STUDENTS' CONCEPTUALIZATIONS OF

MULTIVARIABLE LIMITS

By

BRIAN CLIFFORD FISHER

Bachelor of Science in Mathematics
Oklahoma State University
Stillwater, Oklahoma
2000

Master of Science in Mathematics
Oklahoma State University
Stillwater, Oklahoma
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Dissertation Approved:

Dr. Douglas B. Aichele

Dissertation Adviser

Dr. John Wolfe

Dr. James R. Choike

Dr. Alan Noell

Dr. William Warde

Dr. A. Gordon Emslie

Dean of the Graduate College

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER I

INTRODUCTION

Statement of the Problem

It is well accepted that the limit concept plays a foundational role in present-day calculus education. At the same time, there is widespread agreement among both educators and researchers that most students struggle to develop a solid understanding of the limit concept (for example: Vinner, 1991). This may be due to the actual depth of concept. Tall (1992) refers to Cornu (1983) and states that "this is the first mathematical concept that students meet where one does not find the result by a straightforward mathematical computation. Instead it is 'surrounded with mystery,' in which 'one must arrive at one's destination by a circuitous route'" (Tall, 1992, p.501).

The importance of limits in undergraduate calculus, combined with the difficulty students experience in grasping the concept has resulted in much attention from mathematics education researchers. Several researchers have worked to understand popular misconceptions about the limit concept (Davis and Vinner, 1986; Williams, 1991). It has been suggested by several researchers that a well developed notion of limit could be constructed using the metaphor of motion (Carlsen et al., 2002; Kaput, 1979; Monk, 1992; Tall, 1992; Thompson, 1994b). Furthermore, Williams (1991) found that a significant number (30%) of second semester calculus students contained a dynamic view of limit, and that this dynamic viewpoint was extremely resistant to change.

For these reasons, I have decided to examine how students who hold a dynamic view of limit conceptualize the multivariable limit concept. For functions of two variables, motion must take place on a surface instead of along a curve. It is not automatically clear how students will interpret motion in this new setting. Furthermore, the application of motion in multivariable calculus has the potential to create an epistemological obstacle in the sense of Cornu (1991) and require students to restructure their understanding of limits. I expect this restructuring to take place in a form of generalization similar to that described by Harel and Tall (1989).

With this in mind I have created the following problem statement for this study:

> *Describe how students with a dynamic view of limit generalize their understanding of the limit concept in a multivariable environment.*

As the problem statement suggests, this is a qualitative research study which results in a description of student behavior. I will focus the description on the following goals:

1. Describe what type of generalization students in this setting tend to experience with respect to the schema outlined by Harel and Tall (1989) which emphasizes three modes of generalization: expansive generalization, reconstructive generalization, and disjunctive generalization.
2. Describe the role of motion in students' understanding of multivariable limits. Does it create a cognitive obstacle, or are students able to apply this imagery to the new multivariable situation?

3. Describe how students respond to studying multivariable limits in four different contexts: traditional symbolic manipulation, symbolic manipulation involving polar coordinates, three-dimensional graphing, and contour graphing. Of particular interest is whether some of these contexts tend to allow students to reconstruct their understanding of the limit concept to more closely resemble the formal definition.

Problem Context

History of the Limit Concept

The concept of limit can trace its history back to ancient Greece. The Greek mathematicians spent most of their energy solving geometry problems. The solutions to many of these problems involved limiting concepts. One of the earliest such solutions was provided by Hippocrates of Chios (not to be confused with the famous doctor, Hippocrates of Cos). He proved that "the ratio of two circles is equal to the ratio of the squares of their diameters" (Edwards, 1979, p. 7). He accomplished this by inscribing polygons inside the circle and showing that the relationship is true for all such polygons. He then concluded that since this is true for all such polygons, it must also be true for a circle. However, Hippocrates had no limit concept capable of finishing his argument.

In general, the Greek mathematicians were bothered by the infinite ideas inherent in the limit concept, and soon they began developing methods that could be used to avoid the "horror of the infinite." Mathematicians such as Eudoxus, Archimedes, and Euclid began using the *method of exhaustion* to perform calculations such as that of Hippocrates. This method used contradiction to rigorously prove a statement. It depended on the principle that any magnitude can be made smaller than a second magnitude by repeatedly dividing the first magnitude in half. Using these methods, the Greeks (especially Archimedes) were able to solve many modern day calculus problems. Ultimately, these notions gave rise the formation of calculus as we know it today. However, it is important to note that the ancient Greeks contained no explicit concept of limit. In addition they were unable to generalize their methods, and instead chose to start from scratch to solve each problem they faced. Additionally, they were unable to make the connections between problems of areas and tangents which gave rise to modern day calculus (Baron, 1969; Edwards, 1979).

For hundreds of years after the era of Greek mathematics, mathematicians were unable to approach the ideas of calculus as understood by the Greeks. Prior to the sixteenth century, the works of the Greek mathematicians were “not always generally accessible and never fully mastered” (Edwards, 1979, p. 98). However, there were many important developments prior to the sixteenth century that made it possible for later mathematicians to approach a new way of understanding limit. Among those developments were various graphical representations of what we would call functions. These ideas were introduced in the fourteenth century by Nicole Oresme (Edwards, 1979; Babb, 2005). However, these graphical representations were not intended to be thought of as a set of corresponding values, like a modern day function graph. Instead, Oresme intended for each vertical height of his graph to represent the ‘intensity’ of a quantity (Edwards, 1979; Thompson, 1994a; Babb, 2005). As he wrote in his *Treatise on the Configuration of Qualities and Motions*, “every intensity which can be acquired successively ought to be imagined by a straight line perpendicularly erected on some point of the space or subject of the intensible thing," (Grant, 1974, as quoted in Edwards, 1979, p.88).

By the middle of the seventeenth century these graphical representations had developed quite a bit. Many noted mathematicians of the time used an idea of motion to understand these representations. In fact, Newton "regarded the curve, $f(x,y) = 0$ as the locus of intersection of two moving lines, one vertical and the other horizontal," (Edwards, 1979, p. 191). Newton's use of motion was no doubt influenced by his mentor, Isaac Barrow. While Newton was a student under Barrow's guidance, Barrow gave an important series of lectures on time and motion. Barrow perceived a line as a "trace of a point moving forward… the trace of a moment continuously flowing" (Baron, 1969, p. 240).

This use of the metaphor of motion to understand graphical representations characterized much of Newton's work. In fact, Bardi (2006) states that "Newton's big breakthrough was to view geometry in motion…" (p. 30), and in one of Newton's first attempts to compile his early works, *To Resolve Problems By Motion* in 1666, he "deliberately elects to make the concept of motion the fundamental basis" of his work (Baron, 1969, p. 263). Using these ideas of motion allowed Newton to solve many problems in the development of the calculus; however, a precise definition of limit was still several hundred years away.

It was not until the nineteenth century's increased focus on mathematical rigor that the formal limit definition as we know it today came into being. One issue that had to be confronted was the notion of infinitesimals. The idea was not a new one. In fact, Fermat came very close to modern limit calculations when he substituted $x + e$ for the variable *x* and after simplification removed *e* from the expression (Baron, 1969; Edwards, 1979). It is important to note that Fermat did not consider the value *e* to approach zero or even become zero (he did not even imply that *e* should be small), instead he simply removed expressions containing *e*. It should be noted that even at the time this was questioned by such mathematicians as Rene Descartes (Baron, 1969). However, these ideas and the use of infinitesimal values continued to be popular for hundreds more years. Finally, it was Cauchy who developed "the first comprehensive treatment of mathematical analysis to be based from the outset on a reasonably clear definition of the limit concept" (Edwards, 1979, p. 310). Cauchy's notion of limit was based on an infinitely small variable, which he also called an infinitesimal. This is different from the view that an infinitesimal is an infinitely small *quantity*; instead, according to Cauchy it is a *variable* whose value decreases indefinitely.

Even at this time, the limit concept was "tinged with connotations of continuous motion" (Edwards, 1979, p. 333). The close of the nineteenth century saw the precise construction of the real number system, and with it Weierstrass was able to develop the definition of limit that is commonly used today. His disapproval of the dynamic view of limits led him to create a static formulation in terms of ε and δ which became popular throughout the twentieth century.

In summary, I would like to observe several themes which ran through the historical development of the limit concept:

- The Greeks had problems "passing to the limit." They much preferred static arguments that did not contain notions of the infinite.

- Newton (and many others) found the metaphor of motion to be a powerful tool in understanding the concepts of calculus. However, these ideas of motion were unable to provide a rigorous definition of limit, and were eventually replaced with Weierstrass's static definition.
- For many centuries different mathematicians struggled with the meaning of infinitesimals. These ideas usually contained some sense of infinitely small quantities until Cauchy used a dynamic view of infinitesimal to create a more coherent meaning of the concept of limit.

In conclusion, it should be pointed out that the limit concept was understood in many ways throughout history. Each of these ways of thinking made it possible for mathematicians to understand limits in a useful manner, but each way of thinking also created a barrier towards understanding the limit concept in the way we know it today. In that respect, these ways of thinking created epistemological obstacles in the sense of Cornu (1991).

Curriculum Change and the Limit Concept

The first half of the twentieth century saw calculus generally reserved for undergraduate education and rarely discussed in high school settings. This time period was marked by an emphasis on two-track high school mathematics programs (Jones, 1970). With the Great Depression came decreased college enrollment, and educators responded by focusing on functional competence as the key objective of high school mathematics. Often times mathematics classes became electives, and as a result there was a trend for colleges to lessen their mathematics requirements for admission (Jones, 1970). The affect of this atmosphere on teaching the limit concept is not entirely clear;

however, it might be reasonable to conclude that approaches to limits, as well as other calculus concepts, mirrored that of other subjects in their focus on functional competence. In that sense, we would assume that the limit concept was taught primarily as a procedure by which a certain result could be obtained.

After World War II the educational climate in the United States began to change dramatically. Technological advances made during and immediately following World War II revitalized the status of mathematics and science in the country. Colleges saw an increase in enrollment partially due to returning soldiers attending college on the "GI Bill" (Jones, 1970). At the same time, the country began recognizing that its population was not prepared to meet the demands of a new technological society. Accompanying all this with the feeling that America was beginning to fall behind the rest of the world scientifically, emphasized by the Soviet launch of Sputnik in 1957. The United States began a period of reexamining the way mathematics was taught at all levels throughout the country. The resulting period from the early 1960s to the mid 1970s became known as the "New Math Era" and was marked by an increased focus on abstraction and mathematical rigor (Bosse, 1995). An early introduction to key mathematical ideas also marked this period resulting in a push for calculus to be introduced to students while in high school. The effect this had on teaching calculus and the limit concept was significant. Calculus was approached in a more rigorous manner than before and became a more common element in a student's education.

One of the most controversial reports coming from this time was the Cambridge Conference on School Mathematics (1970). The Cambridge Conference was considered an ambitious goal set out to challenge the mathematics education community on what can be accomplished (Adler, 1970). This report called for calculus to be taught using "precise formulations" rather than what it refers to as "loose calculus," which "deals with 'variables' (in a Leibnizian sense) rather than functions," (Cambridge Conference on School Mathematics, 1970, p. 40). The Cambridge Conference set forth two proposed curricular programs, both of which featured a rigorous treatment of calculus in the final two years of high school. However, there was some disagreement whether calculus should first be introduced at an earlier time on a more intuitive basis. The argument against an introduction was that

> The student who has already developed some taste for mathematical rigor will be dissatisfied with only half the story in calculus when the fundamental concepts are not carefully defined and precisely used. Because he cannot carry his arguments back to well-defined concepts, he will not fully understand what calculus is about. Finally, one often forms wrong impressions in an intuitive approach which are hard to "unlearn" later, and the luster is worn off the subject when one has to return to it later to tie together loose ends (Cambridge Conference on School Mathematics, 1970, p. 47).

On the other hand, the Cambridge Conference recognized the historical significance of the calculus and wanted all students to be able to appreciate it whether or not they completed the final years of the mathematics program.

In the late 1960s and early 1970s a backlash against the "New Math Era" began. This resulted in several different movements, including the "Back to the Basics" movement. Importantly, most reform movements after the mid-1970s called for a decreased emphasis on mathematical rigor. In their publication, *Agenda for Action*, The

National Council of Teachers of Mathematics (NCTM) laid out its recommendations for school mathematics in the 1980s. Among other things, this publication called for "the use of imagery, visualization, and spatial concepts" (1980, p. 3) to understand mathematical ideas. This is clearly a different emphasis than the precise definitions used by the Cambridge Conference. In addition, there became a question as to the need and relevance of calculus. In the same publication, the NTCM challenges mathematics educators and college mathematicians to "reevaluate the role of calculus" (ibid, p. 21) in school curriculum. Importantly, it was suggested that perhaps calculus should not be the focal point of college preparatory mathematics and that other branches of the mathematical sciences should be encouraged in its place. A few years later, Shirley Hill made the case for a new curriculum that suggested an alternative path for capable students which "would stress statistics and computer science rather than calculus" (Hill, 1982, p. 116).

By the late 1980s and early 1990s educators began to focus again on calculus as a foundation of mathematical learning. In *A Call for Change*, the Mathematical Association of American (MAA) set out recommendations for teacher preparation. In this they called for teachers to model real world problems using calculus and to explore the concepts of calculus both on an intuitive basis and in depth (Leitzel, ed., 1991). From the perspective of this publication, the emphasis is clearly on gaining an intuitive understanding of calculus. It writes,

> Historically, while investigating continuous processes, many of the ideas and techniques of calculus were developed and used on an intuitive basis before the theory was made rigorous… By building an intuitive base for analyzing

> continuous processes, these teachers might be more willing to take intellectual risks in their own classrooms. The actual material covered is less important than developing conceptual understanding of the ideas (ibid, p. 35).

In 1989, the NCTM released their *Curriculum and Evaluation Standards for School Mathematics*. In this publication, the NCTM "does *not* advocate the *formal* study of calculus in high school for all students or even for college-intending students. Rather, if calls for opportunities for students to systematically, but informally, investigate the central ideas of calculus" (NCTM, 1989, p. 180). Therefore, by the early 1990s the trend in mathematics education was to teach calculus on an informal, intuitive basis rather than using the precise formulations and rigor of the "New Math Era."

The calculus reform movement is generally considered to have begun in 1986 with the Tulane calculus conference (Schoenfeld, 1995). This conference resulted in The MAA's publication of *Toward a Lean and Lively Calculus* (Douglas, 1986) which aimed to slim down the calculus curriculum by teaching fewer topics but covering them in greater depth. This began a period marked by numerous projects all aimed at reforming the calculus curriculum. There was significant variation between these different projects, but most incorporated an increased use of technology, an emphasis on applications, and the use of multiple representations (Ganter, 1999).

The increased emphasis on technology in mathematical teaching and learning was nearly inevitable with the increased availability of technology in society. In many ways, this emphasis of technology spurred on the other major changes during the calculus reform movement. The use of technology in mathematical learning allowed students to encounter problems in real world settings that would have been impossible before. This allowed the reform movement to place an emphasis on application problems, often projects spanning over multiple class periods. In addition, the use of technology allowed for easy transitions between symbolic, numerical, and graphical representations of functions. This ease of transition allowed the reformers to place a greater emphasis on multiple representations in the classroom.

This emphasis on multiple representations is of a particular interest to this study. This is one of the foundations of the Harvard Consortium's hallmark textbook (Hughes-Hallett, et al. 1994). This text emphasized the "Rule of Three," which pushed for all concepts, in particular function concepts, to be studied in graphical, numerical, and analytical settings. The group later reformed this concept to the "Rule of Four" which added verbal representations to the list (Schoenfeld, 1995). This notion of multiple representations found its way into publications beyond just those of the calculus reform movement. For example, *A Call for Change*, published by The Mathematical Association of America, was written as a recommendation for the curriculum of teachers of mathematics (Leitzel (Ed.), 1991). This publication called for teachers to be able to "represent functions as symbolic expressions, verbal descriptions, tables, and graphs and move from one representation to another" (p. 31). This push for representing functions in multiple ways brought with it a notion of function that was broader than before. Instead of restricting the notion of function to its definition, there is now an emphasis on thinking of functions in a wide variety of manners. The purpose of the present study is to understand how students connect and generalize these different representations of the limit concept. In particular, how do they generalize a graphical notion of dynamic motion in multivariable settings and how does the representation of the multivariable function impact this generalization?

Importantly, we see that calculus education has undergone several major changes during the past century. The major characteristics of each period were:

- Pre-World War II was marked by an emphasis on “functional mathematics”
- The “New Math Era” came after World War II and encouraged an increased focus on mathematical rigor and precision and an earlier introduction to mathematical topics.
- The backlash to the “New Math Era” resulted in calls to return to the basics in teaching math. This resulted in a decreased emphasis in mathematical rigor. During this time there also came a reevaluation of the role of calculus in education with some experts calling for programs which emphasize statistics and computer science over calculus.
- During the 1990s, the central ideas of calculus again became a center piece of mathematics education. Experts called for these central ideas to be approached, at least throughout high school, though informal intuition rather than with a formal calculus course.
- The Calculus Reform Movement began in 1986 and was marked by an increased emphasis on technology, applications, and multiple representations. In particular, the emphasis on multiple representations called for students to be able to connect symbolic, numeric, and graphical representations of a function.

In conclusion, it should be noted that the precise, formal definition was once a foundation of calculus education during the “New Math Era.” However, today, formal definitions have been replaced by intuition and informal understanding using multiple representations. It is in this spirit that I will explore these informal notions of limit that are developed by students and how these notions manifest themselves in a multivariable environment.

CHAPTER II

RELEVANT LITERATURE

Dynamic Imagery and Covariational Reasoning

One common issue in research on students' understanding of calculus is the use of dynamic imagery to represent functions. It was observed that students struggled to understand the function concept in the traditional correspondence manner. As a result, researchers began studying the use of a dynamic understanding of function. Monk (1992) labeled these modes of thinking as "pointwise" and "across-time." He observed that for some problems it was advantageous for students to use "across-time" thinking to make sense of the situation. As a result of more study, a number of scholars including Kaput (1994) began pushing for a more dynamic view of function in school curriculum. As summarized by Thompson (1994a), "today's static picture of function hides many of the intellectual achievements that gave rise to our current conceptions." (p. 29)

Over the next few years more studies in the vein of Monk's 1992 study were conducted and several authors began referring to this type of "across-time" thinking about functions as *covariational reasoning*. Confrey and Smith (1994, 1995) wrote some of the first publications referring to covariational reasoning. They mixed their notion of covariation, which was for students to "coordinate values in two different columns"

(1995, p.78), with an emphasis on multiplication as "splitting" instead of repeated addition. They noted that using covariation to understand functions "makes the rate of change concept more visible and at the same time, more critical" (1994, p.138).

This connection to rate of change is central throughout the research on functions as covariation. Thompson (1994b) studied the relationship between students' understanding of the fundamental theorem of calculus and their concepts of rate of change. He suggested that student's difficulties with the fundamental theorem of calculus are rooted in their poor understanding of rate of change and their inability to develop an image of function as covariation. During this time the notion of covariation evolved from the idea of coordinating the values in two columns of data to one of holding two values of a function in mind simultaneously. In 1998, Saldanha and Thompson further explained their view of covariation by noting that "In early development one coordinates two quantities' values – think of one, then the other, then the first, then the second, and so on. Later images of covariation entail understanding time as a continuous quantity, so that, in one's image, the two quantities' values persist" (p. 298). In this sense, understanding function as covariation is more than a special way to look at a table or graph. It is a way of thinking that includes two different changing values which simultaneously depend on each other.

Cottrill et al. (1995) also noticed these simultaneous changing values while exploring how students come to understand the limit concept. They used the theoretical perspective of APOS theory (to be explained in the "theoretical perspectives" section) to create a description of how people come to learn about the concept of limits. From their perspective, one of the key difficulties in coming to understand the limit concept lies in the complexity of the concept itself. They argue that successfully constructing a limit schema involves coordinating two different processes together through complicated existential and universal quantifiers, which is a task that remains inaccessible to most students. So, similar to Saldanha and Thompson's (1998) view that students' understanding of covariation must simultaneously coordinate two changing values in their minds, Cottrill et al. (1995) found that understanding the limit concept requires coordinating two simultaneous processes.

In the study done by Saldanha and Thompson (1998) the researchers observed an eighth grade student as he dealt with covarying quantities. Two important elements came from this study. The first element was that coming to understand functions as covariational quantities is a non-trivial task. The second was that the notion of covariation is developmental. It is in that vein that Carlson et al. (2002) developed a framework for studying functions as covariation. In this study they developed five mental actions and five corresponding levels of reasoning which could be associated with varying degrees of understanding functions as covarying quantities. In the lowest levels of understanding, students are able to coordinate the change in one variable with change in another, but with little understanding of the degree to which changing one quantity will affect another. Meanwhile the highest levels of understanding require that a student can hold in his/her mind the instantaneous rate of change of one quantity with respect to another and realize this as a continuously changing rate as the value of the independent variable changes. It was found in this study that many students understood functions as covarying quantities on a lower level, but few achieved a high level of covariational understanding. It was also suggested that this type of dynamic reasoning might be an example of *transformational reasoning* as described by Simon (1996).

Several studies have looked at the existence of such dynamic function concepts with respect to the concept of limit. Williams (1991) found that a dynamic view of limit was common among students and that it was extremely resistant to change. Many other authors tend to agree "that cognitively the strongest images are the dynamic ones," (Mamona-Downs, 2001, p. 264). On the other hand Oehrtman (2002, 2003) classified motion as a "weak metaphor." He found that while students frequently refer to situations using the language of motion, they are not describing something which is actually moving. These two studies will be discussed in greater detail in the next section, and it is one of the sub-focuses of this study to examine the nature and strength of such dynamic images in students while they encounter multivariable limits.

Students' Conceptions of the Limit Concept

Over the past twenty-five years, there have been many studies on how students understand the limit concept. They vary in many ways. Some focus on limits as understood in an introductory calculus class (for example, Williams, 1991) while others focus on limits of sequences and series (for example, Alcock and Simpson, 2004). Some attempt to characterize common misconceptions (for example, Davis and Vinner, 1986) while others attempt to describe how a students comes to understand the concept (for example, Cottrill, et al., 1995). However all studies share the findings that the limit concept is difficult for students to grasp and a complete understanding of the limit concept is rare.

Many authors have emphasized the language used when referring to limits. It is common in beginning calculus courses to use the phrase *approaching* when referring to a limit; however, it has been pointed out that in everyday language phrases such as "approaches," "tend towards," or "gets close to," carry a connotation that the point is never actually reached (Schwarzenberger and Tall, 1978). In this sense, the words used to refer to limit concepts often carry everyday meanings that are in conflict with their mathematical meanings (Monaghan, 1991). Davis and Vinner (1986) suggest that this influence of language is an unavoidable obstacle towards understanding the limit concept.

Beyond the obstacle of language, Davis and Vinner (1986) attempted to characterize several predominant misconceptions found among students studying limits of sequences and series. Several of these misconceptions are clearly related to understanding limits in beginning calculus. Among the related misconceptions are: A sequence can never reach its limit; a limit is a bound on the sequence; and a sequence must have a final term. The first two of these misconceptions were studied by Williams (1991). He found that 70% of the students in his study agreed that "a limit is a number or point the function gets close to but never reaches," and 33% agreed that "a limit is a number or point past which a function cannot go," (p. 221). From this we see that these misconceptions, and in particular the misconception of limit as being unreachable, are prevalent among students studying limits in an introductory calculus course.

It has also been shown that students often hold to naïve beliefs about limits. Tall (1992) describes the "generic limit property" which is a belief that if every term of a sequence contains a common property, it can be assumed that the limit of that sequence will also contain that property. As Tall (1992) notes, this belief has its roots in the history of mathematics, “as in Cauchy’s belief that the limit of continuous functions must again be continuous” (p. 502). He contends that this naïve belief is a cause of the popular misconception that 0.99999… is strictly less than one.

Cornu (1991) looked at limits through the lens of epistemological obstacles. From his perspective “it is useful to study the history of the concept to locate periods of slow development and the difficulties which arose which may indicate the presence of epistemological obstacles” (p. 159). From analyzing the history of the limit concept, Cornu located four such obstacles:

1. “The failure to link geometry with numbers” (p. 159). This was evidenced by the Greeks’ study of the limit concept. They were able to use very sophisticated geometric limiting arguments to solve a variety of interesting problems. However, in their studies, each problem was approached in its own geometrical context. They were only able to apply these ideas to magnitudes, not numbers, and therefore they were unable to generalize their efforts to a unifying concept of limit.
2. “The notion of the infinitely large and infinitely small” (p. 160). Many great mathematicians, including Isaac Newton and Augistin-Louis Cauchy, struggled with the notion of infinitesimal quantities. The idea of an infinitely small quantity was freely used by Leonhard Euler to solve a variety of interesting problems. However, it was not until Cauchy described the infinitesimal as a variable which tends to zero, and Weierstrauss developed a static definition of limit that the current limit concept came to being.

3. "The metaphysical aspect of the notion of limit" (p.161). The limit concept has often times more closely resembled a philosophical subject rather than a mathematical one. Many great mathematicians, from the Greek mathematicians to Joseph Louis Lagrange and the European mathematicians of the 18th century, expressed horror at the metaphysical aspects of limit. Cornu states that many students today find themselves in a similar situation when they are able to compute using limits but fail to understand it as "real" mathematics.
4. "Is the limit attained or not?" (p. 161). This question was debated for centuries among top mathematicians. Some believed that a quantity can only be made as close as we like to its limit, while others believed that at some point the infinitely small quantities "vanished" allowing the quantity to actually achieve its limit. As observed by Davis and Vinner (1986) and Williams (1991) this is still a question among students. Although, it should be said that these two studies viewed it from the vantage point that the limit *may* be attained (such as for the limit of a constant function) while Cornu viewed the phrase "attaining the limit" to mean that the limit *must* be attained as the limit point is approached.

Both Sierpinska (1987) and Williams (1991) also studied epistemological obstacles related to limits. Sierpinska noted four sources of epistemological obstacles: scientific knowledge, infinity, function, and real number. Through her study she attempted to cause cognitive conflict in students who have various cognitive obstacles in an effort to help each student overcome his/her obstacles. She found that none of the obstacles had been completely overcome, however some cognitive conflict did take place. In her opinion, attitudes towards scientific and mathematical knowledge created another obstacle which was difficult for these students to overcome.

In another study about students' attitudes, Szyldlik (2000) studied the relationship between students' understanding of the limit concept and their sources of conviction. She found that students who hold internal sources of conviction are more likely to see calculus as logical and consistent. These students are better able to use definitions and logic to make sense of mathematics, more likely to have a coherent understanding of the limit concept, and more likely to hold a static conception of limit throughout the interview. On the other hand, she found that students who hold external sources of conviction tend to view mathematics as a collection of procedures and rules to be memorized and applied in the appropriate situations. To these students, mathematical theory, including definitions, proofs, and counterexamples, is unlikely to play an important role in their understanding. These students are more likely to give incomplete or contradictory explanations of the limit concept, more likely to hold common misconceptions about limits, and less likely to have the ability to explain the procedures they are using.

In perhaps the most comprehensive study to date regarding the understanding of the limit concept, Williams (1991) compared students' limit models to six limit characterizations and explored a variety of materials intended to cause cognitive conflict within the students. He did this by asking students to complete a questionnaire containing several common beliefs as shown in prior research (see Figure 1).

A. Please mark the following six statements about limits as being true or false:

1. T F A limit describes how a function moves as x moves toward a certain point.
2. T F A limit is a number or point past which a function cannot go.
3. T F A limit is a number that the y-values of a function can be made arbitrarily close to by restricting the x-values.
4. T F A limit is a number or point the function gets close to but never reaches.
5. T F A limit is an approximation that can be made as accurate as you wish.
6. T F A limit is determined by plugging in numbers closer and closer to a given number until the limit is reached.

B. Which of the above statements best describes a limit as you understand it? (Circle one)

1 2 3 4 5 6 None

C. Please describe in a few sentences what you understand a limit to be. That is, describe what it means to say that the limit of a function f as $x \to s$ is some number L.

Figure 1. Questionnaire used by Williams (1991, p. 221).

According to Williams, "statements 1 – 6 can be characterized as describing limit respectively as (a) dynamic-theoretical, (b) acting as a boundary, (c) formal, (d) unreachable, (e) acting as an approximation, and (f) dynamic-practical" (p. 221). From the results of this questionnaire, he found "that students often describe their understandings of limit in terms of two or more of these informal ideas" (p. 225). In addition, the most popular characterizations from his questionnaire were: 1. the dynamic-theoretical model, selected as "true" by 80% of respondents and selected the best description by 30% of respondents; 2. the unreachable model, selected as "true" by 70% of respondents and selected the best description by 36% of respondents; and 3. the formal model, selected as "true" by 66% of respondents and selected the best description by 19% of respondents.

After administering the questionnaire, Williams selected a small group of students to take part in the second phase of the study which consisted of tasks designed to create cognitive conflict within students possessing these informal models of limit. Of most interest to my study are the results of the students identified as possessing the dynamic-theoretical model of limit. None of the students containing dynamic-theoretical models changed their view of limit during the course of the study. When these students encountered functions that contradicted their current model of limit, often they would dismiss the contradiction as irrelevant – an anomaly that does not pertain to most situations. In this way, Williams noticed that students' attitudes towards mathematical truth played a key role in determining their reactions towards the study's tasks. Several students made statements that they do not believe a general description of limit exists. As stated by one of the study participants, "I don't think there is a definition that is going to fulfill every function there is" (p. 232). Williams found several aspects of models that students valued that might have contributed to their resistance to change their viewpoints when faced with cognitive conflicts. Two of these aspects discussed are expediency and simplicity.

Where Williams found that the dynamic-theoretical model of limit was common among calculus students and relatively resistant to change, Oehrtman (2002, 2003) classified motion as a "weak" metaphor, stating that language referring to motion was frequently not intended to be a description of something actually moving. In one example which is particularly relevant to this study, students were asked to "Explain what it means for a function of two variables to be continuous" (Oehrtman, 2003, p. 399). In response to this prompt, six of the twenty-five participants actually discussed an object in motion, while another eleven students used motion language without applying that language to an actual object. For the six students who described something as moving, Oehrtman argues "that motion tended to be simply superimposed on another conceptual image that actually carried the structure and logic of their thinking" (p. 402). For example, one student describes a continuous two variable function as a board in which a mouse can run around on without falling through. In this case, the motion of the mouse was not the primary imagery; rather, the primary image was that of a board without any holes in it.

Much of Oehtman's 2003 work was based on the use of metaphors studied by Lakoff and Nunez (2000). The relevance of this theory for the present study will be discussed in the section "mental representations and conceptual metaphors." From this perspective, Nunez (1999) points out that there are inherent differences between two conceptualizations of continuity, which he refers to as *natural continuity* and *Cauchy-Weierstrauss continuity*. From Nunez's perspective, natural continuity arises from a natural metaphor "a line IS the motion of a traveler tracing that line" (p. 56). From this perspective, the motion creates the line, and continuity is the result of fluid motion. Contrasting this view of continuity is the Cauchy-Weierstrauss view of continuity, which is the result of the 19th century formal mathematics. From Nunez's perspective, Cauchy-Weierstrauss continuity is built upon three conceptual metaphors: "A line IS a set of points; Natural continuity IS gaplessness; Approaching a limit IS preservation of closeness near a point" (p. 57). To Nunez, these three metaphors create a separate conceptualization of continuity that contrasts significantly with the conceptualization of natural continuity.

Just as Nunez observed a cognitive difference between natural continuity and the formal definition of continuity, other authors have noticed significant difference between students' understanding of limits and the formal definition of limit. One significant difference is described by Kyeong Roh Hah (2005) as *reversibility* or *reverse thinking*. These words describe one inherent difference in the formal limit approach to the intuitive approach often used by students. Introductory calculus often teaches the limit concept as the result of the function as the independent variable gets closer and closer to the limit point. Notice, in this case it is the independent variable which is made close to the limit point and the value of the dependent variable is observed. This contrasts the formal definition of limit which requires that the dependent variable can be made arbitrarily close to the limit value for all values of the independent variable within some neighborhood of the limit point. In this case it is the dependent variable that is being made close to the limit value instead of the independent variable being made close to the limit point. Roh Hah studied this type of reverse thinking in the context of infinite sequences and described it as "the ability to think of the infinite process in defining the limit in terms of the index and simultaneously to reverse the process by finding an appropriate index in terms of an arbitrarily chosen error bound" (pp. 20-21).

Theoretical Perspectives

In the 1950s and 1960s, the "new math" movement brought with it an increased emphasis on clear definitions and mathematical rigor. However, by the late 1970s and early 1980s, mathematics education researchers began observing sharp differences between mathematical concepts as they were taught in class and the concepts as they were understood by the students. In particular, several authors noted key difficulties understanding such concepts as limits of functions (Ervynck, 1981; Sierpinska, 1987) and limits of sequences and series (Davis and Vinner, 1986).

At this time researchers began distinguishing between mathematical ideas as they are presented in formal mathematics and those same ideas as they are understood by students. The terms concept image and concept definition were created to describe this difference (Tall and Vinner, 1981; Vinner and Hershkowitz, 1980). The term *concept image* was introduced to describe the "total cognitive structure that is associated with the concept, which includes mental pictures and associated properties and processes" (Tall and Vinner, 1981, p.152). On the other hand, the *concept definition* was created to refer to a formal definition of a concept, such as a definition found in a textbook. In line with constructivist learning theories, this idea focuses learning and understanding on the individual and his or her conceptions rather than on the formal words used to describe a concept.

Cognitive Obstacles

Other researchers showed that large parts of students' concept images are built on intuition and experiences gained outside the formal teaching of a subject. Cornu (1991) refers to these conceptions of an idea obtained from daily experience prior to formal instruction as *spontaneous conceptions*. These conceptions can be quite powerful and do not disappear when formal ideas are presented. Instead, these spontaneous conceptions and any new knowledge obtained from instruction may coexist independently or they may intermingle to form new conceptions in the student. This occurs even if the different ideas conflict with each other. According to Papert (1980), "Sometimes the conflicting pieces of knowledge can be reconciled, sometimes one or the other must be abandoned, and sometimes the two can both be 'kept around' if safely maintained in separate mental compartments" (Papert, 1980, p.121). Importantly, it was observed that learning a new idea, in itself, was not enough to change a students' prior conceptions. Instead, the student might simultaneously hold on to both ideas and then select which one to use in any given situation. Students may even retrieve combinations of the two ideas, with detrimental results (Davis and Vinner, 1986).

In response to these observations, researchers began looking for models of cognitive change which might describe how a student may conceptually reorganize a concept. One such model requires that three criteria must be met before a student will be willing to undergo conceptual reorganization. First, the student must be dissatisfied with the current organization of a concept. Second, an alternative conception must be available which the student finds both reasonable and understandable. Third, the student must come to view this alternative conception as useful or valuable (Posner, Strike, Hewson and Gertzog, 1982). In a similar vein, Nussbaum and Novick (1982) propose an instructional method which will allow students to create conceptual change. They propose that, first, the student take part in an *exposing event* created to help students become acquainted with their own current conceptions. The student is then exposed to a *discrepant event* created to cause dissatisfaction towards the student's current conceptions. Finally a *resolution* is provided which gives the student an opportunity to interact with new, alternative conceptions.

From these ideas of cognitive change, several researchers in mathematics education began studying cognitive obstacles. A cognitive obstacle can be described as a conception that creates a barrier to further student understanding. Cornu (1991) describes several different types of obstacles: "genetic and psychological obstacles which occur as a result of the personal development of the student, didactical obstacles which occur because of the nature of the teaching and the teacher, and epistemological obstacles which occur because of the nature of the mathematical concepts themselves" (Cornu, 1991, p.158). Epistemological obstacles are of particular interest to this study since they tend to be conceptions that prove to be quite useful in one domain but create an obstacle when translated into another, similar domain. Furthermore, epistemological obstacles are often unavoidable and essential to learning, and they are frequently found in the historical development of the concept (Cornu, 1991).

Mental Representations and Conceptual Metaphors

It is important to consider how students come to understand mathematics. One common theoretical perspective is that of mental representations (Williams, 2001). This viewpoint holds that mathematical learning places ideas, facts, and procedures as part of an internal network of mental representations, and the depth of understanding is determined by connections with other representations within this network (Hiebert and Carpenter, 1992). From this perspective, a mathematical idea is understood when its mental representation has a large number of strong, robust connections with other representations.

In contrast to this theory is the idea of conceptual metaphors developed by Lakoff and Nunez (2000). From the perspective of these authors, there is an “intimate relation between cognition, mind, and living body experience in the world” (Nunez, 1999, p. 49). In this way, all mathematics is considered to be a result of our embodied experiences, and the meaning of mathematics is built upon conceptual metaphors which project meaning onto new, abstract domains from previously understood, more concrete concepts.

It is from this second perspective that this study will be conducted. This would emphasize the belief that knowledge is not only based on connections between related concepts, but that students will actually create meaning for new, abstract concepts based on their understanding of other well-understood ideas. In relation to this study on students' understanding of limits in three-dimensional calculus, we would expect the students to have previously developed a strong understanding of the limit concept in single variable calculus and they would attempt to project this understanding onto the new, multivariable limit problem.

One powerful metaphor used to understand limits is the *fictive motion metaphor* (Talmy, 1988). This provides a metaphorical means of conceptualizing a static curve as the result of dynamic motion. This metaphor is common throughout the English language. A statement such as “the trail goes to the peak of the mountain” uses dynamic language to capture essence of a static object, a trail. In this way, the fictive motion metaphor is used in mathematics to perceive a graph not as a set of points but a path created by dynamic motion. Since this study focuses on students’ use of dynamic imagery in multivariable calculus, the fictive motion metaphor has the potential to play an important role in the description of students’ conceptualization of the multivariable limit concept.

Metaphorical thinking has been studied in regard to limits in particular and within mathematics education as a whole. Of particular interest are the studies by Oehrtman (2002, 2003) in which he analyzed the written and verbal language of first-year calculus students' reasoning about limits. In his research he divides metaphors into weak and strong metaphors. Strong metaphors are ones that "force the relevant concepts involved to change in response to one another" (Oehrtman, 2003, p.398). These metaphors are active and support creative thinking in new domains.

Abstraction and Generalization

The concept of reflective abstraction was introduced and discussed by Piaget (see, for example Piaget, 1985) to describe the development of logico-mathematical structures in a child during cognitive development. Reflective abstraction is considered to be entirely internal, as opposed to empirical abstraction and pseudo-empirical abstraction which derive from properties of objects and actions of those objects, respectively. Piaget considered four different kinds of mental constructions that could take place during reflective abstraction: *interiorization,* or the construction of an internal process in order to represent a perceived phenomena; *coordination* of two or more processes into a single new one; *encapsulation* of a dynamic process into a static object; and *generalization,* or the application of existing knowledge to new phenomena.

Generalization is discussed in Harel and Tall (1989) and he distinguishes three different types of generalization that may occur. *Expansive generalization* extends an individual's thinking from one domain to another without changing the original ideas. *Reconstructive generalization* extends an individual’s thinking while at the same time reconstructing the existing concepts in order to make the generalization reasonable. Harel and Tall also describe *disjunctive generalization* in which new ideas are created without an attempt to connect them with prior understanding. This is generalization in the sense that the student is familiar with a larger range of concepts, but this could not be considered a mental reconstruction of the student's knowledge in the way that Piaget discusses it.

Infinite Processes

Many difficulties in understanding the limit concept are found in the infinite nature of limits. The limits concept is often conceptualized as an infinite process which can never be completed in its entirety. However, even though this process cannot be completed, mathematicians are capable of speaking about the limit process as a coherent whole, and they are capable of using the result of a limit as an object to create more sophisticated processes. This means of understanding a process as an object is not unique to the limit concept, but plays a vital role in all of mathematics. Tall et al. (2000) give a thorough description of several authors' descriptions of the cognitive processes involved in converting a mathematical process to an encapsulated object. I will briefly describe several of these viewpoints below.

