A STUDY ON THE SOLUTIONS OF KAWAHARA, AND COMPLEX-VALUED BURGERS AND KDV-BURGERS EQUATIONS

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CHAPTER 1

Introduction

The Korteweg-de Vries (KdV) equation is a nonlinear partial differential equation. This equation remains the focus of active mathematical research mainly for two important reasons. Firstly, it has a very rich and interesting mathematical theory behind it; secondly, it models many physical phenomena such as waves on shallow water surfaces, gas dynamics, hydromagnetic, plasma physics, blood flow in arteries, and so on.

1.1 Historical background

The KdV equation is named after the two Dutch mathematicians Diederik Johannes Korteweg and Gustav de Vries. Under the supervision of Korteweg, de Vries wrote his doctoral dissertation and presented at the university of Amsterdam on December 1, 1894. This was where they introduced this famous equation for the first time. It was their effort to give a theoretical treatment of John Scott Russell's observation of the solitary wave in 1834 [39]. During these 60 years, further investigations were undertaken by Airy [1], Boussinesq [4], and Rayleigh [38] to understand the phenomenon observed by Russell.

The model that Korteweg and de Vries derived for the propagation of waves in one direction on the free surface of a shallow canal was

$$\frac{\partial \eta}{\partial t} = \frac{3}{2}\sqrt{\frac{g}{l}}\frac{\partial}{\partial x}\left(\frac{2}{3}\alpha\eta + \frac{1}{2}\eta^2 + \frac{1}{3}\sigma\frac{\partial^2 \eta}{\partial x^2}\right)$$

where l is the normal depth of the canal, η is the surface elevation above the equilibrium level, α is arbitrary constant, g is acceleration due to gravity, ρ the density of fluid, T is the surface tension [27], and $\sigma = l^3 - \dfrac{Tl}{\rho g}$. This was the original version of the KdV equation. Using the transformation

$$t' = \frac{1}{2}\sqrt{\frac{g}{l\sigma}}t, \; x' = -\frac{x}{\sqrt{\sigma}}, \; u = -\frac{1}{2}\eta - \frac{1}{3}\alpha,$$

one obtains the standard KdV equation

$$u_t + 6uu_x + u_{xxx} = 0. \tag{1.1}$$

The active life of the KdV equation began in 1960 when Gardner and Morikawa [15] derived it again in an analysis of collisionless hydromagnetic waves. The results of Fermi, Pasta, and Ulam [14] while exploring the heat transfer in crystal lattices with nonlinear interaction motivated Zabusky and Kruskal [49] to study the KdV equation. Similarly, Shen [41], Miura [33], Taniuti and Wei, Su and Gardner ([18]), and other researchers have shown that KdV equation models diverse wave phenomena in the theory of solids, liquids, gases, and plasmas.

In the last four decades, many researchers have devoted their time to the construction of exact solutions of the KdV equation. Zabusky and Kruskal were the first to give a name *solitons* to describe particle-like solitary waves that can collide and preserve their identities after the collision [49]. Many methods, including perturbation techniques, inverse scattering transform, the Bäcklund transform, and the Lax method, were devised to study the solitonic behavior of a wide class of nonlinear equations such as Schrödinger, Boussinesq, Burgers, KdV, and modified KdV(mKdV) equations. For the KdV equation given in (1.1), we assume the travelling wave solution of the form $u(x,t) = f(x - vt)$, and finally obtain a soliton solution

$$u(x,t) = \frac{v}{2}\,\mathrm{sech}^2\,(x - vt - x_0)$$

which represents a single hump of elevation $\frac{v}{2}$ travelling with velocity v to the right without changing its shape , where the propagation speed is proportional to the wave amplitude, with larger waves moving faster.

Kawahara equation is a fifth order generalized KdV equation named after the Japanese mathematician Takuji Kawahara. He published a paper "*Oscillatory Solitary Waves in Dispersive Media*" on 1971 [21] and investigated the steady solutions of

$$f_t + \frac{3}{2} f f_x + \alpha f_{xxx} - \beta f_{xxxxx} = 0 \tag{1.2}$$

on the basis of numerical calculation and concluded that both oscillatory and solitary wave solutions are possible to exist. This type of fifth order equation was also obtained in the following two cases. Kakutani and Ono [21] showed that when the angle ϕ between the propagation direction of a magneto-acoustic wave in a cold collision-free plasma and the external magnetic field becomes a critical angle, the third order derivative term in the KdV equation vanishes and is replaced by the fifth order derivative term, so that as the propagation nears the critical angle, one may obtain, with proper re-scalling, an equation with both third order and fifth order derivative terms. Hasomoto [21] also obtained such an equation for shallow water waves near a critical value of the surface tension when the effect of surface tension is taken into account.

The complex KdV equation does not have a long history, but this equation has recently attracted a tremendous amount of attention because of its solutions which blow up in finite time. This equation is found to be more sophisticated than its real counterpart. Since the complex KdV is equivalent to a system of two nonlinearly coupled equations, the conservation laws no longer allow the deduction of global bounds and does not lead to the global boundedness of the L^2-norm of its solutions. Many researchers also studied complex-valued Burgers and KdV-Burgers equations to see the effect of dispersion and dissipation on the solutions of complex KdV. We remark that the study of complex-valued Burgers and KdV-Burgers equations can be justified both physically and mathematically. Physically, these complex equations do arise in the modeling of several physical phenomena ([24],[28],[29]). Mathematically, these equations exhibit some remarkable features and admit solutions with much richer structures than those of their real-valued ones.

1.2 Literature review

There is a large body of literature on the study of smoothing properties of initial value problems (IVP) and initial-boundary-value problems (IBVP) of the real-valued KdV equation. Consider the initial value problem

$$\begin{cases} u_t + u_x + uu_x + u_{xxx} = 0, x \in \mathbb{R} \,\text{or}\, \mathbb{T}, t > 0, \\ u(x,0) = u_0(x), x \in \mathbb{R} \,\text{or}\, \mathbb{T}, \end{cases} \tag{1.3}$$

where $\mathbb{T}$ is a periodic domain. The hierarchy of infinite conservation laws provides global time bounds for its solutions in any Sobolev space H^k with $k \geq 0$. In fact, the recent work of Kenig et al.[23] showed that the equation (1.3) posed on real line is locally well-posed for any initial datum $u_0 \in H^k(\mathbb{R})$ with $k > -\frac{3}{4}$, and that the periodic IVP for the real KdV is locally well-posed for $u_0 \in H^k(\mathbb{T})$ with any $k > -\frac{1}{2}$. Collinder et al. [10] strengthened the local well-posedness for the periodic IVP to include the case $k = -\frac{1}{2}$, and also showed that the global well-posedness follows from the local well-posedness through successive iterations.

The IBVP of the KdV equation has also received a considerable amount of attention as compared to the IVP. The KdV equation on the half line fits well with the laboratory studies, wherein the waves are generated by a wave-maker at the left end [3]. However, it is difficult to implement numerical methods in unbounded domains. Therefore, the use of a finite interval with a suitable set of boundary conditions is an alternative choice, and such two-point boundary-value problems can still be used to model the laboratory studies of waves generated by a wavemaker at the left end which propagate down the channel before the waves reach the other end of the channel. Bona et al. [3] studied the local and global well-posedness of IBVP for the real

KdV equation

$$\begin{cases} u_t + u_x + uu_x + u_{xxx} = 0, x \in (0,1), t > 0, \\ u(x,0) = u_0(x), \\ u(0,t) = h_1(t), u(1,t) = h_2(t), u_x(1,t) = h_3(t) \end{cases} \tag{1.4}$$

defined in the bounded domain $[0,1]$, and established that the IBVP is locally well-posed for any $u_0 \in H^k$ and $(h_1, h_2, h_3) \in H^{\frac{(1+k)}{3}} \times H^{\frac{(1+k)}{3}} \times H^{\frac{k}{3}}$ with $k \geq 0$. They also solved the global well-posedness of (1.4) for $u_0 \in H^k, k \geq 0$ and more regular boundary data.

The Kawahara equation, a fifth order KdV equation, has been studied analytically and numerically by many mathematicians (see [7],[9],[19],[20],[22],[37],[44]). In general, it is difficult to compute solutions of the fifth-order KdV equations numerically, due to the fifth-order term. In [40], Shen proposed the dual-Petrov-Galerkin algorithm for third and higher odd-order differential equations that involves an innovative choice of test and trial functions, which allow free integration by parts without generating boundary terms. This algorithm is equivalent to spectral-Galerkin approximation in weighted spaces. Numerical experiments involving the usual third-order KdV equation

$$\begin{cases} \alpha u_t + \beta u_x + \gamma u u_x + u_{xxx} = 0, x \in (-1,1), t \in (0,T], \\ u(x,0) = u_0(x), x \in (-1,1), \\ u(-1,t) = g(t), u(1,t) = u_x(1,t) = 0, t \in [0,T] \end{cases} \tag{1.5}$$

in [40] indicate that the dual-Petrov-Galerkin algorithm is very accurate and efficient. In a recent work [16], Goubet and Shen studied the IBVP for the third-order KdV equation

$$\begin{cases} u_t - \beta u_x + uu_x + u_{xxx} = 0, x \in (-1,1), t > 0, \\ u(x,0) = u_0(x), \\ u(-1,t) = u(1,t) = u_x(1,t) = 0 \end{cases} \tag{1.6}$$

in a functional framework based on the dual-Petrov-Galerkin method. More precisely, they established the existence and uniqueness of solutions to this IBVP in weighted Sobolev spaces.

The dual-Petrov-Galerkin algorithm was recently further developed and implemented for a fifth-order KdV equation

$$\begin{cases} \alpha u_t + \mu u_x + \gamma u u_x + \beta u_{xxx} - u_{xxxxx} = 0,\ x \in (-1,1),\ t \in (0,T], \\ u(x,0) = u_0(x), x \in (-1,1), \\ u(-1,t) = g(t), u_x(-1,t) = h(t), u(1,t) = u_x(1,t) = u_{xx}(1,t) = 0,\ t \in [0,T] \end{cases} \tag{1.7}$$

in [48] where $\alpha \geq 0$, μ, γ, and β are rescaled parameters depending on the physical parameters and scaling. The authors applied the above-mentioned algorithm and provided a rigorous error analysis and demonstrated the effectiveness of the algorithm by computing some challenging solitary and oscillatory-solitary waves.

The study of solutions of the Kawahara and modified Kawahara equations on the whole real line in Sobolev spaces of negative indices was recently carried out by Chen, Li, Miao, and Wu [9]. They established the local well-posedness of the IVP for Kawahara equation

$$\begin{cases} u_t + u u_x + \alpha u_{xxx} + \beta u_{xxxxx} = 0, x, t \in \mathbb{R} \\ u(x,0) = u_0(x) \end{cases} \tag{1.8}$$

in $H^k(\mathbb{R})$ with $k > -\frac{7}{4}$ and the local well-posedness for the modified Kawahara equation

$$\begin{cases} u_t + u^2 u_x + \alpha u_{xxx} + \beta u_{xxxxx} = 0, x, t \in \mathbb{R} \\ u(x,0) = u_0(x) \end{cases} \tag{1.9}$$

in $H^k(\mathbb{R})$ with $k \geq -\frac{1}{4}$, and have improved the existing low regularity well-posedness results.

As given in the historical background, complex-valued Burgers, KdV-Burgers, and other complex-valued partial differential equations have recently attracted a good deal of attention. A lot of effort has been devoted to the important issue of whether or not their solutions can blow up in a finite time. In 1987, B. Birnir studied the solutions of the complex KdV equation represented by Weierstrass $\mathcal{P}$-function, and proved that they blow up in finite time as a second order pole. He used the inverse scattering transform to study these singularities (see [5] and [6]). In [2], J. Bona and F. Weissler presented some criteria to imply that the solutions of nonlinear, dispersive evolution equations lose regularity in finite time. The papers of Yuan and Wu ([45],[46],[47]) treated the complex KdV and KdV-Burgers equations as systems of two nonlinearly coupled equations and clarified how the potential singularities of the real part are related to those of the imaginary part. In [47], the solution of the form

$$u(x,t) = \sum_{k=1}^{\infty} a_k(t) e^{ikx} \tag{1.10}$$

was studied to the IVP of the complex KdV equation

$$u_t + 2uu_x + u_{xxx} = 0$$

defined in the periodic domain $\mathbb{T} = [0, 2\pi]$, where u_0 is assumed to have the form

$$u_0(x) = \sum_{k=1}^{\infty} a_{0k} e^{ikx}. \tag{1.11}$$

The authors showed that there is no regular global series solution if the first initial mode $a = a_1(0)$ of the initial datum $u_0(x)$ satisfies $a \geq 6$. Very recently, Y. Li [30] obtained simple explicit formulas for finite time blow-up solutions of the complex KdV equation through Darboux transform. In fact, he constructed a solution

$$u(x,t) = i + \frac{8\exp(-12(t-1) + i(8t+2x))}{[\exp(-12(t-1) + i(8t+2x)) + 1]^2}, \tag{1.12}$$

of the complex KdV equation

$$u_t - 6uu_x + u_{xxx} = 0$$

in periodic case. When $t = 0$, $u(x,0) \in C^\infty$ and $t = 1$, finite-time blow up is developed with two singularities at $x = \frac{3}{2}\pi - 4$ and $x = \frac{5}{2}\pi - 4$.

The IVP of the complex Burgers equation

$$\begin{cases} u_t - 6uu_x - \nu u_{xx} = 0, & x \in \mathbb{R},\, t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.13}$$

can be solved explicitly through the Hopf-Cole transform, and the solution is given by

$$u(x,t) = -3\frac{\int_{-\infty}^{\infty} \frac{x-y}{t} \exp\left[-\frac{|x-y|^2}{2\nu t} - \frac{1}{2\nu}\int_{-\infty}^{y} u_0(s)\,ds\right] dy}{\int_{-\infty}^{\infty} \exp\left[-\frac{|x-y|^2}{2\nu t} - \frac{1}{2\nu}\int_{-\infty}^{y} u_0(s)\,ds\right] dy}. \tag{1.14}$$

The singularities of the solution u in (1.14) correspond to the zeros of v in the Cole-Hopf transformation $u = -\dfrac{2v_x}{v}$. For $\nu = 1$, the equation (1.13) has a global smooth solution u if $u_0 \in L^1(\mathbb{R})$ and $\int_{\mathbb{R}} |\Im(u_0)| \leq 2\pi$. But, for $\delta > 0$, there exists a smooth compactly supported u_0 with $\int_{\mathbb{R}} |u_0| < 2\pi + \delta$ such that the solution blows up. The continuity argument can also be used to show that u develops a singularity whenever $u_0 \in L^1(\mathbb{R})$ satisfies $|\int_{\mathbb{R}} \Im(u_0)| > 2\pi$ and $\int_{\mathbb{R}} \Im(u_0)$ is not of the form $2\pi + 4k\pi$, where k is an integer. More about this may be found in [36].

As one may expect, the behavior of solutions to the complex KdV-Burgers equations is more sophisticated due to the presence of the nonlinearity, dispersion and dissipation. In [47], Yuan and Wu showed the finite-time bound for the L^2-norm of its solutions and remarked that either increasing dissipation or decreasing L^2-norm of the initial datum lengthens the time. In the same paper, they bound the Sobolev norm $\|u(\cdot,t)\|_{H^k}(k \geq 1)$ in terms of $\|u(\cdot,t)\|_{L^2}$. This result implies that any possible finite-time singularity must develop in the L^2-norm. In another paper (see [45]), the authors proved that when the dissipation dominates, such as in the case when ν is comparable to the size of the initial data, the solution is then global in time and decays exponentially for large time. In addition, extensive numerical experiments were performed to reveal the blowup structures.

There are several examples that show the significant differences between the real-valued and complex-valued solutions. One important example is the Navier-Stokes equations. It remains open as to whether or not classical solutions of the 3D incompressible Navier-Stokes equations can develop finite-time singularities. However, Li and Sinai [31] recently showed that the complex solutions of the 3D Navier-Stokes equations corresponding to large parameter family of initial data blow up in finite time. Their work motivated the study of Poláčik and Šverák [36] on the complex-valued solutions of the Burgers equation, as mentioned before.

1.3 Statement of problems and results

This dissertation is aimed at studying two important issues. The first is to study the solution of the Kawahara equation

$$u_t + uu_x + \beta u_{xxx} - u_{xxxxx} = f, \tag{1.15}$$

in weighted Sobolev spaces, where β is related to the Bond number in the presence of surface tension and $\beta = 0$ corresponds to the critical Bond number $\frac{1}{3}$ (see [43]).

Attention will be focused on IBVP of (1.15) in the spatial domain $I = (-1, 1)$ with the boundary and initial conditions

$$\begin{cases} u(-1,t) = g(t),\ u_x(-1,t) = h(t),\ u(1,t) = u_x(1,t) = u_{xx}(1,t) = 0, \quad t \geq 0, \\ u(x,0) = u_0(x), \quad x \in I. \end{cases} \tag{1.16}$$

Since (1.15) and (1.16) can be reformulated as an equivalent problem with homogeneous boundary conditions, it will be assumed that $g(t) = h(t) \equiv 0$. In fact, through the transform

$$u(x,t) = v(x,t) + B(x,t)$$

with

$$B(x,t) = -\frac{(x-1)^3}{4}(2g(t) + h(t)) - \frac{(x-1)^4}{16}(3g(t) + 2h(t)),$$

(1.15) and (1.16) can be converted into an IBVP with homogeneous boundary condition

$$\begin{cases} v_t + B_x v + (v+B)v_x + \beta v_{xxx} - v_{xxxxx} = -B_t - BB_x - \beta B_{xxx}, \\ v(-1,t) = v_x(-1,t) = v(1,t) = v_x(1,t) = v_{xx}(1,t) = 0. \end{cases}$$

In principle, this homogeneous problem can be studied in a similar fashion as (1.15) and (1.16).

The goal here is to build a theory similar to that of Goubet and Shen, cited earlier, on the existence and uniqueness of solutions to the IBVP (1.15) and (1.16) in weighted Sobolev spaces. Their approach [16] will be followed, but the situation here is more complicated. The dispersive part consists of two terms $\beta u_{xxx} - u_{xxxxx}$ and its corresponding weak formulation fails to be coercive for $\beta \leq -\frac{3}{80}$ (see Section 3.2 for more details). For $\beta > -\frac{3}{80}$, the IBVP (1.15) and (1.16) is shown to possess a unique global solution for u_0 in weighted Sobolev spaces with increasing regularity (Theorem 3.5). If, in addition, the L^2-norm of u_0 is small, then the solution in these weighted Sobolev spaces decay exponentially in time.

The IBVP (1.15) and (1.16) is also studied numerically to complement the theoretical results. In fact, the numerical solutions of a slightly more general problem than (1.15) and (1.16) will be studied. This problem involves two parameters β_1 and β_2, and the existence and uniqueness theory applies to the case when

$$\beta_1 > -\frac{3}{80}\beta_2. \tag{1.17}$$

The solutions of the general problem will be computed corresponding to β_1 and β_2 in different ranges and plotted their standard L^2-norms and weighted L^2-norms. The graphs show that a solution corresponding to β_1 and β_2 violating (1.17) may not exist for all time and thus the IBVP (1.15) and (1.16) with β violating the condition may not be globally well-posed. Comparisons are also made between the weighted Sobolev norms and the standard Sobolev norms.

The second aim is to study the solutions of complex-valued Burgers and KdV-Burgers equations. This work addresses the blow up for complex-valued Burgers equation

$$u_t - 6uu_x - \nu u_{xx} = 0 \tag{1.18}$$

and the global regularity issue on solutions of the complex KdV-Burgers equations

$$u_t - 6uu_x + \alpha u_{xxx} - \nu u_{xx} = 0, \tag{1.19}$$

where $\nu \geq 0$ and $\alpha \geq 0$ are parameters and $u = u(x,t)$ is a complex-valued function. Attention will be focused on the spatially periodic solutions, namely $x \in \mathbb{T} = [0, 2\pi]$, and we supplement (1.18) and (1.19) with a given initial data

$$u(x,0) = u_0(x), \quad x \in \mathbb{T}. \tag{1.20}$$

The first major result is for the complex Burgers equation (1.18), and it asserts that for any sufficiently large time T, there exists an explicit smooth initial data u_0 such that its corresponding solution blows up at $t = T$ (Theorem 4.3). This result was partially motivated by a recent paper of Poláčik and Šverák [36], in which the complex-valued Burgers equation on the whole line was shown to develop finite-time singularities for compactly supported smooth data. Their proof takes advantage of the explicit solution formula obtained via the Hopf-Cole transform. The finite-time singular solutions constructed in this paper assume the form

$$u(x,t) = \sum_{k=1}^{\infty} a_k(t)\, e^{ikx} \tag{1.21}$$

and correspond to the initial data $u_0(x) = a\, e^{ix}$. It will be emphasized that solutions of the form (1.21) are locally well-posed in the usual Sobolev space H^k with a suitable index k (see Theorem 4.1 for more details). For any $T \geq T_0$ (a fixed number depending only on ν), one obtains a lower bound for $|a_k(T)|$ through a careful observation of the pattern that $a_k(t)$s exhibit, and the finite time singularity of (1.21) in L^2 then follows if one takes a as appearing in u_0 to be sufficiently large. This result reveals a fundamental difference between the real-valued solutions of the Burgers equation and their complex counterparts. The diffusion in the case of complex-valued solutions no longer dissipates the L^2-norm, which can blow up in a finite time. However, if one knows that the L^2-norm of a complex-valued solution is bounded, then there would be no finite-time singularity (Theorem 4.2).

