ADAPTIVE OUTPUT FEEDBACK CONTROL OF NONLINEAR SYSTEMS

By

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NOMENCLATURE

M^H	Complex conjugate transpose of the matrix or vector M.
$M^\top$	Transpose of the matrix or vector M.
$\bar{c}$	Complex conjugate of the complex scalar c.
$M > 0$ (≥ 0)	M is a positive definite (positive semi-definite) matrix.
$M < 0$ (≤ 0)	M is a negative definite (negative semi-definite) matrix.
$M > N$ $(M \geq N)$	$M - N > 0$ (≥ 0).
$M < N$ $(M \leq N)$	$M - N < 0$ (≤ 0).
$\mathrm{tr}(M)$	The trace of the matrix M.
$\lambda(M)$ $(\lambda_i(M))$	The eigenvalue set (the i-th eigenvalue) of the matrix M.
$\lambda_{min}(M)$	The minimal eigenvalue of the matrix M.
$\lambda_{max}(M)$	The maximal eigenvalue of the matrix M.
$\mathrm{Re}(c)$	The real part of the complex scalar c.
$\mathrm{Im}(c)$	The imaginary part of the complex scalar c.
m_{ij}	The ij-th entry of the matrix M.
i	$\sqrt{-1}$ or a real number.
$\|c\|$	The absolute value of the complex scalar c.
$\|M\|$	The Euclidean norm of the matrix M, or the length of the vector M defined by $\sqrt{M^H M}$.
$\mathbb{C}$	Field of complex numbers.
$\mathbb{R}$	Field of real numbers.
$\mathbb{F}$	Filed of real or complex numbers.
I	Identity matrix of appropriate dimension.

0	Scalar zero or zero matrix of appropriate dimensions.
$\sigma(M)$	$\triangleq \sqrt{\lambda(M^H M)}$, the singular value of the matrix M.
$\sigma_{min}(M)$ ($\sigma_{max}(M)$)	The minimal (maximal) singular value of M.
$\langle M, N \rangle$	$\triangleq M^H N$, the inner product of the column vectors M and N.
$\mu_M(M)$	$\triangleq \lambda_{max}((M + M^H)/2)$, the maximal eigenvalue of the matrix $(M + M^H)/2$.
$\mu_m(M)$	$\triangleq \lambda_{min}((M + M^H)/2)$, the minimal eigenvalue of the matrix $(M + M^H)/2$.
s	Laplace operator or differential operator.
$\text{adj}(M)$	The adjoint matrix of the matrix M.
$\det(M)$	The determinant of the matrix M.
$\text{diag}(a_1, a_2, \ldots, a_n)$	Diagonal matrix with a_i, $i = 1 : n$ as diagonal elements. a_i may be a matrix.

CHAPTER 1

INTRODUCTION

It is common that whenever a practical system is modeled, there are uncertainties in the model due to limited knowledge of the system. Uncertainties can be one or combination of the following: (1) unknown parameters (time invariant or time-varying), (2) unknown dynamics, (3) unmeasurable states. In literature, these three uncertainties are commonly known as parametric uncertainty, dynamic uncertainty, and static uncertainty, respectively.

To control a system with uncertainties, there are several possible avenues. The first possible way is to depart from the idea of exploiting *a priori* information on the system as much as possible. The most common used knowledge is the linear parametrization property. The second possible way is the case where all uncertainties are treated via a worst-case design by ignoring the uncertainty structure. The third possible way is to go through a combination of the first two where some knowledge of the structure of the uncertainty is used and other uncertainties are treated via a worst-case design. All these avenues have their own advantages and disadvantages.

Robust adaptive control refers to control of partially known systems. It is very effective for controlling a system with uncertainties, and it has been successfully applied to applications such as autopilots for aircrafts and ships, cement mills, paper machines, and power systems.

It is common that a control designer does not have access to all states of the controlled system. To relax the requirement for full state measurement is practically important. Adaptive output feedback control only uses the measured output information of the system and has the capability to handle uncertainties in the system. It has been extensively studied in literature.

Due to diverse considerations and various behaviors of controlled systems, adaptive feedback control design is a complex process, and there are still many open problems. For example, when a nonlinear system does not satisfy matching conditions, that is, disturbances do not enter the control channel, the following question arises: how to design adaptive feedback controllers to stabilize the closed-loop nonlinear system? Obviously, this problem is not always solvable. It is clear that there is no universal adaptive output feedback controllers for general nonlinear systems. A realistic way is to specify certain classes of nonlinear systems which are practically relevant and design stable adaptive output feedback controllers for those systems. This thesis considers adaptive output feedback control of such nonlinear systems.

1.1 Output feedback control of nonlinear systems

The output feedback control problem for nonlinear systems has received, and continues to receive, considerable attention in literature due to its importance in many practical applications where measurement of all the state variables is not possible. There are two major classes of output feedback control schemes: static output feedback scheme and observer-based output feedback scheme.

In static output feedback control, no observer is used to estimate the unmeasurable states. The static output feedback control is classified into two major classes: the direct output feedback control and the dynamic compensator. In the direct output feedback control, the control law is given by a linear ([1], [2], [3]) or nonlinear ([4]) functions of the output of the controlled system. This method can only be applied to very limited systems, like linear time-invariant systems with known dynamics. In dynamic compensator ([5], [6], [7], [8], [9], [10], [11]), a compensator, which consists of some linear differential equations, is added to the original system. The output feedback gain is computed based on the augmented system and desired poles of the closed-loop system. A constructive procedure of this method for linear time-invariant systems can be found in [5]. Other work on static output feedback control design can be found in ([12], [13], [14], [15], [16]).

In observer-based output feedback control, an observer is designed to estimate unmeasurable states, and the output of the system as well as the estimated states are used to design the control law. Early work on the observer-based output feedback control design for linear time-invariant systems can be found in ([17], [18], [19], [20], [21]). After that, a large amount of work was done by using observer design technique, adaptive control technique, and robust control technique to design the output feedback controller for linear and nonlinear systems. Both adaptive control and robust control are capable of dealing with uncertainties. In adaptive control, an adaptation law is designed to estimate unknown parameters in the system dynamics ([22], [23], [24]). In robust control, the uncertainties are considered by using some sort of knowledge on the plant dynamics, such as bounds or bounding functions of the uncertainties ([25], [26], [27]). The combination of adaptive control and robust control results in the robust adaptive control design technique ([28], [29], [30], [31], [32], [33], [34]). In most cases, an observer is required to implement a robust adaptive control because only the output of the system is available for control design. To solve the output tracking problem for certain classes of systems, variable structure control schemes were used in ([35], [36], [37], [38], [39], [40], [41], [42], [43]), and robust adaptive control schemes were used in ([44], [45], [46], [41], [47], [48], [49], [50]).

Unlike linear systems, separation principle does not generally hold for nonlinear systems. Therefore, the output feedback control problem for nonlinear systems is much more challenging than stabilization using full-state feedback. It is well known that the observer design problem for nonlinear systems by itself is quite challenging. One has to often consider special classes of nonlinear systems to solve the observer design problem as well as the output feedback control problem. Due to their practical significance, two special classes of systems that were often considered in the literature are nonlinear systems with a triangular structure and Lipschitz nonlinear systems.

A systematic approach to the development of observers for nonlinear systems was given in [51]; a nonlinear coordinate transformation was used to transform the original nonlinear system to a linear system with the addition of an output injection term. The nonlinear state transformations were also employed in [52, 53, 54] to obtain linear canonical forms that can be used for observer design. A comparative study of four techniques that appeared in the 1980's for observing the states of nonlinear systems was given in [55]. In [56], a new approach was given for the nonlinear observer design problem; a general set of necessary and sufficient conditions was derived by using the Lyapunov's auxiliary theorem.

Observer design techniques for Lipschitz nonlinear systems were considered in [57, 38, 58, 59, 60]. The observer design techniques proposed in these papers are based on quadratic Lyapunov functions and thus depend on the existence of a positive definite solution to an Algebraic Ricatti Equation (ARE). In [58], insights into the complexity of designing observers for Lipschitz nonlinear systems were given; it was discussed that in addition to choosing the observer gain in their nonlinear Luenberger-like observer, one has to make sure that the eigenvectors of the closed-loop observer system matrix must also be well-conditioned to ensure asymptotic stability. The existence of a stable observer for Lipschitz nonlinear systems was addressed in [59]; a sufficient condition was given on the Lipschitz constant. Some of the results of [59] were recently corrected by [60]. For the nonlinear observer of [59], it was shown in [60] that two sufficient conditions are required to guarantee that the observer is exponentially stable.

In [61], counterexamples were given to discuss the problem of global asymptotic stabilization by output feedback; a phenomenon called "unboundedness unobservability" was defined; it means that some unmeasured state components may escape in finite time whereas the measurements remain bounded. Recent research has focused on considering a selective class of nonlinear systems by placing some structural conditions on the nonlinearities to solve the output feedback problem. Global stabilization by dynamic output feedback of nonlinear systems which can be transformed to the output feedback form was given in [62]. Output feedback control of nonlinear systems in triangular form with nonlinearities satisfying certain growth conditions was considered in [63, 64]. In [65], it was shown that global stabilization of nonlinear systems is possible using linear feedback for a class of systems which have triangular structure and nonlinearities satisfy certain norm bounded growth conditions. A backstepping design procedure for dynamic feedback stabilization for a class of triangular Lipschitz nonlinear systems with unknown time-varying parameters was given in [66]. Output feedback control of nonlinear systems has been extensively studied in recent literature [48, 67, 68, 69].

Many practical applications require estimation of the states and parameters that can be used in designing a stable control algorithm; the unmeasurable states and parameters are generally estimated based on the knowledge of the physical system, such as a model, and the available measurements. Design of a stable adaptive observer that simultaneously estimates the unmeasurable state and the unknown parameters for a general class of nonlinear systems is still an open problem. This has led to continued strong interest over the years in the development of stable adaptive observers. Early work on stable adaptive observers for linear time-invariant systems can be found in [70, 71]. A large number of results in adaptive control of linear and nonlinear systems can be found in [72, 73, 74]. In [49], an extensive survey of adaptive output feedback control methods for nonlinear systems without derived input signal measurements was given. [75] contains an extensive literature survey of reference model based adaptive control of linear systems.

Design of adaptive observers for nonlinear systems with exponential rate of convergence was given in [76]. A new method for the design of locally convergent observers using the backstepping method was proposed in [77]. A discussion of persistent excitation in adaptive systems was given in [78]. In [56], design of a nonlinear observer for nonlinear systems using Lyapunov's auxiliary theorem was proposed; the proposed nonlinear observer design is analogous to the linear Luenberger observer theory. A dual-observer structure to estimate the unmeasurable state of the Lugre dynamic friction model was proposed in [79]; an adaptive controller and observer was designed to simultaneously estimate the unknown friction parameters and the unmeasurable friction state.

Design of a stable adaptive observer that simultaneously estimates the unmeasurable state and the unknown parameters for a general class of nonlinear systems is still an open problem. This has led to continued strong interest over the years in the development of stable adaptive observers. Adaptive observer design for nonlinear systems is usually restricted to a certain class of systems. In [80], the linear adaptive observer derived in [81] has been modified and extended to a class of nonlinear time-varying systems, in which the nonlinear system is considered to be transformed into an adaptive observer canonical form. In the adaptive observer canonical form, the unmeasured states and unknown parameters appear linearly in known functions in the dynamics. An adaptive observer, which is driven by auxiliary filters, was developed; stable convergence of the estimates were shown under certain persistency of excitation conditions.

Necessary and sufficient conditions for transforming a general nonlinear system into a canonical form that is nonlinear purely in the output variables can be found in [24]. Based on the early work of [80, 51], considerable work on adaptive nonlinear observers has been reported by Marino et. al. in a series of papers; see [82] and the references there-in; Marino et. al. studied adaptive observers for nonlinear systems that can be transformed via a global state space diffeomorphism into the form which is similar to that given in [80]; the considered system is linear in the unknown parameters and the nonlinearities are functions of the known output and input variables only.

1.2 Decentralized output feedback control of large-scale systems

Large-scale interconnected systems can be found in such diverse fields as power systems, space structures, manufacturing processes, transportation, and communication. An important motivation for the design of decentralized schemes is that the information exchange between sub-systems of a large-scale system is not needed; thus, the individual sub-system controllers are simple and use only locally available information. Decentralized control of large-scale systems has received considerable interest in the systems and control literature. A large body of literature in decentralized control of large-scale systems can be found in [83]. In [84], a survey of early results in decentralized control of large-scale systems was given. Decentralized control schemes that can achieve desired robust performance in the presence of uncertain interconnections can be found in [85, 86, 87]. A decentralized control scheme for robust stabilization of a class of nonlinear systems using the Linear Matrix Inequalities (LMI) framework was proposed in [88].

In many practical situations, complete state measurements are not available at each individual sub-system for decentralized control; consequently, one has to consider decentralized feedback control based on measurements only or design decentralized observers to estimate the state of individual sub-systems that can be used for estimated state feedback control. There has been a strong research effort in literature towards development of decentralized control schemes based on output feedback via construction of decentralized observers. Early work in this area can be found in [89, 85, 83]. Subsequent work in [90, 91, 92, 93] has focused on the decentralized output feedback problem for a number of special classes of nonlinear systems. Design of an observer-based output feedback controller is a challenging problem for nonlinear systems. It is well known that the separation principle may not be applicable to nonlinear systems [68]. In [94], the decentralized controller and observer design problems were formulated in the LMI framework for large-scale systems with nonlinear interconnections that are quadratically bounded. Autonomous linear decentralized observer-based output feedback controllers for all sub-systems were obtained. The existence of a stabilizing controller and observer depended on the feasibility of solving an optimization problem in the LMI framework; further, for a solution to exist, this formulation also required, for each sub-system, that the number of control inputs must be equal to the dimension of the state.

1.3 Adaptive control of time-varying systems

It is evident from a study of the literature that an important motivation for designing adaptive controllers is in dealing with time-varying parameters. Even though research in identification and control of time-varying systems has been active during the past two decades, adaptive estimation of time-varying parameters in linearly parameterized systems is still an open problem. Most adaptive estimation algorithms, such as the least-squares and the gradient algorithms and a number of variations of them, have nice stability and convergence properties in the ideal case when the parameters are constant[72, 33]. But these algorithms fail to retain most of their properties when the parameters are time-varying.

The amount of adaptive control research for systems with uncertain constant parameters is much larger than systems which have uncertain time-varying parameters. As pointed out in [72], one of the compelling reasons for considering adaptive methods in practical applications is to compensate for large variations in plant parameter values. Adaptive control of a class of slowly time-varying discrete-time systems is considered in [95]; it is shown that the traditional gradient algorithm designed for the estimation of the constant parameters can maintain stability when the plant parameters are slowly time-varying. In [96], time-varying linear systems in linear parameterized form with modeling error is considered for adaptive control design; gradient algorithm with projection is used to estimate the time-varying parameters; it is shown that the parameter estimation error is bounded under the assumption that the parameter variations are uniformly small and the modeling errors are bounded by a small exponentially weighted sum of plant inputs and outputs. Model reference adaptive control with slowly time-varying plants can be found in [97]. A number of results in adaptive control of linear-time varying plants can be found in [98]. In [99], a comparative survey with respect to performance and robustness between recursive and direct least-squares estimation algorithms is presented; a non-recursive algorithm that improves robustness to bounded disturbances for the case of slowly time-varying parameters is given.

In [100], it is shown via simulation results that applying local regression in traditional least-squares with a forgetting factor algorithm can reduce the estimation error in the mean-square sense for systems with slowly time-varying parameters. Adaptive control of discrete-time systems with time-varying parameters can be found in [101, 102]; polynomial approximation of the time-varying parameters in a discrete-sense is used in the parameter estimation algorithms. Nonparameteric regression techniques to various statistical problems, using local polynomial modeling, are discussed extensively in [103]. In [104], an adaptive controller is developed for time-varying mechanical systems based on polynomial approximations of time-varying parameters and disturbances; experimental results of the adaptive controller on a planar robot are given to verify the proposed adaptive controller.

High performance tracking control of mechanical systems is essential in a number of industrial applications; examples include, material handling and parts assembly. In many of the industrial applications, the mechanical system dynamics is time-varying due to a time-varying payload and/or time-varying disturbances. Examples of such applications include pouring and filling operations using robots. There has been an increase in recent research activity in adaptive control of time-varying systems. But most of this research has focused on assuming worst case bounds for time-varying parameters and/or their derivatives; an amalgam of adaptive control and robust control techniques has been used in the control designs with the controller gains chosen based on worst case bounds. The resulting controllers, although stable, give rise to large and often practically unbounded control inputs.

In [105], a robust switching controller was designed for robot manipulators with time-varying parameters performing path tracking tasks; properties of the element by element product of matrices was used to isolate the time-varying parameters from the inertia matrix. A robust adaptive controller for robot manipulators consisting of slowly time-varying parameters was presented in [106]. A smooth robust adaptive sliding mode controller was given in [107]. A robust adaptive control algorithm subject to bounded disturbances and bounded and (possibly) time-varying parameters was given in [108]; it was shown that the controller achieves asymptotic tracking if the disturbances vanish and the parameters are constant. In [109], an adaptive controller for time-varying mechanical systems was proposed based on the assumption that the time-varying parameters are given by a group of known bounded time functions and unknown constants. A time-scaling technique of mapping one cycle period of the desired trajectory into a unit interval was proposed to provide robustness to the parameter adaptation algorithms. A novel experimental platform consisting of a two-link manipulator with time-varying payload that mimics filling and pouring operations was built to verify the proposed adaptive algorithm experimentally.

A number of results in adaptive control of linear-time varying plants can be found in [98]. Adaptive control of discrete-time linear systems with time-varying parameters can be found in [101, 102]. In [102], the problem of estimating the unknown time-varying parameters is transformed to the problem of observing an unknown state of a linear discrete-time system using the Taylor's formula. In [100], it is shown that applying local regression in traditional least-squares algorithm with a forgetting factor can reduce the estimation error in the mean-square sense for systems with slowly time-varying parameters. Regressions techniques and their applications using local polynomial modelling are discussed in great detail in [103].

1.4 Contributions

The contributions of this thesis can be summarized as follows.

1. The output feedback control problem for unmatched Lipschitz nonlinear systems is investigated. A new observer design and output feedback control law are proposed, and sufficient conditions under which the proposed method can be applied are given. An illustrative example on a flexible link robot is provided to illustrate the design procedures an verify the proposed method. The proposed solution to the output feedback control problem for unmatched Lipschitz is the first result on this topic.

2. Decentralized output feedback control problem for large-scale interconnected nonlinear systems is considered. The nonlinear interconnection function of each subsystem is assumed to satisfy a quadratic constraint on the entire state of the large-scale system. A decentralized estimated state feedback controller and a decentralized observer are designed for each subsystem. Sufficient conditions, for each subsystem, under which the proposed controller and observer can achieve exponential stabilization of the overall large-scale system are developed. An LMI approach is also used to design a decentralized output feedback control for the large-scale interconnected system considered. It is shown that the proposed LMI approach is feasible. Further, the proposed LMI approach does not require

the invertibility of the input matrix of each subsystem, which was the case in a recent paper in the literature [94]. Simulation results on a numerical example are given to verify the proposed design.

3. Output feedback control of a class of nonlinear systems, which contains product terms of unknown parameters and unmeasurable states, is studied. By representing the dynamics of the original nonlinear system in a suitable form, a new observer design and output feedback control law are designed based on a parameter-dependent Lyapunov function. Numerical examples are given to illustrate the proposed design. Also, experiments on the dynamic friction on a two-link robot is provided. The simulation and experimental results are discussed.

4. On-line estimation of time-varying parameters in dynamic systems, which can be represented by linearly parameterized model, is studied. The problem of estimating time-varying parameters in such systems is transformed to the problem of estimating time-invariant parameters in small time intervals. Modification of the least-squares algorithm and gradient algorithm are proposed to estimate time-varying parameters, and a resetting strategy for estimates at the beginning of each interval is provided. Based on the proposed method for estimating time-varying parameters, an adaptive output feedback controller for mechanical systems with time-varying parameters and disturbances is designed. A novel experiment on a two-link robot is designed and conducted to verify the proposed design.

5. Matrix equations, such as linear differential matrix equations, algebraic Riccati equations, and Lyapunov equations, which play an important role in systems and control theory, are investigated. Important results from literature are reviewed. Some useful and easily computable necessary conditions for the exis-

tence of a positive semi-definite solution to the algebraic Riccati equation are derived; an upper bound on the solution of ARE is also derived. Further, upper and lower bounds for the trace of the solution of the time-varying linear matrix differential equation are obtained.

1.5 Organization of the report

The rest of the thesis is organized as follows.

- Chapter 2 considers the output feedback control of Lipschitz nonlinear systems.
- Chapter 3 investigates the decentralized output feedback control for large-scale interconnected nonlinear systems.
- In Chapter 4, a class of nonlinear systems which contain products of unknown parameters and unmeasurable states are studied; an adaptive output feedback controller is designed based on a parameter–dependent Lyapunov function.
- Adaptive control of mechanical systems with unknown time-varying parameters and unknown time-varying disturbances is addressed in Chapter 5.
- Some well known matrix equations in systems and control theory are reviewed and investigated in Chapter 6.
- Summary and future work are given in Chapter 7.

CHAPTER 2

CONTROLLER AND OBSERVER DESIGN FOR LIPSCHITZ NONLINEAR SYSTEMS

In this chapter, a solution to the output feedback control problem for Lipschitz nonlinear systems under some sufficient conditions on the Lipschitz constant is provided. Systems with Lipschitz nonlinearity are common in many practical applications. Many nonlinear systems satisfy the Lipschitz property at least locally by representing them by a linear part plus a Lipschitz nonlinearity around their equilibrium points. First, a linear full-state feedback controller is proposed and a sufficient condition under which exponential stabilization of the closed-loop system is achieved with full-state feedback is derived. Second, a Luenberger-like observer is proposed, which is shown to be an exponentially stable observer under a sufficient condition. Given that the sufficient conditions of the controller and observer problem are satisfied, it is shown that the proposed controller with estimated state feedback from the proposed observer will achieve global exponential stabilization, that is, the proposed controller and observer designs satisfy the separation principle.

The rest of the chapter is organized as follows. In Section 2.1, the class of Lipschitz nonlinear systems considered, the assumptions, the notation used, and some prior results that will be useful for the developments in the chapter are given. The full-state feedback control problem, the observer design problem, and the output feedback control problem are considered in Sections 2.2, 2.3, and 2.4, respectively. Section 2.5 gives an algorithmic procedure for computing the controller and observer gains while satisfying the sufficient conditions. An illustrative example is provided with simulation results in Section 2.6 to verify the proposed methods. Section 2.7 gives summary and some relevant future research.

2.1 Preliminaries

Consider the problem of controller and observer design for the following class of Lipschitz nonlinear systems:

$$\dot{x} = Ax + Bu + \Phi(x, u), \tag{2.1a}$$

$$y = Cx \tag{2.1b}$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}^p$, and $y \in \mathbb{R}^q$ are the system state, input, and output, respectively. It is assumed, without loss of generality, that $x = 0$ is the equilibrium point of the system (2.1).

Assume that the system (2.1) satisfies the following.

Assumption A2.1 *$\Phi(x, u)$ is Lipschitz with respect to the state x, uniformly in the control u, that is, there exists a constant γ such that*

$$\|\Phi(x_1, u) - \Phi(x_2, u)\| \leq \gamma \|x_1 - x_2\| \tag{2.2}$$

for all $x_1, x_2 \in \mathcal{D} \subset \mathbb{R}^n$ and $u \in \mathbb{R}^p$.

Assumption A2.2 *$\Phi(x, u)$ is such that $\|\Phi(x, u)\| \leq \gamma \|x\|$ for all $u \in \mathbb{R}^p$.*

Assumption A2.3 *The pair (A, B) is controllable.*

Assumption A2.4 *The matrix A is Hurwitz. If the matrix A is not Hurwitz, since (A, B) is controllable, a preliminary control can be used to make it Hurwitz.*

Assumption A2.5 *The pair (C, A) is observable.*

Definition 2.1 *The number $\delta(M, N)$ is defined as*

$$\delta(M, N) = \min_{\omega \in \mathbb{R}} \sigma_{min} \begin{pmatrix} i\omega I - M \\ N \end{pmatrix} \tag{2.3}$$

where $i = \sqrt{-1}$ and I is an identity matrix with appropriate dimension.

The distance between a pair (A, C) and the set of pairs with an unobservable purely imaginary mode is given by $\delta(A, C)$. Similarly, $\delta(A^\top, B^\top)$ gives the distance between the pair (A, B) and the set of pairs with an uncontrollable purely imaginary mode. See [60] for a discussion of the number δ and a bisection algorithm for computing it.

The distance $\delta(A, C)$ can be computed by the bisection algorithm with precision prec in MATLAB as follows:

```
a=0;
b=norm(A,2)+norm(C,2);
choose N such that b/2^N < prec
    for i=1:N
    gamma=(a+b)/2;
    form H_gamma;
    if H_gamma is hyperbolic  %see Lemma 2.2
        a=gamma;
    else
        b=gamma;
    end
end
```

Lemma 2.1 *Consider the Algebraic Ricatti Equation (ARE)*

$$A^\top P + PA + PRP + Q = 0. \tag{2.4}$$

If R is symmetric positive semi-definite, Q is symmetric positive definite, A is Hurwitz, and the associated Hamiltonian matrix

$$\mathcal{H} = \begin{bmatrix} A & R \\ -Q & -A^\top \end{bmatrix} \tag{2.5}$$

is hyperbolic, that is, $\mathcal{H}$ has no eigenvalues on the imaginary axis, then there exists a unique symmetric positive definite matrix P, which is the solution of the ARE (2.4).

Proof. See [60]. ■

Lemma 2.2 *Let $\gamma \geq 0$ and define*

$$\mathcal{H}_\gamma = \begin{bmatrix} A & I \\ C^\top C - \gamma^2 I & -A^\top \end{bmatrix}.$$

Then $\gamma < \delta(A, C)$ if and only if $\mathcal{H}_\gamma$ is hyperbolic.

Proof. See [59, 60]. ■

The results on this chapter will be shown in the following order.

(1) A linear full-state feedback controller is designed and sufficient conditions under which the equilibrium is exponentially stable are provided.

(2) A 'Luenberger-like' nonlinear observer for state estimation is proposed, and a sufficient condition under which the observer is exponentially stable is given.

(3) An output feedback controller for the Lipschitz nonlinear systems (2.1) is designed by using the results from that of the controller of (1) and observer of (2).

2.2 Full-state feedback controller design

In this section, the regulation problem for the system (2.1a) with full-state linear feedback under the Assumptions A2.2, A2.3, and A2.4, is considered. Consider the following control input:

$$u = -\frac{K}{\|B\|^2} x \tag{2.6}$$

where K is the feedback gain matrix. With this control input, the closed-loop dynamics is

$$\begin{aligned}\dot{x} &= \left(A - \frac{BK}{\|B\|^2}\right)x + \Phi(x,u) \\ &\triangleq \bar{A}_c x + \Phi(x,u).\end{aligned} \tag{2.7}$$

To determine K, we consider the following Lyapunov function candidate

$$V_c(x) = x^\top P_c x \tag{2.8}$$

where P_c is a symmetric positive definite matrix. The time derivative of the Lyapunov function candidate along the trajectories of (2.7) is

$$\begin{aligned}\dot{V}_c(x) &= x^\top \left(\bar{A}_c^\top P_c + P_c\bar{A}_c\right)x + 2x^\top P_c\Phi(x,u) \\ &\leq x^\top \left(\bar{A}_c^\top P_c + P_c\bar{A}_c\right)x + 2\gamma\|P_c x\|\|x\| \\ &\leq x^\top \left(\bar{A}_c^\top P_c + P_c\bar{A}_c + P_cP_c + \gamma^2 I\right)x\end{aligned} \tag{2.9}$$

where the first inequality is a consequence of assumption A2.2 and the second inequality is obtained by using the inequality

$$2\gamma\|P_c x\|\|x\| \leq x^\top P_cP_c x + \gamma^2 x^\top x.$$

For any $\eta_c > 0$, if

$$\bar{A}_c^\top P_c + P_c\bar{A}_c + P_cP_c + \gamma^2 I = -\eta_c I, \tag{2.10}$$

then

$$\dot{V}_c \leq -\eta_c x^\top x. \tag{2.11}$$

Using the definition of $\bar{A}_c$, the ARE (2.10) can be simplified to

$$A^\top P_c + P_cA - \frac{K^\top B^\top P_c}{\|B\|^2} - \frac{P_cBK}{\|B\|^2} + P_cP_c + (\gamma^2 + \eta_c)I = 0. \tag{2.12}$$

The choice of the control gain matrix

$$K = \frac{1}{2}B^\top P_c \tag{2.13}$$

results in the following ARE:

$$A^\top P_c + P_c A + P_c \left(I - \frac{BB^\top}{\|B\|^2} \right) P_c + (\gamma^2 + \eta_c) I = 0. \tag{2.14}$$

Now we consider the problem of the existence of a symmetric positive definite matrix P_c, which is the solution to the ARE (2.14). Since A is Hurwitz, the matrix $\left(I - \frac{BB^\top}{\|B\|^2} \right)$ is positive semi-definite, and the matrix $(\gamma^2+\eta_c)I$ is positive definite, by Lemma 2.1, the problem reduces to showing that the associated Hamiltonian matrix

$$\mathcal{H}_c = \begin{bmatrix} A & I - \frac{BB^\top}{\|B\|^2} \\ -(\gamma^2 + \eta_c) I & -A^\top \end{bmatrix} \tag{2.15}$$

is hyperbolic. The following lemma gives the condition under which the Hamiltonian $\mathcal{H}_c$ is hyperbolic.

Lemma 2.1 *$\mathcal{H}_c$ is hyperbolic if and only if*

$$\sqrt{\gamma^2 + \eta_c} < \delta \left(A^\top, \frac{\sqrt{\gamma^2 + \eta_c}}{\|B\|} B^\top \right). \tag{2.16}$$

Proof. Consider the determinant of the matrix $(sI - \mathcal{H}_c)$ given by

$$\begin{aligned} \det(sI - \mathcal{H}_c) &= \det \begin{bmatrix} sI - A & -I + \frac{BB^\top}{\|B\|^2} \\ (\gamma^2 + \eta_c) I & sI + A^\top \end{bmatrix} \\ &= (-1)^n \det \begin{bmatrix} (\gamma^2 + \eta_c) I & sI + A^\top \\ sI - A & -I + \frac{BB^\top}{\|B\|^2} \end{bmatrix}. \end{aligned} \tag{2.17}$$

Since $(\gamma^2 + \eta_c)I$ is non-singular, using the formula for determinant of block matrices [110, p. 650], we obtain

$$\begin{aligned} \det(sI - \mathcal{H}_c) &= (-1)^n (\gamma^2 + \eta_c)^n \det \left[\left(-I + \frac{BB^\top}{\|B\|^2} \right) - (sI - A)(\gamma^2 + \eta_c)^{-1} (sI + A^\top) \right] \\ &= (-1)^n \det \left[(\gamma^2 + \eta_c) \left(-I + \frac{BB^\top}{\|B\|^2} \right) - (sI - A)(sI + A^\top) \right]. \end{aligned} \tag{2.18}$$

Define

$$G(s) = (\gamma^2 + \eta_c) \left(-I + \frac{BB^\top}{\|B\|^2} \right) - (sI - A)(sI + A^\top). \tag{2.19}$$

From the equations (2.18) and (2.19), s is an eigenvalue of $\mathcal{H}_c$ if and only if $G(s)$ is singular. Hence, to prove that $\mathcal{H}_c$ is hyperbolic, one can prove that $G(-i\omega)$ is non-singular for all $\omega \in \mathbb{R}$. Notice that

$$\begin{aligned}\Delta_c(-i\omega) &\triangleq -(-i\omega I - A)(-i\omega I + A^\top) + (\gamma^2 + \eta_c)\frac{BB^\top}{\|B\|^2} \\ &= \begin{bmatrix} i\omega I - A^\top \\ \dfrac{\sqrt{\gamma^2+\eta_c}}{\|B\|}B^\top \end{bmatrix}^H \begin{bmatrix} i\omega I - A^\top \\ \dfrac{\sqrt{\gamma^2+\eta_c}}{\|B\|}B^\top \end{bmatrix}.\end{aligned} \tag{2.20}$$

Therefore, if

$$\delta\left(A^\top, \frac{\sqrt{\gamma^2+\eta_c}}{\|B\|}B^\top\right) > \sqrt{\gamma^2+\eta_c}\ ,$$

then

$$G(-i\omega) = -(\gamma^2+\eta_c)I + \Delta_c(-i\omega) > 0 \tag{2.21}$$

for all $\omega \in \mathbb{R}$. Thus, $\mathcal{H}_c$ is hyperbolic. This completes the sufficiency part of the proof. The necessary part of the proof is similar to that of Lemma 2.2; it can be shown in [60] and omitted here. ■

Remark 2.1 *Since* $\delta\left(A^\top, \frac{\gamma}{\|B\|}B^\top\right)$ *is a continuous function of* γ*, the function* $f(\gamma) \triangleq \gamma - \delta\left(A^\top, \frac{\gamma}{\|B\|}B^\top\right)$ *is also a continuous function of* γ*. Therefore, if* $f(\gamma) < 0$*, then there exists a* $\gamma_1 > \gamma$ *such that* $f(\gamma_1) < 0$*. Hence, if* $f(\gamma) < 0$*, then there exists an* $\eta_c > 0$ *such that* $f(\sqrt{\gamma^2+\eta_c}\ ) < 0$*. Consequently, the condition for* $\mathcal{H}_c$ *being hyperbolic given by (2.16) can be simplified to be* $\gamma < \delta\left(A^\top, \frac{\gamma}{\|B\|}B^\top\right)$*.*

The following theorem summarizes the results pertaining to the full-state feedback controller design.

Theorem 2.1 *For the nonlinear system* (2.1)*, with the Assumptions A2.2, A2.3, A2.4, and with the control input given by* (2.6)*, the equilibrium* $x = 0$ *is exponentially* *stable if the following condition is satisfied:*

$$\gamma < \delta\left(A^\top, \frac{\gamma}{\|B\|}B^\top\right). \tag{2.22}$$

2.3 Observer design

First consider the following recent result from [60].

Theorem 2.2 *Consider the n-dimensional system*

$$\dot{x} = Ax + \Phi(x, u)$$
$$y = Cx$$

where the matrix A is stable, (C, A) is observable, the non-linearity $\Phi(x, u)$ is globally Lipschitz with respect to the state x, uniformly in control u, with a Lipschitz constant γ. If

$$\gamma < \delta(A, C) \tag{2.23}$$

and

$$\gamma < \delta\left(A, \frac{\gamma}{\|C\|}C\right) \tag{2.24}$$

then there exists a gain matrix L such that the observer

$$\dot{\hat{x}} = A\hat{x} + \Phi(\hat{x}, u) - L(C\hat{x} - y)$$

is an exponential observer for the system.

In the following, a different exponentially stable observer for the system (2.1) will be proposed. It will also be shown that one of the two conditions given by (2.23) and (2.24), is sufficient for the proposed observer to be exponentially stable.

Consider the following observer for the system (2.1):

$$\dot{\widehat{x}} = A\widehat{x} + Bu + \Phi(\widehat{x}, u) + \frac{\gamma^2 + \varepsilon_o}{\|C\|^2}L(y - C\widehat{x}) \tag{2.25}$$

where $\varepsilon_o \geq -\gamma^2$ and L is the observer gain matrix. Define the estimation error $\widetilde{x} = x - \widehat{x}$. The error dynamics is

$$\begin{aligned}\dot{\widetilde{x}} &= \left(A - \frac{\gamma^2 + \varepsilon_o}{\|C\|^2} LC\right) \widetilde{x} + \Phi(x, u) - \Phi(\widehat{x}, u) \\ &\triangleq \bar{A}_o \widetilde{x} + \Phi(x, u) - \Phi(\widehat{x}, u).\end{aligned} \tag{2.26}$$

To find L we consider the following Lyapunov function candidate

$$V_o(\widetilde{x}) = \widetilde{x}^\top P_o \widetilde{x}. \tag{2.27}$$

The time derivative of the Lyapunov function candidate along the trajectories of (2.26) is

$$\dot{V}_o(\widetilde{x}) = \widetilde{x}^\top \left(\bar{A}_o^\top P_o + P_o \bar{A}_o\right) \widetilde{x} + 2\widetilde{x}^\top P_o \left(\Phi(x, u) - \Phi(\widehat{x}, u)\right). \tag{2.28}$$

Simplifying along the same lines as done in the full-state feedback controller case, we obtain

$$\dot{V}_o(\widetilde{x}) \leq \widetilde{x}^\top \left(A^\top P_o + P_o A - \frac{\gamma^2 + \varepsilon_o}{\|C\|^2}(C^\top L^\top P_o + P_o LC) + P_o P_o + \gamma^2 I\right) \widetilde{x}. \tag{2.29}$$

Choosing

$$L = \frac{1}{2} P_o^{-1} C^T \tag{2.30}$$

we obtain the following. If

$$A^\top P_o + P_o A + P_o P_o - \frac{\gamma^2 + \varepsilon_o}{\|C\|^2} C^\top C + \gamma^2 I = -\eta_o I \tag{2.31}$$

for some $\eta_o > \max(\varepsilon_0, 0)$, then

$$\dot{V}_o(\widetilde{x}) \leq -\eta_o \widetilde{x}^\top \widetilde{x}. \tag{2.32}$$

Now the problem reduces to finding conditions under which there exists a positive definite solution P_o to the ARE

$$A^\top P_o + P_o A + P_o P_o + (\gamma^2 + \eta_o) I - \frac{\gamma^2 + \varepsilon_o}{\|C\|^2} C^\top C = 0. \tag{2.33}$$

Since A is Hurwitz and the matrix $(\gamma^2 + \eta_o)I - \frac{\gamma^2 + \varepsilon_o}{\|C\|^2} C^\top C$ is positive definite, by Lemma 2.1, we need to show that the associated Hamiltonian

$$\mathcal{H}_o = \begin{bmatrix} A & I \\ -(\gamma^2 + \eta_o)I + \frac{\gamma^2 + \varepsilon_o}{\|C\|^2} C^\top C & -A^\top \end{bmatrix} \tag{2.34}$$

is hyperbolic. The following lemma gives the condition under which $\mathcal{H}_o$ is hyperbolic.

Lemma 2.3 *$\mathcal{H}_o$ is hyperbolic if and only if*

$$\sqrt{\gamma^2 + \eta_o} < \delta\left(A, \frac{\sqrt{\gamma^2 + \varepsilon_o}}{\|C\|} C\right). \tag{2.35}$$

Proof. Similar to the Lemma 2.2. ■

Notice that the arguments of Remark 2.1 also hold for Lemma 2.3. The following theorem summarizes the results of this section.

Theorem 2.3 *For the nonlinear system given by (2.1), with the Assumptions A2.1, A2.4, and A2.5, if*

$$\gamma < \delta\left(A, \frac{\gamma}{\|C\|} C\right) \tag{2.36}$$

is satisfied, then the observer (2.25) is an exponentially stable observer for the system (2.1).

Remark 2.2 *Notice that the proposed observer, (2.25), requires only one sufficient condition, (2.36), as opposed to the two sufficient conditions for the observer given in Theorem 2.2. The two conditions are required because: (1) the observer structure does not guarantee that the "Q" matrix in the ARE (2.4) is positive definite and (2) the associated Hamiltonian matrix must be hyperbolic. The proposed observer, (2.25), guarantees that the "Q" matrix in the ARE (2.4) is positive definite.*

Remark 2.3 *Notice that the conditions given by (2.22) and (2.36) guarantee the existence of $\eta_c > 0$ and $\eta_o > 0$, but not their values. Instead, we can check the condi-*

tions (2.16) and (2.35) with specified η_c and η_o, which give the rate of convergence of controller and observer, respectively.

2.4 Output feedback controller design

Combining the full-state feedback control design of Section 2.2 and the observer design of Section 2.3, one can design an output feedback controller for the system (2.1), which is illustrated by the following theorem.

Theorem 2.4 *Consider the system (2.1) with the Assumptions A2.1, A2.2, A2.3, A2.4, and A2.5. If the conditions (2.22) and (2.36) hold, then the equilibrium $x = 0$ of the system (2.1) is exponentially stable, with*

$$u = -\frac{1}{\|B\|^2} K\widehat{x} \tag{2.37}$$

where $\widehat{x}$ is the estimate of x generated by (2.25), K is the gain matrix given by (2.13), and P_c is the solution to the ARE (2.14). Further, the observation error, $\widetilde{x} = x - \widehat{x}$, exponentially converges to zero.

Proof. Substituting the output feedback control law given by (2.37) in (2.1) and simplifying it is obtained that

$$\dot{x} = \Big(A - \frac{1}{2}\frac{BB^\top}{\|B\|^2} P_c\Big)x + \Phi(x, u) + \frac{1}{2}\frac{BB^\top}{\|B\|^2} P_c \widetilde{x}. \tag{2.38}$$

Notice that (2.38) is the same as (2.7) except for an additional term in (2.38). The time derivative of the Lyapunov function candidate $V_c(x)$ given by (2.8) along the trajectories of (2.38) is

$$\dot{V}_c(x) \leq x^\top \left(A^\top P_c + P_c A + P_c\Big(I - \frac{BB^\top}{\|B\|^2}\Big)P_c + \gamma^2 I\right) x + \frac{1}{\|B\|^2} x^\top P_c BB^\top P_c \widetilde{x} \tag{2.39}$$

Since P_c is the solution to the ARE (2.14), one has

$$\dot{V}_c(x) \leq -\eta_c x^\top x + \zeta_c \|x\| \|\widetilde{x}\| \tag{2.40}$$

where $\zeta_c = \|P_c\|^2$.

Now consider the function

$$W(x, \widetilde{x}) = \zeta V_c(x) + V_o(\widetilde{x}) \tag{2.41}$$

where $\zeta > 0$ and $V_o(\widetilde{x})$ is as given by (2.27). The time derivative of $W(x, \widetilde{x})$ is given by

$$\dot{W}(x, \widetilde{x}) \leq -\zeta\eta_c\|x\|^2 + \zeta\zeta_c\|x\|\|\widetilde{x}\| - \eta_o\|\widetilde{x}\|^2. \tag{2.42}$$

Choosing $\zeta = \eta_c\eta_o/\zeta_c^2$ results in

$$\dot{W}(x, \widetilde{x}) \leq -\frac{1}{2}\zeta\eta_c\|x\|^2 - \frac{1}{2}\eta_o\|\widetilde{x}\|^2. \tag{2.43}$$

Therefore, x and $\widetilde{x}$ exponentially converge to zero. ∎

Remark 2.4 *The number δ is realization dependent, that is, it depends on A, B, C. If A is unstable to begin with, then any preliminary control used to stabilize A will affect δ. Since δ and γ depend on the realization, appropriate coordinate transformations as discussed in [59], in some cases, can be used to increase δ and reduce γ.*

Remark 2.5 *The bisection algorithm given in [60] can be used to compute δ; it was suggested that 0 and $\|A\|$ be used as the initial guess for the lower and upper bounds, respectively, for $\delta(A, C)$. It is possible that the value of δ may be greater than $\|A\|$. The upper bound must be changed to $\sigma_{min}\begin{pmatrix} A \\ C \end{pmatrix}$ because*

$$\delta(A, C) = \min_{\omega\in\mathbb{R}} \sigma_{min}\begin{pmatrix} i\omega I - A \\ C \end{pmatrix} \leq \sigma_{min}\begin{pmatrix} -A \\ C \end{pmatrix} = \sigma_{min}\begin{pmatrix} A \\ C \end{pmatrix}.$$

Remark 2.6 *If $\Phi(x, u)$ is globally Lipschitz with respect to x, then the three results given by Theorem 2.1, 2.3, and 2.4 will be applicable globally.*

2.5 Algorithm to compute gain matrices

In the following, a systematic procedure is provided to compute the observer and controller gain matrices with respect to the original system (2.1) in the event of the use of the preliminary control and coordinate transformations.

2.5.1 Observer gain matrix

1. **Pole placement**

   Rewrite (2.1) in the following form

$$\dot{x} = (A - L_1 C)x + Bu + L_1 y + \Phi(x, u), \tag{2.44a}$$

$$y = Cx \tag{2.44b}$$

   where L_1 is chosen such that $(A - L_1 C)$ is stable.

2. **Similarity transformation**

   Let $x = T_o x'$. Then (2.44) becomes

$$\begin{aligned}\dot{x}' &= T_o^{-1}(A - L_1 C)T_o x' + T_o^{-1}(Bu + L_1 y) + T_o^{-1}\Phi(T_o x', u)\\ &\triangleq A'x' + B'u + T_o^{-1}L_1 y + T_o^{-1}\Phi(T_o x', u),\end{aligned} \tag{2.45a}$$

$$y = CT_o x' \triangleq C'x' \tag{2.45b}$$

   where $T_o \in \mathbb{R}^{n\times n}$ is a nonsingular matrix. The new Lipschitz gain γ' is obtained from the following inequality

$$\|T_o^{-1}\Phi(T_o x_1, u) - T_o^{-1}\Phi(T_o x_2, u)\| \leq \gamma_o'\|x_1 - x_2\| \quad \forall x_1, x_2 \in \mathbb{R}^n, u \in \mathbb{R}^p. \tag{2.46}$$

   The observer for (2.45) is given by

$$\dot{\widehat{x'}} = A'\widehat{x'} + B'u + T_o^{-1}L_1 y + T_o^{-1}\Phi(T_o\widehat{x'}, u) + \frac{\gamma'^2 + \varepsilon_o}{\|C'\|^2}L'(y - C'\widehat{x'}), \tag{2.47a}$$

$$\widehat{y} = C'\widehat{x'}. \tag{2.47b}$$

After choosing $\varepsilon_o \geq -{\gamma'_o}^2$ and $\eta_o > \max(\varepsilon_o, 0)$, check the condition

$$ {\gamma'_o}^2 + \eta_o < \delta^2 \left(A', \frac{\sqrt{{\gamma'}^2 + \varepsilon_o}}{\|C'\|} C' \right) \tag{2.48} $$

for the existence of the solution P_o to the ARE

$$ A'^{\top} P_o + P_o A' + P_o P_o + ({\gamma'_o}^2 + \eta_o) I - ({\gamma'_o}^2 + \varepsilon_o) \frac{C'^{\top} C'}{\|C'\|^2} = 0. \tag{2.49} $$

If (2.48) is satisfied, the observer gain is chosen to be $L' = \frac{1}{2} P_o^{-1} C'^{\top}$ where $P_o = P_o^{\top} > 0$ is the solution of (2.49).

3. **Observer gain matrix for the original system**

   Notice that if one defines $\widehat{x} = T_o \widehat{x'}$ as the estimate of x, the system (2.47) can be rewritten in terms of $\widehat{x}$ by the following equations.

$$ \dot{\widehat{x}} = A\widehat{x} + Bu + \Phi(\widehat{x}, u) + L(y - C\widehat{x}), \tag{2.50a} $$

$$ \widehat{y} = C\widehat{x} \tag{2.50b} $$

   where

$$ L = L_1 + \frac{{\gamma'_o}^2 + \varepsilon_o}{\|CT_o\|^2} T_o L'. \tag{2.51} $$

2.5.2 Controller gain matrix

1. **Pole placement**

   Rewrite (2.1) in the following form

$$ \dot{x} = (A - BK_1)x + B(u + K_1 x) + \Phi(x, u) \tag{2.52} $$

   where K_1 is chosen such that $(A - BK_1)$ is stable.

2. **Similarity transformation**

   Let $x = T_c x'$, (2.52) becomes

$$ \begin{aligned} \dot{x}' &= T_c^{-1}(A - BK_1)T_c x' + T_c^{-1}B(u + K_1 T x') + T_c^{-1}\Phi(T_c x', u) \\ &\triangleq A'x' + B'(u + K_1 T_c x') + T_c^{-1}\Phi(T_c x', u) \end{aligned} \tag{2.53} $$

where $T_c \in \mathbb{R}^{n\times n}$ is a nonsingular matrix. The new Lipschitz gain γ_c' is obtained from the following inequality

$$\|T_c^{-1}\Phi(T_c x_1, u) - T_c^{-1}\Phi(T_c x_2, u)\| \le \gamma_c' \|x_1 - x_2\| \quad \forall x_1, x_2 \in \mathbb{R}^n, u \in \mathbb{R}^p. \tag{2.54}$$

Choosing the control for (2.53) as $u = -K_1 T_c x' - \frac{1}{\|B'\|^2} K' x'$ results in the following closed-loop system

$$\dot{x}' = \Big(A' - \frac{1}{\|B'\|^2} B' K'\Big) x' + T_c^{-1}\Phi(T_c x', u). \tag{2.55}$$

Choose $\eta_c > 0$ and check the condition

$$\gamma_c'^2 + \eta_c < \delta^2\Big(A'^\top, \frac{\sqrt{\gamma_c'^2 + \eta_c}}{\|B'\|} B'^\top\Big) \tag{2.56}$$

for the existence of the solution P_c to the ARE

$$A'^\top P_c + P_c A' + P_c\Big(I - \frac{1}{\|B'\|^2} B' B'^\top\Big) P_c + \big(\gamma'^2 + \eta_c\big) I = 0. \tag{2.57}$$

If (2.56) is satisfied, the control gain is chosen to be $K' = B'^\top P_c / 2$ where $P_c = P_c^\top > 0$ is the solution of (2.57).

3. **The control gain matrix for the original system**

   The gain matrix used in the full-state feedback controller or output feedback controller is

$$K = K_1 + \frac{1}{\|T_c^{-1} B\|^2} K' T_c^{-1}. \tag{2.58}$$

Remark 2.7 *The changes of coordinates for the observer design and the full-state feedback control design may not be the same.*

Remark 2.8 *The transformed system, for instance, (2.45) in the observer design case, has the Lipschitz constant γ_o', which depends on the transformation matrix T_o.*

It is possible to choose a suitable transformation T_0 such that $\gamma_o' < \gamma$. From Theorem 2.3, it is seen that, to satisfy the condition (2.36), it is better to decrease γ and increase $\delta\left(A, \frac{\gamma}{\|C\|}C\right)$. To reduce the Lipschitz gain by using similarity transformation may increase $\delta(\cdot)$ also.

The argument above holds for the full-state feedback control design also.

2.6 An illustrative example: a flexible link robot

In this section, consider the observer and controller design for a flexible link robot [38, 59, 60, 111].

Example 2.1 *The dynamics of the robot is described by the following state space representation:*

$$\dot{x} = Ax + bu + \Phi(x,u), \tag{2.59a}$$

$$y = Cx \tag{2.59b}$$

where

$$x = \begin{bmatrix} \theta_m \\ \omega_m \\ \theta_1 \\ \omega_1 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -48.6 & -1.25 & 48.6 & 0 \\ 0 & 0 & 0 & 1 \\ 19.5 & 0 & -19.5 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 21.6 \\ 0 \\ 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \Phi(x,u) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -3.33\sin(x_3) \end{bmatrix},$$

and θ_m is the angular position of the motor; ω_m is the angular velocity of the motor; θ_1 is the angular position of the link; and ω_1 is the angular velocity of the link.

In the following, the procedure for the observer design and output feedback control design are presented.

Observer design: Since A is not stable, we design a preliminary gain L_1 such that $(A - L_1 C)$ is stable with poles at $-9.3275, -8.9203, -9.6711$ and -4.7722. The gain L_1 is found to be

$$L_1 = \begin{bmatrix} 9.3275 & 1.0000 \\ -48.7804 & 22.1136 \\ -0.0524 & 3.1994 \\ 19.4066 & -0.9032 \end{bmatrix}.$$

The Lipschitz constant of $\Phi(x, u)$ with respect to x is $\gamma = 3.33$. Using the similarity transformation $x = T_o x'$, transform the system (2.59) with $T_o = \mathrm{diag}(1, 1, 1, 10)$. The new Lipschitz constant is $\gamma'_o = 0.333$. Choose constants $\varepsilon_o = 0.1111$ and $\eta_o = 0.1211$, and check the condition given by (2.48). It is computed that $\delta\left(A', \frac{\sqrt{\gamma'^2 + \varepsilon_o}}{\|C'\|} C'\right) = 0.8389$, so (2.48) is satisfied. Solving the ARE (2.49) results in

$$P_o = \begin{bmatrix} 18.6546 & -0.0234 & 0.0396 & 0.0012 \\ -0.0234 & 5.9522 & -12.5731 & 1.9503 \\ 0.0396 & -12.5731 & 30.8320 & -8.8656 \\ 0.0012 & 1.9503 & -8.8656 & 9.7302 \end{bmatrix}.$$

Then, the observer gain matrix is given by

$$L' = \frac{1}{2} P_o^{-1} C'^{\top} = \begin{bmatrix} 0.0268 & 0.0003 \\ 0.0003 & 1.1392 \\ 0.0001 & 0.5405 \\ 0.0000 & 0.2641 \end{bmatrix}.$$

The observer for the flexible link robot (2.59) is in the form of (2.50) with

$$L = \begin{bmatrix} 9.3334 & 1.0001 \\ -48.7804 & 22.3665 \\ -0.0524 & 3.3194 \\ 19.4066 & -0.3167 \end{bmatrix}$$

where (2.51) is used.

The simulation results of the observer, (2.59), are shown in Figures 2.1 and 2.2. In the simulation, the initial value of x, $x(0)$, is chosen to be $\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^\top$; the initial value of $\widehat{x}$, $\widehat{x}(0)$, is chosen to be $\begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^\top$. The system is assumed to be under no control, that is, $u = 0$. Figure 2.1 shows the motor angular position, motor angular velocity, and their estimates. Figure 2.2 shows the link angular position, the link angular velocity and their estimates. From both the figures, one can see that the estimates converge to the true states.

