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PROJECTION FUNCTIONS AND CLASSES
OF CENTRALLY SYMMETRIC CONVEX BODIES

A DISSERTATION
APPROVED FOR THE DEPARTMENT OF MATHEMATICS

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ABSTRACT

We study various classes of centrally symmetric bodies. The starting point is provided by zonotopes. These are vector sums of line segments and are characterized by the high degree of symmetry that they exhibit. Zonoids are limits of zonotopes and are therefore also highly symmetric. They can be characterized from two rather different points of view. The first is via an integral representation for their support functions. The second is in terms of orthogonal bodies. Associated with each zonoid Z there is an orthogonal body O such that the one dimensional projections of Z have a length equal to the $(d-1)$-dimensional volumes of the projections of O onto the orthogonal space. Many extensions of these ideas have appeared in the literature. These extensions aim to provide a heirarchy of classes of centrally symmetric bodies stretching from the zonoids to the class of all centrally symmetric bodies. We show that these various techniques of extension lead to different heirarchies – contrary to conjectures that have been made. The extensions mentioned above require us to transform results involving functions on the sphere to functions on Grassmann manifolds of higher rank. To deal with the added complications that this involves, we make extensive use of harmonic analysis and representation theory of rotation groups.

We also study averages of projections and show that all convex bodies in $\mathbb{E}^d$ are determined by averages of their three dimensional projections. This is in contrast to the case of two dimensional projections where there is a single dimension in which such determination is not possible. On the other hand, it is similar to the case of k-dimensional projections with $k \geq d/2$. These results also involve harmonic analysis, this time in the more classical realm of spherical harmonics.

§1. Statement of Problems and Background

In this section, we will collect the basic notations and results used in this thesis. We will also state the main problems studied in this thesis and give background information on these problems. By $\mathbb{E}^d$ we denote d-dimensional Euclidean space ($d \geq 2$). By S^{d-1} we denote the $(d-1)$-dimensional unit sphere in $\mathbb{E}^d$. A set $A \subset \mathbb{E}^d$ is convex if for any two points x, $y \in \mathbb{E}^d$ and $0 \leq \lambda \leq 1$, we have that $(1-\lambda)x + \lambda y \in A$. A nonempty, compact and convex subset of $\mathbb{E}^d$ is called a convex body. The class of all convex bodies in $\mathbb{E}^d$ will be denoted by $\mathcal{K}^d$. If a convex body has interior points it is said to be d-dimensional. By $\mathcal{K}_0^d$ we denote the class of all d-dimensional convex bodies in $\mathbb{E}^d$.

If $K, L \in \mathcal{K}^d$, the *Minkowski addition* $K + L$ is defined by

$$K + L = \{x + y \mid x \in K, \ \ y \in L\}$$

and $K + L$ is again in $\mathcal{K}^d$. If λ is a real number, λK is defined by

$$\lambda K = \{\lambda x \mid x \in K\}.$$

If $\lambda = -1$, we usually write $-K$ instead of $(-1)K$. For any point $p \in \mathbb{E}^d$, we usually write $K + p$ instead of $K + \{p\}$. The convex body K is said to be centrally symmetric with respect to a point p if $-(K - p) = K - p$. If K is centrally symmetric with respect to the origin, then K is called a centered body. The class of all centrally symmetric convex bodies in $\mathbb{E}^d$ will be denoted by $\mathcal{C}^d$. Similarly, the class of all d-dimensional centrally symmetric convex bodies in $\mathbb{E}^d$ will be denoted by $\mathcal{C}_0^d$.

A polytope is the bounded intersection of finitely many closed half-spaces of $\mathbb{E}^d$. It can also be defined as the convex hull of finitely many points of $\mathbb{E}^d$. It is clear that a polytope is a convex body. Again, if a polytope in $\mathbb{E}^d$ has interior points it is said to be d-dimensional.

One of the most important functions related to a convex body K is the so called support function $h(K,\cdot)$. It is defined on $\mathbb{E}^d$ by

$$h(K,x) := \sup\{\langle x,y\rangle \mid y \in K\} \qquad \text{for each} \qquad x \in \mathbb{E}^d. \tag{1.1}$$

We often restrict the domain of a support function to S^{d-1} and refer to the support function defined on $\mathbb{E}^d$ as an extended support function. But we usually simply say support function if it is clear from the context whether a given support function should be viewed as a function of S^{d-1} or as the corresponding extended function on $\mathbb{E}^d$.

A function h defined on $\mathbb{E}^d$ is said to be convex if for all λ with $0 \le \lambda \le 1$ and all $x, y \in \mathbb{E}^d$, it satisfies the inequality

$$h((1-\lambda)x + \lambda y) \le (1-\lambda)h(x) + \lambda h(y).$$

A function h on $\mathbb{E}^d$ is said to be positively homogeneous of degree m if for any $\mu \ge 0$

$$h(\mu x) = \mu^m h(x).$$

We next list two properties of support function. These properties can be found in Groemer [1996] or Schneider [1993].

1. *The extended support function h of a convex body is a real valued convex function and positively homogeneous of degree 1. Conversely, if h is any real valued convex function on $\mathbb{E}^d$ that is positively homogeneous of degree 1, then there is exactly one convex body whose support function is h.*

2. *Let $h(K,u)$ and $h(L,u)$ be the support functions of K and L respectively. For any $\alpha \ge 0$, $\beta \ge 0$,*

$$h(\alpha K + \beta L, u) = \alpha h(K,u) + \beta h(L,u) \quad \textit{for each} \quad u \in S^{d-1}.$$

If X is $\mathbb{E}^d$ or S^{d-1}, we let $C(X)$ denote the set of all continuous functions on X, and $C_e(X)$ the set of all even continuous functions on X. We denote by $C^k(X)$

$(C^\infty(X))$ the set of k-times continuously (infinitely) differentiable functions on X. By $C_e^k(X)$ $(C_e^\infty(X))$ we denote the set of even functions in $C^k(X)$ $(C^\infty(X))$. When we say that K is a *smooth convex body (of degree k)*, we mean that its support function $h(K,\cdot)$ is in $C^k(S^{d-1})$, $k \geq 1$.

For the set of nonempty compact subsets of $\mathbb{E}^d$, one can define the Hausdorff metric. The Hausdorff metric is very convenient for analytic and geometric applications and can be found in various books in analysis and geometry. In what follows, we will restrict ourselves to the set $\mathcal{K}^d$ and define the Hausdorff metric on $\mathcal{K}^d$.

For $K, L \in \mathcal{K}^d$, the Hausdorff distance is defined by

$$\delta(K,L) := \max\{\sup_{x\in K}\inf_{y\in L}\|x-y\|, \sup_{x\in L}\inf_{y\in K}\|x-y\|\}$$

or, equivalently, by

$$\delta(K,L) := \min\{\lambda \geq 0 | K \subset L + \lambda B^d, L \subset K + \lambda B^d\}.$$

Here B^d is the unit ball in $\mathbb{E}^d$ with its center at the origin. It can be shown that δ is a metric and that $\mathcal{K}^d$ is a complete metric space (see for example Schneider [1993], p.47). The famous Blaschke selection theorem states that for each bounded sequence of convex bodies one can select a subsequence converging to a convex body. In the following sections, if we do not state otherwise, all metrical and topological notions of $\mathcal{K}^d$ will refer to the Hausdorff metric and the topology induced by it. The Hausdorff metric can be expressed in terms of support functions

$$\delta(K,L) = \sup\{|h(K,u) - h(L,u)| : u \in S^{d-1}\} := \| h(K,\cdot) - h(L,\cdot) \|_\infty$$

where $\| \,.\, \|_\infty$ denotes the maximum norm for real functions on S^{d-1}. Define a map

$$\begin{aligned} \phi: \quad & \mathcal{K}^d \to C(S^{d-1}) \\ & K \mapsto h(K,\cdot). \end{aligned}$$

This provides an isometric embedding of $\mathcal{K}^d$ into $C(S^{d-1})$, the space of real-valued continuous functions on the unit sphere.

We now have a metric on $\mathcal{K}^d$ and thus we can consider approximations. One of the most basic facts on approximation is that any convex body $K \in \mathcal{K}^d$ can be approximated by a sequence of polytopes. On the other hand, a convex body can also be approximated by sufficiently smooth bodies (see Schneider [1993], p.158). Thus for some of the problems we often work with polytopes or sufficiently smooth bodies and then obtain the results for general convex bodies by applying an approximation process. We can also talk about continuity of various functions defined on $\mathcal{K}^d$. For example, the volume function V_d is continuous on $\mathcal{K}^d$.

A (positive) measure on S^{d-1} is a non-negative, real-valued, σ-additive function on the σ-algebra $\mathcal{B}(S^{d-1})$ of Borel subsets of S^{d-1}. A signed measure on S^{d-1} is a real-valued, σ-additive function on the σ-algebra $\mathcal{B}(S^{d-1})$ of Borel subsets of S^{d-1}. A measure or a signed measure on S^{d-1} is said to be even if it is invariant under reflection in the origin.

By λ_{d-1} we denote the *spherical Lebesgue measure* on S^{d-1}. The volume of the unit ball B^d in $\mathbb{E}^d$ is given by

$$\kappa_d = \frac{\pi^{d/2}}{\Gamma(1+\frac{d}{2})},$$

and its surface area is

$$\omega_d = \lambda_{d-1}(S^{d-1}) = d\kappa_d.$$

Here $\Gamma(1) = 1$, $\Gamma(1/2) = \sqrt{\pi}$ and, for $x > 0$, $\Gamma(1+x) = x\Gamma(x)$.

For any convex body $K \in \mathcal{K}^d$ and $j \in \{1, 2, \cdots, d-1\}$, the j-th area measure $S_j(K)$ plays an important role in the study of convex geometry. We next will give a brief introduction to these surface area measures.

Let $K \in \mathcal{K}^d$. Since K is a convex body in $\mathbb{E}^d$, for any point $x \in \mathbb{E}^d \setminus K$ there corresponds a unique point $p(K, x) \in K$ that is nearest to x; the unit vector pointing from $p(K, x)$ towards x is denoted by $u(K, x)$. For $\omega \in \mathcal{B}(S^{d-1})$ and

$\epsilon > 0$ we let $V_\epsilon(K;\omega)$ denote the Lebesgue measure of the *local parallel set*

$$B_\epsilon(K;\omega) := \{x \in \mathbb{E}^d : 0 < \| x - p(K,x) \| \le \epsilon \quad \text{and} \quad u(K,x) \in \omega\}.$$

Thus $V_\epsilon(K;\omega)$ can be considered as a function of ϵ. It can be shown that this function can be written as a polynomial

$$V_\epsilon(K;\omega) = \frac{1}{d}\sum_{j=0}^{d-1} \binom{d}{j} \epsilon^{d-j} S_j(K;\omega).$$

It turns out that the set function $S_j(K,\cdot)$ thus defined is a positive measure on $\mathcal{B}(S^{d-1})$ and it is called the j-th area measure of K. If K is class C^2_+, that is, the boundary ∂K of K is twice continuously differentiable and has positive curvature, then

$$S_j(K,\omega) = \int_\omega s_j(u)\lambda_{d-1}(du),$$

where s_j is j-th normalized elementary symmetric function of the principal radii of curvature of ∂K. One of the main problems concerning area measures is the following: what are necessary and sufficient conditions for a measure on $\mathcal{B}(S^{d-1})$ to be the j-th area measure of a convex body? We will follow Firey and call it the Christoffel-Minkowski problem (see Firey [1975]). The extreme cases $j = 1$ and $j = d-1$ were first studied by E. Christoffel and H. Minkowski. The answer is known for $j = 1$ (see Berg [1969] and Firey [1968]) and for $j = d-1$ (see Aleksandrov [1938] and Fenchel-Jessen [1938]). The necessary and sufficient conditions for the first-order area measure given by Berg and Firey are not easy to handle. To solve this problem, Schneider [1977a] gave an independent treatment to the Christoffel-Minkowski problem for polytopes using some elementary methods. The necessary and sufficient conditions for any finite Borel measure μ in S^{d-1} to be $S_{d-1}(K,\cdot)$ for some convex body K are:

(a) μ is not concentrated on any great subsphere of S^{d-1} and

(b)

$$\int_{S^{d-1}} \langle u,v\rangle \mu(dv) = 0 \quad \text{for all} \quad u \in S^{d-1}. \tag{1.2}$$

This result is often called *Minkowski's existence theorem* (see Gardner [1995], p.374). It is almost trivial that condition (b) is necessary (see Schneider [1993], p.281). The beautiful part of this theorem is that condition (b) is also sufficient in order that μ be the $(d-1)$-st area measure of a convex body $K \in \mathcal{K}_0^d$. This follows from the fact that if K is of d-dimensional, μ cannot be concentrated on a great sphere. For the intermediate cases $(2 \leq j \leq d-2)$, the Christoffel-Minkowski problem is still open. However, Firey [1970] gave an answer to the Christoffel-Minkowski problem for sufficiently smooth convex bodies of revolution. The uniqueness question for Christoffel-Minkowski problem has long been solved: If K and K' have the same j-th area measure, then K and K' differ at most by a translation. For details about the uniqueness question see Buseman [1958, p.70]. We will not study the Cristoffel-Minkowski problem in this thesis. Instead we will use the results, especially Firey's results concerning the smooth bodies of revolution in our investigation of orthogonal bodies in Section 4.

A zonotope is a finite *Minkowski sum* of line segments. Zonotopes are the polytopes for which all two-dimensional faces have a center of symmetry (see Schneider [1993], p.182). By definition, a zonoid in $\mathbb{E}^d$ is a convex body that can be approximated, in the Hausdorff metric, by a sequence of zonotopes. It is clear from the definition that zonotopes are centrally symmetric. Zonoids are also centrally symmetric since central symmetry is preserved under limits in the Hausdorff metric.

Let Z be any centered zonotope. Then Z can be written as

$$Z = [\alpha_1 v_1, -\alpha_1 v_1] + \cdots + [\alpha_k v_k, -\alpha_k v_k],$$

where $\alpha_i > 0$, $v_i \in S^{d-1}$ and $1 \leq i \leq k$. By an easy calculation, we see that the support function of the zonotope Z is

$$h(Z, u) = \sum_{i=1}^{k} \alpha_i |\langle u, v_i \rangle| \qquad \text{for all } u \in S^{d-1}.$$

Conversely, if the support function of a convex body Z has the above form then Z must be a centered zonotope. The next theorem, which can be found in Schneider [1993, p.183], is a natural generalization of the above characterization of zonotopes.

Theorem 1.1. *A convex body $K \in \mathcal{K}^d$ is a zonoid with center at the origin if and only if its support function can be represented in the form*

$$h(K,x) = \int_{S^{d-1}} |\langle x, v\rangle| \rho(K, dv) \qquad \text{for each} \quad x \in \mathbb{E}^d \tag{1.3}$$

with some even measure $\rho(K, \cdot)$ on S^{d-1}

The next theorem (see Schneider [1993], p.184) shows that the even measure ρ given by (1.3) is unique. This measure is called the *generating measure* of K.

Theorem 1.2. *If ρ is an even signed measure on S^{d-1} with*

$$\int_{S^{d-1}} |\langle u, v\rangle| \rho(dv) = 0 \tag{1.4}$$

for all $u \in S^{d-1}$, then $\rho = 0$. Furthermore, if $f \in C_e^k(S^{d-1})$, where $k \geq d+1$ is even, there exists an even continuous function g on S^{d-1} such that

$$f(u) = \int_{S^{d-1}} |\langle u, v\rangle| g(v) \lambda_{d-1}(dv) \qquad \text{for each} \quad u \in S^{d-1}. \tag{1.5}$$

Note that the second part of the above theorem shows that if a centered convex body K is sufficiently smooth then there is a signed measure ρ such that

$$h(K,u) = \int_{S^{d-1}} |\langle u, v\rangle| \rho(dv) \qquad \text{for each} \quad u \in S^{d-1}. \tag{1.6}$$

The existence of the signed measure ρ is a motivation for considering the generalized zonoids that are defined as follows:

If K is a convex body whose support function can be represented in the form of (1.6) with an even signed measure $\rho(K, \cdot)$, then K and each of its translates is called a *generalized zonoid.* The even signed measure ρ is called the *generating measure* of the generalized zonoid K and is often written as $\rho(K, \cdot)$.

By Z_d we denote the class of zonoids in $\mathbb{E}^d$ and by $\mathcal{G}_d$ we denote the class of generalized zonoids in $\mathbb{E}^d$. It is clear that

$$Z_d \subseteq \mathcal{G}_d \subseteq \mathcal{C}^d.$$

We next want to show Z_d, $\mathcal{G}_d$ and $\mathcal{C}^d$ are different classes. Shephard [1964] showed that the only polytopal zonoids are zonotopes. Furthermore, Schneider [1970] showed that a polytope is a generalized zonoid only if it is a zonotope. It follows that a centrally symmetric polytope with non-symmetric two-dimensional faces, for example an octahedron, is not a generalized zonoid. Shephard and Schneider's results tell us that it is not possible to find a polytopal generalized zonoid that is not a zonoid. However, Schneider [1970, p.69] gave explicit examples that are generalized zonoids but not zonoids. Since these examples motivate our further study about certain convex body classes, we will describe them as follows:

Let $K_\alpha \in \mathcal{K}^3$, $\alpha \in [-2/5, 1/2]$, be a convex body whose support function is given by

$$h(K_\alpha, u) = 1 + \alpha P_2^3(\langle e_3, u\rangle) \quad \text{for all} \quad u \in S^2,$$

where $e_3 \in S^2$ is fixed and $P_2^3(x) = (3x^2-1)/2$. It can be shown that K_α is, indeed, a convex body. Also it is a generalized zonoid if and only if $\alpha \in [-2/5, 1/2]$ and it is a zonoid if and only if $\alpha \in [-1/4, 1/2]$. So $K_\alpha \in \mathcal{G}^d \setminus Z^d$ for all $\alpha \in [-2/5, -1/4)$. Therefore Z_d, $\mathcal{G}_d$ and $\mathcal{C}^d$ are different convex body classes.

It follows from the definition that Z_d is a closed subset of $\mathcal{C}^d$. Z_d is not dense in $\mathcal{C}^d$. This can be justified by the fact that the octahedron, a centrally symmetric convex body, is not a zonoid. In contrast to this, $\mathcal{G}^d$, the class of generalized zonoids, is dense in $\mathcal{C}^d$ (see Schneider [1993], p.189). $\mathcal{G}^d$ is not closed in $\mathcal{C}^d$, since $\mathcal{G}^d \neq \mathcal{C}^d$.

By $\mathcal{L}_j^d$ we denote the Grassmann manifold of all j-dimensional subspaces of $\mathbb{E}^d$. By ν_j^d we denote the rotation invariant probability measure on $\mathcal{L}_j^d$. If $1 \leq j \leq d-1$, the *j-th projection function* of K is a function on the Grassmannian $\mathcal{L}_j^d$ and is denoted by $V_j(K|\cdot)$. Its value at $E \in \mathcal{L}_j^d$ is the j-dimensional volume of $K|E$, where $K|E$ denotes the orthogonal projection of K on E. For each E, $F \in \mathcal{L}_j^d$, $|\langle E, F\rangle|$ denotes the absolute value of the determinant of the orthogonal projection of E onto F. By $E^\perp$ we denote the orthogonal space of E and thus $E^\perp \in \mathcal{L}_{d-j}^d$. It is easy to show that $|\langle E,F\rangle| = |\langle E^\perp, F^\perp\rangle|$ (see Weil [1982]). For a measure ψ on $\mathcal{L}_j^d$, the measure $\psi^\perp$ on $\mathcal{L}_{d-j}^d$ is defined by

$$\psi^\perp(A) := \psi(\{E \in \mathcal{L}_j^d : E^\perp \in A\}),$$

where A is any Borel set in $\mathcal{L}_{d-j}^d$.

For $j \in \{1, 2, \ldots, d\}$, we denote by $L^2(\mathcal{L}_j^d)$ the space of square-integrable function in $\mathcal{L}_j^d$. If f is a function on $\mathcal{L}_j^d$, we denote by $f^\perp$ the function on $\mathcal{L}_{d-j}^d$ defined by $f^\perp(E) = f(E^\perp)$. We often identify $\mathcal{L}_{d-j}^d$ with $\mathcal{L}_j^d$ (via orthogonality). Thus $f \in L^2(\mathcal{L}_j^d)$ if and only if $f^\perp \in L^2(\mathcal{L}_{d-j}^d)$. It is clear that

$$\int_{\mathcal{L}_j^d} f(E)\psi(dE) = \int_{\mathcal{L}_{d-j}^d} f^\perp(F)\psi^\perp(dF),$$

and that

$$\int_{\mathcal{L}_j^d} f(E)\nu_j^d(dE) = \int_{\mathcal{L}_{d-j}^d} f^\perp(F)\nu_{d-j}^d(dF).$$

The 1-st projection function of a convex body K can be expressed in terms of its support function. That is,

$$V_1(K|u) = h(K,u) + h(K,-u) \qquad \text{for each} \quad u \in S^{d-1}. \tag{1.7}$$

If K is centered, the 1-st projection function is just twice its support function. The *Cauchy projection formula* gives a very nice representation for the $(d-1)$-st projection function of a convex body K.

$$V_{d-1}(K|u^\perp) = \frac{1}{2}\int_{S^{d-1}} |\langle u, v\rangle| S_{d-1}(K, dv) \qquad \text{for each} \quad u \in S^{d-1}. \tag{1.8}$$

There are several places in this thesis where we need to convert an integral on $\mathcal{L}_1^d$ to an integral on S^{d-1}. For any $u \in S^{d-1}$, denote by $[u]$ the one-dimensional subspace of $\mathbb{E}^d$ parallel to u. We define the following map

$$\phi : S^{d-1} \to \mathcal{L}_1^d, \qquad \phi : \pm u \mapsto [u].$$

If μ is a measure in S^{d-1}, then we have a measure ρ in $\mathcal{L}_1^d$ such that $\rho(\mathcal{E}) = \mu(\phi^{-1}(\mathcal{E}))$ for any Borel set $\mathcal{E}$ in $\mathcal{L}_1^d$. On the other hand, if ρ is a measure in $\mathcal{L}_1^d$ then we have an even measure μ in S^{d-1} such that $\mu(\omega) = \rho(\phi(\omega))$ for any Borel set ω in S^{d-1}. It should be noted that ϕ maps λ_{d-1} to a rotation invariant measure in $\mathcal{L}_1^d$ and thus the image measure is, up to a positive constant C_d, the probability measure ν_1^d. If f is an even continuous function on S^{d-1} (f can also be thought as a function on $\mathcal{L}_1^d$), then

$$\int_{S^{d-1}} f(u)\lambda_{d-1}(du) = C_d \int_{\mathcal{L}_1^d} f([u])\nu_1^d([u]). \tag{1.9}$$

Put $f = 1$ and we get that $C_d = \omega_d = d\kappa_d$.

Let K be a centered zonoid with generating measure $\rho(K, \cdot)$ and let $u \in S^{d-1}$. Then there exists a positive measure $\rho_1(K, \cdot)$ on $\mathcal{L}_1^d$ such that

$$\begin{aligned} V_1(K|[u]) &= h(K,u) + h(K,-u) = 2h(K,u) \\ &= 2\int_{S^{d-1}} |\langle u,v\rangle| \rho(K,dv) \\ &= \int_{\mathcal{L}_1^d} |\langle [u],[v]\rangle| \rho_1(K,d[v]). \end{aligned}$$

On the other hand, if there is a positive measure $\rho_1(K, \cdot)$ on $\mathcal{L}_1^d$ such that

$$V_1(K|[u]) = \int_{\mathcal{L}_1^d} |\langle [u],[v]\rangle| \rho_1(K,d[v]) \quad \text{for each } \ u \in S^{d-1},$$

then we have an even measure $\rho(K, \cdot)$ on S^{d-1} such that

$$h(K,u) = \int_{S^{d-1}} |\langle u,v\rangle| \rho(K,dv) \quad \text{for each } \ \in S^{d-1}.$$

Thus K is a centered zonoid. The above characterization of centered zonoid motivates the definition of the classes $\mathcal{K}(j)$ for $j \in \{1, 2, \ldots, d\}$.

The convex body class $\mathcal{K}(j)$ contains all the centrally symmetric bodies K for which there is a positive measure $\rho_j(K, \cdot)$ on $\mathcal{L}_j^d$ such that the j-th projection function $V_j(K|\cdot)$ has the integral representation

$$V_j(K|E) = \int_{\mathcal{L}_j^d} |\langle E, F\rangle| \rho_j(K, dF) \quad \text{for each } \; E \in \mathcal{L}_j^d. \tag{1.10}$$

The measure $\rho_j(K, \cdot)$ on $\mathcal{L}_j^d$ is called a *j-th projection generating measure* of K.

Clearly $\mathcal{K}(1)$ is just the class of zonoids. We claim that $\mathcal{K}(d-1)$ comprises all centrally symmetric convex bodies. In fact, by definition, any convex body $K \in \mathcal{K}(d-1)$ is centrally symmetric. If $K \in \mathcal{C}^d$ and $u \in S^{d-1}$, then, by (1.8), we have

$$\begin{aligned} V_{d-1}(K|u^{\perp}) &= \frac{1}{2} \int_{S^{d-1}} |\langle u, v\rangle| S_{d-1}(K, dv) \\ &= \int_{\mathcal{L}_1^d} \langle [u], [v]\rangle | \mu(K, d[v]) \\ &= \int_{\mathcal{L}_{d-1}^d} \langle u^{\perp}, v^{\perp}\rangle | \rho_{d-1}(K, dv^{\perp}). \end{aligned}$$

Here $\mu(K, \cdot)$ is the image measure of $\frac{1}{2} S_{d-1}(K, \cdot)$ under the map $S^{d-1} \to \mathcal{L}_1^d$, $u \mapsto [u]$ and $\rho_{d-1}(K, \cdot)$ is the image measure of μ under the map $\mathcal{L}_1^d \to \mathcal{L}_{d-1}^d$, $[u] \mapsto u^{\perp}$. So $K \in \mathcal{K}(d-1)$ and therefore $\mathcal{K}(d-1)$ is the class of all centrally symmetric convex bodies.

The measures $\rho_1(K, \cdot)$ and $\rho_{d-1}(K, \cdot)$ are uniquely determined by K . These results were rediscovered several times (see Schneider [1993], p.192 for details). Matheron [1975] conjectured that $\rho_j(K, \cdot)$ is uniquely determined by K for all $1 \le j \le d-1$. This conjecture was disproved by Goodey and Howard [1990] in the cases $1 < j < d-1$.

The convex body classes $\mathcal{K}(j)$ appear in many papers. Some results about the classes $\mathcal{K}(j)$ are obtained by using functional analytic techniques (see Goodey and

Weil [1991]). Weil [1982] showed that

$$\mathcal{K}(1) \subseteq \mathcal{K}(j) \subseteq \mathcal{K}(d-1) \quad \text{for} \quad 1 \leq j \leq d-1. \tag{1.11}$$

So the classes $\mathcal{K}(j)$, $j = 1, 2, \cdots, d-1$, form a natural hierarchy ranging from the zonoids to all centrally symmetric convex bodies. It is interesting to find more relations among these classes. But very little is known about these relations (see Schneider [1997b]).

The first question about the convex body classes is , for $1 < j < d-1$, whether there are non-zonoids in $\mathcal{K}(j)$ and whether there is a centrally symmetric convex body K such that K is not in $\mathcal{K}(j)$. In other words, we want to know whether the strict inclusions

$$\mathcal{K}(1) \subset \mathcal{K}(j) \subset \mathcal{K}(d-1) \tag{1.12}$$

are true for all $1 < j < d-1$. In section 4, we will give a class of bodies of revolution defined by their j-th area measures. These bodies will be in $\mathcal{K}(d-1)$ but not in $\mathcal{K}(j)$. This will lead to the second half of the strict inclusion of (1.12). Furthermore, we will give a class of bodies K_α defined by their support functions and calculate the j-th projection generating functions explicitly. This leads to a positive answer to the strict inclusions (1.12). It is natural to believe that the following inclusions are true

$$\mathcal{K}(1) \subset \mathcal{K}(2) \subset ... \subset \mathcal{K}(j) \subset ... \subset \mathcal{K}(d-1) \tag{1.13}$$

However, this is still an open question.

