j$ and $\zeta = \mu, \epsilon$, by Lemma 3.4.4, we have $\langle \nu, \zeta^\vee \rangle = 2\langle \nu, \zeta \rangle / ||\zeta||^2 = 1$, as μ and ϵ are long roots. Therefore, $2\langle \nu, \zeta \rangle = ||\zeta||^2$, and so,

$$||(\mu - \alpha) + (\epsilon - \gamma_j)||^2 = ||\alpha||^2 + ||\gamma_j||^2 - ||\mu||^2 - ||\epsilon||^2.$$

Since μ and ϵ are assumed to be long roots, this shows that $||(\mu - \alpha) + (\epsilon - \gamma_j)||^2 \leq 0$, which contradicts that $(\mu - \alpha) + (\epsilon - \gamma_j)$ is a root. Hence, $\langle \gamma_j, \alpha \rangle > 0$. ■

Lemma 6.3.12 *If $V(\mu + \epsilon)$ is a special constituent of $\mathfrak{l} \otimes \mathfrak{z}(\mathfrak{n})$ of type 1a or type 2 then, for any $\gamma_j \in \Delta_{\mu+\epsilon}(\mathfrak{z}(\mathfrak{n}))$,*

$$\Delta_{\mu+\epsilon}(\mathfrak{g}(1)) \subset \Delta_{\gamma_j}(\mathfrak{g}(1)).$$

In particular, $\Delta_{\gamma_j}(\mathfrak{g}(1)) \neq \emptyset$ for any $\gamma_j \in \Delta_{\mu+\epsilon}(\mathfrak{z}(\mathfrak{n}))$.

Proof. If $V(\mu + \epsilon)$ is of type 1a then the assertion follows from Lemma 6.3.11. If $V(\mu + \epsilon)$ is of type 2 then Lemma 6.3.6 implies that $\Delta_{\mu+\epsilon}(\mathfrak{g}(1)) = \{\mu\}$. Now this lemma follows from Lemma 6.3.9 by taking $\delta = \gamma_j \in \Delta_{\mu+\epsilon}(\mathfrak{z}(\mathfrak{n}))$. ■

If $V(\mu+\epsilon)$ is a special constituent of $\mathfrak{l} \otimes \mathfrak{z}(\mathfrak{n})$ then, for $\beta \in \Delta$, we write

$$\theta(\beta) = (\mu+\epsilon) - \beta.$$

Lemma 6.3.13 *If $V(\mu+\epsilon)$ is a special constituent of $\mathfrak{l} \otimes \mathfrak{z}(\mathfrak{n})$ of type 1a or type 2 then, for any $\gamma_j \in \Delta_{\mu+\epsilon}(\mathfrak{z}(\mathfrak{n}))$,*

$$\Delta_{\theta(\gamma_j)}(\mathfrak{g}(1)) \neq \emptyset.$$

Proof. Since $\gamma_j \in \Delta_{\mu+\epsilon}(\mathfrak{z}(\mathfrak{n}))$, we have $(\mu+\epsilon) - \gamma_j \in \Delta$. As $V(\mu+\epsilon)$ is assumed to be of type 1a or type 2, by definition, it follows that $\mu + \epsilon \notin \Delta$. Thus, by Lemma 6.3.8, we have $\mu - \gamma_j \in \Delta$ and $\epsilon - \gamma_j \in \Delta$. Then,

$$\theta(\gamma_j) - \mu = (\mu+\epsilon) - \gamma_j - \mu = \epsilon - \gamma_j \in \Delta;$$

that is, $\mu \in \Delta_{\theta(\gamma_j)}(\mathfrak{g}(1))$. ■

Lemma 6.3.14 *If $V(\mu+\epsilon)$ is a special constituent of type 1a or type 2 then*

$$\sum_{\gamma_j \in \Delta_{\mu+\epsilon}(\mathfrak{z}(\mathfrak{n}))} N_{\mu,\epsilon-\gamma_j} N_{-\mu,\gamma_j-\epsilon} N_{\epsilon,-\gamma_j} N_{-\epsilon,\gamma_j} > 0,$$

where $N_{\alpha,\beta}$ are the structure constants for $\alpha, \beta \in \Delta$, defined in Section 4.1.

Proof. It follows from Property (H7) of our normalizations in Section 4.1 that

$$N_{\mu,\epsilon-\gamma_j} N_{-\mu,\gamma_j-\epsilon} = -\frac{q_{\mu,\epsilon-\gamma_j}(1 + p_{\mu,\epsilon-\gamma_j})}{2} ||\mu||^2$$

and

$$N_{\epsilon,-\gamma_j} N_{-\epsilon,\gamma_j} = -\frac{q_{\epsilon,-\gamma_j}(1 + p_{\epsilon,-\gamma_j})}{2} ||\epsilon||^2.$$

In particular, by (4.1.1) in Section 4.1, we have $N_{\mu,\epsilon-\gamma_j} N_{-\mu,\gamma_j-\epsilon} \leq 0$ and $N_{\epsilon,-\gamma_j} N_{-\epsilon,\gamma_j} \leq 0$. By Lemma 6.3.7 and Lemma 6.3.9, $\Delta_{\mu+\epsilon}(\mathfrak{z}(\mathfrak{n})) \neq \emptyset$ and $\gamma_j - \epsilon \in \Delta$ for any $\gamma_j \in \Delta_{\mu+\epsilon}(\mathfrak{z}(\mathfrak{n}))$. Therefore, for all $\gamma_j \in \Delta_{\mu+\epsilon}(\mathfrak{z}(\mathfrak{n}))$, we have

$$N_{\mu,\epsilon-\gamma_j} N_{-\mu,\gamma_j-\epsilon} N_{\epsilon,-\gamma_j} N_{-\epsilon,\gamma_j} > 0.$$

■

Lemma 6.3.15 *If $V(\mu+\epsilon)$ is a special constituent of type 1a then, for any $\alpha \in \Delta_{\mu+\epsilon}(\mathfrak{g}(1))\backslash\{\mu,\epsilon\}$ and any $\gamma_j \in \Delta_{\mu+\epsilon}(\mathfrak{z}(\mathfrak{n}))$,*

$$[X_{-\gamma_j}, X_{\alpha-\mu}] = [X_{\theta(\gamma_j)}, X_{\alpha-\mu}] = 0.$$

Proof. We show that $-\gamma_j+\alpha-\mu$ and $\theta(\gamma_j)+\alpha-\mu$ are neither zero nor roots. First of all, if $-\gamma_j+\alpha-\mu=0$ then $\gamma_j=\mu-\alpha\in\Delta(\mathfrak{l})$, which contradicts that $\gamma_j\in\Delta(\mathfrak{z}(\mathfrak{n}))$. Next, if $\theta(\gamma_j)+\alpha-\mu=0$ then since $\theta(\gamma_j)+\alpha-\mu=\epsilon+\alpha-\gamma_j$, we would have $\alpha+\epsilon=\gamma_j\in\Delta$. On the other hand, as $V(\mu+\epsilon)$ is assumed to be of type 1a, ϵ is a long root. As $\alpha\in\Delta_{\mu+\epsilon}(\mathfrak{g}(1))\backslash\{\mu,\epsilon\}$, by Lemma 6.3.9, we have $\alpha-\epsilon\in\Delta$. Then, by Lemma 3.4.4, it follows that $\alpha+\epsilon\notin\Delta$, which is a contradiction.

To show $\gamma_j+\alpha-\mu$ is not a root, observe that, by Lemma 3.4.4, we have

$$\langle -\gamma_j+\alpha-\mu, \mu^\vee\rangle = -1+1-2=-2.$$

Thus, if $-\gamma_j+\alpha-\mu\in\Delta$ then $(-\gamma_j+\alpha-\mu)+2\mu$ would be a root. However, since μ is a long root, it is impossible. The fact that $\theta(\gamma_j)+\alpha-\mu\notin\Delta$ can be shown in a similar manner. ■

Lemma 6.3.16 *If $V(\mu+\epsilon)$ is a special constituent of type 1a then, for any $\alpha \in \Delta_{\mu+\epsilon}(\mathfrak{g}(1))\backslash\{\mu,\epsilon\}$ and any $\gamma_j \in \Delta_{\mu+\epsilon}(\mathfrak{z}(\mathfrak{n}))$,*

$$p_{\mu-\gamma_j,\alpha-\mu}=0 \quad \text{and} \quad q_{\mu-\gamma_j,\alpha-\mu}=1,$$

where $p_{\alpha,\beta}$ and $q_{\alpha,\beta}$ are the constants defined in (4.1.1) in Section 4.1. In particular, we have

$$N_{\mu-\gamma_j,\alpha-\mu}N_{-(\mu-\gamma_j),-(\alpha-\mu)} = -\frac{||\mu-\gamma_j||^2}{2}. \tag{6.3.17}$$

Proof. Observe that, by Lemma 6.3.11, $(\alpha-\mu)+(\mu-\gamma_j)=\gamma_j-\alpha$ is a root. As $V(\mu+\epsilon)$ is assumed to be of type 1a, μ is a long root. By Remark 6.2.15, the root γ_j is also a long root. Therefore $\mu-\gamma_j$ is a long root. Now the first part of the lemma follows immediately from Lemma 3.4.4, and the second follows from Property (H7) in our normalizations in Section 4.1. ■

Lemma 6.3.18 *If $V(\mu+\epsilon)$ is a special constituent of type 1a or type 2 then, for any $\alpha \in \Delta_{\mu+\epsilon}(\mathfrak{g}(1))$ and any $\gamma_j \in \Delta_{\mu+\epsilon}(\mathfrak{z}(\mathfrak{n}))$,*

$$N_{\alpha,-\gamma_j} N_{-\theta(\gamma_j),\theta(\alpha)} = N_{\theta(\alpha),-\gamma_j} N_{-\theta(\gamma_j),\alpha}.$$

Proof. Observe that, by Property (H3) in Section 4.1, we have $\kappa(X_\alpha, X_{-\alpha}) = 1$ for all $\alpha \in \Delta$. Thus, $N_{-\theta(\gamma_j),\theta(\alpha)} = \kappa([X_{-\theta(\gamma_j),}, X_{\theta(\alpha)}], X_{\alpha-\gamma_j})$. Then, we have

$$\begin{aligned} N_{-\theta(\gamma_j),\theta(\alpha)} &= \kappa([X_{-\theta(\gamma_j),}, X_{\theta(\alpha)}], X_{\alpha-\gamma_j}) \\ &= \frac{1}{N_{\alpha,-\gamma_j}} \kappa([X_{-\theta(\gamma_j)}, X_{\theta(\alpha)}], [X_\alpha, X_{-\gamma_j}]) \\ &= \frac{1}{N_{\alpha,-\gamma_j}} \kappa([X_{-\gamma_j}, [X_{-\theta(\gamma_j)}, X_{\theta(\alpha)}]], X_\alpha). \end{aligned}$$

Since $V(\mu+\epsilon)$ is assumed to be of type 1a or type 2, we have $(-\gamma_j) + (-\theta(\gamma_j)) = -(\mu+\epsilon) \notin \Delta$. Thus, $[X_{-\gamma_j}, X_{-\theta(\gamma_j)}] = 0$. Hence, by the Jacobi identity,

$$\begin{aligned} N_{-\theta(\gamma_j),\theta(\alpha)} &= \frac{1}{N_{\alpha,-\gamma_j}} \kappa([X_{-\gamma_j}, [X_{-\theta(\gamma_j)}, X_{\theta(\alpha)}]], X_\alpha) \\ &= \frac{1}{N_{\alpha,-\gamma_j}} \kappa([X_{-\theta(\gamma_j)}, [X_{-\gamma_j}, X_{\theta(\alpha)}]], X_\alpha) \\ &= \frac{1}{N_{\alpha,-\gamma_j}} N_{-\theta(\gamma_j),\theta(\alpha)-\gamma_j} N_{-\gamma_j,\theta(\alpha)}. \end{aligned} \quad (6.3.19)$$

We have $N_{-\gamma_j,\theta(\alpha)} = -N_{\theta(\alpha),-\gamma_j}$. Moreover, since $-\theta(\gamma_j) + (\theta(\alpha) - \gamma_j) + \alpha = 0$, by Property (H6) of our normalizations, it follows that $N_{-\theta(\gamma_j),\theta(\alpha)-\gamma_j} = -N_{-\theta(\gamma_j),\alpha}$. Therefore, by (6.3.19), we have

$$N_{\alpha,-\gamma_j} N_{-\theta(\gamma_j),\theta(\alpha)} = N_{-\theta(\gamma_j),\theta(\alpha)-\gamma_j} N_{-\gamma_j,\theta(\alpha)} = N_{\theta(\alpha),-\gamma_j} N_{-\theta(\gamma_j),\alpha}.$$

■

Lemma 6.3.20 *If $V(\mu+\epsilon)$ is a special constituent of type 1a then, for any $\alpha \in \Delta_{\mu+\epsilon}(\mathfrak{g}(1)) \backslash \{\mu, \epsilon\}$ and any $\gamma_j \in \Delta_{\mu+\epsilon}(\mathfrak{z}(\mathfrak{n}))$, we have the following:*

(1) $N_{\alpha,-\gamma_j} N_{\mu,-\alpha} = N_{\mu,-\gamma_j} N_{\alpha-\mu,\mu-\gamma_j}$, *and*

(2) $N_{-\theta(\gamma_j),\theta(\alpha)}N_{-\theta(\alpha),\theta(\mu)} = N_{-\theta(\gamma_j),\theta(\mu)}N_{-(\mu-\gamma_j),-(\alpha-\mu)}$.

Proof. By Lemma 6.3.9, we know that $\alpha - \mu \in \Delta$. Therefore, it holds that $X_\alpha = (1/N_{\alpha-\mu,\mu})[X_{\alpha-\mu}, X_\mu]$. Then we have

$$[X_\alpha, X_{-\gamma_j}] = \frac{1}{N_{\alpha-\mu,\mu}}[[X_{\alpha-\mu}, X_\mu], X_{-\gamma_j}].$$

By Lemma 6.3.15, it follows that $[X_{-\gamma_j}, X_{\alpha-\mu}] = 0$, and the Jacobi identity gives

$$\begin{aligned}[X_\alpha, X_{-\gamma_j}] &= \frac{1}{N_{\alpha-\mu,\mu}}[[X_{\alpha-\mu}, X_\mu], X_{-\gamma_j}] \\ &= \frac{1}{N_{\alpha-\mu,\mu}}[X_{\alpha-\mu}, [X_\mu, X_{-\gamma_j}]] \\ &= \frac{N_{\mu,-\gamma_j}}{N_{\alpha-\mu,\mu}}N_{\alpha-\mu,\mu-\gamma_j}X_{\alpha-\gamma_j}.\end{aligned}$$

Note that Lemma 6.3.11 is applied to have $\alpha - \gamma_j \in \Delta$ from line two to line three. Since $[X_\alpha, X_{-\gamma_j}] = N_{\alpha,-\gamma_j}X_{\alpha-\gamma_j}$, we obtain

$$N_{\alpha,-\gamma_j} = \frac{N_{\mu,-\gamma_j}}{N_{\alpha-\mu,\mu}}N_{\alpha-\mu,\mu-\gamma_j}. \tag{6.3.21}$$

Since, by Property (H6) of our normalizations, we have $N_{\alpha-\mu,\mu} = N_{\mu,-\alpha}$, Now Statement (1) follows from multiplying both sides of (6.3.21) by $N_{\mu,-\alpha}$. Since Statement (2) may be shown similarly, we skip the proof. ■

Lemma 6.3.22 *Let $\mathfrak{q}$ be a two-step nilpotent parabolic subalgebra of non-Heisenberg type, listed in (3.3.2) or (3.3.3), and $\alpha_{\mathfrak{q}}$ be the simple root that parametrizes the parabolic subalgebra $\mathfrak{q}$. If $V(\mu+\epsilon)$ is a special constituent of type 1a then, for any $\alpha \in \Delta_{\mu+\epsilon}(\mathfrak{g}(1))\backslash\{\mu, \epsilon\}$ and any $\gamma_j \in \Delta_{\mu+\epsilon}(\mathfrak{z}(\mathfrak{n}))$,*

$$N_{\alpha,-\gamma_j}N_{\mu,-\alpha}N_{-\theta(\gamma_j),\theta(\alpha)}N_{-\theta(\alpha),\theta(\mu)} = N_{\mu,\epsilon-\gamma_j}N_{\epsilon,-\gamma_j}\frac{||\alpha_{\mathfrak{q}}||^2}{2}. \tag{6.3.23}$$

Proof. By Lemma 6.3.20, we have

$$\begin{aligned}N_{\alpha,-\gamma_j}N_{\mu,-\alpha}N_{-\theta(\gamma_j),\theta(\alpha)}N_{-\theta(\alpha),\theta(\mu)} &= N_{\mu,-\gamma_j}N_{\alpha-\mu,\mu-\gamma_j}N_{-\theta(\gamma_j),\theta(\mu)}N_{-(\mu-\gamma_j),-(\alpha-\mu)} \\ &= N_{\mu,-\gamma_j}N_{-\theta(\gamma_j),\theta(\mu)}N_{\alpha-\mu,\mu-\gamma_j}N_{-(\mu-\gamma_j),-(\alpha-\mu)} \\ &= N_{\mu,-\gamma_j}N_{-\theta(\gamma_j),\theta(\mu)}\frac{||\mu-\gamma_j||^2}{2}.\end{aligned}$$

Note that (6.3.17) is applied from line two to line three. Since $-\theta(\gamma_j)+\theta(\mu)+(\mu-\gamma_j) = 0$ with $\theta(\mu) = (\mu+\epsilon)-\mu = \epsilon$, by Property (H6) of our normalizations, we have $N_{-\theta(\gamma_j),\theta(\mu)} = N_{\epsilon,\mu-\gamma_j}$. By Lemma 6.3.8 with $\alpha = \mu$, $\beta = \epsilon$, and $\delta = \gamma_j$, it follows that $N_{\epsilon,\mu-\gamma_j} N_{\mu,-\gamma_j} = N_{\mu,\epsilon-\gamma_j} N_{\epsilon,-\gamma_j}$. Therefore,

$$N_{\mu,-\gamma_j} N_{-\theta(\gamma_j),\theta(\mu)} = N_{\mu,-\gamma_j} N_{\epsilon,\mu-\gamma_j} = N_{\mu,\epsilon-\gamma_j} N_{\epsilon,-\gamma_j}.$$

Remark 6.2.15 shows that γ_j and $\alpha_{\mathfrak{q}}$ are long roots, when $V(\mu+\epsilon)$ is of type 1a. Since μ is assumed to be a long root, the root $\mu-\gamma_j$ is a long root. Thus $||\mu-\gamma_j||^2 = ||\alpha_{\mathfrak{q}}||^2$. Hence,

$$\begin{aligned} N_{\alpha,-\gamma_j} N_{\mu,-\alpha} N_{-\theta(\gamma_j),\theta(\alpha)} N_{-\theta(\alpha),\theta(\mu)} &= N_{\mu,-\gamma_j} N_{-\theta(\gamma_j),\theta(\mu)} \frac{||\mu-\gamma_j||^2}{2} \\ &= N_{\mu,\epsilon-\gamma_j} N_{\epsilon,-\gamma_j} \frac{||\alpha_{\mathfrak{q}}||^2}{2}. \end{aligned}$$

■

CHAPTER 7

The Ω_2 Systems

We continue with $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{g}(1) \oplus \mathfrak{z}(\mathfrak{n})$ a maximal two-step nilpotent parabolic subalgebra of non-Heisenberg type, listed in (3.3.2) or (3.3.3). In this chapter, by using the preliminary results from Chapter 6, we shall determine the complex parameter $s_2 \in \mathbb{C}$ for the line bundle $\mathcal{L}_{-s}$ so that the Ω_2 systems are conformally invariant on $\mathcal{L}_{s_2}$. This is done in Theorem 7.3.6.

7.1 Covariant Map τ_2

As we have observed in Section 2.5 and as we have done in Section 4.2 for the Ω_1 system, to construct the $\Omega_2|_{V^*}$ system, we use the covariant map τ_2 and the associated L-intertwining operator $\tilde{\tau}_2|_{V^*}$, where V^* is an irreducible constituents of $\mathfrak{l}^* \otimes \mathfrak{z}(\mathfrak{n})^* = \mathfrak{g}(0)^* \otimes \mathfrak{g}(2)^*$. The purpose of this section is to show that the covariant map τ_2 is not identically zero, and also that the L-intertwining operators $\tilde{\tau}_2|_{V^*}$ are not identically zero for certain irreducible constituents V. We keep on using the normalizations from Section 4.1.

We start by showing that τ_2 is not identically zero. The covariant map τ_2 is given by

$$
\begin{aligned}
\tau_2 : \mathfrak{g}(1) &\to \mathfrak{l} \otimes \mathfrak{z}(\mathfrak{n}) \\
X &\mapsto \frac{1}{2}\,\mathrm{ad}(X)^2 \omega_0
\end{aligned}
$$

with $\omega_0 = \sum_{\gamma_j \in \Delta(\mathfrak{z}(\mathfrak{n}))} X_{-\gamma_j} \otimes X_{\gamma_j}$. The following technical lemma will make a certain argument simpler in later proofs.

Lemma 7.1.1 *If $V(\mu+\epsilon)$ is a special constituent of type 1a or type 2 then*

$$\tau_2(X_\mu + X_\epsilon) = a_{\mu,\epsilon}\, \mathrm{ad}(X_\mu)\, \mathrm{ad}(X_\epsilon)\, \omega_0, \tag{7.1.2}$$

where $a_{\mu,\epsilon} = 1 + \delta_{\mu,\epsilon}$ with $\delta_{\mu,\epsilon}$ the Kronecker delta.

Proof. It is clear that (7.1.2) holds if $\mu+\epsilon$ is of type 2. Indeed, if $\epsilon = \mu$ then we have

$$\tau_2(2X_\mu) = 4\tau_2(X_\mu) = 2\mathrm{ad}(X_\mu)^2\omega_0.$$

If $\mu+\epsilon$ is of type 1a then, by definition, $\mu+\epsilon \notin \Delta$ and both μ and ϵ are long roots. Thus, in the case, $\mathrm{ad}(X_\mu)\,\mathrm{ad}(X_\epsilon) = \mathrm{ad}(X_\epsilon)\,\mathrm{ad}(X_\mu)$. Moreover, by Lemma 3.4.4, we have $\mathrm{ad}(X_\mu)^2 X_{-\gamma_j} = \mathrm{ad}(X_\epsilon)^2 X_{-\gamma_j} = 0$ for any $\gamma_j \in \Delta(\mathfrak{z}(\mathfrak{n}))$. Hence,

$$\tau_2(X_\mu + X_\epsilon) = (1/2)(2\,\mathrm{ad}(X_\mu)\,\mathrm{ad}(X_\epsilon))\,\omega_0 = \mathrm{ad}(X_\mu)\,\mathrm{ad}(X_\epsilon)\,\omega_0.$$

■

Proposition 7.1.3 *Let $\mathfrak{q}$ be a maximal two-step nilpotent parabolic subalgebra of non-Heisenberg type listed in (3.3.2) or (3.3.3). Then the covariant map τ_2 is not identically zero.*

Proof. To prove that τ_2 is not identically zero, it suffices to show that there exists a vector $X \in \mathfrak{g}(1)$ so that $\tau_2(X) \neq 0$. Observe that, for each $\mathfrak{q}$ under consideration, $\mathfrak{l} \otimes \mathfrak{z}(\mathfrak{n})$ has at least one special constituent $V(\mu+\epsilon)$ of type 1a or type 2 (see Table 6.5). Therefore, $\Delta(\mathfrak{g}(1))$ always contains a root ϵ so that $V(\mu+\epsilon)$ is such a special constituent. Then, to prove this proposition, we show that $\tau_2(X_\mu + X_\epsilon) \neq 0$, where X_μ and X_ϵ are root vectors for μ and ϵ, respectively, with $\mu+\epsilon$ the highest weight for a special constituent of type 1a or type 2.

Let $\mu+\epsilon$ be the highest weight of a special constituent of type 1a or type 2. By

Lemma 7.1.1 we have

$$\begin{aligned}\tau_2(X_\mu + X_\epsilon) &= a_{\mu,\epsilon}\ \mathrm{ad}(X_\mu)\ \mathrm{ad}(X_\epsilon)\ \omega_0 \\ &= a_{\mu,\epsilon} \sum_{\gamma_j \in \Delta(\mathfrak{z}(\mathfrak{n}))} \mathrm{ad}(X_\mu)\ \mathrm{ad}(X_\epsilon)\ X_{-\gamma_j} \otimes X_{\gamma_j}\end{aligned} \tag{7.1.4}$$

with $a_{\mu,\epsilon} = 1 + \delta_{\mu,\epsilon}$. If there were a root $\gamma_j \in \Delta(\mathfrak{z}(\mathfrak{n}))$ such that $\epsilon - \gamma_j = -\mu$ then $\mu + \epsilon = \gamma_j \in \Delta$, which contradicts the assumption that $\mu + \epsilon$ is of type 1a or type 2. By Lemma 6.3.9, if $\mu + \epsilon - \gamma_j \in \Delta$ then $\epsilon - \gamma_j \in \Delta$. Then, for all $\gamma_j \in \Delta(\mathfrak{z}(\mathfrak{n}))$,

$$\mathrm{ad}(X_\mu)\ \mathrm{ad}(X_\epsilon)\ X_{-\gamma_j} = \begin{cases} N_{\mu,\epsilon-\gamma_j} N_{\epsilon,-\gamma_j} X_{\mu+\epsilon-\gamma_j} & \text{if } \mu + \epsilon - \gamma_j \in \Delta \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, we have

$$\begin{aligned}\tau_2(X_\mu + X_\epsilon) &= a_{\mu,\epsilon} \sum_{\gamma_j \in \Delta(\mathfrak{z}(\mathfrak{n}))} \mathrm{ad}(X_\mu)\ \mathrm{ad}(X_\epsilon)\ X_{-\gamma_j} \otimes X_{\gamma_j} \\ &= a_{\mu,\epsilon} \sum_{\gamma_j \in \Delta_{\mu+\epsilon}(\mathfrak{z}(\mathfrak{n}))} N_{\mu,\epsilon-\gamma_j} N_{\epsilon,-\gamma_j} X_{\mu+\epsilon-\gamma_j} \otimes X_{\gamma_j}.\end{aligned}$$

Since $\{X_{\mu+\epsilon-\gamma_j} \otimes X_{\gamma_j} \mid \gamma_j \in \Delta_{\mu+\epsilon}(\mathfrak{z}(\mathfrak{n}))\}$ is a linearly independent set, this shows that $\tau_2(X_\mu + X_\epsilon) \neq 0$. ■

Next we identify irreducible constituent $V(\nu)^*$ so that $\tilde{\tau}_2|_{V(\nu)^*}$ is not identically zero. In Section 6.1, we observed that, given an irreducible constituent $V(\nu)^*$, the L-intertwining operator $\tilde{\tau}_2|_{V(\nu)^*} \in \mathrm{Hom}_L(V(\nu)^*, \mathcal{P}^2(\mathfrak{g}(1)))$ is given by

$$\tilde{\tau}_2|_{V(\nu)^*}(Y^*)(X) = Y^*(\tau_2(X)), \tag{7.1.5}$$

where $\mathcal{P}^2(\mathfrak{g}(1))$ is the space of polynomials on $\mathfrak{g}(1)$ of degree 2. By Proposition 6.1.5, we know that if $\tilde{\tau}_2|_{V(\nu)^*}$ is not identically zero then $V(\nu)$ is a special constituent of $\mathfrak{l} \otimes \mathfrak{z}(\mathfrak{n})$. We now show that the converse of Proposition 6.1.5 also holds, when the special constituent $V(\nu)$ is of type 1a or type 2. If $l \in L$ and $Z \in \mathfrak{l}$ then we denote the action of the group and its Lie algebra on $X_\alpha \otimes X_{\gamma_j}$ by $l \cdot (X_\alpha \otimes X_{\gamma_j})$ and $Z \cdot (X_\alpha \otimes X_{\gamma_j})$, respectively.

Proposition 7.1.6 *If $V(\mu+\epsilon)$ is a special constituent of $\mathfrak{l} \otimes \mathfrak{z}(\mathfrak{n})$ of type 1a or type 2 then the following hold:*

1. *The vector $\tau_2(X_\mu + X_\epsilon)$ is a highest weight vector for $V(\mu+\epsilon)$.*

2. *The L-intertwining operator $\tilde{\tau}_2|_{V(\mu+\epsilon)^*}$ is not identically zero.*

Proof. We have shown that in the proof for Proposition 7.1.3 that $\tau_2(X_\mu + X_\epsilon) \neq 0$. Moreover, Lemma 7.1.1 gives that $\tau_2(X_\mu + X_\epsilon) = a_{\mu,\epsilon} \operatorname{ad}(X_\mu) \operatorname{ad}(X_\epsilon)\omega_0$ with $a_{\mu,\epsilon} = 1 + \delta_{\mu,\epsilon}$. For $l \in L$, we have $l \cdot \omega_0 = \omega_0$ (see Corollary 2.5.3) and so

$$l \cdot \tau_2(X_\mu + X_\epsilon) = a_{\mu,\epsilon} \operatorname{ad}(\operatorname{Ad}(l)X_\mu) \operatorname{ad}(\operatorname{Ad}(l)X_\epsilon)\, \omega_0.$$

By replacing l by $\exp(tZ)$ with $Z \in \mathfrak{l}$, differentiating, and setting $t = 0$, we obtain

$$Z \cdot \tau_2(X_\mu + X_\epsilon) = a_{\mu,\epsilon} \big(\operatorname{ad}([Z, X_\mu]) \operatorname{ad}(X_\epsilon) + \operatorname{ad}(X_\mu) \operatorname{ad}([Z, X_\epsilon])\big)\, \omega_0. \qquad (7.1.7)$$

In particular, if $Z = H \in \mathfrak{h}$ in (7.1.7) then

$$H \cdot \tau_2(X_\mu + X_\epsilon) = (\mu+\epsilon)(H)\tau_2(X_\mu + X_\epsilon).$$

Therefore $\tau_2(X_\mu + X_\epsilon)$ is a weight vector with weight $\mu+\epsilon$. To show that $\tau_2(X_\mu + X_\epsilon)$ is a highest weight vector, we replace Z in (7.1.7) by X_α with $\alpha \in \Delta^+(\mathfrak{l})$. Since μ is the highest weight for $\mathfrak{g}(1)$, we have

$$X_\alpha \cdot \tau_2(X_\mu + X_\epsilon) = a_{\mu,\epsilon} \operatorname{ad}(X_\mu) \operatorname{ad}([X_\alpha, X_\epsilon])\, \omega_0.$$

If $\mu+\epsilon$ is of type 2 then, as $\epsilon = \mu$ in the case, clearly $X_\alpha \cdot \tau_2(X_\mu + X_\epsilon) = 0$. The case that $\mu+\epsilon$ is of type 1a follows from Lemma 6.3.3.

To prove the second statement, it is enough to show that there exist $Y^* \in V(\mu+\epsilon)^*$ and $X \in \mathfrak{g}(1)$ so that $\tilde{\tau}_2(Y^*)(X) \neq 0$. Let Y_l^* be a lowest weight vector for $V(\mu+\epsilon)^*$.

Observe that if Y_h is a highest weight vector for $V(\mu+\epsilon)$ then $Y_l^*(Y_h) \neq 0$. Since $\tau_2(X_\mu + X_\epsilon)$ is a highest weight vector for $V(\mu+\epsilon)$, we have

$$\tilde{\tau}_2|_{V(\mu+\epsilon)^*}(Y_l^*)(X_\mu + X_\epsilon) = Y_l^*(\tau_2(X_\mu + X_\epsilon)) \neq 0.$$

■

7.2 The $\Omega_2|_{V(\mu+\epsilon)^*}$ Systems

Proposition 7.1.6 shows that the L-intertwining operator $\tilde{\tau}_2|_{V(\mu+\epsilon)^*}$ is not identically zero, when $V(\mu+\epsilon)$ is a special constituent of $\mathfrak{l} \otimes \mathfrak{z}(\mathfrak{n})$ of type 1a or type 2. In this section, we thus construct the $\Omega_2|_{V(\mu+\epsilon)^*}$ system corresponding to irreducible constituents $V(\mu+\epsilon)$ of type 1a or type 2. Here it may be helpful to recall some notation introduced in Section 6.3. For any $\mathrm{ad}(\mathfrak{h})$-invariant subspace $W \subset \mathfrak{g}$ and any weight $\nu \in \mathfrak{h}^*$, we write

$$\Delta_\nu(W) = \{\alpha \in \Delta(W) \mid \nu - \alpha \in \Delta\}.$$

Recall from Section 6.3 that when $V(\mu+\epsilon)$ is a special constituent of $\mathfrak{l} \otimes \mathfrak{z}(\mathfrak{n})$, we write

$$\theta(\beta) = (\mu+\epsilon) - \beta.$$

As indicated in Section 2.5, the L-intertwining operator $\tilde{\tau}_2|_{V(\mu+\epsilon)^*}$ yields a system of differential operators. We have denoted such operators by $\Omega_2|_{V(\mu+\epsilon)^*}(Y^*)$ with $Y^* \in V(\mu+\epsilon)^*$, where $\Omega_2|_{V(\mu+\epsilon)^*} : V(\mu+\epsilon)^* \to \mathbb{D}(\mathcal{L}_{-s})^{\bar{\mathfrak{n}}}$ is $\mathcal{U}(\mathfrak{l})$-equivariant. Because of such equivariance, the system is totally determined, once $\Omega_2|_{V(\mu+\epsilon)^*}(Y_l^*)$ is constructed, where Y_l^* is a lowest weight vector in $V(\mu+\epsilon)^*$.

The first step is to explicitly describe $Y_l^* \in V(\mu+\epsilon)^*$. Observe that we have a non-zero map

$$\begin{aligned} \bar{\tau}_2 : \mathfrak{g}(-1) &\to \mathfrak{l} \otimes \mathfrak{z}(\bar{\mathfrak{n}}) \\ \bar{X} &\mapsto \frac{1}{2}\mathrm{ad}(X)^2\bar{\omega}_0 \end{aligned}$$

with $\bar{\omega}_0 = \sum_{\gamma_t \in \Delta(\mathfrak{z}(\mathfrak{n}))} X_{\gamma_t} \otimes X_{-\gamma_t}$. One checks, as in the proofs for Lemma 2.5.4 and Proposition 7.1.3, that $\bar{\tau}_2$ is a non-zero L-equivariant map. Moreover, if $V(\mu+\epsilon)$ is a special constituent of type 1a or type 2 then, as in Lemma 7.1.1,

$$\bar{\tau}_2(X_{-\mu} + X_{-\epsilon}) = a_{\mu,\epsilon} \operatorname{ad}(X_{-\mu}) \operatorname{ad}(X_{-\epsilon})\, \bar{\omega}_0$$

with $a_{\mu,\epsilon} = 1 + \delta_{\mu,\epsilon}$. Arguing as in Proposition 7.1.6, we can show that $\bar{\tau}_2(X_{-\mu} + X_{-\epsilon})$ is a lowest weight vector for $V(\mu+\epsilon)^*$ with lowest weight $-\mu-\epsilon$. Thus,

$$\begin{aligned} Y_l^* &= \operatorname{ad}(X_{-\mu})\operatorname{ad}(X_{-\epsilon})\bar{\omega}_0 \\ &= \sum_{\gamma_t \in \Delta_{\mu+\epsilon}(\mathfrak{z}(\mathfrak{n}))} N_{-\mu,\gamma_t-\epsilon} N_{-\epsilon,\gamma_t} X_{-\theta(\gamma_t)} \otimes X_{-\gamma_t} \end{aligned} \tag{7.2.1}$$

is a lowest weight vector for $V(\mu+\epsilon)^*$. Observe that, by Lemma 6.3.9, we have $\gamma_t - \epsilon \in \Delta$ for $\gamma_t \in \Delta_{\mu+\epsilon}(\mathfrak{z}(\mathfrak{n}))$.

For Y_l^* as in (7.2.1), we have

$$\begin{aligned} Y_l^*(\tau_2(X)) &= \frac{1}{2} \sum_{\substack{\gamma_t \in \Delta_{\mu+\epsilon}(\mathfrak{z}(\mathfrak{n})) \\ \gamma_j \in \Delta(\mathfrak{z}(\mathfrak{n}))}} N_{-\mu,\gamma_t-\epsilon} N_{-\epsilon,\gamma_t} \kappa(X_{-\theta(\gamma_t)}, \operatorname{ad}(X)^2 X_{-\gamma_j}) \kappa(X_{-\gamma_t}, X_{\gamma_j}) \\ &= \frac{1}{2} \sum_{\gamma_t \in \Delta_{\mu+\epsilon}(\mathfrak{z}(\mathfrak{n}))} N_{-\mu,\gamma_t-\epsilon} N_{-\epsilon,\gamma_t} \kappa(X_{-\theta(\gamma_t)}, \operatorname{ad}(X)^2 X_{-\gamma_t}). \end{aligned} \tag{7.2.2}$$

Write $X = \sum_{\alpha \in \Delta(\mathfrak{g}(1))} \eta_\alpha X_\alpha$ and let $\gamma_t \in \Delta_{\mu+\epsilon}(\mathfrak{z}(\mathfrak{n}))$. Then,

$$\begin{aligned} \kappa(X_{-\theta(\gamma_t)}, \operatorname{ad}(X)^2 X_{-\gamma_t}) &= \sum_{\alpha,\beta \in \Delta(\mathfrak{g}(1))} \eta_\alpha \eta_\beta \kappa(X_{-\theta(\gamma_t)}, [X_\beta, [X_\alpha, X_{-\gamma_t}]]) \\ &= \sum_{\alpha,\beta \in \Delta(\mathfrak{g}(1))} \eta_\alpha \eta_\beta \kappa([X_{-\theta(\gamma_t)}, X_\beta], [X_\alpha, X_{-\gamma_t}]) \\ &= \sum_{\substack{\alpha \in \Delta_{\gamma_t}(\mathfrak{g}(1)) \\ \beta \in \Delta_{\theta(\gamma_t)}(\mathfrak{g}(1))}} \eta_\alpha \eta_\beta N_{\alpha,-\gamma_t} N_{-\theta(\gamma_t),\beta} \kappa(X_{\beta-\theta(\gamma_t)}, X_{\alpha-\gamma_t}). \end{aligned}$$

Observe that, by Lemma 6.3.12 and Lemma 6.3.13, the sets $\Delta_{\gamma_t}(\mathfrak{g}(1))$ and $\Delta_{\theta(\gamma_t)}(\mathfrak{g}(1))$ are non-empty. By the normalization (H3) in Section 4.1, if $\kappa(X_{\beta-\theta(\gamma_t)}, X_{\alpha-\gamma_t}) \neq 0$ then $\beta - \theta(\gamma_t) = \gamma_t - \alpha$. Thus $\kappa(X_{\beta-\theta(\gamma_t)}, X_{\alpha-\gamma_t}) = 0$ unless $\beta = (\mu+\epsilon) - \alpha = \theta(\alpha)$.

Therefore,

$$\begin{aligned}\kappa(X_{-\theta(\gamma_t)}, \operatorname{ad}(X)^2 X_{-\gamma_t}) &= \sum_{\substack{\alpha\in\Delta_{\gamma_t}(\mathfrak{g}(1))\\ \beta\in\Delta_{\theta(\gamma_t)}(\mathfrak{g}(1))}} \eta_\alpha\eta_\beta N_{\alpha,-\gamma_t}N_{-\theta(\gamma_t),\beta}\kappa(X_{\beta-\theta(\gamma_t)}, X_{\alpha-\gamma_t})\\ &= \sum_{\alpha\in\Delta_{\gamma_t}(\mathfrak{g}(1))\cap\Delta_{\mu+\epsilon}(\mathfrak{g}(1))} N_{\alpha,-\gamma_t}N_{-\theta(\gamma_t),\theta(\alpha)}\eta_\alpha\eta_{\theta(\alpha)}\\ &= \sum_{\alpha\in\Delta_{\mu+\epsilon}(\mathfrak{g}(1))} N_{\alpha,-\gamma_t}N_{-\theta(\gamma_t),\theta(\alpha)}\kappa(X, X_{-\alpha})\kappa(X, X_{-\theta(\alpha)}).\end{aligned} \tag{7.2.3}$$

Lemma 6.3.12 is used in line three to show that $\Delta_{\gamma_t}(\mathfrak{g}(1))\cap\Delta_{\mu+\epsilon}(\mathfrak{g}(1)) = \Delta_{\mu+\epsilon}(\mathfrak{g}(1))$. Hence, by (7.2.2) and (7.2.3), $\tilde{\tau}_2|_{V(\mu+\epsilon)^*}(Y_l^*)(X) = Y_l^*(\tau_2(X))$ is

$$\begin{aligned}&\tilde{\tau}_2|_{V(\mu+\epsilon)^*}(Y_l^*)(X)\\ &= \frac{1}{2}\sum_{\substack{\alpha\in\Delta_{\mu+\epsilon}(\mathfrak{g}(1))\\ \gamma_t\in\Delta_{\mu+\epsilon}(\mathfrak{z}(\mathfrak{n}))}} (N_{-\mu,\gamma_t-\epsilon}N_{-\epsilon,\gamma_t})(N_{\alpha,-\gamma_t}N_{-\theta(\gamma_t),\theta(\alpha)})\kappa(X, X_{-\alpha})\kappa(X, X_{-\theta(\alpha)}).\end{aligned}$$

Now, via the composition of maps

$$V(\mu+\epsilon)^* \stackrel{\tilde{\tau}_2|_{V(\mu+\epsilon)^*}}{\to} \mathcal{P}^2(\mathfrak{g}(1)) \to \operatorname{Sym}^2(\mathfrak{g}(-1)) \stackrel{\sigma}{\hookrightarrow} \mathcal{U}(\bar{\mathfrak{n}}) \stackrel{R}{\to} \mathbb{D}(\mathcal{L}_{-s})^{\bar{\mathfrak{n}}},$$

for $Y_l^* \in V(\mu+\epsilon)^*$, the second-order differential operator $\Omega_2(Y_l^*) \in \mathbb{D}(\mathcal{L}_{-s})^{\bar{\mathfrak{n}}}$ is given by

$$\Omega_2(Y_l^*) = \frac{1}{2}\sum_{\substack{\alpha\in\Delta_{\mu+\epsilon}(\mathfrak{g}(1))\\ \gamma_t\in\Delta_{\mu+\epsilon}(\mathfrak{z}(\mathfrak{n}))}} (N_{-\mu,\gamma_t-\epsilon}N_{-\epsilon,\gamma_t})(N_{\alpha,-\gamma_t}N_{-\theta(\gamma_t),\theta(\alpha)})\ulcorner R(X_{-\alpha})R(X_{-\theta(\alpha)})\urcorner,$$

where $\ulcorner ab\urcorner = (1/2)(ab+ba)$. By Lemma 6.3.18, no symmetrization is needed. Therefore we obtain

$$\Omega_2(Y_l^*) = \frac{1}{2}\sum_{\substack{\alpha\in\Delta_{\mu+\epsilon}(\mathfrak{g}(1))\\ \gamma_t\in\Delta_{\mu+\epsilon}(\mathfrak{z}(\mathfrak{n}))}} (N_{-\mu,\gamma_t-\epsilon}N_{-\epsilon,\gamma_t})(N_{\alpha,-\gamma_t}N_{-\theta(\gamma_t),\theta(\alpha)})R(X_{-\alpha})R(X_{-\theta(\alpha)}). \tag{7.2.4}$$

7.3 Special Values of the $\Omega_2|_{V(\mu+\epsilon)^*}$ Systems

In this section we determine the special values of the line bundle $\mathcal{L}_{-s}$ for which the $\Omega|_{V(\mu+\epsilon)^*}$ system is conformally invariant, under the assumption that $V(\mu+\epsilon)$ is a special constituent of type 1a or type 2.

Choose a basis of weight vectors $Y_1^*, \ldots, Y_n^*$ for $V(\mu+\epsilon)^*$ and let $Y_l^* = Y_1^*$ be a lowest weight vector. We study $\Omega_2(Y_1^*), \ldots, \Omega_2(Y_n^*)$. To show that the list of differential operators $\Omega_2(Y_1^*)$, ..., $\Omega_2(Y_n^*)$ is conformally invariant on the bundle $\mathcal{L}_{-s}$, we need to prove that in $\mathbb{D}(\mathcal{L}_{-s})^{\bar{\mathfrak{n}}}$,

$$[\pi_s(X), \Omega_2(Y_i^*)] \in \mathrm{span}_{C^\infty(\bar{N}_0)}\{\Omega_2(Y_1^*), \ldots, \Omega_2(Y_n^*)\} \tag{7.3.1}$$

for all $X \in \mathfrak{g}$ and all i. By Proposition 2.3.14, (7.3.1) holds if

$$[\pi_s(X), \Omega_2(Y_i^*)]_e \in \mathrm{span}_{\mathbb{C}}\{\Omega_2(Y_1^*)_e, \cdots, \Omega_2(Y_n^*)_e\} \tag{7.3.2}$$

holds for all $X \in \mathfrak{g}$ and all i. Here, for $D \in \mathbb{D}(\mathcal{L}_{-s})$, $D_{\bar{n}}$ denotes the linear functional $f \mapsto (D \bullet f)(\bar{n})$ for $f \in C^\infty(\bar{N}_0, \mathbb{C}_{\chi^{-s}})$. We show that a simplification of (7.3.2) implies (7.3.1).

