GENERALIZING THE THEOREM OF NAKAGAWA ON BINARY CUBIC FORMS TO NUMBER FIELDS

By

JORGE DIOSES

Bachelor of Science in Mathematics
Pontifical Catholic University of Peru
Lima, Lima, Peru
1997

Licentiate in Mathematics
Pontifical Catholic University of Peru
Lima, Lima, Peru
2000

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Dissertation Approved:

Dr. David Wright

Dissertation Advisor

Dr. Heidi Hoffer

Dr. Anthony Kable

Dr. Chris Francisco

Dr. Sheryl Tucker

Dean of the Graduate College

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CHAPTER 1

Introduction

1.1 Overview

A fundamental problem in number theory is to describe in as precise a manner as possible the collection of algebraic number fields, meaning the extensions of finite degree of the rational number field $\mathbb{Q}$. For example, all the quadratic extensions may be precisely described as $\mathbb{Q}(\sqrt{d})$ where d ranges over all square-free integers not equal to 1. One corollary of this is that there is a simple correspondence between fields of positive discriminant $\mathbb{Q}(\sqrt{d})$ and fields of negative discriminant $\mathbb{Q}(\sqrt{-d})$. Generalizing these simple statements to even just cubic fields is highly nontrivial. For instance, the smallest negative discriminant of an extension of degree 3 of $\mathbb{Q}$ is -23, corresponding to the polynomial $x^3 - x + 1$, while the smallest positive discriminant is 49 corresponding to $x^3 - x^2 - 2x + 1$. Tables of discriminants of cubic fields, both positive and negative, were calculated in the late nineteenth and early twentieth century, and there was no apparent correlation between the two lists of positive and negative discriminants. In [14] Ohno found a correspondence between these fields, which was most easily explained in terms of class numbers of integral binary cubic forms. This correspondence was stated as a conjecture which was later proved in [13] by Nakagawa using class field theory.

The goal of this thesis is to investigate possible generalizations of this result to the case of quadratic and cubic extensions of an arbitrary number field. It relies on the work of Datskovsky and Wright in [5] where the original definitions are extended to global fields of characteristic not equal to 2 or 3.

In the second section of Chapter 1, we revise the original statements of the theorem of Nakagawa when the base field is $\mathbb{Q}$. We also state in simple terms a conjecture that would generalize this result to any number field. The conjecture is shown as an equation between Dirichlet series, considered as sums over number fields of degree at most 3.

We present in Chapter 2 the basic definitions and notations for number fields, binary cubic forms, zeta functions, and Dirichlet series. We also define the appropriate Haar measures that are needed throughout this thesis.

In Chapter 3, we evaluate certain local integrals that play an important role in the generalization process. These local integrals define the Dirichlet series of one of the sides of our conjecture. We work under the assumption that 3 is unramified in the given number field.

The calculation of the residues at its poles of the Dirichlet series mentioned above is given in Chapter 4. We use a technique given by Datskovsky and Wright to accomplish this. Based on this, we are able to give a justification for our conjecture.

In Chapter 5, we introduce additional definitions that allow us to decompose our conjectured identity into a family of identities between sums over fields. Each side of these identities corresponds to a special field. We also introduce some terminology and review some basic facts of class field theory. They will be invoked in the last chapter of this thesis.

We test numerically the validity of the conjecture in Chapter 6. In order to do so, we make use of available tables of cubic and quadratic fields. We do this for the quadratic imaginary field $\mathbb{Q}(i)$. This provides strong evidence for the proposed identity. We also verify Nakagawa's theorem.

Finally, in Chapter 7, we reduce both sides of our conjectured identity to sums over characters of an idele class group. We proved some results mentioned in the first chapter. Using the results from Chapter 5, we express these Dirichlet series as sums containing only ideles.

1.2 Statement of results

In 1972, Shintani created the theory of zeta functions associated to the space of binary cubic forms in the paper [17], and used that theory to establish the basic analytic properties of two Dirichlet series

$$\xi_1(s) = \sum_{m=1}^{\infty} \frac{h_1(m) + \frac{1}{3}h_2(m)}{m^s} \qquad\qquad \xi_2(s) = \sum_{m=1}^{\infty} \frac{h(-m)}{m^s} \tag{1.1}$$

where $h(m)$ denotes the number of $\mathrm{SL}_2(\mathbb{Z})$-equivalence classes of integral binary cubic forms of discriminant m, and, for $m > 0$, $h_1(m)$ and $h_2(m)$ denote the numbers of classes of discriminant m with isotropy group of order 1 and 3, respectively. Shintani showed that these Dirichlet series converge absolutely for $\mathrm{Re}(s) > 1$, that they have meromorphic continuations to the entire s-plane, and that they satisfy a functional equation. In addition, he proved that they were holomorphic except for simple poles at $s = 1$ and $s = 5/6$, and he gave formulas for the residues of the Dirichlet series at these poles. With this information, Shintani was able to improve a theorem of Davenport [7] on the mean-value of class-numbers of integral binary cubic forms, by giving a more precise formula for the error term.

The functional equation that Shintani discovered actually expresses $\xi_1(1-s)$ and $\xi_2(1-s)$ as linear combinations of the dual Dirichlet series

$$\hat{\xi}_1(s) = \sum_{m=1}^{\infty} \frac{\hat{h}_1(m) + \frac{1}{3}\hat{h}_2(m)}{m^s} \qquad\qquad \hat{\xi}_2(s) = \sum_{m=1}^{\infty} \frac{\hat{h}(-m)}{m^s}$$

where the dual class-numbers are the numbers of $\mathrm{SL}_2(\mathbb{Z})$-equivalence classes of integral binary cubic forms

$$F_x(u,v) = x_1u^2 + x_2u^2v + x_3uv^2 + x_4v^3$$

where the middle coefficients x_2, x_3 are both divisible by 3. This turns out to be the natural dual lattice to the lattice of integral binary cubic forms.

Datskovsky and Wright introduced adelic terminology and notation into Shintani's work in the papers [19, 5, 6, 4], and generalized Shintani's results to the space of binary cubic forms over an arbitrary global field of characteristic different from 2 and 3. In [5], it was observed in Proposition 4.1 that Shintani's functional equation has a natural diagonalization. To state this diagonalization, we write

$$\xi_{\pm}(s) = \xi_1(s) \pm \frac{1}{\sqrt{3}}\,\xi_2(s) \qquad\qquad \hat{\xi}_{\pm}(s) = \hat{\xi}_1(s) \pm \frac{1}{\sqrt{3}}\,\hat{\xi}_2(s)$$

We define the associated gamma factors to be

$$r_{\pm}(s) = \frac{2^s\, 3^{3s}}{\pi^{2s}}\,\Gamma(s)\Gamma\left(\frac{s}{2} + \frac{1}{4} \mp \frac{1}{6}\right)\Gamma\left(\frac{s}{2} + \frac{1}{4} \mp \frac{1}{3}\right).$$

The diagonalized functional equations are then

$$r_{\pm}(1-s)\xi_{\pm}(1-s) = \pm 3\, r_{\pm}(s)\hat{\xi}_{\pm}(s)$$

for either choice of sign $\pm$.

In [14], Ohno observed that these diagonalized functional equations would be especially symmetric if there were an identity between the original Dirichlet series and the dual Dirichlet series. This suggested an identity between class-numbers for the lattice of integral forms and class-numbers for its dual lattice. By extensive computation of class-numbers, Ohno verified numerically the conjecture that for the original Shintani series (1.1)

$$\hat{\xi}_1(s) = 3^{-3s}\xi_2(s) \qquad\qquad \hat{\xi}_2(s) = 3^{1-3s}\xi_1(s)$$

These identities imply that

$$\hat{\xi}_{\pm}(s) = \pm 3^{\frac{1}{2}-3s}\,\xi_{\pm}(s).$$

With that identity, the diagonalized functional equations become

$$\varepsilon_{\pm}(1-s)\xi_{\pm}(1-s) = \varepsilon_{\pm}(s)\xi_{\pm}(s)$$

where

$$\varepsilon_{\pm}(s) = \frac{2^s\, 3^{\frac{3}{2}s}}{\pi^{2s}}\,\Gamma(s)\Gamma\left(\frac{s}{2}+\frac{1}{4}\mp\frac{1}{6}\right)\Gamma\left(\frac{s}{2}+\frac{1}{4}\mp\frac{1}{3}\right).$$

Ohno also proved by means of Shintani's functional equation that his two conjectured identities above are logically equivalent. Once Ohno's conjecture was stated, work on the truth of it was swift, and Nakagawa proved the conjecture in 1998 in [13]. Nakagawa's proof is based partly on an idea of Scholz [16] relating the 3-class group of quadratic fields $\mathbb{Q}(\sqrt{d})$ to the 3-class group of $\mathbb{Q}(\sqrt{-3d})$.

The research presented here is concerned with the generalization of Ohno's conjecture and Nakagawa's theorem to Shintani Dirichlet series for the space of binary cubic forms over a number field k. Since the ring of integers $\mathfrak{o}$ of the number field k need not have class number 1, the direct generalization of Shintani's Dirichlet series to number fields is more easily expressed in terms of field extensions k'/k of degree at most 3 than it is in terms of class-numbers of binary cubic forms. To explain this, we first present an identity proved in [5] for Shintani's original series. Let $\zeta(s)$ denote Riemann's zeta function, and $\zeta_k(s)$ the Dedekind zeta function of the number field k. Let d_k denote the absolute value of the discriminant of $k/\mathbb{Q}$. For a field k of degree at most 3 over $\mathbb{Q}$, we define $o(k) = 6$ if $k = \mathbb{Q}$ and $o(k) = [k:\mathbb{Q}]$ otherwise. We also define

$$R_k(s) = \begin{cases} \zeta(s)^3 & \text{if } k = \mathbb{Q}, \\ \zeta(s)\zeta_k(s) & \text{if } [k:\mathbb{Q}] = 2, \\ \zeta_k(s) & \text{if } [k:\mathbb{Q}] = 3. \end{cases}$$

Then Shintani's Dirichlet series are proved in [5] to be equal to

$$\xi_1(s) = 2\,\zeta(4s)\zeta(6s-1) \sum_{k \text{ tot. real}} \frac{d_k^{-s}}{o(k)} \frac{R_k(2s)}{R_k(4s)}$$

$$\xi_2(s) = 2\,\zeta(4s)\zeta(6s-1) \sum_{k \text{ complex}} \frac{d_k^{-s}}{o(k)} \frac{R_k(2s)}{R_k(4s)}$$

where the first series ranges over totally real $k/\mathbb{Q}$ of degree at most 3, and the second series ranges over $k/\mathbb{Q}$ with one complex infinite place and degree at most 3.

Nakagawa also provided a proof of this identity in [12].

We can directly use this terminology to state the generalization of Shintani's series to number fields k. First, the series range over the extensions k'/k of degree at most 3. Again we define $o(k') = 6$ if $k' = k$ and $o(k') = [k' : k]$ otherwise. Define $d_{k'/k}$ to be the absolute norm of the relative discriminant of the extension k'/k, and put

$$R_{k'}(s) = \begin{cases} \zeta_k(s)^3 & \text{if } k' = k, \\ \zeta_k(s)\zeta_{k'}(s) & \text{if } [k' : k] = 2, \\ \zeta_{k'}(s) & \text{if } [k' : k] = 3. \end{cases}$$

To state the generalized Shintani series over a number field k, it remains to define the notion of signature of an extension k'/k of degree less or equal to 3. For each real place v (or real embedding) of k, either the places of k' lying over v are either all real, in which case we say $\alpha_v(k'/k) = +$, or there is a unique complex place w lying over v, and then we say $\alpha_v(k'/k) = -$. This situation is unique to extensions of degree at most 3. We call the vector $\alpha(k'/k) = (\alpha_v(k'/k))_{v \text{ real}}$ the **signature** of k'/k. If r_1 is the number of real places of k, then there are 2^{r_1} possible signatures; we denote the set of such signatures by A. For each possible signature α of k, we denote the set of extensions k'/k of degree at most 3 which have signature α by $\mathscr{K}_\alpha$. At last, we may state the Shintani series for each signature $\alpha \in A$ as

$$\xi_\alpha(s) = \zeta_k(4s)\zeta_k(6s-1) \sum_{k' \in \mathscr{K}_\alpha} \frac{d_{k'/k}^{-s}}{o(k')} \frac{R_{k'}(2s)}{R_{k'}(4s)} \tag{1.2}$$

where the sum ranges over all extensions k' in $\mathscr{K}_\alpha$. (Note that the factor of 2 has disappeared; in [19, 5] forms are considered equivalent relative to the group GL_2 rather than SL_2, and that accounts for the factor of 2.) Datskovsky and Wright prove in [5] these series extend to meromorphic functions of s which are holomorphic everywhere except for simple poles at $s = 1$ and $s = 5/6$. That paper also provides precise expressions for the residues of these series at 1 and 5/6. Finally, that paper also proves a functional equation expressing $\xi_\alpha(1-s)$ as linear combinations of **dual Dirichlet series** $\hat{\xi}_\alpha(s)$.

By utilizing the adelic approach of [5], an expression nearly identical to equation (1.2) may be derived for the dual Dirichlet series $\hat{\xi}_\alpha(s)$. (Note: To be consistent with Nakagawa's theorem and Shintani's original notation, we modify the definition of $\hat{\xi}_\alpha(s)$ in Datskovky-Wright by a constant factor, to be explained in Chapter 2.) The only differences are due to new local factors at primes v lying over 3. These new local factors were calculated for $\mathbb{Q}$ by Nakagawa in Lemma 3.6, p. 121, of [13]. In order to give the formula for $\hat{\xi}_\alpha(s)$, we first need to define the **splitting type** of the prime v of k in an extension k' of degree at most 3:

Type (1): $k' \otimes_k k_v \cong k_v^3$, or k_v^2, or k_v,

Type (2u): $k' \otimes_k k_v \cong k_v \oplus F$, or F, where F/k_v is quadratic unramified,

Type (2r): $k' \otimes_k k_v \cong k_v \oplus F$, or F, where F/k_v is quadratic ramified,

Type (3u): $k' \otimes_k k_v \cong F$ where F/k_v is cubic unramified,

Type (3r): $k' \otimes_k k_v \cong F$ where F/k_v is cubic ramified.

Then we use the technique of Datskovsky-Wright to prove the following:

Theorem 1.1 *Let k be a number field of degree n in which 3 is unramified. Then the dual Dirichlet series for each signature α over k may be expressed as*

$$\hat{\xi}_\alpha(s) = \zeta_k(4s)\zeta_k(6s-1) \sum_{k' \in \mathscr{K}_\alpha} \frac{d_{k'/k}^{-s}}{o(k')} \frac{R_{k'}(2s)}{R_{k'}(4s)} \prod_{v|3} T_{k',v}(s) \tag{1.3}$$

where for each prime $v \mid 3$ we have $q_v = |3|_v^{-1}$, and we define the rational functions

$$T_{k',v}(s) = \begin{cases} q_v^{-4s}\,\dfrac{1+q_v^{1-2s}+2q_v^{1-4s}}{(1+q_v^{-2s})^2} & \text{if } v \text{ is of type (1) in } k', \\ q_v^{-4s}\,\dfrac{1+q_v^{1-2s}}{1+q_v^{-4s}} & \text{if } v \text{ is of type (2u) in } k', \\ q_v^{-2s}\,\dfrac{1+q_v^{1-4s}}{1+q_v^{-2s}} & \text{if } v \text{ is of type (2r) in } k', \\ q_v^{-4s}\,\dfrac{1+q_v^{1-2s}-q_v^{1-4s}}{1-q_v^{-2s}+q_v^{-4s}} & \text{if } v \text{ is of type (3u) in } k', \\ 1 & \text{if } v \text{ is of type (3r) in } k'. \end{cases}$$

The proof of this theorem appears in Chapter 3. We expect that a similar theorem holds when 3 is ramified in k, but we have not yet completed the necessary local calculations.

The goal of this thesis is to relate the collection of dual Dirichlet series $\hat{\xi}_\alpha(s)$ to the collection of original series $\xi_\alpha(s)$. Just as in Nakagawa's theorem, it will emerge that the proper generalization relates $\hat{\xi}_\alpha(s)$ to $\xi_{-\alpha}(s)$, where the signature $-\alpha$ is the negative of α in the sense that for each real place v of k we have $\alpha_v = +$ if and only if $(-\alpha)_v = -$. At the end of this introduction, we shall explain why this involution should occur. Datskovsky and Wright proved that the series $\xi_\alpha(s)$ and $\hat{\xi}_\alpha(s)$ have meromorphic continuations to the entire s-plane which are holomorphic except for simple poles at $s = 1$ and $s = 5/6$, and they gave explicit formulas for the residues of $\xi_\alpha(s)$ at 1 and 5/6. In order to test the relationship between $\hat{\xi}_\alpha(s)$ and $\xi_\alpha(s)$, in Chapter 4 we use the results of Datskovsky-Wright to calculate similar formulas for the residues of $\hat{\xi}_\alpha(s)$ at 1 and 5/6. Based on those formulas, we prove the following:

Theorem 1.2 *Let k be a number field of degree n with r_1 real places and r_2 complex places. For every signature α over k for which we have m real places v with $\alpha_v = +$, the identity*

$$\hat{\xi}_{-\alpha}(s) = 3^{r_2+m-3ns}\,\xi_\alpha(s)$$

is true at $s = 1$ and $5/6$. Moreover, this is the only expression of the form 3^{A+Bs} for which this theorem is true.

Notice that this theorem makes no condition on whether 3 is ramified or not in k. That led us to conjecture this generalization of the Ohno-Nakagawa identity:

Conjecture 1.1 (Generalized Ohno Conjecture) *Let k be a number field of degree n with r_1 real places and r_2 complex places. For every signature α over k for which we have m real places v with $\alpha_v = +$, we have:*

$$\hat{\xi}_{-\alpha}(s) = 3^{r_2+m-3ns}\,\xi_\alpha(s) \tag{1.4}$$

To see how this is consistent with Nakagawa's theorem, in that case we have $n = 1$, $r_1 = 1$ and $r_2 = 0$. Then in this conjecture ξ_+ corresponds to ξ_1 in Shintani's notation, and ξ_- corresponds to ξ_2. For signature $\alpha = +$, we have $m = 1$ and thus $\hat{\xi}_2(s) = 3^{1-3s}\,\xi_1(s)$, while for $\alpha = -$ we have $m = 0$ and $\hat{\xi}_1(s) = 3^{-3s}\,\xi_2(s)$. This is precisely Nakagawa's theorem.

The remainder of this thesis is dedicated to the reduction of this conjecture to manageable pieces which we hope may be proved by class field theory and the ideas of Nakagawa. First, by direct substitution of equations (1.2) and (1.3) into conjecture (1.4), we see that a number of factors directly cancel out, leaving only

$$\sum_{k' \in \mathscr{K}_{-\alpha}} \frac{d_{k'/k}^{-s}}{o(k')} \frac{R_{k'}(2s)}{R_{k'}(4s)} \prod_{v|3} T_{k',v}(s) = 3^{r_2+m-3ns} \sum_{k' \in \mathscr{K}_{\alpha}} \frac{d_{k'/k}^{-s}}{o(k')} \frac{R_{k'}(2s)}{R_{k'}(4s)} \tag{1.5}$$

In examining this proposed identity, we see that the Euler products multiplying $d_{k'/k}^{-s}$ are all of the form $\sum_{\mathfrak{a}} \frac{c_{\mathfrak{a}}}{\mathrm{N}(\mathfrak{a})^{2s}}$ where $\mathfrak{a}$ ranges over the integral ideals of k, $\mathrm{N}(\mathfrak{a})$ denotes the absolute norm of $\mathfrak{a}$, and the coefficients $c_{\mathfrak{a}}$ are ordinary rational numbers. The important point here is the factor 2 in the exponent. Since $3^{-3ns} = \mathrm{N}(3\mathfrak{o})^{-3s}$ for the ideal generated by 3 in k, the difference must be made up in the norms of the relative discriminants $d_{k'/k}$. To be more precise, consider an extension k_1'/k counted on the left side of the identity with a corresponding term equal to a constant multiple of

$\left(d_{k'_1/k}\,\mathrm{N}(\mathfrak{a}_1)^2\right)^{-s}$. This may be 'counteracted' by a term $\left(d_{k'_2/k}\,\mathrm{N}(3\mathfrak{o})^3\,\mathrm{N}(\mathfrak{a}_2)^2\right)^{-s}$ on the right side if and only if the ratio of norms of the relative discriminants $d_{k'_2/k}/d_{k'_1/k}$ is divisible by an odd power of $\mathrm{N}(3\mathfrak{o})$. This is one suggestion that the conjectured identity (1.5) may be split into a sequence of identities by restricting the fields k' included in the summations on each side.

That idea is also promoted by the one of the key ideas of Nakagawa's proof, which in turn was based on a theorem of Scholz [16]. For an extension k'/k, let L/k be its Galois closure. If k'/k has degree at most 3, then L/k contains a unique subextension $F = k(\sqrt{\delta})$ of degree at most 2, generated by the square root of the discriminant δ of any generating element of k' over k. We call F the **resolvent field** of k'/k. Note that $F = k$ if k'/k is trivial or a cyclic cubic extension. We define the **dual resolvent field** to F to be $\hat{F} = k(\sqrt{-3\delta})$. Thus, F and $\hat{F}$ are subfields of the extension $F(\sqrt{-3})$ obtained by adjoining the cube roots of unity to F. The relative discriminants of F/k and $\hat{F}/k$ are, up to multiplication by the square of an ideal of k, equal to δ and -3δ, respectively. It will turn out that the odd power of 3 in the identity (1.5) will be accounted by restricting the terms on the left to those extensions k'/k with resolvent field equal to $\hat{F}$ and the terms on the right to those k'/k with resolvent field F. This motivates the following refined conjecture:

Conjecture 1.2 (Resolvent Field Identity) *Let k be a number field of degree n with r_1 real places and r_2 complex places. Let δ be a nonzero element of k, and define $\mathscr{C}(\delta)$ to be the set of extensions k'/k of degree at most 3 whose resolvent field is $k(\sqrt{\delta})$. Let m be the number of real embeddings of k for which the image of δ is a positive number. Then, using the notation for Shintani's series which we have previously developed, we have the identity*

$$\sum_{k'\in\mathscr{C}(-3\delta)} \frac{d_{k'/k}^{-s}}{o(k')}\frac{R_{k'}(2s)}{R_{k'}(4s)} \prod_{v|3} T_{k',v}(s) = 3^{r_2+m-3ns} \sum_{k'\in\mathscr{C}(\delta)} \frac{d_{k'/k}^{-s}}{o(k')}\frac{R_{k'}(2s)}{R_{k'}(4s)} \tag{1.6}$$

Since the generalized Ohno conjecture is just the sum of all these resolvent field identities for δ ranging over the nonzero elements of k modulo multiplication by squares (we will denote this set by $k^\times/k^2$), we may at least state:

Theorem 1.3 *If the resolvent field identity is true for all nonzero δ in k, then the generalized Ohno conjecture* (1.4) *is true for k.*

For the one known case $k = \mathbb{Q}$, we shall explain in detail why the converse is true, thus proving:

Theorem 1.4 *For $k = \mathbb{Q}$, the resolvent field identity is true for all nonzero δ in $\mathbb{Q}$.*

The last part of this thesis is devoted to the analysis necessary to eventually complete the proof of the resolvent field identity in terms of class field theory. For nonzero $\delta \in k$, the fields $k' \in \mathscr{C}(\delta)$ have Galois closure L containing $F = k(\sqrt{\delta})$. If $[k' : k] \leq 2$, then $F = k'$. If $[k' : k] = 3$, then L is a cyclic cubic extension of F. If k'/k is cyclic cubic, then $L = k'$ and $F = k$. If k'/k is noncyclic cubic, then L is an S_3-extension of k containing F, and L contains three cubic extensions of k all conjugate to k'. Class field theory implies the abelian Galois extensions of a number field F correspond to the open subgroups (and hence characters) of the idele class group $\mathcal{J}_F$ of F. Basic notation and statements of class field theory will be provided in a later section. In this case, the cyclic cubic extensions L/F correspond bijectively to the open subgroups of index 3 in the idele class group $\mathcal{J}_F$. These open subgroups in turn correspond in a one-to-two way to nontrivial complex characters χ of $\mathcal{J}_F$ satisfying $\chi^3 = 1$. These observations will allow us to rewrite the resolvent field series in Theorem 1.3 as a sum over idele class characters of order dividing 3.

To state this precisely, for any resolvent field F let $\mathcal{X}(F)$ denote the group of characters χ of the idele class group $\mathcal{J}_F$ of F such that $\chi^3 = 1$ and also $\chi \circ \mathrm{N}_{F/k} = 1$ if $[F : k] = 2$, where $\mathrm{N}_{F/k}$ denotes the relative norm. If $F = k$, the latter condition is omitted. The conductor of such a character is an integral ideal $\mathfrak{f}_\chi$ in F. Let $\mathrm{N}(\mathfrak{f}_\chi)$ denote the absolute norm of the conductor of χ. Then for the extension k'/k

corresponding to the kernel of χ, we have $d_{k'/k} = d_{F/k}\,\mathrm{N}(\mathfrak{f}_\chi)$. The idele class character χ induces a character on all prime ideals $\mathfrak{P}$ of F where $\mathfrak{P} \nmid \mathfrak{f}_\chi$. By abuse of notation, we will write this character's value as $\chi(\mathfrak{P})$. For prime ideal factors of the conductor, we will extend this definition by setting $\chi(\mathfrak{P}) = 0$. Then we will prove:

Theorem 1.5 *Let k be a number field, δ be a nonzero element of k, and $F = k(\sqrt{\delta})$. Then, using the notation we have previously developed for the collection $\mathscr{C}(\delta)$ of extensions k'/k and for the character group $\mathcal{X}(F)$, we have the identity*

$$\sum_{k' \in \mathscr{C}(\delta)} \frac{d_{k'/k}^{-s}}{o(k')} \frac{R_{k'}(2s)}{R_{k'}(4s)} = \frac{d_{F/k}^{-s}}{o(F)} \sum_{\chi \in \mathcal{X}(F)} \mathrm{N}(\mathfrak{f}_\chi)^{-s} \prod_{\mathfrak{P}} \left(1 + \chi(\mathfrak{P})\ \mathrm{N}(\mathfrak{P})^{-2s}\right) \tag{1.7}$$

where $o(F) = 3$ if $F = k$ and 1 otherwise, and the product is taken over all prime ideals of F.

We also will produce a similar expression for the dual discriminant field series as a sum over cubic characters of the idele class group $\mathcal{J}_{\hat{F}}$, although this is complicated by the presence of the Euler factors $T_{k',v}$ for places $v|3$. The cubic characters of $\mathcal{J}_F$ and $\mathcal{J}_{\hat{F}}$ are related to the cubic characters of the compositum $F\hat{F}$. Using the fact that the compositum $F\hat{F}$ contains the cube roots of unity, Scholz was able to use this relationship to deduce a relationship between the three-class-numbers of F and $\hat{F}$ in [16]. The completion of our project and the proof of the generalized Ohno conjecture at least for fields k where 3 is unramified would follow from refining Scholz' ideas to establish the resolvent field series identity for all nonzero δ in k. Our future research will be directed toward providing this proof.

CHAPTER 2

Basic notation and review of zeta functions of binary cubic forms

2.1 Notation for number fields and local fields

Our notation for number fields, local fields, adele rings and idele rings, etc., will be fairly consistent with that presented in [19, 5, 6] and somewhat with Weil [18]. Let k be a number field of degree n over the rational number field $\mathbb{Q}$. For any place v of k, we use k_v to denote the completion of k at the place v. Let r_1 and r_2 be the numbers of real and complex, respectively, places of k, i.e. places such that $k_v \cong \mathbb{R}$ and $\mathbb{C}$, respectively. Then $r_1 + 2r_2 = n$. These together form the set of infinite places of k, and we shall write $v \mid \infty$ for these places.

For all local fields k_v, we normalize the absolute value $|\cdot|_v$ to be the modulus of any additive Haar measure on k_v. This means that $\mu(aU) = |a|_v\mu(U)$ for any open subset U of k_v, any Haar measure μ of k_v, and any nonzero element a of k_v. For real places v, this means $|x|_v$ is the customary absolute value on $\mathbb{R}$, while for complex places v this means $|x|_v$ is the square of the usual absolute value on $\mathbb{C}$.

For real places v, we choose the Haar measure d_vx on k_v such that $\int_0^1 d_vx = 1$, i.e. the usual Lebesgue measure on $\mathbb{R}$. For complex places v, we choose d_vx so that the measure of the unit circle $\{x \in \mathbb{C} : |x|_v \leq 1\}$ is 2π. That means d_vx is twice the usual Lebesgue measure on $\mathbb{C}$; the reason for this choice is that it is more convenient for d_vx to represent the differential form $|dx \wedge d\overline{x}|$ in integration formulas.

For any finite place v of k, we write $v \nmid \infty$, and we denote the maximal compact subring of k_v by o_v. The unique prime ideal of o_v is denoted p_v, and we choose a generator of the principal ideal p_v and name it π_v, called a uniformizer of k_v. The modulus q_v of k_v is the order of the finite field o_v/p_v. The additive Haar measure $d_v x$ on k_v is normalized so that $\int_{o_v} d_v x = 1$. The absolute value $|a|_v$ on k_v is normalized so that $d_v(ax) = |a|_v d_v x$, which implies that $|\pi_v|_v = q_v^{-1}$.