Another way to compare various classes of centrally symmetric bodies can be obtained by considering projections. Recall that a polytope is zonotope if and only if all its 2-faces have a center of symmetry. Therefore if all three-dimensional projections of a polytope P are zonotopes, then P is a zonotope. It is natural to ask whether the analogous result for zonoids is true. However, Weil [1982] showed that this is not the case. He defined convex body classes Z_j, $2 \leq j \leq d$, to be the set of convex bodies K such that all the j-dimensional projections of K are zonoids. In that paper, Weil showed that the following inclusions are strict

$$Z_d \subset Z_{d-1} \subset \cdots \subset Z_2. \tag{1.14}$$

Of course Z_d comprises all zonoids. Also Z_2 is exactly the class of centrally symmetric bodies. This follows from the fact that a convex body in $\mathbb{E}^2$ is a zonoid if and only if it is centrally symmetric (see Gardner [1994], p.145) and the fact that a convex body is centrally symmetric if and only if all its two-dimensional projections are centrally symmetric. Therefore, we have

$$\mathcal{K}(1) = Z_d \qquad \text{and} \qquad \mathcal{K}(d-1) = Z_2.$$

Weil [1982] conjectured (see also Schneider and Weil [1983]) that $\mathcal{K}(j) = Z_{d-j+1}$ for all $1 \le j \le d-1$. It is explained above that the conjecture is true if $j = 1$ or $j = d-1$. But Goodey and Weil [1991] showed that the conjecture is false if $j = d-2$. They considered the polytopal members of these classes and used non-zonotopal polytopes (constructed by McMullen [1970]) with centrally symmetric facets. It is interesting to know whether $\mathcal{K}(j) = Z_{d-j+1}$ is true or not for the case $j \in \{2, 3, ..., d-3\}$. We find that it is false for the case $j \in \{2, 3\}$ and furthermore it is false for any $j > 1$ if d is big enough. In contrast to the counterexample (polytopal) given by Goodey and Weil, we will give counterexamples comprising smooth convex bodies of revolution.

McMullen [1984] showed a very interesting result that can be stated as follows: If C^d is the unit cube in $\mathbb{E}^d$ and $1 \le j \le d$ then

$$V_j(C^d|E) = V_{d-j}(C^d|E^{\perp}) \qquad \text{for all} \quad E \in \mathcal{L}_j^d. \tag{1.15}$$

It is natural to ask whether there are any other convex bodies that satisfy the above equation. For details about this problem, see McMullen [1984 and 1987] and Schnell [1994]. A variation of this problem is to study convex body pairs

$(K, K') \in \mathcal{K}^d \times \mathcal{K}^d$ such that $V_j(K|E) = V_{d-j}(K'|E^\perp)$ for all $E \in \mathcal{L}_j^d$ and all $j \in \{1, 2, \ldots, d-1\}$. Schneider [1997a] studied this problem for polytopes and characterized the conditions for which these convex body pairs exist. Variations of this problem have also been investigated by Weil [1971] and Goodey [1997].

Let $j \in \{1, 2, \ldots, d-1\}$ be fixed and let $K \in \mathcal{C}_0^d$. If there exists a convex body $K' \in \mathcal{C}_0^d$ such that

$$V_j(K|E) = V_{d-j}(K'|E^\perp) \qquad \text{for all} \quad E \in \mathcal{L}_j^d \tag{1.16}$$

then we say that K' is a *j-th orthogonal body* of K. Denote

$$\mathcal{O}_j =: \{K \in \mathcal{C}_0^d : K \text{ has a j-th orthogonal body}\}. \tag{1.17}$$

Of course, for any $j \in \{1, 2, \ldots, d-1\}$ the unit cube is in $\mathcal{O}_j$ and the cube is a j-th orthogonal body of itself. The polytopes given by Schneider [1997a] are also in $\mathcal{O}_j$. Here we will be interested in finding non-polytopal convex bodies that are in $\mathcal{O}_j$. Note that a full dimensional ball is in $\mathcal{O}_j$ for all $1 \leq j \leq d-1$. It is natural to ask whether $\mathcal{O}_j$ contains any other convex bodies. In Section 4, we will construct a class of bodies of revolution $L_{\beta,j}$ that are in $\mathcal{O}_j$. In the meantime, we find that for these bodies of revolution, $\mathcal{O}_j$ is closely related to $\mathcal{K}(j)$. This is not surprising for $j = 1$ or $j = d-1$. We will show next that $\mathcal{O}_{d-1} = \mathcal{C}_0^d$ and $\mathcal{O}_1$ is just the class of d-dimensional zonoids in $\mathbb{E}^d$.

Let $K \in \mathcal{K}^d$. The *projection body* ΠK of K is the centered body such that

$$h(\Pi K, u) = V_{d-1}(K|u^\perp) = \frac{1}{2}\int_{S^{d-1}} |\langle u, v\rangle| S_{d-1}(K, v) \quad \text{for all} \;\; u \in S^{d-1}.$$

The last equality of the above definition is from *Cauchy's projection formula.* It is easy to check that $h(\Pi K, \cdot)$ is a convex function and positively homogeneous of degree 1. Thus it is indeed a support function. A projection body is a centered zonoid. Conversely, every centered d-dimensional zonoid in $\mathbb{E}^d$ is the projection body of a unique centered convex body (see Gardner [1994], p.141).

Note that for any $K \in \mathcal{C}_0^d$ and $u \in S^{d-1}$,

$$V_{d-1}(K|u^{\perp}) = h(\Pi K, u) = \frac{1}{2}V_1(\Pi K, u) = V_1(\frac{1}{2}(\Pi K), u).$$

This shows that $\mathcal{C}_0^d \subseteq \mathcal{O}_{d-1}$ and thus $\mathcal{C}_0^d = \mathcal{O}_{d-1}$.

If $K \in \mathcal{O}_1$ is centered then there exists $K' \in \mathcal{C}_0^d$ such that

$$V_1(K|u) = V_{d-1}(K'|u^{\perp}) = 2h(K, u).$$

It follows that K is a projection body of K'. Hence K is a centered zonoid. This shows that $\mathcal{O}_1 \subseteq \mathcal{K}(1)$. On the other hand, if K is a d-dimensional centered zonoid, then K is a projection body of a unique centered convex body K'. Therefore, $K \in \mathcal{O}_1$. We will investigate relationships between the classes $\mathcal{K}(j)$ and $\mathcal{O}_j$. For $j \in \{2, 3, \ldots, d-2\}$, we will construct a class of bodies of revolution $W_{\gamma,j}$ such that $W_{\gamma,j} \in \mathcal{O}_j \setminus \mathcal{K}(j)$. Therefore, $\mathcal{O}_j \neq \mathcal{K}(j)$ if $j \in \{2, 3, \ldots, d-2\}$.

The layout of this thesis is as follows: in Section 2, we will give a short introduction to the harmonic analysis that relates to the techniques we will use. The main results of this thesis and their proofs can be found in Section 3 and Section 4. In section 5, we will turn our attention to averages of projections and give some results concerning mean projection bodies. In Section 6, we will discuss some open problems related to this thesis.

For clarity, we would like to outline the main results obtained in Section 3 and Section 4 (the result in Section 5 is independent of these results and is not listed here):

(1) If K is a generalized zonoid in $\mathbb{E}^d$ with generating function $\rho(K, \cdot)$, then for any $j \in \{1, 2, \ldots, d-1\}$, we can give a j-th projection generating measure of K explicitly and this measure turns out to be a function on $\mathcal{L}_j^d$.

(2) If $K \in \mathcal{K}^d$ is a sufficiently smooth body of revolution and $\rho_j(K, \cdot)$ is a j-th projection generating function of K having the same rotational symmetry as K, then

$$K \in \mathcal{K}(j) \qquad \textit{if and only if} \qquad \rho_j(K, E) \geq 0 \qquad \textit{for all} \qquad E \in \mathcal{L}_j^d.$$

(3) Let $j \in \{2, 3, \dots, d-2\}$ and K be a sufficiently smooth body of revolution. If $K \in \mathcal{K}(j-1)$ and $K \in Z_{d-j+1}$, then $K \in \mathcal{K}(j)$.

(4) For any $j \in \{2, 3, \dots, d-2\}$, the inclusions $\mathcal{K}(1) \subset \mathcal{K}(j) \subset \mathcal{K}(d-1)$ are strict.

(5) If $j = 2$ or $j = 3$, there exist convex bodies K_α such that $K_\alpha \in \mathcal{K}(j) \setminus Z_{d-j+1}$ for all $d > j+1$; Furthermore, for any given $j > 1$, there exist convex bodies K_α such that $K_\alpha \in \mathcal{K}(j) \setminus Z_{d-j+1}$ if d is large enough.

(6) For any $j \in \{2, 3, \dots, d-2\}$, we can construct bodies of revolution $W_{\gamma,j}$ such that $W_{\gamma,j} \in \mathcal{O}_j \setminus \mathcal{K}(j)$. Therefore, $\mathcal{O}_j \neq \mathcal{K}(j)$.

§2. Harmonic Analysis Aspects

The use of spherical harmonic analysis in geometry can be traced back to the early 1900s. In 1901, Adolf Hurwitz proved the isoperimetric inequality for domains in the 2-dimensional Euclidean plane using Fourier series and later he used spherical harmonics to show an analogous inequality for three-dimensional convex bodies. In 1911, Hermann Minkowski studied three-dimensional convex bodies of constant width. He showed that if a three-dimensional convex body has the property that all its projections have the same perimeter, then the original body has constant width. The techniques that Minkowski used were (three-dimensional) spherical harmonics. Since then, many geometric results have been found using harmonic analysis. Harmonic analysis has now become an indispensable tool in the study of convex geometry.

If F is a function whose domain is a subset of $\mathbb{E}^d$ that contains S^{d-1}, we write $F^\wedge$ for the restriction of F to S^{d-1}. If, on the other hand, F is defined on S^{d-1} we let $F^\vee$ denote the radial extension of F to $\mathbb{E}^d \setminus \{0\}$. Thus

$$F^\vee(x) = F(x/ \parallel x \parallel) \qquad \text{for all } \ x \in \mathbb{E}^d \setminus \{0\}. \tag{2.1}$$

A function F on S^{d-1} is said to be n times differentiable (or n times continuously differentiable) if the partial derivatives of $F^\vee$ of order n exist (or exist and continuous). The Laplace operator $\triangle$ is defined by

$$\triangle = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_d^2}. \tag{2.2}$$

The Laplace operator can be defined on S^{d-1} using the above extension procedure. The operator will be denoted by

$$\triangle_o F = (\triangle F^\vee)^\wedge. \tag{2.3}$$

The operator $\triangle_o$ is usually called the Laplace-Betrami operator. A function F is said to be harmonic if $\triangle F = 0$. A spherical harmonic of degree n on S^{d-1} is a harmonic, homogeneous polynomial of degree n in d variables restricted to the unit sphere. The set $\mathcal{H}_n^d$ of spherical harmonics of degree n is a vector subspace of $C(S^{d-1})$ of dimension

$$N(d,n) = \binom{d+n-1}{n} - \binom{d+n-3}{n-2}. \tag{2.4}$$

The spherical harmonics are eigenfunctions of the Laplace-Betrami operator $\triangle_o$; each $Y_n^d \in \mathcal{H}_n^d$ satisfies

$$\triangle_o Y_n^d = -n(n+d-2)Y_n^d. \tag{2.5}$$

Denote by $L^2(S^{d-1})$ the Hilbert space of square-integrable real functions on S^{d-1} (with the usual identification of functions coinciding almost everywhere) with inner product $\langle\cdot,\cdot\rangle$ defined by

$$\langle f,g\rangle = \int_{S^{d-1}} f(u)g(u)\lambda_{d-1}(du). \tag{2.6}$$

The induced L^2–norm is denoted by $\|\cdot\|$. The spherical harmonics on S^{d-1} are clearly in $L^2(S^{d-1})$. Thus the properties of spherical harmonics can be stated using the Hilbert space $L^2(S^{d-1})$.

The spherical harmonics of different degrees are orthogonal, e.g. if $f \in \mathcal{H}_n^d$, $g \in \mathcal{H}_m^d$ and $n \neq m$, then $\langle f,g\rangle = 0$. The system of spherical harmonics is complete. That is, if for any fixed $f \in L^2(S^{d-1})$, we have

$$\langle f,Y\rangle = 0 \quad \text{for all } Y \in \mathcal{H}_n^d, \quad \text{and all } n \geq 0 \tag{2.7}$$

implies that $f = 0$ almost everywhere.

The Legendre polynomials P_j^d of dimension d and degree j can be defined by means of the generating function

$$(1-2tx+x^2)^{-\frac{d-2}{2}} = \sum_{j=0}^{\infty} \binom{d+j-3}{d-3} P_j^d(t)x^j. \tag{2.8}$$

The Legendre polynomials P_j^d have the property that

$$P_j^d(t) = \frac{(-1)^j}{2^j(\vartheta+1)(\vartheta+2)\cdots(\vartheta+j)}(1-t^2)^{-\vartheta}\left(\frac{d}{dt}\right)^j(1-t^2)^{\vartheta+j}, \tag{2.9}$$

where $\vartheta = \frac{d-3}{2}$ and $d \geq 3$. This identity is called *Rodrigues' formula.*

A real function f on S^{d-1} is said to be *a zonal function with pole e_d* if $f(u)$ depends only on the distance of u from e_d. Here e_d is a fixed vector in S^{d-1}. It is well known that for any fixed $e_d \in S^{d-1}$, $P_n^d(\langle u, e_d\rangle) \in \mathcal{H}_n^d$. So $P_n^d(\langle u, e_d\rangle)$ is a zonal spherical harmonic in $\mathcal{H}_n^d$. On the other hand, it can be shown that any zonal spherical harmonic with pole e_d in $\mathcal{H}_n^d$ is of the form $cP_n^d(\langle u, e_d\rangle)$, where c is a constant independent of u (see for example Groemer [1996], p.119).

Denote by $SO(d)$ the rotation group of $\mathbb{E}^d$. For any $\rho \in SO(d)$ and $f \in L^2(S^{d-1})$, define ρf by $(\rho f)(u) = f(\rho^{-1}u)$ for each $u \in S^{d-1}$. A subspace $\mathcal{H} \subset L^2(S^{d-1})$ is said to be *invariant under group actions* of $SO(d)$ if $\rho f \in \mathcal{H}$ for all $\rho \in SO(d)$ and $f \in \mathcal{H}$. $H \subset L^2(S^{d-1})$ is said to be *irreducible* if the only invariant subspaces of $\mathcal{H}$ are $\mathcal{H}$ and $\{0\}$. It is well known that $\mathcal{H}_0^d$, $\mathcal{H}_1^d, \ldots$ are the only non-trivial, finite-dimensional, invariant and irreducible subspaces of $L^2(S^{d-1})$.

For reference in the following sections, we explicitly list the Legendre polynomials of degree up to 4 for all dimensions $d \geq 3$:

$$P_0^d(t) = 1; \qquad P_1^d(t) = t; \tag{2.10}$$

$$P_2^d(t) = \frac{1}{d-1}(dt^2 - 1); \tag{2.11}$$

$$P_3^d(t) = \frac{1}{d-1}t((d+2)t^2 - 3); \tag{2.12}$$

$$P_4^d(t) = \frac{1}{d^2-1}\left[(d+4)(d+2)t^4 - (6d+12)t^2 + 3\right] \tag{2.13}$$

Another important class of orthogonal polynomials comprises the Gegenbauer polynomials C_n^ν. These can be expressed in terms of Legendre polynomials, namely

$$C_n^{\frac{d-2}{2}}(t) = \binom{n+d-3}{d-3} P_n^d(t) = \frac{\Gamma(n+d-2)}{\Gamma(d-2)\Gamma(n+1)} P_n^d(t), \tag{2.14}$$

where

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt \qquad \text{for } x > 0. \tag{2.15}$$

Recall from Section 1 that $x\Gamma(x) = \Gamma(x+1)$, $\Gamma(1) = 1$ and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

We next list a couple of recurrence formulas for Gegenbauer polynomials that can be found in Gradshteyn and Ryhzik [1994, §8.933].

$$nC_n^v(x) = 2v[xC_{n-1}^{v+1}(x) - C_{n-2}^{v+1}(x)]. \tag{2.16}$$

$$(2v+n)C_n^v(x) = 2v[xC_n^{v+1}(x) - C_{n-1}^{v+1}(x)]. \tag{2.17}$$

The hypergeometric series is defined as

$$F(a,b;c;x) = 1 + \sum_{\mu=1}^{\infty} \frac{a(a+1)\cdots(a+\mu-1)}{1.2\cdots\mu} \frac{b(b+1)\cdots(b+\mu-1)}{c(c+1)\cdots(c+\mu-1)} x^\mu \tag{2.18}$$

The Jacobi polynomials can be expressed in terms of hypergeometric series (see Szegö [1939], p.62) as follows:

$$P_n^{(\alpha,\beta)}(x) = \binom{n+\alpha}{n} F(-n, n+\alpha+\beta+1; \alpha+1; \frac{1-x}{2}) \tag{2.19}$$

Note that the hypergeometric series stops if a or b is a negative integer. Thus if $n \geq 1$, $P_n^{(\alpha,\beta)}(x)$ is a polynomial of degree n. If $\alpha = \beta = \frac{d-2}{2}$ then $P_n^{(\alpha,\beta)}(x)$ is a multiple of Legendre polynomial $P_n^d(x)$.

The next theorem, the Funk-Hecke Theorem, is particularly powerful and will be used in several places in this thesis.

Theorem 2.1 (Funk-Hecke Theorem). *If Φ is a bounded integrable function on $[-1,1]$ and $H \in \mathcal{H}_n^d$, then $\Phi(\langle u,v\rangle)$ is (for some fixed $u \in S^{d-1}$) an integrable function on S^{d-1} and*

$$\int_{S^{d-1}} \Phi(\langle u,v\rangle) H(v) \lambda_{d-1}(dv) = \alpha_{d,n}(\Phi) H(u) \tag{2.20}$$

with

$$\alpha_{d,n}(\Phi) = \omega_{d-1} \int_{-1}^{1} \Phi(t) P_n^d(t) (1-t^2)^\vartheta dt, \tag{2.21}$$

where $\vartheta = (d-3)/2$ *and* P_n^d *is the Legendre polynomial of dimension* d *and degree* n.

The *cosine transform* $T : C(S^{d-1}) \to C(S^{d-1})$ is defined by

$$(Tf)(u) = \int_{S^{d-1}} |\langle u, v\rangle| f(v)\lambda_{d-1}(dv). \tag{2.22}$$

An important property of the cosine transform is that it is injective on $C_e(S^{d-1})$. Theorem 1.2 tells us that if $k \geq d+2$ is even and $f \in C_e^k(S^{d-1})$, then there is a $g \in C_e(S^{d-1})$ such that $f = Tg$. Therefore the cosine transform T is a bijection of $C_e^\infty(S^{d-1})$ to itself.

If $H \in \mathcal{H}_n^d$, then by *the Funk-Hecke Theorem* there exist numbers $\lambda_{d,n}$ such that

$$T(H)(u) = \int_{S^{d-1}} |\langle u, v\rangle| H(v)\lambda_{d-1}(dv) = \lambda_{d,n}H(u) \quad \text{for all } u \in S^{d-1} \tag{2.23}$$

For our applications, we need to know $\lambda_{d,n}$ explicitly. The next theorem can be found in Groemer [1996, p.102].

Theorem 2.2. *Let* $\lambda_{d,n}$ *be defined as (2.23). Then*

(1) $\lambda_{d,n} = 0$ *if* n *is odd;*

(2)

$$\lambda_{d,n} = (-1)^{\frac{n-2}{2}} 2 \frac{1 \cdot 3 \cdots (n-3)}{(d+1)(d+3)\cdots(d+n-1)} \kappa_{d-1}$$

if n *is even* ($\lambda_{d,0} = 2\kappa_{d-1}$, $\lambda_{d,2} = \frac{2}{d+1}\kappa_{d-1}$).

A related cosine transform can be defined on the square integrable functions on the Grassmannian $\mathcal{L}_i^d$. For $1 \leq i \leq d-1$, *the generalized cosine transform* $T_i^d : L^2(\mathcal{L}_i^d) \to L^2(\mathcal{L}_i^d)$ is defined by

$$(T_i^d f)(E) = \int_{\mathcal{L}_i^d} |\langle E, F\rangle| f(F)\nu_i^d(dF) \qquad f \in L^2(\mathcal{L}_i^d),\ E \in \mathcal{L}_i^d. \tag{2.24}$$

For $1 \leq i \neq j \leq d-1$ the *Radon transform* $R^d_{i,j} : L^2(\mathcal{L}^d_i) \to L^2(\mathcal{L}^d_j)$ is defined by

$$(R^d_{i,j} f)(E) = \int_{\mathcal{L}^d_i(E)} f(F) \nu^E_i(dF) \qquad f \in L^2(\mathcal{L}^d_i),\ E \in \mathcal{L}^d_j, \tag{2.25}$$

where

$$\mathcal{L}^d_i(E) = \begin{cases} \{F \in \mathcal{L}^d_i : F \subset E\} & i < j \\ \{F \in \mathcal{L}^d_i : F \supset E\} & i > j, \end{cases}$$

and ν^E_i is the invariant probability measure on $\mathcal{L}^d_i(E)$. If $1 \leq i \neq j \leq d-1$, then

$$(R^d_{i,j} f)^\perp = R^d_{d-i,d-j} f^\perp \quad \text{and} \quad (T^d_i f)^\perp = T^d_{d-i} f^\perp. \tag{2.26}$$

If $1 \leq i < j < k \leq d-1$, then

$$R^d_{i,k} = R^d_{j,k} R^d_{i,j} \quad \text{and} \quad R^d_{k,i} = R^d_{j,i} R^d_{k,j}. \tag{2.27}$$

Both (2.26) and (2.27) can be found in Goodey and Zhang [1998, p.348]. Furthermore, they showed that for $1 \leq i \neq j \leq d-1$,

$$R^d_{i,j} T^d_i = \alpha^{i,d-i}_{j,d-j} T^d_j R^d_{i,j}, \tag{2.28}$$

where

$$\alpha^{i,j}_{k,l} = \frac{i!\kappa_i j!\kappa_j}{k!\kappa_k l!\kappa_l} \tag{2.29}$$

Denote by $V_i(K)$ the *i-th intrinsic volume* of a convex body K in $\mathbb{E}^d$. It can be defined by

$$V_i(K) = \alpha^{i,d-i}_{d,0} \int_{\mathcal{L}^d_i} V_i(K|E) \nu^d_i(dE). \tag{2.30}$$

The *Cauchy-Kubota formulas* can be written in terms of Radon Transforms (see Schneider and Weil [1992])

$$\alpha^{0,j}_{i,j-i} (R^d_{i,j} V_i(K|\cdot)(F) = V_i(K|F) \quad \text{for all} \quad F \in \mathcal{L}^d_j \tag{2.31}$$

Both cosine and Radon transforms satisfy a certain duality relation. Let $f, g \in L^2(\mathcal{L}_i^d)$ and $h \in L^2(\mathcal{L}_j^d)$. Then

$$\int_{\mathcal{L}_i^d} (T_i^d f)(X) g(X) \nu_i^d(dX) = \int_{\mathcal{L}_i^d} f(X)(T_i^d g)(X) \nu_i^d(dX). \tag{2.32}$$

This follows immediately from the definition of cosine transforms and Fubini theorem. The next identity can be found in Schneider and Weil [1992, p.150].

$$\int_{\mathcal{L}_j^d} (R_{i,j}^d f)(Y) h(Y) \nu_j^d(dY) = \int_{\mathcal{L}_i^d} f(X)(R_{j,i}^d h)(X) \nu_i^d(dX). \tag{2.33}$$

The cosine transform T_i^d can be further extended to distributions on $\mathcal{L}_i^d$ (see Goodey and Zhang [1998]). With this extension, T_i^d maps a measure μ on $\mathcal{L}_i^d$ to a continuous function $T_i^d \mu$ such that

$$(T_i^d \mu)(X) = \int_{\mathcal{L}_i^d} |\langle X, Y \rangle| \mu(dY), \qquad \text{for each } X \in \mathcal{L}_i^d. \tag{2.34}$$

We now bring together some major definitions and results about group representations and harmonic analysis on $\mathcal{L}_i^d$ necessary for our purposes. Most of these definitions and results can be found in Boerner [1963].

A *representation* D of a group G is a collection of matrices acting on a vector space V. For each $g \in G$ there is a $D(g) : V \to V$ such that

$$D(gh) = D(g)D(h) \qquad \text{and} \qquad D(e) = I_N;$$

here e is the identity of G, $\dim V = N$ and I_N is the $N \times N$ unit matrix.

We assume D_1 and D_2 are representations of G on the spaces V_1, V_2. If there is a non-singular matrix $P : V_2 \to V_1$ such that

$$D_1(g)P = PD_2(g) \qquad \text{for all } g \in G,$$

then D_1 and D_2 are said to be *equivalent* representations.

The representation D on V of G is said to be *irreducible* if there is no proper subspace W of V such that

$$D(g) : W \to W \qquad \text{for all } g \in G.$$

Lemma 2.3 (Schur's Lemma)

PART 1. *Let D_1 and D_2 be two irreducible representations of G acting on V_1 and V_2 (of dimensions N and M). Let $P : V_2 \to V_1$ be a matrix with*

$$D_1(g)P = PD_2(g) \quad \text{for all } g \in G.$$

Then, either $P = 0$ or P is non-singular; in the latter case $n = m$ and D_1 and D_2 are equivalent.

PART 2. *If D is an irreducible representation of G on V and if P is a matrix $P : V \to V$ with*

$$D(g)P = PD(g) \qquad \text{for all } g \in G,$$

then $P = \lambda I_N$ for some $\lambda \in \mathbb{C}$.

A compact Lie group admits an invariant (Haar) measure. This can be used to show that every representation of a compact Lie group is equivalent to a unitary representation (one comprising only unitary matrices). If D is a representation of a compact Lie group G then the *character* of D is the continuous function $\chi_D : G \to \mathbb{C}$, defined by

$$\chi_D(g) = \text{trace} D(g).$$

It is clear that equivalent representations have the same character. Schur's orthogonality relations (see below) can be used to establish the converse.

Let D_1 and D_2 be irreducible unitary representations of G on the spaces V_1, V_2 of dimensions M and N. For each $g \in G$, let $D_1(g) = (a_{ij}(g))$ and $D_2(g) = (b_{hk}(g))$. *Schur's Orthogonality Relations* are

$$\int_G a_{ij}(g)\overline{b_{kh}(g)}\, dg = \begin{cases} 0 & D_1,\ D_2 \text{ not equivalent} \\ \frac{1}{N}\delta_{jh}\delta_{ik} & D_1 = D_2. \end{cases}$$

As a consequence

$$\int_G \chi_{D_1}(g)\overline{\chi_{D_2}(g)}\, dg = \begin{cases} 0 & D_1,\ D_2 \text{ not equivalent} \\ 1 & D_1,\ D_2 \text{ equivalent.} \end{cases}$$

It follows that representations with the same character are equivalent.

The orthogonality conditions show that the functions $a_{ij} \in L^2(G)$. The *Peter-Weyl Theorem* shows that they form a complete orthogonal system in $L^2(G)$. Consequently there is an orthogonal decomposition of $L^2(G)$ into invariant irreducible subspaces indexed by equivalence classes of irreducible representations.

