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A GENERIC PICARD-VESSIOT EXTENSION FOR GL_n AND THE INVERSE DIFFERENTIAL GALOIS PROBLEM

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A DISSERTATION
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ABSTRACT

Let F be a differential field with field of constants the algebraically closed field C. Let Y_{ij} be differential indeterminates and let $R = F\{Y_{ij}\}[X_{11}, \ldots, X_{nn}]$ be a differential ring with derivation $D(X_{ij}) = \sum_{k=1}^{n} Y_{ik}X_{kj}$. We show that the quotient field $Q = F\langle Y_{ij}\rangle(X_{ij})$ of R is a Picard-Vessiot extension of $F(Y_{ij})$ for $\mathrm{GL}_n(C)$. Moreover, we show that Q is a generic extension in the following sense: If $E \supset F$ is a Picard-Vessiot extension with differential Galois group G=$\mathrm{GL}_n(C)$ then E is isomorphic to $F(X_{ij})$ as a G-module and as an F-module. Under this isomorphism D_E, the derivation of E, goes to a G-equivariant derivation which has a form similar to D with the coefficients Y_{ij} specialized to elements f_{ij} from F. The differential subfield $C\langle f_{ij}\rangle(X_{ij}) \subset F(X_{ij})$ is shown to be a Picard-Vessiot extension of $C\langle f_{ij}\rangle$ with group $\mathrm{GL}_n(C)$. From this, one can retrieve the Picard-Vessiot extension $F(X_{ij}) \supset F$ by extension of scalars from $C\langle f_{ij}\rangle$ to F.

Conversely, if given F we specialize the Y_{ij} to f_{ij} in F so that the corresponding extension $C\langle f_{ij}\rangle(X) \supset C\langle f_{ij}\rangle$ has no new constants we obtain a solution to the inverse differential Galois problem for $\mathrm{GL}_n(C)$. In the second part of this dissertation we show necessary and sufficient conditions for such a specialization to exist.

Chapter 0. Introduction

The results in this thesis are aimed at solving the inverse differential Galois problem for the General Linear Group. In the most general setting the inverse problem consists in determining what differential field extensions are Picard-Vessiot extensions of a base differential field with a given algebraic group over its field of constants the differential Galois group for the extension.

We structure this introduction in three parts. The first two are quick reviews of the main definitions and results in Differential Galois Theory and Gröbner Bases Theory that will be used in this thesis. In the third part we explain our results.

Henceforth we will use the terminology and notation employed in [M], which is also the source used for the summary in 0.1. For a thorough survey on the direct and inverse differential Galois problems one may refer to [S].

0.1 What is Differential Galois Theory?

Let F be a field. A *derivation* on F is an additive map $D : F \to F$ such that $D(ab) = D(a)b + aD(b)$ for any elements $a, b \in F$. A *differential field* is a pair (F, D), where F is a field and D is a derivation on F.

A *constant* in a differential field F is an element whose derivative is zero. The set of constants of F is a differential subfield, usually denoted by C.

We fix a field F of characteristic 0 and assume that its field of constants C is algebraically closed.

A *differential extension field* of F is a differential field $E \supseteq F$ such that the restriction of the derivation on E coincides with the derivation on F.

In the same fashion one defines *differential ring* and *differential extension ring*.

If (R, D) is a differential ring then an ideal $\mathcal{I} \subseteq R$ is a *differential ideal* if $D(\mathcal{I}) \subseteq \mathcal{I}$.

For an element $y \in F$ we will often use $y^{(i)}$ to denote $D^i(y)$.

Consider a countable set of indeterminates over the differential field F, denoted by $Y^{(i)}$, $i = 0, 1, 2, \ldots$. Let $Y = Y^{(0)}$. Define a derivation on the polynomial ring $F[Y, Y^{(1)}, Y^{(2)}, \ldots]$ by $D(Y^{(i)}) = Y^{(i+1)}$ and $D(f) = D_F(f)$, for $f \in F$, where D_F is the derivation of F. The ring $F[Y, Y^{(1)}, Y^{(2)}, \ldots]$ with the above derivation is called *the ring of differential polynomials in* Y and is denoted by $F\{Y\}$. Y is called a *differential indeterminate.* Likewise one can adjoin a finite number of differential indeterminates $Y_1, \ldots, Y_n$ to F and consider the differential ring $F\{Y_1, \ldots, Y_n\}$. The quotient field of $F\{Y_1, \ldots, Y_n\}$ will be denoted by $F\langle Y_1, \ldots, Y_n\rangle$.

A *linear homogeneous differential operator* over F is an operator of the form

$$L = Y^{(\ell)} + a_{\ell-1}Y^{(\ell-1)} + \cdots + a_0Y^{(0)}$$

where $a_i \in F$. For an element $y \in E$, $L(y) = y^{(\ell)} + a_{\ell-1}y^{(\ell-1)} + \cdots + a_0y^{(0)}$ and the set of solutions of $L = 0$ in E is $\{y \in E | L(y) = 0\}$.

Given a field F and a differential operator $L = Y^{(\ell)} + a_{\ell-1}Y^{(\ell-1)} + \cdots + a_0Y^{(0)}$ over F there is a proper differential extension field $E \supset F$ in which the equation $L = 0$ has ℓ solutions algebraically independent over F. This differential extension can be realized as the quotient field of the polynomial ring over F in ℓ^2 indeterminates $R = F[y_{1,0}, \ldots, y_{\ell,\ell-1}]$ with derivation D_R given by

$$D_R(y_{i,j}) = \begin{cases} y_{i,j+1} & \text{for } j \leqslant \ell - 1 \\ -\sum_{k=0}^{\ell-1} a_k y_{i,k} & \text{for } j = \ell \end{cases}.$$

If E is a differential extension of F and L is a linear operator over F then the set of solutions of $L = 0$ in E is a vector space over the field of constants of E.