Output feedback control: As done in the observer design case, we first use a preliminary control to make $(A - BK_1)$ stable with poles at -5.8989, -5.6390, -4.9245 and -8.9109. The gain K_1 is found to be

$$K_1 = \begin{bmatrix} 7.8092 & 1.1168 & -4.3436 & 1.12 \end{bmatrix}.$$

Then, a similarity transformation, $x = T_c x'$, is used to reduce the Lipschitz gain with $T_c = \text{diag}(1, 1, 1, 10)$. The new Lipschitz constant is $\gamma_c' = 0.333$. Choose constants $\eta_c = 3.7947(10^{-4})$, and check the condition given by (2.56). It is computed that $\delta\left(A'^\top, \frac{\sqrt{\gamma'^2 + \varepsilon_c}}{\|B'\|} B'^\top\right) = 0.3552$, so (2.56) is satisfied. Solving the ARE (2.57) results in

$$P_c = \begin{bmatrix} 13.4725 & 1.4496 & -8.8421 & 17.387 \\ 1.4496 & 0.18736 & -0.99393 & 1.8462 \\ -8.8421 & -0.99393 & 6.1806 & -11.1047 \\ 17.387 & 1.8462 & -11.1047 & 26.2607 \end{bmatrix},$$

which in turn results in

$$K' = \frac{1}{2}B'^{\top}P_c = \left[\begin{array}{cccc} 15.6553 & 2.0235 & -10.7345 & 19.9385 \end{array} \right].$$

The control input for the flexible link robot (2.59) is $u = -K\widehat{x}$ with

$$K = \left[\begin{array}{cccc} 7.8428 & 1.1212 & -4.3666 & 1.1243 \end{array} \right]$$

where (2.58) is used.

The simulation results for regulating the states of the flexible robot (2.59) to zero are shown in Figures 2.3 and 2.4. In this simulation, the initial values of x and $\widehat{x}$ are chosen to be the same as those in the simulation for the observer in the previous simulation. Figure 2.3 shows the motor angular position, motor angular velocity, and their estimates. Figure 2.4 shows the link angular position, the link angular velocity and their estimates. Comparing Figures 2.3 and 2.4 with Figures 2.1 and 2.2, it is clearly seen that, under the output feedback control, four states of the robot ($\theta_m, \omega_m, \theta_1$ and ω_1) converge to zero rapidly; whereas, without control, the states converge to zero very slowly. Also, the convergence of the estimated states to their true values is observed.

2.7 Summary

In this chapter, the full-state feedback control problem, the observer design problem, and the output feedback control problem for a class of Lipschitz nonlinear systems are considered. A linear full-state feedback controller and a nonlinear observer are proposed, and sufficient conditions under which exponential stability is achieved are given. Generally, for nonlinear systems, stabilization by state feedback plus observability does not imply stabilization by output feedback, that is, separation principle usually does not hold for nonlinear systems. However, for the class of nonlinear systems considered in this chapter, by using the proposed full-state linear feedback controller and the proposed nonlinear observer, it is shown that the separation principle holds; that is, the same gain matrix which was obtained in the design of the full-state linear feedback controller can be used with the estimated state, where the estimates are obtained from the proposed observer.

Systems with Lipschitz nonlinearity are common in many practical applications. Many nonlinear systems satisfy the Lipschitz property at least locally by representing them by a linear part plus a Lipschitz nonlinearity around their equilibrium points. Hence, the class of systems considered in this chapter cover a fairly large number of systems in practice.

There are some challenging problems that need to be addressed in the future. It is clear that the number δ is realization dependent. So, a natural question to ask is which realization gives the maximum value for δ and further, how does one transform the system given in any arbitrary form to this particular realization. Moreover, it is also not clear as to how one can, in general, find transformations that increase δ and decrease γ simultaneously.

It is also emphasized here that the conditions for both full-state feedback and output feedback stabilization are sufficient conditions; how to satisfy these two sufficient conditions is a challenging problem which needs to be investigated in the future. The problem essentially reduces to the following: how are the eigenvalues of A and the singular values of $(i\omega I - A)$ related. Further, how does one influence the singular values of $(i\omega I - A)$ if we have control over arbitrary assignment of eigenvalues of A.

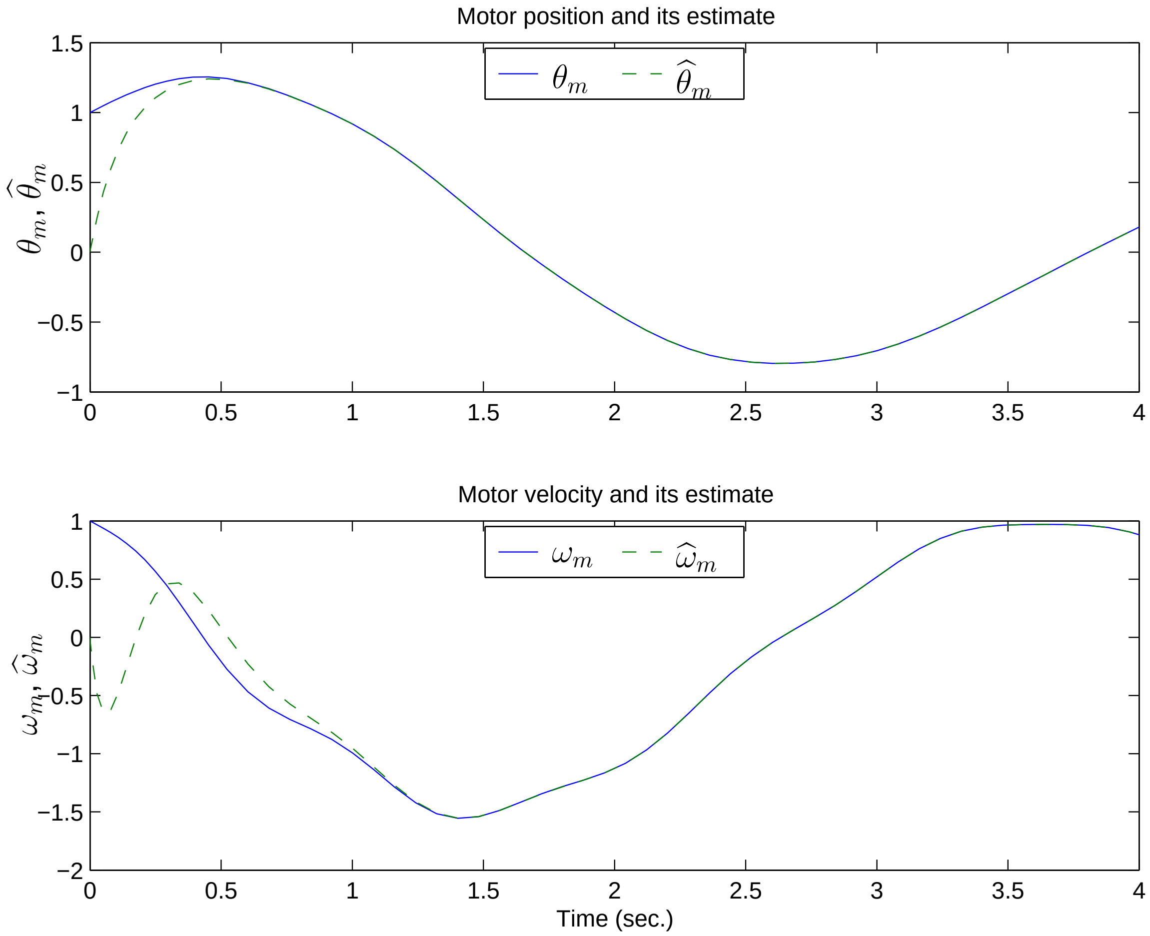

Figure 2.1: Simulation result of Example 2.1. The motor angular position θ_m and its estimate $\widehat{\theta}_m$ are shown in the top plot. The motor angular velocity ω_m and its estimated $\widehat{\omega}_m$ are shown in the bottom plot. The control is $u = 0$.

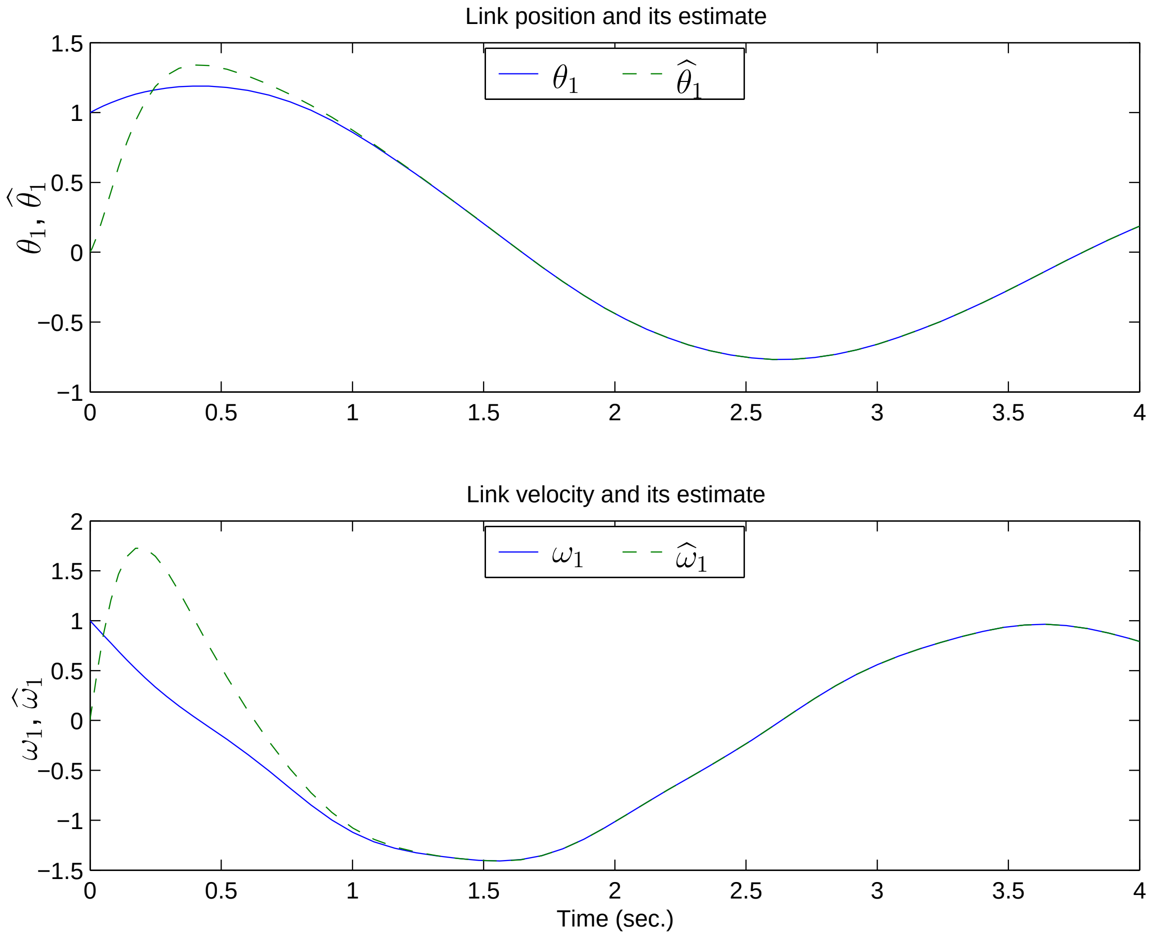

Figure 2.2: Simulation result of Example 2.1. The link angular position θ_1 and its estimate $\widehat{\theta}_1$ are shown in the top plot. The link angular velocity ω_1 and its estimate $\widehat{\omega}_1$ are shown in the bottom plot. The control is $u = 0$.

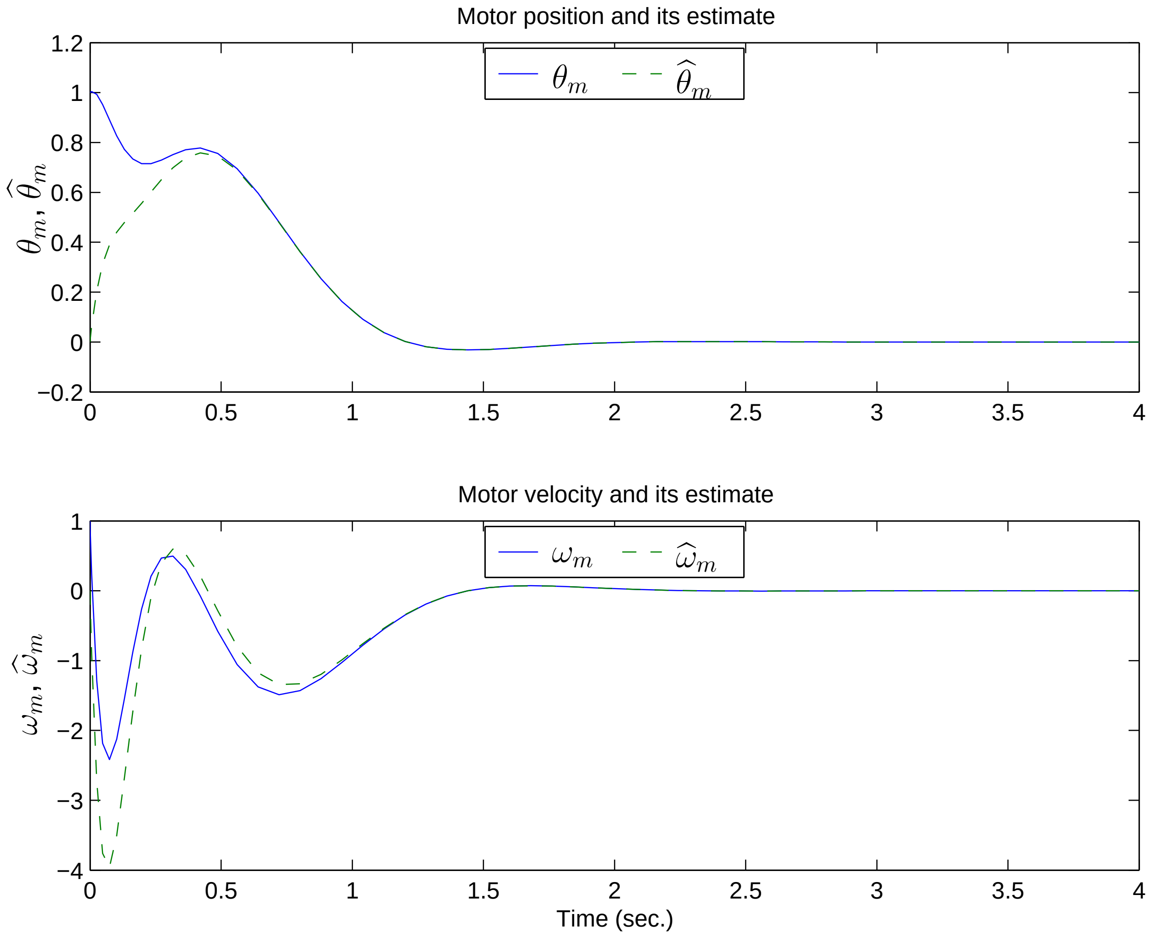

Figure 2.3: Simulation result of Example 2.1. The motor angular position θ_m and its estimate $\widehat{\theta}_m$ are shown in the top plot. The motor angular velocity ω_m and its estimated $\widehat{\omega}_m$ are shown in the bottom plot. The control is $u = -K\widehat{x}$.

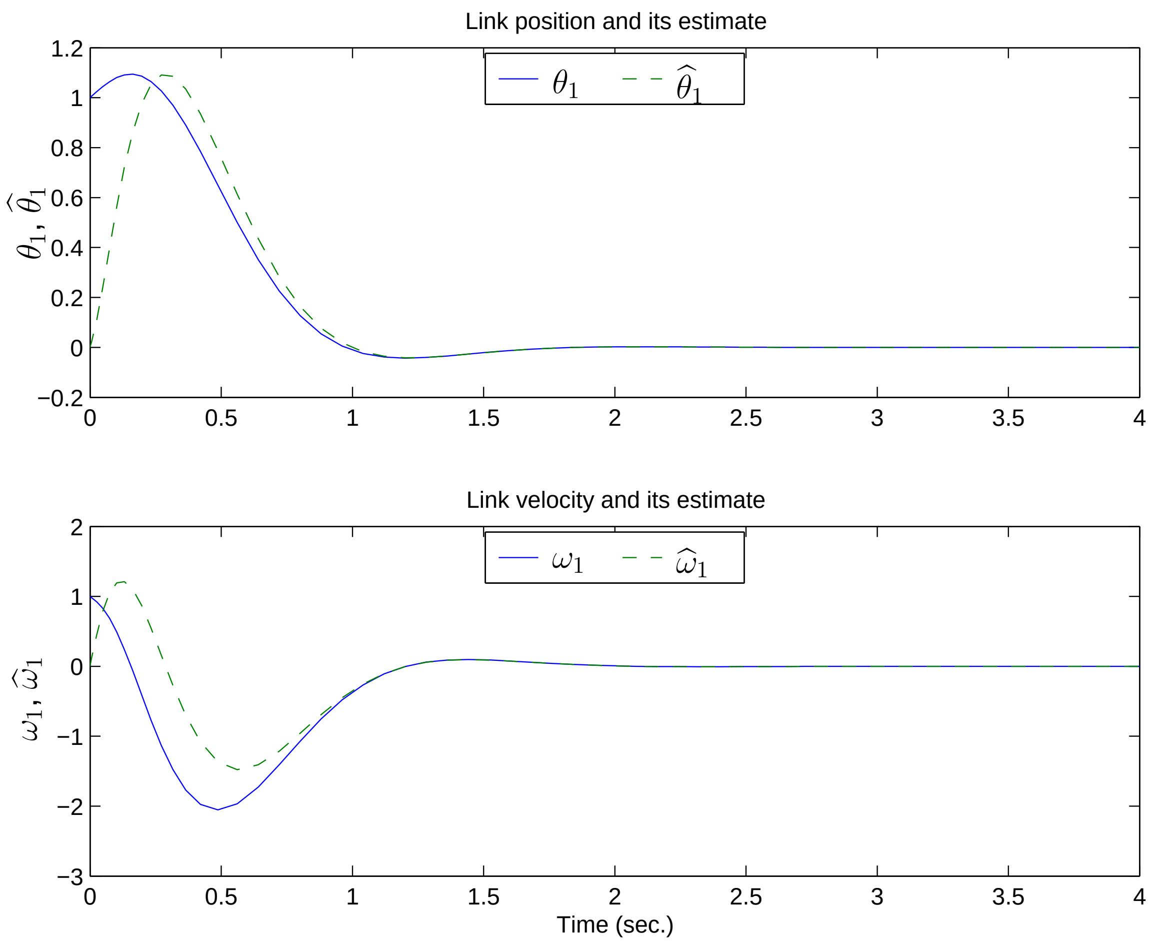

Figure 2.4: Simulation result of Example 2.1. The link angular position θ_1 and its estimate $\widehat{\theta}_1$ are shown in the top plot. The link angular velocity ω_1 and its estimate $\widehat{\omega}_1$ are shown in the bottom plot. The control is $u = -K\widehat{x}$.

CHAPTER 3

DECENTRALIZED OUTPUT FEEDBACK CONTROL OF LARGE-SCALE INTERCONNECTED NONLINEAR SYSTEMS

The primary motivation for designing decentralized control laws for large-scale systems is that they do not require information exchange between individual subsystems. Only information from the local system can be used to design a controller to stabilize the overall system. This constraint renders design of stable decentralized control laws for large-scale systems difficult. Decentralized output feedback control design is more challenging because only the output information of the local system can be used to design local controllers. In this chapter, a solution is provided for designing decentralized output feedback controller for a class of large-scale nonlinear systems with quadratically bounded nonlinear interconnections. Designs using the LMI approach and the ARE approach are addressed. Exponential stabilization of the overall system under the proposed decentralized output feedback control is achieved.

The rest of the chapter is organized as follows. In Section 3.1, the class of considered large-scale systems is given with a discussion of the problem. In Section 3.2, related results available in literature and their limitation by using Linear Matrix Inequality (LMI) approach are discussed and a new LMI approach is proposed to design a decentralized output feedback controller. In Section 3.3, another decentralized controller/observer structure based on the existence of solutions to AREs is proposed; sufficient conditions under which exponential stabilization is achieved are derived. Simulation results on an example are given in Section 3.4. Section 3.5 summarizes this chapter and highlights some future research topics on the problem.

3.1 Problem formulation

The following class of large-scale interconnected nonlinear systems is considered:

$$\dot{x}_i(t) = A_i x_i(t) + B_i u_i(t) + h_i(t, x), \quad x_i(t_0) = x_{i0}, \tag{3.1a}$$

$$y_i(t) = C_i x_i(t) \tag{3.1b}$$

where $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^{m_i}$, $y_i \in \mathbb{R}^{l_i}$, $h_i \in \mathbb{R}^{n_i}$, t_0, and x_{i0} are the state, input, output, nonlinear interconnection function, initial time, and initial state of the i-th sub-system, and $x = \begin{bmatrix} x_1^\top & x_2^\top & \dots & x_N^\top \end{bmatrix}^\top$ is the state of the overall system. The term $h_i(t, x)$ is called the interconnection of the i-th sub-system with other sub-systems plus the uncertainty dynamics from the i-th sub-system itself, and it is assumed that the exact expression of $h_i(t, x)$ is unknown. The large-scale system is composed of N sub-systems, that is, $i = 1, 2, \dots, N$. The objective of this chapter is to design a totally decentralized observer-based linear controller that robustly regulates the state of the overall system without any information exchange between sub-systems, that is, the local controller u_i is constrained to use only local output signal y_i. One specific practical application whose system model conforms to (3.1) with quadratic interconnection bounds (3.2) is a multimachine power system consisting of N interconnected machines with steam valve control; the dynamic model is discussed in [112].

To make the problem tractable, we specify each sub-system given by (3.1) by the following assumptions.

Assumption A3.1 *The interconnections are piecewise-continuous functions in both variables, and satisfy the quadratic constraints [94]*

$$h_i^T(t, x) h_i(t, x) \leq \alpha_i^2 x^\top H_i^\top H_i x \tag{3.2}$$

where $\alpha_i > 0$ are interconnection bounds, H_i are bounding matrices. Also, α_i and $\|H_i\|$ are known.

Assumption A3.2 *(A_i, B_i) is a controllable pair and (C_i, A_i) is an observable pair. Without loss of generality, it is assumed that A_i is stable, that is, all eigenvalues of A_i have negative real parts.*

Remark 3.1 *Comparing the assumption on the interconnection $h_i(\cdot)$ given by (3.2) and the assumption on the nonlinearity $\Phi(\cdot)$ given by (2.2) in Chapter 2, one will find a similarity between them. If $\Phi(0, u) = 0$, then the $Phi(x, u)$ also satisfies the quadratic condition given by (3.2). This can be seen from the following. Since*

$$\|\Phi(x, u) - \Phi(0, u)\| = \|\Phi(x, u)\| \leq \gamma \|x\|,$$

one has

$$\Phi^{\top}(x, u)\Phi(x, u) \leq \gamma^2 x^{\top} x,$$

which is an inequality in the form of (3.2). However, if $\Phi(0, u) \neq 0$, conditions given by (2.2) and (3.2) are generally different. The condition (2.2) says that the slope of any two points on the trajectory $\Phi(\cdot)$ should not exceed γ, whereas, the condition (3.2) says that the norm of the trajectory $h_i(\cdot)$ should be linearly bounded by the norm of x, that is, $\|h_i(t, x)\| \leq \alpha_i \|x\|$.

Notice that, because of the nature of the interconnection, $h_i(t, x)$, in some cases, system (3.1) may not be stabilizable even with full-state feedback control. For example, consider the first sub-system in the following form

$$\dot{x}_1 = \underbrace{\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}}_{A_1} x_1 + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B_1} u_1 + \gamma_1 \begin{bmatrix} x_{11} - x_{12} \\ x_{21} \end{bmatrix}, \tag{3.3a}$$

$$y_1 = \begin{bmatrix} 1 & 0 \end{bmatrix} x_1. \tag{3.3b}$$

If $\gamma_1 = 1$, then the first state of x_1, x_{11}, has the following dynamics

$$\dot{x}_{11} = x_{11},$$

which is unstable and we lose controllability of the system. Therefore, one cannot design a controller to stabilize the system (3.3) with the given interconnection, although (A_1, B_1) is a controllable pair. From the example, it is clear that the structure and bounds of the interconnections will affect controllability of the sub-system. The same holds true for observability of the system. So, there must be some conditions on the system matrices and the interconnections under which the controllability and observability properties are preserved.

Two broad methods are used to design observer-based decentralized output feedback controllers for large-scale systems: (1) Design local observer and controller for each sub-system independently, and check the stability of the overall closed-loop system. In this method, the interconnection in each sub-system is regarded as an unknown input [89, 91]. (2) Design the observer and controller by posing the output feedback stabilization problem as an optimization problem. The optimization approach using LMIs can be found in [94]. In the next two sections, both approaches are investigated.

3.2 The LMI approach

In this section, the linear matrix inequality is briefly introduced. An LMI approach to design a decentralized output feedback controller for the large-scale system (3.1) is proposed. The proposed approach does not require the invertibility of the $B_i, i = 1, \ldots, N$. Feasibility of the proposed LMI solution is also shown.

3.2.1 Preliminaries

A very wide variety of problems in system and control theory can be reduced to a few standard convex or quasiconvex optimization problems involving linear matrix inequalities. These optimization problems can be solved numerically very efficiently using recently developed interior-point methods (e.g. MATLAB LMI solvers).

A linear matrix inequality has the form

$$F(x) \triangleq F_0 + \sum_{i=1}^{m} x_i F_i > 0 \tag{3.4}$$

where $x \in \mathbb{R}^m$ is the variable and the symmetric matrices $F_i = F_i^\top \in \mathbb{R}^{n\times n}, i = 0, 1, \ldots, m$, are given. The inequality symbol in (3.4) means that $F(x)$ is positive-definite. Of course, the LMI (3.4) is equivalent to a set of n polynomial inequalities in x, that is, the leading principal minors of $F(x)$ must be positive. Nonstrict LMIs have the form

$$F(x) \geq 0.$$

It is usual to encounter the constraint that some quadratic function (or quadratic form) be negative whenever some other quadratic function (or quadratic forms) are all negative. In some cases, this constraint can be expressed as an LMI in the data defining the quadratic functions or forms; in other cases, one can form an LMI that is conservative but often useful approximation of the constraint. This approximation by LMI is called the S-procedure. The following two lemmas [113, p. 23] describe the S-procedure for quadratic functions and nonstrict inequalities, and for quadratic functions and strict inequalities, respectively.

Lemma 3.1 *Let $F_0, \ldots, F_p$ be quadratic functions of the variable $\xi \in \mathbb{R}^n$:*

$$F_i(\xi) \triangleq \xi^\top T_i \xi + 2w_i^\top \xi + v_i, \quad i = 0, \ldots, p$$

where $T_i = T_i^\top$. Consider the following condition on $F_0, \ldots, F_p$:

$$F_0(\xi) \geq 0 \text{ for all } \xi \text{ such that } F_i(\xi) \geq 0, \quad i = 0, \ldots, p. \tag{3.5}$$

Then, if

$$\text{there exist } \tau_1 \geq 0, \ldots, \tau_p \geq 0 \text{ such that for all } \xi,$$
$$F_0(\xi) - \sum_{i=1}^{p} \tau_i F_i(\xi) \geq 0,$$

then (3.5) holds. When $p = 1$, the converse holds, provided that there is some ξ_0 such that $F_1(\xi_0) > 0$.

Lemma 3.2 *Let $T_0, \ldots, T_p \in \mathbb{R}^{n\times n}$ be symmetric matrices. Consider the following condition on $T_0, \ldots, T_p$:*

$$\xi^\top T_0 \xi > 0 \text{ for all } \xi \neq 0 \text{ such that } \xi^\top T_i \xi \geq 0, \quad i = 1, \ldots, p. \tag{3.6}$$

If

$$\text{there exists } \tau_1 \geq 0, \ldots, \tau_p \geq 0 \text{ such that } T_0 - \sum_{i=1}^{p} \tau_i T_i > 0,$$

then (3.6) holds. When $p = 1$, the converse holds, provided that there is some ξ_0 such that $\xi_0^\top T_1 \xi_0 > 0$.

3.2.2 Decentralized output feedback controller design by the LMI approach

The overall system (3.1) can be rewritten as

$$\dot{x}(t) = A_D x(t) + B_D u(t) + h(t, x), \quad x(t_0) = x_0, \tag{3.7a}$$

$$y(t) = C_D x(t) \tag{3.7b}$$

where

$$A_D = \operatorname{diag}(A_1, \ldots, A_N), \qquad B_D = \operatorname{diag}(B_1, \ldots, B_N),$$

$$C_D = \operatorname{diag}(C_1, \ldots, C_N), \qquad u = \begin{bmatrix} u_1^\top & \ldots & u_N^\top \end{bmatrix}^\top,$$

$$y = \begin{bmatrix} y_1^\top & \ldots & y_N^\top \end{bmatrix}^\top, \qquad h = \begin{bmatrix} h_1^\top & \ldots & h_N^\top \end{bmatrix}^\top.$$

The nonlinear interconnections $h(t, x)$ are bounded as follows:

$$h^\top(t, x) h(t, x) \leq x^\top \Gamma^\top \Gamma x \tag{3.8}$$

where

$$\Gamma^\top \Gamma = \sum_{i=1}^{N} \alpha_i^2 H_i^\top H_i. \tag{3.9}$$

The pair (A_D, B_D) is controllable and the pair (C_D, A_D) is observable, which is a direct result of each subsystem being controllable and observable.

Since the system (3.7) is linear with nonlinear interconnections, a common question to ask is under what conditions can we design a decentralized linear controller and a decentralized linear observer that will stabilize the system in the presence of bounded interconnections. Towards solving this problem, one can consider the following linear decentralized controller and observer:

$$u(t) = K_D \widehat{x}(t), \tag{3.10}$$

$$\dot{\widehat{x}}(t) = A_D \widehat{x}(t) + B_D u(t) + L_D (y(t) - C_D \widehat{x}(t)) \tag{3.11}$$

where

$$K_D = \operatorname{diag}(K_1, \ldots, K_N), \tag{3.12}$$

$$L_D = \operatorname{diag}(L_1, \ldots, L_N) \tag{3.13}$$

are the controller gain matrix and the observer gain matrix, respectively. Rewriting (3.7) and (3.11) in the coordinates $x(t)$ and $\widetilde{x}(t)$, where $\widetilde{x}(t) \triangleq x(t) - \widehat{x}(t)$ is the estimation error, the closed-loop dynamics is

$$\dot{x}(t) = (A_D + B_D K_D) x(t) - B_D K_D \widetilde{x}(t) + h(t, x), \tag{3.14a}$$

$$\dot{\widetilde{x}}(t) = (A_D - L_D C_D) \widetilde{x}(t) + h(t, x). \tag{3.14b}$$

Let

$$A_c \triangleq A_D + B_D K_D, \quad A_o \triangleq A_D - L_D C_D. \tag{3.15}$$

Consider the following Lyapunov function candidate

$$V(x, \widetilde{x}) = x^\top \bar{P}_c x + \widetilde{x}^\top \bar{P}_o \widetilde{x}. \tag{3.16}$$

The time derivative of $V(x, \widetilde{x})$ along the trajectories of (3.14) is given by

$$\begin{aligned}
\dot{V}(x, \widetilde{x}) &= x^\top (A_c^\top \bar{P}_c + \bar{P}_c A_c) x - x^\top \bar{P}_c B_D K_D \widetilde{x} - \widetilde{x}^\top K_D^\top B_D^\top \bar{P}_c x + x^\top \bar{P}_c h + h^\top \bar{P}_c x \\
&\quad \widetilde{x}^\top (A_o^\top \bar{P}_o + \bar{P}_o A_o) \widetilde{x} + \widetilde{x}^\top \bar{P}_o h + h^\top \bar{P}_o \widetilde{x} \\
&= \begin{bmatrix} x \\ \widetilde{x} \\ h \end{bmatrix}^\top \begin{bmatrix} A_c^\top \bar{P}_c + \bar{P}_c A_c & -\bar{P}_c B_D K_D & \bar{P}_c \\ -K_D^\top B_D^\top \bar{P}_c & A_o^\top \bar{P}_o + \bar{P}_o A_o & \bar{P}_o \\ \bar{P}_c & \bar{P}_o & 0 \end{bmatrix} \begin{bmatrix} x \\ \widetilde{x} \\ h \end{bmatrix}.
\end{aligned} \tag{3.17}$$

The nonlinear interconnection condition given by (3.8) is equivalent to

$$\begin{bmatrix} x \\ \widetilde{x} \\ h \end{bmatrix}^\top \begin{bmatrix} -\Gamma^\top \Gamma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} x \\ \widetilde{x} \\ h \end{bmatrix} \leq 0. \tag{3.18}$$

The stabilization of the system (3.14) requires that

$$\dot{V}(x, \widetilde{x}) < 0 \tag{3.19}$$

for all $x, \widetilde{x} \neq 0$; together with the condition given by (3.18), one can apply Lemma 3.2 and obtain that if

$$\begin{bmatrix} A_c^\top \bar{P}_c + \bar{P}_c A_c & -\bar{P}_c B_D K_D & \bar{P}_c \\ -K_D^\top B_D^\top \bar{P}_c & A_o^\top \bar{P}_o + \bar{P}_o A_o & \bar{P}_o \\ \bar{P}_c & \bar{P}_o & 0 \end{bmatrix} - \tau \begin{bmatrix} -\Gamma^\top \Gamma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{bmatrix} < 0, \tag{3.20a}$$

$$\bar{P}_c > 0, \; \bar{P}_o > 0, \; \tau > 0, \tag{3.20b}$$

then, the inequality (3.19) is satisfied. Let

$$P_c \triangleq \frac{\bar{P}_c}{\tau}, \quad P_o \triangleq \frac{\bar{P}_o}{\tau}.$$

The condition given by (3.20) is equivalent to

$$\begin{bmatrix} A_c^\top P_c + P_c A_c + \Gamma^\top \Gamma & -P_c B_D K_D & P_c \\ -K_D^\top B_D^\top P_c & A_o^\top P_o + P_o A_o & P_o \\ P_c & P_o & -I \end{bmatrix} < 0, \; P_c > 0, \; P_o > 0. \tag{3.21}$$

The Schur complement result:

$$\begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} < 0$$

is equivalent to

$$Q < 0 \quad \text{and} \quad R - S^\top Q^{-1} S < 0.$$

Considering (3.8) and (3.15), and applying the Schur complement to the inequality (3.21), results in

$$P_c > 0, P_o > 0, \tag{3.22a}$$

$$\begin{bmatrix} W_C & -P_c B_D K_D & P_c & \alpha_1 H_1^\top & \dots & \alpha_N H_N^\top \\ -(P_c B_D K_D)^\top & W_O & P_o & 0 & \dots & 0 \\ P_c & P_o & -I & 0 & \dots & 0 \\ \alpha_1 H_1 & 0 & 0 & -I & \dots & 0 \\ \vdots & 0 & \vdots & \vdots & \ddots & \vdots \\ \alpha_N H_N & 0 & 0 & 0 & \dots & -I \end{bmatrix} < 0 \tag{3.22b}$$

where

$$W_C \triangleq A_D^\top P_c + P_c A_D + (P_c B_D K_D)^\top + (P_c B_D K_D),$$
$$W_O \triangleq A_D^\top P_o + P_o A_D - P_o L_D C_D - (P_o L_D C_D)^\top.$$

Rearranging entries and scaling corresponding columns and rows related to $H_i, i =$

$1, \ldots, N$, on the left hand side matrix (3.22b), one obtains

$$P_c > 0, P_o > 0, \tag{3.23a}$$

$$\begin{bmatrix} W_C & H_1^\top & \ldots & H_N^\top & -P_c B_D K_D & P_c \\ H_1 & -\gamma_1 I & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & 0 & 0 \\ H_N & 0 & \ldots & -\gamma_N I & 0 & 0 \\ -(P_c B_D K_D)^\top & 0 & 0 & 0 & W_O & P_o \\ P_c & 0 & 0 & 0 & P_o & -I \end{bmatrix} < 0 \tag{3.23b}$$

where $\gamma_i = \dfrac{1}{\alpha_i^2} > 0$. Now the problem of stabilization of the large-scale system (3.1) by decentralized output feedback control is transferred to the problem of finding finding $\gamma_i > 0, i = 1, \ldots, N$, such that inequalities in (3.23) are satisfied. Further, if the following optimization problem

$$\text{Minimize} \sum_{i=1}^{N} \gamma_i \text{ subject to Equation (3.23)} \tag{3.24}$$

is feasible, the selection of the control gain matrix K_D and observer gain matrix L_D not only stabilizes the overall system (3.14), but also simultaneously maximizes the interconnection bounds α_i.

In the optimization problem given by (3.24), variables are P_c, P_o, K_D, L_D and $\gamma_i, i = 1, \ldots, N$. Since there are coupled term of matrix variables P_c and K_D, and P_o and L_D in the matrix inequality (3.23b), the optimization (3.24) is not on a convex set. One has to find a way to transform the inequality (3.23b) to a form which is affine in variables. To achieve this, one can introduce variables

$$M_D \triangleq P_c B_D K_D, \quad N_D \triangleq P_o L_D. \tag{3.25}$$

Then, the optimization problem (3.24) becomes

$$\text{Minimize } \sum_{i=1}^{N} \gamma_i \text{ subject to}$$

$$P_c > 0, P_o > 0, \tag{3.26a}$$

$$\begin{bmatrix} W_C & H_1^\top & \dots & H_N^\top & -M_D & P_c \\ H_1 & -\gamma_1 I & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & 0 & 0 \\ H_N & 0 & \dots & -\gamma_N I & 0 & 0 \\ -M_D^\top & 0 & 0 & 0 & W_O & P_o \\ P_c & 0 & 0 & 0 & P_o & -I \end{bmatrix} < 0 \tag{3.26b}$$

The solution to the optimization problem (3.26) gives rise to M_D and N_D. The controller and observer gain matrices were obtained from M_D and N_D in [94] in the following manner. The observer gain matrix L_D can be computed using (3.25) as

$$L_D = P_o^{-1} N_D.$$

However, controller gain matrix K_D can be obtained only in the case when B_D is invertible, that is,

$$K_D = B_D^{-1} P_c^{-1} M_D.$$

Obviously, invertibility of B_D requires that $B_i, i = 1, \dots, N$ be invertible, which is too restrictive. When all the B_i are not invertible, it is not possible to obtain the control gain matrix K_D from the optimization problem (3.26). The following addresses the LMI solution to the case when B_i are not invertible.

One can pre-multiply and post-multiply the left hand side of (3.23b) by

$$\text{diag}(P_c^{-1}, I, I, \dots, I)$$

and define $Y = P_c^{-1}$ to obtain following conditions which are equivalent to (3.23):

$$Y > 0, P_o > 0, \tag{3.27a}$$

$$\begin{bmatrix} W'_C & YH_1^\top & \dots & YH_N^\top & -B_DK_D & I \\ H_1Y & -\gamma_1 I & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & 0 & 0 \\ H_NY & 0 & \dots & -\gamma_N I & 0 & 0 \\ -(B_DK_D)^\top & 0 & 0 & 0 & W_O & P_o \\ I & 0 & 0 & 0 & P_o & -I \end{bmatrix} < 0 \tag{3.27b}$$

where $W'_C \triangleq YA_D^\top + A_DY + (B_DK_DY)^\top + (B_DK_DY)$.

Let

$$\bar{M}_D \triangleq K_DY, \quad \begin{bmatrix} S_1 & S_2 \end{bmatrix} \triangleq \begin{bmatrix} -B_DK_D & I \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}.$$

Now, the problem is to find Y, P_o, K_D, L_D, and $\gamma_i, i = 1, \dots, N$, which can be found by the following two steps.

Step 1. Maximize the interconnection bounds α_i by solving the following optimization problem:

$$\text{Minimize } \sum_{i=1}^{N} \gamma_i \text{ subject to}$$

$$Y > 0, F_{opt} = \begin{bmatrix} W'_C & YH_1^\top & \dots & YH_N^\top \\ H_1Y & -\gamma_1 I & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ H_NY & 0 & \dots & -\gamma_N I \end{bmatrix} < 0. \tag{3.28}$$

Step 2. Find P_o and N_D by using K_D obtained from Step 1 and solving the following optimization problem

$$\text{Minimize} \sum_{i=1}^{N} \beta_i \text{ subject to}$$

$$P_o > 0, \Lambda > 0, \begin{bmatrix} \Lambda F_{opt} & S_1 & S_2 \\ S_1^\top & W_O & P_o \\ S_2^\top & P_o & -I \end{bmatrix} < 0. \tag{3.29}$$

where $\Lambda = \text{diag}(\beta_1 I_1, \beta_2 I_2, \ldots, \beta_N I_N, \beta_1 I_1, \beta_2 I_2, \ldots, \beta_N I_N)$, and I_i denotes the $n_i \times n_i$ identity matrix. The matrices F_{opt} and S_1 in Step 2 are obtained from Step 1.

The control gain K_D is obtained from Step 1 as

$$K_D = \bar{M}_D Y^{-1}, \tag{3.30}$$

and the observer gain L_D is obtained from Step 2 as

$$L_D = P_o^{-1} N_D. \tag{3.31}$$

Remark 3.2 *Unlike the case when B_D is invertible, inequalities given by* (3.28) *and* (3.29) *cannot be solved simultaneously. The optimization problem* (3.28) *of step 1 must be solved followed by step 2.*

Remark 3.3 *Since Y, A_D, B_D and $\bar{M}_D$ are all block diagonal matrices, it is not difficult to show that $\Lambda F_{opt} = \Lambda^{1/2} F_{opt} \Lambda^{1/2} < 0$ when $F_{opt} < 0$, and further in this case, $\Lambda F_{opt} < F_{opt} < 0$ if $\beta_i > 1, i = 1, 2, \ldots, N$.*

Remark 3.4 *If $\Lambda = I$, the LMI* (3.29) *may not be feasible for the selection of F_{opt} and K_D resulting from the optimization problem* (3.28)*. On the other hand, by choosing Λ as a matrix variable, the LMI* (3.29) *becomes feasible, which will be shown in the following.*

The following lemmas illustrate the feasibility of the LMI problems (3.28) and (3.29).

Lemma 3.3 *The optimization problem given by* (3.28) *is feasible if* $(A_i, B_i), i = 1, \ldots, N$*, is a controllable pair.*

Proof. To prove the LMI optimization problem (3.28) is feasible, one needs to show that there exists a solution that satisfies the inequality (3.28). In view of (3.28) and H_i being constant matrices, to show that there exist $Y > 0$, $\bar{M}_D$, $\gamma_i > 0, i = 1, \ldots, N$, such that $F_{opt} < 0$, it is sufficient to show that

$$\begin{aligned} &\textit{there exists a } Y > 0,\ \bar{M}_D \textit{ such that} \\ &W'_C < 0 \end{aligned} \tag{3.32}$$

because of the existence of large enough γ_i to dominate the off-diagonal block elements H_i in (3.28). Notice that

$$\begin{aligned} W'_C =& Y A_D^\top + A_D Y + (B_D \bar{M}_D)^\top + B_D \bar{M}_D \\ =& P_c^{-1} A_D^\top + A_D P_c^{-1} + (B_D K_D P_c^{-1})^\top + B_D K_D P_c^{-1} \\ =& P_c^{-1}\big((A_D + B_D K_D)^\top P_c + P_c(A_D + B_D K_D)\big) P_c^{-1}. \end{aligned}$$

Since (A_i, B_i) is a controllable pair, there exist a $P_c > 0$ and a K_D such that

$$(A_D + B_D K_D)^\top P_c + P_c(A_D + B_D K_D) < 0.$$

Therefore, the statement (3.32) is true. This completes the proof. ■

Lemma 3.4 *If* $(A_i, C_i), i = 1, \ldots, N$ *is an observable pair, the optimization problem* (3.29) *is feasible.*

Proof. We first prove that

$$\begin{aligned} &\textit{there exists a } P_o > 0 \textit{ and } N_D \textit{ such that} \\ &\begin{bmatrix} W_O & P_o \\ P_o & -I \end{bmatrix} < 0. \end{aligned} \tag{3.33}$$

Applying the Schur complement to the above matrix inequality yields the following equivalent inequality

$$W_O + P_o P_o < 0. \tag{3.34}$$

Recall that $N_D = P_o L_D$ and $W_O = A_D^\top P_o + P_o A_D - P_o L_D C_D - (P_o L_D C_D)^\top$. Equation (3.34) can be rewritten as

$$P_o\big((A_D - L_D C_D)Y_o + Y_o(A_D - L_D C_D)^\top + I\big)P_o < 0 \tag{3.35}$$

where $Y_o = P_o^{-1}$. Since (A_i, C_i) is an observable pair, there exists a $Y_o > 0$ and an L_D such that

$$(A_D - L_D C_D)Y_o + Y_o(A_D - L_D C_D)^\top + I < 0.$$

Hence, the statement (3.33) is true.

Since $F_{opt} < 0$ and the statement (3.33) is true, all the principal minors of the matrix on the left hand side of (3.29) are negative. Since S_1 and S_2 are constant matrices after solving the optimization given in Step 1, to guarantee that (3.29) holds, it is sufficient to let the principal minor ΛF_{opt} dominate the off-diagonal block elements S_1 and S_2; this can be achieved by a large $\Lambda > 0$. This completes the proof. ■

Remark 3.5 *The final uncertainty gains* γ_i, $i = 1, \ldots, N$, *is* $\beta_i \gamma_i$ *where* γ_i *is obtained from the optimization problem* (3.28) *and* β_i *is obtained from* (3.29).

The LMI optimization problems given by (3.28) and (3.29) do not pose any restrictions on the size of the matrix variables Y, $\bar{M}_D$, P_o and N_D. Consequently, the results of these two optimization problems may yield very large controller and observer gain matrices K_D and L_D, respectively. In view of (3.30) and (3.31), one can restrict K_D and L_D by posing constraints on the matrices Y, $\bar{M}_D$, P_o and N_D, and a further constraint on γ_i [88] as

$$\gamma_i - \frac{1}{\bar{\alpha}_i^2} < 0, \bar{\alpha}_i > 0; \quad Y_i^{-1} < \kappa_{Y_i} I, \kappa_{Y_i} > 0;$$

$$\bar{M}_{D_i} \bar{M}_{D_i}^\top < \kappa_{\bar{M}_{D_i}} I, \kappa_{\bar{M}_{D_i}} > 0; \tag{3.36}$$

$$\beta_i - \bar{\beta}_i > 0, \bar{\beta}_i > 0; \quad P_{o_i}{}^{-1} < \kappa_{P_{o_i}} I, \kappa_{P_{o_i}} > 0;$$

$$N_{D_i}^\top N_{D_i} < \kappa_{N_{D_i}} I, \kappa_{N_{D_i}} > 0 \tag{3.37}$$

where $\bar{M}_{D_i}$ and N_{D_i} are the i-th diagonal blocks of $\bar{M}_D$ and N_D, respectively. The constraints given by (3.36) and (3.37) place restrictions on the size of the control gain matrix K_D and observer gain matrix L_D, respectively. Equations (3.36) and (3.37) are respectively equivalent to

$$\gamma_i - \frac{1}{\bar{\alpha}_i^2} < 0, \quad \begin{bmatrix} -Y_i & -I \\ -I & -\kappa_{Y_i} I \end{bmatrix} < 0,$$

$$\begin{bmatrix} -\kappa_{\bar{M}_{D_i}} I & \bar{M}_{D_i} \\ \bar{M}_{D_i}^\top & -I \end{bmatrix} < 0, \quad \kappa_{Y_i}, \kappa_{\bar{M}_{D_i}} > 0, \tag{3.38}$$

$$\beta_i - \bar{\beta}_i > 0, \quad \begin{bmatrix} -P_{o_i} & -I \\ -I & -\kappa_{P_{o_i}} I \end{bmatrix} < 0,$$

$$\begin{bmatrix} -\kappa_{N_{D_i}} & N_{D_i}^\top \\ N_{D_i} & -I \end{bmatrix} < 0, \quad \kappa_{N_{D_i}}, \kappa_{P_{o_i}} > 0. \tag{3.39}$$

Combining (3.28) and (3.38), (3.29) and (3.39), and changing the optimization objectives to the minimization of $\sum_{i=1}^N (\gamma_i + \kappa_{Y_i} + \kappa_{\bar{M}_{D_i}})$ and $\sum_{i=1}^N (\beta_i + \kappa_{P_{o_i}} + \kappa_{N_{D_i}})$, respectively, results in the following two LMI optimization problems:

Step 1′. Maximize the interconnection bounds α_i by solving the following optimization problem:

$$\text{Minimize} \sum_{i=1}^{N} (\gamma_i + \kappa_{Y_i} + \kappa_{\bar{M}_{D_i}}) \text{ subject to}$$

$$\text{Equations (3.28) and (3.38).} \tag{3.40}$$

Step 2′. Find P_o and N_D by using K_D obtained from Step 1′ and solving the following optimization problem

$$\text{Minimize } \sum_{i=1}^{N}(\beta_i + \kappa_{P_{o_i}} + \kappa_{N_{D_i}}) \text{ subject to}$$

$$\text{Equations (3.29) and (3.39).} \tag{3.41}$$

Similar to Lemmas 3.3 and 3.4, it can be shown that the optimization problems (3.40) and (3.41) are feasible when all the subsystems are controllable and observable, provided that $\bar{\alpha}_i$ is chosen sufficiently small. This is because one can choose large $\bar{\beta}$, $\kappa_{\bar{M}_{D_i}}$, κ_{Y_i}, $\kappa_{N_{D_i}}$ and $\kappa_{P_{o_i}}$, and small $\bar{\alpha}_i$ to satisfy (3.38) and (3.39).

The results of the LMI solution to the decentralized output feedback control problem for the large scale system (3.1) are summarized in the following theorem.

Theorem 3.1 *Consider the large scale system* (3.1) *with the observer given by* (3.11) *and the controller given by* (3.10). *If*

$$\alpha_i \leq \min\left(\frac{1}{\sqrt{\gamma_i}}, \frac{1}{\sqrt{\beta\gamma_i}}\right) \tag{3.42}$$

where γ_i and β are solutions to the optimization problems (3.40) *and* (3.41), *then the selection of controller and observer gain matrices as given by* (3.30) *and* (3.31) *results in a stable closed-loop system.*

3.3 The ARE approach

In this section, the problem of decentralized exponential stabilization of the large-scale system via output feedback will be reduced to that of the existence of symmetric positive definite solutions of two Algebraic Ricatti Equations (AREs). Further, sufficient conditions for the existence of symmetric positive definite solutions will be derived; the conditions are developed using the concepts of distance to controllability and distance to observability [114] and their related results.

Consider the following linear decentralized controller and observer for the i-th sub-system:

$$u_i(t) = K_i \widehat{x}_i(t), \tag{3.43}$$

$$\dot{\widehat{x}}_i(t) = A_i \widehat{x}_i(t) + B_i u_i(t) + L_i(y_i(t) - C_i \widehat{x}_i(t)) \tag{3.44}$$

where K_i and L_i are the controller and observer gain matrices. Substituting these into the system, (3.1), one obtains

$$\dot{x}_i(t) = (A_i + B_i K_i) x_i(t) - B_i K_i \widetilde{x}_i(t) + h_i(t, x), \tag{3.45}$$

$$\dot{\widetilde{x}}_i(t) = (A_i - L_i C_i) \widetilde{x}_i(t) + h_i(t, x) \tag{3.46}$$

where $\widetilde{x}_i = x_i - \widehat{x}_i$. For simplicity define the following:

$$A_{Bi} = B_i K_i \text{ and } \; A_{Ci} = L_i C_i. \tag{3.47}$$

To find the controller and observer gain matrices, consider the the following quadratic Lyapunov function candidate:

$$V(x, \widetilde{x}) = \sum_{i=1}^{N} \Big(x_i^\top P_i x_i + \widetilde{x}_i^\top \widetilde{P}_i \widetilde{x}_i \Big). \tag{3.48}$$

The time derivative of $V(x, \widetilde{x})$ along the trajectories of (3.45) and (3.46), after some simplification, is given by

$$\begin{aligned} \dot{V}(x, \widetilde{x}) = \sum_{i=1}^{N} \Big\{ & x_i^\top [(A_i + A_{Bi})^\top P_i + P_i (A_i + A_{Bi})] x_i \\ & + \widetilde{x}_i^\top [(A_i - A_{Ci})^\top \widetilde{P}_i + \widetilde{P}_i (A_i - A_{Ci})] \widetilde{x}_i \\ & \underbrace{-\widetilde{x}_i^\top A_{Bi}^\top P_i x_i - x_i^\top P_i A_{Bi} \widetilde{x}_i} + \underbrace{h_i^\top P_i x_i + x_i^\top P_i h_i} + \underbrace{h_i^\top \widetilde{P}_i \widetilde{x}_i + \widetilde{x}_i^\top \widetilde{P}_i h_i} \Big\}. \end{aligned} \tag{3.49}$$

To simplify the terms with under braces in (3.49), one can use the following well known inequality. For any two real matrices X and Y, which are of the same dimension,

$$X^\top Y + Y^\top X \leq X^\top X + Y^\top Y. \tag{3.50}$$

Using the inequality (3.50) for the terms with under braces in (3.49), yields

$$\widetilde{x}_i^\top(-A_{Bi})^\top P_i x_i + x_i^\top P_i(-A_{Bi})\widetilde{x}_i \le \widetilde{x}_i^\top(-A_{Bi})^\top(-A_{Bi})\widetilde{x}_i + x_i^\top P_i P_i x_i, \tag{3.51a}$$

$$h_i^\top P_i x_i + x_i^\top P_i h_i \le h_i^\top h_i + x_i^\top P_i P_i x_i, \tag{3.51b}$$

$$h_i^\top \widetilde{P}_i \widetilde{x}_i + \widetilde{x}_i^\top \widetilde{P}_i h_i \le h_i^\top h_i + \widetilde{x}_i^\top \widetilde{P}_i \widetilde{P}_i \widetilde{x}_i. \tag{3.51c}$$

Each interconnection function, $h_i(t,x)$ satisfies

$$h_i^\top(t,x)h_i(t,x) = \alpha_i^2 x^\top H_i^\top H_i x \le \alpha_i^2 \nu_i x^\top x, \tag{3.52}$$

where $\nu_i = \lambda_{max}(H_i^\top H_i)$. Further, one has

$$\begin{aligned}\sum_{i=1}^N 2h_i^\top h_i &\le \sum_{i=1}^N 2\alpha_i^2 v_i(x_1^\top x_1 + \ldots + x_N^\top x_N) \\ &= \gamma^2(x_1^\top x_1 + \ldots + x_N^\top x_N)\end{aligned} \tag{3.53}$$

where $\gamma^2 \triangleq \sum_{i=1}^N 2\alpha_i^2 v_i$. Using inequalities (3.51) and (3.53) in (3.49), one obtains

$$\begin{aligned}\dot{V}(x,\widetilde{x}) \le \sum_{i=1}^N \Big\{ & x_i^\top \Big[(A_i + A_{Bi})^\top P_i + P_i(A_i + A_{Bi}) + 2P_iP_i + \gamma^2 I\Big] x_i \\ & + \widetilde{x}_i^\top \Big[(A_i - A_{Ci})^\top \widetilde{P}_i + \widetilde{P}_i(A_i - A_{Ci}) + A_{Bi}^\top A_{Bi} + \widetilde{P}_i\widetilde{P}_i\Big] \widetilde{x}_i \Big\}.\end{aligned} \tag{3.54}$$

Choose the following gain matrices:

$$K_i = -(B_i^\top B_i)^{-1} B_i^\top P_i, \tag{3.55}$$

$$L_i = \frac{1}{2}\varepsilon_i \widetilde{P}_i^{-1} C_i^\top. \tag{3.56}$$

where $\varepsilon_i > 0$. Substituting the gains into the derivative of the Lyapunov function candidate results in

$$\begin{aligned}\dot{V}(x,\widetilde{x}) \le \sum_{i=1}^N \Big\{ & x_i^\top \Big[A_i^\top P_i + P_i A_i + 2P_i(I - B_i(B_i^\top B_i)^{-1} B_i^\top)P_i + \gamma^2 I\Big] x_i \\ & + \widetilde{x}_i^\top \Big[A_i^\top \widetilde{P}_i + \widetilde{P}_i A_i + \widetilde{P}_i\widetilde{P}_i + \widetilde{Q}_{i1} - \varepsilon_i C_i^\top C_i\Big] \widetilde{x}_i \Big\}.\end{aligned} \tag{3.57}$$

where $\widetilde{Q}_{i1} \triangleq K_i^\top B_i^\top B_i K_i$.