We will be particularly interested in the representations of the rotation group $SO(d)$. For convenience we write

$$A(\tau) = \begin{pmatrix} \cos 2\pi\tau & -\sin 2\pi\tau \\ \sin 2\pi\tau & \cos 2\pi\tau \end{pmatrix}.$$

We assume $g \in SO(d)$. If $d = 2p$, there is an element of $SO(d)$ conjugate to g of the form

$$t = \begin{pmatrix} A(\tau_1) & 0 & \dots & 0 \\ 0 & A(\tau_2) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A(\tau_p) \end{pmatrix}.$$

If $d = 2p+1$, there is a conjugate element of the form

$$t = \begin{pmatrix} A(\tau_1) & 0 & \dots & 0 & 0 \\ 0 & A(\tau_2) & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & A(\tau_p) & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

For such an element t and a representation D, we have

$$D(t) = \begin{pmatrix} e^{2\pi i\phi_1} & 0 & \dots & 0 \\ 0 & e^{2\pi i\phi_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e^{2\pi i\phi_N} \end{pmatrix},$$

where

$$\phi_j(\tau_1, \dots, \phi_p) = m_{j_1}\tau_1 + m_{j_2}\tau_2 + \dots + m_{j_p}\tau_p \qquad (m_{j_k} \in \mathbf{Z}).$$

The p-tuple $(m_{j_1}, \dots, m_{j_p})$ is a weight of the representation D. The highest weight is one for which

$$\begin{aligned} &m_1 \geq m_2 \geq \dots \geq m_{p-1} \geq |m_p| \qquad &\text{if } d = 2p \\ &m_1 \geq m_2 \geq \dots \geq m_p \geq 0 \qquad &\text{if } d = 2p+1. \end{aligned} \tag{2.34}$$

A theorem of Cartan shows that a representation is determined by its highest weight. Consequently we have

$$L^2(SO(d)) = \bigoplus_{(m_1,\ldots,m_p)} H^d_{(m_1,\ldots,m_p)},$$

where $(m_1,\ldots,m_p)$ satisfies (2.34) and $H^d_{(m_1,\ldots,m_p)}$ are irreducible, invariant subspaces of $L^2(SO(d))$.

If K is a subgroup of G and D is an irreducible representation of G then D is also a representation of K, but it is not necessarily irreducible. The Branching Theorem addresses this point. For $SO(d)$, it states that

$$H^{2p+1}_{(m_1,\ldots,m_p)} \simeq \bigoplus_{(m'_1,\ldots,m'_p)} H^{2p}_{(m'_1,\ldots,m'_p)}$$

where $m_1 \geq m'_1 \geq m_2 \geq m'_2 \geq \cdots \geq m_p \geq |m'_p|$;

$$H^{2p}_{(m_1,\ldots,m_p)} \simeq \bigoplus_{(m'_1,\ldots,m'_{p-1})} H^{2p-1}_{(m'_1,\ldots,m'_{p-1})}$$

where $m_1 \geq m'_1 \geq m_2 \geq m'_2 \geq \cdots \geq m'_{p-1} \geq |m_p|$.

The Branching Theorem shows that the spaces $H^d_{(n,0,\ldots,0)}$ are isomorphic to the spaces $\mathcal{H}^d_n$ of spherical harmonics. More generally, if $\mathcal{L}^d_i$ denotes the Grassmannian of i-subspaces of $\mathbb{E}^d$ then

$$L^2(\mathcal{L}^d_i) \simeq \bigoplus_{(2m_1,\ldots,2m_i,0,\ldots,0)} H^d_{(2m_1,\ldots,2m_i,0,\ldots,0)},$$

assuming $2i \leq d$. The other values of i are taken care of by the identification (via orthogonality) $\mathcal{L}^d_{d-i} = \mathcal{L}^d_i$; see Strichartz [1975], Sugiura [1962] and Takeuchi [1973].

The spaces $H^d_{(m_1,\ldots,m_{p-1},0)}$ are closed under complex conjugation and so the real parts of these functions yield an invariant irreducible subspace of $L^2(SO(d))$, which henceforth denotes the space of real valued square integrable functions. If $m_p \neq 0$, the conjugates of the functions in $H^d_{(m_1 \dots, m_p)}$ comprise the space $H^d_{(m_1 \dots, m_{p-1}, -m_p)}$. Consequently, the real and imaginary parts of these functions give a basis for the real part of

$$H^d_{(m_1 \dots, m_p)} \oplus H^d_{(m_1 \dots, m_{p-1}, -m_p)}$$

and this is a real irreducible invariant subspace. We write

$$L^2(\mathcal{L}^d_i) = \bigoplus_{(2m_1, \dots, 2m_i, 0, \dots, 0)} H^{d,i}_{(2m_1, \dots, 2m_i, 0, \dots, 0)},$$

to denote the real decomposition of $L^2(\mathcal{L}^d_i)$, for $2i \leq d$.

We have defined the cosine transform T^d_i and Radon transform $R^d_{i,j}$. Both of these transforms intertwine the group action of $SO(d)$. This means

$$T^d_i(D(g)f) = D(g)T^d_i f \qquad \text{for all } f \in L^2(\mathcal{L}^d_i),$$

and

$$R^d_{i,j}(D(g)f) = D(g)R^d_{i,j} f \qquad \text{for all } f \in L^2(\mathcal{L}^d_i).$$

It follows from Schur's Lemma that there are multipliers $\lambda^{d,i}_{(2m_1, \dots, 2m_i, 0, \dots, 0)}$ such that

$$T^d_i f = \lambda^{d,i}_{(2m_1, \dots, 2m_i, 0, \dots, 0)} f \qquad \text{for all } f \in H^{d,i}_{(2m_1, \dots, 2m_i, 0, \dots, 0)}.$$

For $1 \leq i \leq d-1$, we put

$$\mathcal{R}^d_i = \bigoplus_{m=0}^{\infty} H^{d,i}_{(2m, 0, \dots, 0)},$$

and

$$\mathcal{T}^d_i = \bigoplus_{\lambda^{d,i}_{(2m_1, \dots, 2m_i, 0, \dots, 0)} \neq 0} H^{d,i}_{(2m_1, \dots, 2m_i, 0, \dots, 0)}.$$

The following three lemmas can be found in Goodey and Zhang [1998].

Lemma 2.4. *The inclusion $\mathcal{R}_i^d \subset \mathcal{T}_i^d$ holds for all $1 \leq i \leq d-1$ and is strict precisely when $1 < i < d-1$.*

Lemma 2.5. *For $1 \leq i \neq j \leq d-1$, the cosine and Radon transforms*

$$T_i^d : \mathcal{T}_i^d \to \mathcal{T}_i^d \quad \textit{and} \quad R_{i,j}^d : \mathcal{R}_i^d \to \mathcal{R}_j^d$$

are injective.

Lemma 2.6. *For $1 \leq i \leq d-1$, the cosine transform*

$$T_i^d : \mathcal{R}_i^d \cap C^\infty(\mathcal{L}_i^d) \to \mathcal{R}_i^d \cap C^\infty(\mathcal{L}_i^d)$$

is bijective.

A function $f \in L^2(\mathcal{L}_i^d)$ is said to be *rotationally symmetric* if $\rho f = f$ for all $\rho \in SO(d-1)$. Here we think of $SO(d-1)$ as a subgroup of $SO(d)$. Goodey and Zhang [1998] observed that any rotationally symmetric function $f \in L^2(\mathcal{L}_i^d)$ is a member of $\mathcal{R}_i^d$. In fact, if $f \in H_{(m_1,\dots,m_p)}^{d,i}$ is non-trivial and rotationally symmetric, then the restriction of f to $\mathcal{L}_i^{d-1}$ is a constant. Thus the restriction of f is a member of $H_{(0,\dots,0)}^{d-1,i}$. By the Branching Theorem, $m_1 \geq 0 \geq m_2, \cdots, \geq 0$ and therefore $m_2 = m_3 = \cdots = m_p = 0$. This shows that $f \in \mathcal{R}_i^d$.

§3. Projection Generating Functions

In this section, we will restrict ourselves to $\mathcal{G}^d$, the class of generalized zonoids in $\mathbb{E}^d$. Let $K \in \mathcal{G}^d$ and let $\rho(K,\cdot)$ be the unique generating measure of K. It is explained in Goodey and Weil [1993, section 6] that from $\rho(K,\cdot)$ one can derive, for $j \in \{1,2,\dots,d-1\}$, a *j-th projection generating measure* $\overline{\rho}_j(K,\cdot)$ on $\mathcal{L}_j^d$ such that

$$V_j(K|E) = \int_{\mathcal{L}_j^d} |\langle E,F\rangle| \overline{\rho}_j(K,dF) \qquad \text{for each} \qquad E \in \mathcal{L}_j^d. \tag{3.1}$$

We next will describe how this projection generating measure is obtained. But before we do this, we need to introduce some notations. For $u_1,u_2,\dots,u_j \in S^{d-1}$, we denote by $D_j(u_1,\dots,u_j)$ the j-dimensional volume of the parallelepiped spanned by $\{u_1,u_2,\dots,u_j\}$. If $L_1,\dots,L_j \in \mathcal{L}_1^d$, we put

$$D_j(L_1,L_2,\dots,L_j) := D_j(u_1,\dots,u_j),$$

where $u_1,\dots,u_j \in S^{d-1}$ and u_i is parallel to L_i. Note that this definition is independent of the choice of the unit vectors u_i. Also $D_j(u_1,\dots,u_j) = 0$ if and only if $u_1,u_2,\dots,u_j$ are linearly dependent. By $L(u_1,\dots,u_j)$ we denote the linear subspace of $\mathbb{E}^d$ spanned by $\{u_1,\dots,u_j\}$.

For simplicity, we will, in this section, use $\rho(\cdot)$ instead of $\rho(K,\cdot)$ to denote the unique generating measure of K.

It was shown in Weil [1979, p.176] that for each $E \in \mathcal{L}_j^d$

$$V_j(K|E) = \frac{2^j}{j!}\int_{S^{d-1}}\cdots\int_{S^{d-1}} |\langle E, L(u_1,\dots,u_j)\rangle| D_j(u_1,\dots,u_j)\rho(du_1)\dots\rho(du_j).$$

Define a measure Ψ_j on the Cartesian product $(S^{d-1})^j$ such that for each Borel set $\mathcal{B} \subseteq (S^{d-1})^j$

$$\Psi_j(\mathcal{B}) := \frac{2^j}{j!}\int_{\mathcal{B}} D(u_1,\dots,u_j)\rho^j(du_1,\dots,du_j),$$

where

$$\rho^j := \underbrace{\rho\otimes\cdots\otimes\rho}_{\text{j factors}}.$$

Now define a map $t_j : (S^{d-1})^j \to \mathcal{L}_j^d$ such that $t_j : (u_1, \dots, u_j) \mapsto L(u_1, \dots, u_j)$ if $\dim(L(u_1, \dots, u_j)) = j$ and define it arbitrarily (but measurable) otherwise. The image measure $\overline{\rho}_j(K, \cdot) := t_j(\Psi_j)$ is a j-th projection generating measure that satisfies (3.1). We have

$$\overline{\rho}_j(K, A) = \int_{t_j^{-1}(A)} D(u_1, \dots, u_j)\rho(du_1)\dots\rho(du_j), \tag{3.2}$$

where A is any Borel set in $\mathcal{L}_j^d$; see Goodey and Weil [1991].

In this thesis, we will always denote by $\overline{\rho}_j(K, \cdot)$ the j-th projection generating measure of K derived from (3.2). $\rho_j(K, \cdot)$ will be used to denote a j-th projection generating measure of K in general if we do not state otherwise.

If g is a function defined on S^{d-1} such that for each Borel set $\omega \subseteq S^{d-1}$

$$\rho(\omega) = \int_\omega g(u)\lambda_{d-1}(du), \tag{3.3}$$

then g is called a generating function of K. We often write g as ρ. Thus if ρ is the generating function of K, then (3.2) can be written as

$$\overline{\rho}_j(K, A) = \int_{t_j^{-1}(A)} D(u_1, \dots, u_j)\rho(u_1)\dots\rho(u_j)\lambda_{d-1}(du_1)\dots\lambda_{d-1}(du_j). \tag{3.4}$$

Theorem 3.1. *Let K be a generalized zonoid such that its generating measure ρ is a function on S^{d-1}. For any $E \in \mathcal{L}_j^d$, define*

$$I_{j,n}(K, E) := \underbrace{\int_{u_1} \cdots \int_{u_j}}_{\mathrm{span}(u_1,\dots,u_j)=E} D_j^n(u_1, \dots, u_j)\rho(u_1)\dots\rho(u_j)\lambda_{j-1}(du_1)\dots\lambda_{j-1}(du_j). \tag{3.5}$$

Then

$$\overline{\rho}_j(K, E) = C_{d,j} I_{j,d-j+1}(K, E), \quad \textit{where } C_{d,j} = \frac{\omega_{d-j+1}\dots\omega_d}{\omega_1\dots\omega_j} \tag{3.6}$$

is the j-th projection generating measure (function) of K such that, for each $E \in \mathcal{L}_j^d$,

$$V_j(K|E) = \int_{\mathcal{L}_j^d} |\langle E, F\rangle| \overline{\rho}_j(K, F)\nu_j^d(dF). \tag{3.7}$$

Proof:

Let A be any Borel set in $\mathcal{L}_j^d$ and χ_A denote the characteristic function of A. Then by (3.4) we have

$$\begin{aligned}
\overline{\rho}_j(K,A) &= \int_{t_j^{-1}(A)} D(u_1,\ldots,u_j)\rho(u_1)\ldots\rho(u_j)\lambda_{d-1}(du_1)\ldots\lambda_{d-1}(du_j) \\
&= \int_{S^{d-1}}\cdots\int_{S^{d-1}} \chi_A(t(u_1,\ldots,u_j)D(u_1,\ldots,u_j)\times \\
&\qquad\qquad \rho(u_1)\ldots\rho(u_j)\lambda_{d-1}(du_1)\ldots\lambda_{d-1}(du_j) \\
&= (\omega_d)^j \int_{\mathcal{L}_1^d}\cdots\int_{\mathcal{L}_1^d} \chi_A(t(L_1,\ldots,L_j))D(L_1,\ldots,L_j)\times \\
&\qquad\qquad \rho(L_1)\ldots\rho(L_j)\nu_1^d(dL_1)\ldots\nu_1^d(dL_j).
\end{aligned}$$

Here we have identified the unit vector u_i with the one dimensional subspace L_i parallel to u_i. The constant $(\omega_d)^j$ is obtained from (1.9). Note that

$$\overline{\rho}_j(K,A) = \int_{\mathcal{L}_j^d} \chi_A(E)\overline{\rho}_j(K,dE).$$

Therefore,

$$\begin{aligned}
\int_{\mathcal{L}_j^d} \chi_A(E)\overline{\rho}_j(K,dE) = (\omega_d)^j \int_{\mathcal{L}_1^d}\cdots\int_{\mathcal{L}_1^d} \chi_A(t(L_1,\ldots,L_j))\times \\
D(L_1,\ldots,L_j)\rho(L_1)\ldots\rho(L_j)\nu_1^d(dL_1)\ldots\nu_1^d(dL_j).
\end{aligned}$$

By standard approximation arguments, we have, for each continuous function g on $\mathcal{L}_j^d$, that

$$\begin{aligned}
\int_{\mathcal{L}_j^d} g(E)\overline{\rho}_j(K,dE) =& (\omega_d)^j \int_{\mathcal{L}_1^d}\cdots\int_{\mathcal{L}_1^d} g(t(L_1,\ldots,L_j))\times \\
& D(L_1,\ldots,L_j)\rho(L_1)\ldots\rho(L_j)\nu_1^d(dL_1)\ldots\nu_1^d(dL_j).
\end{aligned}$$

Next, we will use a result in Schneider and Weil [1992, Theorem 6.16, p.156] and we state it as follows:

For any measurable real function $f\colon (\mathcal{L}_1^d)^j \to \mathbb{R}^1$, we have

$$\int_{\mathcal{L}_1^d}\cdots\int_{\mathcal{L}_1^d} f(L_1,\ldots,L_j)\nu_1^d(dL_1)\ldots\nu_1^d(dL_j) = C_{d,j}\Big(\frac{\omega_j}{\omega_d}\Big)^j \times$$
$$\times \int_{\mathcal{L}_j^d}\Big\{\int_{\mathcal{L}_1^L}\cdots\int_{\mathcal{L}_1^L} f(L_1,\ldots,L_j)D^{d-j}(L_1,\ldots,L_j)\nu_1^L(dL_1)\ldots\nu_1^L(dL_j)\Big\}\nu_j^d(dL). \tag{3.8}$$

Here $\mathcal{L}_1^L$ is the Grassmannian of all one-dimensional subspaces of $L \in \mathcal{L}_j^d$ and ν_1^L is the rotation invariant probability measure on $\mathcal{L}_1^L$.

Put

$$f(L_1,\ldots,L_j) := g(t(L_1,\ldots,L_j))D(L_1,\ldots,L_j)\rho(L_1)\ldots\rho(L_j).$$

It is clear that f is a measurable, real function on $(\mathcal{L}_1^d)^j$. Therefore,

$$\begin{aligned}
&\int_{\mathcal{L}_j^d} g(E)\overline{\rho}_j(K,dE)\\
&= (\omega_d)^j \int_{\mathcal{L}_1^d}\cdots\int_{\mathcal{L}_1^d} f(L_1,\ldots,L_j)\nu_1^d(dL_1)\ldots\nu_1^d(dL_j)\\
&= C_{d,j}(\omega_j)^j \int_{\mathcal{L}_j^d}\Big\{\int_{\mathcal{L}_1^L}\cdots\int_{\mathcal{L}_1^L} g(t(L_1,\ldots,L_j))D^{d-j+1}(L_1,\ldots,L_j)\times\\
&\qquad\qquad \rho(L_1)\ldots\rho(L_j)\nu_1^L(dL_1)\ldots\nu_1^L(dL_j)\Big\}\nu_j^d(dL)\\
&= C_{d,j}(\omega_j)^j \int_{\mathcal{L}_j^d} g(L)\Big\{\int_{\mathcal{L}_1^L}\cdots\int_{\mathcal{L}_1^L} D^{d-j+1}(L_1,\ldots,L_j)\times\\
&\qquad\qquad \rho(L_1)\ldots\rho(L_j)\nu_1^L(dL_1)\ldots\nu_1^L(dL_j)\Big\}\nu_j^d(dL)\\
&= C_{d,j}(\omega_j)^j \int_{\mathcal{L}_j^d} g(L)M_{j,d}(K,L)\nu_j^d(dL),
\end{aligned}$$

where

$$M_{j,d}(K,L) := \int_{\mathcal{L}_1^L}\cdots\int_{\mathcal{L}_1^L} D^{d-j+1}(L_1,\ldots,L_j)\rho(L_1)\ldots\rho(L_j)\nu_1^L(dL_1)\ldots\nu_1^L(dL_j)$$

is a function on $\mathcal{L}_j^d$. Since g is an arbitrary function in $C(\mathcal{L}_j^d)$, $\overline{\rho}_j(K,\cdot)$ is, in fact, a function on $\mathcal{L}_j^d$ and for each $E \in \mathcal{L}_j^d$,

$$\begin{aligned}
\overline{\rho}_j(K,E) &= C_{d,j}(\omega_j)^j M_{j,d}(K,E) \\
&= C_{d,j}(\omega_j)^j \int_{\mathcal{L}_1^E} \cdots \int_{\mathcal{L}_1^E} D^{d-j+1}(L_1,\ldots,L_j)\times \\
&\qquad\qquad \rho(L_1)\ldots\rho(L_j)\nu_1^E(dL_1)\ldots\nu_1^E(dL_j) \\
&= C_{d,j} \int_{S^{d-1}\cap E} \cdots \int_{S^{d-1}\cap E} D^{d-j+1}(u_1,\ldots,u_j)\times \\
&\qquad\qquad \rho(u_1)\ldots\rho(u_j)\lambda_{j-1}(du_1)\ldots\lambda_{j-1}(du_j)\times \\
&= C_{d,j} \underbrace{\int_{u_1}\cdots\int_{u_j}}_{\mathrm{span}(u_1,\ldots,u_j)=E} D^{d-j+1}(u_1,\ldots,u_j)\times \\
&\qquad\qquad \rho(u_1)\ldots\rho(u_j)\lambda_{j-1}(du_1)\ldots\lambda_{j-1}(du_j) \\
&= C_{d,j} I_{j,d-j+1}(K,E).
\end{aligned}$$

Here we have used the fact that $D(L_1,\ldots,L_j) = 0$ if $L_1,\ldots,L_j$ are linearly dependent in E. $\square$

Note that if K is a sufficiently smooth centrally symmetric convex body, to be precise, if its support function $h(K,\cdot) \in C^n(S^{d-1})$, where $n \geq d+2$ is even, then it is a generalized zonoid. It follows from Theorem 1.2 that the generating measure $\rho(K,\cdot)$ is, in fact, a function on S^{d-1}. Therefore, by Theorem 2.1, we can derive a j-th projection generating function $\overline{\rho}_j(K,\cdot)$ of K. Recall that $\overline{\rho}_j(K,\cdot)$ is not unique if $j \in \{2,3,\ldots,d-2\}$. However, we will show that if K is also a body of revolution, then $\overline{\rho}_j$ determines whether K is in $\mathcal{K}(j)$ or not.

Theorem 3.2. *Let K be a body of revolution such that its support function $h(K,\cdot)$ is in $C^n(S^{d-1})$, where $n \geq d+2$ is even. Let $\rho_j(K,\cdot)$ be any j-th projection generating function of K having the same rotational symmetry as K. Then*

$$K \in \mathcal{K}(j) \qquad \textit{if and only if} \qquad \rho_j(K,E) \geq 0 \qquad \textit{for all} \qquad E \in \mathcal{L}_j^d.$$

Proof:

First, let

$$\rho_j(K,E) \geq 0 \qquad \text{for all} \qquad E \in \mathcal{L}_j^d.$$

For any Borel set ω on $\mathcal{L}_j^d$, let

$$\mu(K,\omega) =: \int_\omega \rho_j(K,E)\nu_j^d(dE). \tag{3.9}$$

Then μ is a clearly a positive measure on $\mathcal{L}_j^d$ and satisfies

$$V_j(K|E) = \int_{\mathcal{L}_j^d} |\langle E,F\rangle|\mu(K,dF) \qquad \text{for all} \qquad E \in \mathcal{L}_j^d. \tag{3.10}$$

This shows that $K \in \mathcal{K}(j)$.

Now let $K \in \mathcal{K}(j)$. Then there exists a positive measure μ on $\mathcal{L}_j^d$ such that μ satisfies (3.10). Thus for all $E \in \mathcal{L}_j^d$,

$$V_j(K|E) = (T_j^d\mu)(E) = (T_j^d\rho_j)(E). \tag{3.11}$$

Here, for simplicity, we use ρ_j to denote $\rho_j(K,\cdot)$.

Our objective is to prove that $\rho_j \geq 0$. To this end, let $e_d \in S^{d-1}$ be parallel to the axis of revolution of K and $SO(d-1)$ be the group of rotations in $e_d^\perp$. We may think $SO(d-1)$ as a subgroup of $SO(d)$ and denote the rotation invariant probability measure on $SO(d-1)$ as ν. By the definition of ρ_j, it is invariant under $SO(d-1)$. This is to say, for each $\theta \in SO(d-1)$ and $E \in \mathcal{L}_j^d$,

$$\rho_j(E) = \rho_j(\theta E).$$

If $f \in C^\infty(\mathcal{L}_j^d)$ with $f \geq 0$ and $\theta \in SO(d-1)$, then

$$\begin{aligned}\int_{\mathcal{L}_j^d} \rho_j(E)f(E)\nu_j^d(dE) &= \int_{\mathcal{L}_j^d} \rho_j(\theta^{-1}F)f(\theta^{-1}F)\nu_j^d(dF)\\ &= \int_{\mathcal{L}_j^d} \rho_j(F)f(\theta^{-1}F)\nu_j^d(dF).\end{aligned}$$

Hence

$$\begin{aligned}\int_{\mathcal{L}_j^d} \rho_j(E) f(E) \nu_j^d(dE) &= \int_{SO(d-1)} \int_{\mathcal{L}_j^d} \rho_j(F) f(\theta^{-1}F) \nu_j^d(dF) \nu(d\theta) \\ &= \int_{\mathcal{L}_j^d} \rho_j(F) \int_{SO(d-1)} f(\theta^{-1}F) \nu(d\theta) \nu_j^d(dF) \\ &= \int_{\mathcal{L}_j^d} \rho_j(F) \bar{f}(F) \nu_j^d(dF),\end{aligned}$$

where $\bar{f}(F) = \int_{SO(d-1)} f(\theta^{-1}F)\nu(d\theta)$. It is clear that $\bar{f} \geq 0$ and $\bar{f} \in C^\infty(\mathcal{L}_j^d)$. Also $\bar{f}$ is clearly rotationally symmetric and thus, by the observation at the end of Section 2, $\bar{f} \in \mathcal{R}_j^d$. By Lemma 2.6, there exists $g \in \mathcal{R}_j^d$ such that $\bar{f} = T_j^d g$. Thus

$$\begin{aligned}\int_{\mathcal{L}_j^d} \rho_j(E) f(E) \nu_j^d(dE) &= \int_{\mathcal{L}_j^d} \rho_j(F)(T_j^d g)(F) \nu_j^d(dF) \\ &= \int_{\mathcal{L}_j^d} (T_j^d \rho_j)(F) g(F) \nu_j^d(dF) = \int_{\mathcal{L}_j^d} (T_j^d \mu)(F) g(F) \nu_j^d(dF) \\ &= \int_{\mathcal{L}_j^d} (T_j^d g)(F) \mu(dF) = \int_{\mathcal{L}_j^d} \bar{f}(F) \mu(dF) \geq 0.\end{aligned}$$

Here we have used the duality relation (2.32) for T_j^d. Since f is an arbitrary function in $C^\infty(\mathcal{L}_j^d)$ with $f \geq 0$, ρ_j is a positive distribution. Note that positive distributions are positive measures. Thus $\rho_j \geq 0$ as required. □

Lemma 3.3. *Let $I_{j,n}(K,\cdot)$ be the function on $\mathcal{L}_j^d$ defined in (3.5). Then there exists a positive constant c_j' such that for each $E \in \mathcal{L}_j^d$*

$$I_{j,n}(K,E) = \\ c_j' \int_{S^{d-1}\cap E} \left\{ I_{j-1,n+1}(K, v^\perp) \int_{S^{d-1}\cap E} \rho(u) |\langle u,v\rangle|^n \lambda_{j-1}(du) \right\} \lambda_{j-1}(dv), \tag{3.12}$$

where $v \in E \cap S^{d-1}$ and $v^\perp$ is the $(j-1)$-dimensional subspace in E orthogonal to v and ρ is the generating function of K.

Proof:

Let $E \in \mathcal{L}_j^d$ be fixed. Let $u_1, \ldots, u_j \in E \cap S^{d-1}$ be such that $E = L(u_1, \ldots, u_j)$. It was shown in Weil [1979, Proposition 2.1] that for each i, $1 < i < j$,

$$D_j(u_1, \ldots, u_j) = D_i(u_1, \ldots, u_i) D_{j-i}(u_{i+1}, \ldots, u_j) \cdot \\ \cdot |\langle L(u_1, \ldots, u_i), L^{\perp}(u_{i+1}, \ldots, u_j)\rangle|,$$

where $L^{\perp}(u_{i+1}, \ldots, u_j)$ is the orthogonal space of $L(u_{i+1}, \ldots, u_j)$ in E. In particular,

$$\begin{aligned} D_j(u_1, \ldots, u_j) &= D_{j-1}(u_1, \ldots, u_{j-1})|\langle v^{\perp}, u_j^{\perp}\rangle| \\ &= D_{j-1}(u_1, \ldots, u_{j-1})|\langle v, u_j\rangle|. \end{aligned} \tag{3.13}$$

Here, and in what follows, v denotes a unit vector in $E \cap S^{d-1}$ such that $v^{\perp} = L(u_1, \ldots, u_{j-1})$.