Differential Galois Theory studies the nature of the differential extensions generated by the set of solutions of a differential equation over a differential base field. *Picard-Vessiot extensions* are the differential analogues of Galois extensions in the classical Galois Theory of fields. They are defined as follows:

Definition 0.1.1. *Let* $L = Y^{(\ell)} + a_{\ell-1}Y^{(\ell-1)} + \cdots + a_0Y^{(0)}$ *be a linear* *homogeneous differential operator over F. A differential extension field $E \supseteq F$ is a* Picard-Vessiot extension *of F for L if:*

(1) *E is generated over F as a differential field by the solutions of $L = 0$ in E.*

(2) *The constants of E are the constants of F.*

(3) *$L = 0$ has ℓ solutions in E that are linearly independent over the constants.*

A differential extension satisfying only part (2) of the above definition is called a *no new constant extension.* This condition guarantees that we are not adding superfluous solutions to our extension.

Given a differential base field F and a linear homogeneous differential operator $L = Y^{(\ell)} + a_{\ell-1}Y^{(\ell-1)} + \cdots + a_0 Y^{(0)}$ over F there is a differential field extension $E \supset F$ that is a Picard-Vessiot extension of F for L. It can be constructed as follows:

Let $S = F[y_{1,0}, \ldots, y_{\ell,\ell-1}][w^{-1}]$ be the localization of the polynomial ring over F in ℓ^2 indeterminates $R = F[y_{1,0}, \ldots, y_{\ell,\ell-1}]$ at $w = \det[y_{i,j}]$. As above, one can define a derivation on R by

$$D_R(y_{i,j}) = \begin{cases} y_{i,j+1} & \text{for } j \leqslant \ell - 1 \\ -\sum_{k=0}^{\ell-1} a_k y_{i,k} & \text{for } j = \ell \end{cases} .$$

This derivation can be extended to a derivation on S in the obvious way.

S is called the ***full universal solution algebra*** for (the differential equation) $L(=0)$ (abbreviated, FUSA-L).

Let $P \subset S$ be a maximal differential ideal. Then the fraction field $E = Q(S/P) \supset F$ is a Picard-Vessiot extension of F for L. This result is the content of Theorem 3.4 in [M].

Moreover, Picard-Vessiot extensions of a given differential base field F for a fixed L are unique up to isomorphism.

Next we define *differential homomorphisms*.

Definition 0.1.2. *Let F_1, F_2 be differential fields with respective derivations D_{F_1} and D_{F_2}. A* homomorphism of differential fields *is a field homomorphism $\psi : F_1 \to F_2$ such that $D_{F_2}(\psi(a)) = \psi(D_{F_1}(a))$ for all $a \in F_1$.*

Let $E \supset F$ be a Picard-Vessiot extension of F for L. The set $G(E/F)$ of differential automorphisms of E over F is a group. Moreover, it has the structure of an algebraic group over the field of constants C of F and therefore it can be viewed as a subgroup of the general linear group $\mathrm{GL}_n(C)$ of $n \times n$ matrices with non-zero determinant, for some n. We take full advantage of this algebraic group structure in our construction of a generic Picard-Vessiot extension for $\mathrm{GL}_n(C)$.

In Differential Galois Theory, the so-called Galois correspondence also holds. It is the content of the following theorem:

Fundamental Theorem of Galois Theory. *Let E be a Picard-Vessiot extension of F for the linear homogeneous differential operator L. Then there is a lattice-inverting bijective correspondence between*

$$\{F \subseteq K \subseteq E | K \textit{is a differential subfield}\}$$

and

$$\{G \leq G(E/F) | G \textit{ is a Zariski closed subgroup}\}$$

given by

$$K \longmapsto G(E/K) \qquad \textit{and} \qquad G \longmapsto E^G.$$

Picard-Vessiot intermediate field extensions correspond to normal subgroups.

0.2 The Multivarible Division Algorithm.

The machinery of Gröbner Bases provides polynomial rings in several variables with a division algorithm that mimics the Euclidean algorithm in the one variable case. We will present some basic definitions and results that appear in [AL] as we introduce the algorithm.

As usual, let $k[X_1, \ldots, X_n]$ denote the ring of polynomials $f(X_1, \ldots, X_n)$ in n variables with coefficients in the field k. Let

$$\mathbb{T}^n = \{X_1^{\beta_1} \cdots X_n^{\beta_n} \,|\, \beta_i \in \mathbb{N},\, i = 1, \ldots, n\}$$

be the set of monic monomials in the X_i, henceforth called *power products*. For convenience, $X_1^{\beta_1}\cdots X_n^{\beta_n}$ will be denoted by X^β, where $\beta = (\beta_1,\cdots,\beta_n) \in \mathbb{N}^n$.

First, in order to be able to perform a division of polynomials in n variables, we need to put an *ordering* on $\mathbb{T}^n$ as follows:

Definition 0.2.1. *A* term order *on $\mathbb{T}^n$ is a total order $<$ on $\mathbb{T}^n$ satisfying the following two conditions:*

(i) $1 < X^\beta$ *for all* $X^\beta \in \mathbb{T}^n$, $X^\beta \neq 1$.

(ii) *If* $X^\alpha < X^\beta$ *then* $X^\alpha X^\gamma < X^\beta X^\gamma$, *for all* $X^\gamma \in \mathbb{T}^n$.