Proposition 7.3.3 *Let $V(\mu+\epsilon)^*$ be the dual module of a special constituent $V(\mu+\epsilon)$ of $\mathfrak{l} \otimes \mathfrak{z}(\mathfrak{n})$ with respect to the Killing form. Suppose that the operator $\Omega_2|_{V(\mu+\epsilon)^*}$: $V(\mu+\epsilon)^* \to \mathbb{D}(\mathcal{L}_{-s})^{\bar{\mathfrak{n}}}$ is non-zero. If X_h is a highest weight vector for $\mathfrak{g}(1)$ and if we have*

$$[\pi_s(X_h), \Omega_2(Y_l^*)]_e \in \mathrm{span}_{\mathbb{C}}\{\Omega_2(Y_1^*)_e, \cdots, \Omega_2(Y_n^*)_e\}$$

for a lowest weight vector Y_l^* and a basis $\{Y_1^*, \ldots, Y_n^*\}$ for $V(\mu+\epsilon)^*$ then the $\Omega_2|_{V(\mu+\epsilon)^*}$ system is a conformally invariant system.

Proof. By Remark 2.5.11, the $\Omega_k|_{V(\mu+\epsilon)^*}$ system satisfies the condition (S1) of Definition 2.1.4. We need to prove that (7.3.2) holds for all $X \in \mathfrak{g} = \bar{\mathfrak{n}} \oplus \mathfrak{l} \oplus \mathfrak{n}$. Note that, by definition, we have $\Omega_2(Y_i^*) \in \mathbb{D}(\mathcal{L}_{-s})^{\bar{\mathfrak{n}}}$. Hence (7.3.2) holds for $X \in \bar{\mathfrak{n}}$ trivially. The

L_0-equivariance of $\Omega_2|_{V(\mu+\epsilon)^*}$ shows that (7.3.2) holds for $X \in \mathfrak{l}$. Furthermore, Lemma 2.6.4 established (7.3.2) when $X \in \mathfrak{g}(1)$. Now we handle the case when $X \in \mathfrak{z}(\mathfrak{n})$.

If $X \in \mathfrak{z}(\mathfrak{n})$ then, since $\mathfrak{z}(\mathfrak{n}) = [\mathfrak{g}(1), \mathfrak{g}(1)]$, it is of the form $X = [X_1, X_2]$ for some $X_1, X_2 \in \mathfrak{g}(1)$. Then, by the Jacobi identity, we have

$$\begin{aligned}&[\pi_s(X), \Omega_2(Y_i^*)]\\&= [\pi_s(X_1), [\pi_s(X_2), \Omega_2(Y_i^*)]] - [\pi_s(X_2), [\pi_s(X_1), \Omega_2(Y_i^*)]].\end{aligned}$$

By Proposition 2.3.3, we have $\pi_s(X_j)_e = 0$ for $j = 1, 2$. It follows from Lemma 2.6.4 that for $j = 1, 2$ and all i, we have

$$\big[\pi_s(X_j), \Omega_2(Y^*)\big]_e \in \mathrm{span}_{\mathbb{C}}\{\Omega_2(Y_1^*)_e, \ldots, \Omega_2(Y_n^*)_e\}.$$

Therefore, by Lemma 2.6.1,

$$\begin{aligned}&[\pi_s(X), \Omega_2(Y_i^*)]_e\\&= [\pi_s(X_1), [\pi_s(X_2), \Omega_2(Y_i^*)]]_e - [\pi_s(X_2), [\pi_s(X_1), \Omega_2(Y_i^*)]]_e\\&\in \mathrm{span}_{\mathbb{C}}\{\Omega_2(Y_1^*)_e, \ldots, \Omega_2(Y_k^*)_e\}.\end{aligned}$$

■

Proposition 7.3.4 *If μ is the highest weight for $\mathfrak{g}(1)$ and $\alpha, \beta \in \Delta(\mathfrak{g}(1))$ then*

$$\begin{aligned}&[\pi_s(X_\mu), R(X_{-\alpha})R(X_{-\beta})]_e\\&= R([[X_\mu, X_{-\alpha}], X_{-\beta}]]_e + s\lambda_{\mathfrak{q}}([X_\mu, X_{-\alpha}])R(X_{-\beta})_e + s\lambda_{\mathfrak{q}}([X_\mu, X_{-\beta}])R(X_{-\alpha})_e.\end{aligned}$$

Proof. This simply follows by substituting $Y = X_\mu$, $X_1 = X_{-\alpha}$, and $X_2 = X_{-\beta}$ in Proposition 2.4.5, and evaluating at $\bar{n} = e$. ■

If $V(\mu+\epsilon)$ is a special constituent of $\mathfrak{l} \otimes \mathfrak{z}(\mathfrak{n})$ of type 1a or type 2 then we write

$$C(\mu, \epsilon) = \sum_{\gamma_t \in \Delta_{\mu+\epsilon}(\mathfrak{z}(\mathfrak{n}))} N_{\mu,\epsilon-\gamma_t} N_{-\mu,\gamma_t-\epsilon} N_{\epsilon,-\gamma_t} N_{-\epsilon,\gamma_t}. \tag{7.3.5}$$

By Lemma 6.3.14, we have $C(\mu, \epsilon) \neq 0$.

Theorem 7.3.6 *Let $\mathfrak{g}$ be a complex simple Lie algebra and let $\mathfrak{q}$ be a maximal two-step nilpotent parabolic subalgebra of non-Heisenberg type, listed in (3.3.2) or (3.3.3). If Y_l^* is the lowest weight vector defined in (7.2.1) for the dual module $V(\mu+\epsilon)^*$ of a special constituent $V(\mu+\epsilon)$ of type 1a or type 2 with respect to the Killing form, and if $\alpha_{\mathfrak{q}}$ is the simple root that parametrizes $\mathfrak{q}$ then the following hold:*

1. *If $V(\mu+\epsilon)$ is of type 1a then*

$$[\pi_s(X_\mu), \Omega_2(Y_l^*)]_e = \frac{||\alpha_{\mathfrak{q}}||^2}{2} C(\mu,\epsilon)(s+s_2)R(X_{-\epsilon})_e, \tag{7.3.7}$$

with $s_2 = \dfrac{|\Delta_{\mu+\epsilon}(\mathfrak{g}(1))|}{2} - 1$, where $|\Delta_{\mu+\epsilon}(\mathfrak{g}(1))|$ is the cardinality of $\Delta_{\mu+\epsilon}(\mathfrak{g}(1))$.

2. *If $V(\mu+\epsilon)$ is of type 2 then*

$$[\pi_s(X_\mu), \Omega_2(Y_l^*)]_e = \frac{||\alpha_{\mathfrak{q}}||^2}{2} C(\mu,\mu)(s-1)R(X_{-\mu})_e. \tag{7.3.8}$$

Proof. We start by showing that (7.3.7) holds. It follows from (7.2.4) that

$$\begin{aligned}&[\pi_s(X_\mu), \Omega_2(Y_l^*)]_e \\ &= \frac{1}{2} \sum_{\substack{\alpha\in\Delta_{\mu+\epsilon}(\mathfrak{g}(1)) \\ \gamma_t\in\Delta_{\mu+\epsilon}(\mathfrak{z}(\mathfrak{n}))}} (N_{-\mu,\gamma_t-\epsilon}N_{-\epsilon,\gamma_t})(N_{\alpha,-\gamma_t}N_{-\theta(\gamma_t),\theta(\alpha)})[\pi_s(X_\mu), R(X_{-\alpha})R(X_{-\theta(\alpha)})]_e.\end{aligned} \tag{7.3.9}$$

We use Proposition 7.3.4 to compute $[\pi_s(X_\mu), R(X_{-\alpha})R(X_{-\theta(\alpha)})]_e$. This is

$$\begin{aligned}&[\pi_s(X_\mu), R(X_{-\alpha})R(X_{-\theta(\alpha)})]_e \\ &= R([[X_\mu, X_{-\alpha}], X_{-\theta(\alpha)}])_e + s\lambda_{\mathfrak{q}}([X_\mu, X_{-\alpha}])R(X_{-\theta(\alpha)})_e + s\lambda_{\mathfrak{q}}([X_\mu, X_{-\theta(\alpha)}])R(X_{-\alpha})_e.\end{aligned}$$

We consider the contributions from each term in (7.3.9), separately. Recall here that, as we defined in Section 3.2, our parabolic subalgebra $\mathfrak{q}$ is parametrized by the simple root $\alpha_{\mathfrak{q}} \in \Pi$ and that $\lambda_{\mathfrak{q}}$ is the fundamental weight for $\alpha_{\mathfrak{q}}$.

First we study the contribution from the second term. It is

$$T_2 = \frac{s}{2} \sum_{\substack{\alpha\in\Delta_{\mu+\epsilon}(\mathfrak{g}(1)) \\ \gamma_t\in\Delta_{\mu+\epsilon}(\mathfrak{z}(\mathfrak{n}))}} (N_{-\mu,\gamma_t-\epsilon}N_{-\epsilon,\gamma_t})(N_{\alpha,-\gamma_t}N_{-\theta(\gamma_t),\theta(\alpha)})\lambda_{\mathfrak{q}}([X_\mu, X_{-\alpha}])R(X_{-\theta(\alpha)})_e.$$

As $\mathfrak{g}(1)$ is the 1-eigenspace of $\mathrm{ad}(H_{\mathfrak{q}})$ with $H_{\mathfrak{q}}$ defined in (3.2.1), the set $\Delta(\mathfrak{g}(1))$ is

$$\Delta(\mathfrak{g}(1)) = \big\{\beta \in \Delta \mid \frac{2\langle \lambda_{\mathfrak{q}}, \beta\rangle}{||\alpha_{\mathfrak{q}}||^2} = 1\big\}. \tag{7.3.10}$$

Therefore, by the normalization (H4) in Section 4.1, for $\beta \in \Delta(\mathfrak{g}(1))$, we have

$$\lambda_{\mathfrak{q}}(H_\beta) = \langle \lambda_{\mathfrak{q}}, \beta\rangle = \frac{||\alpha_{\mathfrak{q}}||^2}{2}.$$

Thus,

$$\lambda_{\mathfrak{q}}([X_\mu, X_{-\alpha}]) = \frac{||\alpha_{\mathfrak{q}}||^2}{2}\delta_{\alpha,\mu} \tag{7.3.11}$$

with $\delta_{\alpha,\mu}$ the Kronecker delta. So the contribution from this term is

$$\begin{aligned} T_2 &= \frac{s}{2} \sum_{\substack{\alpha\in\Delta_{\mu+\epsilon}(\mathfrak{g}(1)) \\ \gamma_t\in\Delta_{\mu+\epsilon}(\mathfrak{z}(\mathfrak{n}))}} (N_{-\mu,\gamma_t-\epsilon}N_{-\epsilon,\gamma_t})(N_{\alpha,-\gamma_t}N_{-\theta(\gamma_t),\theta(\alpha)})\lambda_{\mathfrak{q}}([X_\mu, X_{-\alpha}])R(X_{-\theta(\alpha)})_e \\ &= \frac{s||\alpha_{\mathfrak{q}}||^2}{4} \sum_{\gamma_t\in\Delta_{\mu+\epsilon}(\mathfrak{z}(\mathfrak{n}))} (N_{-\mu,\gamma_t-\epsilon}N_{-\epsilon,\gamma_t})(N_{\mu,-\gamma_t}N_{-\theta(\gamma_t),\theta(\mu)})R(X_{-\theta(\mu)})_e \\ &= \frac{s||\alpha_{\mathfrak{q}}||^2}{4} \sum_{\gamma_t\in\Delta_{\mu+\epsilon}(\mathfrak{z}(\mathfrak{n}))} (N_{-\mu,\gamma_t-\epsilon}N_{-\epsilon,\gamma_t})(N_{\mu,-\gamma_t}N_{\epsilon,\mu-\gamma_t})R(X_{-\epsilon})_e. \end{aligned}$$

We showed in Lemma 6.3.8 that $N_{\mu,-\gamma_t}N_{\epsilon,\mu-\gamma_t} = N_{\mu,\epsilon-\gamma_t}N_{\epsilon,-\gamma_t}$. Hence,

$$\begin{aligned} T_2 &= \frac{s||\alpha_{\mathfrak{q}}||^2}{4} \sum_{\gamma_t\in\Delta_{\mu+\epsilon}(\mathfrak{z}(\mathfrak{n}))} (N_{-\mu,\gamma_t-\epsilon}N_{-\epsilon,\gamma_t})(N_{\mu,-\gamma_t}N_{\epsilon,\mu-\gamma_t})R(X_{-\epsilon})_e \\ &= \frac{s||\alpha_{\mathfrak{q}}||^2}{4} \sum_{\gamma_t\in\Delta_{\mu+\epsilon}(\mathfrak{z}(\mathfrak{n}))} (N_{-\mu,\gamma_t-\epsilon}N_{-\epsilon,\gamma_t})(N_{\mu,\epsilon-\gamma_t}N_{\epsilon,-\gamma_t})R(X_{-\epsilon})_e \\ &= \frac{s||\alpha_{\mathfrak{q}}||^2}{4} \sum_{\gamma_t\in\Delta_{\mu+\epsilon}(\mathfrak{z}(\mathfrak{n}))} (N_{\mu,\epsilon-\gamma_t}N_{-\mu,\gamma_t-\epsilon}N_{\epsilon,-\gamma_t}N_{-\epsilon,\gamma_t})R(X_{-\epsilon})_e \\ &= \frac{s||\alpha_{\mathfrak{q}}||^2}{4}C(\mu,\epsilon)R(X_{-\epsilon})_e. \end{aligned}$$

The same argument shows that the contribution from the third term is

$$\begin{aligned} T_3 &= \frac{s}{2} \sum_{\substack{\alpha\in\Delta_{\mu+\epsilon}(\mathfrak{g}(1)) \\ \gamma_t\in\Delta_{\mu+\epsilon}(\mathfrak{z}(\mathfrak{n}))}} (N_{-\mu,\gamma_t-\epsilon}N_{-\epsilon,\gamma_t})(N_{\alpha,-\gamma_t}N_{-\theta(\gamma_t),\theta(\alpha)})\lambda_{\mathfrak{q}}([X_\mu, X_{-\theta(\alpha)}])R(X_{-\alpha})_e \\ &= \frac{s||\alpha_{\mathfrak{q}}||^2}{4}C(\mu,\epsilon)R(X_{-\epsilon})_e. \end{aligned}$$

Now we consider the contribution from the first term. It is

$$T_1 = \frac{1}{2} \sum_{\substack{\alpha\in\Delta_{\mu+\epsilon}(\mathfrak{g}(1))\\ \gamma_t\in\Delta_{\mu+\epsilon}(\mathfrak{z}(\mathfrak{n}))}} (N_{-\mu,\gamma_t-\epsilon}N_{-\epsilon,\gamma_t})(N_{\alpha,-\gamma_t}N_{-\theta(\gamma_t),\theta(\alpha)})R([[X_\mu, X_{-\alpha}], X_{-\theta(\alpha)}])_e.$$

We claim that if $\alpha = \epsilon$ or μ then $[[X_\mu, X_{-\alpha}], X_{-\theta(\alpha)}] = 0$, where $\theta(\alpha)$ denotes $\theta(\alpha) = (\mu+\epsilon)-\alpha$. If $\alpha = \epsilon$ then, by Remark 6.2.14, $[X_\mu, X_{-\alpha}] = [X_\mu, X_{-\epsilon}] = 0$. If $\alpha = \mu$ then

$$[[X_\mu, X_{-\mu}], X_{-\theta(\mu)}] = [[X_\mu, X_{-\mu}], X_{-\epsilon}] = \epsilon(H_\mu)X_{-\epsilon} = 0.$$

Note that Remark 6.2.14 is applied to obtain $\epsilon(H_\mu) = \langle \epsilon, \mu\rangle = 0$. Moreover, by Remark 6.3.10, we have $\Delta_{\mu+\epsilon}(\mathfrak{g}(1))\backslash\{\mu,\epsilon\} \neq \emptyset$. The contribution from T_1 is

$$\begin{aligned}
T_1 &= \frac{1}{2} \sum_{\substack{\alpha\in\Delta_{\mu+\epsilon}(\mathfrak{g}(1))\\ \gamma_t\in\Delta_{\mu+\epsilon}(\mathfrak{z}(\mathfrak{n}))}} (N_{-\mu,\gamma_t-\epsilon}N_{-\epsilon,\gamma_t})(N_{\alpha,-\gamma_t}N_{-\theta(\gamma_t),\theta(\alpha)})R([[X_\mu, X_{-\alpha}], X_{-\theta(\alpha)}])_e \\
&= \frac{1}{2} \sum_{\substack{\alpha\in\Delta_{\mu+\epsilon}(\mathfrak{g}(1))\backslash\{\mu,\epsilon\}\\ \gamma_t\in\Delta_{\mu+\epsilon}(\mathfrak{z}(\mathfrak{n}))}} (N_{-\mu,\gamma_t-\epsilon}N_{-\epsilon,\gamma_t})(N_{\alpha,-\gamma_t}N_{-\theta(\gamma_t),\theta(\alpha)})R([[X_\mu, X_{-\alpha}], X_{-\theta(\alpha)}])_e \\
&= \frac{1}{2} \sum_{\substack{\alpha\in\Delta_{\mu+\epsilon}(\mathfrak{g}(1))\backslash\{\mu,\epsilon\}\\ \gamma_t\in\Delta_{\mu+\epsilon}(\mathfrak{z}(\mathfrak{n}))}} (N_{-\mu,\gamma_t-\epsilon}N_{-\epsilon,\gamma_t})(N_{\alpha,-\gamma_t}N_{-\theta(\gamma_t),\theta(\alpha)})(N_{\mu,-\alpha}N_{\mu-\alpha,-\theta(\alpha)})R(X_{-\epsilon})_e \\
&= \frac{1}{2} \sum_{\substack{\alpha\in\Delta_{\mu+\epsilon}(\mathfrak{g}(1))\backslash\{\mu,\epsilon\}\\ \gamma_t\in\Delta_{\mu+\epsilon}(\mathfrak{z}(\mathfrak{n}))}} (N_{-\mu,\gamma_t-\epsilon}N_{-\epsilon,\gamma_t})(N_{\alpha,-\gamma_t}N_{-\theta(\gamma_t),\theta(\alpha)})(N_{\mu,-\alpha}N_{-\theta(\alpha),\theta(\mu)})R(X_{-\epsilon})_e \\
&= \frac{1}{2} \sum_{\substack{\alpha\in\Delta_{\mu+\epsilon}(\mathfrak{g}(1))\backslash\{\mu,\epsilon\}\\ \gamma_t\in\Delta_{\mu+\epsilon}(\mathfrak{z}(\mathfrak{n}))}} (N_{-\mu,\gamma_t-\epsilon}N_{-\epsilon,\gamma_t})(N_{\alpha,-\gamma_t}N_{\mu,-\alpha}N_{-\theta(\gamma_t),\theta(\alpha)}N_{-\theta(\alpha),\theta(\mu)})R(X_{-\epsilon})_e.
\end{aligned}$$

Note that, from line three to line four, we use that $N_{\mu-\alpha,-\theta(\alpha)} = N_{-\theta(\alpha),\theta(\mu)}$, as $(\mu-\alpha)+(-\theta(\alpha))+\theta(\mu) = 0$ (see Property (H6) in Section 4.1). By Lemma 6.3.22, we have

$$N_{\alpha,-\gamma_t}N_{\mu,-\alpha}N_{-\theta(\gamma_t),\theta(\alpha)}N_{-\theta(\alpha),\theta(\mu)} = N_{\mu,\epsilon-\gamma_t}N_{\epsilon,-\gamma_t}\frac{||\alpha_\mathfrak{q}||^2}{2}.$$

Therefore,

$$\begin{aligned}
T_1 &= \frac{1}{2} \sum_{\substack{\alpha \in \Delta_{\mu+\epsilon}(\mathfrak{g}(1)) \setminus \{\mu,\epsilon\} \\ \gamma_t \in \Delta_{\mu+\epsilon}(\mathfrak{z}(\mathfrak{n}))}} (N_{-\mu,\gamma_t-\epsilon} N_{-\epsilon,\gamma_t})(N_{\alpha,-\gamma_t} N_{\mu,-\alpha} N_{-\theta(\gamma_t),\theta(\alpha)} N_{-\theta(\alpha),\theta(\mu)}) R(X_{-\epsilon})_e \\
&= \frac{||\alpha_{\mathfrak{q}}||^2}{4} \sum_{\substack{\alpha \in \Delta_{\mu+\epsilon}(\mathfrak{g}(1)) \setminus \{\mu,\epsilon\} \\ \gamma_t \in \Delta_{\mu+\epsilon}(\mathfrak{z}(\mathfrak{n}))}} (N_{-\mu,\gamma_t-\epsilon} N_{-\epsilon,\gamma_t})(N_{\mu,\epsilon-\gamma_t} N_{\epsilon,-\gamma_t}) R(X_{-\epsilon})_e \\
&= \frac{||\alpha_{\mathfrak{q}}||^2}{4} \sum_{\substack{\alpha \in \Delta_{\mu+\epsilon}(\mathfrak{g}(1)) \setminus \{\mu,\epsilon\} \\ \gamma_t \in \Delta_{\mu+\epsilon}(\mathfrak{z}(\mathfrak{n}))}} (N_{\mu,\epsilon-\gamma_t} N_{-\mu,\gamma_t-\epsilon} N_{\epsilon,-\gamma_t} N_{-\epsilon,\gamma_t}) R(X_{-\epsilon})_e \\
&= \frac{||\alpha_{\mathfrak{q}}||^2}{4} C(\mu,\epsilon) \sum_{\alpha \in \Delta_{\mu+\epsilon}(\mathfrak{g}(1)) \setminus \{\mu,\epsilon\}} R(X_{-\epsilon})_e \\
&= \frac{||\alpha_{\mathfrak{q}}||^2}{4} C(\mu,\epsilon)(|\Delta_{\mu+\epsilon}(\mathfrak{g}(1))| - 2) R(X_{-\epsilon})_e.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
[\pi_s(X_\mu), \Omega_2(Y_l^*)]_e &= T_1 + T_2 + T_3 \\
&= \frac{||\alpha_{\mathfrak{q}}||^2}{2} C(\mu,\epsilon)\big(s + \frac{|\Delta_{\mu+\epsilon}(\mathfrak{g}(1))|}{2} - 1\big) R(X_{-\epsilon})_e.
\end{aligned}$$

Now we are going to prove the equation (7.3.8). If $V(\mu+\epsilon)$ is of type 2 then $\mu+\epsilon = 2\mu$; in particular, $\theta(\mu) = (2\mu) - \mu = \mu$. By Lemma 6.3.6, $\Delta_{2\mu}(\mathfrak{g}(1)) = \{\mu\}$. Thus, (7.2.4) becomes

$$\begin{aligned}
\Omega_2(Y_l^*) &= \frac{1}{2} \sum_{\substack{\alpha \in \Delta_{2\mu}(\mathfrak{g}(1)) \\ \gamma_t \in \Delta_{2\mu}(\mathfrak{z}(\mathfrak{n}))}} (N_{-\mu,\gamma_t-\epsilon} N_{-\epsilon,\gamma_t})(N_{\alpha,-\gamma_t} N_{-\theta(\gamma_t),\theta(\alpha)}) R(X_{-\alpha}) R(X_{-\theta(\alpha)}) \\
&= \frac{1}{2} \sum_{\gamma_t \in \Delta_{2\mu}(\mathfrak{z}(\mathfrak{n}))} (N_{-\mu,\gamma_t-\mu} N_{-\mu,\gamma_t})(N_{\mu,-\gamma_t} N_{-\theta(\gamma_t),\theta(\mu)}) R(X_{-\mu}) R(X_{-\theta(\mu)}) \\
&= \frac{1}{2} \sum_{\gamma_t \in \Delta_{2\mu}(\mathfrak{z}(\mathfrak{n}))} (N_{-\mu,\gamma_t-\mu} N_{-\mu,\gamma_t})(N_{\mu,-\gamma_t} N_{-\theta(\gamma_t),\mu}) R(X_{-\mu})^2. \qquad (7.3.12)
\end{aligned}$$

Since $(-\theta(\gamma_t)) + \mu + (\mu - \gamma_t) = 0$, we have $N_{-\theta(\gamma_t),\mu} = N_{\mu,\mu-\gamma_t}$. Thus,

$$
\begin{aligned}
&\frac{1}{2}\sum_{\gamma_t\in\Delta_{2\mu}(\mathfrak{z}(\mathfrak{n}))}(N_{-\mu,\gamma_t-\mu}N_{-\mu,\gamma_t})(N_{\mu,-\gamma_t}N_{-\theta(\gamma_t),\mu})R(X_{-\mu})^2\\
&=\frac{1}{2}\sum_{\gamma_t\in\Delta_{2\mu}(\mathfrak{z}(\mathfrak{n}))}(N_{-\mu,\gamma_t-\mu}N_{-\mu,\gamma_t})(N_{\mu,-\gamma_t}N_{\mu,\mu-\gamma_t})R(X_{-\mu})^2\\
&=\frac{1}{2}\sum_{\gamma_t\in\Delta_{2\mu}(\mathfrak{z}(\mathfrak{n}))}(N_{\mu,\mu-\gamma_t}N_{-\mu,\gamma_t-\mu}N_{\mu,-\gamma_t}N_{-\mu,\gamma_t})R(X_{-\mu})^2\\
&=\frac{1}{2}C(\mu,\mu)R(X_{-\mu})^2. \qquad (7.3.13)
\end{aligned}
$$

Therefore,

$$[\pi_s(X_\mu),\Omega_2(Y_l^*)]_e=\frac{1}{2}C(\mu,\mu)[\pi_s(X_\mu),R(X_{-\mu})^2]_e.$$

It follows from (7.3.11) that $\lambda_{\mathfrak{q}}([X_\mu,X_{-\mu}])=||\alpha_{\mathfrak{q}}||^2/2$. Then, by Proposition 7.3.4 with $\alpha=\beta=\mu$, we have

$$
\begin{aligned}
[\pi_s(X_\mu),R(X_{-\mu})^2]_e &= R([[X_\mu,X_{-\mu}],X_{-\mu}])_e+2s\lambda_{\mathfrak{q}}([X_\mu,X_{-\mu}])R(X_{-\mu})_e\\
&=-\mu(H_\mu)R(X_{-\mu})_e+2s\cdot\frac{||\alpha_{\mathfrak{q}}||^2}{2}R(X_{-\mu})_e\\
&=(s||\alpha_{\mathfrak{q}}||^2-||\mu||^2)R(X_{-\mu})_e.
\end{aligned}
$$

Observe that Table 6.5 shows that a special constituent of type 2 occurs only when $\mathfrak{q}$ is of type $B_n(n)$, type $C_n(i)$ or $F_4(4)$. Appendix C and Appendix D show that when $\mathfrak{q}$ is of these types, we have $||\mu||^2=||\alpha_{\mathfrak{q}}||^2$. Therefore,

$$[\pi_s(X_\mu),R(X_{-\mu})^2]_e=(s||\alpha_{\mathfrak{q}}||^2-||\mu||^2)R(X_{-\mu})_e=||\alpha_{\mathfrak{q}}||^2(s-1)R(X_{-\mu})_e.$$

Hence, we obtain

$$
\begin{aligned}
[\pi_s(X_\mu),\Omega_2(Y_l^*)]_e &= \frac{1}{2}C(\mu,\mu)[\pi_s(X_\mu),R(X_{-\mu})^2]_e\\
&=\frac{||\alpha_{\mathfrak{q}}||^2}{2}C(\mu,\mu)(s-1)R(X_{-\mu})_e.
\end{aligned}
$$

■

To emphasize the fundamental weight $\lambda_{\mathfrak{q}}$, we write $\mathcal{L}(-s\lambda_{\mathfrak{q}})$ for the line bundle $\mathcal{L}_{-s}$. Now, by combining Proposition 7.3.3 and Theorem 7.3.6, we conclude the following.

Corollary 7.3.14 *Under the same hypotheses in Theorem 7.3.6, we have:*

1. *If $V(\mu+\epsilon)^*$ is of type 1a then the $\Omega_2|_{V(\mu+\epsilon)^*}$ system is conformally invariant on the line bundle $\mathcal{L}(s_2\lambda_{\mathfrak{q}})$, where s_2 is the constant given in Theorem 7.3.6.*

2. *If $V(\mu+\epsilon)^*$ is of type 2 then the $\Omega_2|_{V(\mu+\epsilon)^*}$ system is conformally invariant on the line bundle $\mathcal{L}(-\lambda_{\mathfrak{q}})$.*

Proof. This corollary follows from Proposition 7.3.3 and Theorem 7.3.6. ■

As we defined in Definition 6.2.11, we denote by $V(\mu+\epsilon_\gamma)$ the special constituent of $\mathfrak{l}\otimes\mathfrak{z}(\mathfrak{n})$ so that $V(\mu+\epsilon_\gamma)\subset\mathfrak{l}_\gamma\otimes\mathfrak{z}(\mathfrak{n})$, and denote by $V(\mu+\epsilon_{n\gamma})$ the special constituent so that $V(\mu+\epsilon_{n\gamma})=\mathfrak{l}_{n\gamma}\otimes\mathfrak{z}(\mathfrak{n})$. See Table 6.5 for the types of $V(\mu+\epsilon_\gamma)$ and $V(\mu+\epsilon_{n\gamma})$ for each case. Table 7.1 below summarizes the line bundles $\mathcal{L}(s_0\lambda_{\mathfrak{q}})$ on which the Ω_2 systems are conformally invariant. Here, a dash indicates that there does not exist the special constituent $V(\mu+\epsilon_{n\gamma})$. When $\mathfrak{q}$ is of type $B_n(n-1)$, the special constituent $V(\mu+\epsilon_{n\gamma})$ is of type 1b, and when $\mathfrak{q}$ is of type $C_n(i)$, the special constituent $V(\mu+\epsilon_\gamma)$ is of type 3. Therefore, we put a question mark for these cases in the table.

By Corollary 2.7.7, if an Ω_2 system is conformally invariant over the line bundle $\mathcal{L}(s_0\lambda_{\mathfrak{q}})$ then the generalized Verma module $M_{\mathfrak{q}}[\mathbb{C}_{-s_0\lambda_{\mathfrak{q}}}]=\mathcal{U}(\mathfrak{g})\otimes_{\mathcal{U}(\mathfrak{q})}\mathbb{C}_{-s_0\lambda_{\mathfrak{q}}}$ is reducible. Table 7.2 summarizes the generalized Verma modules that correspond to the line bundles in Table 7.1.

Table 7.1: Line Bundles with Special Values

Parabolic subalgebra $\mathfrak{q}$	$\Omega_2\|_{V(\mu+\epsilon_\gamma)^*}$	$\Omega_2\|_{V(\mu+\epsilon_{n\gamma})^*}$
$B_n(i), 3 \leq i \leq n-2$	$\mathcal{L}\big((n-i-\frac{1}{2})\lambda_i\big)$	$\mathcal{L}(\lambda_i)$
$B_n(n-1)$	$\mathcal{L}\big(\frac{1}{2}\lambda_{n-1}\big)$	?
$B_n(n)$	$\mathcal{L}(-\lambda_n)$	–
$C_n(i), 2 \leq i \leq n-1$	?	$\mathcal{L}(-\lambda_i)$
$D_n(i), 3 \leq i \leq n-3$	$\mathcal{L}\big((n-i-1)\lambda_i\big)$	$\mathcal{L}(\lambda_i)$
$E_6(3)$	$\mathcal{L}(\lambda_3)$	$\mathcal{L}(2\lambda_3)$
$E_6(5)$	$\mathcal{L}(\lambda_5)$	$\mathcal{L}(2\lambda_5)$
$E_7(2)$	$\mathcal{L}(2\lambda_2)$	–
$E_7(6)$	$\mathcal{L}(\lambda_6)$	$\mathcal{L}(3\lambda_6)$
$E_8(1)$	$\mathcal{L}(3\lambda_1)$	–
$F_4(4)$	$\mathcal{L}(-\lambda_4)$	–

Table 7.2: The Generalized Verma Modules corresponding to $\mathcal{L}(s_0\lambda_{\mathfrak{q}})$ in Table 7.1

Parabolic subalgebra $\mathfrak{q}$	$\Omega_2\vert_{V(\mu+\epsilon_\gamma)^*}$	$\Omega_2\vert_{V(\mu+\epsilon_n\gamma)^*}$
$B_n(i), 3 \leq i \leq n-2$	$M_{\mathfrak{q}}\left[\mathbb{C}_{-(n-i-\frac{1}{2})\lambda_i}\right]$	$M_{\mathfrak{q}}[\mathbb{C}_{-\lambda_i}]$
$B_n(n-1)$	$M_{\mathfrak{q}}\left[\mathbb{C}_{-\frac{1}{2}\lambda_{n-1}}\right]$	?
$B_n(n)$	$M_{\mathfrak{q}}[\mathbb{C}_{\lambda_n}]$	–
$C_n(i), 2 \leq i \leq n-1$	?	$M_{\mathfrak{q}}[\mathbb{C}_{\lambda_i}]$
$D_n(i), 3 \leq i \leq n-3$	$M_{\mathfrak{q}}[\mathbb{C}_{-(n-i-1)\lambda_i}]$	$M_{\mathfrak{q}}[\mathbb{C}_{-\lambda_i}]$
$E_6(3)$	$M_{\mathfrak{q}}[\mathbb{C}_{-\lambda_3}]$	$M_{\mathfrak{q}}[\mathbb{C}_{-2\lambda_3}]$
$E_6(5)$	$M_{\mathfrak{q}}[\mathbb{C}_{-\lambda_3}]$	$M_{\mathfrak{q}}[\mathbb{C}_{-2\lambda_3}]$
$E_7(2)$	$M_{\mathfrak{q}}[\mathbb{C}_{-2\lambda_2}]$	–
$E_7(6)$	$M_{\mathfrak{q}}[\mathbb{C}_{-\lambda_6}]$	$M_{\mathfrak{q}}[\mathbb{C}_{-3\lambda_6}]$
$E_8(1)$	$M_{\mathfrak{q}}[\mathbb{C}_{-3\lambda_1}]$	–
$F_4(4)$	$M_{\mathfrak{q}}[\mathbb{C}_{\lambda_4}]$	–

CHAPTER 8

The Homomorphisms between Generalized Verma Modules induced by the Ω_1 System and Ω_2 Systems

By [2], attached to the $\Omega_k|_{W^*}$ system conformally invariant on the line bundle $\mathcal{L}(s_0\lambda_\mathfrak{q})$, there is a non-zero $\mathcal{U}(\mathfrak{g})$-homomorphism $\varphi_{\Omega_k} : \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{q})} F(\Omega_k|_{W^*}) \to \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{q})} \mathbb{C}_{-s_0\lambda_\mathfrak{q}}$, where $F(\Omega_k|_{W^*})$ is a finite dimensional simple $\mathfrak{l}$-submodule occurring in $\big(\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{q})} \mathbb{C}_{-s_0\lambda_\mathfrak{q}}\big)^\mathfrak{n}$ (see Section 2.7). The aim of this chapter is to determine whether or not the homomorphisms φ_{Ω_k} are standard for $k = 1$, 2, when $\mathfrak{q}$ is a maximal two-step nilpotent parabolic subalgebra of non-Heisenberg type.

It is important to recall that there are irreducible constituents W of $\mathfrak{g}(-2+k) \otimes \mathfrak{g}(2)$ with $k = 1, 2$ so that the systems under consideration consist of $\dim_\mathbb{C}(W)$ differential operators. As in Section 2.7, for a basis $\{Y_1^*, \ldots, Y_m^*\}$ of W^*, we write $\Omega_k(Y_1^*), \ldots, \Omega_k(Y_m^*)$ for the system of operators. Each $\Omega_k(Y_j^*)$ acts on the space of smooth sections $\Gamma(G_0/Q_0, \mathcal{L}(-s\lambda_\mathfrak{q}))$ for the line bundle $\mathcal{L}(-s\lambda_\mathfrak{q})$ by right differentiation. Indeed, if $\sigma : \mathrm{Sym}(\bar{\mathfrak{n}}) \to \mathcal{U}(\bar{\mathfrak{n}})$ is the symmetrization map then there are elements $\omega_k(Y_j^*) \in \sigma(\mathrm{Sym}^k(\bar{\mathfrak{n}}))$ so that $\Omega_k(Y_j^*) \bullet f = R(\omega_k(Y_j^*)) \bullet f$ for $f \in \Gamma(G_0/Q_0, \mathcal{L}(-s\lambda_\mathfrak{q}))$. If W^* has highest weight ν and if the system $\Omega_k|_{W^*} = \Omega_k(Y_1^*), \ldots, \Omega_k(Y_m^*)$ of differential operators is conformally invariant on the line bundle $\mathcal{L}(s_0\lambda_\mathfrak{q})$ then

$$F(\Omega_k|_{W^*}) = \mathrm{span}_\mathbb{C}\{\omega_k(Y_j^*) \otimes 1 \mid j = 1, \ldots, m\} \tag{8.0.1}$$

is the simple $\mathfrak{l}$-submodule of $M_\mathfrak{q}[\mathbb{C}_{-s_0\lambda_\mathfrak{q}}]^\mathfrak{n} = \big(\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{q})} \mathbb{C}_{-s_0\lambda_\mathfrak{q}}\big)^\mathfrak{n}$ with highest weight $\nu - s_0\lambda_\mathfrak{q}$. Then the inclusion map $\iota \in \mathrm{Hom}_L\big(F(\Omega_k|_{W^*}), M_\mathfrak{q}[\mathbb{C}_{-s_0\lambda_\mathfrak{q}}]\big)$ induces a non-zero $\mathcal{U}(\mathfrak{g})$-homomorphism $\varphi_{\Omega_k} \in \operatorname{Hom}_{\mathcal{U}(\mathfrak{g}),L}\big(M_{\mathfrak{q}}[F(\Omega_k|_{W^*})], M_{\mathfrak{q}}[\mathbb{C}_{-s_0\lambda_{\mathfrak{q}}}]\big)$ between generalized Verma modules, that is given by

$$\begin{aligned} M_{\mathfrak{q}}[F(\Omega_k|_{W^*})] &\stackrel{\varphi_{\Omega_k}}{\to} M_{\mathfrak{q}}[\mathbb{C}_{-s_0\lambda_{\mathfrak{q}}}] \\ u \otimes \big(\omega_k(Y^*) \otimes 1\big) &\mapsto u \cdot \iota\big(\omega_k(Y^*) \otimes 1\big). \end{aligned} \tag{8.0.2}$$

We want to determine whether or not the maps φ_{Ω_k} are standard. To do so, it is convenient to parametrize generalized Verma modules by their infinitesimal characters. Therefore, for the rest of this chapter, we write

$$M_{\mathfrak{q}}[F(\Omega_k|_{W^*})] = M_{\mathfrak{q}}(\nu - s_0\lambda_{\mathfrak{q}} + \rho)$$

and

$$M_{\mathfrak{q}}[\mathbb{C}_{-s_0\lambda_{\mathfrak{q}}}] = M_{\mathfrak{q}}(-s_0\lambda_{\mathfrak{q}} + \rho),$$

where ρ is half the sum of the positive roots. Then (8.0.2) is expressed by

$$\begin{aligned} M_{\mathfrak{q}}(\nu - s_0\lambda_{\mathfrak{q}} + \rho) &\stackrel{\varphi_{\Omega_k}}{\to} M_{\mathfrak{q}}(-s_0\lambda_{\mathfrak{q}} + \rho) \\ u \otimes v &\mapsto u \cdot \iota(v) \end{aligned} \tag{8.0.3}$$

with $v = \omega_k(Y^*) \otimes 1$.

8.1 The Standard Map between Generalized Verma Modules

In this section we first recall the notion of the standard maps between generalized Verma modules. We then show when the standard map of the generalized Verma modules in (8.0.3) is zero. This is done in Proposition 8.1.6.

For $\eta \in \mathfrak{h}^*$, let $M(\eta)$ be the (ordinary) Verma module with infinitesimal character η. Write

$$\mathbf{P}_{\mathfrak{l}}^+ = \{\zeta \in \mathfrak{h}^* \mid \langle \zeta, \alpha^\vee \rangle \in 1 + \mathbb{Z}_{\geq 0} \text{ for all } \alpha \in \Pi(\mathfrak{l})\}.$$

For $\eta, \zeta \in \mathbf{P}_{\mathfrak{l}}^{+}$, suppose that there exists a non-zero $\mathcal{U}(\mathfrak{g})$-homomorphism $\varphi : M(\eta) \to M(\zeta)$. If $K(\eta)$ is the kernel of the canonical projection $\mathrm{pr}_{\eta} : M(\eta) \to M_{\mathfrak{q}}(\eta)$ then, by Proposition 3.1 in [22], it follows that $\varphi(K(\eta)) \subset K(\zeta)$. Thus the map φ induces a $\mathcal{U}(\mathfrak{g})$-homomorphism $\varphi_{std} : M_{\mathfrak{q}}(\eta) \to M_{\mathfrak{q}}(\zeta)$ so that the diagram

$$\begin{array}{ccc} M(\eta) & \xrightarrow{\varphi} & M(\zeta) \\ {\scriptstyle \mathrm{pr}_{\eta}}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_{\zeta}} \\ M_{\mathfrak{q}}(\eta) & \xrightarrow{\varphi_{std}} & M_{\mathfrak{q}}(\zeta) \end{array}$$

commutes. The map φ_{std} is called the **standard map** from $M_{\mathfrak{q}}(\eta)$ to $M_{\mathfrak{q}}(\zeta)$. These maps were first studied by Lepowsky [22]. Of course φ_{std} could be zero. Note that since $\dim \mathrm{Hom}_{\mathcal{U}(\mathfrak{g})}(M(\eta), M(\zeta)) \leq 1$, the standard maps φ_{std} are uniquely determined up to scalar multiples. Not every homomorphism between generalized Verma modules is standard and the classification of all homomorphisms between generalized Verma modules is an open problem.

If $\nu = -(1 - s_0)\alpha_{\mathfrak{q}}$ with $1 - s_0 \in 1 + \mathbb{Z}_{\geq 0}$ then one can show that the standard map φ_{std} from $M_{\mathfrak{q}}(-(1-s_0)\alpha_{\mathfrak{q}} - s_0\lambda_{\mathfrak{q}} + \rho)$ to $M_{\mathfrak{q}}(-s_0\lambda_{\mathfrak{q}} + \rho)$ is non-zero by computing $\varphi_{std}(1 \otimes v^{+})$, where $1 \otimes v^{+}$ is a highest weight vector of $M_{\mathfrak{q}}(-(1-s_0)\alpha_{\mathfrak{q}} - s_0\lambda_{\mathfrak{q}} + \rho)$ for weight $-(1 - s_0)\alpha_{\mathfrak{q}} - s_0\lambda_{\mathfrak{q}}$. To prove it, we will use the following well-known result. (See for example [10, Proposition 1.4].)

Proposition 8.1.1 *Given $\lambda \in \mathfrak{h}^*$ and $\alpha \in \Pi$, suppose that $n = \langle \lambda + \rho, \alpha^{\vee} \rangle \in 1 + \mathbb{Z}_{\geq 0}$. If $1 \otimes v^{+}$ is a highest weight vector of weight λ in $M(\lambda + \rho)$ then $X_{-\alpha}^{n} \cdot (1 \otimes v^{+})$ is a highest weight vector of weight $-n\alpha + \lambda$.*

Observe that, since $M_{\mathfrak{q}}(\nu - s_0\lambda_{\mathfrak{q}} + \rho) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{q})} F(\Omega_k)$ and $M_{\mathfrak{q}}(-s_0\lambda_{\mathfrak{q}} + \rho) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{q})} \mathbb{C}_{-s_0\lambda_{\mathfrak{q}}}$, if v_h and $1_{-s_0\lambda_{\mathfrak{q}}}$ are highest weight vectors for $F(\Omega_k)$ and $\mathbb{C}_{-s_0\lambda_{\mathfrak{q}}}$, respectively, then $1 \otimes v_h$ and $1 \otimes 1_{-s_0\lambda_{\mathfrak{q}}}$ are highest weight vectors for $M_{\mathfrak{q}}(\nu - s_0\lambda_{\mathfrak{q}} + \rho)$ of weight $\nu - s_0\lambda_{\mathfrak{q}}$ and for $M_{\mathfrak{q}}(-s_0\lambda_{\mathfrak{q}} + \rho)$ of weight $-s_0\lambda_{\mathfrak{q}}$, respectively.