From (3.57), one has the following result. For some $\eta_i > 0$ and $\widetilde{\eta}_i > 0$, if there exist positive definite solutions to the AREs

$$A_i^\top P_i + P_i A_i + 2P_i(I - B_i(B_i^\top B_i)^{-1} B_i^\top)P_i + \gamma^2 I + \eta_i I = 0, \tag{3.58}$$

$$A_i^\top \widetilde{P}_i + \widetilde{P}_i A_i + \widetilde{P}_i \widetilde{P}_i + \widetilde{Q}_{i1} + \widetilde{\eta}_i I - \varepsilon_i C_i^\top C_i = 0, \tag{3.59}$$

then

$$\dot{V}(x, \widetilde{x}_i) \leq -\sum_{i=1}^{N} \left[\eta_i x_i^\top x_i + \widetilde{\eta}_i \widetilde{x}_i^\top \widetilde{x}_i\right] \tag{3.60}$$

where ε_i and $\widetilde{\eta}_i$ are chosen such that $\widetilde{Q}_{i1} + \widetilde{\eta}_i I - \varepsilon_i C_i^\top C_i > 0$. As a result, if there are positive definite solutions to the AREs (3.58) and (3.59), then $V(x, \widetilde{x})$ is a Lyapunov function; that is, $V(x, \widetilde{x})$ is positive and $\dot{V}(x, \widetilde{x})$ is negative for $x, \widetilde{x} \neq 0$.

Remark 3.6 *The control gain matrix K_i given by (3.55) requires that $B_i^\top B_i$ is invertible; $B_i^\top B_i$ is invertible if B_i has full column rank; if two or more columns in B_i are dependent, then one can always combine the control signals corresponding to the dependent columns.*

Remark 3.7 *If A_i is not stable, then we can use pre-feedback to stabilize A_i by changing K_i and L_i given by equations (3.55) and (3.56), respectively, to the following:*

$$K_i = -(B_i^\top B_i)^{-1} B_i^\top P_i - \bar{K}_i, \tag{3.61}$$

$$L_i = \frac{1}{2}\varepsilon_i \widetilde{P}_i^{-1} C_i^\top + \bar{L}_i. \tag{3.62}$$

where $\bar{K}_i$ and $\bar{L}_i$ are pre-feedback gains such that $A_i^c \triangleq A_i - B_i \bar{K}_i$ and $A_i^o \triangleq A_i - \bar{L}_i C_i$ are Hurwitz. In such a case, A_i in (3.58) and (3.59) must be replaced by A_i^c and A_i^o, respectively.

Remark 3.8 *Notice that one cannot design the controller and observer independently, that is, the separation principle does not hold; the ARE (3.59) depends on the control gain matrix K_i. It is well known that the separation principle generally* *does not hold for nonlinear systems. But it should be noted that the above reduction procedure has yielded the following: one can design the controller gain independent of the observer and further, only the first ARE, (3.58), explicitly depends on the interconnection bounds.*

The problem of designing a stable controller and stable observer for the large-scale system (3.1) now reduces to the following: What are the conditions under which there exist positive definite solutions to the AREs (3.58) and (3.59). In the following sufficient conditions which guarantee the existence of positive definite solutions to the two AREs, for each sub-system, are developed, and the main theorem about the overall closed-loop system is shown.

3.3.1 Sufficient conditions

We first consider the ARE (3.58). The associated Hamiltonian matrix is given by

$$\mathcal{H}_i = \begin{bmatrix} A_i & R_i \\ -Q_i & -A_i^\top \end{bmatrix} \tag{3.63}$$

where

$$R_i = 2\left(I - B_i(B_i^\top B_i)^{-1}B_i^\top\right) \geq 0 \text{ and } Q_i = \left(\gamma^2 + \eta_i\right) I > 0.$$

The following lemma gives a condition under which $\mathcal{H}_i$ is hyperbolic; thus, by lemma 2.1, it gives a sufficient condition for the existence of a unique symmetric positive definite solution to the ARE (3.58).

Lemma 3.5 *$\mathcal{H}_i$ is hyperbolic if and only if*

$$\delta\Big(A_i^\top, \sqrt{2(\gamma^2+\eta_i)}\ (B_i^\top B_i)^{-1/2}B_i^\top\Big) > \sqrt{2(\gamma^2+\eta_i)}\ . \tag{3.64}$$

Proof. Consider the determinant of the matrix $(sI - \mathcal{H}_i)$ given by

$$\det(sI - \mathcal{H}_i) = \det \begin{bmatrix} sI - A_i & -2\left(I - B_i(B_i^\top B_i)^{-1}B_i^\top\right) \\ (\gamma^2 + \eta_i)I & sI + A_i^\top \end{bmatrix}$$

$$= (-1)^{n_i} \det \begin{bmatrix} (\gamma^2 + \eta_i)I & sI + A_i^\top \\ sI - A_i & -2\left(I - B_i(B_i^\top B_i)^{-1}B_i^\top\right) \end{bmatrix} \tag{3.65}$$

Since $(\gamma^2 + \eta_i)I$ is non-singular, using the formula for determinant of block matrices [110, p. 650], we obtain

$$\begin{aligned} &\det(sI - \mathcal{H}_i) \\ =&(-1)^{n_i}(\gamma^2 + \eta_i)^{n_i} \det\left[-2\left(I - B_i(B_i^\top B_i)^{-1}B_i^\top\right) - (sI - A_i)(\gamma^2 + \eta_i)^{-1}(sI + A_i^\top)\right] \\ =&(-1)^{n_i} \det\left[-2(\gamma^2 + \eta_i)\left(I - B_i(B_i^\top B_i)^{-1}B_i^\top\right) - (sI - A_i)(sI + A_i^\top)\right]. \end{aligned} \tag{3.66}$$

Define

$$G(s) = -2(\gamma^2 + \eta_i)\left(I - B_i(B_i^\top B_i)^{-1}B_i^\top\right) - (sI - A_i)(sI + A_i^\top). \tag{3.67}$$

From the equations (3.66) and (3.67), s is an eigenvalue of $\mathcal{H}_i$ if and only if $G(s)$ is singular. Hence, to prove that $\mathcal{H}_i$ is hyperbolic, one can prove that $G(-i\omega)$ is non-singular for all $\omega \in \mathbb{R}$. Notice that

$$\begin{aligned} \Delta_c(-i\omega) &\triangleq -(-i\omega I - A_i)(-i\omega I + A_i^\top) + 2(\gamma^2 + \eta_i)B_i(B_i^\top B_i)^{-1}B_i^\top \\ &= \begin{bmatrix} i\omega I - A_i^\top \\ \sqrt{2(\gamma^2 + \eta_i)}\,(B_i^\top B_i)^{-1/2}B_i^\top \end{bmatrix}^H \begin{bmatrix} i\omega I - A_i^\top \\ \sqrt{2(\gamma^2 + \eta_i)}\,(B_i^\top B_i)^{-1/2}B_i^\top \end{bmatrix}. \end{aligned} \tag{3.68}$$

Therefore, if

$$\delta\Big(A_i^\top, \sqrt{2(\gamma^2 + \eta_i)}\,(B_i^\top B_i)^{-1/2}B_i^\top\Big) > \sqrt{2(\gamma^2 + \eta_i)}\,,$$

then

$$G(-i\omega) = -2(\gamma^2 + \eta_i)I + \Delta_c(-i\omega) > 0 \tag{3.69}$$

for all $\omega \in \mathbb{R}$. Thus, $\mathcal{H}_i$ is hyperbolic. This completes the sufficiency part of the proof. The necessary part of the proof is similar to that of Lemma 2.2. ■

Now consider the ARE (3.59). The Hamiltonian matrix associated with the ARE (3.59) is

$$\widetilde{\mathcal{H}}_i = \begin{bmatrix} A_i & \widetilde{R}_i \\ -\widetilde{Q}_i & -A_i^\top \end{bmatrix} \tag{3.70}$$

where

$$\widetilde{R}_i = I > 0 \text{ and } \widetilde{Q}_i = \widetilde{Q}_{i1} + \widetilde{\eta}_i I - \varepsilon_i C_i^\top C_i.$$

Choose $\widetilde{\eta}_i > 0$ and $\varepsilon_i > 0$ such that $\widetilde{Q}_i > 0$. The following lemma gives a condition under which $\widetilde{\mathcal{H}}_i$ is hyperbolic; thus, by Lemma 2.1, it gives a sufficient condition for the existence of a symmetric positive definite solution to the ARE (3.59).

Lemma 3.6 *$\widetilde{\mathcal{H}}_i$ is hyperbolic if and only if*

$$\sqrt{\lambda_{max}(\widetilde{Q}_{i1}) + \widetilde{\eta}_i} < \delta(A_i, C_i). \tag{3.71}$$

Proof. Similar to the Lemma 3.5. ■

Notice that, by using the arguments of Remark 2.1 to Lemmas 3.5 and 3.6, the two conditions given by (3.64) and (3.71) can be respectively simplified to

$$\sqrt{2}\,\gamma < \delta\Big(A_i^\top, \sqrt{2}\,\gamma(B_i^\top B_i)^{-1/2}B_i^\top\Big), \tag{3.72}$$

$$\sqrt{\lambda_{max}(\widetilde{Q}_{i1})} < \delta(A_i, C_i). \tag{3.73}$$

Now, it is ready to introduce the main theorem on the results of the proposed method.

Theorem 3.2 *For the large-scale system given by (3.1) or (3.7), the decentralized controller and observer as given by (3.43) and (3.44) will result in exponential stabilization of the overall large-scale system, if (3.72) and (3.73) are satisfied for all $i = 1, 2 \ldots, N$.*

Proof. If (3.64) and (3.71) are satisfied for all $i = 1, 2, \ldots, N$, then from Lemmas 3.5, 3.6 and 2.1, the AREs (3.58) and (3.59) have symmetric positive definite solutions, P_i and $\widetilde{P}_i$, respectively. Consequently, one can choose $V(x, \widetilde{x})$ given by (3.48) as the Lyapunov function of the overall system (3.7). Thus, exponential stabilization of the overal closed-loop system is achieved. ∎

3.3.2 Remarks

Remark 3.9 *If the sufficient condition, given in Lemma 3.5 or by (3.72), is satisfied, then there exists a symmetric positive definite matrix P_i that satisfies (3.58) and the state feedback gain matrix K_i can be obtained by (3.55). As a special case, when the matrix B_i is invertible, we have the following result.*

Lemma 3.7 *If B_i is invertible, then there always exists a symmetric positive definite solution P_i to the ARE (3.58).*

Proof. When B_i is invertible, $I - B_i(B_i^\top B_i)^{-1} B_i^\top = 0$, as a result, the ARE (3.58) reduces to following Lyapunov equation

$$A_i^\top P_i + P_i A_i + (\gamma^2 + \eta_i) I = 0.$$

Since $(\gamma^2 + \eta_i) I > 0$ and A_i is stable, the above Lyapunov equation always has a symmetric positive-definite solution P_i for any positive γ and η_i. ∎

Remark 3.10 *When C_i is a square matrix and with independent columns, that is, the state x_i can be uniquely determinated from the output y_i, the ARE (3.59) always has a symmetric positive-definite solution if ε_i is chosen large enough. This is illustrated in the following lemma.*

Lemma 3.8 *If C_i is invertible, there always exists a symmetric positive definite matrix $\widetilde{P}_i$ to the ARE (3.59).*

Proof. Because $\widetilde{Q}_{i1} + \widetilde{\eta}_i I$ is a constant matrix, ε_i can be chosen large enough such that

$$\widetilde{Q}_i \triangleq \widetilde{Q}_{i1} + \widetilde{\eta}_i I - \varepsilon_i C_i^\top C_i < 0. \tag{3.74}$$

Notice that $-\widetilde{Q}_i$ is a symmetric positive definite matrix. Using Cholesky factorization, one can find a real symmetric positive definite invertible matrix $\widetilde{G}_i$ such that $-\widetilde{Q}_i = \widetilde{G}_i \widetilde{G}_i^\top$. Now, the problem of finding a symmetric positive definite solution to the ARE (3.59) reduces to the problem of finding a symmetric positive definite solution $\widetilde{P}_i$ to the following ARE:

$$(-A_i)^\top \widetilde{P}_i + \widetilde{P}_i(-A_i) - \widetilde{P}_i \widetilde{P}_i + \widetilde{G}_i \widetilde{G}_i^\top = 0. \tag{3.75}$$

Since $(I, -A_i^\top)$ is observable and $(-A_i^\top, \widetilde{G}_i)$ is controllable, the ARE (3.75) has a unique positive definite solution $\widetilde{P}_i$ [115]. ■

It is possible that for some sub-systems the matrix C_i is invertible, that is, all state variables of the i-th sub-system are available for feedback, then the condition for existence of a positive definite solution to the ARE (3.59) is given by (3.74) instead of (3.71).

The above lemmas, Lemmas 3.7 and 3.8, can be understood in the following ways: (1) If B_i is invertible, then the interconnection $h_i(x)$ satisfies matching condition, therefore the stabilization of the i-th sub-system can always be achieved as long as stable estimates of states of the overall system are provided. (2) If C_i is invertible, then the state of i-th sub-system can be obtained for the output of the i-th sub-system, hence, the state of the i-th sub-system is always observable.

Remark 3.11 *Since the constant ε_i affects the convergence rate of the observation error and the stability of the overall system, a natural question to ask is what happens if we increase/decrease the value of ε_i. The following lemma gives a result related to this.*

Lemma 3.9 *If the sufficient condition (3.71) is satisfied for a particular* ε_i*, then there exists a symmetric positive definite solution to the ARE (3.59) for any* $\varepsilon_i{}' \geq \varepsilon_i$ *instead of* ε_i*. Moreover, the solution corresponding to* ε_i' *for the ARE (3.59),* $\widetilde{P}_i'$*, satisfies* $\widetilde{P}_i' \geq \widetilde{P}_i$*.*

Proof. Lemma 3.9 is the direct result of the following lemma, Lemma 3.10. ■

Lemma 3.10 *[116]: Let* A, Q_2 *and* R *be given* $n \times n$ *matrices such that* Q_2 *is symmetric and* R *is symmetric positive-definite. Furthermore, assume that* P_2 *is a symmetric positive-definite matrix satisfying*

$$A^\top P_2 + P_2 A + P_2 R P_2 + Q_2 = 0$$

and Q_1 *is a symmetric matrix such that* $Q_1 \leq Q_2$*. Then there exists a symmetric positive-definite matrix* P_1*, such that* $P_1 \geq P_2$*, and*

$$A^\top P_1 + P_1 A + P_1 R P_1 + Q_1 = 0.$$

Remark 3.12 *The convergence rate of each sub-system observer can be increased by amplifying the observer gain matrix* L_i *obtained from (3.56) by* $\varepsilon_i'/\varepsilon_i$*. Let* $L_i = \frac{1}{2}\varepsilon_i'\widetilde{P}_i^{-1}C_i^\top$*, where* $\widetilde{P}_i$ *is the symmetric positive definite solution to the ARE (3.59) obtained with* ε_i*. Then the inequality (3.60) becomes*

$$\dot{V}(x,\widetilde{x}) \leq -\sum_{i=1}^{N}\left[\eta_i x_i^\top x_i + \widetilde{\eta}_i \widetilde{x}_i^\top \widetilde{x}_i + (\varepsilon_i' - \varepsilon_i)\widetilde{x}_i^\top C_i^\top C_i \widetilde{x}_i\right]. \tag{3.76}$$

Since $\varepsilon_i' - \varepsilon_i > 0$*, the convergence rate of* $\widetilde{x}_i$ *to zero is increased.*

Remark 3.13 *The inequality (3.50) used in separating the terms can be quite conservative. Instead of (3.50), one can use the following inequality*

$$X^\top Y + Y^\top X \leq \frac{1}{\varepsilon}X^\top X + \varepsilon Y^\top Y \tag{3.50$'$}$$

where ε *is a real positive scalar. The disadvantage of this approach is that one has to choose the constants* ε *in the design also. Using (3.50′) for the terms with under* *braces in (3.49), we obtain*

$$\widetilde{x}_i^\top(-A_{Bi})^\top P_i x_i + x_i^\top P_i(-A_{Bi})\widetilde{x}_i \le \frac{1}{\varepsilon_{i1}}\widetilde{x}_i^\top(-A_{Bi})^\top(-A_{Bi})\widetilde{x}_i + \varepsilon_{i1}x_i^\top P_i P_i x_i, \quad (3.51'\text{a})$$

$$h_i^\top P_i x_i + x_i^\top P_i h_i \le \frac{1}{\varepsilon_{i2}}h_i^\top h_i + \varepsilon_{i2}x_i^\top P_i P_i x_i, \quad (3.51'\text{b})$$

$$h_i^\top \widetilde{P}_i\widetilde{x}_i + \widetilde{x}_i^\top \widetilde{P}_i h_i \le \frac{1}{\varepsilon_{i3}}h_i^\top h_i + \varepsilon_{i3}\widetilde{x}_i^\top \widetilde{P}_i\widetilde{P}_i\widetilde{x}_i. \quad (3.51'\text{c})$$

Then by choosing same observer matrix as given by (3.56) and the controller gain matrix as follows

$$K_i = -\frac{\varepsilon_{i1}+\varepsilon_{i2}}{2}(B_i^\top B_i)^{-1}B_i^\top P_i, \quad (3.55')$$

the two AREs give by (3.58) and (3.59), respectively, become

$$A_i^\top P_i + P_i A_i + (\varepsilon_{i1}+\varepsilon_{i2})P_i(I - B_i(B_i^\top B_i)^{-1}B_i^\top)P_i + \gamma^2 I + \eta_i I = 0, \quad (3.58')$$

$$A_i^\top \widetilde{P}_i + \widetilde{P}_i A_i + \varepsilon_{i3}\widetilde{P}_i\widetilde{P}_i + \widetilde{Q}_{i1} + \widetilde{\eta}_i I - \varepsilon_i C_i^\top C_i = 0. \quad (3.59')$$

The sufficient conditions for the existence of the symmetric positive definite solutions to (3.58′) and (3.59′), respectively, are given by

$$\delta\Big(A_i^\top, \sqrt{(\gamma^2+\eta_i)(\varepsilon_{i1}+\varepsilon_{i2})}\,(B_i^\top B_i)^{-1/2}B_i^\top\Big) > \sqrt{(\gamma^2+\eta_i)(\varepsilon_{i1}+\varepsilon_{i2})}\,. \quad (3.64')$$

and

$$\sqrt{\varepsilon_{i3}(\lambda_{max}(\widetilde{Q}_{i1})+\widetilde{\eta}_i)} < \delta(A_i, \sqrt{\varepsilon_{i3}\varepsilon_i}\,C_i). \quad (3.71')$$

where $\widetilde{Q}_{i1} \triangleq \frac{1}{4\varepsilon_{i1}}(\varepsilon_{i1}+\varepsilon_{i2})^2 P_i B_i(B_i^\top B_i)^{-\top}B_i^\top P_i$. *Notice that with this approach, one has to also choose three more constants,* $\varepsilon_{i1}, \varepsilon_{i2}, \varepsilon_{i3}$, *for each sub-system in the design of the decentralized controller and observer.*

3.4 Numerical example and simulation

Example 3.1 *Consider the following large-scale system composed of two sub-systems:*

$$\dot{x}_1 = \begin{bmatrix} 0 & 1 \\ -125 & -22.5 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_1 + h_1(t, x), \quad x_1(0) = \begin{bmatrix} 5 \\ 5 \end{bmatrix}, \tag{3.77a}$$

$$y_1 = \begin{bmatrix} 1 & 0 \end{bmatrix} x_1 \tag{3.77b}$$

and

$$\dot{x}_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -37.5 & -50 & -13.5 \end{bmatrix} x_2 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u_2 + h_2(t, x)x, \quad x_2(0) = \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix}, \tag{3.78a}$$

$$y_2 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x_2 \tag{3.78b}$$

where

$$x_1 = \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix}, \quad x_2 = \begin{bmatrix} x_{21} \\ x_{22} \\ x_{23} \end{bmatrix}, \quad x = \begin{bmatrix} x_1^\top \\ x_2^\top \end{bmatrix},$$

$$h_1(x) = \alpha_1 \cos(x_{22}) H_1 x, \quad h_2(x) = \alpha_1 \cos(x_{11}) H_2 x, \quad \alpha_1 = \alpha_2 = 0.2,$$

$$H_1 = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \quad H_2 = \frac{1}{\sqrt{15}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

where H_1 and H_2 are normalized matrices.

3.4.1 The LMI approach

Choosing $\bar{\alpha}_i = 0.001, \beta_i = 0.0001, i = 1, 2$, and solving the optimization problems (3.40) and (3.41) results in

$$M_D = \begin{bmatrix} 6.9444 & -0.00219 & 0 & 0 & 0 \\ 0 & 0 & 0.02921 & 0.05512 & -0.01157 \end{bmatrix},$$

$$Y = \begin{bmatrix} 5.0306 & -7.50042 & 0 & 0 & 0 \\ -7.5004 & 442.7348 & 0 & 0 & 0 \\ 0 & 0 & 64.4923 & -36.8027 & -8.94841 \\ 0 & 0 & -36.8027 & 32.8764 & -10.34002 \\ 0 & 0 & -8.94841 & -10.34 & 387.4311 \end{bmatrix},$$

$$N_D = \begin{bmatrix} 0.048412 & 0 \\ -0.22122 & 0 \\ 0 & 0.14179 \\ 0 & -0.03029 \\ 0 & -0.27011 \end{bmatrix},$$

$$P_o = \begin{bmatrix} 38.8031 & 6.9728 & 0 & 0 & 0 \\ 6.97282 & 1.5802 & 0 & 0 & 0 \\ 0 & 0 & 9.13782 & 10.0134 & 2.9448 \\ 0 & 0 & 10.0134 & 13.4936 & 3.6363 \\ 0 & 0 & 2.94483 & 3.63628 & 1.273 \end{bmatrix},$$

$$\gamma_1 = 38.2554, \quad \gamma_2 = 6.9444, \quad \beta_1 = 0.5227, \quad \beta_2 = 0.5628.$$

Gain matrices K_D and L_D are found to be

$$K_D = \begin{bmatrix} 1.4162 & 0.02399 & 0 & 0 & 0 \\ 0 & 0 & 0.00411 & 0.00635 & 0.00023 \end{bmatrix},$$

$$L_D = \begin{bmatrix} 0.12752 & 0 \\ -0.70268 & 0 \\ 0 & 0.25635 \\ 0 & 0.10646 \\ 0 & -1.1093 \end{bmatrix}$$

by (3.30) and (3.31), respectively. It is easy to check that the condition given by (3.42) is satisfied. Hence, according to Theorem 3.1, the closed-loop system is quadratically stable.

The simulation results are shown in Figures 3.1 and 3.2. In Figure 3.1, the state x_{11} and its estimate $\widehat{x}_{11}$, the state x_{12} and its estimate $\widehat{x}_{12}$, and the control u_1 are shown in the first, second and third plot, respectively. Figure 3.2 shows the states x_2, their estimates $\widehat{x}_2$, and the control u_2. It can be observed from both the figures that the state of the overall system, x, and their estimates, $\widehat{x}$, converge to zero.

3.4.2 The ARE approach

The gain γ is computed based on the values of α_1, α_2, H_1 and H_2 as $\gamma = 0.4$. The following constant gains are chosen in the simulation.

$$\varepsilon_1 = 0.5, \qquad \varepsilon_2 = 0.125, \qquad \eta_1 = 0.1,$$

$$\widetilde{\eta}_1 = 0.5, \qquad \eta_2 = 0.01, \qquad \widetilde{\eta}_2 = 0.2.$$

It can be checked that the conditions given by (3.64) and (3.71) are satisfied for both sub-systems. Thus, there exist positive definite solutions to (3.58) and (3.59). These solutions are

$$P_1 = \begin{bmatrix} 0.93171 & 0.00798 \\ 0.00798 & 0.00614 \end{bmatrix}, \qquad \widetilde{P}_1 = \begin{bmatrix} 1.6912 & 0.01144 \\ 0.01144 & 0.01163 \end{bmatrix},$$

$$P_2 = \begin{bmatrix} 0.78946 & 0.66216 & 0.03058 \\ 0.66216 & 0.8394 & 0.0378 \\ 0.03058 & 0.0378 & 0.00927 \end{bmatrix}, \quad \widetilde{P}_2 = \begin{bmatrix} 0.43093 & 0.38521 & 0.00547 \\ 0.38521 & 0.61093 & 0.01494 \\ 0.00547 & 0.01494 & 0.00853 \end{bmatrix}.$$

The control gain matrices and observer gain matrices are obtained from (3.55) and (3.56) as

$$K_1 = \begin{bmatrix} -0.00798 & -0.00614 \end{bmatrix}, \qquad K_2 = \begin{bmatrix} -0.03058 & -0.03780 & -0.00927 \end{bmatrix},$$

$$L_1 = \begin{bmatrix} 0.14882 \\ -0.14645 \end{bmatrix}, \qquad L_2 = \begin{bmatrix} 0.33579 \\ -0.2157 \\ 0.1625 \end{bmatrix}.$$

To increase the convergence rate of the observers, we choose $100L_1$ and $10L_2$ as the observer gain matrices for the first and second sub-system, respectively, in the simulation.

The simulation results are shown in Figures 3.3 and 3.4. In Figure 3.3, the state x_{11} and its estimate $\widehat{x}_{11}$, the state x_{12} and its estimate $\widehat{x}_{12}$, and the control u_1 are shown in the first, second and third plot, respectively. Figure 3.4 shows the states x_2, their estimates $\widehat{x}_2$, and the control u_2. It can be observed from both the figures that the state of the overall system, x, and their estimates, $\widehat{x}$, converge to zero.

3.5 Summary

In this chapter, a decentralized output feedback controller and observer for a class of large-scale interconnected nonlinear systems are proposed. The interconnecting nonlinearity of each sub-system was assumed to be bounded by a quadratic form of states of the overall system. Local output signals from each sub-system are required to generate the local feedback controller and exact knowledge of the nonlinear interconnection is not required for designing the proposed decentralized controller and observer. The LMI approach and ARE approach are investigated. In the ARE approach, sufficient conditions for the existence of the decentralized controller and observer are given via the analysis of two AREs. Simulation results on a numerical example verify the proposed design.

There are some challenging problems related to the quantity δ. The quantities $\delta(A, C)$ or $\delta(A^\top, B^\top)$ are realization dependent. The properties of δ as a function of various state-space realizations is of importance. In particular, finding the realization of the state-space maximizes the value of δ will be useful.

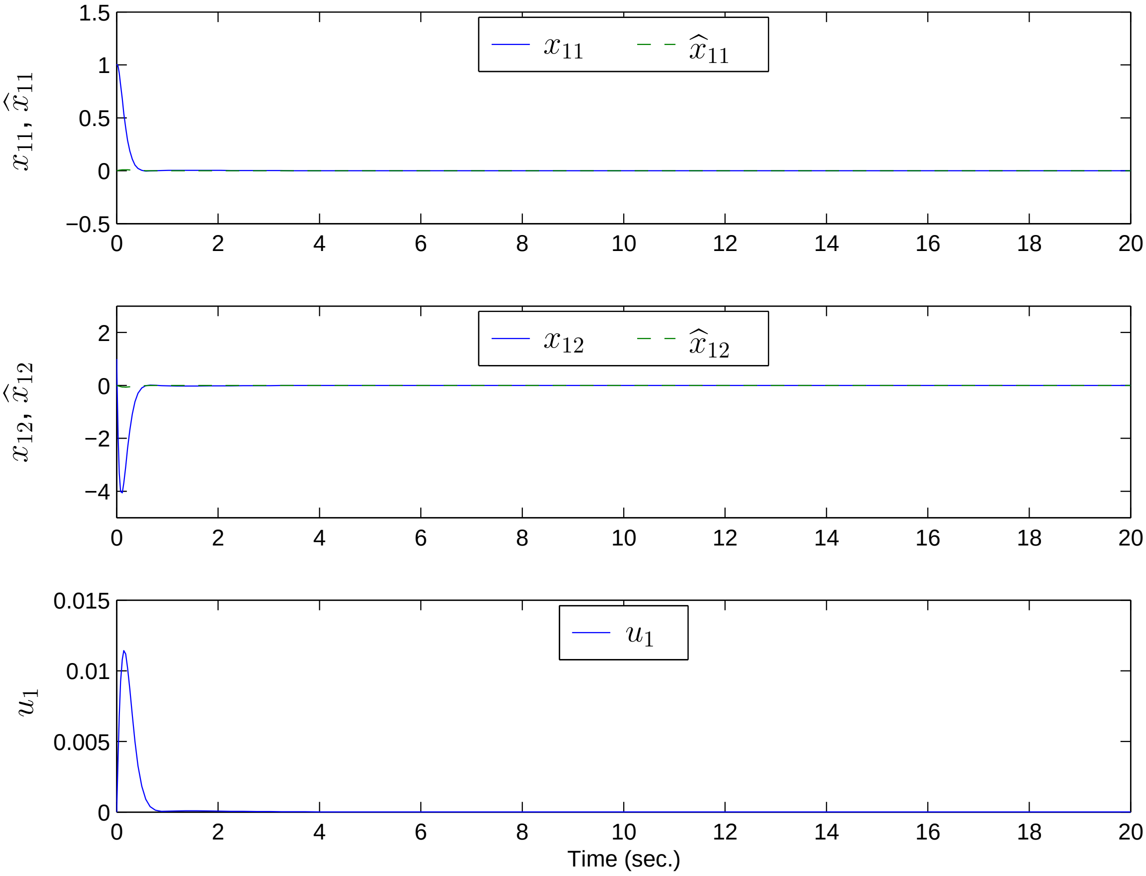

Figure 3.1: Simulation result of Example 3.1 for the first sub-system (3.77) from the LMI approach. The top plot shows the first state x_{11} and its estimate $\widehat{x}_{11}$. The middle plot shows the second state x_{12} and its estimate $\widehat{x}_{12}$. The bottom plot shows the control u_1.

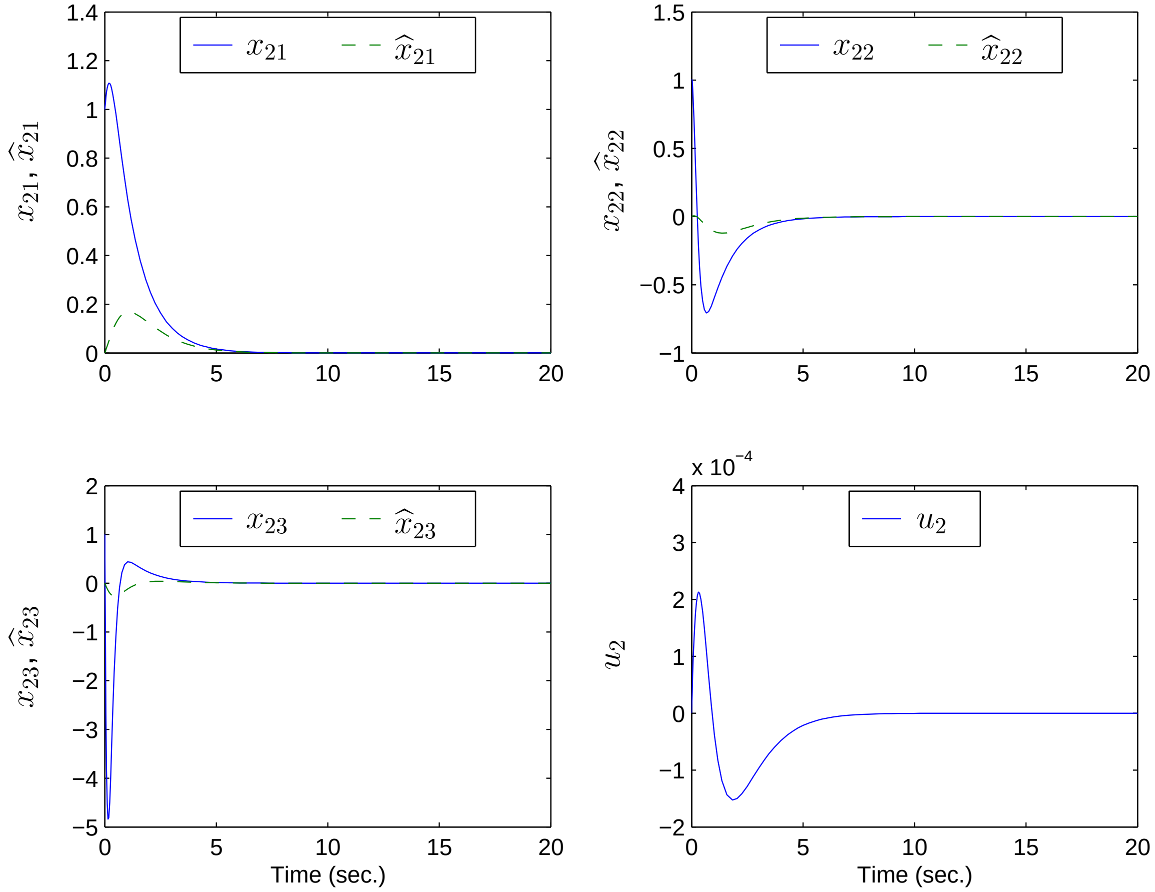

Figure 3.2: Simulation result of Example 3.1 for the second sub-system (3.78) from the LMI approach. The top plot at the left column shows the first state x_{21} and its estimate $\widehat{x}_{21}$. The top plot at the right column shows the second state x_{22} and its estimate $\widehat{x}_{22}$. The bottom plot at the left column shows the third state x_{23} and its estimate $\widehat{x}_{23}$. The bottom plot shows the control u_2.

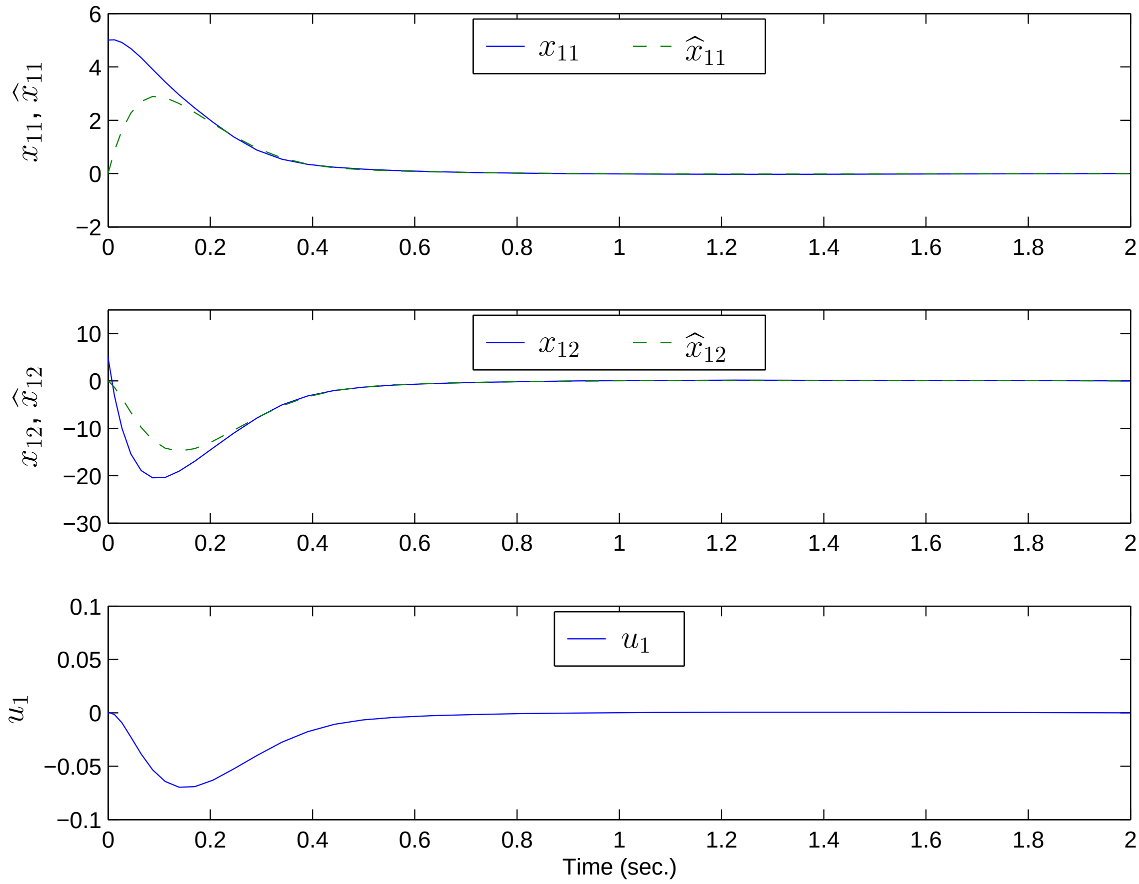

Figure 3.3: Simulation result of Example 3.1 for the first sub-system (3.77) from the ARE approach. The top plot shows the first state x_{11} and its estimate $\widehat{x}_{11}$. The middle plot shows the second state x_{12} and its estimate $\widehat{x}_{12}$. The bottom plot shows the control u_1.

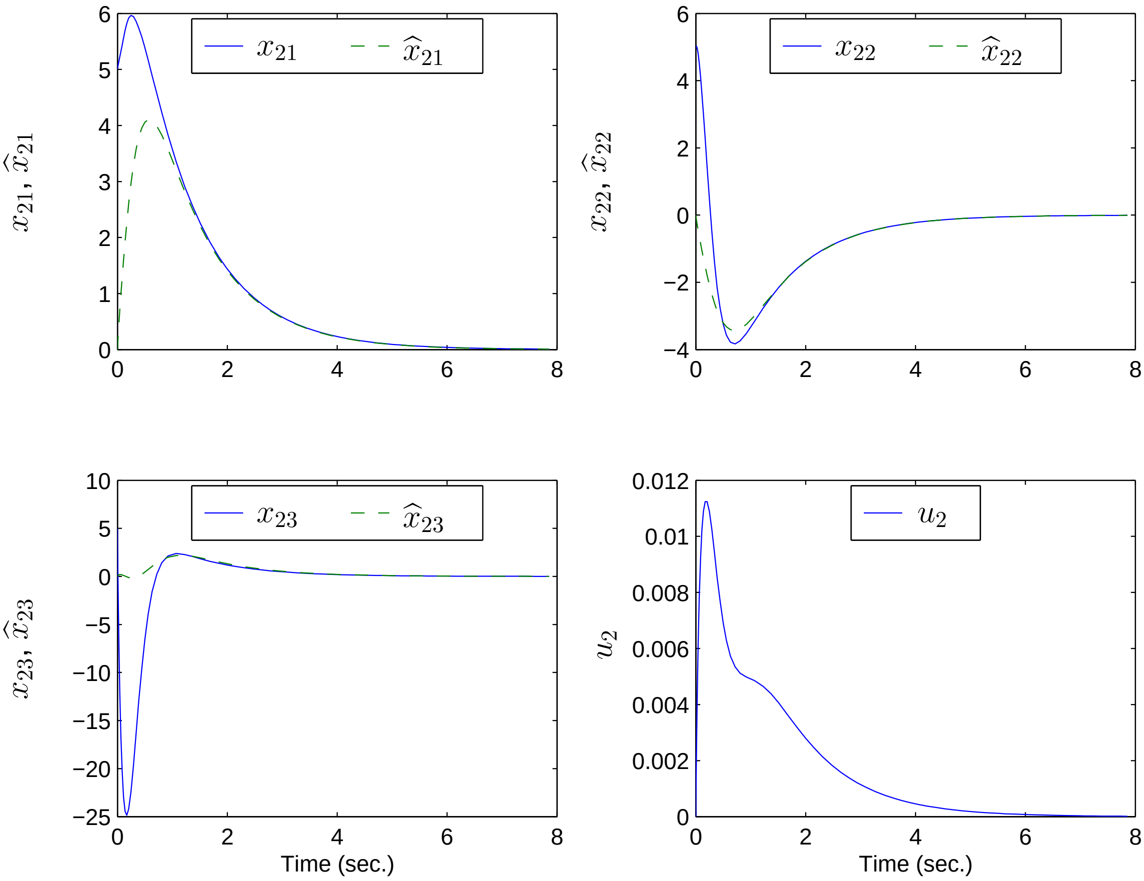

Figure 3.4: Simulation result of Example 3.1 for the second sub-system (3.78) from the ARE approach. The top plot at the left column shows the first state x_{21} and its estimate $\widehat{x}_{21}$. The top plot at the right column shows the second state x_{22} and its estimate $\widehat{x}_{22}$. The bottom plot at the left column shows the third state x_{23} and its estimate $\widehat{x}_{23}$. The bottom plot shows the control u_2.

CHAPTER 4

ADAPTIVE CONTROLLER AND OBSERVER DESIGN FOR A CLASS OF NONLINEAR SYSTEMS

In this chapter, observer and controller design for a class of nonlinear systems, which contain coupled unknown parameters and unmeasurable states, is considered. Unlike prior research on adaptive observer design, existence of a transformation that can transform the original system to a system where unknown parameters appear linearly with known signals is not assumed.

The rest of the chapter is organized as follows. Section 4.1 gives introduction to representative adaptive observer design. The problem formulation is given in Section 4.2. The dynamics, assumptions on the dynamics, and the control objective are also given in Section 4.2. Section 4.3 gives the procedure for obtaining the modified form of the dynamics of the original system. Based on the modified form of the dynamics, the adaptive controller and observer design are presented in Section 4.4. Examples on reducing the dimension of the modified dynamics, simulation on a nonlinear system, and experimental results on a single link with dynamic friction compensation are given in Section 4.5. Summary of the chapter is given in Section 4.6.

4.1 Representative work on adaptive observer design

Many practical applications require estimation of the states and parameters in designing a stable control algorithm; the unmeasurable states and parameters are generally estimated based on the knowledge of the physical system, such as a model, and the available measurements. The adaptive scheme which would identify unknown

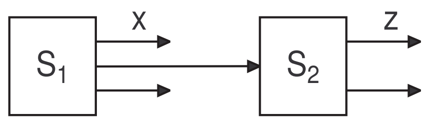

Figure 4.1: A simple observer.

parameters and observe unknown states in a dynamic system is usually called the adaptive observer.

A Luenberger observer [20, 19, 117, 118] allows asymptotic reconstruction of the state of a linear time-invariant system from its its input and output, provided that (1) the system parameters are known, (2) the structure of the system is known, and (3) the system is observable. The basic theory of the Luenberger observer is that "almost any system is an observer" [118]. Consider a cascaded system shown in Figure 4.1. S_1 is a free system described by

$$\dot{x}(t) = Ax(t) \tag{4.1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $A \in \mathbb{R}^{n \times n}$ is a constant matrix. S_2 is described by

$$\dot{z}(t) = Fz(t) + Hx(t) \tag{4.2}$$

where $z(t) \in \mathbb{R}^m$ is the state, $F \in \mathbb{R}^{m \times m}$ and $H \in \mathbb{R}^{m \times n}$ are constant matrices. S_2 is driven by the available outputs $Hx(t)$ from S_1. It is shown (Theorem 1, [118]) that if there is a transformation T satisfying $TA - FT = H$, then

$$z(t) = Tx(t) + e^{Ft}[z(0) - Tx(0)] \tag{4.3}$$

It is noted that, if A and F have no common eigenvalues, there is a unique solution T to the equation $TA - FT = H$ [20].

Equation (4.3) shows the relationship between $x(t)$, the state of the first system S_1, and z(t), the state of the second system S_2. This relationship indicates that the second system will almost always serve as an observer of the first system in that its state will tend to track a linear transformation of the state of the first system, provided matrix F is negative definite. In this case, $\lim_{t\to\infty} z(t) = Tx(t)$. The second system can be freely constructed as long as two conditions are satisfied: (1) F is a negative definite matrix, and (2) A and F will not share same eigenvalues. If the first system is a forced system, that is,

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{4.4}$$

where $u \in \mathbb{R}^q$ is the input and $B \in \mathbb{R}^{n\times q}$ is a constant matrix, the second system can be chosen as

$$\dot{z}(t) = Fz(t) + Hx(t) + TBu(t). \tag{4.5}$$

A similar result as in (4.3) can be achieved.

From the basic theory of the Luenberger observer, it can be seen that to design an observer for the first system is to design the second system by using the knowledge of the first system (matrices A, B and H). For the case where no *priori* knowledge of the system parameters is available, which occurs for example in model reference adaptive control design, an adaptive observer is applied. The basic idea of the adaptive observer design is to use a Luenberger observer to observe the state, while the parameters of the Luenberger observer are continuously adapted such that the observation error asymptotically approaches to zero. Notice that the estimated parameters may not converge to their true values.

Early work on stable adaptive observers for linear time-invariant systems can be found in [119, 120, 81, 70]; Lyapunov synthesis technique is applied to derive the parameter adaptation law and Luenberger observer technique is used to design the state observer. Certain auxiliary filters are used to generate signals from the input and output signals and those filtered signals are fed into the observer. Global asymptotic convergence of the observation process is achieved. One essential feature in the design of an adaptive observer for a linear time-invariant system is choosing a suitable representation of the original system, which is illustrated in the following by a single-input single-output system. Consider a dynamic system which has one input $u(t)$ and one output $y(t)$, described by

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \tag{4.6a}$$

$$y(t) = h^\top x(t) \tag{4.6b}$$

where $x(t) \in \mathbb{R}^n$ is a state vector, $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^n$, and $h \in \mathbb{R}^n$ are unknown constant matrices. The adaptive observer design problem is: (1) estimate the state vector x, and (2) identify the parameters of the triple (A, B, h) under the following assumptions:

Assumption A4.1 *The order of the system, n, is known.*

Assumption A4.2 *Only $u(t)$ and $y(t)$ are available for feedback.*

Assumption A4.3 *The system (4.6) is completely observable.*

The coupling term $Ax(t)$ is an obstacle for designing adaptive observer because the the product of unknown parameter in A and unknown state in x appears in the dynamics. Further, $h^\top x(t)$ also has coupling terms. One way to solve this problem is to transform the representation of system (4.6) into another form such that the coupling terms disappear. It is shown in [117, 118, 120] that under Assumption A4.3, system (4.6) can be represented by

$$\begin{bmatrix} \dot{y} \\ \cdots \\ \dot{z} \end{bmatrix} = \begin{bmatrix} & \vdots & r^\top \\ a & \vdots & \cdots \\ & \vdots & F \end{bmatrix} \begin{bmatrix} y \\ \cdots \\ z \end{bmatrix} + \begin{bmatrix} b \end{bmatrix} u, \quad \begin{matrix} y(0) = y_0 \\ z(0) = z_0 \end{matrix} \tag{4.7}$$

where $z \in \mathbb{R}^{n-1}$ is the unknown state, $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^n$ are two unknown parameter vectors to be identified, $r \in \mathbb{R}^{n-1}$ and $F \in \mathbb{R}^{(n-1)\times(n-1)}$ are known such that $(r^\top, F)$ is a completely observable pair. Since F is chosen by the designer and eigenvalues of F are eigenvalues of the observer for state vector z, the convergence speed of the observation of z can be arbitrarily selected. A particular choice [120] of r and F can be $r = \begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix}^\top$ and $F = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_{n-1}), \lambda_i < 0, i = 1, 2, \dots, (n-1)$. The reason that the system (4.6) can be presented in the form of (4.7) is that the system (4.6) can be described by an n-th order transfer function, and therefore only $2n$ parameters are needed to be identified, and there is a great amount of freedom in choosing the internal state representation x, or equivalently, in the choice of the triple (A, b, h). It is noted that in the new representation (4.7), there are no coupling terms. All unknown parts (state z, parameter vectors a and b) are coupled with known variables in that z with F, a with y, and b with u. Based on this canonical form, a large amount of work ([71, 70, 121]) on adaptive observer design for linear time-invariant system was conducted.

Design of a stable adaptive observer that simultaneously estimates the unmeasurable state and the unknown parameters for a general class of nonlinear systems is still an open problem. This has led to continued strong interest over the years in the development of stable adaptive observers. Adaptive observer design for nonlinear systems is usually restricted to a certain class of systems. In [80], the linear adaptive observer derived in [81] has been modified and extended to a class of nonlinear time-varying systems, in which the nonlinear system is considered to be transformed into an adaptive observer canonical form given by

$$\dot{x}(t) = Rx(t) + \Omega(\omega(t))\theta(t) + g(t), \tag{4.8a}$$

$$y(t) = x_1(t) \tag{4.8b}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}, y \in \mathbb{R}, \Omega(\omega(t)) \in \mathbb{R}^{n\times p}$, $\omega(t)$ is a vector of known functions of $u(t)$ and $y(t)$, $\theta(t) \in \mathbb{R}^p$ is a vector of unknown parameters, $g(t) \in \mathbb{R}^n$ is a vector of known functions, and $R \in \mathbb{R}^{n\times n}$ is a known constant matrix. Notice that the state $x(t)$ and parameter $\theta(t)$ appear linearly in known functions in the dynamics. An adaptive observer, which is driven by a $p(n-1)$ dimensional auxiliary filter, was developed for (4.8); stable convergence of the estimates is shown under certain persistency of excitation conditions.

Necessary and sufficient conditions for transforming a general nonlinear system into a canonical form that is nonlinear purely in the output variables can be found in [24]. Based on the early work of [80, 51], considerable work on adaptive nonlinear observers has been reported by Marino et. al. in a series of papers; see [82] and the references there-in; Marino et. al. studied adaptive observers for nonlinear systems that can be transformed via a global state space diffeomorphism into

$$\begin{aligned} \dot{x}(t) &= A_c x(t) + \psi_0(y(t), u(t)) + b\psi^T(y(t), u(t))\theta, \\ y(t) &= C_c x(t) \end{aligned} \tag{4.9}$$

where $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}$, $u(t) \in \mathbb{R}^m$, $\psi(y(t), u(t)) \in \mathbb{R}^p$ is a known smooth function of the output, $y(t)$, and the input vector, $u(t)$, $\theta \in \mathbb{R}^p$ is an unknown parameter vector, and $A_c = \begin{bmatrix} 0 & I_{n-1} \\ 0 & 0 \end{bmatrix}$, $C_c = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}$. Notice again that the system is linear in the unknown parameters and the nonlinearities are functions of the known output and input variables only.

In [122], global adaptive output feedback controller for a class of nonlinear systems which consist of a set of unknown constant parameters and unmeasurable state variables was considered. Under certain assumptions and parameter dependent filtered transformations (see [122]), the nonlinear system was transformed into the following form

$$\dot{z} = F(\bar{x}, x, \omega, t)z, \tag{4.10a}$$

$$\dot{\bar{z}} = S(\bar{x}, x, \omega, t)\bar{z}, \tag{4.10b}$$

$$\dot{\omega} = \Omega(\omega, \bar{x}, x, t), \tag{4.10c}$$

$$\dot{\bar{x}} = A\bar{x} + bx, \tag{4.10d}$$

$$\begin{aligned}\dot{x} &= u + \pi(\bar{x}, x, \omega, t) + \bar{z}^\top l(\bar{x}, x, t) \\ &\quad + \bar{\pi}(\bar{x}, x, \omega, t)\theta + p^\top(z)\delta(\bar{x}, x, \omega, t) + p^\top(z)\bar{\delta}(\bar{x}, x, \omega, t)\theta\end{aligned} \tag{4.10e}$$

where $z \in \mathbb{R}^{v_1}$ and $\bar{z} \in \mathbb{R}^{v_2}$ are unmeasurable states; $\omega \in \mathbb{R}^{v_1+v_2+d-r}$, $\bar{x} \in \mathbb{R}^r$ and $x \in \mathbb{R}$ are measured states, $A \in \mathbb{R}^{r\times r}$ is a stable matrix and $b \in \mathbb{R}^r$ is known, $u \in \mathbb{R}$ is the control input, matrix functions F, S, Ω, π, l, $\bar{\pi}$, δ, and $\bar{\delta}$ are known, $\theta \in \mathbb{R}^{\bar{q}}$ is an unknown constant parameter vector, $p(z) \in \mathbb{R}^s$ is a vector function whose entries are products of entries of vector z. Under the assumption that there exist symmetric positive-definite matrices P_z and $P_{\bar{z}}$ such that

$$P_z F(\bar{x}, x) + F^\top(\bar{x}, x)P_z \leq -I_{v_1}, \tag{4.11}$$

$$P_{\bar{z}} S(\bar{x}, x) + S^\top(\bar{x}, x)P_{\bar{z}} \leq 0 \tag{4.12}$$

for all $(\bar{x}, x) \in \mathbb{R}^{r+1}$, it was proved that the bounded estimation of θ, the boundedness of $\bar{z}$, and the convergence to zero of z, $\bar{x}$ and x were achieved. Equations (4.11) and (4.12) are equivalent to requiring that the z-dynamics and $\bar{z}$-dynamics be asymptotically stable and stable, respectively. Notice that the the unmeasurable state z is coupled with the unknown parameter θ, but $\bar{z}$ is not; the method proposed in [122] cannot be applied to systems which do not have asymptotically stable z-dynamics.