If we work with a nonarchimedean local field K without reference to a global field k, we shall denote the maximal compact subring by O, the unique prime ideal by P, a uniformizer by π, the modulus by q, the normalized Haar measure by dx, and the normalized absolute value by $|\cdot|$.

For any ring R (always commutative with identity), we denote the subgroup of invertible elements by $R^\times$. Hence, for a field K, $K^\times$ denotes the nonzero elements of K. Thus, the unit subgroup of o_v is denoted by $o_v^\times$, and this is the same as the subgroup of elements x satisfying $|x|_v = 1$.

For a number field k, we denote the ring of adeles by $\mathbb{A} = \mathbb{A}_k = \prod'_v k_v$ and the group of ideles by $\mathbb{A}_k^\times = \prod'_v k_v^\times$, where these are restricted direct products in the usual sense. The ideles correspond to units in the ring of adeles, but the restricted product topology is not the same as the subspace topology. The idele norm $|a|_{\mathbb{A}}$ of an idele $a \in \mathbb{A}_k^\times$ is the modulus of multiplication by a relative to any Haar measure on the adeles $\mathbb{A}_k$. This means $|a|_{\mathbb{A}} = \prod_v |a_v|_v$, where a_v denotes the component of a at the place v. For all $x \in k^\times$ embedded along the diagonal in $\mathbb{A}_k^\times$, we have the product formula $|x|_{\mathbb{A}} = 1$.

We choose for the Haar measure on the additive group of adeles $\mathbb{A}_k$ the restricted tensor product measure $d_{\mathbb{A}} x = \bigotimes_v d_v x_v$, where $x = (x_v)_v$. The number field k embeds along the diagonal as a discrete subgroup of $\mathbb{A}$ such that $\mathbb{A}/k$ is compact. With this choice of measure, the induced measure of the quotient is $\int_{\mathbb{A}/k} d_{\mathbb{A}} x = d_k^{1/2}$, in terms of the absolute value of the discriminant of k (see Prop. V.4.7 in [18]).

On the multiplicative group $k_v^\times$ of nonzero elements, we choose Haar measures $d_v^\times x = \dfrac{d_v x}{|x|_v}$ if v is an infinite place, and for finite places we choose $d_v^\times x$ so that the unit group $o_v^\times$ has measure 1. On the ideles $\mathbb{A}_k^\times$, we choose the Haar measure

$d_{\mathbb{A}}^{\times}x = \bigotimes_v d_v^{\times}x_v$. By Prop. V.4.9 of [18], the measure of the set $C(m)$, the image in $\mathbb{A}_k^{\times}/k^{\times}$ of all ideles x satisfying $1 \leq |x|_{\mathbb{A}} \leq m$, is $\rho_k \log m$ with

$$\rho_k = \frac{2^{r_1}(2\pi)^{r_2} h_k R_k}{e_k},$$

where h_k is the class number of k, R_k is the regulator of k, and e_k is the number of roots of unity in k.

Let $\mathbb{A}^1$ denote the subgroup of ideles x with idele norm $|x|_{\mathbb{A}} = 1$. Then $k^{\times}$ is a subgroup of $\mathbb{A}^1$, and the quotient group $\mathbb{A}^1/k^{\times}$ is compact. (See Theorem IV.4.6 in [18].) There is an embedding of the group of positive real numbers $\mathbb{R}_+$ into the ideles $\mathbb{A}^{\times}$, which we will denote $z(t)$, satisfying $|z(t)|_{\mathbb{A}} = t$ for all $t \in \mathbb{R}_+$. One such embedding is defined as the idele $z(t) = (z_v(t))_v$ where $z_v(t) = 1$ for all finite places v and $z_v(t) = t^{1/n}$ for all infinite places v, with $n = [k : \mathbb{Q}]$. Then we may decompose the Haar measure for $\mathbb{A}^{\times}$ in the following way

$$\int_{\mathbb{A}^{\times}} \phi(x)\, d_{\mathbb{A}}^{\times}x = \int_0^{\infty} \left(\int_{\mathbb{A}^1} \phi(z(t)x)\, d_{\mathbb{A}}^1 x \right) \frac{dt}{t},$$

for any integrable function ϕ. Then the measure of $\mathbb{A}^1/k^{\times}$ induced by $d_{\mathbb{A}}^1 x$ is ρ_k.

Finally, we introduce our notation for the Dedekind zeta function $\zeta_k(s)$. Let $\mathfrak{o} = \mathfrak{o}_k$ denote the ring of integers of k, and for each integral ideal $\mathfrak{a}$ of $\mathfrak{o}$ let $\mathrm{N}(\mathfrak{a})$ denote the absolute norm of $\mathfrak{a}$, i.e. the cardinality of the quotient ring $\mathfrak{o}/\mathfrak{a}$. The Dedekind zeta function is

$$\zeta_k(s) = \sum_{\mathfrak{a}} \mathrm{N}(\mathfrak{a})^{-s} = \prod_{\mathfrak{p}} \left(1 - \mathrm{N}(\mathfrak{p})^{-s}\right)^{-1}$$

where the sum extends over all integral ideals $\mathfrak{a}$ and the product extends over all prime ideals $\mathfrak{p}$. Equivalently, this may be written as a product over all finite places of v as

$$\zeta_k(s) = \prod_{v \nmid \infty} (1 - q_v^{-s})^{-1}.$$

This zeta function converges locally uniformly for $\mathrm{Re}(s) > 1$ and has a meromorphic continuation to the entire s-plane which is holomorphic except for a simple pole at

$s = 1$ with residue

$$\operatorname*{Res}_{s=1} \zeta_k(s) = \frac{\rho_k}{d_k^{1/2}}.$$

These facts and more may be found in VII.6 of [18].

2.2 Binary cubic forms

This notation is based on [17, 19, 5]. A binary cubic form is an expression $F_x(u,v) = x_1u^3 + x_2u^2v + x_3uv^2 + x_4v^3$ where x_j, $1 \leq j \leq 4$ are the coefficients. We will generally think of a four-dimensional vector $x = (x_1, x_2, x_3, x_4)$ as a binary cubic form. The module of binary cubic forms with coefficients in a ring R is denoted by V_R. There is a natural representation of the group $G = \mathrm{GL}_2$ on V given by

$$F_{g\cdot x}(u,v) = \frac{1}{\det g}\, F_x\left((u,v)\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$$

for $x \in V$ and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$. We say two forms x, y are G-equivalent if $y = g \cdot x$ for some $g \in G$.

The discriminant of the form x is a homogeneous polynomial $P(x)$ of degree 4 (see p. 35 in [5]). This polynomial satisfies $P(g \cdot x) = (\det g)^2 P(x)$ for $g \in G$ and $x \in V$. A binary cubic form x is defined to be singular if and only if $P(x) = 0$. We denote the hypersurface of singular forms in V by S, and the subset of nonsingular forms by V'. Both subsets are G-invariant.

Just as in Shintani [17], we define the bilinear form

$$[x,y] = x_1y_4 - \frac{1}{3}x_2y_3 + \frac{1}{3}x_3y_2 - x_4y_1$$

on V. For the involution $g^\iota = (\det g)^{-1}g$ on G, this form satisfies $[g \cdot x, g^\iota \cdot y] = [x,y]$ for all $x, y \in V$ and $g \in G$.

We next turn to binary cubic forms over a field K. The splitting field $K(x)$ of a binary cubic form $x \in V_K$ is the smallest extension of K (in a given algebraic closure of K) for which the form factors into linear factors defined over $K(x)$. The splitting field is either K, a quadratic or cubic cyclic extension of K, or an S_3-extension of K. The key connection between the space of binary cubic forms and field extensions of K is the following fact:

Proposition 2.1 *Two nonsingular binary cubic forms $x, y \in V_K$ are G_K-equivalent if and only if their splitting fields are the same $K(x) = K(y)$.*

When we wish to refer to the points of G and V defined over a ring R, we will write G_R and V_R, for example, G_k, V_k, G_{k_v}, V_{k_v}, $G_{\mathbb{A}}$, $V_{\mathbb{A}}$, etc. We shall choose the Haar measure $d_v x$ on V_{k_v} to be simply the product of the four coordinate measures $d_v x_j$ and similarly for the measure $d_{\mathbb{A}} x$ on $V_{\mathbb{A}}$.

We should point out that *this choice is different* from that of Datskovsky-Wright [5], where the self-dual measure was chosen relative to a standard bi-invariant for on $V_{\mathbb{A}}$. We are changing this choice so that the definition of the dual Dirichlet series will be in agreement with the papers of Shintani, Ohno, and Nakagawa in the case of $k = \mathbb{Q}$.

For a local field K, we choose the invariant measure dg on $g \in G_K$ in the same manner as in [5]. To review, we define U_K to be the maximal compact subgroups the orthogonal group $O(2)$ if $K \cong \mathbb{R}$, the unitary group $U(2)$ if $K \cong \mathbb{C}$, and the group G_O if K is nonarchimedean with maximal compact subring O. We define B_K to be the Borel subgroup of lower triangular matrices in G_K. Then by the Iwasawa decomposition $G_K = U_K B_K$. We choose the invariant measure du on $u \in U_K$ so that U_K has measure 1. We define a right-invariant measure db on $b \in B_k$ satisfying

$$\int_{B_K} \phi(b)\, db = \int_{K^\times} d^\times t_1 \int_{K^\times} d^\times t_2 \int_K dc\, \phi(n(c)a(t_1, t_2))$$

where the Haar measures on K and $K^\times$ are as previously defined and we use the notation $n(c) = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$ and $a(t_1, t_2) = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}$. Here ϕ is any integrable function on G_K. Then the measure dg on G_K is defined by

$$\int_{G_K} \phi(g)\, dg = \int_{U_K} du \int_{B_K} db\, \phi(ub).$$

In the case of a nonarchimedean field K with maximal compact subring O, if ϕ is the characteristic function of the maximal compact subgroup G_O, then

$$\begin{aligned}\int_{G_K} \phi(g)\, dg &= \int_{U_K} du \int_{B_O} db \\ &= \int_{B_O} db \\ &= \int_{O^\times} d^\times t_1 \int_{O^\times} d^\times t_2 \int_O dc = 1,\end{aligned}$$

by our choice of Haar measure on K and $K^\times$. This confirms this measure satisfies the condition that G_O has measure 1.

On the adelizations $V_{\mathbb{A}}$ and $G_{\mathbb{A}}$, we take the invariant measures to be the restricted product measures $d_{\mathbb{A}}x = \bigotimes_v d_v x_v$ and $d_{\mathbb{A}}g = \bigotimes_v d_v g_v$. Then V_k is a discrete cocompact subgroup of $V_{\mathbb{A}}$ such that $\int_{V_{\mathbb{A}}/V_k} d_{\mathbb{A}}x = d_k^2$. On the quotient group $G_{\mathbb{A}}/G_k$, the set $F(m)$ of all points g with $1 \le |\det g|_{\mathbb{A}} \le m$ has measure $\gamma_k \log m$, where

$$\gamma_k = \left(\frac{2}{\pi}\right)^{r_1} \frac{h_k R_k}{e_k}\, d_k^{3/2}\, \zeta_k(2).$$

(This constant comes from the volume calculation after Prop. 6.3, p. 528, in [19], and the volume normalization on p. 66 in [5].)

Let $G^1_{\mathbb{A}}$ denote the subgroup of all $g \in G_{\mathbb{A}}$ with $|\det g|_{\mathbb{A}} = 1$. Then G_k is a discrete subgroup of $G_{\mathbb{A}}$ with quotient $G^1_{\mathbb{A}}/G_k$ of finite invariant volume. Define an embedding $w(t)$ of $t \in \mathbb{R}_+$ into $G_{\mathbb{A}}$ by $w(t) = a(z(\sqrt{t}), z(\sqrt{t}))$, using the embedding $z : \mathbb{R}_+ \to \mathbb{A}^\times$ defined in the previous section. Then we have $|\det w(t)|_{\mathbb{A}} = t$, and we may decompose the invariant volume on $G_{\mathbb{A}}$ as follow

$$\int_{G_{\mathbb{A}}/G_k} \phi(g)\, d_{\mathbb{A}}g = \int_0^\infty \left(\int_{G^1_{\mathbb{A}}/G_k} \phi(w(t)g)\, d^1_{\mathbb{A}}g \right) \frac{dt}{t},$$

for integrable functions $\phi(g)$ on $G_{\mathbb{A}}/G_k$. With this definition, we see that the measure of $G^1_{\mathbb{A}}/G_k$ with respect to $d^1_{\mathbb{A}}g$ is γ_k.

Finally, we shall describe the orbits of nonsingular binary cubic forms over a local field k_v. By Proposition 2.1, these correspond to the possible splitting fields $k_v(x)$ over k_v. For a complex place v, there is only one such splitting field, namely, $\mathbb{C}$, and thus there is only one nonsingular orbit. For a real place v, the splitting field may be $\mathbb{R}$ or $\mathbb{C}$ corresponding to whether the discriminant $P(x)$ is positive or negative in k_v. For a finite place v, there are finitely many possible splitting fields $k_v(x)$ which we group into five basic types:

Type (1): $k_v(x) = k_v$,

Type (2u): $k_v(x)/k_v$ is quadratic unramified,

Type (2r): $k_v(x)/k_v$ is quadratic ramified,

Type (3u): $k_v(x)/k_v$ is cubic unramified,

Type (3r): $k_v(x)/k_v$ is the Galois closure of a ramified cubic extension K_v/k_v.

In the first four types, we abbreviate $K_v = k_v(x)$, while in the last type K_v is a possibly non-Galois cubic extension whose Galois closure is $k_v(x)$. There is only one orbit of each type (1), (2u) and (3u), while there may be more than one orbits of types (2r) and (3r). We shall adopt the description of these given on pp. 35-36 in [5]. In particular, for each orbit α we shall choose orbital representatives $x_\alpha \in V_{k_v}$ as described in that paper. Those are arranged so that $P(x_\alpha)$ is a relative discriminant $\Delta(K_v/k_v)$ of K_v/k_v. Occasionally, we shall use A_v to denote the set of orbits $G_{k_v}\backslash V'_{k_v}$.

2.3 Zeta functions of binary cubic forms and Dirichlet series

The adelic version of Shintani's zeta function associated to the space (G, V) of binary cubic forms is

$$Z(\omega, \Phi) = \int_{G_{\mathbb{A}}/G_k} \omega(\det g) \sum_{x \in V'_k} \Phi(g \cdot x)\, d_{\mathbb{A}} g,$$

where ω is a quasicharacter on $\mathbb{A}_k^\times$ which is trivial on $k^\times$ and $\Phi(x)$ is a Schwartz-Bruhat function on the adelization $V_{\mathbb{A}}$. Here G_k is a discrete subgroup embedded along the diagonal in $G_{\mathbb{A}}$, and for every nonsingular form $x \in V_k'$ the stabilizer subgroup $G_{k,x}$ is a finite subgroup of order 1, 2, 3 or 6, depending on the splitting field $k(x)$ as described in [5]. Here we shall simplify the notation because we will not be referring to non-principal quasicharacters. We shall put $\omega(x) = |x|_{\mathbb{A}}^{2s}$, and define

$$Z(s,\Phi) = \int_{G_{\mathbb{A}}/G_k} |\det g|_{\mathbb{A}}^{2s} \sum_{x \in V_k'} \Phi(g \cdot x)\, d_{\mathbb{A}} g,$$

which is proved in [19] to be absolutely and locally uniformly convergent for $\mathrm{Re}(s) > 1$, and to have a meromorphic continuation to the entire s-plane which is holomorphic except for simple poles at $s = 1$ and $s = 5/6$.

By rewriting the inner summation over G_k-equivalence classes and then exchanging summation and integration, we obtain

$$Z(s,\Phi) = \sum_{x \in G_k \backslash V_k'} \frac{1}{|G_{k,x}|} \int_{G_{\mathbb{A}}} |\det g|_{\mathbb{A}}^{2s}\, \Phi(g \cdot x)\, d_{\mathbb{A}} g,$$

where x ranges over representatives of the G_k-equivalence classes of nonsingular forms in V_k'. Assuming the Schwartz-Bruhat function $\Phi = \otimes_v \Phi_v$ is of product form, the integral above has an Euler product

$$\int_{G_{\mathbb{A}}} |\det g|_{\mathbb{A}}^{2s}\, \Phi(g \cdot x)\, d_{\mathbb{A}} g = \prod_v \int_{G_{k_v}} |\det g_v|_v^{2s}\, \Phi_v(g_v \cdot x)\, d_v g_v.$$

For each place v, the form x belongs to one α_v of finitely many orbits in V_{k_v}' over k_v. Thus, there is an element $g_{x,v} \in G_{k_v}$ such that $x_{\alpha_v} = g_{x,v} \cdot x$, using the standard orbital representatives mentioned in Section 2.2 (see page 67 of [5]). Note that $P(x_{\alpha_v}) = P(g_{x,v} \cdot x) = (\det g_{x,v})^2\, P(x)$. Then the Euler factor may be rewritten by substitution as

$$\begin{aligned}
\int_{G_{k_v}} |\det g_v|_v^{2s}\, \Phi_v(g_v \cdot x)\, d_v g_v &= \int_{G_{k_v}} |\det g_v|_v^{2s}\, \Phi_v(g_v g_{x,v}^{-1} \cdot x_{\alpha_v})\, d_v g_v \\
&= |\det g_{x,v}|_v^{2s} \int_{G_{k_v}} |\det g_v|_v^{2s}\, \Phi_v(g_v \cdot x_{\alpha_v})\, d_v g_v \\
&= \frac{|P(x_{\alpha_v})|_v^s}{|P(x)|_v^s} \int_{G_{k_v}} |\det g_v|_v^{2s}\, \Phi_v(g_v \cdot x_{\alpha_v})\, d_v g_v
\end{aligned}$$

Note that for $x \in V'_k$ we have $|P(x)|_{\mathbb{A}} = \prod_v |P(x)|_v = 1$ by the product formula. Also, if k'/k is an extension of degree at most 3 whose Galois closure is the splitting field $k(x)$ of x, then $P(x_{\alpha_v})$ is the v-adic component of the adelic relative discriminant $\Delta(k'/k)$ as defined in [8], and so

$$\prod_v |P(x_{\alpha_v})|_v = d_{k'/k}^{-1}$$

in terms of the absolute norm of the relative discriminant of k'/k. We introduce the following notation for the local zeta functions of the space of binary cubic forms

$$Z_{\alpha_v}(s, \Phi_v) = \int_{G_{k_v}} |\det g_v|_v^{2s}\, \Phi_v(g_v \cdot x_{\alpha_v})\, d_v g_v.$$

Then returning to the Euler product, we have

$$\begin{aligned} \int_{G_{\mathbb{A}}} |\det g|_{\mathbb{A}}^{2s}\, \Phi(g \cdot x)\, d_{\mathbb{A}} g &= \prod_v \int_{G_{k_v}} |\det g_v|_v^{2s}\, \Phi_v(g_v \cdot x)\, d_v g_v \\ &= d_{k'/k}^{-s} \prod_v Z_{\alpha_v}(s, \Phi_v). \end{aligned}$$

In this notation, α_v denotes the local orbit corresponding to the form x or its corresponding extension k'/k of degree at most 3.

At this point, we are ready to convert the zeta function from a sum over orbits $x \in G_k \backslash V'_k$ to a sum over extensions k'/k of degree at most 3. We shall put $o(k'/k) = |G_{k,x}|$ in all cases where $k(x)/k$ is a Galois extension of degree at most 3. When the splitting field $k(x)/k$ is an S_3-extension, it is the Galois closure of any of three conjugate cubic subextensions k'/k. We shall allow our series to include the same term for each of those cubic subextensions, and to compensate we set $o(k') = o(k'/k) = 3$ instead of $|G_{k,x}| = 1$. With these conventions, we now have the following series expansion of the adelic zeta function

$$Z(s, \Phi) = \sum_{k'/k} \frac{d_{k'/k}^{-s}}{o(k')} \prod_v Z_{\alpha_v}(s, \Phi_v). \tag{2.1}$$

Again the local orbits α_v depend on the local type of k'/k over v.

The Euler products in (2.1) include factors for the infinite places v. Our next step toward producing Shintani's Dirichlet series is to factor out the local zeta functions for the infinite places. This is where the signatures come into play. If v is a complex place, there is only one nonsingular orbit. If v is a real place, there are two nonsingular orbits: one for forms of positive discriminant and one for forms of negative discriminant. We shall denote these choices by $\alpha_v = +$ and $-$ just as in the definition of signature in Chapter 1.2. We write the signature as a vector $\alpha = (\alpha_v)_{v|\infty}$ of these choices for all infinite places, and we denote the set of signatures α by A (or A_∞ if we wish to emphasize that this is a choice of orbits at the infinite places). For $\alpha \in A_\infty$, we define

$$Z_\alpha(s, \Phi_\infty) = \prod_{v|\infty} Z_{\alpha_v}(s, \Phi_v).$$

where Φ_∞ denotes the part of the Schwartz-Bruhat function Φ corresponding to the infinite places of k. Thus, Φ_∞ is a rapidly decreasing C_∞-function on the real vector space $\prod_{v|\infty} k_v$ of dimension n. As in Chapter 1.2, for each signature α, we define $\mathscr{K}_\alpha$ to be the collection of all extensions k'/k of degree at most 3 which have signature α. Then at last we have the decomposition of the adelic zeta function as follows

$$Z(s, \Phi) = \sum_{\alpha \in A_\infty} Z_\alpha(s, \Phi_\infty)\ \xi_\alpha(s, \Phi_0) \tag{2.2}$$

with

$$\xi_\alpha(s, \Phi_0) = \sum_{k' \in \mathscr{K}_\alpha} \frac{d_{k'/k}^{-s}}{o(k')} \prod_{v \nmid \infty} Z_{\alpha_v}(s, \Phi_v) \tag{2.3}$$

where $\Phi_0 = \otimes_{v\nmid\infty} \Phi_v$. Again, take note that the orbit α_v for each finite place is determined by the extension k'/k.

For each finite place v, the Schwartz-Bruhat function Φ_v is a locally constant function of compact support on k_v. Thus, $\xi_\alpha(s, \Phi_0)$ does turn out to be essentially a Dirichlet series. To obtain Shintani's series in particular, we make a standard choice of these Schwartz-Bruhat functions at finite places, namely, $\Phi_v = \Phi_{0,v}$, the characteristic function of the submodule V_{o_v} of all binary cubic forms with coefficients in o_v, the maximal compact subring of k_v. With this choice, in the case of $k = \mathbb{Q}$ (or any field k of class number 1) the sum over $V'_{\mathbb{Q}}$ in the definition of the zeta function $Z(s, \Phi)$ reduces to a sum over $V'_{\mathbb{Z}}$, the set of nonsingular integral binary cubic forms, and we have the identity

$$Z(s, \Phi) = Z_\infty(s, \Phi_\infty) = \int_{G_{\mathbb{R}}/G_{\mathbb{Z}}} |\det g|^s \sum_{x \in V'_{\mathbb{Z}}} \Phi_\infty(g \cdot x)\, d_{\mathbb{R}} g,$$

exactly the zeta function defined by Shintani in [17]. Thus, the natural generalization of Shintani's Dirichlet series is

$$\xi_\alpha(s) = \sum_{k' \in \mathscr{K}_\alpha} \frac{d_{k'/k}^{-s}}{o(k')} \prod_{v \nmid \infty} Z_{\alpha_v}(s, \Phi_{0,v})$$

Datskovsky-Wright calculated these local integrals to be

$$Z_{\alpha_v}(s, \Phi_{0,v}) = (1 - q_v^{1-6s})^{-1}(1 - q_v^{-2s})^{-1} \cdot \begin{cases} (1 + q_v^{-2s})^2, & \text{if } \alpha_v \text{ is of type (1)}, \\ 1 + q_v^{-4s}, & \text{if } \alpha_v \text{ is of type (2u)}, \\ 1 + q_v^{-2s}, & \text{if } \alpha_v \text{ is of type (2r)}, \\ 1 - q_v^{-2s} + q_v^{-4s}, & \text{if } \alpha_v \text{ is of type (3u)}, \\ 1, & \text{if } \alpha_v \text{ is of type (3r)}, \end{cases} \tag{2.4}$$

where q_v denotes the module of the finite place v.

2.4 Fourier transforms and the dual Dirichlet series

The analytic properties of Shintani's Dirichlet series are derived by means of the Poisson Summation Formula and the use of Fourier transforms. We will establish our conventions for Fourier transform in this section and then review the functional equation for the adelic zeta function defined in the previous section. Then we will extract the dual Dirichlet series, which are the other components to the Ohno-Nakagawa identity.

We have to define standard nontrivial additive characters $\langle\cdot\rangle_v$ on the local fields k_v for all places v of k and the adelic additive character $\langle x\rangle = \prod_v \langle x_v\rangle_v$ on adeles $x = (x_v)_v \in \mathbb{A}_k$. First, for $\mathbb{Q}$, we choose standard additive characters on $\mathbb{Q}_\infty = \mathbb{R}$ and $\mathbb{Q}_p$ for any prime p as follows

$$\langle x\rangle_{\mathbb{R}} = \exp(2\pi i\, x)$$
$$\langle x\rangle_p = \exp(-2\pi i\, \{x\}_p)$$

where $\{x\}_p$ is defined as $\sum_{j=-m}^{-1} a_j\, p^j \in \mathbb{Q}$ in terms of the standard p-adic expansion $x = \sum_{j=-m}^{\infty} a_j\, p^j$ with coefficients $0 \le a_j \le p-1$. Note that all these characters are trivial on $\mathbb{Z}$. Also, if $x \in \mathbb{Q}$, then by the theorem of partial fractions we have $x - \sum_p \{x\}_p \in \mathbb{Z}$, where the sum extends over all prime numbers p. That proves that the adelic character $\langle x\rangle = \langle x\rangle_{\mathbb{A}} = \prod_v \langle x_v\rangle_v$ is trivial on $\mathbb{Q}$ embedded along the diagonal in $\mathbb{A}_{\mathbb{Q}}$.

To define the additive characters on extensions k_v, we use the trace from k_v to $\mathbb{R}$ or $\mathbb{Q}_p$. Thus, for infinite places v, $\langle x\rangle_v = \langle \mathrm{Tr}_{k_v/\mathbb{R}}(x)\rangle_{\mathbb{R}}$, and for finite places $v \mid p$ we have $\langle x\rangle_v = \langle \mathrm{Tr}_{k_v/\mathbb{Q}_p}(x)\rangle_p$. For $x \in k$, we have

$$\mathrm{Tr}_{k/\mathbb{Q}}(x) = \sum_{v\mid\infty} \mathrm{Tr}_{k_v/\mathbb{R}}(x)$$

and

$$\mathrm{Tr}_{k/\mathbb{Q}}(x) = \sum_{v\mid p} \mathrm{Tr}_{k_v/\mathbb{Q}_p}(x)$$

for all finite primes p. This implies that the adelic additive character $\langle x\rangle = \prod_v \langle x_v\rangle_v$ on $\mathbb{A}_k$ is trivial on k embedded along the diagonal in $\mathbb{A}_k$. As in Weil [18], Defn. VII.2.4, we choose a differental idele $\delta = (\delta_v)_v$ such that, for all $v \mid \infty$ we have $\delta_v = 1$ and, for all $v \nmid \infty$ the largest ideal contained in the kernel of $\langle \cdot \rangle_v$ is $\delta_v^{-1} o_v$. Then $\delta_v o_v$ is the different of $k_v/\mathbb{Q}_p$. By Prop. VII.2.6 in Weil [18], we have $|\delta|_{\mathbb{A}} = d_k^{-1}$.

Using the measure $d_v x$ defined in Section 2.1, we define the Fourier transform on Schwartz-Bruhat functions Φ on k_v by

$$\hat{\Phi}(x) = \int_{k_v} \Phi(y) \langle yx \rangle_v \, d_v y.$$

Then $\hat{\Phi}$ is also a Schwartz-Bruhat function on k_v, and we have the inversion formula

$$\Phi(x) = |\delta_v|_v^{-1} \int_{k_v} \hat{\Phi}(y) \langle -yx \rangle_v \, d_v y.$$

Similarly, for an adelic Schwartz-Bruhat function Φ, we define

$$\hat{\Phi}(x) = \int_{\mathbb{A}_k} \Phi(y) \langle yx \rangle_{\mathbb{A}} \, d_{\mathbb{A}} y,$$

with inversion formula

$$\Phi(x) = d_k \int_{\mathbb{A}_k} \hat{\Phi}(y) \langle -yx \rangle_{\mathbb{A}} \, d_{\mathbb{A}} y.$$

The self-dual measure on $\mathbb{A}_k$ would then be $d_k^{-1/2} \, d_{\mathbb{A}} x$.

To extend these concepts to the vector space V of binary cubic forms, we compose the above-defined additive characters with the bilinear alternating form $[x, y]$ on V defined in Section 2.2. We use the notation

$$\langle x, y \rangle = \langle [x, y] \rangle$$

with subscripts v and $\mathbb{A}$ as needed. With Fourier transform for functions on V_{k_v} defined by

$$\hat{\Phi}(x) = \int_{V_{k_v}} \Phi(y) \, \langle x, y \rangle_v \, d_v y$$

we have the inversion formula

$$\Phi(x) = |3|_v^{-1} \, |\delta_v|_v^2 \int_{V_{k_v}} \hat{\Phi}(y) \, \langle x, y \rangle_v \, d_v y,$$

due to the coefficients $\frac{1}{3}$ in the bilinear form. A negative sign is not needed due to the bilinear form being alternating. For the adelic Fourier transform $\hat{\Phi}$, we have

$$\Phi(x) = d_k^{-2} \int_{V_{\mathbb{A}}} \hat{\Phi}(y) \langle x, y\rangle_{\mathbb{A}}\, d_{\mathbb{A}} y,$$

since the product formula implies $|3|_{\mathbb{A}} = 1$. It follows that the self-dual measure on $V_{\mathbb{A}}$ is $d_k^{-2}\, d_{\mathbb{A}} x$, where $d_{\mathbb{A}} x$ is the measure chosen in Section 2.2.