Proposition 8.1.2 *If $1 - s_0 \in 1 + \mathbb{Z}_{\geq 0}$ then the standard map φ_{std} from $M_{\mathfrak{q}}(-(1 - s_0)\alpha_{\mathfrak{q}} - s_0\lambda_{\mathfrak{q}} + \rho)$ to $M_{\mathfrak{q}}(-s_0\lambda_{\mathfrak{q}} + \rho)$ maps*

$$1 \otimes v_h \mapsto cX_{-\alpha_{\mathfrak{q}}}^{1-s_0} \otimes 1_{-s_0\lambda_{\mathfrak{q}}} \neq 0$$

for some non-zero constant c. In particular, the standard map φ_{std} is non-zero.

Proof. Write $n = 1 - s_0$ and denote by $1 \otimes 1_{-n\alpha_{\mathfrak{q}} - s_0\lambda_{\mathfrak{q}}}$ a highest weight vector for $M(-n\alpha_{\mathfrak{q}} - s_0\lambda_{\mathfrak{q}} + \rho)$ of weight $-n\alpha_{\mathfrak{q}} - s_0\lambda_{\mathfrak{q}}$. Observe that since $\langle \lambda_{\mathfrak{q}}, \alpha_0^\vee \rangle = \langle \rho, \alpha_0^\vee \rangle = 1$, we have $n = 1 - s_0 = \langle -s_0\lambda_{\mathfrak{q}} + \rho, \alpha_0^\vee \rangle$. Hence $-n\alpha_{\mathfrak{q}} - s_0\lambda_{\mathfrak{q}} + \rho = s_{\alpha_{\mathfrak{q}}}(-s_0\lambda_{\mathfrak{q}} + \rho)$. By hypothesis, we have $n = 1 - s_0 \in 1 + \mathbb{Z}_{\geq 0}$. It then follows from Proposition 8.1.1 that the map $\varphi : M_{\mathfrak{q}}(-n\alpha_{\mathfrak{q}} - s_0\lambda_{\mathfrak{q}} + \rho) \to M_{\mathfrak{q}}(-s_0\lambda_{\mathfrak{q}} + \rho)$ is given by

$$\varphi(1 \otimes 1_{-n\alpha_{\mathfrak{q}} - s_0\lambda_{\mathfrak{q}}}) = cX_{-\alpha_{\mathfrak{q}}}^n \otimes 1$$

with $c \neq 0$. As $\alpha_{\mathfrak{q}} \in \Pi \backslash \Pi(\mathfrak{l})$, if $\mathrm{pr}_{-s_0\lambda_{\mathfrak{q}} + \rho} : M(-s_0\lambda_{\mathfrak{q}} + \rho) \to M_{\mathfrak{q}}(-s_0\lambda_{\mathfrak{q}} + \rho)$ is the canonical projection then $\mathrm{pr}_{-s_0\lambda_{\mathfrak{q}} + \rho}(X_{-\alpha_{\mathfrak{q}}}^n \otimes 1) \neq 0$. Then the universal property of $M_{\mathfrak{q}}(-n\alpha_{\mathfrak{q}} - s_0\lambda_{\mathfrak{q}} + \rho)$ in the relative category $\mathcal{O}^{\mathfrak{q}}$ (see for example Section 9.4 in [10]) guarantees that $\mathrm{pr}_{-s_0\lambda_{\mathfrak{q}} + \rho} \circ \varphi$ factors through a non-zero map $\varphi_{std} : M_{\mathfrak{q}}(-n\alpha_{\mathfrak{q}} - s_0\lambda_{\mathfrak{q}} + \rho) \to M_{\mathfrak{q}}(-s_0\lambda_{\mathfrak{q}} + \rho)$. ■

In order to determine if φ_{std} is non-zero in a more general setting, we will use the following theorem by Lepowski.

Theorem 8.1.3 [22, Proposition 3.3] *Let $\eta, \zeta \in \mathbf{P}_{\mathfrak{l}}^+$, and assume that $M(\eta) \subset M(\zeta)$. Then the standard map φ_{std} from $M_{\mathfrak{q}}(\eta)$ to $M_{\mathfrak{q}}(\zeta)$ is zero if and only if $M(\eta) \subset M(s_\alpha\zeta)$ for some $\alpha \in \Pi(\mathfrak{l})$.*

Theorem 8.1.3 reduces the existence problem of the non-zero standard map φ_{std} between generalized Verma modules to that of the non-zero map between appropriate

Verma modules. It is very well-known when a non-zero $\mathcal{U}(\mathfrak{g})$-homomorphism between Verma modules exists. To describe the condition efficiently, we first introduce the definition of a *link* of two weights.

Definition 8.1.4 (Bernstein-Gelfand-Gelfand) *Let* $\lambda, \delta \in \mathfrak{h}^*$ *and* $\beta_1, \ldots, \beta_t \in \Delta^+$. *Set* $\delta_0 = \delta$ *and* $\delta_i = s_{\beta_i} \cdots s_{\beta_1}\delta$ *for* $1 \leq i \leq t$. *We say that the sequence* $(\beta_1, \ldots, \beta_t)$ ***links*** δ *to* λ *if*

(1) $\delta_t = \lambda$ *and*

(2) $\langle \delta_{i-1}, \beta_i^\vee \rangle \in \mathbb{Z}_{\geq 0}$ *for* $1 \leq i \leq t$.

Theorem 8.1.5 (BGG-Verma) *Let* $\lambda, \delta \in \mathfrak{h}^*$. *The following conditions are equivalent:*

1. $M(\lambda) \subset M(\delta)$
2. $L(\lambda)$ *is a composition factor of* $M(\delta)$
3. *There exists a sequence* $(\beta_1, \ldots, \beta_t)$ *with* $\beta_i \in \Delta^+$ *that links* δ *to* λ,

where $L(\lambda)$ *is the unique irreducible quotient of* $M(\lambda)$.

It is important to observe that if there is a non-zero $\mathcal{U}(\mathfrak{g})$-homomorphism (not necessarily standard) from $M_{\mathfrak{q}}(\eta)$ to $M_{\mathfrak{q}}(\zeta)$ then $M(\eta) \subset M(\zeta)$. Indeed, if there exists a non-zero $\mathcal{U}(\mathfrak{g})$-homomorphism $f : M_{\mathfrak{q}}(\eta) \to M_{\mathfrak{q}}(\zeta)$ then $M_{\mathfrak{q}}(\eta)/\ker(f)$ is embedded into $M_{\mathfrak{q}}(\zeta)$. Observe that, as $L(\eta)$ is a unique irreducible quotient of $M(\eta)$, it is also a unique irreducible quotient of $M_{\mathfrak{q}}(\eta)$ and so of $M_{\mathfrak{q}}(\eta)/\ker(f)$. In particular, via the embedding $M_{\mathfrak{q}}(\eta)/\ker(f) \hookrightarrow M_{\mathfrak{q}}(\zeta)$, the irreducible quotient $L(\eta)$ is a composition factor of $M_{\mathfrak{q}}(\zeta)$. Since the composition factors of $M_{\mathfrak{q}}(\zeta)$ are those of $M(\zeta)$, this shows that $L(\zeta)$ is a composition factor of $M(\zeta)$. Now it follows from Theorem 8.1.5 that $M(\eta) \subset M(\zeta)$. Taking into account Theorem 8.1.5 and this observation, in our setting, Theorem 8.1.3 is equivalent to the following proposition.

Proposition 8.1.6 *Let $M_{\mathfrak{q}}(\nu - s_0\lambda_{\mathfrak{q}} + \rho)$ and $M_{\mathfrak{q}}(-s_0\lambda_{\mathfrak{q}} + \rho)$ be the generalized Verma modules in (8.0.3). Then the standard map from $M_{\mathfrak{q}}(\nu - s_0\lambda_{\mathfrak{q}} + \rho)$ to $M_{\mathfrak{q}}(-s_0\lambda_{\mathfrak{q}} + \rho)$ is zero if and only if there exists $\alpha \in \Pi(\mathfrak{l})$ so that $-\alpha - s_0\lambda_{\mathfrak{q}} + \rho$ is linked to $\nu - s_0\lambda_{\mathfrak{q}} + \rho$.*

Proof. First observe that since there exists a non-zero $\mathcal{U}(\mathfrak{g})$-homomorphism φ_{Ω_k} from $M_{\mathfrak{q}}(\nu - s_0\lambda_{\mathfrak{q}} + \rho)$ to $M_{\mathfrak{q}}(-s_0\lambda_{\mathfrak{q}} + \rho)$, by the observation right above this proposition, we have $M(\nu - s_0\lambda_{\mathfrak{q}} + \rho) \subset M(-s_0\lambda_{\mathfrak{q}} + \rho)$. Therefore, by Theorem 8.1.3 and Theorem 8.1.5, the standard map from $M_{\mathfrak{q}}(\nu - s_0\lambda_{\mathfrak{q}} + \rho)$ to $M_{\mathfrak{q}}(-s_0\lambda_{\mathfrak{q}} + \rho)$ is zero if and only if there exists $\alpha \in \Pi(\mathfrak{l})$ so that $s_\alpha(-s_0\lambda_{\mathfrak{q}} + \rho)$ is linked to $\nu - s_0\lambda_{\mathfrak{q}} + \rho$. As $\langle \lambda_{\mathfrak{q}}, \alpha^\vee \rangle = 0$ and $\langle \rho, \alpha^\vee \rangle = 1$ for $\alpha \in \Pi(\mathfrak{l})$, we have $s_\alpha(-s_0\lambda_{\mathfrak{q}} + \rho) = -\alpha - s_0\lambda_{\mathfrak{q}} + \rho$. Now this proposition follows. ∎

8.2 The Homomorphism φ_{Ω_1} induced by the Ω_1 System

In this section we show that the map φ_{Ω_1} that is induced by the Ω_1 system is standard when $\mathfrak{q}$ is a maximal two-step nilpotent parabolic subalgebra of non-Heisenberg type. We keep the notation from Section 8.1.

The Ω_1 system is $R(X_{-\alpha_1}), \ldots, R(X_{-\alpha_m})$ for $\Delta(\mathfrak{g}(1)) = \{\alpha_1, \ldots, \alpha_m\}$. This system is conformally invariant on the line bundle $\mathcal{L}(s_0\lambda_{\mathfrak{q}})$ with $s_0 = 0$. It yields a finite dimensional simple $\mathfrak{l}$-submodule $F(\Omega_1)$ in $\left(\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{q})} \mathbb{C}_0\right)^{\mathfrak{n}} = M_{\mathfrak{q}}(\rho)^{\mathfrak{n}}$. Since $-\alpha_{\mathfrak{q}}$ is the highest weight of $\mathfrak{g}(-1) = W^*$, the simple $\mathfrak{l}$-module $F(\Omega_1)$ has highest weight $\nu - s_0\lambda_{\mathfrak{q}} = -\alpha_{\mathfrak{q}}$. Therefore the inclusion map $F(\Omega_1) \hookrightarrow M_{\mathfrak{q}}(\rho)$ induces a non-zero $\mathcal{U}(\mathfrak{g})$-homomorphism $\varphi_{\Omega_1} : M_{\mathfrak{q}}(-\alpha_{\mathfrak{q}} + \rho) \to M_{\mathfrak{q}}(\rho)$.

Proposition 8.2.1 *If $\mathfrak{q}$ is a maximal two-step nilpotent parabolic subalgebra of non-Heisenberg type then the standard map φ_{std} from $M_{\mathfrak{q}}(-\alpha_{\mathfrak{q}} + \rho)$ to $M_{\mathfrak{q}}(\rho)$ is non-zero. Moreover, there exists $c \neq 0$ so that $\varphi_{std}(1 \otimes v_h) = cX_{-\alpha_{\mathfrak{q}}} \otimes 1_0$.*

Proof. This follows from Proposition 8.1.2 with $s_0 = 0$. ∎

Theorem 8.2.2 *If $\mathfrak{q}$ is a maximal two-step nilpotent parabolic subalgebra of non-Heisenberg type then the map φ_{Ω_1} is standard.*

Proof. Since $\varphi_{\Omega_1}(1 \otimes v_h) = 1 \cdot v_h = v_h$, to prove that φ_{Ω_1} is standard, by Proposition 8.2.1, it suffices to show that $v_h = cX_{-\alpha_{\mathfrak{q}}} \otimes 1_0$ with some non-zero constant c. To do so, as v_h is a highest weight vector for $F(\Omega_1)$, we show that $X_{-\alpha_{\mathfrak{q}}} \otimes 1_0$ is a highest weight vector for $F(\Omega_1)$. Since the Ω_1 system is $R(X_{-\alpha_1}), \ldots, R(X_{-\alpha_m})$ for $\Delta(\mathfrak{g}(1)) = \{\alpha_1, \ldots, \alpha_m\}$, it is clear that the elements $\omega_1(X_{-\alpha_j}) \in \sigma(\mathrm{Sym}^1(\bar{\mathfrak{n}})) = \bar{\mathfrak{n}}$ that correspond to $R(X_{-\alpha_j})$ under R are $\omega_1(X_{-\alpha_j}) = X_{-\alpha_j}$. Then it follows from (8.0.1) that

$$F(\Omega_1) = \mathrm{span}_{\mathbb{C}}\{X_{-\alpha} \otimes 1_0 \mid \alpha \in \Delta(\mathfrak{g}(1))\}.$$

Therefore $X_{-\alpha_{\mathfrak{q}}} \otimes 1_0$ is a highest weight vector for $F(\Omega_1)$. ■

8.3 The Homomorphisms φ_{Ω_2} induced by the Ω_2 Systems

In this section, by using the results in Table 7.1, we determine whether or not the homomorphisms φ_{Ω_2} that are induced by the Ω_2 systems are standard, when $\mathfrak{q}$ is a maximal two-step nilpotent parabolic subalgebra of non-Heisenberg type, listed in (3.3.2) or (3.3.3). The results are summarized in Table 8.1 at the end of this section.

Recall from Definition 6.2.12 that we classify the special constituents $V(\mu + \epsilon)$ as type 1a, type 1b, type 2, and type 3. If we observe Table 6.5 and Table 7.1 then we see that each $\Omega_2|_{V(\mu+\epsilon)^*}$ system satisfies exactly one of the following:

1. The special constituent $V(\mu + \epsilon)$ is of type 2.

2. The special value s_0 is a positive integer.

3. The parabolic $\mathfrak{q}$ is of type $B_n(i)$ for $3 \leq i \leq n-1$ and $V(\mu + \epsilon) = V(\mu + \epsilon_\gamma)$.

We shall consider these three cases separately.

8.3.1 The Type 2 Case

We first study the homomorphism attached to the special constituent $V(\mu+\epsilon)$ of type 2. By Table 6.5, we consider the following three cases:

1. $V(\mu+\epsilon_\gamma)$ for $B_n(n)$,
2. $V(\mu+\epsilon_{n\gamma})$ for $C_n(i)$ $(2 \leq i \leq n-1)$, and
3. $V(\mu+\epsilon_\gamma)$ for $F_4(4)$.

If $V(\mu+\epsilon)$ is a type 2 special constituent then, by definition, we have $V(\mu+\epsilon) = V(2\mu)$. Thus, $V(\mu+\epsilon)^* = V(2\mu)^* = V(-2\alpha_{\mathfrak{q}})$. Therefore ν in (8.0.3) is $\nu = -2\alpha_{\mathfrak{q}}$. Moreover, by Theorem 7.3.6, the $\Omega_2|_{V(2\mu)^*}$ system is conformally invariant on the line bundle $\mathcal{L}(-\lambda_{\mathfrak{q}})$. Thus $s_0 = -1$. Therefore we have $\varphi_{\Omega_2} : M_{\mathfrak{q}}(-2\alpha_{\mathfrak{q}} + \lambda_{\mathfrak{q}} + \rho) \to M_{\mathfrak{q}}(\lambda_{\mathfrak{q}} + \rho)$.

Proposition 8.3.1 *If $\mathfrak{q}$ is the maximal two-step nilpotent parabolic subalgebra of type $B_n(n)$, $C_n(i)$ for $2 \leq i \leq n-1$, or $F_4(4)$ then the standard map φ_{std} from $M_{\mathfrak{q}}(-2\alpha_{\mathfrak{q}} + \lambda_{\mathfrak{q}} + \rho)$ to $M_{\mathfrak{q}}(\lambda_{\mathfrak{q}} + \rho)$ is non-zero. Moreover, there exists $c \neq 0$ so that $\varphi_{std}(1 \otimes v_h) = cX^2_{-\alpha_{\mathfrak{q}}} \otimes 1_{\lambda_{\mathfrak{q}}}$.*

Proof. This follows from Proposition 8.1.2 with $s_0 = -1$. ∎

Observe that if Y_l^* is the lowest weight vector for $V(2\mu)^*$ defined in (7.2.1) then, by (7.3.12) and (7.3.13), the differential operator $\Omega_2(Y_l^*)$ is

$$\Omega_2(Y_l^*) = \frac{1}{2}C(\mu,\mu)R(X_{-\mu})^2,$$

where $C(\mu,\mu)$ is the constant defined in (7.3.5). Therefore, the element $\omega_2(Y_l^*)$ in $\sigma(\mathrm{Sym}^2(\bar{\mathfrak{n}}))$ that corresponds to $\Omega_2(Y_l^*)$ under R is

$$\omega_2(Y_l^*) = \frac{1}{2}C(\mu,\mu)X^2_{-\mu}. \tag{8.3.2}$$

In particular, the simple $\mathfrak{l}$-submodule $F(\Omega_2|_{V(2\mu)^*})$ of $M_{\mathfrak{q}}(\lambda_{\mathfrak{q}}+\rho)^{\mathfrak{n}} = \big(\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{q})} \mathbb{C}_{\lambda_{\mathfrak{q}}}\big)^{\mathfrak{n}}$ has lowest weight $X^2_{-\mu} \otimes 1_{\lambda_{\mathfrak{q}}}$ with μ the highest weight for $\mathfrak{g}(1)$.

Theorem 8.3.3 *Let $\mathfrak{q}$ be a maximal two-step nilpotent parabolic subalgebra of non-Heisenberg type, listed in (3.3.2) or (3.3.3). If the special constituent $V(\mu+\epsilon)$ is of type 2 then the map φ_{Ω_2} is standard.*

Proof. In order to prove that φ_{Ω_2} is standard, by Proposition 8.3.1, it suffices to show that $X^2_{-\alpha_{\mathfrak{q}}} \otimes 1_{\lambda_{\mathfrak{q}}}$ is a highest weight vector for $F(\Omega_2|_{V(2\mu)^*})$. Since $F(\Omega_2|_{V(2\mu)^*})$ has highest weight $\nu - s_0\lambda_{\mathfrak{q}} = -2\alpha_{\mathfrak{q}} + \lambda_{\mathfrak{q}}$, it is enough to show that $X^2_{-\alpha_{\mathfrak{q}}} \otimes 1_{\lambda_{\mathfrak{q}}}$ is in $F(\Omega_2|_{V(2\mu)^*})$. We know that a lowest weight vector for $F(\Omega_2|_{V(2\mu)^*})$ is $X^2_{-\mu} \otimes 1_{\lambda_{\mathfrak{q}}}$. This will allow us to show that $X^2_{-\alpha_{\mathfrak{q}}} \otimes 1_{\lambda_{\mathfrak{q}}}$ is in $F(\Omega_2|_{V(2\mu)^*})$. We do so in a case-by-case manner. Recall that we have to consider the following three cases:

1. $V(\mu+\epsilon_\gamma)$ for $B_n(n)$,
2. $V(\mu+\epsilon_{n\gamma})$ for $C_n(i)$ $(2 \leq i \leq n-1)$, and
3. $V(\mu+\epsilon_\gamma)$ for $F_4(4)$.

We start with the case $V(\mu+\epsilon_\gamma)$ for $B_n(n)$. In the standard realization of the roots we have $\mu = \varepsilon_1$, $\alpha_{\mathfrak{q}} = \alpha_n = \varepsilon_n$, and

$$\Delta^+(\mathfrak{l}) = \{\varepsilon_j - \varepsilon_k \mid 1 \leq j < k \leq n\}$$

(see Appendix C). Thus,

$$X^2_{-\mu} \otimes 1_{\lambda_{\mathfrak{q}}} = X^2_{-\varepsilon_1} \otimes 1_{\lambda_n} \quad \text{and} \quad X^2_{-\alpha_{\mathfrak{q}}} \otimes 1_{\lambda_{\mathfrak{q}}} = X^2_{-\varepsilon_n} \otimes 1_{\lambda_n}.$$

A direct computation shows that

$$X^2_{\varepsilon_1-\varepsilon_n} \cdot (X^2_{-\varepsilon_1} \otimes 1_{\lambda_n}) = 2N^2_{\varepsilon_1-\varepsilon_n,-\varepsilon_1} X^2_{-\varepsilon_n} \otimes 1_{\lambda_n}.$$

Therefore, $X^2_{-\alpha_{\mathfrak{q}}} \otimes 1_{\lambda_{\mathfrak{q}}} = X^2_{-\varepsilon_n} \otimes 1_{\lambda_n}$ is in $F(\Omega_2|_{V(2\mu)^*})$ since $X_{\varepsilon_1-\varepsilon_n} \in \mathfrak{l}$.

Next, we handle the case that $V(\mu+\epsilon_{n\gamma})$ for $C_n(i)$ for $2 \leq i \leq n-1$. In the standard realization of the roots we have $\mu = \varepsilon_1 + \varepsilon_{i+1}$, $\alpha_{\mathfrak{q}} = \alpha_i = \varepsilon_i - \varepsilon_{i+1}$, and $\Delta^+(\mathfrak{l}) = \Delta^+(\mathfrak{l}_\gamma) \cup \Delta^+(\mathfrak{l}_{n\gamma})$ with

$$\Delta^+(\mathfrak{l}_\gamma) = \{\varepsilon_j - \varepsilon_k \mid 1 \leq j < k \leq i\}$$

and

$$\Delta^+(\mathfrak{l}_{n\gamma}) = \{\varepsilon_j \pm \varepsilon_k \mid i+1 \leq j < k \leq n\} \cup \{2\varepsilon_j \mid i+1 \leq j \leq n\}$$

(see Appendix C). Thus,

$$X^2_{-\mu} \otimes 1_{\lambda_\mathfrak{q}} = X^2_{-(\varepsilon_1+\varepsilon_{i+1})} \otimes 1_{\lambda_i} \text{ and } X^2_{-\alpha_\mathfrak{q}} \otimes 1_{\lambda_\mathfrak{q}} = X^2_{-(\varepsilon_i-\varepsilon_{i+1})} \otimes 1_{\lambda_i}.$$

A direct computation shows that

$$X^2_{\varepsilon_1-\varepsilon_i} X^2_{2\varepsilon_{i+1}} \cdot (X^2_{-(\varepsilon_1+\varepsilon_{i+1})} \otimes 1_{\lambda_i}) = 4N^2_{2\varepsilon_{i+1},-(\varepsilon_1+\varepsilon_{i+1})} N^2_{\varepsilon_1-\varepsilon_i,-(\varepsilon_1-\varepsilon_{i+1})} X^2_{-(\varepsilon_i-\varepsilon_{i+1})} \otimes 1_{\lambda_i}.$$

Therefore, $X^2_{-\alpha_\mathfrak{q}} \otimes 1_{\lambda_\mathfrak{q}} = X^2_{-(\varepsilon_i-\varepsilon_{i+1})} \otimes 1_{\lambda_i}$ is in $F(\Omega_2|_{V(2\mu)^*})$.

For the last case that $V(\mu + \epsilon_\gamma)$ for $F_4(4)$, observe that we have $\mu = \alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4$ and $\alpha_\mathfrak{q} = \alpha_4$ (see Appendix C). Thus,

$$X^2_{-\mu} \otimes 1_{\lambda_\mathfrak{q}} = X^2_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)} \otimes 1_{\lambda_4} \text{ and } X^2_{-\alpha_\mathfrak{q}} \otimes 1_{\lambda_\mathfrak{q}} = X^2_{-\alpha_4} \otimes 1_{\lambda_4}.$$

The roots in $\Delta^+(\mathfrak{l})$ are the positive roots in which α_4 has multiplicity zero. Therefore α_3 and $\alpha_1 + 2\alpha_2 + 2\alpha_3$ are in $\Delta^+(\mathfrak{l})$. A direct computation shows that

$$\begin{aligned} &X^2_{\alpha_3} X^2_{\alpha_1+2\alpha_2+2\alpha_3} \cdot (X^2_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)} \otimes 1_{\lambda_4}) \\ &= 4N^2_{\alpha_1+2\alpha_2+2\alpha_3,-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)} N^2_{\alpha_3,-(\alpha_3+\alpha_4)} X^2_{-\alpha_4} \otimes 1_{\lambda_4}. \end{aligned}$$

Therefore, $X^2_{-\alpha_\mathfrak{q}} \otimes 1_{\lambda_\mathfrak{q}} = X^2_{-\alpha_4} \otimes 1_{\lambda_4}$ is in $F(\Omega_2|_{V(2\mu)^*})$. ■

8.3.2 The Positive Integer Special Value Case

Next we handle the case that the special value s_0 is a positive integer.

Theorem 8.3.4 *Let $\mathfrak{q}$ be a maximal two-step nilpotent parabolic subalgebra of non-Heisenberg type, listed in (3.3.2) or (3.3.3). If the special value s_0 is a positive integer then the standard map from $M_\mathfrak{q}(\nu - s_0\lambda_\mathfrak{q} + \rho)$ to $M_\mathfrak{q}(-s_0\lambda_\mathfrak{q} + \rho)$ is zero. Consequently, the map φ_{Ω_2} is non-standard.*

Proof. By Proposition 8.1.6, to show that the standard map is zero, it suffices to show that there exists $\alpha \in \Pi(\mathfrak{l})$ so that $-\alpha - s_0\lambda_{\mathfrak{q}} + \rho$ is linked to $\nu - s_0\lambda_{\mathfrak{q}} + \rho$. We achieve it by a case-by-case observation. By Table 7.1, the following are the cases under consideration:

1. $V(\mu + \epsilon_{n\gamma})$ for $B_n(i)$ $(3 \leq i \leq n-2)$

2. $V(\mu + \epsilon_{\gamma})$ and $V(\mu + \epsilon_{n\gamma})$ for $D_n(i)$ $(3 \leq i \leq n-3)$

3. $V(\mu + \epsilon_{\gamma})$ and $V(\mu + \epsilon_{n\gamma})$ for $E_6(3)$

4. $V(\mu + \epsilon_{\gamma})$ and $V(\mu + \epsilon_{n\gamma})$ for $E_6(5)$

5. $V(\mu + \epsilon_{\gamma})$ for $E_7(2)$

6. $V(\mu + \epsilon_{\gamma})$ and $V(\mu + \epsilon_{n\gamma})$ for $E_7(6)$

7. $V(\mu + \epsilon_{\gamma})$ for $E_8(1)$

Our strategy is to first observe that the highest weight ν for $V(\mu+\epsilon)^*$ is of the form

$$\nu = -2\beta - \alpha' - \alpha''$$

for some $\beta \in \Delta(\mathfrak{g}(1))$ and $\alpha', \alpha'' \in \Pi(\mathfrak{l})$. We then show that the sequence (α', β) links $-\alpha'' - s_0\lambda_{\mathfrak{q}} + \rho$ to $(-2\beta - \alpha' - \alpha'') - s_0\lambda_{\mathfrak{q}} + \rho$. Since the argument that shows that (α', β) links $-\alpha'' - s_0\lambda_{\mathfrak{q}} + \rho$ to $(-2\beta - \alpha' - \alpha'') - s_0\lambda_{\mathfrak{q}} + \rho$ is the same for each case, we will describe the detail of the computation only for the case $V(\mu + \epsilon_{n\gamma})$ of $B_n(i)$ and omit the computation for other cases.

<u>1. $V(\mu + \epsilon_{n\gamma})$ for $B_n(i)$ for $3 \leq i \leq n-2$:</u> Since, by Table 7.1, the special value s_0 is $s_0 = 1$, we want to show that there is $\alpha \in \Pi(\mathfrak{l})$ so that $-\alpha - \lambda_i + \rho$ is linked to $\nu - \lambda_i + \rho$. First we find the highest weight ν for $V(\mu + \epsilon_{n\gamma})^*$. Observe that we have $\Delta^+(\mathfrak{l}) = \Delta^+(\mathfrak{l}_\gamma) \cup \Delta^+(\mathfrak{l}_{n\gamma})$ with

$$\Delta^+(\mathfrak{l}_\gamma) = \{\varepsilon_j - \varepsilon_k \mid 1 \leq j < k \leq i\}$$

and

$$\Delta^+(\mathfrak{l}_{n\gamma}) = \{\varepsilon_j \pm \varepsilon_k \mid i+1 \leq j < k \leq n\} \cup \{\varepsilon_j \mid i+1 \leq j \leq n\}$$

in the standard realization of the roots (see Appendix C). Since

$$\Delta(\mathfrak{z}(\mathfrak{n})) = \{\varepsilon_j + \varepsilon_k \mid 1 \leq j < k \leq i\},$$

the simple $\mathfrak{l}$-module $\mathfrak{z}(\mathfrak{n})$ has lowest weight $\varepsilon_{i-1} + \varepsilon_i$. As $V(\mu + \epsilon_{n\gamma}) = \mathfrak{l}_{n\gamma} \otimes \mathfrak{z}(\mathfrak{n})$, we have

$$V(\mu + \epsilon_{n\gamma})^* = \mathfrak{l}_{n\gamma} \otimes \mathfrak{z}(\mathfrak{n})^*.$$

Since $\mathfrak{l}_{n\gamma}$ has highest weight $\varepsilon_{i+1} + \varepsilon_{i+2}$, this shows that the highest weight ν for $V(\mu + \epsilon_{n\gamma})^*$ is

$$\nu = (\varepsilon_{i+1} + \varepsilon_{i+2}) - (\varepsilon_{i-1} + \varepsilon_i) = -\varepsilon_{i-1} - \varepsilon_i + \varepsilon_{i+1} + \varepsilon_{i+2}.$$

Observe that

$$-\varepsilon_{i-1} - \varepsilon_i + \varepsilon_{i+1} + \varepsilon_{i+2} = -2(\varepsilon_i - \varepsilon_{i+1}) - (\varepsilon_{i-1} - \varepsilon_i) - (\varepsilon_{i+1} - \varepsilon_{i+2})$$

with $\varepsilon_i - \varepsilon_{i+1} \in \Delta(\mathfrak{g}(1))$ and $\varepsilon_{i-1} - \varepsilon_i$, $\varepsilon_{i+1} - \varepsilon_{i+2} \in \Pi(\mathfrak{l})$ (see Appendix C). Now we claim that $(\varepsilon_{i-1} - \varepsilon_i, \varepsilon_i - \varepsilon_{i+1})$ links $-(\varepsilon_{i+1} - \varepsilon_{i+2}) - \lambda_i + \rho$ to $-2(\varepsilon_i - \varepsilon_{i+1}) - (\varepsilon_{i-1} - \varepsilon_i) - (\varepsilon_{i+1} - \varepsilon_{i+2}) - \lambda_i + \rho$. This is to show that

$$s_{\varepsilon_i - \varepsilon_{i+1}} s_{\varepsilon_{i-1} - \varepsilon_i}(-(\varepsilon_{i+1} - \varepsilon_{i+2}) - \lambda_i + \rho) = -2(\varepsilon_i - \varepsilon_{i+1}) - (\varepsilon_{i-1} - \varepsilon_i) - (\varepsilon_{i+1} - \varepsilon_{i+2}) - \lambda_i + \rho$$

with

$$\langle -(\varepsilon_{i+1} - \varepsilon_{i+2}) - \lambda_i + \rho, (\varepsilon_{i-1} - \varepsilon_i)^\vee \rangle \in \mathbb{Z}_{\geq 0}$$

and

$$\langle s_{\varepsilon_{i-1} - \varepsilon_i}(-(\varepsilon_{i+1} - \varepsilon_{i+2}) - \lambda_i + \rho), (\varepsilon_i - \varepsilon_{i+1})^\vee \rangle \in \mathbb{Z}_{\geq 0}$$

(See Definition 8.1.4). Observe that, as $\varepsilon_{i-1} - \varepsilon_i \in \Pi(\mathfrak{l})$, we have $\langle \lambda_i, (\varepsilon_{i-1} - \varepsilon_i)^\vee \rangle = 0$. Since $\langle \rho, (\varepsilon_{i-1} - \varepsilon_i)^\vee \rangle = 1$, it follows that

$$\langle -(\varepsilon_{i+1} - \varepsilon_{i+2}) - \lambda_i + \rho, (\varepsilon_{i-1} - \varepsilon_i)^\vee \rangle = 1 \in \mathbb{Z}_{\geq 0}.$$

Thus,

$$s_{\varepsilon_{i-1}-\varepsilon_i}(-(\varepsilon_{i+1}-\varepsilon_{i+2})-\lambda_i+\rho) = -(\varepsilon_{i-1}-\varepsilon_i)-(\varepsilon_{i+1}-\varepsilon_{i+2})-\lambda_i+\rho.$$

Next, as $\varepsilon_i-\varepsilon_{i+1}$ is the simple root that determines the parabolic $\mathfrak{q}$, we have $\langle\lambda_i,(\varepsilon_i-\varepsilon_{i+1})^\vee\rangle = 1$. Since $\langle\rho,(\varepsilon_i-\varepsilon_{i+1})^\vee\rangle = 1$, it follows that

$$\begin{aligned}
&\langle s_{\varepsilon_{i-1}-\varepsilon_i}(-(\varepsilon_{i+1}-\varepsilon_{i+2})-\lambda_i+\rho),(\varepsilon_i-\varepsilon_{i+1})^\vee\rangle\\
&=\langle -(\varepsilon_{i-1}-\varepsilon_i)-(\varepsilon_{i+1}-\varepsilon_{i+2})-\lambda_i+\rho,(\varepsilon_i-\varepsilon_{i+1})^\vee\rangle\\
&=2\in\mathbb{Z}_{\geq 0}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&s_{\varepsilon_i-\varepsilon_{i+1}}s_{\varepsilon_{i-1}-\varepsilon_i}(-(\varepsilon_{i+1}-\varepsilon_{i+2})-\lambda_i+\rho)\\
&=s_{\varepsilon_i-\varepsilon_{i+1}}(-(\varepsilon_{i-1}-\varepsilon_i)-(\varepsilon_{i+1}-\varepsilon_{i+2})-\lambda_i+\rho)\\
&=-2(\varepsilon_i-\varepsilon_{i+1})-(\varepsilon_{i-1}-\varepsilon_i)-(\varepsilon_{i+1}-\varepsilon_{i+2})-\lambda_i+\rho.
\end{aligned}$$

2. <u>$V(\mu+\epsilon_\gamma)$ and $V(\mu+\epsilon_{n\gamma})$ for $D_n(i)$ for $3\leq i\leq n-3$</u>: We start with $V(\mu+\epsilon_\gamma)$. Since, by Table 7.1, the special value s_0 is $s_0=n-i-1$, we want to show that there is $\alpha\in\Pi(\mathfrak{l})$ so that $-\alpha-(n-i-1)\lambda_i+\rho$ is linked to $\nu-(n-i-1)\lambda_i+\rho$. By Table 6.1, we have $\mu+\epsilon_\gamma=2\varepsilon_1$. Observe that if $\alpha_j=\varepsilon_j-\varepsilon_{j+1}$ and $w_j=s_{\alpha_1}s_{\alpha_2}\cdots s_{\alpha_j}$ for $1\leq j\leq i-1$ then the longest element w_0 of the Weyl group of type A_{i-1} may be expressed as $w_0=w_{i-1}w_{i-2}\cdots w_1$. Since $V(\mu+\epsilon_\gamma)$ is an $\mathfrak{l}_\gamma$-submodule of $\mathfrak{l}_\gamma\otimes\mathfrak{z}(\mathfrak{n})$ with $\mathfrak{l}_\gamma$ of type A_{i-1}, the highest weight ν for $V(\mu+\epsilon_\gamma)$ is

$$\nu=-w_0(2\varepsilon_1)=-2\varepsilon_i.$$

We have

$$-2\varepsilon_i=-2(\varepsilon_i-\varepsilon_{n-1})-(\varepsilon_{n-1}-\varepsilon_n)\ -(\varepsilon_{n-1}+\varepsilon_n)$$

with $\varepsilon_i-\varepsilon_{n-1}\in\Delta(\mathfrak{g}(1))$ and $\varepsilon_{n-1}-\varepsilon_n$, $\varepsilon_{n-1}+\varepsilon_n\in\Pi(\mathfrak{l})$ (see Appendix C). Then a direct computation shows that $(\varepsilon_{n-1}-\varepsilon_n,\varepsilon_i-\varepsilon_{n-1})$ links $-(\varepsilon_{n-1}+\varepsilon_n)-(n-i-1)\lambda_i+\rho$ to $-2\varepsilon_i-(n-i-1)\lambda_i+\rho$.

Next we consider $V(\mu+\epsilon_{n\gamma})$. Since, by Table 7.1, the special value s_0 is $s_0=1$, we want to show that there is $\alpha\in\Pi(\mathfrak{l})$ so that $-\alpha-\lambda_i+\rho$ is linked to $\nu-\lambda_i+\rho$. As for the case for $V(\mu+\epsilon_{n\gamma})$ of $B_n(i)$, the highest weight ν for $V(\mu+\epsilon_{n\gamma})^*$ of $D_n(i)$ is

$$\begin{aligned}\nu &= -\varepsilon_{i-1}-\varepsilon_i+\varepsilon_{i+1}+\varepsilon_{i+2}\\ &= -2(\varepsilon_i-\varepsilon_{i+1})-(\varepsilon_{i-1}-\varepsilon_i)-(\varepsilon_{i+1}-\varepsilon_{i+2})\end{aligned}$$

with $\varepsilon_i-\varepsilon_{i+1}\in\Delta(\mathfrak{g}(1))$ and $\varepsilon_{i-1}-\varepsilon_i$, $\varepsilon_{i+1}-\varepsilon_{i+2}\in\Pi(\mathfrak{l})$ (see Appendix C). A direct computation shows that $(\varepsilon_{i-1}-\varepsilon_i,\varepsilon_i-\varepsilon_{i+1})$ links $-(\varepsilon_{i+1}-\varepsilon_{i+2})-\lambda_i+\rho$ to $(-\varepsilon_{i-1}-\varepsilon_i+\varepsilon_{i+1}+\varepsilon_{i+2})-\lambda_i+\rho$.

3. $V(\mu+\epsilon_\gamma)$ and $V(\mu+\epsilon_{n\gamma})$ for $E_6(3)$: We start with $V(\mu+\epsilon_\gamma)$. Since, by Table 7.1, the special value s_0 is $s_0=1$, we want to show that there is $\alpha\in\Pi(\mathfrak{l})$ so that $-\alpha-\lambda_3+\rho$ is linked to $\nu-\lambda_3+\rho$. By Table 6.2, we have

$$\mu+\epsilon_\gamma=\alpha_1+2\alpha_2+2\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6.$$

As $V(\mu+\epsilon_\gamma)$ is a simple $\mathfrak{l}_\gamma$-submodule of $\mathfrak{l}_\gamma\otimes\mathfrak{z}(\mathfrak{n})$, if w_0 is the longest element of the Weyl group of $\mathfrak{l}_\gamma$ then, by using LiE, the highest weight ν for $V(\mu+\epsilon_\gamma)^*$ is given by

$$\begin{aligned}\nu &= -w_0(\alpha_1+2\alpha_2+2\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6)\\ &= -2\alpha_3-\alpha_1-\alpha_4.\end{aligned}$$

with $\alpha_3\in\Delta(\mathfrak{g}(1))$ and α_1, $\alpha_4\in\Pi(\mathfrak{l})$. Now a direct computation shows that (α_1,α_3) links $-\alpha_4-\lambda_3+\rho$ to $(-2\alpha_3-\alpha_1-\alpha_4)-\lambda_3+\rho$.

Next we consider $V(\mu+\epsilon_{n\gamma})$. Since, by Table 7.1, the special value s_0 is $s_0=2$, we want to show that there is $\alpha\in\Pi(\mathfrak{l})$ so that $-\alpha-2\lambda_3+\rho$ is linked to $\nu-2\lambda_3+\rho$. Observe that $\mathfrak{l}_{n\gamma}$ has highest weight α_1 (see Appendix C) and $\mathfrak{z}(\mathfrak{n})$ has lowest weight $\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5$. Since $V(\mu+\epsilon_{n\gamma})^*=\mathfrak{l}_{n\gamma}\otimes\mathfrak{z}(\mathfrak{n})^*$, the highest weight ν for $V(\mu+\epsilon_{n\gamma})^*$ is

$$\begin{aligned}\nu &= (\alpha_1)-(\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5)\\ &= -2(\alpha_3+\alpha_4)-\alpha_2-\alpha_5\end{aligned}$$

with $\alpha_3+\alpha_4 \in \Delta(\mathfrak{g}(1))$ and α_2, $\alpha_5 \in \Pi(\mathfrak{l})$. A direct computation shows that $(\alpha_2, \alpha_3+\alpha_4)$ links $-\alpha_5 - 2\lambda_3 + \rho$ to $(-2(\alpha_3+\alpha_4) - \alpha_2 - \alpha_5) - 2\lambda_3 + \rho$.

4. $V(\mu+\epsilon_\gamma)$ and $V(\mu+\epsilon_{n\gamma})$ for $E_6(5)$: We start with $V(\mu+\epsilon_\gamma)$. Since, by Table 7.1, the special value s_0 is $s_0 = 1$, we want to show that there is $\alpha \in \Pi(\mathfrak{l})$ so that $-\alpha - \lambda_5 + \rho$ is linked to $\nu - \lambda_5 + \rho$. By Table 6.2, we have

$$\mu + \epsilon_\gamma = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 2\alpha_5 + \alpha_6.$$

As $V(\mu+\epsilon_\gamma)$ is a simple $\mathfrak{l}_\gamma$-submodule of $\mathfrak{l}_\gamma \otimes \mathfrak{z}(\mathfrak{n})$, if w_0 is the longest element of the Weyl group of $\mathfrak{l}_\gamma$ then, by using LiE, the highest weight ν for $V(\mu+\epsilon_\gamma)^*$ is given by

$$\begin{aligned}\nu &= -w_0(2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 2\alpha_5 + \alpha_6)\\ &= -2\alpha_5 - \alpha_4 - \alpha_6.\end{aligned}$$

with $\alpha_5 \in \Delta(\mathfrak{g}(1))$ and α_4, $\alpha_6 \in \Pi(\mathfrak{l})$. Now a direct computation shows that (α_4, α_5) links $-\alpha_6 - \lambda_5 + \rho$ to $(-2\alpha_5 - \alpha_4 - \alpha_6) - \lambda_5 + \rho$.

Next we consider $V(\mu+\epsilon_{n\gamma})$. By Table 7.1, the special value s_0 is $s_0 = 2$, we want to show that there is $\alpha \in \Pi(\mathfrak{l})$ so that $-\alpha - 2\lambda_5 + \rho$ is linked to $\nu - 2\lambda_5 + \rho$. Observe that $\mathfrak{l}_{n\gamma}$ has highest weight α_6 (see Appendix C) and $\mathfrak{z}(\mathfrak{n})$ has lowest weight $\alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6$. Since $V(\mu+\epsilon_{n\gamma})^* = \mathfrak{l}_{n\gamma} \otimes \mathfrak{z}(\mathfrak{n})^*$, the highest weight ν for $V(\mu+\epsilon_{n\gamma})^*$ is

$$\begin{aligned}\nu &= (\alpha_6) - (\alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6)\\ &= -2(\alpha_4+\alpha_5) - \alpha_2 - \alpha_3\end{aligned}$$

with $\alpha_4+\alpha_5 \in \Delta(\mathfrak{g}(1))$ and α_2, $\alpha_3 \in \Pi(\mathfrak{l})$. A direct computation shows that $(\alpha_2, \alpha_4+\alpha_5)$ links $-\alpha_3 - 2\lambda_5 + \rho$ to $(-2(\alpha_4+\alpha_5) - \alpha_2 - \alpha_3) - 2\lambda_5 + \rho$.