Theorem 1.4.6 in [AL] establishes that every term order on $\mathbb{T}^n$ is a well-ordering.

The next three definitions provide examples of term orders.

Definition 0.2.2. *The* lexicographical order *on $\mathbb{T}^n$ with $X_1 > X_2 > \cdots > X_n$ is defined as follows: For*

$$\alpha = (\alpha_1,\ldots,\alpha_n),\ \beta = (\beta_1,\ldots,\beta_n) \in \mathbb{N}^n$$

we define

$$X^\alpha < X^\beta \text{ if and only if } \begin{cases} \text{the first coordinates } \alpha_i \text{ and } \beta_i \text{ in } \alpha \text{ and } \beta \\ \text{from the left which are different satisfy } \alpha_i < \beta_i \end{cases}.$$

So, in the two variable case we have

$$1 < X_2 < X_2^2 < X_2^3 < \cdots < X_1 < X_1X_2 < X_1X_2^2 < \cdots < X_1^2 \cdots.$$

This order is denoted by *lex*.

Definition 0.2.3. *The* degree lexicographical order *on* $\mathbb{T}^n$ *with* $X_1 > X_2 > \cdots > X_n$ *is defined as follows: For*

$$\alpha = (\alpha_1, \ldots, \alpha_n),\ \beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$$

we define

$$X^\alpha < X^\beta \text{ if and only if } \begin{cases} \displaystyle\sum_{i=1}^{n} \alpha_i < \sum_{i=1}^{n} \beta_i, \text{ or} \\ \displaystyle\sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} \beta_i, \text{ and } X^\alpha < X^\beta \text{ with respect to} \\ \text{lex with } X_1 > X_2 > \cdots > X_n. \end{cases}$$

So, in the two varible case we have

$$1 < X_2 < X_1 < X_2^2 < X_1X_2 < X_1^2 < X_2^3 < X_1X_2^2 < X_1^2X_2 < X_1^3 < \cdots .$$

This order is denoted by *deglex*.

Definition 0.2.4. *The* degree reverse lexicographical order *on* $\mathbb{T}^n$ *with* $X_1 > X_2 > \cdots > X_n$ *is defined as follows: For*

$$\alpha = (\alpha_1, \ldots, \alpha_n),\ \beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$$

we define

$$X^\alpha < X^\beta \text{ if and only if } \begin{cases} \displaystyle\sum_{i=1}^{n} \alpha_i < \sum_{i=1}^{n} \beta_i, \text{ or} \\ \displaystyle\sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} \beta_i, \text{ and the first coordinates } \alpha_i \\ \text{and } \beta_i \text{ in } \alpha \text{ and } \beta \text{ from the right, which are} \\ \text{different, satisfy } \alpha_i > \beta_i. \end{cases}$$

This order is denoted by *degrevlex*.

In the two variable case the degree lexicographical and the degree reverse lexicographical orders are the same. It is no longer the case, however, if there are three or more variables:

$$X_1^2X_2X_3 > X_1X_2^3 \text{ with respect to deglex with } X_1 > X_2 > X_3$$

but

$$X_1^2X_2X_3 < X_1X_2^3 \text{ with respect to degrevlex with } X_1 > X_2 > X_3.$$

We say that we have a term order on $k[X_1, \cdots, X_n]$ if we have a term order on $\mathbb{T}^n$.

Now, if we fix a term order on $k[X_1, \cdots, X_n]$ we can write any non-zero polynomial $f \in k[X_1, \cdots, X_n]$ as

$$f = a_1X^{\alpha_1} + a_2X^{\alpha_2} + \cdots + a_rX^{\alpha_r},$$

where $0 \neq a_i \in k$, $X^{\alpha_i} \in \mathbb{T}^n$, and $X^{\alpha_1} > X^{\alpha_2} > \cdots > X^{\alpha_r}$. In this setting, the following are defined:

- $\text{lp}(f)=X^{\alpha_1}$, the *leading power product* of f;
- $\text{lc}(f)=a_1$, the *leading coefficient* of f;
- $\text{lt}(f)=a_1X^{\alpha_1}$, the *leading term* of f.

Definition 0.2.5. *Given f, g, h in $k[X_1, \dots, X_n]$, with $g \neq 0$, we say that f reduces to h modulo g in one step, written*

$$f \xrightarrow{g} h,$$

if and only if $lp(g)$ divides a non-zero term $\mathbf{X} = a_\alpha X^\alpha$ that appears in f and

$$h = f - \frac{\mathbf{X}}{lt(g)} g.$$

For $f = 0$ we define $\mathrm{lp}(0) = \mathrm{lc}(0) = \mathrm{lt}(0) = 0$.

Definition 0.2.6. *Let f, h and $f_1, \dots, f_s$ be polynomials in $k[X_1, \dots, X_n]$, with $f_i \neq 0$ $(0 \leqslant i \leqslant s)$, and let $F = \{f_1, \dots, f_s\}$. We say that f* reduces *to h modulo F, denoted*

$$f \xrightarrow{F}_{+} h,$$

if and only if there exists a sequence of indices $i_1, i_2, \dots, i_t \in \{1, \dots, s\}$ and a sequence of polynomials $h_1, \dots, h_{t-1} \in k[X_1, \dots, X_n]$ such that

$$f \xrightarrow{f_{i_1}} h_1 \xrightarrow{f_{i_2}} h_2 \xrightarrow{f_{i_3}} \cdots \xrightarrow{f_{i_{t-1}}} h_{t-1} \xrightarrow{f_{i_t}} h.$$