4.2 Problem statement

In this chapter, we consider the following class of systems that contain the product of the unmeasurable state variables as well as unknown parameters:

$$\dot{x} = Mx + hu + h\left(d(x) + f_\theta^\top(x)\theta + f_z^\top(x)z + \theta^\top G_z(x)z\right), \tag{4.13a}$$

$$\dot{z} = B_z(x)z + a_z(x) \tag{4.13b}$$

where $x \in \mathbb{R}^n$ is the measured state, $z \in \mathbb{R}^m$ is the unmeasured state, $u \in \mathbb{R}$ is the control input, $\theta \in \mathbb{R}^p$ is an unknown constant parameter vector, $M \in \mathbb{R}^{n\times n}$ is a known constant matrix, $h \in \mathbb{R}^n$ is a known constant vector, and $d(x) \in \mathbb{R}$, $f_\theta(x) \in \mathbb{R}^p$, $f_z(x) \in \mathbb{R}^m$, $G_z(x) \in \mathbb{R}^{p\times m}$, $a_z(x) \in \mathbb{R}^m$, and $B_z(x) \in \mathbb{R}^{m\times m}$ are known smooth functions of x.

To specify the class of nonlinear systems, it is assumed that the system dynamics described by (4.13) satisfies the following four assumptions:

Assumption A4.4 *The pair (M, h) is controllable.*

Assumption A4.5 *There exists a symmetric positive definite matrix $P_z \in \mathbb{R}^{m\times m}$ such that $B_z^\top(x)P_z + P_zB_z(x) \leq -Q_z$ for all $x \in \mathbb{R}^n$, where Q_z is a positive semi-definite matrix. Also, for every bounded $x(t)$, the solution $z(t)$ is bounded for any initial condition $z(t_0)$.*

Assumption A4.6 *The sign of each parameter, $\theta_i, i = 1, 2, \ldots, p$, in the parameter vector θ is known, and θ_i is bounded.*

Assumption A4.7 *The functions $d(x)$, $f_\theta(x)$, $f_z(x)$, $G_z(x)$, $a_z(x)$ and $B_z(x)$ are bounded functions of x.*

We have the following remarks on the above four assumptions.

Remark 4.1 *Assumption A4.4 guarantees the existence of a control gain vector $c \in$*

$\mathbb{R}^n$ *that can stabilize the linear part of the* x*-dynamics; that is, there exists a symmetric positive definite solution,* P*, to the Lyapunov equation* $(M - hc^\top)^\top P + P(M - hc^\top) = -Q$*, where* Q *is a symmetric positive definite matrix and* c *is a feedback gain vector.*

Remark 4.2 *Assumption A4.5 ensures that* z *has a stable dynamics provided* $x(t)$ *is bounded.*

Remark 4.3 *Assumption A4.6 is the only required knowledge of the unknown parameters and is reasonable for many practical plants. The sign of the unknown parameters are used in a parameter dependent Lyapunov function candidate during the design process.*

Remark 4.4 *Assumption A4.7 is a common and reasonable assumption in the control design for nonlinear systems.*

The system given by (4.13) has the following features.

- The uncertainties in the (4.13a) satisfy the matching condition, that is, they enter the control channel. This is different with systems discussed in Chapter 2 and Chapter 3, where uncertainties are unmatched. One can incorporate unmatched uncertainty in the (4.13a) and use similar method applied in Chapter 1 and Chapter 2 to design controller and observer plus the proposed method in the subsequent sections of this chapter.

- The last term in the x-dynamics in (4.13) is a product of the unmeasurable state z and the unknown parameter vector θ.

- The unknown dynamics, given by (4.13b), is driven by known vector function of the measured state x.

The objective of this chapter is to design an adaptive controller and observer for the system (4.13) such that asymptotic regulation of the measurable state, asymptotic convergence of the state estimation errors to zero or their boundedness based on certain conditions, and boundedness of the estimated parameters are achieved. Instead of using the parameter-dependent filtered transformation as in [122], the approach proposed in this chapter is to cast the system in a modified form which can be used to design a control algorithm based on a parameter dependent Lyapunov function. The process of casting the original nonlinear system into the modified form is constructive and is always possible.

4.3 Modified form of the system dynamics

In this section, the procedure of expressing the system given by (4.13) into a modified form will be described. The controller and adaptive observer design for the system (4.13) in subsequent sections will be based on the modified form. The modified form has larger dimension than the original system given by (4.13). However, the modified form still describes the same system given by (4.13). Similar assumptions on the modified form will be obtained based on Assumptions A4.5 to A4.7.

The nonlinear system (4.13) can be cast in the following form:

$$\dot{x} = Mx + h\left(u + d(x) + f^{\top}(x)\Theta + Z^{\top}G(x)\Theta\right), \tag{4.14a}$$

$$\dot{Z} = a(x) + B(x)Z \tag{4.14b}$$

where

$$Z^\top = [\underbrace{z^\top, \ \dots, \ z^\top, \ \dots, \ z^\top}_{(p+1)-times}], \tag{4.15a}$$

$$\Theta^\top = [\underbrace{\theta_1, \ \dots, \ \theta_1}_{m-times}, \ \underbrace{\theta_2, \ \dots, \ \theta_2}_{m-times}, \ \dots, \ \underbrace{\theta_p, \ \dots, \ \theta_p}_{m-times}, \ \underbrace{1, \ \dots, \ 1}_{m-times}], \tag{4.15b}$$

$$f^\top(x) = [f_{\theta_1}(x), \ \underbrace{0, \ \dots, \ 0}_{(m-1)-times}, \dots, \ f_{\theta_p}(x), \ \underbrace{0, \ \dots, \ 0}_{(m-1)-times}, \underbrace{0, \ \dots, \ 0}_{m-times}], \tag{4.15c}$$

$$a^\top(x) = [\underbrace{a_z^\top(x), \ \dots, \ a_z^\top(x), \ \dots, \ a_z^\top(x)}_{(p+1)-times}], \tag{4.15d}$$

$$B(x) = \text{diag}(\underbrace{B_z(x), \ \dots, \ B_z(x), \ \dots, \ B_z(x)}_{(p+1)-times}), \tag{4.15e}$$

$$G(x) = \text{diag}(g_{z11}(x), \ \dots, \ g_{z1m}(x), \ \dots, \ g_{zp1}(x), \ \dots, \ g_{zpm}(x), \ f_{z1}(x), \ \dots, \ f_{zm}(x)) \tag{4.15f}$$

where $g_{zij}(x), \ i = 1, 2, \dots, p, \ j = 1, 2, \dots, m$, is the ij-th element of $G_z(x)$, $f_{zi}(x), \ i = 1, 2, \dots, m$, is the i-th element of $f_z(x)$, and $f_{\theta i}, \ i = 1, 2, \dots, p$, is the i-th element of $f_\theta(x)$.

Equations (4.13) and (4.14) describe the same system. The unknown parameter vector Θ and the unmeasurable state vector Z in (4.14) are of larger dimension than that of θ and z in (4.13); $Z \in \mathbb{R}^{m(p+1)}$ is a vector cascaded by $(p+1)$ z's; $\Theta \in \mathbb{R}^{m(p+1)}$ is a vector cascaded by m times θ_i's, $i = 1, 2, \dots, p$, and an m-vector with each entry equal to 1; $a(x) \in \mathbb{R}^{m(p+1)}$ is cascaded by $(p+1)$ $a_z(s)$'s; $G(x) \in \mathbb{R}^{m(p+1)\times m(p+1)}$ is a diagonal matrix whose diagonal entries are the entries of $G_z(x)$ and $f_z(x)$; $f(x) \in \mathbb{R}^{m(p+1)}$ is a vector whose $[(j-1)m+1]$-th element is $f_{\theta j}(x), j = 1, 2, \dots, p$ and the other elements equal to zero.

The motivation for casting the nonlinear system described by (4.13) in the form given by (4.14) with a new parameter vector Θ and a new state vector Z is to account for the non-zero off-diagonal entries in $G_z(x)$ and the non-zero entries in $f_z(x)$; as a result of this, the proposed stable adaptive controller and observer design is feasible. A non-zero off-diagonal entry in $G_z(x)$ means that two unknown parameters are coupled with the same unmeasurable state variable (or two unmeasurable state variables are coupled with the same parameter). Assuming that none of the elements of the matrix $G_z(x)$ and vector $f_z(x)$ are zero, then $Z \in \mathbb{R}^q$ and $\Theta \in \mathbb{R}^q$, where $q = m(p+1)$. If some entries in $G_z(x)$ and/or $f_z(x)$ are zero, it is possible to reduce the dimension of the vector Z. Correspondingly, the dimensions of $G(x)$, Θ, $f(x)$, $a(x)$ and $B(x)$ are also reduced. An example is given in Section 4.5 to illustrate the reduction procedure.

With the assumptions on the original system described by (4.13), it can be proved that the following three assumptions, which correspond respectively to Assumptions A4.5, A4.6, and A4.7 on the original system (4.13), are true for the system in the modified form (4.14):

Assumption A4.5′ *There exists a symmetric positive definite matrix* $P_Z = P_Z^\top > 0$ *such that* $B^\top(x)P_Z + P_Z B(x) \leq Q_Z$. *Further,* $P_Z = diag(P_z, \ldots, P_z, \ldots, P_z)$, *and* $Q_Z = diag(Q_z, \ldots, Q_z, \ldots, Q_z)$. *Also, for every bounded* $x(t)$, *the solution of* $Z(t)$ *is bounded for any initial condition* $Z(t_0)$.

Assumption A4.6′ *The sign of each parameter,* Θ_i, $i = 1, 2, \ldots, q$, *in the parameter vector* Θ *is known, and* Θ_i, *is bounded.*

Assumption A4.7′ *The functions* $d(x)$, $f(x)$, $G(x)$, $a(x)$ *and* $B(x)$ *are bounded functions of* x.

The proof of Assumptions A4.6′ and A4.7′ is straightforward. In the following, we prove the Assumption A4.5′.

Proof of Assumption A4.5′. From the Assumption A4.5, one has

$$B_z^\top(x)P_z + P_z B_z(x) \leq -Q_z, \quad \forall x \in \mathbb{R}^n. \tag{4.16}$$

From (4.16),

$$
\begin{aligned}
&\mathrm{diag}(B_z(x),\ldots,B_z(x),\ldots,B_z(x))\mathrm{diag}(P_z,\ldots,P_z,\ldots,P_z)\\
\leq&\mathrm{diag}(Q_z,\ldots,Q_z,\ldots,Q_z), \quad \forall x\in\mathbb{R}^n.
\end{aligned}
\tag{4.17}
$$

Let $P_Z \triangleq \mathrm{diag}(P_z,\ldots,P_z,\ldots,P_z)$ and $Q_Z \triangleq \mathrm{diag}(Q_z,\ldots,Q_z,\ldots,Q_z)$, and considering (4.15e), one has

$$
B^\top(x)P_Z + P_Z B(x) \leq Q_Z, \quad \forall x\in\mathbb{R}^n.
$$

Also, $P_Z = P_Z^\top > 0$ and $Q_Z = Q_Z^\top \geq 0$. ■

4.4 Adaptive controller and observer design

In this section, the adaptive controller and observer design for the system (4.13) will be proposed. The actual design will be based on the modified representation of (4.13), given by (4.14), as both representations, (4.13) and (4.14), describe the same nonlinear system. First, the design strategy is illustrated by a simple example, then the design for the general case is proposed.

4.4.1 Design for a simple example

In this section, a simple example is considered to show the design process using Lyapunov's method. The example is given by the following equations:

$$
\begin{aligned}
\dot{x} &= u + f(x)\theta + g(x)z\theta,\\
\dot{z} &= b(x)z
\end{aligned}
\tag{4.18}
$$

where $x, u, \theta, z, f(x), g(x), b(x) \in \mathbb{R}$. The sign of θ is known and $b(x) \leq -\epsilon$, $\epsilon > 0$, for all $x \in \mathbb{R}$. The goal is to design a control algorithm such that x converges asymptotically to zero; this involves design of the control input u, observer design to estimate z, and design of an adaptation scheme for θ.

Choose the following control input:

$$u = -cx - f(x)\widehat{\theta} - g(x)\widehat{z}\widehat{\theta} \tag{4.19}$$

where $c > 0$. Substituting the control input (4.19) into the x-dynamics of (4.18) results in the equation:

$$\dot{x} = -cx - f(x)\widetilde{\theta} - g(x)\widehat{z}\widetilde{\theta} - g(x)\widetilde{z}\theta \tag{4.20}$$

where $\widehat{(*)}$ is the estimate of $(*)$, and $\widetilde{(*)} = \widehat{(*)} - (*)$ is the estimation error of $(*)$. To design an observer for z and a parameter adaptation algorithm for θ, the following Lyapunov function candidate is chosen:

$$V = \frac{1}{2}x^2 + V_{\widetilde{\theta}} + V_{\widetilde{z}} \tag{4.21}$$

where $V_{\widetilde{\theta}}$ is a radially unbounded positive function of $\widetilde{\theta}$ and $V_{\widetilde{z}}$ is a positive function of $\widetilde{z}$ and θ, and is radially unbounded with respect to $\widetilde{z}$. The time derivative of V along the trajectory of (4.19) is

$$\dot{V} = -cx^2 - f(x)x\widetilde{\theta} - g(x)x\widehat{z}\widetilde{\theta} - g(x)x\widetilde{z}\theta + \dot{V}_{\widetilde{\theta}} + \dot{V}_{\widetilde{z}}. \tag{4.22}$$

A sufficient condition for $\dot{V} \leq -cx^2$ is

$$\dot{V}_{\widetilde{\theta}} + \dot{V}_{\widetilde{z}} \leq f(x)x\widetilde{\theta} + g(x)x\widehat{z}\widetilde{\theta} + g(x)x\widetilde{z}\theta. \tag{4.23}$$

One possible choice of $\dot{V}_{\widetilde{\theta}}$ and $\dot{V}_{\widetilde{z}}$ that satisfy inequality (4.23) is

$$\dot{V}_{\widetilde{\theta}} = f(x)x\widetilde{\theta} + g(x)x\widehat{z}\widetilde{\theta}, \tag{4.24}$$

$$\dot{V}_{\widetilde{z}} \leq g(x)x\widetilde{z}\theta. \tag{4.25}$$

The following choice of $V_{\widetilde{\theta}}$ and $\dot{\widetilde{\theta}}$ can satisfy (4.24):

$$V_{\widetilde{\theta}} = \frac{1}{2\gamma_1}\widetilde{\theta}^2, \tag{4.26}$$

$$\dot{\widetilde{\theta}} = \gamma_1(f(x) + g(x)\widehat{z})x \tag{4.27}$$

where $\gamma_1 > 0$. Equation (4.25) can be rewritten as

$$\frac{\partial V_{\tilde{z}}}{\partial \tilde{z}} \dot{\tilde{z}} \leq g(x) x \tilde{z} \theta. \tag{4.28}$$

Substituting the z-dynamics of (4.18) into (4.28) yields

$$\frac{\partial V_{\tilde{z}}}{\partial \tilde{z}} (\dot{\hat{z}} - b(x) z) \leq g(x) x \tilde{z} \theta. \tag{4.29}$$

Now consider the following choice for $V_{\tilde{z}}$:

$$V_{\tilde{z}} = \frac{1}{2\gamma_2} |\theta| \tilde{z}^2 \tag{4.30}$$

where $\gamma_2 > 0$, and $|\theta| > 0$ is assumed. Equation (4.29) becomes

$$\tilde{z}[\dot{\hat{z}} - b(x) z - \gamma_2 \operatorname{sgn}(\theta) g(x) x] \leq 0. \tag{4.31}$$

Therefore, if

$$\dot{\hat{z}} = b(x) \hat{z} + \gamma_2 \operatorname{sgn}(\theta) g(x) x \tag{4.32}$$

then

$$\dot{V}_{\tilde{z}} = b(x) \tilde{z}^2 \leq 0. \tag{4.33}$$

Thus, for the given system, by choosing the control input given by (4.19), the parameter adaptation scheme given by (4.27) for θ and the observer (4.32) for z, a Lyapunov function for the system is

$$V = \frac{1}{2} x^2 + \frac{1}{2\gamma_1} \tilde{\theta}^2 + \frac{1}{2\gamma_2} |\theta| \tilde{z}^2. \tag{4.34}$$

The time derivative of V is

$$\dot{V} = -c x^2 + \frac{1}{\gamma_2} b(x) |\theta| \tilde{z}^2 \leq -c x^2 - \frac{1}{\gamma_2} \epsilon |\theta| \tilde{z}^2. \tag{4.35}$$

Hence, $x(t)$ and $\tilde{z}(t)$ converge asymptotically to zero, and the parameter estimate $\hat{\theta}(t)$ is bounded.

4.4.2 Design for the general case

The design process outlined in the previous section for the simple example can be extended to the general case given by (4.14). The following theorem illustrates the main result of this paper.

Theorem 4.1 *Consider the plant described by (4.14), the following control law (4.36), parameter estimation algorithm (4.37), and observer (4.38)*

$$u = -c^\top x - d(x) - f^\top(x)\widehat{\Theta} - \widehat{Z}^\top G(x)\widehat{\Theta}, \tag{4.36}$$

$$\dot{\widehat{\Theta}} = 2\Gamma\left(G(x)\widehat{Z} + f(x)\right)h^\top Px, \tag{4.37}$$

$$\dot{\widehat{Z}} = a(x) + B(x)\widehat{Z} + P_Z^{-1}G(x)sgn(\Theta)h^\top Px \tag{4.38}$$

where $\Gamma = \Gamma^\top \in \mathbb{R}^{q\times q} > 0$, $c \in \mathbb{R}^n$, $sgn(\Theta) = \begin{bmatrix} sgn(\Theta_1) & \dots & sgn(\Theta_q) \end{bmatrix}^\top$, *and* c *is chosen such that* P *is the symmetric positive definite solution of the Lyapunov equation*

$$(M - hc^\top)^\top P + P^\top(M - hc^\top) = -Q, \tag{4.39}$$

for any given positive definite matrix Q. *Then, the closed-loop system has the following properties.*

i) $u(t)$, $\widehat{\Theta}(t)$, $\widetilde{\Theta}(t)$, $\widehat{Z}(t)$, *and* $\widetilde{Z}(t)$ *are bounded.*

ii) $\lim_{t\to\infty} x(t) = 0$.

iii) If $Q_z > 0$, $\lim_{t\to\infty} \widetilde{Z}(t) = 0$.

Proof. Using the control input and the observer given by (4.36) and (4.38), respectively, the x-dynamics and the state estimation error dynamics are

$$\dot{x} = (M - hc^\top)x - h\left(f^\top(x) + \widehat{Z}^\top G(x)\right)\widetilde{\Theta} - h\Theta^\top G(x)\widetilde{Z}, \tag{4.40}$$

$$\dot{\widetilde{Z}} = B(x)\widetilde{Z} + P_Z^{-1}G(x)\mathrm{sgn}(\Theta)h^\top Px. \tag{4.41}$$

Consider the following Lyapunov function candidate:

$$V(x, \widetilde{\Theta}, \widetilde{Z}, \Theta) = x^\top P x + \frac{1}{2}\widetilde{\Theta}^\top \Gamma^{-1} \widetilde{\Theta} + \widetilde{Z}^\top \Lambda_{|\Theta|} P_Z \widetilde{Z} \tag{4.42}$$

where $\Lambda_{|\Theta|}$ is a diagonal matrix whose i-th diagonal element is the absolute value of the i-th element of the parameter vector Θ. From (4.15b), one obtains

$$\begin{aligned}\Lambda_{|\Theta|} &= \operatorname{diag}(|\Theta_1|, |\Theta_2|, \ldots, |\Theta_q|) \\ &= \operatorname{diag}(|\theta_1| I_m, \ldots, |\theta_i| I_m, \ldots, |\theta_p| I_m).\end{aligned}$$

Notice that $V(x, \widetilde{\Theta}, \widetilde{Z}, \Theta)$ is indeed a Lyapunov function candidate because $\Lambda_{|\Theta|} P_Z$ is a symmetric positive definite matrix, which can be seen from the following:

$$\begin{aligned}\Lambda_{|\Theta|} P_Z &= \operatorname{diag}(|\theta_1| I_m, \ldots, |\theta_i| I_m, \ldots, |\theta_p| I_m) \operatorname{diag}(P_z, \ldots, P_z, \ldots, P_z) \\ &= \operatorname{diag}(|\theta_1| P_z, \ldots, |\theta_i| P_z, \ldots, |\theta_p| P_z) \\ &= \operatorname{diag}(\sqrt{|\theta_1|}\, P_z, \ldots, \sqrt{|\theta_i|}\, P_z, \ldots, \sqrt{|\theta_p|}\, P_z) \\ &= \Lambda_{\sqrt{|\Theta|}} P_Z \Lambda_{\sqrt{|\Theta|}}\end{aligned} \tag{4.43}$$

Since P_Z is a symmetric positive definite matrix, from (4.43) we can see that $\Lambda_{|\Theta|} P_Z$ is a symmetric positive definite matrix.

Taking the the time derivative of $V(x, \widetilde{\Theta}, \widetilde{Z}, \Theta)$, and simplifying using (4.37), (4.40), and (4.41), we obtain

$$\begin{aligned}\dot{V} &= \dot{x}^\top P x + x^\top P \dot{x} + \widetilde{\Theta}^\top \Gamma^{-1} \dot{\widetilde{\Theta}} + 2\widetilde{Z}^\top \Lambda_{|\Theta|} P_Z \dot{\widetilde{Z}} \\ &= \; x^\top \left((M - hc^\top)^\top P + P(M - hc^\top) \right) x + \widetilde{Z}^\top \big(B^\top(x) P_Z \Lambda_{|\Theta|} \\ &\quad + \Lambda_{|\Theta|} P_Z B(x) \big) \widetilde{Z} + 2\widetilde{Z}^\top \Lambda_{|\Theta|} G(x) \operatorname{sgn}(\Theta) h^\top P x - 2\widetilde{Z}^\top G(x) \Theta h^\top P x\end{aligned} \tag{4.44}$$

Since $G(x)$ and $\Lambda_{|\Theta|}$ are diagonal, we have

$$\Lambda_{|\Theta|} G(x) \operatorname{sgn}(\Theta) = G(x) \Lambda_{|\Theta|} \operatorname{sgn}(\Theta) = G(x)\Theta.$$

Also,

$$
\begin{aligned}
&B^\top(x)P_Z\Lambda_{|\Theta|} + \Lambda_{|\Theta|}P_Z B(x) \\
&= \text{diag}\left(|\theta_1|(B_z^\top(x)P_z + P_z B_z(x)), \ldots, |\theta_p|(B_z^\top(x)P_z + P_z B_z(x))\right) \\
&\leq \text{diag}\left(-|\theta_1|Q_z, \ldots, -|\theta_p|Q_z\right) \\
&= -\Lambda_{|\Theta|}Q_Z.
\end{aligned}
$$

Notice that $\Lambda_{|\Theta|}Q_Z$ is a positive semi-definite matrix. Therefore, we have

$$
\begin{aligned}
\dot{V} &= -x^\top Q x - \widetilde{Z}^\top \Lambda_{|\Theta|} Q_Z \widetilde{Z} \\
&\leq -\lambda_{min}(Q)x^\top x - \lambda_{min}(\Lambda_{|\Theta|}Q_Z)\widetilde{Z}^\top \widetilde{Z}
\end{aligned}
\tag{4.45}
$$

where $\lambda_{min}(\cdot)$ denotes the minimum eigenvalue of a matrix.

Hence, (4.42) is a Lyapunov function for the closed-loop system, which guarantees that x, $\widetilde{\Theta}$ and $\widetilde{Z}$ are bounded; $\widehat{\Theta}$ is bounded because $\widehat{\Theta} = \widetilde{\Theta} + \Theta$ and Θ is bounded; Z is bounded by Assumptions A4.5 and A4.5, which in turn guarantees that $\widehat{Z}$ $(= Z + \widetilde{Z})$ is bounded; the control input $u(t)$ is bounded as it is a function of all bounded variables. From equations (4.40) and (4.41), both $\dot{\widetilde{Z}}$ and $\dot{x}$ are bounded. Therefore, $\widetilde{Z} \in \mathcal{L}_\infty$, $\dot{\widetilde{Z}} \in \mathcal{L}_\infty$, $x \in \mathcal{L}_\infty \cap \mathcal{L}_2$ and $\dot{x} \in \mathcal{L}_\infty$. By invoking Barbalat's Lemma [73], we obtain $\lim_{t\to\infty} x = 0$. Moreover, if Q_z is positive definite, $\widetilde{Z} \in \mathcal{L}_2$ in addition to $\widetilde{Z} \in \mathcal{L}_\infty$, $\dot{\widetilde{Z}} \in \mathcal{L}_\infty$; therefore, $\lim_{t\to\infty} \widetilde{Z} = 0$. ■

Remark 4.5 *Theorem 4.1 addresses the regulation problem for the class of nonlinear systems described by (4.13). This design process can be extended to the tracking problem as well, which is shown in Example 4.3 of the next section. Further, one can also extend the proposed design to multiple-input systems.*

Remark 4.6 *Notice that the original system described by (4.13) contains m state variables and p parameters that are to be estimated. In the proposed design, if none of the elements of the vector $f_z(x)$ and the matrix $G_z(x)$ is zero, we require $m(p+1)$*

filters for the estimation of the unmeasurable states and mp filters for the estimation of the unknown parameters.

Remark 4.7 *The estimated parameters are not guaranteed to converge to their true values. From the control point of view, it is not necessary that the estimated parameters converge to their true values. Since the nonlinear system studied in this paper cannot be expressed in the form of a standard parametric model (see [33]), it is difficult to obtain the persistency of excitation conditions under which the estimated parameter* $\widehat{\Theta}$ *converges to its true value* Θ. *To enhance the robustness of the closed-loop system due to parameter drift, we can use the* σ*-modification procedure given in [33]. In such a case, the parameter estimation algorithm (4.37) can be changed to*

$$\dot{\widehat{\Theta}} = -\sigma\Gamma\widehat{\Theta} + 2\Gamma\left(G(x)\widehat{Z} + f(x)\right)h^\top Px. \tag{4.46}$$

Choosing the same Lyapunov function candidate V *as in (4.42), the time derivative of* V *becomes*

$$\dot{V} \leq -x^\top Qx - \lambda_{min}(\Lambda_{|\Theta|}Q_Z)\widetilde{Z}^\top\widetilde{Z} - 2\sigma\widetilde{\Theta}^\top\widehat{\Theta}. \tag{4.47}$$

Since $-2\sigma\widetilde{\Theta}^\top\widehat{\Theta} \leq -\sigma\widetilde{\Theta}^\top\widetilde{\Theta} + \sigma\|\Theta\|^2$, *we have*

$$\dot{V} \leq -\lambda_{min}(Q)x^\top x - \lambda_{min}(\Lambda_{|\Theta|}Q_Z)\widetilde{Z}^\top\widetilde{Z} - \sigma\widetilde{\Theta}^\top\widetilde{\Theta} + \sigma\|\Theta\|^2. \tag{4.48}$$

Since the parameter Θ *is bounded, it follows that* $x, \widetilde{\Theta}$ *and* Z *converge to a residual set whose radius is proportional to the square root of the upper bound of* $\sigma\|\Theta\|^2$.

4.5 Numerical examples with simulations and experimentation on a two-link robot

In this section, three examples are presented. The first example illustrates the reduction procedure for the case when some elements of $G_z(x)$ and/or $f_z(x)$ in the original dynamics (4.13) are zero. The second and the third examples verify the adaptive controller and observer design via numerical simulations. In the second example, the system has two measurable states, two unmeasurable states and two unknown parameters. The objective is regulation of states. In the third example, we consider the tracking problem for a mechanical system with dynamic friction.

Example 4.1 *This example will illustrate the reduction procedure. Consider the system described by (4.13) with $m = 2$, $p = 3$, and $B_z(x)$ diagonal. The modified system has*

$$
\begin{aligned}
Z &= \begin{bmatrix} z_1 & z_2 & z_1 & z_2 & z_1 & z_2 & z_1 & z_2 \end{bmatrix}^\top, \\
\Theta &= \begin{bmatrix} \theta_1 & \theta_1 & \theta_2 & \theta_2 & \theta_3 & \theta_3 & 1 & 1 \end{bmatrix}^\top, \\
G(x) &= diag(g_{z11}(x),\ g_{z12}(x),\ g_{z21}(x),\ g_{z22}(x),\ g_{z31}(x),\ g_{z32}(x),\ f_{z1}(x),\ f_{z2}(x)), \\
a(x) &= \begin{bmatrix} a_{z1}(x) & a_{z2}(x) & a_{z1}(x) & a_{z2}(x) & a_{z1}(x) & a_{z2}(x) & a_{z1}(x) & a_{z12}(x) \end{bmatrix}^\top, \\
B(x) &= diag(B_z(x),\ B_z(x),\ B_z(x),\ B_z(x)), \\
f(x) &= \begin{bmatrix} f_{\theta 1}(x) & 0 & f_{\theta 2}(x) & 0 & f_{\theta 3}(x) & 0 & 0 & 0 \end{bmatrix}^\top.
\end{aligned}
$$

If $f_z^\top(x) = [0,\ 0]$, discard the last two rows of Z, Θ, $a(x)$ and $f(x)$, and the last two rows and columns of $G(x)$ and $B(x)$, which will result in $q = 6$, which is less than the maximum size of eight. If $g_{z12}(x) = 0$, then the second row of Z, Θ, $a(x)$ and $f(x)$, and the second row and the second column of $G(x)$ and $B(x)$ can be discarded, which gives $q = 7$. ■

Example 4.2 *Consider the system*

$$\dot{x} = Mx + hu + h(d(x) + f_\theta^\top(x)\theta + f_z^\top(x)z + \theta^\top G_z(x)z), \tag{4.49a}$$

$$\dot{z} = a_z(x) + B_z(x)z \tag{4.49b}$$

where

$$M = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad h = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad d(x) = 0,$$

$$f_\theta(x) = \begin{bmatrix} f_{\theta_1} \\ f_{\theta_2} \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}, \quad f_z(x) = \begin{bmatrix} f_{z1} \\ f_{z2} \end{bmatrix} = \begin{bmatrix} x_2 \\ 0 \end{bmatrix},$$

$$G_z(x) = \begin{bmatrix} g_{z11} & g_{z12} \\ g_{z21} & g_{z22} \end{bmatrix} = \begin{bmatrix} \frac{x_1}{2-e^{-|x_2|}} & 5 \\ \frac{x_2}{2-e^{-|x_1|}} & 0 \end{bmatrix}, \quad a_z(x) = \begin{bmatrix} a_{z1} \\ a_{z2} \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_2 \end{bmatrix},$$

$$B_z(x) = \begin{bmatrix} b_{z11} & b_{z12} \\ b_{z21} & b_{z22} \end{bmatrix} = \begin{bmatrix} -1.5 - \cos(x_1) & 0 \\ 0 & -1.5 - \sin(x_2) \end{bmatrix}.$$

The system described by (4.49) can be represented in the following form:

$$\dot{x} = Mx + hu + h[f^\top(x)\Theta + Z^\top G(x)\Theta],$$
$$\dot{Z} = a(x) + B(x)Z$$

where

$$f(x) = \begin{bmatrix} f_{\theta 1} & 0 & f_{\theta_2} & 0 & 0 \end{bmatrix}^\top,$$
$$G(x) = diag(g_{z11},\ g_{z12},\ g_{z21},\ f_{z1},\ f_{z2}),$$
$$a(x) = \begin{bmatrix} a_{z1} & a_{z2} & a_{z1} & a_{z1} & a_{z2} \end{bmatrix}^\top,$$
$$B(x) = diag(b_{z11},\ b_{z22},\ b_{z11},\ b_{z11},\ b_{z22}),$$
$$Z = \begin{bmatrix} Z_1 & Z_2 & Z_3 & Z_4 & Z_5 \end{bmatrix}^\top = \begin{bmatrix} z_1 & z_2 & z_1 & z_1 & z_2 \end{bmatrix}^\top,$$
$$\Theta = \begin{bmatrix} \Theta_1 & \Theta_2 & \Theta_3 & \Theta_4 & \Theta_5 \end{bmatrix}^\top = \begin{bmatrix} \theta_1 & \theta_1 & \theta_2 & 1 & 1 \end{bmatrix}^\top,$$

and $\Theta_4 = \Theta_5 = 1$*;* Θ_4 *and* Θ_5 *are not estimated. The following values are chosen:*

$$c = \begin{bmatrix} 25 & 10 \end{bmatrix}^\top, \quad P_z = 10I, \quad \Gamma = 0.1I$$

where I is the 5×5 identity matrix. The true parameter vector is

$$\theta = \begin{bmatrix} 2 & 4 \end{bmatrix}^\top.$$

The initial values used in the simulation are:

$$x(0) = \begin{bmatrix} 1 & 1 \end{bmatrix}^\top, \qquad \widehat{\Theta}(0) = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \end{bmatrix}^\top,$$
$$Z(0) = \begin{bmatrix} -1 & -1 & -1 & -1 & -1 \end{bmatrix}^\top, \qquad \widehat{Z}(0) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix}^\top.$$

Simulation results are shown in Figures 4.2 through 4.4; it can be observed that x asymptotically converges to zero; the estimated state $\widehat{Z}$ converges to Z; and the estimated parameter $\widehat{\Theta}$ is bounded.

Example 4.3 *Consider a single-link mechanical system described by*

$$J\ddot{s} = u - f_f \tag{4.50}$$

where J is the inertia of the link, s is the angular position of the link, $\dot{s}$ is the angular velocity of the link, u is the control input, and f_f is the friction torque described by the following LuGre dynamic friction model [123]:

$$\dot{z} = \dot{s} - \frac{\sigma|\dot{s}|}{g(\dot{s})}z, \tag{4.51}$$

$$f_f = \theta_1 z + \theta_2 \dot{z} + \theta_3 \dot{s}, \quad g(\dot{s}) = F_c + (F_s - F_c)e^{-(\dot{s}/\omega_s)^2} \tag{4.52}$$

where σ, θ_1, θ_2, θ_3, F_s, F_c and ω_s are positive friction coefficients; σ, F_s, F_c and ω_s are generally identified by experiments off-line and are assumed to be known for this simulation. J is known. The objective is to control the link such that the position and velocity of the link track a predefined trajectory s_d and $\dot{s}_d$, respectively. It is assumed that that s_d and $\dot{s}_d$ are bounded, and the angular position and the angular velocity are measurable.

Combining (4.50), (4.51) and (4.52) and representing in matrix form yields

$$\dot{\zeta} = M\zeta + hu + h(f_\theta^\top \theta + \theta^\top G_z(\zeta)z), \tag{4.53a}$$

$$\dot{z} = a_z(\zeta) + B_z(\zeta)z \tag{4.53b}$$

where

$$\zeta = \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} = \begin{bmatrix} s \\ \dot{s} \end{bmatrix}, \quad M = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad f_\theta = \begin{bmatrix} 0 \\ -\zeta_2 \\ -\zeta_2 \end{bmatrix},$$

$$h = \begin{bmatrix} 0 \\ \frac{1}{J} \end{bmatrix}, \quad G_z(\zeta) = \begin{bmatrix} -1 \\ \frac{\sigma|\zeta_2|}{g(\zeta_2)} \\ 0 \end{bmatrix}, \quad \theta = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix},$$

$$a_z(\zeta) = \zeta_2, \qquad B_z(\zeta) = -\frac{\sigma|\zeta_2|}{g(\zeta_2)}.$$

Defining the trajectory vector $x_d = \begin{bmatrix} s_d & ds_d \end{bmatrix}^\top$ *and representing (4.53a) and (4.53b) in terms of the tracking error* $x \triangleq \begin{bmatrix} x_1 & x_2 \end{bmatrix}^\top = \zeta - x_d$ *results in the following error dynamics and z-dynamics:*

$$\dot{x} = Mx + hu + h(f_\theta^\top(x + x_d)\theta + \theta^\top G_z(x + x_d)z), \tag{4.54a}$$

$$\dot{z} = a_z(x + x_d) + B_z(x + x_d)z. \tag{4.54b}$$

The above two equations can be re-written in the following form suitable for the adaptive controller and observer design:

$$\dot{x} = Mx + hu + h(f^\top(x)\Theta + Z^\top G(x)\Theta), \tag{4.55a}$$

$$\dot{Z} = a(x) + B(x)Z \tag{4.55b}$$

where

$$\Theta \triangleq \begin{bmatrix} \Theta_1 \\ \Theta_2 \\ \Theta_3 \end{bmatrix} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}, \qquad Z \triangleq \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix} = \begin{bmatrix} z \\ z \\ z \end{bmatrix},$$

$$f(x) = \begin{bmatrix} 0 \\ -x_2 - \dot{s}_d \\ -x_2 - \dot{s}_d \end{bmatrix}, \qquad G(x) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & \frac{\sigma|x_2+\dot{s}_d|}{g(x_2+\dot{s}_d)} & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$a(x) = \begin{bmatrix} x_2 + \dot{s}_d \\ x_2 + \dot{s}_d \\ x_2 + \dot{s}_d \end{bmatrix}, \qquad B(x) = \begin{bmatrix} -\frac{\sigma|x_2+\dot{s}_d|}{g(x_2+\dot{s}_d)} & 0 & 0 \\ 0 & -\frac{\sigma|x_2+\dot{s}_d|}{g(x_2+\dot{s}_d)} & 0 \\ 0 & 0 & -\frac{\sigma|x_2+\dot{s}_d|}{g(x_2+\dot{s}_d)} \end{bmatrix}.$$

Notice that x $(= \zeta - x_d)$ *is available because* ζ *is measurable and* x_d *is known; and* Θ *and* Z *are estimated;* Z_3 *need not be estimated because* $g_{33}(x) = 0$.

Experiments were conducted on the base link of a two-link NSK manipulator shown in Figure 4.5. The base link is controlled to follow a sinusoidal trajectory,

$$x_d(t) = \begin{bmatrix} \sin(0.4\pi t) & 0.4\pi\cos(0.4\pi t) \end{bmatrix}^\top.$$

The inertia of the base-link is $J = 3.4$. *The following values are chosen:*

$$c = 3.4 * \begin{bmatrix} 2500 \\ 100 \end{bmatrix}, \quad P = \begin{bmatrix} 626.25 & 0.01 \\ 0.01 & 0.2501 \end{bmatrix}, \quad \Gamma = diag(5,5,5), \quad P_z = 0.1I.$$

The parameters in z*-dynamics are:*

$$\sigma = 340, \quad F_s = 11, \quad F_c = 1.557, \quad \omega_s = 0.14.$$

The initial values are chosen as

$$\widehat{\Theta}^\top(0) = [0,0,0], \quad \widehat{Z}^\top(0) = [0,0,0].$$

Experimental results are shown in Figures 4.6 through 4.8. Figure 4.6 shows the desired position trajectory, position tracking error and velocity tracking error from *the top plot to the bottom plot, respectively. The estimates of the unmeasured state* $Z(t)$, $\widehat{Z}_1(t)$ *and* $\widehat{Z}_2(t)$, *are shown in Figure 4.7. Parameter estimates are shown in Figure 4.8.* ■

4.6 Summary

A new adaptive controller and a nonlinear observer are designed for a class of nonlinear systems which contain the products of an unmeasured state and an unknown parameter. A stable adaptive controller and a stable nonlinear observer are designed using a parameter dependent Lyapunov function. The proposed design is verified via simulation and experimental examples, the results of which are shown and discussed. Future work should focus on the inclusion of coupled terms of the unknown parameters and unmeasured states in the unmeasurable state dynamics. Future research should also focus on the investigation of the existence of parameter independent state diffeomorphisms that will transform a general nonlinear systems to the class of systems considered in this chapter.

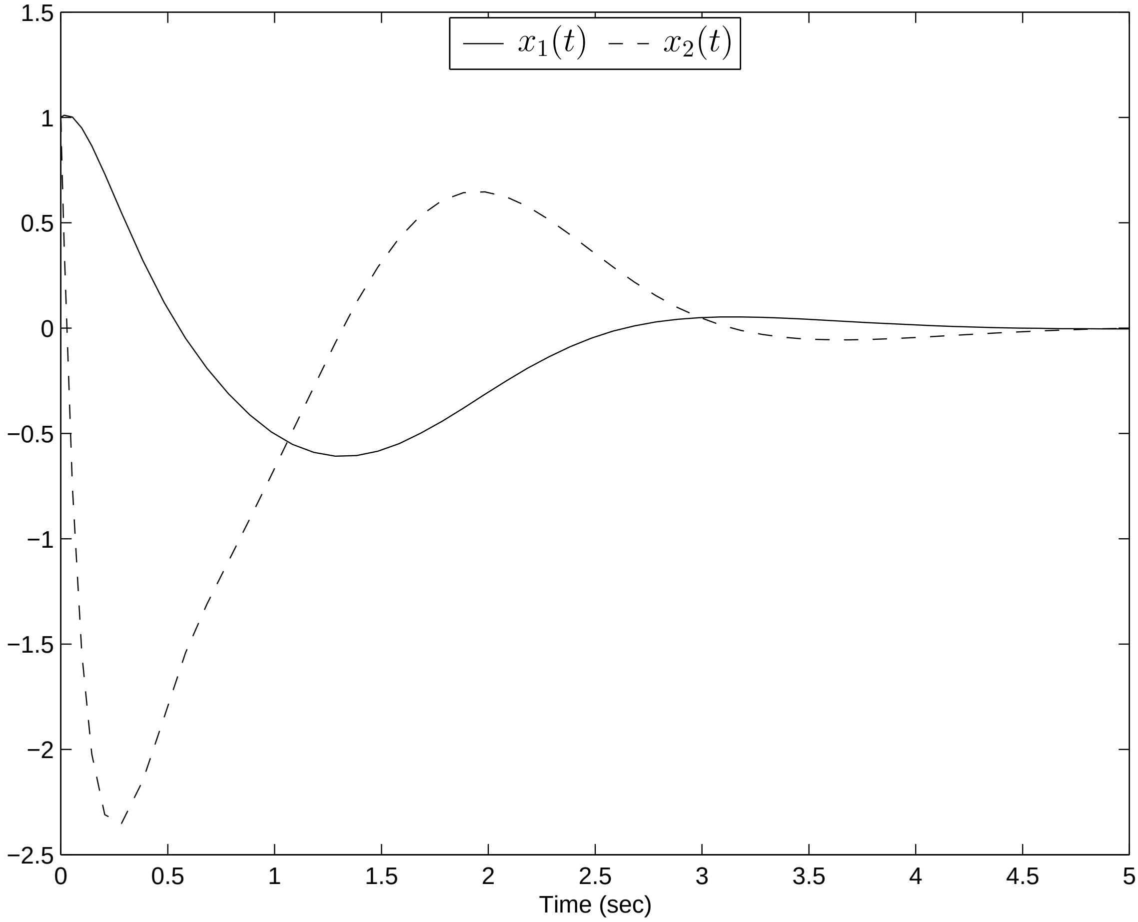

Figure 4.2: Simulation result of Example 4.2. Trajectory of $x(t)$ is shown.

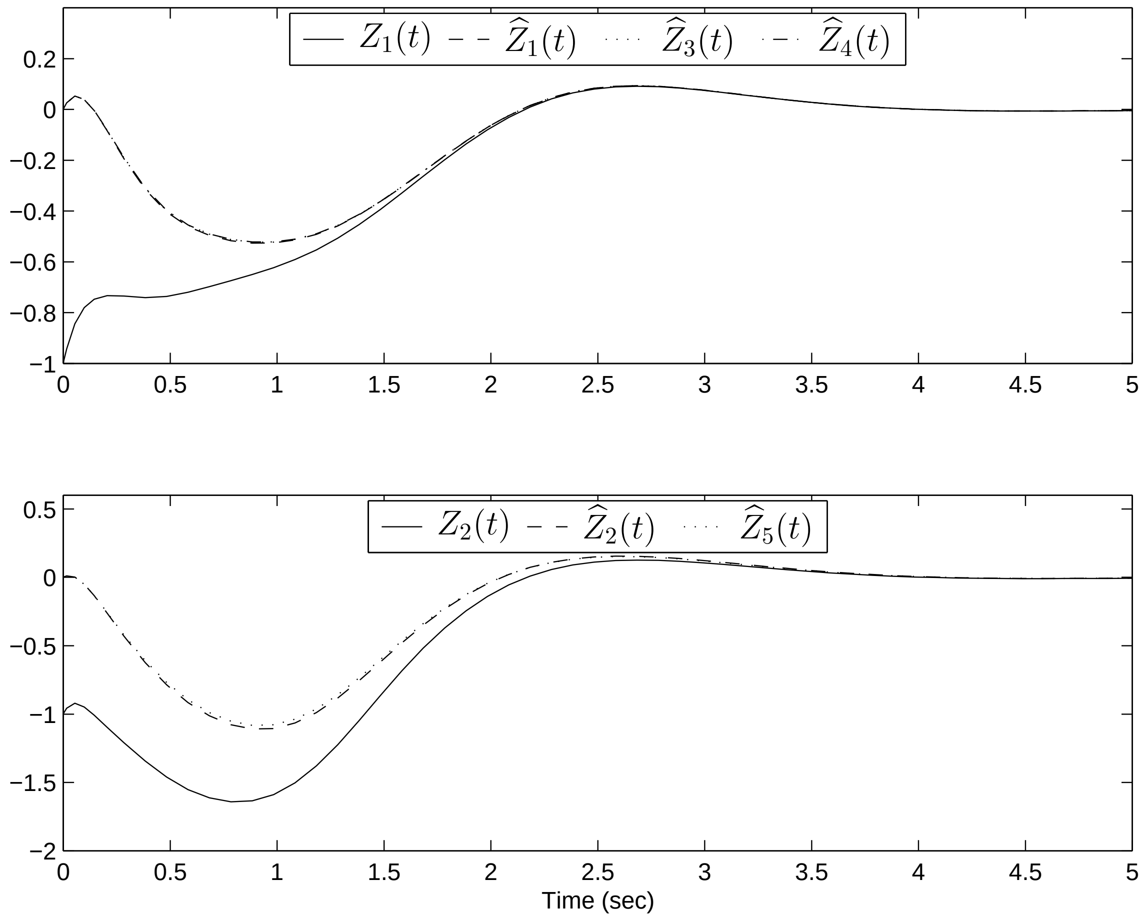

Figure 4.3: Simulation result of Example 4.2. The top plot shows the trajectory of $Z_1(t)$ and its estimates $\widehat{Z}_1(t)$, $\widehat{Z}_4(t)$ and $\widehat{Z}_5(t)$. The bottom plot shows the trajectory of $Z_2(t)$ and its estimates $\widehat{Z}_2(t)$ and $\widehat{Z}_4(t)$.

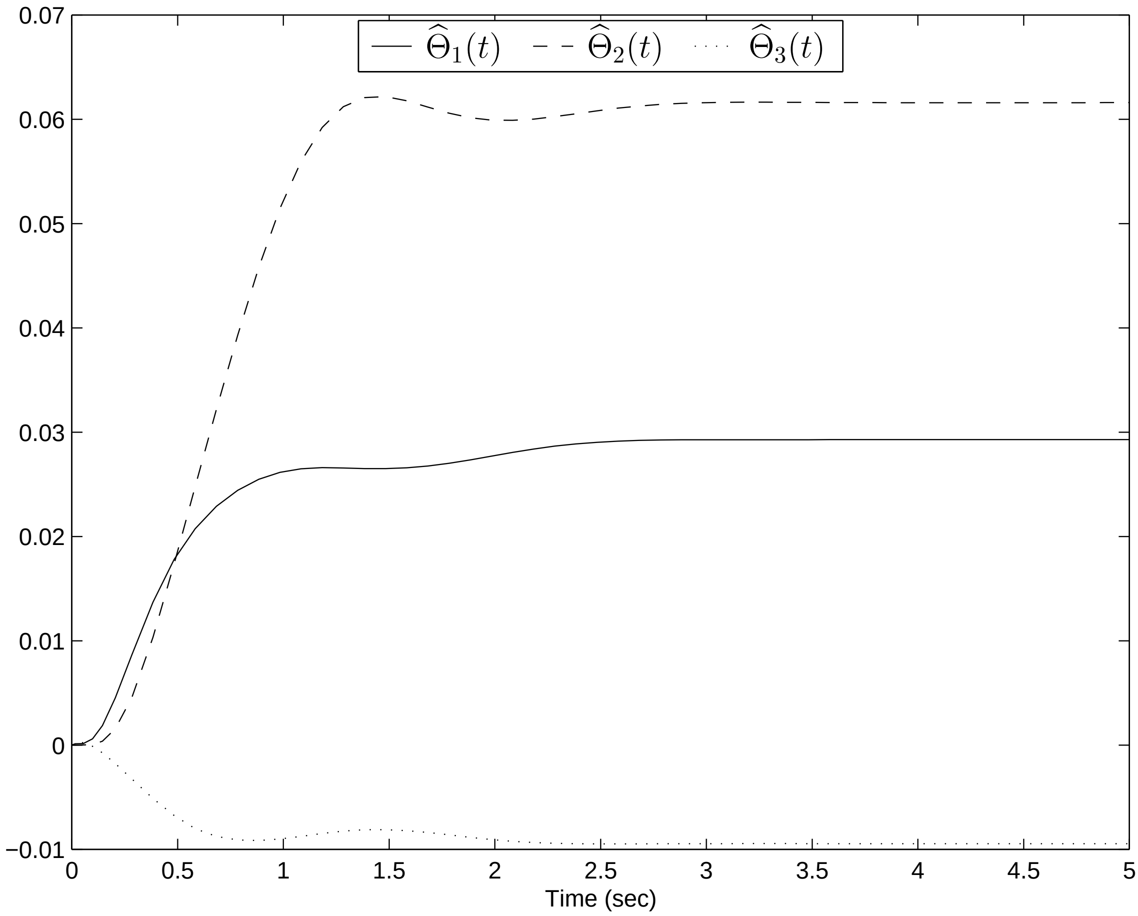

Figure 4.4: Simulation result of Example 4.2. Estimated parameters, $\widehat{\Theta}_1(t), \widehat{\Theta}_2(t)$, and $\widehat{\Theta}_3(t)$ are shown.

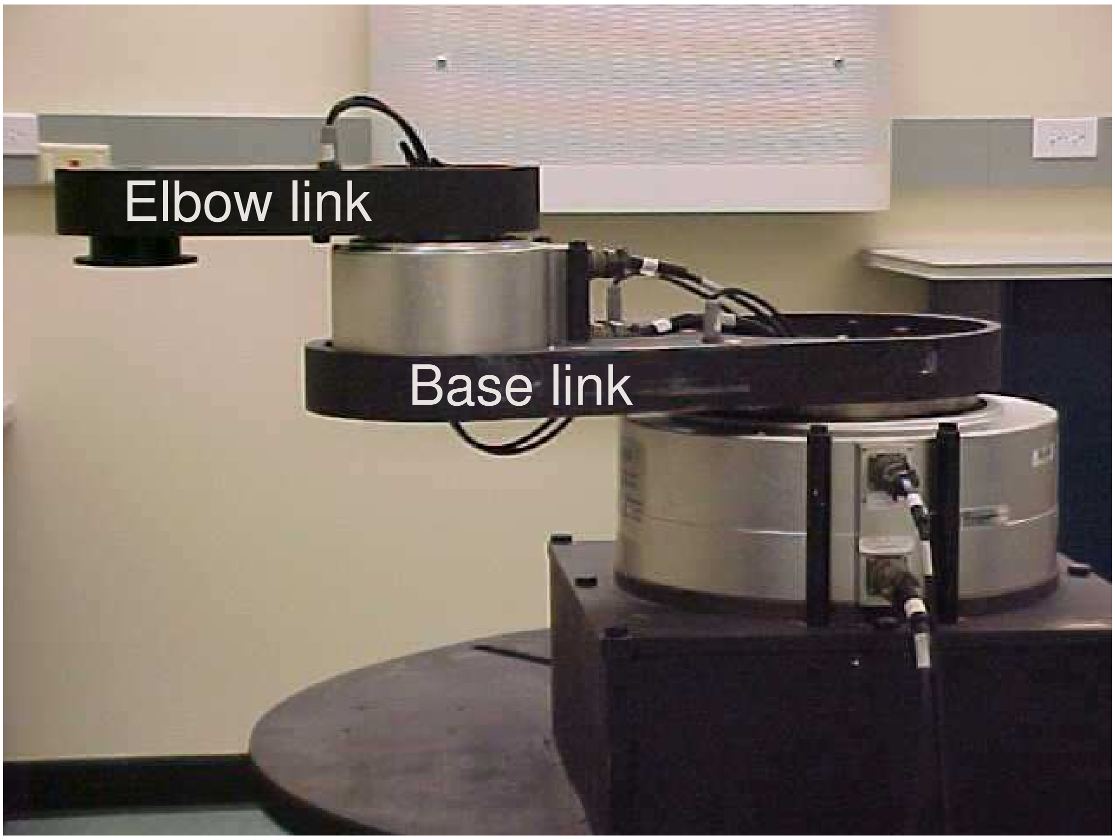

Figure 4.5: Picture of the experimental platform in Example 4.3.

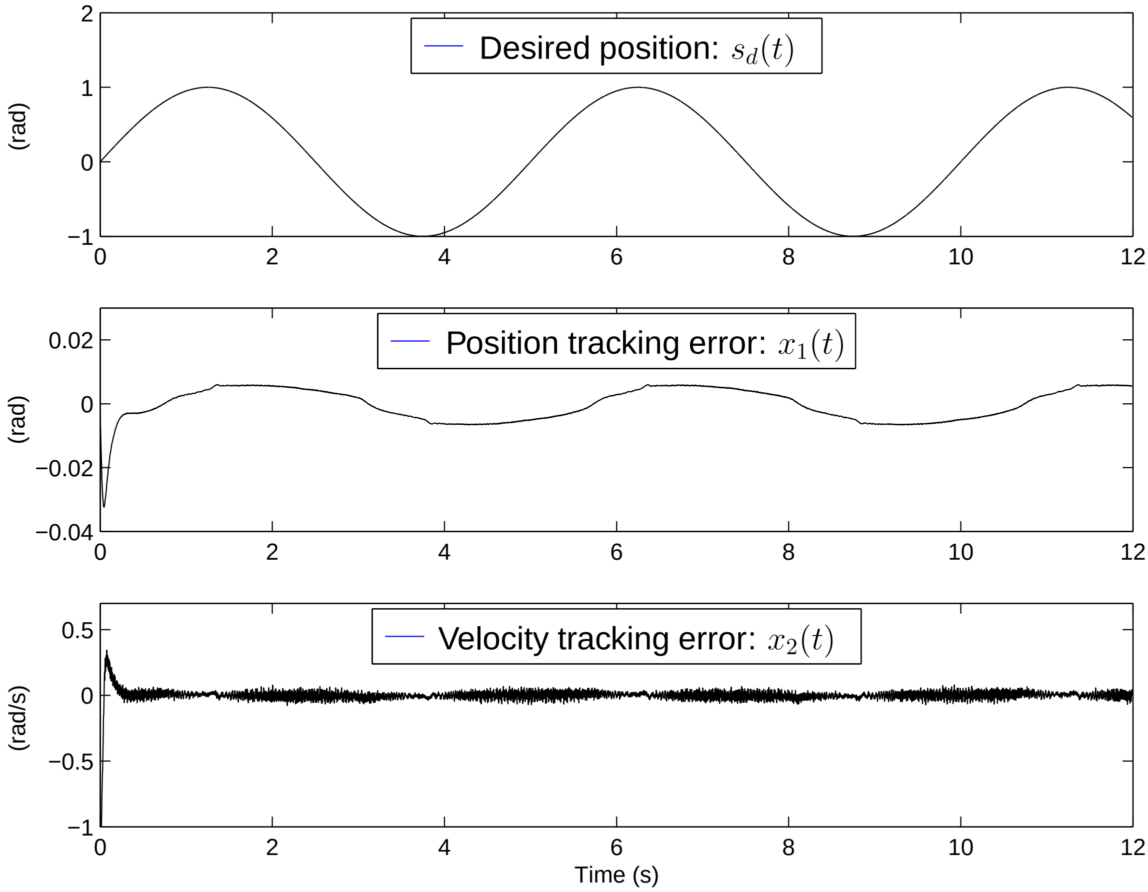

Figure 4.6: Experimental result of Example 4.3. Desired position, $s_d(t) = \sin(0.4\pi t)$ (top plot), angular position tracking error, $x_1(t)$ (middle plot), and angular velocity tracking error, $x_2(t)$ (bottom plot), are shown.