In Theorem 6.1 of [19], Wright generalized Shintani's proof for $\mathbb{Q}$ to establish the functional equation

$$Z(s, \hat{\Phi}) = Z(1-s, \Phi)$$

for the adelic zeta function defined in Section 2.3. In this functional equation $\hat{\Phi}$ is defined relative to the self-dual measure on $V_{\mathbb{A}}$. To comply with our choice of measure, we must replace $\hat{\Phi}$ by $d_k^{-2}\hat{\Phi}$, which gives the functional equation

$$Z(s, \hat{\Phi}) = d_k^2 Z(1-s, \Phi).$$

We next carry out the same unfolding process for $Z(s, \hat{\Phi})$ into Dirichlet series that we did for $Z(s, \Phi)$ in Section 2.3. If $\Phi = \otimes_v \Phi_v$ has product form, then $\hat{\Phi} = \otimes_v \hat{\Phi}_v$ also has product form, and in the end we find that

$$Z(s, \hat{\Phi}) = \sum_{\alpha \in A_\infty} Z_\alpha(s, \hat{\Phi}_\infty)\, \xi_\alpha(s, \hat{\Phi}_0)$$

with

$$\hat{\xi}_\alpha(s, \hat{\Phi}_0) = \sum_{k' \in \mathscr{K}_\alpha} \frac{d_{k'/k}^{-s}}{o(k')} \prod_{v \nmid \infty} Z_{\alpha_v}(s, \hat{\Phi}_v)$$

with all the same notational conventions as in Section 2.3. We now again make the choice that for all finite places v we have $\Phi_v = \Phi_{0,v}$ is the characteristic function of V_{o_v}. Then the Fourier transform $\hat{\Phi}_{0,v}$ may be computed to be the characteristic function of $(\delta_v^{-1} o_v) \times (3\delta_v^{-1} o_v) \times (3\delta_v^{-1} o_v) \times (\delta_v^{-1} o_v) \subset V_{k_v}$, or more briefly $\hat{\Phi}_{0,v}(x) = \Phi_{0,v}(\delta_v(x_1, \frac{1}{3}x_2, \frac{1}{3}x_3, x_4))$, as mentioned on p. 69 in [5] (with a slight typographical error entering 3 instead of $\frac{1}{3}$ as it should be). We would like to factor out the differental element as much as possible. First we define $\Psi_{0,v}(x) = \Phi_{0,v}(x_1, \frac{1}{3}x_2, \frac{1}{3}x_3, x_4)$, so that $\hat{\Phi}_{0,v}(x) = \Psi_{0,v}(\delta_v x)$. Note that $\begin{pmatrix} \delta_v & 0 \\ 0 & \delta_v \end{pmatrix} \cdot x = \delta_v x$ by the definition of our representation of GL_2 on V. Then by changing variables in the integral defining the local zeta function, we have

$$\begin{aligned} Z_{\alpha_v}(s, \hat{\Phi}_{0,v}) &= \int_{G_{k_v}} |\det g_v|_v^{2s} \ \ \hat{\Phi}_{0,v}(g_v \cdot x_{\alpha_v})\, d_v g_v \\ &= \int_{G_{k_v}} |\det g_v|_v^{2s} \ \ \Psi_{0,v}(\begin{pmatrix} \delta_v & 0 \\ 0 & \delta_v \end{pmatrix} g_v \cdot x_{\alpha_v})\, d_v g_v \\ &= |\delta_v|_v^{-4s} \int_{G_{k_v}} |\det g_v|_v^{2s} \ \ \Psi_{0,v}(g_v \cdot x_\alpha)\, d_v g_v \\ &= |\delta_v|_v^{-4s} \ Z_{\alpha_v}(s, \Psi_{0,v}). \end{aligned}$$

Using the fact that $\prod_v |\delta_v|_v = d_k^{-1}$, we have, for this choice of $\Phi_v = \Phi_{0,v}$ for all finite places v,

$$Z(s, \hat{\Phi}) = d_k^{4s} \sum_{\alpha \in A_\infty} Z_\alpha(s, \hat{\Phi}_\infty) \ \hat{\xi}_\alpha(s)$$

with

$$\hat{\xi}_\alpha(s) = \sum_{k' \in \mathscr{K}_\alpha} \frac{d_{k'/k}^{-s}}{o(k')} \prod_{v \nmid \infty} Z_{\alpha_v}(s, \Psi_{0,v}).$$

This completes our definition of the **dual Dirichlet series** $\hat{\xi}_\alpha(s)$. It is our major goal now to work out the relationship between $\hat{\xi}_\alpha(s)$ and $\xi_\alpha(s)$ conjectured in Chapter 1.2.

As a consequence of the functional equation for the adelic zeta function, we have the relation

$$\sum_{\alpha \in A_\infty} Z_\alpha(s, \hat{\Phi}_\infty) \ \hat{\xi}_\alpha(s) = d_k^{2-4s} \sum_{\alpha \in A_\infty} Z_\alpha(1-s, \Phi_\infty) \ \xi_\alpha(1-s). \tag{2.5}$$

Later, as needed, we shall review the known facts about the residues of these Dirichlet series, and the functional equation satisfied by the local zeta functions at infinite places. In the next chapter, we calculate explicit expressions for $Z_{\alpha_v}(s, \Psi_{0,v})$, comparable to those produced for $Z_{\alpha_v}(s, \Phi_{0,v})$ in [5]. Note that if v is a finite place that does not lie over the prime 3, then 3 is a unit in k_v and it follows that $\Psi_{0,v} = \Phi_{0,v}$. Thus, in that case, the evaluation of the local zeta function is given by equation (2.4). In the next chapter, we shall consider exclusively places $v \mid 3$.

CHAPTER 3

Local integrals of a Fourier transform

3.1 Statement of the integral to be calculated

In this chapter, we address the problem of calculating for a finite place v the local zeta function $Z_{\alpha_v}(s, \Psi_{0,v})$ as described at the end of Section 2.4. Since our work in this chapter will be exclusively over a local field, we shall simplify our notation by letting K be a nonarchimedean local field and using the notation of Sections 2.1 and 2.2. To review, we denote by O, P, π, q, respectively, the maximal compact subring of O, the unique prime ideal P of O, a uniformizer or any generator of P so that $P = \pi O$, and the order of the finite field O/P, respectively. We normalize the absolute value $|a|$ for $a \in K$ to be the modulus of multiplication by a with respect to any additive Haar measure on K. Thus, $|\pi| = q^{-1}$. We denote by dy the additive Haar measure on K for which the measure of O is 1, and by dg the invariant measure on G_K so that G_O has measure 1.

Our goal is to evaluate the local integral

$$Z_\alpha(s, \Psi_0) = \int_{G_K} |\det g|^{2s}\, \Psi_0(g \cdot x_\alpha)\, dg$$

where s is a complex number with $\mathrm{Re}(s) > 1$, α is an orbit of nonsingular binary cubic forms in V_K, x_α is the standard representative of α, and $\Psi_0(x) = \Phi_0(x_1, \frac{1}{3}x_2, \frac{1}{3}x_3, x_4)$ where Φ_0 is the characteristic function of the compact subset V_O. Hence, Ψ_0 is the characteristic function of the subset $O \times 3O \times 3O \times O \subset V_O$. If 3 is a unit in K, then $\Psi_0 = \Phi_0$, and this evaluation was completed in [5]. Thus, in this chapter we assume that K is a 3-field of characteristic not equal to 3. For the definition of Fourier transform given in Section 2.4 and a differental element $\delta \in K$, we saw in that section that

$$Z_\alpha(s, \hat{\Phi}_0) = |\delta|^{-4s}\, Z_\alpha(s, \Psi_0)$$

which explains how this integral arises as a Fourier transform. The result we will prove in this chapter is the following:

Theorem 3.1 *Assuming that* 3 *is a uniformizer of* K*, we have*

$$Z_\alpha(s, \Psi_0) = (1 - q^{1-6s})^{-1}(1 - q^{-2s})^{-1} \cdot \begin{cases} q^{-4s}(1 + q^{1-2s} + 2q^{1-4s}) & \text{if } \alpha \text{ is of type (1)}, \\ q^{-4s}(1 + q^{1-2s}) & \text{if } \alpha \text{ is of type (2u)}, \\ q^{-2s}(1 + q^{1-4s}) & \text{if } \alpha \text{ is of type (2r)}, \\ q^{-4s}(1 + q^{1-2s} - q^{1-4s}) & \text{if } \alpha \text{ is of type (3u)}, \\ 1 & \text{if } \alpha \text{ is of type (3r)}. \end{cases}$$

3.2 Reductions of the local integral

We begin with some simple reductions in this calculation. First, we write $G_K = U_K B_K$ using the Iwasawa decomposition defined in Section 2.2 of Chapter 2. For any u in $U_K = G_O$, we have that $u \cdot x$ is in V_O if and only if x is in V_O. Hence $\Phi_0(u \cdot x) = \Phi_0(x)$ for all $x \in V_K$. The same holds for Ψ_0. Then, considering the measure on G_K defined above, we have

$$\begin{aligned} Z_\alpha(s, \Psi_0) &= \int_{U_K} du \int_{B_K} db\, |\det(ub)|^{2s}\, \Psi_0(ub \cdot x_\alpha) \\ &= \int_{U_K} du\, |\det u|^{2s} \int_{B_K} db\, |\det b|^{2s}\, \Psi_0(u \cdot (b \cdot x_\alpha)) \\ &= \int_{U_K} du \cdot \int_{B_K} db\, |\det b|^{2s}\, \Psi_0(b \cdot x_\alpha) \\ &= \int_{B_K} db\, |\det b|^{2s}\, \Psi_0(b \cdot x_\alpha)\,. \end{aligned}$$

The last integral gives us a motivation to define, for any Schwartz-Bruhat function Ψ, the integral transform

$$I_\alpha(s,\Psi) = \int_{B_K} db\,|\det b|^{2s}\,\Psi(b\cdot x_\alpha)\,.$$

Since any b in B_K can be written as $b = n(y)a(t,u)$ for some $y \in K$ and t, u in $K^\times$, we can express this last integral as

$$I_\alpha(s,\Psi) = \int_{K^\times} d^\times t \int_{K^\times} d^\times u \int_K dy\,|tu|^{2s}\Psi(n(y)a(t,u)\cdot x_\alpha)\,.$$

Considering the operator D on the space of Schwartz-Bruhat functions defined by $D\Psi = \Psi - a(\pi^2,\pi)\cdot\Psi$, or more explicitly, $(D\Psi)(x) = \Psi(x) - \Psi(a(\pi^{-2},\pi^{-1})\cdot x)$, we conclude that

$$\begin{aligned}
I_\alpha(s,D\Psi) &= I_\alpha(s,\Psi) - \int_{K^\times} d^\times t \int_{K^\times} d^\times u \int_K dy\,|tu|^{2s}\,\Psi(a(\pi^{-2},\pi^{-1})\cdot(n(y)a(t,u)\cdot x_\alpha)) \\
&= I_\alpha(s,\Psi) - \int_{K^\times} d^\times t \int_{K^\times} d^\times u \int_K dy\,|tu|^{2s}\,\Psi(n(\pi y)a(\pi^{-2}t,\pi^{-1}u)\cdot x_\alpha) \\
&= I_\alpha(s,\Psi) - |\pi|^{-1+6s} I_\alpha(s,\Psi) \\
&= (1-q^{1-6s})\,I_\alpha(s,\Psi)\,,
\end{aligned}$$

where we have made the substitutions $y \mapsto \pi^{-1}y$, $t \mapsto \pi^2 t$, and $u \mapsto \pi u$.

In particular, for $\Psi = \Psi_0$ and putting $\Psi_1 = D\Psi_0$ we obtain

$$Z_\alpha(s,\Psi_0) = (1-q^{1-6s})^{-1}\,I_\alpha(s,\Psi_1)\,.$$

So in order to calculate the value of this zeta function explicitly, we simply need to evaluate $I_\alpha(s,\Psi_1)$. By definition,

$$\begin{aligned}
\Psi_1(x) &= \Psi_0(x) - \Psi_0(\pi^{-3}x_1,\pi^{-2}x_2,\pi^{-1}x_3,x_4) \\
&= \Phi_0(x_1,\tfrac{1}{3}x_2,\tfrac{1}{3}x_3,x_4) - \Phi_0(\pi^{-3}x_1,\tfrac{1}{3}\pi^{-2}x_2,\tfrac{1}{3}\pi^{-1}x_3,x_4)\,.
\end{aligned}$$

In other words, Ψ_1 is the characteristic function of $(O\times 3O\times 3O\times O)\setminus(P^3\times 3P^2\times 3P\times O)$.

In cases (1) and (2) we take $x_\alpha = (0, 1, \lambda, \mu)$. Case (1) corresponds to $\lambda = 0$ and $\mu = 1$ and so $x_\alpha = (0, 1, 1, 0)$. The integral $I_\alpha(s, \Psi_1)$ becomes now

$$I_\alpha(s, \Psi_1) = \int_{K^\times} d^\times t \int_{K^\times} d^\times u \int_K dy\, |t|^{2s-1}\, |u|^{2s}\, \Psi_1\left(0, t, 2y + u, \frac{y(y+u)}{t}\right) .$$

Case (2) corresponds to $\lambda = \theta$ and $\mu = \overline{\theta}$ where $O' = O[\theta]$ is the maximal compact subring of a quadratic extension $K' = K(\theta)$ of K and so $x_\alpha = (0, 1, \theta + \overline{\theta}, \theta\overline{\theta})$. If the extension is unramified, case (2u), we take θ to be a unit not congruent to any unit in O module π. For the ramified extension, case (2r), we take θ to be the uniformizer π. Given this choice of x_α we can write the integral $I_\alpha(s, \Psi_1)$ for case (2), after making an appropriate substitution, as

$$I_\alpha(s, \Psi_1) = \int_{K^\times} d^\times t \int_{K^\times} d^\times u \int_K dy\, |t|^{2s-1}\, |u|^{2s}\, \Psi_1\left(0, t, \mathrm{Tr}(y + u\theta), \frac{\mathrm{N}(y + u\theta)}{t}\right) ,$$

where Tr and N are the relative trace and norm, respectively, for the extension K' over K.

In case (3) we take $x_\alpha = (1, \theta + \theta' + \theta'', \theta\theta' + \theta\theta'' + \theta'\theta'', \theta\theta'\theta'')$ where $O' = O[\theta]$ is the maximal compact subring of a cubic extension $K' = K(\theta)$ of K. For case (3u) we take θ to be a unit not congruent to any unit in O module π. For the case (3r) we take θ to be the uniformizer π. For this choice of x_α we can write $I_\alpha(s, \Psi_1)$, following a change of variables, as

$$\begin{aligned} I_\alpha(s, \Psi_1) = \int_{K^\times} d^\times t \int_{K^\times} d^\times u \int_K dy\, |t|^{-2s-1} |u|^{6s} \\ \cdot \Psi_1\left(t, \mathrm{Tr}(y + u\theta), \frac{\mathrm{S}(y + u\theta)}{t}, \frac{\mathrm{N}(y + u\theta)}{t^2}\right) , \end{aligned}$$

where Tr, S, and N are the relative trace, second symmetric function, and norm, respectively, for this cubic extension.

In the next section we will calculate the value of the integral $I_\alpha(s, \Psi_1)$. By the reductions done in this section we know that this will suffice to prove Theorem 3.1. The calculation will be done assuming that 3 is a uniformizer in K, i.e. $3O = P$.

When that is the case, Ψ_1 becomes the characteristic function of $(O \times P \times P \times O) \setminus (P^3 \times P^3 \times P^2 \times O)$.

3.3 Evaluation of the local integral

Let us complete the task of evaluating $I_\alpha(s, \Psi_0)$ for each of the possible types of α.

3.3.1 Type (1)

In the corresponding integral, since 2 is a unit, we can make the following change of variables $(t, u, y) \mapsto (t/4, u, (y-u)/2)$ to obtain

$$I_\alpha(s, \Psi_1) = \int_{K^\times} d^\times t \int_{K^\times} d^\times u \int_K dy \, |t|^{2s-1} \, |u|^{2s} \, \Psi_1 \left(0, t, y, \frac{y^2 - u^2}{t}\right) .$$

The integral is nonzero only if $\left(t, y, \frac{y^2-u^2}{t}\right)$ belongs to $(P \times P \times O) \setminus (P^3 \times P^2 \times O)$ and in this case its value is

$$I_\alpha(s, \Psi_1) = \int_{K^\times} |t|^{2s-1} \, d^\times t \int_{K^\times} |u|^{2s} \, d^\times u \int_K dy \, .$$

The integral is therefore nonzero in the following cases:

(a) $t \in P^\times$, $y \in P$, $y^2 - u^2 \in tO$. These conditions imply that $u^2 \in P$ and so $u \in P = \pi O$. Thus, the integral in this case becomes

$$\begin{aligned} I_{(a)} &= \int_{\pi O^\times} |t|^{2s-1} \, d^\times t \int_P |u|^{2s} \, d^\times u \int_P dy \\ &= q^{1-2s} \cdot \sum_{l=1}^{\infty} \int_{\pi^l O^\times} |u|^{2s} \, d^\times u \cdot q^{-1} \\ &= q^{-2s} \sum_{l=1}^{\infty} q^{-2sl} \\ &= \frac{q^{-4s}}{1-q^{-2s}} \, . \end{aligned}$$

(b) $t \in \pi^2 O^\times$, $y \in P$, $y^2 - u^2 \in tO$. Again as before, $u^2 \in P$ and so $u \in P = \pi O$. Hence, the integral is now

$$\begin{aligned} I_{(b)} =& \int_{\pi^2 O^\times} |t|^{2s-1}\, d^\times t \int_P |u|^{2s}\, d^\times u \int_P dy \\ =& q^{2-4s} \cdot \sum_{l=1}^{\infty} \int_{\pi^l O^\times} |u|^{2s}\, d^\times u \cdot q^{-1} \\ =& q^{1-4s} \sum_{l=1}^{\infty} q^{-2sl} \\ =& \frac{q^{1-6s}}{1-q^{-2s}}\,. \end{aligned}$$

(c) $t \in \pi^l O^\times$, $y \in \pi O^\times$, $y^2 - u^2 \in tO$, $l \geq 3$. From these conditions it follows that $(y-u)(y+u) \in P^l \subset P$. But P is a prime ideal so either $y - u \in P$ or $y + u \in P$. In the former case, since $y + u = 2y - (y - u) \in \pi O^\times$, we conclude that $y - u \in P^{l-1}$ and so $u \in y + P^{l-1} = y(1 + P^{l-2})$. In the latter case, since $y - u = 2y - (y + u) \in \pi O^\times$, we obtain that $y + u \in P^{l-1}$ and so $u \in -y + P^{l-1} = -y(1 + P^{l-2})$. This means that $u \in y(1+P^{l-2}) \sqcup -y(1+P^{l-2})$ for a fixed y.

Before we continue we observe that according to the definition of the measures given in Chapter 2, we have

$$d^\times u = \frac{1}{1-q^{-1}} \frac{du}{|u|}\,.$$

Moreover, since $O^\times/(1+P) \cong (O/P)^\times$ and $(1+P^{m-1})/(1+P^m) \cong O/P$ for $m \geq 2$, we have that the measure $d^\times u$ satisfies

$$\int_{1+P^m} d^\times u = \frac{1}{(q-1)q^{m-1}} \quad m \geq 1\,.$$

We can finally calculate the integral for this case.

$$
\begin{aligned}
I_{(c)} =& \sum_{l=3}^{\infty} \int_{\pi^l O^\times} |t|^{2s-1}\, d^\times t \int_{\pi O^\times} dy \left(\int_{y(1+P^{l-2})} |u|^{2s}\, d^\times u + \int_{-y(1+P^{l-2})} |u|^{2s}\, d^\times u \right) \\
=& \sum_{l=3}^{\infty} \int_{\pi^l O^\times} |t|^{2s-1}\, d^\times t \cdot \int_{\pi O^\times} dy \ \cdot 2q^{-2s} \int_{1+P^{l-2}} |u|^{2s}\, d^\times u \\
=& \sum_{l=3}^{\infty} q^{(1-2s)l} \cdot (1-q^{-1})q^{-1} \cdot \frac{2q^{-2s}}{(q-1)q^{l-3}} \\
=& 2q^{1-2s} \sum_{l=3}^{\infty} q^{-2sl} \\
=& \frac{2q^{1-8s}}{1-q^{-2s}} \,.
\end{aligned}
$$

Now, combining these results we obtain the desired result for type (1)

$$
I_\alpha(s, \Psi_1) = I_{(a)} + I_{(b)} + I_{(c)} = \frac{q^{-4s}(1+q^{1-2s}+2q^{1-4s})}{1-q^{-2s}} \,.
$$

3.3.2 Type (2u)

When $K' = K(\theta)$ is a unramified quadratic extension of K with maximal compact subring $O' = O[\theta]$, we have that θ is a unit and π is also a uniformizer of K'. Hence, $y + u\theta$ is in $(P')^m$ if and only if both y and u are in P. The corresponding integral is nonzero only if $\left(t, \mathrm{Tr}(y+u\theta), \dfrac{\mathrm{N}(y+u\theta)}{t}\right)$ is in $(P \times P \times O) \setminus (P^3 \times P^2 \times O)$ and its value is given by

$$
I_\alpha(s, \Psi_1) = \int_{K^\times} |t|^{2s-1}\, d^\times t \int_{K^\times} |u|^{2s}\, d^\times u \int_K dy \,.
$$

The integral is nonzero for the following cases:

(a) $t \in \pi O^\times$, $\mathrm{Tr}(y+u\theta) \in P$, $\mathrm{N}(y+u\theta) \in tO$. Under these conditions, $y + u\theta \in P'$ and so $y \in P$ and $u \in P$. This integral was calculated above, type (1) case (a), and its value is

$$
\begin{aligned}
I_{(a)} =& \int_{\pi O^\times} |t|^{2s-1}\, d^\times t \int_P |u|^{2s}\, d^\times u \int_P dy \\
=& \frac{q^{-4s}}{1-q^{-2s}} \,.
\end{aligned}
$$

(b) $t \in \pi^2 O^\times$, $\mathrm{Tr}(y+u\theta) \in P$, $\mathrm{N}(y+u\theta) \in tO$. With these conditions, as before, $y \in P$ and $u \in P$. We have already calculated this integral, type (1) case (b), and we found that

$$\begin{aligned} I_{(b)} =& \int_{\pi^2 O^\times} |t|^{2s-1}\, d^\times t \int_P |u|^{2s}\, d^\times u \int_P dy \\ =& \frac{q^{1-6s}}{1-q^{-2s}}\,. \end{aligned}$$

Hence, we have found the value of the integral for type (2u)

$$I_\alpha(s,\Psi_1) = I_{(a)} + I_{(b)} = \frac{q^{-4s}(1+q^{1-2s})}{1-q^{-2s}}\,.$$

3.3.3 Type (2r)

When $K' = K(\theta)$ is a ramified quadratic extension of K with maximal compact subring $O' = O[\theta]$, we have that θ is a uniformizer of K' and $\mathrm{N}(\theta)$ is a uniformizer of K. Then $\mathrm{N}(y+u\theta)$ is in P^m if and only $|y| \le q^{-m/2}$ and $|u| \le q^{-(m-1)/2}$. The associated integral is nonzero only if $\left(t, \mathrm{Tr}(y+u\theta), \dfrac{\mathrm{N}(y+u\theta)}{t}\right)$ is in $(P\times P\times O)\backslash(P^3\times P^2\times O)$ and this case it reduces to

$$I_\alpha(s,\Psi_1) = \int_{K^\times} |t|^{2s-1}\, d^\times t \int_{K^\times} |u|^{2s}\, d^\times u \int_K dy\,.$$

The integral is nonzero for the following cases:

(a) $t \in \pi O^\times$, $\mathrm{Tr}(y+u\theta) \in P$, $\mathrm{N}(y+u\theta) \in tO$. These conditions imply that $|y| \le q^{-1/2}$ and $|u| \le 1$ and so $y \in P$ and $u \in O$. Then the integral is

$$\begin{aligned} I_{(a)} =& \int_{\pi O^\times} |t|^{2s-1}\, d^\times t \int_O |u|^{2s}\, d^\times u \int_P dy \\ =& q^{1-2s} \cdot \sum_{l=0}^{\infty} \int_{\pi^l O^\times} |u|^{2s}\, d^\times u \cdot q^{-1} \\ =& q^{-2s} \sum_{l=0}^{\infty} q^{-2sl} \\ =& \frac{q^{-2s}}{1-q^{-2s}}\,. \end{aligned}$$

(b) $t \in \pi^2 O^\times$, $\mathrm{Tr}(y+u\theta) \in P$, $\mathrm{N}(y+u\theta) \in tO$. These conditions imply that $|y| \leq q^{-1}$ and $|u| \leq q^{-1/2}$ and so $y \in P$ and $u \in P$. Then the value of the integral is already known, see type (1) case (b).

$$\begin{aligned} I_{(b)} =& \int_{\pi^2 O^\times} |t|^{2s-1}\, d^\times t \int_P |u|^{2s}\, d^\times u \int_P dy \\ =& \frac{q^{1-6s}}{1-q^{-2s}}\,. \end{aligned}$$

Therefore, we have calculated the value of the integral for type (2r)

$$I_\alpha(s, \Psi_1) = I_{(a)} + I_{(b)} = \frac{q^{-2s}(1+q^{-4s})}{1-q^{-2s}}\,.$$

3.3.4 Type (3u)

If $K' = K(\theta)$ is a unramified cubic extension of K with maximal compact subring $O' = O[\theta]$, then θ is a unit and π is a uniformizer of K'. Thus, $y + u\theta$ is in $(P')^m$ if and only if both y and u are in P^m. Additionally, we can assume that $\mathrm{Tr}(\theta) = 0$ and $\mathrm{S}(\theta)$ and $\mathrm{N}(\theta)$ are units of K. And for any y and u in K, we have

$$\begin{aligned} \mathrm{Tr}(y+u\theta) &= 3y \\ \mathrm{S}(y+u\theta) &= 3y^2 + u^2\,\mathrm{S}(\theta) \\ \mathrm{N}(y+u\theta) &= y^3 + yu^2\,\mathrm{S}(\theta) + u^3\,\mathrm{N}(\theta)\,. \end{aligned}$$

The corresponding integral is nonzero only if $\left(t, \mathrm{Tr}(y+u\theta), \frac{\mathrm{S}(y+u\theta)}{t}, \frac{\mathrm{N}(y+u\theta)}{t^2}\right)$ is in $(O \times P \times P \times O) \setminus (P^3 \times P^3 \times P^2 \times O)$ and its value is

$$I_\alpha(s, \Psi_1) = \int_{K^\times} |t|^{-2s-1}\, d^\times t \int_{K^\times} |u|^{6s}\, d^\times u \int_K dy\,.$$

Therefore, the integral is nonzero in the following cases:

(a) $t \in O^\times$, $\mathrm{Tr}(y+u\theta) \in P$, $\mathrm{S}(y+u\theta) \in tP$, $\mathrm{N}(y+u\theta) \in t^2 O$. The first and last conditions implies that $y + u\theta \in O'$ and so $y \in O$ and $u \in O$. But to satisfy the third condition, we actually require $u \in P$. So the integral becomes

$$\begin{aligned} I_{(a)} &= \int_{O^\times} |t|^{-2s-1}\, d^\times t \int_P |u|^{6s}\, d^\times u \int_O dy \\ &= 1 \cdot \sum_{l=1}^{\infty} \int_{\pi^l O^\times} |u|^{6s}\, d^\times u \cdot 1 \\ &= \sum_{l=1}^{\infty} q^{-6sl} \\ &= \frac{q^{-6s}}{1-q^{-6s}}. \end{aligned}$$

(b) $t \in \pi O^\times$, $\mathrm{Tr}(y+u\theta) \in P$, $\mathrm{S}(y+u\theta) \in tP$, $\mathrm{N}(y+u\theta) \in t^2 O$. These conditions imply that $y+u\theta \in P'$ and so $y \in P$ and $u \in P$. With this the integral is now

$$\begin{aligned} I_{(b)} &= \int_{\pi O^\times} |t|^{-2s-1}\, d^\times t \int_P |u|^{6s}\, d^\times u \int_P dy \\ &= q^{2s+1} \cdot \sum_{l=1}^{\infty} \int_{\pi^l O^\times} |u|^{6s}\, d^\times u \cdot q^{-1} \\ &= q^{2s} \sum_{l=1}^{\infty} q^{-6sl} \\ &= \frac{q^{-4s}}{1-q^{-6s}}. \end{aligned}$$

(c) $t \in \pi^2 O^\times$, $\mathrm{Tr}(y+u\theta) \in P$, $\mathrm{S}(y+u\theta) \in tP$, $\mathrm{N}(y+u\theta) \in t^2 O$. With these conditions we have $y+u\theta \in P^2$ and so $y \in P^2$ and $u \in P^2$. And the integral in this case is given by

$$\begin{aligned} I_{(c)} &= \int_{\pi^2 O^\times} |t|^{-2s-1}\, d^\times t \int_{P^2} |u|^{6s}\, d^\times u \int_{P^2} dy \\ &= q^{4s+2} \cdot \sum_{l=2}^{\infty} \int_{\pi^l O^\times} |u|^{6s}\, d^\times u \cdot q^{-2} \\ &= q^{4s} \sum_{l=2}^{\infty} q^{-6sl} \\ &= \frac{q^{-8s}}{1-q^{-6s}}. \end{aligned}$$

(d) $t \in \pi^3 O^\times$, $(\mathrm{Tr}(y+u\theta), \mathrm{S}(y+u\theta)) \in (P \times tP) \setminus (P^3 \times tP^2)$, $\mathrm{N}(y+u\theta) \in t^2 O$. The first and last conditions imply that $y+u\theta \in (P')^2$ and so $y \in P^2$ and

$u \in P^2$. However, the remaining condition will be valid only if $u \in \pi^2 O^\times$. So the integral becomes

$$\begin{aligned} I_{(d)} =& \int_{\pi^3 O^\times} |t|^{-2s-1}\, d^\times t \int_{\pi^2 O^\times} |u|^{6s}\, d^\times u \int_{P^2} dy \\ =& q^{6s+3} \cdot q^{-12s} \cdot q^{-2} \\ =& q^{1-6s}\,. \end{aligned}$$

Putting together these partial results gives us the value of the integral for type (3u)

$$I_\alpha(s, \Psi_1) = I_{(a)} + I_{(b)} + I_{(c)} + I_{(d)} = \frac{q^{-4s}(1 + q^{1-2s} - q^{1-4s})}{1 - q^{-2s}}\,.$$

3.3.5 Type (3r)

If $K' = K(\theta)$ is a ramified cubic extension of K with maximal compact subring $O' = O[\theta]$, then θ is a uniformizer of K' and $\mathrm{N}(\theta)$ is a uniformizer of K. Hence, $\mathrm{N}(y + u\theta)$ is in P^m if and only if $|y| \leq q^{-m/3}$ and $|u| \leq q^{-(m-1)/3}$. Moreover, we assume that $\mathrm{Tr}(\theta) \in P$, $\mathrm{S}(\theta) \in P$, and $\mathrm{N}(\theta) \in \pi O^\times$. For any y and u in K we have

$$\begin{aligned} \mathrm{Tr}(y + u\theta) &= 3y + u\,\mathrm{Tr}(\theta) \\ \mathrm{S}(y + u\theta) &= 3y^2 + 2yu\,\mathrm{Tr}(\theta) + t^2\,\mathrm{S}(\theta) \\ \mathrm{N}(y + u\theta) &= y^3 + y^2 u\,\mathrm{Tr}(\theta) + yu^2\,\mathrm{S}(\theta) + u^3\,\mathrm{N}(\theta)\,. \end{aligned}$$

As before, the integral is nonzero only if $\left(t, \mathrm{Tr}(y + u\theta), \dfrac{\mathrm{S}(y + u\theta)}{t}, \dfrac{\mathrm{N}(y + u\theta)}{t^2}\right)$ is in $(O \times P \times P \times O) \setminus (P^3 \times P^3 \times P^2 \times O)$ and its value is

$$I_\alpha(s, \Psi_1) = \int_{K^\times} |t|^{-2s-1}\, d^\times t \int_{K^\times} |u|^{6s}\, d^\times u \int_K dy\,.$$

Therefore, to have a nonzero integral we have to consider following cases:

(a) $t \in O^\times$, $\mathrm{Tr}(y + u\theta) \in P$, $\mathrm{S}(y + u\theta) \in tP$, $\mathrm{N}(y + u\theta) \in t^2 O$. From these assumptions, we conclude that $|y| \leq 1$ and $|u| \leq q^{1/3}$, i.e. $y \in O$ and $u \in O$.