Definition 0.2.7. *A polynomial r is called reduced with respect to a set of non-zero polynomials $F = \{f_1, \dots, f_s\}$ if $r = 0$ or no power product that appears in r is divisible by any one of the $lp(f_i)$, $i = 1, \dots, s$. In other words, r cannot be reduced modulo F.*

Definition 0.2.8. *If $f \xrightarrow{F}_+ r$ and r is reduced with respect to F, then we call r a remainder for f with respect to F.*

The reduction process can be viewed as a generalization of the long division in one variable. The Multivariable Division Algorithm (see box on next page) is defined as a sequence of reductions. The following theorem shows that the Multivariable Division Algorithm actually works:

Theorem 0.2.9. *Given a set of non-zero polynomials $F = \{f_1, \ldots, f_s\}$ and $f \in k[X_1, \ldots, X_n]$, the Multivariable Division Algorithm produces polynomials $u_1, \ldots, u_s, r \in k[X_1, \ldots, X_n]$ such that*

$$f = u_1 f_1 + \cdots + u_s f_s + r,$$

with r reduced with respect to F and

$$\mathrm{lp}(f) = \max\{\max_{1 \leqslant i \leqslant s}(\mathrm{lp}(u_i)\mathrm{lp}(f_i)), \mathrm{lp}(r)\}.$$

Proof. First note that the algorithm terminates: at each step, the leading term of the polynomial h is subtracted off until it cannot longer be done. This gives a sequence $h_1, h_2, \ldots$ of the h's in the algorithm, where h_{i+1} is obtained from h_i by subtracting off $\mathrm{lt}(h_i)$ and possibly some smaller terms: $h_{i+1} = h_i - (\mathrm{lt}(h_i) + \text{lower terms})$. This is because h_{i+1} is computed from h_i by subtracting off $\frac{\mathrm{lt}(h_i)}{\mathrm{lt}(f_i)} f_i = \mathrm{lt}(h_i) + \text{lower terms}$ (in case some $\mathrm{lp}(f_j)$ divides

$\mathrm{lp}(h_i)$) or subtracting off $\mathrm{lt}(h_i)$ (in case no $\mathrm{lp}(f_j)$ divides $\mathrm{lp}(h_i)$). Thus, for all i, $\mathrm{lp}(h_{i+1}) < \mathrm{lp}(h_i)$. Since the term order is a well-ordering, the list of the h_i's must stop.

MULTIVARIABLE DIVISION ALGORITHM

INPUT: $f, f_1, \ldots, f_s \in k[X_1, \ldots, X_n]$ with $f_i \neq 0$ $(1 \leqslant i \leqslant s)$

OUTPUT: $u_1, \ldots, u_s, r$ such that $f = u_1 f_1 + \cdots + u_s f_s + r$
and r is reduced with respect to $\{f_1, \cdots, f_s\}$ and
$\max\{\mathrm{lp}(u_1)\mathrm{lp}(f_1), \ldots, \mathrm{lp}(u_s)\mathrm{lp}(f_s), \mathrm{lp}(r)\} = \mathrm{lp}(f)$.

INITIALIZATION: $u_1 := 0, u_2 := 0, \ldots, u_s := 0, r := 0, h := f$

WHILE $h \neq 0$ **DO**

IF there exist i such that $\mathrm{lp}(f_i)$ divides $\mathrm{lp}(h)$ **THEN**

choose i least such that $\mathrm{lp}(f_i)$ divides $\mathrm{lp}(h)$

$$u_i := u_i + \frac{\mathrm{lt}(h)}{\mathrm{lt}(f_i)}$$

$$h := h - \frac{\mathrm{lt}(h)}{\mathrm{lt}(f_i)} f_i$$

ELSE

$$r := r + \mathrm{lt}(h)$$

$$h := h - \mathrm{lt}(h)$$

For the second part observe that since $f = h$ at the beginning of the algorithm and we are subtracting off the leading power product of h at every stage we must have at every stage $\mathrm{lp}(h)\leqslant\mathrm{lp}(f)$. Now, for each i, u_i is obtained by adding terms $\frac{\mathrm{lt}(h)}{\mathrm{lt}(f_i)}$ where $\frac{\mathrm{lt}(h)}{\mathrm{lt}(f_i)}f_i$ cancels the leading term of h. It follows that $\mathrm{lp}(u_i)\mathrm{lp}(f_i)\leqslant\mathrm{lp}(f)$. Since r is obtained by adding in terms $\mathrm{lt}(h)$, we have that $\mathrm{lp}(r)\leqslant\mathrm{lp}(f)$, as well.□

Examples and exercises that illustrate the algorithm can be found in [AL].

0.3 The Generic Extension for $\mathbf{GL_n(C)}$.

As we pointed out at the beginning of this introduction the problem that motivates our construction is the following inverse problem:

Given a differential field F with algebraically closed field of constants C, find a Picard-Vessiot extension of F whose differential Galois group is $GL_n(C)$.

Let Y_{ij} be differential inderminates. A *generic Picard-Vessiot extension* for $\mathrm{GL}_n(C)$ is a Picard-Vessiot extension of the *generic field* $C\langle Y_{11},\cdots,Y_{nn}\rangle$, with differential Galois group $\mathrm{GL}_n(C)$ and the property that any Picard-Vessiot extension of F with differential Galois group $\mathrm{GL}_n(C)$ is obtained from it by first specializing the Y_{ij} to $f_{ij}\in F$ and then extending scalars to F over C.