Note that (3.8) can be written as

$$\begin{aligned} &\int_{\mathcal{L}_1^j} \cdots \int_{\mathcal{L}_1^j} f(L_1, \ldots, L_{j-1}) \nu_1^j(dL_1) \ldots \nu_1^j(dL_{j-1}) \\ &\quad = c_j \int_{\mathcal{L}_{j-1}^j} \left\{ \int_{\mathcal{L}_1^L} \cdots \int_{\mathcal{L}_1^L} f(L_1, \ldots, L_{j-1}) \times \right. \\ &\qquad\qquad \left. D(L_1, \ldots, L_{j-1}) \nu_1^L(dL_1) \ldots \nu_1^L(dL_{j-1}) \right\} \nu_{j-1}^j(dL), \end{aligned}$$

where c_j is a positive constant. It follows that there exists positive constant c_j' such that

$$\begin{aligned} &\int_{S^{d-1}\cap E} \cdots \int_{S^{d-1}\cap E} f(u_1, \ldots, u_{j-1}) \lambda_{j-1}(du_1) \ldots \lambda_{j-1}(du_{j-1}) \\ &\quad = c_j' \int_{S^{d-1}\cap E} \left\{ \int_{S^{d-1}\cap v^{\perp}} \cdots \int_{S^{d-1}\cap v^{\perp}} f(u_1, \ldots, u_{j-1}) D(u_1, \ldots, u_{j-1}) \right. \\ &\qquad\qquad \left. \lambda_{j-2}(du_1) \ldots \lambda_{j-2}(du_{j-1}) \right\} \lambda_{j-1}(dv). \end{aligned} \tag{3.14}$$

Now fix u_j and put

$$f(u_1, \ldots, u_{j-1}) = D_{j-1}^n(u_1, \ldots, u_{j-1})|\langle u_j, v\rangle|^n \rho(u_1) \ldots, \rho(u_{j-1}). \tag{3.15}$$

Apply (3.15) to (3.14) and we get

$$\begin{aligned}
&\int_{S^{d-1}\cap E}\cdots\int_{S^{d-1}\cap E} D^n_{j-1}(u_1,\ldots,u_{j-1})|\langle u_j,v\rangle|^n \\
&\qquad\qquad \rho(u_1)\ldots\rho(u_{j-1})\lambda_{j-1}(du_1)\cdots\lambda_{j-1}(du_{j-1}) \\
&= c'_j\int_{S^{d-1}\cap E}\left\{\int_{S^{d-1}\cap v^\perp}\cdots\int_{S^{d-1}\cap v^\perp} D^{n+1}_{j-1}(u_1,\ldots,u_{j-1})\times\right. \\
&\qquad \left.|\langle u_j,v\rangle|^n\rho(u_1)\ldots\rho(u_{j-1})\lambda_{j-2}(du_1)\ldots\lambda_{j-2}(du_{j-1})\right\}\lambda_{j-1}(dv) \\
&= c'_j\int_{S^{d-1}\cap E}|\langle u_j,v\rangle|^n I_{j-1,n+1}(K,v^\perp)\lambda_{j-1}(dv)
\end{aligned} \tag{3.16}$$

It follows from (3.13) and (3.16) that

$$\begin{aligned}
&I_{j,n}(K,E) \\
&= \underbrace{\int_{u_1}\cdots\int_{u_j}}_{\mathrm{span}(u_1,\ldots,u_j)=E} D^n_j(u_1,\ldots,u_j)\rho(u_1)\ldots\rho(u_j)\lambda_{j-1}(du_1)\ldots\lambda_{j-1}(du_j) \\
&= \int_{S^{d-1}\cap E}\cdots\int_{S^{d-1}\cap E} D^n_j(u_1,\ldots,u_j)\rho(u_1)\ldots\rho(u_j)\lambda_{j-1}(du_1)\ldots\lambda_{j-1}(du_j) \\
&= \int_{S^{d-1}\cap E}\rho(u_j)\left\{\int_{S^{d-1}\cap E}\cdots\int_{S^{d-1}\cap E} D^n_{j-1}(u_1,\ldots,u_{j-1})|\langle u_j,v\rangle|^n\right. \\
&\qquad\qquad \left.\rho(u_1)\ldots\rho(u_{j-1})\lambda_{j-1}(du_1)\ldots\lambda_{j-1}(du_{j-1})\right\}\lambda_{j-1}(du_j) \\
&= c'_j\int_{S^{d-1}\cap E}\rho(u_j)\left\{\int_{S^{d-1}\cap E}|\langle u_j,v\rangle|^n I_{j-1,n+1}(K,v^\perp)\lambda_{j-1}(dv)\right\}\lambda_{j-1}(du_j) \\
&= c'_j\int_{S^{d-1}\cap E}\left\{I_{j-1,n+1}(K,v^\perp)\int_{S^{d-1}\cap E}\rho(u)|\langle u,v\rangle|^n\lambda_{j-1}(du)\right\}\lambda_{j-1}(dv)
\end{aligned}$$

as required. □

Theorem 3.4. *Let K be a body of revolution such that its support function $h(K,\cdot)$ is in $C^n(S^{d-1})$, where $n \geq d+2$ is even. If $K \in \mathcal{K}(j-1)$ and $K \in Z_{d-j+1}$, then $K \in \mathcal{K}(j)$.*

Proof: By Lemma 3.3, there exists a positive constant c'_j such that

$$\begin{aligned}
&I_{j,n}(K,E) = \\
&\qquad c'_j\int_{S^{d-1}\cap E}\left\{I_{j-1,n+1}(K,v^\perp)\int_{S^{d-1}\cap E}\rho(u)|\langle u,v\rangle|^n\lambda_{j-1}(du)\right\}\lambda_{j-1}(dv).
\end{aligned}$$

Here $v \in E \cap S^{j-1}$ and $v^{\perp}$ is the $(j-1)$-dimensional subspace in E orthogonal to v. Let $n = d - j + 1$. It follows from Theorem 3.1 that there exists a positive constant c_j'' such that for each $E \in \mathcal{L}_j^d$

$$\begin{aligned} &\overline{\rho}_j(K, E) \\ &= c_j'' \int_{S^{d-1} \cap E} \rho_{j-1}(K, v^{\perp}) \left\{ \int_{S^{d-1} \cap E} \rho(u) |\langle u, v \rangle|^{d-j+1} \lambda_{j-1}(du) \right\} \lambda_{j-1}(dv). \end{aligned} \tag{3.18}$$

Weil [1976] showed that $K \in Z_{d-j+1}$ if and only if for all $v \in S^{j-1}$

$$\int_{S^{j-1}} \rho(K, u) |\langle u, v \rangle|^{d-j+1} \lambda_{j-1}(du) \geq 0. \tag{3.19}$$

Since K is a sufficiently smooth body of revolution and $K \in \mathcal{K}(j-1)$, by Theorem 2.2, $\overline{\rho}_{j-1}(K, \cdot)$ is positive. Because K is in Z_{d-j+1}, by (3.17), the inner integral of (3.18) is also positive. Therefore, $\overline{\rho}_j(K, \cdot)$ is positive. This shows that $K \in \mathcal{K}(j)$. □

§4. The Classes $\mathcal{K}(j)$, $\mathcal{O}_j$ and Z_{d-j+1}

In this section, we would like to investigate various problems concerning the classes $\mathcal{K}(j)$, $\mathcal{O}_j$ and Z_{d-j+1} $(1 \le j \le d-1)$. Our first motivation is to answer the question whether the inclusions $\mathcal{K}(1) \subset \mathcal{K}(j) \subset \mathcal{K}(d-1)$ are strict if $1 < j < d-1$ (see Schneider [1997b, p.364]). We will give a positive answer to this question. We will also show that, contrary to conjectures that have appeared in the literature, $\mathcal{K}(j)$, $\mathcal{O}_j$ and Z_{d-j+1} are generally different classes if $1 < j < d-1$. In the meantime, we will provide evidences for the inclusions $Z_{d-j+1} \subset \mathcal{K}(j) \subset \mathcal{O}_j$. Although general convex bodies are mentioned in some places, we will, in this section, mainly work with smooth bodies of revolution in $\mathbb{E}^d$, especially convex bodies K_α, $L_{\beta,j}$ and $W_{\gamma,j}$.

Let e_d be a fixed unit vector in S^{d-1}. Let K_α be the convex body whose support function is given by

$$h(K_\alpha, u) = 1 + \alpha P_2^d(\langle e_d, u\rangle) \qquad \text{for each } u \in S^{d-1}. \tag{4.1}$$

The convex bodies $L_{\beta,j}$ and $W_{\gamma,j}$ are defined by their j-th area measures, namely

$$S_j(L_{\beta,j}, u) = 1 + \beta P_2^d(\langle e_d, u\rangle) \qquad \text{for each } u \in S^{d-1} \tag{4.2}$$

and

$$S_j(W_{\gamma,j}, u) = 1 + \gamma P_4^d(\langle e_d, u\rangle) \qquad \text{for each } u \in S^{d-1}. \tag{4.3}$$

Here $P_2^d(x)$ and $P_4^d(x)$ are Legendre polynomials given by (2.11) and (2.13).

The convexity of K_α follows from elementary results. But the convexity of $L_{\beta,j}$ and $W_{\gamma,j}$ are more complicated and rely on the following result of Firey [1970, p.387]:

Theorem. *(Firey) In order for a function Φ over S^{d-1} to be the j-th elementary symmetric function of the principal radii of a strictly convex body in $\mathbb{E}^d$ which* *is a body of revolution, it is necessary and sufficient that, in some system of geographic coordinates on S^{d-1}, Φ is a function ϕ of the latitude ϑ alone and , over $-\frac{\pi}{2} < \vartheta < \frac{\pi}{2}$:*

(a)$\phi(\vartheta)$ is continuous and has finite limits as ϑ tends to $\pm\frac{\pi}{2}$;

(b)$\int_{\vartheta}^{\frac{\pi}{2}} \phi(x) \cos^{d-2} x \sin x dx > 0$ and is zero for $\vartheta = -\frac{\pi}{2}$;

(c)For any $\vartheta \in (-\pi/2, \pi/2)$,

$$\phi(\vartheta) > (d-j-1) \int_{\vartheta}^{\frac{\pi}{2}} \phi(x) \cos^{d-2} x \sin x dx / \cos^{d-1} \vartheta,$$

where $d \geq 3$ and $1 \leq j \leq d-1$.

We will show that if α, β and γ lie on certain ranges then K_α, $L_{\beta,j}$ and $W_{\gamma,j}$ are indeed convex bodies. These ranges will be given explicitly. It is clear that K_α, $L_{\beta,j}$ and $W_{\gamma,j}$ are smooth bodies of revolution if α, β and γ lie in these ranges.

For our purpose, we need to give *j-th projection generating measures* of K_α, $L_{\beta,j}$ and $W_{\gamma,j}$ explicitly. The j-th projection generating measure of K_α is derived using results in Section 3 and relies on various tensor product formulas derived by Goodey and Weil [1991]. For $L_{\beta,j}$ and $W_{\gamma,j}$, a more direct approach is used involving cosine and Radon transforms. In this approach, we will also be able to give the j-th projection functions of $L_{\beta,j}$ and $W_{\gamma,j}$. These projection functions will be used in our investigation of orthogonal bodies. We next want to describe this approach.

For each $u \in S^{d-1}$, the *j-th intrinsic volume* of the projection of K onto a hyperplane $u^\perp$ is given by

$$V_j(K|u^\perp) = \frac{\binom{d-1}{j}}{2\kappa_{d-j-1}} \int_{S^{d-1}} |\langle u, v\rangle| S_j(K, dv), \tag{4.4}$$

or, in terms of cosine transforms,

$$\begin{aligned} V_j(K|u^\perp) &= \frac{\binom{d-1}{j}}{2\kappa_{d-j-1}} \Big(T^d_{d-1}(S_j^\perp(K,\cdot)\Big)(u^\perp) \\ &= \frac{\binom{d-1}{j}}{2\kappa_{d-j-1}} \Big(T^d_1(S_j(K,\cdot)\Big)^\perp (u^\perp). \end{aligned} \tag{4.5}$$

The formula (4.4) is called *the generalized Cauchy projection formula* and can be found, for example, in Gardner [1994, p.378]. The Cauchy-Kubota formulas (2.31), together with (4.5), give

$$\begin{aligned} R^d_{j,d-1}V_j(K|\cdot) &= \frac{\binom{d-1}{j}}{2\kappa_{d-j-1}\alpha^{0,d-1}_{j,d-j-1}} T^d_{d-1}(S^\perp_j(K,\cdot)) \\ &= \frac{\kappa_j}{2\kappa_{d-1}} T^d_{d-1}(S^\perp_j(K,\cdot)). \end{aligned} \tag{4.6}$$

Since $V_j(K|\cdot) = T^d_j(\rho_j(K,\cdot))$, (4.6) can be written as

$$R^d_{j,d-1}(T^d_j(\rho_j(K,\cdot)) = \frac{\kappa_j}{2\kappa_{d-1}} T^d_{d-1}(S^\perp_j(K,\cdot)). \tag{4.7}$$

It follows from (2.28) that

$$\alpha^{j,d-j}_{d-1,1} T^d_{d-1}\Big(R^d_{j,d-1}(\rho_j(K,\cdot)\Big) = \frac{\kappa_j}{2\kappa_{d-1}} T^d_{d-1}(S^\perp_j(K,\cdot)). \tag{4.8}$$

So far, (4.4) to (4.8) are true for all convex bodies $K \in \mathcal{K}^d$. Now let $\overline{K}$ be $L_{\beta,j}$ or $W_{\gamma,j}$. First we have

$$\big(R^d_{j,d-1}(\|e_d|\cdot\|^{2n})\big)(u^\perp) = \sigma_{j,n}\|e_d|u^\perp\|^{2n} \qquad \text{for each } u \in S^{d-1}, \tag{4.9}$$

where $\sigma_{j,n}$ is a constant, which can be calculated explicitly, depending on n and j. We will prove (4.9) later in this section. Since $\overline{K}$ is a body of revolution, $V_j(\overline{K}|\cdot)$ is a rotationally symmetric function. By the observation at the end of Section 2, $V_j(\overline{K}|\cdot) \in \mathcal{R}^d_j$. Thus $R^d_{j,d-1}$ is injective if we restrict to $\mathcal{R}^d_j$ (Lemma 2.5). By definition, $S_j(\overline{K},\cdot)$ is a polynomial in $\langle e_d,\cdot\rangle^2$. It is easy to see that $\big(T^d_{d-1}S_j(\overline{K},\cdot)\big)(u^\perp)$ is a polynomial in $\langle e_d,u^\perp\rangle^2$. It follows from (4.6) and (4.9) that $V_j(\overline{K}|\cdot)$ is a polynomial in $\|\,e_d|\cdot\,\|^2$. So $(R^d_{j,d-1})^{-1}$ makes sense for $\overline{K}$. By (4.6) we have

$$V_j(\overline{K}|\cdot) = \frac{\kappa_j}{2\kappa_{d-1}}(R^d_{j,d-1})^{-1}(T^d_{d-1}S^\perp_j(\overline{K},\cdot)). \tag{4.10}$$

Note that $S^\perp_j(\overline{K},\cdot)$ is an even measure and that T^d_{d-1} is injective on even measures. It follows from (4.8) that

$$\alpha^{j,d-j}_{d-1,1} R^d_{j,d-1}(\rho_j(\overline{K},\cdot)) = \frac{\kappa_j}{2\kappa_{d-1}} S^\perp_j(\overline{K},\cdot).$$

Therefore, we have

$$\begin{aligned}\rho_j(\overline{K},\cdot) &= (\alpha^{j,d-j}_{d-1,1})^{-1}\frac{\kappa_j}{2\kappa_{d-1}}(R^d_{j,d-1})^{-1}(S_j^{\perp}(\overline{K},\cdot)) \\ &= \binom{d}{j}\frac{1}{d\kappa_{d-j}}(R^d_{j,d-1})^{-1}(S_j^{\perp}(\overline{K},\cdot)).\end{aligned} \tag{4.11}$$

The convex bodies K_α, $L_{\beta,j}$ and $W_{\gamma,j}$ are smooth bodies of revolution. For each of these bodies, we can calculate a j-th projection generating function, which is a polynomial in $\| e_d|\cdot \|^2$, explicitly. By Theorem 3.2, these j-th projection generating functions (polynomials) will determine the ranges of α, β or γ for which K_α, $L_{\beta,j}$ or $W_{\gamma,j}$ is in $\mathcal{K}(j)$.

For orthogonal bodies, we will work with $L_{\beta,j}$ and $W_{\gamma,j}$. We will use (4.2) and (4.3) to define bodies $L_{\beta',d-j}$ and $W_{\gamma',d-j}$ respectively. We then use (4.10) to calculate the j-th projection functions (polynomials) of $L_{\beta,j}$, $L_{\beta',d-j}$, $W_{\gamma,j}$ and $W_{\gamma',d-j}$. These polynomials, together with Firey's result, will be used to find the ranges of β or γ for which $L_{\beta,j}$ or $W_{\gamma,j} \in \mathcal{O}_j$. We will show that β' is related to β and $L_{\beta',d-j}$ is the orthogonal body of $L_{\beta,j}$ if β is in an appropriate range. The same thing can be said about $W_{\gamma,j}$ and $W_{\gamma',d-j}$.

For the classes Z_{d-j+1} we return to K_α since it is more convenient to work with support functions for these bodies.

Lemma 4.1. *K_α is a convex body if and only if*

$$-\frac{d-1}{2d-1} \le \alpha \le \frac{d-1}{d+1}. \tag{4.12}$$

Proof:

First we recall a fact from Groemer [1996, p.23] that a function h on S^{d-1} is a support function of a convex body in $\mathbb{E}^d$ if and only if for every two-dimensional subspace E the restriction of h on $S = E \cap S^{d-1}$, say h_S, is a support function on S.

Now consider a two-dimensional subspace E in $\mathbb{E}^d$. Choose two mutually orthogonal unit vectors v_0 and w_0 in $S = E \cap S^{d-1}$ such that w_0 is orthogonal to

e_d. Denote by θ the angle between a given vector $u \in S$ and v_0. We have

$$u = v_0 \cos\theta + w_0 \sin\theta.$$

Then the support function of K_α (restricted to E) can be written as

$$\begin{aligned} h(K_\alpha,\theta) &= 1 + \frac{\alpha}{d-1}(d\langle e_d, u\rangle^2 - 1) \\ &= 1 + \frac{\alpha}{d-1}(d\langle e_d, v_0\rangle^2 \cos^2\theta - 1). \end{aligned}$$

Note that

$$h'(K_\alpha,\theta) = -\frac{d\alpha}{d-1}\langle e_d, v_0\rangle^2 \sin 2\theta$$

and

$$h''(K_\alpha,\theta) = 2\frac{d\alpha}{d-1}\langle e_d, v_0\rangle^2(1 - 2\cos^2\theta).$$

Thus

$$\begin{aligned} & h(K_\alpha,\theta) + h''(K_\alpha,\theta) \\ &\quad = 1 + \frac{\alpha}{d-1}(d\langle e_d, v_0\rangle^2 \cos^2\theta - 1) + 2\frac{d\alpha}{d-1}\langle e_d, v_0\rangle^2(1 - 2\cos^2\theta) \\ &\quad = 1 - \frac{\alpha}{d-1} + \frac{2d\alpha}{d-1}\langle e_d, v_0\rangle^2 - \frac{3d\alpha}{d-1}\langle e_d, v_0\rangle^2 \cos^2\theta. \end{aligned}$$

The condition for $h(K_\alpha,\theta)$ (restricted to E) to be a support function of a convex body is (see Groemer [1996, p.22])

$$h(K_\alpha,\theta) + h''(K_\alpha,\theta) \geq 0. \tag{4.13}$$

Note that $0 \leq \cos^2\theta \leq 1$ and thus the above inequality is equivalent to

$$1 - \frac{\alpha}{d-1} - \frac{d\alpha}{d-1}\langle e_d, v_0\rangle^2 \geq 0, \qquad \text{if } \alpha \geq 0 \tag{4.14}$$

and

$$1 - \frac{\alpha}{d-1} + \frac{2d\alpha}{d-1}\langle e_d, v_0\rangle^2 \geq 0, \qquad \text{if } \alpha < 0. \tag{4.15}$$

It follows from our early remarks in this proof that if $\alpha \geq 0$, then $h(K,u)$ is the support function of $K \in \mathcal{K}^d$ if and only if (4.15) holds for all $0 \leq \langle e_d, v_0 \rangle^2 \leq 1$. That is,

$$\alpha \leq \frac{d-1}{1+d\langle e_d, v_0\rangle^2} \qquad \text{or} \qquad \alpha \leq \frac{d-1}{d+1}.$$

Similarly, if $\alpha < 0$ then , by (4.15), $h(K,u)$ is the support function of $K \in \mathcal{K}^d$ if and only if

$$\alpha \geq -\frac{d-1}{2d-1}.$$

Therefore we get (4.12) as required. □

It is clear that $h(K_\alpha, \cdot)$ is an even function on S^{d-1} and $h(K_\alpha, \cdot) \in C^\infty(S^{d-1})$. By Theorem 1.2, the convexity condition is also the condition that K_α is a generalized zonoid. The generating measure of K_α is, in fact, a function on S^{d-1}.

Lemma 4.2.

(a)The generating function of K_α is

$$\rho(K_\alpha, u) = \frac{1}{2\kappa_{d-1}}(a + b\langle e_d, u\rangle^2) \quad \textit{for each} \quad u \in S^{d-1}, \tag{4.16}$$

where

$$a = 1 - \frac{d+1}{d-1}\alpha \quad \textit{and} \quad b = \frac{(d+1)d}{d-1}\alpha. \tag{4.17}$$

(b) K_α is a zonoid if and only if

$$-\frac{1}{d+1} \leq \alpha \leq \frac{d-1}{d+1}. \tag{4.18}$$

Proof:

Since $h(K_\alpha, \cdot)$ is an even function on S^{d-1} and $h(K_\alpha, \cdot) \in C^\infty(S^{d-1})$, by Theorem 1.2, there exists a continuous function, the generating function, $\rho(K_\alpha, \cdot)$ on S^{d-1} such that

$$h(K_\alpha, u) = 1 + \alpha P_2^d(\langle e_d, u\rangle) = \int_{S^{d-1}} |\langle u, v\rangle| \rho(K_\alpha, v)\lambda_{d-1}(dv).$$

By the Funk-Heck Theorem,

$$\rho(K_\alpha, u) = \lambda_{d,0}^{-1} + \lambda_{d,2}^{-1}\frac{\alpha}{d-1}(d\langle e_d, u\rangle^2 - 1),$$

where $\lambda_{d,0} = 2\kappa_{d-1}$ and $\lambda_{d,2} = \frac{2\kappa_{d-1}}{d+1}$ (see Theorem 2.2). Put

$$a := 1 - \frac{d+1}{d-1}\alpha \qquad \text{and} \qquad b := \frac{(d+1)d}{d-1}\alpha.$$

Therefore, we have

$$\rho(K_\alpha, u) = \frac{1}{2\kappa_{d-1}}(a + b\langle e_d, u\rangle^2).$$

Recall that K_α is a zonoid if and only if $\rho(K_\alpha, u) \geq 0$ for all $u \in S^{d-1}$. Put $X = \langle e_d, u\rangle^2$ and thus K_α is a zonoid if and only if α satisfies

$$1 - \frac{d+1}{d-1}\alpha + \frac{(d+1)d}{d-1}\alpha X \geq 0 \quad \text{for all} \quad 0 \leq X \leq 1. \tag{4.19}$$

The inequality (4.19) is true if and only if it is true for both $X = 0$ and $X = 1$. That is

$$1 - \frac{d+1}{d-1}\alpha \geq 0 \quad \text{and} \quad 1 + (d+1)\alpha \geq 0. \tag{4.20}$$

Clearly (4.20) gives (b) as required. □

We next want to calculate $\rho_j(K_\alpha, \cdot)$ explicitly. Instead of working with K_α directly, we will work with any generalized zonoid K whose generating function has the form

$$\rho(K, u) = a + b\langle e_d, u\rangle^2, \qquad \text{for each} \quad u \in S^{d-1},$$

where a is a positive number and b is any real number.

Lemma 4.3. *Let K be the generalized zonoid in $\mathbb{E}^d$ with generating function*

$$\rho(K, u) = a + b\langle e_d, u\rangle^2 \quad \textit{where} \quad u \in S^{d-1}.$$

Let $I_{j,n}(K, \cdot)$ be defined as in (3.5). Then, up to a positive multiplicative constant,

$$I_{j,n}(K, E) = \sum_{i=0}^{j} C(j, i, n)\left(\frac{b}{a}\right)^i \| e_d|E \|^{2i}, \qquad \textit{for each} \;\; E \in \mathcal{L}_j^d, \tag{4.21}$$

where $C(j,0,n) = C(j,1,n) = 1$ *and*

$$C(j,i,n) = \binom{j}{i} \frac{1}{(n+j)^i} \prod_{k=0}^{i-1} \frac{n+j+2k}{j+2k} \qquad \textit{for all} \quad i \geq 1. \tag{4.22}$$

Proof:

We will use induction on j. First, let $j = 2$ and n be any positive integer. For $E \in \mathcal{L}_2^d$,

$$I_{2,n}(K,E) = \underbrace{\int_{u_1}\int_{u_2}}_{\mathrm{span}(u_1,u_2)=E} D_2^n(u_1,u_2)\rho(K,u_1)\rho(K,u_2)\lambda_1(du_1)\lambda_1(du_2). \tag{4.23}$$

Let

$$u_E = \frac{e_d|E}{\| e_d|E \|} \in S^{d-1} \cap E.$$

Choose an orthonormal basis $\{u_E, v_E\}$ of E. Then for any u_1, $u_2 \in S^{d-1} \cap E$, we have

$$u_1 = v_E \cos\theta + u_E \sin\theta$$

and

$$u_2 = v_E \cos\phi + u_E \sin\phi.$$

Here θ is the angle from u_1 to v_E and ϕ is the angle from u_2 to v_E. Note that v_E is orthogonal to e_d. Thus

$$\rho(K_\alpha, u_1) = a + b\langle e_d, u_1\rangle^2 = a + b\| e_d|E \|^2 \sin^2\theta \tag{4.24}$$

and

$$\rho(K_\alpha, u_2) = a + b\langle e_d, u_2\rangle^2 = a + b\| e_d|E \|^2 \sin^2\phi. \tag{4.25}$$

Also

$$D_2^n(u_1,u_2) = | \sin^n(\theta - \phi) | .$$

Put

$$\mathcal{I}_n := \int_0^{2\pi} | \sin^n\phi | \, d\phi.$$

Then

$$\mathcal{I}_n = 2\pi^{\frac{1}{2}}\frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n+2}{2})} = 2\pi^{\frac{1}{2}}\frac{n-1}{n}\frac{\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n}{2})} = \frac{n-1}{n}\mathcal{I}_{n-2}.$$

Apply (4.24) and (4.25) to (4.23) and we have

$$\begin{aligned}
&I_{2,n}(K,E)\\
&= \int_0^{2\pi}\int_0^{2\pi} |\sin(\theta-\phi)|^n(a+b\| e_d|E\|^2\sin^2\theta)(a+b\| e_d|E\|^2\sin^2\phi)d\theta d\phi\\
&= a^2A + 2abB\| e_d|E\|^2 + b^2C\| e_d|E\|^4.
\end{aligned}$$

Here

$$\begin{aligned}
A &= \int_0^{2\pi}\int_0^{2\pi} |\sin^n(\theta-\phi)|\, d\phi d\theta = 2\pi\mathcal{I}_n;\\
B &= \int_0^{2\pi}\int_0^{2\pi} |\sin^n(\theta-\phi)|\sin^2\theta d\phi d\theta\\
&= \int_0^{2\pi} |\sin^n\phi|\, d\phi\int_0^{2\pi}\sin^2\theta d\theta = \pi\mathcal{I}_n = \frac{1}{2}A;\\
C &= \int_0^{2\pi}\int_0^{2\pi} |\sin^n(\theta-\phi)|\sin^2\theta\sin^2\phi d\theta d\phi\\
&= \int_0^{2\pi}\sin^2\phi\left\{\int_0^{2\phi} |\sin^n(\theta-\phi)|\sin^2\theta d\theta\right\}d\phi\\
&= \int_0^{2\pi}\sin^2\phi\left\{\int_{-\phi}^{2\pi-\phi}\sin^2(t+\phi)\,|\sin^n t|\, dt\right\}d\phi\\
&= \int_0^{2\pi}\sin^2\phi\left\{\cos^2\phi\int_{-\phi}^{2\pi-\phi} |\sin^n t|\sin^2 t dt + \sin^2\phi\int_{-\phi}^{2\pi-\phi} |\sin^n t|\cos^2 t dt\right.\\
&\qquad\qquad \left. + 2\sin\phi\cos\phi\int_{-\phi}^{2\pi-\phi} |\sin^n t|\cos t\sin t dt\right\}d\phi\\
&=\mathcal{I}_{n+2}\int_0^{2\pi}\sin^2\phi\cos^2\phi d\phi + \mathcal{I}_n\int_0^{2\pi}\sin^4\phi d\phi - \mathcal{I}_{n+2}\int_0^{2\pi}\sin^4\phi d\phi\\
&=\frac{\pi}{4}\mathcal{I}_{n+2} + \frac{3\pi}{4}\mathcal{I}_n - \frac{3\pi}{4}\mathcal{I}_{n+2} = \frac{n+4}{8(n+2)}A.
\end{aligned}$$

Hence

$$\begin{aligned}
I_{2,n}(K,E) &= a^2A + abA\| e_d|E\|^2 + \frac{n+4}{8(n+2)}b^2A\| e_d|E\|^4\\
&= a^2A\left\{1 + \frac{b}{a}\| e_d|E\|^2 + \frac{n+4}{8(n+2)}\left(\frac{b}{a}\right)^2\| e_d|E\|^4\right\}
\end{aligned}$$

Set $C(2,0,n) = C(2,1,n) = 1$ and $C(2,2,n) = \frac{n+4}{8(n+2)}$. Consequently the lemma is true if $j = 2$.