5. $V(\mu+\epsilon_\gamma)$ for $E_7(2)$: Since, by Table 7.1, the special value s_0 is $s_0 = 2$, we want to show that there is $\alpha \in \Pi(\mathfrak{l})$ so that $-\alpha - 2\lambda_2 + \rho$ is linked to $\nu - 2\lambda_2 + \rho$. By Table 6.2, we have

$$\mu + \epsilon_\gamma = 2\alpha_1 + 2\alpha_2 + 4\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7.$$

As $V(\mu+\epsilon_\gamma)$ is a simple $\mathfrak{l}_\gamma$-submodule of $\mathfrak{l}_\gamma \otimes \mathfrak{z}(\mathfrak{n})$, if w_0 is the longest element of the Weyl group of $\mathfrak{l}_\gamma$ then, by using LiE, the highest weight ν for $V(\mu+\epsilon_\gamma)^*$ is given by

$$\begin{aligned}\nu &= -w_0(2\alpha_1 + 2\alpha_2 + 4\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7)\\ &= -2(\alpha_2+\alpha_4) - \alpha_3 - \alpha_5\end{aligned}$$

with $\alpha_2+\alpha_4 \in \Delta(\mathfrak{g}(1))$ and $\alpha_3,\ \alpha_5 \in \Pi(\mathfrak{l})$. Now a direct computation shows that $(\alpha_3, \alpha_2+\alpha_4)$ links $-\alpha_5 - 2\lambda_2 + \rho$ to $(-2(\alpha_2+\alpha_4) - \alpha_3 - \alpha_5) - 2\lambda_2 + \rho$.

6. $V(\mu+\epsilon_\gamma)$ and $V(\mu+\epsilon_{n\gamma})$ for $E_7(6)$: We start with $V(\mu+\epsilon_\gamma)$. Since, by Table 7.1, the special value s_0 is $s_0 = 1$, we want to show that there is $\alpha \in \Pi(\mathfrak{l})$ so that $-\alpha - \lambda_6 + \rho$ is linked to $\nu - \lambda_6 + \rho$. By Table 6.2, we have

$$\mu + \epsilon_\gamma = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 4\alpha_5 + 2\alpha_6 + \alpha_7.$$

As $V(\mu+\epsilon_\gamma)$ is a simple $\mathfrak{l}_\gamma$-submodule of $\mathfrak{l}_\gamma \otimes \mathfrak{z}(\mathfrak{n})$, if w_0 is the longest element of the Weyl group of $\mathfrak{l}_\gamma$ then, by using LiE, the highest weight ν for $V(\mu+\epsilon_\gamma)^*$ is given by

$$\begin{aligned}\nu &= -w_0(2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 4\alpha_5 + 2\alpha_6 + \alpha_7)\\ &= -2\alpha_6 - \alpha_5 - \alpha_7\end{aligned}$$

with $\alpha_6 \in \Delta(\mathfrak{g}(1))$ and $\alpha_5,\ \alpha_7 \in \Pi(\mathfrak{l})$. Now a direct computation shows that (α_5, α_6) links $-\alpha_7 - \lambda_6 + \rho$ to $(-2\alpha_6 - \alpha_5 - \alpha_7) - \lambda_6 + \rho$.

Next we consider $V(\mu+\epsilon_{n\gamma})$. By Table 7.1, the special value s_0 is $s_0 = 3$, we want to show that there is $\alpha \in \Pi(\mathfrak{l})$ so that $-\alpha - 3\lambda_6 + \rho$ is linked to $\nu - 3\lambda_6 + \rho$. Observe that $\mathfrak{l}_{n\gamma}$ has highest weight α_7 (see Appendix C) and $\mathfrak{z}(\mathfrak{n})$ has lowest weight $\alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7$. Since $V(\mu+\epsilon_{n\gamma})^* = \mathfrak{l}_{n\gamma} \otimes \mathfrak{z}(\mathfrak{n})^*$, the highest weight ν for $V(\mu+\epsilon_{n\gamma})^*$ is

$$\begin{aligned}\nu &= (\alpha_7) - (\alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7)\\ &= -2(\alpha_4+\alpha_5+\alpha_6) - \alpha_2 - \alpha_3\end{aligned}$$

with $\alpha_4+\alpha_5+\alpha_6 \in \Delta(\mathfrak{g}(1))$ and $\alpha_2,\ \alpha_3 \in \Pi(\mathfrak{l})$. A direct computation shows that $(\alpha_2, \alpha_4+\alpha_5+\alpha_6)$ links $-\alpha_3 - 3\lambda_6 + \rho$ to $(-2(\alpha_4+\alpha_5+\alpha_6) - \alpha_2 - \alpha_3) - 3\lambda_6 + \rho$.

7. $V(\mu+\epsilon_\gamma)$ for $E_8(1)$: Since, by Table 7.1, the special value s_0 is $s_0 = 3$, we want to show that there is $\alpha \in \Pi(\mathfrak{l})$ so that $-\alpha - 3\lambda_1 + \rho$ is linked to $\nu - 3\lambda_1 + \rho$. By Table 6.2, we have

$$\mu + \epsilon_\gamma = 2\alpha_1 + 4\alpha_2 + 5\alpha_3 + 8\alpha_4 + 7\alpha_5 + 6\alpha_6 + 4\alpha_7 + 2\alpha_8.$$

As $V(\mu+\epsilon_\gamma)$ is a simple $\mathfrak{l}_\gamma$-submodule of $\mathfrak{l}_\gamma \otimes \mathfrak{z}(\mathfrak{n})$, if w_0 is the longest element of the Weyl group of $\mathfrak{l}_\gamma$ then, by using LiE, the highest weight ν for $V(\mu+\epsilon_\gamma)^*$ is given by

$$\begin{aligned}\nu &= -w_0(2\alpha_1 + 4\alpha_2 + 5\alpha_3 + 8\alpha_4 + 7\alpha_5 + 6\alpha_6 + 4\alpha_7 + 2\alpha_8)\\ &= -2(\alpha_1+\alpha_3+\alpha_4) - \alpha_2 - \alpha_5\end{aligned}$$

with $\alpha_1+\alpha_3+\alpha_4 \in \Delta(\mathfrak{g}(1))$ and $\alpha_2,\ \alpha_5 \in \Pi(\mathfrak{l})$. Now a direct computation shows that $(\alpha_2, \alpha_1+\alpha_3+\alpha_4)$ links $-\alpha_5 - 3\lambda_1 + \rho$ to $(-2(\alpha_1+\alpha_3+\alpha_4) - \alpha_2 - \alpha_5) - 3\lambda_1 + \rho$.

■

8.3.3 The $V(\mu+\epsilon_\gamma)$ Case for $B_n(i)$ for $3 \le i \le n-1$

Now we consider the case $V(\mu+\epsilon_\gamma)$ of $B_n(i)$ for $3 \le i \le n-1$. By Table 7.1, the special value s_0 is $s_0 = n - i - (1/2)$ for $1 \le i \le n-1$ (note that when $i = n-1$, we have $s_0 = 1/2 = n - (n-1) - (1/2)$). By the same argument used for the case $V(\mu+\epsilon_\gamma)^*$ of $D_n(i)$ in the proof of Theorem 8.3.4, the highest weight ν for $V(\mu+\epsilon_\gamma)^*$ is $\nu = -2\varepsilon_i$. Therefore we have

$$\varphi_{\Omega_2} : M_{\mathfrak{q}}(-2\varepsilon_i - (n-i-(1/2))\lambda_i + \rho) \to M_{\mathfrak{q}}(-(n-i-(1/2))\lambda_i + \rho). \qquad (8.3.5)$$

We first show that the standard map φ_{std} is non-zero.

Proposition 8.3.6 *If $\mathfrak{q}$ is the maximal two-step nilpotent parabolic subalgebra of type $B_n(i)$ with $3 \le i \le n-1$ then the standard map φ_{std} from $M_{\mathfrak{q}}(-2\varepsilon_i - (n-i-(1/2))\lambda_i + \rho)$ to $M_{\mathfrak{q}}(-(n-i-(1/2))\lambda_i + \rho)$ is non-zero.*

Proof. By Proposition 8.1.6, to prove this proposition, it suffices to show that there is no $\alpha \in \Pi(\mathfrak{l})$ so that $-\alpha-(n-i-(1/2))\lambda_i+\rho$ is linked to $-2\varepsilon_i-(n-i-(1/2))\lambda_i+\rho$. For simplicity we write

$$\delta(i) = -(n-i-(1/2))\lambda_i + \rho.$$

Since $\varepsilon_i = \sum_{j=i}^n \alpha_j$ with α_j simple roots in the standard numbering, we want to show that there is no $\alpha \in \Pi(\mathfrak{l})$ so that $-\alpha+\delta(i)$ is linked to $-2\varepsilon_i+\delta(i) = -2\sum_{j=i}^n \alpha_j+\delta(i)$. Suppose that such $\alpha' \in \Pi(\mathfrak{l})$ exists. Let $(\beta_1, \ldots, \beta_m)$ be a link from $-\alpha'+\delta(i)$ to $-2\sum_{j=i}^n \alpha_j + \delta(i)$. Without loss of generality, we assume that for all $j = 1, \ldots, m$,

$$\langle s_{\beta_{j-1}} \cdots s_{\beta_1}(-\alpha' + \delta(i)), \beta_j^\vee \rangle \neq 0.$$

(If $j = 1$ then set $s_{\beta_0} = e$, the identity.) By the property (2) in Definition 8.1.4, this means that we assume that

$$\langle s_{\beta_{j-1}} \cdots s_{\beta_1}(-\alpha' + \delta(i)), \beta_j^\vee \rangle \in 1 + \mathbb{Z}_{\geq 0} \tag{8.3.7}$$

for all $j = 1, \ldots, m$. Observe that it follows from the property (2) in Definition 8.1.4 that any weight linked from $-\alpha' + \delta(i)$ is of the from

$$(-\sum_{\alpha \in \Pi} n_\alpha \alpha) - \alpha' + \delta(i) \text{ with } n_\alpha \in \mathbb{Z}_{\geq 0}. \tag{8.3.8}$$

We have $\Delta^+ = \Delta^+(\mathfrak{l}) \cup \Delta(\mathfrak{g}(1)) \cup \Delta(\mathfrak{z}(\mathfrak{n}))$, where $\Delta^+(\mathfrak{l})$, $\Delta(\mathfrak{g}(1))$, and $\Delta(\mathfrak{z}(\mathfrak{n}))$ are the sets of the positive roots in which α_i has multiplicity zero, one, and two, respectively. As $(\beta_1, \ldots, \beta_m)$ is a link from $-\alpha' + \delta(i)$ to $-2\sum_{j=i}^n \alpha_j + \delta(i)$, we have

$$s_{\beta_m} \cdots s_{\beta_1}(-\alpha' + \delta(i)) = -2\sum_{j=i}^n \alpha_j + \delta(i). \tag{8.3.9}$$

If $\beta_j \in \Delta^+(\mathfrak{l})$ for all j then we would have

$$\begin{aligned} -2\sum_{j=i}^n \alpha_j + \delta(i) &= s_{\beta_m} \cdots s_{\beta_1}(-\alpha' + \delta(i)) \\ &= (-\sum_{\alpha \in \Pi(\mathfrak{l})} k_\alpha \alpha) - \alpha' + \delta(i) \end{aligned}$$

for some $k_\alpha \in \mathbb{Z}_{\geq 0}$. This implies that

$$-2\alpha_i - 2\sum_{j=i+1}^{n} \alpha_j = (-\sum_{\alpha\in\Pi(\mathfrak{l})} k_\alpha \alpha) - \alpha'. \tag{8.3.10}$$

This is absurd, because, as $\Pi(\mathfrak{l}) = \Pi\backslash\{\alpha_i\}$ and $\alpha' \in \Pi(\mathfrak{l})$, the simple root α_i does not contribute to the right hand side of (8.3.10). Thus, there must exist at least one β_j in $(\beta_1, \ldots, \beta_m)$ with $\beta_j \in \Delta(\mathfrak{g}(1)) \cup \Delta(\mathfrak{z}(\mathfrak{n}))$.

Now we show that any β_j in $(\beta_1, \ldots, \beta_m)$ cannot belong to $\Delta(\mathfrak{g}(1)) \cup \Delta(\mathfrak{z}(\mathfrak{n}))$. First, suppose that there exists β_r in $(\beta_1, \ldots, \beta_m)$ with $\beta_r \in \Delta(\mathfrak{z}(\mathfrak{n}))$. Observe that $\Delta(\mathfrak{z}(\mathfrak{n}))$ consists of the positive roots $\varepsilon_j + \varepsilon_k$ for $1 \leq j < k \leq i$ (see Appendix C). So β_r is $\beta_r = \varepsilon_s + \varepsilon_t$ for some $1 \leq s < t \leq i$. Since each $\varepsilon_l = \sum_{j=l}^{n} \alpha_j$ with α_j simple roots, the positive root $\beta_r = \varepsilon_s + \varepsilon_t$ with $1 \leq s < t \leq i$ can be expressed as

$$\beta_r = \varepsilon_s + \varepsilon_t = \sum_{j=s}^{t-1} \alpha_j + 2\sum_{j=t}^{n} \alpha_j.$$

If $c = \langle s_{\beta_{r-1}} \cdots s_{\beta_1}(-\alpha' + \delta(i)), \beta_r^\vee \rangle$ then

$$\begin{aligned} s_{\beta_r} \cdots s_{\beta_1}(-\alpha' + \delta(i)) &= s_{\beta_{r-1}} \cdots s_{\beta_1}(-\alpha' + \delta(i)) - c\beta_r \\ &= s_{\beta_{r-1}} \cdots s_{\beta_1}(-\alpha' + \delta(i)) - c\big(\sum_{j=s}^{t-1} \alpha_j + 2\sum_{j=t}^{n} \alpha_j\big). \end{aligned} \tag{8.3.11}$$

Observe that, by (8.3.8), $s_{\beta_{r-1}} \cdots s_{\beta_1}(-\alpha' + \delta(i))$ is of the form

$$s_{\beta_{r-1}} \cdots s_{\beta_1}(-\alpha' + \delta(i)) = (-\sum_{\alpha\in\Pi} m_\alpha \alpha) - \alpha' + \delta(i) \tag{8.3.12}$$

for some $m_\alpha \in \mathbb{Z}_{\geq 0}$. Moreover, as $s_{\beta_m} \cdots s_{\beta_1}(-\alpha' + \delta(i))$ is a weight linked from $s_{\beta_r} \cdots s_{\beta_1}(-\alpha' + \delta(i))$, the weight $s_{\beta_m} \cdots s_{\beta_1}(-\alpha' + \delta(i))$ is of the form

$$s_{\beta_m} \cdots s_{\beta_1}(-\alpha' + \delta(i)) = (-\sum_{\alpha\in\Pi} m'_\alpha \alpha) + s_{\beta_r} \cdots s_{\beta_1}(-\alpha' + \delta(i)) \tag{8.3.13}$$

for some $m'_\alpha \in \mathbb{Z}_{\geq 0}$. By combining (8.3.11), (8.3.12), and (8.3.13), we have

$$\begin{aligned}
&s_{\beta_m} \cdots s_{\beta_1}(-\alpha' + \delta(i)) \\
&= (-\sum_{\alpha \in \Pi} m'_\alpha \alpha) + s_{\beta_r} \cdots s_{\beta_1}(-\alpha' + \delta(i)) \\
&= (-\sum_{\alpha \in \Pi} m'_\alpha \alpha) + s_{\beta_{r-1}} \cdots s_{\beta_1}(-\alpha' + \delta(i)) - c\big(\sum_{j=s}^{t-1} \alpha_j + 2\sum_{j=t}^{n} \alpha_j\big) \\
&= (-\sum_{\alpha \in \Pi} m'_\alpha \alpha) + (-\sum_{\alpha \in \Pi} m_\alpha \alpha) - c\big(\sum_{j=s}^{t-1} \alpha_j + 2\sum_{j=t}^{n} \alpha_j\big) - \alpha' + \delta(i) \qquad (8.3.14)
\end{aligned}$$

with $m_\alpha, m'_\alpha \in \mathbb{Z}_{\geq 0}$. By (8.3.7), we have

$$c = \langle s_{\beta_{r-1}} \cdots s_{\beta_1}(-\alpha' + \delta(i)), \beta_r^\vee \rangle \in 1 + \mathbb{Z}_{\geq 0}.$$

Therefore, by (8.3.14), the weight $s_{\beta_m} \cdots s_{\beta_1}(-\alpha' + \delta(i))$ is of the form

$$s_{\beta_m} \cdots s_{\beta_1}(-\alpha' + \delta(i)) = -\sum_{\alpha \in \Pi} n_\alpha \alpha - \sum_{j=s}^{t-1} \alpha_j - 2\sum_{j=t}^{n} \alpha_j - \alpha' + \delta(i)$$

for some $n_\alpha \in \mathbb{Z}_{\geq 0}$. By (8.3.9), this implies that

$$2\sum_{j=i}^{n} \alpha_j = \sum_{\alpha \in \Pi} n_\alpha \alpha + \sum_{j=s}^{t-1} \alpha_j + 2\sum_{j=t}^{n} \alpha_j + \alpha'.$$

Since $s < t \leq i$, we then have

$$\begin{aligned}
0 &= \sum_{\alpha \in \Pi} n_\alpha \alpha + \sum_{j=s}^{t-1} \alpha_j + 2\sum_{j=t}^{n} \alpha_j + \alpha' - 2\sum_{j=i}^{n} \alpha_j \\
&= \begin{cases} \sum_{\alpha \in \Pi} n_\alpha \alpha + \sum_{j=s}^{t-1} \alpha_j + 2\sum_{j=t}^{i-1} \alpha_j + \alpha' & \text{if } t < i \\ \sum_{\alpha \in \Pi} n_\alpha \alpha + \sum_{j=s}^{t-1} \alpha_j + \alpha' & \text{if } t = i. \end{cases} \qquad (8.3.15)
\end{aligned}$$

This is a contradiction, because, as $n_\alpha \in \mathbb{Z}_{\geq 0}$, (8.3.15) cannot be zero. Therefore no β_j in $(\beta_1, \ldots, \beta_m)$ is a root in $\Delta(\mathfrak{z}(\mathfrak{n}))$.

Next we suppose that there exists β_r in $(\beta_1, \ldots, \beta_m)$ with $\beta_r \in \Delta(\mathfrak{g}(1))$. There are long roots and short roots in $\Delta(\mathfrak{g}(1))$. We handle these cases separately. We first suppose that β_r is a long root in $\Delta(\mathfrak{g}(1))$. The long roots in $\Delta(\mathfrak{g}(1))$ are $\varepsilon_j \pm \varepsilon_k$ for

$1 \leq j \leq i$ and $i+1 \leq k \leq n$ (see Appendix C). The roots $\varepsilon_j \pm \varepsilon_k$ may be expressed in terms of simple roots as

$$\varepsilon_j + \varepsilon_k = \sum_{l=j}^{n} \alpha_l + \sum_{l=k}^{n} \alpha_l = \sum_{l=j}^{i-1} \alpha_l + \alpha_i + \sum_{l=i+1}^{k-1} \alpha_l + 2\sum_{l=k}^{n-1} \alpha_l + 2\alpha_n$$

and

$$\varepsilon_j - \varepsilon_k = \sum_{l=j}^{n} \alpha_l - \sum_{l=k}^{n} \alpha_l = \sum_{l=j}^{i-1} \alpha_l + \alpha_i + \sum_{l=i+1}^{k-1} \alpha_l.$$

We show that if $\beta_r = \varepsilon_j \pm \varepsilon_k$ then $\langle s_{\beta_{r-1}} \cdots s_{\beta_1}(-\alpha' + \delta(i)), \beta_r^\vee \rangle \notin \mathbb{Z}$. Observe that since α_n is the only short simple root, the coroot $(\varepsilon_j + \varepsilon_k)^\vee$ can be expressed as

$$\begin{aligned}
&(\varepsilon_j + \varepsilon_k)^\vee \\
&= \Big(\sum_{l=j}^{i-1} \alpha_l + \alpha_i + \sum_{l=i+1}^{k-1} \alpha_l + 2\sum_{l=k}^{n-1} \alpha_l + 2\alpha_n \Big)^\vee \\
&= \sum_{l=j}^{i-1} \frac{2\alpha_l}{||\varepsilon_j + \varepsilon_k||^2} + \frac{2\alpha_i}{||\varepsilon_j + \varepsilon_k||^2} + \sum_{l=i+1}^{k-1} \frac{2\alpha_l}{||\varepsilon_j + \varepsilon_k||^2} + 2\sum_{l=k}^{n-1} \frac{2\alpha_l}{||\varepsilon_j + \varepsilon_k||^2} + 2 \cdot \frac{2\alpha_n}{||\varepsilon_j + \varepsilon_k||^2} \\
&= \sum_{l=j}^{i-1} \alpha_l^\vee + \alpha_i^\vee + \sum_{l=i+1}^{k-1} \alpha_l^\vee + 2\sum_{l=k}^{n-1} \alpha_l^\vee + \alpha_n^\vee.
\end{aligned}$$

Similarly, we have

$$(\varepsilon_j - \varepsilon_k)^\vee = \sum_{l=j}^{i-1} \alpha_l^\vee + \alpha_i^\vee + \sum_{l=i+1}^{k-1} \alpha_l^\vee.$$

Now observe that, as λ_i is the fundamental weight for α_i, for $\alpha \in \Pi$, we have

$$\begin{aligned}
\langle \delta(i), \alpha^\vee \rangle &= \langle -(n - i - (1/2))\lambda_i + \rho, \alpha^\vee \rangle \\
&= \begin{cases} -n + i + (3/2) & \text{if } \alpha = \alpha_i \\ 1 & \text{otherwise.} \end{cases}
\end{aligned} \tag{8.3.16}$$

Thus,

$$
\begin{aligned}
&\langle\delta(i), (\varepsilon_j+\varepsilon_k)^\vee\rangle \\
&= \langle\delta(i), \sum_{l=j}^{i-1}\alpha_l^\vee + \alpha_i^\vee + \sum_{l=i+1}^{k-1}\alpha_l^\vee + 2\sum_{l=k}^{n-1}\alpha_l^\vee + \alpha_n^\vee\rangle \\
&= \sum_{l=j}^{i-1}\langle\delta(i),\alpha_l^\vee\rangle + \langle\delta(i),\alpha_i^\vee\rangle + \sum_{l=i+1}^{k-1}\langle\delta(i),\alpha_l^\vee\rangle + 2\sum_{l=k}^{n-1}\langle\delta(i),\alpha_l^\vee\rangle + \langle\delta(i),\alpha_n^\vee\rangle \\
&= (i-1-(j-1)) + (-n+i+(3/2)) + (k-1-i) + 2(n-1-(k-1)) + 1 \\
&= n-k+i-j+(3/2).
\end{aligned}
$$

Similarly,

$$\langle\delta(i), (\varepsilon_j-\varepsilon_k)^\vee\rangle = -n+k+i-j+(1/2).$$

Hence, for $\beta_r = \varepsilon_j \pm \varepsilon_k$, we have $\langle\delta(i), \beta_r^\vee\rangle \notin \mathbb{Z}$. Now, by (8.3.12), we have

$$
\begin{aligned}
\langle s_{\beta_{r-1}}\cdots s_{\beta_1}(-\alpha'+\delta(i)), \beta_r^\vee\rangle &= \langle(-\sum_{\alpha\in\Pi} m_\alpha\alpha) - \alpha' + \delta(i), \beta_r^\vee\rangle \\
&= -\sum_{\alpha\in\Pi} m_\alpha\langle\alpha,\beta_r^\vee\rangle - \langle\alpha',\beta_r^\vee\rangle + \langle\delta(i),\beta_r^\vee\rangle
\end{aligned}
$$

with $m_\alpha \in \mathbb{Z}$. Since $m_\alpha, \langle\alpha,\beta_r^\vee\rangle, \langle\alpha',\beta_r^\vee\rangle \in \mathbb{Z}$ and $\langle\delta(i),\beta_r^\vee\rangle \notin \mathbb{Z}$, this shows that $\langle s_{\beta_{r-1}}\cdots s_{\beta_1}(-\alpha'+\delta(i)), \beta_r^\vee\rangle \notin \mathbb{Z}$.

Next we suppose that β_r is a short root in $\Delta(\mathfrak{g}(1))$. The short roots in $\Delta(\mathfrak{g}(1))$ are ε_j for $1 \le j \le i$ (see Appendix C). Thus β_r is $\beta_r = \varepsilon_l$ for some $1 \le l \le i$. Since ε_l is of the form $\varepsilon_l = \sum_{j=l}^n \alpha_j$, (8.3.9) forces that $l = i$; otherwise, $s_{\beta_m}\cdots s_{\beta_1}(-\alpha'+\delta(i))$ would have a contribution from some $\alpha_j \in \Pi$ with $1 \le j \le i-1$. Thus $\beta_r = \varepsilon_i = \sum_{j=i}^n \alpha_j$. Since β_r is a short root, the coroot $\beta_r^\vee = (\sum_{j=i}^n \alpha_j)^\vee$ can be expressed as

$$\beta_r^\vee = \big(\sum_{j=i}^n \alpha_j\big)^\vee = \sum_{j=i}^n \frac{2\alpha_j}{||\beta_r||^2} = \frac{2\alpha_i}{||\beta_r||^2} + \sum_{j=i+1}^{n-1}\frac{2\alpha_j}{||\beta_r||^2} + \frac{2\alpha_n}{||\beta_r||^2} = 2\alpha_i^\vee + 2\sum_{j=i+1}^{n-1}\alpha_j^\vee + \alpha_n^\vee.$$

It then follows from (8.3.16) that

$$
\begin{aligned}
\langle\delta(i),\beta_r^\vee\rangle &= \langle -(n-i-(1/2))\lambda_i+\rho,\big(\sum_{j=i}^{n}\alpha_j\big)^\vee\rangle\\
&= \langle -(n-i-(1/2))\lambda_i+\rho, 2\alpha_i^\vee+2\sum_{j=i+1}^{n-1}\alpha_j^\vee+\alpha_n^\vee\rangle\\
&= 2\langle -(n-i-(1/2))\lambda_i+\rho,\alpha_i^\vee\rangle+2\sum_{j=i+1}^{n-1}\langle -(n-i-(1/2))\lambda_i+\rho,\alpha_j^\vee\rangle\\
&\quad +\langle -(n-i-(1/2))\lambda_i+\rho,\alpha_n^\vee\rangle\\
&= 2(-n+i+(3/2))+2(n-1-i)+1\\
&= 2.
\end{aligned}
$$

Thus, by (8.3.12), we have

$$
\begin{aligned}
\langle s_{\beta_{r-1}}\cdots s_{\beta_1}(-\alpha'+\delta(i)),\beta_r^\vee\rangle &= \langle(-\sum_{\alpha\in\Pi}m_\alpha\alpha)-\alpha'+\delta(i),\beta_r^\vee\rangle\\
&= \langle-\sum_{\alpha\in\Pi}m_\alpha\alpha-\alpha',\beta_r^\vee\rangle+2 \qquad (8.3.17)
\end{aligned}
$$

with $m_\alpha\in\mathbb{Z}_{\geq 0}$. Thus, as $\beta_r=\sum_{j=i}^n\alpha_j$, if $d=\langle-\sum_{\alpha\in\Pi}m_\alpha\alpha-\alpha',\beta_r^\vee\rangle+2$ then $s_{\beta_r}\cdots s_{\beta_1}(-\alpha'+\delta(i))$ is of the form

$$s_{\beta_r}\cdots s_{\beta_1}(-\alpha'+\delta(i)) = s_{\beta_{r-1}}\cdots s_{\beta_1}(-\alpha'+\delta(i))-d\sum_{j=i}^n\alpha_j.$$

By (8.3.12) and (8.3.13), we have

$$
\begin{aligned}
s_{\beta_m}\cdots s_{\beta_1}(-\alpha'+\delta(i)) &= (-\sum_{\alpha\in\Pi}m'_\alpha\alpha)+s_{\beta_r}\cdots s_{\beta_1}(-\alpha'+\delta(i))\\
&= (-\sum_{\alpha\in\Pi}m'_\alpha\alpha)+s_{\beta_{r-1}}\cdots s_{\beta_1}(-\alpha'+\delta(i))-d\sum_{j=i}^n\alpha_j\\
&= (-\sum_{\alpha\in\Pi}m'_\alpha\alpha)+(-\sum_{\alpha\in\Pi}m_\alpha\alpha)-d\sum_{j=i}^n\alpha_j-\alpha'+\delta(i)
\end{aligned}
$$

with $m_\alpha, m'_\alpha\in\mathbb{Z}_{\geq}$. Therefore, $s_{\beta_m}\cdots s_{\beta_1}(-\alpha'+\delta(i))$ can be expressed as

$$s_{\beta_m}\cdots s_{\beta_1}(-\alpha'+\delta(i)) = -\sum_{\alpha\in\Pi}n_\alpha\alpha-d\sum_{j=i}^n\alpha_j-\alpha'+\delta(i)$$

for some $n_\alpha \in \mathbb{Z}_{\geq 0}$. By (8.3.9), this implies that

$$2\sum_{j=i}^{n} \alpha_j = \sum_{\alpha\in\Pi} n_\alpha \alpha + d\sum_{j=i}^{n} \alpha_j + \alpha'. \tag{8.3.18}$$

By comparing the coefficients of α_i in the both sides, we have

$$n_{\alpha_i} + d = 2. \tag{8.3.19}$$

By (8.3.7) and (8.3.17), we have $d = \langle -\sum_{\alpha\in\Pi} m_\alpha \alpha - \alpha', \beta_r^\vee\rangle + 2 \in 1 + \mathbb{Z}_{\geq 0}$. Since $n_{\alpha_i} \in \mathbb{Z}_{\geq 0}$, (8.3.19) forces that

$$d = 2 \text{ or } d = 1.$$

If $d = 2$ then (8.3.18) becomes

$$2\sum_{j=i}^{n} \alpha_j = \sum_{\alpha\in\Pi} n_\alpha \alpha + 2\sum_{j=i}^{n} \alpha_j + \alpha'.$$

Therefore,

$$\sum_{\alpha\in\Pi} n_\alpha \alpha + \alpha' = 0, \tag{8.3.20}$$

which is a contradiction, because as $\alpha' \in \Pi$ and $k'_\alpha \in \mathbb{Z}_{\geq 0}$, the left hand side of (8.3.20) cannot be zero. If $d = 1$ then, since $d = \langle -\sum_{\alpha\in\Pi} m_\alpha \alpha - \alpha', \beta_r^\vee\rangle + 2$, we have

$$\langle -\sum_{\alpha\in\Pi} m_\alpha \alpha - \alpha', \beta_r^\vee\rangle + 2 = 1.$$

Thus,

$$\langle \sum_{\alpha\in\Pi} m_\alpha \alpha + \alpha', \beta_r^\vee\rangle = 1. \tag{8.3.21}$$

Observe that, as $\beta_r = \varepsilon_i$ in the standard realization, if $\langle \alpha, \beta_r^\vee\rangle \neq 0$ for $\alpha \in \Pi$ then α must be $\alpha = \varepsilon_{i-1} - \varepsilon_i$ in $\Pi(\mathfrak{l})$ or $\alpha = \varepsilon_i - \varepsilon_{i+1}$ in $\Pi\backslash\Pi(\mathfrak{l})$. Since $\langle \varepsilon_{i-1} - \varepsilon_i, \varepsilon_i^\vee\rangle = -2$, $\langle \varepsilon_i - \varepsilon_{i+1}, \varepsilon_i^\vee\rangle = 2$, and $\alpha' \in \Pi(\mathfrak{l})$, the left hand side of (8.3.21) is

$$\begin{aligned}
\langle \sum_{\alpha\in\Pi} m_\alpha \alpha + \alpha', \beta_r^\vee\rangle &= m_{\varepsilon_{i-1}-\varepsilon_i}\langle \varepsilon_{i-1} - \varepsilon_i, \varepsilon_i^\vee\rangle + m_{\varepsilon_i-\varepsilon_{i+1}}\langle \varepsilon_i - \varepsilon_{i+1}, \varepsilon_i^\vee\rangle + \langle \alpha', \varepsilon_i^\vee\rangle \\
&= -2m_{\varepsilon_{i-1}-\varepsilon_i} + 2m_{\varepsilon_i-\varepsilon_{i+1}} - 2\delta_{\alpha',\varepsilon_{i-1}-\varepsilon_i} \\
&= 2(m_{\varepsilon_i-\varepsilon_{i+1}} - m_{\varepsilon_{i-1}-\varepsilon_i} - \delta_{\alpha',\varepsilon_{i-1}-\varepsilon_i}),
\end{aligned}$$

where $\delta_{\alpha',\varepsilon_{i-1}-\varepsilon_i}$ is the Kronecker delta. As $m_{\varepsilon_i-\varepsilon_{i+1}}$, $m_{\varepsilon_{i-1}-\varepsilon_i}$, and $\delta_{\alpha',\varepsilon_{i-1}-\varepsilon_i}$ are integers, this shows that $\langle \sum_{\alpha\in\Pi} m_\alpha \alpha + \alpha', \beta_r^\vee \rangle \neq 1$, which contradicts (8.3.21). Therefore, no β_r in $(\beta_1, \ldots, \beta_m)$ is a short root in $\Delta(\mathfrak{g}(1))$. Hence there is no link from $-\alpha' + \delta(i)$ to $-2\sum_{j=i}^n \alpha_j + \delta(i)$. ■

Now we are going to show that the map

$$\varphi_{\Omega_2} : M_{\mathfrak{q}}(-2\varepsilon_i - (n-i-(1/2))\lambda_i + \rho) \to M_{\mathfrak{q}}(-(n-i-(1/2))\lambda_i + \rho)$$

is standard. First recall that we have $M_{\mathfrak{q}}(-2\varepsilon_i - (n-i-(1/2))\lambda_i + \rho) = \mathcal{U}(\mathfrak{g}) \otimes F(\Omega_2|_{V(\mu+\epsilon_\gamma)^*})$, where $F(\Omega_2|_{V(\mu+\epsilon_\gamma)^*})$ is the finite dimensional simple $\mathfrak{l}$-submodule of $M_{\mathfrak{q}}(-(n-i-(1/2))\lambda_i + \rho)$ induced by the $\Omega_2|_{V(\mu+\epsilon_\gamma)^*}$ system. If v_h is a highest weight vector for $F(\Omega_2|_{V(\mu+\epsilon_\gamma)^*})$ then $\varphi_{\Omega_2}(1\otimes v_h) = 1\cdot v_h = v_h$. On the other hand, if $1\otimes v^+$ is a highest weight vector for $M(-2\varepsilon_i - (n-i-(1/2))\lambda_i + \rho)$ with weight $-2\varepsilon_i - (n-i-(1/2))\lambda_i$ and $\mathrm{pr} : M(-(n-i-(1/2))\lambda_i+\rho) \to M_{\mathfrak{q}}(-(n-i-(1/2))\lambda_i+\rho)$ is the canonical projection then $\varphi_{std}(1\otimes v_h) = (\mathrm{pr}\circ\varphi)(1\otimes v^+)$, where φ is an embedding of $M(-2\varepsilon_i - (n-i-(1/2))\lambda_i + \rho)$ into $M(-(n-i-(1/2))\lambda_i + \rho)$. Note that, by Proposition 8.3.6, we have $(\mathrm{pr}\circ\varphi)(1\otimes v^+) = \varphi_{std}(1\otimes v_h) \neq 0$. We want to show that v_h is a scalar multiple of $(\mathrm{pr}\circ\varphi)(1\otimes v^+)$. Moreover, since $M_{\mathfrak{q}}(-(n-i-(1/2))\lambda_i+\rho) \cong \mathcal{U}(\bar{\mathfrak{n}}) \otimes \mathbb{C}_{-(n-i-(1/2))\lambda_i+\rho}$ as an $\mathfrak{l}$-module and since $F(\Omega_2|_{V(\mu+\epsilon_\gamma)^*})$ is an $\mathfrak{l}$-submodule of $M_{\mathfrak{q}}(-(n-i-(1/2))\lambda_i + \rho)$, we have

$$v_h = u_h \otimes 1_{-(n-i-(1/2))\lambda_i} \tag{8.3.22}$$

and

$$(\mathrm{pr}\circ\varphi)(1\otimes v^+) = \tilde{u} \otimes 1_{-(n-i-(1/2))\lambda_i} \tag{8.3.23}$$

for some $u_h, \tilde{u} \in \mathcal{U}(\bar{\mathfrak{n}})\backslash\{0\}$. Hence, to show that v_h is a scalar multiple of $(\mathrm{pr}\circ\varphi)(1\otimes v^+)$, it suffices to show that u_h in (8.3.22) is a scalar multiple of $\tilde{u}$ in (8.3.23).

Observe that since $v_h = u_h \otimes 1_{-(n-i-(1/2))\lambda_i}$ is a highest weight vector for the simple $\mathfrak{l}$-submodule $F(\Omega_2|_{V(\mu+\epsilon_\gamma)^*})$ of $\mathcal{U}(\bar{\mathfrak{n}}) \otimes \mathbb{C}_{-(n-i-(1/2))\lambda_i+\rho}$, for all $\alpha \in \Pi(\mathfrak{l})$, we have $X_\alpha \cdot (u_h \otimes 1_{-(n-i-(1/2))\lambda_i}) = 0$. Therefore $\mathrm{ad}(X_\alpha)(u_h) = 0$ for all $\alpha \in \Pi(\mathfrak{l})$. Moreover, as $F(\Omega_2|_{V(\mu+\epsilon_\gamma)^*})$ has highest weight $-2\varepsilon_i-(n-i-(1/2))\lambda_i$ and is spanned by the elements of the form $u\otimes 1_{-(n-i-(1/2))\lambda_i}$ with $u \in \sigma(\mathrm{Sym}^2(\bar{\mathfrak{n}}))$, it follows that u_h is in $\sigma(\mathrm{Sym}^2(\bar{\mathfrak{n}}))$ with weight $-2\varepsilon_i$, where $\sigma : \mathrm{Sym}(\bar{\mathfrak{n}}) \to \mathcal{U}(\bar{\mathfrak{n}})$ is the symmetrization map.

Definition 8.3.24 *For $u \in \mathcal{U}(\bar{\mathfrak{n}})$, we say that u satisfies* Condition (H) *if u satisfies the following conditions:*

(1) $u \in \sigma(Sym^2(\bar{\mathfrak{n}}))$,

(2) u has weight $-2\varepsilon_i$, and

(3) $ad(X_\alpha)(u) = 0$ for all $\alpha \in \Pi(\mathfrak{l})$.

It follows from the observation made before Definition 8.3.24 that $u_h \in \mathcal{U}(\bar{\mathfrak{n}})$ in (8.3.22) satisfies Condition (H). Our first goal is to show that any element in $\mathcal{U}(\bar{\mathfrak{n}})$ that satisfies Condition (H) is a scalar multiples of u_h.

Lemma 8.3.25 *For any $\beta \in \Delta^+(\mathfrak{l}) \cup \Delta(\mathfrak{z}(\mathfrak{n}))$, we have $2\varepsilon_i - \beta \notin \Delta^+$.*

Proof. This lemma follows from a direct observation (see Appendix C for $\Delta^+(\mathfrak{l}) = \Delta^+(\mathfrak{l}_\gamma) \cup \Delta^+(\mathfrak{l}_{n\gamma})$ and $\Delta(\mathfrak{z}(\mathfrak{n}))$). ■

We write $\mathfrak{u} = \bigoplus_{\alpha\in\Delta^+} \mathfrak{g}_\alpha$ for the nilradical of $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{u}$ and we denote by $\bar{\mathfrak{u}}$ the opposite nilradical of $\mathfrak{u}$. Recall that, as $\mathfrak{n}$ is the nilradical of the parabolic subalgebra $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{n}$, we have $\mathfrak{n} \subset \mathfrak{u}$.

Lemma 8.3.26 *If u is in $Sym^2(\bar{\mathfrak{u}})$ with weight $-2\varepsilon_i$ then u is of the form*

$$AX_{-\varepsilon_i}^2 + \sum_{k=i+1}^{n} B_k X_{-(\varepsilon_i+\varepsilon_k)} X_{-(\varepsilon_i-\varepsilon_k)}$$

for some constants A and B_k. In particular, we have $u \in Sym^2(\bar{\mathfrak{n}})$.

Proof. If $u \in \sigma(\mathrm{Sym}^2(\bar{\mathfrak{u}}))$ with weight $-2\varepsilon_i$ then u is of the from

$$u = \sum c_\beta X_{-\beta} X_{-2\varepsilon_i+\beta}$$

for some constants c_β, where the sum runs over the roots $\beta \in \Delta^+ = \Delta^+(\mathfrak{l}) \cup \Delta(\mathfrak{g}(1)) \cup \Delta(\mathfrak{z}(\mathfrak{n}))$ so that $2\varepsilon_i - \beta \in \Delta^+$. By Lemma 8.3.25, the roots β must be in $\Delta(\mathfrak{g}(1))$. Thus if $\Delta_{2\varepsilon_i}(\mathfrak{g}(1)) = \{\beta \in \Delta(\mathfrak{g}(1)) \mid 2\varepsilon_i - \beta \in \Delta\}$ then

$$u = \sum_{\beta \in \Delta_{2\varepsilon_i}(\mathfrak{g}(1))} c_\beta X_{-\beta} X_{-2\varepsilon_i+\beta}.$$

By Appendix C, we have

$$\Delta(\mathfrak{g}(1)) = \{\varepsilon_j \pm \varepsilon_k \mid 1 \leq j \leq i \text{ and } i+1 \leq k \leq n\} \cup \{\varepsilon_j \mid 1 \leq j \leq i\}.$$

Thus,

$$\begin{aligned} \Delta_{2\varepsilon_i}(\mathfrak{g}(1)) &= \{\beta \in \Delta(\mathfrak{g}(1)) \mid 2\varepsilon_i - \beta \in \Delta\} \\ &= \{\varepsilon_i \pm \varepsilon_k \mid i+1 \leq k \leq n\} \cup \{\varepsilon_i\}. \end{aligned}$$

Therefore u is of the form

$$\begin{aligned} u &= \sum_{\beta \in \Delta_{2\varepsilon_i}(\mathfrak{g}(1))} c_\beta X_{-\beta} X_{-2\varepsilon_i+\beta} \\ &= c_{\varepsilon_i} X_{-\varepsilon_i}^2 + \sum_{k=i+1}^{n} c_{\varepsilon_i+\varepsilon_k} X_{-(\varepsilon_i+\varepsilon_k)} X_{-(\varepsilon_i-\varepsilon_k)} + \sum_{k=i+1}^{n} c_{\varepsilon_i-\varepsilon_k} X_{-(\varepsilon_i-\varepsilon_k)} X_{-(\varepsilon_i+\varepsilon_k)} \\ &= c_{\varepsilon_i} X_{-\varepsilon_i}^2 + \sum_{k=i+1}^{n} (c_{\varepsilon_i+\varepsilon_k} + c_{\varepsilon_i-\varepsilon_k}) X_{-(\varepsilon_i+\varepsilon_k)} X_{-(\varepsilon_i-\varepsilon_k)}. \end{aligned}$$

If $A = c_{\varepsilon_i}$ and $B_k = c_{\varepsilon_i+\varepsilon_k} + c_{\varepsilon_i-\varepsilon_k}$ then u can be expressed as

$$u = AX_{-\varepsilon_i}^2 + \sum_{k=i+1}^{n} B_k X_{-(\varepsilon_i+\varepsilon_k)} X_{-(\varepsilon_i-\varepsilon_k)}.$$

■

Proposition 8.3.27 *If $u \in \mathcal{U}(\bar{\mathfrak{n}})$ satisfies Condition (H) then u is a scalar multiple of u_h.*

Proof. We will show that any vector u that satisfies Condition (H) is of the form

$$u_0 = X_{-\varepsilon_i}^2 + \sum_{j=i+1}^{n} b_j X_{-(\varepsilon_i+\varepsilon_j)} X_{-(\varepsilon_i-\varepsilon_j)}, \tag{8.3.28}$$

where

$$b_j = 2(-1)^{n-j+1} \prod_{k=j}^{n} \frac{N_{\varepsilon_k-\varepsilon_{k+1},-(\varepsilon_i-\varepsilon_{k+1})} N_{\varepsilon_n,-\varepsilon_i}}{N_{\varepsilon_k-\varepsilon_{k+1},-(\varepsilon_i+\varepsilon_k)} N_{\varepsilon_n,-(\varepsilon_i+\varepsilon_n)}} \quad \text{for } j = i+1, \ldots, n-1 \tag{8.3.29}$$

and

$$b_n = -\frac{2N_{\varepsilon_n,-\varepsilon_i}}{N_{\varepsilon_n,-(\varepsilon_i+\varepsilon_n)}}. \tag{8.3.30}$$

If u satisfies Condition (H) then $u \in \sigma(\mathrm{Sym}^2(\bar{\mathfrak{n}})) \subset \sigma(\mathrm{Sym}^2(\bar{\mathfrak{u}}))$ and has weight $-2\varepsilon_i$. Thus it follows from Lemma 8.3.26 that u is of the from

$$u = AX_{-\varepsilon_i}^2 + \sum_{k=i+1}^{n} B_k X_{-(\varepsilon_i+\varepsilon_k)} X_{-(\varepsilon_i-\varepsilon_k)} \tag{8.3.31}$$

for some constants A and B_k. Now observe that, by the condition (3) in Definition 8.3.24, we have $\mathrm{ad}(X_\alpha)(u) = 0$ for all $\alpha \in \Pi(\mathfrak{l})$. Therefore, as $\varepsilon_j - \varepsilon_{j+1}$ and ε_n are in $\Pi(\mathfrak{l})$ for $j = i+1, \ldots, n-1$, we have

$$\mathrm{ad}(X_{\varepsilon_j-\varepsilon_{j+1}})(u) = 0 \quad \text{and} \quad \mathrm{ad}(X_{\varepsilon_n})(u) = 0$$

for $j = i+1, \ldots, n-1$. By (8.3.31), this means that for $j = i+1, \ldots, n-1$,

$$\mathrm{ad}(X_{\varepsilon_j-\varepsilon_{j+1}})\big(AX_{-\varepsilon_i}^2 + \sum_{k=i+1}^{n} B_k X_{-(\varepsilon_i+\varepsilon_k)} X_{-(\varepsilon_i-\varepsilon_k)}\big) = 0$$

and

$$\mathrm{ad}(X_{\varepsilon_n})\big(AX_{-\varepsilon_i}^2 + \sum_{k=i+1}^{n} B_k X_{-(\varepsilon_i+\varepsilon_k)} X_{-(\varepsilon_i-\varepsilon_k)}\big) = 0,$$

which are

$$B_j \,\mathrm{ad}(X_{\varepsilon_j-\varepsilon_{j+1}})(X_{-(\varepsilon_i+\varepsilon_j)} X_{-(\varepsilon_i-\varepsilon_j)}) + B_{j+1} \,\mathrm{ad}(X_{\varepsilon_j-\varepsilon_{j+1}})(X_{-(\varepsilon_i+\varepsilon_{j+1})} X_{-(\varepsilon_i-\varepsilon_{j+1})}) = 0$$

and

$$A \,\mathrm{ad}(X_{\varepsilon_n})(X_{-\varepsilon_i}^2) + B_n \,\mathrm{ad}(X_{\varepsilon_n})(X_{-(\varepsilon_i+\varepsilon_n)} X_{-(\varepsilon_i-\varepsilon_n)}) = 0.$$

By solving the system of linear equations, we obtain $B_j = b_j A$ for $j = i+1, \ldots, n$ with b_j in (8.3.29) and (8.3.30). Therefore, by (8.3.28) and (8.3.31), we get $u = Au_0$. ∎

By Proposition 8.3.27, to prove that φ_{Ω_2} in (8.3.5) is standard, it suffices to show that $\tilde{u}$ in (8.3.23) satisfies Condition (H). As $(\mathrm{pr} \circ \varphi)(1 \otimes v^+) = \tilde{u} \otimes 1_{-(n-i-(1/2))\lambda_i}$ is a highest weight vector with weight $-2\varepsilon_i - (n-i-(1/2))\lambda_i$, one can easily see that $\tilde{u}$ satisfies the conditions (2) and (3) in Definition 8.3.24. So we want to show that $\tilde{u}$ is in $\sigma(\mathrm{Sym}^2(\bar{\mathfrak{n}}))$. To do so we need to show several technical lemmas.