Dubinsky (1991) and his colleagues developed a theory of conceptual development based on the creation of actions, processes, objects, and schemas. This theory has become known as APOS theory. In this model of student learning, the student first understands a mathematical concept as an *action* to be performed. After some experience with the action, the student is able to perceive the action as a *process* and speak of its result without being required to perform the action. Eventually this process will be encapsulated into an *object* which can in turn be used to create more sophisticated mathematical actions. The student then gathers these related actions, processes and objects into a coherent collection called a *schema*. The concept of schema is similar to that of a concept image, with the exception that a schema is required to be coherent while a concept image is not.

From the perspective of APOS theory, an action is a step-by-step mathematical procedure. In order for a concept to progress from an action to process, the individual must become aware of the various steps involved in the action and have the ability to reflect on them. It is this ability to think about the action without actually performing it that distinguishes a process from an action. The primary difference between a process and an object is the ability to conceptualize the process as a whole and perform actions with it.

Sfard (1991) prefers to the use the word *reification* over *encapsulation* to describe the process of understanding a mathematical process as an object. For Sfard the act of reification is a movement from an operational understanding to a structural understanding. Sfard's description of the transformation of a process to an object takes place in three steps. She describes the adoption of a familiar process as the *interiorization* of that process. Once interiorized, the process can be compacted and understood as a whole; which she refers to as *condensation*. To Sfard a condensed process is still operational and the individual will interact with the process in an operational manner. It is the process of *reification* that transitions the individual from dealing with an operational process to a structural object. To Sfard it is precisely this transition from operational to structural understanding that signifies the transition from a condensed process to a reified object.

Eddie Gray and David Tall (1994) used the word *procept* to describe their understanding of how a mathematical concept can take the form of both a process and an object simultaneously. From their perspective the important development is the creation of a symbol to represent both the process and the object is an essential part of the development of a procept.

Many of these authors developed their theories using finite procedures, such as counting or addition. The infinite nature of the limit concept makes it particularly challenging for an individual to progress from viewing the concept as a process to viewing it as an object. Tall et al. (2000) discussed this difficulty and pointed out that in "the peculiar case of the limit concept where the (potentially infinite) process of computing a limit may not have a finite algorithm at all [...] a procept may exist, which has both a process (tending to a limit) and a concept (of limit), yet there is no procedure to compute the desired result" (p. 226).

CHAPTER III

METHODS

The methods of this study were developed with the problem statement in mind: "Describe how students with a dynamic view of limit generalize their understanding of the limit concept in a multivariable environment." In particular, the study was designed to accomplish the following objectives; the study should:

- Identify students with a dynamic view of limit.
- Provide an opportunity to analyze participants' prior understanding of the limit concept.
- Allow participants to encounter the limit concept in multivariable environments.
- Provide an opportunity to observe participants' generalization of the limit concept in these multivariable environments.

In this chapter I will begin by explaining why qualitative research methods were used to design this study. I will then describe the methods of the study, including the participants, settings, interviewee selection method, and data collection methods. Finally, I will conclude this chapter with a discussion about the researcher's role and perspective.

Why a Qualitative Study?

Data collection for the study took place during the fall 2007 semester at a large state university. It was determined that a series of in-depth interviews using qualitative analysis would be required to adequately respond to the study's problem statement.

Maxwell (1996) describes five research purposes which are specifically suited to qualitative analysis.

- “Understanding the *meaning*, for participants in the study, of the events, situations, and actions that are involved with and of the accounts that they give of their lives and experiences” (p. 17).
- “Understanding the particular *context* within which the participants act, and the influence that this context has on their actions” (p. 17).
- “Identifying *unanticipated* phenomena and influences, and generating new grounded theories about the latter” (p. 19).
- “Understanding the *process* by which events and actions take place” (p. 19).
- Developing *causal explanations*” (p. 20).

An examination of each of these five research purposes creates a strong argument for this study to be qualitative in nature.

The purpose of this study is to create a description of students’ conceptualizations of the limit concept. Internal conceptualizations are, by their nature, not observable. Therefore, observable data must take the form of written responses, mathematical calculations, and verbal descriptions of mathematical concepts. The primary interest of the study is not in reporting the observable data, but rather in understanding the *meaning* of the observable data in terms of the students’ internal conceptualizations of the limit concept. Because of this, the *context* of the observable data and the *process* which students use to create mathematical conclusions play an essential role in understanding the *meaning* of the observable data.

The fact that this is a first study on students' conceptualizations of multivariable limits is an important factor in choosing a qualitative study. The lack of previous research reports on the topic requires an ability to respond to *unanticipated* events. A quantitative study cannot be prepared to deal with unanticipated behavior, but a qualitative study using grounded theory techniques can describe unanticipated behavior in the context in which it happens.

Finally, the purpose of this study asks a *causal question* that requires qualitative methods to answer. The causal question in this study is "what role does prior understanding of the limit concept have in students' conceptualizations of multivariable limits?" This question is qualitative in the sense that it seeks to describe the influence of certain cognitive events on other events. This contrasts a quantitative question which seeks to explain current events in terms of the variance of a previous set of events.

For the above reasons a qualitative study was developed that allowed the researcher to observe and interview students interacting with multivariable limits. The study was developed to contain three key components, a) an initial questionnaire which provided a means of selecting interview participants and comparing those participants to the student population as a whole, b) an interview probing students' understanding of single variable calculus, and c) a series of two interviews involving multivariable limits in four different settings.

Participants and Setting

The goal of this study was to analyze the changes in student thinking about the limit concept as they encounter multivariable limits. For this reason, it was important to observe students who are familiar with both multivariable functions and the limit concept but have yet to study multivariable limits. The university where this study took place teaches calculus as a three-part sequence with multivariable calculus being a main focus of the third semester in this sequence. Therefore, participants for the study were chosen from this third semester calculus course.

Participants for the study were chosen using a process of purposeful selection. The selection of a *purposeful sampling* seeks to choose uniquely qualified individuals capable of providing information necessary to answer the study's research questions as described by Maxwell (1996). For a small-scale study, purposeful selection is often preferred to a random sample. Random samples are necessary to externally generalize the findings of the study; however, since external generalization requires a sufficiently large sample size, in the case of a small sample size it is preferred to purposefully select participants likely to provide useful information towards answering the study's research questions. It is important to note that the sample chosen for this study was not what is often called a 'convenience sample.' Rather than choosing participants based on convenience, participants were chosen using the predetermined set of guidelines outlined in the next section.

The purposeful selection process of this study sought to find students who tend to conceptualize the limit concept in a dynamic manner. Since the goal of this project is to describe student's cognitive behavior, preference was given to students who demonstrated a strong ability to express themselves in a clear manner. For these reasons, a questionnaire was given to all willing students who participated in third semester calculus in the fall of 2007. This questionnaire had two useful purposes. Most importantly, it provided an opportunity to analyze a large number of students' understanding of the limit concept, allowing those students with the preferences described above to be selected for the interviews. It also created a description of the entire population of students enrolled in this course, indicating how well our selected students represented the course's population as a whole.

Study Questionnaire

The development of the questionnaire (see Appendix A) was inspired by Williams (1991). Part A consisted of six definitions each representing a different theoretical model of limit commonly held by students studying calculus. It is important to note that actual models of limit held by students are extremely complex cognitive structures, and they are not expected to precisely line up with any theoretical model given here. For this reason, I will use the phrase *theoretical model* to refer to those theoretical models of limit that I believe a student may possess, and I will use the phrase *personal model* to refer the actual model of limit held by an individual student.

Similar to Williams, the beginning questionnaire attempted to gauge to what degree students agree with various theoretical models. Each of these models was inspired by either research on students' understandings of the limit concept or an historical development of the limit concept. The model represented by each question is given below:

Question 1. "A limit describes how a function moves as you approach a given point."

Dynamic Model. This model is based on the dynamic imagery that the graph of a function is the path of a point swept out over time. This imagery was used by Newton when he developed the calculus (Edwards, 1979) and has been shown to be common among calculus students studying the limit concept (Williams, 1991).

This is the theoretical model we are most interested in and represents what Williams refers to as a dynamic-theoretical model.

Question 2. "A limit can be found by plugging in a number infinitely close to a point."

Infinitesimal Model. This model is based on the existence of infinitely small quantities. Cornu (1991) discussed this as an epistemological obstacle towards the development of a formal limit concept.

Question 3. "A limit is a number that a function can be made arbitrarily close to by taking values sufficiently close to a certain point."

Formal Model. This model is based on the modern, Cauchy-Weierstrass definition of limit, and closely mimics the definition of limit given in many introductory calculus textbooks.

Question 4. "A limit is a number or point the function gets close to but never reaches."

Unreachable Model. This model is based on the popular misconception that a limit can never be attained (Davis and Vinner, 1986).

Question 5. "A limit is an approximation that can be made as accurate as you wish."

Approximation Model. Based on the notion of limit as an approximation. This was used in the study by Williams (1991).

As observed by Williams, students rarely possess a personal model of limit closely aligned to one of these theoretical models. Instead, students' personal models of limit tended to be complex combinations of these theoretical models. For this reason, classification of students into distinct categories is extremely difficult. Questions 6 - 8 were included to better distinguish which theoretical model (or models) best represented each student's personal model of limit (see Appendix A).

In our process of purposeful selection, the following criteria were used to select which students would take part in the remainder of the study.

1. Students should select "somewhat agree" or "strongly agree" to question 1 of the questionnaire.
2. Students should circle the number 1 on question 6 of the questionnaire.
3. Students should use dynamic language in their responses to questions 1, 7, and 8. Dynamic language is considered to be language that emphasizes the use of motion in understanding the limit concept. Such language might include key phrases such as "moves towards," "approaches," or "gets closer to."
4. Students should provide written descriptions that demonstrate an ability to express themselves in a clear manner.

From these four criteria, students were invited to participate in the interview portion of the study in the following manner: Questionnaires were collected from all students indicating interest in participating in the interviews. Questionnaires that failed to meet #1 above were removed from consideration. The remaining questionnaires were analyzed and those that did not meet #2 or #3 above were removed from consideration. The remaining questionnaires were analyzed along both #3 and #4 above, and students were judged as to how strongly they met each of the criteria. Students who were judged to have used strong dynamic language in their responses as described in #3 above and who were judged to demonstrate a strong ability to express their thinking as describes in #4 above were invited to participate in the interviews. In total nine students were invited to participate in the interviews. Seven of the nine students agreed to participate in the interviews and all seven completed interview process.

Data Collection

Recalling that the goal of this study is to create a description of student behavior, it was decided that a qualitative study using task-based interviews would be the best method for data collection. Thomas (1998) described various interview strategies used in qualitative studies. The interviews in this study contained questions of two types: general questions discussing the meaning of the limit concept and task-based questions centered on specific limit problems. The general questions can be described as *loose questions* in the sense that their goal is to "reveal the variable ways respondents interpret a general question" (Thomas, 1998, p. 129). Task-based questions can be described as *response-guided questions* in the sense that they "consist of the interviewer beginning with a prepared question, then spontaneously creating follow-up queries relating to the interviewee's answer to the opening question" (p. 132). In the case of this study, the initial questions take the form of a mathematics problem and follow-up questions are asked to clarify meaning about students' responses while solving the mathematical problem.

Data from the interviews took three forms: written work in response to mathematical limit problems, verbal responses to questions throughout the interview, and observation of student behavior throughout the interview (important behaviors might include pointing at a graph or the use of hand gestures). In order to capture both verbal and observational data throughout the course of the interviews, it was decided that videotaping would be a primary source of data collection. Each video segment was stored on DVD disks and viewed only by those involved in overseeing the study.

In order to analyze students' prior understanding of the limit concept, the first interview session (see Appendix B) focused solely on the students' understanding of limits in a traditional introductory calculus course.

Participants for this study were carefully chosen so that they possessed a personal model of limit that involves some element of dynamic imagery. Question 1 and question 2 of the first interview session were designed to evaluate the strength and nature of this dynamic imagery in each student's personal model of limit.

> Question 1: Review your answers to the questionnaire given earlier. Would you like to change any of your answers? Are there any answers that you would like to clarify?
>
> Question 2: When describing a function as "approaching" or "getting close to" a point, this idea would best be explained as:
>
>> a) Evaluating a function at different numbers over time with those numbers successively getting closer to the point in question.
>>
>> b) Mentally envisioning a point on a graph moving closer and closer to the limit point.

Question 1 reviewed with each student his/her responses to the questionnaire, providing an opportunity for students to explain their responses in detail. Question 2 was designed to evaluate the dynamic nature of each student's conception of limit by providing them with two options, one which involves examining a function at various points over time and a second which involves mentally envisioning a point in motion along a graph. Williams (1991) would refer to those students who select (a) as having a dynamic-practical model of limit and those selecting (b) as having a dynamic-theoretical model of limit. The remainder of the first interview session was designed to observe students' behavior on traditional introductory calculus limit problems in order to determine how the students' personal model of limit is put into practice on actual problems (see Appendix B). This portion of the interview involved both symbolic and graphical limit problems from introductory calculus, and a calculator was made available to students for use with these problems.

The remaining interview sessions were designed to provide students with the opportunity to encounter multivariable limits in four different settings: traditional symbolic manipulation, symbolic manipulation involving polar coordinates, three-dimensional graphing, and contour graphing. These were designed as four separate treatments (see Appendices C and D). Each treatment introduces students to the multivariable setting in question, asks students to describe how to determine whether a multivariable limit exists or not in that setting, asks students to explain why they believe their method should work, and observes students using this method on problems from this setting. It was decided that for purposes of time, these four treatments would take place over the course of two interview sessions, with each session containing a symbolic and graphing portion. The first session contained traditional symbolic manipulation and three-dimensional graphing while the second session contained symbolic manipulation involving polar coordinates and contour graphing. In order to allow the students to interact with a larger number of multivariable functions, a computer was used to experience graphs during the three dimensional graphing portion of the interviews and computer generated graphs were printed out and provided during the contour graphing portion of the interview. The mathematical software package Maple 11© was used to create all the graphs used in this study.

In addition to these four treatments, students were asked several interview questions following each multivariable limit experience designed to help them reflect on their experiences with multivariable limits. This portion of the interviews was created to provide students with an opportunity to discuss their overall understanding of the limit concept and the connection between single variable and multivariable limits.

Researcher's Perspective

In qualitative research it is necessary that the researcher be actively involved in the setting of the research study, and as a consequence the collection and interpretation of the data will be effected by the role the researcher plays. Holliday (2002) explains, "The presence of the researcher in the research setting is unavoidable and must be treated as a resource" (p. 173). Because of this fact, the perspective that I bring with me into the study should be assessed in order to put the data from the interviews in context. For this reason, I will spend the remainder of this section discussing my beliefs and expectations prior to the collection of data for the study.

In this study I took the position that students would enter the study with an initial personal model of the limit concept. I use the word *model* much in the same sense as Williams (1991) to be a collection of cognitive structures which has an internal meaning to the student and carries with it some predictive qualities. I use the phrase *personal model* in contrast to the phrase *theoretical model* which captures a hypothetical conceptualization of the limit concept. Throughout the study I expected students to be able to coherently express their beliefs about their personal model of limit and use the model to determine the truthfulness of related mathematical statements. I also expected the students to be able to use this model to make sense of related mathematical ideas; in particular, I expect the students to try to use their personal model of limit to make sense of limit problems in a multivariable setting. I anticipated that the students' models of limit would be relatively static, until confronted with a discrepant event, in the sense of Nussbaum and Novick (1982).

At the same time, I expected students involved in the interviews to be in the early developmental stages in terms of their personal model of multivariable limits. I expected this notion of multivariable limits to be less coherent and less consistent than their personal model of single variable limits. However, I anticipated that students would use their single variable limit model to interpret multivariable limits, providing a basis for their actions and statements in the new context.

It is worthwhile to note that I use the term *model* to mean something very similar to Dubinsky's (1991) use of the word *schema*. The primary difference is my emphasis on the predictive qualities of a students' model of limit, while Dubinsky emphasizes the collection of actions, processes and objects which are contained in a limit schema. However, both these notions are more specific than Tall and Vinner's (1981) notion of a student's concept image. Both a model and a schema are intended to be coherent in the sense that students are, to some degree, aware of these structures and able to use them in productive ways. Meanwhile, the term *concept image* is the collection of *all* cognitive structures connected with the concept. This concept image may not be coherent and a student may have little awareness of it or ability to use it productively.

Many past studies have looked at students' abilities to use the imagery of motion to understand the limit concept (Monk, 1992; Thompson, 1994b; Carlson et al, 2002). Williams (1991) found that 30 percent of the students in his study contained what he called a dynamic-theoretical view of limit, while 80% of the students believed that a dynamic-theoretical definition of limit was true. This dynamic-theoretical view of limit is marked by a student's use of motion to understand the limit concept. Students involved in this study were carefully chosen to have a personal model of limit similar to the dynamic-theoretical model of limit and I expected this aspect of their thinking to influence the way they conceptualize multivariable limits.

In a multivariable setting, the idea of motion easily assists showing that a limit does not exist, since a limit such as $\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$ can be shown to not exist by observing that when moving along the *y*-axis the limit tends to 1, while when moving along the *x*-axis, the limit tends to -1. Since these two directional limits are unequal, the limit does not exist. This should be somewhat familiar to students; since, in the two-dimensional case you can show a limit does not exist by showing that $\lim_{x\to a^+} f(x) \neq \lim_{x\to a^-} f(x)$. However, unlike in the two-dimensional case, the multivariable limit is not solved by simply showing that the two directional limits are equal. In fact, in the three-dimensional case there are uncountably many directional limits, following any possible path to the point (*a*,*b*), that must all be equal for $\lim_{(x,y)\to(a,b)} f(x,y)$ to exist. Therefore, the imagery of motion towards a point is insufficient to show that a multivariable limit exists.

Because this use of motion is insufficient to completely understand the multivariable limit problem, I expected students to encounter a cognitive obstacle from their application of motion into multivariable limits. From there, I anticipated an effort on the part of the students to reconstruct their understanding of the limit concept to allow for a complete understanding of multivariable limits. The nature of this anticipated reconstruction process is one of the primary focuses in this study.

These expectations color the way I interacted with students during the interviews. As I engaged in response-guided questioning, they affected the types of questions I asked and the manner in which these questioned were presented. I do not believe my role in the interviews should be perceived as a negative aspect of the study design; rather, I believe that my expectations enabled me to guide the interviews towards a line of discourse that would be profitable for answering the study's research questions.

CHAPTER IV

DATA ANALYSIS

In this chapter I will describe how the collected data was analyzed. I will begin with a description of the questionnaire results which show how the seven selected interview participants compare to the entire calculus III student population. Then I will describe how the interview data was analyzed qualitatively. Finally, I will describe how this analysis of the transcripts along with an analysis of formal mathematics led to the development of three models of limit: neighborhood, dynamic, and topological. These three models will be used to code the interview data and shape the results provided in chapter V.

Questionnaire Results

During the first week of the fall semester, the study questionnaire was distributed in all five sections of calculus III offered by the university. All willing students completed the questionnaire at this time and a total of 208 students returned their responses. The results on questions one through five are given in Tables 1 and 2 below.

	Strongly Agree	Somewhat Agree	Neither	Somewhat Disagree	Strongly Disagree	Likert Average (4 = strongly agree)
Statement 1	105	68	3	22	10	3.13
Statement 2	46	97	19	33	13	2.62
Statement 3	63	93	28	14	5	2.96
Statement 4	84	51	20	30	23	2.69
Statement 5	32	66	35	37	37	2.09

Table 1: Questionnaire Results (Cumulative)

	Strongly Agree	Somewhat Agree	Neither	Somewhat Disagree	Strongly Disagree
Statement 1	50%	33%	1%	11%	5%
Statement 2	22%	47%	9%	16%	6%
Statement 3	31%	46%	14%	7%	2%
Statement 4	40%	25%	10%	14%	11%
Statement 5	15%	32%	17%	18%	18%

Table 2: Questionnaire Results (Percentage)

As opposed to a Likert Scale analysis, the result can also be viewed as binomial data, as in tables 3 and 4 below.

	Agree	Disagree
Statement 1	173	32
Statement 2	143	46
Statement 3	156	19
Statement 4	135	53
Statement 5	98	74

Table 3: Questionnaire Results (Binomial)

	Agree	Disagree
Statement 1	83%	15%
Statement 2	69%	22%
Statement 3	77%	9%
Statement 4	65%	25%
Statement 5	47%	36%

Table 4: Questionnaire Results (Binomial Percentage)

It can be observed from the tables above that the respondents have a strong tendency towards agreeing with the statements as presented. This corresponds with the finding from Williams (1991) that students are often capable of believing several models of limit simultaneously. This result, however, should be treated carefully due to the known phenomenon of acquiescence bias that students tend to agree with statements as presented. Due to this known fact, students were asked which model best described the way they understood the limit concept and the results are presented in tables 5 and 6 below.

Statement:	1	2	3	4	5	None
	78	15	37	58	5	8

Table 5: Question #6 Results (Cumulative)

Statement:	1	2	3	4	5	None
	39%	7%	18%	29%	2%	4%

Table 6: Question #6 Results (Percentage)

From the 208 students who completed the study questionnaire, 36 agreed to take part in the interview portion of the study. Their responses follow in tables 7 and 8.

	Strongly Agree	Somewhat Agree	Neither	Somewhat Disagree	Strongly Disagree	Likert Average (4 = strongly agree)
Statement 1	20	12	1	3	0	3.36
Statement 2	7	15	6	7	1	2.56
Statement 3	10	17	3	2	3	2.83
Statement 4	13	7	5	6	5	2.47
Statement 5	7	11	3	6	9	2.03

Table 7: Questionnaire Results, Interview Volunteers

Statement:	1	2	3	4	5	None
	14	4	6	9	0	1

Table 8: Question #6 Results, Interview Volunteers

These responses are closely aligned with the responses of the student population as a whole, as tables 9 and 10 show.

	Agree		Disagree	
	36 Volunteers	208 Students	36 Volunteers	208 Students
Statement 1	89%	83%	8%	15%
Statement 2	63%	69%	23%	22%
Statement 3	84%	77%	16%	9%
Statement 4	65%	65%	35%	25%
Statement 5	67%	47%	56%	36%

Table 9: Questionnaire Results, Volunteer Comparison

Statement	1	2	3	4	5	None
36 Volunteers	42%	12%	18%	27%	0%	3%
208 Students	39%	7%	18%	29%	2%	4%

Table 10: Question #6 Results, Volunteer Comparison

As the above tables show, with a few slight variations the two groups responded to the questionnaire in a similar fashion. The only exception was found in the response to question 5 which was found to be significantly different ($p < .05$). Since the emphasis of this study is on students with a dynamic understanding of limit, this difference in the students' opinions of the approximations model of limit was deemed to be insignificant in light of the study's goals. Therefore, this difference was not further explored in this study.

From these 36 volunteers 9 were contacted to take part in the individual interview sessions, and of those 9, 7 participated in the interviews. All participants who began the interview process completed all 3 interviews.

A total of three males and four females took part in the interview portion of the study. These seven interview participants will be given the pseudonyms Mike, Jessica, Jennifer, Amanda, Josh, Ashley, and Chris for the remainder of this study. These seven students were chosen using a method of purposeful selection so that their questionnaires showed a tendency towards understanding the limit using dynamic imagery. These seven students' responses to the questionnaire are shown in tables 11 and 12.

	Strongly Agree	Somewhat Agree	Neither	Somewhat Disagree	Strongly Disagree
Statement 1	5	2	0	0	0
Statement 2	2	3	0	1	1
Statement 3	1	3	1	0	2
Statement 4	2	2	1	1	1
Statement 5	1	1	0	2	3

Table 11: Questionnaire Results, Interview Participants

Model:	1	2	3	4	5	None
	6	0	1	0	0	0

Table 12: Question #6 Results, Interview Participants

The above tables demonstrate the clear preference for participants who agree with statement 1 on the questionnaire. As shown by the entire class results, these seven students represent a sizeable portion of the class. From table 4, 83% of the students agree with statement 1 and from table 6, 38% of the students believe statement one best describes how they understand the concept of limit. Both of these numbers were the highest recorded for any of the five statements.

Qualitative Data Analysis

Qualitative data was analyzed using a grounded theory approach to data analysis. Maxwell (1996) describes grounded theory when he says, “The theory is grounded in the actual data collected, in contrast to a theory that is developed conceptually and then simply tested against empirical data” (p. 33). The fact that little previous research had been recorded on multivariable limits requires that a grounded theory approach be employed.

This process of theory development led to the realization that the initial theoretical models of limit created for the study questionnaire were inadequate to describe students’ interactions with the multivariable limit concept. This led to the creation of three new theoretical models of limit which were observed as part of students’ descriptions of multivariable limits. These models grew out of the observation that students tend to conceptualize limits using either a) a sense of ‘closeness,’ b) a dynamic process, or c) an examination of external features. In addition these models of limit were closely tied to the formal mathematics of the limit concept.

Investigating these three models led to a coding scheme for examining how students understand multivariable limits. This coding scheme was used to analyze the data and provide a description of student behavior throughout the interviews. In the remainder of this chapter, the development of the three models of limit will be described, resulting in a coding scheme which shall be used to bring clarity to the interview data.

Definitions of Limit in Formal Mathematics

In the effort to describe how students understand the multivariable limit concept, it is worthwhile to consider the formal mathematics behind the concept of limit. In this section, we will discuss how the concept of limit is developed formally and what cognitive structures might be necessary to understand this formal development.

In formal mathematics, there are two ways to define the concept of limit in $\mathbb{R}^n$. The traditional definition requires the use of universal and existential quantifiers.

Formal Limit Definition: If there exists a value $\boldsymbol{L}$ such that for every positive number, $\varepsilon > 0$, there exists a value, $\delta > 0$ such that $|\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{L}| < \varepsilon$ whenever $0 < |\boldsymbol{x} - \boldsymbol{a}| < \delta$, then we say that the limit of the function, *f*, as $\boldsymbol{x}$ approaches $\boldsymbol{a}$ is $\boldsymbol{L}$, and we write $\lim_{\boldsymbol{x} \to \boldsymbol{a}} \boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{L}$.

In contrast to this formal definition there is a definition based on sequences.

Sequential Limit Definition: If for every sequence $(\boldsymbol{x}_n : \boldsymbol{x}_i \neq \boldsymbol{a})$ with $\lim_{n \to \infty}(\boldsymbol{x}_n) = \boldsymbol{a}$ we have that $\lim_{n \to \infty} \boldsymbol{f}(\boldsymbol{x}_n) = \boldsymbol{L}$, then we say that the limit of the function, *f*, as $\boldsymbol{x}$ approaches $\boldsymbol{a}$ is $\boldsymbol{L}$, and we write $\lim_{\boldsymbol{x} \to \boldsymbol{a}} \boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{L}$.

For a function, $f\colon \mathbb{R}^n \to \mathbb{R}^m$, these two definitions can be proven to be equivalent.

However, in practice, using each definition involves inherently different cognitive processes. To compare the two definitions, I will describe the possible cognitive processes necessary to develop an understanding of each definition. A possible description of the cognitive structures required for understanding the formal limit definition is given below:

1. Mentally construct a neighborhood of values around the point $\boldsymbol{L}$.
2. Mentally construct a corresponding neighborhood of values around the point $\boldsymbol{a}$.
    a. Coordinate these two constructions such that the neighborhood around $\boldsymbol{a}$ is mapped into the neighborhood around $\boldsymbol{L}$.
3. Construct a process of reducing the size of the neighborhoods around $\boldsymbol{L}$ while maintaining corresponding, coordinated neighborhoods around $\boldsymbol{a}$.

Note that coordinating these two neighborhoods requires reverse thinking as described by Roh Hah (2005).

In contrast to the formal definition, a possible description of the cognitive structures required for understanding limits using the sequential definition of limit is given below:

1. Construct a schema for evaluating the limit using a single sequence.
    a. This involves mentally constructing a sequence, $(\boldsymbol{x_n})$, that approaches the point $\boldsymbol{a}$.
    b. Then constructing the resulting sequence, $(\boldsymbol{f}(\boldsymbol{x_n}))$.
    c. Finally, evaluating the result of this infinite sequence, the point $\boldsymbol{L}$.

2. Mentally construct an infinite process of evaluating the limit of these sequences successively.
3. Capture this infinite process into a coherent understanding of the limit of all possible sequences simultaneously.

Capturing infinite processes is an inherently difficult task as described in chapter IV, "Infinite Processes."

Just as Nunez and his colleagues (1999) found that the concept of continuity has two cognitively different conceptualizations, the two notions of limit based on the two formal definitions above appear to be built on inherently different conceptualizations. The formal definition must first be grounded on a notion of 'closeness' that can be used to create mathematical neighborhoods. This conceptualization is static and relies heavily on the conceptualization of the real number system. On the other hand, the sequential definition is built upon a process of examining points along a sequence. The conceptualization of this process is significantly different from that of mathematical neighborhoods in that it is both dynamic in nature and infinite.

Although students rarely understand the limit concept using one of these formal definitions, the two definitions do provide a blueprint for different cognitive structures that can be used to develop an understanding of the limit concept. The idea of 'closeness' used in the formal definition will form the basis for the neighborhood model of limit, and the dynamic process used in the sequential definition will form a basis for the dynamic model of limit. In the following sections, I will describe these models along with a third model, the topographical model, which is not based in formal mathematics.

The Neighborhood Model of Limit

It is well documented that students struggle to understand the formal definition of limit. This is, in part, due to the need for "reverse thinking" (Roh Hah, 2005). A description of limit using reverse thinking was observed in only one student throughout the course of the study. This discussion occurred in response to question 5 on the questionnaire which posed an approximation view of the limit concept.

Excerpt 1

> JESSICA: Yes, because if you get closer to it, like, if you like, *x*, well you can make it as close to 2 as you want. I mean, yeah, it could take you like years to figure it out but you could get as close as you wanted to depending on what the number that you put in.

Jessica's use of the phrases "you can make it as close to 2 as you want" and "you could get as close as you wanted to depending on what the number that you put in" both show some evidence of reverse thinking. They show that Jessica has an awareness that goes beyond simple closeness to an understanding that this closeness can be controlled. Oehrtman (2003) has argued that a metaphor of approximation can be used to help students better understand the definition of limit, and Jessica's response seems to support this notion. However, this was the only incident of her using language that demonstrated the use of reverse thinking, and it seems unlikely that this notion plays a vital role in her understanding of limits.

Statement 3 on the questionnaire was designed to illustrate the formal definition of limit; however, no students used reverse thinking when responding to this statement. Amanda shows her confusion towards the problem in the excerpt below

Excerpt 2

> AMANDA: Alright, number three, ‘A limit is a number that a function can be made arbitrarily close to by taking values sufficiently close to a certain point.’ This one I read over and over several times and I was like, ‘Huh?’ I know it’s given other people problems.
>
> INTERVIEWER: Yeah.
>
> AMANDA: See, the arbitrarily close to, isn’t that like, you can make a function close to certain point? Like, to me it’s, it looks like you’re choosing where the function is going. But, I don’t really know. See, I don’t think you can really choose where you’re… where you’re limit is. But, maybe I interpreted that wrong?

Amanda takes the notion of reverse thinking and interprets it as “choosing where the function is going.” To her, the dynamic nature of the limit concept requires that the independent variable be considered first, and the dependent variable is analyzed as a result. This perspective shuts out the ability to apply reverse thinking to the limit concept.

I argue that “reverse thinking” is only part of the cognitive structure needed to develop an understanding of the formal definition of limit. It is also important for students to develop a sense of ‘closeness’ that can lead to the construction of mathematical neighborhoods. In Nunez’s (1999) description of continuity, he points out that a key metaphor for Cauchy-Weierstrauss continuity is “limit IS preservation of closeness near a point” (p. 58). Using this metaphor, dynamic language such as the words “approaching” or “tending to” lose their meaning and are replaced by the static elements of a neighborhood consisting of values near the limit point.

This idea of “closeness” will be the basis for the *neighborhood model* of limit discussed throughout this study. The phrase “neighborhood thinking” will refer to the use of ‘closeness’ by a student to describe his/her conceptualizations of the limit concept. In practical terms, neighborhood thinking can manifest itself in both graphical and symbolic settings. In a graphical sense, students “look around” the function to analyze the behavior of its graph near the limit point to create arguments about the value of the limit. In a symbolic way, students may calculate values very near the limit point to draw conclusions about the limit itself. Mathematicians use inequalities to create symbolic arguments involving ‘closeness’; however, there was no evidence that students associated the use of inequalities with neighborhood thinking during this study.

The Dynamic Model of Limit

There is very little chance that students involved in this study had previously encountered the sequential definition of limit. Even though the study of infinite sequences was developed the prior semester, there is little in the curriculum to connect the ideas of sequences with those of finding traditional limits in calculus one. However, the related infinite process is a concept that is encountered by almost every student in introductory calculus. This process tends to manifest itself in two ways, symbolically using a process of evaluating the function at successive points and visually using a metaphor of motion.

The symbolic process of analyzing limits is generally introduced early in an introductory calculus course. This process is often conceptualized in the form of a table of values in which the independent variable gets successively closer to the limit point and the corresponding dependent variable is analyzed. This process is intended to be both dynamic and infinite in nature, with the independent variable getting closer and closer to the limit point with each iteration of the process. In action, however, this is never treated as an infinite process, but rather the process is ended after some finite number of steps and a conclusion is drawn about the limit of the function. In this way, as noted by Tall and his colleagues (2000) the concept of limit comes to be understood as both a process of approaching a point and the conclusion of that infinite process, while the process itself can never be carried out to its conclusion.

Visual representations of this dynamic motion have been referred to by others as the ‘fictive motion metaphor’ (Talmy, 1988; Lakoff and Nunez, 2000). Using this metaphor, the static curve of a graph actually represents the results of motion. Colloquial language supports the use of static objects embodying motion. For example, the phrase “this road goes to the lake” describes a static object, a road, as conveying a sense of motion. The road itself is not in motion, but motion is visualized on top of this structure. In a similar way, students may visualize motion in the static graph of a function. In the case of a single variable limit, this motion takes place from either the right or the left, creating the right and left hand limits. This visualization is different from the process of analyzing a function using a sequence of points in the fact that the visualization does not carry the same infinite nature as the process. Instead, the fluid motion from either side can be conceptualized as a single action and the conclusion can be visualized without using an infinite process. Similar to the symbolic process, the term ‘limit’ can refer to both the visualized motion towards the limit point and the conclusion of that motion.

Although the two manifestations above are conceptualized in significantly different ways, they both describe a dynamic process, either symbolic or visual, which results in the value of the limit. This process forms the basis of the *dynamic model* of limit discussed throughout this study. The phrase “dynamic thinking” will refer to students' use of a dynamic process, either symbolically or visually, which is intended to result in the limit value.

The Topographical Model of Limit

A student in calculus I can be quite successful solving most limit problems encountered by only looking at the surface features of a function and never developing a sense of limit related to that of either the formal definition or the sequential definition of limit. This way of thinking will be referred to as the *topographical model* of limit. It begins with an ability to recognize and classify discontinuities in a function, and through this classification procedure, it is possible to know the resulting limit value. A possible description of the cognitive structures required for developing topographical thinking in calculus I is given below:

1. The function must be viewed as an object with inherent characteristics and properties.
2. Points must be classified as either continuous or discontinuous. If it is discontinuous, then the type of discontinuity (removeable, jump, infinite) must be classified.
3. The result of the limit must be deduced from the classification process.
    a. A continuous function's limit is given by evaluating the function at the point.
    b. A removable discontinuity must be 'removed' before the limit can be evaluated. That is, using a Gestalt-type viewpoint, a point must be identified which can fill in the hole on the function.
    c. A jump discontinuity has no limit.

d. An infinite discontinuity must be examined to determine if a sign change takes place.

The term “function” in stage 1 is used in a loose sense. It is quite possible for students to treat external elements of the function, such as an equation or a graph, as the object being encountered. Even though for some students these external elements are not necessarily connected to the concept of function, they may still be used in a productive manner for solving most calculus I limit problems. However, it should be noted that this method does not cover every possible function that a student could encounter. For example, the function $f(x) = \sin\left(\frac{1}{x}\right)$ cannot be classified under the above system at $x = 0$.

This means of conceptualizing the limit concept, by itself, potentially weakens the students’ ability to understand important limiting situations in calculus. For example, Carlsen, Oehrtman, and Thompson (2007) argue that a topographical understanding of limits does not provide an understanding of the limiting processes necessary to understand differentiation and integration. It can also be observed that this conceptualization may lead to difficulties in understanding limits at infinity, which require an inherently different classification system in order to be understood topographically.

It is also important to observe that the topographical understanding of limit directly contradicts with the limit concept as introduced in formal mathematics. In formal mathematics, the notion of continuity is defined using the definition of limit as its basis. The topographical understanding of limit described here requires an understanding of continuity prior to the concept of limit, which is opposite that of the formal mathematics.

Throughout this study, the term "topographical thinking" will refer to the use of external characteristics of the function to make decisions about the value of the limit. These external characteristics tend to be visual in the form of a graph; however, it is also possible for students to use the external characteristics of a symbolic expression, for example the value of a function at a single point, to draw conclusions about the limit. This type of topographical thinking has several characteristics that distinguish it from both the neighborhood and dynamic model. Topographical thinking is static in the sense that it refers to the external features of a function as opposed to dynamic thinking which envisions motion involved in a function, and topographical thinking also places an emphasis on classifying functions based on external characteristics. This classification process need not be well defined; rather it can be based on loose feelings about the external characteristics. However, it is different from dynamic and neighborhood thinking in the fact that it aims to classify and not analyze the features of the function.

Textbook Treatment of the Limit Concept

Students involved in this study used the textbook *Calculus* by James Stewart (2003). This textbook introduces the concept of limit with the following definition.

Definition 1: We write

$$\lim_{x \to a} f(x) = L$$

and say "the limit of $f(x)$, as x approaches a, equals L"

if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by taking x to be sufficiently close to a (on either side of a) but not equal to a.

(p. 71)

Immediately following this definition, the text states that "roughly speaking, this says that the values of $f(x)$ get closer and closer to the number L as x gets closer to the number a (from either side of a) but $x \neq a$" (p. 71).

These two statements seem to be an attempt to connect students with the two formal definitions of limit discussed previously. The words "arbitrarily" and "sufficiently" appear to be an attempt to capture the notion of 'closeness' used in the formal definition of limit. The words "closer and closer" appear to be an attempt to capture the dynamic nature of the sequential definition.

Students using this textbook are introduced to both the neighborhood and the dynamic model of limit. The implication from the written textual material is that the neighborhood model of limit should be considered the 'official' understanding of limit, while the dynamic model is described as a "rough" description of the neighborhood model. There is no mention of understanding the limit using conceptualizations connected to the topographical model, and there is no implication that the two models of limit described are, in any way, cognitively different.

Coding Scheme

The three models of limit described were used to create a coding scheme for analyzing the qualitative interview data. Interview statements were analyzed, and when applicable, categorized into one of the three categories in Table 13.

Category	Description
Dynamic	Uses motion to develop meaning in a function.
Neighborhood	Uses a sense of 'closeness' to develop meaning in a function.
Topographical	Uses the shape of a graph to develop meaning in a function.

Table 13: Three Models of Limit

The analysis of the transcripts frequently revealed language relfelcting cognitive structures inherent in the three models of limit. Applicable text was sorted into 'instances' of language supporting one of the three categories. An 'instance' is understood to be a statement describing one complete thought. Statements were divided into two or more instances when the interviewee appeared to switch the focus of his/her description from one thought to another. For example in Ashley's statement below while referring to a multivariable contour graph. Her statement was considered to have two separate instances.