I also explore the conditions under which solutions of (1.18) are global in time. A simple example of the global solutions of (1.18) corresponds to the initial data $u_0(x) = a_0 e^{ix}$ with $|a_0| < 1$, provided ν and α satisfy a suitable condition, say $\nu^2 + 4\alpha^2 \geq 9$ (see Theorem 4.5). For general initial data of the form

$$u_0(x) = \sum_{k=1}^{\infty} a_{01}\, e^{ikx}$$

with $|a_{01}| < 1$, (1.19) possesses a unique local solution (1.21) with $a_k(t)$ given by a finite sum of terms that can be made explicit through an inductive relation. To show the convergence of (1.21) for large time, it is necessary to estimate $|a_k(t)|$, and the approach is to count the total number of terms that it contains. This counting problem is closely related to the number of nonnegative integer solutions to the equation

$$j_1 + 2j_2 + 3j_3 + \cdots + kj_k = k$$

for a fixed integer $k > 0$. Using a result by Hardy and Ramanujan [17], one may establish the global regularity of (1.21) under a mild assumption (see Theorem 4.4). In addition, $\|u(\cdot,t)\|_{H^k}$ decays exponentially in t for large t for any $k \geq 0$.

Inspired by a recent work of Sinai on the Navier-Stokes equations [42], a study is undertaken for the series solution of (1.19) that can be written as

$$u(x,t) = \sum_{k \in \mathbb{Z}\setminus\{0\}} \frac{c(k,t)}{|k|^{\gamma}}\, e^{ikx}, \tag{1.22}$$

where $\gamma > 1$ is a real number, and $c(k,t)$ is bounded uniformly in terms of k and t. If $T > 0$ and $R_0 = \sup\limits_{k \in \mathbb{Z}\setminus\{0\}} |c(k,0)|$ satisfy

$$R_0\sqrt{T} \leq C(\gamma)\sqrt{\nu}$$

for some suitable constant $C(\gamma)$, it is shown that u in (1.22) is a classical solution of (1.19) on $[0, T]$ (Theorems 4.6, 4.7 and 4.8). This is achieved through three major steps. The first step establishes the existence of $c(k, t)$ such that

$$\widehat{u}(k,t) = \frac{c(k,t)}{|k|^\gamma}$$

solves the Fourier transform of the complex KdV-Burgers equation. The second step verifies that u in (1.22) is a weak solution, in the distributional sense, while the third step proves the bound

$$|c(k,t)| \leq \frac{C}{|k|^{\gamma+l}}$$

where $l > 0$ is any fixed integer. A combination of the last two steps especially implies that u in (1.22) is a classical solution.

1.4 Organization of the dissertation

The organization of this dissertation is as follows. The second chapter is dedicated to relevant notation and definitions. The main work of this dissertation will be presented in the third and fourth chapters.

The third chapter is divided into different sections and subsections. Section 3.2 focuses on a weak formulation of the stationary and linearized equation

$$\beta u_{xxx} - u_{xxxxx} = f$$

and establishes the existence and uniqueness of solutions to this formulation with any f in a weighted L^2-space (Theorem 3.1). In particular, the solution operator is shown to be the generator of a contraction semi-group. Section 3.3 presents the existence and uniqueness results for the full IBVP (3.1) and (3.2) (Theorems 3.5). Section 3.5 contains the numerical results.

The fourth chapter is devoted to the study of solutions of complex-valued Burgers and KdV-Burgers equations. It is divided into three primary sections. The second section focuses on the complex Burgers equation, and presents Theorems 4.1, 4.2, and 4.3. The third section details the global regularity results concerning the complex KdV-Burgers equations for two types of series solutions of the form (1.21) and (1.22).

CHAPTER 2

Preliminaries

In this chapter, we briefly discuss some of the preliminaries to get equipped with the necessary spaces and inequalities that we require for our study in the next two chapters. The detail of the definitions and inequalities given below can be found in [12], [13], and [35].

2.1 Notations and definitions of some spaces

Standard notations, and the definitions of some important spaces, will be discussed in this section.

Definition 2.1 (*Norm and Normed Space*) *Let $\mathbb{X}$ be a vector space over $\mathbb{R}$ or $\mathbb{C}$. A norm is a function $x \mapsto \|x\|$ from $\mathbb{X}$ to $[0, \infty)$ such that*

- $\|x + y\| \leq \|x\| + \|y\|$
- $\|\lambda x\| \leq |\lambda|\|x\|$
- $\|x\| = 0 \Leftrightarrow x = 0,$

and the vector space $\mathbb{X}$ equipped with a norm $\|\cdot\|$ is called a normed vector space.

Definition 2.2 (*Banach Space and Hilbert Space*) *A normed vector space that is complete with respect to its norm is called a Banach space. An inner product space that is complete with respect to the norm $\|x\| = \sqrt{x \cdot x}$ is called a Hilbert space.*

Definition 2.3 **(L^p Space)** *The L^p-norm of a function on $\mathbb{X}$, $0 < p < \infty$, is defined by*

$$\|f\|_{L^p(\mathbb{X})} = \left(\int_{\mathbb{X}} |f|^p dx\right)^{\frac{1}{p}}$$

and the L^p space is a space of funtions defined as

$$L^p(\mathbb{X}) = \{f : \mathbb{X} \to \mathbb{C} \quad \textit{such that} \quad \|f\|_{L^p(\mathbb{X})} < \infty\}.$$

Assume $U \subset \mathbb{R}^n$ is open and $0 \leq \gamma \leq 1$. A *Lipschitz continuous function* is a real-valued function on U satisfying

$$|u(x) - u(y)| \leq C|x - y| \quad (x, y \in U).$$

If u satisfies

$$|u(x) - u(y)| \leq C|x - y|^\gamma \quad (x, y \in U)$$

for some constant C, then u is said to be *Hölder continuous* with exponent γ.

Definition 2.4 **(*Hölder Space*)** *The Hölder space $C^{k,\gamma}(\bar{U})$ is a Banach space that consists of those functions u that are k-times continuously differentiable, and whose k-th partial derivatives are bounded and Hölder continuous with exponent γ, i.e.*

$$C^{k,\gamma}(\bar{U}) = \{u : \|u\|_{C^{k,\gamma}(\bar{U})} = \sum_{|\alpha| \leq k} \|D^\alpha u\|_{C(\bar{U})} + \sum_{|\alpha| \leq k} [D^\alpha u]_{C^{0,\gamma}(\bar{U})} < \infty\},$$

where $\|u\|_{C(\bar{U})} = \sup_{x \in U} |u(x)|$ *and* $[u]_{C^{0,\gamma}(\bar{U})} = \sup_{x \neq y} \left(\frac{|u(x) - u(y)|}{|x - y|^\gamma}\right)$.

Definition 2.5 **(*Sobolev Space*)** *The Sobolev space $W^{k,p}(U)$ consists of all locally summable functions $u : U \to \mathbb{R}$ such that, for each α with $|\alpha| \leq k$, $D^\alpha u$ exists in the weak sense and belongs to $L^p(U)$. If $u \in W^{k,p}$, then*

$$\|u\|_{W^{k,p}(U)} = \left(\sum_{|\alpha| \leq k} \int_U |D^\alpha u|^p dx\right)^{\frac{1}{p}} \quad (1 \leq p \leq \infty).$$

1. If $p = 2, W^{k,2}(U) = H^k(U)$; therefore, if $f \in H^k(U)$ then $D^k f \in L^2(U)$. When $k = 0$, $H^0(U)$ is the usual L^2 space.

2. The closure of $C_c^\infty(U)$ in $W^{k,p}(U)$ is denoted by $W_0^{k,p}(U)$. Thus, $u \in W_0^{k,p}(U)$ if and only if there exist functions $u_m \in C_c^\infty(U)$ such that $u_m \to u$ in $W^{k,p}(U)$. We interpret $W_0^{k,p}(U)$ as consisting of those functions $u \in W^{k,p}(U)$ such that $D^\alpha u = 0$ on ∂U for all $|\alpha| \leq k-1$.

3. If $s \in \mathbb{R}$, the Sobolev space $H^s(\mathbb{R})$ is defined via the Fourier transform in the standard fashion. The function f is in $H^s(\mathbb{R})$ if

$$\|f\|_{H^s(\mathbb{R})} = C\int_{\mathbb{R}} (1+k^2)^s |\hat{f}(k)|^2 dk < \infty.$$

4. Let $\mathbb{X}$ be a Banach space with norm $\|\cdot\|$. The following are the spaces involving time.

Definition 2.6 *The space $L^p(0,T;\mathbb{X})$ consists of all functions $u : [0,T] \to \mathbb{X}$ with*

$$\|u\|_{L^p(0,T;\mathbb{X})} := \left(\int_0^T \|u(t)\|^p dt\right)^{\frac{1}{p}} < \infty \quad \textit{for} \quad 1 \leq p < \infty \quad \textit{and}$$

$$\|u\|_{L^\infty(0,T;\mathbb{X})} := \operatorname{ess}\sup_{0\leq t\leq T} \|u(t)\| < \infty.$$

Definition 2.7 *The space $C([0,T];\mathbb{X})$ comprises all continuous functions $u : [0,T] \to \mathbb{X}$ with*

$$\|u\|_{C([0,T];X)} := \max_{0\leq t\leq T} \|u(t)\| < \infty.$$

2.2 Some elementary inequalities

Several inequalities are used in this dissertation. Some of them are as follows.

1. **Cauchy's Inequality** $ab \leq \frac{a^2}{2} + \frac{b^2}{2}$. $(a, b \in \mathbb{R})$

2. **Cauchy's Inequality with ϵ** $ab \leq \epsilon a^2 + \frac{b^2}{4\epsilon}$. $(a, b \in \mathbb{R}^+, \epsilon > 0)$

3. **Young's Inequality** Let $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, then

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}. \quad (a, b > 0)$$

4. **Young's Inequality with ϵ** Let $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, then

$$ab \le \epsilon a^p + \frac{b^q}{(\epsilon p)^{\frac{q}{p}} q}. \quad (a, b > 0, \epsilon > 0)$$

5. **Hölder's Inequality** Assume $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, then if $u \in L^p(U), v \in L^q(U)$,

$$\int_U |uv| dx \le \|u\|_{L^p(U)} \|v\|_{L^q(U)}.$$

6. **Minkowski's Inequality** Assume $1 \le p < \infty$, $u, v \in L^p(U)$, then

$$\|u + v\|_{L^p(U)} \le \|u\|_{L^p(U)} + \|v\|_{L^p(U)}.$$

7. **Interpolation Inequality for L^p-norms** Assume $1 \le s \le r \le t \le \infty$ and $\frac{1}{r} = \frac{\theta}{s} + \frac{(1-\theta)}{t}$. Suppose $u \in L^s(U) \cap L^t(U)$. Then $u \in L^r(U)$, and

$$\|u\|_{L^r(U)} \le \|u\|^{\theta}_{L^s(U)} \|u\|^{1-\theta}_{L^t(U)}.$$

8. **Cauchy-Schwartz Inequality** $|x \cdot y| \le |x||y|. \quad (x, y \in \mathbb{R}^n)$

9. **Gagliardo-Nirenberg-Sobolev Inequality** Assume $1 \le p < n$. There exists a constant C depending only on p and n such that

$$\|u\|_{L^{p*}(\mathbb{R}^n)} \le C \|Du\|_{L^p(\mathbb{R}^n)}, \ (u \in C^1_c(\mathbb{R}^n))$$

where $p* = \frac{np}{n-p}$ is the *Sobolev conjugate* of p.

10. **General Sobolev Inequality** Let U be a bounded open subset of $\mathbb{R}^n$ with a C^1 boundary. Assume $u \in W^{k,p}(U)$. If $k < \frac{n}{p}$ then $u \in L^q(U)$ where $\frac{1}{q} = \frac{1}{p} - \frac{k}{n}$. In addition,

$$\|u\|_{L^q(U)} \le C \|u\|_{W^{k,p}(U)}.$$

11. **Gronwall's Inequality(*differential form*)** Suppose $f(t)$ is a non-negative, absolutely continuous function on $[0, T]$ whose derivative is bounded according to

$$\frac{df}{dt} \leq g(t)f(t) + h(t)$$

for some non-negative functions $g(t)$ and $h(t)$. Then $f(t)$ is bounded pointwise in time according to

$$f(t) \leq \exp\left(\int_0^t g(r)dr\right)\left(f(0) + \int_0^t h(s)ds\right)$$

for all $0 \leq t \leq T$.

12. **Gronwall's Inequality(*integral form*)** Suppose $f(t)$ is a non-negative, summable function on $[0, T]$ which satisfies the integral inequality

$$f(t) \leq C + K\int_0^t f(s)ds$$

for all t in $[0, T]$ where C and K are positive constants. Then

$$f(t) \leq C\exp(Kt).$$

At last, we state a theorem called the Fixed Point Theorem (or Contraction Mapping Principle). This theorem is useful in showing the existence and uniqueness of solutions of a given differential equation.

Definition 2.8 (*Contraction Mapping*) *Let $\mathbb{X}$ be a Banach space. If $G : \mathbb{X} \to \mathbb{X}$ satisfies*

$$\|G(u) - G(v)\| \leq c\|u - v\|$$

for all u and $v \in \mathbb{X}$ with $0 < c < 1$, then G is called a contraction mapping.

Theorem 2.1 (*Contraction Mapping Principle*) *Let $\mathbb{X}$ be a Banach space and $G : \mathbb{X} \to \mathbb{X}$ a contraction mapping. Then there exists a unique $u \in \mathbb{X}$ such that $G(u) = u$.*

CHAPTER 3

The Kawahara equation in weighted Sobolev spaces

3.1 Overview

Fifth-order Korteweg-de Vries type equations

$$u_t - u_{xxxxx} = F(x, t, u, u_x, u_{xx}, u_{xxx})$$

arise naturally in modeling many wave phenomena. In particular, the Kawahara equation

$$u_t + uu_x + \beta u_{xxx} - u_{xxxxx} = f \tag{3.1}$$

has been derived to model magneto-acoustic waves in plasmas [21] and shallow water waves with surface tension [19]. In this equation, β is either negative, zero, or positive, and is related to the Bond number in the presence of surface tension. If $\beta > 0$, a solitary wave of depression or elevation appears. For $\beta < 0$, solutions of (3.1) exhibit highly oscillatory behaviors, and $\beta = 0$ corresponds to the critical Bond number $\frac{1}{3}$ (see [43]).

For application and computational purposes, it is better to study the initial and boundary-value problems rather than pure initial-value problems. So, the attention will be focused on the initial- and boundary-value problem (IBVP) of (3.1) in the spatial domain $I = (-1, 1)$ with the boundary and initial conditions given by

$$\begin{cases} u(-1,t) = g(t),\ u_x(-1,t) = h(t),\ u(1,t) = u_x(1,t) = u_{xx}(1,t) = 0, & t \geq 0, \\ u(x,0) = u_0(x), \quad x \in I. \end{cases} \tag{3.2}$$

For application purpose, one can consider an example of waves generated by a wavemaker naturally set on a semi-infinite interval, and (3.1) and (3.2) serve as a good approximate model before the waves reach the right boundary. For computational purpose, one must reduce the problem in an infinite domain to a finite domain. As explained in the section 1.3, (3.1) and (3.2) can be reformulated as an equivalent problem with homogeneous boundary conditions. Henceforth, it will be assumed that $g(t) = h(t) \equiv 0$. First, the weak formulation of the stationary linear equation will be explained, and then shown the existence and uniqueness of solutions of this weak formulation in suitable weighted spaces.

3.2 Weak formulation of the stationary linear equation

In this section, a weak formulation of the boundary-value problem for the stationary equation

$$\begin{cases} \beta u_{xxx} - u_{xxxxx} = f, \quad x \in (-1,1), \\ u(-1) = u(1) = u_x(-1) = u_x(1) = u_{xx}(1) = 0 \end{cases} \tag{3.3}$$

is presented, and a theory on the existence and uniqueness of solutions to this formulation is established. First, some notations are introduced, and some Hardy-type inequalities are discussed.

3.2.1 Notations

The following notations will be used for some weighted Sobolev spaces. For the non-negative weight ω, define

$$\begin{aligned} &I = (-1,1), \\ &H(I) = L^2_\omega = \left\{ u \in L^1_{loc}(I) : \int_I u^2(x)\,\omega(x)\,dx < \infty \right\}, \\ &V(I) = \left\{ u \in H^2_0(I) : \; u_{xx} \in L^2_{\omega'} \right\}, \\ &W(I) = \left\{ u \in V(I) : \; u_{xxx} \in L^2_{\frac{\omega^2}{\omega'}} \right\}, \end{aligned} \tag{3.4}$$

where $H^k(I)$ is the usual L^2 based Sobolev space and $H_0^k(I)$ denotes the completion of $C_0^\infty(I)$ under H^k-norm. In addition, denote the inner product in H by $(\cdot,\cdot)_H$.

For the purpose of eliminating boundary terms, we choose $\omega(x) = \frac{1+x}{1-x}$. Correspondingly, $\omega'(x) = \frac{2}{(1-x)^2}$, $\omega''(x) = \frac{4}{(1-x)^3}$, $\omega'''(x) = \frac{12}{(1-x)^4}$, and $\frac{\omega^2(x)}{\omega'(x)} = \frac{(1+x)^2}{2}$.

3.2.2 Hardy-type inequalities

In this section, some density results and Hardy-type inequalities in weighted Sobolev spaces will be discussed. The following lemma explains these relations in detail.

Lemma 3.1 *The space V, endowed with the norm $\|u_{xx}\|_{L^2_{\omega'}}$, and W endowed with the norm $\|u_{xxx}\|_{L^2_{\frac{\omega^2}{\omega'}}}$ are Hilbert spaces. The embedding relations*

$$C_0^\infty \hookrightarrow W \hookrightarrow V \hookrightarrow H$$

are dense and continuous, and the following Hardy type inequalities hold:

$$\int_I \frac{u^2}{(1-x)^6}dx \le \frac{4}{25}\int_I \frac{u_x^2}{(1-x)^4}dx, \quad \int_I \frac{u_x^2}{(1-x)^4}dx \le \frac{4}{9}\int_I \frac{u_{xx}^2}{(1-x)^2}dx \quad \forall u \in V, \tag{3.5}$$

$$r^2\int_I \frac{u_{xx}^2}{(1-x)^2}dx-(2r+3qr-q^2)\int_I \frac{u_x^2}{(1-x)^4}dx+(1-5q+20r)\int_I \frac{u^2}{(1-x)^6}dx \ge 0 \tag{3.6}$$

for any real number r and q, and

$$\int_I \frac{u_{xx}^2}{(1-x)^2}dx \le \int_I u_{xxx}^2(1+x)^2dx \quad \forall u \in W. \tag{3.7}$$

The general form in (3.6) is very useful and can be tailored for special needs. For example, by letting $(r,q) = (\frac{1}{2}, \frac{3}{2})$ and $(r,q) = (\frac{1}{2}, 1)$, one has

$$\int_I \frac{u_x^2}{(1-x)^4}dx \le \frac{1}{4}\int_I \frac{u_{xx}^2}{(1-x)^2}dx + \frac{7}{2}\int_I \frac{u^2}{(1-x)^6}dx,$$

$$\int_I \frac{u_x^2}{(1-x)^4}dx \le \frac{1}{6}\int_I \frac{u_{xx}^2}{(1-x)^2}dx + 4\int_I \frac{u^2}{(1-x)^6}dx,$$

respectively. The proof of this lemma follows the ideas of Goubet and Shen [16] and Shen [40].

Proof of Lemma 3.1.

First of all, $\|u_{xx}\|_{L^2_{w'}}$ and $\|u_{xxx}\|_{L^2_{\frac{w^2}{w'}}}$ are clearly norms in V and W, respectively. To show that C_0^∞ is dense in V, it suffices to show $C_0^\infty(I)^\perp = \{0\}$. To this end, let $u \in C_0^\infty(I)^\perp$, namely

$$\int_I u_{xx}\phi_{xx}\omega'(x)dx = 0 \quad \text{for all } \phi \in C_0^\infty.$$

Since $\omega'(x) = \dfrac{2}{(1-x)^2}$, one obtains by integration by parts the identity

$$\partial_{xx}\left(u_{xx}(1-x)^{-2}\right) = 0,$$

which implies

$$u_{xx} = a(1-x)^3 + b(1-x)^2.$$

Therefore, for some constants c and d,

$$u(x) = \frac{1}{20}a(1-x)^5 + \frac{b}{12}(1-x)^4 + c(1-x) + d.$$

The boundary conditions $u(\pm 1) = u_x(\pm 1) = u_{xx}(1) = 0$ imply that $a = b = c = d = 0$, which is to say that $u = 0$. Similar arguments show that C_0^∞ is dense in H and in W.

Now the inequalities (3.5), (3.6) and (3.7) will be proved. Since C_0^∞ are dense in W, V and H, it suffices to prove them for $u \in C_0^\infty$. To prove (3.5), for any number a,

$$\begin{aligned} 0 &\le \int_I \left(\frac{u}{1-x} + au_x\right)^2 \frac{1}{(1-x)^4}dx \\ &= \int_I \frac{u^2}{(1-x)^6}dx + 2a\int_I \frac{u\,u_x}{(1-x)^5}dx + a^2\int_I \frac{(u_x)^2}{(1-x)^4}dx. \end{aligned}$$

Integration by parts in the second term lead to

$$2a\int_I \frac{u\,u_x}{(1-x)^5}dx = -5a\int_I \frac{u^2}{(1-x)^6}dx.$$

Taking $a = \frac{2}{5}$ yields the first inequality in (3.5). Similarly, one can show

$$\int_I \frac{u_x^2}{(1-x)^4}dx \le \frac{4}{9}\int_I \frac{u_{xx}^2}{(1-x)^2}dx \quad \text{for all } u \in V.$$

To prove (3.6), it suffices to consider

$$\begin{aligned} 0 \;&\le \int_I \left(\frac{u}{(1-x)^2} + \frac{qu_x}{(1-x)} + ru_{xx}\right)^2 \frac{1}{(1-x)^2}dx \\ &= (1-5q+20r)\int_I \frac{u^2}{(1-x)^6}dx - (2r+3qr-q^2)\int_I \frac{u_x^2}{(1-x)^4}dx + r^2\int_I \frac{u_{xx}^2}{(1-x)^2}dx. \end{aligned}$$

In particular, when $q = \frac{3}{2}$ and $r = \frac{1}{2}$, one has

$$0 \le \frac{7}{2}\int_I \frac{u^2}{(1-x)^6}dx - \int_I \frac{u_x^2}{(1-x)^4}dx + \frac{1}{4}\int_I \frac{u_{xx}^2}{(1-x)^2}dx.$$

Inequality (3.7) is obtained by considering

$$\begin{aligned} 0 \;&\le \int_I \left(u_{xxx}(1+x) + \frac{u_{xx}}{1-x}\right)^2 dx \\ &= \int_I u_{xxx}^2(1+x)^2dx + 2\int_I \frac{1+x}{1-x}u_{xx}u_{xxx}dx + \int_I \frac{u_{xx}^2}{(1-x)^2}dx \end{aligned}$$

and integrating by parts in the second term.