The integral reduces in this case to

$$\begin{aligned} I_{(a)} =& \int_{O^\times} |t|^{-2s-1}\, d^\times t \int_O |u|^{6s}\, d^\times u \int_O dy \\ =& 1 \cdot \sum_{l=0}^{\infty} \int_{\pi^l O^\times} |u|^{6s}\, d^\times u \cdot 1 \\ =& \sum_{l=0}^{\infty} q^{-6sl} \\ =& \frac{1}{1-q^{-6s}} . \end{aligned}$$

(b) $t \in \pi O^\times$, $\mathrm{Tr}(y+u\theta) \in P$, $\mathrm{S}(y+u\theta) \in tP$, $\mathrm{N}(y+u\theta) \in t^2 O$. Using these assumptions, we obtain $|y| \le q^{-2/3}$ and $|u| \le q^{-1/3}$, i.e. $y \in P$ and $u \in P$. The integral, which was calculated for type (3u) case (b), is

$$\begin{aligned} I_{(b)} =& \int_{\pi O^\times} |t|^{-2s-1}\, d^\times t \int_P |u|^{6s}\, d^\times u \int_P dy \\ =& \frac{q^{-4s}}{1-q^{-6s}} . \end{aligned}$$

(c) $t \in \pi^2 O^\times$, $\mathrm{Tr}(y+u\theta) \in P$, $\mathrm{S}(y+u\theta) \in tP$, $\mathrm{N}(y+u\theta) \in t^2 O$. These assumptions imply that $|y| \le q^{-4/3}$ and $|u| \le q^{-1}$, i.e. $y \in P^2$ and $u \in P$. So our integral is

$$\begin{aligned} I_{(c)} =& \int_{\pi^2 O^\times} |t|^{-2s-1}\, d^\times t \int_P |u|^{6s}\, d^\times u \int_{P^2} dy \\ =& q^{4s+2} \cdot \sum_{l=1} \int_{\pi^l O^\times} |u|^{6s}\, d^\times u \cdot q^{-2} \\ =& q^{4s} \sum_{l=1}^{\infty} q^{-6sl} \\ =& \frac{q^{-2s}}{1-q^{-6s}} . \end{aligned}$$

These results can be combined to get the integral for type (3r)

$$I_\alpha(s, \Psi_1) = I_{(a)} + I_{(b)} + I_{(c)} = \frac{1}{1-q^{-2s}} .$$

This concludes the evaluation of $I_\alpha(s, \Psi_0)$ in all cases, and therefore completes the proof of Theorem 3.1.

3.4 Verification of a simple identity

In this section we use an identity which was shown to be true by Datskovsky and Wright in order to verify the validity of Theorem 3.1. Consider any locally integrable function ϕ on V_K. By formula (2.4) on page 38 of [5] we have the following identity

$$\int_{V_K} \phi(x)\,dx = \sum_\alpha b_K c_\alpha \int_{G_K} |\det g|^2\,\phi(g\cdot x_\alpha)\,dg\,,$$

where the sum is taken over all the G_K–orbits in V_K, $b_K = q^{1-2e}(1-q^{-1})(1-q^{-2})$, e is the order of δ in K, and $c_\alpha = \dfrac{|P(x_\alpha)|}{o(\alpha)}$ where $P(x_\alpha)$ is the discriminant of x_α and $o(\alpha)$ is the order of the stabilizer in G_K of any $x\in\alpha$. In particular, we can take ϕ to be $\hat{\Phi}_0$. The left hand side is simply

$$\int_{V_K} \hat{\Phi}_0(x)\,dx = |3|\,|\delta|^{-2}\,\Phi_0(0) = |3|\,|\delta|^{-2}$$

by the Fourier inversion formula and our choice of the measure dx on V_K. The right hand side becomes

$$\sum_\alpha b_K c_\alpha\, Z_\alpha(1,\hat{\Phi}_0) = b_K|\delta|^{-4}\sum_\alpha c_\alpha\, Z_\alpha(1,\Psi_0)$$

by the definition of the orbital zeta function.

These formulas are valid for any p–field whose characteristic is not 2 or 3. We want to very the above identity for the kind of fields we have been considering in this chapter, that is for 3–fields for which 3 is a uniformizer. Therefore, under these assumptions using Theorem 3.1 and after canceling out common terms, the identity we are trying to show reduces to

$$\sum_\alpha c_\alpha\, Z_\alpha(1,\Psi_0) = b_K^{-1}|3|\,|\delta|^2 = q^{-2}(1-q^{-1})^{-1}(1-q^{-2})^{-1}\,.$$

For convenience, we write $\alpha_{(j)}$ to indicate that α is an orbit of type (j), where j take the values 1, 2u, 3u, 2r, 3r. We recall that $c_{\alpha_{(1)}} = 1/6$, $c_{\alpha_{(2u)}} = 1/2$, and $c_{\alpha_{(3u)}} = 1/3$. Moreover,

$$\sum_{\alpha_{(2r)}} c_{\alpha_{(2r)}} = \frac{1}{q} \qquad \text{and} \qquad \sum_{\alpha_{(3r)}} c_{\alpha_{(3r)}} = \frac{1}{q^2}\,,$$

where the sums are taken over all the orbits α of type (2r) and (3r), respectively.

Since $|\pi| = q^{-1}$, the orbital zeta function of Ψ_0 at 1 can be written by Theorem 3.1 and the definition of δ, as

$$Z_\alpha(1, \Psi_0) = (1 - q^{-5})^{-1}(1 - q^{-2})^{-1} \cdot \begin{cases} q^{-4}(1 + q^{-1} + 2q^{-3}) & \text{if } \alpha \text{ is of type (1),} \\ q^{-4}(1 + q^{-1}) & \text{if } \alpha \text{ is of type (2u),} \\ q^{-2}(1 + q^{-3}) & \text{if } \alpha \text{ is of type (2r),} \\ q^{-4}(1 + q^{-1} - q^{-3}) & \text{if } \alpha \text{ is of type (3u),} \\ 1 & \text{if } \alpha \text{ is of type (3r).} \end{cases}$$

We can now use all this information to calculate

$$\begin{aligned}\sum_\alpha c_\alpha\, Z_\alpha(1, \Psi_0) =& \frac{1}{6}\, Z_{\alpha_{(1)}}(1, \Psi_0) + \frac{1}{2}\, Z_{\alpha_{(2u)}}(1, \Psi_0) + \frac{1}{3}\, Z_{\alpha_{(3u)}}(1, \Psi_0) \\ &+ \frac{1}{q}\, Z_{\alpha_{(2r)}}(1, \Psi_0) + \frac{1}{q^2}\, Z_{\alpha_{(3r)}}(1, \Psi_0) \\ =&(1 - q^{-5})^{-1}(1 - q^{-2})^{-1}\left[\frac{1}{6} \cdot q^{-4}(1 + q^{-1} + 2q^{-3}) + \frac{1}{2} \cdot q^{-4}(1 + q^{-1}) \right. \\ &\left. + \frac{1}{3} \cdot q^{-4}(1 + q^{-1} - q^{-3}) + \frac{1}{q} \cdot q^{-2}(1 + q^{-3}) + \frac{1}{q^2} \cdot 1\right] \\ =&(1 - q^{-5})^{-1}(1 - q^{-2})^{-1}(q^{-2} + q^{-3} + q^{-4} + q^{-5} + q^{-6}) \\ =&q^{-2}(1 - q^{-5})^{-1}(1 - q^{-2})^{-1}(1 + q^{-1} + q^{-2} + q^{-3} + q^{-4}) \\ =&q^{-2}(1 - q^{-1})^{-1}(1 - q^{-2})^{-1}\end{aligned}$$

which concludes the verification of the identity.

CHAPTER 4

Residues of the Dirichlet series and generalizing Ohno-Nakagawa

In [5], formulas were calculated for the residues of the Dirichlet series $\xi_\alpha(s)$ at its poles $s = 1$ and $s = 5/6$, but the calculation of the residue formulas for the dual Dirichlet series $\hat{\xi}_\alpha(s)$ was left incomplete. The method of that paper required the calculation of local integrals of a Schwartz-Bruhat function, which was completed for nonarchimedean local fields for the characteristic functions Φ_0, but not for its Fourier transform $\hat{\Phi}_0$. As we saw in the previous chapter, these new local integrals are more difficult to calculate. We shall use instead the filtration method of [6] to calculate the residues of the dual Dirichlet series, and thus our results in this chapter will be valid for all number fields k, and not simply those where 3 is unramified.

At the end of this chapter, we shall use the complete set of residues of both $\xi_\alpha(s)$ and $\hat{\xi}_\alpha(s)$ to deduce what the correct generalization of Ohno's conjecture should be. Our method will verify this conjecture at least at $s = 1$ and $s = 5/6$.

4.1 Filtrations of the Dirichlet series

It will be necessary to generalize the decomposition of the adelic zeta function given in equation (2.2) on page 22. Instead of singling out just the infinite places, we shall now allow an arbitrary finite set S of places of k to be distinguished. We shall assume that S contains all infinite places as well as possibly some finite places. For a place v of k, let A_v denote the finite set of G_{k_v}-orbits of nonsingular forms in V_{k_v}. Let $A_S = \prod_{v \in S} A_v$. Then $\alpha = (\alpha_v)$ will denote a choice from A_S, meaning a choice of an orbit at each place in S. We will call these choices **orbit vectors** over S.

Then just as in Section 2.3 of Chapter 2, assuming the Schwartz-Bruhat function $\Phi = \otimes_v \Phi_v$ has product type and that $\Phi_v = \Phi_{0,v}$ for all finite places $v \notin S$, the adelic zeta function may be decomposed as

$$Z(s,\Phi) = \sum_{\alpha \in A_S} Z_{\alpha,S}(s,\Phi)\,\xi_{\alpha,S}(s)$$

where now

$$\begin{aligned} Z_{\alpha,S}(s,\Phi) &= \prod_{v \in S} Z_{\alpha_v}(s,\Phi_v), \qquad (4.1)\\ Z_{\alpha_v}(s,\Phi_v) &= \int_{G_{k_v}} |\det g_v|_v^{2s}\,\Phi_v(g_v \cdot x_{\alpha_v})\,d_v g_v \\ \xi_{\alpha,S}(s) &= \sum_{k' \in \mathscr{K}_\alpha} \frac{d_{k'/k}^{-s}}{o(k')} \prod_{v \notin S} Z_{\alpha_v}(s,\Phi_{0,v}). \end{aligned}$$

Here x_{α_v} is the standard choice of representative for the chosen orbit α_v of binary cubic forms over k_v (described in Prop. 2.1, p. 35 of [5]). Also, $\mathscr{K}_\alpha$ denotes the set of extensions k'/k of degree at most 3 so that $k' \otimes_k k_v$ corresponds to the choice of orbit α_v for each place $v \in S$. The choice of S is built into the choice of the orbit vector $\alpha \in A_S$, but we will indicate S explicitly in the notation $\xi_{\alpha,S}$ because the technique we wish to use in this chapter is to extend the distinguished set S of places until relations between the various Dirichlet series are easier to detect.

In particular, suppose T is a finite set of places of k that contains S. We denote the set of places in T which do not belong to S as $T \setminus S$. Then there is a natural restriction mapping from orbit vectors in A_T to orbit vectors in A_S. We will generally use α to denote a choice of orbits in A_S and β to denote a choice of orbits in A_T. If $\beta_v = \alpha_v$ for all $v \in S$, we say $\beta|_S = \alpha$, meaning that β restricted to S agrees with α.

We can decompose the Dirichlet series $\xi_{\alpha,S}(s)$ in terms of the series $\xi_{\beta,T}(s)$ with

$\beta|_S = \alpha$ as follows

$$\xi_{\alpha,S}(s) = \sum_{k' \in \mathscr{K}_\alpha} \frac{d_{k'/k}^{-s}}{o(k')} \prod_{v \notin S} Z_{\alpha_v}(s, \Phi_{0,v})$$

$$= \sum_{\beta|_S=\alpha} \sum_{k' \in \mathscr{K}_\beta} \frac{d_{k'/k}^{-s}}{o(k')} \prod_{v \in T \setminus S} Z_{\beta_v}(s, \Phi_{0,v}) \prod_{v \notin T} Z_{\beta_v}(s, \Phi_{0,v})$$

where the β sum ranges over those $\beta \in A_T$ whose restriction to S is α, and we have explicitly written the Euler product in terms of the local zeta integral factors. Recall that for $v \notin T$, β_v is determined as the orbit corresponding to k'/k at v. Thus, it should be kept in mind that for $v \notin T$ the orbit β_v is a function of the extension k'/k. Rearranging the above sum and product gives

$$\xi_{\alpha,S}(s) = \sum_{\beta|_S=\alpha} \left[\prod_{v \in T \setminus S} Z_{\beta_v}(s, \Phi_{0,v}) \right] \xi_{\beta,T}(s) \tag{4.2}$$

This is the main filtration formula for the original Dirichlet series.

We can extend these ideas to the dual Dirichlet series as well and obtain the formulas

$$\hat{\xi}_{\alpha,S}(s) = \sum_{k' \in \mathscr{K}_\alpha} \frac{d_{k'/k}^{-s}}{o(k')} \prod_{v \notin S} Z_{\alpha_v}(s, \Psi_{0,v}) \tag{4.3}$$

$$= \sum_{\beta|_S=\alpha} \left[\prod_{v \in T \setminus S} Z_{\beta_v}(s, \Psi_{0,v}) \right] \hat{\xi}_{\beta,T}(s) \tag{4.4}$$

The main idea exploited in this chapter is that if the set of places T contains all places lying over 3, then 3 is a unit for all $v \notin T$, and therefore $\Psi_{0,v} = \Phi_{0,v}$ for all $v \notin T$. Consequently $\hat{\xi}_{\beta,T}(s) = \xi_{\beta,T}(s)$ for all $\beta \in A_T$. Thus, the component series in the two decompositions of $\xi_{\alpha,S}$ and $\hat{\xi}_{\alpha,S}(s)$ are the same, and the sole difference lies in the finitely many local zeta function factors $Z_{\beta_v}(s, \Psi_{0,v})$ for $v \in T \setminus S$.

4.2 Poles and residues

First, let us review the slightly more general framework of the earlier papers of Datskovsky-Wright. In [19, 4], it is proved that the adelic zeta function

$$Z(\omega, \Phi) = \int_{G_{\mathbb{A}}/G_k} \omega(\det g) \sum_{x \in V_k'} \Phi(g \cdot x)\, d_{\mathbb{A}} g,$$

has a meromorphic continuation to the entire complex manifold Ω_k of quasicharacters ω on $\mathbb{A}^\times/k^\times$. This continuation is holomorphic except for simple poles at $\omega = \omega_0$, ω_2, $\chi\omega_{1/3}$ and $\chi\omega_{5/3}$ where χ is any character satisfying $\chi^3 = 1$. In this thesis, we are restricting the quasicharacters to principal ones $\omega = \omega_{2s} = |\cdot|_{\mathbb{A}}^{2s}$ for complex s. Thus,

$$Z(s, \Phi) = \int_{G_{\mathbb{A}}/G_k} |\det g|_{\mathbb{A}}^{2s} \sum_{x \in V_k'} \Phi(g \cdot x)\, d_{\mathbb{A}} g,$$

is holomorphic in the entire s-plane with the exception of simple poles at $s = 0$, $\frac{1}{6}$, $\frac{5}{6}$, and 1.

The decomposition (2.2) of the zeta function $Z(s, \Phi)$ in terms of the Dirichlet series $\xi_\alpha(s)$ allows us to prove that the Dirichlet series have meromorphic continuations to the entire s-plane which are holomorphic except for simple poles at $s = 1$ and $s = 5/6$. Very general residue formulas are stated in Theorem 6.2, p. 71, in [5]. The measure on $G_{\mathbb{A}}$ used in that paper is not the tensor product measure defined in Section 2 of Chapter 2. Thus, after tracing through the notation presented in [19, 5], we find the following residue formulas

$$\operatorname*{Res}_{s=1} \xi_{\alpha,S}(s) = \frac{\rho_k}{2\sqrt{d_k}}\, \rho_S\, \zeta_{k,S}(2)\, c_\alpha\, [1 + b_\alpha] \tag{4.5}$$

$$\operatorname*{Res}_{s=5/6} \xi_{\alpha,S}(s) = \frac{\rho_k}{6 d_k}\, \rho_S\, \zeta_{k,S}\left(\frac{1}{3}\right)\, c_\alpha\, a_\alpha$$

where ρ_k and d_k are defined in Chapter 2,

$$\rho_S = \prod_{\substack{v \in S \\ v \nmid \infty}} (1 - q_v^{-1}),$$

$$\zeta_{k,S}(s) = \prod_{v \notin S} (1 - q_v^{-s})^{-1}, \qquad \text{(partial Dedekind zeta function, analytically continued)}$$

and the $a_\alpha = \prod_{v\in S} a_{\alpha_v}$, $b_\alpha = \prod_{v\in S} b_{\alpha_v}$, and $c_\alpha = \prod_{v\in S} c_{\alpha_v}$ are constants describing the structure of the orbit α_v defined on pages 58, 61 and 38 of [5].

To describe a_{α_v}, b_{α_v} and c_{α_v}, suppose that the orbit α_v corresponds to the local extension k'_w/k_v of degree at most 3 (up to conjugacy) or equivalently the simple algebra $k' \otimes_k k_v$ which has dimension 3 over k_v. In [5], these extensions are classified into five types: (1), (2u), (2r), (3u) and (3r), where the number is the degree of the extension, and the letter indicates whether that extension is unramified or ramified. Let $\Delta_{\alpha_v} = \Delta(k'_w/k_v)$ be the relative discriminant, which is an element of $k_v^\times$ determined uniquely modulo multiplication by squares of units, and let $o(\alpha_v)$ be the number of automorphisms of $k' \otimes k_v$ over k_v. For types (1), (2), and (3), respectively, we have $o(\alpha_v) = 6$, 2, and 3 or 1, respectively, the last depending on whether the local extension is Galois or not. Then from [5], page 38 and top of page 36, we have

$$c_{\alpha_v} = \frac{|\Delta_{\alpha_v}|_v}{o(\alpha_v)}. \tag{4.6}$$

On page 61 of [5], b_{α_v} is simply defined as 3, 1, 0, resp. for types (1), (2), (3), resp. The definition of a_{α_v} is the most involved and is described in a chart on page 58 of [5].

To obtain the residue formulas for the dual Dirichlet series $\hat{\xi}_{\alpha,S}(s)$, we choose a set T of places that contains S and all finite places $v \mid 3$. Then, as we mentioned before, for all orbit vectors $\beta \in A_T$, we have $\hat{\xi}_{\beta,T}(s) = \xi_{\beta,T}(s)$. Our filtration formulas now give the following relations among the residues of all these series, for $r = 1$ and $5/6$:

$$\begin{aligned}
\operatorname*{Res}_{s=r} \xi_{\alpha,S}(s) &= \sum_{\beta|_S=\alpha} \left[\prod_{v\in T\setminus S} Z_{\beta_v}(r, \Phi_{0,v}) \right] \operatorname*{Res}_{s=r} \xi_{\beta,T}(s) \qquad (4.7)\\
\operatorname*{Res}_{s=r} \hat{\xi}_{\alpha,S}(s) &= \sum_{\beta|_S=\alpha} \left[\prod_{v\in T\setminus S} Z_{\beta_v}(r, \Psi_{0,v}) \right] \operatorname*{Res}_{s=r} \xi_{\beta,T}(s)
\end{aligned}$$

In the next sections, we shall use properties of the local zeta functions together with these filtrations to calculate formulas for the residues of the dual Dirichlet series.

4.3 Residue of the dual Dirichlet series at $s = 1$

In this section, we abbreviate $\epsilon_k = \dfrac{\rho_k}{2\sqrt{d_k}}$, all the factors in the residue at $s = 1$ that depend on k but on nothing else. The filtration formula (4.7) together with the residue formulas (4.5) yield

$$\begin{aligned}
\operatorname*{Res}_{s=1} \hat{\xi}_{\alpha,S}(s) &= \sum_{\beta|_S=\alpha} \left[\prod_{v\in T\setminus S} Z_{\beta_v}(1, \Psi_{0,v}) \right] \operatorname*{Res}_{s=1} \xi_{\beta,T}(s) \\
&= \sum_{\beta|_S=\alpha} \left[\prod_{v\in T\setminus S} Z_{\beta_v}(1, \Psi_{0,v}) \right] \epsilon_k\, \rho_T\, \zeta_{k,T}(2)\, c_\beta\, [1 + b_\beta] \\
&= \epsilon_k\, \rho_T\, \zeta_{k,T}(2) \sum_{\beta|_S=\alpha} \left[\prod_{v\in T\setminus S} Z_{\beta_v}(1, \Psi_{0,v}) \right] c_\beta\, [1 + b_\beta].
\end{aligned}$$

Note that the local zeta function was shown in [5] to be holomorphic for $\mathrm{Re}(s) > 1/6$, and thus we can simply substitute $s = 1$ in the residue calculation.

By the organizing principle that a sum of products may be rearranged as a product of sums, we will now manipulate the above residue formulas. Both c_β and b_β factor as products over the places $v \in T$; however, before factoring out the terms corresponding to places in S, we must split the residue formula and then factor, using the facts that

$$\zeta_{k,T}(s) = \zeta_{k,S}(s) \prod_{v\in T\setminus S} (1 - q_v^{-s})$$

$$\rho_T = \rho_S \prod_{v\in T\setminus S} (1 - q_v^{-1})$$

$$c_\beta = c_\alpha \prod_{v\in T\setminus S} c_{\beta_v}$$

$$b_\beta = b_\alpha \prod_{v\in T\setminus S} b_{\beta_v}$$

This leads to

$$\operatorname*{Res}_{s=1} \hat{\xi}_{\alpha,S}(s) = \epsilon_k \, \rho_T \, \zeta_{k,T}(2) \cdot$$

$$\left\{ \sum_{\beta|_S=\alpha} \left[\prod_{v\in T\setminus S} Z_{\beta_v}(1,\Psi_{0,v}) \right] \, c_\beta + \sum_{\beta|_S=\alpha} \left[\prod_{v\in T\setminus S} Z_{\beta_v}(1,\Psi_{0,v}) \right] \, c_\beta b_\beta \right\}$$

$$= \epsilon_k \, \rho_S \, \zeta_{k,S}(2) \cdot$$

$$\left\{ c_\alpha \sum_{\beta|_S=\alpha} \left[\prod_{v\in T\setminus S} (1-q_v^{-1})(1-q_v^{-2}) c_{\beta_v} \, Z_{\beta_v}(1,\Psi_{0,v}) \right] + \right.$$

$$\left. c_\alpha b_\alpha \sum_{\beta|_S=\alpha} \left[\prod_{v\in T\setminus S} (1-q_v^{-1})(1-q_v^{-2}) c_{\beta_v} b_{\beta_v} \, Z_{\beta_v}(1,\Psi_{0,v}) \right] \right\}$$

$$= \epsilon_k \, \rho_S \, \zeta_{k,S}(2) \cdot$$

$$\left\{ c_\alpha \prod_{v\in T\setminus S} \left[(1-q_v^{-1})(1-q_v^{-2}) \sum_{\beta_v\in A_v} c_{\beta_v} \, Z_{\beta_v}(1,\Psi_{0,v}) \right] + \right.$$

$$\left. c_\alpha b_\alpha \prod_{v\in T\setminus S} \left[(1-q_v^{-1})(1-q_v^{-2}) \sum_{\beta_v\in A_v} c_{\beta_v} b_{\beta_v} \, Z_{\beta_v}(1,\Psi_{0,v}) \right] \right\}$$

The sums that appear can be simplified by means of Fourier transform formulas proved in [5]. Keeping in mind that we are using the measure such that V_{o_v} has measure 1, the formula (2.4) on page 38 in [5], restated as Proposition 5.1 on page 52, implies that, for finite places v,

$$(1-q_v^{-1})(1-q_v^{-2}) \sum_{\alpha_v\in A_v} c_{\alpha_v} Z_{\alpha_v}(1,\Psi_{0,v}) = \int_{V_{k_v}} \Psi_{0,v}(x) \, d_v x \, .$$

Since $\Psi_{0,v}$ is the characteristic function of $o_v \times 3o_v \times 3o_v \times o_v \subset V_{k_v}$, the integral is $|3|_v^2$. Thus, we have the first sum in the residue formula equal to

$$(1-q_v^{-1})(1-q_v^{-2}) \sum_{\alpha_v\in A_v} c_{\alpha_v} Z_{\alpha_v}(1,\Psi_{0,v}) = |3|_v^2.$$

The other part of the residue at 1 involves Theorem 5.2, Proposition 5.2 and the definition of the singular invariant distributions Σ_3 from [5]. Assume throughout that v is a finite place and k_v is a p-field. Theorem 5.2 on page 61 says that

$$(1-q_v^{-1}) \sum_{\alpha_v\in A_v} c_{\alpha_v} b_{\alpha_v} \, Z_{\alpha_v}(1,\Psi_{0,v}) = \Sigma_3(\Psi_{0,v})$$

and the distribution Σ_3 is defined on page 54 as

$$\Sigma_3(\Phi_v) = \Sigma_3(2, \Phi_v) = \int_{k_v^\times} d_v^\times t \int_{k_v} d_v x \int_{k_v} d_v y \, |t|_v^2 \, \Phi_v(0, t, x, y),$$

for G_{o_v}-symmetric functions Φ_v, where o_v is the maximal compact subring in k_v. Then it is straightforward to evaluate

$$\begin{aligned}\Sigma_3(\Psi_{0,v}) &= \int_{k_v^\times} d_v^\times t \int_{k_v} d_v x \int_{k_v} d_v y \, |t|_v^2 \, \Psi_{0,v}(0, t, x, y) \\ &= \left(\int_{3o_v} |t|_v^2 \, d_v^\times t \right) |3|_v = |3|_v^3 \, (1 - q_v^{-2})^{-1}\end{aligned}$$

Combining this with our earlier equation for $\Sigma_3(\Psi_{0,v})$ produces

$$(1 - q_v^{-1}) \sum_{\alpha_v \in A_v} c_{\alpha_v} \, b_{\alpha_v} \, Z_{\alpha_v}(1, \Psi_{0,v}) = |3|_v^3 \, (1 - q_v^{-2})^{-1}.$$

This gives the second sum in our residue formula at 1 as

$$(1 - q_v^{-1})(1 - q_v^{-2}) \sum_{\alpha_v \in A_v} c_{\alpha_v} \, b_{\alpha_v} \, Z_{\alpha_v}(1, \Psi_{0,v}) = |3|_v^3.$$

This leads to the full residue formula at $s = 1$:

$$\operatorname*{Res}_{s=1} \hat{\xi}_{\alpha,S}(s) = \epsilon_k \, \rho_S \, \zeta_{k,S}(2) \, c_\alpha \cdot \prod_{v \in T \setminus S} |3|_v^2 \cdot \left[1 + b_\alpha \prod_{v \in T \setminus S} |3|_v \right] .$$

Our last task is to account for our assumption that T contains all the places $v \mid 3$. Define $|x|_S = \prod_{v \in S} |x|_v$ for any $x \in k$. By our assumptions on T, we have $|3|_T = |3|_{\mathbb{A}} = 1$, by the idele product formula. Then $\prod_{v \in T \setminus S} |3|_v = |3|_T / |3|_S = |3|_S^{-1}$. Then our final residue formula at 1 is

$$\operatorname*{Res}_{s=1} \hat{\xi}_{\alpha,S}(s) = \frac{\rho_k}{2\sqrt{d_k}} \rho_S \, \zeta_{k,S}(2) \, c_\alpha \, |3|_S^{-2} \, \left[1 + b_\alpha \, |3|_S^{-1} \right] . \tag{4.8}$$

4.4 Residue of the dual Dirichlet series at $s = 5/6$

Next, we turn to the residue formula at 5/6. Here we abbreviate $\epsilon_k = \dfrac{\rho_k}{6 d_k}$, all the factors in the residue at $s = 5/6$ that depend on k but on nothing else. We use the same notation and arguments at the beginning of Section 4.3 along with the formula $a_\beta = a_\alpha \prod_{v \in T \setminus S} a_{\beta_v}$, but start with the residue formula for $\xi_{\alpha,S}$ at 5/6. This leads to

$$\begin{aligned}
\operatorname*{Res}_{s=5/6} \hat{\xi}_{\alpha,S}(s) &= \sum_{\beta|_S=\alpha} \left[\prod_{v\in T\setminus S} Z_{\beta_v}(5/6, \Psi_{0,v}) \right] \operatorname*{Res}_{s=5/6} \xi_{\beta,T}(s) \\
&= \sum_{\beta|_S=\alpha} \left[\prod_{v\in T\setminus S} Z_{\beta_v}(5/6, \Psi_{0,v}) \right] \epsilon_k \, \rho_T \, \zeta_{k,T} \left(\frac{1}{3}\right) \, c_\beta \, a_\beta \\
&= \epsilon_k \, \rho_T \, \zeta_{k,T} \left(\frac{1}{3}\right) \sum_{\beta|_S=\alpha} \left[\prod_{v\in T\setminus S} Z_{\beta_v}(5/6, \Psi_{0,v}) \right] c_\beta \, a_\beta \\
&= \epsilon_k \, \rho_S \, \zeta_{k,S} \left(\frac{1}{3}\right) c_\alpha \, a_\alpha \sum_{\beta|_S=\alpha} \left[\prod_{v\in T\setminus S} (1-q_v^{-1})(1-q_v^{-1/3}) \, c_{\beta_v} \, a_{\beta_v} \, Z_{\beta_v}(5/6, \Psi_{0,v}) \right] \\
&= \epsilon_k \, \rho_S \, \zeta_{k,S} \left(\frac{1}{3}\right) c_\alpha \, a_\alpha \prod_{v\in T\setminus S} \left[(1-q_v^{-1})(1-q_v^{-1/3}) \sum_{\beta_v \in A_v} c_{\beta_v} \, a_{\beta_v} \, Z_{\beta_v}(5/6, \Psi_{0,v}) \right]
\end{aligned}$$

after exchanging the sum and product in exactly the same way as in the preceding section.