Excerpt 3

> ASHLEY: I'm trying to think back to the pattern idea I had with those, so there's a very distinct movements and this one seems to have a more distinct movement. Let's see… This one probably exists, just because again it has that upward sloping of values and similar line patterns. I would say this one doesn't exist.

The first instance is her description of how motion plays a role in her understanding of multivariable limits. The phrase “there’s a very distinct movements and this one seems to have a more distinct movement” was judged to be an instance of dynamic thinking since the statement emphasized the “movement” of the graph. However, it was judged that in the second part of her statement she switched her focus from describing her use of motion in general to describing her thought process on one particular problem. So it was judged that the statement, “because again it has that upward sloping of values and similar line patterns” constituted a second instance which was classified as topographical thinking because its central focus is that of exterior features of the contour graph.

In some cases, however students used a very long statement to describe a single thought. For example, Chris made the following statement when reflecting on multivariable limits.

Excerpt 4

> CHRIS: It depends which direction you’re looking at as to where the limit is coming from, where it’s going, what kind of thing, how is moving or approaching, using those two words that are confusing. But, yeah, in the first two, from a different direction, from a different x or a different y direction it changes like where it is or where it’s moving to. But on this one, on the third one, from every direction it’s the same.

This statement was judged to be a single instance since Chris maintained focus on describing his thought about which direction the limit was coming from throughout his statement. Even though he mentioned three separate problems, it was judged that he was not switching his focus from one problem to the next, but rather he was using the three problems as examples to illustrate his thinking. For these reasons, the above text was considered one instance of dynamic thinking since the focus of the entire passage was on the use of motion from different directions on the graph.

Statements which were not easily associated with one of the three primary categories were not recorded as an instance. Josh's statement below is an example of a non-instance.

Excerpt 5

> JOSH: Ok, First one (limit describes how a function moves as you approach a given point) the way my calc I teacher […] described it was just like, it was the behavior of the function. That you may have a discontinuous function that has a point above but if you look at the limit your analyzing the behavior on each side, you're not just evaluating a value at that given point. So that's why I thought this one was the most accurate description of it because it kind of described the behavior.

Josh's statement above had elements of all three coding categories in it. He used motion, closeness, and the shape of the graph to describe his thinking; however, the main emphasis of the description surrounded the word "behavior" which was used ambiguously in this statement. Since the statement did not contain a strong description of any of the three categories, it was considered a non-instance and was not recorded as part of the coding scheme.

An important issue when coding the interview results is that of conventional mathematics language usage. In calculus, common language about limits involves words such as "approaches" or "goes towards." Although these words carry with them a connotation of dynamic motion, the meaning a student may give to them can vary significantly. For this reason, the use of common mathematical language alone is not considered sufficient evidence that a student is using dynamic reasoning. Below is an example of Amanda's description of a multivariable limit

Excerpt 6

> AMANDA: Ok, so a limit from a line is kind of like how the line approaches the point. So, say zero, if the line 2x is how it approaches zero that just, you just have to worry about the x and y variables, same with in a curve, like a parabola or something, same thing, you just have to worry about the x and y coordinates and you can easily see just the x and y coordinates on a graph. But in multivariable, you have several different variables that are approaching the same point, and so you can't exactly see how that happens on a graph, easily, so you have to take several slices and look at those curves at the slices and put it all together and analyze it that way.

In her description, Amanda frequently used the word "approaches," but it is not clear that Amanda was actually evoking a sense of motion to describe the multivariable limit. Instead, it is quite plausible that Amanda was using the word "approaches" as similar to the word "behavior" which does not necessarily connote dynamic motion.

In general, a conscious effort was made during the coding process to err on the side of exclusion rather than inclusion. In that sense, a statement such as Excerpt 4 above might actually refer to dynamic motion, but was excluded because reasonable doubt of its appropriateness remained.

Upon completion of the initial coding process, all coded text was reexamined to determine accuracy in coding. Instances from different interviews and involving different students were compared to determine that the coding scheme was implemented in a consistent basis. Any discrepancies in the coded instances were addressed and changed as necessary.

In all, 283 total instances were recorded with the majority describing dynamic thinking. Table 14 below shows the number of recorded instances for each coded category.

Coded Category	Instances (Percentage)
Dynamic Thinking	160 (56.5%)
Neighborhood Evaluation	41 (14.5%)
Topographical Examination	82 (29.0%)
Total	283 Instances

Table 14: Instances for Each Coded Category

CHAPTER V

STUDY RESULTS

Qualitative Study Results

The results of the qualitative data collected through the task-based interviews with the seven selected students are presented in this chapter. The coding scheme developed in the previous chapter will be used to color these results and create a picture of how students in this study came to conceptualize the multivariable limit concept.

These results are reported in two sections. The first section follows the study through its five primary settings: introductory calculus, multivariable limits using symbolic manipulation, multivariable limits using three dimensional graphing, multivariable limits using polar coordinates, and multivariable limits using contour graphs. This section discusses how different students experienced these settings using the three models of limit as a background for this description. The second description classifies ten misconceptions about multivariable limits encountered by students during the interviews. These misconceptions will be discussed and connections to the three models of limit will be made when appropriate.

The Concept of Limit in Different Settings

The coding scheme developed in Chapter IV was used to describe how students conceptualize the limit concept in different settings. The number and type of instances recorded during each setting is summarized in table 15 below.

Setting	Number of "Dynamic" Instances	Number of "Neighborhood" Instances	Number of "Topographical" Instances	Total Number of Recorded Instances
Introductory Calculus	46	16	21	83
Multivariable Symbolic Manipulation	18	0	5	23
Three Dimensional Graphing	41	2	29	72
Use of Polar Coordinates	4	2	7	13
Contour Graphing	33	17	19	69
Other Instances	18	4	1	23
Total	160	41	82	283

Table 15: Recorded Instances by Setting

Introductory Calculus

There were 83 recorded instances from the first session of interviews covering single variable calculus. Dynamic motion was the dominant imagery during these interviews, resulting in 46 of the 83 instances (55%) (See table 15). These instances can then be further divided into two categories. The first category being those images depicting fluid motion along a path as in the excerpt below.

Excerpt 7

> INTERVIEWER: So, this first one, um, you said strongly agree and… I'll let you talk for a little bit.

> JENNIFER: I think on this one I was just thinking of how when a limit says as x approaches certain thing it is showing how the function is moving towards a given point. So, to say x equals one, it's showing how that function is going as it approaches…

The second category contains those images constructed by successively choosing points along a given path such as in the following excerpt.

Excerpt 8

> CHRIS: Well, um, a lot of times in calc one I think that's where they don't really teach you the easy way of doing things for a long time, so you do, you can plug in that exact point but you can get something completely different than what that limit's actually… So you end up one way, you go a half and then a quarter and then you just keep getting closer and closer and you'll see the curve show up.

The dynamic model of limit has two natural manifestations that were discussed in the previous chapter, symbolic manifestations and visual manifestations. Those instances of dynamic thinking categorized as 'fluid motion' generally reflected a visual representation of the limit concept while dynamic thinking involved a process of successively choosing points on a path generally reflected a symbolic representation of the limit concept. Most students involved in the study used dynamic language involving both of these categories at different times in the interviews.

Dubinsky (1991) described the development of internal processes using APOS theory. In this theory, a function is first seen as an action to be evaluated at distinct points, and later as a process that gives the value of the function of each point simultaneously. A student who successfully uses both visual and symbolic images of limit and connects these two representations in a meaningful way has necessarily achieved a process conception of the limit concept. For this student, it is the process of simultaneously evaluating all points along the path which allows the visualized motion along the path. Sfard (1991) used the term *condensation* to refer this process of mentally capturing an operational process in a compact form that can be understood as a whole. Throughout the remainder of this chapter I will use Sfard's terminology and refer to *condensed* processes as processes in which the student is aware of the (possibly infinite) steps in the process and is able to reason about all the steps without performing them in a sequential manner. Using this terminology, I argue that for some students, a visualized limit process represents a condensed sequential limit process.

The relationship between these two internalizations – visual and symbolic – of the limit concept for each student is not always clear. As an interviewer, I aimed to ask students about these separate conceptualizations of dynamic motion. To achieve this I asked all students the following question:

Question

> When describing a function as "approaching" or "getting close to" a point, this idea would best be explained as:
>
> Choice A) Evaluating a function at different numbers over time with those numbers successively getting closer to the point in question.
>
> Choice B) Mentally envisioning a point on a graph moving closer and closer to the limit point.

Most responses contained little substance, perhaps due to the weak nature of the students' awareness of their own thought processes or to the student's inability to understand the question as stated. However, several students did respond with meaningful discussion. The following excerpts show how different students responded to this question.

Excerpt 9

> JESSICA: Probably… Well, normally B would make sense more… Well, like what I do, because… yeah, because like when I figure out, like a limit

> of one, like I think of going from zero and going, ok .5, ok .75, ok .1, err, 1. So, probably B.

Jessica described precisely the idea behind choice A, but then used that description as justification for selecting choice B. For Jessica, I question how strongly this statement represented conceptual thinking as opposed to procedural thinking.

Excerpt 10

> AMANDA: Um, they both kind of work, but I think, when I think of it approaching a point, I see the graph going towards the point, but to figure it out I plug in the numbers. So, it's more like, 'A' is more practical.
>
> INTERVIEWER: Ok, so A is more practical?
>
> AMANDA: But B kind of helps you think about it, I suppose. I kind of see a little car moving down a line…

Amanda appeared to confirm my belief that evaluating successive points can represent more of a procedure than a conceptualization. It seems that B is her actual conceptualization of the limit concept; whereas, A is the procedure she uses to actually solve problems. Notice that the motion involved does not carry the structure of her thinking, rather a physical structure – the graph or line – carries the body of her thinking and the two dynamic notions – visual and symbolic – are means of exploring that structure. In this case, I would argue that for Amanda, fluid, visual motion does not represent a condensation of the symbolic process, but rather the fluid motion defines the limit process for her, and the symbolic process is a loosely connected procedure used for computation.

Excerpt 11

> JOSH: If I had to pick one, I would probably say A – because I've never heard of envisioning a point getting closer – I don't know I'll have to think about that (pause) it's almost both, I don't know…

INTERVIEWER: That's fine.

JOSH: It's hard to… I guess it would be a point – no it wouldn't be a point, because it's not one single point it's evaluating different points. It's kind of like the bouncing ball thing. You start off on either side and it gets closer and closer and closer. So it would be envisioning points that are getting closer to the point in question.

INTERVIEWER: So you seem…

JOSH: It's more A.

Like Amanda, Josh explored a physical structure using motion. However, to Josh it is important that this structure contains many points. His statement "no it wouldn't be a point, because it's not one single point it's evaluating different points," indicated that he was bothered by the image of a single point moving along a line, because the line is not a single solid structure, but rather a collection of points. This is important in that it shows that Josh viewed the graph of a function, and therefore the limit concept, in an inherently different manner than Amanda. This difference will be explored in greater detail in Chapter VI.

During the introductory calculus interviews, the notion of analyzing successive points along a path was often coupled with the use of neighborhoods to conceptualize limits. Students often blended these two ideas together in the same discussion as in the following excerpt.

Excerpt 12

JESSICA: there's some, like, limits that you can't actually plug in the point

INTERVIEWER: Correct.

> JESSICA: But you can do it like really close, like 1.999 and get a really close answer. So if you put in one and then like 1.5 and then like .8 and then like 1.9, you'll get closer to the value that you're supposed to get as the limit…

Here Jessica first stated that you can use really close numbers "like 1.999 and get a really close answer." This is evidence of analyzing the function using a notion of closeness similar to the concept of neighborhoods. However, her next statement is that "if you put in one and then like 1.5 and then like .8 and then like 1.9, you'll get closer to the value that you're supposed to get as the limit." This statement is indicative of a dynamic approach created by analyzing the function along successive points. This shows that, for Jessica, these two conceptualizations – using closeness and analyzing along successive points – are connected when dealing with single variable limits. I believe that this connection is a potentially important one, and will discuss it more in detail below and in chapter VI.

One situation in which a variety of conceptualizations occur is when using calculators to solve limit problems. In using calculators, students have several options to analyze the limit, they can simply look at the shape of the graph, they can use the 'trace' function or create a table to look at the function along successive points, or they can use the 'zoom' function to look at a small area around the point in question. These different approaches each connect with the different categories for analyzing functions used in this study. Table 16 summarizes this relationship.

Use of Calculator	*Associated Category*
Analyze the Shape of the Graph	Topographical
Use the “Trace” Function	Dynamic, Using Fluid Motion along a Path
Create a Table	Dynamic, Analyzed along Successive Points
Use the “Zoom” Function	Neighborhood

Table 16: The Graphing Calculator and the Three Models of Limit

In the following excerpt, Chris used a calculator to find the limit of a function. He used imagery that suggests both dynamic and neighborhood conceptualizations of the limit concept.

Excerpt 13

> INTERVIEWER: And now, what about the limit? What do you believe about the limit?
>
> CHRIS: The limit, um… You would plug in points on either side of that to see. Or we can just trace it and see where it goes. And, pretty much it’s going to go… what you can do with this is you can plug in – you could plug in, like, points close to one, or you can just plug in one. And you have x equals, x goes to one, and you get one over two. And, so, I would say that one over two is the limit.

Chris’s use of the ‘trace’ feature on the calculator along with the statement that he will “see where it goes” make use of dynamic imagery to understand the limit of the function. However, shortly afterward, Chris decided to “plug in, like, points close to one.” This carries with it a connotation of closeness. For Chris, these two ideas are closely related.

It is reasonable to believe that students such as Chris see the idea of closeness as a conclusion of dynamic motion. That is, the dynamic motion will move them closer and closer – either through fluid motion or successive points – and eventually they will become “close enough.” I believe that this notion of “close enough” is the notion that translates into closeness. There is evidence of this in the following statement by Jennifer.

Excerpt 14

> JENNIFER: I like to picture everything in my head. What it looks like and the limit. I guess, going back to how I found limits, is you can do that as you get really close. Like, .01, .001, …

In this statement, Jennifer's language suggested that of closeness with the phrase "as you get really close;" however, her interpretation of "really close" is ".01, .001, …" This statement indicated of a process of successively analyzing points in a dynamic way.

This interpretation of the result of a dynamic event being a "really close" point comes from the inherent difficulty in condensing an infinite process. The process of moving towards a point, whether in a fluid manner or using successive points, is infinite in nature and potentially difficult to grasp. The two excerpts above indicated that a metaphor of closeness can be used to condense this process in order to understand the end result of the limit computation.

It is possible for students to circumvent this infinite process by using exterior features of the function to determine the result of the limit. These fall under the category of topographical information outlined earlier. Topographical conceptualizations are naturally tied to the graph of a function. For students, this graph is an object with different features and characteristics. In the spirit of Gestalt psychology, students look at the missing portions of a graph in an attempt to understand the limit of the function at those points. The following quote from Ashley demonstrates this manner of looking at a graph.

Excerpt 15

> ASHLEY: As x approaches zero would be zero.
>
> INTERVIEWER: And the reason why it is zero is because…?

> ASHLEY: Because that's where that empty hole is and you're getting closer on each side.

Ashley looked at the graph of a function with a removable discontinuity at zero. Her response was to locate the hole in the graph and use that hole to justify her reasoning about the limit. Her topographical understanding is then justified using the dynamic characterization that "you're getting closer on each side."

Topographical considerations are also used when the limit does not exist. The following excerpts come from students encountering a graph of a function with a jump discontinuity.

Excerpt 16

> JENNIFER: Uh, the limit of that one wouldn't exist because… well, x at negative 5 would equal… I think the limit wouldn't exist because the two points aren't together.

Excerpt 17

> AMANDA: Uh, four, the limit as the function approaches four does not exist, because it's disjointed here, but if you do it from, say, the negative side and the positive side, you get a limit.

Both Jennifer and Amanda used external features to describe why the limit should not exist. Jennifer looked at the location of the discontinuity and observed that there appear to be two points which are not positioned together. This makes it impossible to locate a spot that successfully completes the limit. Amanda choose to use the term "disjointed," which is more descriptive of a jump in the graph. Both students pointed to external features to make their decisions about the limit of the function. Amanda went further and connected these external features with the process of finding the limit symbolically.

Excerpt 18

> INTERVIEWER: I'm actually curious. How does doing this problem relate to the description of limit you gave over here? Of it, kind of relating to…. So, we mentioned some things like you said, 'I would draw a picture and explain a limit is a point a curve gets closer and closer to but never reaches .' I'm curious how this problem relates to this explanation that we were given earlier on this assignment.
>
> AMANDA: Ok, can I use this? (referring to graphing problem)
>
> INTERVIEWER: Yeah, you can use anything, that's fine.
>
> AMANDA: It's basically like this problem right here. That limit, to me. Because you've taken out, you've factored out the x minus one, the point where it's undefined, so there's a hole in the graph, and so what's remaining is it approaching.

To Amanda, the hole in the graph is the result of a denominator that equals zero. If she can "factor out" the part that equals zero in the denominator, she can, in a sense, remove the hole and evaluate the function as though it had no hole in it. She reconfirmed this belief later when using polar coordinates.

Excerpt 19

> INTERVIEWER: Ok, so we've successfully re-written these as polar coordinates. So my question is can we find the limit as *r* goes to zero of these functions?
>
> AMANDA: Ok.
>
> INTERVIEWER: Do you kind of know what that would be?
>
> AMANDA: Well, in this one it wouldn't matter, since there is no *r* to have in the limit. I guess you cancelled it out, though, so it's like removing the hole, type thing.
>
> INTERVIEWER: So, kind of tell me about what you mean when you say 'removing the hole.'
>
> AMANDA: So, here, there's *r*'s in the denominator in all of them, so if *r* is going to zero it will be undefined. And so by simplifying, you can

factor out something from the denominator and numerator and remove the undefined piece from the limit.

INTERVIEWER: Ok.

AMANDA: Leaving you with some other information to comprehend the rest of the limit.

In the above discussion, Amanda explained how she translated this idea of topographical thinking when she is dealing with symbolic expressions. Her notions of "removing the hole" and "[removing] the undefined piece from the limit" mirror what takes place when a student visualizes a graph and looks at where the function should be in order to determine its limit.

In the second half of this chapter we will discuss misconceptions students encountered while studying multivariable limits. It turns out that a significant number of the documented mistakes took place when students used topographical thinking. The tendency among the students was to focus too closely on the point and make a conclusion about the limit of the function based on whether or not the function existed at a given point.

It should be pointed out that topographical information played a vital role in students' understanding of limits in the introductory calculus interviews. Students who made effective use of dynamic and neighborhood imagery did so by superimposing these images onto a preexisting structure of the function. In a sense, they searched this structure for important information, but in analyzing this structure they looked beyond the external characteristics and used either dynamic or neighborhood imagery. Chris demonstrated this ability to analyze the characteristics of a function using different forms of imagery.

Excerpt 20

> INTERVIEWER: Ok, good, let's think about this third one here, at x = 4.
> CHRIS: Where it jumps. Um, I think it is not continuous. I would put it as 'does not exist.'
>
> INTERVIEWER: So, you said it does not exist because it is not continuous?
>
> Chris: Right.
>
> Interviewer: What do you mean when you say that?
>
> CHRIS: Because it jumps right here. The left hand limit is equal to three and the right hand limit is equal to negative three, or negative four, whatever that is. But, the function is actually equal to three at that point. I would say that because the left hand limit and the right hand limit don't equal each other – they are two different numbers – the limit isn't going towards one point.

Twice Chris mentioned a "jump" that he saw in the graph, which is classified as a topographical external characteristic. He also used the word "continuous" which for him described a function which has a limit at every point (this will be discussed in the section on "Misconceptions"). This use of the word "continuous" is also a description of external features of the graph. However, when he was pressed to explain his thinking, he used dynamic imagery to analyze the function and says "the limit isn't going towards one point." This shows how Chris was able to reason with topographical thinking and then used dynamic imagery on top of the external features to make conclusions about the limit of the function.

Symbolic Manipulation of Multivariable Functions

The first portion of the second interview session examined multivariable limits using symbolic methods. There were a total of 23 coded instances during these interviews. Of these 23 instances, 18 (78%) were categorized as dynamic thinking (See

Table 15). This is not surprising, since the selection process for this study emphasized students with a dynamic view of limit. However, two external factors likely played a role in the prevalence of dynamic thinking during this portion of the interview as well: classroom experience, and the presentation of the first interview task.

The students' classroom experiences were undocumented, so definitive statements about their experiences are impossible. However, Josh gives us some insight into what he learned from the classroom at the beginning of his second interview.

Excerpt 21

> INTERVIEWER: So, today we kind of start doing calculus III stuff. So let me ask, you've said you've kind of covered multivariable limits?
>
> JOSH: Yes.
>
> INTERVIEWER: Is that right? So why don't we just start by telling me what you've studied about them, what you've learned about them, things like that.
>
> JOSH: Ok, basically, the same kind of general concept that you're looking, instead of just one function that you're looking at the limit, you're looking at this surface that, unfortunately, is a little more difficult. When we were doing one variable limit what we were doing was, 'do we approach from the left' or 'do we approach from the right' and we compared those two, whereas now, we can approach from an infinite amount of directions, there's different paths and approaches that we can take. We can take left, right, parabola, we can circle around it, there's an infinite amount, so it's a little more difficult to prove the limit does exist; whereas it's a little bit easier to say it doesn't exist, so if you can take a limit from one path and compare it to another path and get two different numbers, you can easily say it doesn't exist, whereas to prove it does exist is a little more difficult…

We can see that Josh interpreted his classroom experiences with multivariable limits as a transition from single variable limits where, instead of two directions there are now "an infinite amount of directions." In the following excerpt, we see that Jessica had a similar in-class experience.

Excerpt 22

> INTERVIEWER: So, like I told you last time, we're going to actually do some multivariable stuff today, and I guess you've studied multivariable limits?
>
> JESSICA: Um, we did a little in class.
>
> INTERVIEWER: Ok. Well, would you mind telling me just a little bit about what you learned, or kind of describe what a multivariable limit is?
>
> JESSICA: Well, it's, ok… So, we learned that, ok, in calc one, there's like a graph and you have a line, or a curve, and then the limit is evaluated from the left or the right. But in three dimensions, it's a surface and it can be found from… it can be evaluated from any direction, so it makes it really complicated to find if it exists or not, so…

Similar to Josh, Jessica viewed the multivariable limit as an extension of the single variable limit with the exception that the single variable limit "is evaluated from the left or the right," while the multivariable limit "can be evaluated from any direction." Considering the imagery brought into the interview by Josh and Jessica, it seems natural for them to use dynamic motion when analyzing the limit at a point.

In addition to the views brought from the classroom, there is the viewpoint implied by the materials used during the interview. Just as the interviewer is considered an active part of the study, so too should all materials be considered an active part of the study. The material for the first task in the study emphasized analyzing the function along different paths. This was presented to the students using the following language.

Find $\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$ along the line $x = 0$.

The phrase “along the line $x = 0$” is highly suggestive of dynamic motion, and possibly plays a role in explaining why students used dynamic reasoning to such a large extent during this portion of the study.

During the interviews, this dynamic imagery was quite effective at showing that a multivariable limit does not exist.

Excerpt 23

> INTERVIEWER: So, looking at these results, can you decide anything about whether or not this limit exists?
>
> JESSICA: Um, it doesn’t exist because this… it would exist if you go… it would exist if as you go from different directions towards (0,0) if it equals the same number. But, since it’s like negative one, one, and zero, it doesn’t exist because they’re all giving you different values, so like it jumps around – so it doesn’t exist.

In the above statement, Jessica used dynamic imagery “from different directions” to convince herself that the limit does not exist. However, just like in the single variable case, she superimposed this motion on top of another structure. In this case, her structure is the “different directions” and the motion takes place on top of those directions. Her conclusion that the values “jump around” is an indication of a topographical structure to her thinking, that the different directions “jump around” and so the limit does not exist. It is not exactly clear what Jessica visualized when she used the phrase “jump around,” but she appeared to be describing a visual image which resulted from an analysis of the function using dynamic imagery.

On the other hand, students were not successful at showing that a limit does exist. Several misconceptions were exposed and will be discussed in greater detail in the second half of this chapter on “Misconceptions.” Several misconceptions revolved around the difficulty of condensing the infinite process of analyzing the multivariable function along every possible path. Students created different solution methods to capture this infinite process, from assuming a finite number of paths was sufficient to understand the limit to analyzing all linear paths by inserting $y = mx$ into the equation.

Two students, Jessica and Josh, were aware of their inability to interpret the infinite process of analyzing the function along every possible path.

Excerpt 24

> JESSICA: I decided that the limit's zero, because they're all zero. I mean, that's, it's…. It might not be right, because I didn't do it from every direction. But, I guess these are the best three to do – to generally figure it out – so I think it's zero. And then the same thing for the second one… (works for awhile) it's zero as well.
>
> INTERVIEWER: Ok, so you think this one is zero also? Let me ask you, so you're taking these lines… you said earlier that you weren't one hundred percent sure it was zero but you thought it was?
>
> JESSICA: Well, yeah, because you're only taking three lines, you're not taking infinity lines, because there's infinity directions that you can come from to get to (0,0) – if it's a surface.

Excerpt 25

> JOSH: I would say from this, because I'm taking a lot of different things and trying them and I'm ending up with the same answer, I'm going to say that this limit, I need to show that this limit exists. But this is where I kind of fall off. We didn't do any of that in the homework and looked at it, and it's like looking at the examples, what it does is like sigma, sigma delta.
>
> INTERVIEWER: Epsilon is the one, epsilon delta.
>
> JOSH: Sorry, epsilon delta argument. It makes sense doing the examples, but actually going in and doing it, I get completely lost.
>
> INTERVIEWER: So, it's not real clear to you how to show that this exists? You believe it exists, but it's not real clear to you…

> JOSH: I think I could show that it exists, I don't know for a fact that it does – I wouldn't say confidently that it does exist – but I know that there's…
>
> INTERVIEWER: You suspect that is does?
>
> JOSH: Yes, there's the word that I was looking for. But the epsilon delta argument, it doesn't make a lot of sense in my mind.
>
> INTERVIEWER: So is that kind of, so I was going to ask, how do you go about showing one these exists, if it does exist?
>
> JOSH: You use the epsilon-delta argument…
>
> INTERVIEWER: So, what's the real idea of the epsilon-delta argument? Does that kind of make any sense to you or are you just following…
>
> JOSH: Yeah, it's most of like, here's this formula that shows a limit of several variables will exist, and it's kind of, it doesn't make a lot of sense to me, it's just kind of playing with the formula and putting in arbitrary thing and then, viola. It's just kind really unclear and muddy to me. It's like, if I see an example done then I'm like, 'that makes sense,' but if I'm presented with like, 'do this,' like in this case, then I'm kind of lost.

Jessica and Josh both realized the need to analyze the function along an infinite number of paths in order to show that the limit exists. Jessica, even though she only analyzed a finite number of paths, displayed confidence that her conclusion was correct, but held back the exception that there was possibly another direction that would change the result. Josh, on the other hand, showed very little confidence that the limit exists. He was prepared to attempt to show that it might exist, but his only method of doing so was the epsilon-delta definition, and that definition did not make sufficient sense to Josh for him to use it. In both cases, their use of dynamic imagery left them unable to show that a multivariable limit does exist.

During this portion of the interviews there were five instances which were not classified as dynamic thinking. All five of those instances were classified as topographical thinking with no instances of neighborhood thinking. Of the five instances of topographical thinking, three discussed the value of the function at the point (0,0). These are considered topographical because they focus on external characteristic of the function, namely that is undefined at (0,0). These are classified as misconceptions and will be discussed in greater detail in the second half of the chapter on "Misconceptions."

It is important to observe that during this portion of the interviews, no students displayed neighborhood thinking. Josh explicitly mentioned being shown the epsilon-delta definition of limit in class, and it is likely that other students encountered this definition as well. This definition, if internalized by the student, should lead to a conceptualization of the limit concept using the neighborhood model. It is apparent that none of the students that took part in the interviews successfully constructed an understanding of the limit concept based on the epsilon-delta definition. Posner, Strike, Hewson, & Gertzog (1982) argue that for a student to reconstruct his/her thinking he/she needs to a) develop a sense of dissatisfaction with the concept as currently understood, b) possess an understandable alternative conception, and c) have a reason to believe this alternative conception is valuable. In that sense, both Josh and Jessica demonstrated a sense of dissatisfaction with the way they currently understood the multivariable limit concept as they attempted to show that a multivariable limit does exist, but neither possessed an understandable alternative conception in order to reconstruct their thinking. Josh showed that he was exposed to such an alternative conception, but clearly it was not understandable for him, and he was not able to use it to reconstruct his conceptualization of the multivariable limit concept.

Three Dimensional Graphing of Multivariable Functions

The second portion of the second interview session analyzed multivariable limits using three dimensional graphing. Students encountered three dimensional graphs using the mathematics software program, Maple. This program allowed students to analyze graphs from many different angles by rotating the image on the screen. This allowed the students to examine the surfaces from any vantage point they choose in order to better understand the shape of the surface. There were a total of 72 instances recorded during this portion of the study. Of these recorded instances, 41 (57%) were classified under the dynamic model, two under the neighborhood model, and 29 (40%) under the topographical model (See Table 15).

Of those instances characterized as topographical thinking, 21 (72%) were references to the holes created in the surfaces by the Maple graphing program. These holes were drawn larger than singularities are traditionally drawn, and this might have played a key role in the students' tendency to discuss the holes in the graph. Amanda's conversation below shows her reaction to the holes in the graph.

Excerpt 26

> INTERVIEWER: What do you think, would you say that this limit exists or not?
>
> AMANDA: As it approaches zero?
>
> INTERVIEWER: Yeah as it approached (0,0).
>
> AMANDA: This looks the same from above as the other one did… Um, no.
>
> INTERVIEWER: Ok, and you say no because…

AMANDA: Again, there's another hole right where there's (0,0). There's just no way there's a limit there because there's no function in the middle to go with…

At this point in the interview, Amanda concluded that the limit does not exist because of the hole she sees in the graph. When she said, "looks the same from above," she referred to Figure 3 below, which is the view of the surface from the positive *z*-axis.

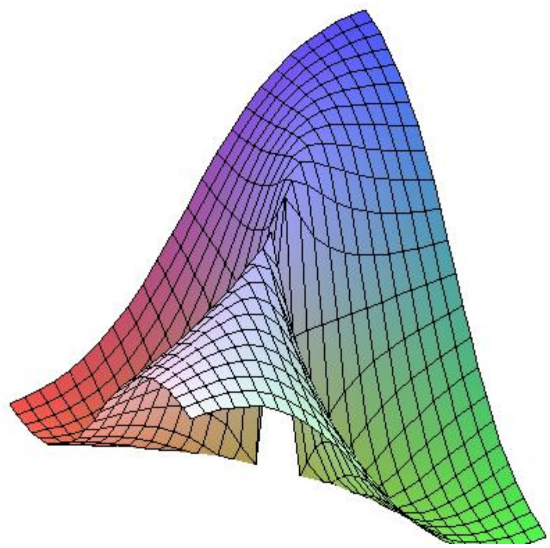


Figure 2: Three Dimensional Surface Examined by Amanda

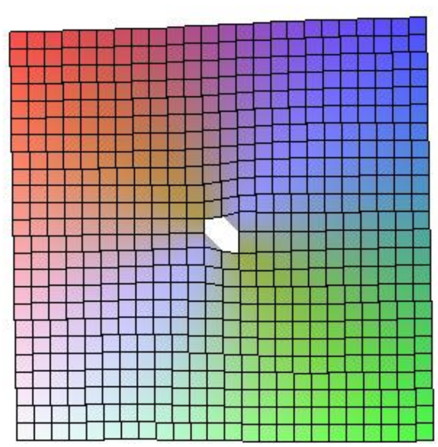


Figure 3: Figure 2 as Viewed from the Positive z-axis

Her statement could possibly be interpreted as a belief that the limit of the function is the value of the function at the point; however, later in the conversation she clarified the meaning of the hole in the graph and changed the way she evaluated the multivariable limit.

Excerpt 27

> AMANDA: Well, it's all, on this one at least they're all going to that similar point, but there's just like a little hole of discontinuity there.
>
> INTERVIEWER: So what do you believe about the limit as (x,y) approaches (0,0) of this function? Or do you have a belief?
>
> AMANDA: Well, it wouldn't exist. Ok, is this just one point? This little hole?
>
> INTERVIEWER: So the hole – it's not supposed to be a big hole, it's supposed to be like a one point hole. The graph, the computer has trouble drawing that.
>
> AMANDA: Ok, so actually it could exist because it doesn't have to be equal to it at that one point.
>
> INTERVIEWER: And if you look at this function, it would make sense for it to exist at (0,0)?
>
> AMANDA: So all the little holes are, like, one point holes?
>
> INTERVIEWER: Yes, yes would you like to look at the other ones?
>
> AMANDA: Yes, I would.

This discussion took place while Amanda analyzed Figure 4 below.

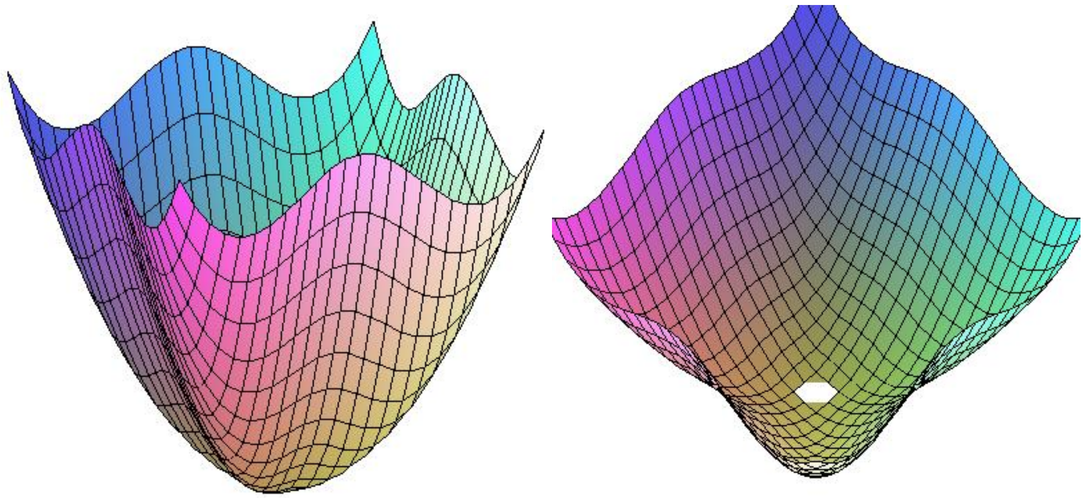


Figure 4: Two Views of the Surface Discussed in Excerpt 21

Amanda explained that her earlier confusion was a result of misinterpreting the hole in the graph. She previously viewed it as a large empty space, a set of many points surrounding the point (0,0) instead of the point (0,0) alone. Her desire to review her previous answers demonstrates her realization that she had experienced a shift in thinking towards the graphs.

As the discussion with Amanda continued, she reviewed those earlier graphs and used dynamic imagery to make sense of the multivariable limit.

Excerpt 28

> AMANDA: So, this one, I still don't think the limit exists.
>
> INTERVIEWER: And why do you say the limit does not exist?
>
> AMANDA: Because at (0,0) there's, say from this direction, it's approaching negative infinity… yeah, and from this direction it's going up. It's just, they don't… To me, it doesn't seem like they're all, all the sides are approaching the same point.

Like other students interviewed, her dynamic imagery is superimposed upon the existing structure of the surface. She initially began by applying dynamic motion one direction at a time. This can be seen when she stated, “from this direction it’s approaching negative infinity… yeah, and from this direction it’s going up.” As she made this statement, she swept out paths along the surface with the cursor. These paths are depicted in Figure 5 below.

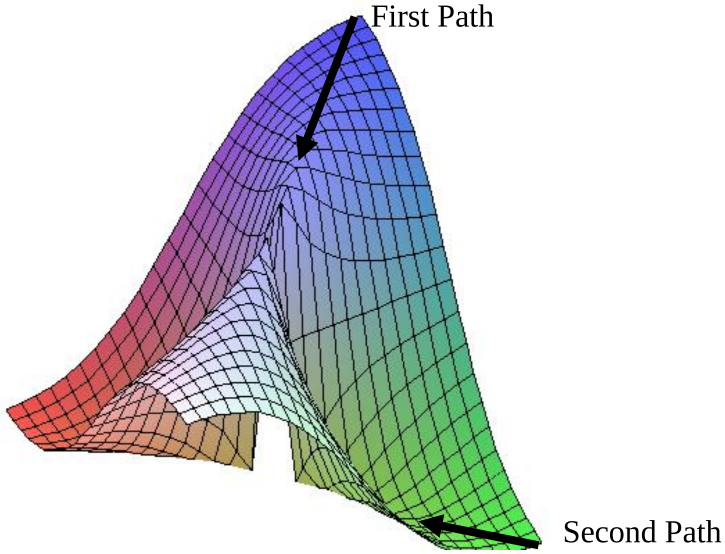


Figure 5: Paths Amanda Creates with the Cursor

However, once she began discussing the next surface, her language subtly changed.

Excerpt 29

> INTERVIEWER: Ok, what about this one?
>
> AMANDA: I don’t think this one exists either.
>
> INTERVIEWER: Ok, and what do you see that makes you think it doesn’t exist?
>
> AMANDA: It is… well, these lines of the curve, I mean, they’re going, they’re not at all going towards (0,0) and to me it seems like the function would all have to converge, or like go towards this one point for it to actually have a limit, and this isn’t happening here. But on this one…

Instead of discussing motion along different directions, one direction at a time, she discussed motion of the function as a whole, presumably describing motion along every possible path simultaneously. This gets even stronger as she discussed a function which has a limit that exists at (0,0).

Excerpt 30

> INTERVIEWER: So this one is different.
>
> AMANDA: Right.
>
> INTERVIEWER: So what is different about this one?
>
> AMANDA: Well the whole function is kind of approaching this one point, there actually could be a limit described.

Phrases such as, “whole function is kind of approaching this one point,” and “the function would all have to converge,” demonstrated that Amanda no longer viewed motion along one path at a time, but rather attempted to condense this infinite process of analyzing different paths into a single action of the “whole function… approaching.”

The above statements still do not make it clear that Amanda connected the process of using motion along different paths and her notion of the whole function approaching a point. As the interviewer, I pressed her to clarify these statements and this resulted in the following conversation.

Excerpt 31

> AMANDA: Um, it’s harder when they’re so complicated, three dimensional objects, but I think it’s just that, again they’re all going towards a similar point.
>
> INTERVIEWER: And when you say ‘they all’ what do you mean by ‘they all’? I’m just trying to harass you.

AMANDA: All of the function, all of the different slices of, of different ways of looking at it.

To Amanda the phrase "All of the function" and the phrase "all of the different slices" were synonymous. This strengthens the claim that her statements were a result of her attempts to condense this infinite process in order to understand the multivariable limit.

Topographical information was often used in conjunction with dynamic motion in order to draw conclusions about the multivariable limit. The conversation below demonstrated how Jennifer used topographical thinking in conjunction with dynamic motion to make decisions about the multivariable limit.

Excerpt 32

JENNIFER: This one, I think, I would say that the limit does exist.

INTERVIEWER: You say it does exist? Why would you say it does exist?

JENNIFER: Because, even though they don't all meet at that point, they are all going towards that point. Or, all the planes around that shape seem to all go to that point if it's at (0,0).

INTERVIEWER: So, now let me ask, if I were to give you some other graph how would you go about deciding whether the limit existed at (0,0) or not of that graph? What would you be looking for?