Lemma 8.3.32 *No polynomial in Sym*$^r(\bar{\mathfrak{n}})$ *for* $r \geq 3$ *has weight* $-2\varepsilon_i$.

Proof. Observe that the simple root $\alpha_{\mathfrak{q}} = \alpha_i$ has multiplicity ≥ 1 in the roots $\beta \in \Delta(\mathfrak{n})$. Therefore α_i has multiplicity greater than or equal to $r \leq 3$ in the weights for any polynomials in $\mathrm{Sym}^r(\bar{\mathfrak{n}})$. Since α_i has multiplicity 2 in $-2\varepsilon_i = -2\sum_{j=i}\alpha_j$, no polynomial in $\mathrm{Sym}^r(\bar{\mathfrak{n}})$ has weight $-2\varepsilon_i$. ■

Corollary 8.3.33 *Any non-zero polynomials in Sym*$^r(\bar{\mathfrak{u}})$ *with weight* $-2\varepsilon_i$ *for* $r \geq 3$ *have contributions from root vectors* $X_{-\alpha}$ *for* $\alpha \in \Delta^+(\mathfrak{l})$.

Proof. Since $\Delta(\mathfrak{u}) = \Delta^+(\mathfrak{l}) \cup \Delta(\mathfrak{n})$, this is an immediate consequence of Lemma 8.3.32. ■

Lemma 8.3.34 *If* $u \in \mathcal{U}(\bar{\mathfrak{u}})$ *has weight* $-2\varepsilon_i$ *then* u *can be expressed as*

$$u = AX_{-\varepsilon_i}^2 + \sum_{k=i+1}^{n} B_k X_{-(\varepsilon_i+\varepsilon_k)} X_{-(\varepsilon_i-\varepsilon_k)} + \sum_{\alpha \in \Delta^+(\mathfrak{l})} u^\alpha X_{-\alpha} \tag{8.3.35}$$

for some constants A *and* B_k, *and some elements* $u^\alpha \in \mathcal{U}(\bar{\mathfrak{u}})$.

Proof. If

$$\mathcal{U}_r(\bar{\mathfrak{u}}) = \{u \in \mathcal{U}(\bar{\mathfrak{u}}) \mid u \text{ has degree at most } r\}$$

then $\mathcal{U}(\bar{\mathfrak{u}}) = \bigcup_{r=1}^{\infty} \mathcal{U}_r(\bar{\mathfrak{u}})$ and $\mathcal{U}_{r+1}(\bar{\mathfrak{u}})/\mathcal{U}_r(\bar{\mathfrak{u}}) \cong \mathrm{Sym}^{r+1}(\bar{\mathfrak{u}})$. We show this lemma by induction on the degree r for $\mathcal{U}_r(\bar{\mathfrak{u}})$. First observe that since $-2\varepsilon_i \notin \Delta$, the element u cannot be in $\mathcal{U}_1(\bar{\mathfrak{u}}) = \bar{\mathfrak{u}}$. Thus if $u \in \mathcal{U}_2(\bar{\mathfrak{u}})$ then $u \in \mathrm{Sym}^2(\bar{\mathfrak{u}}) \cong \mathcal{U}_2(\bar{\mathfrak{u}})/\bar{\mathfrak{u}}$. Therefore, by Lemma 8.3.26, if $u \in \mathcal{U}_2(\bar{\mathfrak{u}})$ then $u = AX_{-\varepsilon_i}^2 + \sum_{k=i+1}^{n} B_k X_{-(\varepsilon_i+\varepsilon_k)} X_{-(\varepsilon_i-\varepsilon_k)}$ for some constants A and B_k. Now assume that this lemma holds for $u \in \mathcal{U}_r(\bar{\mathfrak{u}})$ for $3 \leq r \leq t$, and suppose that $u \in \mathcal{U}_{t+1}(\bar{\mathfrak{u}})$. By Corollary 8.3.33, any polynomials in $\mathcal{U}_{t+1}(\bar{\mathfrak{u}})/\mathcal{U}_t(\bar{\mathfrak{u}}) \cong \mathrm{Sym}^{t+1}(\bar{\mathfrak{u}})$ with weight $-2\varepsilon_i$ have contributions from root vectors in $\mathfrak{l}$. By permuting the root vectors, in $\mathcal{U}_{t+1}(\bar{\mathfrak{u}})$, those polynomials can be expressed as

$$(\text{some polynomial in } \mathcal{U}_t(\bar{\mathfrak{u}})) + \sum_{\alpha \in \Delta^+(\mathfrak{l})} v^\alpha X_{-\alpha}$$

with some $v^\alpha \in \mathcal{U}_t(\bar{\mathfrak{u}})$. Therefore the element $u \in \mathcal{U}_{t+1}(\bar{\mathfrak{u}})$ is of the form

$$u = p + \sum_{\alpha \in \Delta^+(\mathfrak{l})} v^\alpha X_{-\alpha}$$

for some $p, v^\alpha \in \mathcal{U}_t(\bar{\mathfrak{u}})$. By the induction hypothesis, the polynomial $p \in \mathcal{U}_t(\bar{\mathfrak{u}})$ can be expressed as

$$p = AX^2_{-\varepsilon_i} + \sum_{k=i+1}^{n} B_k X_{-(\varepsilon_i+\varepsilon_k)} X_{-(\varepsilon_i-\varepsilon_k)} + \sum_{\alpha \in \Delta^+(\mathfrak{l})} \hat{u}^\alpha X_{-\alpha}$$

for some constants A and B_k, and some elements $\hat{u}^\alpha \in \mathcal{U}_{t-1}(\bar{\mathfrak{u}})$. If $u^\alpha = \hat{u}^\alpha + v^\alpha$ then u is of the form in (8.3.35). By induction, this lemma follows. ∎

Now we are ready to show that the map φ_{Ω_2} in (8.3.5) is standard. Recall that if $1 \otimes v^+$ is a highest weight vector for $M(-2\varepsilon_i - (n-i-(1/2))\lambda_i + \rho)$ with weight $-2\varepsilon_i - (n-i-(1/2))\lambda_i$ and $\mathrm{pr} : M(-(n-i-(1/2))\lambda_i + \rho) \to M_{\mathfrak{q}}(-(n-i-(1/2))\lambda_i + \rho)$ is the canonical projection then $\varphi_{std}(1 \otimes v_h) = (\mathrm{pr} \circ \varphi)(1 \otimes v^+)$, where φ is an embedding of $M(-2\varepsilon_i - (n-i-(1/2))\lambda_i + \rho)$ into $M(-(n-i-(1/2))\lambda_i + \rho)$. By Proposition 8.3.6, we have $(\mathrm{pr} \circ \varphi)(1 \otimes v^+) = \varphi_{std}(1 \otimes v_h) \neq 0$.

Theorem 8.3.36 *If $\mathfrak{q}$ is the maximal two-step nilpotent parabolic subalgebra of type $B_n(i)$ for $3 \leq i \leq n-1$ then the map φ_{Ω_2} induced by the $\Omega_2|_{V(\mu+\epsilon_\gamma)^*}$ system is standard.*

Proof. Observe that, as $M(-(n-i-(1/2))\lambda_i + \rho) \cong \mathcal{U}(\bar{\mathfrak{u}}) \otimes \mathbb{C}_{-(n-i-(1/2))\lambda_i}$, the vector $\varphi(1 \otimes v^+)$ is of the form $\varphi(1 \otimes v^+) = u' \otimes 1_{-(n-i-(1/2))\lambda_i}$ for some $u' \in \mathcal{U}(\bar{\mathfrak{u}})$. Since

$\varphi(1 \otimes v^+)$ has weight $-2\varepsilon_i - (n-i-(1/2))\lambda_i$, the element u' has weight $-2\varepsilon_i$. Thus, by Lemma 8.3.34, we have

$$u = AX^2_{-\varepsilon_i} + \sum_{k=i+1}^{n} B_k X_{-(\varepsilon_i+\varepsilon_k)} X_{-(\varepsilon_i-\varepsilon_k)} + \sum_{\alpha \in \Delta^+(\mathfrak{l})} u^\alpha X_{-\alpha}$$

for some constants A and B_k, and some elements $u^\alpha \in \mathcal{U}(\bar{\mathfrak{u}})$. Observe that $X_{-\varepsilon_i}$, $X_{-(\varepsilon_i+\varepsilon_k)}$, and $X_{-(\varepsilon_i-\varepsilon_k)}$ are not in $\mathfrak{l}$. Thus, if $\tilde{u} = AX^2_{-\varepsilon_i} + \sum_{k=i+1}^{n} B_k X_{-(\varepsilon_i+\varepsilon_k)} X_{-(\varepsilon_i-\varepsilon_k)}$ then

$$\varphi_{std}(1 \otimes v_h) = (\mathrm{pr} \circ \varphi)(1 \otimes v^+) = \tilde{u} \otimes 1_{-(n-i-(1/2))\lambda_i}. \tag{8.3.37}$$

Clearly we have $\tilde{u} \in \sigma(\mathrm{Sym}^2(\bar{\mathfrak{n}}))$. Moreover, as $(\mathrm{pr} \circ \varphi)(1 \otimes v^+)$ is a highest weight vector for weight $-2\varepsilon_i - (n-i-(1/2))\lambda_i$, the element $\tilde{u}$ satisfies the conditions (2) and (3) in Definition 8.3.24; hence, it satisfies Condition (H). Thus, by Proposition 8.3.27, there exists a constant c so that $\tilde{u} = cu_h$ with u_h in (8.3.22). By Proposition 8.3.6, we have $\tilde{u} \neq 0$; thus $c \neq 0$. Since $\varphi_{\Omega_2}(1 \otimes v_h) = v_h = u_h \otimes 1_{-(n-i-(1/2))\lambda_i}$, it follows from (8.3.37) that $\varphi_{\Omega_2}(1 \otimes v_h) = (1/c)\varphi_{std}(1 \otimes v_h)$. ■

In Table 8.1 below we summarize the classification on the maps φ_{Ω_2}.

Table 8.1: The Homomorphism φ_{Ω_2} for the Non-Heisenberg Case

Parabolic subalgebra $\mathfrak{q}$	$\Omega_2\|_{V(\mu+\epsilon_\gamma)^*}$	$\Omega_2\|_{V(\mu+\epsilon_{n\gamma})^*}$
$B_n(i), 3 \leq i \leq n-2$	standard	non-standard
$B_n(n-1)$	standard	?
$B_n(n)$	standard	–
$C_n(i), 2 \leq i \leq n-1$	?	standard
$D_n(i), 3 \leq i \leq n-3$	non-standard	non-standard
$E_6(3)$	non-standard	non-standard
$E_6(5)$	non-standard	non-standard
$E_7(2)$	non-standard	–
$E_7(6)$	non-standard	non-standard
$E_8(1)$	non-standard	–
$F_4(4)$	standard	–

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APPENDIX A

Reducibility Points

By Corollary 2.7.7, if an Ω_k system is conformally invariant over the line bundle $\mathcal{L}_{s_0}$ then the corresponding generalized Verma module is reducible. Then, in this appendix, to support our results in Table 7.1, we shall show all the parameters of $t \in \mathbb{C}$ for which the generalized Verma modules of $\mathfrak{q}$ listed in (3.3.3) are reducible. We achieve it in Theorem A.5.1.

Here we recall some notation. For any $\mathrm{ad}(\mathfrak{h})$-invariant proper subspace $V \subset \mathfrak{g}$, we denote by $\Delta(V)$ the set of roots α so that $\mathfrak{g}_\alpha \subset V$. We write $\Delta^+(V) = \Delta^+ \cap \Delta(V)$. If $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{n}$ is a standard parabolic of $\mathfrak{g}$ then let $\Pi(\mathfrak{l})$ and $W(\mathfrak{l})$ denote the simple system of $\Delta^+(\mathfrak{l})$ and the Weyl group of $\Delta(\mathfrak{l})$, respectively. We identify $W(\mathfrak{l})$ with the subgroup of the Weyl group W of $\mathfrak{g}$ generated by $\{s_\alpha \mid \alpha \in \Pi(\mathfrak{l})\}$. We write $\mathfrak{z}(\mathfrak{l})$ for the center of $\mathfrak{l}$. Let ρ denote half the sum of positive roots of $\mathfrak{g}$.

A.1 Verma modules and Generalized Verma Modules

The aim of this section is to review on the Verma modules and the generalized Verma modules. We start by defining the Verma modules. For $\lambda \in \mathfrak{h}^*$, let $\mathbb{C}_\lambda$ be the one-dimensional $\mathcal{U}(\mathfrak{b})$-module defined by

$$H \cdot 1 = \lambda(H)1 \quad \text{for all } H \in \mathfrak{h}$$
$$X \cdot 1 = 0 \quad \text{for all } X \in \mathfrak{u},$$

where $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{u}$ with $\mathfrak{u} = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$. The **Verma module** $M(\lambda)$ with highest weight

$\lambda - \rho$ is the left $\mathcal{U}(\mathfrak{g})$-module given by

$$M(\lambda) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C}_{\lambda-\rho}.$$

We denote by $L(\lambda)$ its unique irreducible quotient.

Let $\mathcal{O}$ be the BGG category and $\mathcal{O}_\lambda$ be the full subcategory of $\mathcal{O}$ consisting of the modules of $\mathcal{O}$ with generalized infinitesimal character λ. For module M of $\mathcal{O}$ we denote by $[M]$ its formal character. The formal character of Verma module $M(\lambda)$ is given by

$$[M(\lambda)] = D^{-1}e^{\lambda},$$

where D is the Weyl denominator, namely, $D = e^{\rho}\prod_{\alpha\in\Delta^+}(1 - e^{-\alpha})$, and e^{λ} is the $\mathbb{Z}$-valued function on $\mathfrak{h}^*$ that takes the value one at λ and zero elsewhere. If W_λ is the integral Weyl group of λ then $\{[M(w\lambda)] : w \in W_\lambda\}$ and $\{[L(w\lambda)] : w \in W_\lambda\}$ form $\mathbb{Z}$-bases for the Grothendieck group $K(\mathcal{O}_\lambda)$.

Fix $\mathfrak{q}$ a parabolic subalgebra containing $\mathfrak{b}$ and write $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{n}$. For applications of $\mathfrak{q}$ to representation theory, the relative category $\mathcal{O}^{\mathfrak{q}}$ is often used. The basis of the corresponding Grothendieck group $K(\mathcal{O}^{\mathfrak{q}})$ is given by generalized Verma modules. These modules are defined as follows.

Define

$$\mathbf{P}^+_{\mathfrak{l}} = \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, \alpha^\vee \rangle \in 1 + \mathbb{Z}_{\geq 0} \text{ for all } \alpha \in \Pi(\mathfrak{l})\}.$$

For $\nu \in \mathbf{P}^+_{\mathfrak{l}}$, let $V(\nu - \rho)$ be the irreducible finite dimensional $\mathcal{U}(\mathfrak{l})$-module with highest weight $\nu - \rho$.[1] Extend $V(\nu - \rho)$ to be a $\mathcal{U}(\mathfrak{q})$-module by letting $\mathfrak{n}$ act trivially. Then define the **generalized Verma module** $M_{\mathfrak{q}}(\nu)$ with highest weight $\nu - \rho$ by means of

$$M_{\mathfrak{q}}(\nu) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{q})} V(\nu - \rho).$$

[1]See Section 3.2 what we mean by a highest weight of a finite dimensional representation of reductive algebra $\mathfrak{l}$.

It is clear that $M_{\mathfrak{q}}(\nu)$ is a highest weight $\mathcal{U}(\mathfrak{g})$-module. So it follows from the universal property of the Verma module $M(\nu)$ that $M_{\mathfrak{q}}(\nu)$ is a quotient of $M(\nu)$; in particular, $L(\nu)$ is its unique simple quotient. The formal character of $M_{\mathfrak{q}}(\nu)$ is given by

$$[M_{\mathfrak{q}}(\nu)] = D^{-1} \sum_{w \in W(\mathfrak{l})} (-1)^{l(w)} e^{w\nu},$$

where $l(w)$ is the length of $w \in W(\mathfrak{l})$.

Define

$$\mathbf{P}_{\mathfrak{l}}^{+}(1) = \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, \alpha^\vee \rangle = 1 \text{ for all } \alpha \in \Pi(\mathfrak{l})\}. \tag{A.1.1}$$

It is easy to see that $\dim(E(\nu - \rho)) = 1$ if and only if $\nu \in \mathbf{P}_{\mathfrak{l}}^{+}(1)$. In this case the generalized Verma module $M_{\mathfrak{q}}(\nu)$ is called a **scalar generalized Verma module** due to Boe [3].

In Section A.2, we shall state a criterion due to Jantzen that determines whether or not a given generalized Verma module is irreducible. To conclude this section we summarize some technical results so that the criterion can be introduced easily.

We start by simple necessary and sufficient conditions on the irreducibility of generalized Verma modules. Set $\Lambda_{\mathfrak{l}}^{+} = \{\nu \in \mathfrak{h}^* \mid \langle \nu, \alpha^\vee \rangle \in \mathbb{Z}_{\geq 0} \text{ for all } \alpha \in \Pi(\mathfrak{l})\}$.

Theorem A.1.2 [10, Theorem 9.12] *Let $\nu \in \mathfrak{h}^*$ with $\nu - \rho \in \Lambda_{\mathfrak{l}}^{+}$. Then if $\langle \nu, \beta^\vee \rangle \notin \mathbb{Z}_{>0}$ for all $\beta \in \Delta(\mathfrak{n})$ then $M_{\mathfrak{q}}(\nu)$ is irreducible. The converse also holds if ν is regular.*

Remark A.1.3 *Our convention on $M_{\mathfrak{q}}(\lambda)$ is different from [10] by the ρ-shift.*

In order to state Jantzen's criterion we need introduce extra notation. For $\lambda \in \mathfrak{h}^*$, define

$$Y(\lambda) = D^{-1} \sum_{w \in W(\mathfrak{l})} (-1)^{l(w)} e^{w\lambda}$$

with D the Weyl denominator. It is clear from the definition of $Y(\lambda)$ that we have $Y(\lambda) = [M_{\mathfrak{q}}(\lambda)]$ if $\lambda \in \mathbf{P}_{\mathfrak{l}}^{+}$. Moreover $Y(\lambda)$ has the following properties.

Proposition A.1.4 [23, Corollary 2.2.10] *We have the following properties:*

(1) If $\lambda \in \mathfrak{h}^*$ is $\Delta(\mathfrak{l})$-singular then $Y(\lambda) = 0$.

(2) For $\lambda \in \mathfrak{h}^*$ and $w \in W(\mathfrak{l})$ we have $Y(\lambda) = (-1)^{l(w)}Y(w\lambda)$.

As we defined in Section 5.2, a weight $\lambda \in \mathfrak{h}^*$ is said to be $\Delta(\mathfrak{l})$-dominant if $\langle \lambda, \alpha \rangle \geq 0$ for all $\alpha \in \Delta^+(\mathfrak{l})$, and $\Delta(\mathfrak{l})$-regular if $\langle \lambda, \alpha^\vee \rangle \neq 0$ for all $\alpha \in \Delta(\mathfrak{l})$. If $\lambda \in \mathfrak{h}^*$ is not $\Delta(\mathfrak{l})$-regular then we say that λ is $\Delta(\mathfrak{l})$-singular. The following corollary then shows that the converse of Proposition A.1.4 (1) holds if λ is an integral $\Delta(\mathfrak{l})$-regular weight.

Corollary A.1.5 *If $\lambda \in \mathfrak{h}^*$ satisfies $\langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}\backslash\{0\}$ for all $\alpha \in \Delta(\mathfrak{l})$ then $Y(\lambda) \neq 0$.*

Proof. If $\langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}\backslash\{0\}$ for all $\alpha \in \Delta(\mathfrak{l})$ then there exists $w \in W(\mathfrak{l})$ so that $w\lambda$ is an element of $\mathbf{P}^+_{\mathfrak{l}}$. By Proposition A.1.4 (2), we have

$$Y(\lambda) = (-1)^{l(w)}Y(w\lambda) = (-1)^{l(w)}[M_{\mathfrak{q}}(w\lambda)] \neq 0.$$

■

A.2 Jantzen's Criterion

The purpose of this section is to introduce the irreducibility criterion due to Jantzen for generalized Verma modules. We only state a specialization for scalar generalized Verma modules of maximal parabolic subalgebra $\mathfrak{q}$. If V is an $\mathrm{ad}(\mathfrak{h})$-invariant proper subspace of $\mathfrak{g}$ then we write $\rho(V)$ for half the sum of positive roots in $\Delta(V)$.

Let $\mathfrak{g}$ be a complex simple Lie algebra with rank greater than one, and let $\mathfrak{q}$ be the maximal parabolic subalgebra of $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{n}$ determined by a simple root $\alpha_{\mathfrak{q}} \in \Pi$. As $\mathfrak{g}$ has rank greater than one and $\mathfrak{q}$ is a maximal parabolic subalgebra determined by $\alpha_{\mathfrak{q}}$, the center $\mathfrak{z}(\mathfrak{l}) = \bigcap_{\alpha \in \Pi\backslash\{\alpha_{\mathfrak{q}}\}} \ker(\alpha)$ has dimension one. Since $\mathfrak{z}(\mathfrak{l})^* = \mathbb{C}\lambda_{\mathfrak{q}}$ with $\lambda_{\mathfrak{q}}$ the fundamental weight of $\alpha_{\mathfrak{q}}$, the set $\mathbf{P}^+_{\mathfrak{l}}(1)$ defined in (A.1.1) becomes

$$\mathbf{P}^+_{\mathfrak{l}}(1) = \{t\lambda_{\mathfrak{q}} + \rho(\mathfrak{l}) \mid t \in \mathbb{C}\}.$$

Therefore, if

$$\Theta_t = t\lambda_{\mathfrak{q}} + \rho(\mathfrak{l}) \quad \text{with } t \in \mathbb{C}$$

then any scalar generalized Verma modules of $\mathfrak{q}$ may be parametrized by $t \in \mathbb{C}$ as

$$M_{\mathfrak{q}}(\Theta_t) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{q})} \mathbb{C}_{\Theta_t - \rho} \tag{A.2.1}$$

with infinitesimal character Θ_t. Moreover, since $\rho(\mathfrak{n}) = \rho - \rho(\mathfrak{l})$, we have $\rho(\mathfrak{n}) \in \mathfrak{z}(\mathfrak{l})^*$ and so $\rho(\mathfrak{n}) = c_0\lambda_{\mathfrak{q}}$ for some $c_0 \in \mathbb{C}$. Thus the scalar generalized Verma module $M_{\mathfrak{q}}(\Theta_t)$ may be expressed as

$$M_{\mathfrak{q}}(\Theta_t) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{q})} \mathbb{C}_{(t-c_0)\lambda_{\mathfrak{q}}}$$

with infinitesimal character

$$\Theta_t = (t - c_0)\lambda_{\mathfrak{q}} + \rho.$$

Observe that the weight $2\rho(\mathfrak{n})$ is integral and $\langle \rho(\mathfrak{n}), \alpha_0^\vee \rangle \geq 1$, so it follows that $c_0 \in \frac{1}{2}\mathbb{Z}_{>0} = (\frac{1}{2} + \mathbb{Z}_{\geq 0}) \cup (1 + \mathbb{Z}_{\geq 0})$.

In [11] Jantzen introduced a very powerful criterion that determines whether or not given generalized Verma module is irreducible. Although the criterion works for any generalized Verma modules, we only state here the specialization of the criterion to the present situation. For the general statement of Jantzen's criterion see for instance Satz 3 of [11] or Theorem 9.13 of [10].

If

$$S_t = \{\beta \in \Delta(\mathfrak{n}) \mid \langle \Theta_t, \beta^\vee \rangle \in 1 + \mathbb{Z}_{\geq 0}\}$$

then Jantzen's criterion for scalar generalized Verma modules of a maximal parabolic subalgebra $\mathfrak{q}$ reads as follows. This specialization of the criterion is from [23, Theorem 2.2.11].

Theorem A.2.2 (Jantzen's criterion) [11, Satz 3] *Let $\mathfrak{q}$ be a maximal parabolic subalgebra. Then the scalar generalized Verma module $M_{\mathfrak{q}}(\Theta_t)$ is irreducible if and only if*

$$\sum_{\beta \in S_t} Y(s_\beta(\Theta_t)) = 0. \tag{A.2.3}$$

To use Jantzen's criterion we need to determine whether or not $\sum_{\beta \in S_t} Y(s_\beta(\Theta_t))$ is zero. Then it is useful to know when terms $Y(s_\beta(\Theta_t))$ cancel out in (A.2.3). Proposition A.2.4 below deals with this issue. For $w \in W$ we say that w is an **odd** (resp. **even**) element if its length $l(w)$ is an odd (resp. even) integer.

Proposition A.2.4 *Let $\beta_0 \in S_t$ and assume that $Y(s_{\beta_0}(\Theta_t)) \neq 0$. Then $Y(s_{\beta_0}(\Theta_t))$ cancels out in (A.2.3) if and only if there exists $\beta \in S_t \backslash \{\beta_0\}$ with $Y(s_\beta(\Theta_t)) \neq 0$ so that $s_{\beta_0}(\Theta_t)$ and $s_\beta(\Theta_t)$ are conjugate by an odd element of $W(\mathfrak{l})$.*

Proof. If $s_{\beta_0}(\Theta_t)$ and $s_\beta(\Theta_t)$ are conjugate by an odd element of $W(\mathfrak{l})$ then, by Proposition A.1.4, $Y(s_{\beta_0}(\Theta_t))$ cancels out. Then suppose that $Y(s_{\beta_0}(\Theta_t))$ cancels out in (A.2.3). Then there exist $\beta_1, \ldots, \beta_n \in S_t$ with $Y(s_{\beta_i}(\Theta_t)) \neq 0$, so that

$$Y(s_{\beta_0}(\Theta_t)) + \sum_{i=1}^{n} Y(s_{\beta_i}(\Theta_t)) = 0. \tag{A.2.5}$$

Since $Y(s_{\beta_i}(\Theta_t)) \neq 0$, by Proposition A.1.4, the weights $s_{\beta_i}(\Theta_t)$ are all $\Delta(\mathfrak{l})$-regular. Thus we have $\langle s_{\beta_i}(\Theta_t), \alpha^\vee \rangle \in \mathbb{Z}\backslash\{0\}$ for all $\alpha \in \Delta(\mathfrak{l})$. This implies that for each $i = 0, \ldots, n$, there exists $w_i \in W(\mathfrak{l})$ so that $w_i(s_{\beta_i}(\Theta_t)) \in \mathbf{P}^+_{\mathfrak{l}}$. If $\lambda_i = w_i(s_{\beta_i}(\Theta_t))$ then it follows from Proposition A.1.4 that (A.2.5) becomes

$$(-1)^{l(w_0)} Y(\lambda_{\mathfrak{q}}) + \sum_{i=1}^{n} (-1)^{l(w_i)} Y(\lambda_i) = 0. \tag{A.2.6}$$

Moreover, by combining the same terms, (A.2.6) may be written as

$$m_0 Y(\lambda_{\mathfrak{q}}) + \sum_{k=1}^{r} m_k Y(\lambda_{i_k}) = 0 \tag{A.2.7}$$

with $m_0, \ldots, m_r$ some integers. Since $\lambda_{\mathfrak{q}}$ and λ_{i_k} are elements in $\mathbf{P}_{\mathfrak{l}}^+$, we have $Y(\lambda_{\mathfrak{q}}) = [M_{\mathfrak{q}}(\lambda_{\mathfrak{q}})]$ and $Y(\lambda_{i_k}) = [M_{\mathfrak{q}}(\lambda_{i_k})]$ for all $k = 1, \ldots, r$. Thus (A.2.7) is

$$m_0[M_{\mathfrak{q}}(\lambda_{\mathfrak{q}})] + \sum_{k=1}^{r} m_k[M_{\mathfrak{q}}(\lambda_{i_k})] = 0. \tag{A.2.8}$$

Since $[M_{\mathfrak{q}}(\lambda_{\mathfrak{q}})]$ and $[M_{\mathfrak{q}}(\lambda_{i_k})]$ for $k = 1, \ldots, r$ are linearly independent, by (A.2.8), we obtain that $m_k = 0$ for all $k = 0, \ldots, r$.

If $E = \{\beta \in \{\beta_1, \ldots, \beta_n\} \mid w_\beta(s_\beta(\Theta_t)) = \lambda_{\mathfrak{q}}\}$, where w_β is the element of $W(\mathfrak{l})$ such that $w_\beta(s_\beta(\Theta_t)) \in \mathbf{P}_{\mathfrak{l}}^+$, then m_0 may be expressed as

$$m_0 = (-1)^{l(w_0)} + \sum_{\beta \in E} (-1)^{l(w_\beta)}.$$

Since we have $m_0 = 0$, there exists $\delta \in E$ such that $(-1)^{l(w_0)} + (-1)^{l(w_\delta)} = 0$. Moreover, we have $w_0(s_{\beta_0}(\Theta_t)) = \lambda_{\mathfrak{q}} = w_\delta(s_\delta(\Theta_t))$ with $w_0, w_\delta \in W(\mathfrak{l})$. So $s_{\beta_0}(\Theta_t)$ and $s_\delta(\Theta_t)$ are $W(\mathfrak{l})$-conjugate. If those are conjugate by an even element of $W(\mathfrak{l})$ then Proposition A.1.4 implies that $Y(s_{\beta_0}(\Theta_t)) + Y(s_\delta(\Theta_t)) \neq 0$. On the other hand, by the equality $(-1)^{l(w_0)} + (-1)^{l(w_\delta)} = 0$ and the condition $w_0(s_{\beta_0}(\Theta_t)) = w_\delta(s_\delta(\Theta_t))$, we have

$$(-1)^{l(w_0)} Y(w_0(s_{\beta_0}(\Theta_t))) + (-1)^{l(w_\delta)} Y(w_\delta(s_\delta(\Theta_t))) = 0,$$

which is, by Proposition A.1.4, equivalent to

$$Y(s_{\beta_0}(\Theta_t)) + Y(s_\delta(\Theta_t)) = 0.$$

This is a contradiction. Therefore $s_{\beta_0}(\Theta_t)$ and $s_\delta(\Theta_t)$ are conjugate by an odd element of $W(\mathfrak{l})$. ■

To complete this section we give a couple of technical statements that will be used in later sections. Observe that parabolic subalgebra $\mathfrak{q}$ is the one corresponding to the subset $\Pi \backslash \{\alpha_0\} = \{\alpha \in \Pi \mid \langle \lambda_{\mathfrak{q}}, \alpha^\vee \rangle = 0\}$.

Lemma A.2.9 *Let $\beta_1, \beta_2 \in \Delta$. If $s_{\beta_1}(\Theta_t)$ and $s_{\beta_2}(\Theta_t)$ are $W(\mathfrak{l})$-conjugate then*

$$\langle \Theta_t, \beta_1^\vee \rangle \frac{2\langle \beta_1, \lambda_\mathfrak{q} \rangle}{||\alpha_\mathfrak{q}||^2} = \langle \Theta_t, \beta_2^\vee \rangle \frac{2\langle \beta_2, \lambda_\mathfrak{q} \rangle}{||\alpha_\mathfrak{q}||^2}.$$

Proof. Write $s_{\beta_1}(\Theta_t) = ws_{\beta_2}(\Theta_t)$ with $w \in W(\mathfrak{l})$. Then, by applying $2\langle \cdot, \lambda_\mathfrak{q} \rangle / ||\alpha_\mathfrak{q}||^2$ for both sides of $s_{\beta_1}(\Theta_t) = ws_{\beta_2}(\Theta_t)$, we obtain

$$\frac{2\langle s_{\beta_1}(\Theta_t), \lambda_\mathfrak{q} \rangle}{||\alpha_\mathfrak{q}||^2} = \frac{2\langle ws_{\beta_2}(\Theta_t), \lambda_\mathfrak{q} \rangle}{||\alpha_\mathfrak{q}||^2}. \tag{A.2.10}$$

Since the Weyl group action preserves the inner product and $W(\mathfrak{l})$ acts on $\lambda_\mathfrak{q}$ trivially, we have $\langle ws_{\beta_2}(\Theta_t), \lambda_\mathfrak{q} \rangle = \langle s_{\beta_2}(\Theta_t), \lambda_\mathfrak{q} \rangle$. Therefore, (A.2.10) is

$$\frac{2\langle s_{\beta_1}(\Theta_t), \lambda_\mathfrak{q} \rangle}{||\alpha_\mathfrak{q}||^2} = \frac{2\langle s_{\beta_2}(\Theta_t), \lambda_\mathfrak{q} \rangle}{||\alpha_\mathfrak{q}||^2}. \tag{A.2.11}$$

Now the proposed equation follows from $s_\alpha(\Theta_t) = \Theta_t - \langle \Theta_t, \alpha^\vee \rangle \alpha$ for $\alpha \in \Delta$. ■

Proposition A.2.12 *Let $\beta_0 \in S_t$ with $Y(s_{\beta_0}(\Theta_t)) \neq 0$. Assume that $\beta_1, \ldots, \beta_k$ are all the weights in $S_t \backslash \{\beta_0\}$ that satisfy both $Y(s_{\beta_j}(\Theta_t)) \neq 0$ and*

$$\langle \Theta_t, \beta_0^\vee \rangle \frac{2\langle \beta_0, \lambda_\mathfrak{q} \rangle}{||\alpha_\mathfrak{q}||^2} = \langle \Theta_t, \beta_j^\vee \rangle \frac{2\langle \beta_j, \lambda_\mathfrak{q} \rangle}{||\alpha_\mathfrak{q}||^2}.$$

If k is even then $M_\mathfrak{q}(\Theta_t)$ is reducible.

Proof. Set $E = \{\beta_0, \beta_1, \ldots, \beta_k\}$. By Lemma A.2.9, there is no $\delta \in S_t \backslash E$ with $Y(s_\delta(\Theta_t)) \neq 0$ so that $s_\delta(\Theta_t)$ is $W(\mathfrak{l})$-conjugate to $s_{\beta_i}(\Theta_t)$ for $\beta_i \in E$. Therefore, by Proposition A.2.4, the term $Y(s_{\beta_i}(\Theta_t))$ with $\beta_i \in E$ cancels out in (A.2.3) if and only if there exists $\beta_j \in E \backslash \{\beta_i\}$ so that $s_{\beta_i}(\Theta_t)$ and $s_{\beta_j}(\Theta_t)$ are $W(\mathfrak{l})$-conjugate by an odd element of $W(\mathfrak{l})$.

If k is even then since E contains an odd number of elements and we have $Y(s_{\beta_i}(\Theta_t)) \neq 0$ for all $\beta_i \in E$, there exists $\beta_j \in E$ so that $Y(s_{\beta_j}(\Theta_t))$ does not cancel out. Now Jantzen's criterion concludes that $M_\mathfrak{q}(\Theta_t)$ is reducible. ■

Proposition A.2.12 gives a sufficient condition on $M_\mathfrak{q}(\Theta_t)$ to be reducible. However, in general, it takes time to find out all the weights $\beta_1, \ldots, \beta_k$ that satisfy the conditions in the hypothesis. If $\mathfrak{g}$ is simply laced and if $\mathfrak{q}$ is a maximal two-step nilpotent parabolic of non-Heisenberg type then we can check the reducibility of $M_{\mathfrak{q}}(\Theta_t)$ more efficiently. It will be achieved in Section A.4.

A.3 Necessary Conditions of the Reducibility of $M_{\mathfrak{q}}(\Theta_t)$

Although Jantzen's criterion is very powerful, it is in general not easy to determine whether or not (A.2.3) is zero. The purpose of this short section is to introduce a couple of statements that reduce the number of parameters $t \in \mathbb{C}$ for $M_{\mathfrak{q}}(\Theta_t)$ that need to be checked by Jantzen's criterion for certain parabolics $\mathfrak{q}$. Hereafter, we assume that $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{n}$ is a maximal two-step nilpotent parabolic subalgebra of non-Heisenberg type with $\mathfrak{n} = \mathfrak{g}(1) \oplus \mathfrak{z}(\mathfrak{n})$. Note that the decomposition (3.3.1) of $\mathfrak{l}$ is irrelevant, the case that $\mathfrak{q}$ is of type $D_n(n-2)$ is included.

We begin this section with a technical lemma that will be used later. Observe that $\mathfrak{g}(1) = \{\beta \in \Delta^+ \mid \beta(H_{\mathfrak{q}}) = 1\}$ and $\mathfrak{z}(\mathfrak{n}) = \{\beta \in \Delta^+ \mid \beta(H_{\mathfrak{q}}) = 2\}$ with $H_{\mathfrak{q}} = \frac{2}{||\alpha_{\mathfrak{q}}||^2} H_{\lambda_{\mathfrak{q}}}$ in (3.2.1).

Lemma A.3.1 *Let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{g}(1) \oplus \mathfrak{z}(\mathfrak{n})$ be a maximal parabolic subalgebra of non-Heisenberg type. For $\beta \in \Delta(\mathfrak{n})$, we have $\langle \lambda_{\mathfrak{q}}, \beta^\vee \rangle = \frac{1}{2}$, 1, or 2.*

Proof. Since $\mathfrak{n} = \mathfrak{g}(1) \oplus \mathfrak{z}(\mathfrak{n})$, we have

$$\frac{2\langle \lambda_{\mathfrak{q}}, \beta \rangle}{||\alpha_{\mathfrak{q}}||^2} = \begin{cases} 1 & \text{if } \beta \in \Delta(\mathfrak{g}(1)) \\ 2 & \text{if } \beta \in \Delta(\mathfrak{z}(\mathfrak{n})). \end{cases}$$

Thus, the lemma is obvious when $||\beta||^2 = ||\alpha_{\mathfrak{q}}||^2$ or $||\beta||^2 = 2||\alpha_{\mathfrak{q}}||^2$. If $2||\beta||^2 = ||\alpha_{\mathfrak{q}}||^2$ then, by inspection, such β is always in $\Delta(\mathfrak{g}(1))$, and hence $\langle \lambda_{\mathfrak{q}}, \beta^\vee \rangle = 2$. ■

Proposition A.3.2 *Let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{g}(1) \oplus \mathfrak{z}(\mathfrak{n})$ be a maximal parabolic subalgebra of non-Heisenberg type. If $M_{\mathfrak{q}}(\Theta_t)$ is reducible then $t \in \frac{1}{2}\mathbb{Z}$.*

Proof. Observe that $\langle \Theta_t - \rho, \alpha^\vee \rangle = 0 \in \mathbb{Z}_{\geq 0}$ of all $\alpha \in \Pi(\mathfrak{l})$. If $M_\mathfrak{q}(\Theta_t)$ is reducible then, by Theorem A.1.2, there exists $\beta_0 \in \Delta(\mathfrak{n})$ with $\langle \Theta_t, \beta_0^\vee \rangle = \langle (t-c_0)\lambda_\mathfrak{q} + \rho, \beta_0^\vee \rangle = k \in \mathbb{Z}_{>0}$. Hence,

$$t = \frac{k - \langle \rho, \beta_0^\vee \rangle}{\langle \lambda_\mathfrak{q}, \beta_0^\vee \rangle} + c_0,$$

where $k - \langle \rho, \beta_0^\vee \rangle \in \mathbb{Z}$ and $c_0 \in \frac{1}{2}\mathbb{Z}_{>0}$. By Lemma A.3.1, we have $\langle \lambda_\mathfrak{q}, \beta_0^\vee \rangle = 1/2$, 1, or 2. Therefore $t \in \frac{1}{2}\mathbb{Z}$. ■

An irreducible $\mathfrak{l}$-submodule F of a generalized Verma module is called a *leading $\mathfrak{l}$-type* if $\mathfrak{n}$ acts on it trivially. Suppose that F is a leading $\mathfrak{l}$-type of $M_\mathfrak{q}(\Theta_t)$ that is not isomorphic to $\mathbb{C}_{(t-c_0)\lambda_\mathfrak{q}}$. If we write $\mathfrak{h} = \mathbb{C}H_{\lambda_\mathfrak{q}} \oplus \mathfrak{h}_{ss}$ with $\mathfrak{h}_{ss}$ a Cartan subalgebra of the semisimple part of $\mathfrak{l}$ then the highest weight of F has the form $z\lambda_\mathfrak{q} + \nu$ with $z \in \mathbb{C}$ and $\nu \perp \lambda_\mathfrak{q}$. On the other hand, a highest weight vector of F is of the form $u \otimes 1$ with $u \in \mathcal{U}_j(\bar{\mathfrak{n}})$ for some j, and as we observed in Section 3.2, $H_\mathfrak{q} = \frac{2}{||\alpha_\mathfrak{q}||^2} H_{\lambda_\mathfrak{q}}$ induces the 2-grading on $\mathfrak{g}$ and acts by -1 or -2 on $\bar{\mathfrak{n}}$. Therefore we have

$$\frac{2}{||\alpha_\mathfrak{q}||^2} H_{\lambda_\mathfrak{q}} \cdot (u \otimes 1) = \big(-m + \frac{2(t-c_0)}{||\alpha_\mathfrak{q}||^2} \lambda_\mathfrak{q}(H_{\lambda_\mathfrak{q}}) \big)(u \otimes 1)$$

with some $m \in 1 + \mathbb{Z}_{\geq 0}$, which is equivalent to

$$H_{\lambda_\mathfrak{q}} \cdot (u \otimes 1) = \big(-\frac{m}{2} \cdot \frac{||\alpha_\mathfrak{q}||^2}{||\lambda_\mathfrak{q}||^2} + (t - c_0) \big) \lambda_\mathfrak{q}(H_{\lambda_\mathfrak{q}})(u \otimes 1).$$

This shows that the highest weight of F is of the form

$$\big(\nu - \frac{m}{2} \cdot \frac{||\alpha_\mathfrak{q}||^2}{||\lambda_\mathfrak{q}||^2} \lambda_\mathfrak{q}\big) + (t - c_0)\lambda_\mathfrak{q}.$$

Proposition A.3.3 *Let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{g}(1) \oplus \mathfrak{z}(\mathfrak{n})$ be a maximal parabolic subalgebra of non-Heisenberg type. If $M_\mathfrak{q}(\Theta_t)$ is reducible then $t > 0$.*

Proof. Observe that if $M_\mathfrak{q}(\Theta_t)$ is reducible then there exists a leading $\mathfrak{l}$-type F in $M_\mathfrak{q}(\Theta_t)$, that is not isomorphic to $\mathbb{C}_{(t-c_0)\lambda_\mathfrak{q}}$. Then we have $\mathrm{Hom}_{\mathcal{U}(\mathfrak{g})}(\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{q})}$

$F, M_{\mathfrak{q}}(\Theta_t)) \neq 0$. In particular, these two generalized Verma modules have the same infinitesimal character. By the above observation, there exist a constant $m \in 1+\mathbb{Z}_{\geq 0}$ and a weight ν with $\nu \perp \lambda_{\mathfrak{q}}$ so that the infinitesimal character of F is of the form $(\nu - \frac{mk}{2}\lambda_{\mathfrak{q}}) + (t-c_0)\lambda_{\mathfrak{q}} + \rho = (\nu - \frac{mk}{2}\lambda_{\mathfrak{q}}) + t\lambda_{\mathfrak{q}} + \rho(\mathfrak{l})$ with $k = ||\alpha_{\mathfrak{q}}||^2/||\lambda_{\mathfrak{q}}||^2$. Therefore, the weights $(\nu - \frac{mk}{2}\lambda_{\mathfrak{q}}) + t\lambda_{\mathfrak{q}} + \rho(\mathfrak{l})$ and Θ_t are W-conjugate, which in particular implies that

$$||(\nu - \frac{mk}{2}\lambda_{\mathfrak{q}}) + t\lambda_{\mathfrak{q}} + \rho(\mathfrak{l})||^2 = ||\Theta_t||^2.$$

By expanding the both sides and solving for t, one obtains that

$$t = \frac{mk}{4} + \frac{1}{mk||\lambda_{\mathfrak{q}}||^2}\big(||\nu||^2 + 2\langle \nu, \rho(\mathfrak{l})\rangle\big) > 0.$$

■

By combining Proposition A.3.2 and Proposition A.3.3, we conclude the following statement.