The sum in the above formula corresponds to a second Fourier inversion formula. For finite places v, Theorem 5.1 in [5] states that

$$(1-q_v^{-1}) \sum_{\alpha_v \in A_v} c_{\alpha_v} \, a_{\alpha_v} \, Z_{\alpha_v}(5/6, \Psi_{0,v}) = \Sigma_4(\Psi_{0,v}),$$

for the distribution Σ_4 defined on pp. 33 and 34 of [5] by the integral

$$\Sigma_4(\Phi_v) = \Sigma_4(1/3, \Phi_v) = \int_{k_v^\times} d_v^\times t \int_{k_v} d_v x \int_{k_v} d_v y \int_{k_v} d_v z \, |t|_v^{1/3} \, \Phi_v(t,x,y,z)$$

if Φ_v is any G_{o_v}-symmetric function. Replacing Φ_v by $\Psi_{0,v}$, this is easy to calculate as

$$\begin{aligned}
\Sigma_4(\Psi_{0,v}) &= \int_{k_v^\times} d_v^\times t \int_{k_v} d_v x \int_{k_v} d_v y \int_{k_v} d_v z \, |t|_v^{1/3} \, \Psi_{0,v}(t,x,y,z) \\
&= \left(\int_{o_v} |t|_v^{1/3} \, d_v^\times t \right) |3|_v^2 = |3|_v^2 (1-q_v^{-1/3})^{-1}
\end{aligned}$$

Therefore, the sum in the residue formula becomes

$$(1-q^{-1})(1-q^{-1/3}) \sum_{\alpha_v \in A_v} c_{\alpha_v}\, a_{\alpha_v}\, Z_{\alpha_v}(5/6, \Psi_{0,v}) = |3|_v^2.$$

Putting this altogether gives

$$\operatorname*{Res}_{s=5/6} \hat{\xi}_{\alpha,S}(s) = \epsilon_k\, \rho_S\, \zeta_{k,S}\left(\frac{1}{3}\right) c_\alpha\, a_\alpha \cdot \prod_{v \in T \setminus S} |3|_v^2\,.$$

Assuming again that T contains all places lying over 3, we can finally give the formula for the residue at 5/6:

$$\operatorname*{Res}_{s=5/6} \hat{\xi}_{\alpha,S}(s) = \frac{\rho_k}{6d_k}\, \rho_S\, \zeta_{k,S}\left(\frac{1}{3}\right) c_\alpha\, a_\alpha\, |3|_S^{-2}. \tag{4.9}$$

4.5 Generalizing Ohno's conjecture

Ohno's conjecture (see [14, 13]) takes the form

$$\hat{\xi}_1(s) = 3^{-3s}\xi_2(s) \qquad\qquad \hat{\xi}_2(s) = 3^{1-3s}\xi_1(s)$$

where $\xi_1(s)$, $\xi_2(s)$ are Shintani's Dirichlet series (see [17]) corresponding to integral binary cubic forms of positive and negative discriminant, respectively, and $\hat{\xi}_1(s)$, $\hat{\xi}_2(s)$ are the analogous series for the dual lattice. This is the case $k = \mathbb{Q}$ with the set of places S limited to just the one infinite place $v = \infty$. The completion of $\mathbb{Q}$ at the place ∞ is simply the real numbers $\mathbb{R}$, and there are two $G_{\mathbb{R}}$-orbits of nonsingular real binary cubic forms, namely, the totally real forms of positive discriminant, and the complex forms of negative discriminant.

We will attempt to generalize this pattern to any number field k by taking S to be the set of infinite places $v \mid \infty$ of k. For any choice of orbits $\alpha \in A_\infty$, we will define $-\alpha$ as another choice in the following way. First, we decompose A_∞ as a direct product $A_\infty = \prod_{v|\infty} A_v$ and $\alpha = (\alpha_v)_v$. If $k_v = \mathbb{C}$, there is only one $G_{\mathbb{C}}$-orbit, and we define $-\alpha_v = \alpha_v$, the lone orbit. If $k_v = \mathbb{R}$, there are two $G_{\mathbb{R}}$-orbits, and we simply define $-\alpha_v$ to be the other orbit besides α_v. Then for $\alpha \in A_\infty$, we define $-\alpha$ in the natural componentwise fashion. Our goal is to establish a formula of the form

$$\hat{\xi}_{-\alpha}(s) = 3^{A+Bs}\,\xi_\alpha(s)$$

for any $\alpha \in A_\infty$, for some constants A and B dependent on k and α.

The reason for the comparison of series for the orbit types α and $-\alpha$ lies in the theorem of Scholz in [16] about the relationship between the 3-class numbers of quadratic field of positive and negative discriminant. There the key tool is to adjoin the cube roots of unity to the Galois closure of a noncyclic cubic field, with quadratic resolvent field of discriminant D. The extended field now contains another family of conjugate cubic fields with quadratic resolvent field of discriminant $-3D$. That correspondence changes the sign of the discriminants of the cubic fields. Hopefully, this mechanism will be made more precise in the course of our research.

First, we collect the residue calculations of this chapter as well as the original calculations of Datskovsky-Wright into a convenient reference theorem:

Theorem 4.1 *For a finite set S of places of the number field k containing all infinite places, the residues of the Shintani Dirichlet series $\xi_{\alpha,S}(s)$ and the dual series $\hat{\xi}_{\alpha,S}(s)$, as defined in Sections 2.3 and 2.4, are given by the following formulas:*

$$\begin{aligned}
\underset{s=1}{Res}\,\xi_{\alpha,S}(s) &= \frac{\rho_k}{2\sqrt{d_k}}\,\rho_S\,\zeta_{k,S}(2)\,c_\alpha\,[1+b_\alpha] \\
\underset{s=1}{Res}\,\hat{\xi}_{\alpha,S}(s) &= \frac{\rho_k}{2\sqrt{d_k}}\,\rho_S\,\zeta_{k,S}(2)\,c_\alpha\,|3|_S^{-2}\,\left[1+b_\alpha\,|3|_S^{-1}\right] \\
\underset{s=5/6}{Res}\,\xi_{\alpha,S}(s) &= \frac{\rho_k}{6d_k}\,\rho_S\,\zeta_{k,S}\left(\frac{1}{3}\right)\,c_\alpha\,a_\alpha \\
\underset{s=5/6}{Res}\,\hat{\xi}_{\alpha,S}(s) &= \frac{\rho_k}{6d_k}\,\rho_S\,\zeta_{k,S}\left(\frac{1}{3}\right)\,c_\alpha\,a_\alpha\,|3|_S^{-2}
\end{aligned}$$

When S is the set of infinite places, we have $|3|_S = |3|_\infty = 3^n$, where $n = [k:\mathbb{Q}]$.

Then, using the preceding theorem, we have

$$\frac{\hat{\xi}_{-\alpha}(1)}{\xi_\alpha(1)} = 3^{-2n}\,\frac{c_{-\alpha}}{c_\alpha}\,\frac{1+b_{-\alpha}\,3^{-n}}{1+b_\alpha}$$
$$\frac{\hat{\xi}_{-\alpha}(5/6)}{\xi_\alpha(5/6)} = 3^{-2n}\,\frac{c_{-\alpha}}{c_\alpha}\,\frac{a_{-\alpha}}{a_\alpha}\,.$$

The next step in simplifying these formulas is to manipulate the formulas for a_α, b_α and c_α when α corresponds to an orbit type over $\mathbb{R}$ or $\mathbb{C}$.

Since we are only considering places $v \mid \infty$, there are only three local orbit types α_v to consider, which we will denote as $\alpha_v = 0$ if $k_v = \mathbb{C}$, $\alpha_v = +$ if $k_v = \mathbb{R}$ and α_v corresponds to the binary cubic forms of positive discriminant, and $\alpha_v = -$ if $k_v = \mathbb{R}$ and α_v corresponds to the binary cubic forms of negative discriminant. Recall that in Section 4.2, equation (4.6), we reviewed the definitions of b_α and c_α established in [5], and we give these again strictly for the archimedean places

$$c_0 = \frac{1}{6}, \qquad c_+ = \frac{1}{6}, \qquad c_- = \frac{1}{2},$$
$$b_0 = 3, \qquad b_+ = 3, \qquad b_- = 1.$$

Considering ratios, we have

$$\frac{c_{-0}}{c_0} = 1, \qquad \frac{c_+}{c_-} = \frac{1}{3}, \qquad \frac{c_-}{c_+} = 3.$$

For $\alpha = (\alpha_v)_v \in A_\infty$, suppose that for m of the real places v we have $\alpha_v = +$ and then for the other $r_1 - m$ places we have $\alpha_v = -$. Then by taking products of the formulas from the previous paragraph we get

$$\frac{c_{-\alpha}}{c_\alpha} = 3^m\left(\frac{1}{3}\right)^{r_1-m} = 3^{2m-r_1}, \qquad b_\alpha = 3^{r_2+m}, \qquad b_{-\alpha} = 3^{r_2+r_1-m}.$$

Then

$$\frac{1+b_{-\alpha}\,3^{-n}}{1+b_\alpha} = \frac{1+3^{r_1+r_2-m-n}}{1+3^{r_2+m}} = \frac{1+3^{-r_2-m}}{1+3^{r_2+m}} = 3^{-r_2-m},$$

since $n = r_1 + 2r_2$. It is noteworthy that under our choices this ratio simplifies to just a power of 3. Then combining these results we obtain

$$\frac{\hat{\xi}_{-\alpha}(1)}{\xi_\alpha(1)} = 3^{-2n}\,3^{2m-r_1}\,3^{-r_2-m} = 3^{m-2n-r_1-r_2} \tag{4.10}$$

For the value of the ratio at 5/6, we need the formulas for a_α given on page 58 in [5].

$$a_0 = \frac{3\sqrt{3}}{4\pi^2}\,\Gamma\left(\frac{1}{3}\right)^6, \qquad a_+ = \frac{3\sqrt{3}}{2\pi}\,\Gamma\left(\frac{1}{3}\right)^3, \qquad a_- = \frac{3}{2\pi}\,\Gamma\left(\frac{1}{3}\right)^3.$$

Then the ratios are

$$\frac{a_{-0}}{a_0} = 1, \qquad \frac{a_+}{a_-} = \sqrt{3}, \qquad \frac{a_-}{a_+} = \frac{1}{\sqrt{3}}.$$

For $\alpha \in A_\infty$ with m real places v such that $\alpha_v = +$, as before, we have

$$\frac{a_{-\alpha}}{a_\alpha} = \left(\frac{1}{\sqrt{3}}\right)^m (\sqrt{3})^{r_1-m} = 3^{\frac{1}{2}r_1-m}.$$

Then

$$\frac{\hat{\xi}_{-\alpha}(5/6)}{\xi_\alpha(5/6)} = 3^{-2n}\,3^{2m-r_1}\,3^{\frac{1}{2}r_1-m} = 3^{m-2n-\frac{1}{2}r_1}. \tag{4.11}$$

Both values are consistent with $\hat{\xi}_{-\alpha}(s)/\xi_\alpha(s)$ being powers of 3. We may now solve for the constants A, B in this conjectural form:

$$\frac{\hat{\xi}_{-\alpha}(1)}{\xi_\alpha(1)} = 3^{A+B} = 3^{m-2n-r_1-r_2}$$
$$\frac{\hat{\xi}_{-\alpha}(5/6)}{\xi_\alpha(5/6)} = 3^{A+\frac{5}{6}B} = 3^{m-2n-\frac{1}{2}r_1}.$$

This leads to the linear equations:

$$A + B = m - 2n - r_1 - r_2 \qquad A + \frac{5}{6}B = m - 2n - \frac{1}{2}r_1,$$

which have the unique solution

$$A = m + r_2 \qquad B = -3n.$$

Then the proposed generalization of Ohno's Conjecture (and Nakagawa's theorem) is

$$\frac{\hat{\xi}_{-\alpha}(s)}{\xi_\alpha(s)} = 3^{m+r_2-3ns}. \tag{4.12}$$

The calculations presented here establish Theorem 1.2 and motivate Conjecture 1.1.

Finally, let us compare this conjecture to Nakagawa's theorem for $k = \mathbb{Q}$. In that case we have $n = [\mathbb{Q} : \mathbb{Q}] = 1$, $r_1 = 1$, and $r_2 = 0$. Then our conjecture would say

$$\frac{\hat{\xi}_2(s)}{\xi_1(s)} = 3^{1+0-3s} = 3^{1-3s}, \qquad \frac{\hat{\xi}_1(s)}{\xi_2(s)} = 3^{0+0-3s} = 3^{-3s}.$$

This is exactly the theorem of Nakagawa mentioned at the beginning of this section.

CHAPTER 5

Decomposing the Dirichlet series according to the resolvent field

Datskovsky and Wright established the expression of Shintani Dirichlet $\xi_\alpha(s)$ series as a sum over extensions k'/k of degree at most 3, as mentioned in the introduction, and in Chapter 3 of this thesis we established the analogous formula for the dual Dirichlet series $\hat{\xi}_\alpha(s)$. Then after cancelling common factors, as described in Chapter 1.2 at equation (1.5), our generalization of Ohno's conjecture becomes

$$\sum_{k' \in \mathscr{K}_{-\alpha}} \frac{d_{k'/k}^{-s}}{o(k')} \frac{R_{k'}(2s)}{R_{k'}(4s)} \prod_{v|3} T_{k',v}(s) = 3^{r_2+m-3ns} \sum_{k' \in \mathscr{K}_\alpha} \frac{d_{k'/k}^{-s}}{o(k')} \frac{R_{k'}(2s)}{R_{k'}(4s)}$$

In this chapter, we shall decompose this identity according to the resolvent fields of the extensions k'/k, and give the proofs of Theorems 1.3 and 1.4 in Chapter 1.2.

5.1 The resolvent field of an extension k'/k of degree at most 3

If k'/k has degree strictly less than 3, we simply define the **resolvent field** to be $F = k'$. If k'/k is a cubic extension, it is either cyclic, in which case we define the **resolvent field** to be $F = k$, or it is noncyclic and its Galois closure over k contains a unique quadratic field F, which is called the **resolvent field** in that case.

Each resolvent field F has degree at most 2 over k, and thus can be expressed in the form $F = k(\sqrt{\delta})$ for some nonzero element δ of k. By Kummer theory, $k(\sqrt{\delta_1}) = k(\sqrt{\delta_2})$ if and only if $\delta_1/\delta_2 \in k^2$, the subgroup of squares in $k^\times$. Thus, the possible resolvent fields F of k'/k bijectively correspond to the cosets in $k^\times/k^2$. For each $\delta \in k^\times$, define $\mathscr{C}(\delta)$ to be the set of all extensions k'/k of degree at most 3 which have resolvent field equal to $F = k(\sqrt{\delta})$.

For each real embedding $\sigma : k \to \mathbb{R}$ of k, and for any $\delta \in k^\times$, either $\delta^\sigma > 0$ or $\delta^\sigma < 0$. Thus, we can define a **signature** α of δ by setting $\alpha_v = +$ if $\delta^\sigma > 0$ and $\alpha_v = -$ otherwise. This signature is the same as the signature of the extension $k(\sqrt{\delta})/k$ as defined in Chapter 1.2. If $\alpha_v = +$, then $k(\sqrt{\delta}) \otimes_k k_v \cong \mathbb{R} \oplus \mathbb{R}$, and if $\alpha_v = -$, then $k(\sqrt{\delta}) \otimes_k k_v \cong \mathbb{C}$. For any extension $k'/k \in \mathscr{C}(\delta)$, since the Galois closure of k' over k contains $k(\sqrt{\delta})$, this shows that $k' \otimes_k k_v$ must be a direct sum of three copies of $\mathbb{R}$. Similar reasoning in case $\alpha_v = -$ proves that the signature of any $k' \in \mathscr{C}(\delta)$ is the same as the signature of δ. Thus, for any signature α, the set of extensions $\mathscr{K}_\alpha$ is the disjoint union of $\mathscr{C}(\delta)$ over representatives of all cosets $\delta \in k^\times/k^2$ with signature α.

Finally, for any resolvent field $F = k(\sqrt{\delta})$, we define the **dual resolvent field** to be $\hat{F} = k(\sqrt{-3\delta})$. The reason for this choice of dual is that the compositum $F\hat{F}$ must contain the field $F_0 = k(\sqrt{-3})$ generated by the cube roots of unity over k. Kummer theory says that any cyclic cubic extension F'/F for which F contains the cube roots of unity must be of the form $F' = F(\sqrt[3]{\gamma})$ for some $\gamma \in F^\times$, and that fact plays a special role in Scholz' reflection theorem in [16] and Nakagawa's proof. Note that duality is symmetric in that the dual field of $\hat{F}$ is just F. If the signature of δ is α, then clearly the signature of -3δ is $-\alpha$.

We summarize these observations about the resolvent fields in the following proposition:

Proposition 5.1 *For any signature α of the field k and its negative $-\alpha$, the sets of extensions $\mathscr{K}_\alpha$ and $\mathscr{K}_{-\alpha}$, respectively, are the disjoint unions of the subsets $\mathscr{C}(\delta)$ and $\mathscr{C}(-3\delta)$ as δ ranges over representatives of each coset in $k^\times/k^2$ which has signature α.*

By summing over the coset representatives $\delta \in k^\times/k^2$ with signature α, this proposition directly proves what we stated as Theorem 1.3 in Chapter 1.2.

Theorem 5.1 *If for every $\delta \in k^\times$, we have*

$$\sum_{k' \in \mathscr{C}(-3\delta)} \frac{d_{k'/k}^{-s}}{o(k')} \frac{R_{k'}(2s)}{R_{k'}(4s)} \prod_{v|3} T_{k',v}(s) = 3^{r_2+m-3ns} \sum_{k' \in \mathscr{C}(\delta)} \frac{d_{k'/k}^{-s}}{o(k')} \frac{R_{k'}(2s)}{R_{k'}(4s)}$$

then the generalized Ohno conjecture (1.4) *is true.*

Later in this chapter, we shall explore the truth of the converse of this theorem. We next turn to a more detailed discussion of Scholz' reflection. Let k'/k be an extension of degree 3 with resolvent field equal to $F = k(\sqrt{\delta})$ (which has degree 1 or 2 over k). The main idea of Scholz' reflection is that cubic extensions k'/k with resolvent field F roughly correspond to cubic extensions $\hat{k}'/k$ with resolvent field $\hat{F}$. This comes about as follows. The compositum $L = k'F$ is a cyclic cubic extension of F. Let B be the field $B = F(\sqrt{-3}) = F\hat{F}$. Then the degree $[B : k]$ is a divisor of 4, and the compositum $N = k'B$ is a cyclic cubic extension of B. Since B contains the cube roots of unity, by Kummer theory the cubic extension N/B has the form $N = B(\alpha^{1/3})$ for some nonzero $\alpha \in B$.

5.2 Conductors and discriminants of cubic extensions

In this section, we establish the basic notation of conductors, differents, and discriminants of cubic extensions. This material is derived from Hasse [9] and Martinet-Payans [11]. Before we continue, we need to recall a few results from class field theory that would allow us to analyze our Dirichlet series identities.

Theorem 5.2 (Isomorphism Theorem) *There is a one-to-one correspondence between the finite abelian extensions L of F and the open subgroups $U = U_L$ of the idele class group $\mathscr{J}_F = \mathbb{A}_F^\times / F^\times$ such that the Galois group* $\mathrm{Gal}(L/F)$ *is isomorphic to $\mathscr{J}_F/U$. Moreover, if L/F is a finite abelian extension and K is an intermediate field $L \supset K \supset F$, then the corresponding subgroups satisfy $F^\times \subset U_L \subset U_K \subset U_F \subset \mathbb{A}_F^\times$.*

For each character χ of the group $\mathscr{J}_F$ trivial on U, let $\mathfrak{f}_\chi$ denote its conductor. It is an integral ideal in F.

Theorem 5.3 (Conductor-Discriminant Formula) *Let L/F be a finite abelian extension of number fields corresponding to the open subgroup U of the idele class group $\mathcal{J}_F$. Then the relative discriminant $\mathfrak{D}_{L/F}$ of L/F is given by*

$$\mathfrak{D}_{L/F} = \prod_{\chi} \mathfrak{f}_\chi,$$

where χ ranges over all the characters of $\mathcal{J}_F$ trivial on U.

Let k'/k be an extension of degree ≤ 3 with Galois closure L and resolvent field F. The relative discriminants of k'/k, L/k, and F/k respectively, considered as ideals in $\mathfrak{o}_k$, are denoted by $\mathfrak{D}_{k'/k}$, $\mathfrak{D}_{L/k}$, and $\mathfrak{D}_{F/k}$, respectively. The differents of these extensions, as ideals in the rings of the integers of the corresponding overfield, are denoted by $\mathfrak{d}_{k'/k}$, $\mathfrak{d}_{L/k}$, and $\mathfrak{d}_{F/k}$, respectively. The relative discriminants and differents are related by means of the relative norms

$$\mathfrak{D}_{k'/k} = \mathrm{N}_{k'/k}(\mathfrak{d}_{k'/k}), \qquad \mathfrak{D}_{L/k} = \mathrm{N}_{L/k}(\mathfrak{d}_{L/k}), \qquad \mathfrak{D}_{F/k} = \mathrm{N}_{F/k}(\mathfrak{d}_{F/k}).$$

According to the notation introduced in Chapter 1, we can write

$$d_{k'/k} = \mathrm{N}(\mathfrak{D}_{k'/k}), \qquad d_{L/k} = \mathrm{N}(\mathfrak{D}_{L/k}), \qquad d_{F/k} = \mathrm{N}(\mathfrak{D}_{F/k}).$$

If k'/k is noncyclic, then F/k turns out to be quadratic and L/F cyclic cubic. By the isomorphism theorem, this extension corresponds to an open subgroup U of index 3 in $\mathcal{J}_F$. There are two nontrivial cubic characters χ and χ^2 with kernel equal to U. By the conductor-discriminant formula, the discriminant of L/F is $\mathfrak{D}_{L/F} = \mathfrak{f}_\chi^2$, since χ and χ^2 have the same conductor. Therefore by the tower law for discriminants (see Prop. 13 of Chap. VII-4 in [18]) we have

$$\mathfrak{D}_{L/k} = \mathfrak{D}_{F/k}^3 \, \mathrm{N}_{F/k}(\mathfrak{f}_\chi)^2.$$

Next, we shall discuss the concepts of conductors and discriminants over the idele class group. For a place v of F, let i_v be the natural injection of $F_v^\times$ into the idele class group $\mathcal{J}_F$. Thus, $i_v(x)$ is the coset of the idele with v component equal to x and all other components equal to 1.

Let χ be a character of $\mathcal{J}_F$ which is trivial on U. We define the v component to be $\chi_v(x) = \chi(i_v(x))$ for $x \in F_v^\times$. For a finite place v, the kernel of χ_v contains either the full unit group $o_v^\times$ of F (in which case we set $f_v = 0$) or some subgroup $1 + \varpi_v^{f_v} o_v$ for a smallest positive integer f_v, where ϖ_v is a uniformizer in F_v. Then the conductor of χ_v is $\varphi_{\chi_v} = \varpi_v^{f_v}$. For an infinite place v, the kernel of a finite order character χ_v is either all of $F_v^\times$ in which case we set $\varphi_{\chi_v} = 1$, or possibly just the positive real numbers $\mathbb{R}_+$ in the event v is real. In the latter case, we set $\varphi_{\chi_v} = -1$. The **idelic conductor** φ_χ is defined to be the idele with v component equal to φ_{χ_v} for all places v. The conductor is well-defined as an element of $\mathbb{A}_F^\times / \mathbb{A}_{F,\infty}^0$, where

$$\mathbb{A}_{F,\infty}^0 = \prod_{v|\infty} F_v^0 \times \prod_{v\nmid\infty} o_v^\times$$

where F_v^0 represents the connected component of 1 in $F_v^\times$. Hence, $F_v^0 = \mathbb{C}^\times$ if v is complex and $\mathbb{R}_+$ if v is real.

For a place v of F and a place w of L lying above v, let $\{\theta_1, \ldots, \theta_m\}$ be a basis of L_w over F_v. Let $\theta_j^{(i)}$ range over the m conjugates of θ_j. For infinite places v, we define the relative discriminant Δ_{L_w/F_v} of the extension L_w/F_v to be the square of the determinant of the matrix $(\theta_j^{(i)})$, which is an element of $F_v^\times$. This relative discriminant is well-defined only modulo multiplication by elements of F_v^2, the group of squares of elements of $F_v^\times$. By convention, we stipulate $\Delta_{\mathbb{C}/\mathbb{C}} = \Delta_{\mathbb{R}/\mathbb{R}} = 1$ and $\Delta_{\mathbb{C}/\mathbb{R}} = -1$. For finite places, the maximal compact subring of L_w is a free o_v-module, and thus we may select a basis $\{\theta_j\}$. Then Δ_{L_w/F_v} is defined using this basis. With this special choice of the θ_j, the relative discriminant is well-defined module o_v^2, the group of squares of elements of $o_v^\times$. The v-adic part $\Delta_{v,L/F}$ of the relative discriminant of L/F is defined by

$$\Delta_{v,L/F} = \prod_{w|v} \Delta_{L_w/F_v} \in F_v^\times.$$

The **idelic relative discriminant** $\Delta_{L/F}$ is taken to be the idele whose v-adic component is $\Delta_{v,L/F}$. That this is an idele is a consequence of the fact that $\Delta_{v,L/F} \in o_v^\times$ for almost all v. This discriminant is well-defined modulo multiplication by elements of

$$\mathbb{A}_{F,0}^2 = \prod_{v \nmid \infty} o_v^2.$$

This idelic definition of discriminant was first advanced in [8], where many basic properties are established.