The terminology "*generic*" dates back to the works of E. Noether in Galois theory of algebraic equations [N]. Following her approach, L. Goldman [G] introduces the notion of *generic equation with group G* for a homogeneous differential equation of order n. Briefly, given an algebraic subgroup G of $\mathrm{GL}_n(C)$, Goldman sought an nth order homogeneous linear differential polynomial $L(t,y) \in C\langle t_1,\dots,t_n\rangle\{y\}$ where $(t_1,\dots,t_n)$ is a family of differential indeterminates over C such that there exists a fundamental system of zeros $(y_1,\dots,y_n)$ of $L(t,y)$ with the following properties:

(1) $C\langle t_1,\dots,t_n,y_1,\dots,y_n\rangle$ is a Picard-Vessiot extension of $C\langle t_1,\dots,t_n\rangle$ with group of automorphisms G.

(2) For any specialization $(t_1,\dots,t_n) \mapsto (\bar{t}_1,\dots,\bar{t}_n)$ over C which can be extended to a specialization $(t_1,\dots,t_n,y_1,\dots,y_n) \mapsto (\bar{t}_1,\dots,\bar{t}_n,\bar{y}_1,\dots,\bar{y}_n)$ with $C\langle\bar{t}_1,\dots,\bar{t}_n,\bar{y}_1,\dots,\bar{y}_n\rangle$ a Picard-Vessiot extension of $C\langle\bar{t}_1,\dots,\bar{t}_n\rangle$, the differential Galois group H of $C\langle\bar{t}_1,\dots,\bar{t}_n,\bar{y}_1,\dots,\bar{y}_n\rangle$ over $C\langle\bar{t}_1,\dots,\bar{t}_n\rangle$ is a subgroup of G.

(3) If $\mathfrak{F}$ is a differential field with field of constants C and if $\mathfrak{F}\langle\lambda_1,\dots,\lambda_n\rangle$ is a Picard-Vessiot extension of $\mathfrak{F}$ with group $H \subseteq G$, where $(\lambda_1,\dots,\lambda_n)$ is a fundamental system of zeros of a homogeneous linear differential polynomial $L(y) \in \mathfrak{F}\{y\}$ of order n, there exists a specialization $(t_1,\dots,t_n) \mapsto (\bar{t}_1,\dots,\bar{t}_n)$ over C such that $\bar{t}_i \in \mathfrak{F}$ $(i = 1,\dots,n)$ and $L(\bar{t},y) = L(y)$.

Let $t_1,\dots,t_r,\bar{t}_1,\dots,\bar{t}_r$ be elements of some differential field extension of $\mathfrak{F}$. In Goldman's terminology, $(t_1,\dots,t_r) \mapsto (\bar{t}_1,\dots,\bar{t}_r)$ is a specialization over $\mathfrak{F}$ if for any differential polynomial $F(y_1, \ldots, y_r) \in \mathfrak{F}\{y_1, \ldots, y_r\}$ such that $F(t_1, \ldots, t_r) = 0$ we have $F(\bar{t}_1, \ldots, \bar{t}_r) = 0$.

For the general linear group, Goldman constructs a generic equation as follows:

Let

$$P_i(y_1, \ldots, y_n) = \frac{W_i(y_1, \ldots, y_n)}{W_0(y_1, \ldots, y_n)} \qquad (i = 1, \ldots, n),$$

where

$$W_i = (-1)^i \begin{vmatrix} y_1^{(0)} & \cdots & y_n^{(0)} \\ \vdots & & \vdots \\ y_1^{(n-i-1)} & \cdots & y_n^{(n-i-1)} \\ y_1^{(n-i+1)} & \cdots & y_n^{(n-i+1)} \\ \vdots & & \vdots \\ y_1^{(n)} & \cdots & y_n^{(n)} \end{vmatrix}$$

and the y_i are differential indeterminates. Let $t_i = P_i(y_1, \ldots, y_n)$ then

$$L(t, y) = y^{(n)} + P_1(y_1, \ldots, y_n)y^{(n-1)} + \cdots + P_n(y_1, \ldots, y_n)y^{(0)}$$

is a generic equation with group $\mathrm{GL}_n(C)$. Goldman's construction is equivalent to Example 5.26, p. 72 in [M].

By the second property of Goldman's generic equation, if a specialization $(t_1, \ldots, t_r) \mapsto (\bar{t}_1, \ldots, \bar{t}_r)$ is found such that $C\langle \bar{t}_1, \ldots, \bar{t}_n, \bar{y}_1, \ldots, \bar{y}_n \rangle$ is a Picard-Vessiot extension of $C\langle \bar{t}_1, \ldots, \bar{t}_n \rangle$ the differential Galois group H of $C\langle \bar{t}_1, \ldots, \bar{t}_n, \bar{y}_1, \ldots, \bar{y}_n \rangle$ over $C\langle \bar{t}_1, \ldots, \bar{t}_n \rangle$ is only a *subgroup* of $\mathrm{GL}_n(C)$. That is, Goldman's construction does not guarantee that after a suitable specialization of the t_i an equation over $\mathfrak{F}$ is obtained with group $\mathrm{GL}_n(C)$. The reason why the differential Galois group may become smaller is that when the specialization of the coefficients of the generic equation is carried out there is no control on the changes in the derivation of the differential field extension associated to that equation and as a result the derivation in the extension associated to the new equation with the coefficients specialized need not to be G-equivariant. Also, this approach neglects other possible derivations that produce the same equation.