Next, we assume that the lemma is true for $j-1$ and all positive n. We want to show that it is true for j and all positive n.

By Lemma 3.3, there exists a positive constant c_j' such that for each $E \in \mathcal{L}_j^d$

$$\begin{aligned} &I_{j,n}(K,E) \\ &= c_j' \int_{S^{d-1}\cap E} I_{j-1,n+1}(K,v^{\perp}) \left\{ \int_{S^{d-1}\cap E} \rho(K,u) |\langle u,v\rangle|^n \lambda_{j-1}(du) \right\} \lambda_{j-1}(dv), \end{aligned} \tag{4.26}$$

where $v \in E \cap S^{d-1}$ and $v^{\perp}$ is the $(j-1)$-dimensional subspace in E orthogonal to v. Define

$$G(E,v) := \int_{S^{d-1}\cap E} \rho(K,u) |\langle u,v\rangle|^n \lambda_{j-1}(du).$$

We next want to calculate $G(E,v)$. To this end, let

$$u_E = \frac{e_d|E}{\| e_d|E \|} \in S^{d-1} \cap E.$$

Then for any $u \in S^{d-1} \cap E$ we have

$$\langle e_d, u\rangle^2 = \langle u_E, u\rangle^2 \| e_d|E \|^2.$$

Note that

$$\begin{aligned} \rho(K,u) &= a + b\langle e_d, u\rangle^2 \\ &= a + \frac{b}{j}\left[(j-1)P_2^j(\langle u_E, u\rangle) + 1\right] \|e_d|E\|^2 \\ &= a + \frac{b}{j}\|e_d|E\|^2 + \frac{(j-1)b}{j} P_2^j(\langle u_E, u\rangle)\|e_d|E\|^2. \end{aligned}$$

Thus

$$\begin{aligned} G(E,v) =& \left(a + \frac{b}{j}\|e_d|E\|^2\right) \int_{S^{j-1}} |\langle u,v\rangle|^n \lambda_{j-1}(du) + \\ &+ \frac{(j-1)b}{j}\|e_d|E\|^2 \int_{S^{j-1}} |\langle u,v\rangle|^n P_2^j(\langle u_E, u\rangle)\lambda_{j-1}(du) \end{aligned} \tag{4.27}$$

The calculation is related to the beta function $B(x, y)$ defined by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \tag{4.28}$$

for $x,\ y > 0$. An easy calculation gives

$$\int_0^1 x^p(1-x^2)^q\,dx = \frac{1}{2}\frac{\Gamma((p+1)/2)\Gamma(q+1)}{\Gamma(q+1+(p+1)/2)}. \tag{4.29}$$

Note also that

$$\omega_{j-1} = (j-1)\kappa_{j-1} = \frac{\pi^{(j-1)/2}}{\Gamma((j-1)/2)}.$$

Therefore,

$$\begin{aligned}\int_{S^{j-1}} |\langle u,v\rangle|^n\,\lambda_{j-1}(du) &= \omega_{j-1}\int_{-1}^1 |x|^n(1-x^2)^{(j-3)/2}\,dx \\ &= 2\pi^{(j-1)/2}\frac{\Gamma((n+1)/2)}{\Gamma((n+j)/2)}.\end{aligned} \tag{4.30}$$

By the Funk-Hecke Theorem, there are numbers $\lambda_{j,n}$ such that

$$\int_{S^{j-1}} |\langle u,v\rangle|^n P_2^j(\langle u_E, v\rangle)\,dv = \lambda_{j,n}P_2^j(\langle u_E, u\rangle), \tag{4.31}$$

where

$$\begin{aligned}\lambda_{j,n} &= \omega_{j-1}\int_{-1}^1 |x|^n P_2^j(x)(1-x^2)^{(j-3)/2}\,dx \\ &= 2\kappa_{j-1}\int_0^1 x^n(jx^2-1)(1-x^2)^{(j-3)/2}\,dx \\ &= 2\kappa_{j-1}\Big[j\int_0^1 x^{(n+2)}(1-x^2)^{(j-3)/2}\,dx - \int_0^1 x^n(1-x^2)^{(j-3)/2}\,dx\Big] \\ &= 2\kappa_{j-1}\Big[j\frac{\Gamma((n+3)/2)\Gamma((j-1)/2)}{\Gamma((n+j+2)/2)} - \frac{\Gamma((n+1)/2)\Gamma((j-1)/2)}{\Gamma((n+j)/2)}\Big] \\ &= \frac{2\pi^{(j-1)/2}}{j-1}\frac{\Gamma((n+1)/2)}{\Gamma((n+j)/2)}\Big[j\frac{n+1}{n+j} - 1\Big] \\ &= 2\pi^{(j-1)/2}\frac{n}{n+j}\frac{\Gamma((n+1)/2)}{\Gamma((n+j)/2)}.\end{aligned}$$

Combining this with (4.27) and (4.30) gives

$$
\begin{aligned}
&G(E,v)\\
&= 2\pi^{(j-1)/2}\Big[a+\frac{b}{j}\|e_d|E\|^2\Big]\frac{\Gamma((n+1)/2)}{\Gamma((n+j)/2)}\\
&\qquad + 2\pi^{(j-1)/2}\frac{n(j-1)b}{j(n+j)}\|e_d|E\|^2\frac{\Gamma((n+1)/2)}{\Gamma((n+j)/2)}P_2^j(\langle u_E,v\rangle)\\
&= 2\pi^{(j-1)/2}\frac{\Gamma((n+1)/2)}{\Gamma((n+j)/2)}\Big[a+\frac{b}{j}\|e_d|E\|^2\\
&\qquad + \frac{nb}{j(n+j)}\big[j\langle u_E,v\rangle^2-1\big]\|e_d|E\|^2\Big]\\
&= 2\pi^{(j-1)/2}\frac{\Gamma((n+1)/2)}{\Gamma((n+j)/2)}\Big[a+\frac{b}{n+j}\|e_d|E\|^2+\frac{nb}{n+j}\langle u_E,v\rangle^2\|e_d|E\|^2\Big]\\
&= 2a\pi^{(j-1)/2}\frac{\Gamma((n+1)/2)}{\Gamma((n+j)/2)}\Big[1+\frac{1}{n+j}\frac{b}{a}\|e_d|E\|^2+\frac{n}{n+j}\frac{b}{a}\langle u_E,v\rangle^2\|e_d|E\|^2\Big].
\end{aligned}
$$

By inductive assumption, up to a positive multiplicative constant,

$$
\begin{aligned}
I_{j-1,n+1}(K,v^\perp) &= \sum_{i=0}^{j-1} C(j-1,i,n+1)\Big(\frac{b}{a}\Big)^i \| e_d|v^\perp \|^{2i}\\
&= \sum_{i=0}^{j-1} C(j-1,i,n+1)\Big(\frac{b}{a}\Big)^i \big(1-\langle u_E,v\rangle^2\big)^i \| e_d|E \|^{2i},
\end{aligned}
\tag{4.32}
$$

where we have used the fact that

$$
\| e_d|v^\perp \|^2 = \{1-\langle u_E,v\rangle^2\}\| e_d|E \|^2.
$$

Combining (4.26) and (4.32) gives, up to a positive multiplicative constant,

$$
\begin{aligned}
I_{j,n}(K,E) = &\sum_{i=0}^{j-1}\int_{S^{j-1}} C(j-1,i,n+1)\big(1-\langle u_E,v\rangle^2\big)^i\Big(\frac{b}{a}\Big)^i \| e_d|E \|^{2i}\times\\
&\times\Big[1+\frac{1}{n+j}\frac{b}{a}\| e_d|E \|^2+\frac{n}{n+j}\frac{b}{a}\langle u_E,v\rangle^2\| e_d|E \|^2\Big]\lambda_{j-1}(dv).
\end{aligned}
$$

Put

$$
\begin{aligned}
\beta_i^j &:= \int_{S^{j-1}} \big(1-\langle u_E,v\rangle^2\big)^i\lambda_{j-1}(dv)\\
&= \omega_{j-1}\int_{-1}^{1}(1-x^2)^i(1-x^2)^{(j-3)/2}dx = 2\omega_{j-1}\int_0^1 (1-x^2)^{(2i+j-3)/2}dx\\
&= \omega_{j-1}\pi^{\frac{1}{2}}\frac{\Gamma((2i+j-1)/2)}{\Gamma((2i+j)/2)}.
\end{aligned}
$$

For simplicity, put $X := \frac{b}{a}\| e_d|E \|^2$. Then

$$
\begin{aligned}
&I_{j,n}(K,E)\\
&= \sum_{i=0}^{j-1} C(j-1,i,n+1)\left\{\beta_i^j X^i + \frac{n+1}{n+j}\beta_i^j X^{i+1} - \frac{n}{n+j}\beta_{i+1}^j X^{i+1}\right\}\\
&= C(j-1,0,n+1)\beta_0^j\\
&\quad+\sum_{i=1}^{j-1}\left\{C(j-1,i,n+1)\beta_i^j + \left(\frac{n+1}{n+j}\beta_{i-1}^j - \frac{n}{n+j}\beta_i^j\right)C(j-1,i-1,n+1)\right\}X^i\\
&\quad+\left(\frac{n+1}{n+j}\beta_{j-1}^j - \frac{n}{n+j}\beta_j^j\right)C(j-1,j-1,n+1)X^j.
\end{aligned}
\tag{4.33}
$$

We first calculate the middle part of (4.33). Note that

$$
\begin{aligned}
\beta_i^j &= \omega_j \pi^{\frac{1}{2}} \frac{\Gamma((2i+j-1)/2)}{\Gamma((2i+j)/2)} = \frac{2i+j-3}{2i+j-2}\beta_{i-1}^j\\
&= \frac{2i+j-3}{2i+j-2}\frac{2i+j-5}{2i+j-4}\cdots\frac{j-1}{j}\beta_0^j = \prod_{k=0}^{i-1}\frac{j+2k-1}{j+2k}\beta_0^j.
\end{aligned}
$$

For $1 \le i \le j-1$, we have

$$
\begin{aligned}
&C(j-1,i,n+1)\beta_i^j + \left(\frac{n+1}{n+j}\beta_{i-1}^j - \frac{n}{n+j}\beta_i^j\right)C(j-1,i-1,n+1)\\
&= \binom{j-1}{i}\left(\frac{1}{n+j}\right)^i \beta_i^j \prod_{k=0}^{i-1}\frac{n+j+2k}{j+2k-1}\\
&\qquad+\binom{j-1}{i-1}\left(\frac{1}{n+j}\right)^{i-1}\left\{\frac{n+1}{n+j}\beta_{i-1}^j - \frac{n}{n+j}\beta_i^j\right\}\prod_{k=0}^{i-2}\frac{n+j+2k}{j+2k-1}\\
&= \binom{j-1}{i}\left(\frac{1}{n+j}\right)^i \beta_i^j \prod_{k=0}^{i-1}\frac{n+j+2k}{j+2k-1}\\
&\qquad+\binom{j-1}{i-1}\left(\frac{1}{n+j}\right)^{i}\left\{\frac{n+j+2i-2}{j+2i-3}\beta_i^j\right\}\prod_{k=0}^{i-2}\frac{n+j+2k}{j+2k-1}\\
&= \left(\frac{1}{n+j}\right)^i\left\{\binom{j-1}{i} + \binom{j-1}{i-1}\right\}\beta_0^j \prod_{k=0}^{i-1}\frac{n+j+2k}{j+2k}\\
&= \binom{j}{i}\left(\frac{1}{n+j}\right)^i \beta_0^j \prod_{k=0}^{i-1}\frac{n+j+2k}{j+2k}\\
&= \beta_0^j C(j,i,n).
\end{aligned}
$$

We then calculate the last part of (4.33)

$$\begin{aligned}
&\left(\frac{n+1}{n+j}\beta^j_{j-1} - \frac{n}{n+j}\beta^j_j\right)C(j-1,j-1,n+1) \\
&= \left(\frac{1}{n+j}\right)^{j-1}\left\{\frac{n+1}{n+j}\beta^j_{j-1} - \frac{n}{n+j}\beta^j_j\right\}\prod_{k=0}^{j-2}\frac{n+j+2k}{j+2k-1} \\
&= \left(\frac{1}{n+j}\right)^{j}\beta^j_0\prod_{k=0}^{j-1}\frac{n+j+2k}{j+2k} \\
&= \beta^j_0 C(j,j,n).
\end{aligned}$$

Note that

$$C(j-1,0,n+1) = 1 \quad \text{and} \quad \beta^j_0 > 0.$$

Thus for j and all positive n, we have, up to a positive multiplicative constant,

$$I_{j,n}(K,E) = \sum_{i=0}^{j} C(j,i,n)(\frac{b}{a})^i \parallel e_d|E \parallel^{2i},$$

where

$$C(j,i,n) = \binom{j}{i}\frac{1}{(n+j)^i}\prod_{k=0}^{i-1}\frac{n+j+2k}{j+2k} \qquad \text{for all} \quad i > 0$$

and $C(j,0,n) = 1$. This shows that the lemma is true for j and all positive integer n. By induction, the lemma is proved. □

Lemma 4.4. *For any j, $1 < j \leq d-1$, we have, up to a positive multiplicative constant,*

$$\overline{\rho}_j(K_\alpha, E) = \sum_{i=0}^{j}\binom{j}{i}\frac{1}{(d+1)^i}\prod_{k=0}^{i-1}\frac{d+2k+1}{j+2k}(\frac{b}{a})^i \parallel e_d|E \parallel^{2i} \quad \textit{for all } E \in \mathcal{L}^d_j, \tag{4.34}$$

where

$$a = 1 - \frac{d+1}{d-1}\alpha \quad \textit{and} \quad b = \frac{d(d+1)}{d-1}\alpha$$

Proof:

By Theorem 3.1, we have, up to a positive multiplicative constant,

$$\overline{\rho}_j(K_\alpha, E) = I_{j,d-j+1}(K, E).$$

Since the generating function of K_α is

$$\rho(K_\alpha, u) = \frac{1}{2\kappa_d}\left[a + b\langle e_d, u\rangle^2\right],$$

this lemma follows immediately from Lemma 4.3. □

In Section 1, we introduced the classes Z_j of convex bodies for $2 \le j \le d$. Z_j is the set of convex bodies K such that all the j-dimensional projections of K are zonoids. Weil [1982] conjectured that $\mathcal{K}(j) = Z_{d-j+1}$ for all $1 \le j \le d-1$. He showed that the conjecture is true if $j = 1$ or $j = d-1$. Goodey and Weil [1991] gave a negative answer for the case $j = d-2$. They gave polytopes that are in $\mathcal{K}(d-2)$ but not in $Z(3)$. Our next objective is to investigate whether the conjecture is true or not for j, $1 < j < d-2$. In contrast to the polytopes studied by Goodey and Weil, we will work with the smooth bodies of revolution K_α.

Lemma 4.5. *$K_\alpha \in Z_j$ if and only if*

$$-\frac{d-1}{dj-1} \le \alpha \le \frac{d-1}{d+1}. \tag{4.35}$$

Proof:

Let $E \in \mathcal{L}_j^d$. Note that the support function of $K_\alpha|E$ is

$$h(K_\alpha|E, u) = 1 + \alpha P_2^d(\langle e_d, u\rangle),$$

where $u \in S^{d-1} \bigcap E$. We first assume that $\| e_d|E \| > 0$. Let

$$u_E := \frac{e_d|E}{\| e_d|E \|}.$$

Then

$$\begin{aligned} h(K_\alpha|E, u) &= 1 + \alpha P_2^d(\langle e_d, u\rangle) \\ &= 1 - \frac{\alpha}{d-1} + \frac{d\alpha}{(d-1)j}\| e_d|E \|^2 + \frac{d\alpha(j-1)}{(d-1)j}\| e_d|E \|^2 P_2^j(\langle u_E, u\rangle). \end{aligned}$$

The generating function $\rho(K_\alpha|E,\cdot)$ of $K_\alpha|E$ satisfies

$$h(K_\alpha|E,u) = \int_{S^{d-1}\cap E} |\langle u,v\rangle|\rho(K_\alpha|E,v)dv.$$

Using the Funk-Hecke Theorem we get

$$\begin{aligned}
&\rho(K_\alpha|E,u) \\
&= \frac{1}{2\kappa_{j-1}}\Big\{1-\frac{\alpha}{d-1}+\frac{d\alpha}{(d-1)j}\| e_d|E \|^2+ \\
&\qquad\qquad + (j+1)\frac{d\alpha(j-1)}{(d-1)j}\| e_d|E \|^2 P_2^j(\langle u_E,u\rangle)\Big\} \\
&= \frac{1}{2\kappa_{j-1}}\Big\{1-\frac{\alpha}{d-1}-\frac{d\alpha}{d-1}\| e_d|E \|^2+\frac{d(j+1)\alpha}{(d-1)}\| e_d|E \|^2\langle u_E,u\rangle^2\Big\}.
\end{aligned}$$

Put

$$X =: \frac{1}{d-1}+\frac{d}{d-1}\| e_d|E \|^2 - \frac{d(j+1)}{(d-1)}\| e_d|E \|^2\langle u_E,u\rangle^2.$$

Then

$$\frac{-dj+1}{d-1} \leq X \leq \frac{d+1}{d-1}.$$

$K_\alpha|E$ is a zonoid for all $E\in\mathcal{L}_j^d$ with $\|e_d|E\|>0$ if and only if

$$\rho(K_\alpha|E,u) = \frac{1}{2\kappa_{j-1}}(1-\alpha X)\geq 0$$

for all X such that

$$\frac{-dj-1}{d-1} \leq X \leq \frac{d+1}{d-1}.$$

This gives

$$-\frac{d-1}{dj-1} \leq \alpha \leq \frac{d-1}{d+1}.$$

The zonoids form a closed class of convex bodies and so, by approximation, $K_\alpha\in Z_j$ if and only if the above inequality holds. □

Let

$$\alpha_0 = -\frac{d-1}{d^2-jd+d-1}$$

and consider convex bodies K_α where α nears α_0. By Lemma 4.7, $K_\alpha \in Z_{d-j+1}$ if and only if

$$-\frac{d-1}{d^2-jd+d-1} \le \alpha \le \frac{d-1}{d+1}.$$

Thus $K_{\alpha_0} \in Z_{d-j+1}$. Now

$$\frac{b}{a} = \frac{d(d+1)\alpha}{(d-1)-(d+1)\alpha} = -\frac{d+1}{d-j+2}.$$

Recall that the j-th generating function of K_α is

$$\overline{\rho}_j(K_\alpha, E) = \sum_{i=0}^{j} \binom{j}{i} \frac{1}{(d+1)^i} \prod_{k=0}^{i-1} \frac{d+2k+1}{j+2k} (\frac{b}{a})^i \parallel e_d|E \parallel^{2i}.$$

Thus

$$\overline{\rho}_j(K_{\alpha_0}, \cdot) = \sum_{i=0}^{j} \binom{j}{i} (-1)^i \frac{1}{(d-j+2)^i} \prod_{k=0}^{i-1} \frac{d+2k+1}{j+2k} X^i,$$

where $X = \|e_d| \cdot \|^2$. For $j \ge i \ge 1$, let

$$c_i = \binom{j}{i} \frac{1}{(d-j+2)^i} \prod_{k=0}^{i-1} \frac{d+2k+1}{j+2k}$$

and put $c_0 = 1$. Then

$$\overline{\rho}_j(K_{\alpha_0}, \cdot) = \sum_{i=0}^{j} (-1)^i c_i X^i.$$

Now, let us assume that for some j, $1 < j < d-1$

$$\overline{\rho}_j(K_{\alpha_0}, \cdot) = \sum_{i=0}^{j} (-1)^i c_i X^i > 0 \quad \text{for all} \quad X \in [0,1].$$

Let $\alpha < \alpha_0$ and $\alpha \to \alpha_0$. Then

$$\overline{\rho}_j(K_\alpha, X) \to \overline{\rho}_j(K_{\alpha_0}, X)$$

uniformly on [0,1] as $\alpha \to \alpha_0$. If $\overline{\rho}_j(K_{\alpha_0}, X) > 0$ on $[0,1]$, then there exists $\delta_1 > 0$ such that $\overline{\rho}_j(K_\alpha, x) \ge 0$ for all $\alpha \in [\alpha_0 - \delta_1, \alpha_0]$ and all $x \in [0,1]$. Recall that K_α is a convex body if and only if $-\frac{d-1}{2d-1} \le \alpha \le \frac{d+1}{d-1}$. Put

$$\delta_2 := \alpha_0 - \big(-\frac{d-1}{2d-1}\big) = \frac{d-1}{2d-1} - \frac{d-1}{d^2-jd+d-1} > 0$$

and let $\delta := \min\{\delta_1, \delta_2\} > 0$. Then clearly $K_\alpha \in \mathcal{K}(j) \setminus Z_{d-j+1}$ if $\alpha \in [\alpha_0 - \delta, \alpha_0)$. In other words, there are bodies in $\mathcal{K}(j) \setminus Z_{d-j+1}$ if $\overline{\rho}_j(K_{\alpha_0}, X) > 0$ for all $X \in [0,1]$. This leads to our next theorem.

Theorem 4.6.

(a) There are bodies in $\mathcal{K}(j) \setminus Z_{d-j+1}$ if $j = 2$ or $j = 3$ for all $d > j+1$;

(b) For any given $j > 1$, there are bodies in $\mathcal{K}(j) \setminus Z_{d-j+1}$ if d is big enough.

Proof:

It is easy to see that

$$\overline{\rho}_2(K_{\alpha_0}, \cdot) = 1 - \frac{d+1}{d}X + \frac{(d+1)(d+3)}{8d^2}X^2.$$

Note that $\overline{\rho}_2(K_{\alpha_0}, \cdot)$ reaches minimum value at $X = \frac{4d}{d+3} > 1$ and so $\overline{\rho}_2(K_{\alpha_0}, E)$ is decreasing for X, $0 \leq X \leq 1$. Thus

$$\begin{aligned}\overline{\rho}_2(K_{\alpha_0}, \cdot) &\geq 1 - \frac{d+1}{d} + \frac{(d+1)(d+3)}{8d^2} \\ &= \frac{(d-2)^2}{8d^2} > 0 \qquad \text{for all } E \in \mathcal{L}_2^d.\end{aligned}$$

This gives the proof for the case $j = 2$.

For $j = 3$, we let $Y = \frac{X}{d-1}$ and thus $0 \leq Y \leq \frac{1}{d-1}$. Then

$$\overline{\rho}_3(K_{\alpha_0}, Y) = 1 - (d+1)Y + \frac{1}{5}(d+1)(d+3)Y^2 - \frac{1}{105}(d+1)(d+3)(d+5)Y^3.$$

Note that

$$\overline{\rho}_3{}'(K_{\alpha_0}, Y) = (d+1)\big[-1 + \frac{2}{5}(d+3)Y - \frac{1}{35}(d+3)(d+5)Y^2\big]$$

and that $\overline{\rho}_3{}'$ reaches its maximum value at $Y_0 = \frac{7}{d+5} > \frac{1}{d-1}$. Now

$$\begin{aligned}\frac{35(d-1)^2}{d+1}\overline{\rho}_3{}'(K_{\alpha_0}, \frac{1}{d-1}) &= -35(d-1)^2 + 14(d-1)(d+3) - (d+3)(d+5) \\ &= -22d^2 + 90d - 92 < 0 \quad \text{if} \quad d > 4.\end{aligned}$$

This, together with $\overline{\rho}_3{}'(K_{\alpha_0}, 0) = -(d+1) < 0$, gives that $\overline{\rho}_3{}'(K_{\alpha_0}, Y) < 0$ for all $0 \leq Y \leq \frac{1}{d-1}$. Thus the minimum point of $\overline{\rho}_3(K_{\alpha_0}, Y)$ can only be reached at

$Y = \frac{1}{d-1}$. To show that $\overline{\rho}_3(K_{\alpha_0}, Y) > 0$ for all $0 \leq Y \leq \frac{1}{d-1}$, we need only check that

$$\overline{\rho}_3(K_{\alpha_0}, \frac{1}{d-1})$$
$$= 1 - \frac{d+1}{d-1} + \frac{1}{5}\frac{(d+1)(d+3)}{(d-1)^2} - \frac{1}{105}\frac{(d+1)(d+3)(d+5)}{(d-1)^3} > 0.$$

In fact,

$$105(d-1)^3\overline{\rho}_3(K_{\alpha_0}, \frac{1}{d-1})$$
$$= 105(d-1)^3 - 105(d+1)(d-1)^2 + 21(d+1)(d+3)(d-1) - (d+1)(d+3)(d+5)$$
$$= 20d^3 - 156d^2 + 376d - 288 > 0 \qquad \text{if} \quad d > 4.$$

This shows (a).