JENNIFER: I would be looking for if there was some specific spot or point that the graph was going towards… If both sides move towards it, and if they didn't then it wouldn't exist. If they kind of did this thing (moved the graph)…

In the above discussion, Jennifer used dynamic imagery, but this imagery took place on top of her topographical image of the function. In the statement "I would be looking for if there was some specific spot or point that the graph was going towards…" Jennifer's description centered on "some specific spot" and dynamic motion was used to analyze the nature of that specific spot. This shows the inherent relationship between the topographical and dynamic models of thinking.

Like Amanda, Jennifer began this portion of the interview emphasizing the holes in the middle of each graph, but also like Amanda, her attitude changed once she encountered a function whose multivariable limit did exist. It is possible that these students' perception of the limit of a multivariable function did not contain a coherent topographical image of a multivariable limit before this exercise began. When encountering the function whose limit does not exist, the shape of the surface did not match any preconceived topographical shape that the students had experienced. It was not until they experienced the function whose limit did exist that they saw a surface that looked familiar to them.

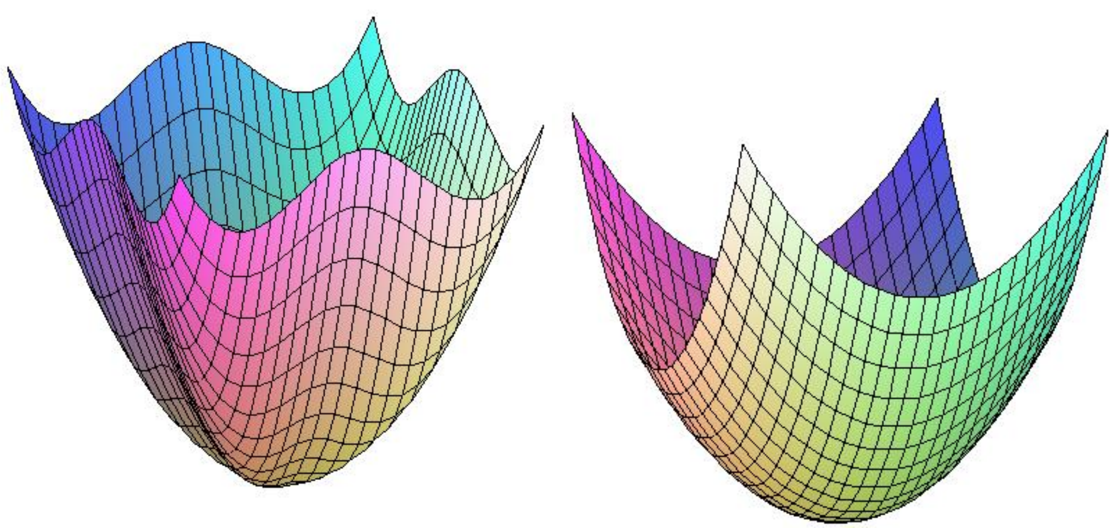


Figure 6: Comparison of the Function Amanda and Jennifer Experienced to a Parabloid

The surface on the left bears a strong resemblance to the surface of the parabloid on the right, with the exception that the point (0,0) is undefined. This seems like a natural generalization of a removable discontinuity, and therefore, it is a natural place for the students to begin building a topographical model of the concept of multivariable limit. Later in the interview, when each student encountered a multivariable limit that does not exist they no longer focused on the hole, but rather used a combination of topographical and dynamic thinking to determine the value of the limit.

Two students, Mike and Josh both struggled with visually condensing the infinite process into a compact whole. Mike circumvented the infinite process by using dynamic imagery only along the two axes. This led him to several mistakes that will be discussed in greater detail in the second half of this chapter on “Misconceptions.” Josh, on the other hand, showed awareness that he was dealing with an infinite process, yet struggled to make sense of the multivariable limit in light of this infinite process.

Excerpt 33

> INTERVIEWER: Do you want to look at any more, or should we..? Let’s look at, I don’t know, I always thought this was a fun looking one right here.
>
> JOSH: Yeah, all the trig functions always have fun looking graphs.
>
> INTERVIEWER: It’s kind of crazy isn’t it?
>
> JOSH: Yeah, it’s like a mold of some sort. Ok, this is kind of the same, look at it from one perspective and then kind of try to look at it from another. But, with these it’s just, it’s kind of if I take one approach, it doesn’t matter which approach I take it appears that no matter what approach I take it comes to this, this central point here in the middle, that the limit would approach this value, this point, here at the center of that cone shaped thing. So, visually it looks like the limit exists, but I wouldn’t say confidently either way, so I’m not sure…

In the above statement, Josh used dynamic motion to explore the graph along different "approaches." He considered these approaches one at a time. Josh failed to take a viewpoint that allowed him to consider all approaches simultaneously, and as a result he remained unconfident about the multivariable limit. Notice that Josh superimposed dynamic motion on top of the shape of the graph, and made a topographical statement about "this central point here in the middle," yet he remained unable to draw any conclusions about the multivariable limit.

Two of the students mentioned in this section, Josh and Amanda, had similar experiences analyzing multivariable limits using three dimensional graphs. Both students used a combination of topographical and dynamic information to analyze the functions along different paths. However, Amanda was able to visualize a condensed dynamic motion of "the whole function… approaching," while Mike struggled to look past the process of analyzing motion one path at a time. I believe this difference can be explained in terms of each students' overall approach to understanding the limit concept. In Chapter VI I will look closely at how different students internalize the limit concept – for example, Amanda uses visual imagery and Mike uses mathematical processes – and describe how these viewpoints impacted their experiences with multivariable limits.

Multivariable Limits Using Polar Coordinates

The third interview session began by analyzing multivariable limits using polar coordinates. This experience yielded significantly fewer recorded instances than the other multivariable environments (See table 15). There were a total of thirteen recorded instances; however, most of these instances referred to the challenge of finding the single variable limit with *r* as the variable. Only five of the thirteen recorded instances took place while students attempted to conceptualize the multivariable limit. Overall, the students struggled to interpret the multivariable limit meaningfully in terms of polar coordinates.

There are many possible factors which prevented students from making sense of multivariable limits using polar coordinates. We will focus on two factors which arose during the interviews: the meaning of the symbols r and θ, and the understanding of the process of changing coordinates.

Using polar coordinates to understand multivariable limits involves an understanding of the relationship between the limit concept and the notion of distance. There was little evidence that students made meaningful connections between these two ideas. The use of distance to understand the limit concept can be conveyed using the neighborhood model of limit, but to this point in the interview sessions, few instances of neighborhood thinking have been recorded.

It is not clear that students connected the use of polar coordinates as an attempt to analyze distance in a function. Of the seven interview participants, three translated the symbol r as representing “the radius,” two translated it as being the “length,” one described the symbol using the language of vectors and one was unable to describe it at all. Those who used the word “radius” to describe polar coordinates seemed to view polar coordinates as a transition from a rectangular to a circular coordinate system.

Excerpt 34

> INTERVIEWER: So let me, let’s see, today we’re going to talk about polar coordinates. Do you remember polar coordinates?
>
> ASHLEY: A little bit.

INTERVIEWER: So, what do you remember about polar coordinates, what's the big idea there?

ASHLEY: Polar coordinates are in radians I believe?

INTERVIEWER: So… yeah…

ASHLEY: Ok, it's been a long time, so… Because that was over a year ago, I guess when we to learned about those… Don't remember a whole lot about it, but it was when you were using *r* and theta to get those…

INTERVIEWER: Ok, so I provided just a real quick summary of polar coordinates.

ASHLEY: There we go.

INTERVIEWER: And it's *r* and theta, and what does the *r* and the theta stand for? I guess you can look and see…

ASHLEY: *r* is the radius and theta is the angle from the positive *x* axis.

Ashley's statement that "polar coordinates are in radians" demonstrated a connection with polar coordinates and analyzing the function using circles. It also, perhaps, explains why the idea of *r* as a "radius" is so strong with the students, because the letter *r* is used to represent radius in other forms of mathematics, and students probably take the chosen letter, "r," as a suggestion of the meaning of the symbol. It is not clear that those students who associated the symbol *r* as being the "radius" associated that representation with the distance from the origin. There was no evidence in these students that letting *r* approach zero represented the analysis of the function near the origin.

Even the students who described the *r* as representing "the length" showed little evidence of using this imagery to understand the multivariable limit. Instead, students struggled to understand a difference in using polar coordinates and Cartesian coordinates. An example of this is in Josh's discussion below. Prior to this discussion, Josh used polar coordinates to find the limit of each function as *r* approached zero, and he found the limits of each function using symbolic methods as in the first part of the second interview session.

Excerpt 35

> JOSH: As *r* approached zero, I got zero for the first two. The third one I didn't know about, but when I went back and looked at the original function, it was obvious that the limit did not exist. So, if I was forced to give an answer on these first two, I kept getting zero for both of them on each test. From different approaches, I kept getting zero, when I rewrote it in polar, and then took the limit as *r* equaled zero of the function, I also got zero, so I would say that those do kind of, might support each other with the limit of those two functions as zero.
>
> INTERVIEWER: So, let me kind of ask. You said it might support… this might… The fact that the limit of this as *r* went to zero supports the fact that these seemed to be showing that the limit was zero.
>
> JOSH: Yeah it continued to be getting zero.
>
> INTERVIEWER: Ok, does that make… Can you kind of tell me why you believe that, and why you think those are connected?
>
> JOSH: It's kind of, it's just more like a different approach to looking at the same thing. Which I kind of said it earlier, with polars, I was unsure of these, if the limit was actually equal to zero. I kept getting zero, but there might be this one path which is a counterexample and the limit doesn't exist. So it kind of, you're looking at the same problem under a different light, to maybe reveal a little more about it and by taking the limit as *r* equals zero, maybe that revealed the final answer, I guess, I don't know. I've never done that with looking at a multivariable in polar coordinates before.

Josh translated the polar coordinate experience as an extension of the symbolic approach from the beginning of the second interview session. Josh translated the act of taking the limit as *r* approaches zero to be analogous to that of analyzing the multivariable limit from different paths. He revealed a lack of confidence in the result of finding the limit with polar coordinates when he stated that "there might be this one path which is a counterexample and the limit doesn't exist." This showed that he did not view the process of letting *r* approach zero as one of taking successively smaller neighborhoods, but rather as one of following a strange "path" that appears when using polar coordinates.

Perhaps Jessica made the most progress in understanding the multivariable limit using polar coordinates. In the following discussion, Jessica had just finished finding the limits of different multivariable functions as *r* approached zero.

Excerpt 36

> INTERVIEWER: That's fine. Now, my question is, if we were to look at these limits, do you think we could make any decisions about whether or not the limit as (x,y) approaches (0,0) of f(x,y) exists? Based on these answers?
>
> JESSICA: Yeah, because you're substituting in, so whatever r goes to, it's the same limit as if x and… yeah, because as r goes to zero, ok, you're going to the point (x,y) at (0,0), so 'yes' it would be the same limit.

Jessica did successfully make a connection between *r* approaching zero and (*x*,*y*) approaching (0,0); however, this connection seemed to be based upon the fact that $r = 0$ represented the origin. I would argue that Jessica was not connecting this process to any use of neighborhoods around the origin, but rather her emphasis is on the process of substituting and the result of the substitution. As I probed her for more information about her understanding of polar coordinates, she gave the following description.

Excerpt 37

> INTERVIEWER: So, if I asked you to use polar coordinates to solve a problem like this, how would you explain that?
>
> JESSICA: Ok, well I would explain it like you, you're substituting in, like you're changing the variables to make it easier, because it simplifies the x and y's because you're just looking for one variable instead of x and y. So, you substitute in the polar coordinates and only look for when r is

> going to zero and not x and y. So, it's easier because you don't have to come from, like, infinity directions.

It seems that the focus of Jessica's description is on the substitution from Cartesian coordinates to polar coordinates. I believe her statement that you "only look for when *r* is going to zero and not *x* and *y*," is not evidence of her condensing the process of analyzing the limit along many paths into a process of analyzing along successive neighborhoods. Rather, I believe that, similar to Josh, she viewed the polar coordinates as providing another direction to approach the origin. Her statement that "you don't have to come from, like, infinity directions," suggested that she had not condensed the infinite process, but believed that she has found a way to avoid the infinite process. Unlike Josh, Jessica had already shown a tendency to believe that a small number of paths are sufficient to understand the multivariable limit. Therefore, I believe that she conceptualized a "path" as *r* approaches zero, and believed that this "path" was sufficient to understand the multivariable limit, but I do not see evidence that this "path" represented the process of analyzing the function using neighborhood thinking.

Contour Graphs of Multivariable Functions

The second portion of the third interview session focused on analyzing multivariable limits using contour graphs. There were a total of 69 recorded instances during this portion of the interviews. These instances represent the most diverse set of conceptualizations found during the study. Of the 69 recorded instances, 33 were classified according to the dynamic model, 17 according to the topographical model, and 19 according to the neighborhood model (see table 15).

This interview session exposed several interesting misconceptions about multivariable limits using contour diagrams. Jennifer visualized the lines on the contour graph as depicting motion and made conclusions about the limit based on the results of her visualized motion. Amanda and Ashley both used arguments involving the symmetry of the graph to explain the existence or non existence of a multivariable limit. These misconceptions will each be covered in greater detail in the second half of this chapter on "Misconceptions."

Similar to the experience with three dimensional graphs, the students tended to conceptualize dynamic motion in one of two ways: by analyzing motion along successive paths one path at a time, or by attempting to conceptualize motion along all possible paths simultaneously. Below is Josh's first response to analyzing contour graphs to make sense of multivariable limits.

Excerpt 38

> JOSH: Ok, basically looking at a contour graph, I kind of have to construct the 3D model from the colors. Lighter is higher, correct?
>
> INTERVIEWER: I think… I don't remember on this one. Yeah, I think yellow tends to be bigger and red tends to be smaller.
>
> JOSH: So from there, as you can see in all these graphs the point (0,0) is a hole, there is some discontinuity there. So, once again just approach from different paths, and see what value I get when I approach from the right, the left, up, down, left, and right, they all appear to be converging at this central point there. So, I would say from that that the limit would exist; whereas, the contrast would be this third one. If I approach from this diagonal here, I appear to be getting close to this low point on the surface; whereas, if I approach from the right, the top right and bottom left diagonals, it appears to be at a higher point. That I'm going to approach this high point here and then approach this low point on either side, there. It's kind of the contrast to it, so I would say that this one does not exist; whereas, this, since each path seems to be approaching the same level on

> the contour, does exist. And the same concept applied here too, that each path or approach appears to be coming to this same surface level.

Josh appeared to be analyzing the contour graph by using dynamic motion successively along different paths. His statement, "So, once again just approach from different paths, and see what value I get when I approach from the right, the left, up, down, left, and right, they all appear to be converging at this central point there," is evidence that he used motion along each path, one path at a time. However, he was able to make conclusions about every path simultaneously with the statement, "they all appear to be converging at this central point there." This demonstrated that Josh was beginning to condense the dynamic process of analyzing the graph along successive paths into a single, coherent action.

Similarly to Josh, Mike used dynamic imagery to understand the multivariable limit.

Excerpt 39

> INTERVIEWER: So, my question is, if you were to look at these graphs… um, if you were to look at the graphs of these functions, could you decide which ones you think the limit exists at (0,0) and which ones the limit doesn't exist at (0,0)?
>
> MIKE: I think these two would exist, just because it's approaching the same point from, like, all directions on both of these. On this one, I don't think the limit would exist just because even though it's approaching the same point from two directions, from these other directions it doesn't look like it's all approaching the same point.
>
> INTERVIEWER: So, if I were to give you some other graphs, how would you decide whether or not the limit exists at (0,0)?
>
> MIKE: I just look around the point and see if, from all directions, if they're all approaching about the same point. Like this – on all sides it's clearly approaching like a single point; whereas, on this from like these

> two directions it looks like it's approaching a completely different point than from the other two.

In his description, Mike handled the infinite process of analyzing the function along all possible paths by categorizing the paths according to different "directions." These "directions" seemed to represent sections of the graph as in the image shown below.

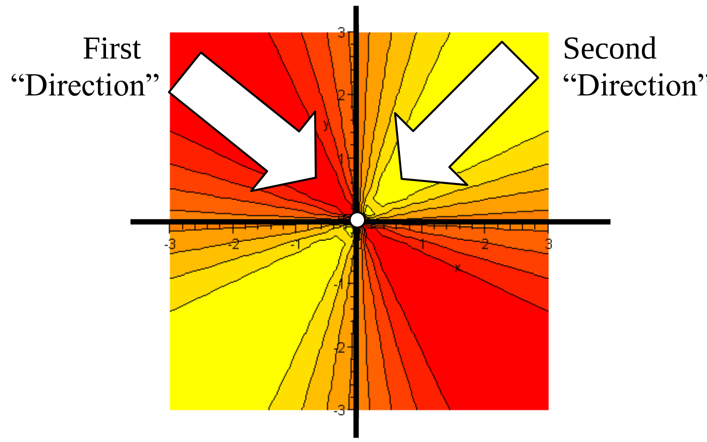


Figure 7: Mike's Use of the Different "Directions"

As Mike continued describing his idea of using various directions to understand the limit, he began speaking of all directions simultaneously. He stated, "I just look around the point and see if, from all directions, if they're all approaching about the same point." In this statement, his focus is on the missing point, and he used dynamic language to describe the function in reference to this one missing point. This description used the same sense of dynamic motion placed on top of a topographical image as was seen when students encountered the three dimensional surfaces in the second part of the second interview session. Later, Mike used the words "I just look around the point" to convey a thought that correlates with the neighborhood model of limit. He never continued to explore this thought, instead moving back to a description of motion from different directions.

Unlike Mike, who never made his sense of closeness explicit, Chris used neighborhood thinking to create a method for examining the multivariable limit. Like Mike, Chris began his discussion using dynamic motion.

Excerpt 40

> INTERVIEWER: So you're talking about this first one?
>
> CHRIS: Yes.
>
> INTERVIEWER: And so, you say it does exist?
>
> CHRIS: Yes and the reason is because the red's smaller and the yellow is the higher numbers, and so what it looks like is from this bottom left coordinate it looks like it's coming up into it. Same with that one I think they look like they're going down towards the same value. And it looks like this is all an equal value right here along the axis. So it looks like everywhere that it does hit the (0,0) point it looks like it's at the same area.

Chris used the word "coordinate" in much the same way that Mike used the word "direction," to represent sections of the contour graph. Chris's language differed slightly, however, in that Chris seemed to be focusing primarily on each of the four quadrants, and likely refered to one of those quadrants as a "coordinate." This view is supported by the following excerpt.

Excerpt 41

> INTERVIEWER: And the third one?
>
> CHRIS: This third one I would say no, because it looks like…
>
> INTERVIEWER: Why would you say no?
>
> CHRIS: It looks like these two coordinates, or these two quadrants are headed towards, at (0,0), they're headed towards a different value. I don't know if they're headed towards a different value, or…

In excerpt 33, Chris used the phrase, “it looks like it’s at the same area.” The idea of being in “the same area” describes the thinking necessary to create a neighborhood model of limit. Unlike Mike, Chris came back to this conceptualization of the limit concept later in the discussion.

Excerpt 42

> INTERVIEWER: So if I asked you to describe a way that you could look at a contour graph and decide if the limit exists at a point, what type of description, what type of way would you come up with?
>
> CHRIS: Like a way to describe if the limit exists on one of these graphs? What I would use is, if it’s goes, if it’s heading, if they’re all heading towards the same point. Pretty much, kind of like…
>
> INTERVIEWER: When you said ‘they’re all’ what are you referring to there?
>
> CHRIS: All the different sections of the graph. Pretty much everywhere around the point that you’re looking at. Make sure there aren’t any, like, jumps or discontinuities or that kind of thing. But it looks like here I would just say you look at all the quadrants from every direction and see if they’re all heading towards the same area. And the way you tell here is just the color, that it’s all the same color, the same point. The same with these, that’s how I would prove this one to be not existent, I guess.

Chris began this description using a dynamic model of limit. The statement, “if they’re all heading towards the same point,” used dynamic imagery to describe how to understand when a limit exists. As an interviewer, I pressed Chris to describe what he meant by the words, and this resulted in a shift in his description. He used a combination of neighborhood and topographical descriptions to explain his thought process about the multivariable limit. He began with the phrase “everywhere around the point that you’re looking at,” which suggested that he used a sense of closeness to describe the function around the origin. Then he used topographical thinking to explain exactly what he was looking for in the graph, "Make sure there aren't any, like, jumps or discontinuities or that kind of thing." Finally, he went back to the neighborhood model when he says, "the way you tell here is just the color, that it's all the same color, the same point." This last statement is significant in that it not only alluded to neighborhood thinking, but also described a method for analyzing the function using this thinking.

The observation that the multivariable limit can be analyzed from looking at the color of the graph around the limit point marked a breakthrough moment for three of the students involved in the interviews. In the above statement we saw how Chris moved from mostly dynamic thinking to developing a strong sense of both neighborhood and topographical thinking when dealing with multivariable limits in contour graphs. From that point in the interview forward, Chris used the combined notions of closeness and external characteristics to describe how to analyze multivariable limits.

The two other students who concluded that the multivariable limit could be evaluated using the color around the origin were Josh and Jessica. For Chris it was not clear if the decision to analyze the functions using the colors near the origin represents a condensation of the infinite process of analyzing each path towards the origin; however, for Josh and Jessica there is more evidence of the students attempting to justify their use of nearby colors using dynamic imagery.

Excerpt 43

> INTERVIEWER: So, my question is can we actually look at these contour graphs – you know I actually noticed that these are the same functions that we used a second ago – can we look at these contour graphs and decide whether or not the limit exists at (0,0) or not?
>
> JESSICA: Yeah, because if it's not shaded in a color, then it's not there. The numbers aren't there, so looking at these, it looks like none of them

> exist at (0,0)… But, yeah, because… Well, this one looks like it might be… k, they both, these two… alright, the limit doesn't exist, but it's approaching a number, so that's what the limit is, so even though the value isn't there – the graph doesn't exist there – it's approaching a number, since the colors are similar around it, it means it's approaching the same number. So, it exists, but this one doesn't because the numbers around it are different. So, it's like not… it's like an asymptote kind of thing.
>
> INTERVIEWER: So, if I was to ask you, how can you tell from looking at a contour graph whether or not the limit exists, what would you tell me?
>
> JESSICA: I would say that if the color around it is the same, like all the way around it, then it would exist at whatever point is nearer to it.
>
> INTERVIEWER: Ok, and so why do you think that that's important, for the color to be the same around that point?
>
> JESSICA: Because otherwise if you're coming, because that way you can come from any direction and it will be the same, but on this one, since you're coming from, like, a high number and this is low, if you approach it from this side you'll get a different value as if you approach it from this side.

Initially, Jessica analyzed the function by looking at the origin and observing that the function does not exist at that point. Her initial reaction was that "The numbers aren't there, so looking at these, it looks like none of them exist at (0,0)…" During the experience involving three dimensional graphing, Jessica took the same stance that the function must be defined at the given point in order for the limit to exist at that point. However, unlike during the three dimensional graphing experience, Jessica reconstructed her thinking to correct her mistake. Initially she was torn by the fact that the function does not exist at the origin, but the contour graph seemed to suggest to her that the limit should exist. She stated that "the limit doesn't exist, but it's approaching a number, so that's what the limit is, so even though the value isn't there – the graph doesn't exist there – it's approaching a number." This showed her internal struggle with the fact that the function did not exist at the origin, yet the function seemed to “approach” a number at the origin. She resolved this conflict in her next statement, “since the colors are similar around it, it means it’s approaching the same number. So, it exists, but this one doesn’t because the numbers around it are different.” In this statement, Jessica used the neighborhood model of limit to convince herself that the multivariable limit must exist.

Like Chris, she developed a method for analyzing multivariable functions using the color of the contour graph around the point. When pressed to justify this method, she explained it using a dynamic model of limit, “because that way you can come from any direction and it will be the same.” This showed that Jessica made a direct connection between her earlier statement that for a multivariable limit you must approach from “infinite directions” and her current argument that the contour graph must be the same color “all the way around.” For Jessica, the act of looking “all the way around” the origin of the function represented the condensation of the “infinity directions” in which the function can approach the origin.

Josh has tended very strongly towards the dynamic model of limit throughout the entire interview. During his three interviews, there were 49 recorded instances of which 41 (84%) were classified under the dynamic model. Furthermore, of his eight non-dynamic recorded instances, five of them occurred while discussing contour graphs and were classified under the neighborhood model. When Josh began analyzing contour graphs, he used dynamic thinking, as before, to describe how he understood the multivariable limit.

Excerpt 44

> JOSH: So from there, as you can see in all these graphs the point (0,0) is a hole, there is some discontinuity there. So, once again just approach from different paths, and see what value I get when I approach from the right, the left, up, down, left, and right, they all appear to be converging at this central point there. So, I would say from that that the limit would exist; whereas, the contrast would be this third one. If I approach from this diagonal here, I appear to be getting close to this low point on the surface; whereas, if I approach from the right, the top right and bottom left diagonals, it appears to be at a higher point. That I'm going to approach this high point here and then approach this low point on either side, there. It's kind of the contrast to it, so I would say that this one does not exist; whereas, this, since each path seems to be approaching the same level on the contour, does exist. And the same concept applied here too, that each path or approach appears to be coming to this same surface level.

At this point Josh used dynamic imagery to analyze the multivariable limit. He began in a very sequential manner, analyzing each path in succession. By the end of his statement he stated that "each path seems to be approaching the same level on the contour," which showed that that he was starting to analyze all paths simultaneously. From this simultaneous analysis, he noticed the behavior of the function near the origin, that it appeared to be on the "same level."

Soon afterward, Josh observed that he can analyze the multivariable limit using the color around the origin.

Excerpt 45

> JOSH: So, same thing right here. I will just try to take this cross section of this graph, which kind of gives me all this area here in the middle. If I approach from any path they're all going to converge at this central point so I would say the limit does exist. This function, however, if I take a cross-section I'm kind of going to get, I'll get this value, or this area and this area are kind of on the same surface – I guess it would actually be these two, if the color shades are right – but that same cross sections would also have different, different levels of surfaces. So, kind of like that other graph, if I approach from the top left or bottom right I'll get but if I approach from the top right or bottom left I'll get a completely separate

> value. So, it doesn't exist. This one, the same thing, if I kind of take a cross-section, the tiny ring around the outside that it appears that it's getting steeper, kind of a volcano I guess, if I want to think of it in some sort of real life scenario. But it just appears that from each path that it's going to converge at that central point. So, I guess if I wanted to assign, like, the most simple definition, it would really be, 'is it the same color all the way around?' I guess would be the easiest way to visually do it with a contour. So…

As he began this statement, Josh still primarily used dynamic imagery along successive "cross-sections" of the graph. However, at the end of this discussion, Josh observed that "it would really be, 'is it the same color all the way around?'" This switch to neighborhood thinking starkly contrasted the previous dynamic thinking that Josh demonstrated up to this point in the interviews. When pressed to explain this new idea, Josh gave the following response.

Excerpt 46

> INTERVIEWER: So, ok, let's spend a second and… so what do you mean by this, 'if it's the same color all the way around,' and how does that makes sense to you?
>
> JOSH: Because of the contour map that a color is assigned to a certain depth or a height that from there it's just kind of, well if you think of multivariable limits in general, that you're going to converge from whatever path at this same value at this point. Well, since the value, the surface height, it basically assigned a color in this, so if all around the point that I'm taking the limit at is the same color then it's going to be the same height at that point. Like this graph here, it's the same color all the way around, so if I approach right, left, any approach whatsoever I take, a parabola approach, anything, it's going to be the same value. So kind of the most basic for a contour graph looking at the limit would be, 'is it the same color all the way the point we're taking the limit at?' I don't know if I described enough.

Josh justified using the color around the point by relating it to the dynamic model of limit that he used frequently throughout the interviews. Josh argued that since the contour graph was the same color around the point then the function will have "the same height" near that point. He further argued that every path towards the origin must pass through that same region making the result of each path "the same value." This argument captured several of the key ideas behind the formal mathematics proof that the sequential definition of limit is equivalent to the formal definition of limit. For Josh, I am convinced that this description symbolized a condensation of the infinite process of using dynamic imagery to approach along any possible path into the single action of analyzing the value of the function in a neighborhood of the origin.

Misconceptions about Multivariable Limits

Interview transcripts were analyzed for student misconceptions about the limit concept. These misconceptions were labeled and similar misconceptions were grouped together. The results are summarized in Table 17 below.

	Mike	Jessica	Jennifer	Amanda	Josh	Ashley	Chris
Symbolic Manipulation of Multivariable Limits							
Generic existence of limits	x						
Consideration of Only a Finite Number of Paths	x	x				x	x
Consideration of Only Linear Paths		x				x	
Misuse of the Word “Continuous”		x		x	x	x	
Failure to Evaluate Directional Limits			x	x			x
Using Partial Differentiation		x	x			x	
Undefined Functions Have No Limit		x				x	
3D Graphs of Multivariable Functions							
Misuse of the Word “Continuous”	x		x				
Consideration of Only a Finite Number of Paths	x						
Undefined Functions Have No Limit		x	x	x		x	x
Multivariable Limits Using Polar Coordinates							
Misconceptions involving constant functions	x					x	
Contour Graphs of Multivariable Functions							
Misuse of the Word “Asymptote”		x					
Contour Lines in Motion			x				
Analyzing the symmetry of the graph				x		x	
Undefined Functions Have No Limit		x				x	
Misuse of the Word “Continuous”							x

Table 17: Student Misconceptions of Multivariable Limits

I analyzed each misconception for its connection to the three models for understanding limits described earlier (dynamic, neighborhood, topographical). A number of the misconceptions are not directly connected to any of the three categories, but, rather, originate from documented misconceptions in introductory calculus. Table 18 illustrates how the misconceptions were categorized in terms of the models of limit described in the previous chapter.

Uncategorized Misconceptions Generic existence of limits Misconceptions involving constant functions Failure to Evaluate Directional Limits Using Partial Differentiation
Misconceptions Involving the Dynamic Model Consideration of Only a Finite Number of Paths Consideration of Only Linear Paths Contour Lines in Motion
Misconceptions Involving the Neighborhood Model <none recorded>
Misconceptions Involving the Topographical Model Misuse of the words "Continuous" and "Asymptote" Undefined Functions Have No Limit Analyzing the symmetry of the graph

Table 18: Misconceptions Categorized According to Limit Models

Uncategorized Misconceptions

Generic Existence of Limits

Tall (1992) describes a "generic limit property" for the limit of a sequence. That is, if every term of a sequence has a property then the limit of that sequence must also have that property. Mike decided to analyze the multivariable limit by looking at the limit along different paths. The exchange below arose following an analysis of the limit along three different paths.

Excerpt 47

> INTERVIEWER: Now, looking through these answers, would we say this limit as (x,y) approaches (0,0) of this function, would you say that this limit exists or not?

> MIKE: (long pause) I don't really know. I would assume that it does exist, but the limits do approach different numbers from, like, the three answers, but… I wouldn't… I'm not sure to be completely honest. I would just assume that since the limit does exist for all three, I would assume that the limit exists for the overall one.

From this exchange, it appears that Mike attempted to apply a "generic limit property" to the multivariable limit problem. In particular, Mike argued that if the limit along each path exists, then the limit of the multivariable function must exist without requiring the limits along each path to be equal.

Misconceptions Involving Constant Functions

Harel and Tall (1989) observed that constant functions tend to cause conceptual problems for students. Students seem distracted by the lack of an independent variable, and often fail to view the resulting expression as a function. In this study, there were several situations requiring the student to evaluate the limit of a function that is constant except at a single point. Most of these situations involved analyzing a multivariable function along a given path where the value of the function was constant. Students showed no signs of struggling conceptually in these settings. However, when using polar coordinates to examine multivariable limits, students struggled over the meaning of the limit as *r* approaches zero when no *r* can be found in the expression. Ashley's attempt to make sense of this limit is described below.

Excerpt 48

> ASHLEY: This one (an earlier problem), I think it's still going to be zero, just for the same reason of, like, when r is point two five, then it's still going to be a fraction of… because I'm trying to think of theta as almost a constant here. I don't know if that's the right way to go about this.
>
> INTERVIEWER: Now what about this third one, is this third one different?

> ASHLEY: The third one has to be different because there is no r in the equation, when it's simplified down, anyway. So it's dependent upon theta and I think it's going to be one. Just because I'm trying to think of… Well, not always one… Well, no, because, like, if you have, ok… like square root of three over two, one half, just the simple ones, that's square root of three over four, but then if you had square root of two over square root of two, that's going to be one half.
>
> INTERVIEWER: One half, so different values…
>
> ASHLEY: So, ok, that's where it's troubling me at. Because I think the limit still has to exist and everything, so that's not really an issue. I just don't know.

As we can see, Ashley struggled to make sense of an expression without the limiting variable involved in it. She wanted to view theta as simply a constant and take the limit with respect to *r* but struggles when there is no *r* in the expression. She seemed to realize that using different values for theta gives different values in the limit, but seemed unable to realize why this development is relevant in terms of multivariable limits. It is worth noting that she was unable to make sense of the meaning of limit in this context, yet confidently concluded that "the limit still has to exist."

Mike also struggled to make sense of this limit situation in the following excerpt.

Excerpt 49

> MIKE: Uh, the first one would just approach zero, the second one would also approach zero, the third one, since you could reduce it and get all the r's out of it… I wouldn't know what to do with the last one… I wouldn't know that last one.
>
> INTERVIEWER: That's fine, um, but we said that we've got zero, zero and it sounds to me like you think this one is not zero? Like, maybe, what are you thinking it might be?
>
> MIKE: I would assume that it would be zero, because it would mean that there is no radius, or it's not based on any radius.

INTERVIEWER: Oh, so you think this one might also be zero?

MIKE: Uh huh.

Mike's response to the fact that the *r* variable can be cancelled out of the third expression is to state that it must have "no radius." He then took this statement to be justification that the limit must equal zero. In this way we can see Mike's impoverished view of polar coordinates. That this transformation from Cartesian to polar coordinates left Mike unsure what the function represented, and concluded that the function expressed something about the value of the radius.

Failure to Evaluate Directional Limits

It is interesting that students generally succeeded at examining the limit of constant functions when taking limits along various paths of a multivariable function, yet several students struggled to make sense of the limit of a polar function without the limiting variable in the expression. This could be due to the fact that the polar function took the form of $\lim_{r\to 0}\cos\theta\sin\theta$; whereas, previous limits took the form of $\lim_{x\to 0}1$ (or something similar). The polar function looked much more sophisticated to a student and perhaps required them to second guess their own thinking.

However, a second explanation relates to the actual process of finding a multivariable limit. As a student examines the limit along a path, he/she must carry out a computation. Several students seemed to view this computation as a terminal computation which would result in the limit of the function along that path. Therefore, when a student noticed that along the path $x = 0$, they find $f(x,y) = 1$, they see the numeral one as a result of the computation (perhaps even a limit computation) rather than a constant with a required limit calculation.

The following excerpt is of Amanda's attempt to make sense of paths along a multivariable function in which the value of the function is not constant along that path.

Excerpt 50

> AMANDA: Ok, so I took a similar approach, just because these are the easiest things to plug in. So, I found zero, the limit as the line x equals zero and the line y equals zero to zero. So, that's the same, and that would work out, but then x equals y, you get something completely different, y over one plus y squared. And so…
>
> INTERVIEWER: Does that seem to tell you anything? Or what do you think about that result?
>
> AMANDA: See, again, that just seems like the limit does not exist, because it's not approaching the same point. But I can't remember if I'm missing something from the lesson. It's bugging me… I'm going to work out this one and see if I get a limit that exists.
>
> INTERVIEWER: Ok, that's a good idea.
>
> AMANDA: [Works silently] Interesting…
>
> INTERVIEWER: So what do you notice about this problem?
>
> AMANDA: Well, they're all of similar form. So, if x and y equals each other, then it seems like the limit would exist, because they're the same variable, kind of… It just seems like I'm missing something, the key part of why it won't all fit together in my mind…

Amanda observed a difference between paths on which the function is constant and paths on which the function is given by an expression. She failed to recognize the need to evaluate the limit along each path in order to examine its value. We can see this in her observation that on two of the paths the function is constant with the value zero, while on the third path the function is given by the expression $\frac{y}{1+y^2}$. This expression also has a limit of zero as y approaches zero, but Amanda seemed to view this as a different result from the prior two paths which result in constant functions.

On the third problem, Amanda again failed to recognize the need to evaluate a limit along the path. However, when analyzing the function along two different paths she ended with the expressions x^2 and y^2, respectively. As in the previous problem, Amanda failed to see a need to evaluate the limit along each path. Instead she compared the function along the entire path to determine if the limit should exist. In this way she concluded that x^2 and y^2 are similar, but with different variables, and that this supports the fact that the limit may exist.

It should be noticed that Amanda was never satisfied with her conclusions from this experience. She seemed aware that something about her thinking was incomplete, but was unable to make the connection to taking limits along the different paths. It is possible that in calculating the value of the function along each path, she had already viewed the limit computations as having been completed.

Using Partial Differentiation

Often students tend to look for a procedure to implement when posed with math problems. Students in this study tended to look towards the procedure of calculating partial derivatives. This seems to be connected to the fact that L'Hospital's Rule provides a succinct procedure for finding many single variable limits, and several students seemed to be searching for a multivariable analog of that rule. The following statement by Jennifer indicates that she was searching for a way to implement L'Hospital's Rule:

Excerpt 51

> INTERVIEWER: Why don't you run through this problem with me? Tell me what you would do, and I'll ask you some more questions about it.
>
> JENNIFER: Um, I feel like I've looked at this… Oh, since we're along the line of x, would you do… Would you do L'Hospital's Rule and take the derivative? With respect to x? Maybe I'm thinking…

She interpreted the task of finding the limit along the line *x* equal zero to be the same as that of using L'Hospital's Rule with partial derivatives with respect to *x*. Ashley also sought to use partial derivatives to make sense of the multivariable limit.

Excerpt 52

> ASHLEY: Here we go, we'll just switch those. Ok, so, then we took the derivative of that – partial derivative, that's the word I was trying to think of – so from here we can say there is a limit…
>
> INTERVIEWER: Ok, now would you mind telling me a little bit about what you wrote here and why you did these things?
>
> ASHLEY: Well, just from here to here I was taking the partial derivative…

Ashley, also looking for a procedure, decided to calculate partial derivatives in order to find the multivariable limit. By her next interview, Ashley recognized that this procedure was not effective and found a new procedure to take its place.

Misconceptions Involving the Dynamic Model

Consideration of Only Linear Paths

Ashley, at the beginning of the third interview session had this conversation about finding multivariable limits.

Excerpt 53

> ASHLEY: Well, multivariable limits, I remembered that you're supposed to set *y* equal to *mx* or something, and then solve it along that. So, like, with this one [works problem]… so it's fine. So, the limit does not exist, because it's not unique.
>
> INTERVIEWER: Oh, ok, so what do you mean by unique?
>
> ASHLEY: By unique, it's dependent upon *m*, and for a limit to exist it has to be unique and the same all the time. And then, it has to be continuous and I think there's one other thing that I'm forgetting. But the main thing here is, it's dependent upon m so it does not exist.
>
> INTERVIEWER: Ok, so last time I think you were talking about these partial derivatives and so I guess you've kind of changed the way you think about these things?
>
> ASHLEY: Yes, partial derivatives are more used for, let's see, well just finding the slope at a certain point, and it's not so much as for finding the limit.

Ashley decided that the procedure of finding partial derivatives is insufficient for finding multivariable limits, but her response was to simply replace this procedure with another one. Her new procedure was that of replacing *y* with *mx* and then calculating the limit as *x* goes to zero. This is equivalent to evaluating the limit along all linear paths without considering nonlinear paths. However, it is not clear that Ashley saw this connection to paths, but rather she appeared to simply view this as the correct procedure to perform in order to find multivariable limits.

Jessica, on the other hand, seemed to recognize that she was approaching from many different paths and points out that the paths worth considering are all linear. In the following discussion, Jessica evaluated the multivariable limit along three paths and found the directional limit to be zero in each one.