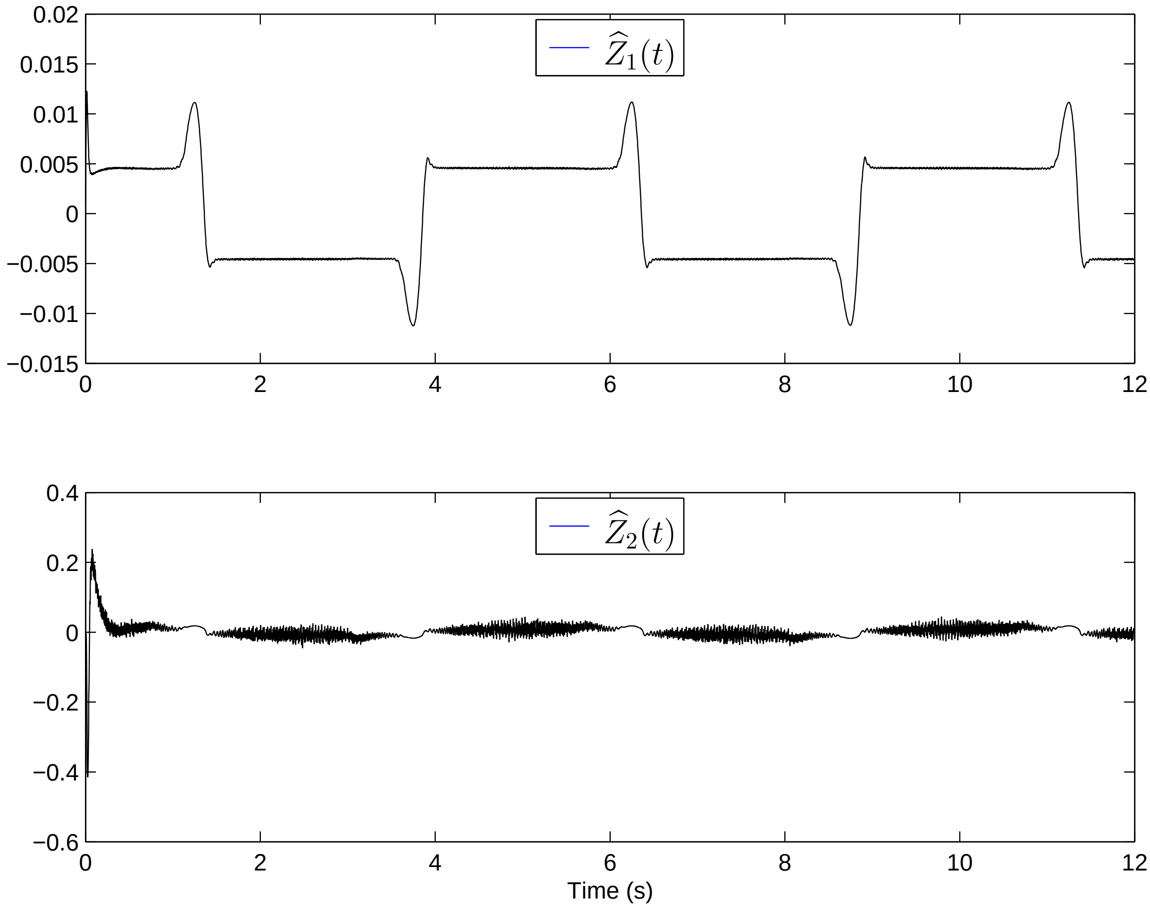

Figure 4.7: Experimental result of Example 4.3. The estimated of $Z(t)$, $\widehat{Z}_1(t)$ and $\widehat{Z}_2(t)$, are shown.

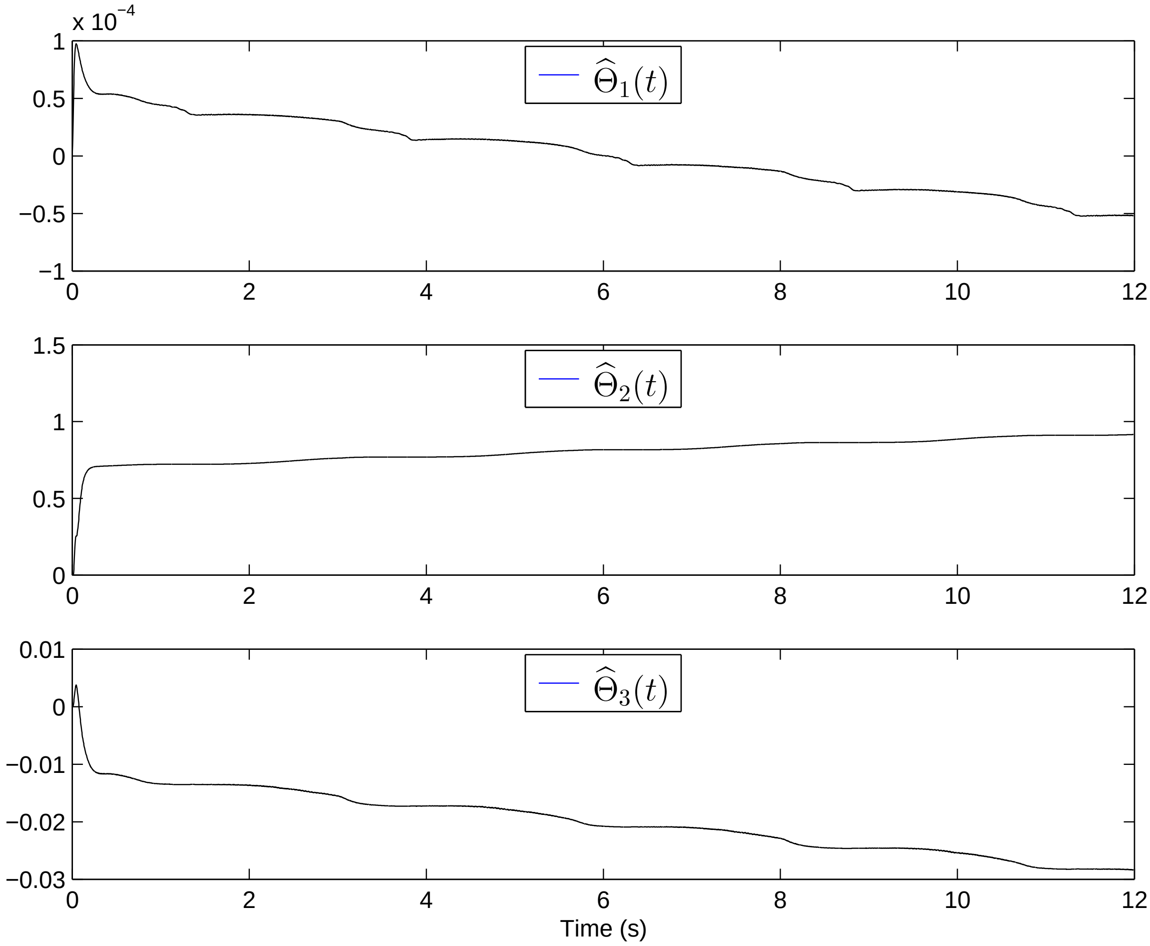

Figure 4.8: Experimental result of Example 4.3. Estimated parameters: $\widehat{\Theta}_1(t), \widehat{\Theta}_2(t)$ and $\widehat{\Theta}_3(t)$ are shown.

CHAPTER 5

ADAPTIVE CONTROL OF MECHANICAL SYSTEMS WITH TIME-VARYING PARAMETERS AND DISTURBANCES

The adaptive control design for mechanical systems is considered in this chapter. It is assumed that the mechanical system is subject to unknown time-varying parameters and disturbances. As mechanical systems can be linearly parameterized, the adaptive estimation of time-varying parameters in linearly parametric model is first considered. Local polynomial approximation in a finite time interval is used to represent the unknown time-varying parameters. The coefficients of the polynomials are estimated locally instead of the unknown time-varying parameter. The accuracy of the approximation depends on the order of the polynomial and the width of the time interval, which can be chosen. The polynomial coefficients vary from one interval to the other, but within an interval they are constant. Thus, each time-varying parameter is approximated independently in each interval by a set of constant coefficients. Based on the approximation, modifications to traditional least-squares algorithm with covariance resetting and the gradient algorithm are provided for the linear time-varying parametric model. Stability of the modified algorithms is shown and discussed. Comparative simulation results for the two algorithms on an example are presented. Adaptive control design for time-varying mechanical systems using the adaptive estimation algorithm proposed is investigated.

The rest of this chapter is organized as follows. Section 5.1 proposes adaptive estimation algorithms for estimating time-varying parameters in linearly parameterized system. Section 5.1.1 introduces how to obtain a linear parametric model for a class of systems. Representation of the time-varying parameters via local polynomials is discussed in Section 5.1.2. The linear time-varying parametric model in terms of the local polynomial approximations is described in Section 5.1.3. Two estimation algorithms, the modified least-squares with covariance resetting and the modified gradient algorithm, are discussed in Section 5.1.4. Section 5.1.5 gives simulation results for an example. In Section 5.2, an adaptive controller is proposed for mechanical systems with time-varying parameters and time-varying disturbances. Section 5.2.1 gives the dynamics of mechanical systems. Design of an adaptive controller is given in Section 5.2.2. Experimental setup, including how to generate time-varying dynamics to the base link, conditions, and results are discussed in Section 5.2.3. Summary of this chapter is given in Section 5.3.

5.1 Adaptive estimation of time-varying parameters using local polynomial approximation

Given a plant, its behavior is determined by its dynamic structure and its parameters. When model parameters, time-varying or time-invariant, are known, the control design process is generally straightforward. In practice, we usually do not have all the information about the model. Instead, partial information may be available, such as the structure of the model and the features of the model parameters. When the parameters are unknown, they have to be deduced by observing the system's response to certain inputs if these parameters are required in control design. In the case when the parameters are fixed, that is, they are constant all the time, it is easier to use frequency or time domain techniques to estimate them. Whenever the model parameters are time-invariant, the estimated parameters can be used instead of their true value. However in many applications, the parameters are time-varying because of changes in operating conditions, aging of equipment, etc. In such cases, off-line estimation results cannot be applied directly. A frequent estimation of the parameters based on the input/output response is required. Contrast to the off-line estimation, this is usually called on-line estimation. The methodology of processing the measurable signals to arrive at the estimations of the parameters yields many estimation algorithms.

The essential idea behind on-line identification is using the difference between the observed system output $y(t)$ and estimated system output $\widehat{y}(t)$ to modify the estimated parameter $\widehat{\theta}(t)$ continuously so that the difference between $y(t)$ and $\widehat{y}(\widehat{\theta}, t)$ becomes small, that is, force $\widehat{y}(\widehat{\theta}, t)$ to approach $y(t)$ as time t increases. The model for generating $\widehat{y}(\widehat{\theta}, t)$ is usually chosen such that it has a similar structure to that of the system under study. The stability properties are usually derived by using Lyapunov method.

The design of the on-line estimation algorithm involves mainly three steps. (1) The first step is to parameterize the plant model. This is very important because some plant models are more convenient than others. The linear parametric model is among the most prevalent system models chosen for on-line estimation design. In this type of model, the unknown parameters are organized into the parameter vector θ. The model is in the linear form of the parameter vector θ, that is, in the form of $y(t) = \theta^\top \phi$ where $y(t)$ is the system output and ϕ is the regression matrix which is composed of signals which can be measured or computed. As ϕ may contain high order time derivatives of the input and/or output of the system, filter technique is usually used so that only the input and output of the system can be used to generate ϕ. If the estimated output $\widehat{y}(\widehat{\theta}, t)$ is generated from the system model with $\widehat{\theta}$ instead of θ then the difference between $y(t)$ and $\widehat{y}(\widehat{\theta}, t)$ is linear in the parameter error $\theta - \widehat{\theta}$. The output difference reflects how close the estimated parameters are to the true parameters. This is the main advantage of the linear parametric model. (2) The second step is to design an adaptation law for the estimated parameter vector. The adaptation law is usually a differential equation whose state is $\widehat{\theta}$ and is designed using stability considerations or simple optimization techniques to minimize a cost function with respect to $\widehat{\theta}$ at each time t. This cost function usually contains the information of the difference between $y(t)$ and $\widehat{y}(\widehat{\theta}, t)$. (3) The last step is to design the input such that the adaptation law has the property that the estimated parameter $\widehat{\theta}(t)$ approaches the unknown system parameter θ as time t approaches infinity. This is important if the objective of the adaptation law is to find the true value of the system parameters. To achieve this objective, the input signal should be able to excite all modes in the plant. In other words, the input signal must have all frequency components which exist in the plant. In the situation where control is the main objective and estimation is just for providing estimation of the unknown parameters, the input signal is the control signal which is generated by the controller. The input signal is determined by the control law and the predefined trajectory of the output of the system. Consequently the input signal may not contain all frequency components needed for estimating the unknown plant parameters. In this case, the properties of the adaptation law should imply that the estimated parameters are bounded.

In this section, the on-line estimation of time-varying parameters in the parametrizable system is considered. The system under study can be expressed in the parametric model:

$$z(t) = \theta^{*\top}(t)\Phi(t) \tag{5.1}$$

where $\theta^*(t) = \begin{bmatrix} \theta_0^*(t) & \theta_1^*(t) & \dots & \theta_m^*(t) \end{bmatrix}^\top \in \mathbb{R}^m$ is the unknown time-varying parameter vector, $\Phi(t) = \begin{bmatrix} \Phi_1(t) & \Phi_2(t) & \dots & \Phi_m(t) \end{bmatrix}^\top \in \mathbb{R}^m$ is the known signal vector, and $z(t) \in \mathbb{R}$ is the measured output. It is assumed that $\theta^*(t)$ belongs to the class of piecewise continuous m-times differentiable functions, that is,

$$\theta^*(t) \in \left\{\theta^{*(m)}(t) \in \mathcal{L}_\infty,\ m = 1, \dots, p\right\} \tag{5.2}$$

where $\theta^{*(m)}(t)$ denotes the m-th time-derivative of $\theta^*(t)$.

Remark 5.1 *If the parameters are constant, then their time-derivative is zero; hence the control design problem with on-line estimation is generally straightforward. Standard method for the estimation of the unknown parameters can be found in may textbook (e.g. [33, 24, 73, 124, 125, 126]).*

Remark 5.2 *We make an assumption that system under control can be parameterized linearly in parameters. This is not valid for all systems. However, for a class of linear and nonlinear systems, such as mechanical systems, linear parameterization is possible. On the other hand, if the system cannot be linearly parameterized, it can be represented by a linear parametric model plus a modelling error, and the modelling error can be considered as a disturbance to the system. In the next section, the method to obtain the linear parametric model for a system described by a differential equation is briefly introduced.*

Remark 5.3 *A continuous-time model rather than a discrete-time model is chosen because results in the continuous-time domain can be extended to the discrete-time domain easily under the assumption of fast sampling, however the converse is generally not feasible. As such, all systems considered in this report are continuous-time systems.*

5.1.1 Linear parametric models

The parametric models and their properties are crucial in parameter identification and adaptive control problems to be studied in subsequent sections. We introduce parameterization methods of dynamic systems with time-invariant and time-varying parameters here. The objective of the parameterization is to represent the plant in a form such that the coefficients of the polynomials in the transfer function description are separated from signals formed by filtering the system input and output. Parameterization is important when the system under study can only provide the measurements of the input and output. The parameterization of a time-invariant system will provide a linear parametric model without modelling error. In contrast, the parameterization of a time-varying system will give rise to a modelling error. This modelling error depends on the rate of the time-varying parameter. The methods described here are based on the developments in [33, 127, 98, 128].

Linear parametric model of time invariant systems

Consider a system described by the following nth-order differential equation

$$y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_0 y = b_{n-1}u^{(n-1)} + b_{n-2}u^{(n-2)} + \ldots + b_0 u \tag{5.3}$$

where u and y are input and output of the system, respectively. $u^{(i)}$ and $y^{(i)}$ denote the i-th derivative of u and y, respectively; a_i and b_i are constant coefficients.

Lumping all the parameters in (5.3) in the parameter vector θ^* and all the input/output signals and their derivatives in the signal vector Y yields

$$\theta^* = \begin{bmatrix} b_{n-1} & b_{n-2} & \ldots & b_0 & a_{n-1} & a_{n-2} & \ldots & a_0 \end{bmatrix}^\top, \tag{5.4}$$

$$\begin{aligned} Y &= \begin{bmatrix} u^{n-1} & u^{n-2} & \ldots & u & -y^{n-1} & -y^{n-2} & \ldots & -y \end{bmatrix}^\top \\ &= \begin{bmatrix} \alpha_{n-1}^\top(s)u & -\alpha_{n-1}^\top y \end{bmatrix}^\top \end{aligned} \tag{5.5}$$

where $\alpha_i(s) \triangleq \begin{bmatrix} s^i & s^{i-1} & \ldots & 1 \end{bmatrix}^\top$, s denotes the Laplace operator. Equation (5.3) can be expressed in the following compact form

$$y^{(n)} = \theta^{*\top} Y. \tag{5.6}$$

Equation (5.3) is represented linear in the parameter vector θ^*. The linear representation of (5.6) is crucial for designing parameter estimation algorithm for θ^* from $y^{(n)}$ and the signal vector Y. However in most applications, only the input signal u and output y of the system are available, and computation of the time derivatives of u

and y is not desirable. $y^{(n)}$ and the time derivative signals in the signal vector Y should be avoided. The common approach to solve this problem is to filter both sides of (5.6) with an n-th order stable filter $\frac{1}{\Lambda(s)}$, which results in

$$z = \theta^{*\top} \phi \tag{5.7}$$

where

$$\begin{aligned} z &= \frac{s^n}{\Lambda(s)} y, \\ \phi &= \begin{bmatrix} \frac{\alpha_{n-1}^\top(s)}{\Lambda(s)} u & -\frac{\alpha_{n-1}^\top(s)}{\Lambda(s)} y \end{bmatrix}^\top, \\ \Lambda(s) &= s^n + \lambda_{n-1} s^{n-1} + \ldots + \lambda_0. \end{aligned}$$

$\Lambda(s)$ is an arbitrary Hurwitz polynomial in s, and it can be expressed as

$$\Lambda(s) = s^n + \lambda^\top \alpha_{n-1}(s) \tag{5.8}$$

where $\lambda = \begin{bmatrix} \lambda_{n-1} & \lambda_{n-2} & \ldots & \lambda_0 \end{bmatrix}^\top$. Using (5.8) we have

$$z = \frac{s^n}{\Lambda(s)} y = \frac{\Lambda(s) - \lambda^\top \alpha_{n-1}(s)}{\Lambda(s)} y = y - \lambda^\top \frac{\alpha_{n-1}(s)}{\Lambda(s)} y. \tag{5.9}$$

Hence,

$$y = z + \lambda^\top \frac{\alpha_{n-1}(s)}{\Lambda(s)} y. \tag{5.10}$$

Because $z = \theta^{*\top} \phi = \theta_1^{*\top} \phi_1 + \theta_2^{*\top} \phi_2$ where

$$\begin{aligned} \theta_1^* &= \begin{bmatrix} b_{n-1} & b_{n-2} & \ldots & b_0 \end{bmatrix}^\top, \\ \theta_2^* &= \begin{bmatrix} a_{n-1} & a_{n-2} & \ldots & a_0 \end{bmatrix}^\top, \\ \phi_1 &= \frac{\alpha_{n-1}(s)}{\Lambda(s)} u, \\ \phi_2 &= -\frac{\alpha_{n-1}(s)}{\Lambda(s)} y, \end{aligned}$$

it follows that

$$y = \theta_\lambda^{*\top} \phi \tag{5.11}$$

where $\theta_\lambda^* = \begin{bmatrix} \theta_1^{*\top} & \theta_2^{*\top} - \lambda^\top \end{bmatrix}^\top$ and $\phi = \begin{bmatrix} \phi_1^\top & \phi_2^\top \end{bmatrix}^\top$.

The state-space representation for generating (5.7) and (5.11) may be obtained by using the identity

$$\mathrm{adj}(sI - \Lambda_c))]l = \alpha_{n-1}(s) \tag{5.12}$$

where adj$((\cdot))$ denotes the adjoint matrix of a matrix, and

$$\Lambda_c = \begin{bmatrix} -\lambda_{n-1} & -\lambda_{n-2} & \dots & -\lambda_0 \\ 1 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & 0 \end{bmatrix}, \; l = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{5.13}$$

which implies that

$$\det(sI - \Lambda_c) = \Lambda(s), \quad (sI - \Lambda_c)^{-1} l = \frac{\alpha_{n-1}(s)}{\Lambda(s)} \tag{5.14}$$

where det$(\cdot)$ denotes the determinant of a matrix. Therefore, the following implementation can be applied to generate the signal vector without using the time derivatives of the input and output of the system

$$\dot{\phi}_1 = \Lambda_c \phi_1 + lu, \; \phi_1 \in \mathbb{R}^n, \tag{5.15a}$$

$$\dot{\phi}_2 = \Lambda_c \phi_2 - ly, \; \phi_2 \in \mathbb{R}^n, \tag{5.15b}$$

$$y = \theta_\lambda^{*\top} \phi, \tag{5.15c}$$

$$z = \theta^{*\top} \phi. \tag{5.15d}$$

Notice that the vector ϕ is generated from the filtered signals of the system input u and system output y; the differentiations of the input and output signal of the system is avoided in the representation given by (5.15), which is important in practical applications because the differentiation of a signal usually results in a very noisy one. Hence, the linear parametric model given by (5.15c) or (5.15d) contains only the information from the input and output of the system. As stated in [33], (5.15) only considers the case when the initial state in (5.3) is zero. If the initial state of (5.3) is not zero, a minor modification on (5.15) can be made. Since Λ_c is a stable matrix, the effect of the nonzero initial state will converge to zero exponentially.

Linear parametric model of time-varying systems

In this section, the parameterization of the linear system described by ordinary differential equations with time-varying parameters is considered. The time-varying linear system is characterized by the following equation

$$y^{(n)} + a_{n-1}(t)y^{(n-1)} + \ldots + a_0(t)y = b_{n-1}(t)u^{(n-1)} + b_{n-2}(t)u^{(n-2)} + \ldots + b_0(t)u \quad (5.16)$$

where $a_i(t)$ and $b_i(t)$, $i = 0, \ldots, (n-1)$, are time-varying parameters.

By defining the following differential operators

$$\begin{aligned} s &\triangleq \frac{d}{dt}, \\ s^i &\triangleq \frac{d^i}{dt^i}, \\ L_f &\triangleq f_n(t)s^n + f_{n-1}(t)s^{n-1} + \ldots + f_0(t) \end{aligned}$$

where $f_i(t)$ is an arbitrary scalar time-varying parameter for each i, the system characterized by (5.16) can be written in the compact form

$$L_a[y] = L_b[u] \quad (5.17)$$

where $L_a \triangleq s^n + a_{n-1}(t)s^{n-1} + \ldots + a_0(t)$ and $L_b \triangleq b_{n-1}(t)s^{n-1} + b_{n-2}(t)s^{n-2} + \ldots + d_0(t)$. We want to derive the linear parametric model for the system described by (5.17). Notice here s denotes the differential operator which differs from the notation used in the previous section. The linear parametric model is desired to have a similar form as that of the linear time invariant case, that is, it is modeled as follows:

$$y(t) = \theta_\lambda^{*\top}(t)\phi + \eta(t) \quad (5.18)$$

where $\theta_\lambda^{*\top}(t)$ is the parameter vector containing the time-varying parameters $a_i(t)$ and $b_i(t)$, ϕ is a vector whose element is generated from the system input and output without differentiations, and $\eta(t)$ is the modeling error term. Notice that different from the linear parametric model for the time invariant system shown in (5.11), there exists an additional term $\eta(t)$ in (5.18). It can be seen from the following analysis that $\eta(t)$ depends on the time derivatives of the parameters.

Denoting $L_\lambda \triangleq \Lambda(s)$ and rewriting (5.17) yields

$$\begin{aligned} L_\lambda[y] &= (L_\lambda - L_a)[y] + L_b[u] \\ &= (L_\lambda - L_a)L_\lambda L_\lambda^{-1}[y] + L_b L_\lambda L_\lambda^{-1}[u]. \end{aligned} \tag{5.19}$$

Multiplying both sides of (5.19) by L_λ^{-1} gives

$$y = L_\lambda^{-1}(L_\lambda - L_a)L_\lambda L_\lambda^{-1}[y] + L_\lambda^{-1}L_b L_\lambda L_\lambda^{-1}[u] \tag{5.20}$$

$$\begin{aligned} =&(L_\lambda - L_a)L_\lambda^{-1}y + L_b L_\lambda^{-1}[u] \\ &+ L_\lambda^{-1}[(L_\lambda - L_a)L_\lambda - L_\lambda(L_\lambda - L_a)]L_\lambda^{-1}[y] + L_\lambda^{-1}[L_b L_\lambda - L_\lambda L_b]L_\lambda^{-1}[u] \end{aligned} \tag{5.21}$$

$$\begin{aligned} =&(L_\lambda - L_a)L_\lambda^{-1}[y] + L_b L_\lambda^{-1}[u] \\ &+ L_\lambda^{-1}[L_\lambda L_a - L_a L_\lambda]L_\lambda^{-1}[y] + L_\lambda^{-1}[L_b L_\lambda - L_\lambda L_b]L_\lambda^{-1}[u] \end{aligned} \tag{5.22}$$

Let

$$\theta_\lambda^{*\top}(t)\phi(t) \triangleq (L_\lambda - L_a)L_\lambda^{-1}[y] + L_b L_\lambda^{-1}[u] \tag{5.23}$$

$$\eta(t) \triangleq L_\lambda^{-1}[L_\lambda L_a - L_a L_\lambda]L_\lambda^{-1}[y] + L_\lambda^{-1}[L_b L_\lambda - L_\lambda L_b]L_\lambda^{-1}[u], \tag{5.24}$$

equation (5.22) becomes

$$y = \theta_\lambda^{*\top}(t)\phi(t) + \eta(t) \tag{5.25}$$

where

$$\theta_\lambda^*(t) = \left[\begin{array}{cc} \theta_1^{*\top}(t) & \theta_2^{*\top}(t) - \lambda^\top \end{array} \right]^\top,$$

$$\theta_1^*(t) = \left[\begin{array}{cccc} b_{n-1}(t) & b_{n-2}(t) & \dots & b_0(t) \end{array} \right]^\top,$$

$$\theta_2^*(t) = \left[\begin{array}{cccc} a_{n-1}(t) & a_{n-2}(t) & \dots & a_0(t) \end{array} \right]^\top.$$

Clearly, if $\eta(t)$ in (5.25) is negligible, $y(t)$ can be generated by using the same implementation given in (5.15) as in the case of linear time invariant system.

From (5.24), it can be seen that the modeling error $\eta(t)$ is caused by the switching between L_λ and L_a, and the switching between L_λ and L_b. Generally, two differential systems with linear time-varying parameters cannot be commuted, that is,

$$L_{\eta_1} \stackrel{\triangle}{=} L_\lambda L_a - L_a L_\lambda \neq 0, \tag{5.26}$$

$$L_{\eta_2} \stackrel{\triangle}{=} L_b L_\lambda - L_\lambda L_b \neq 0. \tag{5.27}$$

Consequently the modeling error $\eta(t)$ will not be zero.

It is not easy to determine exactly how much contribution to the modeling error, $\eta(t)$, is coming from the control input $u(t)$, the Schwartz polynomial $\Lambda(s)$, and variations of $a_i(t)$ and $b_i(t)$, because it involves the manipulation of differential equations. In general, combining two differential equations to obtain a third differential equation, as one might wish to do if two systems characterized by known scalar-differential equations are considered, is not a trivial matter. In [129], the techniques for combining and rules of manipulation of differential equations are developed in the form of an algebra of linear transformations; Two basic operations: multiplication and addition, of two systems are considered. In following, the techniques and methods introduced in [129] are applied to derive the relationship between the modeling error $\eta(t)$ and the input $u(t)$, time-varying parameters $a_i(t)$ and $b_i(t)$, and the Schwartz polynomial $\Lambda(s)$.

It is possible to obtain $\eta(t)$ by using the two basic operations of the differential equations: multiplication and addition introduced in [129]. Figure 5.1 gives a step-by-step procedure to obtain the following differential equation to describe the I/O relationship between the input $u(t)$ and the modeling error $\eta(t)$:

$$L_A[\eta(t)] = L_B[u(t)]. \tag{5.28}$$

In Figure 5.1, diagram (a) describes the system in (5.24); diagram (b) explains L_{η_1} and L_{η_2} as given by (5.26) and (5.27), or the input/output properties given by (5.31) and (5.32) in the following, respectively. In (c), y is substituted in accordance with (5.17). In (d), $L_{a\lambda} = L_a L_\lambda$. In (e), L_{α_1} and L_{δ_1} come from the cascading of two differential equations and are determined by the following equality:

$$L_{\alpha_1} L_{\eta_1} = L_{\delta_1} L_{a\lambda}. \tag{5.29}$$

L_{α_2} and L_{δ_2} also come from the cascading of two differential equations and are determined by the following equality:

$$L_{\alpha_2} L_{\eta_2} = L_{\delta_2} L_{\lambda}. \tag{5.30}$$

In (f), $L_{b'} = L_{\delta_1} L_b$, $L_{a'} = L_{\alpha_1} L_\lambda$, $L_{d'} = L_{\delta_1}$ and $L_{c'} = L_{\alpha_2} L_\lambda$. The system in (f) results from the addition of two systems, whose step-by-step procedure is shown in Figure 5.2.

L_{η_1} and L_{η_2} can be derived as follows. As

$$L_\lambda = \sum_{i=0}^{n} \lambda_i \frac{d^i}{dt^i},$$
$$L_a = \sum_{i=0}^{n} a_i(t) \frac{d^i}{dt^i},$$
$$L_b = \sum_{i=0}^{n-1} b_i(t) \frac{d^i}{dt^i}$$

where $a_n = 1$ and $\lambda_n = 1$, the system characterized by the differential operator L_{η_1} can be expanded as follows:

$$
\begin{aligned}
w_y(t) =& L_{\eta_1}[v_y(t)] \\
=& L_\lambda L_a[v_y(t)] - L_a L_\lambda[v_y(t)] \\
=& \sum_{i=0}^{n} \lambda_i \frac{d^i}{dt^i}\left[\sum_{j=0}^{n} a_j(t)\frac{d^j v_y(t)}{dt^j}\right] - \sum_{i=0}^{n} a_i(t)\frac{d^i}{dt^i}\left[\sum_{j=0}^{n} \lambda_j \frac{d^j v_y(t)}{dt^j}\right] \\
=& \sum_{i=0}^{n}\sum_{j=0}^{n}\sum_{k=0}^{i}\begin{pmatrix} i \\ k \end{pmatrix} \lambda_i \frac{d^{i-k}a_j(t)}{dt^{i-k}}\frac{d^{j+k}v_y(t)}{dt^{j+k}} - \sum_{i=0}^{n}\sum_{j=0}^{n}\sum_{k=0}^{i}\begin{pmatrix} i \\ k \end{pmatrix} a_i(t)\frac{d^{i-k}\lambda_j}{dt^{i-k}}\frac{d^{j+k}v_y(t)}{dt^{j+k}} \\
=& \sum_{i=0}^{n}\sum_{j=0}^{n}\sum_{k=0}^{i-1}\begin{pmatrix} i \\ k \end{pmatrix} \lambda_i \frac{d^{i-k}a_j(t)}{dt^{i-k}}\frac{d^{j+k}v_y(t)}{dt^{j+k}} + \sum_{i=0}^{n}\sum_{j=0}^{n}\lambda_i a_i(t)\frac{d^{j+i}v_y(t)}{dt^{j+i}} \\
&- \sum_{i=0}^{n}\sum_{j=0}^{n} a_i(t)\lambda_j \frac{d^{j+i}v_y(t)}{dt^{j+i}} \\
=& \sum_{i=0}^{n}\sum_{j=0}^{n}\sum_{k=0}^{i-1}\begin{pmatrix} i \\ k \end{pmatrix} \lambda_i \frac{d^{i-k}a_j(t)}{dt^{i-k}}\frac{d^{j+k}v_y(t)}{dt^{j+k}} \qquad (5.31)
\end{aligned}
$$

where $v_y(t)$ and $w_y(t)$ represent the input and the output of the system described by L_{η_1}, respectively. Using the same procedure as in (5.31), the system characterized by the differential operator L_{η_2} can be obtained as

$$
\begin{aligned}
w_u(t) &= L_{\eta_2}[v_u(t)] \\
&= L_b L_\lambda[v_u(t)] - L_\lambda L_b[v_u(t)] \\
&= -\sum_{i=0}^{n}\sum_{j=0}^{n}\sum_{k=0}^{i-1}\begin{pmatrix} i \\ k \end{pmatrix} \lambda_i \frac{d^{i-k}b_j(t)}{dt^{i-k}}\frac{d^{j+k}v_u(t)}{dt^{j+k}}. \qquad (5.32)
\end{aligned}
$$

From (5.31) and (5.32), we have the following.

Remark 5.4 *The time derivatives of the parameters, from the first order to the $(n-1)$-th order time derivatives of $a_i(t)$ and $b_i(t)$, $i = 0 : n$, appear in the differential operators L_{η_1} and L_{η_2}. In other words, the modeling error $\eta(t)$ depends on the time* *derivatives of parameters. Notice that, the modeling error does not depend on the magnitude of the parameters, but depends on the variations of the parameters.*

Remark 5.5 *If these time derivatives are small enough, the modeling error can be negligible. The system in such a case is usually called a "slow time-varying system".*

Remark 5.6 *If the parameters are constant, L_{η_1} and L_{η_2} will become zero which in turn means that $\eta(t) = 0$, that is, there is no modeling error. Actually, the system is linear time invariant system in this case.*

Remark 5.7 *The Schwartz polynomial $\Lambda(s)$ also affects the modeling error $\eta(t)$.*

5.1.2 Representation of time-varying parameters

To represent a time-varying parameter, consider the following result [130]:

Lemma 5.1 *Let I be an open interval in $\mathbb{R}$, and f be a p-times continuously differentiable function of I into $\mathbb{R}$; then, for any pair of points t_0, t in I*

$$f(t) = f(t_0) + \frac{(t-t_0)}{1!}f^{(1)}(t_0) + \ldots + \frac{(t-t_0)^p}{p!}f^{(p)}(t_0) + \int_{t_0}^{t} \frac{(t-\xi)^{p+1}}{(p+1)!}f^{(p+1)}(\xi)d\xi \tag{5.33}$$

where $f^{(i)}(\cdot)$ denotes the i-th derivative of the function $f(\cdot)$.

As a result of Lemma 5.1, the time-varying function can be approximated locally at t_0 as a polynomial of time with constant coefficients, that is,

$$\begin{aligned} f(t) &= a_0(t_0) + a_1(t_0)(t-t_0) + \ldots + a_p(t_0)(t-t_0)^p, \quad t \in [t_0, t_0+T) \\ &= \sum_{i=0}^{p} a_i(t_0)(t-t_0)^i \end{aligned} \tag{5.34}$$

where $a_i(t_0) \triangleq \frac{1}{i!}f^{(i)}(t_0)$, $i = 0, \ldots, p$, $f^{(i)}(t_0)$ is the i-th time derivative evaluated at $t = t_0$, and T is the window length that can be chosen. Assuming the window is

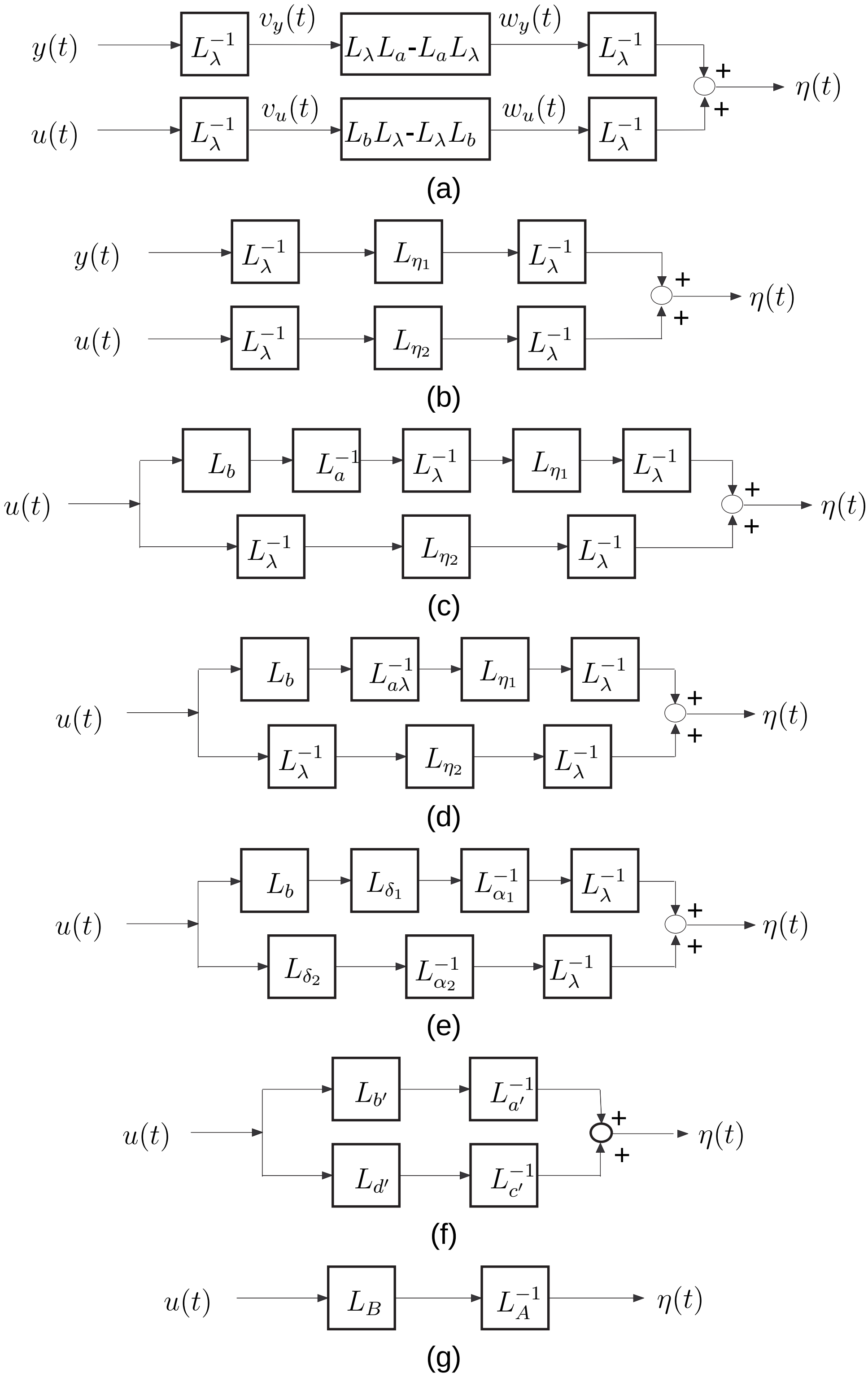

Figure 5.1: Procedure for the derivation of the modeling error $\eta(t)$ given by (5.24).

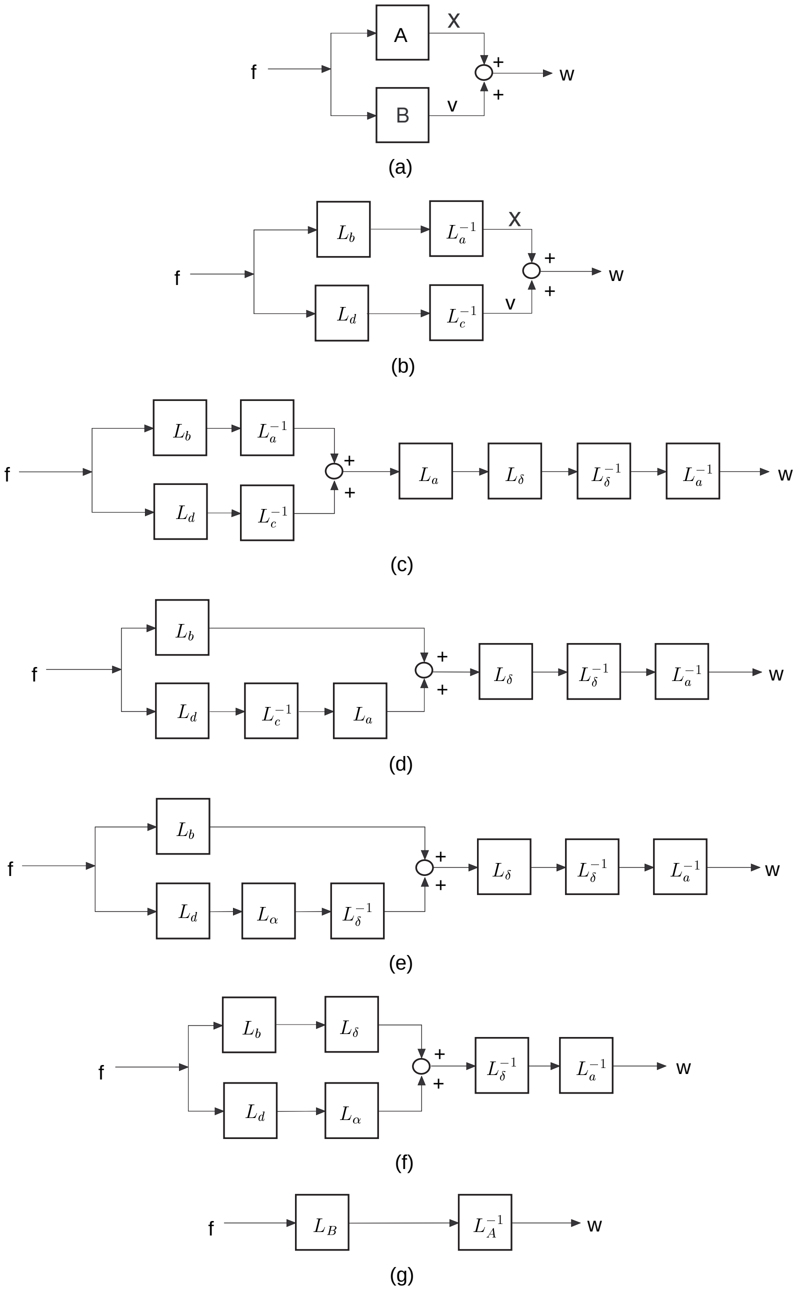

Figure 5.2: Addition of two differential equations. The system A is described by $L_a[x] = L_b[f]$ with $f(t)$ being the input and $x(t)$ the output, and the system B is described by $L_c[v] = L_d[f]$ with $f(t)$ being the input and $v(t)$ the output. The resultant system is described by $L_A[w] = L_B[f]$ with $f(t)$ being the input and $w(t) = x(t) + v(t)$ being the output.

sufficiently small, the last term of (5.33) is neglibible; that is,

$$\delta \triangleq \int_{t_0}^{t} \frac{(t-\xi)^{p+1}}{(p+1)!} f^{(p+1)}(\xi) d\xi$$

is negligible. Suppose that the $(p+1)$-th derivative of $f(t)$ is bounded, that is, $\sup_t \|f^{(p+1)}(t)\| \leq c_p$, then δ can be bounded by

$$|\delta| \leq \frac{c_p (t-t_0)^{p+1}}{(p+1)!} \leq \frac{c_p T^{p+1}}{(p+1)!}, \quad t \in [t_0, t_0 + T). \tag{5.35}$$

The time derivative of δ can be obtained by using Leibnitz rule[1] of differentiating an integral with variable limits, and is given by

$$\dot{\delta} = \int_{t_0}^{t} \frac{(t-\xi)^{p-2}}{(p-2)!} f^{(p)}(\xi) d\xi. \tag{5.36}$$

With the knowledge of the bound on $f^{(p)}(t)$, one can obtain a bound on $\dot{\delta}_f(t, t_0)$ as

$$|\dot{\delta}| \leq \frac{c_p (t-t_0)^{p-1}}{(p-1)!} \leq \frac{c_p T^{p-1}}{(p-1)!}, \quad t \in [t_0, t_0 + T). \tag{5.37}$$

Therefore, it is possible to use (5.34) to approximate $f(t)$ closely by choosing either a higher order polynomial, that is, p large, or a small interval T such that $t - t_0 \leq T$, or both. If we choose t_0 as a nondecreasing sequence of time instants with each difference between adjacent t_0 not more than T, in other words, partition time into segments with the length of each segment not larger than T, then the time-varying function $f(t)$ can be approximated by a number of polynomials of time locally at each t_0 with constant coefficients a_i; Figure 5.3 illustrates the idea, where $f_i(t), i = 0, 1, \ldots,$ locally represents the function $f(t)$ by a polynomial in the i-th window. In general, the coefficients a_i between two intervals are different.

[1] Leibnitz rule

$$\frac{d}{dt}\left[\int_{\theta(t)}^{\psi(t)} f(x,t) dx\right] = \int_{\theta(t)}^{\psi(t)} \frac{\partial f(x,t)}{\partial t} dx - \frac{d\theta(t)}{dt} f(\theta(t), t) + \frac{d\psi(t)}{dt} f(\psi(t), t)$$

The function $f(t)$ can also be approximated locally at $t_r \neq t_0$ by

$$
\begin{aligned}
f(t) &= a_0(t_r) + a_1(t_r)(t - t_r) + \ldots + a_p(t_r)(t - t_r)^p \\
&= \sum_{i=0}^{p} a_i(t_r)(t - t_r)^i
\end{aligned}
\tag{5.38}
$$

where $a_i(t_r) \triangleq \frac{1}{i!} f^{(i)}(t_r)$. To express each $a_j(t_r)$ in terms of $a_i(t_0), i = 0, \ldots, p$, evaluate the j-th derivative of (5.38) and (5.34) at $t = t_r$; notice that one can do this under the assumption that $t_r - t_0 \leq T$. The j-th derivative of (5.34) and (5.38) are:

$$
f^{(j)}(t) = \sum_{i=0}^{p} a_i(t_0) \frac{i!}{(i-j)!} (t - t_0)^{i-j}, \tag{5.39}
$$

$$
f^{(j)}(t) = \sum_{i=0}^{p} a_i(t_r) \frac{i!}{(i-j)!} (t - t_r)^{i-j}. \tag{5.40}
$$

Evaluating (5.39) and (5.40) at $t = t_r$, we obtain

$$
a_j(t_r) = \sum_{i=0}^{p} a_i(t_0) \frac{i!}{j!(i-j)!} (t - t_r)^{i-j}. \tag{5.41}
$$

Therefore, the relationship between $a_j(t_r), j = 0, \ldots, p$, and $a_i(t_0), i = 0, \ldots, p$, is given by

$$
\begin{bmatrix} a_0(t_r) \\ a_1(t_r) \\ \vdots \\ a_p(t_r) \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & t_r - t_0 & \ldots & (t_r - t_0)^p \\ 0 & 1 & \ldots & p(t_r - t_0)^{p-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{bmatrix}}_{A(t_r, t_0)} \begin{bmatrix} a_0(t_0) \\ a_1(t_0) \\ \vdots \\ a_p(t_0) \end{bmatrix}. \tag{5.42}
$$

5.1.3 Local polynomial approximation model

Applying the local polynomial approximation to each element of the time-varying parameter vector $\theta^*(t)$ locally at t_0, that is,

$$
\begin{aligned}
\theta_i^*(t) &= \theta_{i0} + \theta_{i1}(t - t_0) + \ldots + \theta_{ip}(t - t_0)^p \\
&= \theta_i^\top(t_0) L(t, t_0)
\end{aligned}
\tag{5.43}
$$

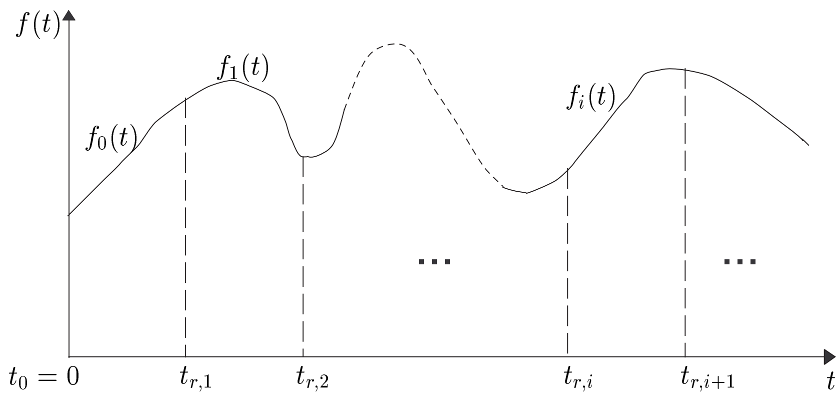

Figure 5.3: Local polynomial approximation of the continuous function $f(t)$.

where $\theta_i(t_0) \triangleq \begin{bmatrix} \theta_{i0}(t_0) & \theta_{i1}(t_0) & \dots & \theta_{ip}(t_0) \end{bmatrix}^\top$ is the unknown constant vector and $L(t, t_0) \triangleq \begin{bmatrix} 1 & (t - t_0) & \dots & (t - t_0)^p \end{bmatrix}^\top$ is a column vector. Notice that $\theta_i^*(t)$ is the original time-varying parameter that is being approximated by the time-polynomial with coefficients $\theta_{i0}, \theta_{i1}, \dots, \theta_{ip}$. If $t_{r,i}$ is defined as the time instant at which the i-th window of the local polynomial approximation begins, then t_0 is given by the sequence $t_0 = \{t_{r,i}\}$ with $i = 0, 1, \dots,$ and $t_{r,i+1} - t_{r,i} = T$. In the following $t_{r,i}$ is referred to as the resetting time, which is the beginning of the i-th window of the local polynomial approximation. Notice that $\theta_i(t_0)$ is constant only within each interval $[t_{r,i},\ t_{r,i+1})$ and in general differs from one interval from another for a time-varying parameter. The parameter vectors at two adjacent resetting times, that is, $\theta_i(t_{r,i})$ and $\theta_i(t_{r,i+1})$, are related by (5.42). The polynomial order p can be chosen for different $\theta_i^*(t)$ based on some a priori knowledge; for convenience, p is chosen to be the same for all the time-varying parameters. Therefore, the original parameter vector $\theta^*(t)$ is related to the approximation polynomial coefficient vector $\theta(t_0)$ by

$$
\begin{aligned}
\theta^*(t) &= \begin{bmatrix} 1 \ \ldots \ (t-t_0)^p & & & \\ & 1 \ \ldots \ (t-t_0)^p & & \\ & & \ddots & \\ & & & 1 \ \ldots \ (t-t_0)^p \end{bmatrix} \theta(t_0) \\
&= \begin{bmatrix} L^\top(t,t_0) & & & \\ & L^\top(t,t_0) & & \\ & & \ddots & \\ & & & L^\top(t,t_0) \end{bmatrix} \theta(t_0) \qquad (5.44) \\
&\triangleq \Lambda(t,t_0)\theta(t_0)
\end{aligned}
$$

where $\Lambda(t, t_0)$ is an $m \times m(p+1)$ matrix. Equation (5.1) and the resetting times can be written as

$$z(t) = \theta^{*\top}(t)\phi(t) = \theta^\top(t_0)\Psi(t,t_0), \qquad (5.45a)$$

$$t_0 = \{t_{r,i}\}, \quad i = 0, 1, 2, \ldots, \qquad (5.45b)$$

where $\Psi(t, t_0) = \Lambda^\top(t, t_0)\Phi(t)$. As $\theta(t_0)$ is now a piecewise constant vector, the problem of estimating the time-varying parameter in (5.1) can be transformed to that of estimating the constant parameter in (5.45a) based on the observations within each interval $[t_0, t_0 + T)$. Consequently, various estimation algorithms designed for estimating constant parameters may be employed with appropriate modifications.

By using (5.42), $\theta(t_{r,i+1})$ and $\theta(t_{r,i})$ are related by the following equation:

$$
\begin{aligned}
\theta(t_{r,i+1}) &= \begin{bmatrix} A(t_{r,i+1},t_{r,i}) & & & \\ & A(t_{r,i+1},t_{r,i}) & & \\ & & \ddots & \\ & & & A(t_{r,i+1},t_{r,i}) \end{bmatrix} \theta(t_{r,i}) \qquad (5.46) \\
&\triangleq B(t_{r,i+1},t_{r,i})\theta(t_{r,i}).
\end{aligned}
$$

Notice that $\theta(t_{r,i})$ is constant in the i-th interval, that is, $\theta(\tau) = \theta(t_{r,i})$ for all $\tau \in [t_{r,i}, t^-_{r,i+1}]$. Equation (5.46) will form the basis for resetting the initial value of the estimate at the beginning of each interval, and equation (5.45a) will be used to identify the constant coefficients of the polynomial in each time interval. In the next section, the least-squares and the gradient algorithms are modified to estimate the time-varying parameter vector by introducing a resetting scheme at the beginning of each interval; the resetting scheme ensures that the estimate of the time-varying parameter vector, $\widehat{\theta}^*(t)$, is continuous consistent with the assumptions on the true time-varying parameters. Stability properties of each identification algorithm with the proposed resetting scheme is shown and discussed.