To see that $V \hookrightarrow H$, one applies (3.5) to obtain

$$\|u\|_H^2 = \int_I u^2(x)\omega(x)dx \le C\int_I \frac{u^2}{(1-x)^6}dx \le C\int_I \frac{u_{xx}^2}{(1-x)^2}dx = C\|u\|_V^2.$$

This concludes the proof of Lemma 3.1.

3.2.3 Weak formulation

Consider the boundary-value problem for the linear fifth-order equation

$$\begin{cases} \beta u_{xxx} - u_{xxxxx} = f, \quad x \in (-1,1), \\ u(-1) = u(1) = u_x(-1) = u_x(1) = u_{xx}(1) = 0. \end{cases} \tag{3.8}$$

For $u \in V$, $v \in W$, consider a bilinear form defined in $V \times W$ by

$$a(u,v) = \int_I u_{xx}(-\beta(v\omega)_x + (v\omega)_{xxx})\,dx, \tag{3.9}$$

then the weak formulation of (3.8) using dual-Petrov-Galerkin method is:

$$a(u, v) = (f, v)_H, \quad \text{for} \quad f \in H. \tag{3.10}$$

Now, the existence and uniqueness of solutions to the weak formulation (3.10) will be established.

3.2.4 Existence and uniqueness of weak formulation

The goal of this subsection is to solve the equation (3.10) for $f \in H$, and define an unbounded operator A by setting $Au = f$. Before stating the main theorem that guarantees the existence and uniqueness of such solutions, the following two lemmas will be given. The proof of the theorem relies on these lemmas.

Lemma 3.2 (The general version of Lax-Milgram Theorem) *Let $W \subset V$ be two Hilbert spaces with W being dense and continuously embedded in V. Let $a(u, v)$ be a bilinear form on $V \times W$ satisfying*

$$a(u, v) \leq M\|u\|_V\|v\|_W \quad \forall u \in V,\, v \in W \tag{3.11}$$

$$a(v, v) \geq m\|v\|_V^2 \quad \forall v \in W, \tag{3.12}$$

where $M > m > 0$ are two constants. Then, for any $f \in V'$, there exists $u \in V$ such that

$$a(u, v) = (f, v) \quad \forall v \in W.$$

If u is also known to be in W, then u is unique.

This general version of Lax-Milgram theorem is due to J.L. Lions [32].

Lemma 3.3 *If $\|u_{xxxxx}(1 + x)^3\|_{L^2(I)} < \infty$, then*

$$\int_I u_{xxxx}^2(1 + x)^4 dx \leq \frac{1}{4}\int_I u_{xxxxx}^2(1 + x)^6 dx. \tag{3.13}$$

Proof. For u satisfying $\|u_{xxxxxx}(1+x)^3\|_{L^2(I)} < \infty$, we consider

$$\begin{aligned}
0 &\le \int_I \left(u_{xxxx}(1+x)^2 + u_{xxxxx}(1+x)^3\right)^2 dx \\
&= \int_I u_{xxxx}^2(1+x)^4 dx + \int_I u_{xxxxx}^2(1+x)^6 dx + \int_I 2u_{xxxx}u_{xxxxx}(1+x)^5 dx \\
&= -4\int_I u_{xxxx}^2(1+x)^4 dx + \int_I u_{xxxxx}^2(1+x)^6 dx, \qquad (3.14)
\end{aligned}$$

which implies (3.13). ■

The next work is to state and prove the main theorem of this section [25].

Theorem 3.1 *For any $\beta > -\frac{3}{80}$ and for any $f \in H$, there exists a unique solution $u \in W$ such that*

$$a(u,v) = (f,v)_H \quad \forall v \in W. \qquad (3.15)$$

As a consequence, one can define an operator A: $D(A) \to H$ by

$$Au = f, \quad \text{where} \quad D(A) = \{u \in W,\ Au \in H\}.$$

Proof. It suffices to show that $a(u,v)$ as defined in (3.9) verifies the conditions of Lemma 3.2. This can be checked directly. For $u \in V$ and $v \in W$, one can write

$$a(u,v) = \int_I u_{xx}(-\beta v_x\omega - \beta v\omega' + v_{xxx}\omega + 3v_{xx}\omega' + 3v_x\omega'' + v\omega''')dx$$

with $\omega'' = \dfrac{4}{(1-x)^3}$ and $\omega''' = \dfrac{12}{(1-x)^4}$. The terms on the right can be bounded as follows.

$$\begin{aligned}
-\beta\int_I u_{xx}v_x\omega dx &\le \frac{16}{27}\sqrt{2}\,|\beta|\left(\int_I \frac{2u_{xx}^2}{(1-x)^2}dx\right)^{1/2}\left(\int_I \frac{v_x^2}{(1-x)^4}dx\right)^{1/2} \\
&\le \frac{32}{81}|\beta|\|u\|_V\|v\|_V, \\
-\beta\int_I u_{xx}v\omega'\,dx &\le 4\sqrt{2}\,|\beta|\left(\int_I \frac{2u_{xx}^2}{(1-x)^2}dx\right)^{1/2}\left(\int_I \frac{v^2}{(1-x)^6}dx\right)^{1/2} \\
&\le \frac{16}{15}|\beta|\|u\|_V\|v\|_V,
\end{aligned} \qquad (3.16)$$

$$
\begin{aligned}
\int_I u_{xx} v_{xxx} \omega dx &\le \left(\int_I u_{xx}^2 \omega' dx\right)^{1/2} \left(\int_I v_{xxx}^2 \frac{\omega^2}{\omega'} dx\right)^{1/2} \\
&= \|u\|_V \|v\|_W, \\
\int_I u_{xx} v_{xx} \omega' dx &\le \|u\|_V \|v\|_V, \\
\int_I u_{xx} v_x \omega'' dx &\le \left(\int_I \frac{2u_{xx}^2}{(1-x)^2} dx\right)^{1/2} \left(\int_I \frac{8v_x^2}{(1-x)^4} dx\right)^{1/2} \\
&\le \frac{4}{3} \|u\|_V \|v\|_V, \\
\int_I u_{xx} v w''' dx &\le \left(\int_I \frac{2u_{xx}^2}{(1-x)^2} dx\right)^{1/2} \left(\int_I \frac{72 v^2}{(1-x)^6} dx\right)^{1/2} \\
&\le \frac{8}{5} \|u\|_V \|v\|_V.
\end{aligned}
$$

Here, Hardy inequalities of Lemma 3.1 have been applied. According to (3.7),

$$
\|v\|_V \le 2\|v\|_W \quad \forall v \in W
$$

and one has thus verified (3.11) with $M = \frac{1184}{405}|\beta| + \frac{91}{5}$.

To prove (3.12), let $v \in W$ and integrate by parts to obtain

$$
\begin{aligned}
a(v,v) &= \int_I v_{xx}(-\beta v_x \omega - \beta v \omega' + v_{xxx}\omega + 3v_{xx}\omega' + 3v_x\omega'' + v\omega''') dx \\
&= \frac{3}{2}\beta \int_I v_x^2 \omega' dx - \frac{1}{2}\beta \int_I v^2 w''' dx \\
&\quad + \frac{5}{2} \int_I v_{xx}^2 \omega' dx - \frac{5}{2} \int_I v_x^2 \omega''' \, dx + \frac{1}{2} \int_I v^2 \omega^{(5)} dx. \qquad (3.17)
\end{aligned}
$$

For $\beta \ge 0$, one applies (3.5) to obtain

$$
\begin{aligned}
\frac{3}{2}\beta \int_I v_x^2 \omega' dx - \frac{1}{2}\beta \int_I v^2 w''' dx &\ge 3\beta \int_I \frac{v_x^2}{(1-x)^2} dx - \frac{8}{3}\beta \int_I \frac{v_x^2}{(1-x)^2} dx \\
&= \frac{1}{3}\beta \int_I \frac{v_x^2}{(1-x)^2} dx.
\end{aligned}
$$

Since $\omega''' = \dfrac{12}{(1-x)^4}$ and $\omega^{(5)} = \dfrac{240}{(1-x)^6}$, one has, for $\beta \ge 0$,

$$
a(v,v) \ge \frac{1}{3}\beta \int_I \frac{v_x^2}{(1-x)^2} dx + 5 \int_I \frac{v_{xx}^2}{(1-x)^2} dx - 30 \int_I \frac{v_x^2}{(1-x)^4} dx + 120 \int_I \frac{v^2}{(1-x)^6} dx.
$$

After ignoring the first term and applying (3.6) with $r = 0.4$ and $q = 1$, one obtains

$$
a(v,v) \ge 0.2 \int_I \frac{v_{xx}^2}{(1-x)^2} dx = 0.1 \|v\|_V^2.
$$

In the case when $\beta < 0$, apply (3.5) to obtain

$$\begin{aligned}\frac{3}{2}\beta\int_I v_x^2\omega' dx - \frac{1}{2}\beta\int_I v^2 w''' dx &\geq 3\beta\int_I \frac{v_x^2}{(1-x)^2}dx \\ &\geq 12\beta\int_I \frac{v_x^2}{(1-x)^4}dx \geq \frac{16}{3}\beta\int_I \frac{v_{xx}^2}{(1-x)^2}dx.\end{aligned}$$

Thus, for $\beta < 0$,

$$a(v,v) \geq \left(5 + \frac{16}{3}\beta\right)\int_I \frac{v_{xx}^2}{(1-x)^2}\,dx - 30\int_I \frac{v_x^2}{(1-x)^4}\,dx + 120\int_I \frac{v^2}{(1-x)^6}\,dx.$$

Applying (3.6) with r and q satisfying

$$1 - 5q + 20r = 4(2r + 3qr - q^2) > 0,$$

we have

$$a(v,v) \geq \left(5 + \frac{16}{3}\beta - \frac{30r^2}{2r + 3qr - q^2}\right)\int_I \frac{v_{xx}^2}{(1-x)^2}\,dx.$$

In order for a to be coercive, β has to satisfy

$$\beta > \frac{15}{16}\left(\frac{6r^2}{2r + 3qr - q^2} - 1\right) = \frac{15(16q^2 - 18q + 5)}{32(5q - 2)}.$$

The optimal range $\beta > -\frac{3}{80}$ is reached when $r = \frac{1}{10}$ and $q = \frac{11}{20}$, and

$$a(v,v) \geq \left(\frac{1}{5} + \frac{16}{3}\beta\right)\int_I \frac{v_{xx}^2}{(1-x)^2}\,dx = \gamma\|v\|_V^2$$

where

$$\gamma = \frac{1}{10} + \frac{8}{3}\beta. \tag{3.18}$$

Lemma 3.2 then implies the existence of $u \in V$ satisfying (3.15).

Now, the uniqueness of u is established. If there are $u_1 \in V$ and $u_2 \in V$ satisfying (3.15), then

$$a(u_1 - u_2, v) = 0 \quad \text{for all } v \in W. \tag{3.19}$$

According to Lemma 3.1, C_0^∞ is densely embededd in V, there is a sequence $v_n \in C_0^\infty$ such that

$$v_n \to u_1 - u_2 \quad \text{in } V.$$

Since $v_n \in C_0^\infty \subset W$, one obtains by (3.19) that,

$$\begin{aligned} \gamma\|v_n\|_V^2 \leq a(v_n, v_n) &= a(u_1 - u_2, v_n) + a(v_n - (u_1 - u_2), v_n) \\ &\leq M\|v_n - (u_1 - u_2)\|_V \|v_n\|_W. \end{aligned} \tag{3.20}$$

Furthermore, for any $v \in W$,

$$a(v_n, v) = a(v_n - (u_1 - u_2), v) \leq M\|v_n - (u_1 - u_2)\|_V \|v\|_W \to 0 \quad \text{as } n \to \infty.$$

In particular, for any $v \in C_0^\infty$,

$$\begin{aligned} a(v_n, v) &= \int_I (v_n)_{xx} \Big(-\beta(v\omega)_x + (v\omega)_{xxx} \Big)\, dx \\ &= \int_I \Big(\beta(v_n)_{xxx} - (v_n)_{xxxxx} \Big)\, v\omega\, dx \to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{3.21}$$

Letting

$$\beta(v_n)_{xxx} - (v_n)_{xxxxx} = f_n \tag{3.22}$$

and choosing $v = f_n$ in (3.21), one has

$$\|f_n\|_H^2 = \int_I f_n^2\, \omega\, dx \to 0 \quad \text{as } n \to \infty. \tag{3.23}$$

The identities in (3.22) and (3.23) allow us to show that

$$\|v_n\|_W^2 \leq C\|v_n\|_V (\|v_n\|_V + \|f_n\|_H). \tag{3.24}$$

Combining (3.20) and(3.24) yields

$$\gamma\|v_n\|_V^2 \leq C\|v_n - (u_1 - u_2)\|_V\, \|v_n\|_V^{\frac{1}{2}} \left(\|v_n\|_V + \|f_n\|_H\right)^{\frac{1}{2}}.$$

Letting $n \to \infty$, one has $v_n \to 0$ in V and consequently

$$\|u_1 - u_2\|_V \leq \|v_n\|_V + \|v_n - (u_1 - u_2)\|_V \to 0 \quad \text{as } n \to \infty.$$

That is, $u_1 = u_2$ in V.

To obtain (3.24), It is used the fact that if u is smooth and satisfies

$$\beta u_{xxx} - u_{xxxxx} = f$$

with the homogeneous boundary condition, then

$$\|u\|_W^2 \le C\|u\|_V(\|u\|_V + \|f\|_H). \tag{3.25}$$

Lets prove this fact. Noting that $\dfrac{\omega^2}{\omega'} = \dfrac{(1+x)^2}{2}$ and integrating by parts, one has

$$\|u\|_W^2 = \int_I u_{xxx}^2 \frac{\omega^2}{\omega'} dx = -\frac{1}{2}\int_I u_{xx}u_{xxxx}(1+x)^2\,dx + \frac{1}{2}\int_I u_{xx}^2\,dx. \tag{3.26}$$

Obviously,

$$\frac{1}{2}\int_I u_{xx}^2\,dx \le \int_I \frac{2u_{xx}^2}{(1-x)^2}dx = \|u\|_V^2. \tag{3.27}$$

By Hölder's inequality, the first term is bounded by

$$\frac{1}{2}\int_I u_{xx}u_{xxxx}(1+x)^2\,dx \le \|u\|_V\left(\frac{1}{8}\int_I u_{xxxx}^2(1+x)^4(1-x)^2\,dx\right)^{1/2}.$$

Applying the inequality in Lemma 3.3, one finds

$$\frac{1}{8}\int_I u_{xxxx}^2(1+x)^4(1-x)^2\,dx \le \frac{1}{2}\int_I u_{xxxx}^2(1+x)^4\,dx \le \frac{1}{8}\int_I u_{xxxxx}^2(1+x)^6\,dx.$$

Since $u_{xxxxx} = \beta u_{xxx} - f$ and

$$\begin{aligned}\frac{1}{2}\int_I(\beta u_{xxx}-f)^2(1+x)^6dx &\le \beta^2\int_I u_{xxx}^2(1+x)^6dx + \int_I f^2(1+x)^6dx\\ &\le 16\beta^2\int_I u_{xxx}^2(1+x)^2dx + 16\int_I f^2\frac{1+x}{1-x}\,dx,\end{aligned}$$

we find

$$\begin{aligned}\frac{1}{2}\int_I u_{xx}u_{xxxx}(1+x)^2\,dx &\le 4\sqrt{2}|\beta|\|u||_V\|u\|_W + 2\|u\|_V\|f\|_H\\ &\le \frac{1}{2}\|u\|_W^2 + C\|u\|_V^2 + 2\|u\|_V\|f\|_H \end{aligned} \tag{3.28}$$

Putting together (3.26), (3.27) and (3.28), (3.25) is concluded.

This uniqueness of u allows us to show that $u \in W$. Since C_0^∞ is dense in H, Lets assume without loss of generality that $f \in C_0^\infty$. Because of the uniqueness, the corresponding solution u is smooth and satisfies

$$\beta u_{xxx} - u_{xxxxx} = f$$

with the homogeneous boundary condition. Therefore, $u \in W$ by (3.25). This completes the proof of Theorem 3.1. ■

3.3 The full initial-and boundary-value problem

This section focuses on the full initial- and boundary-value problem

$$\begin{aligned} &u_t + uu_x + \beta u_{xxx} - u_{xxxxx} = 0, \quad x \in (-1,1),\ t > 0, \\ &u(\pm 1, t) = u_x(\pm 1, t) = u_{xx}(1,t) = 0, \quad t > 0, \\ &u(x,0) = u_0(x), \quad x \in (-1,1). \end{aligned} \tag{3.29}$$

Its solutions are studied at two regularity levels: mild solutions and strong solutions. For this purpose, one first examines the operator A defined in Theorem 3.1. One shows that $-A$ is an infinitesimal generator of a contraction semi-group e^{-At} by using Hille-Yoshida theorem. First, lets discuss semigroup theory and state Hille-Yosida theorem.

3.3.1 Semi-group theory and Hille-Yosida theorem

This section gives a brief introduction of semigroup theory and Hille-Yosida theorem [13]. Semigroup theory is useful to solve the above full IBVP for the Kawahara equation.

Let $\mathbb{X}$ be a Banach space with the norm $\|\cdot\|$.

Definition 3.1 *A one-parameter family* $\{T(t)\}_{0\le t\le\infty}$ *of bounded linear operators from* $\mathbb{X}$ *into* $\mathbb{X}$ *is a semigroup on* $\mathbb{X}$ *if*

1. $T(0) = I$*, where* I *is the identity operator on* $\mathbb{X}$*.*

2. $T(t+s) = T(t)T(s)$ *for every* $t, s \ge 0$ *(the semigroup property).*

3. *The mapping from* $[0,\infty)$ *into* $\mathbb{X}$ *is continuous.*

Remark 3.1 $\{T(t)\}_{0\le t\le\infty}$ *is a contraction semigroup if, in addition,*

$$\|T(t)\| \le 1 \quad (t \ge 0).$$

Assume $\{T(t)\}_{0\leq t\leq\infty}$ is a contraction semigroup on $\mathbb{X}$. The domain of the linear operator A is denoted by $D(A)$ and is defined by

$$D(A) = \{x \in \mathbb{X} : \lim_{t\to 0} \frac{T(t)x - x}{t} \quad \text{exits}\}.$$

The linear operator $A : D(A) \to \mathbb{X}$ is called the infinitesimal generator of the contraction semigroup if

$$Ax = \lim_{t\to 0} \frac{T(t)x - x}{t}, \quad x \in D(A).$$

Remark 3.2 *The domain $D(A)$ is dense in $\mathbb{X}$ and A is a closed operator.*

Now, lets define the resolvent set and a resolvent operator.

Definition 3.2 *Let A be a closed linear operator on $\mathbb{X}$, with domain $D(A)$. The resolvent set $\rho(A)$ of A is the set of all λ for which $(\lambda I - A)$ is invertible. The resolvent operator $R_\lambda : \mathbb{X} \to \mathbb{X}$ is defined by*

$$R_\lambda x = (\lambda I - A)^{-1}x \quad \textit{for} \quad x \in D(A), \lambda \in \rho(A).$$

It is clear, by the Closed Graph Theorem [13], that R_λ is a bounded linear operator.

Now, one may ask: which operators generate contraction semigroup? The following theorem gives the answer to this question.

Theorem 3.2 *(Hille-Yosida) Let A be a closed, densely-defined linear operator on $\mathbb{X}$. Then A is the generator of a contraction semigroup $\{T(t)\}_{0\leq t\leq\infty}$ if, and only if $(0,\infty) \subset \rho(A)$ and $\|R_\lambda\| \leq \frac{1}{\lambda}$ for $\lambda > 0$.*

The contribution of this section to the operator defined in the last section can be summerized in the following theorem. This result is required to write the full IBVP in the integral form.

Theorem 3.3 *Let H, A and $D(A)$ be defined as in the previous section. Then $-A$ is an infinitesimal generator of a contraction semigroup e^{-At}.*

Proof. We apply the Hille-Yosida Theorem (see [34]). It suffices to show that A is closed, $D(A)$ is dense in H and $\|(\lambda I + A)^{-1} f\|_H \leq \frac{1}{\lambda}\|f\|_H$ for any $\lambda > 0$. A is closed can be established by showing that A^{-1} is one-to-one. $D(A)$ is dense in H since $C_0^\infty \subset D(A)$. For $f \in H$, let $u = (\lambda + A)^{-1} f$. Then $(\lambda + A)u = f$ and

$$(f, u)_H = ((\lambda + A)u, u)_H = \lambda\|u\|_H^2 + (Au, u)_H.$$

Since $(Au, u)_H = a(u, u) \geq 0$, we obtain that $\|u\|_H \leq \frac{1}{\lambda}\|f\|_H$. This concludes the proof. ■

3.3.2 Mild solutions

Before giving the definition of mild solutions, lets define the bilinear form on $V \times V$ as

$$B(u, v) = (uv)_x, \quad (u, v) \in V \times V.$$

Let $T > 0$.