There is simple relationship between the conductors and discriminants as ideals with their counterparts as ideles. Let $\mathscr{I}_F$ denote the group of fractional ideals of F. There is a natural homomorphism $\mathrm{id} : \mathbb{A}_F^\times \to \mathscr{I}_F$ described in Chap. V-3 of [18]. Then we have

$$\mathfrak{f}_\chi = \mathrm{id}(\varphi_\chi) \qquad\qquad \mathfrak{D}_{L/F} = \mathrm{id}(\Delta_{L/F}).$$

Moreover, we have an idelic analogous of the conductor-discriminant formula:

Theorem 5.4 (Idelic Conductor-Discriminant Formula) *Let L/F be a finite abelian extension of number fields corresponding to the open subgroup U of the idele class group $\mathcal{J}_F$. Then the idelic relative discriminant $\Delta_{L/F}$ of L/F is given by*

$$\Delta_{L/F} = \prod_\chi \varphi_\chi,$$

where χ ranges over all the characters of $\mathcal{J}_F$ trivial on U.

5.3 The resolvent field identity

Nakagawa's proof establishes by means of Scholz' reflection that the terms corresponding to extensions k'/k in $\mathscr{C}(\delta)$ on one side of Ohno's series identity correspond to the terms for extensions k'/k in $\mathscr{C}(-3\delta)$ on the other side. To simplify our work with these identities, we introduce some notation for the Euler products in the identities.

Following equation (2.4) of Chapter 2, we define the following Euler factors:

$$E_{k',v}(s) = \begin{cases} (1+q_v^{-2s})^2 & \text{if } (k'/k,v) = (1), \\ 1+q_v^{-4s} & \text{if } (k'/k,v) = (2\text{u}), \\ 1+q_v^{-2s} & \text{if } (k'/k,v) = (2\text{r}), \\ 1-q_v^{-2s}+q_v^{-4s} & \text{if } (k'/k,v) = (3\text{u}), \\ 1 & \text{if } (k'/k,v) = (3\text{r}), \end{cases} \tag{5.1}$$

where $(k'/k, v)$ denotes the splitting type of the place v of k in the extension k'/k. We are omitting the common factors that will cancel out in the generalized Ohno-Nakagawa identity. From Theorem 3.1, the dual Euler factors are $\hat{E}_{k',v}(s) = E_{k',v}(s)$ for $v \nmid 3$, and for $v \mid 3$ we have

$$\hat{E}_{k',v}(s) = \begin{cases} q_v^{-4s}(1+q_v^{1-2s}+2q_v^{1-4s}) & \text{if } (k'/k,v) = (1), \\ q_v^{-4s}(1+q_v^{1-2s}) & \text{if } (k'/k,v) = (2\text{u}), \\ q_v^{-2s}(1+q_v^{1-4s}) & \text{if } (k'/k,v) = (2\text{r}), \\ q_v^{-4s}(1+q_v^{1-2s}-q_v^{1-4s}) & \text{if } (k'/k,v) = (3\text{u}), \\ 1 & \text{if } (k'/k,v) = (3\text{r}), \end{cases} \tag{5.2}$$

so long as 3 is unramified in k. Again, we have omitted the factors that cancel out in the conjectured Ohno-Nakagawa identity. Then the cancelled Ohno-Nakagawa identity has the form

$$\sum_{k' \in \mathscr{K}_{-\alpha}} \frac{d_{k'/k}^{-s}}{o(k')} \prod_{v \nmid \infty} \hat{E}_{k',v}(s) = 3^{r_2+m-3ns} \sum_{k' \in \mathscr{K}_{\alpha}} \frac{d_{k'/k}^{-s}}{o(k')} \prod_{v \nmid \infty} E_{k',v}(s). \tag{5.3}$$

By the same reasoning as behind Theorem 5.1, this conjecture is true if and only if

$$\sum_{k' \in \mathscr{C}(-3\delta)} \frac{d_{k'/k}^{-s}}{o(k')} \prod_{v \nmid \infty} \hat{E}_{k',v}(s) = 3^{r_2+m-3ns} \sum_{k' \in \mathscr{C}(\delta)} \frac{d_{k'/k}^{-s}}{o(k')} \prod_{v \nmid \infty} E_{k',v}(s) \tag{5.4}$$

holds for all $\delta \in k^\times/k^2$.

It is important to note that all the exponents of q_v^{-s} in all the Euler factors are even, and yet the exponent of 3^{-s} in the Ohno-Nakagawa identity is odd in a sense we will presently make clear, and that this implies there is a natural splitting of the Ohno-Nakagawa identity. First, we need to express the Euler products as ordinary Dirichlet series. Each nonarchimedean place v of k corresponds to a prime ideal $\mathfrak{p}_v$ in the ring $\mathfrak{o}$ of integers of k, satisfying $q_v = \mathrm{N}(\mathfrak{p}_v) = (\mathfrak{o} : \mathfrak{p}_v)$, the absolute norm of $\mathfrak{p}_v$. The absolute norm of the ideal $3\mathfrak{o}$ generated by 3 in $\mathfrak{o}$ is just 3^n where $n = [k : \mathbb{Q}]$. Then the two sides of our conjecture expand into series of the following form:

$$3^{r_2+m-3ns} \sum_{k' \in \mathscr{K}_\alpha} \frac{d_{k'/k}^{-s}}{o(k')} \prod_{v \nmid \infty} E_{k',v}(s) = \sum_{k' \in \mathscr{K}_\alpha} \sum_{\mathfrak{a}} \frac{C_{k',\mathfrak{a}}}{\mathrm{N}(3^3 \mathfrak{D}_{k'/k} \mathfrak{a}^2)^s}$$

$$\sum_{k' \in \mathscr{K}_{-\alpha}} \frac{d_{k'/k}^{-s}}{o(k')} \prod_{v \nmid \infty} \hat{E}_{k',v}(s) = \sum_{k' \in \mathscr{K}_{-\alpha}} \sum_{\mathfrak{a}} \frac{\hat{C}_{k',\mathfrak{a}}}{\mathrm{N}(\mathfrak{D}_{k'/k} \mathfrak{a}^2)^s},$$

where the coefficients $C_{k',\mathfrak{a}}$, $\hat{C}_{k',\mathfrak{a}}$ are ordinary rational numbers with denominator a divisor of 6. Here the sum over $\mathfrak{a}$ ranges over all integral ideals of $\mathfrak{o}$; however, due to the nature of the Euler products we may assume that the prime power factor of $\mathfrak{a}$ corresponding to any prime ideal $\mathfrak{p}_v$ is $\mathfrak{p}_v^j$ for $0 \le j \le 2$, except for the prime ideals $\mathfrak{p}_v$ lying over 3 in the dual series, which may have exponents $0 \le j \le 4$.

In order for this conjectured identity to hold, the sum of the coefficients $C_{k_1',\mathfrak{a}_1}$ for given $M = \mathrm{N}(3^3 \mathfrak{D}_{k_1'/k} \mathfrak{a}_1^2)$ for varying $k_1' \in \mathscr{K}_\alpha$ and $\mathfrak{a}_1$ in $\mathfrak{o}$ must equal the sum of the coefficients $\hat{C}_{k_2',\mathfrak{a}_2}$ for $M = \mathrm{N}(\mathfrak{D}_{k_2'/k} \mathfrak{a}_2^2)$ and varying $k_2' \in \mathscr{K}_{-\alpha}$ and $\mathfrak{a}_2$ in $\mathfrak{o}$. The terms cancelling in this subidentity would satisfy

$$\mathrm{N}(\mathfrak{D}_{k_2'/k}) = \mathrm{N}(3 \mathfrak{D}_{k_1'/k} \mathfrak{c}^2)$$

for some fractional ideal $\mathfrak{c}$ in k.

In the case $k = \mathbb{Q}$, this equality of norms together with the fact that k_1 and k_2 have opposite splitting types at ∞ implies that $D_{k_2} = -3D_{k_1}$ modulo multiplication by squares, where D_{k_1} and D_{k_2} are the discriminants, as signed integers, of k_1 and k_2

respectively. The resolvent field of k_1 is then $F = \mathbb{Q}(\sqrt{D_{k_1}})$, while the resolvent field of k_2 is the dual $\hat{F} = \mathbb{Q}(\sqrt{-3D_{k_1}})$. This proves that the Ohno-Nakagawa identity for $\mathbb{Q}$ holds only if all the resolvent field identities (5.4) are true. This completes the proof of Theorem 1.4, which we restate as follows:

Theorem 5.5 *For a square-free integer d, let $\mathscr{C}(d)$ denote the collection of all extensions $k/\mathbb{Q}$ of degree at most 3 with resolvent field $\mathbb{Q}(\sqrt{d})$. Then for $d > 0$, we have*

$$\sum_{k \in \mathscr{C}(-3d)} \frac{d_k^{-s}}{o(k)} \prod_p \hat{E}_{k,p}(s) = 3^{1-3s} \sum_{k \in \mathscr{C}(d)} \frac{d_k^{-s}}{o(k)} \prod_p E_{k,p}(s),$$

$$\sum_{k \in \mathscr{C}(3d)} \frac{d_k^{-s}}{o(k)} \prod_p \hat{E}_{k,p}(s) = 3^{-3s} \sum_{k \in \mathscr{C}(-d)} \frac{d_k^{-s}}{o(k)} \prod_p E_{k,p}(s).$$

Everything in the above identities is the same as in the whole Ohno-Nakagawa identity, just split according to the resolvent fields. For general ground fields k, the norm equality $\mathrm{N}(\mathfrak{D}_{k_2'/k}) = \mathrm{N}(3\mathfrak{D}_{k_1'/k}\mathfrak{c}^2)$ does not strictly imply that $\mathfrak{D}_{k_2'/k} = 3\mathfrak{D}_{k_1'/k}\mathfrak{c}^2$ as ideals (since there are possibly different prime ideals of the same norm). However, based on the role of Scholz' reflection in Nakagawa's proof, it is still natural to suppose that the identity splits according to the resolvent fields. After this, we shall work on simplifying the resolvent Ohno-Nakagawa identity (5.4) by means of class field theory.

CHAPTER 6

Examples of the resolvent Ohno-Nakagawa identity

In this chapter, we shall use known tabulations of extensions of degree at most 3 to verify finite analogues of the resolvent Ohno-Nakagawa identity. The examples in this chapter provide precise numerical evidence for Conjectures 1.1 and 1.2; the approach is different from the equalities of class numbers established in Ohno's original paper [14]. Here, instead of calculating class numbers of integral binary cubic forms, we use existing tables of number fields and calculations of their splitting types at different places to check the conjectures recast as an equality of finite sums of finite Euler products. These equalities come from field extensions with bounded ramification, while the discriminants of the binary cubic forms involved may be enormous and far beyond the tables calculated by Ohno.

6.1 The finite Ohno-Nakagawa identity

Just as in Chapter 5, choose $\delta \in k^\times/k^2$, and let $F = k(\sqrt{\delta})$ and $\hat{F} = k(\sqrt{-3\delta})$. Let S be a finite set of places of k containing all infinite places and all places v dividing $3d_{F/k}d_{\hat{F}/k}$. Let $\mathscr{C}_S(\delta) = \mathscr{C}_S(F)$ be the set of all extensions k'/k of degree at most 3 with resolvent field F and which are unramified for all places $v \notin S$. Class field theory implies the set $\mathscr{C}_S(\delta)$ is a finite set of extensions k'/k. Similarly, $\mathscr{C}_S(-3\delta) = \mathscr{C}_S(\hat{F})$ is a finite set of extensions. Thus, for all extensions k'/k in $\mathscr{C}_S(\delta)$ and in $\mathscr{C}_S(-3\delta)$, the relative discriminant $d_{k'/k}$ is divisible only by q_v for places $v \in S$. The terms in the Dirichlet series $\dfrac{a}{M^s}$ for which M is divisible only by q_v for $v \in S$ must cancel out on both sides of the conjectured resolvent field identity (5.4). This proves the following theorem:

Theorem 6.1 *The generalized resolvent field conjecture* (1.6) *is true if and only if for all* $\delta \in k^\times/k^2$ *and all finite sets of places* S *containing all places* $v \mid \infty$ *and* $v \mid 3d_{F/k}d_{\hat{F}/k}$*, where* $F = k(\sqrt{\delta})$*,* $\hat{F} = k(\sqrt{-3\delta})$*, we have*

$$\sum_{k' \in \mathscr{C}_S(\hat{F})} \frac{d_{k'/k}^{-s}}{o(k')} \prod_{v \in S} \hat{E}_{k',v}(s) = 3^{r_2+m-3ns} \sum_{k' \in \mathscr{C}_S(F)} \frac{d_{k'/k}^{-s}}{o(k')} \prod_{v \in S} E_{k',v}(s). \tag{6.1}$$

Recall that $n = [k : \mathbb{Q}]$*,* r_2 *is the number of complex places of* k*,* $o(k')$ *is the automorphism order of* k'/k *(so 1,2,3, or 6) as defined on p. 21,* $E_{k',v}$ *and* $\hat{E}_{k',v}$ *are the Euler factors defined in (5.1) and (5.2), and* m *is the number of real embeddings of* k *for which* δ *is positive. In particular, by Nakagawa's theorem all these finite identities are true when* $k = \mathbb{Q}$*.*

The crucial aspect of this theorem is that there are only finitely many terms on both sides of the identity. We shall call this the **finite Ohno-Nakagawa identity** for S and δ. In this chapter, we confirm this identity for a fair number of cases based on field data from various sources. Section 6.1 presents numerous confirmations of Nakagawa's theorem based on the data of fields of degree at most 3 over $\mathbb{Q}$, while Section 6.2 presents confirmations of our new conjecture over $k = \mathbb{Q}(i)$.

6.2 Resolvent identities over $\mathbb{Q}$

To verify the finite Ohno-Nakagawa identity for a given finite set of places S of k and element $\delta \in k^\times$, we need a list of the extensions k'/k contained in $\mathscr{C}_S(\delta)$, their relative discriminants, and their splitting types at the places $v \in S$. We have obtained this data from several independent sources, which we shall identify below. Again, as we have established, the identities are known consequences of Nakagawa's theorem, but this verification is quite different from Ohno's original data, and we feel these examples of identities are worth describing in detail.

Cohen et al. give a survey of counting number fields in [3]. The Bordeaux computational number theory group has made available tables of number fields of degree at most 7 and discriminants below specific bounds at

`http://pari.math.u-bordeaux1.fr/pub/pari/packages/nftables/`

We originally consulted the files in the Bordeaux archive called `T20.gp`, `T22.gp`, `T31.gp`, `T33.gp`, where the two numbers refer to degree n and number of real places r_1 of the number fields. A typical line in one of these files would be of the form:

```
[321,[1,-1,-4,1],1,[]]
```

which lists the discriminant 321 of the number field, the vector of coefficients of a generating polynomial x^3-x^2-4x+1, the class number and the structure of the class group of the number field. We are only interested in the discriminant and generating polynomial. These provided verification of the identities over $\mathbb{Q}$ at least for small sets of places S. In particular, for $S=\{2,3\}$, we may extract from the files those fields with discriminant of the form 2^a3^b. In addition to $\mathbb{Q}$, there are 7 quadratic fields and 9 cubic fields up to conjugacy. We list these fields in Table 6.1 sorted by the square-free part δ of their discriminant D_k.

As a reminder, these lists include only one of each conjugate triple of noncyclic cubic extensions of $\mathbb{Q}$. Thus, in our identity, we must use $o(k'/k)=3$ for cyclic cubic extensions and $o(k'/k)=1$ for noncyclic cubic extensions. We can identify the cyclic cubic extensions k'/k over $k=\mathbb{Q}$ as those cubic extensions with discriminant equal to a perfect square. Table 6.1 includes only one cyclic cubic field of discriminant 81.

The next task in verifying the identities is to determine the Euler factors $E_{k',p}(s)$ and $\hat{E}_{k',p}(s)$ for each prime $p=2,3$. This requires determining the splitting type of k' over $k_v=\mathbb{Q}_p$. We use the generating polynomial supplied by the Bordeaux tables to carry this out, and we use the `padic` package in Maple to factor this polynomial p-adically. Here is the Maple procedure that accomplishes this

```
# Take a number field of degree <=3 and find its splitting type at p
splittype:=proc(field,p)
     local x,pol,n,m:
     pol:=poly(field,x):
     n:=degree(pol):
     m:=nops([rootp(pol,p)]):
     if n=m then RETURN(1)
     elif ( field[1] mod p ) = 0 then   # ramified
        if n=2 or m=1 then RETURN(3)
        else RETURN(5) fi:
     else              # unramified
        if n=2 or m=1 then RETURN(2)
        else RETURN(4) fi:
     fi:
end:
```

This procedure assumes that `field` is a vector describing the number field as contained in the Bordeaux tables. Thus, according to the Bordeaux format, the entry `field[1]` is the discriminant of the field. First, this procedures uses another procedure `poly(field,x)` that extracts the generating polynomial of the number field with indeterminate `x`. Then it determines the degree `n` of this polynomial (1, 2, or 3), and the number `m` of roots of the polynomial in $\mathbb{Q}_p$ through the `rootp` command in Maple's `padic` package. If $\mathtt{n} = \mathtt{m}$, then the polynomial splits completely over $\mathbb{Q}_p$ and the type is (1). If not, it then tests to see if p is ramified in k' over $k = \mathbb{Q}$ by simply checking whether or not p divides the discriminant `field[1]`. Then in either case, the type is quadratic if $\mathtt{n} = 2$ or $\mathtt{n} = 3$ and $\mathtt{m} = 1$, which means that the generating polynomial has an irreducible quadratic factor over $\mathbb{Q}_p$. In the procedure, types (1), (2u), (2r), (3u) and (3r) are numbered 1, 2, 3, 4, and 5. The results of these calculations are also shown in Table 6.1.

Given the determination of types, our Maple programs determine the Euler factors by substituting `q` $= q_v$ and `x` $= q_v^{-2s}$ into the corresponding entry of the arrays of formats for the Euler factors listed below

```
eulerfac:=[ (1+x)^2,  (1+x^2),  (1+x),  (1-x+x^2),  1]:
eulerfacdual:=
   [ x^2*(1+q*x+2*q*x^2), x^2*(1+q*x),
     x*(1+q*x^2), x^2*(1+q*x-q*x^2),  1]:
```

using the dual factor only for $p = 3$, when $q_v = 3^{f_v}$ under the assumption that 3 is unramified in k_v with residue degree f_v. The original Euler factors were calculated in [5] and presented here in equation (2.4), while the dual Euler factors for $v \mid 3$ were calculated in Theorem 3.1.

These procedures allow us to evaluate the two sides of the finite Ohno-Nakagawa identity in Theorem 6.1. We present them in the form stated in Theorem 6.1 as sums of partial Euler products. The simplest identity covered by our conjecture is the case where $S = \{3\}$ and $\delta = 1$ or -3, since 3 is the only prime dividing $3d_{F/k}d_{\hat{F}/k}$. In that case, the only fields entering the identities are $\mathbb{Q}$, q_1, k_1 and k_3, since these are the only fields for which the absolute value of the discriminant is a power of 3. Then the identities below have the Euler products (for only the prime $p = 3$) for the fields of discriminant -3, -243 on one side and the fields of discriminant 1, 81 on the other side. The Euler factor for 3 is determined by our recipes with the splitting type read from the last column of Table 6.1.

$\boxed{\delta = 1}$

$$\frac{1}{2}\,3^{-s}\,3^{-2\,s}\left(1+3\cdot 3^{-4\,s}\right)+243^{-s}=3^{1-3s}\left[\frac{1}{6}\left(1+3^{-2\,s}\right)^{2}+\frac{1}{3}\,81^{-s}\right]$$

$\boxed{\delta = -3}$

$$\frac{1}{6}\,3^{-4\,s}\left(1+3\cdot 3^{-2\,s}+6\cdot 3^{-4\,s}\right)+\frac{1}{3}81^{-s}=3^{-3s}\left[\frac{1}{2}\,3^{-s}\left(1+3^{-2\,s}\right)+243^{-s}\right]$$

Both of these identities may be easily checked by hand to be true, as Nakagawa's theorem implies.

Next we turn to the full list of fields which are unramified outside $S = \{2, 3\}$. For $\delta = 1, 2, 3, 6$, we have $m = 0$, while for $\delta = -1, -2, -3, -6$ we have $m = 1$. For the purpose of comparison, we shall group the identities in pairs δ and -3δ (mod squares) since these pairs have the same fields on both sides.

$$\boxed{\delta = 1} \quad \frac{1}{2} 3^{-s} \left(1 + 2^{-4s}\right) 3^{-2s} \left(1 + 3 \cdot 3^{-4s}\right) + 108^{-s} + 243^{-s} \left(1 + 2^{-4s}\right) + 2 \cdot 972^{-s}$$
$$= 3^{1-3s} \left[\frac{1}{6} \left(1 + 2^{-2s}\right)^2 \left(1 + 3^{-2s}\right)^2 + \frac{1}{3} 81^{-s} \left(1 - 2^{-2s} + 2^{-4s}\right)\right]$$

$$\boxed{\delta = -3}$$

$$\frac{1}{6} \left(1 + 2^{-2s}\right)^2 3^{-4s} \left(1 + 3 \cdot 3^{-2s} + 6 \cdot 3^{-4s}\right) + \frac{1}{3} 81^{-s} \left(1 - 2^{-2s} + 2^{-4s}\right)$$
$$= 3^{-3s} \left[\frac{1}{2} 3^{-s} \left(1 + 2^{-4s}\right) \left(1 + 3^{-2s}\right) + 108^{-s} + 243^{-s} \left(1 + 2^{-4s}\right) + 2 \cdot 972^{-s}\right]$$

As a small explanation, both of the above identities concern the two fields in Table 6.1 with $\delta = 1$ and the five fields with $\delta = -3$. The Euler factors are determined according to the recipes given above, with the types read off the last two columns of Table 6.1. The remaining pairs of identities follow below

$$\boxed{\delta = 2} \quad \frac{1}{2} 24^{-s} \left(1 + 2^{-2s}\right) 3^{-2s} \left(1 + 3 \cdot 3^{-4s}\right) + 216^{-s} \left(1 + 2^{-2s}\right)$$
$$= 3^{1-3s} \left[\frac{1}{2} 8^{-s} \left(1 + 2^{-2s}\right) \left(1 + 3^{-4s}\right)\right]$$

$$\boxed{\delta = -6} \quad \frac{1}{2} 8^{-s} \left(1 + 2^{-2s}\right) 3^{-4s} \left(1 + 3 \cdot 3^{-2s}\right)$$
$$= 3^{-3s} \left[\frac{1}{2} 24^{-s} \left(1 + 2^{-2s}\right) \left(1 + 3^{-2s}\right) + 216^{-s} \left(1 + 2^{-2s}\right)\right]$$

$$\boxed{\delta = 3} \quad \frac{1}{2} 4^{-s} \left(1 + 2^{-2s}\right) 3^{-4s} \left(1 + 3 \cdot 3^{-2s}\right) + 324^{-s} \left(1 + 2^{-2s}\right)$$
$$= 3^{1-3s} \left[\frac{1}{2} 12^{-s} \left(1 + 2^{-2s}\right) \left(1 + 3^{-2s}\right)\right]$$

$$\boxed{\delta = -1} \qquad \frac{1}{2}\, 12^{-s}\left(1+2^{-2\,s}\right)3^{-2\,s}\left(1+3\cdot 3^{-4\,s}\right)$$
$$= 3^{-3s}\left[\frac{1}{2}\, 4^{-s}\left(1+2^{-2\,s}\right)\left(1+3^{-4\,s}\right)+324^{-s}\left(1+2^{-2\,s}\right)\right]$$

$$\boxed{\delta = 6} \qquad \frac{1}{2}\, 8^{-s}\left(1+2^{-2\,s}\right)3^{-4\,s}\left(1+3\cdot 3^{-2\,s}+6\cdot 3^{-4\,s}\right)+648^{-s}\left(1+2^{-2\,s}\right)$$
$$= 3^{1-3s}\left[\frac{1}{2}\, 24^{-s}\left(1+2^{-2\,s}\right)\left(1+3^{-2\,s}\right)+1944^{-s}\left(1+2^{-2\,s}\right)\right]$$

$$\boxed{\delta = -2} \qquad \frac{1}{2}\, 24^{-s}\left(1+2^{-2\,s}\right)3^{-2\,s}\left(1+3\cdot 3^{-4\,s}\right)+1944^{-s}\left(1+2^{-2\,s}\right)$$
$$= 3^{-3s}\left[\frac{1}{2}\, 8^{-s}\left(1+2^{-2\,s}\right)\left(1+3^{-2\,s}\right)^2+648^{-s}\left(1+2^{-2\,s}\right)\right]$$

These identities may be verified by elementary algebra to be correct; however, we also used Maple's algebraic simplification tools to verify them by computer. We next proceeded to check cases where the set S of places contains all primes up to and including a given prime p. We denote these sets of extensions by S_p. It turn out that for even relatively small primes p such as $p = 11$, the extensions may have discriminant as large as $2^2 3^5 5^2 7^2 11^2 = 144074700$, which is beyond the published Bordeaux tables. To go further, we used the program `cubic` written by Karim Belabas. The algorithm is established in [1], and the source code is available at

`http://www.math.u-bordeaux.fr/~belabas/research/software/cubic-1.2.tgz`

We made some minor modifications in `cubic` to allow it to restrict output to only fields which are unramified for all $p > 11$. Table 6.2 gives the number of fields in $\mathscr{C}_{S_p}$ of both positive and negative discriminant, with the largest discriminant in each set also displayed. Fields are counted only up to conjugacy. With this data and our Maple procedures, we verified the finite Ohno-Nakagawa identity (6.1) for all cases comprised by $\mathscr{C}_{S_{11}}$.

As one simple further example, we shall take $S = \{2, 3, 7\}$ (thus omitting 5) and $\delta = -7$ and $\delta = 21$. The negative discriminants counted are -7 (quadratic) and $-567 = -3^4 7$ (cubic), and the positive discriminants are 21 (quadratic) and $756 = 2^2 3^3 7$ (cubic). The identities (6.1) turn out to be

$$\boxed{\delta = 21}$$

$$\begin{aligned}
&\frac{1}{2}\, 7^{-s} \left(1 + 2^{-2\,s}\right)^2 3^{-4\,s} \left(1 + 3 \cdot 3^{-2\,s}\right) \left(1 + 7^{-2\,s}\right) + 567^{-s} \left(1 - 2^{-2\,s} + 2^{-4\,s}\right) \left(1 + 7^{-2\,s}\right) \\
&\quad = 3^{1-3s} \left[\frac{1}{2}\, 21^{-s} \left(1 + 2^{-4\,s}\right) \left(1 + 3^{-2\,s}\right) \left(1 + 7^{-2\,s}\right) + 756^{-s} \left(1 + 7^{-2\,s}\right)\right]
\end{aligned}$$

$$\boxed{\delta = -7}$$

$$\begin{aligned}
&\frac{1}{2}\, 21^{-s} \left(1 + 2^{-4\,s}\right) 3^{-2\,s} \left(1 + 3 \cdot 3^{-4\,s}\right) \left(1 + 7^{-2\,s}\right) + 756^{-s} \left(1 + 7^{-2\,s}\right) \\
&\quad = 3^{-3s} \left[\frac{1}{2}\, 7^{-s} \left(1 + 2^{-2\,s}\right)^2 \left(1 + 3^{-4\,s}\right) \left(1 + 7^{-2\,s}\right) + 567^{-s} \left(1 - 2^{-2\,s} + 2^{-4\,s}\right) \left(1 + 7^{-2\,s}\right)\right]
\end{aligned}$$

Again, both identities may be verified by elementary algebra, although they must be true due to Nakagawa's theorem.

After using the Bordeaux tables and Belabas' program to complete the above tests of the finite Ohno-Nakagawa identities, we learned of the program of John Jones and David Roberts which enumerates low degree fields with prescribed ramification, which is exactly what we need to test these identities. The Jones-Roberts algorithms are described in [10], and made available at the website

`http://hobbes.la.asu.edu/NFDB/`

We used this program to confirm the list of fields provided by Belabas' program, as well as the Ohno-Nakagawa identities. It can be used to enumerate fields which are unramified except for primes at most 17, and thus provide more confirmation of Nakagawa's theorem.

6.3 Resolvent identities over $\mathbb{Q}(i)$

Here we take $k = \mathbb{Q}(i)$ and consider the extensions k'/k of degree at most 3. Such fields k' have degree 2, 4 or 6, and all infinite places are complex. The Bordeaux tables include files `T40.gp` and `T60.gp` which list all quartic and sextic totally complex fields up to conjugacy and with maximal discriminant 999988 and -199664, respectively. As it turns out, this is not large enough to verify the identity even for $S = \{1+i, 3\}$. For example, by our earlier list for $\mathbb{Q}$, the sextic field $k(\sqrt[3]{3})$ has discriminant $(-4)^3(243)^2 = -3779136$. The papers [15] and [2] provide information about enumerating sextic fields. Fortunately, the Jones-Roberts program allows us to enumerate all fields of degree at most 6 with prescribed ramification at a small set of primes, and as we shall see this allows us to verify the finite Ohno-Nakagawa identity for $k = \mathbb{Q}(i)$ and $S = \{1+i, 3\}$.