5.1.4 Time-varying parameter estimation algorithms

Based on the local polynomial approximation described in the previous section, modified versions of the two classical algorithms, the least-squares and the gradient algorithms, are given for identification of time-varying parameters.

Modified least-squares with covariance resetting

Least-squares algorithm with covariance resetting has been widely used for estimating an unknown constant parameter vector, β, for the following model:

$$y(t) = \beta^\top \phi(t) \tag{5.47}$$

where $\phi(t)$ is a known signal vector. The estimate of β, $\widehat{\beta}$, is given by minimizing the following integral cost function

$$J(\beta) = \frac{1}{2}\int_0^t (y(\tau) - \beta^\top(t)\phi(\tau))^2 d\tau, \tag{5.48}$$

that is, $\widehat{\beta} = \arg\min_{\beta} J(\beta)$. The minimization of (5.48) with covariance resetting results in the following estimation algorithm [33]:

$$\dot{\widehat{\beta}} = \frac{P\phi(y - \widehat{\beta}^T\phi)}{1 + \phi^\top\phi} \tag{5.49a}$$

$$\dot{P} = -\frac{P\phi\phi^\top P}{1 + \phi^\top\phi}, \quad P(t_{cr}^+) = \rho_0 I \tag{5.49b}$$

where t_{cr} is the time at which $\lambda_{min}(P(t)) \leq \rho_1$, $\lambda_{min}(\cdot)$ denotes the smallest eigenvalue of a matrix, I is the identity matrix, and $\rho_0 > \rho_1 > 0$ are some design scalars.

In the following, a modified version of the above algorithm where, in addition to the covariance resetting, the initial value of the estimate is reset at the beginning of each time window of the local polynomial approximation. For the time-varying model given by (5.45a), choose the cost function as follows:

$$J(\widehat{\theta}) = \frac{1}{2}\int_0^t \epsilon^2(t,\ \tau)m^2(\tau)d\tau \tag{5.50}$$

where

$$\epsilon(t,\ \tau) = \frac{z(\tau) - \widehat{\theta}^\top(t)\Psi(\tau)}{m^2(\tau)}$$

is the normalized estimation error, $m^2(\tau) = 1 + n_s^2$ and $n_s^2 = \Psi^\top(\tau)\Psi(\tau)$. The adaptation law is chosen as follows:

$$\dot{\widehat{\theta}}(t) = P(t)\epsilon(t,t)\Psi(t) \tag{5.51a}$$

$$\dot{P}(t) = -\frac{P\Psi\Psi^\top P}{m^2}, \quad P(t_0) = \rho_0 I \tag{5.51b}$$

where $\widehat{\theta}(t)$ is the estimate of $\theta(t_0)$. Further, the covariance matrix is reset as follows:

$$P(t) = \rho_0 I, \quad \text{if} \quad \lambda_{min}(P(t)) \leq \rho_1. \tag{5.52}$$

Equation (5.52) ensures that the covariance matrix is does not get too close to singularity, that is, the covariance matrix is reset within each time window if its minimum eigenvalue becomes less than ρ_1. At the beginning of each window the initial value of the estimate is reset according to the following equation:

$$\widehat{\theta}(t_{r,\ i+1}) = B(t_{r,i+1}, t_{r,i})\widehat{\theta}(t_{r,\ i+1}^-). \tag{5.53}$$

The motivation for resetting the initial value at the beginning of each interval by (5.53) is the following: Since the true parameter $\theta_i^*(t)$ is continuous, the estimate, $\widehat{\theta}_i^*(t)$, should also be continuous; (5.53) guarantees this at the resetting points $t_{r,i}$. The following shows the continuity of $\widehat{\theta}^*(t)$ at the resetting point. Just before resetting for the $(i+1)$-th interval, using (5.44) for the estimate, we obtain

$$\widehat{\theta}^*(t_{r,i+1}^-) = \Lambda(t_{r,i+1}^-, t_{r,i})\widehat{\theta}(t_{r,i+1}^-) \tag{5.54}$$

At the resetting point, again using (5.44) for the estimate with $t_0 = t_{r,i+1}$,

$$\begin{aligned}\widehat{\theta}^*(t_{r,i+1}) &= \Lambda(t_{r,i+1}, t_{r,i+1})\widehat{\theta}(t_{r,i+1}) \\ &= \Lambda(t_{r,i+1}, t_{r,i+1})B(t_{r,i+1}, t_{r,i})\widehat{\theta}(t_{r,i+1}^-)\end{aligned}$$

where (5.53) has been used to obtain the second equality. Notice that

$$\Lambda(t_{r,i+1}, t_{r,i+1}) = \operatorname{diag}(e_1^\top, e_1^\top, \ldots, e_1^\top)$$

where $e_1^\top = \begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix}$. Also, from the definition of $B(t_{r,i+1}, t_{r,i})$ given by (5.46), we obtain

$$\begin{aligned}\Lambda(t_{r,i+1}, t_{r,i+1})B(t_{r,i+1}, t_{r,i}) &= \operatorname{diag}\big(e_1^\top A(t_{r,i+1}, t_{r,i}), e_1^\top A(t_{r,i+1}, t_{r,i}), \ldots, e_1^\top A(t_{r,i+1}, t_{r,i})\big) \\ &= \Lambda(t_{r,i+1}, t_{r,i}).\end{aligned}$$

Therefore, (5.55) becomes

$$\widehat{\theta}^*(t_{r,i+1}) = \Lambda(t_{r,i+1}, t_{r,i})\widehat{\theta}(t_{r,i+1}^-) \tag{5.55}$$

From (5.54) and (5.55), since $\Lambda(t, t_0)$ is a continuous function of t, it can be seen that $\widehat{\theta}^*(t_{r,i+1}^-) = \widehat{\theta}^*(t_{r,i+1})$.

The following theorem gives the stability of the modified least-squares algorithm with covariance resetting for the time-varying model.

Theorem 5.1 *The least-squares algorithm given by (5.51), together with the covariance resetting given by (5.52) and the resetting of the estimate, $\widehat{\theta}(t)$, at the beginning of each interval, given by (5.53), has the following properties:*

(i) $\epsilon, \epsilon n_s, \widehat{\theta}, \dot{\widehat{\theta}} \in \mathcal{L}_\infty$.

(ii) $\epsilon, \epsilon n_s, \dot{\widehat{\theta}} \in \mathcal{L}_2$.

(iii) If $n_s, \Psi \in \mathcal{L}_\infty$ *and* Ψ *satisfies the following persistence of excitation (PE) condition:*

$$\alpha_1 T_0 I \geq \int_t^{t+T_0} \Psi(\tau)\Psi^\top(\tau)d\tau \geq \alpha_0 T_0 I, \quad \forall t \geq 0 \quad \textit{and} \quad T_0 < T, \tag{5.56}$$

for some $0 < \alpha_0 \leq \alpha_1$*, then* $\widehat{\theta}(t)$ *converges exponentially to* $\theta(t_0)$.

(iv) The estimate of $\theta^*(t)$*,* $\widehat{\theta}^*(t)$*, is continuous and bounded. Furthermore, if* Ψ *satisfies the PE condition given in (iii), the estimation error* $\widetilde{\theta}^*(t)$ *exponentially converges to zero within each time interval.*

Proof. Consider the Lyapunov function candidate

$$V(\widetilde{\theta}(t)) = \frac{\widetilde{\theta}^\top(t)P^{-1}(t)\widetilde{\theta}(t)}{2}. \tag{5.57}$$

It can be shown that, within an interval ($t \in [t_0, t_0 + T)$), the derivative of the Lyapunov function candidate satisfies:

$$\dot{V}(t) = -\frac{\epsilon^2 m^2}{2} \leq 0. \tag{5.58}$$

Thus, one can can arrive at (i), (ii) and (iii) of the theorem as given in [33]. Notice that in (iii) there is an additional constraint in the PE condition (5.56), that is, $T_0 < T$. This is necessary for the coefficients of the local polynomial approximation to converge to their true values within any interval. The following gives the proof of (iv).

The estimate of $\theta^*(t)$ for $t \in [t_0, t_0 + T)$, $\widehat{\theta}^*(t)$, is given by

$$\widehat{\theta}^*(t) = \Lambda(t, t_0)\widehat{\theta}(t). \tag{5.59}$$

Hence,

$$\|\widehat{\theta}^*(t)\| \leq \|\Lambda(t,t_0)\| \; \|\widehat{\theta}(t)\| \leq \sqrt{\lambda_{max}(\Lambda^\top(t,t_0)\Lambda(t,t_0))} \; \|\widehat{\theta}(t)\|$$
$$\leq \kappa(T)\|\widehat{\theta}(t)\| \tag{5.60}$$

where $\lambda_{max}(\Lambda^\top(t,t_0)\Lambda(t,t_0)) = 1+(t-t_0)^2+\ldots+(t-t_0)^{2p}$ is the maximum eigenvalue of $\Lambda^\top(t,t_0)\Lambda(t,t_0)$ and $\kappa(T) = \sqrt{1+T^2+\ldots+T^{2p}}$. The boundedness of $\widehat{\theta}^*(t)$ follows from the fact that $\widehat{\theta}(t)$ is bounded. Also, taking the time-derivative of (5.59), we obtain

$$\dot{\widehat{\theta}}^*(t) = \dot{\Lambda}(t,t_0)\widehat{\theta}(t) + \Lambda(t,t_0)\dot{\widehat{\theta}}(t). \tag{5.61}$$

$\dot{\widehat{\theta}}^*(t)$ is bounded within each time interval because $\dot{\Lambda}(t,t_0)$ and $\Lambda(t,t_0)$ are bounded within each time interval, and $\widehat{\theta}(t)$ and $\dot{\widehat{\theta}}(t)$ are bounded (from (i)). Hence, $\widehat{\theta}^*(t)$ is continuous within each time interval. Recall that the continuity of $\widehat{\theta}^*(t)$ at each resetting point is guaranteed by the resetting of the estimate at the beginning of each time interval according to (5.53). Therefore, it follows that $\widehat{\theta}^*(t)$ is uniformly continuous.

Subtracting (5.44) from (5.59) yields

$$\widetilde{\theta}^*(t) = \Lambda(t,t_0)\widetilde{\theta}(t). \tag{5.62}$$

Therefore, the estimation error, $\widetilde{\theta}^*(t)$, is bounded by

$$\|\widetilde{\theta}^*(t)\| \leq \kappa(T)\|\widetilde{\theta}(t)\|. \tag{5.63}$$

Recall that, from (iii), $\widetilde{\theta}(t)$ exponentially converges to zero, which implies that $\widetilde{\theta}^*(t)$ exponentially converges to zero within each interval.

Rate of convergence: In the following an estimate of the rate of convergence of the parameters is derived. The least-squares algorithm, (5.51), satisfies [33]:

$$\widetilde{\theta}(t) = P(t)P^{-1}(t_0)\widetilde{\theta}(t_0), \quad t \in [t_0, t_0+T) \tag{5.64}$$

and

$$P(t) \le \Big[(t - t_0 - T_0)\alpha_0\Big]^{-1} \bar{m} I, \quad \forall t \ge t_0 + T_0 \tag{5.65}$$

where $\bar{m} = \sup_t m^2(t)$. So, the worst case bound of $\widetilde{\theta}(t)$ is given by

$$\begin{aligned} \|\widetilde{\theta}(t)\| &\le \|P(t)\| \, \|P^{-1}(t_0)\| \, \|\widetilde{\theta}(t_0)\| \\ &\le \Big[\rho_0(t - t_0 - T_0)\alpha_0\Big]^{-1} \bar{m} \|\widetilde{\theta}(t_0)\|. \end{aligned} \tag{5.66}$$

At the end of the i-th interval, that is, $t = iT + T^-$, we have

$$\|\widetilde{\theta}(iT + T^-)\| \le \Big[\rho_0(T - T_0)\alpha_0\Big]^{-1} \bar{m} \|\widetilde{\theta}(iT)\|. \tag{5.67}$$

From (5.62), we have

$$\begin{aligned} \|\widetilde{\theta}^*(iT + T)\| &\le \kappa(T) \|\widetilde{\theta}(iT + T^-)\| \\ &\le \kappa(T) \Big[\rho_0(T - T_0)\alpha_0\Big]^{-1} \bar{m} \|\widetilde{\theta}(iT)\|. \end{aligned} \tag{5.68}$$

Notice that, from (5.67) and (5.68), faster convergence of the estimate of the time-varying parameter vector, $\widehat{\theta}^*(t)$, and the vector of coefficients of the polynomial, $\widehat{\theta}(t)$, within a time interval depends on how small T_0 is with respect to T. Further, it also depends on the persistency of excitation level of the signal vector $\Psi(t)$ (α_0) and ρ_0.

Modified gradient algorithm

The gradient algorithm for the time-varying model is given by

$$\dot{\widehat{\theta}}(t) = \Gamma \epsilon \Psi, \quad t \in [t_{r,i}, t_{r,i+1}), i = 0, 1, 2, \ldots, \tag{5.69a}$$

$$\widehat{\theta}(t_{r,\, i+1}) = B(t_{r,i+1}, t_{r,i}) \widehat{\theta}(t_{r,\, i+1}^-), \quad \text{if} \quad t = t_{r,i+1}, \tag{5.69b}$$

where Γ is a constant symmetric positive definite gain matrix. The stability of the modified gradient algorithm can be proved by using a similar procedure as that of the modified least-squares with covariance resetting algorithm with the following Lyapunov function candidate:

$$V(\widetilde{\theta}(t)) = \frac{1}{2} \widetilde{\theta}(t)^\top \Gamma^{-1} \widetilde{\theta}(t). \tag{5.70}$$

With the adaptation law (5.69a) and the resetting algorithm (5.69b), the modified gradient algorithm also has the same properties, $(i) - (iv)$, as given by Theorem 5.1. The stability analysis of the gradient algorithm is similar to that of the least-squares with covariance resetting and is omitted here.

5.1.5 Simulations

Example 5.1 *Consider the following first-order system given in [99]:*

$$z(t) = \theta_1^*(t)u_f(t) + \theta_2^*(t)z_f(t) + n(t) = \theta^{*\top}(t)\phi(t) + n(t)$$

where $\theta^*(t) = \begin{bmatrix} \theta_1^*(t) & \theta_2^*(t) \end{bmatrix}^\top$, $\phi(t) = \begin{bmatrix} u_f(t) & z_f(t) \end{bmatrix}^\top$, *and* $n(t)$ *is the noise introduced into the system. The filtered input and output signals,* $u_f(t)$ *and* $z_f(t)$*, are given by*

$$\dot{u}_f(t) = -300u_f(t) + 300u(t), \;\; \dot{z}_f(t) = -300z_f(t) + 300z(t)$$

where $u(t)$ *and* $z(t)$ *are the input and output of the plant, respectively. The input* $u(t)$ *is chosen to be a random signal with zero mean and a variance of 0.01.*

In the simulation, $\theta_1^*(t)$ is approximated by a sixth order polynomial of time, and $\theta_2^*(t)$ is approximated by first order polynomial of time. The following values are used in the simulations:

$$T = 0.1 \text{ seconds}, \quad \rho_0 = 2400, \quad \rho_1 = 0.005, \quad \Gamma = 2400I.$$

The following five sets of simulations are shown for different sets of time-varying parameters.

Figure 5.4 through Figure 5.8 show the results corresponding to the parameter sets 1 through 5 shown in the table. In each figure, (a) and (b) show the estimation results for the modified least-squares with covariance resetting and (c) and (d) show results for the modified gradient algorithm. Simulation results show that the estimate

Set	$\theta_1^*(t)$	$\theta_2^*(t)$	$n(t)$
1	$\sin(\pi t/10)$	0.5	0
2	$\sin(\pi t)$	0.5	0
3	$\sin(\pi t/10)$	0.5	$N(0, 0.01)$
4	$\sin(\pi t)$	0.5	$N(0, 0.01)$
5	$\sin(\pi t) + \sin(\pi t/5)$	0.5	$N(0, 0.01)$

Table 5.1: Parameters and noises used in simulations

converges to a small region around the true value for both the least-squares and the gradient algorithms.

Without the effect of noise in the model, that is, $n(t) = 0$, the the modified gradient algorithm gives better estimation of parameters than the modified least-squares algorithm. This is because the adaptation of the estimated parameters is driven by instantaneous output error in the gradient algorithm. However, the least-squares algorithm focuses on minimizing an integral function of the normalized output error. Consequently, the least-squares algorithm responds slower to parameter variations than the gradient algorithm. In the presence of noise, modified least-squares algorithm gives smoother estimates than the modified gradient algorithm. Simulation results also show that both algorithms provide stable estimation of time-varying parameters.

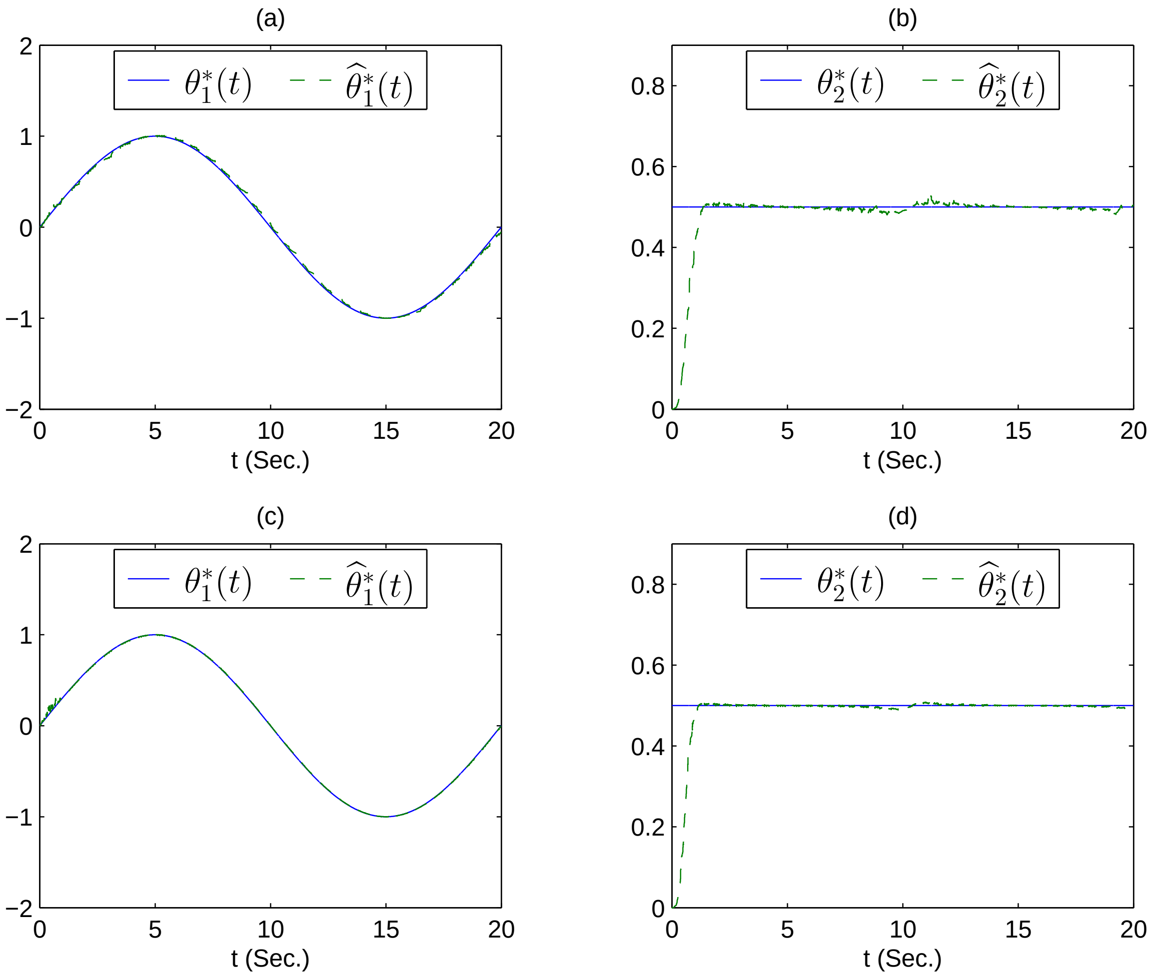

Figure 5.4: Simulation result of Example 5.1. The top two plots, (a) and (b), show the results by using modified least-squared algorithm, and the bottom two plots, (c) and (d), show the results by using the modified gradient algorithm. The true parameters are: $\theta_1^*(t) = \sin(\pi t/10)$ and $\theta_2^*(t) = 0.5$. $n(t) = 0$.

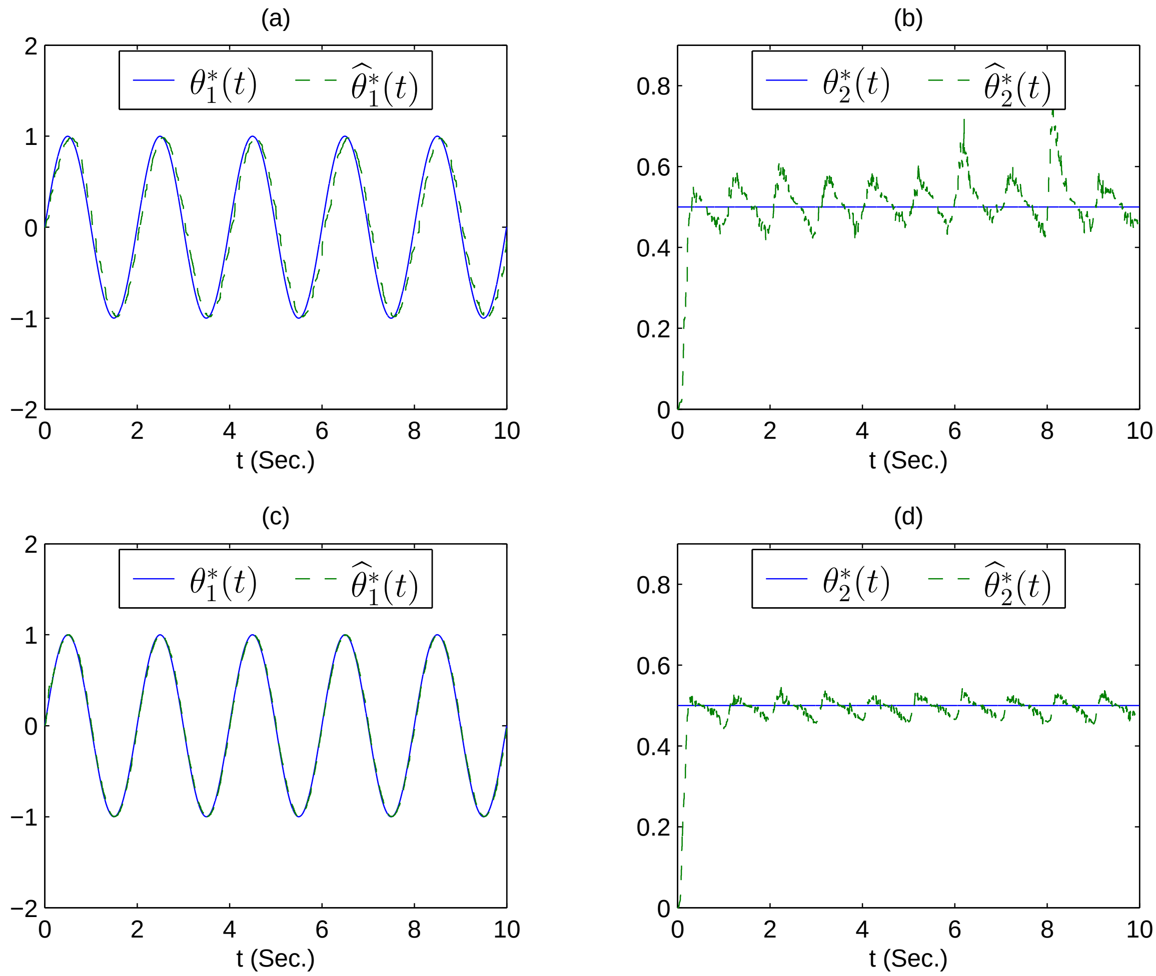

Figure 5.5: Simulation result of Example 5.1. The top two plots, (a) and (b), show the results by using modified least-squared algorithm, and the bottom two plots, (c) and (d), show the results by using the modified gradient algorithm. The true parameters are: $\theta_1^*(t) = \sin(\pi t)$ and $\theta_2^*(t) = 0.5$. $n(t) = 0$.

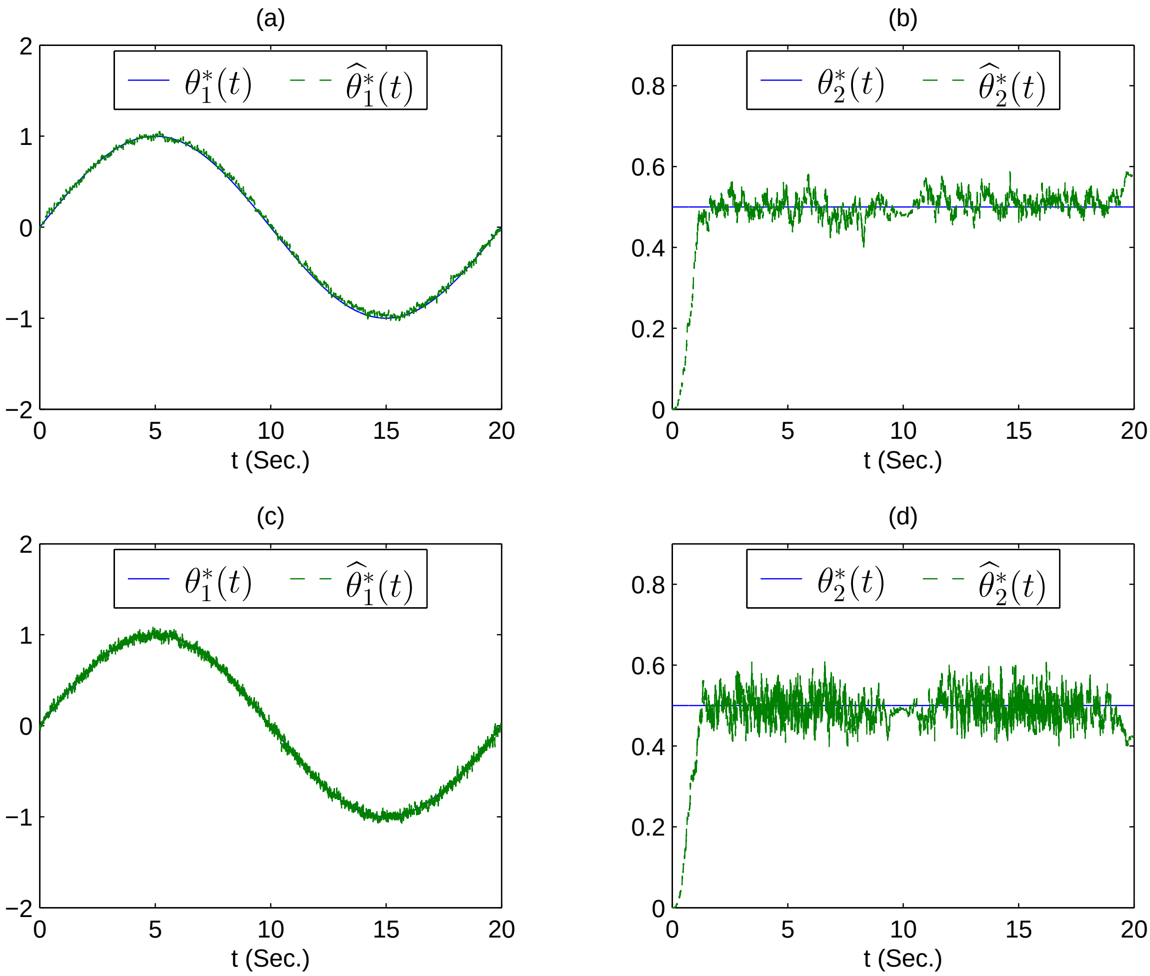

Figure 5.6: Simulation result of Example 5.1. The top two plots, (a) and (b), show the results by using modified least-squared algorithm, and the bottom two plots, (c) and (d), show the results by using the modified gradient algorithm. The true parameters are: $\theta_1^*(t) = \sin(\pi t/10)$ and $\theta_2^*(t) = 0.5$. $n(t) = N(0, 0.01)$.

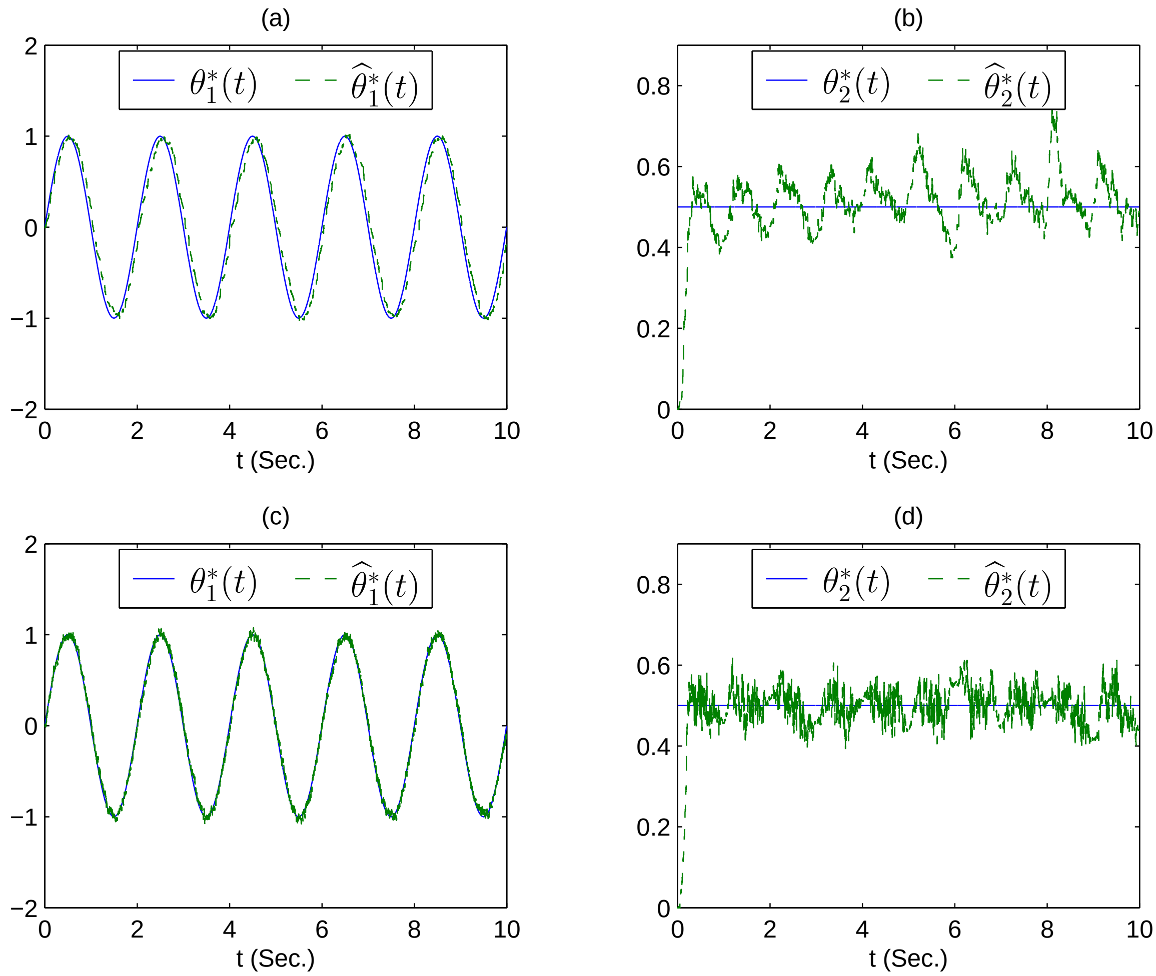

Figure 5.7: Simulation result of Example 5.1. The top two plots, (a) and (b), show the results by using modified least-squared algorithm, and the bottom two plots, (c) and (d), show the results by using the modified gradient algorithm. The true parameters are: $\theta_1^*(t) = \sin(\pi t)$ and $\theta_2^*(t) = 0.5$. $n(t) = N(0, 0.01)$.

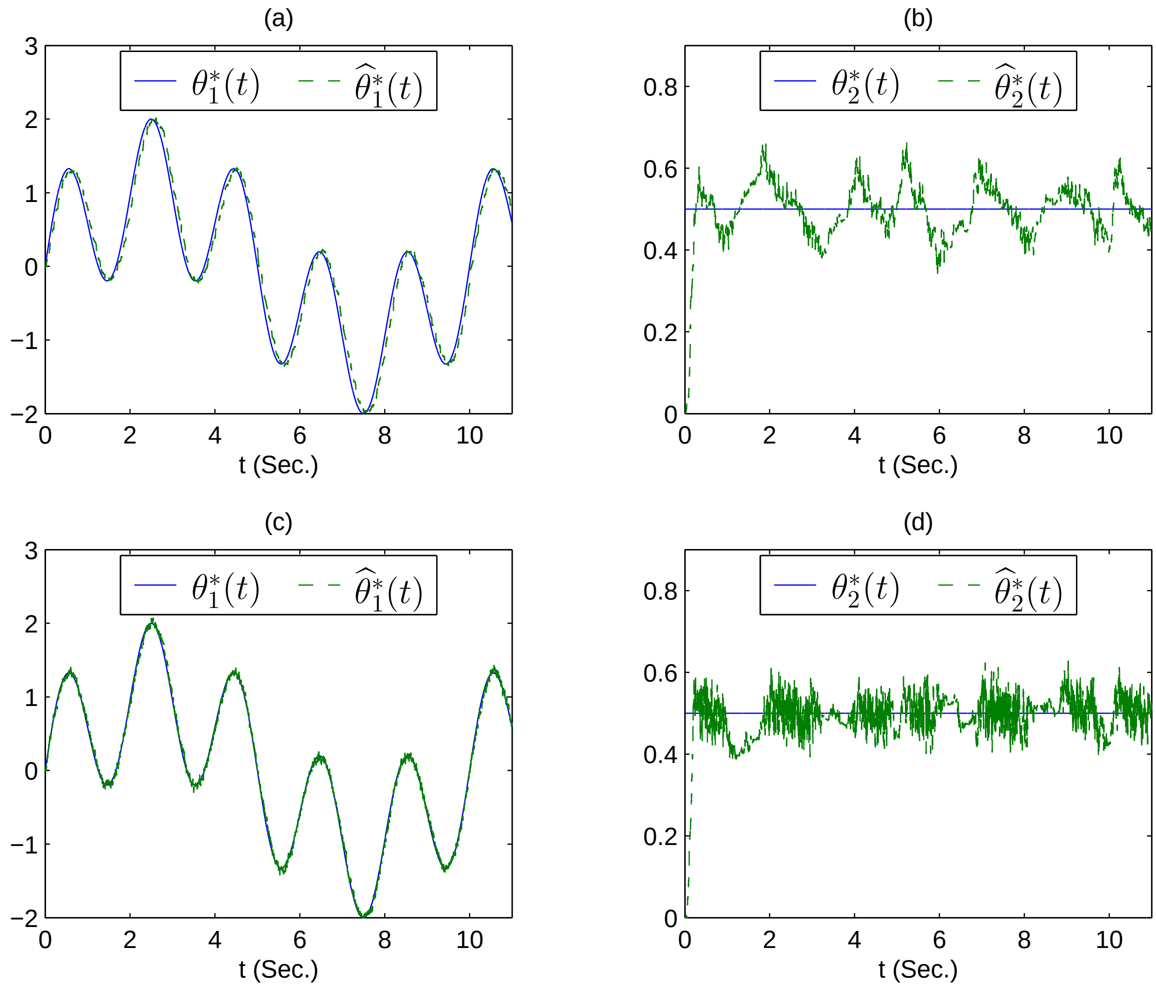

Figure 5.8: Simulation result of Example 5.1. The top two plots, (a) and (b), show the results by using modified least-squared algorithm, and the bottom two plots, (c) and (d), show the results by using the modified gradient algorithm. The true parameters are: $\theta_1^*(t) = \sin(\pi t) + \sin(\pi t/5)$ and $\theta_2^*(t) = 0.5$. $n(t) = N(0, 0.01)$.

5.2 Adaptive control of mechanical systems with time-varying parameters and disturbances

In this section, a new robust adaptive control algorithm for mechanical systems with time-varying parameters and/or time-varying disturbances is proposed and investigated. The proposed method does not assume any structure to the time-varying parameter or disturbance. The adaptive law for unknown time-varying parameters and time-varying disturbance is based on the modified gradient algorithm proposed in Section 5.1. A novel experiment is designed using a two-link mechanical manipulator to investigate the proposed algorithm experimentally. Simulation and experimental results are discussed. The development presented in this section is presented in the paper [104].

The contributions of the section can be summarized by the following: (1) Design of a stable adaptive controller for mechanical systems with time-varying parameters and disturbances using local polynomial approximations, and (2) experimental evaluation of the adaptive controller, and its comparison with an ideal non-adaptive controller.

5.2.1 Dynamics of mechanical systems with time-varying parameters and disturbances

The dynamics of an n degree-of-freedom mechanical system with time-varying parameters and disturbances [109] is given by

$$M(q,\theta^*)\ddot{q} + C(q,\dot{q},\theta^*)\dot{q} + F(q,\dot{\theta}^*)\dot{q} + g(q,\theta^*) = \tau + d(t) \tag{5.71}$$

where $q \in \mathbb{R}^n$ is the vector of generalized coordinates, $M(q,\theta^*) \in \mathbb{R}^{n\times n}$ is the inertia matrix, $C(q,\dot{q},\theta^*) \in \mathbb{R}^{n\times n}$ is the matrix composed of Coriolis and centrifugal terms, $g(q,\theta^*) \in \mathbb{R}^n$ is the gravity vector, $F(q,\dot{\theta}^*) \in \mathbb{R}^{n\times n}$ is a symmetric matrix, which is a consequence of the symmetry of the inertia matrix, $\theta^* \in \mathbb{R}^m$ is the vector of constant and/or time-varying parameters, $\tau \in \mathbb{R}^n$ is the vector of control inputs, and $d(t) \in \mathbb{R}^n$ is the vector of time-varying disturbances.

The properties of the dynamic model (5.71) are given in the following:

Property P5.1 *The inertia matrix, $M(q, \theta^*)$, of the time-varying mechanical system is a symmetric positive definite matrix. Assuming $\theta^*(t)$ is bounded, $M(q, \theta^*)$ is bounded from above and below for all system configurations.*

Property P5.2 *$F(q, \dot{\theta}^*)$ is a symmetric matrix, which is a consequence of the symmetry of the inertia matrix.*

Property P5.3 *The matrix $\dot{M}(q, \theta^*) - 2C(q, \dot{q}, \theta^*) - F(q, \dot{\theta}^*)$ is skew-symmetric. Notice that the skew-symmetry property for the time-varying case is different from that of the time-invariant case [131, 105, 109].*

Property P5.4 *The dynamic model, (5.71), is linear in the unknown parameters, $\theta^*, \dot{\theta}^*$, that is,*

$$M(q, \theta^*)\ddot{q} + C(q, \dot{q}, \theta^*)\dot{q} + F(q, \dot{\theta}^*)\dot{q} + g(q, \theta^*) = Y_1(q, \dot{q}, \ddot{q})\theta^* + Y_2(q, \dot{q})\dot{\theta}^* \tag{5.72}$$

where $Y_1(q, \dot{q}, \ddot{q})$ and $Y_2(q, \dot{q})$ are the regressor matrices corresponding to $\theta^*(t)$ and $\dot{\theta}^*(t)$, respectively.

5.2.2 Adaptive control design

Applying the local polynomial approximation method introduced in Chapter 5.1, we can represent each element of the time-varying parameter vector $\theta^*(t)$ locally at t_0 as follows

$$\theta_i^*(t) = L(t, t_0)\theta_i(t_0) + \delta_{\theta_i^*}(t, t_0) \tag{5.73}$$

where $\theta_i(t_0) \triangleq \begin{bmatrix} \theta_{i0}(t_0) & \theta_{i(p-1)}(t_0) & \ldots & \theta_{ip}(t_0) \end{bmatrix}^\top$ is the unknown constant vector and $L(t,t_0) \triangleq \begin{bmatrix} 1 & (t-t_0) & \ldots & (t-t_0)^{p-1} \end{bmatrix}^\top$ is a row vector. Notice that $\theta_i^*(t)$ is the original time-varying parameter that is being approximated by the time-polynomial with coefficients $\theta_{i0}, \theta_{i1}, \ldots, \theta_{i(p-1)}$. Therefore, the original parameter vector $\theta^*(t)$ is represented by the polynomial coefficient vector $\theta(t_0)$ plus residue vector $\delta_{\theta^*}(t,t_0)$ by

$$\begin{aligned} \theta^*(t,t_0) &= \begin{bmatrix} L(t,t_0) & & & \\ & L(t,t_0) & & \\ & & \ddots & \\ & & & L(t,t_0) \end{bmatrix} \theta(t_0) + \delta_{\theta^*}(t,t_0) \\ &\triangleq \Lambda(t,t_0)\theta(t_0) + \delta_{\theta^*}(t,t_0) \end{aligned} \tag{5.74}$$

where $\Lambda(t,t_0)$ is an $m \times mp$ matrix, $\theta(t_0) \triangleq \begin{bmatrix} \theta_1^\top(t_0) & \ldots & \theta_i^\top(t_0) & \ldots & \theta_m^\top(t_0) \end{bmatrix}^\top \in \mathbb{R}^{mp\times 1}$ and $\delta_{\theta^*}(t,t_0) \triangleq \begin{bmatrix} \delta_{\theta_1^*}(t,t_0) & \ldots & \delta_{\theta_i^*}(t,t_0) & \ldots & \delta_{\theta_m^*}(t,t_0) \end{bmatrix}^\top$ is the m-vector consisting of the residue from approximation of each parameter. The time derivative of $\theta^*(t)$ can be represented by

$$\dot{\theta}^*(t,t_0) = \dot{\Lambda}(t,t_0)\theta(t_0) + \dot{\delta}_{\theta^*}(t,t_0). \tag{5.75}$$

Since each component of the vectors $\delta_{\theta^*}(t,t_0)$ and $\dot{\delta}_{\theta^*}(t,t_0)$ is bounded, they are bounded vectors; assume that the bounds are given by

$$\|\delta_{\theta^*}(t,t_0)\| \le k_{\delta_{\theta^*}}, \quad \forall t \ge 0, \tag{5.76}$$

$$\|\dot{\delta}_{\theta^*}(t,t_0)\| \le k_{\dot{\delta}_{\theta^*}}, \quad \forall t \ge 0. \tag{5.77}$$

As $\theta(t_0)$ is now a piecewise constant vector, the problem of estimating the time-varying parameter $\theta^*(t)$ in the controller design for (5.71) can be transformed to that of estimating the constant parameter $\theta(t_0)$ in (5.74) based on the observations within each interval $[t_0, t_0 + T)$. Consequently, various estimation algorithms designed for estimating constant parameters may be employed with appropriate modifications. By using (5.42), $\theta(t_{r,i+1})$ and $\theta(t_{r,i})$ are related by the following equation:

$$\begin{aligned} \theta(t_{r,i+1}) = & \begin{bmatrix} A(t_{r,i+1}, t_{r,i}) & & & \\ & A(t_{r,i+1}, t_{r,i}) & & \\ & & \ddots & \\ & & & A(t_{r,i+1}, t_{r,i}) \end{bmatrix} \theta(t_{r,i}) \\ \triangleq & B(t_{r,i+1}, t_{r,i})\theta(t_{r,i}). \end{aligned} \tag{5.78}$$

In the following of this section, an adaptive control algorithm is proposed and its stability properties are investigated. A modified gradient projection algorithm given by (5.69) in Chapter 5.1 is used to estimate the time-varying parameter vector.

Consider the trajectory tracking problem for the mechanical system, (5.71), with time-varying parameters and disturbances. Let $q_d(t)$ be the desired trajectory. It is assumed that $q_d(t)$ is twice continuously differentiable. Let $e = q(t) - q_d(t)$ be the joint tracking error, and $e_v = \dot{e} + \Gamma e$ be the reference velocity error. The following notations will be used: $\widehat{(*)}$ is the estimate of $(*)$, and $\widetilde{(*)} = \widehat{(*)} - (*)$ is the estimation error of $(*)$.

Consider the control law, τ, given by

$$\tau = - K_v e_v + M(q, \widehat{\theta}^*)\ddot{q}_r + C(q, \dot{q}, \widehat{\theta}^*)\dot{q}_r + F(q, \widehat{\phi}^*)\frac{\dot{q} + \dot{q}_r}{2} + g(q, \widehat{\theta}^*) + \delta_\tau \tag{5.79}$$

where $\dot{q}_r = \dot{q}_d - \Gamma e$, K_v and Γ are positive definite gain matrices, δ_τ is the additional robust control term which will be designed later, and

$$\widehat{\theta}^*(t, t_0) = \Lambda(t, t_0)\widehat{\theta}(t_0), \tag{5.80}$$

$$\widehat{\phi}^*(t, t_0) = \dot{\Lambda}(t, t_0)\widehat{\theta}(t_0), \tag{5.81}$$

where $\widehat{\theta}(t_0)$ will be generated by the adaptation law. Subtracting (5.74) and (5.75) from (5.80) and (5.81), respectively, results in

$$\widetilde{\theta}^*(t,t_0) = \Lambda(t,t_0)\widetilde{\theta}(t_0) - \delta_{\theta^*}(t,t_0), \tag{5.82}$$

$$\widetilde{\phi}^*(t,t_0) = \dot{\Lambda}(t,t_0)\widetilde{\theta}(t_0) - \dot{\delta}_{\theta^*}(t,t_0), \tag{5.83}$$

where $\widetilde{\phi}^*(t,t_0) \triangleq \widehat{\phi}^*(t,t_0) - \dot{\theta}^*(t,t_0)$. Substitution of the control input (5.79) into the dynamic equation (5.71) and simplifying using the linear parameterization property, Property P5.4, we obtain the error dynamics in terms of e_v as

$$\begin{aligned} M(q,\theta^*)\dot{e}_v + C(q,\dot{q},\theta^*)e_v + \frac{1}{2}F(q,\dot{\theta}^*)e_v + K_v e_v & \\ = M(q,\widetilde{\theta}^*)\ddot{q}_r + C(q,\dot{q},\widetilde{\theta}^*)\dot{q}_r + F(q,\widetilde{\phi}^*)\frac{\dot{q}+\dot{q}_r}{2} + g(q,\widetilde{\theta}^*) + \delta_\tau + d(t) & \quad (5.84) \\ = Y_1(q,\dot{q},\dot{q}_r,\ddot{q}_r)\widetilde{\theta}^* + Y_2(q,\dot{q},\dot{q}_r)\widetilde{\phi}^* + \delta_\tau + d(t) & \end{aligned}$$

where

$$Y_1(q,\dot{q},\dot{q}_r,\ddot{q}_r)\widetilde{\theta}^* = M(q,\widetilde{\theta}^*)\ddot{q}_r + C(q,\dot{q},\widetilde{\theta}^*)\dot{q}_r + g(q,\widetilde{\theta}^*), \tag{5.85}$$

$$Y_2(q,\dot{q},\dot{q}_r)\widetilde{\phi}^* = F(q,\widetilde{\phi}^*)\frac{\dot{q}+\dot{q}_r}{2}. \tag{5.86}$$

Substituting $\widetilde{\theta}^*(t,t_0)$ and $\widetilde{\phi}^*(t,t_0)$ given by (5.82) and (5.83), respectively, into (5.84) yields

$$\begin{aligned} M(q,\theta^*)\dot{e}_v + C(q,\dot{q},\theta^*)e_v + \frac{1}{2}F(q,\dot{\theta}^*)e_v + K_v e_v & \\ = Y_1(q,\dot{q},\dot{q}_r,\ddot{q}_r)(\Lambda(t,t_0)\widetilde{\theta}(t_0) - \delta_{\theta^*}(t,t_0)) + \delta_\tau + d(t) & \\ + Y_2(q,\dot{q},\dot{q}_r)(\dot{\Lambda}(t,t_0)\widetilde{\theta}(t_0) - \dot{\delta}_{\theta^*}(t,t_0)) & \quad (5.87) \\ = Y(q,\dot{q},\dot{q}_r,\ddot{q}_r)\widetilde{\theta}(t_0) + \delta_\tau - Y_1(q,\dot{q},\dot{q}_r,\ddot{q}_r)\delta_{\theta^*}(t,t_0) & \\ - Y_2(q,\dot{q},\dot{q}_r)\dot{\delta}_{\theta^*}(t,t_0) + d(t) & \end{aligned}$$

where $Y(q,\dot{q},\dot{q}_r,\ddot{q}_r) = Y_1(q,\dot{q},\dot{q}_r,\ddot{q}_r)\Lambda(t,t_0) + Y_2(q,\dot{q},\dot{q}_r)\dot{\Lambda}(t,t_0)$.

In the following, for brevity, all the arguments of vectors and matrices are omitted whenever there is no confusion. Consider the following Lyapunov function candidate during each interval, that is, $t \in [t_{r,i}, t_{r,i+1})$,

$$V = \frac{1}{2}e_v^\top M(q,\theta^*)e_v + \frac{1}{2}\widetilde{\theta}^\top \Gamma_1^{-1}\widetilde{\theta}. \tag{5.88}$$

where $\Gamma_1 = \Gamma_1^\top > 0$. The time derivative of V along the trajectories of (5.87) is

$$\begin{aligned}\dot{V} &= e_v^\top M(q,\theta^*)\dot{e}_v + \frac{1}{2}e_v^\top \dot{M}(q,\theta^*)e_v + \widetilde{\theta}^\top \Gamma_1^{-1}\dot{\widetilde{\theta}} \\ &= -e_v^\top K_v e_v + e_v^\top Y\widetilde{\theta} + e_v^\top(\delta_\tau - Y_1\delta_{\theta^*} - Y_2\dot{\delta}_{\theta^*} + d) + \widetilde{\theta}^\top \Gamma_1^{-1}\dot{\widetilde{\theta}}\end{aligned} \tag{5.89}$$

where the Property P5.3 is applied.

To estimate the unknown parameter vector $\widehat{\theta}$, we use the gradient projection algorithm given in [132], which we briefly illustrate in the following. Consider a convex parameter set Π given by

$$\widehat{\theta} = \begin{bmatrix} \widehat{\theta}_1 & \dots & \widehat{\theta}_i & \dots & \widehat{\theta}_{mp} \end{bmatrix}^\top \in \Pi \iff |\widehat{\theta}_i - \rho_i| < \sigma_i, \quad \forall i \in \{1, mp\} \tag{5.90}$$

with ρ_i and σ_i some given real numbers. Consider the function

$$\mathcal{P}(\widehat{\theta}) = \frac{2}{\varepsilon}\left[\sum_{i=1}^{mp}\left|\frac{\widehat{\theta}_i - \rho_i}{\sigma_i}\right|^q - 1 + \varepsilon\right] \tag{5.91}$$

where $0 < \varepsilon < 1$ and $q \geq 2$. Now, consider the "smooth projection" Proj$(\cdot)$, which will be used to estimate $\widehat{\theta}$ while maintaining it in Π:

$$\text{Proj}(\widehat{\theta}, y) = \begin{cases} y, & \text{if } \mathcal{P}(\widehat{\theta}) < 0. \\ y, & \text{if } \mathcal{P}(\widehat{\theta}) = 0 \text{ and } \nabla_{\mathcal{P}}{}^\top y \leq 0. \\ y - \dfrac{\mathcal{P}(\widehat{\theta})\nabla_{\mathcal{P}}\nabla_{\mathcal{P}}{}^\top}{\|\nabla_{\mathcal{P}}\|^2}y, & \text{otherwise.} \end{cases} \tag{5.92}$$

where $\nabla_{\mathcal{P}} = \left[\dfrac{\partial \mathcal{P}(\widehat{\theta})}{\partial \widehat{\theta}}\right]^\top$ is a column vector. Based on the smooth projection defined above, $\widehat{\theta}$ is estimated by

$$\dot{\widehat{\theta}} = \Gamma_1 \text{Proj}(\widehat{\theta}, -Y^\top e_v). \tag{5.93}$$

With the projection algorithm given by (5.93), we have

$$e_v^\top Y\widetilde{\theta} + \widetilde{\theta}^\top \Gamma_1^{-1}\dot{\widetilde{\theta}} = \widetilde{\theta}^\top(Y^\top e_v + Proj(\widehat{\theta}, -Y^\top e_v)) \leq 0. \tag{5.94}$$

Substituting (5.94) into (5.89) results in

$$\dot{V} \leq -e_v^\top K_v e_v + e_v^\top(\delta_\tau - Y_1\delta_{\theta^*} - Y_2\dot{\delta}_{\theta^*} + d). \tag{5.95}$$

Notice that the Lyapunov function candidate, (5.88), and the adaptation law, (5.93), are designed for each time interval, that is, $t \in [t_{r,i}, t_{r,i+1})$. At the beginning of each interval, say $(i+1)$-th interval, the initial value of the estimate is reset according to the following:

$$\widehat{\theta}(t_{r,\ i+1}, t_{r,i+1}) = B(t_{r,i+1}, t_{r,i})\widehat{\theta}(t_{r,\ i+1}^{-}, t_{r,i}). \tag{5.96}$$

The additional robust control term, δ_τ, in (5.95) is chosen as follows:

$$\delta_\tau = \begin{cases} -\Big(k_{\delta_{\theta^*}}\|Y_1\| + k_{\dot{\delta}_{\theta^*}}\|Y_2\| + k_d\Big)\dfrac{e_v}{\|e_v\|}, & \text{if } \|e_v\| \geq \varepsilon_0, \\ -\dfrac{1}{\varepsilon_0}\Big(k_{\delta_{\theta^*}}\|Y_1\| + k_{\dot{\delta}_{\theta^*}}\|Y_2\| + k_d\Big)e_v, & \text{if } \|e_v\| < \varepsilon_0 \end{cases} \tag{5.97}$$

where $\varepsilon_0 > 0$ and $k_d = \sup_{t\geq 0} d(t)$. It can be shown that the system (5.87) is uniformly ultimately bounded [133], and e_v converges in finite time to the set Π_1 defined by

$$\Pi_1 \triangleq \{e_v : \|e_v\| \leq \varepsilon_0\}. \tag{5.98}$$

Since $e_v(t)$ is bounded and $e_v = \dot{e} + \Gamma e$, the tracking error, $e(t)$, and its time derivative, $\dot{e}(t)$, are also uniformly ultimately bounded. Therefore, $q(t)$, $\dot{q}(t)$, $\dot{q}_r(t)$ and $\ddot{q}_r(t)$ are bounded, since $e(t)$, $\dot{e}(t)$, $q_d(t)$, $\dot{q}_d(t)$ and $\ddot{q}_d(t)$ are bounded. The estimated parameters $\widehat{\theta}^*(t)$ and $\widehat{\phi}^*(t)$ are also bounded because $\widehat{\theta}(t)$ and $\Lambda(t, t_0)$ are bounded. From (5.79), the control input $\tau(t)$ is bounded as it is composed of all bounded signals. The following theorem summarizes the results of the analysis.