Excerpt 54

> INTERVIEWER: Right, so what about this limit, have you decided what this limit is or do we need to do more work?
>
> JESSICA: I decided that the limit's zero, because they're all zero. I mean, that's, it's… It might not be right, because I didn't do it from every direction, but I guess these are the best three to do to generally figure it out, so I think it's zero. And then the same thing for the second one [works silently] it's zero as well.
>
> INTERVIEWER: Ok, so you think this one is zero also? Let me ask you, so you're taking these lines… You said earlier that you weren't one hundred percent sure it was zero, but you thought it was?
>
> JESSICA: Well, yeah, because you're only taking three lines, you're not taking infinity lines, because there's infinity directions that you can come from to get to (0,0), if it's a surface.
>
> INTERVIEWER: So, like, what are some other directions you can choose?
>
> JESSICA: Well, you could… x equals negative y? Or, like, because you're coming from this direction, this direction, and this direction, but you could also come from this direction, or… Well, I guess this is the same, I mean, there's like other… There's this line, and this line, and this line, a lot of little lines.

Unlike Ashley, Jessica was aware that she evaluated the multivariable limit along different paths. Moreover, she decided that the only directions that need to be considered were lines passing through the origin.

Consideration of Only a Finite Number of Paths

Similar to the notion that all paths must be linear is the misconception that examining a finite number of paths is sufficient to show that a limit exists. This was a common misconception and appeared in four of the seven interview subjects. As shown by Jessica in Excerpt 8, students made statements about some finite collection of directional limits approaching the same value. This thought convinced them that the multivariable limit existed, while at the same time realizing that not every possible path had been examined. However, for students who seemed to believe that a finite number of directional limits was sufficient, the belief tended to be that directional limits along the two axes were sufficient to give the value of the multivariable limit.

Excerpt 55

> CHRIS: Well, it looks like it's going towards zero from both the left and the right. It does towards zero, but it doesn't actually hit zero, I think… So, therefore, I would just put the limit in, and it's zero, I think.
> Interviewer: And do you think we're able to make any decisions about this limit?
>
> CHRIS: Well, I would say it exists because [inaudible] it just seems like both axes are going towards the same point, and the xy axis is going also towards the same point.
>
> INTERVIEWER: So what if we choose some other line, or something. There's actually one other that we can choose that's interesting, it's y equals x squared… Or maybe it's x equals y squared… Yeah, I think that's what I want. Yeah, so I will just kind of let you look at this and see what you notice. [Student works] Ok, so tell me what you did to get these numbers.
>
> CHRIS: I basically plugged in y squared for x, and now we are back down to the single variable. So it's going to be y equals zero, and we don't have to worry about the x at all. And the thing that changed is the top, it went from, originally, when it was x equals y it was y cubed and now it's just y to the fourth, meaning that in all the places where you plug in numbers, it doesn't matter, negative or positive. I don't really know if it would change the limit, but it would make the zero over a positive sign… It just kind of balances out.
>
> INTERVIEWER: Now let me say something, and that is we need to be careful because x squared you changed into y squared, and I think that was just because you were going quick. Now does that change anything at all?
>
> CHRIS: That changes a lot and it goes to one half. It makes a straight line.
> INTERVIEWER: Now, the question that I have for you is, does, do you think that changes at all the limit, this main limit that we started with?

CHRIS: So did that change, like, the full limit?

INTERVIEW: Yeah, the limit of the multivariable function?

CHRIS: I really, I'm not sure…

We can see that Chris focused his attention on the *x* and *y* axes when deciding which paths to choose to analyze a multivariable limit. The interviewer introduced him to another path along which the limit is different, but that failed to convince Chris that the multivariable limit is undefined. In this next excerpt, we see that Mike had a similar misconception about limits. It is not clear that either student developed a strong sense of what it meant to analyze a limit along a path, and in particular how those limits can affect the value of a multivariable limit.

Excerpt 56

INTERVIEWER: So what do you notice about this one, about this limit? What would you say about it?

MIKE: I would say it exists, because all three of these limits are equal to zero.

INTERVIEWER: Ok, let me have you try something else. This is actually why this is an interesting problem. Try, instead of x equals y, try x equals y squared, and see what happens. (Long pause, student speaks to himself) So, you would say that this one is one half?

MIKE: Uh huh.

INTERVIEWER: Ok, so does that change the way you think about this limit?

MIKE: I wouldn't know how to take it. I wouldn't know what that limit would mean – I don't know from not having it yet.

INTERVIEWER: Ok, so, that's fine, I guess I… I'm just trying to think if there are any more questions to ask about these problems. So, going through the problems, do you think you have and more idea what this limit means and how to find a limit, a multivariable limit?

> MIKE: I understand… I think I understand what the limit means. I have more understanding, but I'm not sure. I'm not quite sure what the, like, if, I'm not quite sure how, like, what… If you place it along a certain line what that does to change, how that can change it, I don't understand. I don't quite understand how that works. But, I understand how this helps, but as for that, I'm not quite sure what that means.

As we can see, Mike struggled to make sense out of the purpose of analyzing the multivariable limit along different paths. He favored the view that the multivariable limit was defined by the behavior of the function along each of the two axes. This is not clear by the above dialogue, but became clearer when we analyze how he evaluated the multivariable limit from a three-dimensional graph.

Excerpt 57

> INTERVIEWER: So, again, I would like you to just kind of look at them and see if you can decide whether or not the limit exists. Kind of tell me what you see and tell me how you see these pictures.
>
> MIKE: This one seems a bit more clear to me because the… Like, it looks like along this axis and this axis (points to both the x and y axes) it… they're both… It's all approaching the point (0,0) right here. It seems like that's a lot more clear than the other one. It didn't look like it was clearly approaching any point. This one looks like it is clearly approaching the point there.
>
> INTERVIEWER: So, I hear you saying that you think maybe this limit exists and the value is somewhere around zero, I suppose, or whatever that value is?
>
> MIKE: Yeah.

This discussion occurred while analyzing the three-dimensional graph in figure 8 below.

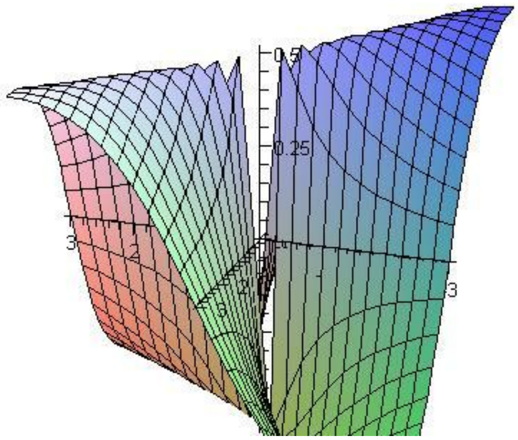


Figure 8: Three Dimensional Graph Discussed in Excerpt 57

Mike made use of dynamic motion to interpret this image. However, this motion seems to only be taking place along the *x* and *y* axes. Since motion along both axes led to the value of zero on the graph, this led Mike to believe that the limit does exist and its value is zero. Whereas, analyzing the graph along any number of other paths would quickly show that the limit does not exist. Mike was unique in this study in that he carried this belief about using motion from the symbolic treatment of multivariable limits to the analysis of multivariable limits using three dimensional graphing. Other students who used a finite number of paths to argue symbolically about limits changed their thinking once they encountered three dimensional graphs of the functions.

Contour Lines in Motion

Just as Mike misused dynamic imagery to make sense of three-dimensional graph of a multivariable function, Jennifer misinterpreted dynamic motion in the context of contour graphing.

Excerpt 58

> INTERVIEWER: Now how do you, what do you see what you look at one of these contour graphs? How do you interpret this picture?
> Jennifer: I kind of see that… They're all… As each of them kind of goes towards zero, they all kind of go away from it. Or all the graphs, all four graphs never cross the x and y axis. They're kind of contour, so…

In this statement, Jennifer showed that she did not view the contour graph as a single depiction of a multivariable function. Rather, she saw it as "four graphs" which each "kind of goes towards zero." As she explained about the multivariable limit, she will gave more information about this "four graphs" and what they actually meant to her.

Excerpt 59

> INTERVIEWER: Ok, so my question is can we look at this contour graph and decide whether the limit exists as (x,y) approaches (0,0)? Looking at these contour graphs.
>
> JENNIFER: Yes
>
> INTERVIEWER: So why don't you just look at these one at a time and tell me, do you think the limit exists or do you think it doesn't exist?
>
> JENNIFER: Here I would say it doesn't exist.
>
> INTERVIEWER: You say it doesn't?
>
> JENNIFER: Because the way they're all behaving, it seems like it will keep going and going and getting really, really close but they won't ever cross the x or y axis… Or the limit is zero… I mean, or…
>
> INTERVIEWER: You can take a second and kind of group your thoughts together. We're not in a rush.
>
> JENNIFER: Because I can see it continuing just to keep going like that, and going and going and going and getting closer, but never crossing zero. And so, maybe, that would make me say that it doesn't exist.
>
> INTERVIEWER: Ok

JENNIFER: But you could also say that it does exist at zero because they all do go towards zero… I don't, there's kind of a… Is it a hyperbola? There… [inaudible]

Jennifer described the lines on the contour graph as a depiction of motion. Her statement that "I can see it continuing just to keep going like that, and going and going and going and getting closer, but never crossing zero" was made in reference to the following contour graph.

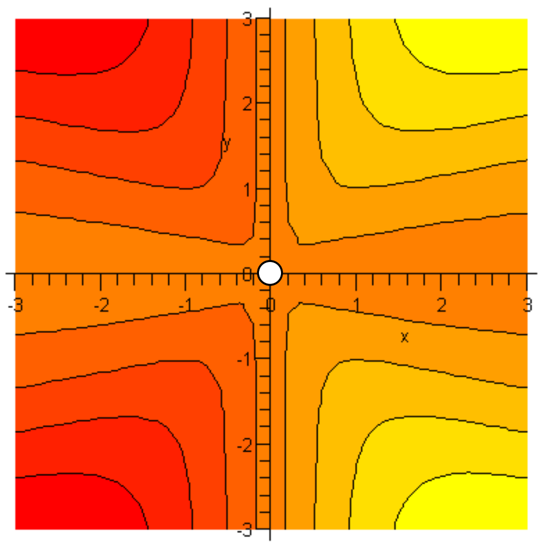


Figure 9: Contour Graph Discussed in Excerpt 59

In her statement Jennifer described a contour line in motion, describing it as "getting closer." I believe she saw each line on the contour graph as a separate instantiation of a fluid line. In this way, the lines depicted the motion of a single line in motion towards the center of the contour graph. She was torn about whether or not the limit exists because on one hand the fluid lines never reach zero, and the lines as a whole do not converge to zero, but on the other hand a singular point on each fluid line seemed to converge to zero.

This explanation of Jennifer's description is supported by her experience with a contour graph whose limit she believed does exist.

Excerpt 60

> INTERVIEWER: Ok, that's fine. I just, I'm wanting to kind of know what you think. So, what about this one? What do you think about this one, the second one?
>
> JENNIFER: This one, I would say it does exist, just because it… There's no, the points meet up, and so, like, as you get closer and closer to zero, that graph gets smaller and smaller… There's no new ones, probably just a dot.

The above discussion is in response to the following contour graph.

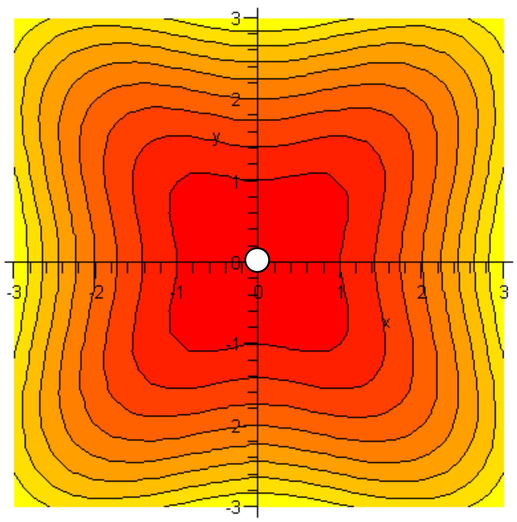


Figure 10: Contour Graph Discussed in Excerpt 60

For this image, Jennifer viewed the series of concentric circle as a depiction of one circle fluidly reducing in size resulting in "just a dot." For her, this is a contour graph which clearly has a limit at the origin, since every point on the circle gets progressively closer to the origin and the resulting "dot" is at the origin.

When asked to describe how to determine the limit using contour graphs, Jennifer gave the following response.

Excerpt 61

> JENNIFER: I would say how, looking at the behavior of all the graphs and then how they are acting as it moves towards zero, and if they're moving away from each other, or if they're moving towards this same spot.

Jennifer's explanation showed that she used dynamic motion to conceptualize the multivariable limit using contour graphs; however, this conceptualization was made by interpreting each contour as a graph depicting motion. In this way, instead of visualizing the contour graph as a static object she viewed it as a representation of a dynamic situation.

Misconceptions Involving the Topographical Model

Misuse of the words "Continuous" and "Asymptote"

Through the course of the interviews, the word "continuous" was misused on numerous occasions; however, it was consistently misused in the same manner. This led to its classification as a misconception.

Amanda first misused the word "continuous" when encountering a single variable limit which does not exist using a graphing calculator

Excerpt 62

> AMANDA: Oh wow, it's already done for me… Um, that jumps quite a bit. So, this is not continuous with just a hole in the graph, it's actually like, like that kind of graph…

From the above conversation, Amanda considered a function which was "not continuous" as different from a function "with just a hole in the graph." To her a function must have a jump or asymptote or something similar to be called "not continuous." When finding multivariable limits using symbolic manipulation, she again used the word "continuous."

Excerpt 63

> AMANDA: I would say that it does not exist.
>
> INTERVIEWER: And why would you say that?
>
> AMANDA: Because these two are both approaching zero. And (0,0) is a common point – one reason, and they have the point negative one, or it's negative one from this side and one from this side and that doesn't, that isn't continuous.

Although this is not directly a misuse of the word "continuous," Amanda seemed to imply with her statement that stating a function is not continuous is justification for stating that a limit does not exist.

Like Amanda, Chris first used the word "continuous" to describe a single variable function with a jump in it.

Excerpt 64

> CHRIS: Where it jumps. Um, I think it is not continuous. I would put it as 'does not exist.'

Similar to Amanda's second statement, Chris seemed to imply that a lack of continuity is equivalent to stating that a limit does not exist. I do not believe, however, that this is the result of Chris actually misunderstanding the relationship between continuity and limits. Rather, I think he used the word "continuous" to mean something much different from the formal mathematical meaning. Chris's next statement was made while evaluating multivariable limits using contour graphs, and I believe it gave a good explanation of what Chris meant when he used the word "continuous."

Excerpt 65

> INTERVIEWER: So if I asked you to describe a way that you could look at a contour graph and decide if the limit exists at a point, what type of description, what type of way would you come up with?

> CHRIS: Like a way to describe if the limit exists on one of these graphs? What I would use is, if it's goes, if it's heading, if they're all heading towards the same point. Pretty much, kind of like…
>
> INTERVIEWER: When you said 'they're all' what are you referring to there?
>
> CHRIS: All the different sections of the graph. Pretty much everywhere around the point that you're looking at. Make sure there aren't any, like, jumps or discontinuities or that kind of thing.

Earlier we saw how this was a pivotal moment in the way that Chris approached multivariable limits using contour graphs. When pushed to describe the meaning behind his dynamic description of a multivariable limit, Chris described the behavior of the function "around the point," and used topographical information to describe different phenomena that might be found there. Chris used the phrase "jumps or discontinuities or that kind of thing" to refer to topographical features that would indicate that a limit does not exist. It appeared that Chris used the word "discontinuous" to mean something like "not normal" or "not nice."

It is for this reason that this misconception was classified as "involving the topographical model," because students appeared to be using the word "continuous" to describe a topographical feature of the function as opposed to a relationship between the function and the limit at that point. This indicated that students are using a visual image to define the word "continuous" rather than a mathematical limiting process.

The word "asymptote" carried a similar connotation for Jessica. The following conversation took place while Jessica used a graphing calculator to examine a single variable function with a jump discontinuity.

Excerpt 66

> INTERVIEWER: So, what does that seem to be saying? You said it does not exist?
>
> JESSICA: Yeah.
>
> INTERVIEWER: So, why would you say it does not exist?
>
> JESSICA: Because, if it jumps up like that, it's an asymptote… I think… Because, like, if something goes from, like, a low number to a high number really fast that means, like, in the calculator it can't actually say that it's an asymptote – so, that's the way it shows it.

Similar to the way Chris used the phrase "not continuous" to describe the topographical features of a function which does not have a limit at a point; Jessica seemed to be using the word "asymptote" to describe a point where the function "jumps up" to a different value. She used this word again in the third interview while using contour graphs to study multivariable limits.

Excerpt 67

> INTERVIEWER: Can we look at these contour graphs and decide whether or not the limit exists at (0,0) or not?
>
> JESSICA: Yeah, because if it's not shaded in a color, then it's not there. The numbers aren't there, so looking at these, it looks like none of them exist at (0,0)… But, yeah, because… Well, this one looks like it might be… ok, they both, these two… alright, the limit doesn't exist, but it's approaching a number, so that's what the limit is, so even though the value isn't there – the graph doesn't exist there – it's approaching a number, since the colors are similar around it, it means it's approaching the same number. So, it exists, but this one doesn't because the numbers around it are different. So, it's like not… it's like an asymptote kind of thing.

Jessica used the word "asymptote" in a similar fashion, this time describing a point where "the numbers around it are different." Like Chris, and possibly Amanda, Jessica seemed to use this word in a manner that described a certain topographical feature of a graph rather than a limiting process.

Undefined Points Have No Limit

In the section "Three Dimensional Graphing of Multivariable Functions," we saw two students, Amanda and Jennifer, who struggled with the meaning of the hole at the origin of the three dimensional surface. These two students appeared to be viewing the hole as a set of points surrounding the origin instead of just the origin itself; however, when they encountered a multivariable function whose limit did exist at (0,0), they changed their interpretation of the hole in the graph and began to use dynamic imagery to determine the existence of the limit. It appeared that Amanda and Jennifer's difficulties arose not from a misconception about the multivariable limit, but rather from a misinterpretation of the graphical information given them. Once they encountered a function that looked familiar, they were able to reinterpret the graphical information and use dynamic imagery to understand the multivariable limit in a successful fashion.

On the other hand, two other students, Jessica and Ashley, consistently pointed to the nonexistence of the function at the origin as evidence of a nonexistent multivariable limit. Ashley's difficulties arose when encountering single variable limits. The following excerpt took place while discussing figure 11 below.

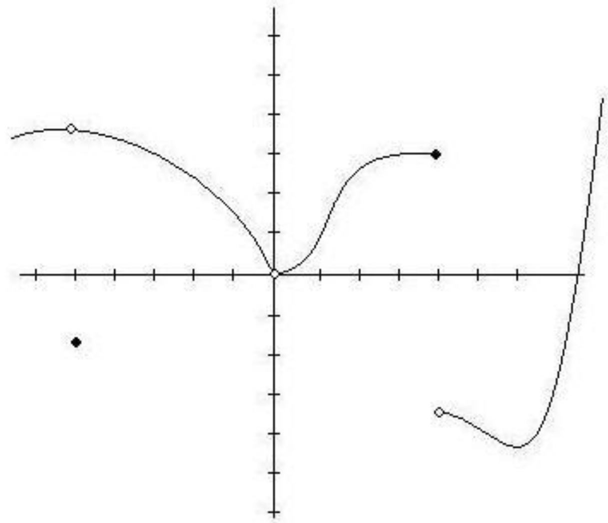
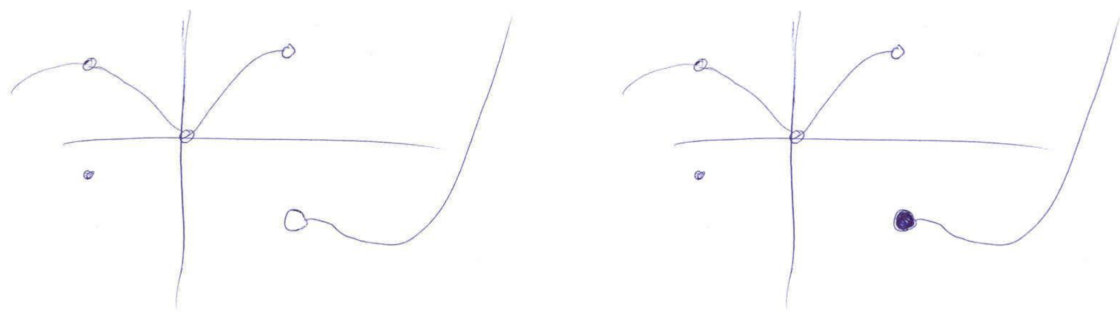


Figure 11: Graph Used for Single Variable Limit Problems

Excerpt 68

> INTERVIEWER: How about this third one, as *x* approaches 4 of *f*(*x*)?
>
> ASHLEY: Well, I think it would be best to break this one up into as *x* is being approached from the positive and then from the negative. But… Cause if you did it from the positive, then it's going to be negative four, and from the negative it's going to be about three. But, ok I forget… I think it's supposed to be three then because it actually reaches that… or… ok, no… I haven't done these in so long.
>
> INTERVIEWER: That's fine, I understand.
>
> ASHLEY: The other thing that I'm thinking right now is that it is undefined because it gets so close on both, but it actually reaches four there, that's the tricky part…

Ashley analyzed this function both by using dynamic motion from each side and by looking at the value at the given point. She saw a conflict between these two ways of analyzing the function and struggled with how they work to give the value of the limit. At this point the interviewer asked her about the 'before' graph shown below and then changed it to look like the 'after' graph shown below.



(Before) (After)

Figure 12: Two Graphs Shown to Ashley

Excerpt 69

> INTERVIEWER: So, say for example we had a similar problem – I'm going just to try to draw this graph, it's going to be a quick hand-sketch thing, so we'll see how it goes… it goes something like this, here's that hole there. But say for example this was an empty hole and this one was empty, would that change the limit at that point? What would be the limit of this function?
>
> ASHLEY: I think it should be undefined, then… because
>
> INTERVIEWER: So, you think this one is undefined?
>
> ASHLEY: Like it doesn't exist because if you're just looking at the hole four and it never actually reaches it on either one… so…
>
> INTERVIEWER: So this one is undefined? So, what do you believe about this one – you believe this one should be three, is that what I heard you say?
>
> ASHLEY: Uh, huh. Because that's that.
>
> INTERVIEWER: And, so if we took this and filled this hole in here, do you think that would change the value of this function?
>
> ASHLEY: Then it would be… negative three and a half…

We saw that Ashley seemed to believe that value of $f(4)$ played a significant role in the limit of the function as x approaches four. However, immediately before this discussion, Ashley spoke about the limit of the function at negative five and zero and in both cases she emphasized that the value of the function at the limit point played no role in the value of the limit.

Excerpt 70

> INTERVIEWER: We have a graph of $f(x)$ and we'd like to figure out what these limits are, so I'd like you to tell me what they are and importantly how you can to that conclusion.
>
> ASHLEY: [talks to herself] Let's see, it's getting closer to… let's see, a little over 3, so 3.4 or 3.5. So, that would be the limit there, but the actual point value for that is somewhere else, but the important thing is the limit.
>
> INTERVIEWER: And so, you said 3.5 or so? Would you mind telling me why, how you came up with 3.5 as your answer?
>
> ASHLEY: [inaudible]
>
> INTERVIEWER: And this one
>
> ASHLEY: As x approaches zero would be zero.
>
> INTERVIEWER: And the reason why it is zero is because…?
>
> ASHLEY: Because that's where that empty hole is and you're getting closer on each side.

I believe that these problems are a result of Ashley relying heavily on topographical information to graphically determine the value of the limit. Ashley seemed to be looking for an "empty hole" in the graph to be the value of the limit. In the cases of excerpts 69 and 70, Ashley could find no such hole and, therefore, she was torn between stating that the limit does not exist and stating that the limit is the value of the function.

While solving a single variable limit symbolically, Ashley revealed more about how she viewed limits.

Excerpt 71

> ASHLEY: [writes solution]
>
> INTERVIEWER: Ok, would you mind telling me how you came up with these solutions here?
>
> ASHLEY: Ok, well, I took out the *x* minus one because that's a difference of squares right there. So, I was able to cancel those out, and … So, I was left with *x* plus one on the bottom and then plugged in one for *x* and got one half.
>
> INTERVIEWER: And one half represents?
>
> ASHLEY: the *y* value…. And, then the limit should also be one half – because, yeah, you can just plug that same thing in.
>
> INTERVIEWER: Now, let's take just a second and ask, 'how did the way that you solved this problem relate to the way that you described limits before, when we were talking about limits?" Does that make sense?
>
> ASHLEY: Yeah, kind of… Let's see… Well, basically, with limits you can plug it into the problem… The big thing is when you have… at a point where the function doesn't exist, so that becomes a problem. And like in this one, that's not the case because you can always have an answer for it, and so it's continuous and then you can just plug in the point to get it.

Ashley did not view the action of cancelling out the factor, $x - 1$, as changing the nature of the function. Instead, she believed that this simplified version of the function is equivalent to the previous version, and importantly, that both functions must exist at the point $x = 1$. More important is Ashley's notion that "with limits you can plug it into the problem." This showed that her primary notion of limit was based on the value of the function at the point at which the limit is evaluated, not on a process of analyzing points near the limit point. However, Ashley realized that the function, as presented in the problem, was not ready to be evaluated until after it was simplified. So, on one hand, she believed it was important to analyze the value of the function at the limit point, but on the other hand, she believed it was important to go through a simplification procedure before inputting the value into the function. This struggle also arose while using symbolic manipulation to find multivariable limits.

Excerpt 72

> INTERVIEWER: Ok, now let us look at this last one, this last one asks to look at the results and what does that say about the limit as (*x*,*y*) equals (0,0), as (*x*,*y*) approaches (0,0) of this function. Does it exist or does it not? And if it exists does it have a value?
>
> ASHLEY: I would say that there is a hole there and that is because, just even when you put (0,0) in then obviously you're going to have zero on the bottom which means that it's undefined, but you still have all these other values, so that would just mean that it's approaching those, or approaching from those sides maybe, but there's still a hole there because it can't have a value.
>
> INTERVIEWER: So, what did you decide about the result, does it exist or not?
>
> ASHLEY: I would say that it doesn't exist, so…
>
> INTERVIEWER: And you would say it does not exist because…
> ASHLEY: There's a hole.
>
> INTERVIEWER: There's a hole. Ok, so I want to ask… I have some more questions here that I didn't put prompts on them. And I was curious, could you just describe some way you would approach a problem to figure out whether or not you think the limit exists at that point.
>
> ASHLEY: Ok.
>
> INTERVIEWER: So, like, if I were to just give you one of these, say this one, say one of those two, just to ask you 'does that limit exist?' how would you go about deciding if it existed of not?
>
> ASHLEY: Normally I would start by just plugging in both values into this and just see if the denominator equals zero, and often it does. So, most likely there's going to be a hole there, but after that I try to pull out any kind of values, like here you can't really do it, but if you can pull out an x from the numerator and denominator and try and cancel out that way you

> can see kind of where that point would be if it were possible for it to exist, but because it's undefined, then you can't… So, in this one then, I would go ahead and take the partial derivatives and then see what those came up with, so…

Ashley tended to be very procedure-oriented. In single variable calculus, that procedure tends towards simplifying symbolic expressions and using topographic information in graphs. Her initial reaction towards using symbolic manipulation to find multivariable limits was to repeat this procedure by first "plugging in both values," and if that does not yield a solution she tried to "cancel out" so that she can "kind of see where that point would be if it were possible for it to exist." I believe that this procedure is actually Ashley's primary image of what it means to evaluate the limit of a function as opposed to a concept image that uses dynamic imagery along different paths to understand the multivariable limit.

When her procedure failed to give a clear answer, Ashley looked for other ways to analyze the limit of the function. We saw earlier that Ashley used partial derivatives as a procedure for finding multivariable limits, perhaps inspired by L'Hospital's Rule from single variable calculus. This use of partial derivatives appeared to be a procedure developed to deal with functions for which the first procedure failed to give a solution to the limit.

Upon returning for the third interview, Ashley became aware that her procedure for analyzing multivariable limits using partial derivatives was inappropriate. At this time, she replaced this procedure with a new procedure of "substituting in $y = mx$" and focused on the uniqueness of this limit. As mentioned earlier, it is not clear from the interviews that Ashley actually saw this procedure as being connected to linear paths through the origin. Instead, it seems more believable that Ashley adopted this procedure as a substitute for the failed procedure using partial derivatives.

When using three dimensional graphing to evaluate multivariable limits, Ashley again looked closely at the hole at the center of the surface. The following excerpt references the two three-dimensional surfaces shown below.

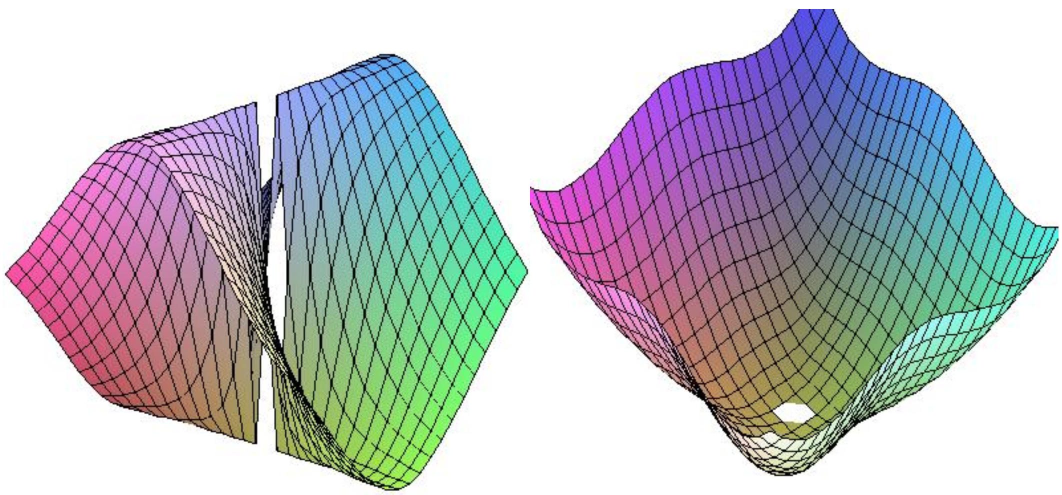


Figure 13: The Two Graphs Discussed in Excerpt 73

Excerpt 73

> INTERVIEWER: So, I was wanting you to kind of look at this function and see if, by looking at it, you can find out if you think the limit should exist or not at the point (0,0). And you might be able to guess, (0,0) is just kind of right there in the middle.
>
> ASHLEY: Ok, well, there appears to be a large gap there, but then looking at it from that angle… Well, there kind of looks like there might be a limit, but this part down here is actually connected, still. So, my guess is now that there's probably not a hole.
>
> INTERVIEWER: Ok.
>
> ASHLEY: But I could be wrong.

INTERVIEWER: Does this say anything to you whether you think the limit exists or not?

ASHLEY: I think it exists.

INTERVIEWER: Ok, can you tell me why you think it might exist?

ASHLEY: Well, because it's still connected and you really don't see any break, the only time you see a break is right here, and I'm not really sure what that's saying, and so that's where I'm getting stuck. Because that could just be a [inaudible].

INTERVIEWER: Let's come on down, now this third one here is the second one we did today. I don't know if that seems familiar to you, but that's the next one we did, so you can kind of look at it and see what you think of it. That's kind of what it looks like.

ASHLEY: So, that definitely has a hole right there. So, I think on this one, I was right in saying, there's a hole. It, kind of, looks like a lampshade.

INTERVIEWER: It kind of does look like a lampshade.

ASHLEY: So, yeah, because that you can really tell that there's supposed to be a hole and there's no values that satisfy that.

INTERVIEWER: So what does that seem to mean? Do you make any conclusions about the limit of the function at (0,0) by looking at this picture?

ASHLEY: I'd say the limit approaches zero but it can never equal zero.

INTERVIEWER: Ok, so can you say what you kind of mean when you say those words?

ASHLEY: Ok, well, all the values are going to be coming closer and closer to zero, but you can never physically reach zero because it would be undefined.

INTERVIEWER: So let's come up and, how is this one different from the one we looked at just a second ago?

ASHLEY: Well, if this one is zero… oh, that looks weird… If this one were to have a limit of zero, I'm still not sure if it does, I kind of think it doesn't.

> INTERVIEWER: So you're thinking it does not have a limit at zero?
>
> ASHLEY: It has… Ok, it doesn't have a hole, it has a limit, which I'm thinking is zero, but it doesn't have hole… See, right there it looks like a hole. So, that's where I'm kind of getting stuck on.

When Ashley analyzed limits using graphs in single variable calculus, she appeared to be examining the graph, looking for an "empty hole" that would become the limit value. When she failed to find such a "hole" she struggled with whether the limit should be the value of the function at the point or whether the limit does not exist. In the above excerpt, Ashley carried some of these beliefs over into her examination of multivariable limits.

To begin, Ashley argued that the function appears to be "actually connected, still" and so she concluded that the limit should exist. At this point, it is not clear how this connected nature of the graph influences her decision whether or not the limit should exist. However, we get more information once she encountered a function whose limit does exist. For this limit she decided that "the limit approaches zero, but it never reaches zero." These statements make important suggestions. The first is that, unlike in the single variable case, Ashley did not have a clear topographical picture of a multivariable limit. In the single variable case, she found a picture with an "empty hole," and that hole would indicate to her that the limit should exist and take the value given by the "empty hole." Therefore, when she searched the multivariable limits using topographical thinking, she immediately began searching for a "hole," but she does not find anything that compares to her topographical image of limits from calculus one. In this way, she was like Jennifer and Amanda, in that their weakness with the three dimensional graphs possible stem from a weak topographical image of multivariable limit.

However, unlike Jennifer and Amanda, Ashley's view of limit carried with it an emphasis placed on the value of the function at the limit point. In Ashley's words, it is important whether or not the function "physically reach(es) zero." It has been documented (David and Vinner, 1986) that students often carry an image of limit as being unreachable, and often this explanation relates to the colloquial use of the word "limit" in modern English. However, Ashley seemed to be struggling with an opposite viewpoint, that the limit must be reached. This caused her struggle with graphical limits in single variable calculus and caused her struggle with multivariable limits using three dimensional graphing. However, I believe that Ashley's struggles derived less from a strong conceptual belief that a limit cannot reach a point and more from her topographical view of what a limit should "look like." Ashley appeared to be struggling for meaning between the two words "hole" and "limit." I believe her previous topographical image of limits equated the idea of limit with a hole in a graph. However, as she encountered multivariable limits, the connection between these two ideas is less clear.

Similar to Ashley, Jessica made the statement while using symbolic manipulation to find multivariable limits that it is important for the function to exist at the point.

Excerpt 74

> INTERVIEWER: So, if I had a limit, if I just said, "I want to know whether this limit exists," what can I do to decide if it exists or not? How would you kind of respond to this?
>
> JESSICA: Well, you can do what you did here and go along the x-axis, y-axis and then different lines that go through the point that you want to find it at. Because it doesn't go through the point then it doesn't make any

> sense. Or, you can kind of see if it's continuous at that point – I think. What I think of it is like, you find where this can't be true, like if this would equal zero, which can't happen because they're both squares and you add them, so it's continuous everywhere. Cause, I mean, that… If you're trying to get to a point that it's not continuous at, or it isn't there at, then it doesn't exist. But, for this case, you would have to do the different directions to get to…

In her statement that "If you're trying to get to a point that it's not continuous at, or it isn't there at, then it doesn't exist," Jessica implied that the function must exist at a point in order for that limit to exist. However, like Ashley, in actuality, Jessica used a procedure to determine the limit value when the function doesn't exist at a point.

When it comes to the three dimensional graphing portion of the interview, Jessica emphasized the presence of a hole in the graph and used that hole as justification for the limit not existing at the point.

Excerpt 75

> INTERVIEWER: Kind of tell me how you understand this and whether or not you think the limit exists
>
> JESSICA: At (0,0)?
>
> INTERVIEWER: At (0,0).
>
> JESSICA: Ok, well, it doesn't exist because there's a hole. And, this part is the surface, so if it's missing, then it's not there, so it doesn't exist, and that's how you can tell. I mean, it doesn't exist along the whole *z*-axis either. So, that's (0,0).
>
> INTERVIEWER: Ok, let's keep looking at some of these other ones. So, come down, here's the second one.
>
> JESSICA: That doesn't exist either. For the same reason it's… there's no surface there.
>
> INTERVIEWER: Ok, let's look at this third one, and see what it looks like.

> JESSICA: It doesn't exist either.
>
> INTERVIEWER: So, if I asked you to describe a way that you would look at a surface and determine whether or not the limit exists as (x,y) approaches (0,0), what would say to me?
>
> JESSICA: Based on a graph?
>
> INTERVIEWER: Yeah, based on a graph.
>
> JESSICA: Well, I would say, look down the z-axis and see if there's a missing place around any area and that would be the place where the limit doesn't exist

Amanda and Jennifer both changed their mind about the role of the hole in the graph after experiencing a multivariable function which has a limit at the origin, and Ashley never rectified her debate between whether it was more important for the function to "reach" the limit point or whether the function should just "approach" the limit point. Jessica, however, was confident in her claim that the limit cannot exist when the function has a hole at the origin. Jessica was consistent in applying this logic throughout the interview and reasserts the connection between the existence of the function at a point and the limit of the function at the point.

Excerpt 76

> JESSICA: It's weird that none of them exist at zero.
>
> INTERVIEWER: Ok, well… Has… So, looking at these graphs, does it affect the way you think about what a multivariable limit is? Or how to understand a multivariable limit?
>
> JESSICA: It kind of proves what I think about it. Like, cause that's what it is, it's where the surface doesn't, isn't there… It's where it's got holes in it or doesn't exist.

This statement combined with her determination that each of the limits fail to exist at the origin demonstrated Jessica's strong inclination towards using the value of the function at the point to determine the limit of the function.

Like Ashley, this view of limits manifested itself during Jessica's first interview on single variable limits. While finding limits using the graph of a single variable function, Jessica made the following statement.

Excerpt 77

> JESSICA: Yeah, but, see… Ok, so I looked, I found the negative five on the *x*-axis and I looked at where the function existed, like the line. So, when it's negative five, normally you would think, like, based on the curve of the line, it should be, like, three. But, since it's open, that means it's not there. So, I just looked for where it does exist, which is like the closed dot… And I did that for all of them.

Ashley and Jessica both frequently struggled with the meaning of a multivariable limit at an undefined point on a graph. They were both able to put together procedures to show symbolically that a multivariable limit does not exist; however, it is likely that both students used these procedures independently of any knowledge about the value of the function at the point. There was a difference, however, in the conviction of each student during the three dimensional graphing portion of the interviews. Jessica was strongly convinced that the nonexistence of the function at the origin implied that the limit also did not exist; while Ashley struggled between the concept of "limit" and that of a "hole."

In each case, the description of the student's cognitive image of limit proved to be extremely difficult and resulted in only a partial explanation of each student's behavior during the interviews. I believe this is due, in large part, to the fact that each student's image of limit, and in particular multivariable limit, was initially weak and experienced growth and change throughout the interview process. As the students grappled with new ways of viewing the limit concept, their image of limit remained in a state of flux, and often contradictions were left unresolved at the end of the interviews. I also think that each student maintained a heavy reliance on procedural thinking, which often led to a disconnect between their behavior on certain problems and their description of the limit concept outside the context of those problems.