At the website `http://hobbes.la.asu.edu/NFDB/`, we first conducted a search for fields of degree 4, $r_1 = 0$, $r_2 = 2$ with arbitrary size discriminant, but ramification possible only at $p_1 = 2$ and $p_2 = 3$. This would include all quadratic extensions of $\mathbb{Q}(i)$ which are unramified outside $S = \{1+i, 3\}$. This produced a list of 29 degree 4 polynomials corresponding to each possible field up to conjugacy. We next used Jones' program to determine the list of sextic fields with $r_1 = 0$ and ramification only at $p_1 = 2$ and $p_2 = 3$. This produced a list of 140 polynomials, which would include all cubic extensions of $\mathbb{Q}(i)$ unramified outside $S = \{1+i, 3\}$. The lists contain one polynomial for each isomorphism class of field matching the conditions imposed.

The next task is to extract from these lists precisely those polynomials generating extensions of $\mathbb{Q}(i)$. For that purpose, we use the following basic fact from field theory:

Lemma 6.1 *Let $L/\mathbb{Q}$ be a finite extension of degree n, and let $K/\mathbb{Q}$ be an extension of degree $m \mid n$. Let α be an element of L such that $L = \mathbb{Q}(\alpha)$. Then K is a subfield of L if and only if the minimal polynomial of α over K has degree $\frac{n}{m}$. This will be a* *factor of the minimal polynomial of* α *over* $\mathbb{Q}$.

Proof. Let $p(x) \in \mathbb{Q}[x]$ be the monic minimal polynomial of $\alpha \in L$; then the degree of $p(x)$ is $[L : \mathbb{Q}] = n$. Let $q(x)$ be the monic minimal polynomial of α over K which is assumed to have degree n/m where $m = [K : \mathbb{Q}]$. Since the compositum LK is the same as the field $K(\alpha)$, then $[LK : K] = [K(\alpha) : K] = n/m$. Then by the tower law $[LK : \mathbb{Q}] = [LK : K][K : \mathbb{Q}] = (n/m)m = n = [L : \mathbb{Q}]$. This proves $LK = L$ and hence that $K \subset L$. By minimality, $q(x)$ is a factor of $p(x)$.

The converse, where we assume $K \subset L$, immediately follows from the tower law $[L : K] = [L : \mathbb{Q}]/[K : \mathbb{Q}]$. ■

Thus, to extract the extensions of $\mathbb{Q}(i)$, we simply have to check if the polynomials in the lists provided by Jones' program factor over $\mathbb{Q}(i)$. For example, the first quartic field in Jones' list is

$$p(x) = x^4 - x^2 + 1.$$

In PARI, the discriminant of the number field generated by a root is calculated by the command `nfinit(p).disc`, where `p` is the polynomial expression. This example has discriminant $D_{k'} = 144$. We can calculate the factorization of $p(x)$ over $\mathbb{Q}(i)$ by means of the command

```
factornf( x^4 -x^2 +1, y^2+1)
```

with the result being

$$p(x) = x^4 - x^2 + 1 = (x^2 - ix - 1)(x^2 + ix - 1)$$

Thus, this number field is a quadratic extension of $\mathbb{Q}(i)$. When we apply this test to the lists of polynomials produced by Jones' program, we find that 5 of the quartic polynomials and 13 of the sextic polynomials generate extensions of $\mathbb{Q}(i)$. These are presented in Tables 6.3 and 6.4. These tables also contain the absolute discriminants $D_{k'}$ and the absolute norms of the relative discriminants $d_{k'/k} = \mathrm{N}(\mathfrak{D}_{k'/k})$

calculated from the tower law given that $D_{\mathbb{Q}(i)} = -4$. From basic facts about discriminants of towers of number fields (see [18], Proposition VIII.4.13), since $D_k = -4$, we have $D_{k'} = D_k^2 \, \mathrm{N}(\mathfrak{D}_{k'/k}) = 16 \, \mathrm{N}(\mathfrak{D}_{k'/k})$ for $[k' : k] = 2$, and $D_{k'} = D_k^3 \, \mathrm{N}(\mathfrak{D}_{k'/k}) = -64 \, \mathrm{N}(\mathfrak{D}_{k'/k})$ for $[k' : k] = 3$. The cubic factors of the sextic polynomials in Table 6.4 are given in Table 6.5. The quadratic and cubic extensions of $k = \mathbb{Q}(i)$ are given as $k(\alpha)$ where α is a root of the quartic or sextic polynomial in our lists. The two factors of the polynomials over k may give non-conjugate extensions of k.

We can also use PARI's command `factornf` to test each of the quadratic and cubic extensions of $\mathbb{Q}(i)$ to see if they contain the quadratic and cubic number fields listed in Table 6.1. The results are presented in the first column of Table 6.3, where three fields k' are identified as the biquadratic fields $q_1(i)$, $q_2(i)$, and $q_3(i)$, and in the fourth column of Table 6.4, where we see 9 of the sextic polynomials factor over the cubic fields k_j, $1 \leq j \leq 9$.

Since $k = \mathbb{Q}(i)$ has class number 1, the relative discriminant of k'/k is of the form $\mathfrak{D}_{k'/k} = D_{k'/k}\mathbb{Z}[i]$, where $D_{k'/k}$ is determined as an element of $(\mathbb{Z}[i] \setminus \{0\})/\{\pm 1\}$. Also, if $k' = k(\alpha)$ for some $\alpha \in \mathbb{Z}[i]$ and the monic minimal polynomial of α over $\mathbb{Z}[i]$ is $q(x)$, then the discriminant $\Delta(q)$ of $q(x)$ as a polynomial is equal to a square of a nonzero element in $\mathbb{Z}[i]$ times the generator of the relative discriminant of k'/k. Thus, $\Delta(q) = u^2 D_{k'/k}$ for some $u \in \mathbb{Z}[i] \setminus \{0\}$. Then $\mathrm{N}(\Delta(q)) = \mathrm{N}(u)^2 \, d_{k'/k}$. The fourth column of Table 6.5 shows the polynomial discriminant $\Delta(q)$ of $q(x)$, and the fifth column shows $\mathrm{N}(u)^2$. The solutions to $\mathrm{N}(u) = 1$, 2, and 4, resp., in $\mathbb{Z}[i]$ are $u \in \{\pm 1, \pm i\}$, $\{\pm 1 \pm i\}$, and $\{\pm 2, \pm 2i\}$, resp. Then $u^2 \in \{\pm 1\}$, $\{\pm 2i\}$, and ± 4, resp. Since $-1 = i^2$ in $\mathbb{Z}[i]$, these calculations allow us to determine $D_{k'/k}$ modulo squares from the calculations of $\Delta(q)$. This is shown in the sixth column of Table 6.5. This is an easier calculation for the quadratic extensions of $\mathbb{Q}(i)$ in that the discriminant of each quadratic polynomial in Table 6.3 is also a relative discriminant of the extension. When the relative discriminant of $q(x)$ is an integer multiple of $1 \pm i$, then the conjugate factor $\overline{q}(x)$ has conjugate relative discriminant. Since $1+i$ does not equal $1-i$ modulo squares in $\mathbb{Z}[i]$, these two factors $q(x)$, $\overline{q}(x)$ give rise to non-conjugate extensions over k. That explains why the factors q and $\overline{q}$ for $d_{k'/k} = 512$, 4608, 23328 and 209952 in Tables 6.3 and 6.5 gives rise to non-conjugate extensions over k.

The resolvent field of each of the listed extensions is $k(\sqrt{D_{k'/k}}) = k(\sqrt{\delta})$ where δ is the square-free part of $D_{k'/k}$. Since we are considering fields unramified outside $S = \{1+i, 3\}$, the integers of $\mathbb{Z}[i]$ which are divisible only by primes over S are equal modulo squares to precisely one of

$$\delta = 1, \quad i, \quad 1 \pm i, \quad 3, \quad 3i, \quad 3(1 \pm i).$$

In our tables, we have factored out squares and identified δ in the last column.

The next issue is to completely determine for the factorization $p(x) = q(x)\overline{q}(x)$ whether the two factor polynomials $q(x)$ and $\overline{q}(x)$ generate conjugate or non-conjugate extensions k'/k. They are conjugate over $\mathbb{Q}$, but not necessarily over k. In the quartic case, the two extensions are $k(\sqrt{\delta})$ and $k(\sqrt{\overline{\delta}})$ where δ is the generator of the relative discriminant modulo squares. By Kummer theory, these are the same extension if and only if $\delta\overline{\delta} \in k^2$. Since $k^2 \cap \mathbb{Q} = \pm\mathbb{Q}^2$, this means $\mathrm{N}(\delta)$ must be a positive square in $\mathbb{Q}$. Thus, for the non-square discriminants 32 and 288, the factors $q(x)$ and $\overline{q}(x)$ generate two different quadratic extensions k'/k, which are not Galois over $\mathbb{Q}$. This explains our notation for the seven quadratic extensions k' of $\mathbb{Q}(i)$ in Table 6.3. There are relations among the splitting fields of the quartic polynomials. The quartics corresponding to Q_3 and Q_5 both have Galois group D_4, while the others have Galois group $C_2 \times C_2$. The splitting fields of both the Q_3 and Q_5 quartics contains Q_2.

In general, suppose $q(x)$ is a monic irreducible polynomial over $k = \mathbb{Q}(i)$, and that $\overline{q}(x)$ is its conjugate polynomial. If α is a root of $q(x)$, then $\overline{\alpha}$ is a root of $\overline{q}(x)$. Let $K = k(\alpha)$ and $\overline{K} = k(\overline{\alpha})$. If K/k is a cyclic cubic extension, then $K = \overline{K}$ if and only if K is a Galois extension of degree 6 of $\mathbb{Q}$, since then conjugation is an automorphism of order 2 of $K/\mathbb{Q}$. The Galois group is then cyclic C_6 or the symmetric group S_3. In the former case, K contains a cyclic cubic extension of $\mathbb{Q}$, which is unramified outside S. The only possibility is the cyclic field k_1 of discriminant 81. From Table 6.4, the second sextic factors over k_1, and thus the splitting field is the compositum $K_2 = k_1\,k = k_1(i)$, which is cyclic of degree 6 over $\mathbb{Q}$. If the Galois group of $K/\mathbb{Q}$ is S_3, then K contains a conjugate triple of nonconjugate cubic fields. Thus, this can be determined again by factoring over the cubic fields in Table 6.1.

The cyclic extensions K/k have relative discriminant generator equal to a square in k, and from our list we see there are just three possible sextics all of which have $D_{K/k} = 81$. The first sextic factors over the cyclic cubic field k_1, while the second one in our list factors over k_7. Thus, both those sextics have the same cubic extensions of k arising from the two cubic factors over k. The other two sextics each give rise to two distinct but conjugate cyclic cubic extensions of k.

All the other cases listed in Table 6.5 correspond to noncyclic cubic extensions of $k = \mathbb{Q}(i)$, since $D_{k'/k}$ is not a square in k. Suppose the three roots of $q(x)$ generate the three conjugate cubic extensions K_1, K_2 and K_3 over k. Suppose that the compositum of these extensions is the S_3-extension L of k. Then the roots of $\overline{q}(x)$ generate the extensions $\overline{K}_1$, $\overline{K}_2$, and $\overline{K}_3$, and their compositum is $\overline{L}$. The triple $\{K_j\}$ is the same as $\{\overline{K}_j\}$ if and only if $\overline{L} = L$, which again means that $L/\mathbb{Q}$ is a Galois extension of degree 12.

Assuming $\overline{L} = L$, the Galois group G of $L/\mathbb{Q}$ contains S_3 as a normal subgroup corresponding to the subfield $k = \mathbb{Q}(i)$, and contains complex conjugation as an order 2 automorphism. Write $S_3 = \langle \sigma, \tau \mid \sigma^3 = \tau^2 = 1, \tau\sigma\tau = \sigma^{-1} \rangle$, with all these automorphisms fixing k. Let ϕ be complex conjugation on k extended to L. Since S_3 is a normal subgroup of G, the three order 2 elements τ, $\sigma\tau$ and $\sigma^2\tau$ are permuted by conjugation by ϕ. Thus, at least one is fixed by conjugation by ϕ. We may relabel τ to be one of the fixed ones so that $\phi\tau = \tau\phi$. This implies that $\langle \tau, \phi \rangle$ is an order

4 subgroup of G, and thus corresponds to a real cubic extension of $\mathbb{Q}$. That means that the original sextic polynomial factors over this cubic field, which would have to be in the list of fields unramified outside S on page 85. We may check which of the sextics in Table 6.4 factors over the cubics listed on page 85, and if so then we conclude $\overline{L} = L$. This accounts for our notation for the 17 cubic extensions of $\mathbb{Q}(i)$ (up to conjugacy) listed in Table 6.5. Also, Table 6.4 shows the cubic fields k_j named in Table 6.1 over which these sextic polynomials factor. The splitting field of the K_4 sextic contain the splitting fields of K_2 and K_3, while the splitting field of the K_{13} sextic contains K_5 and K_{12}. There are no other relations between the splitting fields.

We may now divide the list of $1+7+17 = 25$ extensions of k which are unramified outside S into subsets $\mathscr{C}_S(\delta)$:

$$\begin{aligned}
\mathscr{C}_S(1) &= \{k, \quad K_2, \quad K_3, \quad K_4, \quad \overline{K}_4\} \\
\mathscr{C}_S(3) &= \{Q_1, \quad K_1, \quad K_7, \quad K_{10}, \quad K_{11}\} \\
\mathscr{C}_S(i) &= \{Q_2, \quad K_8\} \\
\mathscr{C}_S(3i) &= \{Q_4, \quad K_5, \quad K_{12}, \quad K_{13}, \quad \overline{K}_{13}\} \\
\mathscr{C}_S(1+i) &= \{Q_3, \quad K_9\} \\
\mathscr{C}_S(3(1+i)) &= \{Q_5, \quad K_6\} \\
\mathscr{C}_S(1-i) &= \{\overline{Q}_3, \quad \overline{K}_9\} \\
\mathscr{C}_S(3(1-i)) &= \{\overline{Q}_5, \quad \overline{K}_6\}
\end{aligned}$$

For $k = \mathbb{Q}(i)$, we have $r_1 = m = 0$ and $n = 2$, and thus the finite Ohno-Nakagawa identity takes the form

$$\sum_{k' \in \mathscr{C}_S(-3\delta)} \hat{Y}_{k',S}(s) = 3^{1-6s} \sum_{k' \in \mathscr{C}_S(\delta)} Y_{k',S}(s),$$

with

$$Y_{k',S}(s) = \frac{d_{k'/k}^{-s}}{o(k')} \prod_{v \in S} E_{k',v}(s) \qquad\qquad \hat{Y}_{k',S}(s) = \frac{d_{k'/k}^{-s}}{o(k')} \prod_{v \in S} \hat{E}_{k',v}(s).$$

For $S \subset \{1+i, 3\}$, we have listed all the extensions and their corresponding values of $D_{k'/k}$. The final piece of information we need to check the identities is the calculation of the splitting type of each place $v \in S$ in the corresponding extensions of k. This would allow us to choose the correct Euler factors $E_{k',v}$ and $\hat{E}_{k',v}$. The splitting type is directly determined from the factorization of the prime ideal $\mathfrak{p}$ corresponding to the place v as a product of prime ideals in the ring of integers of k'. The program PARI contains a procedure `idealfactor` that calculates the prime ideal factorization of a rational prime p in a number field.

As an example, take the extension k' corresponding to $f(x) = x^6 - 2x^3 + 2$. The following PARI commands compute the factorization of $p = 3$ in k'

```
? K=nfinit(x^6-2*x^3+2);\\ Initialize number field K
? fact=idealfactor(K,3);\\ Give prime ideal factorization of 3 in K
? matsize(fact)\\ First dimension = number of prime ideals in fact
% 3 = [1, 2]
? fact[1,2]\\ Second column gives exponent of prime ideal in fact
% 4 = 3
? fact[1,1][3]\\ Ramification order of prime ideal for this row
% 5 = 3
? fact[1,1][4]\\ Residue degree of prime ideal for this row
% 6 = 2
```

The output is a $g \times 2$ matrix where g is the number of distinct prime ideal factors over p in k'. The second column of this matrix gives the power, i.e. ramification order, of that prime ideal for that row in the factorization $p\mathfrak{o}_{k'} = \mathfrak{P}_1^{e_1}\mathfrak{P}_2^{e_2}\cdots\mathfrak{P}_g^{e_g}$. The first entry in the i-th row is a vector of data describing the prime ideal $\mathfrak{P}_i$. The 3rd and 4th entries in that vector are the ramification order e_i and the residue degree f_i of that prime ideal $\mathfrak{P}_i$ in k'. Thus, in our example $3\mathfrak{o}_{k'} = \mathfrak{P}^3$ in this number field where $\mathfrak{P}$ is a prime ideal of residue degree 2. The absolute norm of $\mathfrak{P}$ is then $3^2 = 9$.

The results of this calculation for our complete list of fields is shown in Tables 6.6 and 6.7. The prime 2 splits into at most 3 distinct prime ideals, while the prime 3 splits into at most 2 distinct prime ideals in our list of extensions. Since $2\mathbb{Z}[i] = (1+i)^2\mathbb{Z}[i]$ in k, this explains why all the ramification orders in Table 6.6 are even, and we can deduce the splitting type of $\mathfrak{p} = (1+i)\mathbb{Z}[i]$ in k'/k by just halving the ramification orders. The prime 3 is inert in k with residue degree 2. Thus, we can determine the splitting type in k'/k by halving the residue degrees f_i in Table 6.7.

Before delving into all the cases of our conjecture for $S = \{1+i, 3\}$, the easiest case to check for $k = \mathbb{Q}(i)$ is when $S = \{3\}$ in which case δ may be one of $\{1, 3, i, 3i\}$. In this case, we are reduced to extensions k'/k which are ramified only at the prime 3. That leads to the extensions:

$$\begin{aligned}
\mathscr{C}_S(1) &= \{k, \quad K_2, \quad K_3, \quad K_4, \quad \overline{K}_4\} \\
\mathscr{C}_S(3) &= \{Q_1, \quad K_7\} \\
\mathscr{C}_S(i) &= \emptyset \\
\mathscr{C}_S(3i) &= \emptyset
\end{aligned}$$

All the other extensions in our table are ramified over $1+i$. Then the two Ohno-Nakagawa identities to check are

$$\hat{Y}_{Q_1}(s) + \hat{Y}_{K_7}(s) = 3^{1-6s}\left(Y_k(s) + Y_{K_2}(s) + Y_{K_3}(s) + 2Y_{K_4}(s)\right);$$
$$\hat{Y}_k(s) + \hat{Y}_{K_2}(s) + \hat{Y}_{K_3}(s) + 2\hat{Y}_{K_4}(s) = 3^{1-6s}\left(Y_{Q_1}(s) + Y_{K_7}(s)\right).$$

(Here we abbreviate $Y_K = Y_{K,S}$.) Note that the Y term is the same for extensions K, $\overline{K}$ which are conjugate over $\mathbb{Q}$. For these identities, we need only the Euler factor for the prime 3. Note that 3 has type $(3r)$ in every cubic extension of k, and thus the Euler factor is the same for all these extensions. Our knowledge of the types leads to, knowing that $q_v^{-2s} = 9^{-2s}$ for $v = 3$,

$$
\begin{aligned}
E_{k,3}(s) &= (1+9^{-2s})^2 \\
E_{Q_1,3}(s) &= (1+9^{-2s}) \\
E_{K_2,3}(s) = E_{K_3,3}(s) = E_{K_4,3}(s) = E_{K_7,3}(s) &= 1 \\
\hat{E}_{k,3}(s) &= \ 9^{-4s}(1+9^{1-2s}+2\cdot 9^{1-4s}) \\
\hat{E}_{Q_1,3}(s) &= \ 9^{-2s}(1+9^{1-4s}) \\
\hat{E}_{K_2,3}(s) = \hat{E}_{K_3,3}(s) = \hat{E}_{K_4,3}(s) = \hat{E}_{K_7,3}(s) &= 1
\end{aligned}
$$

The first finite Ohno-Nakagawa identity would then be

$$
\begin{aligned}
\hat{Y}_{Q_1}(s) + \hat{Y}_{K_7}(s) &= 3^{1-6s}\left(Y_k(s) + Y_{K_2}(s) + Y_{K_3}(s) + 2Y_{K_4}(s)\right) \\
\frac{1}{2} 9^{-s}\,\hat{E}_{Q_1,3}(s) + 59049^{-s}\,\hat{E}_{K_7,3}(s) &= \\
3^{1-6s}\left(\frac{1}{6}\,E_{k,3}(s) + \frac{1}{3}\,6561^{-s}\,E_{K_2,3}(s)\right. &\left. + \frac{1}{3}\,6561^{-s}\,E_{K_3,3}(s) + \frac{2}{3}\,6561^{-s}\,E_{K_4,3}(s)\right) \\
\frac{1}{2} 9^{-s} 9^{-2\,s}\left(1+9^{1-4\,s}\right) + 59049^{-s} &= 3^{1-6\,s}\left(\frac{1}{6}\,\left(1+9^{-2\,s}\right)^2 + \frac{4}{3}\,6561^{-s}\right) \\
\frac{1}{2} 3^{-6s}\left(1+3^{2-8\,s}\right) + 3^{-10s} &= 3^{1-6\,s}\left(\frac{1}{6}\,\left(1+3^{-4\,s}\right)^2 + \frac{4}{3}\,3^{-8s}\right) \\
\frac{1}{2}\,3^{-6s} + \frac{1}{2}\,3^{2-14\,s} + 3^{-10s} &= \frac{1}{2}\,3^{-6\,s}\left(1 + 2\cdot 3^{-4\,s} + 3^{-8\,s} + 8\cdot 3^{-8s}\right) \\
&= \frac{1}{2}\,3^{-6\,s} + 3^{-10\,s} + \frac{9}{2}\,3^{-14\,s} \\
0 &= 0,
\end{aligned}
$$

after subtracting terms on both sides, which verifies our conjectured identity.

The second potential identity reduces as follows

$$\begin{aligned}
\hat{Y}_k(s) + \hat{Y}_{K_2}(s) + \hat{Y}_{K_3}(s) + 2\hat{Y}_{K_4}(s) &= 3^{1-6s}\left(Y_{Q_1}(s) + Y_{K_7}(s)\right) \\
\frac{1}{6}\,\hat{E}_{k,3}(s) + \frac{1}{3}\,6561^{-s}\,\hat{E}_{K_2,3}(s) + \frac{1}{3}\,6561^{-s}\,\hat{E}_{K_3,3}(s) + \frac{2}{3}\,6561^{-s}\,\hat{E}_{K_4,3}(s) &= \\
3^{1-6s}\left(\frac{1}{2}9^{-s}\,E_{Q_1,3}(s) + 59049^{-s}\,E_{K_7,3}(s)\right) \\
\frac{1}{6}\,9^{-4s}(1 + 9^{1-2s} + 2\cdot 9^{1-4s}) + \frac{4}{3}\,6561^{-s} &= 3^{1-6s}\left(\frac{1}{2}9^{-s}\,(1 + 9^{-2s}) + 59049^{-s}\right) \\
\frac{1}{6}\,3^{-8s}(1 + 3^{2-4s} + 2\cdot 3^{2-8s}) + \frac{4}{3}\,3^{-8s} &= 3^{1-6s}\left(\frac{1}{2}3^{-2s}\,(1 + 3^{-4s}) + 3^{-10s}\right) \\
\frac{1}{6}\,3^{-8s} + \frac{1}{2}\,3^{1-12s} + 3^{1-16s} + \frac{4}{3}\,3^{-8s} &= 3^{1-6s}\left(\frac{1}{2}\,3^{-2s} + \frac{1}{2}\,3^{-6s} + 3^{-10s}\right) \\
\frac{3}{2}\,3^{-8s} + \frac{1}{2}\,3^{1-12s} + 3^{1-16s} &= \frac{1}{2}\,3^{1-8s} + \frac{1}{2}\,3^{1-12s} + 3^{1-16s} \\
0 &= 0,
\end{aligned}$$

and again our conjectured identity holds in this example.

The next easiest case to check is $\delta = 1 + i$ which comes down to

$$\hat{Y}_{Q_5}(s) + \hat{Y}_{K_6}(s) = 3^{1-6s}\left(Y_{Q_3}(s) + Y_{K_9}(s)\right).$$

Note that $1 + i$ divides all the relative discriminants and so is ramified and in fact of type (2r) in all 4 extensions. The prime 3 is ramified in all but Q_3. It has type (2r) in Q_5 and type (3r) in K_6 and K_9. Then we have

$$\begin{aligned}
\hat{Y}_{Q_5}(s) &= \frac{2^{-5s}3^{-2s}}{2}(1 + 2^{-2s})9^{-2s}(1 + 9^{1-4s}) \\
\hat{Y}_{K_6}(s) &= 2^{-5s}3^{-6s}(1 + 2^{-2s}) \\
Y_{Q_3}(s) &= \frac{2^{-5s}}{2}(1 + 2^{-2s})(1 + 9^{-4s}) \\
Y_{K_9}(s) &= 2^{-5s}3^{-8s}(1 + 2^{-2s})
\end{aligned}$$

After cancelling the common factor of $2^{-5s}(1 + 2^{-2s})$, our conjectured identity boils down to

$$\begin{aligned}
\frac{3^{-2s}}{2} 9^{-2s}(1 + 9^{1-4s}) + 3^{-6s} &= 3^{1-6s}\left(\frac{1}{2}(1 + 9^{-4s}) + 3^{-8s}\right) \\
\frac{1}{2}\,3^{-6s}(1 + 3^{2-8s}) + 3^{-6s} &= 3^{1-6s}\left(\frac{1}{2}(1 + 3^{-8s}) + 3^{-8s}\right) \\
\frac{1}{2}\,3^{-6s} + \frac{1}{2}\,3^{2-14s} + 3^{-6s} &= \frac{3}{2}\,3^{-6s} + \frac{9}{2}\,3^{-14s} \\
0 &= 0.
\end{aligned}$$

Again, the identity holds. All the cases corresponding to the field data for $S = \{1+i, 3\}$ presented in this chapter have been verified to work as well. We believe these examples are fairly compelling for the truth of our conjecture at least for $\mathbb{Q}(i)$, although we intend to present more comprehensive numerical verifications for this and other base fields k in a subsequent work.

#	D_k	k	δ	Type of 2	Type of 3
$\mathbb{Q}$	1	$\mathbb{Q}$	1	(1)	(1)
q_1	-3	$\mathbb{Q}(\sqrt{-3})$	-3	(2u)	(2r)
q_2	8	$\mathbb{Q}(\sqrt{2})$	2	(2r)	(2u)
q_3	-24	$\mathbb{Q}(\sqrt{-6})$	-6	(2r)	(2r)
q_4	12	$\mathbb{Q}(\sqrt{3})$	3	(2r)	(2r)
q_5	-4	$\mathbb{Q}(\sqrt{-1})$	-1	(2r)	(2u)
q_6	24	$\mathbb{Q}(\sqrt{6})$	6	(2r)	(2r)
q_7	-8	$\mathbb{Q}(\sqrt{-2})$	-2	(2r)	(1)
k_1	81	$\mathbb{Q}(\alpha \mid \alpha^3 - 3\alpha - 1 = 0)$	1	(3u)	(3r)
k_2	-108	$\mathbb{Q}(\sqrt[3]{2})$	-3	(3r)	(3r)
k_3	-243	$\mathbb{Q}(\sqrt[3]{3})$	-3	(2u)	(3r)
k_4	-972	$\mathbb{Q}(\sqrt[3]{6})$	-3	(3r)	(3r)
k_5	-972	$\mathbb{Q}(\sqrt[3]{12})$	-3	(3r)	(3r)
k_6	-216	$\mathbb{Q}(\alpha \mid \alpha^3 + 3\alpha - 2 = 0)$	-6	(2r)	(3r)
k_7	-324	$\mathbb{Q}(\alpha \mid \alpha^3 - 3\alpha - 4 = 0)$	-1	(2r)	(3r)
k_8	1944	$\mathbb{Q}(\alpha \mid \alpha^3 - 9\alpha - 6 = 0)$	6	(2r)	(3r)
k_9	-648	$\mathbb{Q}(\alpha \mid \alpha^3 - 3\alpha - 10 = 0)$	-2	(2r)	(3r)

Table 6.1: Fields of degree ≤ 3 unramified outside 2,3.

p	$\#\mathscr{C}^+_{S_p}$	$D_{\max}$	$\#\mathscr{C}^-_{S_p}$	$D_{\max}$
2	2	$8 = 2^3$	2	$-8 = -2^3$
3	6	$1944 = 2^3 3^5$	11	$-972 = -2^2 3^5$
5	14	$16200 = 2^3 3^4 5^2$	34	$-24300 = -2^2 3^5 5^2$
7	39	$793800 = 2^3 3^4 5^2 7^2$	101	$-1190700 = -2^2 3^5 5^2 7^2$
11	105	$96049800 = 2^3 3^4 5^2 7^2 11^2$	319	$-144074700 = -2^2 3^5 5^2 7^2 11^2$

Table 6.2: Number of fields (up to conjugacy) of degree ≤ 3 unramified for primes $> p$

k'	$D_{k'}$	$d_{k'/k}$	$p(x)$	$q(x)$	$D_{k'/k}$	δ
$Q_1 = q_1(i)$	144	$9 = 3^2$	$x^4 - x^2 + 1$	$x^2 - ix - 1$	3	3
$Q_2 = q_2(i)$	256	$16 = 2^4$	$x^4 + 1$	$x^2 + i$	$4i$	i
$Q_3, \overline{Q}_3$	512	$32 = 2^5$	$x^4 - 2\,x^2 + 2$	$x^2 - 1 - i$	$4(1 \pm i)$	$1 \pm i$
$Q_4 = q_3(i)$	2304	$144 = 2^4 3^2$	$x^4 + 9$	$x^2 - 3\,i$	$12i$	$3i$
$Q_5, \overline{Q}_5$	4608	$288 = 2^5 3^2$	$x^4 - 6\,x^2 + 18$	$x^2 - 3 - 3\,i$	$12(1 \pm i)$	$3(1 \pm i)$

Table 6.3: Polynomials generating quadratic extensions of $\mathbb{Q}(i)$ unramified outside $1 + i$, 3.