Our generic extension for $\mathrm{GL}_n(C)$ is the quotient field of the coordinate ring of $\mathrm{GL}_n(C)$ with differential indeterminates Y_{ij} adjoined together with a "*generic*" derivation. The derivation is defined in such a way that after specializing the indeterminates to elements of the base field we still obtain a $\mathrm{GL}_n(C)$-equivariant derivation and therefore the full $\mathrm{GL}_n(C)$ is preserved as the differential Galois group of the new (specialized) extension.

We are now ready to introduce our results.

Let F be a differential field with algebraically closed field of constants C. Let Y_{ij}, $i,j = 1,\ldots,n$, be differential indeterminates over F. We will consider the differential ring $R = F\{Y_{ij}\}[X_{11},\ldots,X_{nn}]$ with derivation given by $D(X_{ij}) = \sum_{\ell=1}^{n} Y_{i\ell}X_{\ell j}$. This derivation on R extends in a natural way to its quotient field $Q = F\langle Y_{ij}\rangle(X_{ij})$. We have the following (Theorem 2.1.1

in Chapter 2):

Theorem 0.3.1. *In the above notation, $Q = F\langle Y_{ij}\rangle(X_{ij})$ is a Picard-Vessiot extension of $F\langle Y_{ij}\rangle$ with differential Galois group $GL_n(C)$. Moreover, Q is a generic Picard-Vessiot extension in the following sense:*

Let $E \supseteq F$ be any Picard-Vessiot extension of F with group $GL_n(C)$, then E is isomorphic to $F(X_{ij})$, the function field of $GL_n(C)$ extended to F, as a $GL_n(C)$-module and as an F-module and there are $f_{ij} \in F$ such that the derivation on E is $D_E = \sum_{\ell=1}^{n} f_{i\ell}X_{\ell j}$ and the Picard-Vessiot extension $F(X_{ij}) \supseteq F$ is obtained from the Picard-Vessiot extension $C\langle f_{ij}\rangle(X_{ij}) \supseteq C\langle f_{ij}\rangle$ by extension of scalars to F over C.

To show that $Q \supseteq F\langle Y_{ij}\rangle$ is Picard-Vessiot we use the following characterization (Proposition 3.9 in [M]):

Proposition 0.3.2. *Let $E \supset F$ be an extension of the differential field F. Assume that the field of constants C of F is algebraically closed. Then E is a Picard-Vessiot extension of F if and only if:*

(1) *$E = F\langle V\rangle$, where $V \subset E$ is a finite-dimensional vector space over C;*

(2) *There is a group G of differential automorphisms of E with $G(V) \subseteq V$ and $E^G = F$;*

(3) *$E \supseteq F$ has no new constants.*

In particular, if the above conditions hold and if $y_1, \ldots, y_n$ is a C-basis of

V, then E is a Picard-Vessiot extension of F for

$$L = \frac{w(Y, y_1, \ldots, y_n)}{w(y_1, \ldots, y_n)}$$

and $L^{-1}(0) = V$.

As usual, $w(f_1, \ldots, f_s)$ denotes the *Wronskian determinant* of $f_1, \ldots, f_s$, defined by

$$w(f_1, \ldots, f_s) = \begin{vmatrix} f_1 & f_2 \ldots & f_s \\ f_1^{(1)} & f_2^{(1)} \ldots & f_s^{(1)} \\ \vdots & \vdots & \vdots \\ f_1^{(s-1)} & f_2^{(s-1)} \ldots & f_s^{(s-1)} \end{vmatrix}.$$

We point out that, as in classical analysis, the non-vanishing of the Wronskian of $f_1, \ldots, f_s \in E$ is a necessary and sufficient condition for their linear independence over the constants of E.

In our case, (1) and (2) follow from the construction, with $G = \mathrm{GL}_n(C)$. However, condition (3) is highly non-trivial and the proof requires a long and delicate argument involving the Multivariable Division Algorithm presented in Section 0.2.

We briefly describe the steps in the proof of (3):

First, we introduce the notion of *Darboux polynomial* (Definition 1.2.1):

Definition 0.3.3. *Let D be a derivation on the polynomial ring $A = k[Y_1, \ldots, Y_s]$. A polyomial $p \in A$ is called a* Darboux polynomial *if there is a polynomial $q \in A$ such that $D(p) = qp$. That is, p divides $D(p)$.*

The definition of Darboux polynomials is motivated by the following fact (in Proposition 1.2.3, Chapter 1, we prove the statement for $A = R$):

Proposition 0.3.4. *Let $p_1, p_2 \in A = k[Y_1, \dots, Y_s]$ with $p_2 \neq 0$ and p_1 and p_2 relatively prime. Suppose that $D(\frac{p_1}{p_2}) = 0$, then p_1 and p_2 are Darboux polynomials. Moreover, if $q_1, q_2 \in A$ are such that $D(p_1) = q_1 p_1$ and $D(p_2) = q_2 p_2$, then $q_1 = q_2$.*

In other words, a new constant in $Q = F\langle Y_{ij}\rangle(X_{ij})$ has to be the quotient of two relatively prime Darboux polynomials in R, which satisfy the Darboux condition for the same polynomial q.

Then we prove that there are no new constants in Q by showing that there are no relatively prime Darboux polynomials in R. We prove this last statement by showing that the only Darboux polynomials in R are, up to a coefficient in F, powers of $\det[X_{ij}]$. This is the content of Theorem 0.3.5 (Theorem 1.3.16 in Chapter 1).

Henceforth we regard the set of power products in $F[X_{ij}]$ ordered with the degree reverse lexicographical order (Definition 0.2.4).