Definition 3.3 *A mild solution of the IBVP (3.29) is a function $u \in C([0, T]; H) \cap L^2(0, T; V)$ satisfying*

$$\frac{du}{dt} + Au = -B(u, u) \quad \textit{in } V' \tag{3.30}$$

$$u(0) = u_0. \tag{3.31}$$

Since $-A$ is the generator of the semigroup e^{-At} , (3.30) and (3.31) can be written in the integral form

$$u(t) = e^{-At}u_0 - \int_0^t e^{-A(t-s)} B(u, u)(s)\, ds.$$

3.3.3 Bilinear estimates

In the proof of theorem given in the next subsection, the following bilinear estimate will be used.

Lemma 3.4 *For any* $(u, v) \in V \times V$*,*

$$\|B(u, v)\|_{V'} \leq C\|u\|_H \|v\|_H^{1/2} \|v\|_V^{1/2},$$

where C *is a constant independent of* u *and* v*.*

Proof. Let $\psi \in V$. one obtains by integrating by parts

$$(B(u, v), \psi)_H = \int_I (uv)_x \psi\omega \, dx = -\int_I uv\psi_x\omega \, dx - \int_I uv\psi\omega' \, dx. \tag{3.32}$$

By the first inequality in Lemma 3.1,

$$\begin{aligned}
\int_I u(x)v(x)\psi_x(x)\omega(x)dx &= \int_I u(x)\omega^{1/2}\psi_x(x)\omega'(x)v(x)\frac{\omega^{1/2}}{\omega'} \, dx \\
&\leq \|u\|_H \|\psi_x(1-x)^{-2}\|_{L^2} \sup_{x\in I} \left| v(x)\frac{\omega^{1/2}}{\omega'} \right| \\
&\leq C\|u\|_H \|\psi\|_V \sup_{x\in I} \left| v(x)\frac{\omega^{1/2}}{\omega'} \right|.
\end{aligned}$$

To complete the estimate, one writes

$$\begin{aligned}
\left| v(x)\frac{\omega^{1/2}}{\omega'} \right|^2 &= \frac{1}{4}v^2(x)(1+x)(1-x)^3 \\
&= \frac{1}{2}\int_{-1}^{x} v(y)v_y(y)(1+y)(1-y)^3 \, dy - \frac{1}{4}\int_{-1}^{x} v^2(y)(1-y)^2(2+4y) \, dy.
\end{aligned}$$

It is clear that these integrals are bounded by $C(\|v\|_H\|v\|_V + \|v\|_H^2)$. Therefore,

$$\int_I u(x)v(x)\psi_x(x)\omega(x)dx \leq C\|u\|_H \|\psi\|_V \|v\|_H^{1/2} \|v\|_V^{1/2}.$$

The second term in (3.32) can be bounded similarly. In fact,

$$\int_I uv\psi\omega' \, dx \leq C\|u\|_H \|v\|_H \sup_{x\in I} |\psi(x)\omega'(x)\omega^{-1}(x)|.$$

To show that $\sup_{x\in I} |\psi(x)\omega'(x)\omega^{-1}(x)| \leq C$, note that

$$\omega'(x)\omega^{-1}(x) = 2(1-x)^{-1}(1+x)^{-1}$$

and show that

$$\frac{\psi^2(x)}{(1-x)^4} \in W^{1,1}(I) \quad \text{and} \quad \frac{\psi^2(x)}{(1+x)^4} \in W^{1,1}(I).$$

Then the embedding $W^{1,1}(I) \subset L^\infty(\bar{I})$ leads to the conclusion. By the first inequality in Lemma 3.1,

$$\int_I \frac{\psi^2(x)}{(1-x)^4}dx \le 4\int_I \frac{\psi^2(x)}{(1-x)^6}dx \le \frac{32}{225}\|\psi\|_V^2.$$

In addition,

$$\partial_x\left(\frac{\psi^2(x)}{(1-x)^4}\right) = \frac{2\psi(x)\psi_x(x)}{(1-x)^4} + \frac{4\psi^2(x)}{(1-x)^5}$$

and

$$\int_I \frac{2\psi(x)\psi_x(x)}{(1-x)^4}dx \le C\|\psi(1-x)^{-3}\|_{L^2}\|\psi_x(1-x)^{-2}\|_{L^2} \le C\|\psi\|_V^2,$$

$$\int_I \frac{4\psi^2(x)}{(1-x)^5}dx \le C\|\psi(1-x)^{-3}\|_{L^2}^2 \le C\|\psi\|_V^2.$$

Therefore, $\dfrac{\psi^2(x)}{(1-x)^4} \in W^{1,1}(I)$ and similarly $\dfrac{\psi^2(x)}{(1+x)^4} \in W^{1,1}(I)$. This concludes the proof of Lemma 3.4. ■

3.3.4 Existence and uniqueness of mild solution

In this subsection, it will be shown that IBVP (3.29) has a unique local (in time) mild solution for any initial data $u_0 \in H$ [25].

Theorem 3.4 *Let $\beta > -\frac{3}{80}$ and let $u_0 \in H$. Then there exists $T = T(\|u_0\|_H)$ such that the initial- and boundary-value problem (3.29) has a unique mild solution u satisfying*

$$u \in C([0,T];H) \cap L^2(0,T;V).$$

In addition, u obeys the bound

$$\|u(t)\|_H^2 + \gamma\int_0^t \|u(\tau)\|_V^2 d\tau \le \|u_0\|_H^2 + \int_0^t \|u(\tau)\|_H^3\|u(\tau)\|_V\, d\tau, \tag{3.33}$$

where γ is as defined in (3.18).

Proof. To prove this theorem, one applies the contraction mapping principle to the integral equation

$$u(t) = e^{-At}u_0 - \int_0^t e^{-A(t-s)}B(u,u)(s)\, ds. \tag{3.34}$$

To this end, let $X = C([0,T]; H) \cap L^2(0,T;V)$ and define, for $u \in X$,

$$\|u\|_X = \sup_{t\in[0,T]} \|u(t)\|_H + \|u\|_{L^2(0,T;V)}.$$

Let $R = \|u_0\|_H$ and $B_{2R} = \{u \in X, \|u\|_X \le 2R\}$.One shows that the right side of (3.34), denoted by $G(u)$, defines a contraction mapping from B_{2R} to B_{2R}.

Let $u \in B_{2R}$. $G(u)$ satisfies

$$\frac{d}{dt}G(u) + AG(u) = -B(u,u)$$

and one obtains after taking the inner product of this equation with $G(u)$ in H,

$$\frac{d}{dt}\|G(u)\|_H^2 + 2a(G(u),G(u)) = -2(B(u,u),G(u))_H.$$

According to the proof of Theorem 3.1 and the bilinear estimate in Lemma 3.4,

$$2a(G(u),G(u)) \ge 2\gamma\|G(u)\|_V^2,$$

$$2|(B(u,u),G(u))_H| \le 2\|B(u,u)\|_{V'}\|G(u)\|_V \le \gamma\|G(u)\|_V^2 + C\|u\|_H^3\|u\|_V.$$

Therefore,

$$\|G(u)\|_H^2 + \gamma\int_0^t \|u(\tau)\|_V^2 d\tau \le \|u_0\|_H^2 + C\int_0^t \|u(\tau)\|_H^3\|u(\tau)\|_V \, d\tau.$$

If we choose $T > 0$ such that $R^2 + 16C\sqrt{T}R^4 < 2\min(1,\gamma)R^2$, then

$$\|G(u)\|_X < 2R.$$

To show G is a contraction, first note that

$$G(u) - G(v) = -\int_0^t e^{-A(t-s)}(B(u-v,u) + B(v,u-v))ds.$$

A similar process as in the estimate of $\|G(u)\|_H$ yields

$$\begin{aligned}
&\|G(u) - G(v)\|_H^2 + \gamma\int_0^t \|G(u) - G(v)\|_V^2 ds \\
&\quad\le \int_0^t (\|u-v\|_H^2\|u\|_H\|u\|_V + \|v\|_H^2\|u-v\|_H\|u-v\|_V)ds \\
&\quad\le C\sqrt{T}\|u-v\|_X^2(\|u\|_X^2 + \|v\|_X^2).
\end{aligned}$$

If T is further restricted to $4C\sqrt{T}R^2 < \min(1,\gamma)$, then

$$\|G(u) - G(v)\|_X \le \nu \|u - v\|_X,$$

where $\nu^2 = (4C\sqrt{T}R^2)/\min(1,\gamma) < 1$. Applying the contraction mapping principle completes the proof of this theorem. ■

3.3.5 Strong solution

Now, the solutions of the IBVP (3.29) are studied in a stronger sense and establish the global existence and uniqueness of such solutions. To this end, define

$$H_1(I) = \{u \in H(I),\ u_x \in H(I)\} \quad \text{and} \quad V_1(I) = \{u \in V(I),\ u_x \in V(I)\}\,.$$

Lets prove the following estimates that will be used to show the global regularity of solutions of the full IBVP.

Lemma 3.5 *The constants C in the bounds are absolute constants.*

1) For any $u \in V$,

$$\left| \int_I u^2 u_x \omega dx \right| \le C \|u\|_{L^2} \|u\|_H \|u\|_V. \tag{3.35}$$

2) For any $u \in H$ and $v \in V$,

$$\left| \int_I (uv_x)_x v\omega \, dx \right| \le C \|u\|_H \|v\|_V^2. \tag{3.36}$$

Proof. Integrating by parts and applying Hölder's inequality, one has

$$\int_I u^2 u_x \omega dx \le ||u||_{L^2} \|u\|_H \sup_{x \in I} |u_x \omega^{1/2}|. \tag{3.37}$$

To bound $\sup\limits_{x\in I} |u_x\omega^{1/2}|$, apply (3.5) in Lemma 3.1 to obtain

$$\begin{aligned} u_x^2(x)\omega(x) &= 2\int_{-1}^{x} u_x u_{xx} \frac{1+x}{1-x}\, dx + \int_{-1}^{x} \frac{2u_x^2}{(1-x)^2}\, dx \\ &\le 16\|u_x(1-x)^{-2}\|_{L^2} \|u_{xx}(1-x)^{-1}\|_{L^2} + 8\|u_x(1-x)^{-2}\|_{L^2} \\ &\le C\|u\|_V^2. \end{aligned}$$

Inserting this bound in (3.37) yields (3.35). The bound in (3.36) can be established similarly. In fact,

$$\begin{aligned}\int_I (uv_x)_x v\omega\, dx &= -\int_I uv_x^2\omega\, dx - \int_I uv\, v_x w'\, dx \\ &\leq C\|u\|_H\|v\|_V \sup_{x\in I}|v_x| + C\|u\|_H\|v\|_V \sup_{x\in I}|v\omega^{-1/2}|. \qquad (3.38)\end{aligned}$$

The bound for $\sup_{x\in I}|v\omega^{-1/2}|$ can be obtained as in Lemma 3.4,

$$\sup_{x\in I}|v\omega^{-1/2}| \leq C\|v\|_V.$$

Also, the bound for $\sup_{x\in I}|v_x|$ can be estimated as follows.

$$\begin{aligned}v_x^2(x) = 2\int_{-1}^{x} v_x v_{xx}\, dx &\leq 16\int_{-1}^{x} \frac{|v_x|}{(1-x)^2}\frac{|v_{xx}|}{(1-x)}\, dx \\ &\leq 16\|v_x(1-x)^{-2}\|_{L^2}\, \|v_{xx}(1-x)^{-1}\|_{L^2} \\ &\leq \frac{16}{3}\|v\|_V^2.\end{aligned}$$

Inserting these bounds in (3.38) leads to (3.36). ■

Theorem 3.5 *Assume $\beta > -\frac{3}{80}$ and $u_0 \in H_1(I)\cap L^2(I)$. Let $T>0$ be arbitrarily fixed. Then the IBVP (3.29) has a unique solution u satisfying*

$$u \in C([0,T]; H_1\cap L^2)\cap L^2(0,T;V_1).$$

Furthermore, if the L^2-norm of u_0 is small in the sense that

$$\|u_0\|_{L^2(I)} \leq C\gamma \qquad (3.39)$$

for some suitable constant C, then $\|u(t)\|_H$ and $\|u_x(t)\|_H$ decay exponentially in time [25].

Proof. Since $u_0\in H$, Theorem 3.4 asserts the existence of a local solution u satisfying

$$u \in C([0,T];H)\cap L^2(0,T;V). \qquad (3.40)$$

Thanks to $u_0 \in L^2(I)$, u obeys the global *a priori* bound

$$\|u(t)\|_{L^2(I)} \le \|u_0\|_{L^2} \quad \text{for all} \quad t > 0. \tag{3.41}$$

This can be established by first noticing that smooth solutions of (3.30) satisfy

$$\|u(t)\|_{L^2}^2 + \int_0^t u_{xx}^2(0,\tau)d\tau = \|u_0\|_{L^2}^2$$

and then going through a limiting process. We now apply (3.41) to show that, for $t \le T$,

$$\|u(t)\|_H \le C(T)\|u_0\|_H, \tag{3.42}$$

where $C(T)$ is a constant depending only on T. Taking the inner product of (3.30) with u in H and applying Lemma 3.5, one obtains

$$\frac{d}{dt}\|u\|_H^2 + 2\gamma\|u\|_V^2 \le C\|u\|_{L^2}\|u\|_H\|u\|_V. \tag{3.43}$$

Inserting the inequality

$$C\|u\|_{L^2}\|u\|_H\|u\|_V \le \gamma\|u\|_V^2 + \frac{1}{4}C^2\gamma^{-1}\|u\|_{L^2}^2\|u\|_H^2$$

in (3.43) and applying Gronwall's inequality, one obtains (3.42). If $\|u_0\|_L^2$ satisfies (3.39), (3.41) and (3.42) imply

$$\frac{d}{dt}\|u\|_H^2 + (2\gamma - C\|u_0\|_{L^2})\,\|u\|_V^2 \le 0,$$

where $\|u\|_H \le \|u\|_V$ has been used. Consequently, $\|u(t)\|_H$ decays exponentially in time.

Lets further show that, for some constant C depending only on T,

$$\|u_x(t)\|_H \le C(T)\|u_{0x}\|_H. \tag{3.44}$$

To prove (3.44), start with the equation

$$\frac{dv}{dt} + Av = -(uv)_x$$

that $v = u_x$ satisfies. Taking then the inner product with v in H and applying Lemma 3.5, one obtains

$$\frac{d}{dt}\|v\|_H^2 + 2\gamma\|v\|_V^2 \le C\|u\|_H\|v\|_V^2.$$

The desired inequality then follows from Gronwall's inequality. This concludes the proof of Theorem 3.5. ■

3.4 More general problem

Finally I want to remark that Theorem 3.4 and Theorem 3.5 can be easily extended to a slightly more general problem than (3.29). In fact, the following corollaries can be established by modifying the proofs of Theorem 3.4 and Theorem 3.5.

Corollary 3.1 *Let $L > 0$ and $J = (-L, L)$. Let $\beta_2 > 0$ and consider*

$$\begin{aligned} &u_t + uu_x + \beta_1 u_{xxx} - \beta_2 u_{xxxxx} = 0, \quad x \in J,\ t > 0,\\ &u(\pm L, t) = u_x(\pm L, t) = u_{xx}(L, t) = 0, \quad t > 0, \\ &u(x, 0) = u_0(x), \quad x \in J. \end{aligned} \tag{3.45}$$

Assume $u_0 \in H(J)$ and

$$L^2\beta_1 > -\frac{3}{80}\beta_2. \tag{3.46}$$

Then there exists $T = T(\|u_0\|_{H(J)})$ such that the IBVP (3.45) has a unique mild solution u satisfying

$$u \in C([0, T]; H(J)) \cap L^2(0, T; V(J)).$$

In addition, u obeys the bound

$$\|u(t)\|_H^2 + \mu \int_0^t \|u(\tau)\|_V^2 d\tau \le \|u_0\|_H^2 + \int_0^t \|u(\tau)\|_H^3 \|u(\tau)\|_V \, d\tau,$$

where $\mu = \frac{5}{14}\beta_2 + \frac{8}{3}L^2 \min(0, \beta_1)$.

Corollary 3.2 *Consider the IBVP (3.45). Assume* β_1 *and* β_2 *satisfying (3.46) and* $u_0 \in H_1(J) \cap L^2(J)$*. Let* $T > 0$ *be arbitrarily fixed. Then the IBVP (3.45) has a unique solution* u *satisfying*

$$u \in C([0,T]; H_1(J) \cap L^2(J)) \cap L^2(0,T; V_1(J)).$$

Furthermore, if the L^2*-norm of* u_0 *is small in the sense that*

$$\|u_0\|_{L^2(J)} \leq C\gamma$$

for some suitable constant C*, then* $\|u(t)\|_H$ *and* $\|u_x(t)\|_H$ *decay exponentially in time.*

3.5 Numerical results

This section studies numerically the behavior of solutions of (3.45) with $L = 1$ and for β_1 and β_2 in different ranges. The numerical scheme is the dual Petrov-Galerkin algorithm that has previously been developed in [40] and [48]. The results presented here clearly indicate that solutions of (3.45) with β_1 and β_2 violating (3.46) may not exist for all time.

First, lets compute the solution of the Kawahara equation

$$u_t + u\, u_x + \frac{1}{M^2} u_{xxx} - \frac{1}{M^4} u_{xxxxx} = 0, \quad x \in (-1,1),\ t \in [0,100] \tag{3.47}$$

with zero boundary data and with the initial datum

$$u(x,0) = u_{ex}(x,0), \tag{3.48}$$

where

$$u_{ex}(x,t) = \frac{105}{169} \operatorname{sech}^4 \left[\frac{M}{2\sqrt{13}} \left(x - \frac{36t}{169} \right) \right] \tag{3.49}$$

is an exact soliton solution of (3.47) before it hits the right boundary.

In the table 3.1 below, listed are the L^2-errors at different times with two different time steps with $M = 200$ and number of modes, $N = 1000$ in the dual-Petrov-Galerkin

Time	L^2- error with $\Delta t = 1.0E - 4$	L^2-error with $\Delta t = 2.0E - 4$	Rate
0.5	3.44E-7	1.374E-6	3.99
1.0	5.926E-7	2.358E-6	3.98
2.0	1.104E-6	4.389E-6	3.98
4.0	2.147E-6	8.494E-6	3.96

Table 3.1: L^2-errors for solitary wave solutions in the Kawahara equation

scheme. Table 3.1 clearly indicates that the Crank-Nicholson-leap-frog scheme is of second-order in time.

In Fig. 3.1, the computed and exact solutions for the above mentioned Kawahara equation are plotted. The computed and exact solutions are virtually indistinguishable.

In (3.47), $\beta_1 = 1/M^2$ and $\beta_2 = 1/M^4$ and they trivially satisfy the condition (3.46). Corollary 3.2 assesses that the IBVP given by (3.47) and (3.48) has a global solution. Fig 3.2 shows the plots of the standard norms $\|u(\cdot,t)\|_{L^2}$ and the weighted norms $\|u(\cdot,t)\|_{L^2_\omega}$ as functions of t by taking $M = 200, N = 1000$, and time step $\Delta t = 0.001$.

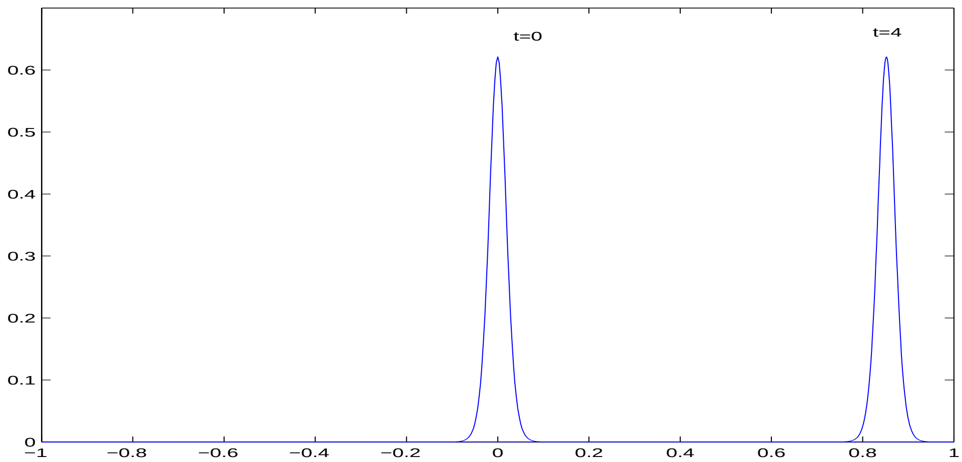

Figure 3.1: solitary wave solutions in the Kawahara equation.

The solution of (3.47) and (3.48) is the solitary wave given by (3.49) and its standard L^2-norm remains a constant before it hits the right boundary. Its L^2-norm starts decaying after it reaches the boundary. The weighted L^2-norm decays exponentially after the time when the wave hits the right boundary.