$D_{k'}$	$d_{k'/k}$	$p(x)$	Factors over
-186624	$2916 = 2^2 3^6$	$x^6 - 2\,x^3 + 2$	k_2
-419904	$6561 = 3^8$	$x^6 + 6\,x^4 + 9\,x^2 + 1$	k_1
-419904	$6561 = 3^8$	$x^6 - 2\,x^3 + 9\,x^2 + 6\,x + 2$	k_7
-419904	$6561 = 3^8$	$x^6 - 2x^3 + 9x^2 - 12x + 5$	none
-746496	$11664 = 2^4 3^6$	$x^6 - 6x^4 + 9x^2 + 4$	k_6
-1492992	$23328 = 2^5 3^6$	$x^6 + 9\,x^2 - 12\,x + 4$	none
-3779136	$59049 = 3^{10}$	$x^6 + 9$	k_3
-6718464	$104976 = 2^4 3^8$	$x^6 - 4x^3 + 9x^2 + 12x + 8$	k_9
-13436928	$209952 = 2^5 3^8$	$x^6 - 4x^3 + 9x^2 - 24x + 20$	none
-15116544	$236196 = 2^2 3^{10}$	$x^6 + 36$	k_4
-15116544	$236196 = 2^2 3^{10}$	$x^6 - 6x^3 + 18$	k_5
-60466176	$944784 = 2^4 3^{10}$	$x^6 + 18x^4 + 81x^2 + 36$	k_8
-60466176	$944784 = 2^4 3^{10}$	$x^6 - 18x^4 - 24x^3 + 81x^2 + 216x + 180$	none

Table 6.4: Sextic polynomials generating cubic extensions of $\mathbb{Q}(i)$ unramified outside $1+i$, 3.

k'	$d_{k'/k}$	$q(x)$	$\Delta(q)$	$N(\Delta(q))/d_{k'/k}$	$D_{k'/k}$	δ
K_1	2916	$x^3 - 1 + i$	$54i$	1	$54i$	3
K_2	6561	$x^3 + 3x + i$	-81	1	81	1
K_3	6561	$x^3 + 3ix - 1 + i$	$162i$	4	81	1
$K_4, \overline{K}_4$	6561	$x^3 - 3ix - 1 + 2i$	81	1	81	1
K_5	11664	$x^3 - 3x - 2i$	216	4	$108i$	$3i$
$K_6, \overline{K}_6$	23328	$x^3 + 3ix - 2i$	$108(1+i)$	1	$108(1 \pm i)$	$3(1 \pm i)$
K_7	59049	$x^3 - 3i$	243	1	243	3
K_8	104976	$x^3 - 3ix - 2 - 2i$	$-324i$	1	$324i$	i
$K_9, \overline{K}_9$	209952	$x^3 + 3ix - 2 - 4i$	$324(1-i)$	1	$324(1 \pm i)$	$1 \pm i$
K_{10}	236196	$x^3 - 6i$	972	4	$486i$	3
K_{11}	236196	$x^3 - 3 + 3i$	$486i$	1	$486i$	3
K_{12}	944784	$x^3 + 9x - 6i$	-1944	4	$972i$	$3i$
$K_{13}, \overline{K}_{13}$	944784	$x^3 - 9x - 12 - 6i$	$-3888i$	16	$972i$	$3i$

Table 6.5: Cubic polynomials generating cubic extensions of $\mathbb{Q}(i)$ unramified outside $1+i$, 3.

k'	$d_{k'/k}$	e_1	f_1	e_2	f_2	e_3	f_3	Type in k'/k
Q_1	9	2	2					(2u)
Q_2	16	4	1					(2r)
$Q_3, \overline{Q}_3$	32	4	1					(2r)
Q_4	144	4	1					(2r)
$Q_5, \overline{Q}_5$	288	4	1					(2r)
K_1	2916	6	1					(3r)
K_2	6561	2	3					(3u)
K_3	6561	2	1	2	1	2	1	(1)
$K_4, \overline{K}_4$	6561	2	3					(3u)
K_5	11664	4	1	2	1			(2r)
$K_6, \overline{K}_6$	23328	4	1	2	1			(2r)
K_7	59049	2	1	2	2			(2u)
K_8	104976	4	1	2	1			(2r)
$K_9, \overline{K}_9$	209952	4	1	2	1			(2r)
K_{10}	236196	6	1					(3r)
K_{11}	236196	6	1					(3r)
K_{12}	944784	4	1	2	1			(2r)
$K_{13}, \overline{K}_{13}$	944784	4	1	2	1			(2r)

Table 6.6: Splitting type of 2 in $k'/\mathbb{Q}$, and corresponding type of $(1+i)\mathbb{Z}[i]$ in k'/k.

k'	$d_{k'/k}$	e_1	f_1	e_2	f_2	Type in k'/k
Q_1	9	2	2			(2r)
Q_2	16	1	2	1	2	(1)
$Q_3, \overline{Q}_3$	32	1	4			(2u)
Q_4	144	2	2			(2r)
$Q_5, \overline{Q}_5$	288	2	2			(2r)
K_1	2916	3	2			(3r)
K_2	6561	3	2			(3r)
K_3	6561	3	2			(3r)
$K_4, \overline{K}_4$	6561	3	2			(3r)
K_5	11664	3	2			(3r)
$K_6, \overline{K}_6$	23328	3	2			(3r)
K_7	59049	3	2			(3r)
K_8	104976	3	2			(3r)
$K_9, \overline{K}_9$	209952	3	2			(3r)
K_{10}	236196	3	2			(3r)
K_{11}	236196	3	2			(3r)
K_{12}	944784	3	2			(3r)
$K_{13}, \overline{K}_{13}$	944784	3	2			(3r)

Table 6.7: Splitting type of 3 in $k'/\mathbb{Q}$, and corresponding type of $3\mathbb{Z}[i]$ in k'/k.

CHAPTER 7

Expressing the resolvent Dirichlet series as sums of idele class group characters

In this last chapter, we shall use class field theory to interpret the resolvent field Ohno-Nakagawa identity (5.4) as an equality of sums over a group of characters of the idele class groups of $F = k(\sqrt{\delta})$ and $\hat{F} = k(\sqrt{-3\delta})$. Scholz' reflection theorem in [16], developed further in Nakagawa [13], proves a relationship between the absolute class groups of $\mathbb{Q}(\sqrt{\delta})$ and $\mathbb{Q}(\sqrt{-3\delta})$. It is our belief that these ideas and the formulas in this chapter will eventually complete the proof of the generalized Ohno-Nakagawa conjecture.

7.1 Simplification of the generalized Ohno-Nakagawa conjecture

The generalized Ohno-Nakagawa conjecture, in the form of the finite resolvent field conjecture, can be restated here as

$$\sum_{k' \in \mathscr{C}_S(-3\delta)} \frac{d_{k'/k}^{-s}}{o(k')} \prod_{v \in S} \hat{E}_{k',v}(s) = 3^{r_2+m-3ns} \sum_{k' \in \mathscr{C}_S(\delta)} \frac{d_{k'/k}^{-s}}{o(k')} \prod_{v \in S} E_{k',v}(s). \tag{7.1}$$

where $\delta \in k^\times/k^2$ and S is a set of places v containing all infinite places and all v that divide $3d_{F/k}$. The general resolvent conjecture is the limit of these as S tends to the set of all places of k

$$\sum_{k' \in \mathscr{C}(-3\delta)} \frac{d_{k'/k}^{-s}}{o(k')} \prod_{v \nmid \infty} \hat{E}_{k',v}(s) = 3^{r_2+m-3ns} \sum_{k' \in \mathscr{C}(\delta)} \frac{d_{k'/k}^{-s}}{o(k')} \prod_{v \nmid \infty} E_{k',v}(s). \tag{7.2}$$

In this chapter, we should make clear that, unlike the introduction, these are sums over extensions up to conjugacy, and thus we take

$$o(k') = \begin{cases} 6 & \text{if } k' = k, \\ 2 & \text{if } [k' : k] = 2, \\ 3 & \text{if } [k' : k] = 3 \text{ and } k'/k \text{ is cyclic,} \\ 1 & \text{if } [k' : k] = 3 \text{ and } k'/k \text{ is non-cyclic.} \end{cases}$$

The extensions in $\mathscr{C}(\delta)$ consist of precisely $F = k(\sqrt{\delta})$ and all cubic extensions k'/k where the Galois closure L of k' over k is a cyclic cubic extension of F. By class field theory, the finite abelian extensions L/F bijectively correspond to the open subgroups U of the idele class group $\mathscr{J}_F = \mathbb{A}_F^\times/F^\times$ such that $\mathscr{J}_F/U \cong \mathrm{Gal}(L/F)$. Thus, the cyclic cubic L/F correspond to the open subgroups U of $\mathbb{A}_F^\times/F^\times$ of index 3. The quotient $\mathscr{J}_F/U$ is a cyclic group of order 3, and is therefore the kernel of precisely two nontrivial cubic characters χ, χ^2 of $\mathscr{J}_F$. Thus, there is a one-to-two correspondence between cyclic cubic extensions L/F and cubic characters χ of $\mathscr{J}_F$.

In our series, if F/k is quadratic, we require the cyclic cubic extensions L/F to be S_3-extensions of the base field k. The Galois group $\mathrm{Gal}(F/k)$ has order two, and we denote a generator by σ. This Galois group acts on $\mathscr{J}_F$ in the usual manner, and the condition that an open subgroup U of index 3 in $\mathscr{J}_F$ should correspond to an S_3-extension L/k is precisely that $U^\sigma = U$ and $x^\sigma = x^{-1} \pmod{U}$ for all $x \in \mathscr{J}_F$. Alternatively, this means that the corresponding cubic character χ of $\mathscr{J}_F$ should satisfy $\chi(xx^\sigma) = \chi(\mathrm{N}_{F/k}(x)) = 1$ for all $x \in \mathscr{J}_F$. If $F = k$, there is no extra condition on χ. We now define $\mathscr{X}(F)$ to be the group of characters χ of $\mathscr{J}_F$ such that $\chi^3 = 1$ and, if $[F : k] = 2$, we have $\chi \circ \mathrm{N}_{F/k} = 1$. Given a finite set S of places of k, we define $\mathscr{X}_S(F)$ to consist of those characters which are unramified at all places outside S.

The trivial character $\chi = 1$ corresponds to the trivial extension F of F. Thus, $k' = F \in \mathscr{C}(\delta)$ corresponds only to $\chi = 1$. If $[k' : k] = 3$, there are two characters

χ, χ^2 of $\mathcal{J}_F$ corresponding to k'. Thus, it makes sense to define

$$o(\chi) = \begin{cases} 6 & \text{if } k' = k, \\ 2 & \text{if } [k' : k] = 2, \\ 6 & \text{if } [k' : k] = 3 \text{ and } k'/k \text{ is cyclic}, \\ 2 & \text{if } [k' : k] = 3 \text{ and } k'/k \text{ is non-cyclic}, \end{cases}$$

to compensate for this doubling of terms in the latter two cases. However, now $o(\chi) = o(F)$ depends only on whether F is k or not. Thus, it factors out of the series altogether. This leads to

$$\sum_{k' \in \mathscr{C}(\delta)} \frac{d_{k'/k}^{-s}}{o(k')} \prod_{v \nmid \infty} E_{k',v}(s) = \frac{1}{o(F)} \sum_{\chi \in \mathcal{X}(F)} d_{k'/k}^{-s} \prod_{v \nmid \infty} E_{k',v}(s).$$

Next, we consider the factors $d_{k'/k}^{-s}$. First, let us assume $[F : k] = 2$. According to Hasse (Satz 3 in [9]), the relative discriminant satisfies $\mathfrak{D}_{k'/k} = \mathfrak{D}_{F/k} \operatorname{N}_{F/k}(\mathfrak{f}_\chi)$ where $\mathfrak{f}_\chi$ is the conductor of the character χ. Taking the absolute norm over $\mathbb{Q}$, we then have $d_{k'/k} = d_{F/k} \operatorname{N}(\mathfrak{f}_\chi)$. Now we may factor our series further

$$\sum_{k' \in \mathscr{C}(\delta)} \frac{d_{k'/k}^{-s}}{o(k')} \prod_{v \nmid \infty} E_{k',v}(s) = \frac{d_{F/k}^{-s}}{o(F)} \sum_{\chi \in \mathcal{X}(F)} \operatorname{N}(\mathfrak{f}_\chi)^{-s} \prod_{v \nmid \infty} E_{k',v}(s).$$

With reference to the dual resolvent field $\hat{F} = k(\sqrt{-3\delta})$, the same analysis applied to the dual Dirichlet series on the left side of (7.2) with the result

$$\sum_{k' \in \mathscr{C}(-3\delta)} \frac{d_{k'/k}^{-s}}{o(k')} \prod_{v \nmid \infty} \hat{E}_{k',v}(s) = \frac{d_{\hat{F}/k}^{-s}}{o(\hat{F})} \sum_{\chi \in \mathcal{X}(\hat{F})} \operatorname{N}(\mathfrak{f}_\chi)^{-s} \prod_{v \nmid \infty} \hat{E}_{k',v}(s).$$

This exhibits both sides of the Ohno-Nakagawa identity as sums over groups of idele class characters. The next task is to rewrite the Euler factors $E_{k',v}$, $\hat{E}_{k',v}$ in terms of the idelic class character χ corresponding to k'.

7.2 Shintani's Dirichlet series as sums of idele class group characters

In this section we provide a proof of formula (1.7), which gives Shintani's Dirichlet series as a sum of idele class group characters. We also deduce a similar formula for the dual Dirichlet series. The factor $E_{k',v}$ is a polynomial function of $q_v^{-2s} = \mathrm{N}(\mathfrak{p}_v)$, where the polynomial depends on the splitting type of v in k'/k. That splitting type may be determined by the splitting type of v in the resolvent extension F/k and the splitting type of any place $w \mid v$ of F in the extension L/F which is the Galois closure of k'/k.

For any character χ of $\mathcal{J}_F$, the homomorphism $\mathrm{id} : \mathbb{A}_F^\times \to \mathscr{I}_F$ introduced in Section 5.2 induces a character, also denoted by χ, on the group of fractional ideals of F that are relatively prime with the conductor $\mathfrak{f}_\chi$. This is defined as follows: If $\mathfrak{a}$ is such an ideal, then we define $\chi(\mathfrak{a})$ to be equal to $\chi(aF^\times)$, for $a \in \mathbb{A}_F^\times$ such that $\mathrm{id}(a) = \mathfrak{a}$.

Let us assume first that F/k is quadratic. If L/F is a cyclic cubic extension corresponding to the idele class group character χ of order 3, then class field theory gives very simple rules for its splitting type:

> The finite place w of F is totally ramified in L/F if and only if the prime ideal $\mathfrak{P}_w \in \mathscr{I}_F$ corresponding to w is a divisor of the conductor of $\mathfrak{f}_\chi$ of χ.
>
> The finite place w splits completely in L/F if and only if $\chi(\mathfrak{P}_w) = 1$.
>
> If $\chi(\mathfrak{P}_w) \neq 1$, then $\mathfrak{P}_w$ remains prime in L, i.e. $\mathfrak{P}_w$ is inert in L/F.

A prime ideal $\mathfrak{p}_v$ in k either remains prime in F: $\mathfrak{p}_v\mathfrak{o}_F = \mathfrak{P}_w$, or is ramified of degree 2 in F: $\mathfrak{p}_v\mathfrak{o}_F = \mathfrak{P}_w^2$, or splits as product of distinct conjugate ideals: $\mathfrak{p}_v\mathfrak{o}_F = \mathfrak{P}_w\overline{\mathfrak{P}_w}$

in F. The splitting type of k'/k at v can be determined in terms of δ and χ as follows

$$(k'/k, v) = \begin{cases} (1) & \text{if } \psi(\delta \mathfrak{o}_k) = 1 \text{ and } \chi(\mathfrak{P}_w) = 1, \\ (2u) & \text{if } \psi(\delta \mathfrak{o}_k) = -1, \\ (2r) & \text{if } \mathfrak{p}_v \mid \delta \mathfrak{o}_k \text{ and } \mathfrak{P}_w \nmid \mathfrak{f}_\chi, \\ (3u) & \text{if } \psi(\delta \mathfrak{o}_k) = 1 \text{ and } \chi(\mathfrak{P}_w) \neq 1, \\ (3r) & \text{if } \mathfrak{P}_w \mid \mathfrak{f}_\chi, \end{cases}$$

where ψ is the quadratic character of $\mathcal{J}_k = \mathbb{A}_k^\times / k^\times$ associated to the extension F/k.

If $F = k$, then k'/k is cyclic and the only splitting types that occur are types (1), (3u), and (3r). The ramified type (3r) occurs if and only if $\mathfrak{p}_v \mid \mathfrak{f}_\chi$. For unramified primes, $\mathfrak{p}_v$ has type (1) if and only if $\chi(\mathfrak{P}_w) = 1$.

For the sake of convenience, we restate here Theorem 1.5 and then proceed to prove it. To complete the proof, it only remains to write the Euler product in terms χ, considered as a character on the group of fractional ideals that are relatively prime to $\mathfrak{f}_\chi$.

Theorem 7.1 *Let k be a number field, δ be a nonzero element of k, and $F = k(\sqrt{\delta})$. Then, for the collection $\mathscr{C}(\delta)$ of extensions k'/k and for the character group $\mathcal{X}(F)$, we have the identity*

$$\sum_{k' \in \mathscr{C}(\delta)} \frac{d_{k'/k}^{-s}}{o(k')} \prod_{v \nmid \infty} E_{k',v}(s) = \frac{d_{F/k}^{-s}}{o(F)} \sum_{\chi \in \mathcal{X}(F)} \mathrm{N}(\mathfrak{f}_\chi)^{-s} \prod_{w \nmid \infty} \left(1 + \chi(\mathfrak{P}_w)\, \mathrm{N}(\mathfrak{P}_w)^{-2s}\right) \tag{7.3}$$

where $o(F) = 3$ if $F = k$ and 1 otherwise, $\mathfrak{P}_w$ is the prime ideal of F corresponding to the finite place w, and the product is taken over all finite places of F.

Proof. First, we assume that $[F : k] = 2$. In the type (1) case, we have $\mathfrak{p}_v \mathfrak{o}_F = \mathfrak{P}_w \overline{\mathfrak{P}_w}$ and $\chi(\mathfrak{P}_w) = \chi(\overline{\mathfrak{P}_w}) = 1$. Since $\mathrm{N}(\mathfrak{P}_w) = q_v$, we have

$$\left(1 + \chi(\mathfrak{P}_w)\, \mathrm{N}(\mathfrak{P}_w)^{-2s}\right) \left(1 + \chi(\overline{\mathfrak{P}_w})\, \mathrm{N}(\mathfrak{P}_w)^{-2s}\right) = (1 + q_v^{-2s})^2.$$

In the type (2u) case, we have $\mathfrak{p}_v \mathfrak{o}_F = \mathfrak{P}_w$ is a prime ideal in F with norm $\mathrm{N}(\mathfrak{P}_w) = q_v^2$. Also, it is automatically true that $\chi(\mathfrak{P}_w) = 1$, by the conditions that $\chi(xx^\sigma) = 1$ and $\chi^3 = 1$. Then

$$1 + \chi(\mathfrak{P}_w)\ \mathrm{N}(\mathfrak{P}_w)^{-2s} = 1 + q_v^{-4s}.$$

In the type (3u) case, we must have $\chi(\overline{\mathfrak{P}_w}) = \chi(\mathfrak{P}_w)^{-1}$ and $\mathrm{N}(\mathfrak{P}_w) = q_v$. Thus, $\chi(\mathfrak{P}_w)$ and $\chi(\overline{\mathfrak{P}_w})$ are the two nontrivial cube roots of unity. Since $\chi(\mathfrak{P}_w) + \chi(\overline{\mathfrak{P}_w}) = -1$, $\chi(\mathfrak{P}_w) \cdot \chi(\overline{\mathfrak{P}_w}) = 1$, it follows that

$$\left(1 + \chi(\mathfrak{P}_w)\ \mathrm{N}(\mathfrak{P}_w)^{-2s}\right)\left(1 + \chi(\overline{\mathfrak{P}_w})\ \mathrm{N}(\mathfrak{P}_w)^{-2s}\right) = 1 - q_v^{-2s} + q_v^{-4s}.$$

The type (3r) case corresponds to prime ideal factors $\mathfrak{P}_w$ of $\mathfrak{f}_\chi$. The usual extension of character χ to ramified primes dividing the conductor is to set $\chi(\mathfrak{P}_w) = 0$. Then

$$\prod_{w|v}\left(1 + \chi(\mathfrak{P}_w)\ \mathrm{N}(\mathfrak{P}_w)^{-2s}\right) = 1.$$

The type (2r) cases correspond to those prime ideals $\mathfrak{p}_v \mid \delta\mathfrak{o}_k$ of k so that $\mathfrak{P}_w \nmid \mathfrak{f}_\chi$, where $\mathfrak{p}_v\mathfrak{o}_F = \mathfrak{P}_w^2$ in F. Since $\overline{\mathfrak{P}_w} = \mathfrak{P}_w$, we conclude that $\chi(\mathfrak{P}_w) = 1$. As $\mathrm{N}(\mathfrak{P}_w) = q_v$, we again have

$$1 + \chi(\mathfrak{P}_w)\ \mathrm{N}(\mathfrak{P}_w)^{-2s} = 1 + q_v^{-2s}.$$

Now, we assume that $F = k$. As we have seen before we can only have types (1), (3u), and (3r). Then all the Euler factors have the form

$$\left(1 + \chi(\mathfrak{p}_v)\ \mathrm{N}(\mathfrak{p}_v)^{-2s}\right)\left(1 + \chi^2(\mathfrak{p}_v)\ \mathrm{N}(\mathfrak{p}_v)^{-2s}\right) = \begin{cases} 1 & \text{if } \mathfrak{p}_v \mid \mathfrak{f}_\chi, \\ (1 + q_v^{-2s})^2 & \text{if } \chi(\mathfrak{p}_v) = 1, \\ 1 - q_v^{-2s} + q_v^{-4s} & \text{if } \chi(\mathfrak{p}_v) \neq 1. \end{cases}$$

This allows to express the complete Euler as

$$\prod_{v\nmid\infty} E_{k',v}(s) = \prod_{w\nmid\infty}\left(1 + \chi(\mathfrak{P}_w)\ \mathrm{N}(\mathfrak{P}_w)^{-2s}\right).$$

Combining this with the results obtained in Section 7.1, we finally complete the proof of the theorem. ■

Now, let us turn our attention to the left side of (7.2). Since $\hat{E}_{k',v}(s) = E_{k',v}(s)$ for the finite $v \mid 3$, the Euler product can be expressed as

$$\prod_{v \nmid \infty} \hat{E}_{k',v}(s) = \prod_{v \nmid \infty} E_{k',v} \cdot \prod_{v \mid 3} \frac{\hat{E}_{k',v}(s)}{E_{k',v}(s)}$$
$$= \prod_{w \nmid \infty} \left(1 + \chi(\mathfrak{P}_w)\ \mathrm{N}(\mathfrak{P}_w)^{-2s}\right) M_\chi(s),$$

where $M_\chi(s)$ is the finite product $\prod_{v \mid 3} \frac{\hat{E}_{k',v}(s)}{E_{k',v}(s)}$.

Theorem 7.2 *Let k be a number field, δ be a nonzero element of k, and $\hat{F} = k(\sqrt{-3\delta})$. Then,*

$$\sum_{k' \in \mathscr{C}(-3\delta)} \frac{d_{k'/k}^{-s}}{o(k')} \prod_{v \nmid \infty} \hat{E}_{k',v}(s) = \frac{d_{\hat{F}/k}^{-s}}{o(\hat{F})} \sum_{\chi \in \mathcal{X}(\hat{F})} \mathrm{N}(\mathfrak{f}_\chi)^{-s} \prod_{w \nmid \infty} \left(1 + \chi(\mathfrak{P}_w)\ \mathrm{N}(\mathfrak{P}_w)^{-2s}\right) M_\chi(s) \tag{7.4}$$

To conclude this discussion, we would like to represent the resolvent field identity in an equivalent way in terms of ideles instead of ideals. Here we used that fact the characters χ can be considered either as characters on the idele class group or characters on the ideal group which are relatively prime to the conductor. Suppose v is a finite place of k and $w \mid v$ is a place of F. Let ϖ_w and $r_w = \mathrm{N}(\mathfrak{P}_w)$ denote a uniformizer and the modulus of F_w, respectively. For $\mathfrak{P}_w \nmid \mathfrak{f}_\chi$, the value of $\chi(\mathfrak{P}_w)$ is, by the definition given in Chapter 5, equal to $\chi_w(\varpi_w)$. We put for each finite place $w \mid v$

$$\Lambda_{\chi_w}(\varpi_w) = \begin{cases} \chi_w(\varpi_w) & \text{if } (k'/k, v) \text{ is of type (1), (2u), (2r), or (3u),} \\ 0 & \text{if } (k'/k, v) = (3r). \end{cases}$$

Moreover, since $\mathfrak{f}_\chi = \mathrm{id}(\varphi_\chi)$ we have $|\varphi_\chi|_{\mathbb{A}_F} = \mathrm{N}(\mathfrak{f}_\chi)^{-1}$. With this notation we can rewrite again the Dirichlet series on the right and left sides of (7.2) and state the following theorem:

Theorem 7.3 *Using the notation developed in this chapter, we have*

$$\sum_{k'\in\mathscr{C}(\delta)} \frac{d_{k'/k}^{-s}}{o(k')} \prod_{v\nmid\infty} E_{k',v}(s) = \frac{d_{F/k}^{-s}}{o(F)} \sum_{\chi\in\mathcal{X}(F)} |\varphi_\chi|_{\mathbb{A}_F}^s \prod_{w\nmid\infty} \big(1 + \Lambda_{\chi_w}(\varpi_w)\, r_w^{-2s}\big). \tag{7.5}$$

and

$$\sum_{k'\in\mathscr{C}(-3\delta)} \frac{d_{k'/k}^{-s}}{o(k')} \prod_{v\nmid\infty} \hat{E}_{k',v}(s) = \frac{d_{\hat{F}/k}^{-s}}{o(\hat{F})} \sum_{\chi\in\mathcal{X}(\hat{F})} |\varphi_\chi|_{\mathbb{A}_{\hat{F}}}^s \prod_{w\nmid\infty} \big(1 + \Lambda_{\chi_w}(\varpi_w)\, r_w^{-2s}\big)\, M_\chi(s) \tag{7.6}$$

where the products range over the all finite places w of $\hat{F}$.

Our future research, aimed at giving the full proof of the generalized Ohno-Nakagawa conjecture, will be based on a more detailed analysis of the extra term $M_\chi(s)$.

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VITA

Jorge Dioses

Candidate for the Degree of

Doctor of Philosophy

Dissertation: GENERALIZING THE THEOREM OF NAKAGAWA ON BINARY CUBIC FORMS TO NUMBER FIELDS

Major Field: Mathematics

Biographical:

Personal Data: Born in Lima, Lima, Peru.

Education:

Received the B.Sc. degree from Pontifical Catholic University of Peru, Lima, Lima, Peru, 1997, in Mathematics

Received the Licentiate diploma from Pontifical Catholic University of Peru, Lima, Lima, Peru, 2000, in Mathematics

Completed the requirements for the degree of Doctor of Philosophy with a major in Mathematics from Oklahoma State University in July, 2012.

Experience:

Grader at Pontifical Catholic University of Peru, Lima, Lima, Peru, from January 1996 to December 2000

Instructor at Pontifical Catholic University of Peru, Lima, Lima, Peru, from April 2001 to December 2002

Teaching Assistant at Oklahoma State University from August 2003 to July 2012.

Name: Jorge Dioses Date of Degree: July, 2012

Institution: Oklahoma State University Location: Stillwater, Oklahoma

Title of Study: GENERALIZING THE THEOREM OF NAKAGAWA ON BINARY CUBIC FORMS TO NUMBER FIELDS

Pages in Study: 100 Candidate for the Degree of Doctor of Philosophy

Major Field: Mathematics

The goal of this thesis is to study possible generalizations of a theorem of Nakagawa, first stated as a conjecture by Ohno, that gives a relationship between cubic fields of positive and negative discriminant. This theorem is described as an equation of Dirichlet series whose coefficients are class numbers of binary cubic forms. Its proof makes an extensive use of class field theory. Our approach for generalizing this result to cubic extensions of an arbitrary number field is to write the series in terms of ideles following the works of Datskovsky and Wright. By comparing the residues at their poles, we are able to deduce a conjecture that is a direct generalization of the original theorem. In the process of refining this generalization, we obtain some results concerning local integrals and series over idele group character. Moreover, we use tables of number fields which are currently available and computer algebra systems to provide strong evidence for the validity of the proposed conjecture.

ADVISOR'S APPROVAL: Dr. David Wright