To prove (b), note that

$$\overline{\rho}_j(K_{\alpha_0}, \cdot) = \sum_{i=0}^{j} \binom{j}{i} (-1)^i \frac{1}{(d-j+2)^i} \prod_{k=0}^{i-1} \frac{d+2k+1}{j+2k} X^i. \tag{4.36}$$

Let

$$Q_j(X) = \sum_{i=0}^{j} \binom{j}{i} (-1)^i \prod_{k=0}^{i-1} \frac{1}{j+2k} X^i.$$

It is easy to see that $\overline{\rho}_j(K_{\alpha_0}, X)$ tends to $Q_j(X)$ uniformly on $[0,1]$ as d tends to $+\infty$. We claim that $Q_j(X) > 0$ on $[0,1]$. In fact, put $c_0 = 1$ and

$$c_i = \binom{j}{i} \prod_{k=0}^{i-1} \frac{1}{j+2k} \tag{4.37}$$

for $i > 0$. Then

$$Q_j(X) = \sum_{i=0}^{j} (-1)^i c_i X^i.$$

Note $c_1 = 1$ and for $i > 1$ we have

$$\frac{c_i}{c_{i-1}} = \frac{j-i+1}{i(j+2i-2)} < 1.$$

This shows that $Q_j(X) > 0$ on $[0,1]$. Let $m_j = \min\{Q_j(X) : X \in [0,1]\}$, then $m_j > 0$. Since $\overline{\rho}_j(K_{\alpha_0}, X) \to Q_j(X)$ uniformly on $[0,1]$, there exists a positive integer N such that for all $d \geq N$

$$|\overline{\rho}_j(K_{\alpha_0}, X) - Q_j(X)| < \frac{m_j}{2} \qquad \text{for all} \qquad 0 \leq X \leq 1.$$

Thus for all $d \geq N$

$$\overline{\rho}_j(K_{\alpha_0}, X) > Q_j(X) - \frac{m_j}{2} \geq \frac{m_j}{2} > 0 \qquad \text{for all} \quad 0 \leq X \leq 1.$$

This completes our proof of this theorem. $\square$

For each $j > 1$ and sufficiently large d, we have found some convex bodies K_α such that $K_\alpha \in \mathcal{K}(j) \setminus Z_{d-j+1}$. The polytopes given by Goodey and Weil are in $\mathcal{K}(j-2) \setminus Z_3$. It is tempting to conjecture that

$$Z_{d-j+1} \subseteq \mathcal{K}(j) \quad \text{for all} \quad 1 \leq j \leq d-1 \tag{4.38}$$

with equality holding if and only if $j = 1$ or $j = d-1$.

We now turn our attention to the convex bodies $L_{\beta,j}$ and $W_{\gamma,j}$. We first show (4.9). That is

$$\big(R^d_{j,d-1}(\|e_d| \cdot \|^{2n})\big)(u^\perp) = \sigma_{j,n} \|e_d|u^\perp\|^{2n} \qquad \text{for each} \;\; u \in S^{d-1}, \tag{4.39}$$

where

$$\sigma_{j,n} = \frac{\omega_j \omega_{d-2} \mu_{d-1,n}}{\omega_{j-1} \omega_{d-1} \mu_{j,n}} \quad \text{and} \quad \mu_{j,n} = \frac{\Gamma((j-1)/2)\Gamma((2n+1)/2)}{\Gamma((2n+j)/2)}.$$

In fact, for almost all $E \in \mathcal{L}^d_j$, denote

$$u_E =: \frac{e_d|E}{\|e_d|E\|} \in S^{d-1} \cap E.$$

Then

$$
\begin{aligned}
\left(R_{1,j}^d(\|e_d|\cdot\|^{2n})\right)(E) &= \int_{\mathcal{L}_1^d(E)} \|e_d|F\|^{2n}\nu_1^E(dF) \\
&= (\omega_j)^{-1}\int_{S^{d-1}\cap E} \|e_d|u\|^{2n}\lambda_{j-1}(du) \\
&= (\omega_j)^{-1}\int_{S^{j-1}} \|e_d|E\|^{2n}|\langle u_E, u\rangle|^{2n}\lambda_{j-1}(du) \\
&= (\omega_j)^{-1}\|e_d|E\|^{2n}\int_{S^{j-1}} |\langle u_E, u\rangle|^{2n}\lambda_{j-1}(du) \\
&= \frac{\omega_{j-1}}{\omega_j}\|e_d|E\|^{2n}\int_{-1}^{1} x^{2n}(1-x^2)^{\frac{j-3}{2}}dx \\
&= \frac{\omega_{j-1}\mu_{j,n}}{\omega_j}\|e_d|E\|^{2n}.
\end{aligned}
$$

Here we used an integral relation

$$\int_{S^{d-1}} \Phi(\langle u,v\rangle)\lambda_{d-1}(du) = \omega_{d-1}\int_{-1}^{1}\Phi(x)(1-x^2)^{\frac{d-3}{2}}dx,$$

where $v \in S^{d-1}$ is fixed and Φ is a continuous function on $[-1,1]$ (see Groemer [1996, p.9]). The claim (4.39) now follows from the fact that $R_{j,d-1}R_{1,j} = R_{1,d-1}$. Since $\nu_1^E(\mathcal{L}_1^d(E)) = 1$, a direct calculation of (4.39) gives $\sigma_{j,0} = 1$. Therefore,

$$\sigma_{j,0} = 1 \quad \text{and} \quad \sigma_{j,n} = \prod_{k=0}^{n-1}\frac{j+2k}{d+2k-1} \text{ if } n \geq 1. \tag{4.40}$$

Lemma 4.7. *For any fixed j, $1 \leq j \leq d-1$, we have*

(a) $L_{\beta,j}$ defines a convex body if and only if

$$-1 \leq \beta \leq \frac{j(d+1)}{2d-j}. \tag{4.41}$$

(b)

$$\rho_j(L_{\beta,j},E) = \binom{d}{j}\frac{1}{d\kappa_{d-j}}\left(1+\beta-\frac{d\beta}{j}\| e_d|E \|^2\right), \quad E \in \mathcal{L}_j^d. \tag{4.42}$$

is a j-th projection generating function of $L_{\beta,j}$ and

$$L_{\beta,j} \in \mathcal{K}(j) \quad \textit{if and only if} \quad -1 \leq \beta \leq \frac{j}{d-j}. \tag{4.43}$$

(c) The j-th projection function of $L_{\beta,j}$ is

$$V_j(L_{\beta,j}|E) = \kappa_j \left(1 + \frac{\beta}{d+1} - \frac{d\beta}{(d+1)j} \|e_d|E\|^2\right), \quad E \in \mathcal{L}_j^d. \tag{4.44}$$

(d) $L_{\beta,j} \in \mathcal{O}_j$ if and only if $L_{\beta,j} \in \mathcal{K}(j)$

Proof:

For any $u \in S^{d-1}$, define an angle $\vartheta \in [-\pi/2, \pi/2]$ such that $\langle e_d, u\rangle = \sin\vartheta$. Then the j-th area measure $S_j(L_{\beta,j}, u)$ can be written as

$$\phi(\vartheta) = 1 + \beta P_2^d(\sin\vartheta) = 1 + \frac{\beta}{d-1}(d\sin^2\vartheta - 1) \quad \text{where} \quad -\frac{\pi}{2} \le \vartheta \le \frac{\pi}{2}.$$

Let

$$f(\vartheta) := \int_\vartheta^{\frac{\pi}{2}} \phi(x)\cos^{d-2} x \sin x dx \text{ where } \quad -\frac{\pi}{2} \le \vartheta \le \frac{\pi}{2}.$$

By Firey's result, $L_{\beta,j}$ is a convex body if and only if

(1) $\phi(\vartheta)$ is continuous and has finite limits as ϑ tends to $\pm\frac{\pi}{2}$;

(2) $f(\vartheta) > 0$ for $-\frac{\pi}{2} < \vartheta < \frac{\pi}{2}$ and is zero for $\vartheta = -\frac{\pi}{2}$;

(3) For any $\vartheta \in (-\pi/2, \pi/2)$,

$$\phi(\vartheta) > \frac{(d-j-1)f(\vartheta)}{\cos^{d-1}\vartheta}.$$

It is clear that (1) is true for all β. Note that

$$\begin{aligned} f(\vartheta) &= \int_\vartheta^{\frac{\pi}{2}} \phi(x)\cos^{d-2} x \sin x dx \\ &= \int_\vartheta^{\frac{\pi}{2}} \left\{1 + \frac{\beta}{d-1}(d\sin^2\vartheta - 1)\right\}\cos^{d-2} x \sin x dx \\ &= \frac{\cos^{d-1}\vartheta}{d-1}[1 + \beta - \frac{d\beta}{d+1}\cos^2\vartheta]. \end{aligned}$$

Thus $f(\vartheta) = 0$ if $\vartheta = -\frac{\pi}{2}$. For $-\frac{\pi}{2} < \vartheta < \frac{\pi}{2}$, $f(\vartheta) > 0$ if and only if

$$1 + \beta - \frac{d\beta}{d+1}\cos^2\vartheta > 0$$

or

$$1+\beta(1-\frac{d}{d+1}\cos^2\vartheta)>0.$$

Since $\frac{1}{d+1}\leq(1-\frac{d}{d+1}\cos^2\vartheta)<1$, this shows that (2) holds if and only if $\beta>-1$.

Now (3) can be written as

$$1+\beta-\frac{d\beta}{d-1}\cos^2\vartheta>\frac{d-j-1}{d-1}\left\{1+\beta-\frac{d\beta}{d+1}\cos^2\vartheta\right\}.$$

That is

$$(1+\beta)j>\frac{d(j+2)\beta}{d+1}\cos^2\vartheta.$$

If $\beta>0$, this is equivalent to $\beta\leq\frac{j(d+1)}{2d-j}$; if $\beta\leq 0$, it is equivalent to $\beta>-1$. Thus (3) is true if and only if

$$-1<\beta<\frac{j(d+1)}{2d-j}.$$

Combining (1), (2) and (3), we have $L_{\beta,j}$ defines a convex body if and only if

$$-1\leq\beta\leq\frac{j(d+1)}{2d-j}.$$

We next calculate $V_j(L_{\beta,j}|\cdot)$ and $\rho_j(L_{\beta,j},\cdot)$. First note that the Funk-Hecke Theorem gives

$$\begin{aligned}T_1^d(S_j(L_{\beta,j},\cdot)) &= \int_{S^{d-1}}|\langle\cdot,v\rangle|S_j(L_{\beta,j},dv)\\ &= \int_{S^{d-1}}|\langle\cdot,v\rangle|\left[1+\beta P_2^d(\langle e_d,v\rangle)\right]\lambda_{d-1}(dv)\\ &= 2\kappa_{d-1}\left[1+\frac{\beta}{d+1}P_2^d(\langle e_d,\cdot\rangle)\right]\\ &= 2\kappa_{d-1}\left[1-\frac{\beta}{(d+1)(d-1)}+\frac{d\beta}{(d+1)(d-1)}\langle e_d,\cdot\rangle^2\right].\end{aligned}$$

Therefore,

$$\begin{aligned}T_{d-1}^d(S_j^\perp(L_{\beta,j},\cdot)) &= \left(T_1^d(S_j(L_{\beta,j},\cdot))\right)^\perp\\ &= 2\kappa_{d-1}\left[1+\frac{\beta}{(d+1)}-\frac{d\beta}{(d+1)(d-1)}\|e_d|\cdot\|^2\right].\end{aligned}$$

Combining this with (4.10) gives, for each $E \in \mathcal{L}_j^d$,

$$V_j(L_{\beta,j}|E) = \kappa_j\big(R^d_{j,d-1}\big)^{-1}\Big(1 + \frac{\beta}{(d+1)} - \frac{d\beta}{(d+1)(d-1)}\|e_d|\cdot\|^2\Big)(E)$$
$$= \kappa_j\Big(1 + \frac{\beta}{(d+1)} - \frac{d\beta}{(d+1)j}\|e_d|E\|^2\Big).$$

This gives (c).

By (4.31),

$$\rho_j(L_{\beta,j}, E) = \binom{d}{j}\frac{1}{d\kappa_{d-j}}\Big(\big(R^d_{j,d-1}\big)^{-1}\big(S_j^{\perp}(L_{\beta,j},\cdot)\big)(E)$$
$$= \binom{d}{j}\frac{1}{d\kappa_{d-j}}\Big(\big(R^d_{j,d-1}\big)^{-1}\big(1 + \beta P_2^d(\langle e^d,\cdot\rangle)\big)^{\perp}\Big)(E)$$

From the above calculations, we have

$$\rho_j(L_{\beta,j}, E) = \binom{d}{j}\frac{1}{d\kappa_{d-j}}\Big(1 + \beta - \frac{d\beta}{j}\|e_d|E\|^2\Big).$$

Note that $L_{\beta,j} \in \mathcal{K}(j)$ if and only if

$$\rho_j(L_{\beta,j}, E) \geq 0 \quad \text{for all} \quad E \in \mathcal{L}_j^d$$

or

$$1 + \beta - \frac{d\beta}{j}\|e_d|E\|^2 \geq 0 \quad \text{for all} \quad E \in \mathcal{L}_j^d.$$

This, together with the convexity condition, gives

$$-1 \leq \beta \leq \frac{j}{d-j}.$$

It remains to show (d). To this end, put

$$\beta' = -\frac{d-j}{j}\beta.$$

It is easy to check from (a) that $L_{\beta,j}$ and $L_{\beta',d-j}$ are convex bodies if and only if $L_{\beta,j} \in \mathcal{K}(j)$.

By (c), we have, for each $E \in \mathcal{L}_j^d$,

$$\begin{aligned}
V_{d-j}(L_{\beta',d-j}|E^\perp) &= \kappa_{d-j}\left(1+\frac{\beta'}{d+1}-\frac{d\beta'}{(d+1)(d-j)}\|e_d|E^\perp\|^2\right)\\
&= \kappa_{d-j}\left(1+\frac{\beta'}{d+1}-\frac{d\beta'}{(d+1)(d-j)}[1-\|e_d|E\|^2]\right)\\
&= \kappa_{d-j}\left(1-\frac{j\beta'}{(d+1)(d-j)}+\frac{d\beta'}{(d+1)(d-j)}\|e_d|E\|^2\right)\\
&= \kappa_{d-j}\left(1+\frac{\beta}{d+1}-\frac{d\beta}{(d+1)j}\|e_d|E\|^2\right)\\
&= \frac{\kappa_{d-j}}{\kappa_j}V_j(L_{\beta,j}|E).
\end{aligned}$$

Define

$$L'_{\beta,j} := \left(\frac{\kappa_j}{\kappa_{d-j}}\right)^{1/(d-j)} L_{\beta',d-j}.$$

It follows from the above calculation that

$$V_j(L_{\beta,j}|E) = V_{d-j}(L'_{\beta,j}|E^\perp) \qquad \text{for all} \qquad E \in \mathcal{L}_j^d.$$

Therefore $L_{\beta,j} \in \mathcal{O}_j$ if and only if $L_{\beta,j} \in \mathcal{K}(j)$. $\square$

For $j \in \{1,2,\ldots,d-1\}$, define

$$k(d,j) = \max\left\{-1, -\frac{j(j+2)}{(d-j+2)(d-j)}\right\}. \tag{4.45}$$

and

$$\begin{aligned}
K(d,j) &= \frac{(d+1)(d+4)(j+4)j}{(j+2)^2(d+2)(d+3)-(d+1)(d+4)(j+4)j}\\
&= \frac{(d+1)(d+4)(j+4)j}{2(2d^2+10d+j^2+4j+12)}.
\end{aligned} \tag{4.46}$$

Define

$$r(d,j) = \frac{(d+4)j}{2(d-j+2)} \quad \text{and} \quad R(d,j) = \min\{K(d,j), \frac{j(j+2)K(d,d-j)}{(d-j)(d-j+2)}\}. \tag{4.47}$$

We claim that

$$r(d,j) = R(d,j) \quad \text{if} \quad j = 1 \text{ or } d-1 \quad \text{and} \quad r(d,j) < R(d,j) \quad \text{otherwise}. \tag{4.48}$$

In fact, if $j = 1$ then

$$r(d,j) = \frac{j(j+2)K(d,d-j)}{(d-j)(d-j+2)} = \frac{d+4}{2(d+1)}$$

and

$$K(d,j) = \frac{5(d+4)(d+1)}{2(2d^2+10d+17)} > \frac{d+4}{2(d+1)}.$$

Thus $r(d,1) = R(d,1)$. If $j = d-1$, then

$$r(d,j) = K(d,j) = \frac{(d+4)(d-1)}{6}$$

and, if $d > 2$,

$$\begin{aligned}\frac{j(j+2)K(d,d-j)}{(d-j)(d-j+2)} &= \frac{(d+1)(d-1)}{3}\frac{5(d+4)(d+1)}{2(2d^2+10d+17)} \\ &= \frac{(d+4)(d-1)}{6}\frac{5(d+1)^2}{2d^2+10d+17} \\ &> \frac{(d+4)(d-1)}{6}.\end{aligned}$$

Thus $r(d,d-1) = R(d,d-1)$. Let $1 < j < d-1$. We first show $r(d,j) < K(d,j)$. This is equivalent to

$$\frac{(d+4)j}{2(d-j+2)} < \frac{(d+1)(d+4)(j+4)j}{(j+2)^2(d+2)(d+3)-(d+1)(d+4)(j+4)j}$$

or

$$\begin{aligned}&2(d-j+2)(d+1)(j+4) - (j+2)^2(d+2)(d+3) + (d+1)(d+4)(j+4)j \\ &= 2(d+2)(j+2)(d-j-1) > 0.\end{aligned}$$

This inequality is clearly true and thus $r(d,j) < K(d,j)$. Next we show

$$r(d,j) < \frac{j(j+2)K(d,d-j)}{(d-j)(d-j+2)}.$$

Similarly, this is equivalent to

$$\begin{aligned}&2(d+1)(j+2)(d-j+4) - (d-j+2)^2(d+2)(d+3) + \\ &\qquad (d+1)(d+4)(d-j+4)(d-j) \\ &= 2(d-j+2)(d+2)(j-1) > 0.\end{aligned}$$

This inequality is, again, clearly true. Therefore, our claim (4.48) is proved.

Lemma 4.8. *For any $j \in \{1, 2, \ldots, d-1\}$, we have*

(a)$W_{\gamma,j}$ is a convex body if and only if

$$-1 \le \gamma \le K(d,j) \tag{4.49}$$

(b)The j-th projection generating function $\rho_j(W_{\gamma,j}, \cdot)$ is given by

$$\begin{aligned}&\rho_j(W_{\gamma,j}, E)\\ &= \binom{d}{j} \frac{1}{d\kappa_{d-j}} \left(1 + \gamma - \frac{2(d+2)\gamma}{j} \|e_d|E\|^2 + \frac{(d+4)(d+2)\gamma}{j(j+2)} \|e_d|E\|^4\right)\end{aligned} \tag{4.50}$$

and $W_{\gamma,j} \in \mathcal{K}(j)$ if and only if

$$k(d,j) \le \gamma \le r(d,j) \tag{4.51}$$

(c)The j-th projection function of $W_{\gamma,j}$ is

$$\begin{aligned}V_j(W_{\gamma,j}|E) = \kappa_j \Big\{ 1 - \frac{\gamma}{(d+1)(d+3)} + \frac{2(d+2)\gamma}{(d+1)(d+3)j} \| e_d|E \|^2 \\ - \frac{(d+4)(d+2)\gamma}{(d+1)(d+3)j(j+2)} \| e_d|E \|^4 \Big\}\end{aligned} \tag{4.52}$$

(d) $W_{\gamma,j} \in \mathcal{O}_j$ if and only if $k(d,j) \le \gamma \le R(d,j)$.

Proof:

First, recall

$$P_4^d(x) = \frac{1}{d^2 - 1}\left[(d+4)(d+2)x^4 - (6d+12)x^2 + 3\right].$$

and observe a fact

$$P_4^d(\sqrt{1-x^2}) = 1 - \frac{2(d+2)}{d-1}x^2 + \frac{(d+4)(d+2)}{d^2-1}x^4. \tag{4.53}$$

For any $u \in S^{d-1}$, define an angle $\vartheta \in [-\pi/2, \pi/2]$ such that $\langle e_d, u\rangle = \sin\vartheta$. Then, by (4.53), $S_j(W_{\gamma,j}, u)$ can be written as

$$\begin{aligned}\phi(\vartheta) &= 1 + \gamma P_4^d(\sin\vartheta)\\ &= 1 + \gamma - \frac{2(d+2)\gamma}{d-1}\cos^2\vartheta + \frac{(d+4)(d+2)\gamma}{d^2-1}\cos^4\vartheta.\end{aligned}$$

Let

$$f(\vartheta) := \int_{\vartheta}^{\frac{\pi}{2}} \phi(x)\cos^{d-2} x \sin x dx \quad \text{where} \quad -\frac{\pi}{2} \leq \vartheta \leq \frac{\pi}{2}.$$

Similar to the proof of the convexity of $L_{\beta,j}$, we will use Firey's result. It is clear that $\phi(\vartheta)$ is continuous and has finite limits as ϑ tends to $\pm\frac{\pi}{2}$. Also $f(-\frac{\pi}{2}) = 0$. Therefore, $W_{\gamma,j}$ is a convex body if and only if

$$f(\vartheta) > 0 \quad \text{and} \quad \phi(\vartheta) > \frac{(d-j-1)f(\vartheta)}{\cos^{d-1}\vartheta} \quad \text{for all} \quad \vartheta \in (-\frac{\pi}{2}, \frac{\pi}{2}). \tag{4.54}$$

Note that

$$\begin{aligned} f(\vartheta) &= \int_{\vartheta}^{\frac{\pi}{2}} \phi(x)\cos^{d-2} x \sin x dx \\ &= \int_{\vartheta}^{\frac{\pi}{2}} \left\{1+\gamma - \frac{2(d+2)\gamma}{d-1}\cos^2 x + \frac{(d+4)(d+2)\gamma}{d^2-1}\cos^4 x\right\} \times \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times \cos^{d-2} x \sin x dx \\ &= \frac{\cos^{d-1}\vartheta}{d-1}\big[1+\gamma - \frac{2(d+2)\gamma}{d+1}\cos^2\vartheta + \frac{(d+4)(d+2)\gamma}{(d+1)(d+3)}\cos^4\vartheta\big]. \end{aligned}$$

For $-\frac{\pi}{2} < \vartheta < \frac{\pi}{2}$, $f(\vartheta) > 0$ if and only if

$$1+\gamma - \frac{2(d+2)\gamma}{d+1}\cos^2\vartheta + \frac{(d+4)(d+2)\gamma}{(d+1)(d+3)}\cos^4\vartheta > 0. \tag{4.55}$$

The second inequality of (4.54) can be written as

$$\begin{aligned} 1+\gamma - &\frac{2(d+2)\gamma}{d-1}\cos^2\vartheta + \frac{(d+4)(d+2)\gamma}{d^2-1}\cos^4\vartheta > \\ &\frac{d-j-1}{d-1}\left\{1+\gamma - \frac{2(d+2)\gamma}{d+1}\cos^2\vartheta + \frac{(d+4)(d+2)\gamma}{(d+1)(d+3)}\cos^4\vartheta\right\}, \end{aligned}$$

or, equivalently,

$$(1+\gamma)j - \frac{2(d+2)(j+2)}{d+1}\gamma\cos^2\vartheta + \frac{(d+4)(d+2)(j+4)}{(d+1)(d+3)}\gamma\cos^4\vartheta > 0. \tag{4.56}$$

Thus $W_{\gamma,j}$ is a convex body if and only if both (4.55) and (4.56) are true for all $\vartheta \in (-\pi/2, \pi/2)$. Consider (4.55) first, put

$$F(x) := 1+\gamma - \frac{2(d+2)\gamma}{d+1}x + \frac{(d+4)(d+2)\gamma}{(d+1)(d+3)}x^2.$$

Since $F(x)$ reaches its extreme value at $x_f = \frac{d+3}{d+4}$, (4.55) is equivalent to

$$F(0) > 0, \qquad F(x_0) > 0 \qquad \text{and } F(1) > 0.$$

Note

$$F(0) > 0 \quad \text{if and only if} \quad \gamma > -1,$$

$$F(1) > 0 \quad \text{if and only if} \quad \gamma < (d+1)(d+3)$$

and

$$F(x_f) > 0 \quad \text{if and only if} \quad \gamma < \frac{(d+1)(d+4)}{2}.$$

Therefore, (4.55) is equivalent to

$$-1 < \gamma < \frac{(d+1)(d+4)}{2}. \tag{4.57}$$

Secondly, consider (4.56). Put

$$G(x) := (1+\gamma)j - \frac{2(d+2)(j+2)}{d+1}\gamma x + \frac{(d+4)(d+2)(j+4)}{(d+1)(d+3)}\gamma x^2.$$

Since $G(x)$ reaches its extreme value at $x_g = \frac{(d+3)(j+2)}{(d+4)(j+4)}$, (4.56) is equivalent to

$$G(0) > 0, \qquad G(x_g) > 0 \qquad \text{and } G(1) > 0.$$

It is easy to check that $G(0) > 0$ if and only if $\gamma > -1$ and that $G(1) > 0$ if and only if

$$\gamma > -\frac{(d+1)(d+3)}{4d-j-8}.$$

Also $G(x_g) > 0$ if and only if

$$\gamma < \frac{(d+1)(d+4)(j+4)j}{(j+2)^2(d+2)(d+3) - (d+1)(d+4)(j+4)j} = K(d,j).$$

Combining these with (4.57) gives(a).