Proposition A.3.4 *Let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{g}(1) \oplus \mathfrak{z}(\mathfrak{n})$ be a maximal parabolic subalgebra of non-Heisenberg type. Then $M_{\mathfrak{q}}(\Theta_t)$ is reducible only if $t \in \frac{1}{2}\mathbb{Z}_{>0} = (1+\mathbb{Z}_{\geq 0}) \cup (\frac{1}{2} + \mathbb{Z}_{\geq 0})$.*

A.4 Reducibility Criteria for Simply-Laced Case

In this section we specialize $\mathfrak{g}$ to be simply laced, and show that in this case a simple condition on the heights of roots significantly reduces the number of cases, for which we need to apply Jantzen's criterion. This is done in Theorem A.4.10, Theorem A.4.15, and Corollary A.4.17.

Let $\mathfrak{g}$ be a complex simple simply laced Lie algebra. We denote by $\mathrm{ht}(\alpha)$ the height of α for $\alpha \in \Delta$ and by $\mathrm{ht}(\mathfrak{l})$ the largest value of the heights of $\alpha \in \Delta(\mathfrak{l})$. We continue to call μ and γ the highest weights of $\mathfrak{g}(1)$ and $\mathfrak{z}(\mathfrak{n})$, respectively.

First we prove a couple of useful properties on the heights of $\alpha \in \Delta$.

Lemma A.4.1 *If $\mathfrak{g}$ is simply laced then $\langle \rho, \alpha^\vee \rangle = ht(\alpha)$ for all $\alpha \in \Delta$.*

Proof. Since $\mathfrak{g}$ is simply laced, the map $\alpha \mapsto \alpha^\vee$ is linear. Therefore for $\sum m_i\alpha_i \in \Delta$ with $\alpha_i \in \Pi$, we have

$$\langle \rho, \big(\sum m_i\alpha_i\big)^\vee\rangle = \sum m_i\langle \rho, \alpha_i^\vee\rangle = \sum m_i = \mathrm{ht}\big(\sum m_i\alpha_i\big).$$

■

Lemma A.4.2 *If $\mathfrak{g}$ is simply laced then, for $\alpha \in \Delta$,*

$$\langle \Theta_t, \alpha^\vee\rangle = \begin{cases} ht(\alpha) & \text{if } \alpha \in \Delta(\mathfrak{l}) \\ (t-c_0) + ht(\alpha) & \text{if } \alpha \in \Delta(\mathfrak{g}(1)) \\ 2(t-c_0) + ht(\alpha) & \text{if } \alpha \in \Delta(\mathfrak{z}(\mathfrak{n})). \end{cases}$$

Proof. Observe that $\Delta(\mathfrak{l})$, $\Delta(\mathfrak{g}(1))$, and $\Delta(\mathfrak{z}(\mathfrak{n}))$ are the sets of roots α so that $2\langle\lambda_{\mathfrak{q}}, \alpha\rangle/||\alpha_{\mathfrak{q}}||^2 = 0, 1, 2$, respectively. Since $\mathfrak{g}$ is simply laced, by the equal-length property of roots, we have

$$\langle \lambda_{\mathfrak{q}}, \alpha^\vee\rangle = \begin{cases} 0 & \text{if } \alpha \in \Delta(\mathfrak{l}) \\ 1 & \text{if } \alpha \in \Delta(\mathfrak{g}(1)) \\ 2 & \text{if } \alpha \in \Delta(\mathfrak{z}(\mathfrak{n})). \end{cases}$$

Now this lemma simply follows from the fact that $\Theta_t = (t-c)\lambda_{\mathfrak{q}} + \rho$ and Lemma A.4.1. ■

Observe that $S_t = \{\beta \in \Delta(\mathfrak{n}) \mid \langle \Theta_t, \beta^\vee\rangle \in 1 + \mathbb{Z}_{\geq 0}\}$. By Proposition A.3.4, we need only consider the reducibility of $M_{\mathfrak{q}}(\Theta_t)$ for $t \in \frac{1}{2}\mathbb{Z}_{>0}$.

Lemma A.4.3 *If $\mathfrak{g}$ is simply laced then, for $t \in \frac{1}{2}\mathbb{Z}_{>0}$, the set S_t is determined as follows:*

1. *If $t - c_0 \notin \mathbb{Z}$ then*

$$S_t = \{\beta \in \Delta(\mathfrak{z}(\mathfrak{n})) \mid ht(\beta) > 2c_0 - 2t\}.$$

2. *If $t - c_0 \in \mathbb{Z}$ then*

$$S_t = \{\beta \in \Delta(\mathfrak{g}(1)) \mid ht(\beta) > c_0 - t\} \cup \{\beta \in \Delta(\mathfrak{z}(\mathfrak{n}) \mid ht(\beta) > 2c_0 - 2t\}.$$

Proof. As $c_0 \in \frac{1}{2}\mathbb{Z}_{>0}$, the proposed equalities follow from Lemma A.4.2. ■

Proposition A.4.4 *Suppose that $\mathfrak{g}$ is simply laced and let $\alpha \in \Delta(\mathfrak{l})$ and $\beta \in S_t$. Then $\langle s_\beta(\Theta_t), \alpha^\vee \rangle = 0$ if and only if $\beta - \alpha \in \Delta$ and $ht(\alpha) = \langle \Theta_t, \beta^\vee \rangle$.*

Proof. By Lemma A.4.2, we have $\langle \Theta_t, \alpha^\vee \rangle = \mathrm{ht}(\alpha)$. Then it follows that $\langle s_\beta(\Theta_t), \alpha^\vee \rangle = \mathrm{ht}(\alpha) - \langle \Theta_t, \beta^\vee \rangle \langle \beta, \alpha^\vee \rangle$. Since $\mathfrak{g}$ is simply laced, by Lemma 3.4.4, we have $\langle \beta, \alpha^\vee \rangle \in \{-1, 0, 1\}$. Observe that $\langle \Theta_t, \beta^\vee \rangle \in 1 + \mathbb{Z}_{\geq 0}$ as $\beta \in S_t$. Thus, if $\langle \beta, \alpha^\vee \rangle = -1$, or 0 then

$$\langle s_\beta(\Theta_t), \alpha^\vee \rangle = \mathrm{ht}(\alpha) - \langle \Theta_t, \beta^\vee \rangle \langle \beta, \alpha^\vee \rangle \geq \mathrm{ht}(\alpha) \neq 0.$$

If $\langle \beta, \alpha^\vee \rangle = 1$ then we have $\langle s_\beta(\Theta_t), \alpha^\vee \rangle = \mathrm{ht}(\alpha) - \langle \Theta_t, \beta^\vee \rangle$. Therefore $\langle s_\beta(\Theta_t), \alpha^\vee \rangle = 0$ if and only if $\langle \beta, \alpha^\vee \rangle = 1$ and $\mathrm{ht}(\alpha) = \langle \Theta_t, \beta^\vee \rangle$. As $\mathfrak{g}$ is simply laced, the condition $\langle \beta, \alpha^\vee \rangle = 1$ is equivalent to $\beta - \alpha \in \Delta$. ■

Proposition A.4.5 *Suppose that $\mathfrak{g}$ is simply laced and let $\beta \in S_t$. If $\langle \Theta_t, \beta^\vee \rangle > ht(\mathfrak{l})$ then $Y(s_\beta(\Theta_t)) \neq 0$.*

Proof. If $\alpha \in \Delta(\mathfrak{l})$ then since $\langle \Theta_t, \beta^\vee \rangle > \mathrm{ht}(\mathfrak{l})$, we have $\langle \Theta_t, \beta^\vee \rangle \neq \mathrm{ht}(\alpha)$. Then it follows from Proposition A.4.4 that $\langle s_\beta(\Theta_t), \alpha^\vee \rangle \neq 0$. So, we have $\langle s_\beta(\Theta_t), \alpha^\vee \rangle \in \mathbb{Z} \backslash \{0\}$. Now Corollary A.1.5 concludes that $Y(s_\beta(\Theta_t)) \neq 0$. ■

Proposition A.4.6 *Suppose that $\mathfrak{g}$ is simply laced and let $\beta \in S_m$ for some $m \in \frac{1}{2}\mathbb{Z}_{>0}$. If $\langle \Theta_m, \beta^\vee \rangle > ht(\mathfrak{l})$ then $\beta \in S_t$ and $Y(s_\beta(\Theta_t)) \neq 0$ for all $t \in m + \mathbb{Z}_{\geq 0}$.*

Proof. There are two cases, namely, $\beta \in \Delta(\mathfrak{g}(1))$ or $\beta \in \Delta(\mathfrak{z}(\mathfrak{n}))$. We only prove the case $\beta \in \Delta(\mathfrak{g}(1))$, since the other case may be proven similarly. If $t \in m + \mathbb{Z}_{\geq 0}$ then $c_0 - m \geq c_0 - t$. On the other hand, since $\beta \in \Delta(\mathfrak{g}(1)) \cap S_m$, Lemma A.4.3 implies that $\mathrm{ht}(\beta) > c_0 - m$. Therefore we obtain $\mathrm{ht}(\beta) > c_0 - t$. Hence, by Lemma A.4.3, $\beta \in S_t$. Moreover, by Lemma A.4.2,

$$\langle \Theta_t, \beta^\vee \rangle = (t - c_0) + \mathrm{ht}(\beta) \geq (m - c_0) + \mathrm{ht}(\beta) = \langle \Theta_m, \beta^\vee \rangle > \mathrm{ht}(\mathfrak{l}).$$

Now Proposition A.4.5 concludes that $Y(s_\beta(\Theta_t)) \neq 0$. ■

Here is a technical lemma that will be used in the proof for Theorem A.4.10 below. Observe that c_0 is the constant so that $\rho(\mathfrak{n}) = c_0 \lambda_{\mathfrak{q}}$.

Lemma A.4.7 *Suppose that $\mathfrak{g}$ is simply laced and that $\langle \Theta_t, \gamma^\vee \rangle > ht(\mathfrak{l})$ for some $t > 0$. If $ht(\gamma) - ht(\mu) + ht(\mathfrak{l}) > c_0$ then $3(t - c_0) + 2ht(\gamma) > ht(\mu)$.*

Proof. By Lemma A.4.2, $\langle \Theta_t, \gamma^\vee \rangle = 2(t - c_0) + \mathrm{ht}(\gamma)$. Therefore, we have

$$3(t - c_0) + 2\mathrm{ht}(\gamma) = \mathrm{ht}(\gamma) + (t - c_0) + \langle \Theta_t, \gamma^\vee \rangle. \tag{A.4.8}$$

On the other hand, since $\langle \Theta_t, \gamma^\vee \rangle > \mathrm{ht}(\mathfrak{l})$ with $t > 0$ and $\mathrm{ht}(\gamma) - \mathrm{ht}(\mu) + \mathrm{ht}(\mathfrak{l}) > c_0$, it follows that

$$\mathrm{ht}(\gamma) + (t - c_0) + \langle \Theta_t, \gamma^\vee \rangle > \mathrm{ht}(\gamma) - c_0 + \mathrm{ht}(\mathfrak{l}) > \mathrm{ht}(\mu). \tag{A.4.9}$$

The proposed inequality now follows by combining (A.4.8) and (A.4.9). ■

The next theorem is our main tool for simply-laced cases to reduce the reducibility parameter t for which Jantzen's criterion needs to be applied.

Theorem A.4.10 *Let $\mathfrak{g}$ be a complex simple simply laced Lie algebra, and $\mathfrak{q}$ be a maximal parabolic subalgebra of non-Heisenberg type. Suppose that $ht(\gamma) - ht(\mu) + ht(\mathfrak{l}) > c_0$. If $\langle \Theta_m, \gamma^\vee \rangle > ht(\mathfrak{l})$ and $\gamma \in S_m$ for some $m \in \frac{1}{2}\mathbb{Z}_{>0}$ then $M_{\mathfrak{q}}(\Theta_t)$ is reducible for all $t \in m + \mathbb{Z}_{\geq 0}$.*

Proof. By Proposition A.4.6, we have $\gamma \in S_t$ and $Y(s_\gamma(\Theta_t)) \neq 0$ for all $t \in m + \mathbb{Z}_{\geq 0}$. We show that $Y(s_\gamma(\Theta_t))$ does not cancel out in (A.2.3). If $Y(s_\gamma(\Theta_t))$ cancels out then, by Proposition A.2.4, there exists $\beta \in S_t \backslash \{\gamma\}$ so that $s_\beta(\Theta_t)$ is $W(\mathfrak{l})$-conjugate to $s_\gamma(\Theta_t)$. Thus, by Lemma A.2.9,

$$\langle \Theta_t, \gamma^\vee \rangle \frac{2\langle \gamma, \lambda_\mathfrak{q} \rangle}{||\alpha_\mathfrak{q}||^2} = \langle \Theta_t, \beta^\vee \rangle \frac{2\langle \beta, \lambda_\mathfrak{q} \rangle}{||\alpha_\mathfrak{q}||^2}. \tag{A.4.11}$$

Since $\gamma \in \Delta(\mathfrak{z}(\mathfrak{n}))$, we have $\frac{2\langle \gamma, \lambda_\mathfrak{q} \rangle}{||\alpha_\mathfrak{q}||^2} = 2$. By Lemma A.4.2, the left hand side of (A.4.11) is then $4(t - c_0) + 2\mathrm{ht}(\gamma)$. Hence,

$$4(t - c_0) + 2\mathrm{ht}(\gamma) = \langle \Theta_t, \beta^\vee \rangle \frac{2\langle \beta, \lambda_\mathfrak{q} \rangle}{||\alpha_\mathfrak{q}||^2}. \tag{A.4.12}$$

If $\beta \in \Delta(\mathfrak{z}(\mathfrak{n}))$ then (A.4.12) is

$$4(t - c_0) + 2\mathrm{ht}(\gamma) = 4(t - c_0) + 2\mathrm{ht}(\beta),$$

which says $\mathrm{ht}(\gamma) = \mathrm{ht}(\beta)$. Since γ is the unique highest root of $\mathfrak{g}$, we obtain $\beta = \gamma$. However, it contradicts the choice of $\beta \in S_t \backslash \{\gamma\}$. If $\beta \in \Delta(\mathfrak{g}(1))$ then, by Lemma A.4.2, (A.4.12) is

$$4(t - c_0) + 2\mathrm{ht}(\gamma) = (t - c_0) + \mathrm{ht}(\beta). \tag{A.4.13}$$

By solving (A.4.13) for $\mathrm{ht}(\beta)$, one obtains that

$$\mathrm{ht}(\beta) = 3(t - c_0) + 2\mathrm{ht}(\gamma). \tag{A.4.14}$$

Then, it follows from Lemma A.4.7 and (A.4.14) that $\mathrm{ht}(\beta) > \mathrm{ht}(\mu)$, which contradicts the choice of $\beta \in \Delta(\mathfrak{g}(1))$. Therefore there is no such $\beta \in S_t \backslash \{\gamma\}$. Hence, $Y(s_\gamma(\Theta_t))$ does not cancel out in (A.2.3), and so $M_\mathfrak{q}(\Theta_t)$ is reducible by Jantzen's criterion. ■

Here is a version of Theorem A.4.10 for the highest weight μ for $\mathfrak{g}(1)$. This theorem shows the reducibility of $M_\mathfrak{q}(\Theta_t)$ for some t, where Theorem A.4.10 cannot apply.

Theorem A.4.15 *Let $\mathfrak{g}$ be a complex simple simply laced Lie algebra, and $\mathfrak{q}$ be a maximal parabolic subalgebra of non-Heisenberg type. Suppose that $\langle \Theta_t, \mu^\vee \rangle > ht(\mathfrak{l})$ for some $t \in \frac{1}{2}\mathbb{Z}_{>0}$. If $\mu \in S_t$ and $ht(\beta) \neq \frac{1}{2}(3(c_0 - t) + ht(\mu))$ for all $\beta \in S_t \cap \Delta(\mathfrak{z}(\mathfrak{n}))$ then $M_\mathfrak{q}(\Theta_t)$ is reducible.*

Proof. The argument of this proof is similar to that for Theorem A.4.10. First, by Proposition A.4.5, $Y(s_\mu(\Theta_t)) \neq 0$. Since $\mu \in S_t$, the term $Y(s_\mu(\Theta_t))$ occurs in (A.2.3). We wish to show that $Y(s_\mu(\Theta_t))$ does not cancel out. If it does then, by Proposition A.2.4, there exists $\beta \in S_t \backslash \{\mu\}$ so that $s_\mu(\Theta_t)$ and $s_\beta(\Theta_t)$ are $W(\mathfrak{l})$-conjugate. Then, as we obtain (A.4.12) in Theorem A.4.10, one can show that

$$(t - c_0) + \mathrm{ht}(\mu) = \langle \Theta_t, \beta^\vee \rangle \frac{2\langle \beta, \lambda_\mathfrak{q} \rangle}{||\alpha_\mathfrak{q}||^2}. \tag{A.4.16}$$

If $\beta \in \Delta(\mathfrak{g}(1))$ then we would end up with $\beta = \mu$, which contradicts the choice of $\beta \in S_t \backslash \{\mu\}$. Now if $\beta \in \Delta(\mathfrak{z}(\mathfrak{n}))$ then, by Lemma A.4.2, (A.4.16) becomes

$$(t - c_0) + \mathrm{ht}(\mu) = 4(t - c_0) + 2\mathrm{ht}(\beta),$$

as $2\langle \lambda_\mathfrak{q}, \beta \rangle / ||\alpha_\mathfrak{q}||^2 = 2$, By solving the equation for $\mathrm{ht}(\beta)$ we obtain that

$$\mathrm{ht}(\beta) = \frac{3(c_0 - t) + \mathrm{ht}(\mu)}{2}.$$

Therefore, if there is no $\beta \in S_t \cap \Delta(\mathfrak{z}(\mathfrak{n}))$ with $\mathrm{ht}(\beta) = \frac{1}{2}(3(c_0 - t) + \mathrm{ht}(\mu))$ then the term $Y(s_\mu(\Theta_t))$ does not cancel out in (A.2.3). Now Jantzen's criterion concludes that $M_\mathfrak{q}(\Theta_t)$ is reducible. ■

Corollary A.4.17 *Let $\mathfrak{g}$ be a complex simple simply laced Lie algebra, and $\mathfrak{q}$ be a maximal parabolic subalgebra listed in (3.3.2) or (3.3.3). Suppose that $\langle \Theta_t, \mu^\vee \rangle > ht(\mathfrak{l})$ for some $t \in \frac{1}{2}\mathbb{Z}_{>0}$. If $\mu \in S_t$ and $\frac{1}{2}(3(c_0 - t) + ht(\mu)) \notin \mathbb{Z}$ then $M_\mathfrak{q}(\Theta_t)$ is reducible.*

A.5 Reducibility Points of $M_\mathfrak{q}(\Theta_t)$ for Exceptional Algebras

Proposition A.3.4 shows that $M_\mathfrak{q}(\Theta_t)$ is reducible only if $t \in \frac{1}{2}\mathbb{Z}_{>0}$, when $\mathfrak{q}$ is a maximal two-step nilpotent parabolic subalgebra of non-Heisenberg type. In this section we shall determine all the values of $t \in \frac{1}{2}\mathbb{Z}_{>0}$ for which $M_\mathfrak{q}(\Theta_t)$ is reducible for $\mathfrak{q}$ listed in (3.3.3), namely,

$$E_6(3),\ E_6(5),\ E_7(2),\ E_7(6),\ E_8(1),\ \text{or}\ F_4(4).$$

Observe that since the deleted Dynkin diagrams for $E_6(3)$ and $E_6(5)$ have symmetry, it suffices to consider only $E_6(3)$.

Now we are going to state the main theorem of this chapter. We mean by the **reducibility points** of $M_{\mathfrak{q}}(\Theta_t)$ all the values of t for which $M_{\mathfrak{q}}(\Theta_t)$ is reducible.

Theorem A.5.1 *If $\mathfrak{q}$ is a maximal two-step nilpotent parabolic subalgebra listed in (3.3.3) then the reducibility points of $M_{\mathfrak{q}}(\Theta_t)$ are given as follows:*

Type	*Reducibility Points*
$E_6(3)$:	$t \in (2 + \mathbb{Z}_{\geq 0}) \cup (\frac{3}{2} + \mathbb{Z}_{\geq 0})$
$E_7(2)$:	$t \in (1 + \mathbb{Z}_{\geq 0}) \cup (\frac{5}{2} + \mathbb{Z}_{\geq 0})$
$E_7(6)$:	$t \in (1 + \mathbb{Z}_{\geq 0}) \cup (\frac{1}{2} + \mathbb{Z}_{\geq 0})$
$E_8(1)$:	$t \in (1 + \mathbb{Z}_{\geq 0}) \cup (\frac{5}{2} + \mathbb{Z}_{\geq 0})$
$F_4(4)$:	$t \in (1 + \mathbb{Z}_{\geq 0}) \cup (\frac{1}{2} + \mathbb{Z}_{\geq 0})$

The proof is given by a case-by-case observation.

We start observing simply laced cases, namely, $E_6(3)$, $E_7(2)$, $E_7(6)$, or $E_8(1)$, since Theorem A.4.10, Theorem A.4.15, or Corollary A.4.17 can be applied. Table A.1 below shows the required constants for those theorems, namely, constant c_0, the values of $\mathrm{ht}(\mathfrak{l})$, $\mathrm{ht}(\mu)$, and $\mathrm{ht}(\gamma)$, for each case. Note that in [23] the constants 3 for E_6 for $k = 2$ on p. 105 and 7 for E_7 for $k = 2$ on p. 107 should read $\frac{9}{2}$ and $\frac{13}{2}$, respectively. Observe that $\mathrm{ht}(\gamma) - \mathrm{ht}(\mu) + \mathrm{ht}(\mathfrak{l}) > c_0$ for each case.

Let m_μ (resp. m_γ) be the least number in $\frac{1}{2}\mathbb{Z}_{>0}$ so that $\langle \Theta_t, \mu^\vee \rangle > \mathrm{ht}(\mathfrak{l})$ for all $t \in m_\mu + \mathbb{Z}_{\geq 0}$ (resp. $\langle \Theta_t, \gamma^\vee \rangle > \mathrm{ht}(\mathfrak{l})$ for all $t \in m_\gamma + \mathbb{Z}_{\geq 0}$). These values are required to apply Theorem A.4.10, Theorem A.4.15, or Corollary A.4.17. In each case of $\mathfrak{q}$ simply laced, we shall often need to observe the heights of certain positive roots. See Appendix C for the heights of positive roots; the lists of the positive roots for exceptional algebras are summarized in the appendix.

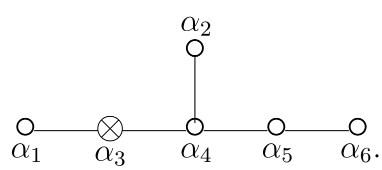
Table A.1:

Type	c_0	ht($\mathfrak{l}$)	ht(μ)	ht(γ)
$E_6(3)$	$\frac{9}{2}$	4	8	11
$E_7(2)$	7	6	13	17
$E_7(6)$	$\frac{13}{2}$	7	12	17
$E_8(1)$	$\frac{23}{2}$	11	22	29

We treat the cases $t \in 1 + \mathbb{Z}_{\geq 0}$ and $t \in \frac{1}{2} + \mathbb{Z}_{\geq 0}$ separately, since elements in S_t are different in those cases.

A.5.1 $E_6(3)$

The deleted Dynkin diagram is



Lemma A.5.2 *We have the following:*

1. $\langle \Theta_t, \mu^\vee \rangle > ht(\mathfrak{l})$ *if and only if* $t > \frac{1}{2}$.

2. $\langle \Theta_t, \gamma^\vee \rangle > ht(\mathfrak{l})$ *if and only if* $t > 1$.

Proof. A direct computation using Lemma A.4.2. ■

Lemma A.5.3 *The set S_t is determined as follows:*

1. *If* $t \in 1 + \mathbb{Z}_{\geq 0}$ *then*

$$S_t = \{\beta \in \Delta(\mathfrak{z}(\mathfrak{n})) \mid ht(\beta) > 9 - 2t\}.$$

2. *If* $t \in \frac{1}{2} + \mathbb{Z}_{\geq 0}$ *then*

$$S_t = \{\beta \in \Delta(\mathfrak{g}(1)) \mid ht(\beta) > (9/2) - t\} \cup \{\beta \in \Delta(\mathfrak{z}(\mathfrak{n})) \mid ht(\beta) > 9 - 2t\}.$$

Proof. This directly follows from Lemma A.4.3 with the value of c_0 in Table A.1. ■

Theorem A.5.4 *Let* $\mathfrak{g}$ *be the complex simple Lie algebra of type* E_6 *and* $\mathfrak{q}$ *be the parabolic subalgebra of* $\mathfrak{g}$ *determined by the simple root* α_3*. For* $t \in 1 + \mathbb{Z}_{\geq 0}$*, the following hold:*

1. $M_{\mathfrak{q}}(\Theta_t)$ *is reducible if* $t \in 2 + \mathbb{Z}_{\geq 0}$*.*

2. $M_{\mathfrak{q}}(\Theta_t)$ *is irreducible if* $t = 1$*.*

Proof. We use Theorem A.4.10 to prove this theorem. We start by checking that the hypotheses in Theorem A.4.10 are satisfied. Observe from Table A.1 that we have $c_0 = \frac{9}{2}$, $\mathrm{ht}(\mathfrak{l}) = 4$, $\mathrm{ht}(\mu) = 8$, and $\mathrm{ht}(\gamma) = 11$. So $\mathrm{ht}(\gamma) - \mathrm{ht}(\mu) + \mathrm{ht}(\mathfrak{l}) > c_0$. Moreover, it follows from Lemma A.5.2 and Lemma A.5.3 that $\langle \Theta_2, \gamma^\vee \rangle > \mathrm{ht}(\mathfrak{l})$ and $\gamma \in S_2$. Now Theorem A.4.10 concludes that $M_{\mathfrak{q}}(\Theta_t)$ is reducible for $t \in 2 + \mathbb{Z}_{\geq 0}$.

If $t = 1$ then direct verification shows that for all $\beta \in S_1$ there exists $\alpha \in \Delta(\mathfrak{l})$ so that $\langle s_\beta(\Theta_1), \alpha^\vee \rangle = 0$. Indeed, if $t = 1$ then Lemma A.5.3 and the attached list of the positive roots shows that

$$\begin{aligned} S_1 &= \{\beta \in \Delta(\mathfrak{z}(\mathfrak{n})) \mid \mathrm{ht}(\beta) > 7\} \\ &= \{\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 \\ &\qquad \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 \\ &\qquad \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 \\ &\qquad \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6\}. \end{aligned}$$

To show $\langle s_\beta(\Theta_1), \alpha^\vee \rangle = 0$, by Proposition A.4.4, it suffices to find $\alpha \in \Delta(\mathfrak{l})$ so that $\beta - \alpha \in \Delta$ and $\mathrm{ht}(\alpha) = \langle \Theta_1, \beta^\vee \rangle$. Observe that if $\beta \in \Delta(\mathfrak{z}(\mathfrak{n}))$ then, by Lemma A.4.2,

$\langle \Theta_t, \beta^\vee \rangle = 2(t - 9/2) + \mathrm{ht}(\beta)$. Thus, for all $\beta \in S_1$,

$$\langle \Theta_1, \beta^\vee \rangle = \mathrm{ht}(\beta) - 7.$$

If

$$\beta_0 = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 \in \Delta$$

then the desired $\alpha \in \Delta(\mathfrak{l})$ are found as follows:

1) $\underline{\beta = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6}$: If $\alpha = \alpha_6 \in \Delta(\mathfrak{l})$ then $\beta - \alpha = \beta_0$ and $\mathrm{ht}(\alpha) = 1 = \langle \Theta_1, \beta^\vee \rangle$.

2) $\underline{\beta = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6}$: If $\alpha = \alpha_5 + \alpha_6 \in \Delta(\mathfrak{l})$ then $\beta - \alpha = \beta_0$ and $\mathrm{ht}(\alpha) = 2 = \langle \Theta_1, \beta^\vee \rangle$.

3) $\underline{\beta = \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6}$: If $\alpha = \alpha_4 + \alpha_5 + \alpha_6 \in \Delta(\mathfrak{l})$ then $\beta - \alpha = \beta_0$ and $\mathrm{ht}(\alpha) = 3 = \langle \Theta_1, \beta^\vee \rangle$.

4) $\underline{\beta = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6}$: If $\alpha = \alpha_2 + \alpha_4 + \alpha_5 + \alpha_6 \in \Delta(\mathfrak{l})$ then $\beta - \alpha = \beta_0$ and $\mathrm{ht}(\alpha) = 4 = \langle \Theta_1, \beta^\vee \rangle$.

Therefore $s_\beta(\Theta_1)$ is $\Delta(\mathfrak{l})$-singular for all $\beta \in S_1$, which implies that, by Proposition A.1.4, $Y(s_\beta(\Theta_1)) = 0$ for all $\beta \in S_1$. Now the irreducibility of $M_{\mathfrak{q}}(\Theta_1)$ follows from Jantzen's criterion. ■

Theorem A.5.5 *Let $\mathfrak{g}$ be the complex simple Lie algebra of type E_6 and $\mathfrak{q}$ be the parabolic subalgebra of $\mathfrak{g}$ determined by the simple root α_3. For $t \in \frac{1}{2} + \mathbb{Z}_{\geq 0}$, the following hold:*

1. *$M_{\mathfrak{q}}(\Theta_t)$ is reducible if $t \in \frac{3}{2} + \mathbb{Z}_{\geq 0}$.*
2. *$M_{\mathfrak{q}}(\Theta_t)$ is irreducible if $t = \frac{1}{2}$.*

Proof. As Theorem A.5.4, the first part is shown by Theorem A.4.10. Indeed, the data in Table A.1 say that $\mathrm{ht}(\gamma) - \mathrm{ht}(\mu) + \mathrm{ht}(\mathfrak{l}) > c_0$. Using $\mathrm{ht}(\gamma) = 11$, Lemma A.5.2

and Lemma A.5.3 show that $\langle \Theta_{\frac{3}{2}}, \gamma^\vee \rangle > \mathrm{ht}(\mathfrak{l})$ and $\gamma \in S_{\frac{3}{2}}$. Then by Theorem A.4.10, $M_{\mathfrak{q}}(\Theta_t)$ is reducible for $t \in \frac{3}{2} + \mathbb{Z}_{\geq 0}$

If $t = \frac{1}{2}$ then $S_{\frac{1}{2}}$ is the union of $\Delta(\mathfrak{g}(1)) \cap S_{\frac{1}{2}}$ and $\Delta(\mathfrak{z}(\mathfrak{n})) \cap S_{\frac{1}{2}}$. By Lemma A.5.3 and the attached list of positive roots, the weights of these are as follows:

$$
\begin{aligned}
S_{\frac{1}{2}} \cap \Delta(\mathfrak{g}(1)) = \{&\beta \in \Delta(\mathfrak{g}(1)) \mid \mathrm{ht}(\beta) > 4\} \\
= \{&\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 \\
&\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 \\
&\alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 \\
&\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 \\
&\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 \\
&\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 \\
&\alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 \\
&\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 \\
&\alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 \\
&\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6\}
\end{aligned}
$$

and

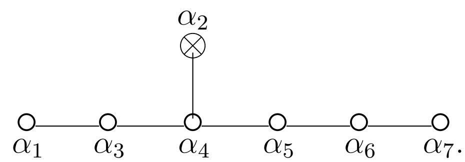
$$
\begin{aligned}
S_{\frac{1}{2}} \cap \Delta(\mathfrak{z}(\mathfrak{n})) = \{&\beta \in \Delta(\mathfrak{z}(\mathfrak{n})) \mid \mathrm{ht}(\beta) > 8\} \\
= \{&\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 \\
&\alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 \\
&\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6\}.
\end{aligned}
$$

One can check that for all $\beta \in S_{\frac{1}{2}}$ there exists $\alpha \in \Delta(\mathfrak{l})$ so that $\langle s_\beta(\Theta_{\frac{1}{2}}), \alpha^\vee \rangle = 0$, as we did in the proof of Theorem A.5.4. Then, by Proposition A.1.4, we have $Y(s_\beta(\Theta_{\frac{1}{2}})) = 0$ for all $\beta \in S_{\frac{1}{2}}$. Now Jantzen's criterion concludes that $M_{\mathfrak{q}}(\Theta_{\frac{1}{2}})$ is irreducible.

■

A.5.2 $E_7(2)$

The deleted Dynkin diagram is



Lemma A.5.6 *We have the following:*

1. $\langle \Theta_t, \mu^\vee \rangle > ht(\mathfrak{l})$ *if and only if* $t > 0$.

2. $\langle \Theta_t, \gamma^\vee \rangle > ht(\mathfrak{l})$ *if and only if* $t > \frac{3}{2}$.

Proof. A direct computation using Lemma A.4.2. ■

Lemma A.5.7 *The set* S_t *is determined as follows:*

1. *If* $t \in 1 + \mathbb{Z}_{\geq 0}$ *then*

$$S_t = \{\beta \in \Delta(\mathfrak{g}(1)) \mid ht(\beta) > 7 - t\} \cup \{\beta \in \Delta(\mathfrak{z}(\mathfrak{n})) \mid ht(\beta) > 14 - 2t\}.$$

2. *If* $t \in \frac{1}{2} + \mathbb{Z}_{\geq 0}$ *then*

$$S_t = \{\beta \in \Delta(\mathfrak{z}(\mathfrak{n})) \mid ht(\beta) > 14 - 2t\}$$

Proof. This directly follows from Lemma A.4.3 with the value of c_0 in Table A.1. ■

Theorem A.5.8 *Let* $\mathfrak{g}$ *be the complex simple Lie algebra of type* E_7 *and* $\mathfrak{q}$ *be the parabolic subalgebra of* $\mathfrak{g}$ *determined by the simple root* α_2*. Then* $M_{\mathfrak{q}}(\Theta_t)$ *is reducible for all* $t \in 1 + \mathbb{Z}_{\geq 0}$.

Proof. The reducibility for $t \in 2 + \mathbb{Z}_{\geq 0}$ is shown by Theorem A.4.10 as Theorem A.5.4. If $t = 1$ then since $\mathrm{ht}(\mu) = 13$, Lemma A.5.6 and Lemma A.5.7 show that $\langle \Theta_1, \mu^\vee \rangle > \mathrm{ht}(\mathfrak{l})$ and $\mu \in S_1$. Moreover we have $\frac{1}{2}(3(c_0 - 1) + \mathrm{ht}(\mu)) = \frac{31}{2} \notin \mathbb{Z}$. Therefore, it follows from Corollary A.4.17 that $M_{\mathfrak{q}}(\Theta_1)$ is reducible. ■

Theorem A.5.9 *Let $\mathfrak{g}$ be the complex simple Lie algebra of type E_7 and $\mathfrak{q}$ be the parabolic subalgebra of $\mathfrak{g}$ determined by the simple root α_2. For $t \in \frac{1}{2} + \mathbb{Z}_{\geq 0}$, the following hold:*

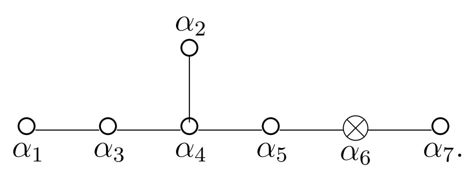
1. *$M_{\mathfrak{q}}(\Theta_t)$ is reducible if $t \in \frac{5}{2} + \mathbb{Z}_{\geq 0}$.*

2. *$M_{\mathfrak{q}}(\Theta_t)$ is irreducible if $t = \frac{1}{2}, \frac{3}{2}$.*

Proof. The first part is shown by Theorem A.4.10 as Theorem A.5.4. If $t = \frac{1}{2}$ or $\frac{3}{2}$ then direct verification as in the proof of Theorem A.5.4 shows that for all $\beta \in S_t$ there exists $\alpha \in \Delta(\mathfrak{l})$ so that $\langle s_\beta(\Theta_t), \alpha^\vee \rangle = 0$. Therefore, by Proposition A.1.4, we have $Y(s_\beta(\Theta_t)) = 0$ for all $\beta \in S_t$. Now Jantzen's criterion concludes that $M_{\mathfrak{q}}(\Theta_t)$ is irreducible if $t = \frac{1}{2}$ or $\frac{3}{2}$. ■

A.5.3 $E_7(6)$

The deleted Dynkin diagram is



Lemma A.5.10 *We have the following:*

1. *$\langle \Theta_t, \mu^\vee \rangle > ht(\mathfrak{l})$ if and only if $t > \frac{3}{2}$.*

2. *$\langle \Theta_t, \gamma^\vee \rangle > ht(\mathfrak{l})$ if and only if $t > \frac{3}{2}$.*

Proof. A direct computation using Lemma A.4.2. ■

Lemma A.5.11 *The set S_t is determined as follows:*

1. *If $t \in 1 + \mathbb{Z}_{\geq 0}$ then*

$$S_t = \{\beta \in \Delta(\mathfrak{z}(\mathfrak{n})) \mid ht(\beta) > 13 - 2t\}.$$

2. *If* $t \in \frac{1}{2} + \mathbb{Z}_{\geq 0}$ *then*

$$S_t = \{\beta \in \Delta(\mathfrak{g}(1)) \mid ht(\beta) > (13/2) - t\} \cup \{\beta \in \Delta(\mathfrak{z}(\mathfrak{n})) \mid ht(\beta) > 13 - 2t\}$$

Proof. This directly follows from Lemma A.4.3 with the value of c_0 in Table A.1. ■

Theorem A.5.12 *Let* $\mathfrak{g}$ *be the complex simple Lie algebra of type* E_7 *and* $\mathfrak{q}$ *be the parabolic subalgebra of* $\mathfrak{g}$ *determined by the simple root* α_6*. Then* $M_{\mathfrak{q}}(\Theta_t)$ *is reducible for all* $t \in 1 + \mathbb{Z}_{\geq 0}$*.*

Proof. The reducibility for $t \in 2 + \mathbb{Z}_{\geq 0}$ follows from Theorem A.4.10 as Theorem A.5.4. To prove the reducibility of $M_{\mathfrak{q}}(\Theta_1)$, we show that there exists $\beta_0 \in S_1$ with $Y(s_{\beta_0}(\Theta_1)) \neq 0$ so that $Y(s_{\beta_0}(\Theta_1))$ does not cancel out in (A.2.3). If $t = 1$ then, by Lemma A.5.11,

$$S_1 = \{\beta \in \Delta(\mathfrak{z}(\mathfrak{n})) \mid \mathrm{ht}(\beta) > 11\}.$$

Set

$$\beta_0 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 \in \Delta(\mathfrak{z}(\mathfrak{n})).$$

Since $\mathrm{ht}(\beta_0) = 15$, we have $\beta_0 \in S_1$. Observe that, by the list of the roots in Appendix D, one can see that β_0 is the unique root of its height. First we check $Y(s_{\beta_0}(\Theta_1)) \neq 0$. To do so, by Corollary A.1.5 and Proposition A.4.4, it suffices to show that $\beta_0 - \alpha \notin \Delta$ for all $\alpha \in \Delta^+(\mathfrak{l})$ of $\mathrm{ht}(\alpha) = \langle \Theta_1, \beta_0^\vee \rangle$. Since $c_0 = 13/2$ and $\beta_0 \in \Delta(\mathfrak{z}(\mathfrak{n}))$ with $\mathrm{ht}(\beta_0) = 15$, Lemma A.4.2 shows that $\langle \Theta_1, \beta_0^\vee \rangle = 4$. There are only three weights in $\Delta^+(\mathfrak{l})$ of height 4, namely,

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$$

$$\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5$$

$$\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5.$$

A direct computation shows that $\beta_0 - \alpha \notin \Delta$ for all $\alpha \in \Delta^+(\mathfrak{l})$ above. Therefore $Y(s_{\beta_0}(\Theta_1)) \neq 0$.

Now we wish to show that $Y(s_{\beta_0}(\Theta_1))$ does not cancel out in (A.2.3). If it does then, by Proposition A.2.4 and Lemma A.2.9, there exists $\beta \in S_1\backslash\{\beta_0\}$ so that

$$\langle \Theta_1, \beta_0^\vee \rangle \frac{2\langle \beta_0, \lambda_\mathfrak{q} \rangle}{||\alpha_\mathfrak{q}||^2} = \langle \Theta_1, \beta^\vee \rangle \frac{2\langle \beta, \lambda_\mathfrak{q} \rangle}{||\alpha_\mathfrak{q}||^2}. \tag{A.5.13}$$

Observe that since $\beta_0, \beta \in S_1 \subset \Delta(\mathfrak{z}(\mathfrak{n}))$, we have $2\langle \beta_0, \lambda_\mathfrak{q}\rangle/||\alpha_\mathfrak{q}||^2 = 2\langle \beta, \lambda_\mathfrak{q}\rangle/||\alpha_\mathfrak{q}||^2 = 2$. Then a direct computation using Lemma A.4.2 shows that (A.5.13) implies $\mathrm{ht}(\beta) = \mathrm{ht}(\beta_0)$. Since β_0 is the unique root of its height, it implies $\beta = \beta_0$. However, it contradicts the choice of $\beta \in S_1\backslash\{\beta_0\}$. Therefore $Y(s_{\beta_0}(\Theta_1))$ does not cancel out in (A.2.3). Hence, by Jantzen's criterion, $M_\mathfrak{q}(\Theta_1)$ is reducible. ■

Theorem A.5.14 *Let $\mathfrak{g}$ be the complex simple Lie algebra of type E_7 and $\mathfrak{q}$ be the parabolic subalgebra of $\mathfrak{g}$ determined by the simple root α_6. Then $M_\mathfrak{q}(\Theta_t)$ is reducible for all $t \in \frac{1}{2} + \mathbb{Z}_{\geq 0}$.*

Proof. The reducibility for $t \in \frac{5}{2} + \mathbb{Z}_{\geq 0}$ follows from Theorem A.4.10 as Theorem A.5.4. For the case of $t = \frac{1}{2}$ or $\frac{3}{2}$, we show that there exists $\beta_0 \in S_t$ with $Y(s_{\beta_0}(\Theta_t)) \neq 0$ so that $Y(s_{\beta_0}(\Theta_t))$ does not cancel out in (A.2.3), as we did in the proof of Theorem A.5.12. Here we only show the case of $t = \frac{1}{2}$, since the other case can be shown similarly.