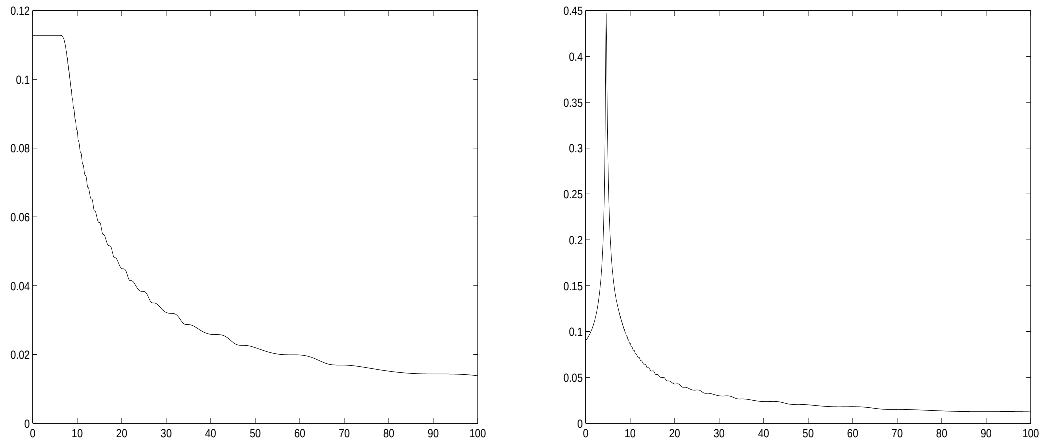

Figure 3.2: Left: $\|u(\cdot,t)\|_{L^2}$ vs. t, Right: $\|u(\cdot,t)\|_{L^2_\omega}$ vs. t

Second, we examine the solution of the Kawahara equation

$$\begin{aligned} &u_t + u\,u_x + \beta_1\,u_{xxx} - \beta_2\,u_{xxxxx} = 0, \quad x \in (-1,1),\ t \in [0,10], \\ &u(x,0) = u_{ex}(x,0) \end{aligned} \tag{3.50}$$

where $\beta_1 = -0.01$ and $\beta_2 = 1/M^4$, and the boundary conditions are set to be homogeneous. It is clear that β_1 and β_2 violate (3.46) when

$$M \geq \left(\frac{15}{4}\right)^{1/4} \approx 1.3916,$$

and the existence and uniqueness theory presented in the previous section does not cover this case. To see how the solution behaves, lets choose $M = 200$ and plot both the L^2-norm $\|u(\cdot,t)\|_{L^2}$ and the weighted L^2-norm $\|u(\cdot,t)\|_{L^2_\omega}$. The graphs in Fig. 3.3 clearly show that both norms quickly grow in time after an initial decay. This is an indication that the IBVP (3.45) may not be globally well-posed when the condition (3.46) is not met.

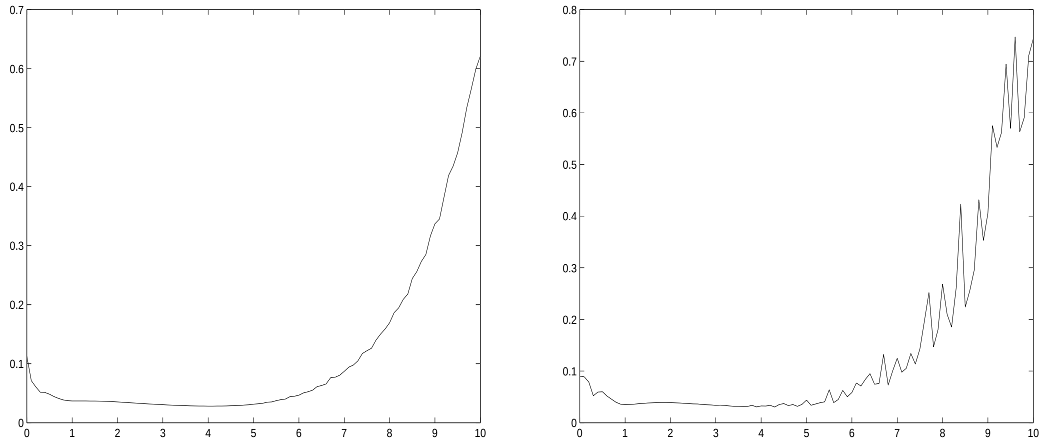

Figure 3.3: Left: $\|u(\cdot,t)\|_{L^2}$ vs. t, Right: $\|u(\cdot,t)\|_{L^2_\omega}$ vs. t

CHAPTER 4

Complex-valued Burgers and KdV-Burgers equations

4.1 Overview

The model problem for this chapter is the IVP of the complex-valued KdV-Burgers equations

$$\begin{cases} u_t - 6uu_x + \alpha u_{xxx} - \nu u_{xx} = 0, \\ u(x,0) = u_0(x), \end{cases} \tag{4.1}$$

where $\nu \geq 0$ and $\alpha \geq 0$ are parameters and $u = u(x,t)$ is a complex-valued function. Attention will be focused on the spatially periodic solutions, namely $x \in \mathbb{T} = [0, 2\pi]$.

This equation reduces to the complex Burgers and complex KdV equations under the following situation.

1. If $\alpha = 0$ in (4.1), the equation reduces to the IVP of the complex Burgers equation

$$\begin{cases} u_t - 6uu_x - \nu u_{xx} = 0,\ x \in \mathbb{T}, t > 0 \\ u(x,0) = u_0(x), x \in \mathbb{T}. \end{cases} \tag{4.2}$$

2. If $\nu = 0$, the equation (4.1) becomes the complex KdV equation

$$\begin{cases} u_t - 6uu_x + \alpha u_{xxx} = 0,\ x \in \mathbb{T}, t > 0 \\ u(x,0) = u_0(x), x \in \mathbb{T}. \end{cases} \tag{4.3}$$

The next section of this chapter is devoted to the blowup for the complex Burgers equation and the other two sections describe the regularity issue of complex KdV-Burgers equation under certain conditions, and the Lax pairs, respectively.

4.2 Blow up for the complex Burgers equation

This section presents three major results. The first one is a local existence and uniqueness result on solutions of the form

$$u(x,t) = \sum_{k=1}^{\infty} a_k(t)\, e^{ikx} \tag{4.4}$$

to the complex-valued KdV-Burgers type equation

$$u_t - 6uu_x + \nu(-\Delta)^\gamma u + \alpha\, u_{xxx} = 0, \tag{4.5}$$

which reduces to the complex Burgers equation when $\gamma = 1$ and $\alpha = 0$. The fractal Laplacian $(-\Delta)^\gamma$ is defined through Fourier transform,

$$\widehat{(-\Delta)^\gamma u}(\xi) = |\xi|^{2\gamma}\, \widehat{u}(\xi).$$

The second result asserts that if the L^2-norm of a solution of (4.5) is bounded on $[0, T]$, then all higher derivatives are bounded and no singularity is possible on $[0, T]$. The third result is a blowup result for the complex Burgers equation (4.2) in a periodic domain $\mathbb{T}= [0, 2\pi]$. It states that for any sufficiently large $T > 0$, there exists an initial data u_0 such that its corresponding solution u blows up at $t = T$. This solution can be represented by (4.4) and the blowup is in the L^2 sense. But before the analytical treatment, lets discuss some computation results for the blow up.

4.2.1 Numerical solutions

In this subsection, we present some results from our numerical experiments performed on the complex Burgers equation. We employed the dual-Petrov-Galerkin algorithm developed by Jie Shen to find the solutions of the equation

$$u_t + \beta uu_x - \nu u_{xx} = 0,$$

where $\beta = 2$. The initial data are of the form

$$u_0(x) = a_1 \exp{(2\pi ix)} + a_2 \exp{(4\pi ix)} + a_3 \exp{(6\pi ix)}$$

having three modes a_1, a_2, and a_3. Appropriate modifications have been made in the algorithm to suit the complex equation. Fixing the values of the modes at $a_1 = 2, a_2 = 4$, and $a_3 = 6$ and taking $\nu = 0.3$, the solutions are computed for different time steps. The following graphs show the solution $u(x,t)$ vs. x in different times. The solid curve represents the real part of u and the dotted curve represents the imaginary part of u.

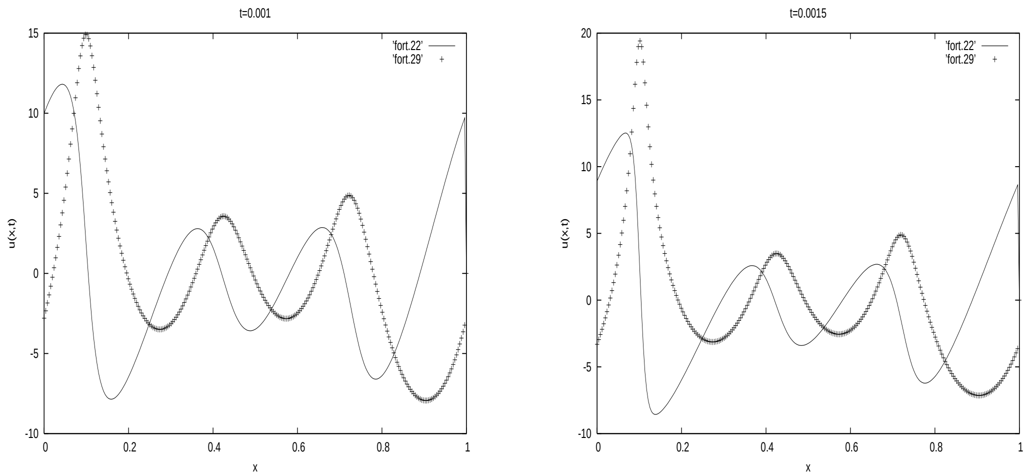

Figure 4.1: Left: $u(x,t)$ vs. x ; Right: $u(x,t)$ vs. x

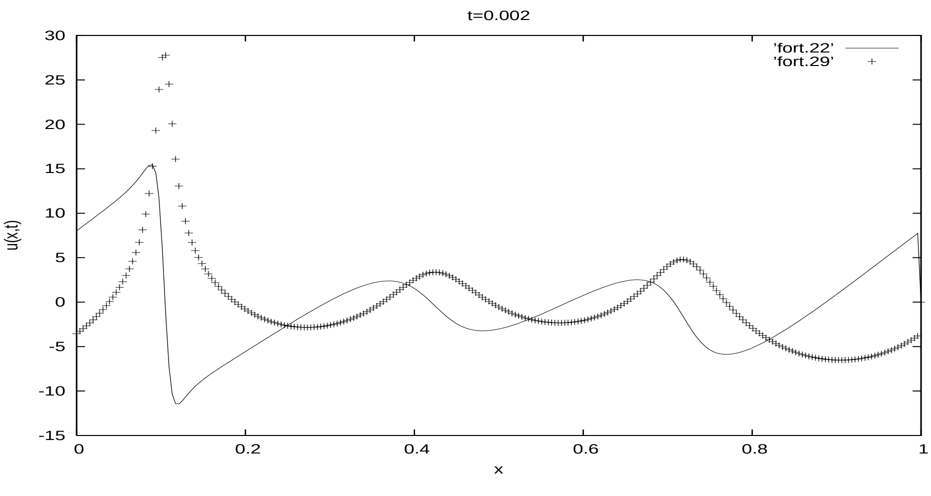

Figure 4.2: $u(x,t)$ vs. x

In figure 4.1, and 4.2 we plot the real and imaginary parts of u at times $t = 0.001, 0.0015$, and 0.002 by taking the mesh number $n = 256$, and time step $\Delta t =$

10^{-6}. The plots clearly show that both real and imaginary parts of the solution $u(x,t)$ quickly lose their shapes and their peaks become unbounded. Both spatial and temporal scales are refined again to $n = 512$, and $\Delta t = 10^{-7}$ and solutions at different times $t = 0.001, 0.002$, and 0.0025 are plotted (Figures 4.3 and 4.4). Similar results are observed and one may suspect a possible singularity in u. These results motivated me to study the blow up solutions of complex Burgers equation rigorously.

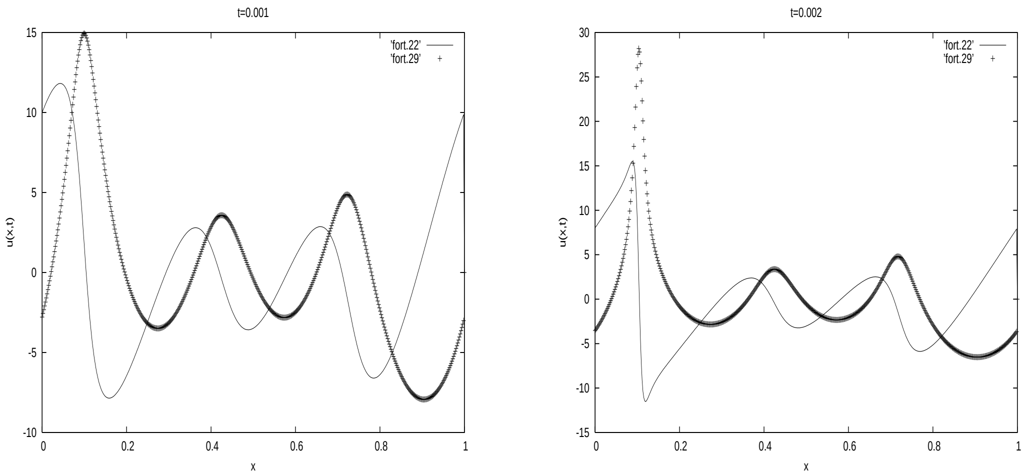

Figure 4.3: Left: $u(x,t)$ vs. x; Right: $u(x,t)$ vs. x

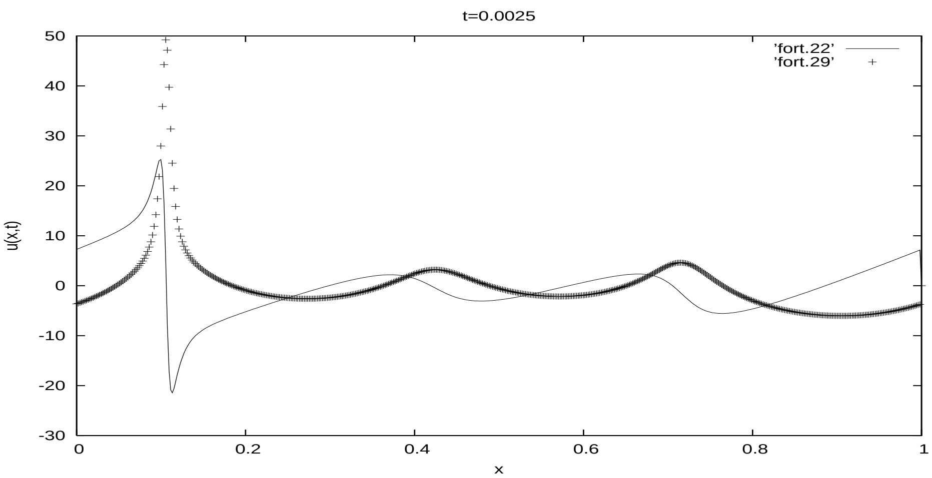

Figure 4.4: $u(x,t)$ vs. x

4.2.2 Local well-posedness

For any $s \in \mathbb{R}$, the homogeneous Sobolev space $\mathring{H}^s(\mathbb{T})$ and the inhomogeneous Sobolev space $H^s(\mathbb{T})$ are defined in the standard fashion. In particular, a function of the form

$$u(x,t) = \sum_{k=1}^{\infty} a_k\, e^{ikx}$$

is in $\mathring{H}^s(\mathbb{T})$ if

$$\|u\|_{\mathring{H}^s(\mathbb{T})} \equiv \sum_{k=1}^{\infty} k^{2s}\, |a_k|^2 < \infty,$$

and in $H^s(\mathbb{T})$ if

$$\|u\|_{H^s(\mathbb{T})} \equiv \sum_{k=1}^{\infty} (1+k^2)^s\, |a_k|^2 < \infty.$$

Clearly, $L^2(\mathbb{T})$ can be identified with $H^0(\mathbb{T})$.

This subsection establishes the following major result.

Theorem 4.1 *Consider (4.5) with $\gamma > \frac{1}{2}$. Let $s > \frac{1}{2}$. Assume $u_0 \in H^s(\mathbb{T})$ has the form*

$$u_0(x) = \sum_{k=1}^{\infty} a_{0k}\, e^{ikx}. \tag{4.6}$$

Then there exists $T = T(\|u_0\|_{H^s})$ such that (4.5) with the initial data u_0 has a unique solution $u \in C([0,T); H^s) \cap L^2([0,T); \mathring{H}^{s+\gamma})$ that assumes the form

$$u(x,t) = \sum_{k=1}^{\infty} a_k(t)\, e^{ikx}.$$

Proof. The existence of such a solution follows from the Galerkin approximation. Let $N \geq 1$ and denote by P_N the projection on the subspace $\{e^{ix}, e^{2ix}, \cdots, e^{iNx}\}$. Let

$$u^N(x,t) = \sum_{k=1}^{N} a_k^N(t)\, e^{ikx}$$

where $a_k^N(t)$ satisfies

$$\begin{aligned} &\frac{d}{dt} a_k^N(t) = 3ik \sum_{k_1+k_2=k} a_{k_1}^N(t)\, a_{k_2}^N(t) + i\alpha\, k^3\, a_k^N(t) - \nu k^{2\gamma}\, a_k^N(t), \\ &a_k^N(0) = a_{0k}^N \equiv a_{0k}. \end{aligned} \tag{4.7}$$

Here $1 \leq k_1 \leq N$ and $1 \leq k_2 \leq N$. From the theory of ordinary differential equations, one knows that (4.7) has a unique local solution $a_k^N(t)$ on $[0,T]$. Lets derive some *a priori* bounds for $u^N(x,t)$. Clearly, $u^N(x,t)$ solves

$$\partial_t u^N = 6P_N(u^N u_x^N) + \alpha u_{xxx}^N - \nu(-\Delta)^\gamma u^N, \quad u^N(x,0) = P_N\, u_0.$$

Lets now show that

$$\frac{d}{dt}\|u^N\|_{H^s}^2 + \nu\|u^N\|_{H^{s+\gamma}}^2 \leq C(\nu,s)\|u^N\|_{H^s}^{\frac{6\gamma-2}{2\gamma-1}}. \tag{4.8}$$

It follows from the equation

$$\frac{d}{dt}a_k^N(t) + \nu k^{2\gamma} a_k^N(t) - i\alpha k^3\, a_k^N(t) = 3ik \sum_{k_1+k_2=k} a_{k_1}^N(t)\, a_{k_2}^N(t)$$

that, after omitting the upper index N for notational convenience,

$$\frac{d}{dt}\sum_{k=1}^{N} k^{2s}\,|a_k(t)|^2 = -2\nu\sum_{k=1}^{N} k^{2(s+\gamma)}\,|a_k(t)|^2 - 6\sum_{k=1}^{N} k^{2s+1}\,\mathcal{I}\left(\bar{a}_k \sum_{k_1+k_2=k} a_{k_1}\, a_{k_2}\right),$$

where $\mathcal{I}$ denotes the imaginary part. To bound the nonlinear term on the right, denoted by J, first, notice that the summation over $k_1 + k_2 = k$ is less than twice the summation over $k_1 + k_2 = k$ with $k_1 \leq k_2$ and $2k_2 \geq k$. Thus,

$$\begin{aligned} J &\leq 6\sum_{k=1}^{N} k^{2s+1}\,|a_k| \sum_{k_1+k_2=k} |a_{k_1}|\,|a_{k_2}| \\ &\leq 12\sum_{k=1}^{N} k^{s+\frac{1}{2}}|a_k| \sum_{k/2\leq k_2\leq k} (2k_2)^{s+\frac{1}{2}}|a_{k_1}|\,|a_{k_2}|. \end{aligned}$$

Applying Hölder's inequality and Young's inequality for series,

$$\begin{aligned} J &\leq 12\left[\sum_{k=1}^{N} k^{2s+1}|a_k|^2\right]^{\frac{1}{2}} \left[\sum_{k=1}^{N}\left(\sum_{k/2\leq k_2\leq k}(2k_2)^{s+\frac{1}{2}}|a_{k_1}|\,|a_{k_2}|\right)^2\right]^{\frac{1}{2}} \\ &\leq 12\left[\sum_{k=1}^{N} k^{2s+1}|a_k|^2\right]^{\frac{1}{2}} \left[\sum_{k_2=1}^{N} k_2^{2s+1}|a_{k_2}|^2\right]^{\frac{1}{2}} \sum_{k_1=1}^{N}|a_{k_1}|. \\ &\leq 12\sum_{k=1}^{N} k^{2s+1}|a_k|^2 \left[\sum_{k_1=1}^{N}|k_1|^{2s}|a_{k_1}|\right]^{\frac{1}{2}} \left[\sum_{k_1=1}^{N} k_1^{-2s}\right]^{\frac{1}{2}} \\ &\leq C(s)\|u^N\|_{\mathring{H}^{s+\frac{1}{2}}}^2\,\|u^N\|_{H^s}. \end{aligned} \tag{4.9}$$

Thus, one obtains

$$\frac{d}{dt}\|u^N\|_{H^s}^2 + 2\nu\|u^N\|_{\mathring{H}^{s+\gamma}}^2 \leq C(s)\|u^N\|_{\mathring{H}^{s+\frac{1}{2}}}^2 \|u^N\|_{H^s}. \tag{4.10}$$

By Hölder's inequality

$$\|u^N\|_{\mathring{H}^{s+\frac{1}{2}}} \leq \|u^N\|_{\mathring{H}^{s+\gamma}}^{\frac{1}{2\gamma}} \|u^N\|_{H^s}^{1-\frac{1}{2\gamma}},$$

one has

$$J \leq C(s)\|u^N\|_{\mathring{H}^{s+\gamma}}^{\frac{1}{\gamma}} \|u^N\|_{H^s}^{3-\frac{1}{\gamma}} \leq \nu\|u^N\|_{\mathring{H}^{s+\gamma}}^2 + C(\nu, s)\|u^N\|_{H^s}^{\frac{6\gamma-2}{2\gamma-1}}. \tag{4.11}$$

(4.10) and (4.11) yield (4.8). With these bounds at our disposal, the existence of a solution u of the form (4.4) is then obtained as a limit of u^N as $N \to \infty$.