Theorem 0.3.5. *Let p and q be polynomials in $R = F\{Y_{ij}\}[X_{ij}]$ that satisfy the Darboux condition $D(p) = qp$. Then there is $a \in \mathbf{N}$ and $\ell \in F$ such that*

$p = \ell \det[X_{ij}]^a$ *and*

$$q = \frac{\ell'}{\ell} + a\sum_{i=1}^{n} Y_{ii}.$$

The first step to prove this theorem is to show that there are no non-trivial Darboux polynomials in $F\{Y_{ij}\}$ (Proposition 1.3.1, Chapter 1):

Proposition 0.3.6. *If $h(Y) \in F\{Y_{ij}\}$ satisfies $h'(Y) = g(Y)h(Y)$, for some $g(Y) \in F\{Y_{ij}\}$ then $h(Y) \in F$.*

Next, we want to find an expression for q in terms of p. For this purpose, for any power product X^α that occurs in p we compute its coefficient in $D(p)$ (Proposition 1.3.5, Chapter 1):

Proposition 0.3.7. *Let $p = \sum_\alpha p_\alpha(Y)X^\alpha \in F\{Y_{ij}\}[X_{ij}]$. Then for any α with $p_\alpha(Y) \neq 0$, the coefficient of X^α in $D(p)$ is*

$$p'_\alpha(Y) + p_\alpha(Y)\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_{ij}Y_{ii} + \sum_{i=1}^{n}\sum_{j=1}^{n}(\alpha_{ij}+1)\sum_{\ell\neq i} p_{\alpha_{ij,\ell}}(Y)Y_{i\ell},$$

where $\alpha_{ij,\ell}$ is the exponent vector of the power product

$$X^{\alpha_{ij,\ell}} = \begin{cases} X_{11}^{\alpha_{11}}\cdots X_{ij}^{\alpha_{ij}+1}\cdots X_{\ell j}^{\alpha_{\ell j}-1}\cdots X_{nn}^{\alpha_{nn}} & \text{if } i < \ell \\ X_{11}^{\alpha_{11}}\cdots X_{\ell j}^{\alpha_{\ell j}-1}\cdots X_{ij}^{\alpha_{ij}+1}\cdots X_{nn}^{\alpha_{nn}} & \text{if } \ell > i \end{cases}.$$

Then, we prove that if a polynomial $q \in F\{Y_{ij}\}[X_{ij}]$ satisfies $D(p) = qp$ for some Darboux polynomial $p \in F\{Y_{ij}\}[X_{ij}]$ then $q \in F\{Y_{ij}\}$, that is, the X_{ij} do not appear in q. Using this fact and Propositions 0.3.6 and 0.3.7 we show that if $p \in F\{Y_{ij}\}[X_{ij}]$ is a Darboux polynomial then it does not contain the Y_{ij}. This last fact is the content of the following (Proposition 1.3.6 in Chapter 1):

Proposition 0.3.8. *Let $p \in F\{Y_{ij}\}[X_{ij}]$ and suppose that $D(p) = qp$, for some $q \in F\{Y_{ij}\}$. Then $p \in F[X_{11}, \ldots, X_{nn}]$.*

So, in summary, if $p, q \in F\{Y_{ij}\}[X_{ij}]$ satisfy $D(p) = qp$ then $p \in F[X_{ij}]$ and $q \in F\{Y_{ij}\}$.

We then show that such a q must be a linear polynomial in the Y_{ij}. We give an expression for q in terms of the coefficients and the exponents of the power products of p and from it deduce that p is a homogeneous polynomial. As a Corollary to this fact we obtain the following expression for q (Corollary 1.3.9 in Chapter 1):

Corollary 0.3.9. *Let $p \in F[X_{11}, \ldots, X_{nn}]$ and suppose that $D(p) = qp$, for some $q \in F\{Y_{ij}\}$. Let $lp(p) = X^\alpha$, $\alpha = (\alpha_{11}, \ldots, \alpha_{nn})$ be the leading power product of p, and let $\ell \in F$ be its coefficient. Then*

$$q = \frac{\ell'}{\ell} + \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{ij} Y_{ii}.$$

With this expression for q we proceed to show that p is, up to a coefficient in F, a power of $\det[X_{ij}]$. This is done by showing that p is the sum of all the power products in the X_{ij} in which for each i_0, $1 \leqslant i_0 \leqslant n$, there is one and only one X_{ij} in the power product with first subindex $i = i_0$ and for each j_0, $1 \leqslant j_0 \leqslant n$, there is one and only one X_{ij} with second subindex $j = j_0$. This characterizes the polynomial $\det[X_{ij}]$ and its powers. For this, the first step is to show that the leading power product of p is one of such power products. We have (Lemma 1.3.11 in Chapter 1):

Lemma 0.3.10. *Let $p \in F[X_{ij}]$ be such that $D(p) = qp, q \in F\{Y_{ij}\}$. Let $lp(p) = X^\alpha$, $\alpha = (\alpha_{11}, \ldots, \alpha_{nn})$, be its leading power product. Then $\alpha_{ij} = 0$ for $j + i \neq n + 1$ and $\alpha_{i,n-i+1} > 0$, $i = 1, \ldots, n$, that is, $lp(p) = X_{1n}^{\alpha_{1n}} X_{2,n-1}^{\alpha_{2,n-1}} \cdots X_{n1}^{\alpha_{n1}}$. Moreover, $\alpha_{i,n-i+1} = \alpha_{1n}$, for $i > 1$. Thus, if we let $a = \alpha_{1n}$, then*

$$lp(p) = (X_{1n} X_{2,n-1} \cdots X_{n1})^a.$$

With this, we can further simplify the expression for q (Corollary 1.3.12 in Chapter 1):