If $t = \frac{1}{2}$ then, by Lemma A.5.11,

$$S_{\frac{1}{2}} = \{\beta \in \Delta(\mathfrak{g}(1)) \mid \mathrm{ht}(\beta) > 6\} \cup \{\beta \in \Delta(\mathfrak{z}(\mathfrak{n})) \mid \mathrm{ht}(\beta) > 12\}.$$

Set

$$\beta_0 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 \in \Delta(\mathfrak{z}(\mathfrak{n})).$$

Since $\mathrm{ht}(\beta_0) = 14$, we have $\beta_0 \in S_{\frac{1}{2}}$. By the list of the positive roots in Appendix D, one can see that β_0 is the unique root of its height. A direct computation as in the proof of Theorem A.5.12 shows that $Y(s_{\beta_0}(\Theta_{\frac{1}{2}})) \neq 0$. Now we are going to show that $Y(s_\beta(\Theta_{\frac{1}{2}}))$ does not cancel out in (A.2.3). Suppose the contrary. Then, by

Proposition A.2.4 and Lemma A.2.9, there exists $\beta \in S_{\frac{1}{2}} \backslash \{\beta_0\}$ so that

$$\langle \Theta_{\frac{1}{2}}, \beta_0^\vee \rangle \frac{2\langle \beta_0, \lambda_\mathfrak{q} \rangle}{||\alpha_\mathfrak{q}||^2} = \langle \Theta_{\frac{1}{2}}, \beta^\vee \rangle \frac{2\langle \beta, \lambda_\mathfrak{q} \rangle}{||\alpha_\mathfrak{q}||^2}. \tag{A.5.15}$$

If $\beta \in S_{\frac{1}{2}} \cap \Delta(\mathfrak{z}(\mathfrak{n}))$ then (A.5.15) implies $\mathrm{ht}(\beta) = \mathrm{ht}(\beta_0)$ as in the proof of Theorem A.5.12. Since β_0 is the unique root of its height, it shows that $\beta = \beta_0$, which contradicts the choice of $\beta \in S_{\frac{1}{2}} \backslash \{\beta_0\}$. If $\beta \in S_{\frac{1}{2}} \cap \Delta(\mathfrak{g}(1))$ then (A.5.15) implies that $\mathrm{ht}(\beta) = 10$, by a direct computation using Lemma A.4.2 with the facts that $2\langle \delta, \lambda_\mathfrak{q} \rangle / ||\alpha_\mathfrak{q}||^2 = 1$ if $\delta \in \Delta(\mathfrak{g}(1))$ and $2\langle \delta, \lambda_\mathfrak{q} \rangle / ||\alpha_\mathfrak{q}||^2 = 2$ if $\delta \in \Delta(\mathfrak{z}(\mathfrak{n}))$. It implies that $s_\beta(\Theta_{\frac{1}{2}})$ for $\beta \in S_1 \backslash \{\beta_0\}$ is $W(\mathfrak{l})$-conjugate to $s_{\beta_0}(\Theta_{\frac{1}{2}})$ only if $\mathrm{ht}(\beta) = 10$. There are only two roots in $\Delta(\mathfrak{g}(1))$ of height 10, namely,

$$\alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$$

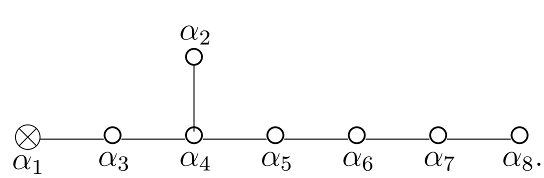
$$\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7.$$

By applying the same argument in the proof of Theorem A.5.4, one can show that $Y(s_\beta(\Theta_{\frac{1}{2}})) = 0$ for those $\beta \in \Delta(\mathfrak{g}(1))$ of height 10. Therefore, by Proposition A.2.4, $Y(s_{\beta_0}(\Theta_{\frac{1}{2}}))$ does not cancel out in (A.2.3). Now Jantzen's criterion concludes that $M_\mathfrak{q}(\Theta_{\frac{1}{2}})$ is reducible.

If $t = \frac{3}{2}$ then one can show by the same argument as above that $Y(s_{\beta_0}(\Theta_{\frac{3}{2}}))$ does not cancel out in (A.2.3) with $\beta_0 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$. Then the reducibility follows from Jantzen's criterion. ■

A.5.4 $E_8(1)$

The deleted Dynkin diagram is



Lemma A.5.16 *We have the following:*

1. $\langle \Theta_t, \mu^\vee \rangle > ht(\mathfrak{l})$ *if and only if* $t > \frac{1}{2}$.

2. $\langle \Theta_t, \gamma^\vee \rangle > ht(\mathfrak{l})$ *if and only if* $t > \frac{5}{2}$.

Proof. A direct computation using Lemma A.4.2. ■

Lemma A.5.17 *The set* S_t *is determined as follows:*

1. *If* $t \in 1 + \mathbb{Z}_{\geq 0}$ *then*

$$S_t = \{\beta \in \Delta(\mathfrak{z}(\mathfrak{n})) \mid ht(\beta) > 23 - 2t\}.$$

2. *If* $t \in \frac{1}{2} + \mathbb{Z}_{\geq 0}$ *then*

$$S_t = \{\beta \in \Delta(\mathfrak{g}(1)) \mid ht(\beta) > (23/2) - t\} \cup \{\beta \in \Delta(\mathfrak{z}(\mathfrak{n})) \mid ht(\beta) > 23 - 2t\}$$

Proof. This directly follows from Lemma A.4.3 with the value of c_0 in Table A.1. ■

Theorem A.5.18 *Let* $\mathfrak{g}$ *be the complex simple Lie algebra of type* E_8 *and* $\mathfrak{q}$ *be the parabolic subalgebra of* $\mathfrak{g}$ *determined by the simple root* α_1*. Then* $M_{\mathfrak{q}}(\Theta_t)$ *is reducible for all* $t \in 1 + \mathbb{Z}_{\geq 0}$.

Proof. The reducibility for $t \in 3 + \mathbb{Z}_{\geq 0}$ follows from Theorem A.4.10 as Theorem A.5.4. If $t = 1$ or 2 then set

$$\beta_0 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8 \quad \text{if } t = 1$$

or

$$\beta_0 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 2\alpha_7 + \alpha_8 \quad \text{if } t = 2.$$

We have $\beta_0 \in S_t$ for both cases that $t = 1$ and $t = 2$. By the same argument as in the proof of Theorem A.5.12, one can check that $Y(s_{\beta_0}(\Theta_t)) \neq 0$ and also that $Y(s_{\beta_0}(\Theta_t))$ does not cancel out in (A.2.3). Now Jantzen's criterion concludes that $M_{\mathfrak{q}}(\Theta_t)$ is reducible if $t = 1$ or 2. ■

Theorem A.5.19 *Let $\mathfrak{g}$ be the complex simple Lie algebra of type E_8 and $\mathfrak{q}$ be the parabolic subalgebra of $\mathfrak{g}$ determined by the simple root α_1. For $t \in \frac{1}{2} + \mathbb{Z}_{\geq 0}$, we have the following:*

1. *$M_{\mathfrak{q}}(\Theta_t)$ is reducible if $t \in \frac{5}{2} + \mathbb{Z}_{\geq 0}$.*

2. *$M_{\mathfrak{q}}(\Theta_t)$ is irreducible if $t = \frac{1}{2}, \frac{3}{2}$.*

Proof. Matumoto shows that $M_{\mathfrak{q}}(\Theta_{\frac{1}{2}})$ is irreducible in Section 4.6 in [23]. So we need only consider $t \in \frac{3}{2} + \mathbb{Z}_{\geq 0}$. The reducibility for $t \in \frac{7}{2} + \mathbb{Z}_{\geq 0}$ follows from Theorem A.4.10 as Theorem A.5.4. If $t = \frac{5}{2}$ then Lemma A.5.16 and Lemma A.5.17 show that $\langle \Theta_{\frac{5}{2}}, \mu^\vee \rangle > \mathrm{ht}(\mathfrak{l})$ and $\mu \in S_{\frac{5}{2}}$. Since $\frac{1}{2}(3(c_0 - \frac{5}{2}) + \mathrm{ht}(\mu)) = \frac{49}{2} \notin \mathbb{Z}$, it follows from Corollary A.4.17 that $M_{\mathfrak{q}}(\Theta_{\frac{5}{2}})$ is reducible.

Now suppose that $t = \frac{3}{2}$. The author wants to emphasize that this case is different from any other cases that we have had above; there are two nonzero terms in (A.2.3). First, Lemma A.5.16 and Lemma A.5.17 show that we have $\langle \Theta_{\frac{3}{2}}, \mu^\vee \rangle > \mathrm{ht}(\mathfrak{l})$ and $\mu \in S_{\frac{3}{2}}$. Thus $Y(s_\mu(\Theta_{\frac{3}{2}})) \neq 0$ by Proposition A.4.5. On the other hand, set

$$\beta_0 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8.$$

One can check that $Y(s_{\beta_0}(\Theta_{\frac{3}{2}})) \neq 0$ by the same argument in the proof of Theorem A.5.12. Moreover, direct verification as in the proof of Theorem A.5.4 shows that $Y(s_\beta(\Theta_{\frac{3}{2}})) = 0$ for all $\beta \in S_{\frac{3}{2}} \backslash \{\mu, \beta_0\}$.

Now we claim that $s_\mu(\Theta_{\frac{3}{2}})$ and $s_{\beta_0}(\Theta_{\frac{3}{2}})$ are conjugate by an odd element of $W(\mathfrak{l})$. Observe that since $Y(s_\mu(\Theta_{\frac{3}{2}})) \neq 0$ and $Y(s_{\beta_0}(\Theta_{\frac{3}{2}})) \neq 0$, by Proposition A.1.4, the weights $s_\mu(\Theta_{\frac{3}{2}})$ and $s_{\beta_0}(\Theta_{\frac{3}{2}})$ are $\Delta(\mathfrak{l})$-regular. We then achieve our claim by computing the $\Delta(\mathfrak{l})$-dominant weight ν_μ (resp. ν_{β_0}) that is $W(\mathfrak{l})$-conjugate to $s_\mu(\Theta_{\frac{3}{2}})$ (resp. $s_{\beta_0}(\Theta_{\frac{3}{2}})$), via the following algorithm: Given $\Delta(\mathfrak{l})$-regular weight $\lambda \in \mathfrak{h}^*$, compute $\langle \lambda, \alpha_j^\vee \rangle$ for all $\alpha_j \in \Pi(\mathfrak{l})$. If λ is $\Delta(\mathfrak{l})$-dominant then nothing to do. If not then there exist simple roots $\alpha_{j_1}, \ldots, \alpha_{j_d} \in \Pi(\mathfrak{l})$ so that $\langle \lambda, \alpha_{j_i}^\vee \rangle < 0$. Apply the simple reflections $s_{\alpha_{j_1}}, \ldots, s_{\alpha_{j_d}}$ to λ. If $s_{\alpha_{j_1}} \ldots s_{\alpha_{j_d}}(\lambda)$ is $\Delta(\mathfrak{l})$-dominant then we stop; otherwise, apply the simple reflections $s_{\alpha_{j_k}}$ of $\alpha_{j_k} \in \Pi(\mathfrak{l})$ with $\langle s_{\alpha_{j_1}} \ldots s_{\alpha_{j_d}}(\lambda), \alpha_{j_k}^\vee \rangle < 0$ to $s_{\alpha_{j_1}} \ldots s_{\alpha_{j_d}}(\lambda)$. We keep these steps until the resulted weight $s_{\alpha_{j_1}}, \ldots, s_{\alpha_{j_r}}(\lambda)$ is $\Delta(\mathfrak{l})$-dominant. For example, to find ν_μ, we first compute $\langle s_\mu(\Theta_{\frac{3}{2}}), \alpha_j^\vee \rangle$ for all $\alpha_j \in \Pi(\mathfrak{l})$. In this case only α_2 makes it negative, so apply s_{α_2} to $s_\mu(\Theta_{\frac{3}{2}})$ and compute $\langle s_{\alpha_2} s_\mu(\Theta_{\frac{3}{2}}), \alpha_j^\vee \rangle$ for all $\alpha_j \in \Pi(\mathfrak{l})$.

By applying the above algorithm to $s_\mu(\Theta_{\frac{3}{2}})$ and $s_{\beta_0}(\Theta_{\frac{3}{2}})$, the $\Delta(\mathfrak{l})$-dominant weights ν_μ and ν_{β_0} are

$$s_3 s_4 s_2 s_5 s_4 s_6 s_3 s_5 s_7 s_8 s_6 s_4 s_7 s_5 s_2 s_6 s_4 s_5 s_3 s_4 s_2 (s_\mu(\Theta_{\frac{3}{2}}))$$

and

$$s_5 s_4 s_3 s_2 s_4 s_5 (s_{\beta_0}(\Theta_{\frac{3}{2}})),$$

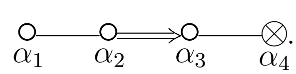
respectively, where $s_j = s_{\alpha_j}$ for $\alpha_j \in \Pi(\mathfrak{l})$. Moreover, a direct computation shows that

$$\begin{aligned} & s_3 s_4 s_2 s_5 s_4 s_6 s_3 s_5 s_7 s_8 s_6 s_4 s_7 s_5 s_2 s_6 s_4 s_5 s_3 s_4 s_2 (s_\mu(\Theta_{\frac{3}{2}})) \\ &= \Theta_{\frac{3}{2}} - 12\alpha_1 - 15\alpha_2 - 21\alpha_3 - 30\alpha_4 - 24\alpha_5 - 18\alpha_6 - 12\alpha_7 - 6\alpha_8 \\ &= s_5 s_4 s_3 s_2 s_4 s_5 (s_{\beta_0}(\Theta_{\frac{3}{2}})). \end{aligned}$$

If $w_1 = s_3 s_4 s_2 s_5 s_4 s_6 s_3 s_5 s_7 s_8 s_6 s_4 s_7 s_5 s_2 s_6 s_4 s_5 s_3 s_4 s_2$ and $w_2 = s_5 s_4 s_3 s_2 s_4 s_5$ then it follows that $s_\mu(\Theta_{\frac{3}{2}}) = w_1^{-1} w_2 \big(s_{\beta_0}(\Theta_{\frac{3}{2}})\big)$. Moreover, the built-in function *length* of LiE shows that $l(w_1^{-1} w_2) = 21$. Therefore $s_\mu(\Theta_{\frac{3}{2}})$ and $s_{\beta_0}(\Theta_{\frac{3}{2}})$ are conjugate by an odd element of $W(\mathfrak{l})$. Now since $Y(s_\beta(\Theta_{\frac{3}{2}})) = 0$ for all $\beta \in S_{\frac{3}{2}} \backslash \{\mu, \beta_0\}$, Proposition A.2.4 and Jantzen's criterion conclude that $M_\mathfrak{q}(\Theta_{\frac{3}{2}})$ is irreducible. ■

A.5.5 $F_4(4)$

The deleted Dynkin diagram is



The constant c_0 in this case is $\frac{11}{2}$. Observe that Theorem A.4.10, Theorem A.4.15, or Corollary A.4.17 cannot be applied for this case, because $\mathfrak{g}$ is not simply laced. So, to compute $\langle \Theta_t, \beta^\vee \rangle$ easily, we choose a specific realization of the root system. As in [9, page 65], we realize $\mathfrak{h}^*$ as $\mathbb{R}^4$ and take $\alpha_1 = e_2 - e_3$, $\alpha_2 = e_3 - e_4$, $\alpha_3 = e_4$, and $\alpha_4 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4)$ with $e_1, \ldots, e_4$ the standard orthonormal basis for $\mathbb{R}^4$. For simplicity we denote by $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$, $(0,0,0,1)$ the orthonormal basis e_1, e_2, e_3, e_4 for $\mathfrak{h}^*$, respectively. By using this realization, the weights in $\Delta^+(\mathfrak{l})$, $\Delta(\mathfrak{g}(1))$, and $\Delta(\mathfrak{z}(\mathfrak{n}))$ are listed as in Table A.2. A direct computation shows that we have $\Theta_t = \left(t, \frac{5}{2}, \frac{3}{2}, \frac{1}{2}\right)$ in this realization. Then, the values of $\langle \Theta_t, \delta^\vee \rangle$ for $\delta \in \Delta(\mathfrak{n})$ are obtained as in Table A.3.

Observe that $S_t = \{\delta \in \Delta(\mathfrak{n}) \mid \langle \Theta_t, \delta^\vee \rangle \in 1 + \mathbb{Z}_{\geq 0}\}$. Then Table A.3 shows that $\mu \in S_t$ when $t \in \frac{1}{2} + \mathbb{Z}_{\geq 0}$, and that $\gamma_4 \in S_t$ for all $t \in (\frac{1}{2} + \mathbb{Z}_{\geq 0}) \cup (1 + \mathbb{Z}_{\geq 0})$.

Theorem A.5.20 *Let $\mathfrak{g}$ be the complex simple Lie algebra of type F_4 and $\mathfrak{q}$ be the parabolic subalgebra of $\mathfrak{g}$ determined by the simple root α_4. Then $M_{\mathfrak{q}}(\Theta_t)$ is reducible for all $t \in 1 + \mathbb{Z}_{\geq 0}$.*

Proof. To prove this theorem we use Theorem A.1.2. Observe that we have $\langle \Theta_t, \alpha^\vee \rangle = \langle \rho(\mathfrak{l}), \alpha^\vee \rangle \neq 0$ for $\alpha \in \Delta(\mathfrak{l})$. Thus it follows from Table A.3 that Θ_t is regular when $t \in 1 + \mathbb{Z}_{\geq 0}$. Therefore, the converse statement of Theorem A.1.2 holds for $t \in 1 + \mathbb{Z}_{\geq 0}$. Now it is clear from Table A.3 that $\langle \Theta_t, \gamma_4^\vee \rangle \in \mathbb{Z}_{>0}$ for all $t \in 1 + \mathbb{Z}_{\geq 0}$. Hence, by Theorem A.1.2, $M_{\mathfrak{q}}(\Theta_t)$ is reducible if $t \in 1 + \mathbb{Z}_{\geq 0}$. ■

Lemma A.5.21 *For $t \in \frac{1}{2} + \mathbb{Z}_{\geq 0}$, we have the following:*

1. *If $t \in \frac{3}{2} + \mathbb{Z}_{\geq 0}$ then $Y(s_\mu(\Theta_t)) \neq 0$.*

2. *If $t \in \frac{1}{2} + \mathbb{Z}_{\geq 0}$ then $Y(s_{\gamma_4}(\Theta_t)) \neq 0$.*

Proof. We only prove (1), since (2) may be proven similarly. A direct computation

Table A.2:

$\Delta^+(\mathfrak{l}) = \big\{(0,1,-1,0)$	$\Delta(\mathfrak{g}(1)) = \big\{\beta_1 \equiv \left(\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right)$	$\Delta(\mathfrak{z}(\mathfrak{n})) = \big\{\gamma_1 \equiv (1,-1,0,0)$
$(0,0,1,-1)$	$\beta_2 \equiv \left(\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right)$	$\gamma_2 \equiv (1,0,-1,0)$
$(0,0,0,1)$	$\beta_3 \equiv \left(\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right)$	$\gamma_3 \equiv (1,0,0,-1)$
$(0,1,0,-1)$	$\beta_4 \equiv \left(\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right)$	$\gamma_4 \equiv (1,0,0,0)$
$(0,0,1,0)$	$\beta_5 \equiv \left(\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$	$\gamma_5 \equiv (1,0,0,1)$
$(0,1,0,0)$	$\beta_6 \equiv \left(\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right)$	$\gamma_6 \equiv (1,0,1,0)$
$(0,0,1,1)$	$\beta_7 \equiv \left(\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right)$	$\gamma \equiv (1,1,0,0)\big\}$,
$(0,1,0,1)$	$\mu \equiv \left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)\big\}$	
$(0,1,1,0)\big\}$		

shows that $s_\mu(\Theta_t) = (\frac{t}{2}-\frac{9}{4}, -\frac{t}{2}+\frac{1}{4}, -\frac{t}{2}-\frac{3}{4}, -\frac{t}{2}-\frac{7}{4})$. Then one can easily check that $\langle s_\mu(\Theta_t), \alpha^\vee\rangle \in \mathbb{Z}\backslash\{0\}$ for all $\alpha \in \Delta(\mathfrak{l})$ if $t \in \frac{3}{2}+\mathbb{Z}_{\geq 0}$. Now this proposition is concluded by Corollary A.1.5. ■

Theorem A.5.22 *Let $\mathfrak{g}$ be the complex simple Lie algebra of type F_4 and $\mathfrak{q}$ be the parabolic subalgebra of $\mathfrak{g}$ determined by the simple root α_4. Then $M_{\mathfrak{q}}(\Theta_t)$ is reducible for all $t \in \frac{1}{2}+\mathbb{Z}_{\geq 0}$.*

We take care of the cases $t \in \frac{5}{2}+\mathbb{Z}_{\geq 0}$ and $t = \frac{1}{2}, \frac{3}{2}$ separately, because the proofs

Table A.3:

	$\langle \Theta_t, \beta_j^\vee \rangle$		$\langle \Theta_t, \gamma_j^\vee \rangle$
β_1	$t - \frac{9}{2}$	γ_1	$t - \frac{5}{2}$
β_2	$t - \frac{7}{2}$	γ_2	$t - \frac{3}{2}$
β_3	$t - \frac{3}{2}$	γ_3	$t - \frac{1}{2}$
β_4	$t + \frac{1}{2}$	γ_4	$2t$
β_5	$t - \frac{1}{2}$	γ_5	$t + \frac{1}{2}$
β_6	$t + \frac{3}{2}$	γ_6	$t + \frac{3}{2}$
β_7	$t + \frac{7}{2}$	γ	$t + \frac{5}{2}$
μ	$t + \frac{9}{2}$		

are slightly different for those cases.

Claim 1: *If $t \in \frac{5}{2} + \mathbb{Z}_{\geq 0}$ then $M_{\mathfrak{q}}(\Theta_t)$ is reducible.*

Proof. First we show that $M_{\mathfrak{q}}(\Theta_t)$ is reducible when $t \in \frac{7}{2} + \mathbb{Z}_{\geq 0}$. It is clear from Table A.3 and Lemma A.5.21 that we have $\gamma_4 \in S_t$ and $Y(s_{\gamma_4}(\Theta_t)) \neq 0$ for any $t \in \frac{1}{2} + \mathbb{Z}_{\geq 0}$. Now suppose that there exists $\delta \in S_t \backslash \{\gamma_4\}$ so that $s_{\gamma_4}(\Theta_t)$ and $s_\delta(\Theta_t)$ are $W(\mathfrak{l})$-conjugate. Then, by using the facts that $\langle \Theta_t, \gamma_4^\vee \rangle = 2t$ and $\gamma_4 \in \Delta(\mathfrak{z}(\mathfrak{n}))$, Lemma A.2.9 gives

$$4t = \langle \Theta_t, \delta^\vee \rangle \frac{2\langle \delta, \lambda_{\mathfrak{q}} \rangle}{||\alpha_{\mathfrak{q}}||^2}. \tag{A.5.23}$$

By using Table A.3, one can check that no $\delta \in \Delta(\mathfrak{n})\backslash\{\gamma_4\}$ satisfies the equation (A.5.23) if $t \in \frac{7}{2} + \mathbb{Z}_{\geq 0}$. Therefore, by Lemma A.2.9 and Proposition A.2.4 that $Y(s_{\gamma_4}(\Theta_t))$ does not cancel out in (A.2.3) for $t \in \frac{7}{2} + \mathbb{Z}_{\geq 0}$. Then Jantzen's criterion concludes that $M_{\mathfrak{q}}(\Theta_t)$ is reducible if $t \in \frac{7}{2} + \mathbb{Z}_{\geq 0}$.

If $t = \frac{5}{2}$ then it is clear from Table A.3 and Lemma A.5.21 that we have $\mu \in S_{\frac{5}{2}}$ and $Y(s_\mu(\Theta_{\frac{5}{2}})) \neq 0$. Then, by using μ instead of γ_4 in the above argument, one can see that there is no $\delta \in S_{\frac{5}{2}} \backslash \{\mu\}$ so that $s_\delta(\Theta_{\frac{5}{2}})$ is $W(\mathfrak{l})$-conjugate to $s_\mu(\Theta_{\frac{5}{2}})$.

Therefore, by Proposition A.2.4 and Jantzen's criterion, $M_{\mathfrak{q}}(\Theta_{\frac{5}{2}})$ is reducible. ■

Claim 2. *If* $t = \frac{1}{2}, \frac{3}{2}$ *then* $M_{\mathfrak{q}}(\Theta_t)$ *is reducible.*

Proof. We start from the case $t = \frac{3}{2}$. By Table A.3 and Lemma A.5.21, we have $\gamma_4 \in S_{\frac{3}{2}}$ and $Y(s_{\gamma_4}(\Theta_{\frac{3}{2}})) \neq 0$. Now suppose that δ is an element in $S_{\frac{3}{2}} \backslash \{\gamma_4\}$ so that $s_\delta(\Theta_{\frac{3}{2}})$ is $W(\mathfrak{l})$-conjugate to $s_{\gamma_4}(\Theta_{\frac{3}{2}})$. Then, as in Claim 1, by using the facts that $\langle \Theta_t, \gamma_4^\vee \rangle = 2t$ and $\gamma_4 \in \Delta(\mathfrak{z}(\mathfrak{n}))$, it follows from Lemma A.2.9 that we have

$$6 = \langle \Theta_{\frac{3}{2}}, \delta^\vee \rangle \frac{2\langle \delta, \lambda_{\mathfrak{q}} \rangle}{||\alpha_{\mathfrak{q}}||^2}. \tag{A.5.24}$$

One can see from Table A.3 that only μ and γ_6 from $\Delta(\mathfrak{n})\backslash\{\gamma_4\}$ satisfy (A.5.24). By Table A.3, it is clear that $\mu, \gamma_6 \in S_{\frac{3}{2}}$ and Lemma A.5.21 shows that $Y(s_\mu(\Theta_{\frac{3}{2}})) \neq 0$. Moreover, a direct computation shows that $\langle s_{\gamma_6}(\Theta_{\frac{3}{2}}), \alpha^\vee \rangle \in \mathbb{Z}\backslash\{0\}$ for all $\alpha \in \Delta(\mathfrak{l})$. Therefore, by Corollary A.1.5, $Y(s_{\gamma_6}(\Theta_{\frac{3}{2}})) \neq 0$. Then Proposition A.2.12 concludes that $M_{\mathfrak{q}}(\Theta_{\frac{3}{2}})$ is reducible.

The case $t = \frac{1}{2}$ can be shown similarly. It follows from Table A.3 and Lemma A.5.21 that we have $\gamma_4 \in S_{\frac{1}{2}}$ and $Y(s_{\gamma_4}(\Theta_{\frac{1}{2}})) \neq 0$. Then one can see that only β_6 and γ_5 satisfy the equation

$$2 = \langle \Theta_{\frac{1}{2}}, \delta^\vee \rangle \frac{2\langle \delta, \lambda_{\mathfrak{q}} \rangle}{||\alpha_{\mathfrak{q}}||^2}.$$

It is clear from Table A.3 that $\beta_6, \gamma_5 \in S_{\frac{1}{2}}$. Direct verification using Corollary A.1.5 shows that $Y(s_{\beta_6}(\Theta_{\frac{1}{2}})) \neq 0$ and $Y(s_{\gamma_5}(\Theta_{\frac{1}{2}})) \neq 0$. Now Proposition A.2.12 concludes that $M_{\mathfrak{q}}(\Theta_{\frac{1}{2}})$ is reducible. ■

A.6 The Special Values and The Reducibility Points

In this section we check that the generalized Verma modules that are corresponding to the line bundles in Table 7.1 for $\mathfrak{q}$ in (3.3.3) are reducible, by using the reducibility points in Theorem A.5.1.

By Corollary 2.7.7, if an Ω_2 system is conformally invariant over the line bundle $\mathcal{L}(s_0\lambda_{\mathfrak{q}})$ then the generalized Verma module $M_{\mathfrak{q}}[\mathbb{C}_{-s_0\lambda_{\mathfrak{q}}}] = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{q})} \mathbb{C}_{-s_0\lambda_{\mathfrak{q}}}$ is reducible. Table A.4 summarizes the generalized Verma modules of $\mathfrak{q}$ in (3.3.3) that correspond to the line bundles in Table 7.1. Here, since the special values and the reducibility points for $\mathfrak{q}$ of type $E_6(5)$ are the same as those for $\mathfrak{q}$ of type $E_6(3)$, we only consider $\mathfrak{q}$ of type $E_6(3)$.

Table A.4:

Type	$\Omega_2\vert_{V(\mu+\epsilon_\gamma)^*}$	$\Omega_2\vert_{V(\mu+\epsilon_{n\gamma})^*}$
$E_6(3)$	$M_{\mathfrak{q}}[\mathbb{C}_{-\lambda_3}]$	$M_{\mathfrak{q}}[\mathbb{C}_{-2\lambda_3}]$
$E_7(2)$	$M_{\mathfrak{q}}[\mathbb{C}_{-2\lambda_2}]$	–
$E_7(6)$	$M_{\mathfrak{q}}[\mathbb{C}_{-\lambda_6}]$	$M_{\mathfrak{q}}[\mathbb{C}_{-3\lambda_6}]$
$E_8(1)$	$M_{\mathfrak{q}}[\mathbb{C}_{-3\lambda_1}]$	–
$F_4(4)$	$M_{\mathfrak{q}}[\mathbb{C}_{\lambda_4}]$	–

To find the corresponding the complex parameter t for $M_{\mathfrak{q}}(\Theta_t)$, observe that the generalized Verma modules $M_{\mathfrak{q}}(\Theta_t) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{q})} \mathbb{C}_{t-c_0\lambda_{\mathfrak{q}}}$ are parametrized by their infinitesimal characters. Therefore, if t_0 is the complex parameter corresponding to $\mathbb{C}_{-s_0\lambda_{\mathfrak{q}}}$ then t_0 is obtained by $t_0 = c_0 - s_0$. Table A.5 collects all the complex parameter t_0 for $\mathfrak{q}$ in (3.3.3).

By Theorem A.5.1, the generalized Verma modules $M_{\mathfrak{q}}(\Theta_t)$ at $t = t_0$ in Table A.5 are reducible. Hence, the special values in Table 7.1 for $\mathfrak{q}$ in (3.3.3) do not contradict Theorem A.5.1.

Table A.5:

Type	$\Omega_2\vert_{V(\mu+\epsilon_\gamma)^*}$	$\Omega_2\vert_{V(\mu+\epsilon_{n\gamma})^*}$
$E_6(3)$	$\frac{7}{2}$	$\frac{5}{2}$
$E_7(2)$	5	–
$E_7(6)$	$\frac{11}{2}$	$\frac{7}{2}$
$E_8(1)$	$\frac{17}{2}$	–
$F_4(4)$	$\frac{13}{2}$	–

APPENDIX B

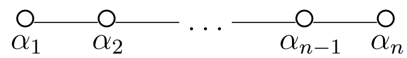
Dynkin Diagrams and Extended Dynkin Diagrams

This appendix collects Dynkin diagrams and the extended Dynkin diagrams for each complex simple Lie algebra. We use the Bourbaki conventions [4] for the numbering of the simple roots for exceptional algebras.

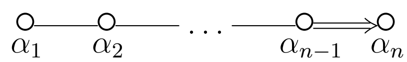
There are three figures in this appendix. Figure B.1 shows the Dynkin diagrams, and Figure B.2 is the Dynkin diagrams with the coefficients of the simple roots in the highest root. The extended Dynkin diagrams are shown in Figure B.3.

Figure B.1: The Dynkin diagrams

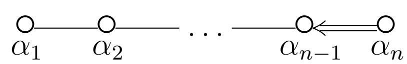
- A_n, $n \geq 2$:



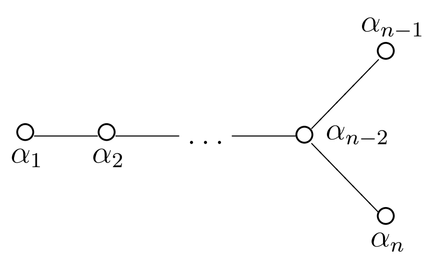
- B_n, $n \geq 3$:



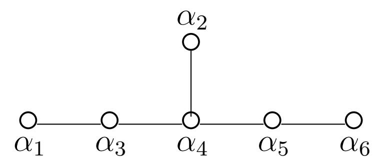
- C_n, $n \geq 2$:



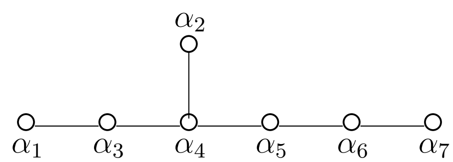
- D_n, $n \geq 4$:



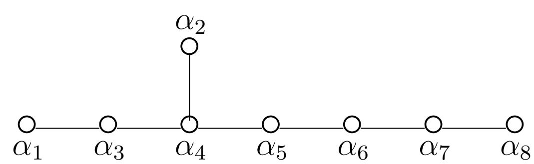
- E_6 :


α_2
α_1 α_3 α_4 α_5 α_6

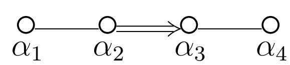
- E_7 :


α_2
α_1 α_3 α_4 α_5 α_6 α_7

- E_8 :


α_2
α_1 α_3 α_4 α_5 α_6 α_7 α_8

- F_4 :


α_1 α_2 α_3 α_4

- G_2 :

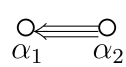
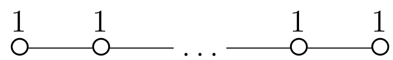
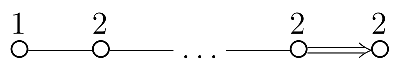

α_1 α_2

Figure B.2: The Dynkin diagrams with the multiplicities of the simple roots in the highest root of $\mathfrak{g}$

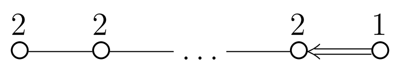
- A_n, $n \geq 2$:



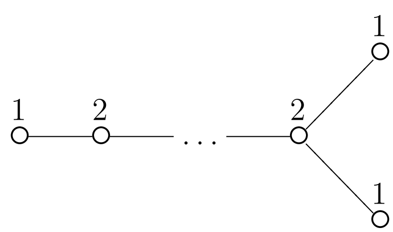
- B_n, $n \geq 3$:



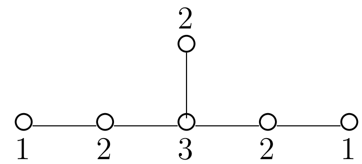
- C_n, $n \geq 2$:



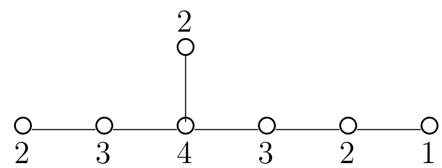
- D_n, $n \geq 4$:



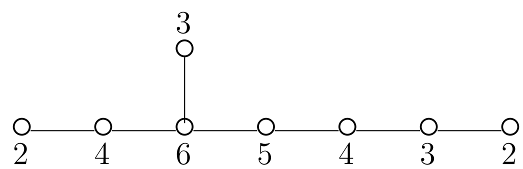
- E_6 :



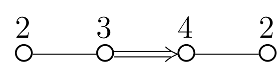
- E_7 :



- E_8 :



- F_4 :



- G_2 :

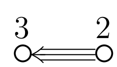
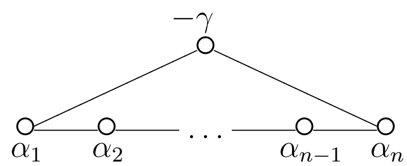
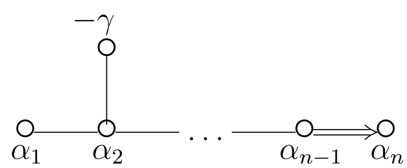


Figure B.3: The extended Dynkin diagrams with γ the highest root of $\mathfrak{g}$

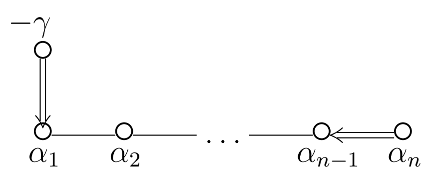
- A_n, $n \geq 2$:



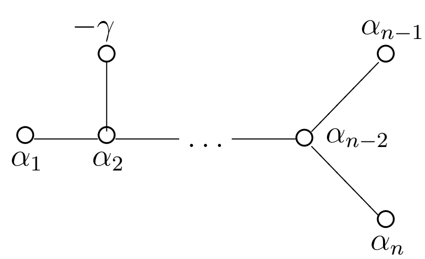
- B_n, $n \geq 3$:



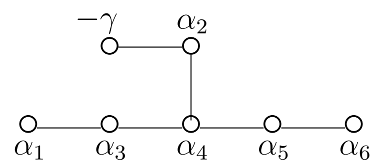
- C_n, $n \geq 2$:



- D_n, $n \geq 4$:

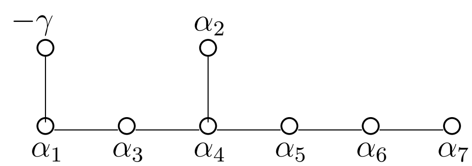


- E_6 :



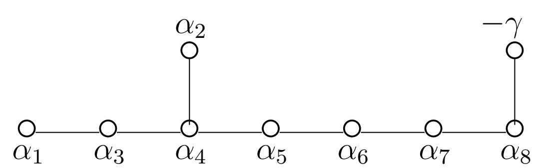
$-\gamma$
α_2
α_1
α_3
α_4
α_5
α_6

- E_7 :



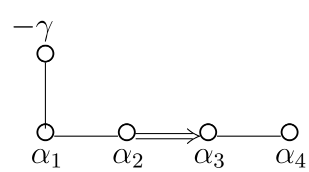
$-\gamma$
α_2
α_1
α_3
α_4
α_5
α_6
α_7

- E_8 :



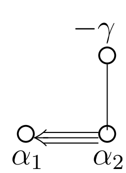
α_2
$-\gamma$
α_1
α_3
α_4
α_5
α_6
α_7
α_8

- F_4 :



$-\gamma$
α_1
α_2
α_3
α_4

- G_2 :



$-\gamma$
α_1
α_2

APPENDIX C

Basic Data

This appendix summarizes the following useful data for maximal two-step nilpotent parabolic subalgebras $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{g}(1) \oplus \mathfrak{z}(\mathfrak{n})$ of non-Heisenberg type:

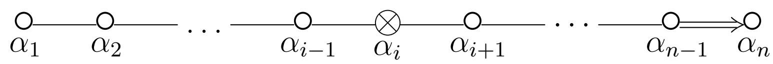
- The deleted Dynkin diagrams
- The simple root α_γ so that $\langle \gamma, \alpha_\gamma \rangle \neq 0$, where γ is the highest root of $\mathfrak{g}$
- The subgraphs of $\mathfrak{l}_\gamma$, $\mathfrak{l}_{n\gamma}$, $\mathfrak{l}_{n-1}$, and $\mathfrak{l}_n$ (if the subalgebras are non-zero)
- The highest weights for $\mathfrak{g}(1)$, $\mathfrak{z}(\mathfrak{n})$, $\mathfrak{l}_\gamma$, $\mathfrak{l}_{n\gamma}$, $\mathfrak{l}_{n-1}$, and $\mathfrak{l}_n$ (if the subalgebras are non-zero)

For the definitions for the deleted Dynkin diagrams and the simple root α_γ, see Section 3.1 and Section 3.3, respectively. Section 3.2 describes about the subspaces $\mathfrak{g}(1)$ and $\mathfrak{z}(\mathfrak{n})$. The definitions for the simple subalgebras $\mathfrak{l}_\gamma$ and $\mathfrak{l}_{n\gamma}$ of $\mathfrak{l}$ are given in Section 3.3. If $\mathfrak{q}$ is of type $D_n(n-2)$ then we denote by $\mathfrak{l}_{n-1}$ (resp. $\mathfrak{l}_n$) the simple subalgebra of $\mathfrak{l}$ whose subgraph is the node for the simple root α_{n-1} (resp. α_n).

The sets of roots contributing to $\mathfrak{g}(1)$, $\mathfrak{z}(\mathfrak{n})$, $\mathfrak{l}_\gamma$, $\mathfrak{l}_{n\gamma}$, $\mathfrak{l}_{n-1}$, and $\mathfrak{l}_n$ are also given for classical algebras. For exceptional algebras one can easily read off such roots from the lists of positive roots in Appendix D. If $\mathfrak{q}$ is determined by $\alpha_{\mathfrak{q}} \in \Pi$ then the roots contributing for $\mathfrak{l}$, $\mathfrak{g}(1)$, and $\mathfrak{z}(\mathfrak{n})$ are the positive roots whose coefficients for $\alpha_{\mathfrak{q}}$ are 0, 1, and 2, respectively.