Lets now turn to the uniqueness. Assume (4.5) has two solutions u_1 and u_2 satisfying

$$u_1,\ u_2 \in C([0,T); H^s) \cap L^2([0,T); \mathring{H}^{s+\gamma}).$$

Then their difference $w = u_1 - u_2$ satisfies

$$w_t + \nu(-\Delta)^\gamma w + \alpha w_{xxx} = 6wu_{1x} + 6u_2 w_x.$$

Applying the same procedure as in the derivation of (4.10), one finds that, for $s > \frac{1}{2}$,

$$\frac{d}{dt}\|w\|_{H^s}^2 + 2\nu\|w\|_{\mathring{H}^{s+\gamma}}^2 \leq C(s)\,\|w\|_{H^s}^2(\|u_1\|_{\mathring{H}^1} + \|u_2\|_{\mathring{H}^1}).$$

The fact that $u_1, u_2 \in L^2([0,T); \mathring{H}^{s+\gamma})$ with $s+\gamma > 1$ and an application of Gronwall's inequality yields the uniqueness. This completes the proof of Theorem 4.1. ■

4.2.3 Boundedness of H^k-norm

In the case when $\gamma \geq 1$, one can actually show that no finite-time singularity is possible if we know that the L^2-norm is bounded *a priori*. In fact, the following theorem states that the L^2-norm controls all higher-order derivatives.

Theorem 4.2 *Let $T > 0$ and let u be a weak solution of (4.5) with $\gamma \geq 1$ on the time interval $[0, T]$. If we know a priori that $u \in L^\infty([0,T]; L^2) \cap L^2([0,T]; \mathring{H}^\gamma)$, namely*

$$M_0 \equiv \sup_{t\in[0,T]} \|u(\cdot,t)\|_{L^2}^2 + \nu \int_0^T \|\Lambda^\gamma u(\cdot,t)\|_{L^2}^2 \, dt < \infty, \tag{4.12}$$

then, for any integer $k > 0$,

$$M_k \equiv \sup_{t\in[0,T]} \|u^{(k)}(\cdot,t)\|_{L^2}^2 + \nu \int_0^T \|\Lambda^{k+\gamma} u(\cdot,t)\|_{L^2}^2 \, dt < \infty.$$

Proof. Lets start with the case $k = 1$. It is easy to verify that

$$\frac{d}{dt}\|u_x(\cdot,t)\|_{L^2}^2 + 2\kappa\|\Lambda^\gamma u_x\|_{L^2}^2 = I_1 + I_2, \tag{4.13}$$

where

$$\begin{aligned} I_1 &= 2\int |u_x|^2 \mathcal{R}(u_x)\, dx, \\ I_2 &= 2\int \mathcal{R}(u\,\overline{u}_x\, u_{xx})\, dx. \end{aligned}$$

Here $\mathcal{R}$ denotes the real part. By the Gagliardo-Nirenberg type equalities,

$$\begin{aligned} |I_1| &\leq 2\|u_x\|_{L^2}^2 \|u_x\|_{L^\infty} \\ &\leq C\|u\|_{L^2}^{\gamma_1} \|u_x\|_{L^2}^2 \|\Lambda^{1+\gamma} u\|_{L^2}^{1-\gamma_1}, \end{aligned}$$

$$\begin{aligned} |I_2| &\leq C\|u\|_{L^\infty} \|u_x\|_{L^2} \|u_{xx}\|_{L^2} \\ &\leq C\|u\|_{L^2}^{\frac{1}{2}} \|u_x\|_{L^2}^{\frac{3}{2}} \|u_{xx}\|_{L^2} \\ &\leq C\|u\|_{L^2}^{\frac{1}{2}+\gamma_2} \|u_x\|_{L^2}^{\frac{3}{2}} \|\Lambda^{1+\gamma} u\|_{L^2}^{1-\gamma_2} \end{aligned}$$

where

$$\gamma_1 = \frac{2\gamma - 1}{2\gamma + 2} \quad \text{and} \quad \gamma_2 = \frac{\gamma - 1}{\gamma + 1}.$$

By Young's inequality,

$$\begin{aligned} |I_1| &\leq \frac{\nu}{2}\|\Lambda^{1+\gamma} u\|_{L^2}^2 + C\nu^{-\frac{1-\gamma_1}{1+\gamma_1}} \|u\|_{L^2}^{\frac{2\gamma_1}{1+\gamma_1}} \|u_x\|_{L^2}^{\frac{4}{1+\gamma_1}}, \\ |I_2| &\leq \frac{\nu}{2}\|\Lambda^{1+\gamma} u\|_{L^2}^2 + C\nu^{-\frac{1-\gamma_2}{1+\gamma_2}} \|u\|_{L^2}^{\frac{1+2\gamma_2}{1+\gamma_2}} \|u_x\|_{L^2}^{\frac{3}{1+\gamma_2}}. \end{aligned}$$

Inserting these inequalities in (4.13) and integrating with respect to t yields

$$\sup_{t\in[0,T]} \|u_x(\cdot,t)\|_{L^2}^2 + \nu \int_0^T \|\Lambda^{1+\gamma} u\|_{L^2}^2\, dt$$
$$\le C(\nu) M_0^{\frac{\gamma_1}{1+\gamma_1}} \int_0^T \|u_x\|_{L^2}^{\frac{4}{1+\gamma_1}}\, dt + C(\nu)\, M_0^{\frac{1+2\gamma_2}{2+2\gamma_2}} \int_0^T \|u_x\|_{L^2}^{\frac{3}{1+\gamma_2}}\, dt,$$

where M_0 is specified in (4.12). By (4.12) and the Gagliardo-Nirenberg type inequality

$$\|u_x\|_{L^2} \le C \|u\|_{L^2}^{1-\frac{1}{\gamma}} \|\Lambda^\gamma u\|_{L^2}^{\frac{1}{\gamma}},$$

one obtains

$$\int_0^T \|u_x\|_{L^2}^{2\gamma}\, dt \le C M_0^\gamma.$$

Therefore,

$$\sup_{t\in[0,T]} \|u_x(\cdot,t)\|_{L^2}^2 + \nu \int_0^T \|\Lambda^{1+\gamma} u\|_{L^2}^2\, dt$$
$$\le C(\nu) M_0^{\frac{4\gamma^2+3\gamma-1}{4\gamma+1}} \sup_{t\in[0,T]} \|u_x(\cdot,t)\|_{L^2}^{\frac{-8\gamma^2+6\gamma+8}{4\gamma+1}}$$
$$+ C(\nu) M_0^{\frac{4\gamma^2+3\gamma-1}{4\gamma}} \sup_{t\in[0,T]} \|u_x(\cdot,t)\|_{L^2}^{\frac{-4\gamma^2+3\gamma+3}{2\gamma}} \tag{4.14}$$

When $\gamma > \frac{3}{4}$, $4\gamma^2 + \gamma - 3 > 0$ and consequently

$$\frac{-8\gamma^2 + 6\gamma + 8}{4\gamma + 1} < 2 \quad \text{and} \quad \frac{-4\gamma^2 + 3\gamma + 3}{2\gamma} < 2.$$

(4.14) then implies that

$$\sup_{t\in[0,T]} \|u_x(\cdot,t)\|_{L^2}^2 + \nu \int_0^T \|\Lambda^{1+\gamma} u\|_{L^2}^2\, dt \le M_1,$$

where M_1 is a constant depending on γ, ν and M_0 alone. L^2-bounds for higher-order derivatives can be obtained through iteration. This completes the proof of Theorem 4.2. ∎

4.2.4 Finite-time blow up

The following theorem details the finite-time blowup solution for the complex Burgers equation.

Theorem 4.3 *For every sufficiently large $T > 0$, there exists an initial data u_0 of the form*

$$u_0(x) = a\, e^{ix} \tag{4.15}$$

such that the corresponding solution u of (4.2) blows up at $t = T$ in the L^2-norm, namely

$$\|u(\cdot, T)\|_{L^2(\mathbb{T})} = \infty. \tag{4.16}$$

For u_0 given by (4.15), the local existence and uniqueness result of the subsection 4.2.2 asserts that the corresponding solution u can be written as

$$u(x,t) = \sum_{k=1}^{\infty} a_k(t)\, e^{ikx}$$

before it blows up. The idea is to choose large a such that

$$\|u(\cdot, T)\|_{L^2}^2 = \sum_{k=1}^{\infty} |a_k(T)|^2 = \infty.$$

Lets attempt to find an explicit representation for $a_k(t)$. It is easy to verify the following iterative formula

$$a_1(t) = a\, e^{-\nu t},\ a_k(t) = 3ik\, e^{-\nu k^2 t} \int_0^t e^{\nu k^2 \tau} \sum_{k_1+k_2=k} a_{k_1}(\tau)\, a_{k_2}(\tau)\, d\tau,\ k = 2, 3, \cdots. \tag{4.17}$$

To see the pattern in $a_k(t)$, lets calculate the first few of them explicitly:

$$a_1(t) = a\, e^{-\nu t}, \tag{4.18}$$

$$a_2(t) = -ia^2\, \nu^{-1} \left[-3e^{-2\nu t} + 3e^{-4\nu t}\right], \tag{4.19}$$

$$a_3(t) = -a^3\, \nu^{-2} \left[9e^{-3\nu t} - \frac{27}{2} e^{-5\nu t} + \frac{9}{2} e^{-9\nu t}\right], \tag{4.20}$$

$$a_4(t) = ia^4\, \nu^{-3} \left[-27e^{-4\nu t} + 54e^{-6\nu t} - \frac{27}{2} e^{-8\nu t} - 18e^{-10\nu t} + \frac{9}{2} e^{-16\nu t}\right], \tag{4.21}$$

$$a_5(t) = a^5\, \nu^{-4} \left[81e^{-5\nu t} - \frac{405}{2} e^{-7\nu t} + \frac{405}{4} e^{-9\nu t} + \frac{135}{2} e^{-11\nu t} - \frac{135}{4} e^{-13\nu t} - \frac{135}{8} e^{-17\nu t} + \frac{27}{8} e^{-25\nu t}\right],$$

$$a_6(t) = -ia^6\,\nu^{-5}\Big[-243e^{-6\nu t} + 729e^{-8\nu t} - \frac{2187}{4}e^{-10\nu t} - \frac{729}{4}e^{-12\nu t}$$
$$+243e^{-14\nu t} + \frac{81}{2}e^{-18\nu t} - \frac{243}{8}e^{-20\nu t} - \frac{243}{20}e^{-26\nu t} + \frac{81}{40}e^{-36\nu t}\Big].$$

The following lemma summarizes the pattern exhibited by $a_k(t)$'s.

Lemma 4.1 *For any $t > 0$,*

$$a_1(t) = a\,b_1(t), \quad a_2(t) = ia^2\,b_2(t), \quad a_3(t) = -a^3\,b_3(t), \quad a_4(t) = -ia^4\,b_4(t) \tag{4.22}$$

and more generally, for $k = 4n + j$ with $n = 0, 1, 2, \cdots$ and $j = 1, 2, 3, 4$,

$$a_k(t) = a_{4n+j}(t) = i^{j-1}\,a^{4n+j}\,b_{4n+j}(t), \tag{4.23}$$

where $b_{4n+j}(t) > 0$ for any $t > 0$.

Remark 4.1 *A special consequence of this lemma is that all terms in the summation in (4.17) have the same sign and thus*

$$|a_k(t)| = 3ke^{\nu k^2 t}\int_0^t e^{\nu k^2\tau}\sum_{k_1+k_2=k}|a_{k_1}(\tau)|\,|a_{k_2}(\tau)|\,d\tau. \tag{4.24}$$

Proof of Lemma 4.1. The identity (4.23) can be shown through induction. For $n = 0$, (4.23) is just (4.22). By (4.17), $a_1(t) = a\,e^{-\nu t}$ and

$$a_2(t) = 6i\,a^2e^{-4\nu t}\int_0^t e^{4\nu\tau}\,b_1^2(\tau)\,d\tau = ia^2b_2(t),$$

where $b_2(t) = 6e^{-4\nu t}\int_0^t e^{4\nu\tau}\,b_1^2(\tau)\,d\tau > 0$. Similarly, $a_3(t) = -a^3\,b_3(t)$ and $a_4(t) = -ia^4\,b_4(t)$ for some $b_3(t) > 0$ and $b_4(t) > 0$.

Lets now consider the general case. Without loss of generality, lets prove (4.23) with $k = 4n + 1$. Assume (4.23) is true for all $k < 4n + 1$. By (4.17),

$$a_k(t) = 3ik\,e^{-\nu k^2\,t}\int_0^t e^{\nu k^2\,\tau}\sum_{k_1+k_2=k}a_{k_1}(\tau)\,a_{k_2}(\tau)\,d\tau.$$

Noticing that $a_{k_1}(\tau)\,a_{k_2}(\tau)$ with $k_1 + k_2 = 4n + 1$ assumes two forms

$$a_{4n_1}(\tau)\,a_{4n_2+1}(\tau) \quad \text{and} \quad a_{4n_1+2}(\tau)\,a_{4n_2-1}(\tau)$$

where $n_1 \geq 0$, $n_2 \geq 0$ and $n_1 + n_2 = n$, one concludes by the inductive assumptions that $a_{k_1}(\tau)\, a_{k_2}(\tau)$ must be of the form $-i\, a^k b_{k1,k_2}(\tau)$ for some positive function $b_{k1,k_2}(\tau) > 0$. Therefore,

$$a_k(t) = a_{4n+1}(t) = a^k\, b_k(t)$$

with

$$b_k(t) = 3k\, e^{-\nu k^2\, t} \int_0^t e^{\nu k^2\, \tau} \sum_{k_1+k_2=k} b_{k1,k_2}(\tau)\, d\tau > 0 \quad \text{for any } t > 0.$$

This completes the proof of Lemma 4.1.

Proof of Theorem 4.3. Without loss of generality, set $\nu = 1$. Assume

$$T \geq T_0 \equiv \sum_{k=2}^{\infty} \frac{1}{k^2} \ln \frac{3k-3}{2k-3} \tag{4.25}$$

and choose a such that

$$A \equiv a\, e^{-T} > 1$$

One proves by induction that

$$|a_k(T)| \geq A^k \quad \text{for} \quad k = 1, 2, 3, \cdots \tag{4.26}$$

which, in particular, yields (4.16). Obviously,

$$|a_1(T)| = a\, e^{-T} = A \geq 1.$$

To prove (4.26) for $k \geq 2$, recall (4.24), namely

$$|a_k(t)| = 3ke^{-k^2 t} \int_0^t e^{k^2 \tau} \sum_{k_1+k_2=k} |a_{k_1}(\tau)|\, |a_{k_2}(\tau)|\, d\tau.$$

Therefore, for $t \geq t_2 \equiv \frac{1}{4} \ln 3$,

$$|a_2(t)| = 6e^{-4t} \int_0^t e^{4\tau} a_1^2(\tau)\, d\tau = \frac{3}{2} A^2 (1 - e^{-4t}) \geq A^2.$$

For $k = 3$, if $t \geq t_3 \equiv t_2 + \frac{1}{9} \ln 2$,

$$\begin{aligned} |a_3(t)| &= 9e^{-9t} \int_0^t e^{9\tau} 2 |a_1(\tau)|\, |a_2(\tau)|\, d\tau \\ &\geq 9e^{-9t} \int_{t_2}^t e^{9\tau} 2 |a_1(\tau)\, a_2(\tau)|\, d\tau \\ &\geq 2A^3\, (1 - e^{-9(t-t_2)}) \geq A^3. \end{aligned}$$

More generally, for any $t \geq t_k = t_{k-1} + \frac{1}{k^2} \ln \frac{3k-3}{2k-3}$,

$$\begin{aligned}
|a_k(t)| &= 3ke^{-k^2 t} \int_0^t e^{k^2\tau} \left(|a_1(\tau)|\,|a_{k-1}(\tau)| + |a_2(\tau)|\,|a_{k-2}(\tau)|\right. \\
&\quad \left. + \cdots + |a_{k-2}(\tau)||a_2(\tau)| + |a_{k-1}(\tau)|\,|a_1(\tau)|\right) d\tau \\
&\geq 3ke^{-k^2 t} \int_{t_{k-1}}^t e^{k^2\tau} \left(|a_1(\tau)|\,|a_{k-1}(\tau)| + |a_2(\tau)|\,|a_{k-2}(\tau)|\right. \\
&\quad \left. + \cdots + |a_{k-2}(\tau)||a_2(\tau)| + |a_{k-1}(\tau)|\,|a_1(\tau)|\right) d\tau \\
&\geq \frac{3k(k-1)}{k^2}(1 - e^{-\nu\, k^2(t-t_{k-1})})A^k \geq A^k.
\end{aligned}$$

If $T \geq T_0$ as defined in (4.25), then $t_k < T$ for any integer $k \geq 1$, and thus

$$|a_k(T)| \geq A^k.$$

This completes the proof of Theorem 4.3.

Lets state and prove a few specific properties for $a_k(t)$.

Proposition 4.1 *Assume u_0 is given by (4.6). For each $k \geq 1$, $a_k(t)$ is of the form*

$$a_k(t) = \sum_{m=k}^{k^2} \alpha_{k,m} e^{-m\nu t}, \tag{4.27}$$

where the complex-valued coefficients $\alpha_{k,m}$ satisfy

$$\sum_{m=k}^{k^2} \alpha_{k,m} = 0 \quad \text{for } k \geq 2, \tag{4.28}$$

$$\alpha_{k,m} = \frac{3ik}{k^2 - m} \sum_{k_1+k_2=k} \sum_{m_1+m_2=m} \alpha_{k_1,m_1}\alpha_{k_2,m_2} \quad \text{for } k \leq m < k^2. \tag{4.29}$$

The indices k_1, k_2, m_1 and m_2 in the summation above obey

$$1 \leq k_1 \leq k-1, \; 1 \leq k_2 \leq k-1, \; k_1 \leq m_1 \leq k_1^2 \quad \text{and} \quad k_2 \leq m_2 \leq k_2^2.$$

Proof. The case in (4.28) is a consequence of the fact that $a_k(0) = 0$ for $k \geq 2$. (4.27) follows from a simple induction. Obviously, $a_1(t) = a\, e^{-\nu t}$. Fix k and assume (4.27) is valid for all integers up to k. Then, for $k_1 \geq 1, k_2 \geq 1, k_1 + k_2 = k+1, k_1 \leq m_1 \leq k_1^2$

and $k_2 \le m_2 \le k_2^2$,

$$\begin{aligned} a_{k+1}(t) &= 3i(k+1) \sum_{k_1+k_2=k+1} \sum_{m_1,m_2} \alpha_{k_1,m_1}\alpha_{k_2,m_2} e^{-\nu(k+1)^2\,t} \int_0^t e^{\nu((k+1)^2-(m_1+m_2))\,\tau}\, d\tau \\ &= \sum_{k_1+k_2=k+1} \sum_{m_1,m_2} \frac{3i(k+1)\,\alpha_{k_1,m_1}\alpha_{k_2,m_2}}{\nu((k+1)^2-(m_1+m_2))} \left(e^{-\nu(m_1+m_2)t} - e^{-\nu(k+1)^2 t}\right). \end{aligned}$$

Since $m_1+m_2 \le k_1^2+k_2^2 \le (k_1+k_2)^2 = (k+1)^2$, this proves (4.27) with (4.29). ∎

Proposition 4.2 *Assume that u_0 is given by (4.6).*

1) Let $k \ge 1$ be an integer. Then

$$\alpha_{k,k} = \left(\frac{3i}{\nu}\right)^{k-1} a^k \quad \textit{and} \quad \alpha_{k,k+2} = -\frac{k}{2}\,\alpha_{k,k}; \tag{4.30}$$

2) Let $k \ge 1$ be an integer. Then, for $n = 1, 3, 5, \cdots$,

$$\alpha_{k,k+n} = 0;$$

3) Let $k \ge 1$ be an integer and let $k^2 > m > U(k) \equiv k^2 - 2k + 2$. Then

$$\alpha_{k,m} = 0. \tag{4.31}$$

Proof. Letting $m_1 = k_1$ and $m_2 = k_2$ in (4.29), we find

$$\alpha_{k,k} = \sum_{k_1+k_2=k} \alpha_{k_1,k_1}\alpha_{k_2,k_2} \frac{3ik}{\nu(k^2-k)} = \frac{3i}{\nu(k-1)} \sum_{k_1+k_2=k} \alpha_{k_1,k_1}\alpha_{k-k_1,k-k_1}.$$

A simple induction allows us to obtain the expression for $\alpha_{k,k}$. To show $\alpha_{k,k+2} = -\frac{k}{2}\,\alpha_{k,k}$, we set $m = k+2$ in (4.29) to obtain

$$\begin{aligned} \alpha_{k,k+2} &= \frac{3ik}{\nu(k^2-k-2)} (\alpha_{1,1}\,\alpha_{k-1,k+1} + \alpha_{2,2}\,\alpha_{k-2,k} + \alpha_{2,4}\,\alpha_{k-2,k-2} \\ &\quad + \cdots + \alpha_{k-2,k-2}\,\alpha_{2,4} + \alpha_{k-2,k}\,\alpha_{2,2} + \alpha_{1,1}\,\alpha_{k-1,k+1}). \end{aligned} \tag{4.32}$$

Inserting the inductive assumptions such as

$$\alpha_{k-1,k+1} = -\frac{k-1}{2}\,\alpha_{k-1,k-1}, \quad \alpha_{k-2,k} = -\frac{k-2}{2}\alpha_{k-2,k-2}, \quad \alpha_{2,4} = -\alpha_{2,2}$$

in (4.32), one obtains

$$\begin{aligned}\alpha_{k,k+2} &= \frac{3ik}{\nu(k^2-k-2)}\left[-\frac{k}{2}\sum_{k_1=1}^{k-1}\alpha_{k_1,k_1}\,\alpha_{k-k_1,k-k_1}+\alpha_{1,1}\,\alpha_{k-1,k-1}\right]\\ &= -\frac{k}{2}\frac{k^2-k}{k^2-k-2}\frac{3ik}{\nu(k^2-k)}\sum_{k_1=1}^{k-1}\alpha_{k_1,k_1}\,\alpha_{k-k_1,k-k_1}\\ &\quad+\frac{3ik}{\nu(k^2-k-2)}\alpha_{1,1}\,\alpha_{k-1,k-1}\\ &= -\frac{k}{2}\frac{k^2-k}{k^2-k-2}\alpha_{k,k}-\frac{k}{2}\frac{-2}{k^2-k-2}\alpha_{k,k}=-\frac{k}{2}\,\alpha_{k,k}.\end{aligned}$$

To show $\alpha_{k,k+1}=0$, we set $m=k+1$ to obtain

$$\alpha_{k,k+1}=\frac{3ik}{\nu(k^2-(k+1))}\left(\alpha_{1,1}\,\alpha_{k-1,k}+\alpha_{2,2}\,\alpha_{k-2,k-1}+\cdots+\alpha_{k-1,k}\,\alpha_{1,1}\right),$$

which equals zero after inserting the inductive assumptions.