We next calculate $V_j(W_{\gamma,j}|\cdot)$ and $\rho_j(W_{\gamma,j},\cdot)$. First note that the Funk-Hecke Theorem gives

$$\begin{aligned}
T_1^d(S_j(W_{\gamma,j},\cdot))(u) &= \int_{S^{d-1}} |\langle u,v\rangle| S_j(W_{\gamma,j},dv) \\
&= 2\kappa_{d-1}\big[1-\frac{\gamma}{(d+1)(d+3)}P_4^d(\langle e_d,u\rangle)\big]
\end{aligned}$$

Therefore,

$$\begin{aligned}
&(2\kappa_{d-1})^{-1}\big(T_1^d(S_j(W_{\gamma,j},\cdot))\big)^{\perp}(u^{\perp}) \\
&= 1-\frac{\gamma}{(d+1)(d+3)}+\frac{2(d+2)}{(d^2-1)(d+3)}\gamma\|\, e_d|u^{\perp}\,\|^2 \\
&\qquad -\frac{(d+4)(d+2)\gamma}{(d^2-1)(d+1)(d+3)}\|\, e_d|u^{\perp}\,\|^4
\end{aligned}$$

It follows from (4.10) that, for $E\in\mathcal{L}_j^d$,

$$\begin{aligned}
V_j(W_{\gamma,j}|E) &= \kappa_j\big(R_{j,d-1}^d\big)^{-1}\big((T_1^d(S_j(W_{\gamma,j},\cdot))^{\perp}\big)(E) \\
&= \kappa_j\Big\{1-\frac{\gamma}{(d+1)(d+3)}+\frac{2(d+2)\gamma}{(d+1)(d+3)j}\|\, e_d|E\,\|^2 \\
&\qquad -\frac{(d+4)(d+2)\gamma}{(d+1)(d+3)j(j+2)}\|\, e_d|E\,\|^4\Big\}.
\end{aligned}$$

This gives (c). By (4.11),

$$\begin{aligned}
\rho_j(W_{\gamma,j},E) &= \binom{d}{j}\frac{1}{d\kappa_{d-j}}\Big((R_{j,d-1}^d)^{-1}(S_j^{\perp}(W_{\gamma,j},\cdot)\Big)(E) \\
&= \binom{d}{j}\frac{1}{d\kappa_{d-j}}\Big((R_{j,d-1}^d)^{-1}(1+\gamma P_4^d(\langle e^d,\cdot\rangle))^{\perp}\Big)(E)
\end{aligned}$$

From the above calculations, we have

$$\rho_j(W_{\gamma,j},E)=\binom{d}{j}\frac{1}{d\kappa_{d-j}}\Big(1+\gamma-\frac{2(d+2)\gamma}{j}\|e_d|E\|^2+\frac{(d+4)(d+2)\gamma}{j(j+2)}\|e_d|E\|^4\Big).$$

Note that $W_{\gamma,j}\in\mathcal{K}(j)$ if and only if

$$\rho_j(W_{\gamma,j},E)\geq 0 \quad \text{for all} \quad E\in\mathcal{L}_j^d$$

or

$$1+\gamma-\frac{2(d+2)\gamma}{j}\|e_d|E\|^2+\frac{(d+4)(d+2)\gamma}{j(j+2)}\|e_d|E\|^4\geq 0 \quad \text{for all} \quad E\in\mathcal{L}^d_j.$$

Put

$$T(x):=1+\gamma-\frac{2(d+2)\gamma}{j}x+\frac{(d+4)(d+2)\gamma}{j(j+2)}x^2.$$

Note that $T(0)\geq 0$ if and only $\gamma\geq -1$ and that $T(1)\geq 0$ if and only if

$$\gamma\geq -\frac{j(j+2)}{(d-j+2)(d-j)}.$$

Note also that $T(x)$ reaches its extreme point at $x_t=\frac{j+2}{d+4}$ and that $T(x_t)\geq 0$ if and only if

$$\gamma\leq\frac{(d+4)j}{2(d-j+2)}=r(d,j)$$

These, together with convex conditions, give that $W_{\gamma,j}\in\mathcal{K}(j)$ if and only if

$$k(d,j)\leq\gamma\leq r(d,j).$$

It remains to show (d). Recall

$$\begin{aligned}V_j(W_{\gamma,j}|E)=\kappa_j\bigg\{1-\frac{\gamma}{(d+1)(d+3)}&+\frac{2(d+2)\gamma}{(d+1)(d+3)j}\|\, e_d|E\,\|^2\\&-\frac{(d+4)(d+2)\gamma}{(d+1)(d+3)j(j+2)}\|\, e_d|E\,\|^4\bigg\}.\end{aligned}$$

Similarly,

$$\begin{aligned}&V_{d-j}(W_{\gamma',d-j}|E^\perp)\\&=\kappa_{d-j}\bigg\{1-\frac{\gamma'}{(d+1)(d+3)}+\frac{2(d+2)\gamma'}{(d+1)(d+3)(d-j)}\|\, e_d|E^\perp\,\|^2\\&\qquad-\frac{(d+4)(d+2)\gamma'}{(d+1)(d+3)(d-j)(d-j+2)}\|\, e_d|E^\perp\,\|^4\bigg\}.\end{aligned}$$

Note that

$$\|\, e_d|E^\perp\,\|^2=1-\|\, e_d|E\,\|^2.$$

Thus we have

$$\begin{aligned}
&V_{d-j}(W_{\gamma',d-j}|E^{\perp}) \\
&= \kappa_{d-j}\Big\{1 - \frac{\gamma'}{(d+1)(d+3)} + \frac{2(d+2)\gamma'}{(d+1)(d+3)(d-j)}(1 - \| e_d|E \|^2) \\
&\qquad - \frac{(d+4)(d+2)\gamma'}{(d+1)(d+3)(d-j)(d-j+2)}(1 - \| e_d|E \|^2)^2\Big\} \\
&= \kappa_{d-j}\Big\{1 - \frac{j(j+2)\gamma'}{(d+1)(d+3)(d-j)(d-j+2)} \\
&\qquad + \frac{2(d+2)(j+2)\gamma'}{(d+1)(d+3)(d-j)(d-j+2)}\| e_d|E \|^2 \\
&\qquad - \frac{(d+4)(d+2)\gamma'}{(d+1)(d+3)(d-j)(d-j+2)}\| e_d|E \|^4\Big\}
\end{aligned}$$

Put

$$\gamma' = \frac{(d-j)(d-j+2)}{j(j+2)}\gamma. \tag{4.58}$$

It is easy to check that

$$V_j(W_{\gamma,j}|E) = \frac{\kappa_j}{\kappa_{d-j}}V_{d-j}(W_{\gamma',d-j}|E^{\perp}) \quad \text{for each } E \in \mathcal{L}_j^d.$$

Therefore, $W_{\gamma,j} \in \mathcal{O}_j$ if both $W_{\gamma,j}$ and $W_{\gamma',d-j}$ are convex bodies. That is

$$-1 \leq \gamma \leq K(d,j) \quad \text{and} \quad -1 \leq \frac{(d-j)(d-j+2)}{j(j+2)}\gamma \leq K(d,d-j),$$

or, equivalently,

$$k(d,j) \leq \gamma \leq R(d,j).$$

This completes the proof of the Lemma. □

Theorem 4.9. *For $j \in \{2,3,\ldots,d-2\}$, $\mathcal{K}(j) \neq \mathcal{O}_j$.*

Proof:

Recall that if $j \in \{2,3,\ldots,d-2\}$ then $r(d,j) < R(d,j)$. So Theorem 4.9 follows immediately from (d) of Lemma 4.8. □

Theorem 4.10. *For any j, $1 < j < d-1$, the inclusions $\mathcal{K}(1) \subset \mathcal{K}(j) \subset \mathcal{K}(d-1)$ are strict.*

Proof:

Consider convex bodies K_α. Recall that $X = (\frac{b}{a})\| e_d|E \|^2$ and that, by Lemma 4.4,

$$\overline{\rho}_j(K_\alpha, E) = \sum_{i=0}^{j} C_i X^i \tag{4.59}$$

where $C_0 = C_1 = 1$ and if $1 \le i \le j \le d$

$$C_i = \binom{j}{i} \frac{1}{(d+1)^i} \prod_{k=0}^{i-1} \frac{d+2k+1}{j+2k}.$$

We first show that $C_i > C_{i+1}$ if $1 \le i \le j$. In fact, if $1 \le i \le j$ then

$$\begin{aligned} \frac{C_{i+1}}{C_i} &= \frac{(j-i)(d+2i+1)}{(i+1)(d+1)(j+2i)} = \left(\frac{j-i}{j+2i}\right)\left(\frac{d+2i+1}{(i+1)(d+1)}\right) \\ &< \frac{d+2i+1}{(i+1)(d+1)} < 1. \end{aligned}$$

This shows that $C_i > C_{i+1}$ if $1 \le i \le j$. Next we claim that there exists $\epsilon_j^d > 0$ such that

$$\sum_{i=0}^{j} C_i X^i \ge 0 \qquad \text{for all} \qquad X \in (-1-\epsilon_j^d, +\infty).$$

If the above statement were false, then there exists a sequence $\{X_n\}$ such that $X_n \to -1$ ($X_n < -1$) and $\sum_{i=0}^{j} C_i {X_n}^i < 0$. Thus $\sum_{i=0}^{j} (-1)^i C_i \le 0$ by the continuity of the polynomial. Since $C_0 = C_1 = 1$ and $C_i > C_{i+1}$ if $i \ge 1$, $\sum_{i=0}^{j} (-1)^i C_i > 0$. This is a contradiction if $j > 1$. Therefore, our claim is proved.

Since $X := (\frac{b}{a})\| e_d|E \|^2$,

$$\sum_{i=0}^{j} C_i X^i \ge 0 \quad \text{if} \quad \frac{b}{a} \in (-1-\epsilon_j^d, \infty).$$

By the definition of $\mathcal{K}(j)$, we have

$$K_\alpha \in \mathcal{K}(j) \quad \text{if} \quad \frac{b}{a} \in (-1-\epsilon_j^d, \infty).$$

Let

$$F(\alpha) := \frac{b}{a} = \frac{(d+1)d\alpha}{(d-1)-(d+1)\alpha}. \tag{4.60}$$

$F(\alpha)$ is clearly a strictly increasing function. By Lemma 4.1, K_α defines a convex body if and only if

$$-\frac{(d-1)}{2d-1} \leq \alpha \leq \frac{d-1}{d+1},$$

or, equivalently,

$$\frac{b}{a} \geq -\frac{d+1}{3}$$

By Lemma 4.3, $K_\alpha \in \mathcal{K}(1)$ if and only if

$$-\frac{1}{d+1} \leq \alpha \leq \frac{d-1}{d+1},$$

or, equivalently,

$$F(\alpha) = \frac{b}{a} \geq -1.$$

Choose $\epsilon_j^d > 0$ such that $\epsilon_j^d < \frac{d-2}{3}$. If $\alpha \in F^{-1}(-1-\epsilon_j^d, -1)$, then $K_\alpha \in \mathcal{K}(j)\backslash\mathcal{K}(1)$. This shows the inclusion $\mathcal{K}(1) \subset \mathcal{K}(j)$ is strict.

Now consider the convex body $L_{\beta,j}$. By Lemma 4.7, $L_{\beta,j}$ is convex if and only if

$$-1 \leq \beta \leq \frac{j(d+1)}{2d-j},$$

and this is equivalent to saying $L_{\beta,j} \in \mathcal{K}(d-1)$. Again by Lemma 4.7, $L_{\beta,j}$ is in $\mathcal{K}(j)$ if and only if

$$-1 \leq \beta \leq \frac{j}{d-j}.$$

Therefore, $L_{\beta,j} \in \mathcal{K}(d-1) \setminus \mathcal{K}(j)$ if

$$\frac{j}{d-j} < \beta \leq \frac{j(d+1)}{2d-j}.$$

This shows the inclusion $\mathcal{K}(j) \subset \mathcal{K}(d-1)$ is also strict. $\square$

§5. Mean Projection Bodies

In this section, we will turn our attention to *Mean Projection Bodies.* Let $K \subset \mathbb{E}^d$ be a convex body and $k \in \{1, \cdots, d-1\}$. The *k-th Mean Projection Body* of K is defined to be the convex body $P_k(K)$ whose support function is given by

$$h(P_k(K), u) = \int_{\mathcal{L}_k^d} h(K|E, u)\nu_k^d(dE) \qquad \text{for all} \qquad u \in S^{d-1}. \tag{5.1}$$

To see that (5.1) indeed defines a convex body in $\mathbb{E}^d$, we define a function h on $\mathbb{E}^d$ such that

$$h(x) := \int_{\mathcal{L}_k^d} h(K|E, x)\nu_k^d(dE) \qquad \text{for all} \qquad x \in \mathbb{E}^d.$$

Since $h(K|E, \cdot)$ is the support function of $K|E$ in $\mathbb{E}^d$, it is convex and positively homogeneous of degree 1. So h is convex and positively homogeneous of degree 1. Therefore h must be the support function of a convex body, say $P_k(K)$, in $\mathbb{E}^d$. $h(P_k(K), \cdot)$ is just the restriction of h to S^{d-1}. If we think of P_k as a geometric operator, it maps a convex body $K \in \mathcal{K}^d$ to $P_k(K) \in \mathcal{K}^d$. It should be noted that although $P_k(K)$ is an average of k-dimensional projections of K, $P_k(K)$ is typically of dimension d. A natural question to ask is how much information about K can be obtained based on knowledge of $P_k(K)$. In particular, we will be interested in the injectivity of P_k for $k \in \{1, 2, ..., d-1\}$. In the following discussions, we also refer to $P_k(K)$ as the Minkowski sum of the k-dimensional projections of K or the average of k-dimensional projections of K.

The operator P_{d-1} was first considered by Schneider [1977b] who showed that if $P_{d-1}(K) = cK$ for some constant c then K is a ball. Spriestersbach [1998] studied the injectivity of P_{d-1}. She showed that P_{d-1} is injective. Furthermore, she gave stability results which show that if $P_{d-1}(K)$ is close to $P_{d-1}(M)$ then K is close to M. Schneider's result can then be used to show that if $P_{d-1}(K)$ is close to cK for some constant c, K is close to a ball. Goodey [1998] used Spriestersbach's techniques and showed that $P_k(K)$ is injective if $k \geq d/2$. The case $k = 1$ is a special one and is mentioned by Goodey [1998] who observes that $P_1(K) = P_1(L)$ if and only if K and L have the same central symmetrand and coincident Steiner points. It follows that P_1 is not injective. Goodey [1998] also studied the case $k = 2$ and showed that, in all dimensions except 14, P_2 is injective. On the other hand, in dimension 14, he showed that if K has sufficiently smooth boundary and positive radii of curvature at all boundary points then there is an $L \neq K$ such that $P_2(L) = P_2(K)$. Thus P_2 is not injective if $d = 14$.

The case $k = 2$ exhibits rather different behaviour from larger values of k. It is interesting to know whether or not the apparently unusual behaviour of P_2 is encountered in other dimensions. Our main result in this section is to show that P_3 is injective for all dimension $d \geq 3$. In fact, we will give the following theorem:

Theorem 5.1. *If K and L are convex bodies in $\mathbb{E}^d$ with $P_3(K) = P_3(L)$ then $K = L$.*

The proof of our theorem will make use of a linear operator

$$p_k : L^2(S^{d-1}) \to L^2(S^{d-1}). \tag{5.2}$$

The definition of p_k was introduced in Goodey, Kiderlen and Weil [1998]. They made use of a spherical projection operator used by Weil [1982] and certain lifting transformations. We refer the reader to Goodey, Kiderlen and Weil [1998] or Goodey [1998] for details about the definition of p_k. We will only give the results about p_k necessary for our discussion. These results, including Lemma 5.2 and Lemma 5.3, can be found in Goodey [1998] and will be included in this thesis for completeness.

Let $k \in \{1, ..., d-1\}$ and $E \in \mathcal{L}_k^d$ be fixed. Let $\pi_E^* : C(S^{d-1} \cap E) \to C(S^{d-1})$ be defined by

$$(\pi_E^* f)(u) = \langle u, v \rangle f(v), \qquad \text{for } u \in S^{d-1} \text{ and } f \in C(S^{d-1} \cap E). \tag{5.3}$$

Here v is the unit vector in E in the direction of $u|E$. If u is orthogonal to E we put $(\pi_E^* f)(u) = 0$. It was shown in Goodey, Kiderlen and Weil [1998] that π_E^* lifts support functions to support functions. In other words, if $K \subset E$ is a convex body and $h_E(K,\cdot)$ is a support function of K in E, then $\pi_E^*(h_E(K,\cdot)) = h(K,\cdot)$ is the support function of K in $\mathbb{E}^d$. Define the linear operator $p_k : C(S^{d-1}) \to C(S^{d-1})$ by

$$(p_k f)(u) = \int_{\mathcal{L}_k^d} (\pi_E^* f|_E)(u)\nu_k^d(dE), \qquad \text{for } u \in S^{d-1} \text{ and } f \in C(S^{d-1}). \quad (5.4)$$

It follows that p_k is the functional equivalent of the geometric operator P_k in the sense that

$$p_k\big(h(K,\cdot)\big) = h(P_k(K),\cdot). \quad (5.5)$$

Lemma 5.2. *For $k \in \{2,3,...,d-1\}$, the operator $p_k : C(S^{d-1}) \to C(S^{d-1})$ has a unique extension to a mapping $p_k : L^2(S^{d-1}) \to L^2(S^{d-1})$ which is linear, bounded, continuous and intertwining.*

Proof:

We first establish the intertwining property of p_k. We let $\rho \in SO(d)$ and $f \in L^2(S^{d-1})$. If $E \in \mathcal{L}_k^d$ and $f \in C(S^{d-1})$ then it follows from (5.3) that for each $u \in S^{d-1}$ and each $\rho \in SO(d)$,

$$(\pi_E^*(\rho f)|_E)(u) = \pi_{\rho^{-1}E}^*(f|_{\rho^{-1}E}(\rho^{-1}u)).$$

Therefore, for each $u \in S^{d-1}$ and each $\rho \in SO(d)$

$$\begin{aligned}(p_k\rho f)(u) &= \int_{\mathcal{L}_k^d} (\pi_E^* f|_E)(u) v_k^d(dE) \\ &= \int_{\mathcal{L}_k^d} (\pi_{\rho^{-1}E}^* f|_{\rho^{-1}E})(\rho^{-1}u) v_k^d(d(\rho^{-1}E)) \\ &= (p_k f)(\rho^{-1}u) = (\rho p_k f)(u).\end{aligned}$$

This shows that $\rho p_k = p_k \rho$. This is the required intertwining property.

If $E \in \mathcal{L}_k^d$, $u \in S^{d-1}$ and $f \in C(S^{d-1})$, then by (5.3)

$$|\ \pi_E^*(f|_E)(u)\ | = \|\ u|E\ \|\,|\ f(u_E)\ | \leq |\ f(u_E)\ |\ .$$

Here u_E is the unit vector in E in the direction of $u|E$. Note that $\nu_k^d(\mathcal{L}_k^d) = 1$. Using Hölder's inequality, we have

$$| p_k(f)(u) |^2 \leq \int_{\mathcal{L}_k^d} \left[\pi_E^*(f|_E)\right]^2(u)\nu_k^d(dE) \leq \int_{\mathcal{L}_k^d} f^2(u_E)\nu_k^d(dE).$$

If ν denotes the unique invariant probability measure on $SO(d)$, we have

$$\begin{aligned}
\| p_k(f) \|^2 &= \int_{SO(d)} \left[\rho p_k(f)\right]^2(u)\nu(d\rho) \\
&= \int_{SO(d)} \left[p_k(\rho f)\right]^2(u)\nu(d\rho) \\
&\leq \int_{SO(d)} \int_{\mathcal{L}_k^d} (\rho f)^2(u_E)\nu_k^d(dE)\nu(d\rho) \\
&= \int_{\mathcal{L}_k^d} \int_{SO(d)} (\rho f)^2(u_E)\nu(d\rho)\nu_k^d(dE) = \| f \|^2
\end{aligned}$$

It follows that p_k is bounded by 1 and that $p_k : C(S^{d-1}) \to L^2(S^{d-1})$ is continuous with the L^2-norm. The uniqueness of the extension is a consequence of denseness of $C(S^{d-1})$ in $L^2(S^{d-1})$. $\square$

For $d = 3, 4 \dots$ and $n = 0, 1, \dots,$ we will denote by $\mathcal{H}_n^d$ the subspace of $L^2(S^{d-1})$ comprising the spherical harmonics of degree n in dimension d. Recall from Section 2 that $\mathcal{H}_n^d$ is irreducible and rotation invariant subspaces of $L^2(S^{d-1})$. It then follows from Schur's Lemma that, for each n, $p_k\mathcal{H}_n^d$ is either trivial or $\mathcal{H}_n^d$ itself. Furthermore, there is a multiplier $\alpha_{n,k,d} \in \mathbb{R}$ such that

$$p_k f = \alpha_{n,k,d} f \qquad \text{for each } f \in \mathcal{H}_n^d. \tag{5.6}$$

Let $f \in L^2(S^{d-1})$ have *condensed spherical harmonic expansion* $f \sim \sum_{n=0}^{\infty} f_n$, where $f_n \in \mathcal{H}_n^d$. Then *Parseval's equation* gives $\| f \|^2 = \sum_{n=0}^{\infty} \| f_n \|^2$ (see Groemer [1996], p.71). By Lemma 5.2, the operator p_k is bounded by 1. So $\alpha_{n,k,d} \leq 1$ and thus

$$\sum_{n=0}^{\infty} {\alpha_{n,k,d}}^2 \| f_n \|^2 \leq \sum_{n=0}^{\infty} \| f_n \|^2 < \infty.$$

It follows that $\sum_{n=0}^{\infty} \alpha_{n,k,d} f_n$ is a spherical harmonic expansion of some function $g \in L^2(S^{d-1})$. If we define $p_k f = g$ then we have

$$p_k f \sim \sum_{n=0}^{\infty} \alpha_{n,k,d} f_n. \tag{5.7}$$

This clearly defines the unique extension of p_k to $L^2(S^{d-1})$. It follows that

$$\ker p_k = \bigoplus_{\substack{n=0 \\ \alpha_{n,k,d}=0}}^{\infty} \mathcal{H}_n^d. \tag{5.8}$$

Lemma 5.3. *Let $d = 3, 4 \ldots$ and $k \in \{2, 3, \ldots, d-1\}$ be fixed. If $\alpha_{n,k,d} \neq 0$ for all $n = 0, 1, \ldots,$ then P_k is injective.*

Proof:

Recall that p_k is the functional equivalent of the geometric operator P_k. We need only show that p_k is injective. If $\alpha_{n,k,d} \neq 0$ for all $n = 0, 1, \ldots,$ by (5.8), $\ker p_k$ is trivial. This gives the injectivity of p_k as required. □

Now we will give a usable formulation of the multipliers $\alpha_{n,k,d}$. This too can be found in Goodey [1998] but is included in this thesis for completeness. If e_d denotes the final vector in the usual basis of $\mathbb{E}^d$ then $P_n^d(\langle e_d, \cdot \rangle)$ is the unique (up to scalar multiplication) member of $\mathcal{H}_n^d$ which is invariant under the group action of $SO(d-1)$; here $\langle x, y \rangle$ denotes the usual scalar product in $\mathbb{E}^d$. Furthermore $P_n^d(1) = 1$. Consequently

$$\begin{aligned} \alpha_{n,k,d} &= \alpha_{n,k,d} P_n^d(\langle e_d, e_d \rangle) = (p_k P_n^d(\langle e_d, \cdot \rangle))(e_d) \\ &= \int_{\mathcal{L}_k^d} (\pi_E^* P_n^d(\langle e_d, \cdot \rangle)|_E))(e_d) \nu_k^d(dE). \end{aligned} \tag{5.9}$$

It follows from (5.3) and (5.9) that

$$\alpha_{n,k,d} = \int_{\mathcal{L}_k^d} \langle e_E, e_d \rangle P_n^d(\langle e_E, e_d \rangle) \nu_k^d(dE), \tag{5.10}$$

where $e_E = \frac{e_d|E}{\|e_d|E\|}$ if $\| e_d|E \| \neq 0$; otherwise, we put $e_E = e_d$. In order to give a better formula for $\alpha_{n,k,d}$, we need an integral geometric formula that is due to Chern [1966, equation 28].

$$\int_{\mathcal{L}_k^d} f(E)\nu_k^d(dE)$$
$$= c_{k,d} \int_{\mathcal{L}_{k-1}^{d-1}} \int_{\mathcal{L}_1^{d-k+1}(F^\perp)} f([u,F])| \langle e_d, u\rangle |^{k-1} \nu_1^{d-k+1}(du)\nu_{k-1}^{d-1}(dF).$$

Here

$$c_{k,d} = \frac{2\Gamma(d/2)}{\Gamma(k/2)\Gamma((d-k)/2)} \quad \text{and} \quad f \in C(\mathcal{L}_k^d).$$

Also $\mathbb{E}^{d-1}$ is the subspace of $\mathbb{E}^d$ orthogonal to e_d. For any $F \in \mathcal{L}_{k-1}^{d-1}(\mathbb{E}^{d-1})$, $[u,F] \in \mathcal{L}_k^d$ denotes the subspace spanned by F and the line $u \in F^\perp$. If we apply this formula to (5.10), we have

$$\begin{aligned}
\alpha_{n,k,d} &= c_{k,d} \int_{\mathcal{L}_{k-1}^{d-1}} \int_{\mathcal{L}_1^{d-k+1}(F^\perp)} P_n^d(| \langle e_d, u\rangle |)| \langle e_d, u\rangle |^k \nu_1^{d-k+1}(du)\nu_{k-1}^{d-1}(dF) \\
&= c_{k,d} \int_{\mathcal{L}_1^{d-k+1}(F^\perp)} P_n^d(| \langle e_d, u\rangle |)| \langle e_d, u\rangle |^k \nu_1^{d-k+1}(du) \int_{\mathcal{L}_{k-1}^{d-1}} \nu_{k-1}^{d-1}(dF) \\
&= c_{k,d} \int_{\mathcal{L}_1^{d-k+1}(F^\perp)} P_n^d(| \langle e_d, u\rangle |)| \langle e_d, u\rangle |^k \nu_1^{d-k+1}(du) \\
&= c_{k,d} \int_0^1 x^k(1-x^2)^{(d-k-2)/2} P_n^d(x)\, dx.
\end{aligned}$$

Therefore, we get

$$\alpha_{n,k,d} = \frac{2\Gamma(d/2)}{\Gamma(k/2)\Gamma((d-k)/2)} \int_0^1 x^k(1-x^2)^{(d-k-2)/2} P_n^d(x)\, dx. \tag{5.11}$$

It is more convenient for us to work with Gegenbauer polynomials C_n^ν, these are related to the Legendre polynomials by

$$C_n^{(d-2)/2}(x) = \binom{n+d-3}{d-3} P_n^d(x).$$

Put

$$t(n,d) =: \int_0^1 x^3(1-x^2)^{(d-5)/2} C_n^{(d-2)/2}(x)\, dx. \tag{5.12}$$

To show Theorem 5.1, it will suffice to show that $\alpha_{n,3,d} \neq 0$ or $t(n,d) \neq 0$ for all $n = 0, 1, \dots$ and $d = 3, 4, \dots$. We may assume that $d \geq 7$ since the case $k \geq d/2$ has already been resolved.