Theorem 5.2 *For the time-varying mechanical system given by (5.71), the proposed adaptive control law given by (5.79), the parameter estimation algorithms given by (5.93), the resetting scheme given by (5.96), and with the knowledge of the bounds given in (5.76) and (5.77), the control input $\tau(t)$, the estimated time-varying parameters $\widehat{\theta}^*(t)$ and $\dot{\widehat{\theta}}^*(t)$ and the tracking error $e(t)$ are uniformly ultimately bounded.*

Remark 5.8 *In the "ideal" case, that is, the unknown parameter vector, $\theta^*(t)$, is constant and the disturbance $d(t) = 0$, we have $\delta_{\theta^*} = 0$, $\dot{\delta}_{\theta^*} = 0$. The time derivative* *of the Lyapunov function candidate given by (5.95) becomes*

$$\dot{V} \leq -e_v^\top K_v e_v + e_v^\top \delta_\tau = -e_v^\top K_v e_v \leq 0. \tag{5.99}$$

Therefore, asymptotic convergence of $e(t)$ to zero is achieved. Thus, the proposed adaptive algorithm can be applied to control of mechanical systems irrespective of whether they involve time-varying parameters or not.

Remark 5.9 *The disturbance vector $d(t)$ can also be approximated locally by polynomials of time. The control input $\tau(t)$ is in the same form as (5.79) except that k_d in (5.97) is replaced by the upper bound of the approximation error $\delta_d(t, t_0)$ given by the following equation:*

$$d(t) = \Lambda'(t, t_0)\theta_d(t_0) + \delta_d(t, t_0)$$

where $\theta_d(t_0)$ is the coefficient vector and $\Lambda'(t, t_0)$ is the matrix that depends on the time interval for approximation. The vector $\theta_d(t_0)$ can be estimated in each interval.

5.2.3 Experiments

To experimentally investigate the proposed control algorithm, a time-varying experiment is designed for a two-link robot, which consists of a two-axis direct drive manipulator as shown in Figure 4.5. The direct drive manipulator operates in the absence of the undesirable factors of mechanical backlash and gear train compliance. Each axis of the manipulator is driven by an NSK Megatorque direct drive servomotor.

The NSK-Megatorque motor system consists of a high torque direct drive brushless actuator, a high-resolution brushless resolver, and a heavy duty precision bearing. The servo-motors are capable of up to 3 revolutions per second maximum velocity and position feedback resolution of up to 156,400 counts per revolution. The base motor delivers up to 240 N-m of rated torque output, and the elbow motor produces up to 40 N-m rated torque output. The real-time system associated with the direct drive manipulator consists of a host computer, a servo DSP card, and a DSP associated with the sensors. For a complete description of the experimental platform we refer the reader to [109].

The elbow link of the planar manipulator is used to generate a time-varying disturbance to the base link. This is done as follows. A constant torque is applied to the elbow link; this has an effect of generating a time-varying payload to the base link; that is, due to the rotation of the elbow link the mass moment of inertia of the base link is varying with time. Further, since the dynamics of both the links is coupled, the motion of the elbow link also causes a disturbance to the base link that is time-varying. Then, the goal is to control the base link, which has a time-varying inertia and is acted on by time-varying disturbances, by using the proposed adaptive controller. The procedure of obtaining the time-varying dynamics for the base link is explained in the following section.

Generation of time-varying dynamics for the base link

The dynamics of the two-link manipulator is given by

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} = \tau - f_f \tag{5.100}$$

where

$$M(q) = \begin{bmatrix} p_1 + 2p_3c_2 & p_2 + p_3c_2 \\ p_2 + p_3c_2 & p_2 \end{bmatrix}, C(q,\dot{q}) = \begin{bmatrix} -p_3\dot{q}_2s_2 & -p_3(\dot{q}_1 + \dot{q}_2)s_2 \\ p_3\dot{q}_1s_2 & 0 \end{bmatrix},$$

q_1 and q_2 are angular positions of the base and the elbow link, respectively, $\tau = [\tau_1, \tau_2]^\top$ is the vector of motor torques, $f_f = [f_1, f_2]^\top$ is the vector of friction torques, $c_2 = \cos(q_2)$ and $s_2 = \sin(q_2)$, and p_1, p_2 and p_3 are coupled inertial parameters. The true values of the coupled inertial parameters without any payload on the elbow link are $p_1 = 3.4$, $p_2 = 0.2$ and $p_3 = 0.15$.

Reducing the two second-order equations given by (5.100) into a single equation results in

$$\begin{aligned}&(p_1p_2 - p_2^2 - p_3^2c_2^2)\ddot{q}_1 - p_3(2p_2\dot{q}_1\dot{q}_2 + p_2\dot{q}_1^2 + p_2\dot{q}_2^2 + p_3c_2\dot{q}_1^2)s_2\\ =&p_2(\tau_1 - f_1) - (p_2 + p_3c_2)(\tau_2 - f_2).\end{aligned} \tag{5.101}$$

Equation (5.101) can be rewritten as

$$I(t)\ddot{q}_1 + \dot{I}(t)\dot{q}_1 + f_1 = \tau_1 + d(t) \tag{5.102}$$

where

$$I(t) = p_1 - p_2 - \frac{p_3^2}{p_2}c_2^2, \tag{5.103}$$

$$d(t) = p_3\big((\dot{q}_1 + \dot{q}_2)^2 + \frac{p_3}{p_2}c_2\dot{q}_1^2 + \frac{2p_3}{p_2}c_2\dot{q}_1\dot{q}_2\big)s_2 - \big(1 + \frac{p_3}{p_2}c_2\big)(\tau_2 - f_2), \tag{5.104}$$

$$f_1 = f_v\dot{q}_1 + f_c\mathrm{sgn}(\dot{q}_1), \tag{5.105}$$

where f_c and f_v are the Coulomb and viscous friction coefficients, respectively. Equation (5.102) represents the dynamics of a single degree-of-freedom system with time-varying inertia ($I(t)$) and time-varying disturbance ($d(t)$). By choosing τ_2, one can introduce a desired $I(t)$ and $d(t)$. In practice, due to the coupling between the base link and the elbow link, the motion of the base link affects the motion of the elbow link, and consequently affects $I(t)$ and $d(t)$. However, a high constant torque applied to the elbow link will generate a high velocity, almost constant, rotation of the elbow link; then, the effect of the motion of the base link on $I(t)$ and $d(t)$ is relatively small, and thus can be neglected.

Experimental conditions

The desired trajectory for the angular position of the base link is chosen to be sinusoidal with an amplitude of 0.5 radians and a frequency of 0.5 Hz; that is, $q_{d1}(t) = 0.5\sin(\pi t)$. The elbow link is used to generate a time-varying disturbance and time-varying moment of inertia to the base link. Data from two sets of experiments is shown in this paper; a constant torque of 4 N-m for the elbow link is used as input in the first case, and a constant torque of 3 N-m is used in the second case. With the applied torques of 4 N-m and 3 N-m, the elbow link will rotate with an angular velocity of around 20 rad/s and 6 rad/s, respectively, after reaching the steady state. A control sampling period of 2 milli-seconds is chosen in all the experiments.

To track the desired trajectory, the torque input to the base link, τ_1, is designed using the proposed adaptive controller (5.79). The parameters $I(t), d(t), f_c$ and f_v are estimated by $\widehat{I}_0 + (t - t_0)\widehat{I}_1, \widehat{d}_0 + (t - t_0)\widehat{d}_1, \widehat{f}_c$ and $\widehat{f}_v$, respectively. Hence, the parameter vector which is estimated in the experiment is

$$\theta = \begin{bmatrix} I_0 & I_1 & d_0 & d_1 & f_v & f_c \end{bmatrix}^\top .$$

The window width for local polynomial approximation is chosen to be 0.1 seconds, that is, $T = 0.1$ seconds. The gain values used in the experiments are

$$\Gamma = 50, \quad K_v = 100, \quad \Gamma_1 = \mathrm{diag}(20, 20, 100, 1000, 5, 10).$$

The constants in the robust control term δ_τ are chosen to be

$$k_{\delta_{\theta^*}} = 0.05, \quad k_{\dot{\delta}_{\theta^*}} = 16, \quad k_d = 20, \quad \epsilon_0 = 0.1.$$

The initial values for the estimate vector $\widehat{\theta}$ is chosen to be

$$\widehat{\theta}(0) = \begin{bmatrix} 3.4 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^\top .$$

The following bounds for the estimated parameters are chosen in the projection algorithm:

$$\widehat{I}_0 \in [1, 10], \quad \widehat{I}_1 \in [-10, 10], \quad \widehat{d}_0 \in [-100, 100], \quad \widehat{d}_1 \in [-2000, 2000].$$

Experimental results

The data shown in all the figures corresponds to $u_2(t) = 4$ N-m during the first 16 seconds and $u_2(t) = 3$ N-m for the remaining 14 seconds; see bottom plot of Figure

5.11. Also, notice that the cycle time of the desired angular position trajectory of the base link is 2 seconds; therefore, the data corresponds to implementation results for 15 cycles.

The time-varying inertia and the disturbance of the base link which are computed by using (5.103) and (5.104) are shown in Figure 5.9. Notice that the time-varying disturbance is periodic with an amplitude of about 50 N-m (with $u_2 = 4$ N-m). The moment of inertia is periodic with an average value of 3.15 Kg-m^2 and a peak-to-peak variation of 0.11 Kg-m^2.

The tracking error of the base link is shown in the top plot of Figure 5.10. It can be observed that the peak tracking error of the base link is less than 0.04 radians even in the presence of time-varying inertia and very large time-varying disturbance; from 16 seconds onwards, when the variation of the inertia and the disturbance are reduced, the tracking error is also reduced. Notice that the motor torque input of the base link, shown as top plot in Figure 5.11, has similar amplitude and frequency as that of the time-varying disturbance. The estimated $d(t)$ and $I(t)$ are shown in Figure 5.12 and Figure 5.13, respectively. Figure 5.14 shows the estimates of the friction coefficients f_v and f_c. It can be observed that all the estimated parameters are within the range defined in the projection algorithm.

Comparison with an ideal non-adaptive controller

To compare the performance of the proposed adaptive controller with a controller that uses true parameter values, an ideal non-adaptive controller is designed and implemented on the experimental platform. Experimental results of the two controllers are compared and discussed.

Equation (5.102) can be rewritten in terms of the tracking error of the base link,

$e_1 \stackrel{\triangle}{=} q_1 - q_{d1}$, as follows:

$$\begin{aligned}
&\ddot{e}_1 + 2\xi\omega_n\dot{e}_1 + \omega_n^2 e_1 \\
=&\frac{1}{I}(\tau_1 - \dot{I}\dot{q}_1 + d - f_1 - I\ddot{q}_{1d} + 2I\xi\omega_n\dot{e}_1 + I\omega_n^2 e_1) \\
=&\frac{1}{I}(\tau_1 - \dot{I}\dot{q}_1 + \bar{d} - I\ddot{q}_{1d} + 2I\xi\omega_n\dot{e}_1 + I\omega_n^2 e_1) + \frac{1}{I}\Big((1 + \frac{p_3}{p_2}c_2)f_2 - f_1\Big)
\end{aligned} \tag{5.106}$$

where ξ, ω_n are two positive constants, and $\bar{d}$ is given by

$$\bar{d} = p_3\big((\dot{q}_1 + \dot{q}_2)^2 + \frac{p_3}{p_2}c_2\dot{q}_1^2 + \frac{2p_3}{p_2}c_2\dot{q}_1\dot{q}_2\big)s_2 - \big(1 + \frac{p_3}{p_2}c_2\big)\tau_2.$$

Now assuming that the true values of all the constant and time-varying parameters are known, an ideal non-adaptive controller is given by

$$\tau_1 = \dot{I}\dot{q}_1 - \bar{d} + I\ddot{q}_{1d} - 2I\xi\omega_n\dot{e}_1 - I\omega_n^2 e_1 + \delta_{\tau_1} \tag{5.107}$$

where δ_{τ_1} is a robustness term to account for the unknown terms involving friction. Notice that the term $\bar{d}$ in the control law can be computed based on the measurements and constant parameters p_1, p_2 and p_3. Substitution of the control law, (5.107), into (5.106) results in

$$\ddot{e}_1 + 2\xi\omega_n\dot{e}_1 + \omega_n^2 e_1 = \frac{1}{I}\Big(\delta_{\tau_1} + (1 + \frac{p_3}{p_2}c_2)f_2 - f_1)\Big). \tag{5.108}$$

In the following, the robustness term δ_{τ_1} will be designed based on bounds on f_1 and f_2. Consider the viscous plus Coulomb friction models for f_1 and f_2. Then f_1 and f_2 can be bounded as given below:

$$|f_1| \leq F_{v_1}|\dot{q}_1| + F_{c_1}$$
$$|f_2| \leq F_{v_2}|\dot{q}_2| + F_{c_2}$$

where $F_{v_1}, F_{c_1}, F_{v_2}$ and F_{c_1} are bounds on the viscous and Coulomb friction coefficients. Therefore, the uncertain term in the right-hand-side of (5.108) can be bounded as given below:

$$\left|\Big(1 + \frac{p_3}{p_2}c_2\Big)f_2 - f_1\right| \leq \mu \stackrel{\triangle}{=} \Big(1 + \frac{p_3}{p_2}\Big)(F_{v_2}|\dot{q}_2| + F_{c_2}) + F_{v_1}|\dot{q}_1| + F_{c_1}.$$

Now, the robustness term in the controller can be chosen as

$$\delta_{\tau_1} = -\frac{\dot{e}_1}{|\dot{e}_1| + \varepsilon}\mu \tag{5.109}$$

where $\varepsilon > 0$ is a small constant.

The experimental results for the non-adaptive controller are shown in Figure 5.15. The parameters used in the experiment are $\xi = 1, \omega_n = 35, F_{v_1} = F_{v_2} = 0.1, \varepsilon = 0.05, F_{c_1} = 8$ and $F_{c_2} = 2$. In Figure 5.15, the top plot shows the tracking error of the base link, the middle plot shows the control input to the base link, and the bottom plot is the input torque to the elbow link.

Comparing with the experimental results of the proposed adaptive controller, we can observe that the tracking error using the ideal non-adaptive controller is smaller as expected since it assumes full knowledge of both the time-varying parameters and disturbances. But the performance improvement is not significant. Further, we can observe that the control inputs are comparable.

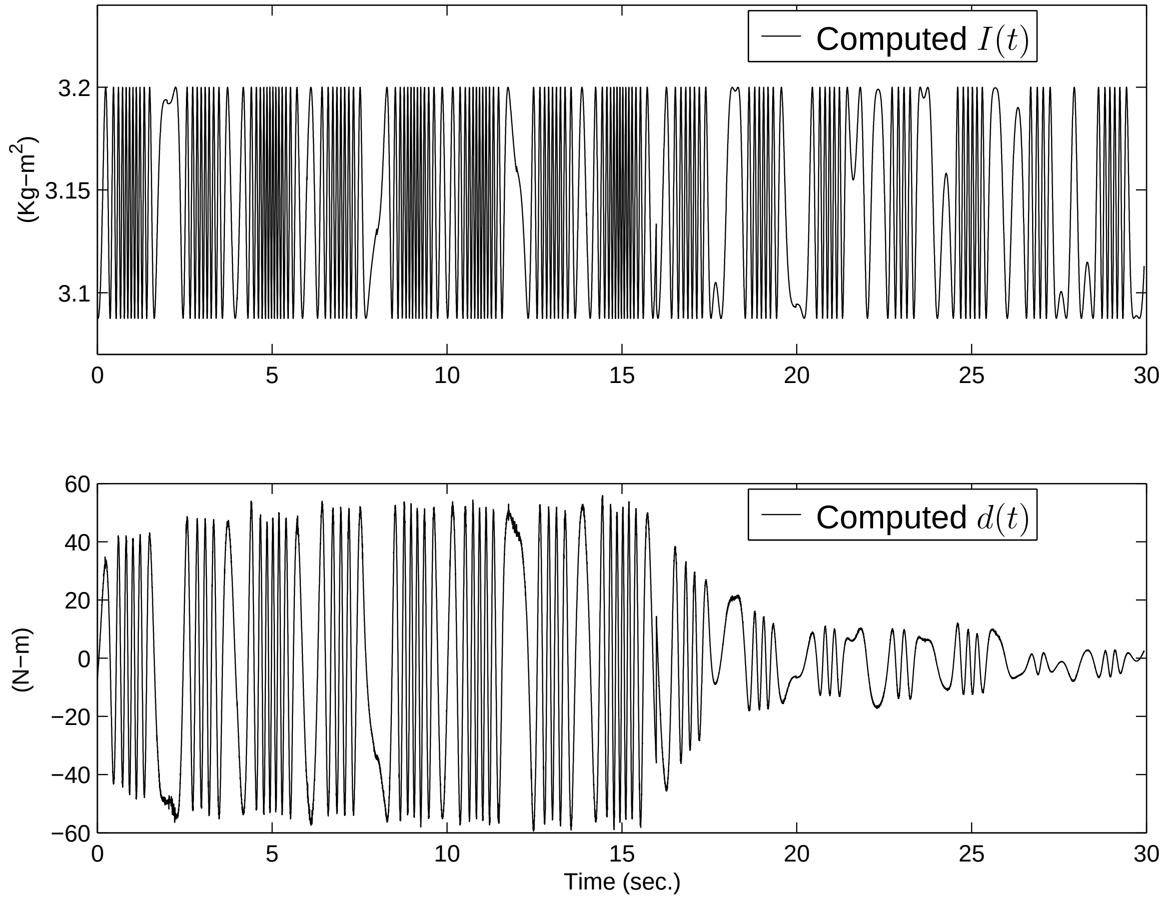

Figure 5.9: The time-varying inertia, $I(t)$ (top plot), and the time-varying disturbance, $d(t)$ (bottom plot) are shown. $I(t)$ and $d(t)$ are computed by using the experimental data of $q_2(t), \dot{q}_2(t), \dot{q}_1(t)$ in (5.103) and (5.104). The data from zero to 16 seconds corresponds to $\tau_2 = 4$ N-m and the data from 16 to 30 seconds corresponds to $\tau_2 = 3$ N-m.

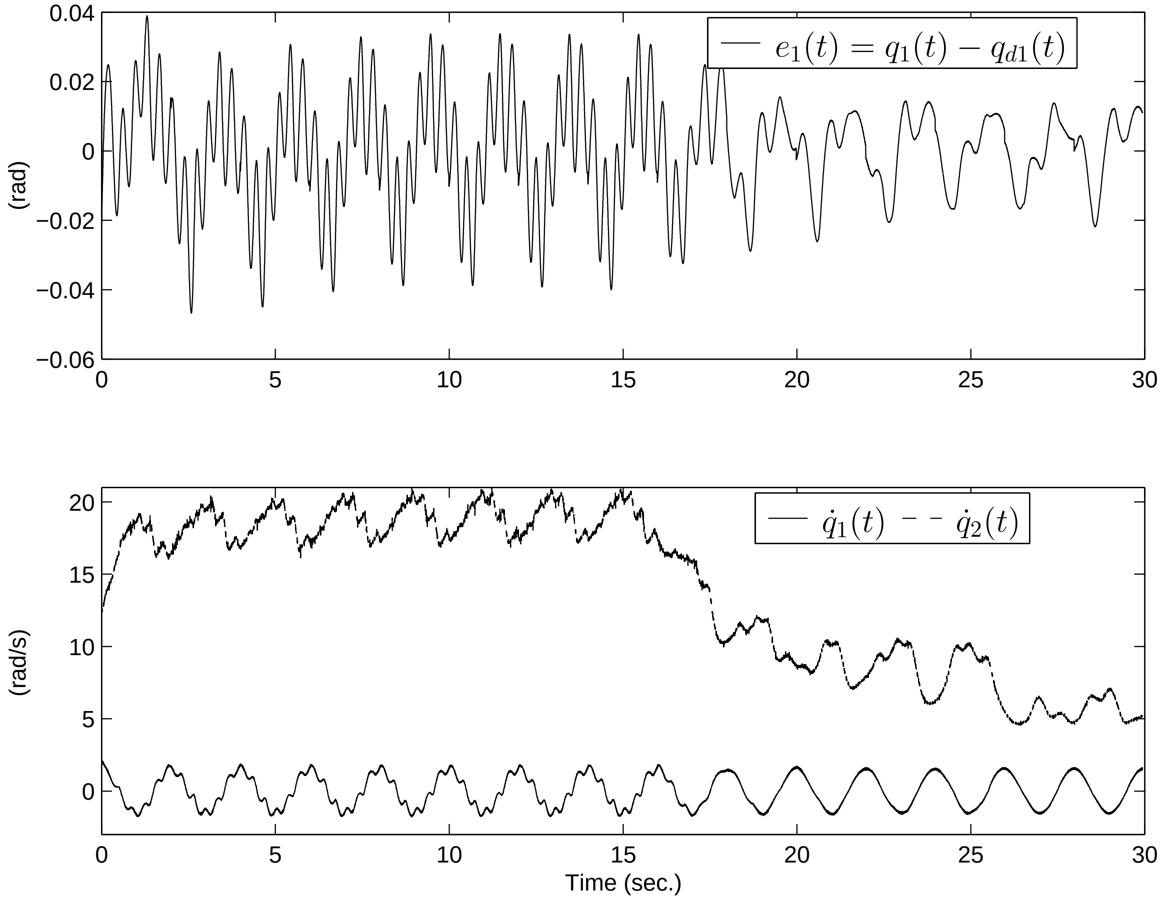

Figure 5.10: Tracking error of the base link ($e_1(t)$, top plot) and the angular velocities of the base link and elbow link ($\dot{q}_1(t)$ and $\dot{q}_2(t)$, bottom plot) are shown.

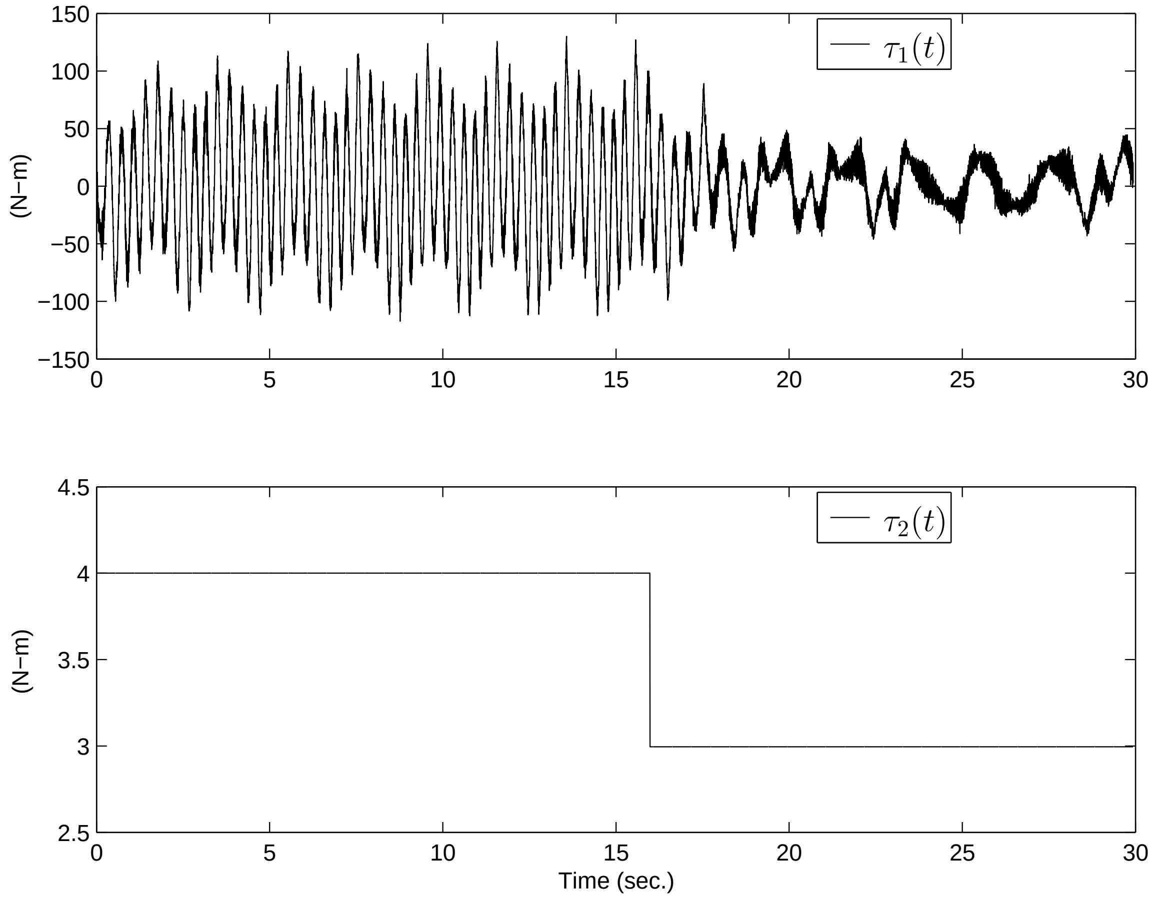

Figure 5.11: Motor control torques of base link ($\tau_1(t)$, top plot) and elbow link ($\tau_2(t)$, bottom plot) are shown.

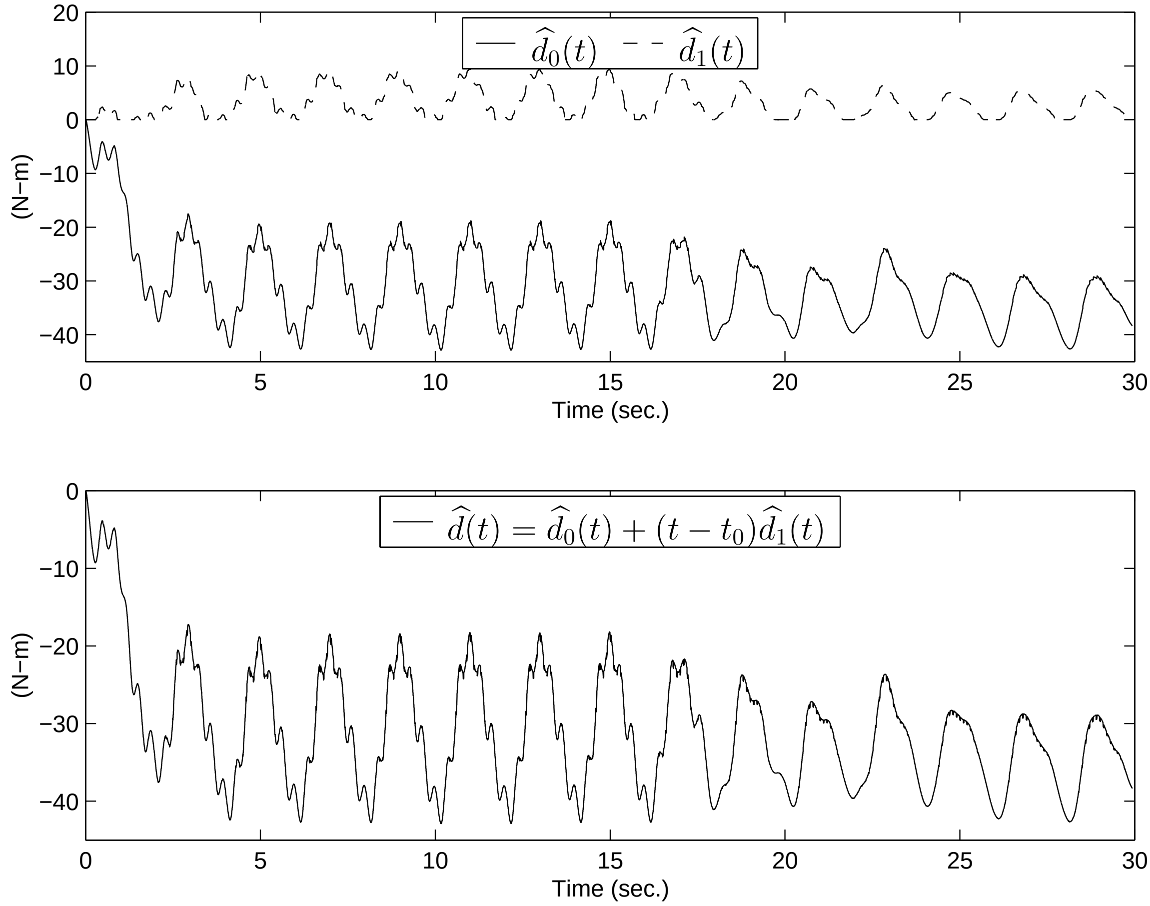

Figure 5.12: Estimated disturbance parameters $\widehat{d}_0(t)$ and $\widehat{d}_1(t)$ are shown in the top plot. The estimate of the disturbance $\widehat{d}(t) = \widehat{d}_0(t) + (t - t_0)\widehat{d}_1(t)$ is shown in the bottom plot.

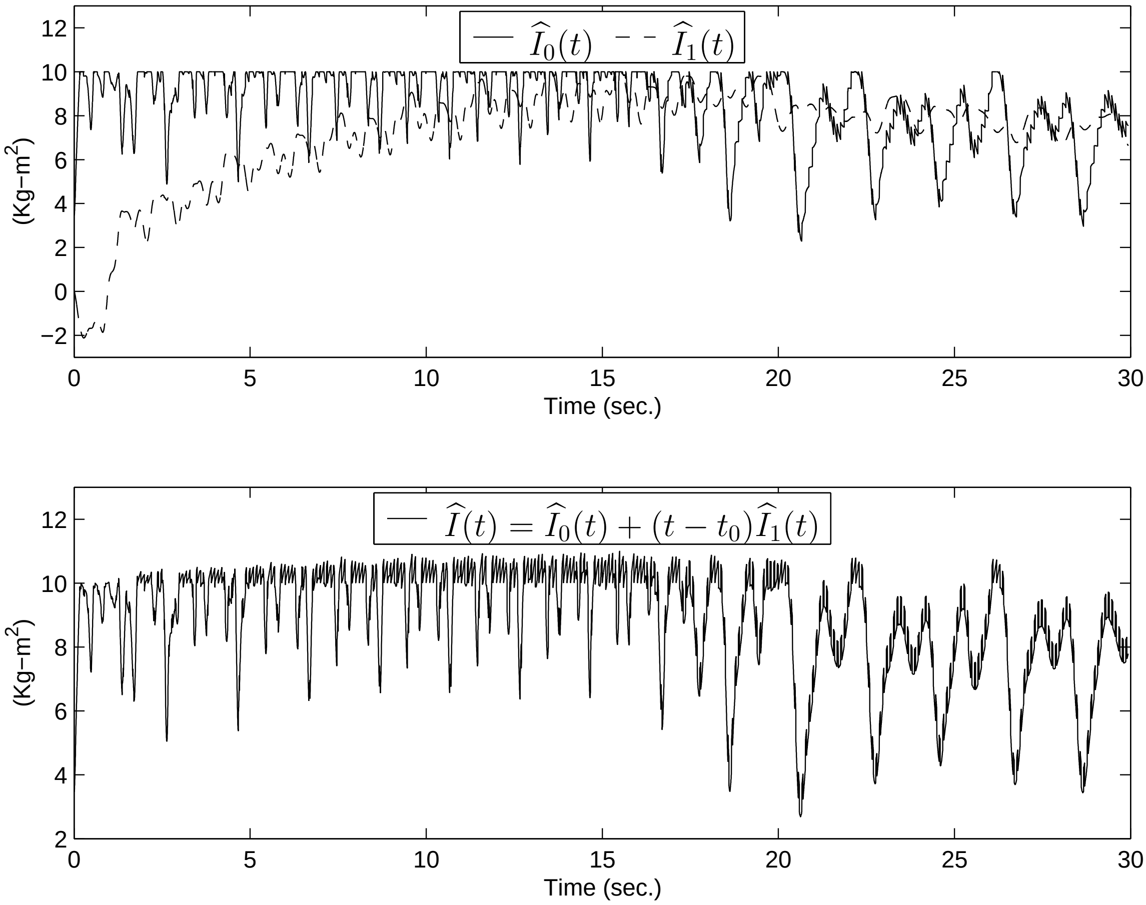

Figure 5.13: Estimated inertia parameters $\widehat{I}_0(t)$ and $\widehat{I}_1(t)$ are shown in the top plot. The estimate of the inertia $\widehat{I}(t) = \widehat{I}_0(t) + (t - t_0)\widehat{I}_1(t)$ is shown in the bottom plot.

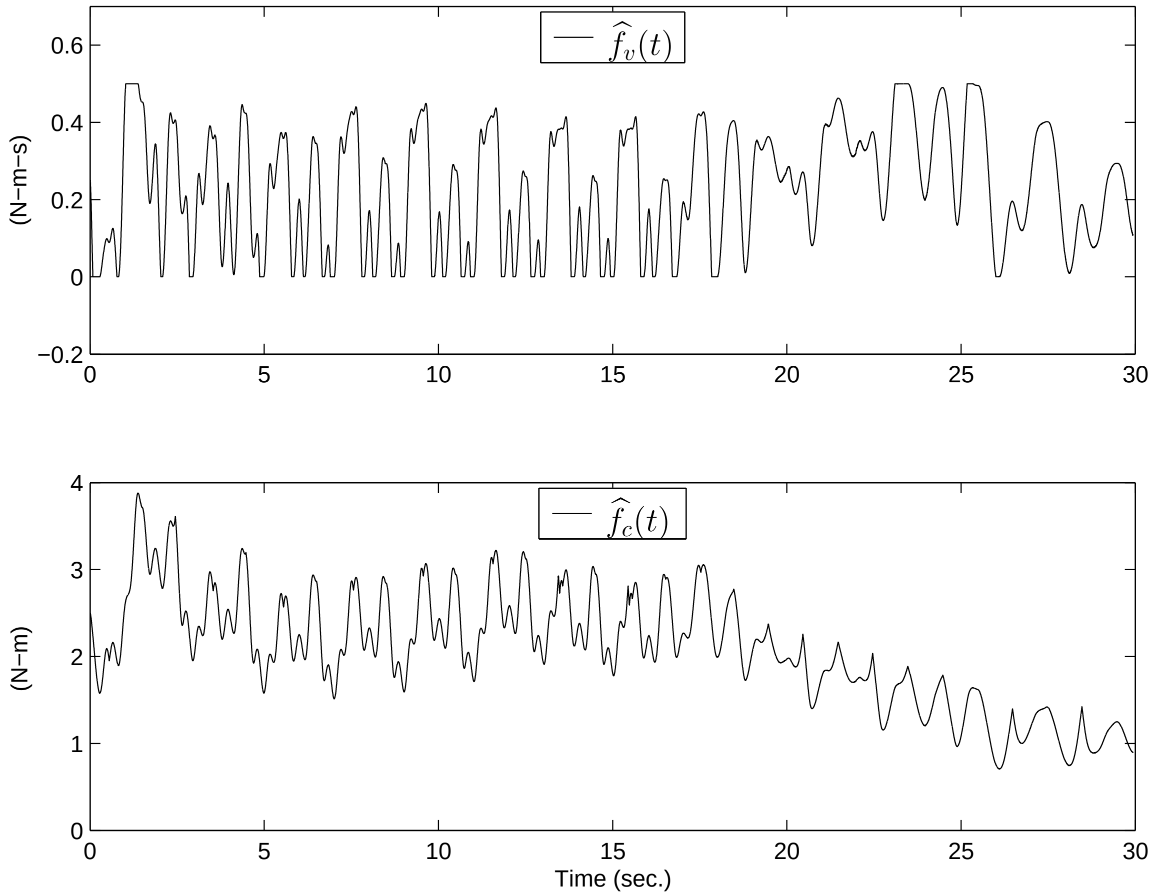

Figure 5.14: Estimated friction parameters $\widehat{f_v}(t)$ and $\widehat{f_c}(t)$ are shown.

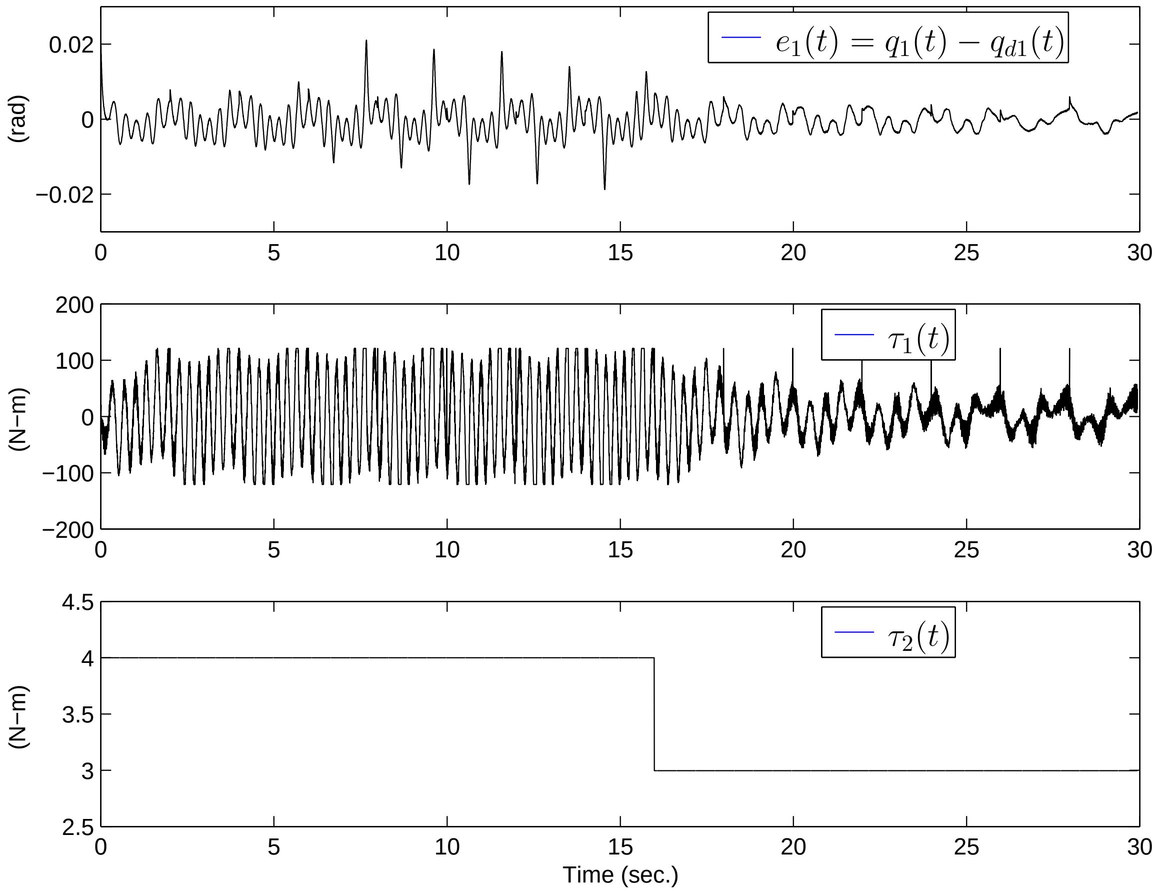

Figure 5.15: Experimental results from the ideal non-adaptive robust controller given by (5.107) and (5.109). Tracking error of the base link ($e_1(t)$, top plot), motor control torques of the base link ($\tau_1(t)$, middle plot) and the elbow link ($\tau_2(t)$, bottom plot) are shown.

5.3 Summary

Modified versions of the traditional least-squares and gradient algorithms for adaptive estimation of unknown time-varying parameters in linear parametric models is proposed. The time-varying parameters were approximated locally in small intervals of time by truncated Taylor series expansion in finite intervals of time. A strategy to reset the initial value of the parameter estimate at the beginning of each time interval is given; this assures that the parameter estimate is continuous at the resetting points. Stability and convergence properties of the proposed estimation algorithms were given. Simulation results conducted on an example verify the proposed algorithms. One particular feature of the method described is that the time-varying parameters are not assumed to be slow time-varying, because both the parameters and their time derivatives are estimated locally. Although the estimation algorithms are developed in the continuous-time domain, they can be extended to the discrete-time domain under the assumption of fast sampling.

A new adaptive controller for mechanical systems with time-varying parameters and disturbances was proposed. Based on the local approximation of the time-varying parameters/disturbances, an adaptive controller was developed for trajectory tracking. The unknown coefficients within each time interval were estimated using a gradient projection algorithm. The tracking error was shown to be ultimately bounded within a certain neighborhood of zero; the size of the neighborhood depends on the choice of the control gains. Using a two-link planar manipulator system, a novel experiment platform was designed to create a time-varying inertia system with time-varying disturbances. This platform was used to validate the proposed adaptive controller experimentally. Further, an ideal non-adaptive controller that assumes full knowledge of the time-varying parameters and disturbances was also implemented.

The performance of the proposed adaptive controller was comparable to an ideal non-adaptive controller.

Future research will focus on robustness of the proposed on-line parameter estimation algorithms in Section 5.1 to the modeling error in the linear time-varying parametric model, that is, consider time-varying systems that cannot be exactly placed in the form given by (5.1).

CHAPTER 6

RICCATI AND LYAPUNOV EQUATIONS

In this chapter, aspects of some matrix equations relevant to systems and control theory will be considered. Matrix equation arises in many problems; especially, Cauchy problems for Riccati operator equations in many linear filtering and prediction [134], transport theory [135], optimization and automatic control problems [136]. In this chapter, the Riccati differential equation in the following form is considered:

$$\dot{P}(t) = A(t)^{\top} P(t) + P(t)A(t) + P(t)R(t)P(t) + Q(t) \tag{6.1}$$

where $A(t)$, $R(t)$, $Q(t)$ and $P(t)$ are all $n \times n$ square matrices. The steady-state solution of (6.1), denoted by P, satisfies the following equation

$$A^{\top} P + PA + PRP + Q = 0 \tag{6.2}$$

where A, R and Q are limits of $A(t)$, $R(t)$ and $Q(t)$, as $t \to \infty$. Equation (6.2) is generally called the Algebraic Riccati Equation (ARE).

If $R(t) = 0$, equations (6.1) and (6.2) reduce to the following two equations, respectively.

$$\dot{P}(t) = A(t)^{\top} P(t) + P(t)A(t) + Q(t), \tag{6.3}$$

$$A^{\top} P + PA + Q = 0. \tag{6.4}$$

The linear matrix equation (6.3) is the special form of (6.1), and is usually called the Lyapunov Matrix Differential Equation or Lyapunov Differential Equation (LDE). Equation (6.4) is a special form of the ARE (6.2), and is usually called the Lyapunov

Matrix Equation or the Lyapunov Equation (6.4). The equation (6.4) is very important in the study of linear systems. The basic properties of it were studied by Liapunov in connection with stability questions.

The rest of this chapter is organized as follows. In Section 6.1, the explicit expression of the solution and bounds on the solution of linear matrix differential equations is studied. Section 6.2 considers the trace bounds on the solution to the Lyapunov equation. In Section 6.3, a class of algebraic Riccati equations are considered; necessary conditions for the existence of a positive semi-definite symmetric matrix as the solution to the ARE are given.

6.1 Linear matrix differential equation

In this section, the linear matrix equation: linear matrix differential equation and Lyapunov matrix equation are considered. Linear matrix equation is encountered in many applications, such as automatic control, optimization, and linear filtering. A motivational example for the application of the linear matrix differential equation is introduced in Section 6.1.1. The solution to the linear matrix differential equation which is in a general form is derived, in which the elements of coefficient matrices of the linear matrix differential equation are assumed to be time-varying. The uniqueness of the solution is proved. Based on the explicit form of the solution to the linear matrix differential equation derived in Section 6.1.2, bounds on this solution are derived in Section 6.1.3.

6.1.1 An application of the linear matrix equation

As an application of a linear differential equation in the form of (6.3), let us consider the problem of evaluating the integral

$$\eta = \int_{t_0}^{t_1} x^\top(\tau) Q(\tau) x(\tau) d\tau \tag{6.5}$$

where $x(t)$ satisfies the following first order differential equation

$$\dot{x}(t) = A(t)x(t). \tag{6.6}$$

The trajectory of (6.6) can be described by

$$x(t) = \Phi(t, t_0)x(t_0) \tag{6.7}$$

where $\Phi(t, t_0)$ is the transition matrix of (6.6). Substituting (6.7) in (6.5) yields

$$\eta(x_0) = x^\top(t_0)\left(\int_{t_0}^{t_1} \Phi^\top(\tau, t_0)Q(\tau)\Phi(\tau, t_0)d\tau\right)x(t_0). \tag{6.8}$$

Define

$$P(t_1, t_0) = \int_{t_0}^{t_1} \Phi^\top(\tau, t_0)Q(\tau)\Phi(\tau, t_0)d\tau.$$

Equation (6.8) becomes

$$\eta(x_0) = x^\top(t_0)P(t_1, t_0)x(t_0). \tag{6.9}$$

From (6.8), it is seen that in order to evaluate η, it is necessary to solve $\Phi(t, t_0)$ first, and then compute $P(t_1, t_0)$ by taking integration over $[t_0, t_1]$. Actually, it is possible to derive a linear differential equation for $P(t_1, t_0)$ itself. In fact, replacing t_0 by t in $P(t_1, t_0)$ and differentiating with respect to t yields

$$\frac{d}{dt}P(t_1, t) = \frac{d}{dt}\int_{t}^{t_1} \Phi^\top(\tau, t)Q(\tau)\Phi(\tau, t)d\tau \tag{6.10}$$

$$= -A^\top(t)P(t_1, t) - P(t_1, t)A(t) - Q(t). \tag{6.11}$$

The value $P(t_1, t_1) = 0$, so one obtains the first order differential equation with boundary condition for $P(t_1, t)$ as

$$\dot{P}(t_1, t) = -A^\top(t)P(t_1, t) - P(t_1, t)A(t) - Q(t), \quad P(t_1, t_1) = 0. \tag{6.12}$$

Equation (6.12) is in the same form as (6.3). We can evaluate η be directly solving the linear differential equation (6.12), which has a boundary condition at the final time.

6.1.2 Solution to the linear matrix equation

The more general linear matrix equation of (6.3) is

$$\dot{P}(t) = A_1(t)P(t) + P(t)A_2(t) + Q(t) \tag{6.13}$$

where $A_1(t) \in \mathbb{R}^{n\times n}$, $A_2(t) \in \mathbb{R}^{n\times n}$ and $Q \in \mathbb{R}^{n\times n}$. The solution of (6.13) can be expressed by the following theorem [137, p. 59].

Theorem 6.1 *The solution of (6.13) with the initial value $P(t_0)$ is given by*

$$P(t) = \Phi_1(t,t_0)P(t_0)\Phi_2^\top(t,t_0) + \int_{t_0}^{t} \Phi_1(t,\tau)Q(\tau)\Phi_2^\top(t,\tau)d\tau \tag{6.14}$$

where $\Phi_1(t,t_0)$ is the transition matrix for the system

$$\dot{x}(t) = A_1(t)x(t)$$

and $\Phi_2(t,t_0)$ is the transition matrix for the system

$$\dot{x}(t) = A_2^\top(t)x(t).$$

Proof. Differentiating both sides of (6.14) with respect to t and using the properties of the transition matrices,

$$\begin{aligned}
\dot{\Phi}_1(t,t_0) &= A_1(t)\Phi_1(t,t_0),\\
\dot{\Phi}_2(t,t_0) &= A_2^\top(t)\Phi_2(t,t_0),\\
\Phi_1(t,t) &= \Phi_2(t,t) = I,
\end{aligned}$$

results in

$$\begin{aligned}
\dot{P}(t) = A_1(t)\Bigg(&\Phi_1(t,t_0)P(t_0)\Phi_2^\top(t,t_0) + \int_{t_0}^{t} \Phi_1(t,\tau)Q(\tau)\Phi_2^\top(t,t_0)d\tau\Bigg)\\
&+ \Bigg(\Phi_1(t,t_0)P(t_0)\Phi_2^\top(t,t_0) + \int_{t_0}^{t} \Phi_1(t,\tau)Q(\tau)\Phi_2^\top(t,t_0)d\tau\Bigg)A_2(t) + Q(t)\\
&= A_1(t)P(t) + P(t)A_2(t) + Q(t).
\end{aligned}$$

Also,

$$P(t_0) = \Phi_1(t_0, T_0)P(t_0)\Phi_2^\top(t, t_0) = P(t_0).$$

This completes the proof. ■

Remark 6.1 *In the case of* $A_1^\top(t) = A_2(t) = A(t) \in \mathbb{R}^{n\times n}$ *and* $Q(t) \in \mathbb{R}^{n\times n}$*, the differential equation (6.13) becomes (6.3), which has the solution*

$$P(t) = \Phi(t, t_0)P(t_0)\Phi^\top(t, t_0) + \int_{t_0}^{t} \Phi(t, \tau)Q(\tau)\Phi^\top(t, \tau)d\tau \tag{6.15}$$

where $\Phi(t, t_0)$ *is the transition matrix for the system* $\dot{x}(t) = A^\top(t)x(t)$*.*

Remark 6.2 *In the case when* $A_1(t) = A_2(t) = A \in \mathbb{R}^{n\times n}$ *and* $Q(t) = Q \in \mathbb{R}^{n\times n}$*, the solution is*

$$P(t) = e^{A^\top(t-t_0)}P(t_0)e^{A(t-t_0)} + \int_{t_0}^{t} e^{A^\top(t-\tau)}Qe^{A(t-\tau)}d\tau. \tag{6.16}$$

Remark 6.3 *Theorem 6.1 does not indicate whether the solution given by (6.14) is unique or not. Further, Theorem 6.1 can be extended to the case where* $A_1(t) \in \mathbb{C}^{n\times n}$*,* $A_2(t) \in \mathbb{C}^{n\times n}$ *and* $Q \in \mathbb{C}^{n\times n}$*. An extension of Theorem 6.1 will be presented regarding the uniqueness of the solution of the linear matrix equation with complex coefficient matrices.*

Eigenvalues of a matrix M are continuous functions of its elements m_{ij}. If all the elements of M are continuous functions of t, then eigenvalues of M are also continuous functions of t. As $\|M\| = \sqrt{\lambda_{max}(M^H M)}$, $\|M\|$ is also a continuous function of t.

Theorem 6.2 *Consider the linear matrix differential equation*

$$\dot{P}(t) = A_1(t)P(t) + P(t)A_2(t) + Q(t), \quad P(t_0) = P_0 \tag{6.17}$$

where $A_1(t) \in \mathbb{C}^{n\times n}$*,* $A_2(t) \in \mathbb{C}^{n\times n}$ *and* $Q \in \mathbb{C}^{n\times n}$*. If all the elements of the matrices* $A_1(t)$ *and* $A_2(t)$ *are continuous functions of time defined on the interval* $t_0 \le t \le t_1$

where t_1 can be infinity, then (6.17) has at most one solution which is defined on the interval $t_0 \le t \le t_1$ with the initial value of P_0. Moreover, this unique solution is given by

$$P(t) = \Phi_1(t, t_0)P(t_0)\Phi_2^H(t, t_0) + \int_{t_0}^{t} \Phi_1(t, \tau)Q(\tau)\Phi_2^H(t, \tau)d\tau \tag{6.18}$$

where $\Phi_1(t, t_0)$ is the transition matrix for the system

$$\dot{x}(t) = A_1(t)x(t), \quad t \ge t_0, \; x(t) \in \mathbb{C}^n$$

and $\Phi_2(t, t_0)$ is the transition matrix for the system

$$\dot{x}(t) = A_2^H(t)x(t), \quad t \ge t_0, \; x(t) \in \mathbb{C}^n.$$

Proof. The uniqueness of the solution to (6.17) will be proved by contradiction. Let $P_1(t)$ and $P_2(t)$ are two distinct solutions to (6.17) with $P_1(t_0) = P_2(t_0) = P_0$. One obtains

$$\dot{P}_1(t) = A_1(t)P_1(t) + P_1(t)A_2(t) + Q(t), \quad P_1(t_0) = P_0, \tag{6.19}$$

$$\dot{P}_2(t) = A_1(t)P_2(t) + P_2(t)A_2(t) + Q(t), \quad P_2(t_0) = P_0. \tag{6.20}$$

Let $Z(t) \triangleq P_2(t) - P_1(t)$. Subtracting (6.19) from (6.20) results in

$$\dot{Z}(t) = A_1(t)Z(t) + Z(t)A_2(t), \quad Z(t_0) = 0. \tag{6.21}$$

Then

$$\frac{d}{dt}(Z^H(t)Z(t)) = Z^H(t)(A_1(t) + A_1^H(t))Z(t) + A_2^H(t)Z^H(t)Z(t) + Z^H(t)Z(t)A_2(t). \tag{6.22}$$

Let

$$A_1'(t) \triangleq A_1(t) + A_1^H(t),$$
$$Z(t) = \begin{bmatrix} z_1(t) & z_2(t) & \dots & z_n(t) \end{bmatrix}$$

where $z_i(t)$, $i = \{1, 2, \ldots, n\}$, is the i-th column of the matrix $Z(t)$. On has the following results

$$\operatorname{tr}\big(Z^H(t)Z(t)\big) = \sum_{i=1}^{n} z_i^H(t)z_i(t), \tag{6.23a}$$

$$\begin{aligned}\operatorname{tr}\big(Z^H(t)(A_1(t) + A_1^H(t))Z(t)\big) &= \sum_{i=1}^{n} z_i^H(t)A_1'(t)z_i(t) \\ &\leq \|A_1'(t)\| \sum_{i=1}^{n} z_i^H(t)z_i(t),\end{aligned} \tag{6.23b}$$

$$\begin{aligned}\operatorname{tr}\big(A_2^H(t)Z^H(t)Z(t) + Z^H(t)Z(t)A_2(t)\big) &= \operatorname{tr}\big(A_2^H(t) + A_2(t)\big)\operatorname{tr}\big(Z^H(t)Z(t)\big) \\ &= 2\sum_{i=1}^{n} \operatorname{Re}(\lambda_i(A_2(t))) \sum_{i=1}^{n} z_i^H(t)z_i(t)\end{aligned} \tag{6.23c}$$

where (6.23c) is obtained by using the trace property of the product of two matrices M and N with appropriate dimensions,

$$\operatorname{tr}(MN) = \operatorname{tr}(NM),$$

the relationship between the trace and eigenvalues of the $n \times n$ matrix M ,

$$\operatorname{tr}(M) = \sum_{i=1}^{n} \lambda_i(M),$$

and the property

$$\lambda_i(M^H) = \bar{\lambda}_i(M), \quad i \in \{1, 2, \ldots, n\}.$$

Let $\eta(t) \triangleq 1 + \|A_1'(t)\| + 2\sum_{i=1}^{n} \operatorname{Re}(\lambda_i(A_2(t)))$ and $z(t) \triangleq \sum_{i=1}^{n} z_i^H(t)z_i(t)$. $\eta(t)$ is a continuous function of t and $\eta(t)$ is a real number. Hence, the following factor, $\rho(t)$,

$$\rho(t) = e^{-\int_{t_0}^{t} \eta(\tau)d\tau},$$

exists and is positive.