In the final interview session, the two students examined multivariable limits using contour graphs. At this point Ashley emphasized her topographical view of the limit concept which will be discussed shortly, while Jessica changed her viewpoint and used both dynamic and neighborhood thinking to evaluate the contour limit.

Excerpt 78

> JESSICA: Yeah, because if it's not shaded in a color, then it's not there. The numbers aren't there, so looking at these, it looks like none of them exist at (0,0)… But, yeah, because… Well, this one looks like it might be… ok, they both, these two… alright, the limit doesn't exist, but it's approaching a number, so that's what the limit is, so even though the value isn't there – the graph doesn't exist there – it's approaching a number, since the colors are similar around it, it means it's approaching the same number. So, it exists, but this one doesn't because the numbers around it are different. So, it's like not… it's like an asymptote kind of thing.

Analyzing the Patterns on a Contour Graph

While using contour graphs to analyze multivariable limits, two students, Ashley and Amanda, used the symmetry of the contour graphs to justify conclusions about the multivariable limit. During the three dimensional graphing portion of the interviews, Ashley struggled with the meaning of the "hole" in the graph versus the "limit" of the graph. At this point she began to make some observations about the structure of the three dimensional surface.

Excerpt 79

> INTERVIEWER: Now, on this other problem you mentioned something about, 'it doesn't exist at the point but it approaches zero.'
>
> ASHLEY: Yes.
>
> INTERVIEWER: Now what about this one, does this one have anything like that going on, where it approaches something?
>
> ASHLEY: I think it's definitely approaching zero at least, but the thing that's throwing me off is this whole gap is missing, but, oh, because it's *x* squared and *y* squared, you're looking at a parabola basically on each side, so then that's how they meet. I bet that doesn't exist at zero. Like the limit is still approaching zero… I think it's… I wasn't looking at it from the parabola perspective.
>
> INTERVIEWER: So, tell me what you're seeing now and what you're thinking.
>
> ASHLEY: Ok, well I'm looking at the different parabolas and how they're occurring, different because like you've got your parabola here, and then another one on this side, so it's how they're meeting together and, so that's why that gap in the middle looks so funny. That would be different values at zero. So, or, for those two equaling zero…

Ashley tried to understand the shape of the surface as a whole. She took a "parabola perspective" which allowed her to get a better grasp on the multivariate function. Even though she did have a better perspective on the function, she held to strict topographical thinking and used the surface features of the graph to make decisions about the limit. In this context, she never viewed the limit concept as a process of analyzing the function near the limit point, but rather she depended on strictly topographical features of the surface.

In a similar way, when Ashley used contour graphs to analyze the multivariable limit she used a topographical argument about the symmetry of the graph to determine the existence of the limit.

Excerpt 80

> INTERVIEWER: So the question is as we look at these contour graphs, is there any way to tell by looking at them whether or not the limit exists as (x,y) approaches (0,0)?
>
> ASHLEY: Well, is this supposed to be a hole right here, or is this just the origin being pointed out?
>
> INTERVIEWER: Oh, well it's supposed to represent a hole.
>
> ASHLEY: Ok, well obviously, because that's (0,0), and I don't think it can exist. Like with these two it might work a little bit just because you can tell it's moving towards it, but like with this one where we determined it didn't exist because it wasn't unique, it doesn't really help so much. These just seem to be more patterned as far as how they go, and that may just have to do with the graphs themselves and not have anything to do with the limits.
>
> INTERVIEWER: Now can you tell me a little bit, what do you mean when you say, 'more patterned?'
>
> ASHLEY: Well, like, in this one it's a really distinct shape over and over, repeating itself and getting closer and closer, and the same kind of thing with this, I mean, it's one in every quadrant but it's that same basic idea; whereas, with this one, yeah there's that same basic shape but it seems to just be more lines than just anything else, and you've got, well, you're low points here and then your high points. Whereas this has a very consistent flow to it, where this is shrinking down and that's increasing, so…

When Ashley continued describing her thoughts on using contour graphs to determine multivariable limits, she further developed her idea of use "patterns" to determine the limit of the function.

Excerpt 81

> ASHLEY: Ok, I'll say this one exists.
>
> INTERVIEWER: Now tell me why you say they exist.
>
> ASHLEY: I'm trying to think back to the pattern idea I had with those, so there's a very distinct movements and this one seems to have a more distinct movement. Let's see… This one probably exists, just because

again it has that upward sloping of values and similar line patterns. I would say this one doesn't exist.

INTERVIEWER: You say it does or does not?

ASHLEY: Does not.

INTERVIEWER: And why would you say that.

ASHLEY: Well, there is a pattern to it and everything. It's almost as if it's just kind of plopped in there and there's not so much rhythm between all the points. So, that's what I'm kind of thinking there. Whereas, like, with this one it's very obvious that there's a flow to it, so on this one I think would be more likely to exist.
INTERVIEWER: Ok.

ASHLEY: And then I would say these two do not exist, because that's jumping around kind of, and this one… I don't know, this one possibly could be just because it's reminding me somewhat of this one

INTERVIEWER: Ok.

ASHLEY: And that's why it's not making sense in my head.

INTERVIEWER: So let me ask you, if I were to ask you to describe a way that you could look at a graph and decide whether or not it exists, how would you describe that way to me?

ASHLEY: Ok, I think there's an obvious flow and cohesion in the graph. So, like on this one, it's very obvious that, like, your corners are your highest points and it's coming in almost like in a cone pattern. Any of these that seem to be just kind of jumping around and not have this very obvious flow because when graphs are moving similarly then they, I think they would be more likely to have an actual limit then, because it would be unique to that; whereas, these it's just like everywhere.

Ashley placed a high importance on the "pattern", "flow" and "cohesion" of the contour graph. However, she never succeeded in pinpointing exactly what she means by these words; instead, she seemed to base her judgments on her gut feeling about the graphs. This is consistent with her use of topographical information throughout the interviews. Her use of topographical information depends on categorizing the shape of the graph and determining the limit based on the categorization. I believe her statements about "flow" and "cohesion" were an attempt to develop a sense of categorization for the contour graphs using the principle that normal-looking graphs tend to have limits that do exist while abnormal-looking graphs tend to have limits that do not exist.

Amanda also used patterns on a contour graph to make sense of multivariable limits.

Excerpt 82

> INTERVIEWER: Ok, so by looking at these graphs, can you tell me whether the limit exists at (0,0)?
>
> AMANDA: Interesting… Yes, I think so.
>
> INTERVIEWER: Ok, well can you tell me why you think that?
>
> AMANDA: It does, it does, it doesn't.
>
> INTERVIEWER: OK, now…
>
> AMANDA: Ok, I'll slow down.
>
> INTERVIEWER: No that's great, now tell me why you believe those things…
>
> AMANDA: So, these two are very symmetrical on the contour graph – well, they look even from all different sides, like they're all, these are all the same all around, like the line spacing and whatnot. And this isn't so much.
>
> INTERVIEWER: So, I'm not real sure exactly what you mean by 'symmetrical' and by 'it looks even' and all that, can you be more detailed about that?
>
> AMANDA: You make me think so much.
>
> INTERVIEWER: Sorry.
>
> AMANDA: That's ok, I signed up for it. Here in the first and the second illustration, the lines are getting, the spacing is even, or the same,

> throughout the four quadrants. And so, reason would say that, therefore, it is increasing or decreasing at the same point and going towards this one thing.
>
> INTERVIEWER: And what about this one? You seem to think that this one does not work?
>
> AMANDA: Yeah.
>
> INTERVIEWER: And can you tell me why this one doesn't work?
>
> AMANDA: Because – also I think the color has to do with the direction.
>
> INTERVIEWER: And, that's true actually, I think way it's interpreted, yellow is big numbers and red is low numbers.
>
> AMANDA: That makes sense. And so, right here, it's all over the place. It's red in the second and fourth, and yellow in the first and third. And, if I remember some of the pictures correctly that we've looked at before, those graphs didn't have limits because they were oscillating, like they never really all came together at one point.

Unlike Ashley, Amanda used her sense of symmetry in conjunction with dynamic imagery in order to determine the multivariable limit. She talked about the symmetry of the graph, but it seemed that the real structure of her argument came from phrases such as "going towards one point" which described dynamic motion taking place on top of the topographical structure of the graph. During this discussion, Amanda referred back to the previous interview when she said, "if I remember some of the pictures correctly that we've looked at before, those graphs didn't have limits because they were oscillating, like they never really all came together at one point." This showed that Amanda had developed a topographical prototype of a multivariable function without a limit. When she spoke about analyzing the symmetry of the graph, I believe she actually attempted to visualize the shape of the graphs and compared it to her topographical knowledge about surfaces. As she encountered more contour graphs, she described this activity of visualizing the three dimensional surface.

Excerpt 83

> INTERVIEWER: So, now if I was to give you more pictures, like these I won't let you see yet, could you describe, maybe, a way that you would look at those pictures and decide whether or not the limit exists at (0,0) or I guess any other point?
>
> AMANDA: I would look at that the line spacing and try, with the contour lines, envision what the actual graph would look like and (inaudible)…
>
> INTERVIEWER: Well, let's go through these and see what you think about them – see which ones you think exist or which ones you think don't exist.
>
> AMANDA: Ok (long pause) so this one is very similar to this, so I would say that it does exist for the same reasoning. This one does exist, because it's like a little circle and goes, up and down type thing, and it'll be going up and down like this, so that's why it's like, positive and then negative and then negative and then positive again, but it is all going to the one point. Same thing for this one, these are just extra little bumps. I don't think these three do, (inaudible).
>
> INTERVIEWER: Ok, and you know what I'm going to say now.
>
> AMANDA: Yes, explain, so this one right here, the lines are so close together, that it's just going all nutso right here, and I just don't think you could, like I think the limit would be too hard to determine.
>
> INTERVIEWER: So, it's 'going nutso'?
>
> AMANDA: Yes, a technical term, yes… Ok, a similar thing here, I can't even, like it's… Well, I guess, I don't know it's just really hard to interpret, but… It has like a little random circle here, and so maybe there's like a little extra spike, and so it just looks like it's messed up here. So, therefore…
>
> INTERVIEWER: Ok, what about this one in the bottom corner?
>
> AMANDA: This one, actually, might have a limit, and… ok, so, it's like, kind of like an ellipsoid here and kind of like an ellipsoid here (hands

> show a saddle shape) and so, it is like meeting in the middle… Yeah, I think this one does.

As Amanda described each of the contour graphs she regularly referred back the shape of the three dimensional surface with phrases such as “it goes up and down,” “there’s like an extra little spike” and “it’s like, kind of like an ellipsoid.”

CHAPTER VI

DISCUSSION AND CONCLUSION

David Tall (2004) described a theory of "three worlds of mathematics." This effort stemmed from Watson, Spirou, and Tall's (2003) work on the development of different approaches to the concept of vector. They described a geometric approach using arrows representing magnitude and direction, a symbolic approach using traditional vector notation, and a formal approach as in the development of vector space mathematics. According to Tall, "we realized that there were not only three distinct types of mathematical concept (geometric, symbolic, and axiomatic), there were actually three very different types of cognitive development which inhabited three distinct mathematical worlds" (Tall, 2004, p. 2). Tall described these three worlds as the 'embodied world,' the 'proceptural world,' and the 'formal world.' In this respect, students in this study inhabited both the 'embodied world' and the 'proceptural world' with no evidence of the 'formal world' playing a role in student conceptualizations.

The 'embodied world' can be described as internal conceptualizations resulting from our perceptions of the world around us. Van Hiele (1986) described how geometric ideas begin with an emphasis on visualizing geometric objects as whole entities, and from there students may develop increasingly sophisticated language in order to describe the properties of various geometric objects and prove statements about them. In this way, Tall argued that mathematical conceptions that take place in the 'embodied world' arrive first from our perceptions and experiences and that these conceptions grow by a means of developing ever-increasing modes of discussing and describing these ideas. The nature of calculus arises from the perceived experiences of rate of change and accumulation, and these experiences are often described in the visual representation of a graph. In the section on "visualizing multivariable limits" we will discuss how students explored this 'embodied world' of mathematics using a combination of topographical and dynamic thinking. In particular we will look closely at Amanda, a student who frequently used the 'embodied world' to reason about limits.

On the other hand, Tall's 'proceptural world' is derived from his use of the word 'procept' to describe the use of a symbol that captures the meaning of both a mathematical process and a concept defined by that process (Gray and Tall,1994) In this world a mathematical action, such as "2 divided by 5", is captured by a symbol, 2/5, which in turn represents both the process of performing division and the result of that process, the ratio two-fifths. In this way, mathematical processes are developed and then encapsulated into objects that may be used for further development of more sophisticated mathematical processes. The limit concept is naturally thought of as both a process of analyzing successive points and as the completion of that process, and for this reason the concept naturally fits within the 'proceptural world' Tall describes. It turns out that the multivariable limit concept, as experienced in this study, involved a process which is slightly different from that of the single variable limit concept. When solving multivariable limits, students analyzed the limit along successive paths. Like the single variable limit concept, this process is infinite in the sense that a student must analyze the function along all possible paths in order to show that a limit does exist. In the section on "infinite processes in multivariable limits" we will discuss in greater detail how students in this study dealt with this infinite process. In particular we will look closely at Josh, a student who frequently uses the 'proceptural world' to reason about limits.

The difference between the approaches to these two 'worlds' of mathematical thought is significant. In the 'embodied world' students use a top-down approach, starting with a perceived object in its entirety and from there students begin to analyze and understand the details of this perceived object. In the 'proceptural world' students use a bottom-up approach that begins with the most basic processes and from those processes, objects are developed and understood and used to develop higher-level processes. In this study of multivariable limits, we find ourselves in a crossroads between these two worlds. Students encounter an infinite process that must be understood in the sense of the 'proceptural world' but at the same time this infinite process describes a visual property of a graph experienced though the 'embodied world.' While most students in the study showed evidence of using both forms of reasoning, I have chosen to highlight two students, Amanda and Josh, in the next two sections. These students were chosen because they strongly favored one of the two 'worlds' described by Tall. Amanda was chosen because her use of visualization allowed her to experience multivariable limits from the perspective of the 'embodied world.' Josh was chosen because his use of mathematical procedures allowed him to experience multivariable limits from the perspective of the 'proceptural world.'

Visualizing Multivariable Limits

Tall's 'embodied world' is founded in perceptions of the world around us. The three models of limit used to frame this study are all instances of embodied experiences. The topographical model gets its foundation from the notion of 'shape,' the dynamic model springs from 'motion,' and the neighborhood model arises from a sense of 'closeness.' In this study, it was found that 'shape' played an important role in understanding the notions of 'motion' and 'closeness.' The conceptualization of visualized objects requires that the student understands the object as a whole first, based on the student's experiences in the embodied world. When I say 'object' I refer to cognitive objects based on external experiences, not only the internalization of tangible objects, but also the internalization of visuospatial imagery. Later, that student can develop an increasing ability to analyze and describe the object in question. The importance of the role of topographical thinking in order to develop dynamic and neighborhood thinking arises from the fact that 'shape' is a cognitive object resulting from the internalization of static images, while 'motion' and 'closeness' result from the internalization of relationships between objects. Students in this study without a strong topographical sense of limit tended to struggle for a context in which to understand ideas of 'motion' or 'closeness.' Whereas, students with a strong topographical foundation of the limit concept were able to use the ideas of 'motion' and 'closeness' effectively in relationship to their topographical image.

Many of the students interviewed used topographical imagery to understand multivariable limits, but one student, Amanda, had a particularly strong emphasis on using visualization. In the next section, I will describe Amanda's experience during the interviews and discuss the relationship between her visualization of the ‘shape’ of a surface and the use of ‘motion’ to understand that surface.

Amanda’s Experience with Multivariable Limits

Throughout the interviews, Amanda favored an image of limit that was founded on a sense of ‘shape’. When asked on the questionnaire to describe the concept of limit to a student who has never studied it before, Amanda wrote, “I would draw a picture and explain that a limit is a point that a curve gets closer and closer to but never reaches.” Her statement emphasized dynamic motion and the notion of limit as unreachable; however, the real structure of her thought was based in the mental “picture” of limit in her head. Without the “picture” of limit, the discussed motion has no context.

The following discussion took place when reviewing the questionnaire during the first interview session.

Excerpt 84

> AMANDA: Ok, umm, ‘A limit describes how a function moves as you approach a given point,’ strongly agree because it’s the right answer.
>
> INTERVIEWER: Right…
>
> AMANDA: Do you want me to talk about that a little?
>
> INTERVIEWER: Yeah, talk about what, what you think when you read this phrase and why you agree with that so strongly.
>
> AMANDA: I think that basically sums up what I think of a limit. It’s how a function looks as it’s approaching a point. That’s like the first thing you learn when you’re learning about limits. You look at the graph and determine whether it’s a limit or not, that sort of thing.

In the above discussion, Amanda showed that her understanding of motion is tied together with the "look" of the function. For her, to analyze the 'motion' of the function is synonymous with analyzing the graph of the function.

Later in the same interview, I pressed Amanda to explain what she meant by the words 'approaches' and 'moves,' and she verbalized this relationship between the shape of the graph and motion along that shape.

Excerpt 85

> INTERVIEWER: I'm actually curious about this one in particular, we see these words 'moves' and 'approaches' in here and I'm kind of wondering what those words mean to you when you read them. Does that make sense? How you interpret those words, 'a function moves' or 'a function approaches.'
>
> AMANDA: Ok, when I see 'a function moves' I see the shape of the graph. And, then kind of see that in my mind. And then 'approaches', I'm following the line of the graph.
>
> INTERVIEWER: So let me ask this. I've written two statements that people might, or might not… Well, maybe no one agrees with them, but maybe you agree with one of these more than the other. "When describing a function as 'approaching' or 'getting close to' a point, this idea would best be explained as: A) Evaluating a function at different numbers over time with those numbers successively getting closer to the point in question, or B) Mentally envisioning a point on a graph moving closer and closer to the limit point." is there one of those you prefer? Or you can say 'I don't like either of them.'
>
> AMANDA: Um, they both kind of work, but I think, when I think of it approaching a point, I see the graph going towards the point, but to figure it out I plug in the numbers. So, it's more like, 'A' is more practical.
>
> INTERVIEWER: Ok, so A is more practical?
>
> AMANDA: But B kind of helps you think about it, I suppose. I kind of see a little car moving down a line…

Here Amanda described what she visualized when she talked about motion with respect to a function. To her, the shape of the graph is ever-present, existing in the background, and motion is superimposed upon that shape. Her visualization was that of "following the line of the graph" perhaps even with some envisioned object, like "a little car." From this statement, we saw that Amanda's use of motion was completely dependent upon her visualization of the graph of the function.

In the above discussion she also mentioned the standard procedure for finding limits by "plug[ing] in the numbers." To her, this procedure was not a means of understanding the limit concept, but rather a means of "figuring [the limit] out." This suggested that Amanda did not view the limit concept as a process, but instead saw the process as a means to finding answers that are captured by the limit when it is envisioned topographically and dynamically.

While solving limits symbolically, Amanda applied another procedure, that of simplifying the expression before solving the limit, but she relied on the shape of the graph to justify her solution.

Excerpt 86

> AMANDA: Ok, so you can factor the bottom to... (starts writing) These two cancel, so the limit as x approaches one of one over x plus one, you plug in and you get one half.
>
> INTERVIEWER: I'm actually curious. How does doing this problem relate to the description of limit you gave over here? Of it, kind of relating to…. So, we mentioned some things like you said, 'I would draw a picture and explain a limit is a point a curve gets closer and closer to but never reaches .' I'm curious how this problem relates to this explanation that we were given earlier on this assignment.
>
> AMANDA: Ok, can I use this? [referring to graphing problem]

> INTERVIEWER: Yeah, you can use anything, that's fine.
>
> AMANDA: It's basically like this problem right here. That limit, to me. Because you've taken out, you've factored out the x minus one, the point where it's undefined, so there's a hole in the graph, and so what's remaining is it approaching.

The procedure that Amanda used did not carry the structure of her thinking, but rather it is used to "figure it out." In this case, her procedure represented a graphical image, which she described at another time as "removing the hole." The structure of her thinking is still carried by the shape of the graph, which she referenced in order to justify her answer.

On the next problem, Amanda encountered a function involving an absolute value. For this function, Amanda had no procedure to find the limit value, so she chose to create meaning for the limit using the shape of the graph.

Excerpt 87

> AMANDA: Let's see. I don't see what would prevent this problem from being any different from this one. To me, as it's approaching one, you plug in… I suppose it would be a little different… [pause] Ok, so, say here's the point it's approaching. Find this graph from this side, it would go up like that. And on the other side it would go up like that. But, when it has the absolute value, this part of the graph would flip up to be positive. So, that's the difference. But, there would still be a limit, because it's still approaching that same point, it would just look different.

When encountering an unfamiliar problem, Amanda relied on her sense of the shape of the graph to draw conclusions about the limit of the function. She drew the figure below to illustrate the relationship between the original function and the function with an absolute value in the numerator.

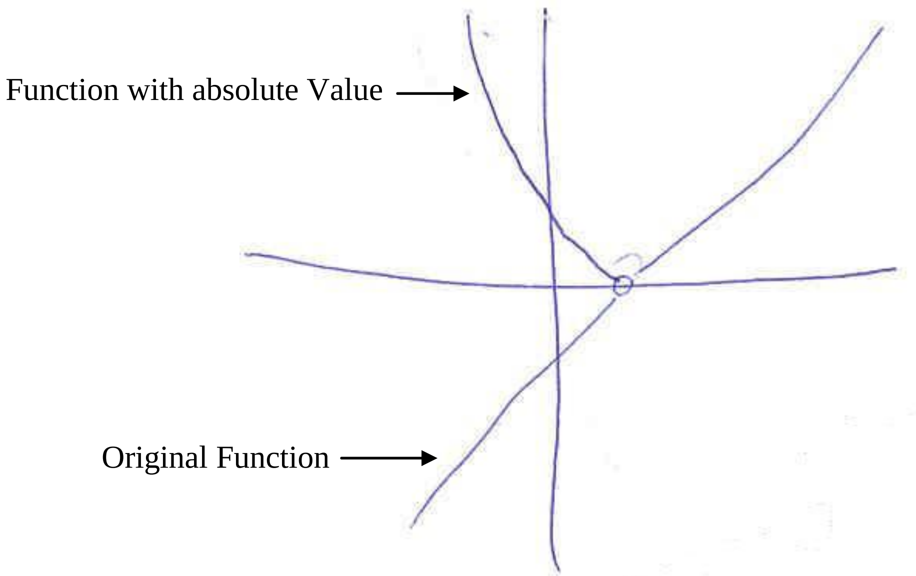


Figure 14: Figure Drawn by Amanda

In the above image, we saw that Amanda interpreted the absolute value in the function as a reflection of the negative portion of the graph across the *x*-axis. After this exchange, I encouraged Amanda to graph the function on a graphing calculator.

Excerpt 88

> INTERVIEWER: Tell you what let's do, let's, would you like to get out the calculator and see how these things… So, why don't we graph some of these in our calculator, and see if that can verify or not our answers.
>
> AMANDA: It's been a while since I've used it.
>
> INTERVIEWER: I can help you if you need to find buttons or something.
>
> AMANDA: Hmm, that's not what I would have envisioned.
>
> INTERVIEWER: Does this kind of support – so you're doing this first one here – does this kind of support what you found as the limit?
>
> AMANDA: No. Because I found it to be one half as it approached one.
>
> INTERVIEWER: So, how would you use a calculator to figure out the limit as you approach one?

> AMANDA: Well you could trace the function… This isn't making any sense to me.
> ...
> AMANDA: Um, that jumps quite a bit. So, this is not continuous, with just a hole in the graph, it's actually like, like that kind of graph [makes hand gesture]… So, that proves that I did something wrong.

When Amanda recognized that her topographical image of the function was incorrect, she became uncomfortable with her previous responses. A large portion of this conversation was omitted where Amanda used different tools on the graphing calculator, the trace and table features, to analyze the different functions. Her conclusion, however, was based on the shape of the resulting graph that it "jumps quite a bit." She then categorized this as a certain "kind of graph" which she described using hand gestures. In this statement, Amanda compared the current graph with an inventory of prototypical graphs that she used to reason about limits. Once she classified her graph within this system, her reasoning about the limit followed. Her use of hand gestures to describe what "kind of graph" was represented is an important development in creating a classification system. The use of a "tag," which according to Meissner (2006) is a symbol, word, or gesture used to represent a concept, allowed Amanda to communicate her internalized classification system with others, and in return allowed her to bring a stronger meaning to it.

In single variable calculus, Amanda possessed a strong topographical image of limit, and she is able to superimpose dynamic motion on top of that image in order to solve limit problems. She developed several procedures to solve single variable limit problems, but all the procedures were understood in the context of her topographical visualizations of limits.

When Amanda encountered multivariable limits, she visualized the surface as being composed of many "slices."

Excerpt 89

> INTERVIEWER: So, like I told you, maybe last time, we're going to be doing calc three stuff now, which is good. So have you got, have you all been studying multivariable limits?
>
> AMANDA: Yes.
>
> INTERVIEWER: Ok, so just kind of, can you tell me kind of what that means and what all goes into the word 'multivariable limit?'
>
> AMANDA: Ok, well, a curve… ok, can we start from a line?
>
> INTERVIEWER: You can say whatever you want to say…
>
> AMANDA: Ok, so a limit from a line is kind of like how the line approaches the point. So, say zero, if the line two x is how it approaches zero that just, you just have to worry about the x and y variables, same with in a curve, like a parabola or something, same thing, you just have to worry about the x and y coordinates and you can easily see just the x and y coordinates on a graph. But in multivariable, you have several different variable that are approaching the same point, and so you can't exactly see how that happens on a graph, easily, so you have to take several slices and look at those curves at the slices and put it all together and analyze it that way.
>
> INTERVIEWER: Ok, now can you tell me what you mean by a 'slice?'
>
> AMANDA: A slice, like, say I have a sphere, or yeah a sphere, and you just go like that [makes a downward cutting motion with her hand] if you cut that open, then you'll just see a circle.

From Amanda's perspective, single variable limits were the result of "seeing" the function and making a conclusion about the limit. However, she currently had no way to "see" the multivariable function in a way that allowed her to understand the multivariable limit; so, she turned to envisioning "slices" of the graph. We learn more about how these different "slices" impacted Amanda's image of the multivariable limit when she discussed how to find multivariable limits symbolically.

When Amanda analyzes $\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$ she used language that appeared to demonstrate that she had a useful conceptualization of the multivariable limit concept.

Excerpt 90

> AMANDA: I would say that it does not exist.
>
> INTERVIEWER: And why would you say that?
>
> AMANDA: Because these two are both approaching zero. And (0,0) is a common point – one reason, and they have the point negative one, or it's negative one from this side and one from this side and that doesn't, that isn't continuous.
>
> INTERVIEWER: So, that is fine…
>
> AMANDA: I don't like this that I don't know if I'm right or not.
>
> INTERVIEWER: Oh, well, I mean, you're making sense of things… So, if I were to ask you, kind of, a way that you would, if I gave you a function like this, how would you know if that limit exists or not?
>
> AMANDA: By picking points and proving that it does not exist. Because there are infinite number of points that it could, that you could test for it to be right, but there's less points that could be wrong…
>
> INTERVIEWER: So, you said 'picking points' what do you mean by this?
>
> AMANDA: Or picking slices, more so.

The above conversation seemed to indicate that Amanda viewed each "slice" as the shape of the surface along a different path towards the origin, and for the multivariable limit to exist it is necessary for the limit along each path to be equal. As she encountered two more limits, $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2 + y^4}$ and, $\lim_{(x,y)\to(0,0)} \frac{x^4 + y^4}{x^2 + y^2}$ we learned what Amanda considered necessary to argue that the limit does exist.

Excerpt 91

> AMANDA: Ok, so I took a similar approach, just because these are the easiest things to plug in. So, I found zero, the limit as the line x equals zero and the line y equals zero to zero. So, that's the same, and that would work out, but then x equals y you get something completely different, y over one plus y squared. And so…
>
> INTERVIEWER: Does that seem to tell you anything? Or what do you think about that result?
>
> AMANDA: See, again, that just seems like the limit does not exist, because it's not approaching the same point. But I can't remember if I'm missing something from the lesson – it's bugging me… I'm going to work out this one and see if I get a limit that exists.
>
> INTERVIEWER: Ok, that's a good idea.
>
> AMANDA: [works] Interesting…
>
> INTERVIEWER: So what do you notice about this problem?
>
> AMANDA: Well, they're all of similar form. So, if x and y equals each other then it seems like the limit would exist. Because they're the same variable, kind of… It just seems like I'm missing something, the key part of why it won't all fit together in my mind…

Amanda saw a contradiction in the fact that two of the "slices" were the constant zero, while the third "slice" was "something completely different," a function whose limit is zero at the origin. Meanwhile, Amanda believed that the two functions x^2 and y^2 are "of similar form" and therefore, it supported the idea that this limit might exist.

Her responses above were discussed in Chapter V in the section on misconceptions, where it was argued that Amanda viewed the computation of the value of a function along a path as a terminal computation, and possibly even a limit computation. In Amanda's case, her reliance on topographical information to evaluate single variable limits may play an important role in explaining why she stopped at this computation. Rather than emphasizing a process of analyzing each "slice" near the origin, Amanda may instead emphasize the shape of each "slice" in a topographical manner. In this way, shape of the constant function zero and the function $\frac{y}{1+y^2}$ would reasonably be interpreted as being different, even though the limit of each function is identical. At the same time, the two functions x^2 and y^2 would be considered to have the same shape.

The next portion of this interview session focused on three dimensional graphing. Initially, Amanda focused her attention on the hole in the center of the surface. In an effort to understand how Amanda connected the ideas of multivariable limit symbolically and graphically, I asked Amanda about the relationship between the value along the path and the three dimensional surface she examined.

Excerpt 92

> INTERVIEWER: So what would this number represent (referring to the path along $x = 0$ calculated in the first portion of the interview)? Or does it have any connection, I'm not sure…
>
> AMANDA: I'm sure it does… This doesn't make sense… Well, I'm assuming, some point like right here where it does change, or there's a point down here and up here. That's the only thing that I can think that's related…

When Amanda examined the three dimensional surface, she was not able translate the notion of "slices" that she developed to understand multivariable limits symbolically. Instead, she refered to the constant path as "some point."

At this point in the interview she was unable to use a dynamic process in the context of the three dimensional surface she was exploring. This is similar to an observation made by Alcock and Simpson (2004) when, referring to a previous study (Alcock, 2001) they observed that "weak non-visualizing students often appeared to lack mathematical cognitive objects, making it seem to them that the mathematics was not 'about' anything" (Alcock and Simpson, 1002, p. 10). For Amanda, this changed once she encountered the graph of a three dimensional function whose limit does exist. As discussed in Chapter V, the familiarity of this function and its relationship to a familiar function (the parabloid) likely made this a natural place for Amanda to develop a topographical understanding of multivariable limits. From this point forward, Amanda used a combination of dynamic reasoning and visual imagery to understand multivariable limits using three dimensional graphing.

Excerpt 93

> INTERVIEWER: Ok, so if I asked you, if I gave you other graphs, and I asked you to decide whether the limit existed at (0,0), or any point, I guess, how would you decide whether or not the limit existed, looking at a graph?
>
> AMANDA: Ok, I would… I guess I would have to do a similar method to this and go along with, like, kind of, go along each axis and see if they're approaching similar points to begin with. And then, I think it's a lot of just eyeing it as well.
>
> INTERVIEWER: So, when you say 'eyeing it' what are you looking for?
>
> AMANDA: Um, it's harder when they're so complicated, three dimensional objects, but I think it's just that, again they're all going towards a similar point.
>
> INTERVIEWER: And when you say 'they all' what do you mean by 'they all'? I'm just trying to harass you.
>
> AMANDA: All of the function, all of the different slices of, of different ways of looking at it.

Amanda described her understanding of multivariable limits as a combination of "approaching" the point using dynamic imagery, and "eyeing it" using topographical imagery. We saw that she reintroduced the idea of "slices" to analyze three dimensional surfaces. This suggested that she was starting to connect the process she learned to solve multivariable limits with her topographical understanding of the shape of the surface. When she solved more problems, she used language that reflects the process used when solving multivariable limits symbolically.

Excerpt 94

> INTERVIEWER: This one? Ok then we hit enter and it will graph it for us. So what I would like you to do is to look at these and tell me if the limit exists at, it ends up being (0,0) on each one.
>
> AMANDA: (long pause) I don't think so.
>
> INTERVIEWER: You don't think so? And why do you not think so?
>
> AMANDA: Because, it's hard to get an idea, but it just appears as if… Ok, so you go from this point and it goes to right about here. But, you go here and you go down and you get on at the bottom of the graph, so if you go here, then you go up here. So, these are three different directions that you're going in you're getting different points.
>
> INTERVIEWER: Ok, let's grab a different one. I don't know which one, just pick one…
>
> AMANDA: Yeah, I think it is.
>
> INTERVIEWER: On this one, so why do you say the limit exists on this one?
>
> AMANDA: Because you come down here to this little hole, and here to that hole, they all seem – even though it's bent it still goes to that one point.
>
> INTERVIEWER: Ok, why don't we choose one more. It doesn't matter which one, they're all… That's a popular one because it looks funny.

> AMANDA: I'm assuming the axis is on top.
>
> INTERVIEWER: I think so. I mean, it goes through the middle, and I'm not sure where…
>
> AMANDA: Yeah, I think it exists.
>
> INTERVIEWER: Ok, because…
>
> AMANDA: Same reasoning. Even though it dips down here at (0,0) a little, it comes up to a similar point.

When examining a multivariable function whose limit does not exist at the origin, Amanda described motion along two separate paths which lead to "different points." This closely mirrored her process of examining "slices" when solving limits symbolically. When Amanda described a multivariable limit that does exist, her language changed. She described a function that "goes to that one point," and a function that "comes up to a similar point." She still used language to describe a dynamic process, but the structure of her thinking seemed to be captured by the "similar point" on the surface. I believe this demonstrated that Amanda had developed a topographical image of the shape of a multivariable function which does not exist at the origin, yet has a limit at the origin. This image appeared to be founded on the visualization of a "similar point." It is not surprising that Amanda tended towards topographical imagery when describing a limit that does exist. The process of analyzing paths to show that that multivariable limit does exist involves an infinite process compared to the finite process of showing that a multivariable limit does not exist.

At the end of the second interview session, Amanda reflected on her experience and described the importance visualization played in her understanding of the multivariable limit concept.

Excerpt 95

> AMANDA: Having a visual is very nice for me. I tend to think of everything visually. And so, this to me didn't make sense on paper, but when I saw it on a graph it all of a sudden clicked that you didn't approach the same direction, so there wasn't a limit. And this one, although it doesn't have the same… like, it's very similar, you know, so I had a feeling there's a limit, so when I looked at the graph it made sense. Anything else?
>
> INTERVIEWER: No, that's fine, I was just wondering... So, now do these numbers, these answers make sense to you? Or do they still seem a little strange? y squared, x squared, and y squared, where they came from?
>
> AMANDA: They make sense. Like, what do you mean, where they came from?
>
> INTERVIEWER: Just what they are and how they relate to finding the limit.
>
> AMANDA: It makes sense to me, and if you look at it, like limit (x,y) goes to (0,0) as both x and y approach zero, it's going to be that same point.

Amanda's conversation above echoed the arguments being laid out in this section, that the dynamic process used when solving multivariable limits symbolically was without context for her, and when she developed an appropriate topographical image the concept "clicked" for her and she realized why the different valued paths forced the limit to not exist. Furthermore, she showed evidence of connecting the symbolic process with the three dimensional graph. Before the graphing portion of the interview, Amanda failed to take the limit of the function along each path, instead comparing the expression given by the function, itself, along each path. Above we see that by the conclusion of the interview Amanda realized the importance of letting "both *x* and *y* approach zero" which incorporates the limiting process into explaining why x^2 and y^2 have the same limit as

(x,y) approaches (0,0). In this way, it "makes sense" to her that the limit should exist, but she relied heavily on topographical information to justify this existence.

The third interview began with the use of polar coordinates to study multivariable limits. Like every other student in the study, Amanda was unable to use polar coordinates to argue affectively about multivariable limits. She initially described polar coordinates as "another way of writing things." I asked her whether the limit in polar coordinates has any bearing on the existence of the multivariable limit.

Excerpt 96

> INTERVIEWER: So, now we look at these limits as r goes to zero, my question, what I want to know, it can – by looking at these limits as r goes to zero – can we understand what would happen to the limits of these guys as (x,y) goes to (0,0)? Are you able to figure those out?
>
> AMANDA: Yeah, I think you could? But it would just be, again polar coordinates is just writing it in a different way, so you would have to convert that back to comprehend the limit of the xy, I feel, because these don't, this isn't exactly, like, you can't say, 'oh, this translates exactly from the simplified version to this version.'
>
> INTERVIEWER: So, by doing this limits, you're not really sure what these limits might be in terms of x and y?
>
> AMANDA: Well, I suppose, ok, I suppose if you find the limit of r, then you could relate it back, using one of these things? But, again, you'd have to interpret it…
>
> INTERVIEWER: So, without any computation…
>
> AMANDA: No, I don't think so.

Amanda's conception of polar coordinates was limited to that of a symbolic conversion. Knowing Amanda's preference for visual imagery, it is likely that she lacked a strong visual representation of polar coordinates. I argue that this weak visualization of polar coordinates plays an important role in explaining Amanda's struggles in using polar coordinates to reason about multivariable limits.

When the interviewer asked Amanda to evaluate multivariable limits using contour graphs, she again placed a strong emphasis on visualizing the multivariable function. As mentioned in Chapter V in the section on misconceptions, Amanda initially focused her attention on the symmetry of the various contour graphs, but later shifted to use dynamic imagery on top of her topographical image of the function.

Excerpt 97

> INTERVIEWER: Ok, so by looking at these graphs, can you tell me whether the limit exists at (0,0)?
>
> AMANDA: Interesting… Yes, I think so.
>
> INTERVIEWER: Ok, well can you tell me why you think that?
>
> AMANDA: It does, it does, it doesn't.
>
> INTERVIEWER: OK, now…
>
> AMANDA: Ok, I'll slow down.
>
> INTERVIEWER: No that's great, now tell me why you believe those things…
>
> AMANDA: So, these two are very symmetrical on the contour graph – well, they look even from all different sides, like they're all, these are all the same all around, like the line spacing and whatnot. And this isn't so much.
>
> INTERVIEWER: So, I'm not real sure exactly what you mean by 'symmetrical' and by 'it looks even' and all that, can you be more detailed about that?
>
> AMANDA: You make me think so much.
>
> INTERVIEWER: Sorry.

AMANDA: That's ok, I signed up for it. Here in the first and the second illustration, the lines are getting, the spacing is even, or the same, throughout the four quadrants. And so, reason would say that, therefore, it is increasing or decreasing at the same point and going towards this one thing.

INTERVIEWER: And what about this one? You seem to think that this one does not work?

AMANDA: Yeah.

INTERVIEWER: And can you tell me why this one doesn't work?

AMANDA: Because – also I think the color has to do with the direction.

INTERVIEWER: And, that's true actually, I think way it's interpreted, yellow is big numbers and red is low numbers.

AMANDA: That makes sense. And so, right here, it's all over the place. It's red in the second and fourth, and yellow in the first and third. And, if I remember some of the pictures correctly that we've looked at before those graphs didn't have limits because they were oscillating, like they never really all came together at one point.