Lemma 5.4. *For all dimensions $d \geq 7$ and $n = 0, 1, 2, \ldots,$ $t_{n,d} \neq 0$.*

Proof:

Put

$$I(k,p,v,n) := \int_0^1 x^k(1-x^2)^p C_n^v(x)dx. \tag{5.13}$$

Recall from §2 the following recursion relations for the Gegenbauer polynomials

$$nC_n^v(x) = 2v[xC_{n-1}^{v+1}(x) - C_{n-2}^{v+1}(x)]$$

and

$$(2v+n)C_n^v(x) = 2v[C_n^{v+1}(x) - xC_{n-1}^{v+1}(x)].$$

Combining these two recursion relations, we have

$$(v+n)I(k,p,v.n) = v[I(k,p,v+1,n) - I(k,p,v+1,n-2)]. \tag{5.14}$$

Let $p := \frac{d-k-2}{2}$, $v := \frac{d-2}{2} - 1$. By (5.14), we have

$$t(n,d) - t(n-2,d) = \frac{v+n}{v} I(3, \frac{d-5}{2}, \frac{d-4}{2}, n)$$

Rodrigues' Formula gives

$$C_n^\nu(x) = \frac{(-1)^n}{2^n} \frac{\Gamma(2\nu+n)\Gamma\left(\frac{2\nu+1}{2}\right)}{\Gamma(2\nu)\Gamma\left(\frac{2\nu+1}{2}+n\right)} \frac{(1-x^2)^{\frac{1}{2}-\nu}}{n!} \frac{d^n}{dx^n}\left[(1-x^2)^{\nu+n-\frac{1}{2}}\right]$$

Consequently,

$$\begin{aligned}
&I(3, \frac{d-5}{2}, \frac{d-4}{2}, n) \\
&= \int_0^1 x^3(1-x^2)^{\frac{(d-2)-3}{2}} C_n^{\frac{(d-2)-2}{2}}(x)dx \\
&= \binom{n+d-5}{d-5} \int_0^1 x^3(1-x^2)^{\frac{(d-2)-3}{2}} P_n^{d-2}(x)dx \\
&= \alpha_{n,d} \int_0^1 x^3 \frac{d^n}{dx^n}(1-x^2)^{n+\frac{d-5}{2}} dx \\
&= \alpha_{n,d}[x^3 \frac{d^{n-1}}{dx^{n-1}}(1-x^2)^{n+\theta}|_0^1 - 3\int_0^1 x^2 \frac{d^{n-1}}{dx^{n-1}}(1-x^2)^{n+\theta}dx] \\
&= -3\alpha_{n,d} \int_0^1 x^2 \frac{d^{n-1}}{dx^{n-1}}(1-x^2)^{n+\theta}dx,
\end{aligned}$$

where

$$\theta = \frac{d-5}{2} \quad \text{and} \quad \alpha_{n,d} = \binom{n+d-5}{d-5} \frac{(-1)^n}{2^n(\theta+1)(\theta+2)...(\theta+n)}.$$

Using integration by parts, we get

$$\begin{aligned}(\alpha_{n,d})^{-1} I(3, \frac{d-5}{2}, \frac{d-4}{2}, n) &= -6\frac{d^{n-4}}{dx^{n-4}}(1-x^2)^{n+\theta}|_0^1 = 6\frac{d^{n-4}}{dx^{n-4}}(1-x^2)^{n+\theta}|_{x=0} \\ &= 6\frac{d^{n-4}}{dx^{n-4}}(1-x^2)^{n-4+\frac{(d+6)-3}{2}}|_{x=0} \\ &= (-1)^{n-4}2^{n-4}(\beta+1)(\beta+2)...(\beta+n-4)P_{n-4}^{d+6}(0)\end{aligned}$$

where $\beta = \frac{d+3}{2}$. It follows that

$$t(5,d) = t(2m+1,d) \quad \text{for all} \quad m = 2,3,\ldots \quad \text{and} \quad d = 7,8,\ldots. \tag{5.15}$$

For any even number $n > 2$, we have

$$t(n,d) - t(n-2,d) = \mu_{d,n}(-1)^{\frac{n}{2}}, \tag{5.16}$$

where

$$\begin{aligned}\mu_{d,n} &:= \binom{n+d-5}{d-5} \frac{(d+2n-4)(\beta+1)(\beta+2)...(\beta+n-4).1.3.5...(n-5)}{2^4(d-4)(\theta+1)(\theta+2)...(\theta+n).(d-1)(d+1)...(d+n-1)} \\ &= \binom{n+d-5}{d-5} \frac{1.3.5\cdots(n-5)(d+2n-4)(-1)^{\frac{n}{2}}}{(d-4)^2(d-3)(d-2)(d-1)(d+1)(d+3)\cdots(d+n-1)}.\end{aligned}$$

Here we have used $\beta = \frac{d+3}{2}$ and $\theta = \frac{d-5}{2}$. Therefore

$$t(n,d) - t(n-4,d) = (-1)^{\frac{n}{2}}\gamma_{d,n}P(d,n), \tag{5.18}$$

where

$$\gamma_{d,n} := \frac{(n+d-7)!1.3.5\cdots(n-7)}{(d-4)(d-1)!(d+1)(d+3)\cdots(d+n-1)n!},$$

and

$$\begin{aligned}&p(n,d) \\ &= (d+2n-4)(n+d-5)(n+d-6)(n-5) - (d+2n-8)n(n-1)(d+n-1) \\ &= (2d-24)n^3 + (3d^2-54d+216)n^2 + (d^3-34d^2+270d-632)n- \\ &\qquad -5d^3+75d^2-370d+600. \qquad (5.19)\end{aligned}$$

We will consider three cases:

Case 1: n is odd.

By direct calculation, we have

$$t(1,d) = \frac{3(d-2)\sqrt{\pi}}{4d}\frac{\Gamma\left(\frac{d-3}{2}\right)}{\Gamma\left(\frac{d}{2}\right)} > 0$$

and

$$t(3,d) = \frac{(d-2)\sqrt{\pi}}{4}\frac{\Gamma\left(\frac{d-3}{2}\right)}{\Gamma\left(\frac{d}{2}\right)} > 0.$$

It follows from (5.15) that for $n = 5, 7, 9, \ldots,$

$$\begin{aligned}
t(n,d) &= t(5,d) \\
&= \frac{d(d-2)\sqrt{\pi}(10\Gamma[1+\frac{d}{2}] + 5d\Gamma[1+\frac{d}{2}] - 6\Gamma[2+\frac{d}{2}] + \Gamma[\frac{d-3}{2}])}{32\Gamma[1+\frac{d}{2}]\Gamma[2+\frac{d}{2}]} \\
&= \frac{d(d-2)\sqrt{\pi}(4\Gamma[1+\frac{d}{2}] + 2d\Gamma[1+\frac{d}{2}] + \Gamma[\frac{d-3}{2}])}{32\Gamma[1+\frac{d}{2}]\Gamma[2+\frac{d}{2}]} > 0
\end{aligned}$$

This shows that $t(n,d)$ is positive for all odd n and completes our discussion of the odd multipliers.

Case 2: n is even and $n \leq 12$

We will show that $t(n,d) > 0$ if n is even and $d \leq 12$. It follows from (5.16) that $t(4m,d) > t(4m-2)$ for $i = 1, 2, \ldots$. Thus it suffices to show that the multipliers of the form $t(4m-2,d)$ are positive. This is achieved by investigation of the polynomials $p(n,d)$ and it is here that we use the fact that $d \leq 12$.

First we have

$$t(0,d) = \frac{2}{(d-3)(d-1)} \qquad \text{and} \qquad t(2,d) = \frac{(3d-1)(d-2)}{(d-3)(d-1)(d+1)}. \tag{5.20}$$

Consequently, the positivity of the even multipliers is reduced to showing that $t(4m-2,d) > 0$ for $m = 2, 3, \ldots$.

For the dimensions $d = 7, \ldots, 12$, we use (5.19) to see that

$$\begin{aligned}
p(n,7) &= -5(2n^3 + 3n^2 + 13n + 6); \\
p(n,8) &= -8(n^3 + 3n^2 + 17n + 15); \\
p(n,9) &= -(6n^3 + 27n^2 + 227n + 300); \\
p(n,10) &= -4(n^3 + 6n^2 + 83n + 150); \\
p(n,11) &= -(2n^3 + 15n^2 + 445n + 1050); \\
p(n,12) &= -560(n+3).
\end{aligned}$$

These polynomials are clearly negative for positive values of n. It follows, therefore, from (3.3) and (3.4) that the numbers $t(4m-2,d)$ increase with m for each dimension $7 \leq d \leq 12$. Combining this with (5.20), we see that the even multipliers are all positive in dimensions $7 \leq d \leq 12$.

Case 3: n is even and $n > 12$

In this case we will encounter some negative multipliers and therefore the arguments are more delicate. We will see that $t(4n,d) > 0$ for $n \geq 0$ and $d > 12$. However, the sequences $\{t(4n-2,d)\}_{n=1}^{\infty}$ will eventually decrease from positive to negative values without ever achieving the value zero. The case of dimension 13 will be rather arduous and cause us to make use of a computer.

First we examine the cubic polynomials $p(n,d)$ for $d \geq 13$. We aim to show that $p(n,d) > 0$ if $n \geq 16$ and $d \geq 13$. This is an easy observation based on the first and second derivatives of $p(n,d)$. The second derivative is $6(d-12)(2n+d-6)$ which is clearly positive for $n \geq 0$ if $d \geq 13$. Consequently the first derivative increases as a function of $n \geq 0$ if $d \geq 13$. This first derivative, evaluated at $n = 8$, is $d^3 + 14d^2 - 210d - 1748$ which, in turn, is an increasing function of $d \geq 13$. It follows that, for each $d \geq 13$, $p(n,d)$ increases as a function of $n \geq 8$. Finally, we note that $p(16,d) = (d+26)(11d^2 + 13d - 2020) > 0$ if $d \geq 13$. This gives the desired result, namely

$$p(n,d) > 0 \qquad \text{for } n \geq 16 \qquad \text{if } d \geq 13. \tag{5.21}$$

It follows from (3.3) and (6.1) that, for each $d \geq 13$, the sequence $\big(t(4n,d)\big)_{n=3}^{\infty}$ increases. Direct calculations give

$$t(0,d) = \frac{2}{(d-3)(d-1)};$$

$$t(4,d) = \frac{(d-2)d(d^2+12d+19)}{4(d-3)(d-1)(d+1)(d+3)};$$

$$t(8,d) = \frac{(d-2)d(d+2)(d+4)(3d^4-32d^3+1218d^2+32288d+138555)}{6720(d-3)(d-1)(d+1)(d+3)(d+5)(d+7)};$$

$$t(12,d) = \frac{(d-2)d(d+2)(d+4)(d+6)(d+8)}{5322240(d-3)(d-1)(d+1)(d+3)(d+5)(d+7)(d+9)(d+11)}\times$$
$$\times\,(7d^6+36d^5+511d^4-12648d^3+768397d^2+32452932d+226313325).$$

It can be checked that these quantities are all positive if $d \geq 13$. Consequently

$$t(4n,d) > 0 \qquad \text{for all } n \geq 0 \qquad \text{if } d \geq 13.$$

It only remains to deal with the multipliers of the form $t(4n+2,d)$ for $d \geq 13$. We will now encounter the possibility of negative multipliers. Here, we deduce from (5.18) and (5.21) that, for each $d \geq 13$, the sequence $\{t(4n+2,d)\}_{n=3}^{\infty}$ decreases. Again, we will calculate some of the early multipliers. We have

$$t(2,d) = \frac{(d-2)(3d-1)}{(d-3)(d-1)(d+1)} > 0 \qquad \text{if } d \geq 13.$$

The next multiplier in the sequence is

$$t(6,d) = -\frac{(d-2)d(d+2)(d+9)(d^2-36d-157)}{120(d-3)(d-1)(d+1)(d+3)(d+5)}.$$

It follows that

$$t(6,d) < 0 \quad \text{for } d \geq 40 \qquad t(6,d) > 0 \quad \text{if } 13 \leq d \leq 39.$$

Similar behaviour is exhibited by the next two multipliers, namely

$$t(10,d) < 0 \quad \text{for } d \geq 26 \qquad t(10,d) > 0 \quad \text{if } 13 \leq d \leq 25$$

$$t(14,d) < 0 \quad \text{for } d \geq 22 \qquad t(14,d) > 0 \quad \text{if } 13 \leq d \leq 21.$$

We deduce that, for each dimension $d \geq 13$, the sequence $\left(t(4n+2,d)\right)_{n=0}^{\infty}$ starts with positive terms and from the term $n = 3$ onwards is decreasing. Furthermore, if it were possible to have $t(4n+2,d) = 0$ we must have $n \geq 4$ and $13 \leq d \leq 21$. This possibility can be ruled out if, for each $13 \leq d \leq 21$ we find a number $n_0 = n_0(d)$ such that $t(4n_0 - 2) > 0$ and $t(4n_0 + 2) < 0$. To this end, we note that

$$n_0(21) = n_0(20) = 3 \quad n_0(19) = 4 \quad n_0(18) = 5 \quad n_0(17) = 7 \quad n_0(16) = 10$$
$$n_0(15) = 19 \quad n_0(14) = 59 \quad n_0(13) = 1606.$$

The case of dimension 13 perhaps deserves some comment. The above results are all derived using a computer. Superficially, one has to deal, in case $d = 13$, with Gegenbauer polynomials of degrees 6426 and 6430. This seems to be beyond the capabilities of many computer algebra systems. In fact, we were only able to do these computations using Mathematica. However, one can use standard recursion formulas for the Gegenbauer polynomials to deduce that

$$t(2n,13) = t(2n-2,13) + (-1)^n 2(2n+9) \frac{\binom{2n+8}{8} \prod_{m=0}^{n-3} (2m+1)}{3 \prod_{m=5}^{n+6} (2m)}.$$

This formula makes the calculations feasible for any mathematical software.

Collecting our results together we have the following:

a). The numbers $t(2n-1,d)$, $n = 1, 2, \ldots$ are positive for all dimensions d;

b). The numbers $t(2n,d)$, $n = 0, 1, \ldots$ are positive for all dimensions $d \leq 12$;

c). The numbers $t(4n,d)$, $n = 0, 1, \ldots$ are positive for all dimensions d;

d). For each dimension $d \geq 13$ there is a number $n_0 = n_0(d)$ such that $t(4n+2) > 0$ if $n < n_0$ and $t(4n+2) < 0$ if $n \geq n_0$.

It follows that $t(n,d) \neq 0$ for all $n \geq 0$ and all dimensions $d \geq 7$. Therefore the Lemma is proved. □

Theorem 5.1 follows immediately from the Lemmas above. This shows, similar to the case $k \geq d/2$, that all convex bodies in $\mathbb{E}^d$ are determined by averages of their three dimensional projections. This is in contrast to the case of two dimensional projections where there is a single dimension ($d = 14$) in which such determination is not possible. This question is still open for the intermediate cases $3 < k < \frac{d}{2}$.

§6. Open Problems

In Section 4, we showed that for any fixed $j \in \{2, \ldots, d-2\}$ and sufficiently large d, $\mathcal{K}(j) \neq Z_{d-j+1}$. We did this by studying the polynomial

$$Q_{d,j}(X) := 1 + \sum_{i=1}^{j} (-1)^i \binom{j}{i} \frac{1}{(d-j+2)^i} \prod_{k=0}^{i-1} \frac{d+2k+1}{j+2k} X^i. \tag{6.1}$$

To be exact, we showed that there exist convex bodies $K_\alpha \in \mathcal{K}(j) \setminus Z_{d-j+1}$ if

$$Q_{d,j}(X) > 0 \qquad \text{for all} \quad 0 \leq X \leq 1 \tag{6.2}$$

and that (6.2) holds if d is sufficiently large. Some calculations for early dimensions suggest that (6.2) is true for all $j \in \{2, \ldots, d-2\}$. Recall also that the polytopes given by Goodey and Weil [1991] are in $K(d-2) \setminus Z_3$. It is tempting to believe that $Z_{d-j+1} \subseteq \mathcal{K}(j)$.

Problem 6.1. *For each* $j \in \{1, 2, \ldots, d-1\}$,

$$Z_{d-j+1} \subseteq \mathcal{K}(j) \tag{6.3}$$

with equality holds if and only if $j = 1$ *or* $j = d-1$.

Next recall our proof of the strict inclusion $\mathcal{K}(1) \subset \mathcal{K}(j)$. Define

$$M_{d,j}(X) := 1 + \sum_{i=1}^{j} \binom{j}{i} \frac{1}{(d+1)^i} \prod_{k=0}^{i-1} \frac{d+2k+1}{j+2k} X^i \tag{6.4}$$

We assume X_j is the biggest (negative) zero of (6.4). It can be seen from the proof of Theorem 4.1 that if $1 \leq j < j+k \leq d-1$ then there are bodies $K_\alpha \in \mathcal{K}(j+k) \setminus \mathcal{K}(j)$ if $X_j > X_{j+k}$. Clearly $X_1 = -1$ and, in fact, Theorem 4.1 is proved by showing $X_j < -1$ for all $j \in \{2, 3, \ldots, d-1\}$. We conjecture the following problem is true.

Problem 6.2. *For each $j \in \{1, 2, \ldots, d-2\}$,*

$$\mathcal{K}(j) \subset \mathcal{K}(j+1). \tag{6.5}$$

Define

$$J_{d,j}(Y) = \sum_{i=0}^{j} \binom{j}{i} \prod_{k=0}^{i-1} \frac{d+2k+1}{j+2k} Y^i. \tag{6.6}$$

It is easy to see that $Q_{d,j}(X)$ and $M_{d,j}(X)$ can be derived from $J_{d,j}(Y)$ by setting

$$Y = \frac{-X}{d-j+2} \quad \text{and} \quad Y = \frac{X}{d+1} \tag{6.7}$$

respectively. Next we want to point out that $J_{d,j}$ is, in fact, a Jacobi polynomial. Therefore both Problem 6.1 and Problem 6.2 are closely related to the study of zeros of this Jacobi polynomial.

It is easy to check (see 2.18) that $J_{d,j}$ is a hypergeometric series

$$J_{d,j}(Y) = F\Big(-j, \frac{d+1}{2}; \frac{j}{2}; -Y\Big). \tag{6.8}$$

By (2.19), $J_{d,j}(Y)$ can be expressed as a Jacobi polynomial

$$J_{d,j}(Y) = \binom{\frac{3j}{2}-1}{j} P_j^{(\frac{j}{2}-1, \frac{d-3j+1}{2})}(1+2Y). \tag{6.9}$$

Therefore, for K_α, Problem 6.1 and Problem 6.2 become questions about the locations of zeros of the Jacobi polynomial $J_{d,j}$. We next want to provide some results about Problem 6.2. Based on these results, we guess that Problem 6.2 has a positive answer, at least for convex bodies K_α. We will use some classical results about Jacobi Polynomials and we refer to Szegö [1939] for information about these polynomials.

Recall that Problem 6.2 is true for bodies K_α if we can show that the biggest (negative) root of

$$P_j^{(\frac{j}{2}-1, \frac{d-3j+1}{2})}(1+2Y) = 0$$

moves to the left as j increases. Fix d and define

$$M_{j,k,m}(Z) = P_j^{(\frac{k}{2}-1,\frac{d-3m+1}{2})}(Z).$$

Let $Z_{j,k,m}$ be the biggest (negative) root of $M_{j,k,m}(Z) = 0$. It follows from Theorem 6.21.1 (Szegö [1939, p.121]) that $Z_{j,j+1,j} < Z_{j,j,j}$ and that $Z_{j,j+1,j+1} < Z_{j,j+1,j}$. But unfortunately $Z_{j,j+1,j+1} < Z_{j+1,j+1,j+1}$. This follows from the interlacing of the roots of Jacobi Polynomials $P_j^{(\alpha,\beta)}$ and $P_{j+1}^{(\alpha,\beta)}$ (see Szegö [1939, Theorem 3.3.2, p.46]). We suspect that

$$Z_{j+1,j+1,j+1} < Z_{j,j,j} \quad \text{for all} \quad j \in \{1,2,\ldots,d-1\}. \tag{6.10}$$

Our guess above is also based on extensive computer experiments that give evidence for the behavior of the roots.

Finally, recall from (3.18) that, for each $E \in \mathcal{L}_j^d$,

$$\rho_j(K,E) \\ = c_j'' \int_{S^{d-1}\cap E} \rho_{j-1}(K,v^\perp)\Big\{\int_{S^{d-1}\cap E} \rho(u)|\,\langle u,v\rangle\,|^{d-j+1}\lambda_{j-1}(du)\Big\}\lambda_{j-1}(dv). \tag{6.11}$$

Here c_j'' is a positive constant. If Problem 6.1 is true and $\rho_{j-1}(K,\cdot)$ is positive, then $K \in Z_{d-j+1}$. Recall from (3.19) that $K \in Z_{d-j+1}$ if and only if

$$\int_{S^{d-1}\cap E} \rho(u)|\,\langle u,v\rangle\,|^{d-j+1}\lambda_{j-1}(du) \geq 0 \tag{6.12}$$

for all $v \in S^{d-1} \cap E$ and all $E \in \mathcal{L}_j^d$. This , in turn, shows that $\rho_j(K,\cdot)$ is positive and that $K \in \mathcal{K}(j)$. This is an indication for a positive answer to Problem 6.2.

REFERENCES

Aleksandrov, A. D.

1937a On the theory of mixed volumes of convex bodies I, Extension of certain concepts in the theory of convex bodies (Russian), *Mat. Sbornik N. S.* **2** (1937), 947–972.

1937b On the theory of mixed volumes of convex bodies II, New inequalities between mixed volumes and their applications (Russian), *Mat. Sbornik N. S.* **2** (1937), 1205–1238.

Berg, C.

1969 Corps convexes et potentiels sphériques, *Danske Vid. Selskab. Mat. - Fys. Medd.* **37**(b), 1–64.

Boerner, H.

1963 *Representations of Groups,* North Holland, Amsterdam 1963.

Busemann, H.

1958 *Convex surfaces,* Interscience, New York 1958.

Chern, S. S.

1966 On the kinematic formula in integral geometry, *J. Math. Mech.* **16** (1966), 101–118.

Eggleston, H. G.

1958 *Convexity,* Cambridge Univ. Press 1958.

Fallert, H., Goodey, P. R. and Weil, W.

1997 Spherical projections and centrally symmetric sets, *Adv. in Math.* **129** (1997), 301–322.

Fenchel, W. and Jessen, B.

1938 Mengenfunktionen und konvexe Korper. *Danske Vid. Selskab. Mat. -Fys. Medd.* **16**, 3.

Firey, W. J.

1968 Christoffel's problem for general convex bodies, *Mathematika* **15**, 7–21.

1970 Intermediate Christoffel-Minkowski Problems for Figures of Revolution, **8** (1970), *Israel J. of Math.*, 384–390.

1975 Some open questions on convex surfaces, In: *Proc. Int. Congr. Math.*, Vancouver 1974 (ed. R. D. James), Canad. Mathematical Congress, 479–484.

Gardner, R.

1995 *Geometric Tomography,* Cambridge University Press, Cambridge 1995.

Goodey, P. R.

1977 Centrally symmetric convex sets and mixed volumes, *Mathematika* **24**, 193–198.

1981 Limits of intermediate surface area measures of convex bodies, *Proc. London Math. Soc.* (3) **43**, 151–168.

1986 Instability of projection bodies, *Geom. Dedicata* **20** (1986), 295–305.

1997 Applications of representation theory to convex bodies. *Suppl. Rend. Circ. Mat. Palermo*, ser. II, **50** (1996), 179–187

1998a Minkowski sums of projections of convex bodies, *Mathematika* **45** (1998), 253–268.

1998b Radon transforms of projection functions, *Math. Proc. Camb. Phil. Soc.* **123** (1998), 159–168.

Goodey, P. R. and Howard, R.

1990 Processes of flats induced by higher dimensional processes, *Adv. in Math.* **80**, 92–109.

Goodey, P. R., Howard, R. and Reeder, M.

1996 Processes of flats induced by higher dimensional processes III, *Geom. Dedicata* **61**, 257–269.

Goodey, P. R., Kiderlen, M. and Weil, W.

1998 Section and projection means of convex bodies, *Monatsh. Math.* **126** (1998), 37–54.

Goodey, P. R. and Schneider, R.

1980 On the intermediate area functions of convex bodies, *Math. Z.* **173** (1980), 185–194.

Goodey, P. R., Schneider, R. and Weil, W.

1997 Projection functions of convex bodies, *Bolyai Society Mathematical Studies* **6** (1997), 23–53.

Goodey, P. R. and Weil, W.

1984 Distributions and valuations, *Proc. London Math. Soc.* (3) **49**, 504–516.

1987 Translative integral formulae for convex bodies, *Aequationes Math.* **34** (1987), 64–77.

1991 Centrally symmetric convex bodies and Radon transformations on higher order Grassmannians, *Mathematika*, 38(1991) 117-133.

1992a The determination of convex bodies from the mean of random sections, *Math. Proc. Camb. Phil. Soc.* **112** (1992), 419 –430.

1992b Centrally symmetric convex bodies and the spherical Radon transform, *J. Differential Geom.* **35** (1992), 675–688.

1992c Integral geometric formulae for projection functions, *Geom. Dedicata* **41** (1992), 117–126.

Goodey, P. R. and Weil, W.
1993 Zonoids and generalizations, *Handbook of convex geometry*, (P. M. Gruber and J. M. Wills, Eds.), North Holland 1993, 1297–1326.

Goodey P. and Zhang G.
1996 Characterizations and inequalities for zonoids, *J. London Math. Soc.* **53** (1996), 184–196.
1998 Inequalities between projection functions of convex bodies. *American Journal of Mathematics* **120** (1998), 345–367

Gradshteyn, I. S. and Ryzhik,I. M.
1994 *Tables of integrals, series and products*, Academic Press, San Diego 1994.

Grinberg, E.
1986 Radon transforms on higher rank Grassmannians, *J. Differential Geom.* **24** (1986), 53–68.

Groemer, H.
1996 *Geometric Applications of Fourier Series and Spherical Harmonics*, Cambridge University Press, Cambridge 1996.

Helgason, S.
1980 *The Radon Transform*, Birkhäuser, Boston, 1980.
1984 *Groups and Geometric Analysis*, Academic Press, Orlando, 1984.

Matheron, G.
1975 *Random Sets and Integral Geometry*, Wiley, New York 1975.

McMullen, P.
1970 Polytopes with centrally symmetric faces, *Israel J. Math.* **8**, 194–196
1984 Volumes of projections of unit cubes, *Bull. London Math. Soc.* **16**, 278–280.
1987 Volumes of complementary projections of convex polytopes, *Monatsh. Math.* **104**, 265–272.

Petty, C. M.
1967 Projection bodies, in: *Proc. Coll. Convexity*, Copengagen 1965 (Københavns Univ. Mat. Inst.) 234–241.

Santaló, L.
1976 *Integral Geometry and Geometric Probability*, Addison–Wesley, Reading, Mass, 1976.

Schneider, R.
1967 Zu einem Problem von Shephard über die Projektionen konvexer Körper, *Math. Z.* **101** (1967), 71–82.

Schneider, R.

1970 über eine Integralgleichung in der Theorie der konvexer Körper, *Math. Nachr.* **44** (1967), 55–75.

1977a Das Christoffel-Problem für Polytope, *Geom. Dedicata* **6**, 81-85

1977b Rekonstruktion eines konvexen Körpers aus seinen Projektionen, *Math. Nachr.* **79** (1977), 325–329.

1993 *Convex Bodies: the Brunn-Minkowski Theory,* Cambridge University Press, Cambridge 1993.

1997a Volumes of projections of polytope pairs, *Suppl. Rend. Circ. Mat. Palermo, ser. II,* **41** (1996), 217–225.

1997b Intrinsic Volumes in Minkowski Spaces, *Suppl. Rend. Circ. Mat. Palermo, ser. II,* **50** (1996), 355-373.

Schneider, R. and Weil, W.

1983 Zonoids and related topics, in: *Convexity and its Applications*, eds P. Gruber and J. M. Wills (Birkhäuser, Basel) 296–317.

1992 *Integralgeometrie*, B.G. Teubner Stuttgart 1992.

Schnell, U.

1994 Volumes of projections of parallelotpes, *Bull. London Math. Soc.* **26** (1994), 181–185.

Shephard, G.

1964 Approximation problems for convex polyhedra, *Mathematika* **11**, 9–18.

1996 Reducible convex sets, *Mathematika* **13** (1966), 49–50.

Spriestersbach, K. K.

1998 Determination of a convex body from the average of projections and stability results, *Math. Proc. Camb. Phil. Soc.* **123** (1998), 561–569.

Strichartz, R.

1975 The explict Fourier decomposition of $L^2(SO(n)/SO(n-m))$, *Can. J. Math.* **27** (1975), 294–310.

1981 L^p estimates for Radon transforms in Euclidean and non-Euclidean spaces, *Duke Math. Jour.* **48** (1981), 699–727.

Sugiura, M.

1962 Representations of compact groups realized by spherical functions on symmetric spaces, *Proc. Japan Acad.* **38** (1962), 111–113.

Szegö, G.

1939 Orthogonal Polynomials, *Amer. Math. Soc.*, (AMS colloquium publications, Vol. **23**), Providence, R. I.

Takeuchi, M.

1973 Polynomial representations associated with symmetric bounded domains, *Osaka J. Math.* **10** (1973), 441–475.

Weil, W.

1971 Über die Projektionenkörper konvexer Polytope, *Arch. Math.* **22**, 664–672.

1976 Centrally symmetric convex bodies and distributions, *Israel J. Math.* **24**, 352–367.

1979 Centrally symmetric convex bodies and distributions, II. *Israel J. Math.* **32**, 173–182.

1982 Zonoide und verwandte Klassen konvexer Körper, *Monatsh. Math.* **94**, 73–84.

1989 Integral Geometry, stochastic geometry and stereology, *Acta Stereologica* **8**, 65–76.

1990 Iterations of translative integral formulae and nonisotropic Poisson processes of particles, *Math. Z.* **205**, 531–549.

Witsenhausen, H. S.

1973 Metric inequalities and zonoid problem, *Proc. Amer. Math. Soc.* **40**, 517–520.

1978 A support characterization of zonotopes, *Mathematika* **25**, 13–16.

Yost, D.

1991 Irreducible convex sets, *Mathematika* **38** (1991), 134–155.