To prove (4.31), it suffices to notice in (4.29) that the second summation is over $m_1+m_2=m$ with $k_1\le m_1\le k_1^2$ and $k_2\le m_2\le k_2^2$. Thus, $m=m_1+m_2\le k_1^2+k_2^2=(k_1+k_2)^2-2k_1k_2\le k^2-2(k-1)$ and $\alpha_{k,m}$ with $U(k)<m<k^2$ is equal to zero. This completes the proof of Proposition 4.2. ■

4.3 Complex KdV-Burgers equation

Lets consider the initial-value problem for the complex KdV-Burgers equation

$$\begin{cases} u_t-6uu_x+\alpha u_{xxx}-\nu u_{xx}=0, & x\in\mathbb{T},\ t>0,\\ u(x,0)=u_0(x), & x\in\mathbb{T},\end{cases}\tag{4.33}$$

and study the global regularity of its series-type solutions of the two forms given in the following two subsections.

4.3.1 Special series-type solutions

Lets consider the solutions of the equation (4.33) of the type

$$u(x,t)=\sum_{k=1}^{\infty}a_k(t)e^{ikx}.\tag{4.34}$$

Suppose that the initial data u_0 is of the form

$$u_0(x) = \sum_{k=1}^{\infty} a_{0k}\, e^{ikx} \tag{4.35}$$

and is in H^s with $s > \frac{1}{2}$. According to Theorem 4.1, (4.33) has a unique local solution $u \in C([0,T); H^s)$ of the form (4.34) for some $T > 0$. To study the global regularity of (4.34), lets explore the structure of $a_k(t)$ and obtain the following two propositions.

Proposition 4.3 *If (4.34) solves (4.33), then $a_k(t)$ can be written as*

$$a_k(t) = \sum_{k\le h\le k^2,\, k\le l\le k^3} a_{k,\,h,\,l}\, e^{-(\nu h - \alpha i l)t} \tag{4.36}$$

where $a_{k,\,h,\,l}$ consists of a finite number of terms of the form

$$C(\alpha,\nu,k,h,l,j_1,\cdots,j_k)\, a_{01}^{j_1}\, a_{02}^{j_2}\, \cdots\, a_{0k}^{j_k} \tag{4.37}$$

with j_1, j_2 ,$\cdots$, j_k non-negative integers satisfying

$$j_1 + 2j_2 + \cdots + kj_k = k. \tag{4.38}$$

Proof. If (4.34) solves (4.33) , then $a_k(t)$ solves the ordinary differential equation

$$\frac{d}{dt}a_k(t) + (\nu k^2 - \alpha i k^3)a_k(t) - 3ik \sum_{k_1+k_2=k} a_{k_1}(t)\, a_{k_2}(t) = 0.$$

The equivalent integral form is given by

$$a_k(t) = e^{-(\nu k^2 - \alpha i k^3)t}\left[a_{0k} + 3ik\int_0^t e^{(\nu k^2-\alpha i k^3)\tau} \sum_{k_1+k_2=k} a_{k_1}(\tau)\, a_{k_2}(\tau)\, d\tau\right]. \tag{4.39}$$

It is easy to show through an inductive process that a_k is of the form (4.36). In addition, for $k \le h < k^2$ and $k \le l < k^3$, the term in (4.37) with fixed j_1, j_2, $\cdots$, j_k satisfying

$$j_1 + 2j_2 + \cdots + kj_k = k$$

can be expressed as

$$\begin{aligned} &C(\alpha,\nu,k,h,l,j_1,\cdots,j_k)\, a_{01}^{j_1}\, a_{02}^{j_2}\, \cdots\, a_{0k}^{j_k} \\ &\quad= \frac{3ik}{\nu(k^2-h) - i\alpha(k^3-l)} \sum_{m_1+n_1=j_1} \cdots \sum_{m_k+n_k=j_k} C(\alpha,\nu,k_1,h_1,l_1,m_1,\cdots,m_{k_1}) \\ &\quad\quad \times C(\alpha,\nu,k_2,h_2,l_2,n_1,\cdots,n_{k_2})\, a_{01}^{m_1+n_1}\, a_{02}^{m_2+n_2}\, \cdots\, a_{0k}^{m_k+n_k} \end{aligned} \tag{4.40}$$

where the indices satisfy

$$1 \le k_1 \le k, \quad 1 \le k_2 \le k, \quad k_1 + k_2 = k,$$
$$k_1 \le h_1 \le k_1^2, \quad k_2 \le h_2 \le k_2^2, \quad h_1 + h_2 = h,$$
$$k_1 \le l_1 \le k_1^3, \quad k_2 \le l_2 \le k_2^3, \quad l_1 + l_2 = l,$$
$$m_1 + n_1 = j_1, \quad m_2 + n_2 = j_2, \quad \cdots, \quad m_k + n_k = j_k.$$
$$(m_r = 0 \quad \text{for } r > k_1 \text{ and } \quad n_r = 0 \quad \text{for } r > k_2)$$
$$m_1 + 2m_2 + \cdots + k_1 m_{k_1} = k_1, \quad n_1 + 2n_2 + \cdots + k_2 n_{k_2} = k_2.$$

When $h = k^2$ and $l = k^3$,

$$C(\alpha, \nu, k, k^2, k^3, j_1, j_2, \cdots, j_k) = \begin{cases} 1 \quad \text{for} \quad (j_1, j_2, \cdots, j_k) = (0, 0, \cdots, 1), \\ -C(\alpha, \nu, k, h, l, j_1, j_2, \cdots, j_k) \,\text{otherwise}, \end{cases} \tag{4.41}$$

for some $h < k^2$ and $l < k^3$. To illustrate these formulas, lets list a_k for $k = 1, 2, 3$,

$$\begin{aligned} a_1(t) &= a_{01}\, e^{-(\nu - i\alpha)t}, \\ a_2(t) &= \frac{6i}{2\nu - 6\alpha i} a_{01}^2\, e^{-(2\nu - 2\alpha i)t} + \left[a_{02} - \frac{6i}{2\nu - 6\alpha i} a_{01}^2 \right] e^{-(4\nu - 8i\alpha)t}, \\ a_3(t) &= \frac{108 a_{01}^3}{(2\nu - 6\alpha i)(6\nu - 24\alpha i)} e^{(-3\nu + 3\alpha i)t} \\ &\quad + \left[\frac{18 i a_{01} a_{02}}{4\nu - 18\alpha i} - \frac{108 a_{01}^3}{(2\nu - 6\alpha i)(4\nu - 18\alpha i)} \right] e^{(-5\nu + 9i\alpha)t} \\ &\quad + \left[a_{03} - \frac{18 i a_{01} a_{02}}{4\nu - 18\alpha i} + \frac{108 a_{01}^3}{(2\nu - 6\alpha i)(4\nu - 18\alpha i)} - \frac{108 a_{01}^3}{(2\nu - 6\alpha i)(6\nu - 24\alpha i)} \right] \\ &\qquad \times e^{(-9\nu + 27\alpha i)t}. \end{aligned}$$

■

Proposition 4.4 *Let $k \ge 1$ be an integer. Let $U(k) = k^2 - 2k + 2$ and $V(k) = k^3 - 3k^2 + 3k$. The coefficients $a_{k,h,l}$ in (4.36) have the following properties*

(1) For $k \le h < k^2$ and $k \le l < k^3$,

$$a_{k,h,l} = \frac{3ik}{\nu(k^2 - h) - i\alpha(k^3 - l)} \sum_{k_1 + k_2 = k} \sum_{h_1 + h_2 = h} \sum_{1_1 + 1_2 = l} \alpha_{k_1,h_1,l_1} \alpha_{k_2,h_2,l_2} \tag{4.42}$$

(2) For $h = k^2$ and $l = k^3$,

$$a_{k,k^2,k^3} = a_k(0) - \sum_{k \le h < k^2} \sum_{k \le l < k^3} a_{k,h,l} \tag{4.43}$$

(3) For $U(k) < h < k^2$ or $V(k) < l < k^3$,

$$a_{k,h,l} = 0. \tag{4.44}$$

Proof. (4.42) follows from a simple induction. (4.43) is obtained by setting $t = 0$ in (4.36). To show (4.44), one notices that the second summation in (4.42) is over $h_1 + h_2 = h$ with $k_1 \le h_1 \le k_1^2$ and $k_2 \le h_2 \le k_2^2$ while the third summation is over $l_1 + l_2 = l$ with $k_1 \le l_1 \le k_1^3$ and $k_2 \le l_2 \le k_2^3$. Thus,

$$h = h_1 + h_2 \le k_1^2 + k_2^2 = k^2 - 2k_1\, k_2 \le k^2 - 2(k-1) = U(k),$$

$$l = l_1 + l_2 \le k_1^3 + k_2^3 = k^3 - 3k\, k_1\, k_2 \le k^3 - 3k(k-1) = V(k).$$

That means, $a_{k,h,l} = 0$ if $U(k) < h < k^2$ and $V(k) < l < k^3$. ■

Theorem 4.4 *Consider (4.33) with $\nu > 0$. Assume $u_0 \in H^s(\mathbb{T})$ with $s > \frac{1}{2}$ can be represented in the form (4.35) with*

$$|a_{0k}| \le 1, \quad k = 1, 2, \cdots \tag{4.45}$$

If there is the uniform bound

$$|C(\alpha, \nu, k, h, l, j_1, \cdots, j_k)| \le C_0(\alpha, \nu) \tag{4.46}$$

for all $k \ge 1$, $k \le h < k^2$, $k \le l < k^3$ and $(j_1, j_2, \cdots, j_k)$ satisfying (4.38), then (4.33) has a unique global solution u given by (4.34). In addition, for any $s \ge 0$, there are $T_0 > 0$ and $\delta > 0$ such that for any $t \ge T_0$,

$$\|u(\cdot, t)\|_{H^s} < \frac{C(\alpha, \nu, s)}{1 - e^{-\nu t}}\, e^{-\delta \nu k t} \tag{4.47}$$

where C is a constant depending only on α, ν and s.

The proof of Theorem 4.4 involves a classical problem in number theory, namely the number of integer solutions $(j_1, j_2, \cdots, j_k)$ to the equation defined in (4.38) for a given positive integer k. This problem is not as simple as it may look like. An upper bound and asymptotic approximation for the number of non-negative solutions are given by G.H. Hardy and S. Ramanujan [17], as stated in the following lemma.

Lemma 4.2 *Let $k > 0$ be an integer and let N_k denote the number of nonnegative solutions to the equation*

$$j_1 + 2j_2 + \cdots + kj_k = k.$$

Then, for some constant C_1,

$$N_k < \frac{C_1}{k}\, e^{2\sqrt{2\,k}}.$$

In addition, N_k has the following asymptotic behavior:

$$N_k \quad \sim \quad \frac{1}{4\sqrt{3}k} e^{\pi\sqrt{\frac{2k}{3}}}, \quad \text{as } k \to \infty.$$

Proof of Theorem 4.4. Applying (4.45) and (4.46), one obtains the following bound for $a_{k,\,h,\,l}$ in (4.36)

$$|a_{k,\,h,\,l}| \le C_0(\alpha, \nu)\, N_k \le \frac{C_2}{k}\, e^{2\sqrt{2\,k}},$$

where $C_2 = C_0 C_1$ and Lemma 4.2 has been used. Therefore

$$\begin{aligned} |a_k(t)| \ &\le \ \sum_{k \le h \le k^2} \sum_{k \le l \le k^3} |a_{k,h,l}|\, e^{-\nu h t} \\ &\le \ C_2\,(k^2 - 1)\, e^{2\sqrt{2}\,\sqrt{k}}\, \frac{e^{-\nu k t}}{1 - e^{-\nu t}}. \end{aligned} \tag{4.48}$$

For any fixed $t > 0$, we can choose $K = K(\nu)$ and $0 < M = M(\nu) < 1$ such that

$$|a_k(t)| \le \frac{C_2}{1 - e^{-\nu t}}\, M^k \quad \text{for } k \ge K.$$

Therefore, u represented by (4.34) converges for any $t > 0$. In addition, $u(\cdot, t) \in H^s$ for any $s \ge 0$. To see the exponential decay of $\|u(\cdot, t)\|_{H^s}$ for large time, choose

$T_0 = T_0(\nu, s)$ such that for any $t \geq T_0$ and $k \geq 1$

$$(1+k^2)^s |a_k(t)|^2 \leq C_2\, M_1^k \, \frac{e^{-\delta\,\nu k t}}{1-e^{-\nu t}},$$

where $M_1 > 0$ and $\delta > 0$ are some constants. This bound then implies (4.47). This completes the proof of Theorem 4.4.

Finally, a special case is considered for which (4.34) is global in time.

Theorem 4.5 *Consider (4.33) with ν and α satisfying $\nu^2 + 4\alpha^2 \geq 9$. If*

$$u_0(x) = a_{01}\, e^{ix} \quad \textit{with} \quad |a_{01}| < 1,$$

then (4.33) has a unique global solution, which can be represented by (4.34). In addition, for any $s \geq 0$, $u(\cdot, t) \in H^s$ for all $t \geq 0$.

Proof. One proves by induction that, for any $t > 0$,

$$|a_k(t)| \leq |a_{01}|^k, \quad k = 1, 2, \cdots. \tag{4.49}$$

Obviously, $|a_1(t)| \leq |a_{01}|$. To prove (4.49) for $k \geq 2$, recall (4.39), namely

$$a_k(t) = 3ik\, e^{-(\nu k^2 - \alpha i k^3)t} \int_0^t e^{(\nu k^2 - \alpha i k^3)\tau} \sum_{k_1+k_2=k} a_{k_1}(\tau)\, a_{k_2}(\tau)\, d\tau.$$

Since $\nu^2 + 4\alpha^2 \geq 9$, one has

$$|a_2(t)| \leq \left|\frac{3}{2\nu - 4\alpha i}\right| |a_{01}|^2 \left(1 - e^{-(4\nu - 8\alpha i)t}\right) \leq |a_{01}|^2$$

and more generally,

$$|a_k(t)| \leq \left|\frac{3(k-1)}{\nu\, k - \alpha i\, k^2}\right| |a_{01}|^k \left(1 - e^{-(\nu\, k^2 - \alpha i\, k^3)t}\right) \leq |a_{01}|^k.$$

It is then clear that (4.34) converges in H^s with $s \geq 0$ for any $t \geq 0$. This completes the proof of Theorem 4.5. ∎

4.3.2 Fourier series-type solutions

This subsection is devoted to full series solutions to the initial-value problem for the complex KdV-Burgers equation (4.33). Suppose that the initial data u_0 is of the form

$$u_0(x) = \sum_{k\neq 0} \frac{c_0(k)}{|k|^\gamma}\, e^{ikx} \tag{4.50}$$

and write its corresponding solution $u = u(x,t)$ as the series

$$u(x,t) = \sum_{k\neq 0} \widehat{u}(k,t)\, e^{ikx}.$$

Then the coefficient $\widehat{u}(k,t)$ satisfies

$$\widehat{u}(k,t) = e^{(-\nu k^2 + i\alpha k^3)t}\widehat{u}_0(k) + 3ik\int_0^t e^{(-\nu k^2+i\alpha k^3)(t-s)} \sum_{j\neq 0, j\neq k} \widehat{u}(j,s)\,\widehat{u}(k-j,s)\,ds$$

and, if $\widehat{u}(k,t) = \dfrac{c(k,t)}{|k|^\gamma}$, then

$$\begin{aligned} c(k,t) \;&=\; e^{(-\nu k^2+i\alpha k^3)t}c_0(k) \\ &\quad +3i\,k|k|^\gamma \int_0^t e^{(-\nu k^2+i\alpha k^3)(t-s)} \sum_{j\neq 0, j\neq k} \frac{c(j,s)}{|j|^\gamma}\,\frac{c(k-j,s)}{|k-j|^\gamma}\,ds. \end{aligned} \tag{4.51}$$

The goal here is to rigorously establish the existence and uniqueness of such solutions and to understand if they solve (4.33) in the classical sense. Lets first define the functional framework.

For $\gamma \geq 0$ and $0 < T \leq \infty$, define $X_{\gamma,T}$ to be the functional space of periodic functions $g = g(x,t)$ on $\mathbb{T}\times[0,T]$ whose fourier coefficient $\widehat{g}(k,t)$ satisfies

$$\widehat{g}(k,t) = \frac{c(k,t)}{|k|^\gamma} \quad \text{for } k\in\mathbb{Z}\setminus\{0\}$$

with

$$\|c\| \equiv \sup_{0\leq t\leq T}\ \sup_{k\in\mathbb{Z}\setminus\{0\}} |c(k,t)| < \infty.$$

It is easily verified that $X_{\gamma,T}$ equipped with the norm

$$\|g\|_{X_{\gamma,T}} = \|c\|$$

is a Banach space. When $T=\infty$, we write X_γ for $X_{\gamma,\infty}$.

With the functional setting at our disposal, lets define the series solution.

Definition 4.1 *Let $\gamma > 1$ and $T > 0$. Assume $u \in X_{\gamma,T}$ has the form*

$$u(x,t) = \sum_{k\neq 0} \widehat{u}(k,t)\, e^{ikx} \quad \textit{with} \quad \widehat{u}(k,t) = \frac{c(k,t)}{|k|^\gamma}.$$

Then u is called a series solution of (4.33) if $c(k,0) = c_0(k)$ and $c(k,t)$ satisfies (4.51) for $t \in [0,T]$.

In this subsection, three theorems are given to guarantee the existence and uniqueness of the series solutions and to show that they are indeed the classical solutions. To prove these theorems, lets start with a lemma.

Lemma 4.3 *For any $\gamma > 1$ and any integer $k \neq 0$,*

$$\sum_{j\neq 0, j\neq k} \frac{1}{|j|^\gamma |k-j|^\gamma} \le \frac{C(\gamma)}{|k|^\gamma},$$

where $C(\gamma)$ is a constant independent of k.

Proof. Without loss of generality, let $k > 0$ and split the sum into three parts,

$$\sum_{j\neq 0, j\neq k} \frac{1}{|j|^\gamma |k-j|^\gamma} = \sum_{j=-\infty}^{-1} \frac{1}{|j|^\gamma |k-j|^\gamma} + \sum_{j=1}^{k-1} \frac{1}{|j|^\gamma |k-j|^\gamma} + \sum_{j=k+1}^{\infty} \frac{1}{|j|^\gamma |k-j|^\gamma}.$$

Obviously, for $\gamma > 1$,

$$\sum_{j=-\infty}^{-1} \frac{1}{|j|^\gamma |k-j|^\gamma} \le \frac{C(\gamma)}{|k|^\gamma} \quad \text{and} \quad \sum_{j=k+1}^{\infty} \frac{1}{|j|^\gamma |k-j|^\gamma} \le \frac{C(\gamma)}{|k|^\gamma}.$$

The middle part can be bounded as follows.

$$\sum_{j=1}^{k-1} \frac{1}{|j|^\gamma |k-j|^\gamma} \le 2 \sum_{1\le j\le \left[\frac{k}{2}\right]} \frac{1}{|j|^\gamma |k-j|^\gamma} = \frac{2}{|k|^\gamma} \sum_{1\le j\le \left[\frac{k}{2}\right]} \frac{1}{|j|^\gamma |1-\frac{j}{k}|^\gamma} \le \frac{C(\gamma)}{|k|^\gamma}.$$

The last summation is bounded uniformly in k via a comparison with a suitable integral. This completes the proof of Lemma 4.3. ■

The following theorem establishes the existence and uniqueness of the series solutions.