Amanda's final statement showed that she developed a means of classifying functions topographically based on the shape of their surface. She used the visual image of a function as "oscillating" and "never [coming] together at one point" as a primary image to describe multivariable limits which do not exist. Later in the interview, when asked how she decided whether or not a limit exists she replied, "I would look at the line spacing and try, with the contour lines, envision what the actual graph would look like…"

Initially, Amanda's description focused on the topographical information of the contour graph with an emphasis on the symmetry of the contour lines; however, her final statement in the above excerpt used dynamic language to describe a function which "never really all [came] together at one point." It is possible that this was affected by the fact that the first two contour graphs she encountered were of functions whose limits existed at the origin while the third function's limit did not exist at the origin. When Amanda analyzed multivariable limits using three dimensional graphing, she had a tendency to describe non existing limits using language depicting a dynamic process; whereas, with limits that did exist, she tended to rely heavily on a topographical "feel" of the function. This is natural, because of the infinite nature of the process involved in showing that multivariable limits exist. Since Amanda placed a strong emphasis on visualizing the function at hand, it is possible that the visualization of a function which did exist prompted Amanda to use dynamic imagery to understand the function as in the case of a three dimensional surface.

As Amanda continued to find multivariable limits using contour graphs, she used a combination of topographical and dynamic imagery to describe her thoughts.

Excerpt 98

> INTERVIEWER: Well, let's go through these and see what you think about them – see which ones you think exist or which ones you think don't exist.
>
> AMANDA: Ok (long pause) so this one is very similar to this, so I would say that it does exist for the same reasoning. This one does exist, because it's like a little circle and goes, up and down type thing, and it'll be going up and down like this, so that's why it's like, positive and then negative and then negative and then positive again, but it is all going to the one point. Same thing for this one, these are just extra little bumps. I don't think these three do, (inaudible).
>
> AMANDA: Ok, and you know what I'm going to say now.
>
> AMANDA: Yes, explain, so this one right here, the lines are so close together, that it's just going all nutso right here, and I just don't think you could, like I think the limit would be too hard to determine.
>
> INTERVIEWER: So, it's 'going nutso'?
>
> AMANDA: Yes, a technical term, yes… Ok, a similar thing here, I can't even, like it's… Well, I guess, I don't know it's just really hard to

> interpret, but… It has like a little random circle here, and so maybe there's like a little extra spike, and so it just looks like it's messed up here. So, therefore…
>
> INTERVIEWER: Ok, what about this one in the bottom corner?
>
> AMANDA: This one, actually, might have a limit, and… ok, so, it's like, kind of like an ellipsoid here and kind of like an ellipsoid here (hands show a saddle shape) and so, it is like meeting in the middle… Yeah, I think this one does.

In summary, I believe Amanda's use of dynamic thinking was often not a 'process' as the word is used by Tall to describe his 'proceptural world.' Instead, I believe that Amanda employed what others have named the "fictive motion metaphor" (Talmy, 1988; Lakoff and Nunez, 2000). This metaphor views a curve as a path of motion, allowing the individual to conceptualize a static object in a dynamic manner. Fictive motion concerns two important visualizations, as described by Ferrara (2003), "a *trajector* (a dynamic entity) and a *landscape* (a static entity, in which the trajector moves" (p. 3). For Amanda, I believe the instances of topographical thinking illustrate her development of a landscape for multivariable limits, and her uses of dynamic thinking are the application of a trajector on that landscape.

I make several important observations from Amanda's experiences with multivariable limits. First, developing a multivariable landscape in which to understand multivariable limits is a nontrivial task. For Amanda, this development did not occur automatically, but rather grew from her experiences exploring the surfaces of multivariable functions using a computerized graphing program. Secondly, without a well developed landscape, there is no context for using a trajector to understand limits. Amanda's preference for visualization left her without a context to generalize her notion of limits from introductory calculus until she adequately established a visual landscape on which she could apply dynamic motion. Finally, Amanda was able to argue successfully about multivariable limits in the two visual portions of the interview, three dimensional graphing and contour graphing. In doing so, it is likely that Amanda developed a context for her symbolic argument that a multivariable limit should not exist. However, there is little reason to believe that Amanda would be able to interpret her visual arguments that a multivariable limit exists into symbolic language, she would likely need to reconstruct some portion of her understanding of multivariable limits before she could argue symbolically that a multivariable limit should exist.

Infinite Processes in Multivariable Limits

Many authors (for example, Davis, 1983; Dubinsky, 1991; Sfard, 1991; Grey and Tall, 1994) have closely studied the relationship between processes and objects in mathematics. These studies all emphasize the origination of a concept as a mathematical action or process. The initial focus is on the computation involved and the steps needed to complete the computation. These studies differ slightly in describing how the transition from process to object takes place in an individual, but they all agree that these processes can be internalized and understood as mathematical objects which can in turn be used to create higher level mathematical processes.

Often the processes being considered are of a finite number of steps, for example, the division algorithm. Initially, division is understood by computing a finite number of steps that result in the output from the process; however, once encapsulated the notion of division can be understood as an object, perhaps in the sense of ratio. The concept of limit, on the other hand, is an infinite process, and this carries with it several conceptual difficulties. As opposed to finite step processes, the conclusion of a limit calculation cannot be determined by carrying out the process to its completion. This makes it possible to discuss both the process of calculating a limit and the conclusion of the limit process without actually possessing a procedure to compute the limit value.

As noted in Chapter V on introductory calculus students appeared to use a metaphor of "close enough" to understand the conclusion of the infinite process. Students used this imagery in connection with the procedure of successively selecting points closer to the point in question. At some point in this procedure, the student considers that he/she has evaluated points "close enough" to the limit point and he/she then draws a conclusion about the result of this procedure. Jessica described this belief while discussing the questionnaire results.

Excerpt 99

> JESSICA: there's some, like, limits that you can't actually plug in the point
>
> INTERVIEWER: Correct.
>
> JESSICA: But you can do it like really close, like 1.999 and get a really close answer. So if you put in one and then like 1.5 and then like .8 and then like 1.9, you'll get closer to the value that you're supposed to get as the limit…

It is possible that Jessica viewed the point 1.999 as "close enough" and once she reached this point she could end the infinite process and make a conclusion about the limit. It is not clear how a student like Jessica determined when she became "close enough" to stop the process or how she determined the result of the infinite process once she stops computing. However, it did appear that Jessica, and other students in the study, believed that there existed a finite number of steps that will give them adequate information to draw conclusions about the infinite limit process.

The classroom experiences of the interviewees were not a formal part of this study. However, as noted in Chapter V in symbolic manipulation of multivariable functions, initial discussions about multivariable limits indicated that several of the students came to the interview with a notion of multivariable limits which relied on using an infinite number of paths through the point in question. This conceptualization of multivariable limits is also the means of introducing the concept in the textbook used for the course (Stewart, 2003). In this context, the multivariable limit concept is an infinite process with each step of the process using a limit calculation along a different path. These paths may be understood visually as curves drawn on a three dimensional surface, or these paths may be understood symbolically as algebraically expressed functions.

Students in this study were successful to use this technique symbolically to show that a multivariable limit does not exist; however, no student made significant progress towards understanding symbolically why a multivariable limit should exist. Since showing that a multivariable limit does not exist is a process that stops in a finite number of steps, it is understandable why students would have success with this process while struggling with the infinite process of deciding that a limit exists. In Chapter V during the discussion on misconceptions, several students were observed concluding that a limit exists after performing only a finite number of steps in the infinite process.

Excerpt 100

> INTERVIEWER: Right, so what about this limit, have you decided what this limit is or do we need to do more work?

> JESSICA: I decided that the limit's zero, because they're all zero. I mean, that's, it's… It might not be right, because I didn't do it from every direction, but I guess these are the best three to do to generally figure it out, so I think it's zero. And then the same thing for the second one [works silently] it's zero as well.
>
> INTERVIEWER: Ok, so you think this one is zero also? Let me ask you, so you're taking these lines… You said earlier that you weren't one hundred percent sure it was zero, but you thought it was?
>
> JESSICA: Well, yeah, because you're only taking three lines, you're not taking infinity lines, because there's infinity directions that you can come from to get to (0,0), if it's a surface.

Jessica was aware of the infinite nature of the process she was exploring, but at the same time she was satisfied with drawing a conclusion about the result of this process after calculating the limit along three separate paths. It is possible that this belief was an analogue to her use of a finite number of points to complete the single variable limit process. In a similar way, Jessica seemed to believe that a finite number of paths provided her with enough information to understand the multivariable limit.

During the interviews covering introductory calculus, two methods of dealing with the infinite nature of limit were observed. As just noted, students developed a sense of "close enough" to halt the infinite process after a finite number of steps and draw a conclusion about the value of the limit. The other method was to visually describe the 'movement' of the function in a fluid way that allows for a single action of moving to a point. In this way, the existence of a single variable limit can be evaluated by using only two actions, movement from the right and movement from the left. However, in multivariable calculus, this action remains infinite, since the movement must take place along every possible path.

A primary difference between the infinite nature of the single variable limit and the multivariable limit is that the single variable limit deals with an infinite collection of points which can be collected into a single object, a line, which can be analyzed in a single action using motion. However, the infinite process involved in multivariable limits involves an infinite collection of paths. It is unclear how this infinite collection can be understood as a single object with a single action to understand multivariable limits. I believe it is the infinite nature of the multivariable limit concept that causes students to struggle with the concept. Three students, Jessica, Josh and Chris, all concluded their interviews with a transition from dynamic thinking to neighborhood thinking to describe the multivariable limit. I believe that this transition may play a key role in understanding the infinite nature of the multivariable limit. One of these students in particular, Josh, appeared to experience some success in capturing the infinite process as a single, understandable conceptualization. In the next section, I will describe Josh's experience with multivariable limits. During this description, I will discuss how he brought coherence to the multivariable limit process and what beliefs played an important role in allowing this development to occur.

Josh's Experience with Multivariable Limits

In the first interview session on introductory calculus, Josh demonstrated a strong, connected understanding of the limit concept. He initially described the limit as "the behavior of the function." It is not clear what Josh meant by his use of the word "behavior" but through the course of the interview, he regularly used dynamic imagery to describe the limit process. I asked him about his use of dynamic language near the beginning of the session.

Excerpt 101

INTERVIEWER: In particular – now, you used the word ‘behavior’ – uh, the sentence used the word ‘moves’ or ‘approaches’ and I’m kind of curious what those words mean to you. When you read that, how do you interpret those words?

JOSH: ‘Moves’ I kind of put as somewhat synonymous with ‘behavior’, because in math behavior kind of is like how the function moves, it’s not an emotion or anything, so those are kind of synonymous to me after looking through the rest that was the closest to what I had associated in my mind – I don’t know – I guess because of my teacher had drilled into my head it was the behavior of the function, that was kind of what I was looking for was a word that was similar to that and moves and approach were kind of the synonyms to that.

INTERVIEWER: now let me ask you have a question, I actually... So, I have two different – so over my time I’ve discovered that people think two different things – actually they think a lot of things, but these are two possible things. So, I’ll just read you this question: “When describing a function as ‘approaching’ or ‘getting close to’ a point, this idea would best be explained as: A) Evaluating a function at different numbers over time with those numbers successively getting closer to the point in question, or B) Mentally envisioning a point on a graph moving closer and closer to the limit point.” I’m curious which one of these you maybe agree with – or neither of them.

JOSH: If I had to pick one, I would probably say A – because I’ve never heard of envisioning a point getting closer – I don’t know I’ll have to think about that (pause) it’s almost both, I don’t know

INTERVIEWER: That’s fine.

JOSH: It’s hard to…I guess it would be a point – no it wouldn’t be a point because it’s not one single point it’s evaluating different points. It’s kind of like the bouncing ball thing. You start off on either side and it gets closer and closer and closer. So it would be envisioning points that are getting closer to the point in question.

INTERVIEWER: So you seem…

JOSH: it’s more A.

In the excerpt above, Josh pointed out that, to him, the word 'behavior' was almost "synonymous" with the word 'moves.' We saw with Amanda that dynamic language carried with it visualized motion on top of a static object. To Josh, however, the graph did not represent anything in motion; rather, it represented a collection of points. To Josh, motion was not captured by the visual imagery of a single object in motion, but by "evaluating different points." In this way, for Josh the limit concept was conceptualized as a process, and the visual representation is "envisioning points" on a graph, where these points represented the process of "evaluating different points."

Although the process of "evaluating different points" was a strong conceptualization for Josh when working single variable limit problems, he also showed that he was comfortable using other techniques to find limits as well, including visualization and algebraic simplification. For Josh, the single variable limit concept could be understood from several different vantage points and the appropriateness of each vantage point depends on the context of the problem.

When Josh began the second interview session, he had developed only one significant view of multivariable limits.

Excerpt 102

> INTERVIEWER: So why don't we just start by telling me what you've studied about them [multivariable limits], what you've learned about them, things like that.
>
> JOSH: Ok, basically, the same kind of general concept that you're looking, instead of just one function that you're looking at the limit, you're looking at this surface that, unfortunately, is a little more difficult. When we were doing one variable limit what we were doing was, 'do we approach from the left' or 'do we approach from the right' and we compared those two, whereas now, we can approach from an infinite amount of directions, there's different paths and approaches that we can

> take. We can take left, right, parabola, we can circle around it, there's an infinite amount, so it's a little more difficult to prove the limit does exist; whereas it's a little bit easier to say it doesn't exist, so if you can take a limit from one path and compare it to another path and get two different numbers, you can easily say it doesn't exist, whereas to prove it does exist is a little more difficult…

Josh had developed an understanding of multivariable limits using a process of analyzing the function along many different paths. He entered the interview with this process well developed, and he recognized the inherent difficulty in showing that a multivariable limit does exist as opposed to showing that it does not exist.

The first multivariable limit Josh encountered had different values as he took the limit along the x-axis and the y-axis. Josh efficiently used his process of analyzing multivariable limits to show that this multivariable limit does not exist. From observing Josh analyze this limit, I have summarized his process as follows:

1. Select a path that passes through the origin
2. Algebraically substitute that path into the multivariable function to create a function of a single variable.
3. Evaluate the limit of the resulting single variable function as (x,y) approaches (a,b). This will be a single variable limit in either x or y.
4. Repeat this process selecting different paths until two paths are found with different single variable limits.

Step four of this procedure created two issues for Josh. First, it is not clear how to go about selecting the next path, especially once the 'obvious' paths (the x-axis, the y-axis, and the line $y = x$) have been chosen. Second, there is no clear way to stop the procedure if the limit, in fact, does exist. The following conversation took place when Josh encountered a multivariable function that required a parabolic path to show that the limit does not exist.

Excerpt 103

> INTERVIEWER: So how about these limits? Does this type of method work on these limits? I think we can just try them – they're just other limits without the prompting.
>
> JOSH: Yes, we kind of, in class my professor kind of set up, like 'here's a strategy to kind of attack the problem' was to go ahead and approach it from, exactly like these other problems did, to approach it from x equals zero, y equals zero and then set them equal to each other and see what happens, and then from there if we can't figure out if it does or doesn't exist then you kind of resort to other methods. But for now I will just go ahead and say, x equals zero as (x,y) approaches (0,0), and then do the same thing, and get zero over y to the fourth is zero, and then I'm going to go back to that same thing setting y equal to zero, and end up with zero over x squared which is zero again. And so far it looks as if the limit exists, because I'm getting the same answer, but because there's an infinite amount of approaches, you can't say from this, 'oh I can tell the limit exists' you have to, you have to do all that extra stuff that's not very fun… And then, this is just kind of the 'ok I'll do the first three things like this' it would be x to the third over… and then from here I'm just going to go ahead and take out x squared, and this reduces to one, so this is what I mean by I do not mathematically correct things, because I haven't taken the limit yet, so… And so from here I've reduced it to one plus x squared and I'm going to go ahead and let it approach zero, and I will plug in zero and get one. So from those first two it seemed maybe the limit does exist, but when I do a different approach I get a different answer, therefore the limit doesn't exist.
>
> INTERVIEWER: Now let me warn you about something. My warning is, just because I think we should… is you have x cubed here and took out the x squared. I think you would have seen that.
>
> JOSH: Oh sorry. Yeah, so it would be, once again I end up with the same thing of zero. So it's zero over that, there's my mathematical error for that problem, so I was wrong and once again end up with the same numbers.

> INTERVIEWER: So what do you think? Where do we go from here on this problem? Have you decided whether you think the limit exists or doesn't exist? And what you do from there?
>
> JOSH: From here what I kind of do is if I end up with the same number for different approaches, I try to basically look at where I can disapprove this, more or less. What I can, If I can play with these numbers and kind of look, well if I can set x or y equal to something or I will be able to get a different number from zero. So, I look at this, if let x equal y squared, I might end up with y to the fourth and the same case here, if x were equal to y squared I would end up with y to the fourth. That would allow me to, kind of, get a different answer. It's like, I look at it and kind of play with it in my head, to 'maybe this would work' and kind of go from there. So from there I would say, I think if I set x equal to y squared, then I might end up with a different answer. The sometimes, if I can't figure it out from just looking at it, then I might just try different approaches. I might do y equals x squared, x equals y squared, try different parabolas. So, from there, I end up with y to the fourth over y to the fourth plus y to the fourth, is one over, oops… Is y to the fourth over two y to the fourth and from there I can say that it is one half. Now I have a different number and I can say that the limit doesn't exist.

Here we see Josh struggled with which path to select next in the process. This struggle separated the multivariable limit process from the single variable limit process. In the single variable limit there is an ordering of the points involved, based on their distance from the limit point so it is possible to systematically choose points that get closer to the limit point. However, with multivariable limits, there is no ordering of the paths that pass through a given point, and so there is no clear means by which to systematically choose paths. Josh was aware that any finite number of paths will be insufficient to show that the limit exists; so, he began to "basically look at where [he] can disprove this." I believe that Josh was mentally experimenting with familiar functions to see if any of them show the potential to disprove the limit. In his words he was "playing with it in [his] head," and if he "can't figure it out from just looking at it, then [he] might just try different approaches." On this problem, he succeeded at finding the function $x = y^2$ and convinced himself that the limit does not exist.

The next problem Josh encountered is a function in which the multivariable limit does exist.

Excerpt 104

> JOSH: This is going to be a… I'll have to think about awhile maybe… I guess I should kind of go through the processes that I do here.
>
> …
>
> JOSH: But, once I go in and do the limit as y equals zero I can do the same things, but for x I'll end up with zero again. And I kind of found myself in the same boat again that I appear to be getting the same number. But, I appear to be getting the same number which I have to go through the same steps again to say, 'well is there something I can plug in here that I'll get a number different than zero?' So, I'll just go ahead and do it. Because if I look and it and it doesn't just click on automatically in my head then I'm just going to keep going with, basically, random things, I'm just going to keep throwing darts at the board and see if I hit anything, so… I'll go ahead and do x equals y and see what happens there. Oops, I keep flip-flopping variables in my head. And once again I'm going to end up with the same thing of eliminating commons factors and I'm going to end up with zero again. Now this is where I'm kind of unsure what to do, because I'm trying to think of something in my head that, 'what can I put in there that will give me something different than zero?' And if I allow, like, x to be y squared or y to be x squared, I'm going to end up with different powers and it will basically be the same problem all over again… Maybe, no it won't. If I let x equal y squared – I'm sorry if I don't make any sense, because I just kind of have to go through an arsenal of tests to see what happens, so… It will be y to the eighth over y to the fourth and from her I can go ahead and factor out some numbers – y squared – will give me… And then if I plug in zero I'm going to end up with the same thing again. This becomes one and then, I would say from this, because I'm taking a lot of different things and trying them and I'm ending up with the same answer, I'm going to say that this limit, I need to show that this limit exists. But this is where I kind of fall off.

Again, Josh seemed to be mentally computing the results of substituting different known functions into the multivariable function. Lacking a path that "clicks" for him, Josh proceeds to select functions in a manner that he describes as "throwing darts at the board." I believe that this process is not random, as Josh describes, but rather I believe that Josh physically substituted familiar functions into the multivariable limit in the case that mentally computing the substitutions becomes overly burdensome.

At the end of this excerpt, Josh appeared to believe that he will not find a path which will show that this limit exists. He would like to apply another way of thinking about the multivariable limit, but his experiences left him stuck. He had been taught the epsilon-delta definition of multivariable limits, but as the following conversation showed, this procedure had little meaning for him.

Excerpt 105

> JOSH: We didn't do any of that [multivariable limits which exist] in the homework and I looked at it, and it's like looking at the examples, what it does is like sigma, sigma-delta.
>
> INTERVIEWER: Epsilon is the one, epsilon-delta.
>
> JOSH: Sorry, epsilon-delta argument. It makes sense doing the examples, but actually going in a doing it, I get completely lost.
>
> INTERVIEWER: So, it's not real clear to you how to show that this exists. You believe it exists, but it's not real clear to you…
>
> JOSH: I think I could show that it exists, I don't know for a fact that it does – I wouldn't say confidently that it does exist – but I know that there's…
>
> INTERVIEWER: You suspect that is does?
>
> JOSH: Yes, there's the word that I was looking for. But the epsilon-delta argument, it doesn't make a lot of sense in my mind.
>
> INTERVIEWER: So is that kind of, so I was going to ask, how do you go about showing one these exists, if it does exist?
>
> JOSH: You use the epsilon-delta argument…
>
> INTERVIEWER: So, what's the real idea of the epsilon-delta argument? Does that kind of make any sense to you or are you just following…

> JOSH: Yeah, it's most of like, here's this formula that shows a limit of several variables will exist, and it's kind of, it doesn't make a lot of sense to me, it's just kind of playing with the formula and putting in arbitrary thing and this, viola. It's just kind really unclear and muddy to me. It's like, if I see an example done then I'm like, 'that makes sense,' but if I'm presented with like, 'do this,' like in this case, then I'm kind of lost.

Unlike single variable limits, where Josh was able to employ a variety of different procedures to understand limit problems, at this point in the interviews he had only shown the ability to use the process of analyzing the functions along different paths to make sense of multivariable limit problems. For Josh, this procedure made it difficult to analyze a limit requiring an unusual path to show that the limit does to exist, and it makes it impossible to analyze a limit that does exist.

When exposed to three dimensional graphs, Josh struggled to make sense of the three dimensional image he saw, but he was able to apply his process of analyzing multivariable limits to the new, visual situation in order to draw conclusions about the limit.

Excerpt 106

> INTERVIEWER: So, I was wondering if you could look at these and based on looking at these graphs tell me whether you believe the limit exists or doesn't exist by looking at the graph.
>
> JOSH: Ok, I'm very new to the – as far as single variable limits go, we did a lot of visual stuff – but I'm kind of new to the whole looking at three dimensional functions and determining visually. But, I kind of take that same approach of 'I'm going to look at it from the left and from the right, from different ways and see if I approach the same point.' This is – I'm terrible at doing…

Notice that Josh was able to apply his dynamic process to the new visual environment without first requiring a strong mental image of the shape of the surface. Josh's struggle was to understand the visual image using his mathematical process, and in the next excerpt he will struggle with how to 'see' the paths on the surface. This is in stark contrast to Amanda, who needed a strong visual understanding of multivariable limits before she could apply dynamic imagery onto that visualized object.

Excerpt 107

> JOSH: So if, sometimes what I have to do… I'm not very good at, like, portraying things in three dimensions in my head, so a lot of time what I'll do is exactly that. Like, kind of, 'OK, I'll take a slice or something' and look at it from a perspective that puts it in a more concise two dimensional way of looking at it and kind of try to analyze it that way. So, I do the same thing, that, well if I do this it looks like a pretty surface or something. So arrange it so I can look at it a more organized way that I can kind analyze it a little bit better, so I'll do the same thing – I'll look it from different paths and see what I get. But, if I just plug in (0,0) it obviously undefined there, so I'm going to get, if it's undefined, the limit is going to approach different things. Well, sorry, I didn't explain that well. If it's undefined, I can't just plug in the point that I'm looking for, so I have to look at it from different directions, so then, from there, I'll try to look at it from a different perspective. If I approach from this direction, I'll get a point, if I approach from the left I'll get a point, and if I look from the other side, they appear to be different points. I don't know, I'm not very good at doing three dimensional visual analysis.
>
> INTERVIEWER: And, I know this is probably the first time you've been asked this. But, that's kind of why I'm asking it. I'm interested in how you make sense of this figure, and how do you make sense of the question, looking at the surface, of 'does the limit exist?' And you're saying some good things, I appreciate you talking.
>
> JOSH: It kind of, I can't just look at it as a whole, I have to go through that whole arsenal of tests, well look at it from this perspective, well it's undefined here, ok, well if it's undefined there then I need to look at it from a different perspective, where now if I approach from different sides I appear to be getting close to this area, but if I look at it from a different yet again, that there's two different points depending on which approach I take. So I would say that the limit is undefined because as I approach, if I'm looking at this perspective after going through those other tests, that I will… If I look at this perspective then I get two separate points, so the limit doesn't exist.

INTERVIEWER: So you say 'two separate points' now tell me what exactly you mean when you say 'I get two separate points?'

JOSH: I mean, if I take one path then I'm going to approach, and ok after taking the limit from this path the function approaches this point, or now I'll take a different approach and approach from the left side and get a different point. So, it's kind of that whole 'well if I go with this step and get one numerical value and then I take a different path and get a different numerical value then I can say the limit does not exist.

Josh interpreted his idea of "different approaches" from analyzing multivariable limits symbolically as viewing the three dimensional graph from "different perspectives." When he said this, he rotated the graph about the *z*-axis, creating "perspectives" such as the images below.

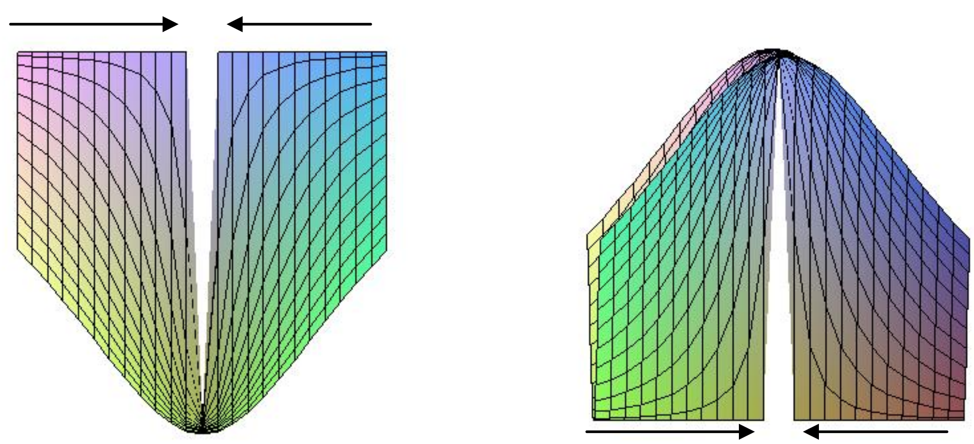


Figure 15: Josh's Two "Perspectives"

At this point, Josh was still analyzing the surface one path at a time. This changed when he encountered a function whose limit exists.

Excerpt 108

JOSH: I'll just kind of go through what I'm thinking. I can tell once again just by looking at the function if I plug in (0,0) I'm going to be undefined, and if I look at the graph I've got the hole in the middle, so it's

> undefined there, but if I kind of… I kind of have to look at it this way, it makes more sense in my mind, that it's kind of, this kind of parabolic looking thing that if I take one approach it appears that I am coming to this point here, if I take the left approach it's the same thing. I'm kind of approaching this same value here in the middle. If I look at it all around, it appears to be coming to this single point, so it almost looks as if it's, they're all kind of converging in the same area, so I would say from that that I think the limit does exist at whatever this value is at the very bottom.

Josh began this description by analyzing the surface one "approach" at a time. He soon noticed that every time he is "approaching this same value here in the middle." At this point, Josh appeared to take a step back and view all the approaches simultaneously. He said, "If I look around, it appears to be coming to this single point […] they're all kind of converging in the same area." Instead of speaking about approaching the origin one path at a time, Josh was now speaking of all paths simultaneously with a phrase like "they're all" referring to all the possible paths to the origin.

This marked an important shift for Josh, since it was the first time he attempted to conceptualize the entire multivariable limit process as a single action. In his attempt to describe all paths to the original simultaneously, Josh looked "all around" the function and observed that that every path goes through "the same area." These statements reflected the neighborhood model of understanding limits. This will become more pronounced when Josh used contour graphs to understand limits. In the next excerpt, Josh encountered another function whose limit exists at the origin, but in this case he applied his multivariable limit process one "approach" at a time and never convinced himself that the limit exists. In this conversation we see that Josh was beginning to develop a sense for what it means to have all paths approach the same value, but he still struggled to evoke this image for every relevant problem.

Excerpt 109

> INTERVIEWER: Do you want to look at any more, or should we..? Let's look at, I don't know, I always thought this was a fun looking one right here.
>
> JOSH: Yeah, all the trig functions always have fun looking graphs.
>
> INTERVIEWER: It's kind of crazy isn't it?
>
> JOSH: Yeah, it's like a mold of some sort. Ok, this is kind of the same, look at it from one perspective and then kind of try to look at it from another. But, with these it's just, it's kind of if I take one approach, it doesn't matter which approach I take it appears that no matter what approach I take it comes to this, this central point here in the middle, that the limit would approach this value, this point, here at the center of that cone shaped thing. So, visually it looks like the limit exists, but I wouldn't say confidently either way, so I'm sure…

In this excerpt Josh was envisioning the limit value along many different "approaches," but because he was analyzing these approaches one at a time he is unable to "say confidently either way" whether the limit exists or not.

During the three dimensional graphing portion of the interviews, Josh was very comfortable using a process of analyzing different paths to explain why a multivariable limit did not exist, yet he found it much more difficult to use this same process to explain why a multivariable limit would exist. Although there was some evidence of Josh condensing the infinite process into a single, understandable whole, the accompanying imagery and language were only applied on occasion and used for every limit which existed. At the beginning of the third interview session, Josh proclaimed that he was "not good at" using three dimensional graphs to understand limits. I believe this was because Josh's visual representation of multivariable limits remained rather weak even though he was capable of applying his process of analyzing different paths to discover the correct answer to most of the problems.

Like the other students in the study, Josh had little success making sense of multivariable limits using polar coordinates. Before examining the problems, I asked Josh to describe his prior experiences with polar coordinates.

Excerpt 110

> INTERVIEWER: So why don't we spend a second, and you can look in here, and you can tell me what you remember about polar coordinates in general, if you remember anything, if not you can read a little bit…
>
> JOSH: I remember that they were a pain in the butt, because you learn this thing all year long, but then all of a sudden it's like, 'now you can do the same thing but in a completely different way.' I just remember it's very confusing at first, instead of thinking of something and, 'oh we'll start with this given value and go consecutively down the values at one two and three,' it's was, 'now we're going to look at it and going around in a circle,' it was difficult to put a grasp on mentally. It was another one of those visual things that you have a problem with, or that I had a problem with.

Like other students in the study, Josh did not describe a relationship between polar coordinates and distance from the origin instead using a description based on circles. Through the interview, Josh showed that he is comfortable converting to polar coordinates, but it is likely that this conversion represents a symbolic process. As Josh himself said, he "had a problem with" the visual aspects of polar coordinates.

After converting the multivariable function to polar coordinates and finding a limit of zero as r approaches zero, Josh decided to find the limit of the multivariable function using the multivariable limit process from the first interview session. In this process, Josh observed that along every path he explored, he gets an answer of zero. I asked him about drawing a conclusion about the multivariable limit.

Excerpt 111

> INTERVIEWER: So if you were forced to answer, you would say that the limit is…
>
> JOSH: If I was forced to answer I would write down this entire page writing down different functions that I could think of. But for now, I guess I would say that it's possible that it does exist.

Josh remained unconvinced about the multivariable limit, but conceded that "it's possible that it does exist." The limit using polar coordinates did not help convince him that the limit should exist. The following conversation took place when Josh was asked about the relationship between limits using polar coordinates and multivariable limit as experienced earlier in the interview.

Excerpt 112

> JOSH: It's kind of, it's just more like a different approach to looking at the same thing. Which I kind of said it earlier, with polars, I was unsure of these, if the limit was actually equal to zero. I kept getting zero, but there might be this one path which is a counterexample and the limit doesn't exist. So it kind of, you're looking at the same problem under a different light, to maybe reveal a little more about it and by taking the limit as r equals zero, maybe that revealed the final answer, I guess, I don't know. I've never done that with looking at a multivariable in polar coordinates before.
>
> INTERVIEWER: So, let me ask – let me get my questions over here – so do you think you have some way that you could use polar coordinates to find out what the limit of a function is?
>
> JOSH: I think I might be able to use it to help, but with the multivariable again, it's that whole, you can take infinity different approaches, so there might be that one example that shows otherwise. I think it might be of some assistance, but I'm not sure that, as I was saying, it revealed the final answer, I don't know if it would actually do that for sure.

As observed in the Chapter V on multivariable limits using polar coordinates, Josh interpreted the limit as r approaches zero as representing another path to the origin.

He saw the polar coordinate computation as no more convincing than any single path chosen during his process of symbolically finding multivariable limits. At this point, Josh has incorporated his experience from polar coordinates into his process understanding of multivariable limits, and likely sees it is some strange path which arises when using polar coordinates.

When Josh explored contour graphs to decide about the limit of multivariable functions, he began by applying the process of analyzing the limit using different approaches.

Excerpt 113

> INTERVIEWER: Ok, so these are contour graphs of the function, and you'll notice they happen to be the same three functions that we just dealt with, so what I would kind of like you to do is to look at these three contour graphs and see if you can make sense of the function and, in particular, see if you can make sense of whether the limit exists as (x,y) approaches (0,0).
>
> JOSH: Ok, basically looking at a contour graph, I kind of have to construct the 3D model from the colors. Lighter is higher, correct?
>
> INTERVIEWER: I think… I don't remember on this one. Yeah, I think yellow tends to be bigger and red tends to be smaller.
>
> JOSH: So from there, as you can see in all these graphs the point (0,0) is a hole, there is some discontinuity there. So, once again just approach from different paths, and see what value I get when I approach from the right, the left, up, down, left, and right, they all appear to be converging at this central point there. So, I would say from that that the limit would exist; whereas, the contrast would be this third one. If I approach from this diagonal here, I appear to be getting close to this low point on the surface; whereas, if I approach from the right, the top right and bottom left diagonals, it appears to be at a higher point. That I'm going to approach this high point here and then approach this low point on either side, there. It's kind of the contrast to it, so I would say that this one does not exist; whereas, this, since each path seems to be approaching the same level on the contour, does exist. And the same concept applied here too, that each path or approach appears to be coming to this same surface level.

To illustrate his use of different approaches, Josh drew lines on the contour graphs, as in the image below.

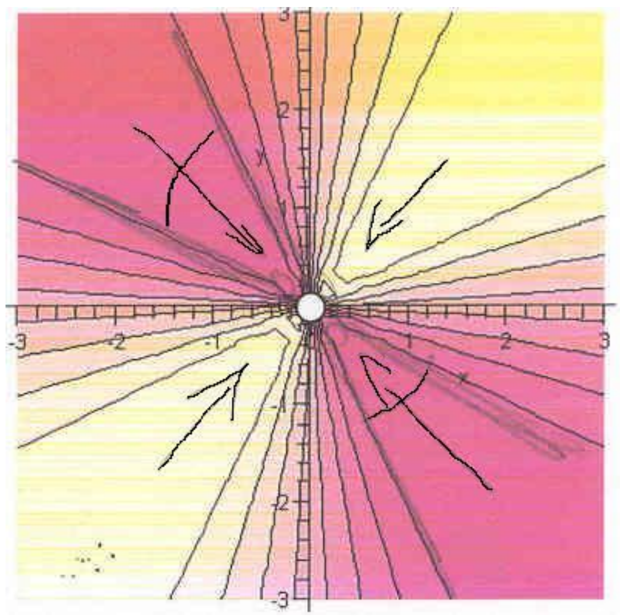


Figure 16: Josh's Use of Paths on a Contour Graph

Josh began this conversation by trying "to construct the 3D model." This visualization brought him back to the three dimensional process where he was able to use different paths to understand the multivariable limit. As when he experienced three dimensional graphing, Josh began exploring the contour graphs one path at a time. However, towards the end of the conversation he began using language demonstrating an attempt to understand all paths simultaneously. He said first that, "each path seems to be approaching the same level on the contour," and then "each path or approach appears to be coming to this same surface level." In his attempt to describe all paths simultaneously, Josh used the phrase "same surface level" which reminds me of the statement "If I look at it all around, it appears to be coming to this single point." In both situations, Josh was attempting to understand the infinite process involved in analyzing multivariable limits along every possible path, and this caused him to move towards understanding the structure of the graph near the origin. As with the three dimensional graphing, this transition originated while Josh encountered multivariable functions with limits that do exist at the origin. This was the beginning of a transition from dynamic thinking to neighborhood thinking for Josh, and as the interview continued he described how to use neighborhood thinking to solve multivariable limit problems using contour graphs.

Excerpt 114

> JOSH: So, same thing right here. I will just try to take this cross section of this graph, which kind of gives me all this area here in the middle. If I approach from any path they're all going to converge at this central point so I would say the limit does exist. This function, however, if I take a cross-section I'm kind of going to get, I'll get this value, or this area and this area are kind of on the same surface – I guess it would actually be these two, if the color shades are right – but that same cross sections would also have different, different levels of surfaces. So, kind of like that other graph, if I approach from the top left or bottom right I'll get but if I approach from the top right or bottom left I'll get a completely separate value. So, it doesn't exist. This one, the same thing, if I kind of take a cross-section, the tiny ring around the outside that it appears that it's getting steeper, kind of a volcano I guess, if I want to think of it in some sort of real life scenario. But it just appears that from each path that it's going to converge at that central point. So, I guess if I wanted to assign, like, the most simple definition, it would really be, 'is it the same color all the way around?' I guess would be the easiest way to visually do it with a contour. So…

In the above excerpt Josh transitioned from using paths approaching the origin to looking at the "area" of the graph near the origin. Finally, he completed his description by providing a means of understanding multivariable limits using contour graphs by asking "is it the same color all the way around?"

In conjunction with this shift in thinking came a change in the illustrations that Josh used to explain his thinking.

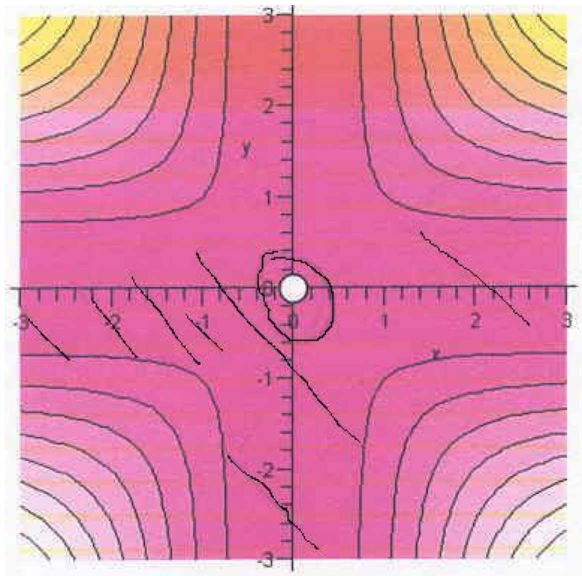


Figure 17: Josh's Use of 'Closeness' on a Contour Graph

Instead of using arrows to depict motion along paths, Josh used a shaded region of the graph that contains a small area around the origin. He had moved from arguing using motion on top of the graph to using regions of the graph. Later in the interview, he showed an ability to use this thinking for limits that do not exist as well as those that do exist.

Excerpt 115

> JOSH: Just kind of look around it, the… well, I have… it goes all the way around it from yellow all the way to a dark red which connotes that it's deeper all the way back to yellow, so from there I'm going from a very high surface back to a very low, like a low value back up, and I'm having several different points, different z values, that if I approach from the top it would be the low value, but if I approach from the right, left, or these lines, to anywhere in between then, I'm going to get higher, kind of the peak value, I mean the peak point, very different values for each approach.

Josh created the following illustration while giving the above explanation about why a multivariable limit should not exist.