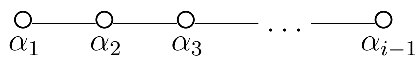
• $\mathrm{B}_n(i)$, $3 \leq i \leq n-2$:

1. The deleted Dynkin diagram:

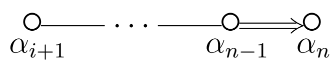


2. $\alpha_\gamma = \alpha_2$

3. The subgraph for $\mathfrak{l}_\gamma$:



4. The subgraph for $\mathfrak{l}_{n\gamma}$:



5. $\mathfrak{g}(1) = V(\mu)$:

   - $\mu = e_1 + e_{i+1}$
   - $\Delta(\mathfrak{g}(1)) = \{e_j \pm e_k \mid 1 \leq j \leq i \text{ and } i+1 \leq k \leq n\} \cup \{e_j \mid 1 \leq j \leq i\}$

6. $\mathfrak{z}(\mathfrak{n}) = V(\gamma)$:

   - $\gamma = e_1 + e_2$
   - $\Delta(\mathfrak{z}(\mathfrak{n})) = \{e_j + e_k \mid 1 \leq j < k \leq i\}$

7. $\mathfrak{l}_\gamma = V(\xi_\gamma)$:

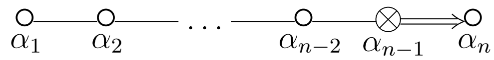
   - $\xi_\gamma = e_1 - e_i$
   - $\Delta^+(\mathfrak{l}_\gamma) = \{e_j - e_k \mid 1 \leq j < k \leq i\}$

8. $\mathfrak{l}_{n\gamma} = V(\xi_{n\gamma})$:

   - $\xi_{n\gamma} = e_{i+1} + e_{i+2}$
   - $\Delta^+(\mathfrak{l}_{n\gamma}) = \{e_j \pm e_k \mid i+1 \leq j < k \leq n\} \cup \{e_j \mid i+1 \leq j \leq n\}$

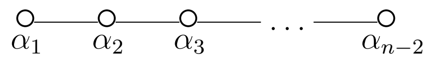
• $\mathrm{B}_n(n-1)$:

1. The deleted Dynkin diagram:



2. $\alpha_\gamma = \alpha_2$

3. The subgraph for $\mathfrak{l}_\gamma$:



4. The subgraph for $\mathfrak{l}_{n\gamma}$:

5. $\mathfrak{g}(1) = V(\mu)$:

   - $\mu = e_1 + e_n$
   - $\Delta(\mathfrak{g}(1)) = \{e_j \pm e_n \mid 1 \leq j \leq n-1\} \cup \{e_j \mid 1 \leq j \leq n-1\}$

6. $\mathfrak{z}(\mathfrak{n}) = V(\gamma)$:

   - $\gamma = e_1 + e_2$
   - $\Delta(\mathfrak{z}(\mathfrak{n})) = \{e_j + e_k \mid 1 \leq j < k \leq n-1\}$

7. $\mathfrak{l}_\gamma = V(\xi_\gamma)$:

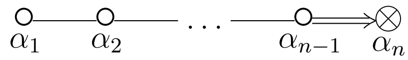
   - $\xi_\gamma = e_1 - e_{n-1}$
   - $\Delta^+(\mathfrak{l}_\gamma) = \{e_j - e_k \mid 1 \leq j < k \leq n-1\}$

8. $\mathfrak{l}_{n\gamma} = V(\xi_{n\gamma})$:

   - $\xi_{n\gamma} = e_n$
   - $\Delta^+(\mathfrak{l}_{n\gamma}) = \{e_n\}$

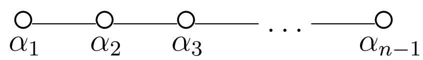
• $\mathrm{B}_n(n)$:

1. The deleted Dynkin diagram:



2. $\alpha_\gamma = \alpha_2$

3. The subgraph for $\mathfrak{l}_\gamma$:



4. $\mathfrak{g}(1) = V(\mu)$:
   - $\mu = e_1$
   - $\Delta(\mathfrak{g}(1)) = \{e_j \mid 1 \leq j \leq n\}$

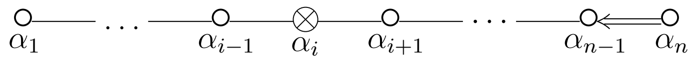
5. $\mathfrak{z}(\mathfrak{n}) = V(\gamma)$:
   - $\gamma = e_1 + e_2$
   - $\Delta(\mathfrak{z}(\mathfrak{n})) = \{e_j + e_k \mid 1 \leq j < k \leq n\}$

6. $\mathfrak{l}_\gamma = V(\xi_\gamma)$:
   - $\xi_\gamma = e_1 - e_n$
   - $\Delta^+(\mathfrak{l}_\gamma) = \{e_j - e_k \mid 1 \leq j < k \leq n\}$

7. $\mathfrak{l}_{n\gamma} = 0$

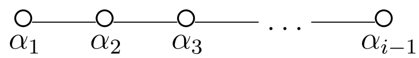
- $\mathrm{C}_n(i)$, $2 \le i \le n-1$:

1. The deleted Dynkin diagram:

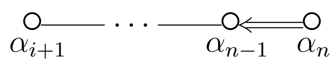


2. $\alpha_\gamma = \alpha_1$

3. The subgraph for $\mathfrak{l}_\gamma$:



4. The subgraph for $\mathfrak{l}_{n\gamma}$:



5. $\mathfrak{g}(1) = V(\mu)$:

   - $\mu = e_1 + e_{i+1}$
   - $\Delta(\mathfrak{g}(1)) = \{e_j \pm e_k \mid 1 \le j \le i \text{ and } i+1 \le k \le n\}$

6. $\mathfrak{z}(\mathfrak{n}) = V(\gamma)$:

   - $\gamma = 2e_1$
   - $\Delta(\mathfrak{z}(\mathfrak{n})) = \{e_j + e_k \mid 1 \le j < k \le i\} \cup \{2e_j \mid 1 \le j \le i\}$

7. $\mathfrak{l}_\gamma = V(\xi_\gamma)$:

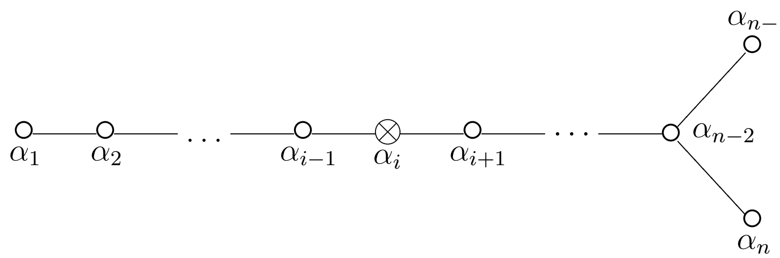
   - $\xi_\gamma = e_1 - e_i$
   - $\Delta^+(\mathfrak{l}_\gamma) = \{e_j - e_k \mid 1 \le j < k \le i\}$

8. $\mathfrak{l}_{n\gamma} = V(\xi_{n\gamma})$:

   - $\xi_{n\gamma} = 2e_{i+1}$
   - $\Delta^+(\mathfrak{l}_{n\gamma}) = \{e_j \pm e_k \mid i+1 \le j < k \le n\} \cup \{2e_j \mid i+1 \le j \le n\}$

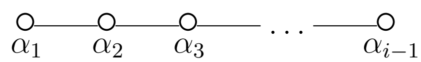
• $\mathrm{D}_n(i)$, $3 \leq i \leq n-3$:

1. The deleted Dynkin diagram:

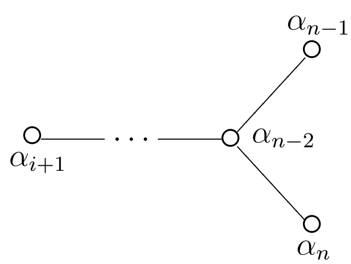


2. $\alpha_\gamma = \alpha_2$

3. The subgraph for $\mathfrak{l}_\gamma$:



4. The subgraph for $\mathfrak{l}_{n\gamma}$:



5. $\mathfrak{g}(1) = V(\mu)$:

   - $\mu = e_1 + e_{i+1}$
   - $\Delta(\mathfrak{g}(1)) = \{e_j \pm e_k \mid 1 \leq j \leq i \text{ and } i+1 \leq k \leq n\}$

6. $\mathfrak{z}(\mathfrak{n}) = V(\gamma)$:

   - $\gamma = e_1 + e_2$
   - $\Delta(\mathfrak{z}(\mathfrak{n})) = \{e_j + e_k \mid 1 \leq j < k \leq i\}$

7. $\mathfrak{l}_\gamma = V(\xi_\gamma)$:

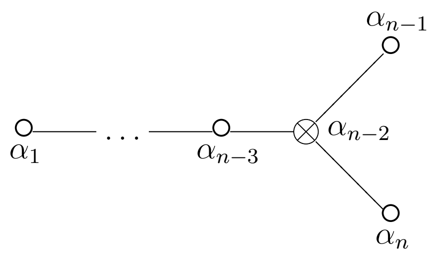
   - $\xi_\gamma = e_1 - e_i$
   - $\Delta^+(\mathfrak{l}_\gamma) = \{e_j - e_k \mid 1 \leq j < k \leq i\}$

8. $\mathfrak{l}_{n\gamma} = V(\xi_{n\gamma})$:

   - $\xi_{n\gamma} = e_{i+1} + e_{i+2}$
   - $\Delta^+(\mathfrak{l}_{n\gamma}) = \{e_j \pm e_k \mid i+1 \leq j < k \leq n\}$

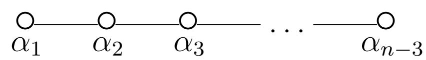
$D_n(n-2)$:

1. the deleted Dynkin diagram:

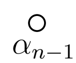


2. $\alpha_\gamma = \alpha_2$

3. the subgraph for $\mathfrak{l}_\gamma$:



4. the subgraph for $\mathfrak{l}_{n-1}$:



5. the subgraph for $\mathfrak{l}_n$:

6. $\mathfrak{g}(1) = V(\mu)$:

   - $\mu = e_1 + e_{n-1}$
   - $\Delta(\mathfrak{g}(1)) = \{e_j \pm e_k \mid 1 \le j \le n-2 \text{ and } k = n-1, n\}$

7. $\mathfrak{z}(\mathfrak{n}) = V(\gamma)$:

   - $\gamma = e_1 + e_2$
   - $\Delta(\mathfrak{z}(\mathfrak{n})) = \{e_j + e_k \mid 1 \le j < k \le n-2\}$

8. $\mathfrak{l}_\gamma = V(\xi_\gamma)$:

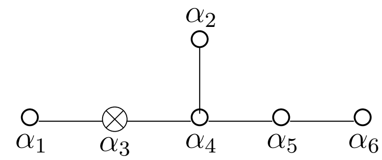
   - $\xi_\gamma = e_1 - e_{n-2}$
   - $\Delta^+(\mathfrak{l}_\gamma) = \{e_j - e_k \mid 1 \leq j < k \leq n-2\}$

9. $\mathfrak{l}_{n-1}$: $\Delta^+(\mathfrak{l}_{n-1}) = \{e_{n-1} - e_n\}$

10. $\mathfrak{l}_n$: $\Delta^+(\mathfrak{l}_n) = \{e_{n-1} + e_n\}$

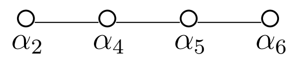
• $E_6(3)$:

1. The deleted Dynkin diagram:

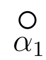


2. $\alpha_\gamma = \alpha_2$

3. The subgraph for $\mathfrak{l}_\gamma$:



4. The subgraph for $\mathfrak{l}_{n\gamma}$:



5. $\mathfrak{g}(1) = V(\mu)$:

   - $\mu = \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6$

6. $\mathfrak{z}(\mathfrak{n}) = V(\gamma)$:

   - $\gamma = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$

7. $\mathfrak{l}_\gamma = V(\xi_\gamma)$:

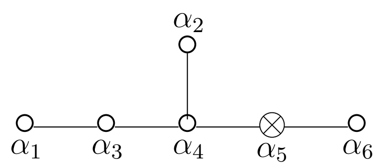
   - $\xi_\gamma = \alpha_2 + \alpha_4 + \alpha_5 + \alpha_6$

8. $\mathfrak{l}_{n\gamma} = V(\xi_{n\gamma})$:

   - $\xi_{n\gamma} = \alpha_1$

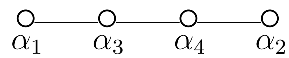
• $\mathrm{E}_6(5)$:

1. The deleted Dynkin diagram:

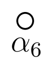


2. $\alpha_\gamma = \alpha_2$

3. The subgraph for $\mathfrak{l}_\gamma$:



4. The subgraph for $\mathfrak{l}_{n\gamma}$:



5. $\mathfrak{g}(1) = V(\mu)$:

   - $\mu = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6$

6. $\mathfrak{z}(\mathfrak{n}) = V(\gamma)$:

   - $\gamma = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$

7. $\mathfrak{l}_\gamma = V(\xi_\gamma)$:

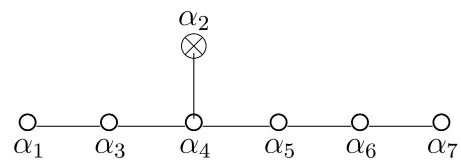
   - $\xi_\gamma = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$

8. $\mathfrak{l}_{n\gamma} = V(\xi_{n\gamma})$:

   - $\xi_{n\gamma} = \alpha_6$

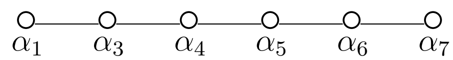
• $\mathrm{E}_7(2)$:

1. The deleted Dynkin diagram:



2. $\alpha_\gamma = \alpha_1$

3. The subgraph for $\mathfrak{l}_\gamma$:



4. $\mathfrak{g}(1) = V(\mu)$:

    - $\mu = \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$

5. $\mathfrak{z}(\mathfrak{n}) = V(\gamma)$:

    - $\gamma = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$

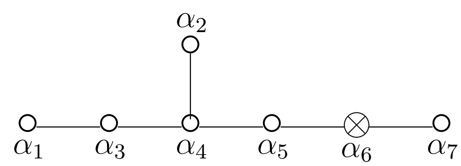
6. $\mathfrak{l}_\gamma = V(\xi_\gamma)$:

    - $\xi_\gamma = \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7$

7. $\mathfrak{l}_{n\gamma} = 0$

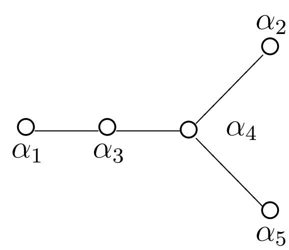
• $E_7(6)$:

1. The deleted Dynkin diagram:

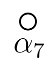


2. $\alpha_\gamma = \alpha_1$

3. The subgraph for $\mathfrak{l}_\gamma$:



4. The subgraph for $\mathfrak{l}_{n\gamma}$:



5. $\mathfrak{g}(1) = V(\mu)$:

   - $\mu = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7$

6. $\mathfrak{z}(\mathfrak{n}) = V(\gamma)$:

   - $\gamma = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$

7. $\mathfrak{l}_\gamma = V(\xi_\gamma)$:

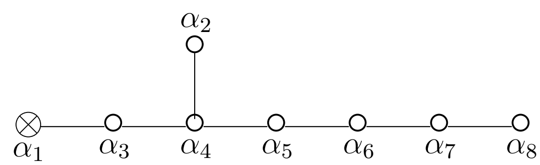
   - $\xi_\gamma = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5$

8. $\mathfrak{l}_{n\gamma} = V(\xi_{n\gamma})$:

   - $\xi_{n\gamma} = \alpha_7$

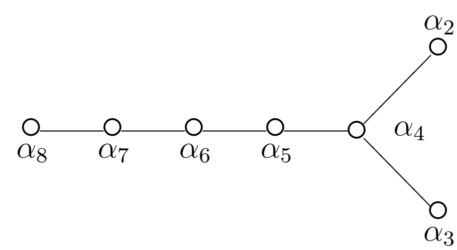
• $\mathrm{E}_8(1)$:

1. The deleted Dynkin diagram:



2. $\alpha_\gamma = \alpha_8$

3. The subgraph for $\mathfrak{l}_\gamma$:



4. $\mathfrak{g}(1) = V(\mu)$:

   - $\mu = \alpha_1 + 3\alpha_2 + 3\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8$

5. $\mathfrak{z}(\mathfrak{n}) = V(\gamma)$:

   - $\gamma = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8$

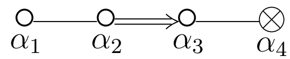
6. $\mathfrak{l}_\gamma = V(\xi_\gamma)$:

   - $\xi_\gamma = \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_8$

7. $\mathfrak{l}_{n\gamma} = 0$

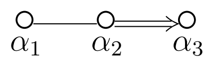
- $\mathrm{F}_4(4)$:

1. The deleted Dynkin diagram:



2. $\alpha_\gamma = \alpha_1$

3. The subgraph for $\mathfrak{l}_\gamma$:



4. $\mathfrak{g}(1) = V(\mu)$:

   - $\mu = \alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4$

5. $\mathfrak{z}(\mathfrak{n}) = V(\gamma)$:

   - $\gamma = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$

6. $\mathfrak{l}_\gamma = V(\xi_\gamma)$:

   - $\xi_\gamma = \alpha_1 + 2\alpha_2 + 2\alpha_3$

7. $\mathfrak{l}_{n\gamma} = 0$

APPENDIX D

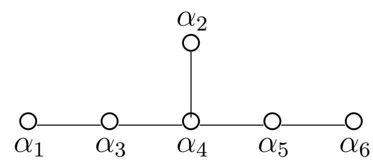
Lists of Positive Roots for Exceptional Algebras

In this appendix the lists of the positive roots for exceptional algebras are collected. The positive roots are given both in terms of simple roots and in a realization of the root system. The height of each positive root is also shown. These lists would be useful, when we find the roots contributing for $\mathfrak{l}$, V^+, and $\mathfrak{z}(\mathfrak{n})$.

E$_6$:

- $\Pi = \{\frac{1}{2}(e_1 - e_2 - e_3 - e_4 - e_5 + \sqrt{3}e_6), e_1 + e_2, e_2 - e_1, e_3 - e_2, e_4 - e_3, e_5 - e_4\}$

- Dynkin diagram:



- $\Delta^+ = \{e_i + e_j\}_{i<j\leq 5} \cup \{e_i - e_j\}_{j<i\leq 5} \cup \{\frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4 \pm e_5 + \sqrt{3}e_6)\}_{\text{number of munus signs even}}$
(36 postive roots)

A list of the postive roots:

Height 1:	α_1	$\frac{1}{2}(e_1 - e_2 - e_3 - e_4 - e_5 + \sqrt{3}e_6)$
	α_2	$e_1 + e_2$
	α_3	$e_2 - e_1$
	α_4	$e_3 - e_2$
	α_5	$e_4 - e_3$
	α_6	$e_5 - e_4$

Height 2:	$\alpha_1 + \alpha_3$	$\frac{1}{2}(-e_1 + e_2 - e_3 - e_4 - e_5 + \sqrt{3}e_6)$
	$\alpha_2 + \alpha_4$	$e_1 + e_3$
	$\alpha_3 + \alpha_4$	$e_3 - e_1$
	$\alpha_4 + \alpha_5$	$e_4 - e_2$
	$\alpha_5 + \alpha_6$	$e_5 - e_3$

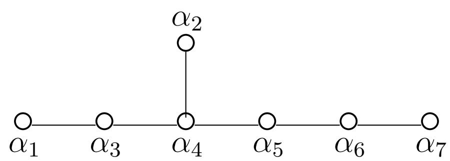
Height 3:	$\alpha_1 + \alpha_3 + \alpha_4$	$\frac{1}{2}(-e_1 - e_2 + e_3 - e_4 - e_5 + \sqrt{3}e_6)$
	$\alpha_2 + \alpha_3 + \alpha_4$	$e_2 + e_3$
	$\alpha_2 + \alpha_4 + \alpha_5$	$e_1 + e_4$
	$\alpha_3 + \alpha_4 + \alpha_5$	$e_4 - e_1$
	$\alpha_4 + \alpha_5 + \alpha_6$	$e_5 - e_2$
Height 4:	$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$	$\frac{1}{2}(e_1 + e_2 + e_3 - e_4 - e_5 + \sqrt{3}e_6)$
	$\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5$	$\frac{1}{2}(-e_1 - e_2 - e_3 + e_4 - e_5 + \sqrt{3}e_6)$
	$\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5$	$e_2 + e_4$
	$\alpha_2 + \alpha_4 + \alpha_5 + \alpha_6$	$e_1 + e_5$
	$\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$	$e_5 - e_1$
Height 5:	$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5$	$\frac{1}{2}(e_1 + e_2 - e_3 + e_4 - e_5 + \sqrt{3}e_6)$
	$\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$	$\frac{1}{2}(-e_1 - e_2 - e_3 - e_4 + e_5 + \sqrt{3}e_6)$
	$\alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5$	$e_3 + e_4$
	$\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$	$e_2 + e_5$
Height 6:	$\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5$	$\frac{1}{2}(e_1 - e_2 + e_3 + e_4 - e_5 + \sqrt{3}e_6)$
	$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$	$\frac{1}{2}(e_1 + e_2 - e_3 - e_4 + e_5 + \sqrt{3}e_6$
	$\alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6$	$e_3 + e_5$
Height 7:	$\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5$	$\frac{1}{2}(-e_1 + e_2 + e_3 + e_4 - e_5 + \sqrt{3}e_6)$
	$\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6$	$\frac{1}{2}(e_1 - e_2 + e_3 - e_4 + e_5 + \sqrt{3}e_6)$
	$\alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6$	$e_4 + e_5$

Height 8:	$\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6$	$\frac{1}{2}(-e_1 + e_2 + e_3 - e_4 + e_5 + \sqrt{3}e_6)$
	$\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6$	$\frac{1}{2}(e_1 - e_2 - e_3 + e_4 + e_5 + \sqrt{3}e_6)$
Height 9:	$\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6$	$\frac{1}{2}(-e_1 + e_2 - e_3 + e_4 + e_5 + \sqrt{3}e_6)$
Height 10:	$\alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$	$\frac{1}{2}(-e_1 - e_2 + e_3 + e_4 + e_5 + \sqrt{3}e_6)$
Height 11:	$\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$	$\frac{1}{2}(e_1 + e_2 + e_3 + e_4 + e_5 + \sqrt{3}e_6)$

E_7:

- $\Pi = \{\frac{1}{2}(e_1 - e_2 - \cdots - e_6 + \sqrt{2}e_7), e_1 + e_2, e_2 - e_1, e_3 - e_2, e_4 - e_3, e_5 - e_4, e_6 - e_5\}$

- Dynkin diagram:



- Δ^+

  $= \{e_i + e_j\}_{i<j\leq 6} \cup \{e_i - e_j\}_{j<i\leq 6} \cup \{\sqrt{2}e_7\} \cup \{\frac{1}{2}(\pm e_1 \pm e_2 \pm \cdots \pm e_6 + \sqrt{2}e_7)\}_{\text{number of munus signs odd}}$

  (63 postive roots)

A list of the postive roots:

Height 1:	α_1	$\frac{1}{2}(e_1 - e_2 - e_3 - e_4 - e_5 - e_6 + \sqrt{2}e_7)$
	α_2	$e_1 + e_2$
	α_3	$e_2 - e_1$
	α_4	$e_3 - e_2$
	α_5	$e_4 - e_3$
	α_6	$e_5 - e_4$
	α_7	$e_6 - e_5$

Height 2:	$\alpha_1 + \alpha_3$	$\frac{1}{2}(-e_1 + e_2 - e_3 - e_4 - e_5 - e_6 + \sqrt{2}e_7)$
	$\alpha_2 + \alpha_4$	$e_1 + e_3$
	$\alpha_3 + \alpha_4$	$e_3 - e_1$
	$\alpha_4 + \alpha_5$	$e_4 - e_2$
	$\alpha_5 + \alpha_6$	$e_5 - e_3$
	$\alpha_6 + \alpha_7$	$e_6 - e_4$
Height 3:	$\alpha_1 + \alpha_3 + \alpha_4$	$\frac{1}{2}(-e_1 - e_2 + e_3 - e_4 - e_5 - e_6 + \sqrt{2}e_7)$
	$\alpha_2 + \alpha_3 + \alpha_4$	$e_2 + e_3$
	$\alpha_2 + \alpha_4 + \alpha_5$	$e_1 + e_4$
	$\alpha_3 + \alpha_4 + \alpha_5$	$e_4 - e_1$
	$\alpha_4 + \alpha_5 + \alpha_6$	$e_5 - e_2$
	$\alpha_5 + \alpha_6 + \alpha_7$	$e_6 - e_3$
Height 4:	$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$	$\frac{1}{2}(e_1 + e_2 + e_3 - e_4 - e_5 - e_6 + \sqrt{2}e_7)$
	$\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5$	$\frac{1}{2}(-e_1 - e_2 - e_3 + e_4 - e_5 - e_6 + \sqrt{2}e_7)$
	$\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5$	$e_2 + e_4$
	$\alpha_2 + \alpha_4 + \alpha_5 + \alpha_6$	$e_1 + e_5$
	$\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$	$e_5 - e_1$
	$\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7$	$e_6 - e_2$
Height 5:	$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5$	$\frac{1}{2}(e_1 + e_2 - e_3 + e_4 - e_5 - e_6 + \sqrt{2}e_7)$
	$\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$	$\frac{1}{2}(-e_1 - e_2 - e_3 - e_4 + e_5 - e_6 + \sqrt{2}e_7)$
	$\alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5$	$e_3 + e_4$
	$\alpha_2 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7$	$e_1 + e_6$
	$\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$	$e_2 + e_5$
	$\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7$	$e_6 - e_1$

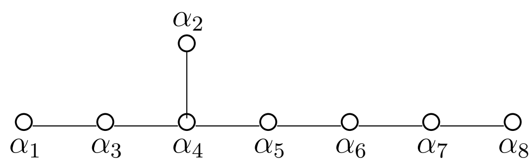
Height 6:	$\alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5$	$\frac{1}{2}(e_1-e_2+e_3+e_4-e_5-e_6+\sqrt{2}e_7)$
	$\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6$	$\frac{1}{2}(e_1+e_2-e_3-e_4+e_5-e_6+\sqrt{2}e_7)$
	$\alpha_1+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7$	$\frac{1}{2}(-e_1-e_2-e_3-e_4-e_5+e_6+\sqrt{2}e_7)$
	$\alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6$	e_3+e_5
	$\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7$	e_2+e_6
Height 7:	$\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5$	$\frac{1}{2}(-e_1+e_2+e_3+e_4-e_5-e_6+\sqrt{2}e_7)$
	$\alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6$	$\frac{1}{2}(e_1-e_2+e_3-e_4+e_5-e_6+\sqrt{2}e_7)$
	$\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7$	$\frac{1}{2}(e_1+e_2-e_3-e_4-e_5+e_6+\sqrt{2}e_7)$
	$\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6$	e_4+e_5
	$\alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6+\alpha_7$	e_3+e_6
Height 8:	$\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+\alpha_6$	$\frac{1}{2}(-e_1+e_2+e_3-e_4+e_5-e_6+\sqrt{2}e_7)$
	$\alpha_1+\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6$	$\frac{1}{2}(e_1-e_2-e_3+e_4+e_5-e_6+\sqrt{2}e_7)$
	$\alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6+\alpha_7$	$\frac{1}{2}(e_1-e_2+e_3-e_4-e_5+e_6+\sqrt{2}e_7)$
	$\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6+\alpha_7$	e_4+e_6
Height 9:	$\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+2\alpha_5+\alpha_6$	$\frac{1}{2}(-e_1+e_2-e_3+e_4+e_5-e_6+\sqrt{2}e_7)$
	$\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+\alpha_6+\alpha_7$	$\frac{1}{2}(-e_1+e_2+e_3-e_4-e_5+e_6+\sqrt{2}e_7)$
	$\alpha_1+\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6+\alpha_7$	$\frac{1}{2}(e_1-e_2-e_3+e_4-e_5+e_6+\sqrt{2}e_7)$
	$\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_7$	e_5+e_6
Height 10:	$\alpha_1+\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6$	$\frac{1}{2}(-e_1-e_2+e_3+e_4+e_5-e_6+\sqrt{2}e_7)$
	$\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+2\alpha_5+\alpha_6+\alpha_7$	$\frac{1}{2}(-e_1+e_2-e_3+e_4-e_5+e_6+\sqrt{2}e_7)$
	$\alpha_1+\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_7$	$\frac{1}{2}(e_1-e_2-e_3-e_4+e_5+e_6+\sqrt{2}e_7)$

Height 11:	$\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$	$\frac{1}{2}(e_1 + e_2 + e_3 + e_4 + e_5 - e_6 + \sqrt{2}e_7)$
	$\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7$	$\frac{1}{2}(-e_1 + e_2 - e_3 - e_4 + e_5 + e_6 + \sqrt{2}e_7)$
	$\alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7$	$\frac{1}{2}(-e_1 - e_2 + e_3 + e_4 - e_5 + e_6 + \sqrt{2}e_7)$
Height 12:	$\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7$	$\frac{1}{2}(e_1 + e_2 + e_3 + e_4 - e_5 + e_6 + \sqrt{2}e_7)$
	$\alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7$	$\frac{1}{2}(-e_1 - e_2 + e_3 - e_4 + e_5 + e_6 + \sqrt{2}e_7)$
Height 13:	$\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7$	$\frac{1}{2}(e_1 + e_2 + e_3 - e_4 + e_5 + e_6 + \sqrt{2}e_7)$
	$\alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$	$\frac{1}{2}(-e_1 - e_2 - e_3 + e_4 + e_5 + e_6 + \sqrt{2}e_7)$
Height 14:	$\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$	$\frac{1}{2}(e_1 + e_2 - e_3 + e_4 + e_5 + e_6 + \sqrt{2}e_7)$
Height 15:	$\alpha_1 + 2\alpha_2 + 2\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$	$\frac{1}{2}(e_1 - e_2 + e_3 + e_4 + e_5 + e_6 + \sqrt{2}e_7)$
Height 16:	$\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$	$\frac{1}{2}(-e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + \sqrt{2}e_7)$
Height 17:	$2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$	$\sqrt{2}e_7$

E_8:

- $\Pi = \{\frac{1}{2}(e_1 - e_2 - \cdots - e_7 + e_8), e_1 + e_2, e_2 - e_1, e_3 - e_2, e_4 - e_3, e_5 - e_4, e_6 - e_5, e_7 - e_6\}$

- Dynkin diagrams:



- $\Delta^+ = \{e_i + e_j\}_{i<j\leq 8} \cup \{e_i - e_j\}_{j<i\leq 8} \cup \{\frac{1}{2}(\pm e_1 \pm e_2 \pm \cdots \pm e_7 + e_8)\}_{\text{number of munus signs even}}$

  (120 postive roots)

A list of the postive roots:

Height 1:	α_1	$\frac{1}{2}(e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 + e_8)$
	α_2	$e_1 + e_2$
	α_3	$e_2 - e_1$
	α_4	$e_3 - e_2$
	α_5	$e_4 - e_3$
	α_6	$e_5 - e_4$
	α_7	$e_6 - e_5$
	α_8	$e_7 - e_6$

Height 2:	$\alpha_1 + \alpha_3$	$\frac{1}{2}(-e_1 + e_2 - e_3 - e_4 - e_5 - e_6 - e_7 + e_8)$
	$\alpha_2 + \alpha_4$	$e_1 + e_3$
	$\alpha_3 + \alpha_4$	$e_3 - e_1$
	$\alpha_4 + \alpha_5$	$e_4 - e_2$
	$\alpha_5 + \alpha_6$	$e_5 - e_3$
	$\alpha_6 + \alpha_7$	$e_6 - e_4$
	$\alpha_7 + \alpha_8$	$e_7 - e_5$
Height 3:	$\alpha_1 + \alpha_3 + \alpha_4$	$\frac{1}{2}(-e_1 - e_2 + e_3 - e_4 - e_5 - e_6 - e_7 + e_8)$
	$\alpha_2 + \alpha_3 + \alpha_4$	$e_2 + e_3$
	$\alpha_2 + \alpha_4 + \alpha_5$	$e_1 + e_4$
	$\alpha_3 + \alpha_4 + \alpha_5$	$e_4 - e_1$
	$\alpha_4 + \alpha_5 + \alpha_6$	$e_5 - e_2$
	$\alpha_5 + \alpha_6 + \alpha_7$	$e_6 - e_3$
	$\alpha_6 + \alpha_7 + \alpha_8$	$e_7 - e_4$
Height 4:	$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$	$\frac{1}{2}(e_1 + e_2 + e_3 - e_4 - e_5 - e_6 - e_7 + e_8)$
	$\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5$	$\frac{1}{2}(-e_1 - e_2 - e_3 + e_4 - e_5 - e_6 - e_7 + e_8)$
	$\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5$	$e_2 + e_4$
	$\alpha_2 + \alpha_4 + \alpha_5 + \alpha_6$	$e_1 + e_5$
	$\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$	$e_5 - e_1$
	$\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7$	$e_6 - e_2$
	$\alpha_5 + \alpha_6 + \alpha_7 + \alpha_8$	$e_7 - e_3$

Height 5:	$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5$	$\frac{1}{2}(e_1 + e_2 - e_3 + e_4 - e_5 - e_6 - e_7 + e_8)$
	$\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$	$\frac{1}{2}(-e_1 - e_2 - e_3 - e_4 + e_5 - e_6 - e_7 + e_8)$
	$\alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5$	$e_3 + e_4$
	$\alpha_2 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7$	$e_1 + e_6$
	$\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$	$e_2 + e_5$
	$\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7$	$e_6 - e_1$
	$\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8$	$e_7 - e_2$
Height 6:	$\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5$	$\frac{1}{2}(e_1 - e_2 + e_3 + e_4 - e_5 - e_6 - e_7 + e_8)$
	$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$	$\frac{1}{2}(e_1 + e_2 - e_3 - e_4 + e_5 - e_6 - e_7 + e_8)$
	$\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7$	$\frac{1}{2}(-e_1 - e_2 - e_3 - e_4 - e_5 + e_6 - e_7 + e_8)$
	$\alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6$	$e_3 + e_5$
	$\alpha_2 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8$	$e_1 + e_7$
	$\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7$	$e_2 + e_6$
	$\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8$	$e_7 - e_1$
Height 7:	$\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5$	$\frac{1}{2}(-e_1 + e_2 + e_3 + e_4 - e_5 - e_6 - e_7 + e_8)$
	$\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6$	$\frac{1}{2}(e_1 - e_2 + e_3 - e_4 + e_5 - e_6 - e_7 + e_8)$
	$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7$	$\frac{1}{2}(e_1 + e_2 - e_3 - e_4 - e_5 + e_6 - e_7 + e_8)$
	$\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8$	$\frac{1}{2}(-e_1 - e_2 - e_3 - e_4 - e_5 - e_6 + e_7 + e_8)$
	$\alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6$	$e_4 + e_5$
	$\alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7$	$e_3 + e_6$
	$\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8$	$e_2 + e_7$

Height 8:	$\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+\alpha_6$	$\frac{1}{2}(-e_1+e_2+e_3-e_4+e_5-e_6-e_7+e_8)$
	$\alpha_1+\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6$	$\frac{1}{2}(e_1-e_2-e_3+e_4+e_5-e_6-e_7+e_8)$
	$\alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6+\alpha_7$	$\frac{1}{2}(e_1-e_2+e_3-e_4-e_5+e_6-e_7+e_8)$
	$\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8$	$\frac{1}{2}(e_1+e_2-e_3-e_4-e_5-e_6+e_7+e_8)$
	$\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6+\alpha_7$	e_4+e_6
	$\alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8$	e_3+e_7
Height 9:	$\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+2\alpha_5+\alpha_6$	$\frac{1}{2}(-e_1+e_2-e_3+e_4+e_5-e_6-e_7+e_8)$
	$\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+\alpha_6+\alpha_7$	$\frac{1}{2}(-e_1+e_2+e_3-e_4-e_5+e_6-e_7+e_8)$
	$\alpha_1+\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6+\alpha_7$	$\frac{1}{2}(e_1-e_2-e_3+e_4-e_5+e_6-e_7+e_8)$
	$\alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8$	$\frac{1}{2}(e_1-e_2+e_3-e_4-e_5-e_6+e_7+e_8)$
	$\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_7$	e_5+e_6
	$\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6+\alpha_7+\alpha_8$	e_4+e_7
Height 10:	$\alpha_1+\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6$	$\frac{1}{2}(-e_1-e_2+e_3+e_4+e_5-e_6-e_7+e_8)$
	$\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+2\alpha_5+\alpha_6+\alpha_7$	$\frac{1}{2}(-e_1+e_2-e_3+e_4-e_5+e_6-e_7+e_8)$
	$\alpha_1+\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_7$	$\frac{1}{2}(e_1-e_2-e_3-e_4+e_5+e_6-e_7+e_8)$
	$\alpha_1+\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6+\alpha_7+\alpha_8$	$\frac{1}{2}(e_1-e_2-e_3+e_4-e_5-e_6+e_7+e_8)$
	$\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8$	$\frac{1}{2}(-e_1+e_2+e_3-e_4-e_5-e_6+e_7+e_8)$
	$\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_7+\alpha_8$	e_5+e_7
Height 11:	$\alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6$	$\frac{1}{2}(e_1+e_2+e_3+e_4+e_5-e_6-e_7+e_8)$
	$\alpha_1+\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6+\alpha_7$	$\frac{1}{2}(-e_1-e_2+e_3+e_4-e_5+e_6-e_7+e_8)$
	$\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_7$	$\frac{1}{2}(-e_1+e_2-e_3-e_4+e_5+e_6-e_7+e_8)$
	$\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+2\alpha_5+\alpha_6+\alpha_7+\alpha_8$	$\frac{1}{2}(-e_1+e_2-e_3+e_4-e_5-e_6+e_7+e_8)$
	$\alpha_1+\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_7+\alpha_8$	$\frac{1}{2}(e_1-e_2-e_3-e_4+e_5-e_6+e_7+e_8)$
	$\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+2\alpha_7+\alpha_8$	e_6+e_7

Height 12:	$\alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6+\alpha_7$	$\frac{1}{2}(e_1+e_2+e_3+e_4-e_5+e_6-e_7+e_8)$
	$\alpha_1+\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+2\alpha_6+\alpha_7$	$\frac{1}{2}(-e_1-e_2+e_3-e_4+e_5+e_6-e_7+e_8)$
	$\alpha_1+\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6+\alpha_7+\alpha_8$	$\frac{1}{2}(-e_1-e_2+e_3+e_4-e_5-e_6+e_7+e_8)$
	$\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_7+\alpha_8$	$\frac{1}{2}(-e_1+e_2-e_3-e_4+e_5-e_6+e_7+e_8)$
	$\alpha_1+\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+2\alpha_7+\alpha_8$	$\frac{1}{2}(e_1-e_2-e_3-e_4-e_5+e_6+e_7+e_8)$
Height 13:	$\alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+2\alpha_6+\alpha_7$	$\frac{1}{2}(e_1+e_2+e_3-e_4+e_5+e_6-e_7+e_8)$
	$\alpha_1+\alpha_2+2\alpha_3+3\alpha_4+3\alpha_5+2\alpha_6+\alpha_7$	$\frac{1}{2}(-e_1-e_2-e_3+e_4+e_5+e_6-e_7+e_8)$
	$\alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6+\alpha_7+\alpha_8$	$\frac{1}{2}(e_1+e_2+e_3+e_4-e_5-e_6+e_7+e_8)$
	$\alpha_1+\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+2\alpha_6+\alpha_7+\alpha_8$	$\frac{1}{2}(-e_1-e_2+e_3-e_4+e_5-e_6+e_7+e_8)$
	$\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+2\alpha_7+\alpha_8$	$\frac{1}{2}(-e_1+e_2-e_3-e_4-e_5+e_6+e_7+e_8)$
Height 14:	$\alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+3\alpha_5+2\alpha_6+\alpha_7$	$\frac{1}{2}(e_1+e_2-e_3+e_4+e_5+e_6-e_7+e_8)$
	$\alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+2\alpha_6+\alpha_7+\alpha_8$	$\frac{1}{2}(e_1+e_2+e_3-e_4+e_5-e_6+e_7+e_8)$
	$\alpha_1+\alpha_2+2\alpha_3+3\alpha_4+3\alpha_5+2\alpha_6+\alpha_7+\alpha_8$	$\frac{1}{2}(-e_1-e_2-e_3+e_4+e_5-e_6+e_7+e_8)$
	$\alpha_1+\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+2\alpha_6+2\alpha_7+\alpha_8$	$\frac{1}{2}(-e_1-e_2+e_3-e_4-e_5+e_6+e_7+e_8)$
Height 15:	$\alpha_1+2\alpha_2+2\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7$	$\frac{1}{2}(e_1-e_2+e_3+e_4+e_5+e_6-e_7+e_8)$
	$\alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+3\alpha_5+2\alpha_6+\alpha_7+\alpha_8$	$\frac{1}{2}(e_1+e_2-e_3+e_4+e_5-e_6+e_7+e_8)$
	$\alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+2\alpha_6+2\alpha_7+\alpha_8$	$\frac{1}{2}(e_1+e_2+e_3-e_4-e_5+e_6+e_7+e_8)$
	$\alpha_1+\alpha_2+2\alpha_3+3\alpha_4+3\alpha_5+2\alpha_6+2\alpha_7+\alpha_8$	$\frac{1}{2}(-e_1-e_2-e_3+e_4-e_5+e_6+e_7+e_8)$
Height 16:	$\alpha_1+2\alpha_2+3\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7$	$\frac{1}{2}(-e_1+e_2+e_3+e_4+e_5+e_6-e_7+e_8)$
	$\alpha_1+2\alpha_2+2\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7+\alpha_8$	$\frac{1}{2}(e_1-e_2+e_3+e_4+e_5-e_6+e_7+e_8)$
	$\alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+3\alpha_5+2\alpha_6+2\alpha_7+\alpha_8$	$\frac{1}{2}(e_1+e_2-e_3+e_4-e_5+e_6+e_7+e_8)$
	$\alpha_1+\alpha_2+2\alpha_3+3\alpha_4+3\alpha_5+3\alpha_6+2\alpha_7+\alpha_8$	$\frac{1}{2}(-e_1-e_2-e_3-e_4+e_5+e_6+e_7+e_8)$

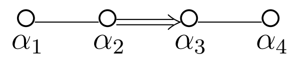
Height 17:	$2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$	$e_8 - e_7$
	$\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 + \alpha_8$	$\frac{1}{2}(-e_1 + e_2 + e_3 + e_4 + e_5 - e_6 + e_7 + e_8)$
	$\alpha_1 + 2\alpha_2 + 2\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_8$	$\frac{1}{2}(e_1 - e_2 + e_3 + e_4 - e_5 + e_6 + e_7 + e_8)$
	$\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8$	$\frac{1}{2}(e_1 + e_2 - e_3 - e_4 + e_5 + e_6 + e_7 + e_8)$
Height 18:	$2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 + \alpha_8$	$e_8 - e_6$
	$\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_8$	$\frac{1}{2}(-e_1 + e_2 + e_3 + e_4 - e_5 + e_6 + e_7 + e_8)$
	$\alpha_1 + 2\alpha_2 + 2\alpha_3 + 4\alpha_4 + 3\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8$	$\frac{1}{2}(e_1 - e_2 + e_3 - e_4 + e_5 + e_6 + e_7 + e_8)$
Height 19:	$2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_8$	$e_8 - e_5$
	$\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8$	$\frac{1}{2}(-e_1 + e_2 + e_3 - e_4 + e_5 + e_6 + e_7 + e_8)$
	$\alpha_1 + 2\alpha_2 + 2\alpha_3 + 4\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8$	$\frac{1}{2}(e_1 - e_2 - e_3 + e_4 + e_5 + e_6 + e_7 + e_8)$
Height 20:	$2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8$	$e_8 - e_4$
	$\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8$	$\frac{1}{2}(-e_1 + e_2 - e_3 + e_4 + e_5 + e_6 + e_7 + e_8)$
Height 21:	$2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8$	$e_8 - e_3$
	$\alpha_1 + 2\alpha_2 + 3\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8$	$\frac{1}{2}(-e_1 - e_2 + e_3 + e_4 + e_5 + e_6 + e_7 + e_8)$
Height 22:	$2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8$	$e_8 - e_2$
	$\alpha_1 + 3\alpha_2 + 3\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8$	$\frac{1}{2}(e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7 + e_8)$
Height 23:	$2\alpha_1 + 2\alpha_2 + 4\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8$	$e_8 - e_1$
	$2\alpha_1 + 3\alpha_2 + 3\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8$	$e_1 + e_8$
Height 24:	$2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8$	$e_2 + e_8$

Height 25:	$2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8$	$e_3 + e_8$
Height 26:	$2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8$	$e_4 + e_8$
Height 27:	$2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 2\alpha_7 + \alpha_8$	$e_5 + e_8$
Height 28:	$2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + \alpha_8$	$e_6 + e_8$
Height 29:	$2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8$	$e_7 + e_8$

F_4:

- $\Pi = \{e_2 - e_3, e_3 - e_4, e_4, \frac{1}{2}(e_1 - e_2 - e_3 - e_4)\}$

- Dynkin diagram:



- $\Delta^+ = \{e_i\} \cup \{e_i + e_j\}_{i<j} \cup \{e_i - e_j\}_{i<j} \cup \{\frac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4)\}$ (24 postive roots)

A list of the postive roots:

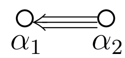
Height 1:	α_1	$e_2 - e_3$
	α_2	$e_3 - e_4$
	α_3	e_4
	α_4	$\frac{1}{2}(e_1 - e_2 - e_3 - e_4)$
Height 2:	$\alpha_1 + \alpha_2$	$e_2 - e_4$
	$\alpha_2 + \alpha_3$	e_3
	$\alpha_3 + \alpha_4$	$\frac{1}{2}(e_1 - e_2 - e_3 + e_4)$
Height 3:	$\alpha_1 + \alpha_2 + \alpha_3$	e_2
	$\alpha_2 + 2\alpha_3$	$e_3 + e_4$
	$\alpha_2 + \alpha_3 + \alpha_4$	$\frac{1}{2}(e_1 - e_2 + e_3 - e_4)$

Height 4:	$\alpha_1 + \alpha_2 + 2\alpha_3$	$e_2 + e_4$
	$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$	$\frac{1}{2}(e_1 + e_2 - e_3 - e_4)$
	$\alpha_2 + 2\alpha_3 + \alpha_4$	$\frac{1}{2}(e_1 - e_2 + e_3 + e_4)$
Height 5:	$\alpha_1 + 2\alpha_2 + 2\alpha_3$	$e_2 + e_3$
	$\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4$	$\frac{1}{2}(e_1 + e_2 - e_3 + e_4)$
	$\alpha_2 + 2\alpha_3 + 2\alpha_4$	$e_1 - e_2$
Height 6:	$\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4$	$\frac{1}{2}(e_1 + e_2 + e_3 - e_4)$
	$\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4$	$e_1 - e_3$
Height 7:	$\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4$	$\frac{1}{2}(e_1 + e_2 + e_3 + e_4)$
	$\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4$	$e_1 - e_4$
Height 8:	$\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$	e_1
Height 9:	$\alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4$	$e_1 + e_4$
Height 10:	$\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$	$e_1 + e_3$
Height 11:	$2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$	$e_1 + e_2$

G_2:

- $\Pi = \{e_1 - e_2, -2e_1 + e_2 + e_3\}$

- Dynkin diagram:



- $\Delta^+ = \{e_1 - e_2,\ -2e_1 + e_2 + e_3,\ -e_1 + e_3,\ -e_2 + e_3,\ e_1 - 2e_2 + e_3,\ -e_1 - e_2 + 2e_3\}$
  (6 postive roots)

A list of the postive roots:

Height 1:	α_1	$e_1 - e_2$
	α_2	$-2e_1 + e_2 + e_3$
Height 2:	$\alpha_1 + \alpha_2$	$-e_1 + e_3$
Height 3:	$2\alpha_1 + \alpha_2$	$-e_2 + e_3$
Height 4:	$3\alpha_1 + \alpha_2$	$e_1 - 2e_2 + e_3,$
Height 5:	$3\alpha_1 + 2\alpha_2$	$-e_1 - e_2 + 2e_3$

VITA

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Candidate for the Degree of

Doctor of Philosophy

Dissertation: CONFORMALLY INVARIANT SYSTEMS OF DIFFERENTIAL OPERATORS ASSOCIATED TO TWO-STEP NILPOTENT MAXIMAL PARABOLICS OF NON-HEISENBERG TYPE

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Pages in Study: 215 Candidate for the Degree of Doctor of Philosophy

Major Field: Mathematics

Scope and Method of Study: The main work of this thesis concerns systems of differential operators that are equivariant under an action of a Lie algebra. We call such systems conformally invariant. The main goal of this thesis is to construct such systems of operators for a homogeneous manifold G_0/Q_0 with G_0 a Lie group and Q_0 a maximal two-step nilpotent parabolic subgroup. We use the invariant theory of a prehomogeneous vector space to build such systems.

Findings and Conclusions: We determined the complex parameters for the line bundles $\mathcal{L}_{-s}$ on which our systems of differential operators are conformally invariant. The systems that we construct yield explicit homomorphisms between appropriate generalized Verma modules. We also determine whether or not these homomorphisms are standard.

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