Taking trace on both sides of (6.22) and using the results given by (6.23), one obtains

$$\frac{d}{dt}(z(t)) \leq \eta(t)z(t). \tag{6.24}$$

Multiplying both sides of (6.24) by $\rho(t)$ and simplifying, we have

$$\frac{d}{dt}(\rho(t)z(t)) \leq 0. \tag{6.25}$$

Integrating (6.25) in the interval $[t_0, t]$ for all $t \leq t_1$ yields

$$\rho(t)z(t) - \rho(t_0)z(t_0) \leq 0. \tag{6.26}$$

Since $Z(t_0) = 0$, $z(t) = \sum_{i=1}^{n} z_i(t)^H z_i(t) = 0$. Also, $\rho(t) > 0$. From (6.26), it is concluded that

$$z(t) = 0, \quad \forall t_0 \leq t \leq t_1.$$

Therefore, $z_i(t) = 0$, $i = \{1, 2, \ldots, n\}$, which in turn implies that $Z(t) = 0$, that is, $P_1(t) = P_2(t)$ for all $t_0 \leq t \leq t_1$. This contradicts with the assumption that $P_1(t) \neq P_2(t)$. Hence, the solution to (6.17) is unique provided it exists.

To prove that (6.18) is the solution of (6.17), one can use the similar method with that used in the proof of Theorem 6.1. Here, the property of $\Phi_2(t, t_0)$,

$$\dot{\Phi}_2(t, t_0) = A_2^H(t)\Phi_2(t, t_0), \tag{6.27}$$

should be used. ■

Remark 6.4 *One can directly apply Theorem 6.2 to the special case of (6.17),*

$$\dot{x}(t) = A(t)x(t) \tag{6.28}$$

where $x(t) \in \mathbb{C}^n$ and $A(t) \in \mathbb{C}^{n\times n}$. Compare to (6.17), (6.28) is a reduced version of (6.17) with $A_1(t) = A(t)$ and $A_2(t) = Q(t) = 0$. Equation (6.28) describes a linear time-varying system. If $A(t)$ satisfies the hypotheses given in Theorem 6.2, the solution of (6.28) is obviously unique.

Theorem 6.2 provides sufficient conditions for the existence of the solution to the linear matrix differential equation (6.17) because the solution is unique if it exists.

From (6.18), it is seen that, if $\Phi_1(\cdot)$ and $\Phi_2(\cdot)$ exist, and if the integral in (6.18) exists, then solution $P(t)$ exists. The following theorem summarizes sufficient conditions for the existence of a solution to (6.17).

Theorem 6.3 *Consider the linear matrix differential equation*

$$\dot{P}(t) = A_1(t)P(t) + P(t)A_2(t) + Q(t) \tag{6.29}$$

where $A_1(t) \in \mathbb{C}^{n\times n}$, $A_2(t) \in \mathbb{C}^{n\times n}$ and $Q \in \mathbb{C}^{n\times n}$. If the following conditions are satisfied:

1. *All elements of the matrices of $A_1(t)$ and $A_2(t)$ are continuous functions of time defined on the interval $t_0 \le t \le t_1$.*

2. *The following two linear time-varying equations are solvable,*

$$\dot{x}(t) = A_1(t)x(t), \quad t \ge t_0, \; x(t) \in \mathbb{C}^n,$$
$$\dot{x}(t) = A_2^H(t)x(t), \quad t \ge t_0, \; x(t) \in \mathbb{C}^n,$$

   and have transition matrices $\Phi_1(t,t_0)$ and $\Phi_2(t,t_0)$, respectively.

3. *The integral*

$$\int_{t_0}^{t} \Phi_1(t,\tau)Q(\tau)\Phi_2^H(t,\tau)d\tau,$$

   exists.

Then, the linear matrix differential equation (6.29) has a unique solution. Moveover, this solution is given by (6.18).

6.1.3 Bounds on the solution of the linear matrix equation

In this section, the bounds on the solution to the linear matrix differential equation (6.17), given by (6.18), and its special case where $A_1(t) = A_2^H(t)$ is considered. It is well known that the linear matrix differential equation of the following form

$$\dot{P}(t) = A^H(t)P(t) + P(t)A(t) + Q(t) \tag{6.30}$$

plays an important role in systems, control and optimization [137]. A number of applications of equation (6.30), and its special cases, can be found in systems and control theory. It is of importance to find bounds on the solution of the equation without explicitly solving it.

Trace and eigenvalue bounds on the solution of the following matrix differential equation, also called the Lyapunov matrix differential equation, can be found in [138] and [139]:

$$\dot{P}(t) = A^\top P(t) + P(t)A + Q \tag{6.31}$$

where $A \in \mathbb{R}^{n\times n}$, $Q = Q^\top \in \mathbb{R}^{n\times n}$, $Q \geq 0$ and A is stable. Notice that equation (6.30) is a more general case of (6.31). Upper and lower bounds for the trace or eigenvalues of the solution to (6.30) have not been reported in the literature. The upper and lower bounds for the trace of the solution to (6.30) will be derived in this section.

Because the solution to the linear matrix differential equation (6.17), given by (6.18), is unique, we can be obtaine the bounds on the solution based on the explicit form of the solution. First introduce several technical lemmas which are required to derive the bounds.

Lemma 6.1 *Let $M = M^H \geq 0$ and $N = N^H$, then*

$$\lambda_{min}(N)tr(M) \leq tr(MN) \leq \lambda_{max}(N)tr(M). \tag{6.32}$$

Proof. Since N is a Hermitian matrix, by Schur triangularization theorem [140, p. 69], there exists a unitary matrix H such that

$$D = UNU^H \tag{6.33}$$

where D is a diagonal matrix whose diagonal elements are the eigenvalues of N. Hence we have

$$\begin{aligned}\mathrm{tr}(MN) &= \mathrm{tr}(UMU^HUNU^H)\\ &= \mathrm{tr}(UMU^HD).\end{aligned}$$

Since $M \geq 0$, which in turn implies that $UMU^H \geq 0$, all diagonal elements of UMU^H are nonnegative real numbers. Hence, we have

$$\lambda_{min}(N)\mathrm{tr}(UMU^H) \leq \mathrm{tr}(UMU^HD) \leq \lambda_{max}(N)\mathrm{tr}(UMU^H). \tag{6.34}$$

Notice that $\mathrm{tr}(UMU^H) = \mathrm{tr}(M)$. Equation (6.32) follows. ■

The inequality (6.32) is well known to hold for the case where both M and N are symmetric positive definite [141, 138, 140, 142], and for the case where both M and N are symmetric and M is positive definite [143]. Lemma 6.2 shows that (6.32) holds for any Hermitian matrix N.

Lemma 6.2 *Let $A(t) \in \mathbb{C}^{n\times n}$, $X \in \mathbb{C}^{n\times n}$, and $\Phi(t, t_0)$ be the transition matrix of the linear time-varying system*

$$\dot{x} = A(t)x(t), \quad t \geq t_0. \tag{6.35}$$

Then, for any $X = X^H \geq 0$, the following is true:

$$tr(X)e^{\int_\tau^t 2\mu_m(A(\xi))d\xi} \leq tr(\Phi(t,\tau)X\Phi^H(t,\tau)) \leq tr(X)e^{\int_\tau^t 2\mu_M(A(\xi))d\xi} \tag{6.36}$$

for all $t \geq \tau \geq t_0$, where $\mu_m(M) \triangleq \lambda_{min}((M + M^H)/2)$ and $\mu_M(M) \triangleq \lambda_{max}((M + M^H)/2)$.

Proof. Using the property of $\Phi(t, \tau)$,

$$\frac{d}{dt}\Phi(t,\tau) = A(t)\Phi(t,\tau), \tag{6.37}$$

and the trace properties of a matrix,

$$\mathrm{tr}(MN) = \mathrm{tr}(NM), \tag{6.38}$$

$$\mathrm{tr}(M+N) = \mathrm{tr}(M) + \mathrm{tr}(N), \tag{6.39}$$

we have

$$\begin{aligned}\frac{d}{dt}\big(\mathrm{tr}\big(\Phi(t,\tau)X\Phi^H(t,\tau)\big)\big) &= \mathrm{tr}\bigg(\frac{d}{dt}\big(\Phi(t,\tau)X\Phi^H(t,\tau)\big)\bigg)\\ &= \mathrm{tr}\big(A(t)\Phi(t,\tau)X\Phi^H(t,\tau) + \Phi(t,\tau)X\Phi^H(t,\tau)A^H(t)\big)\\ &= \mathrm{tr}\big(A(t) + A^H(t))\Phi(t,\tau)X\Phi^H(t,\tau)\big).\end{aligned}$$

Since $\Phi(t,t_0)X\Phi^H(t,t_0) \geq 0$ and $\big(A(t) + A^H(t)\big)$ is a Hermitian matrix, we can apply Lemma 6.1 to the right-side of the above identity to get

$$\frac{d}{dt}\big(\mathrm{tr}\big(\Phi(t,\tau)X\Phi^H(t,\tau)\big)\big) \geq 2\mu_m(A(t))\mathrm{tr}(\Phi(t,\tau)X\Phi^H(t,\tau)), \tag{6.40a}$$

$$\frac{d}{dt}\big(\mathrm{tr}\big(\Phi(t,\tau)X\Phi^H(t,\tau)\big)\big) \leq 2\mu_M(A(t))\mathrm{tr}(\Phi(t,\tau)X\Phi^H(t,\tau)). \tag{6.40b}$$

Notice that $\mathrm{tr}(\Phi(\tau,\tau)X\Phi^H(\tau,\tau)) = \mathrm{tr}(X)$, and $\mu_M(A(t))$ and $\mu_m(A(t))$ are continuous functions. Equation (6.40) consists of two first-order scalar differential inequalities. Solving these two first-order scalar differential inequalities gives rise to (6.36).
■

Remark 6.5 *Lemma 6.2 reduces to the inequality*

$$e^{2\mu_m(A)t} \leq \frac{1}{n}tr\Big(e^{At}e^{A^Ht}\Big) \leq e^{2\mu_M(A)t}, \tag{6.41}$$

when $A(t) = A$. This inequality is used in [138] to derive the upper and lower bounds on the solution to the linear matrix differential equation (6.31).

Remark 6.6 *Since $\Phi(t,t_0)\Phi^H(t,t_0)$ is a Hermitian matrix, its eigenvalues are real numbers. From the relation between the trace and eigenvalues of an $n \times n$ matrix M, $tr(M) = \sum_{i=1}^n \lambda(M)$, and Theorem 6.2, one can obtain the following corollary on the lower and upper bounds on the maximal and minimal eigenvalues of $\Phi(t,t_0)\Phi^H(t,t_0)$.*

Corollary 6.1 *The maximal and minimal eigenvalues of* $\Phi(t,t_0)\Phi^H(t,t_0)$ *are bounded by*

$$\lambda_{max}(\Phi(t,t_0)\Phi^H(t,t_0)) \geq e^{\int_{t_0}^{t} 2\mu_m(A(\tau))d\tau}, \tag{6.42a}$$

$$\lambda_{min}(\Phi(t,t_0)\Phi^H(t,t_0)) \leq e^{\int_{t_0}^{t} 2\mu_M(A(\tau))d\tau}, \tag{6.42b}$$

for all $t \geq t_0$.

Proof. Let $X = I$ in (6.36), one has

$$ne^{\int_{\tau}^{t} 2\mu_m(A(\xi))d\xi} \leq \operatorname{tr}\big(\Phi(t,\tau)\Phi^H(t,\tau)\big) \leq ne^{\int_{\tau}^{t} 2\mu_M(A(\xi))d\xi} \tag{6.43}$$

Using the following inequality

$$n\mu_m(\Phi(t,\tau)\Phi^H(t,\tau)) \leq \operatorname{tr}\big(\Phi(t,\tau)\Phi^H(t,\tau)\big) \leq n\mu_M\Phi(t,\tau)\Phi^H(t,\tau),$$

results in inequalities (6.42). ■

Lemma 6.3 *Let* $M \in \mathbb{C}^{n\times n}$ *and* $N \in \mathbb{C}^{n\times n}$*. Then*

$$\operatorname{Re}(tr(M^H N)) \leq \frac{1}{2}(tr(M^H M) + tr(N^H N)). \tag{6.44}$$

Proof. Since

$$\begin{aligned}
0 &\leq \operatorname{tr}((M-N)^H(M-N)) \\
&= \operatorname{tr}(M^H M + N^H N - M^H N - N^H M) \\
&= \operatorname{tr}(M^H M) + \operatorname{tr}(N^H N) - \operatorname{tr}(M^H N) - \operatorname{tr}(N^H M) \\
&= \operatorname{tr}(M^H M) + \operatorname{tr}(N^H N) - \operatorname{tr}(M^H N) - \overline{\operatorname{tr}(M^H N)} \\
&= \operatorname{tr}(M^H M) + \operatorname{tr}(N^H N) - 2\operatorname{Re}(\operatorname{tr}(M^H N)),
\end{aligned}$$

inequality (6.44) follows. ■

In the following, the upper trace bound on the real part of the solution to the linear matrix differential equation (6.17) is derived based on its solution given by (6.18). The result is illustrated by the the following theorem.

Theorem 6.4 *Consider the linear matrix differential equation which satisfies the hypotheses given in Theorem 6.2. The real part of the solution to the linear matrix differential equation (6.17) is bounded by*

$$\mathrm{Re}(tr(P(t))) \leq \frac{n}{2}\left(e^{\int_{t_0}^{t} 2\mu_M(A_1(\xi)P_0)d\xi} + e^{\int_{t_0}^{t} 2\mu_M(A_2(\xi))d\xi} + \int_{t_0}^{t}\left(e^{\int_{\tau}^{t} 2\mu_M(A_1(\xi)P_0)d\xi}d\tau + e^{\int_{\tau}^{t} 2\mu_M(A_2(\xi))d\xi}\right)d\tau\right) \tag{6.45}$$

for all $t \geq t_0$.

Proof. Taking the real part of the trace on both sides of (6.18) results in

$$\mathrm{Re}(\mathrm{tr}(P(t))) = \mathrm{Re}\left(\mathrm{tr}\left(\Phi_1(t,t_0)P_0\Phi_2^H(t,t_2) + \int_{t_0}^{t}\Phi_1(t,\tau)Q(\tau)\Phi_2^H(t,\tau)d\tau\right)\right) \tag{6.46a}$$

$$\leq \frac{1}{2}\mathrm{tr}\big(\Phi(t,t_0)P_0P_0^H\Phi_1^H(t,t_0)\big) + \frac{1}{2}\mathrm{tr}\big(\Phi_2(t,t_0)\Phi_2^H(t,t_0)\big) \tag{6.46b}$$

$$+\frac{1}{2}\mathrm{tr}\left(\int_{t_0}^{t} Q(\tau)\Phi_1(t,\tau)\Phi_1^H(t,\tau)Q^H(\tau)d\tau\right) \tag{6.46c}$$

$$+\frac{1}{2}\mathrm{tr}\left(\int_{t_0}^{t}\Phi_2(t,\tau)\Phi_2^H(t,\tau)d\tau\right) \tag{6.46d}$$

where Lemma 6.3 is used. By Lemma 6.3, we have

$$\mathrm{tr}\big(\Phi(t,t_0)P_0P_0^H\Phi_1^H(t,t_0)\big) \leq ne^{\int_{t_0}^{t} 2\mu_M(A_1(\xi)P_0)d\xi}, \tag{6.47a}$$

$$\mathrm{tr}\big(\Phi_2(t,t_0)\Phi_2^H(t,t_0)\big) \leq ne^{\int_{t_0}^{t} 2\mu_M(A_2(\xi))d\xi}, \tag{6.47b}$$

$$\mathrm{tr}\left(\int_{t_0}^{t} Q(\tau)\Phi_1(t,\tau)\Phi_1^H(t,\tau)Q^H(\tau)d\tau\right) \leq n\int_{t_0}^{t}\left(e^{\int_{\tau}^{t} 2\mu_M(A_1(\xi)P_0)d\xi}d\tau\right)d\tau, \tag{6.47c}$$

$$\mathrm{tr}\left(\int_{t_0}^{t}\Phi_2(t,\tau)\Phi_2^H(t,\tau)d\tau\right) \leq n\int_{t_0}^{t}\left(e^{\int_{\tau}^{t} 2\mu_M(A_2(\xi))d\xi}d\tau\right)d\tau. \tag{6.47d}$$

Using (6.47) and (6.46) and simplifying results in (6.45). ■

Remark 6.7 *If $A_1(t)$, $A_2(t)$ and $P(t)$ are all real matrices in t, then* $\mathrm{Re}(tr(P(t))) = tr(P(t))$. *In this case, Theorem 6.4 gives the upper bound on the trace of $P(t)$.*

Remark 6.8 *From the proof of Theorem 6.4, it is seen that, since $A_1(t)$ and $A_2(t)$ are assumed to be arbitrary, the completion of square inequality given by Lemma 6.3* *is used to obtain the upper bound of the trace of the product of two matrices. In the case where some restriction on coefficient matrices, $A_1(t)$, $A_2(t)$ and $Q(t)$, and the initial matrix P_0, are posed, we have the following theorem on the two-side bounds on the solution of the linear matrix differential equation.*

Theorem 6.5 *Consider the following linear matrix differential equation*

$$\dot{P}(t) = A^H(t)P(t) + P(t)A(t) + Q(t), \quad P(t_0) = P^H(t_0) = P_0 \geq 0 \tag{6.48}$$

where $A(t) \in \mathbb{C}^{n\times n}$, $Q(t) = Q^H(t) \in \mathbb{C}^{n\times n}$ and $Q(t) \geq 0$ are continuous functions of t. The trace of the solution to (6.48) is bounded by

$$tr(P(t)) \leq tr(P_0)e^{\int_{t_0}^{t} 2\mu_M(A(\xi))d\xi} + \int_{t_0}^{t} tr(Q(\tau))e^{\int_{\tau}^{t} 2\mu_M(A(\xi))d\xi}d\tau, \tag{6.49a}$$

$$tr(P(t)) \geq tr(P_0)e^{\int_{t_0}^{t} 2\mu_m(A(\xi))d\xi} + \int_{t_0}^{t} tr(Q(\tau))e^{\int_{\tau}^{t} 2\mu_m(A(\xi))d\xi}d\tau. \tag{6.49b}$$

for all $t \geq t_0$.

Proof. By Theorem 6.2, the unique solution to (6.48) is given by

$$P(t) = \Phi(t,t_0)P_0\Phi^H(t,t_0) + \int_{t_0}^{t} \Phi(t,\tau)Q(\tau)\Phi^H(t,\tau)d\tau, \quad t \geq t_0 \tag{6.50}$$

where $\Phi(t,t_0)$ is the transition matrix of the linear time-varying system

$$\dot{x}(t) = A^H(t)x(t).$$

Since all eigenvalues of $P(t)$ are real, taking trace on both sides of the solution, we have

$$\mathrm{tr}(P(t)) = \mathrm{tr}(\Phi(t,t_0)P_0\Phi^H(t,t_0)) + \int_{t_0}^{t} \mathrm{tr}(\Phi(t,\tau)Q(\tau)\Phi^H(t,\tau))d\tau. \tag{6.51}$$

Applying Lemma 6.2 to the two terms at the right-hand side of (6.51) yields respectively

$$\mathrm{tr}(P_0)e^{\int_{t_0}^{t} 2\mu_M(A(\xi))d\xi} \geq \mathrm{tr}(\Phi(t,t_0)P_0\Phi^H(t,t_0)) \geq \mathrm{tr}(P_0)e^{\int_{t_0}^{t} 2\mu_m(A(\xi))d\xi}, \quad (6.52a)$$

$$\int_{t_0}^{t} \mathrm{tr}(Q(\tau))e^{\int_{\tau}^{t} 2\mu_M(A(\xi))d\xi} \geq \int_{t_0}^{t} \mathrm{tr}(\Phi(t,\tau)Q(\tau)\Phi^H(t,\tau))d\tau$$
$$\geq \int_{t_0}^{t} \mathrm{tr}(Q(\tau))e^{\int_{\tau}^{t} 2\mu_m(A(\xi))d\xi}d\tau. \quad (6.52b)$$

Applying (6.52) to (6.51) results in (6.49). ■

Remark 6.9 *[138] considers a special case of the linear matrix equation (6.48) given by*

$$\dot{P}(t) = A^\top P(t) + P(t)A + Q, \quad P(0) = P^\top(0) = P_0 > 0 \quad (6.53)$$

where $A \in \mathbb{R}^{n\times n}$, $Q = BB^\top \in \mathbb{R}^{n\times n}$ and A is a stable matrix. The upper and lower bounds for the trace of the solution of (6.53) were given by

$$tr(P(t)) \leq \left(tr(P_0) + \frac{tr(Q)}{2\mu_M(A)}\right)e^{2\mu_M(A)t} - \frac{tr(Q)}{2\mu_M(A)}, \quad (6.54a)$$
$$tr(P(t)) \geq \left(tr(P_0) - \frac{tr(Q)}{2\mu_M(-A)}\right)e^{-2\mu_M(-A)t} + \frac{tr(Q)}{2\mu_M(-A)} \quad (6.54b)$$

for all $t \geq 0$. It is easy to see that (6.49a) is identical to (6.54a) with $t_0 = 0$. Also, since $\mu_M(-A) = -\mu_m(A)$ for any stable real matrix A, (6.49b) is identical to (6.54b). We can also recover the bounds on the steady-state solution to (6.53) given in [138].

Remark 6.10 *It is observed from the inequality (6.49) that the bounds of $P(t)$, either in terms of the trace or eigenvalues, are affected by two factors: (1) the traces of Q and P_0, and (2) the minimal and the maximal eigenvalue of the matrix $A + A^H$.*

Remark 6.11 *If the steady-state solution of (6.48), P_{ss}, that is, the solution given by (6.50) when t approaches to infinity, exists, equation (6.50) evaluated at $t = \infty$ gives the solution to the following Lyapunov matrix equation*

$$0 = A_{ss}^H P_{ss} + P_{ss}A_{ss} + Q_{ss}. \quad (6.55)$$

The lower trace bound of the solution to (6.55) can be obtained from (6.49b) directly by replacing t with ∞. However, the upper trace bound of the steady-state solution may not be able to obtain by directly applying (6.49a) because the right-hand side of (6.49a) may go to infinity which is meaningless to define an upper bound. The trace bound on the solution to the Lyapunov matix equation in the form of (6.55) will be presented by Lemma 6.4 in Section 6.2.

The upper and lower bounds for the trace of the solution to the time-varying linear matrix differential equation are derived in this section. Previous work ([138] and [139]) gave bounds for the time-invariant linear matrix differential equation; the results can be used only for time-invariant systems. Whereas, the results of Theorem 6.5 can be applied to linear time-varying systems.

6.2 Lyapunov equation

In this section, the solution to the Lyapunov matrix equation and the bound on the solution will be considered.

6.2.1 Solutions to the Lyapunov equation

Unlike the linear matrix differential equation in the form of (6.17), the Lyapunov matrix equation with time-varying coefficient matrices in the following general form

$$A_1(t)P(t) + P(t)A_2(t) = Q(t) \tag{6.56}$$

where $A_1(t) \in \mathbb{C}^{n\times n}$, $A_2(t) \in \mathbb{C}^{n\times n}$ and $Q(t) \in \mathbb{C}^{n\times n}$, may not have a unique solution. For example, if $A_1(t) = A_2(t) = Q(t) = 0$, any $P(t)$ can be the solution to (6.56). Also, equation (6.56) may not have a solution.

The case where $A_1(t)$, $A_2(t)$ and $Q(t)$ are constant is of importance. When A_1 and A_2 are stable matrices, one has the following theorem on the solution to the corresponding Lyapunov equation.

Theorem 6.6 *If all eigenvalues of A_1 and A_2 have negative real parts, then the Lyapunov equation*

$$A_1 P + P A_2 + Q = 0 \tag{6.57}$$

where $A_1 \in \mathbb{C}^{n\times n}$, $A_2 \in \mathbb{C}^{n\times n}$ and $Q \in \mathbb{C}^{n\times n}$, has a unique solution. The solution P is given by the convergent integral

$$P = \int_0^\infty e^{A_1 t} Q e^{A_2 t} dt. \tag{6.58}$$

Proof. The integral is convergent since it is a sum of terms of the form $t^k e^{(\lambda_i+\lambda_j)t}$ where λ_i and λ_j are eigenvalues of A_1 and A_2, respectively, and k is a nonnegative integer. Since $\text{Re}(\lambda_i + \lambda_j) < 0$, $\int_0^\infty t^k e^{(\lambda_i+\lambda_j)t} dt$ converges. Hence, P exists and bounded.

Notice that

$$\frac{d}{dt} e^{At} = A e^{At} = e^{At} A, \tag{6.59}$$

where the second equality is obtained by using the the property that e^{At} and A commute. Thus, we have

$$\begin{aligned}
A_1 P + P A_2 + Q &= \int_0^\infty (A_1 e^{A_1 t} Q e^{A_2 t} + e^{A_1 t} Q e^{A_2 t} A_2) dt + Q && (6.60)\\
&= \int_0^\infty \frac{d}{dt} (e^{A_1 t} Q e^{A_2 t})\ dt + Q && (6.61)\\
&= e^{A_1 t} Q e^{A_2 t}|_{t=\infty} - e^{A_1 t} Q e^{A_2 t}|_{t=0} + Q && (6.62)\\
&= 0 && (6.63)
\end{aligned}$$

The uniqueness of the solution can be shown by proving that the equation

$$A_1 P + P A_2 = 0 \tag{6.64}$$

has only one solution $P = 0$. Observe that $L(P) \triangleq A_1 P + P A_2$ is a linear mapping of $\mathcal{C}^{n^2}$ to $\mathcal{C}^{n^2}$. Because there exists a P, given by (6.65), such that $L(P) = Q$ for any given Q, the range space of the mapping $L(P)$ is n^2 dimensional. Hence the dimension of the null space of $L(P)$ is zero, that is, the only solution to $L(P) = 0$ is $P = 0$. This proves that the solution given by (6.58) is unique. ■

A special case of (6.57) is

$$A^\top P + PA + Q = 0 \tag{6.65}$$

where $A \in \mathbb{R}^{n\times n}$ and $Q \in \mathbb{R}^{n\times n}$, and A is stable. The Lyapunov equation (6.65) is very important in linear systems. The following theorem describes important properties of the Lyapunov equation (6.65).

Theorem 6.7 *If A is stable, then the Lyapunov equation (6.65) has a unique solution. Moreover, the solution is given by the convergent integral*

$$P = \int_0^\infty e^{A^\top t} Q e^{At} dt. \tag{6.66}$$

In additional, if $Q = Q^\top$, then $P = P^\top$ has the same sign as Q.

Proof. The proof for (6.66) being the solution to (6.65) and this solution being unique is similar to that for Theorem 6.6. The last statement in Theorem 6.7 is easy to prove since (1) $e^{A^\top t}$ and e^{At} are nonsingular matrices, and (2) Q can be decomposed as $Q = M^\top M$ (if $Q \geq 0$) or $-Q = M^\top M$ (if $Q \leq 0$). Refer to the following Schur triangularization theorem. ■

Theorem 6.8 ***(Schur triangularization theorem** [140, p. 67]) If $M \in \mathbb{C}^{n\times n}$, then there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that $T = UAU^H$ is an upper triangular matrix with the characteristic roots of M along the main diagonal. If $M \in \mathbb{R}^{n\times n}$, then U may be chosen to be a real orthogonal matrix. The matrix M is normal if and only if T is diagonal.*

6.2.2 Trace bounds for the solution to the Lyapunov equation

Lemma 6.4 *Assume that the solution to the Lyapunov algebraic equation*

$$A^H P + PA + Q = 0 \tag{6.67}$$

where $A \in \mathbb{C}^{n\times n}$, $Q = Q^H \in \mathbb{C}^{n\times n}$ *and* $Q \geq 0$ *exists and* $\mu_M(A) < 0$. *Then, the trace of the solution to the above equation is bounded by*

$$-\frac{tr(Q)}{2\mu_M(A)} \leq tr(P(t)) \leq -\frac{tr(Q)}{2\mu_m(A)}. \tag{6.68}$$

Proof. From Theorem 6.6, it is obtained that the solution to (6.67) is given by

$$P = \int_0^\infty e^{A^H t} Q e^{At} dt. \tag{6.69}$$

Then one obtains

$$\mathrm{tr}(P) = \int_0^\infty \mathrm{tr}\Big(e^{A^H t} Q e^{At}\Big) dt. \tag{6.70}$$

Applying Lemma 6.3 to (6.70) results in

$$\mathrm{tr}(Q) e^{\int_0^\infty 2\mu_m(A^H)d\xi} \leq \mathrm{tr}(P) \leq \mathrm{tr}(Q) e^{\int_0^\infty 2\mu_M(A^H)d\xi}. \tag{6.71}$$

Since $\mu_m(A^H) = \mu_m(A)$, $\mu_M(A^H) = \mu_M(A)$, and $\mu_M(A) < 0$ which implies $\mu_m(A) < 0$, it follows that

$$tr(Q) e^{\int_0^\infty 2\mu_m(A)d\xi} = -\frac{\mathrm{tr}(Q)}{2\mu_m(A)}, \tag{6.72a}$$

$$tr(Q) e^{\int_0^\infty 2\mu_M(A)d\xi} = -\frac{\mathrm{tr}(Q)}{2\mu_M(A)}. \tag{6.72b}$$

Combining (6.71) and (6.72) yields (6.68). ■

Other results on the bounds for the solution to (6.67) for the case where A is a real stable matrix can be found in [142, 143, 138].

Remark 6.12 *In Lemma 6.5, if* $Q \leq 0$ *and other hypotheses are the same, then the trace of the solution of (6.67) is bounded by*

$$-\frac{tr(Q)}{2\mu_m(A)} \leq tr(P(t)) \leq -\frac{tr(Q)}{2\mu_M(A)}.$$

These bounds are obtained by considering the following Lyapunov matrix equation

$$A^H(-P) + (-P)A + (-Q) = 0.$$

Lemma 6.4 illustrates the relationship between the traces (eigenvalues) of three matrices, A, P and Q, when (6.67) is satisfied. An important quantity regarding the matrices P and Q for the control problem is the "condition number" which is defined as $\dfrac{\lambda_{min}(Q)}{\lambda_{max}(P)}$. Consider the following problem. Given a system

$$\dot{x} = A_o x + Bu + f(x) \tag{6.73}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $f(x) \in \mathbb{R}^n$, $A_0 \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. Assume (A_o, B) is a controllable pair, and $f(x)$ is unknown but is bounded by $\|f(x)\| \leq \frac{1}{2}c\|x\|$. We want to design a full state feedback control $u = Kx$ such that the closed-loop system

$$\begin{aligned}\dot{x} &= (A_o + BK) + f(x)\\ &\triangleq Ax + f(x)\end{aligned} \tag{6.74}$$

is stable. Obviously, A should be stable. Now, consider the following Lyapunov function candidate

$$V = x^\top P x$$

where $P = P^\top \in \mathbb{R}^n$. The time derivative of V is given by

$$\begin{aligned}\dot{V} &= x^\top(A^\top P + PA)x + 2x^\top P f(x)\\ &\leq x^\top(A^\top P + PA) + c\lambda_{max}(P)x^\top x.\end{aligned} \tag{6.75}$$

Since A is a stable matrix, for any given $Q = Q^\top > 0 \in \mathbb{R}^n$, a unique symmetric positive definite matrix P exists for the following Lyapunov equation

$$A^\top P + PA + Q = 0.$$

Hence, from (6.75), one obtains

$$\dot{V} \leq -x^\top Qx + c\lambda_{max}(P)x^\top x$$
$$\leq -(\lambda_{min}(Q) - c\lambda_{max}(P))x^\top x.$$

Therefore, if

$$\frac{\lambda_{min}(Q)}{\lambda_{max}(P)} > c \tag{6.76}$$

then $\dot{V} \leq -\alpha x^\top x$ where $\alpha > 0$, which implies that the closed-loop system (6.74) is exponentially stable. One interesting question is, for any constant c, is it possible to have (6.76) satisfied? Unfortunately, there is no general solution to this question. The complexity of this problem can be seen from Lemma 6.4. Consider the case when $\lambda_{max}(A) < 0$. From Lemma 6.4, one has

$$-2\mu_m(A) \leq \frac{\mathrm{tr}(Q)}{\mathrm{tr}(P)} \leq -2\mu_M(A). \tag{6.77}$$

By the fact that

$$\lambda_{min}(Q) \leq \frac{\mathrm{tr}(Q)}{n}, \quad \lambda_{max}(P) \geq \frac{\mathrm{tr}(Q)}{n},$$

one has

$$\frac{\lambda_{min}(Q)}{\lambda_{max}(P)} \leq \frac{\mathrm{tr}(Q)}{\mathrm{tr}(P)} \leq -2\mu_M(A). \tag{6.78}$$

From (6.78), it is seen that the condition number for the Lyapunov matrix equation (6.67) is bounded by the maximum eigenvalue of $A + A^H$. It is well-known that if (A_o, B) is a controllable pair, then the eigenvalue of $A = A_o + BK$ can be arbitrarily assigned. However, there is no direct relationship between the eigenvalues of A and the eigenvalues of $A + A^H$.

The following two results [144] pertinent to the condition number of the Lyapunov equation (6.67) are also important.

1. For a given stable matrix $A \in \mathbb{R}^{n\times n}$, the maximum value of $\dfrac{\lambda_{min}(Q)}{\lambda_{max}(P)}$ can be obtained if Q is chosen as an identity matrix.

2. If a stable matrix $A \in \mathbb{R}^{n\times n}$ is in the block diagonal form, that is,

$$A = \text{diag}\left(\begin{bmatrix} -\sigma_1 & -\omega_1 \\ -\omega_1 & -\sigma_1 \end{bmatrix}, \ldots, \begin{bmatrix} -\sigma_i & -\omega_i \\ -\omega_i & -\sigma_i \end{bmatrix}, -\sigma_{2i+1}, \ldots, -\sigma_n \right),$$

then there exists a special pair of matrices P and Q satisfying (6.67) and

$$\frac{\lambda_{min}(Q)}{\lambda_{max}(P)} = 2\min_i\{\sigma_i\}, \quad i = 1, 2, \ldots, n.$$

6.3 Algebraic Riccati Equation

In this section, we consider the characterization of solvability of the algebraic Riccati equation

$$A_1P + PA_2 + PRP + Q = 0 \tag{6.79}$$

where $A_1 \in \mathbb{C}^{n\times n}$, $A_2 \in \mathbb{C}^{n\times n}$, $R \in \mathbb{C}^{n\times n}$ and $Q \in \mathbb{C}^{n\times n}$. The ARE (6.79) permits a functional treatment of two linear operator equations. The results are summarized from [145, 146] and discussed. Later, the case where $A_1 = A_2 = A = A^H$ of (6.79), which is very important in applications, such as optimization theory, control theory and linear filtering theory, is studied from the perspective of existence of positive (negative) solution and stability.

6.3.1 Asymmetric Algebraic Riccati Equation

The necessary and sufficient conditions for the existence of the solution to the ARE (6.79) were investigated in [147, 148, 149, 150, 151, 145, 146]. The main result can be illustrated by the following theorem.

Theorem 6.9 *Let the associated matrix of the ARE (6.79) $\mathcal{H}$ and any polynomial $f(x)$ with complex coefficients and the order less or equal to n be given by*

$$\mathcal{H} = \begin{bmatrix} A_2 & R \\ -Q & -A_1 \end{bmatrix}, \quad f(\mathcal{H}) = \begin{bmatrix} M & E \\ N & F \end{bmatrix}. \tag{6.80}$$

A matrix P is a solution to the ARE (6.79) if and only if either one of the following conditions are satisfied:

1. *P satisfies the identity*

$$P \begin{bmatrix} M \\ E \end{bmatrix} = \begin{bmatrix} N \\ F \end{bmatrix} \tag{6.81}$$

   and M^{-1} and/or E^{-1} exists.

2. *P satisfies the identity*

$$\begin{bmatrix} E \\ F \end{bmatrix} P = \begin{bmatrix} -M \\ -N \end{bmatrix} \tag{6.82}$$

   and E^{-1} and/or F^{-1} exists.

Proof. See [145, 146]. ∎

Remark 6.13 *Two conditions of Theorem 6.9, given by (6.81) and (6.82), respectively, stem from one property of matrix $\mathcal{H}$, that is, $\mathcal{H}$ is similar to the matrix $\begin{bmatrix} A_2 + RP & R \\ 0 & -A_1 - PR \end{bmatrix}$, since*

$$\begin{bmatrix} I & 0 \\ -P & I \end{bmatrix} \begin{bmatrix} A_2 & R \\ -Q & -A_1 \end{bmatrix} \begin{bmatrix} I & 0 \\ P & I \end{bmatrix} = \begin{bmatrix} A_2 + RP & R \\ 0 & -A_1 - PR \end{bmatrix}. \tag{6.83}$$

6.3.2 Symmetric Algebraic Riccati Equation

The solutions to the following ARE was first studied in [152].

$$A^H P + PA - PRP + Q = 0 \tag{6.84}$$

where $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times n}$ and $C \in \mathbb{C}^{n\times n}$.

It is necessary to investigate the solutions of (6.84) because (6.84) is the steady-state form of the Riccati differential equations with constant coefficient matrices. Also the ARE (6.84) arises in the multiwire lines [153], linear filtering and prediction, and optimal control.

Properties of the associated matrix of the ARE

The ARE (6.84) is associated with the following $2n \times 2n$ matrix

$$\mathcal{H} = \begin{bmatrix} A & -R \\ -Q & -A^H \end{bmatrix}. \tag{6.85}$$

We use the notation

$$a_i = \begin{bmatrix} b_i \\ c_i \end{bmatrix}$$

where $b_i \in \mathbb{C}^n$ and $c_i \in \mathbb{C}^n$, for the $2n$-dimensional eigenvector of $\mathcal{H}$ corresponding to the eigenvalue λ_i. The properties of $\mathcal{H}$ can be summarized as follows.

Property P6.1 *If R and Q are hermitian, then H is Hamiltonian, that is, $\mathcal{H}$ satisfies the following equality:*

$$\mathcal{H}^H T + T\mathcal{H} = 0 \tag{6.86}$$

where $T = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$, *$I$ is the identity matrix of dimension $n \times n$, 0 is a zero matrix of appropriate dimension.*

Proof. By direct matrix computation.

Property P6.2 *If R and Q are hermitian, $\mathcal{H}$ has at most n eigenvalues with positive (negative) real parts. Moreover, if λ is an eigenvalue of $\mathcal{H}$, then so is $-\bar{\lambda}$.*

Proof. Assume λ is an eigenvalue of $\mathcal{H}$ and $\begin{bmatrix} x \\ y \end{bmatrix}$ be the corresponding eigenvector, where $x, y \in \mathbb{C}^n$. Thus we have

$$\begin{bmatrix} A & -R \\ -Q & -A^H \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix},$$

which is equivalent to

$$\begin{bmatrix} A^H & -Q \\ -R & -A \end{bmatrix} \begin{bmatrix} -y \\ x \end{bmatrix} = -\lambda \begin{bmatrix} -y \\ x \end{bmatrix}$$

due to the fact that $R = R^H$ and $Q = Q^H$. Hence, $-\lambda$ is an eigenvalue of $\mathcal{H}^H$, which in turn implies that $-\bar{\lambda}$ is an eigenvalue of $\mathcal{H}$ and $\mathcal{H}$ has at most n eigenvalues with positive (negative) real parts.

Property P6.3 *Let $b_1, \ldots, b_n$ be eigenvectors of $\mathcal{H}$ corresponding to eigenvalues $\lambda_1, \ldots, \lambda_n$, and assume that $\begin{bmatrix} b_1 & \ldots & b_n \end{bmatrix}^{-1}$ exists. If $\bar{\lambda}_j \neq -\lambda_k, 1 \leq j, k \leq n$, then*

$$P = \begin{bmatrix} c_1 & \ldots & c_n \end{bmatrix} \begin{bmatrix} b_1 & \ldots & b_n \end{bmatrix}^{-1}$$

is Hermitian.

Proof. See [152, 154].

Explicit expression for the solutions of the Algebraic Riccati Equation

The explicit expression for the solutions to the ARE (6.84) was first investigated in [152], where the solutions are explicitly expressed by the eigenvalues of the associated matrix $\mathcal{H}$ given in (6.85). The matrix $\mathcal{H}$ is assumed to have a diagonal Jordan form. Later, in [154], the assumption on the matrix $\mathcal{H}$ to have a diagonal Jordan form is relaxed to that $\mathcal{H}$ must have a Jordan block form. The solution of (6.84) is described by the Property P6.3.

6.3.3 Trace bounds for the solution to the Algebraic Riccati Equation

Consider the following ARE:

$$A^\top P + PA + PRP + Q = 0 \tag{6.87}$$

where $R = R^\top \geq 0 \in \mathbb{R}^{n\times n}$, $Q = Q^\top \geq 0 \in \mathbb{R}^{n\times n}$ and $A \in \mathbb{R}^{n\times n}$. Finding necessary and sufficient conditions for the ARE is of considerable interest. Notice that, in (6.87), if $R = R^\top \leq 0$ and other matrices remain the same properties, the ARE (6.87) becomes

$$A^\top P + PA - PRP + Q = 0. \tag{6.88}$$

The ARE (6.88) appears in the systems and control areas [141, 155, 143]. There has been a strong interest in determining the bounds on solutions to the Lyapunov equation and the ARE (6.88). The ARE (6.87) is also important in linear control designs for nonlinear systmes; for example, see Chapter 2. Extensive work has been reported on this topic in the literature [156, 157, 158]. In this section, the trace bounds on the solution to the ARE (6.87) are derived. Based on the trace bounds, some useful necessary conditions for the existence of a positive definite solution to the ARE (6.87) are also derived. The necessary conditions obtained are easily computable.

Theorem 6.10 *The trace bounds on the solution to the ARE (6.87) satisfy the following inequalities.*

$$x \leq \frac{n}{\lambda_{min}(R)}\left(-\mu_m(A) + \sqrt{\mu_m(A)^2 - \frac{\lambda_{min}(R)tr(Q)}{n}}\right), \text{ if } \lambda_{min}(R) > 0, \tag{6.89a}$$

$$x \leq -\frac{tr(Q)}{2\mu_m(A)}, \quad \text{if } \lambda_{min}(R) = 0 \tag{6.89b}$$

where $x = trace(P)$.

Proof. Taking trace on both sides of the ARE (6.87) results in

$$\mathrm{tr}(A^\top P) + \mathrm{tr}(PA) + \mathrm{tr}(PRP) + \mathrm{tr}(Q) = 0. \tag{6.90}$$

Using the matrix trace property $\mathrm{tr}(MN) = \mathrm{tr}(NM)$, we obtain

$$\mathrm{tr}(A^\top P) + \mathrm{tr}(PA) = 2\mathrm{tr}\left(P\frac{A + A^\top}{2}\right). \tag{6.91}$$

Now, consider the following inequalities (see Lemma 6.1):

$$\mu_m(A)\mathrm{tr}(P) \leq \mathrm{tr}\left(P\frac{A + A^\top}{2}\right) \leq \mu_M(A)\mathrm{tr}(P), \tag{6.92}$$

$$\lambda_{min}(R)[\mathrm{tr}(P)]^2/n \leq \mathrm{tr}(PRP) \leq \lambda_{max}(R)[\mathrm{tr}(P)]^2 \tag{6.93}$$

where (6.93) is obtained by using Lemma 6.1 twice. Using (6.91), (6.92) and (6.93) in (6.90) yields

$$2\mu_m(A)\mathrm{tr}(P) + \frac{\lambda_{min}(R)}{n}\mathrm{tr}(P)^2 + \mathrm{tr}(Q) \leq 0. \tag{6.94}$$

Equation (6.94) is equivalent to

$$\left(\mathrm{tr}(P) + n\frac{\mu_m(A)}{\lambda_{min}(R)}\right)^2 + \frac{n}{\lambda_{min}(R)^2}\left(\lambda_{min}(R)\mathrm{tr}(Q) - n\mu_m(A)^2\right) \leq 0, \text{ if } \lambda_{min}(R) > 0, \tag{6.95a}$$

$$2\mu_m(A)\mathrm{tr}(P) + \mathrm{tr}(Q) \leq 0, \quad \text{if } \lambda_{min}(R) = 0. \tag{6.95b}$$

Inequalities given by (6.95) give rise to (6.89). ■

From the Theorem 6.10 and its proof, we have the necessary conditions for the existence of a solution to the ARE (6.87), which are given by the following corollary.

Corollary 6.2 *Suppose that $P \geq 0$ is a solution to the ARE (6.87). It is necessary that the following be true:*

$$\lambda_{min}(R)tr(Q) - n{\mu_m}^2(A) < 0, \tag{6.96a}$$

$$\mu_m(A) < 0. \tag{6.96b}$$

Proof. From (6.95a), it is clear that the second term in the left-hand side of the inequality must be negative. Hence, (6.96a) must be true. From (6.89a) and (6.89b), it is clear that (6.96b) must be true. ■

The following lemma gives an interesting result related to the condition (6.96b).

Lemma 6.5 *If A is Hurwitz, then $\mu_m(A) < 0$.*

Proof. Let $v \neq 0$ be an eigenvector of A corresponding to the eigenvalue of A. Then

$$Av = \lambda(A)v. \tag{6.97}$$

Taking the complex conjugate transpose on both sides of (6.97) results in

$$v^H A^\top = \bar{\lambda}(A)v^H. \tag{6.98}$$

Pre-multiplying both sides of (6.97) by v^H and post-multiplying both sides of (6.98) by v, and adding the resulting equations yield

$$v^H(A + A^\top)v = (\lambda(A) + \bar{\lambda}(A))v^H v. \tag{6.99}$$

Because A is Hurwitz, the real part of any eigenvalue of A is negative. Hence

$$(\lambda(A) + \bar{\lambda}(A))v^H v = 2\mathrm{Re}(\lambda(A))v^H v < 0. \tag{6.100}$$

From (6.99), (6.100), and S being symmetric, one has

$$\mu_m(A)v^H v \leq v^H \left(\frac{A + A^\top}{2}\right) v = \mathrm{Re}(\lambda(A))v^H v < 0. \tag{6.101}$$

Therefore, $\mu_m(A) < 0$. ■

Example 6.1 *Consider the following example.*

$$A = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad Q = \varepsilon \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{6.102}$$

where $a_1, a_2 \in \mathbb{R}$ and $\varepsilon > 0$.

Let $P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}$. For any a_1, a_2, one solution of the ARE is

$$\begin{cases} p_1 = -a_1 \pm \sqrt{a_1^2 - \varepsilon} \\ p_2 = 0 \\ p_3 = -a_2 \pm \sqrt{a_2^2 - \varepsilon} \end{cases} \tag{6.103}$$

If $|a_1| = |a_2|$, in addition to (6.103), we also have other solutions given by

$$\begin{cases} p_1 = -a_1 \pm \sqrt{a_1^2 - \varepsilon - p_2^2} \\ p_3 = -a_2 \mp \sqrt{a_2^2 - \varepsilon - p_2^2} \end{cases} \tag{6.104}$$

and p_2 is arbitrary.

For (6.103), the necessary and sufficient conditions for P to be a symmetric positive definite matrix are

$$a_1 < 0, \;\; a_2 < 0, \;\; \varepsilon - a_1^2 < 0, \;\text{ and } \varepsilon - a_2^2 < 0. \tag{6.105}$$

Similarly, the necessary and sufficient conditions for P to be a symmetric positive definite matrix, for the solutions given by (6.104) ($|a_1| = |a_2|$), are

$$a_1 < 0, \;\; a_2 < 0, \;\; \varepsilon - a_1^2 - p_2^2 < 0, \;\text{ and } \varepsilon - a_2^2 - p_2^2 < 0. \tag{6.106}$$

The condition given by (6.96) for this example is equivalent to

$$\varepsilon - a_{min}^2 < 0, \tag{6.107a}$$

$$a_{min} < 0 \tag{6.107b}$$

where $a_{min} = \min(a_1, a_2)$. For (6.105) and (6.106) to be true, it is necessary that (6.107) must be true.

6.4 Summary

In this chapter, explicit expression of the solution and the bounds on the solution of a class of linear matrix differential equations were studied. Trace bounds of solution to Lyapunov matrix equation and algebraic Riccati equation were also derived. A set of easily computable necessary conditions for the existence of solutions to a class of algebraic Riccati equation was given.

CHAPTER 7

SUMMARY AND FUTURE WORK

A chapter by chapter summary of the thesis is given below.

In Chapter 2, stabilization of a class of Lipschitz nonlinear systems via output feedback was considered. This class of nonlinear systems are not required to satisfy matching conditions. A new full-state feedback control design is first addressed, then a new observer is proposed. Further, an output feedback control scheme which combines the results from the full-state feedback control and the observer design is provided. Both linear full-state feedback controller and Luenberger-like observer are exponentially stable. The output feedback controller achieves exponential stabilization of the closed-loop system. Sufficient conditions are developed for the design of the proposed observer and controller, and these sufficient conditions are easy to check. A numerical simulation example is given. To the author's best knowledge, this is the first time that a stable output feedback controller is designed for unmatched Lipschitz nonlinear systems.

In Chapter 3, decentralized output feedback control of large-scale systems with quadratically bounded nonlinear interconnections on the state of the overall system is addressed. The key feature of the decentralized output feedback controller is that the control of each subsystem can use only the output information of the local system. Two approaches are provided: the LMI approach and the ARE approach. In the LMI approach, the decentralized control gain matrices and the decentralized observer gain matrices are obtained by solving two LMI problems. These two LMI problems were shown to be feasible under the assumption that each local system is controllable and observable. The proposed LMI solution in this chapter does not require that the input matrix of each subsystem be invertible. Decentralized output feedback control design is also solved by the ARE approach, in which the problem of finding the control gain matrices and observer gain matrices is reduced to the problem of solving Algebra Riccati Equations under sufficient conditions. In both approaches, exponential stabilization of the overall system under the proposed decentralized output feedback control is achieved. A numerical example is provided to verify the ARE approach.

Chapter 4 investigates the design of stable adaptive controller and observer for a class of nonlinear systems. The class of nonlinear systems considered contain product terms of unmeasurable states and unknown parameters, which are boarder than those systems which only have product terms of unknown parameters and known functions. The nonlinear system is cast into a modified form. The modified representation of the dynamics of the system is always feasible and has the advantage that the number of filters can be reduced, when the controller and observer design are based on the modified dynamics. The design strategy is illustrated by a simple example first and then extended to the general case. A parameter–dependent Lyapunov function is used to design the controller and observer. Asymptotic convergence of the output error is obtained and all signals in the closed-loop system are bounded. Simulation results on examples are shown and discussed for the proposed scheme. The key feature in the proposed design is the relaxation of the requirement on the dynamics of unmeasurable states. Unlike other papers where the nominal part of the dynamics of unmeasurable states is required to be asymptotically stable, the proposed design requires the unmeasurable dynamics to be stable.

On-line estimation of time-varying parameters and adaptive output feedback control design for mechanical systems with time-varying parameters and time-varying disturbances were addressed in Chapter 5. A large amount of literature on time-varying systems is reviewed and relevant topics on estimation and control of time-varying systems are discussed and summarized. The time-varying parameters appear linearly in the dynamics of the system. A strategy for approximating a time-varying parameter locally by a polynomial is presented. The estimation of time-varying parameters in linear plants is transformed to the estimation of time-invariant parameters of the system in different time intervals via local polynomial approximations. The estimation time is divided into small intervals; in each interval the time-varying parameter is approximated by a time polynomial with unknown constant coefficients. A condition for resetting of the estimate at the beginning of each interval is given; this guarantees that the estimate of a time-varying parameter is continuous; and also allows for the coefficients of the polynomial to be different in different time intervals. It is shown that the proposed strategy for the estimation of time-varying parameters is applicable with simple modifications of the least-squares algorithm with covariance resetting and the gradient algorithm. Simulation results of the proposed algorithm on a number of examples with time-varying parameters are shown and discussed. A new adaptive control algorithm for mechanical systems with time-varying parameters and/or time-varying disturbances is proposed and investigated. The proposed method does not assume any structure to the time-varying parameters or disturbances. A novel experiment is designed by using a two-link mechanical manipulator to investigate the proposed algorithm experimentally. Simulation and experimental results are shown and discussed.

In Chapter 6, matrix equations, especially, linear differential matrix equation, Lyapunov equation and algebraic Riccati equation, are considered. A large amount of literature is reviewed. Important issues of matrix equations, such as the conditions for the existence of a solution to matrix equations, the expressions of solutions, and upper and lower bounds of the solution to matrix equations are investigated and new results are given. New results on the bounds obtained in this report are useful since the considered equations are encountered in many applications in systems and control.

There are some challenging open issues related to the problems studied in the thesis.

For the output feedback control of Lipschitz nonlinear systems in Chapter 2, the relaxation of the sufficient conditions can be further investigated. Because the sufficient conditions directly depend on the number δ and the Lipschitz constant γ, finding a way to simultaneously decrease δ and increase γ must be investigated. The LMI technique may be a possible way to solve this problem.

Application of the techniques, studied in Chapter 3, to decentralized output feedback control of large-scale systems must be investigated. Design of decentralized controllers and decentralized observers poses challenging problems due to the nonlinear, and often uncertain, interconnections between subsystems of large-scale systems.

Future work should focus on the inclusion of coupled terms of the unknown parameters and unmeasured states in the unmeasurable state dynamics. Future research should also focus on the investigation of the existence of parameter independent state diffeomorphisms that will transform a general nonlinear systems to the class of systems considered in Chapter 4.

Robustness of the proposed algorithms in Chapter 5 to modeling error in the linear time-varying parametric model must be considered in the future. Output feedback control design for broader nonlinear systems with time-varying parameters and time-varying disturbances should also be investigated.

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