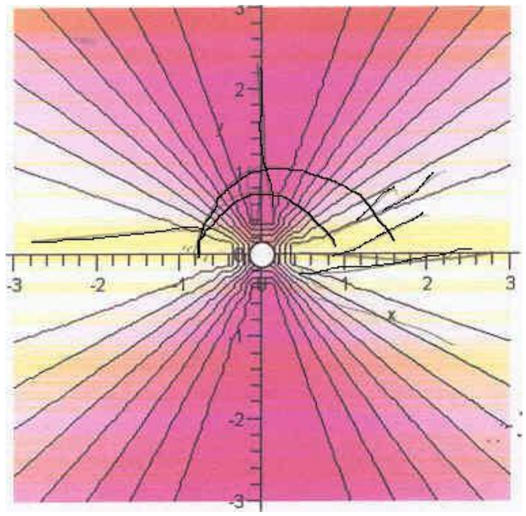


Figure 18: Josh Uses ‘Closeness’ to Explain a Limit that Does Not Exist

For Josh this transition towards neighborhood thinking does not mean an abandonment of his previous dynamic process; rather, the neighborhood thinking is understood in the context of his dynamic process. This was seen in the above excerpt as well as in the following discussion when I pressed him to explain how he makes sense of his new means of understanding limits.

Excerpt 116

> INTERVIEWER: So, ok, let’s spend a second and… so what do you mean by this, ‘if it’s the same color all the way around,’ and how does that makes sense to you?
>
> JOSH: Because of the contour map that a color is assigned to a certain depth or a height that from there it’s just kind of, well if you think of multivariable limits in general, that you’re going to converge from whatever path at this same value at this point. Well, since the value, the surface height, it basically assigned a color in this, so if all around the point that I’m taking the limit at is the same color then it’s going to be the same height at that point. Like this graph here, it’s the same color all the way around, so if I approach right, left, any approach whatsoever I take, a

> parabola approach, anything, it's going to be the same value. So kind of the most basic for a contour graph looking at the limit would be, 'is it the same color all the way the point we're taking the limit at?' I don't know if I described enough.

In the above description I believe that Josh relied on the fact that every path through the origin must pass though a small neighborhood of the origin. In this way, he could take "any approach whatsoever" and know that the approach must be the "same height" because it must pass through that same neighborhood of the contour graph.

Through the course of these interviews, I observed Josh transitioning through the three phases listed below.

1. Initially, Josh conceptualized the limit concept along paths, considering the paths one at a time.

2. During the three dimensional and contour graphing experiences, Josh began describing the behavior of all paths simultaneously.

3. At the conclusion of the interviews, Josh had developed a notion of neighborhood that described the region of the graph around the origin. This region was used to make decisions about the multivariable limit of a function, and it was connected to the dynamic process of analyzing the function along different paths.

Sfard (1991) describes the encapsulation of a process as a three part sequence. "First there must be a process performed on the already familiar objects, then the idea of turning this process into a more compact, self-contained whole should emerge, and finally an ability to view this new entity as a permanent object in its own right must be acquired" (pp. 64-65). Sfard refers to these three steps as interiorization, condensation, and reification. Other authors (for example, Dubinsky 1991) use a three part sequence to describe the encapsulation of a mathematical process. For most of these descriptions, the first and second phases are distinguished in part by the individual's awareness of the entire process. For Grey and Tall (1994), this awareness is captured by a mathematical symbol. For Cottrill et al. (1996) the awareness is demonstrated by the individual possessing conscious control over the various elements of the process.

Josh viewed the multivariable limit concept as a process of analyzing the function along every path through the origin. By any of the above descriptions, he successfully moved his understanding of this process from the first phase to the second phase. By the end of the interviews, he had developed a compact way to view the infinite process, talked about the process as a whole, and showed awareness and control over individual steps of the process. I will use the term from Sfard (1991) and say that Josh *condensed* his understanding of the multivariable limit process.

It is not clear, however, that Josh has, in fact, completed the encapsulation process of his understanding of the multivariable limit concept into a cognitive object. Tall (2000) discusses the meaning of a cognitive object resulting from the encapsulation of a mathematical process. He pointed out that many authors describe the resulting objects in different ways, but from his point of view "what matters more is not what it *is*, but what we can *do* with it." From this perspective, Josh had not been asked to *do* anything that involved the concept of multivariable limit as an object, and so I will not attempt to argue that Josh fully encapsulated this process into a reified object.

What is important about Josh's understanding of the multivariable limit process is that he successfully *condensed* this process internally and this act of condensing the process led him to look at the concept of multivariable limit in a different manner. It was the use and exploration of Josh's dynamic multivariable process that led him to eventually conceptualize the multivariable limit using neighborhood thinking with a sense of 'closeness.' From Josh's perspective, using the area around the origin was sufficient to capture the multivariable limit process because *every* step in this process involved a path that passed though this small region. It was precisely Josh's ability to conceptualize every path simultaneously that led him to draw this conclusion about the multivariable limit.

In Chapter V in the section on "introductory calculus" students were observed evoking an idea of "close enough" to justify the termination of the infinite process of limit that required them to analyze points successively closer to the point in question. The implication was that once "close enough" points are chosen, the result of the process can be inferred without further calculation. It is reasonable to think that this metaphor of "close enough" is a result of students condensing the infinite process of limit into a coherent whole that gives them control over the various aspects of the process. For these reasons, I argue that the notion of "closeness" is a natural response to the condensation of an infinite limiting process, whether that limiting process takes place in single or multivariable calculus.

For Josh, the struggle to show that a multivariable limit exists concluded with a realization about the importance of the area around the limit point. Josh was able to use this argument affectively to say that multivariable limits experiences using three dimensional graphing or contour graphing do exist. I believe that Josh has laid the foundation on which he could develop a symbolic argument that a multivariable limit should exist; however, I believe that such an argument still required significant cognitive development on his part, particularly in developing notation to capture his thinking about the behavior of the function around the limit point.

Answering the Study Questions

This study was developed to answer the problem statement given in chapter one: *Describe how students with a dynamic view of limit generalize their understanding of the limit concept in a multivariable environment.*

Three models of limit were developed to describe how the limit concept could be understood – the topographical model, dynamic model and neighborhood model. The study participants' use of each of these models were documented and described in detail. This description was laid out through the course of Chapter V.

In addition to the original problem statement, three sub questions were created to focus the description in terms of the type of generalization that took place, the use of dynamic imagery, and the role of different multivariable environments in understanding multivariable limits. In the following sections, I would like to address each of these sub questions and discuss the results found by this study.

Describe what type of generalization students in this setting tend to experience with respect to the schema outlined by Harel and Tall (1989) which emphasizes three modes of generalization: expansive generalization, reconstructive generalization, and disjunctive generalization.

Throughout the study, all three modes of generalization were observed, and each student involved in the interviews experienced generalization in their own way. Two students, Amanda and Josh, are of particular interest because they represent two distinct approaches towards understanding the limit concept. Amanda's experience centered on the visualized images of shape and motion; whereas, Josh's experience centered on the creation and understanding of a mathematical process.

For Amanda, in order to generalize her use of dynamic motion to understand multivariable limits she first needed to construct an internal classification system based on her visualizations of various multivariable functions in order to understand the difference in shape between a multivariable limit that does or does not exist. Prior to the construction of these internal visualizations, Amanda was unable to fully apply her image of limit based on dynamic motion, and instead she experienced what Harel and Tall would call *disjunctive generalization*. However, after visualizing different limits, Amanda experienced *expansive generalization* in the sense that she was able to adapt her use of dynamic motion and place it on top of the newly visualized shapes in order to solve multivariable limit problems.

For Josh, the multivariable limit experience began as the creation of a process to examine multivariable limits. This process was an *expansive generalization* of the analysis of limits from the right and the left to the analysis of multivariable limits along every possible path. Josh was successful at using this process to understand multivariable limits that did not exist, but he was unsuccessful at using this process to understand multivariable limits that exist. This created cognitive conflict in Josh which was eventually overcome by analyzing the area around the limit point in order to make conclusions about the multivariable limit. This analysis represented a period of *reconstructive generalization* for Josh. In reconstructing his conceptualization of limits he included a sense of ‘closeness’ that allowed him to describe multivariable limits which exist.

Describe the role of motion in students’ understanding of multivariable limits. Does it create a cognitive obstacle, or are students able to apply this imagery to the new multivariable situation?

In this study, students conceptualized dynamic motion in two ways. The first conceptualization was visualized motion, which took the form of a fluid action of moving along the graph toward a limit point. The second conceptualization was a dynamic process which successively selected points closer to the limit point. For single variable limits, most students used both images fluently at different moments in the interview, and when discussing multivariable limits, most students used language reflecting visualized motion.

For most students, the use of motion, in itself, did not create a cognitive obstacle. The only real instance of motion creating a cognitive obstacle was the visualization of contour lines in motion described by Jennifer in Chapter V in the section on misconceptions. Other struggles related to motion arose from one of two other cognitive obstacles. The first obstacle could be described as the struggle to apply shape to the multivariable limit concept. Students such as Amanda who relied heavily on visualization to understand limits struggled to understand multivariable limits without a strong sense of how the ‘shape’ of a multivariable function affects the existence of a multivariable limit. The second obstacle could be described as the struggle to apply the infinite process of analyzing a multivariable limit along every possible path though the limit point. In using this process, it was not the motion along the paths that caused a struggle; rather, the obstacle was the infinite nature of this process. Several students such as Josh managed to overcome this obstacle by using a notion of ‘closeness’ to the limit point.

Describe how students respond to studying multivariable limits in four different contexts: traditional symbolic manipulation, symbolic manipulation involving polar coordinates, three-dimensional graphing, and contour graphing. Of particular interest is whether some of these contexts tend to allow students to reconstruct their understanding of the limit concept to more closely resemble the formal definition.

Overall, students in this study showed little evidence of a conceptualization of limit compatible with the formal definition of limit. As mentioned in Chapter IV there was only one instance of a student using reverse thinking to describe the limit concept. However, during the multivariable limit experiences several students developed an ability to use a sense of ‘closeness’ reflecting the mathematical concept of neighborhoods to discuss multivariable limits. I regard this use of neighborhoods to be a necessary conceptualization in order to understand the formal definition of limit.

Three significant occurrences of students using neighborhood thinking took place while discussing multivariable limits using contour graphs. All three of these students made the observation that the multivariable limit could be determined using the color of the contour graph around the limit point. Furthermore, these students connected this idea of closeness on the contour graph with an argument involving dynamic motion towards the limit point. An argument of this nature is reminiscent of the formal proof that the sequential and formal definitions of limit are equivalent when the functions involved are real-valued. This connection provides encouraging evidence that the use of contours can be used to help facilitate the condensation of the multivariable limit process and develop a neighborhood-based understanding of the limit concept.

Reflections

In this final section, I would like to reflect on the results of this study. This reflection naturally leads me to address two issues: what limitations did this study have and what future research needs to take place?

First, the small sample size of only seven students makes drawing external generalizations impossible. This limitation is inherent in qualitative research; the small sample size that creates problems for generalization is what enables the study to create a description of students' thinking. Alcock and Simpson (2004, 2005) published two papers categorizing real analysis students using a combination of their visual reasoning and their view of their own role in learning mathematics. I believe that a classification system resembling their work would be relevant for students in multivariable calculus. With the creation of such a classification system, a small sample size would be less problematic with each member of the smaller sample representing a larger category of multivariable calculus students.

Second, mathematics can be considered as socially constructed with individuals constructing meaning through learning communities (Wegner, 1998). From this perspective, a weakness of my study resides in the fact that the classroom experiences of the students were not observed as part of the data, and so the social acquisition of the mathematics in this study was not observed. This results in the study being a description of the students' conclusions about multivariable limits and the way that those conclusions are altered by different multivariable settings. A study describing the social experience of learning multivariable limits would be a logical next step in future research of the multivariable limit concept. A more powerful tool than simple classroom observation is the teaching experiment (Simon, 1995). The description of this study may be used to create a theoretical development of the multivariable limit concept, and then classroom materials may be developed to guide students through this development and student interaction with these materials is observed. The teaching experiment model is designed to expose imperfections in the theoretical development of the concept which in turn allows for a refinement of the theory and the materials.

Two of the multivariable settings, in particular, provided surprising results. The first was the experience with polar coordinates. No student in the study was able to conceptualize the multivariable limit concept in terms of polar coordinates. Originally, I believed that the experience with polar coordinates could potentially lead students to develop a formal understanding of the limit concept. Perhaps this new view of the limit concept is partially responsible for the struggles that students experienced in this setting. However, in retrospect, there is much work that needs to be done describing how students conceptualize polar coordinates. In particular, descriptions of how polar coordinates are visualized and how the variables r and θ are internalized by students are critical in order to understand a multitude of multivariable phenomena. In addition, the use of the concept of distance by students in both polar coordinates and with absolute values is unclear, yet I believe distance plays a vital role in the way mathematicians understand these concepts.

The other surprise came when students encountered multivariable limits using contour graphs. I anticipated the reactions to contour graphs to mirror those reactions found when using three dimensional graphing. However, three of the students experienced a critical shift in their conceptualization of the multivariable concept, from dynamic to neighborhood thinking, when they encountered multivariable limits using contour graphs. A better understanding of how students understand contour graphs and what features of the contour graph allowed these students to break through with their conceptualizations would be insightful for developing curriculum to allow students to understand the limit concept in terms of neighborhoods.

In conclusion, I believe the ways in which students conceptualized motion in this study were significant. As described in Chapter V, multiple representations and informal notions of limit have become the trend in teaching introductory calculus. This increased emphasis has led to many calls for motion to play a key role in school curriculum. From this perspective it is significant that motion did not create a cognitive obstacle for students studying multivariable limits; rather, the primary obstacles were due to visualization of multivariable functions and the struggle with the infinite process of analyzing the limit along every possible path. Importantly, condensing the process of analyzing motion along every path led several students to incorporate neighborhood thinking into their conceptualizations of the limit concept. This indicates that not only does motion not create an unnecessary obstacle for students, but it may be part a vital part of reconstructing students' thinking towards a metaphor of 'closeness.'

Finally, I do believe that the multivariable limit concept has the potential to be a powerful tool towards enabling students to reach an understanding of limit that mirrors the formal definition. I believe the mathematics community should consider its role in the undergraduate curriculum and teach this concept not just as a means to solve several problems but as a means to illuminate a type of thinking beneficial to the students.

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APPENDIX A – STUDY QUESTIONNAIRE

Questionnaire Form # ________

1. A limit describes how a function moves as you approach a given point.

Strongly Agree Somewhat Agree Neither Agree Nor Disagree Somewhat Disagree Strongly Disagree

Why did you choose this answer?

2. A limit can be found by plugging in a number infinitely close to a point.

Strongly Agree Somewhat Agree Neither Agree Nor Disagree Somewhat Disagree Strongly Disagree

Why did you choose this answer?

3. A limit is a number that a function can be made arbitrarily close to by taking values sufficiently close to a certain point.

Strongly Agree Somewhat Agree Neither Agree Nor Disagree Somewhat Disagree Strongly Disagree

Why did you choose this answer?

4. A limit is a number or point the function gets close to but never reaches.

Strongly Agree Somewhat Agree Neither Agree Nor Disagree Somewhat Disagree Strongly Disagree

Why did you choose this answer?

5. A limit is an approximation that can be made as accurate as you wish.

Strongly Agree Somewhat Agree Neither Agree Nor Disagree Somewhat Disagree Strongly Disagree

Why did you choose this answer?

Questionnaire Page One

Form # ________

6. Which of the above statements best describes a limit as you understand it? (circle one)

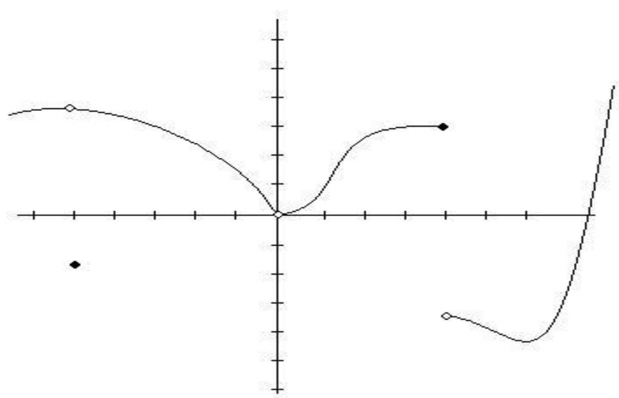
1 2 3 4 5 none

7. Describe what it means to say that $\lim_{x \to 3} f(x) = 4$.

8. How would you describe the idea of limit to someone who has never studied calculus? Please use a few sentences to give your description.

Questionnaire Page Two

APPENDIX B – INTERVIEW 1 MATERIALS



Find the following:

$$\lim_{x \to -5} f(x)$$

$$\lim_{x \to 0} f(x)$$

$$\lim_{x \to 4} f(x)$$

Student Handout: Page One

$$f(x) = \frac{x-1}{x^2-1}$$

What is the value of *f*(1), if it exists?

What is $\lim_{x \to 1} f(x)$, if it exists?

$$f(x) = \frac{|x-1|}{x^2-1}$$

What is the value of *f*(1), if it exists?

What is $\lim_{x \to 1} f(x)$, if it exists?

Student Handout: Page Two

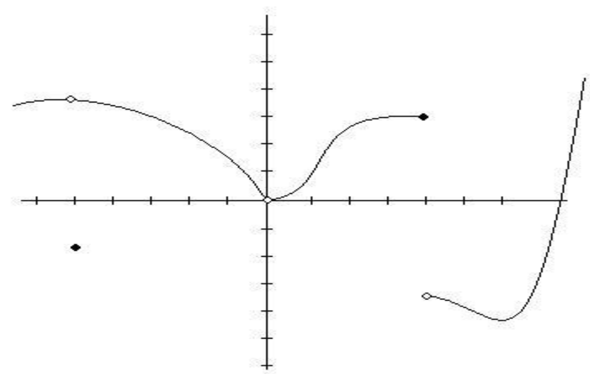
1. Review your answers to the questionnaire given earlier. Would you like to change any of your answers? Are there any answers that you would like to clarify?

2. When describing a function as “approaching” or “getting close to” a point, this idea would best be explained as:

a) Evaluating a function at different numbers over time with those numbers successively getting closer to the point in question.

b) Mentally envisioning a point on a graph moving closer and closer to the limit point.

3. For the following graph of *f*(*x*), find $\lim_{x \to -5} f(x)$, $\lim_{x \to 0} f(x)$, and $\lim_{x \to 4} f(x)$.



4. Consider the function $f(x) = \dfrac{x-1}{x^2-1}$. Find the value of *f*(1), if it exists? Find $\lim_{x \to 1} f(x)$, if it exists? Describe how these solutions relate to your response to Part C of the questionnaire.

5. Consider the function $f(x) = \dfrac{|x-1|}{x^2-1}$. Find is the value of *f*(1), if it exists? Find $\lim_{x \to 1} f(x)$, if it exists? Describe how these solutions relate to your response to Part C of the questionnaire.

Interviewer Notes

APPENDIX C – INTERVIEW 2 MATERIALS

Consider the function $f(x, y) = \dfrac{x^2 - y^2}{x^2 + y^2}$

a) Find $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 - y^2}{x^2 + y^2}$ along the line $x = 0$.

b) Find $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 - y^2}{x^2 + y^2}$ along the line $y = 0$.

c) Find $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 - y^2}{x^2 + y^2}$ along the line $x = y$.

d) Based on your results from a) through c) above, would you say that $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 - y^2}{x^2 + y^2}$ exists? Why or why not?

Student Handout: Page One

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2 + y^4}$$

$$\lim_{(x,y)\to(0,0)} \frac{x^4 + y^4}{x^2 + y^2}$$

Student Handout: Page Two

1. Have you studied multivariable limits before? If so, tell me what you have learned. How would you describe $\lim_{(x,y)\to(a,b)} f(x,y)$?

2. Do the traditional limit problem.

3. In general what can you do to determine whether $\lim_{(x,y)\to(0,0)} f(x,y)$ exists? Why do you believe this should work?

4. Does it work for the limits given?

5. Using a computer graphing program, graph each of the following functions (we have already worked with them).

a. $f(x,y) = \dfrac{x^2 - y^2}{x^2 + y^2}$

b. $f(x,y) = \dfrac{xy^2}{x^2 + y^4}$

c. $f(x,y) = \dfrac{x^4 + y^4}{x^2 + y^2}$

6. Examine the behavior of each function near the origin and determine $\lim_{(x,y)\to(0,0)} f(x,y)$, if it exists, if it does not exist explain why.

7. Describe how you could determine whether $\lim_{(x,y)\to(0,0)} f(x,y)$ exists by analyzing a graph of the function $f(x,y)$. Why do you believe this should work?

8. For each graph given, determine $\lim_{(x,y)\to(0,0)} f(x,y)$, if it exists, if it does not exist explain why.

Interviewer Notes

1. Review your answers to the initial questionnaire. Would you like to change any of your answers? Are there any answers that you would like to clarify?

2. How closely is the idea of limit in the expression $\lim_{(x,y)\to(a,b)} f(x,y)$ related to the idea of limit in the expression $\lim_{x\to a} f(x)$?

a. Very closely related
b. Somewhat closely related
c. Not closely related
d. Not related at all

If you responded that it was somewhat closely related or very closely related, please describe this relationship in a few sentences.

3. Did you feel that your understanding of the idea of limit in the problem $\lim_{x\to a} f(x)$ was affected by studying the idea of limit in the problem $\lim_{(x,y)\to(a,b)} f(x,y)$?

a. It was significantly affected
b. It was somewhat affected
c. It was slightly affected
d. It was not affected at all

If you responded that it was somewhat affected or significantly affected, please describe how your understanding was affected in a few sentences.

Additional Questions

APPENDIX D – INTERVIEW 3 MATERIALS

If you forgot, here is a (brief) summary of polar coordinates:

Polar coordinates are in the form (r , θ), where r represents the distance from the point to the origin, and θ represents the angle between the positive x-axis and the line segment connecting the origin with the point.

The Pythagorean Theorem gives that $r^2 = x^2 + y^2$

And, trigonometry gives us that $\tan\theta = \frac{y}{x}$

We can also solve these for x and y getting the equations: $x = r\cos\theta$, $y = r\sin\theta$

Re-write each function using polar coordinates.

$$f(x, y) = \frac{xy^2}{x^2 + y^2}$$

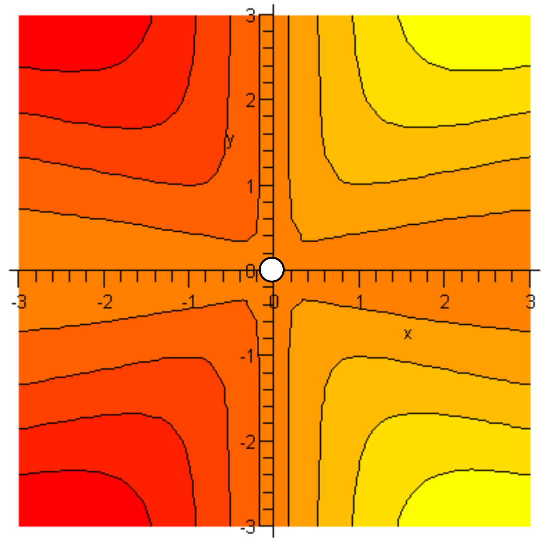
$$f(x, y) = \frac{x^4 + y^4}{x^2 + y^2}$$

$$f(x, y) = \frac{xy}{x^2 + y^2}$$

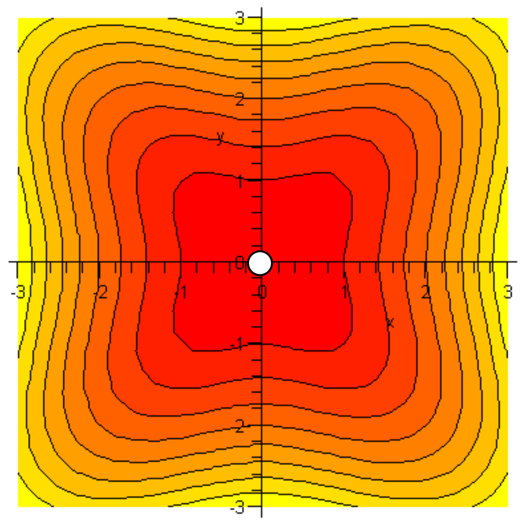
For each function determine $\lim_{r\to 0} f(r,\theta)$. For which functions $f(x, y)$ do you believe $\lim_{(x,y)\to(0,0)} f(x, y)$ exists? Why?

Student Handout: Page One

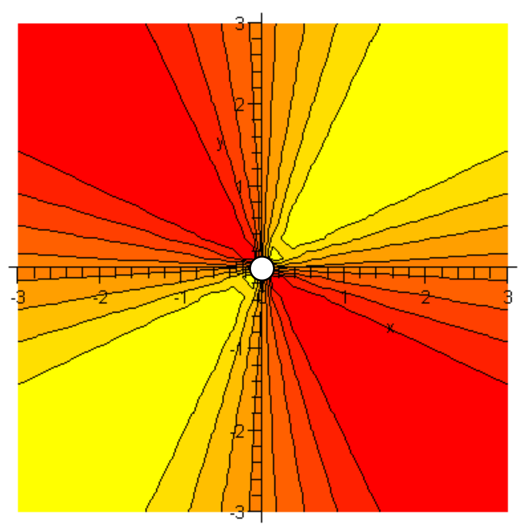
$$f(x,y)=\frac{xy^2}{x^2+y^2}$$



$$f(x,y)=\frac{x^4+y^4}{x^2+y^2}$$

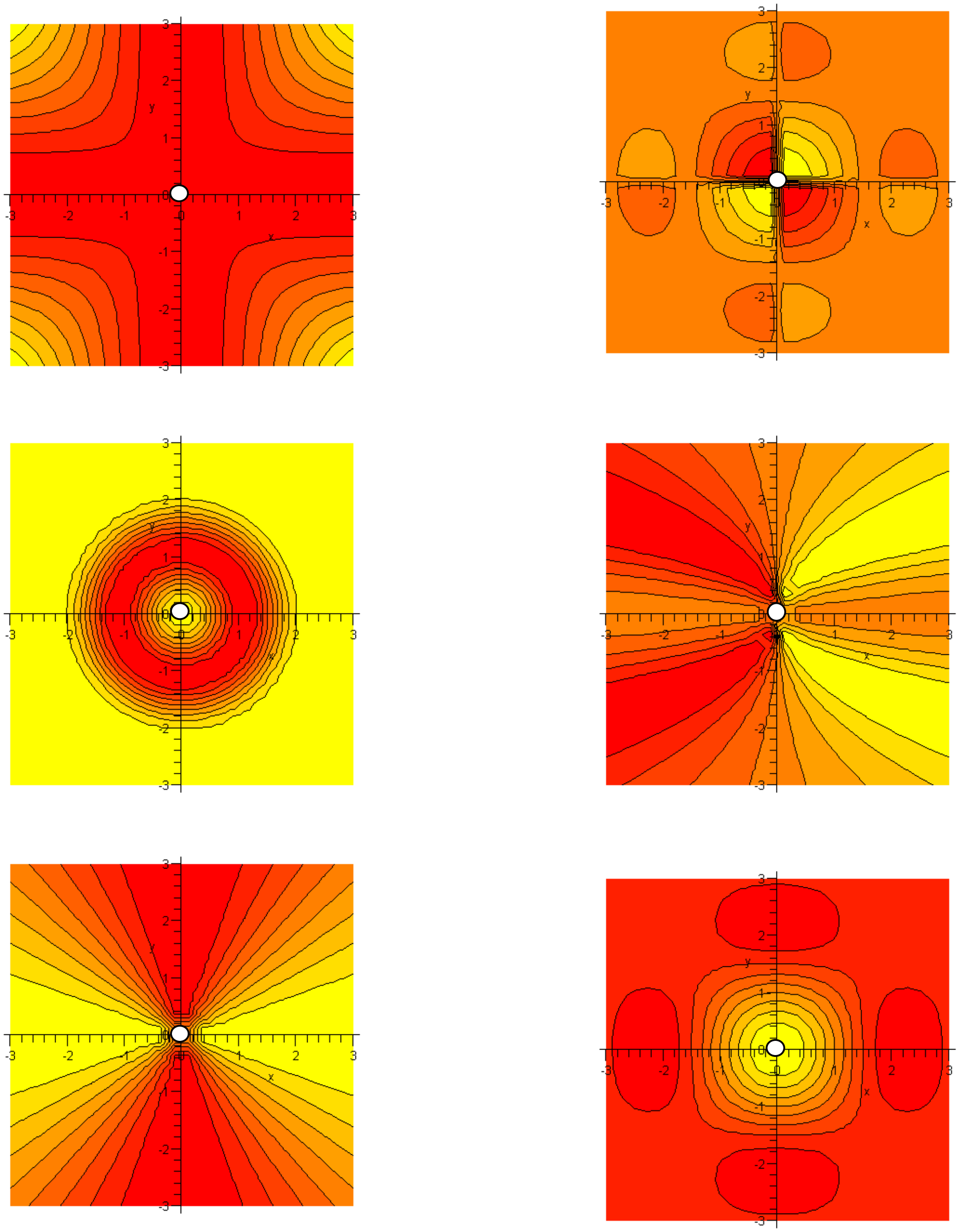


$$f(x,y)=\frac{xy}{x^2+y^2}$$



Student Handout: Page Two

Contour graphs of multivariable functions not defined at (0,0).



Student Handout: Page Three

1. Use polar coordinates to re-write each of the following functions.

2. For each function determine $\lim_{r\to 0} f(r,\theta)$. For which functions $f(x,y)$ do you believe $\lim_{(x,y)\to(0,0)} f(x,y)$ exists? Why?

3. Describe how you can determine whether $\lim_{(x,y)\to(0,0)} f(x,y)$ exists by analyzing the function using polar coordinates? Why do you believe this should work?

4. Using the contour graphs provided, examine the behavior of each function near the origin and determine $\lim_{(x,y)\to(0,0)} f(x,y)$, if it exists, if it does not exist explain why.

5. Describe how you could determine whether $\lim_{(x,y)\to(0,0)} f(x,y)$ exists by analyzing a contour graph of the function $f(x,y)$. Why do you believe this should work?

6. For each contour graph below, determine $\lim_{(x,y)\to(0,0)} f(x,y)$, if it exists, if it does not exist explain why.

Interviewer Notes

1. Review your answers to the initial questionnaire. Would you like to change any of your answers? Are there any answers that you would like to clarify?

2. How closely is the idea of limit in the expression $\lim_{(x,y)\to(a,b)} f(x,y)$ related to the idea of limit in the expression $\lim_{x\to a} f(x)$?

e. Very closely related
f. Somewhat closely related
g. Not closely related
h. Not related at all

If you responded that it was somewhat closely related or very closely related, please describe this relationship in a few sentences.

3. Did you feel that your understanding of the idea of limit in the problem $\lim_{x\to a} f(x)$ was affected by studying the idea of limit in the problem $\lim_{(x,y)\to(a,b)} f(x,y)$?

e. It was significantly affected
f. It was somewhat affected
g. It was slightly affected
h. It was not affected at all

If you responded that it was somewhat affected or significantly affected, please describe how your understanding was affected in a few sentences.

Additional Questions

APPENDIX E – INFORMED CONSENT FORMS

Form # ________

Informed Consent Form

I received the information regarding the purpose and format of this study from the researcher present. I understand that I am under no obligation to take part in this study, and I understand that I may cease my participation in this study at any time without fear of being penalized in any way.

By signing below, I verify that I am over 18 years of age, that I understand my rights as a participant in this study, and that I sign this consent form voluntarily.

Signature of Participant: ____________________ Date: ______________

Are you interested in taking part in the interview portion of this study and would you like the researcher to contact you if are an appropriate candidate for the interviews?

Yes ☐ No ☐

If you marked "yes," please print your name below and provide a method for the researcher to contact you:

Questionnaire Participants' Informed Consent Form

The following script will be read to all students with potential to take part in the study selection process. Afterward, students wishing to take part in this process will complete a short consent form and fill out the initial study questionnaire.

Verbal Script:

Hello, I am doing a study on how students understand limits in multivariable calculus. The purpose of this study is to explore and describe the different ways that students make sense of limits during a multivariable calculus course. Your instructor has given me the next 15 minutes so that willing students can take part in this study. I will be distributing short questionnaires which ask you questions about how you understand the limit concept. Once returned to me, your consent forms will be detached from the questionnaire and the only reference will be the three digit code listed on both your responses and your consent form. By taking part in this study you will be contributing to the understanding mathematics educators have about how students understand a key calculus concept. Through studies of this nature, educators are better able to design courses that students traditionally find very challenging.

In addition, I will be seeking a small number of students to take part in a series of three interviews exploring how they understand limits in multivariable calculus. Those taking part in these interviews will have an opportunity to explore the limit concept beyond what is usually covered in class. In addition, I will be providing free one-on-one tutoring to those involved in the interview portion of the study for the remainder of the semester. If you would be interested in being a volunteer for the interview portion of this study please indicate this at the appropriate place.

It is important to note that in ***no way*** *are you obligated to take part in this study, and you may quit your participation in this study* ***at any time*** *without fear of being penalized in any way.*

Thank You

Verbal Script Accompanying the Questionnaire Participants' Informed Consent Form

Informed Consent Form

Students' conceptualization of the limit concept in a multivariable setting
Research performed by Brian Fisher under his advisor, Dr. Doug Aichele

The primary goal of this study is to create a description of how students make sense of limits in multivariable calculus. As a result, this study seeks students willing to allow the investigator to examine their understanding of limits in different situations. You are being asked to participate in this study by answering interview questions related to limits in a variety of settings. For this reason, the study will take place in a series of three interviews which will each be video recorded. These interviews will take approximately one hour each, and they will be held in a classroom in the math sciences building with only the student and researcher present. Videos of each interview will be stored on DVD disks by the researcher, and any conversations may be transcribed for further use. Your name will not be included on any stored videos or transcriptions, and all video will be viewed only by the researcher and those overseeing this study. It is also possible that the consent process and data collection will be observed by research oversight staff responsible for safeguarding the rights and wellbeing of people who participate in research.

By better understanding the ways in which students make sense of limit concepts, the mathematics education research community is able to improve instruction for classes such as calculus. By taking part in this study you may come to a better understanding of the limit concept yourself. Since this concept plays a critical role in math classes such as calculus, it is possible that this study will help you in your future math studies. In addition to any benefit you may receive by being in the study, free one-on-one math tutoring will be available to you throughout the remainder of the semester. This tutoring may be arranged with the researcher and will be aimed at allowing you to better understand calculus and achieve a higher grade in your course.

It should be emphasized that participation in this study is entirely voluntary and you may discontinue your involvement with this research at any time without risk of being penalized in any way. If you have further questions about the research and your rights as a research volunteer, you may contact Dr. Sue C. Jacobs, IRB Chair, 219 Cordell North, Stillwater, OK 74078, 405-744-1676 or irb@okstate.edu.

Signing below signifies that I am over 18 years of age and have read and fully understand the consent form. I sign it freely and voluntarily. A copy of this form has been given to me.

Signature of Participant: ____________________ Date: ______________

I certify that I have personally explained this document before requesting that the participant sign it.

Signature of Researcher: ____________________ Date: ______________
Researcher's contact information:
Brian Fisher, 311 Math Sciences, Stillwater, Ok 74078. (405) 744-2239. brifishe@math.okstate.edu

Interview Participants' Informed Consent Form

APPENDIX F – IRB APPROVAL FORM

Oklahoma State University Institutional Review Board

Date:	Wednesday, July 11, 2007
IRB Application No	AS0746
Proposal Title:	Students' Conceptualization of the Limit Concept in a Multivariable Setting
Reviewed and Processed as:	Expedited

Status Recommended by Reviewer(s): Approved Protocol Expires: 7/10/2008

Principal Investigator(s

Brian Fisher 1928 NW 176th St. Edmond, OK 73003	Doug Aichele 401 MS Stillwater, OK 74078

The IRB application referenced above has been approved. It is the judgment of the reviewers that the rights and welfare of individuals who may be asked to participate in this study will be respected, and that the research will be conducted in a manner consistent with the IRB requirements as outlined in section 45 CFR 46.

☑ The final versions of any printed recruitment, consent and assent documents bearing the IRB approval stamp are attached to this letter. These are the versions that must be used during the study.

As Principal Investigator, it is your responsibility to do the following:

1. Conduct this study exactly as it has been approved. Any modifications to the research protocol must be submitted with the appropriate signatures for IRB approval.
2. Submit a request for continuation if the study extends beyond the approval period of one calendar year. This continuation must receive IRB review and approval before the research can continue.
3. Report any adverse events to the IRB Chair promptly. Adverse events are those which are unanticipated and impact the subjects during the course of this research; and
4. Notify the IRB office in writing when your research project is complete.

Please note that approved protocols are subject to monitoring by the IRB and that the IRB office has the authority to inspect research records associated with this protocol at any time. If you have questions about the IRB procedures or need any assistance from the Board, please contact Beth McTernan in 219 Cordell North (phone: 405-744-5700, beth.mcternan@okstate.edu).

Sincerely,

Sue C Jacobs

Sue C. Jacobs, Chair
Institutional Review Board

VITA

BRIAN CLIFFORD FISHER

Candidate for the Degree of

Doctor of Philosophy

Thesis: STUDENTS' CONCEPTUALIZATIONS OF MULTIVARIABLE LIMITS

Major Field: Mathematics, with Specialization in Mathematics Education

Biographical:

Personal Data: Born November 7, 1978 in Oklahoma City, son of Cary and Mary Fisher.

Education:
Graduated as Valedictorian of Tuttle High School, 1997. Received a Bachelor of Science Degree in Mathematics from Oklahoma State University, December 2000. Received a Master of Science Degree from Oklahoma State University, July 2002. Completed the requirements for the Doctor of Philosophy in Mathematics with Specialization Mathematics Education at Oklahoma State University, Stillwater, Oklahoma in July, 2008.

Experience:
Mathematics Tutor, Oklahoma State University MLRC, August 1998 – May 2000. Coordinator, Oklahoma State University MLRC, August 2000 – December 2000. Graduate Teaching Assistant, Oklahoma State University Mathematics Department, January 2000 – May 2008. Advisory Board, Oklahoma State University MLRC, August 2004 – Present.

Professional Memberships:
Mathematical Association of America, American Mathematics Society

Name: Brian Clifford Fisher Date of Degree: July, 2008

Institution: Oklahoma State University Location: Stillwater, Oklahoma

Title of Study: STUDENTS' CONCEPTUALIZATIONS OF MULTIVARIABLE LIMITS

Pages in Study: 242 Candidate for the Degree of Doctor of Philosophy

Major Field: Mathematics, with Specialization in Mathematics Education

Scope and Method of Study:

The objective of this study was to describe how students with a dynamic view of limit generalize their understanding of the limit concept in a multivariable setting. This description emphasizes the type of generalization that takes place among the students (Harel and Tall, 1989) and the role of motion among students' conceptualizations.

To achieve these goals, a series of task-based interviews were conducted with seven students enrolled in multivariable calculus. These interviews were analyzed and a coding scheme was developed to describe the data. This coding scheme arose from analysis of the data combined with the role of limit in formal mathematics. It emphasizes three models for understanding the limit concept, the dynamic model, the neighborhood model, and the topographical model.

Findings and Conclusions:

After analyzing the coded data, two important interactions between the three models of limit were described. First, it was found that students superimposed dynamic imagery on top of existing topographical structures in order to understand multivariable limits, and a weak topographical understanding of multivariable limits contributed to students struggling to understand the multivariable limit concept. Second, it was found that students implementing dynamic imagery in the context of multivariable limits confronted an infinite process of analyzing motion along an infinite number of paths. It was found that students' struggles to understand the multivariable limit were connected to their struggles to understand this infinite process. Additionally, it was found that the condensation of this infinite process led several students towards the neighborhood model of limit.

ADVISER'S APPROVAL: Douglas B. Aichele