Theorem 4.6 *Consider the initial-value problem for the complex KdV-Burgers equation (4.33). Let* $\gamma > 1$ *and assume* $u_0 \in X_\gamma$ *has the form (4.50). If* $R_0 \equiv \|u_0\|_{X_\gamma}$ *and* $T > 0$ *satisfy*

$$C(\gamma)\,\sqrt{T}\,R_0 < \sqrt{\nu}$$

for some suitable constant $C = C(\gamma)$*, then (4.33) has a unique series solution* $u \in X_{\gamma,T}$*. In addition,*

$$\|u\|_{X_{\gamma,T}} < 2R_0.$$

Proof. The approach of the proof is the method of successive approximation. For each $k \in \mathbb{Z} \setminus \{0\}$, define for $n = 1, 2, \cdots$,

$$\begin{aligned} c^{(0)}(k,t) &= e^{(-\nu k^2 + i\alpha k^3)t} c_0(k), \\ c^{(n)}(k,t) &= e^{(-\nu k^2 + i\alpha k^3)t} c_0(k) \\ &\quad + 3i\,k|k|^\gamma \int_0^t e^{(-\nu k^2 + i\alpha k^3)(t-s)} \sum_{j\neq 0, j\neq k} \frac{c^{(n-1)}(j,s)}{|j|^\gamma}\,\frac{c^{(n-1)}(k-j,s)}{|k-j|^\gamma}\,ds. \end{aligned}$$

It suffices to show, for some $\theta \in (0,1)$,

$$\|c^{(n)}\| \le 2R_0, \tag{4.52}$$

$$\|c^{(n)} - c^{(n-1)}\| \le \theta \|c^{(n-1)} - c^{(n-2)}\|. \tag{4.53}$$

One proves (4.52) by induction. Assume (4.52) holds for all $n \le m$. Then

$$|c^{(m+1)}(k,t)| \le e^{-\nu k^2 t}\,R_0 + \frac{C}{\nu}\,|k|^{\gamma-1}(1 - e^{-\nu|k|^2 t})\,\|c^{(m)}\|^2 \sum_{j\neq 0, j\neq k} \frac{1}{|j|^\gamma |k-j|^\gamma}.$$

Applying Lemma 4.3 and the inductive assumption, one has

$$|c^{(m+1)}(k,t)| \le e^{-\nu k^2 t}\,R_0 + \frac{C(\gamma)}{\nu}\,|k|^{-1}(1 - e^{-\nu|k|^2 t})\,R_0^2. \tag{4.54}$$

It is easily verified that, for any $k \neq 0$ and $t \ge 0$,

$$|k|^{-1}\,(1 - e^{-\nu|k|^2 t}) \le (\nu\,t)^{\frac{1}{2}}.$$

Consequently,

$$\|c^{(m+1)}\| \leq R_0 + \frac{C(\gamma)}{\nu^{\frac{1}{2}}}\, T^{\frac{1}{2}} R_0^2.$$

If

$$T^{\frac{1}{2}}\, R_0 < \frac{\nu^{\frac{1}{2}}}{C(\gamma)}, \tag{4.55}$$

then

$$\|c^{(m+1)}\| \leq 2R_0.$$

To prove (4.53), consider the difference

$$\begin{aligned} |c^{(n)}(k,t) - c^{(n-1)}(k,t)| &= 3|k|^{\gamma+1}\, e^{-\nu k^2 t} \\ &\times \int_0^t e^{\nu k^2 s} \sum_{j\neq 0, j\neq k} \frac{|c^{(n-1)}(j,s)c^{(n-1)}(k-j,s) - c^{(n-2)}(j,s)c^{(n-2)}(k-j,s)|}{|j|^\gamma |k-j|^\gamma}\, ds. \end{aligned}$$

Writing

$$\begin{aligned} c^{(n-1)}(j,s)\, c^{(n-1)}(k-j,s) &- c^{(n-2)}(j,s)\, c^{(n-2)}(k-j,s) \\ &= \left[c^{(n-1)}(j,s) - c^{(n-2)}(j,s)\right] c^{(n-1)}(k-j,s) \\ &\quad + c^{(n-2)}(j,s) \left[c^{(n-1)}(k-j,s) - c^{(n-2)}(k-j,s)\right] \end{aligned}$$

and estimating as in the proof of (4.52), one obtains

$$|c^{(n)}(k,t) - c^{(n-1)}(k,t)| \leq \frac{C(\gamma)}{\nu^{\frac{1}{2}}}\, t^{\frac{1}{2}} R_0 \|c^{(n-1)} - c^{(n-2)}\|.$$

When (4.55) is satisfied, then

$$\|c^{(n)} - c^{(n-1)}\| \leq \theta \|c^{(n-1)} - c^{(n-2)}\|$$

with

$$\theta = \frac{C(\gamma)}{\nu^{\frac{1}{2}}}\, T^{\frac{1}{2}} R_0 < 1.$$

Inequalities (4.52) and (4.53) allow to construct the limit of $c^{(n)}(k,t)$ as

$$c(k,t) = c^{(1)}(k,t) + \sum_{n=1}^{\infty} \left(c^{(n+1)}(k,t) - c^{(n)}(k,t)\right).$$

Going through a simple limiting process, one can show that $c(k,t)$ satisfies (4.51). In addition, by letting $m \to \infty$ in (4.54), one obtains

$$|c(k,t)| \leq e^{-\nu k^2 t} R_0 + \frac{C(\gamma)}{\nu} |k|^{-1} R_0^2, \tag{4.56}$$

which forms the basis for further regularity estimates. This completes the proof of Theorem 4.6. ∎

One now proves that for any $t > 0$, the series solution $u = u(x,t)$ in Theorem 4.6 is actually a classical solution. Lets divide this process into two steps. First, lets show that it is a weak solution in the standard distributional sense.

Theorem 4.7 *Assume the conditions of Theorem 4.6 and let u be the series solution obtained there. Then u is a weak solution in the sense that*

$$\int_0^T \int_{\mathbb{T}} u\,(\phi_t - 6u\phi_x + \alpha\phi_{xxx} - \nu\phi_{xx})\,dx\,dt - \int_{\mathbb{T}} u_0(x)\phi(x,0)\,dx = 0$$

for any $\phi \in C_0^\infty(\mathbb{T} \times [0,T))$.

Proof. Recall that

$$u(x,t) = \sum_{k\neq 0} \widehat{u}(k,t)e^{ikx} \quad \text{with} \quad \widehat{u}(k,t) = \frac{c(k,t)}{|k|^\gamma}.$$

Let $N > 0$ be an integer. Consider

$$u_N(x,t) = \sum_{|k|\leq N, k\neq 0} \widehat{u}(k,t)\, e^{ikx}.$$

To derive the equation for u_N, multiply (4.51) by $\frac{1}{|k|^\gamma}$ and differentiate with respect to t to get

$$\frac{d}{dt}\widehat{u}(k,t) = (-\nu k^2 + i\alpha k^3)\,\widehat{u}(k,t) + 3i\,k \sum_{j\neq 0, j\neq k} \widehat{u}(j,t)\,\widehat{u}(k-j,t).$$

Multiplying this equation by e^{ikx} and summing over $|k| \leq N$ $(k \neq 0)$, one has

$$\partial_t u_N - 6u_N\,(u_N)_x + \alpha(u_N)_{xxx} - \nu(u_N)_{xx} = R_N, \tag{4.57}$$

where R_N is given by

$$R(x,t) = 3 \sum_{|k|\le N, k\neq 0} (ik)e^{ikx} \sum_{|j|>N} \widehat{u}(j,t)\,\widehat{u}(k-j,t).$$

Multiplying (4.57) by $\phi \in C_0^\infty(\mathbb{T}\times[0,T))$ yields

$$\int_0^T \int_{\mathbf{T}} u_N \left(\phi_t - 6u_N\phi_x + \alpha\phi_{xxx} - \nu\phi_{xx}\right) dxdt$$
$$- \int_{\mathbf{T}} u_0(x)\phi(x,0)\, dx = \int_0^T \int_{\mathbf{T}} R_N \phi \, dxdt.$$

Since $u_N(\cdot,t) \to u(\cdot,t)$ in L^2 uniformly for $t\in[0,T]$, one obtains by letting $N\to\infty$

$$\int_0^T \int_{\mathbb{T}} u \left(\phi_t - 6u\phi_x + \alpha\phi_{xxx} - \nu\phi_{xx}\right) dxdt$$
$$- \int_{\mathbb{T}} u_0(x)\phi(x,0)\, dx = \lim_{N\to\infty}\int_0^T \int_{\mathbb{T}} R_N \phi \, dxdt. \tag{4.58}$$

To show the limit on the right is zero, lets use the basic inequality

$$\int_{\mathbb{T}} R_N\phi\, dx \le \left(\int_{\mathbb{T}} R_N^2(x,t)\, dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{T}} \phi(x,t)\, dx\right)^{\frac{1}{2}}$$

and show $\|R_N\|_{L^2} \to 0$. This can be done as follows. Because of the bound

$$\int_{\mathbb{T}} R_N^2(x,t)\, dx \le C \left[\sum_{k=1}^N k^2 \sum_{|j|>N} \frac{1}{|j|^{2\gamma}|j-k|^{2\gamma}} + \sum_{k=-N}^{-1} k^2 \sum_{|j|>N} \frac{1}{|j|^{2\gamma}|j-k|^{2\gamma}}\right]$$

for some constant C, it suffices to consider

$$\sum_{k=1}^N k^2 \sum_{j>N} \frac{1}{|j|^{2\gamma}|j-k|^{2\gamma}} \le \frac{1}{(N+1)^{2\gamma}} \sum_{k=1}^N \frac{k^2}{(N-k+1)^{2\gamma}} \sum_{j\ge N+1} \frac{1}{(1+\frac{j-N-1}{N-k+1})^{2\gamma}}.$$

The last summation can be bounded as in

$$\sum_{j\ge N+1} \frac{1}{(1+\frac{j-N-1}{N-k+1})^{2\gamma}} \le 1 + \int_0^\infty \frac{1}{(1+\frac{x}{N-k+1})^{2\gamma}}\, dx = 1 + \frac{N-k+1}{2\gamma-1}.$$

Thus,

$$\sum_{k=1}^N k^2 \sum_{j>N} \frac{1}{|j|^{2\gamma}|j-k|^{2\gamma}} \le \frac{C(\gamma)}{(N+1)^{2\gamma}} \sum_{k=1}^N \frac{k^2}{(N-k+1)^{2\gamma-1}} \le \frac{C(\gamma)}{N^{4\gamma-4}}.$$

For $\gamma > 1$, it approaches zero as $N\to\infty$. It then follows from (4.58) that u satisfies the weak formulation. This completes the proof of Theorem 4.7. ■

The following theorem asserts the regularity of u.

Theorem 4.8 *Assume the conditions of Theorem 4.6 and let u be the series solution obtained there. Then, for any $t_0 > 0$ and nonnegative integer m,*

$$u \in C^1([t_0, T); H^m). \tag{4.59}$$

In particular, this regularity result with Theorem 4.7 implies that u is a classical solution of the complex KdV-Burgers equation (4.33).

Proof. Obviously $u \in L^2([0, T); L^2)$. Fix $t \in (0, T)$. Inserting the simple inequality

$$e^{-\nu k^2 t} \leq \frac{1}{|k|} e^{-\nu t} \quad \text{for any } k \in \mathbb{Z} \setminus \{0\}$$

in (4.56), one finds

$$c(k, t) = \frac{\widetilde{c}(k, t)}{|k|} \tag{4.60}$$

with

$$|\widetilde{c}(k, t)| \leq \widetilde{R}_0 \quad \text{for all } k \neq 0 \text{ and } 0 < t < T.$$

Then $u(x, t)$ can be represented as

$$u(x, t) = \sum_{k \neq 0} \frac{\widetilde{c}(k, t)}{|k|^{\gamma+1}} e^{ikx}.$$

In particular, $u(\cdot, t) \in H^1(\mathbb{T})$. An iterative process will allow to show

$$c(k, t) = \frac{\widetilde{c}(k, t)}{|k|^m}, \quad u(x, t) = \sum_{k \neq 0} \frac{\widetilde{c}(k, t)}{|k|^{\gamma+m}} e^{ikx}. \tag{4.61}$$

for any positive integer m, where $\widetilde{c}$ may not be the same as in (4.60). Thus $u(\cdot, t) \in H^m(\mathbb{T})$. To show the regularity of u in t, lets turn to (4.51), which implies that $c(k, t)$ is differentiable in t and

$$\frac{d}{dt} c(k, t) == (-\nu k^2 + i\alpha k^3)\, c(k, t) - i\, k|k|^\gamma \sum_{j \neq 0, j \neq k} \frac{c(j, t) c(k - j, t)}{|j|^\gamma\, |k - j|^\gamma}.$$

It then easily follows from (4.61) and Lemma 4.3 that

$$\left|\frac{d}{dt}c(k,t)\right| \le \frac{C}{|k|}.$$

This together with $u(\cdot,t) \in H^m(\mathbb{T})$ guarantee (4.59). This completes the proof of Theorem 4.8. ■

4.4 The Lax pairs and Darboux transformation for finite-time blow up

The combination of the Lax pairs and Darboux transformation can lead one to find the blowup solution of the complex-valued equations in the context. These two mathematical terms will be briefly introduced and an example will be given to show how their combination leads to finding the blowup solution of complex KdV equation.

4.4.1 The Lax pairs

In 1968, Lax developed a method for solving nonlinear partial differential equations. We consider the initial-value problem for $u(x,t)$ which satisfies the following nonlinear equation

$$\begin{cases} u_t = N(u) \\ u(x,0) = f(x) \end{cases} \tag{4.62}$$

where $u \in \mathbb{Y}\,\forall t$, $\mathbb{Y}$ is an appropriate functional space, and $N : \mathbb{Y} \to \mathbb{Y}$ is a nonlinear operator involving x or derivatives with respect to x. To express the equation (4.62) in operator form, Lax found two linear nonconstant differential operators L and M such that

$$\begin{cases} L\phi = \lambda\phi \\ \phi_t = M\phi \end{cases} \tag{4.63}$$

where L and M depend on an unknown function $u(x,t)$ [11]. L describes the spectral (scattering) problem, with ϕ the usual eigenfunction, and M describes how the eigenfunction evolve in time. One obtains

$$\begin{aligned}
\tfrac{\partial L}{\partial t}\phi &= \frac{\partial}{\partial t}(L\phi) - L\frac{\partial \phi}{\partial t} \\
&= \frac{\partial}{\partial t}(\lambda\phi) - LM\phi \\
&= \frac{\partial \lambda}{\partial t}\phi + \lambda\frac{\partial \phi}{\partial t} - LM\phi \\
&= \frac{\partial \lambda}{\partial t}\phi + \lambda M\phi - LM\phi \\
&= \frac{\partial \lambda}{\partial t}\phi + ML\phi - LM\phi
\end{aligned}$$

Thus, the spectral parameter λ is constant if, and only if

$$\frac{\partial L}{\partial t} + (LM - ML) = 0. \tag{4.64}$$

The equation (4.64) is known as the Lax equation and L and M are called the Lax pairs. This equation contains the nonlinear equation for correctly chosen L and M. The nonlinear equation (4.62) can be expressed as a Lax equation (4.64), and if $L\phi = \lambda\phi$ for $t \geq 0$ and $x \in \mathbb{R}$, then $\dfrac{\partial \lambda}{\partial t} = 0$, and ϕ satisfies $\dfrac{\partial \phi}{\partial t} = M\phi$.

There is no systematic way of finding operators L and M that satisfy the above conditions. The main difficulty is how to check for a given PDE whether or not it has a Lax equation, and if so, how to find the Lax pairs L and M.

Example 4.1 *$L = -\partial_{xx} + u$, and $M = -4\partial_{xxx} + 6u\partial_x + 3u_x$ are the Lax pairs of the standard complex KdV equation*

$$u_t - 6uu_x + u_{xxx} = 0. \tag{4.65}$$

To show this, we simply follow the routine work.

4.4.2 The Darboux transformation

In 1882, G. Darboux studied the eigenvalue problem of a linear second order partial differential equation, known as the Schrödinger equation given by

$$\phi_{xx} + u(x)\phi = \lambda\phi, \tag{4.66}$$

where u is a given function. If $u(x)$ and $\phi(x, \lambda)$ are two functions satisfying (4.66) and $f(x) = \phi(x, \lambda_0)$ is the solution of (4.66) for $\lambda = \lambda_0$, where λ_0 is fixed, then U and Φ defined by

$$U = u - 2(\ln(f))_{xx} \quad \text{and} \quad \Phi(x, \lambda) = \phi_x - \frac{f_x}{f}\phi(x, \lambda)$$

satisfy

$$\Phi_{xx} + U\Phi = \lambda\Phi.$$

The transformation $(u, \phi) \to (U, \Phi)$ is called the Darboux transformation.

Finding Darboux transformation of a given equation is not an easy process. The following example gives the Darboux transformation of the standard complex KdV equation.

Example 4.2 *Let u be a solution of the standard complex KdV equation, and ϕ be a solution to the Lax pair for a particular u and λ. The transformation*

$$U = u - 2\partial_{xx}(\ln(\phi)), \quad \Psi = \psi_x - (\partial_x(\ln(\phi)))\psi,$$

where ψ solves the Lax pair at u and an λ, is the Darboux transformation of the the standard KdV equation.

In fact, $u_t - 6uu_x + u_{xxx} = 0$ is the integrability condition of the Lax pairs

$$\begin{aligned} -\phi_{xx} + u\phi &= \lambda\phi \\ \phi_t &= -4\phi_{xxx} + 6u\phi_x + 3u_x\phi \end{aligned} \tag{4.67}$$

discussed in example 4.1. The first equation is a Schrödinger equation, so the transformation Ψ hold the first equation $\Psi_{xx} + U\Psi = \lambda\Psi$ invariant. U and Ψ also satisfy second equation as well. Therefore $U = u - 2\partial_x x(\ln(\phi)$ is another solution of KdV.

4.4.3 Blowup of complex KdV equation

Using the Lax pairs and Darboux transformation of the complex KdV equation, Y. Charles Li [30] derived a simple explicit formula for the solution of the complex

KdV in periodic domain that blows up in finite time. Consider the periodic boundary condition

$$u(x+2\pi,t) = u(x,t).$$

For any complex constant $a = a_r + ia_i$, where a_r and a_i are real and imaginary part of a, $u(x,t) = a$ is a solution of (4.65). By using the Laplace transform, the Lax pair (4.67) has two independent solutions at $u = a$ and $\lambda = a + k^2$, for any $k \in \mathbb{Z}$. These solutions are given by

$$\psi_1 = \exp(\omega t + ikx) \quad \text{and} \quad \psi_2 = \exp\{-(\omega t + ikx)\}$$

where $\omega = 6iak + 4ik^3$. The real part and imaginary parts of ω are given by

$$\omega_r = -6a_i k \quad \text{and} \quad \omega_i = 6a_r k + 4k^3.$$

The next solution ϕ of (4.67) can be obtained by the linear combination of these two solutions, as in

$$\phi = c_1\psi_1 + c_2\psi_2 = c_1 \exp(\omega t + ikx) + c_2 \exp\{-(\omega t + ikx)\}.$$

where c_1 and c_2 are two arbitrary complex constants. Let $\dfrac{c_1}{c_2} = \exp(\rho + i\gamma)$ with ρ and γ two arbitrary real constants. To use the Darboux transformation, lets evaluate $\partial_x^2(\ln(\phi))$. Now,

$$\partial_x(\ln(\phi)) = ik\left(\frac{\exp(2\omega t + 2ikx + \rho + i\gamma) - 1}{\exp(2\omega t + 2ikx + \rho + i\gamma) + 1}\right) \quad \text{and}$$

$$2\partial_x^2(\ln(\phi)) = -8k^2 \frac{\exp(2\omega t + 2ikx + \rho + i\gamma)}{(\exp(2\omega t + 2ikx + \rho + i\gamma) + 1)^2}.$$

Applying the Darboux transformation, one obtains

$$U = a + 8k^2 \frac{\exp(2\omega t + 2ikx + \rho + i\gamma)}{(\exp(2\omega t + 2ikx + \rho + i\gamma) + 1)^2}.$$

The choice $k = 1, a = i, \rho = 12$, and $\gamma = 0$ gives the explicit formula for the solution of the complex KdV

$$u(x,t) = i + 8\frac{\exp(-12(t-1) + i(8t + 2x))}{[\exp(-12(t-1) + i(8t + 2x)) + 1]^2}.$$

When $t = 0$, $u(x, 0)$ is C^∞. When $t = 1$, the solution $u(x, 1)$ develops two singularities at $x = \frac{3}{2}\pi - 4$ and $x = \frac{5}{2}\pi - 4$.

This idea of finding explicit formula for the blow up solutions may be useful for other complex-valued equations as well. The main difficulty is to find Lax pairs and the Darboux transformation for the specific equation.

CHAPTER 5

Conclusions

The main objectives of this dissertation are to study the solutions of the Kawahara equation in weighted Sobolev spaces, prove that the solution of the complex Burgers equation blows up in finite time, and show the regularity of solutions of complex KdV-Burgers equation for suitable initial data.

The presence of higher order terms in the Kawahara equation makes it difficult to solve. In this work, the weak formulation of the dispersive part of the IBVP of Kawahara equation is first considered and shown that the formulation possesses a unique solution for $\beta > -\frac{3}{80}$. Using this result, the full IBVP is solved in weighted Sobolev spaces and proved that the solutions are well-posed globally in time and if the L^2-norm of u_0 is small, then the solutions in these weighted Sobolev spaces decay exponentially in time. Two numerical experiments are performed to complement the theoretical observations: first one satisfying the condition on β and second one voilating the condition. Numerical results show that the solutions may not be globally well-posed if the condition on β is not met. The theoretical treatment of this problem is still open. This work opens door to study the solutions of several integrable and non-integrable fifth-order KdV type equations.

Motivated by the fact that complex KdV has solutions that blows up in finite time, the solution of complex-valued Burgers equation is investigated for potential singularities. The work in this dissertation asserts that the spatially periodic solutions of the IVP of the complex Burgers equation blows up in finite time for an explicit smooth initial data. The special series type and Fourier series type solutions of the complex

KdV-Burgers equation are studied to find the conditions under which they are regular for all time. The main results examine how the dispersion and dissipation should interact and what condition that the initial data should satisfy for the complex KdV-Burgers equation to have a unique global solution. This dissertation produces some significant results related to complex-valued Burgers and KdV-Burgers equations and directs one to think for several open problems concerning these equations.

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The KdV equation is a nonlinear partial differential equation. The real-valued as well as complex-valued KdV equations have wide physical applications and very rich mathematical theory. The work in this dissertation studies two important problems.

First, the initial- and boundary-value problem for the Kawahara equation, a fifth-order KdV type equation, is studied in weighted Sobolev spaces. This functional framework is based on the dual-Petrov-Galerkin algorithm, a numerical method proposed by Shen to solve third and higher odd-order partial differential equations. The theory presented here includes the existence and uniqueness of a local mild solution and of a global strong solution in these weighted spaces. If the L^2-norm of the initial data is sufficiently small, these solutions decay exponentially in time. Numerical computations are performed to complement the theory.

Second, spatially periodic complex-valued solutions of the Burgers and KdV-Burgers equations are studied in detail. It is shown that for any sufficiently large time T, there exists an explicit initial data such that its corresponding solution of the Burgers equation blows up at T. In addition, the global convergence and regularity of two types of series solutions of the KdV-Burgers equation are established for initial data satisfying mild conditions.

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