Corollary 0.3.11. *Let $p \in F[X_{11}, \ldots, X_{nn}]$ and suppose that $D(p) = qp$, $q \in F\{Y_{ij}\}$. Let a be such that*

$$l_p(p) = (X_{1n} X_{2,n-1} \cdots X_{n1})^a$$

and let $lc(p) = \ell \in F$ be the leading coefficient of p. Then

$$q = \frac{\ell'}{\ell} + a \sum_{i=1}^{n} Y_{ii}.$$

The following (Corollary 1.3.13 in Chapter 1) is an obvious consequence of the above facts:

Corollary 0.3.12. *Let p be as in Corollary 0.3.11. Then p is homogeneous of degree na.*$\square$

Now, note that in particular Lemma 0.3.10 implies that p is not reduced modulo $\det[X_{ij}]$ since the leading power product of p is $\mathrm{lp}(p) = (X_{1n}X_{2,n-1}\cdots X_{n1})^a = \mathrm{lp}(\det[X_{ij}])^a$. This last statement, along with the Multivariable Division Algorithm, allows us to conclude the proof of Theorem 0.3.5.

As a consequence we cannot obtain the Darboux polynomials p_1, p_2 as in Proposition 0.3.4. Condition (3) in Proposition 0.3.2 then follows.

The fact that E is isomorphic to $F(X_{ij})$, as a $\mathrm{GL}_n(C)$-module and as an F-module follows from Kolchin's Theorem on the structure of Picard-Vessiot extensions, proved in [M]. The statement about the form of the derivation D_E on E is a consequence of the fact that the partial derivatives with respect to the X_{ij} are a basis for the Lie algebra of $\mathrm{GL}_n(C)$. The generic property of our extension is hence established. This ends the proof of Theorem 0.3.1.

If we consider a different derivation on $R = F\{Y_{ij}\}[X_{ij}]$ we obtain the same result regarding Darboux polynomials in R (Theorem 1.3.17 in Chapter 1):

Theorem 0.3.13. *Let F be a differential field with algebraically closed field of constants C. Let $D_{E(ij)} \in Lie(GL_n(C))$ be the derivation given by mul-*

tiplication by the matrix $E(ij)$, with 1 in position (i,j) and zeros elsewhere. Let $\mathcal{D}_{st}$, $s,t=1,\ldots,n$, be any other basis of $Lie(GL_n(C))$. Define a derivation on the ring $R=F\{Y_{ij}\}[X_{11},\ldots,X_{nn}]$ by $\mathcal{D}=\sum Y_{st}\mathcal{D}_{st}$. Let p and q be polynomials in R that satisfy the Darboux condition $\mathcal{D}(p)=qp$. Then there is a positive integer a and a scalar $\ell\in F$, such that $p=\ell\det[X_{ij}]^a$ and $q=\frac{\ell'}{\ell}+a\sum_{i=1}^n Y_{ii}$.

In summary, we started with a differential field F whose field of constants C is algebraically closed and wanted to find the Picard-Vessiot extensions of F with Galois group $\mathrm{GL}_n(C)$. With these data we constructed a Picard-Vessiot extension for the "*generic*" field $F\langle Y_{ij}\rangle$, with Galois group $\mathrm{GL}_n(C)$, and showed that the desired Picard-Vessiot extensions, if they exist, are homomorphic images of the generic one, where the homomorphism is given by $Y_{ij}\mapsto f_{ij}$. In order to solve the inverse problem we need to show that such a homomorphism can be found.

Our next set of results provide necessary and sufficient conditions for the homomorphism $Y_{ij}\mapsto f_{ij}$ to yield a Picard-Vessiot extension via the procedure described in Theorem 0.3.1. The field F is assumed to be of finite transcendence degree over its field of constants C, so that it can be viewed as $F=C(t_1,\ldots,t_m)[z_1,\ldots,z_k]$ where the t_i are algebraically independent over C and the z_i are algebraic over $C(t_1,\ldots,t_m)$. The derivation on F extends to $F(X_{ij})$ in such a way that $D(f \otimes X_{ij}) = D(f) \otimes X_{ij} + f \otimes \sum_{\ell=1}^{n} f_{i\ell} X_{\ell j}$ on $F \otimes C[X_{ij}]$. First, we have (Theorem 2.2.2 in Chapter 2):

Theorem 0.3.14. *Let C be an algebraically closed field with zero derivation. Let $F = C(t_1, \ldots, t_m)[z_1, \ldots, z_k]$, where the t_i and z_i are as above. Let $\mathcal{C}$ be the field of constants of $F(X_{ij})$, where the X_{ij} are algebraically independent over C. Then $\mathcal{C} = C$ if and only if the X_{ij} are algebraically independent over $\mathcal{C}$.*

To *check* whether the X_{ij} are algebraically independent over $\mathcal{C}$, we let $X(k)$ denote the set of power products in the X_{ij} of total degree k or less. We then have that the X_{ij} are algebraically independent over $\mathcal{C}$ if and only if, for each k, the set $X(k)$ is linearly independent over $\mathcal{C}$. Fix any ordering in $X(k)$ and let W_k denote the Wronskian of $X(k)$ relative to that ordering (of course a Wronskian computed using any other ordering of the elements of $X(k)$ will only differ from this one by a sign). The condition of Theorem 0.3.14 is equivalent to requesting that W_k is non-zero for all k.

We need to find a specialization (homomorphism) $F\{Y_{ij}\}[X_{ij}] \to F[X_{ij}]$, with $Y_{ij} \mapsto f_{ij}$, so that the differential extension $F(X_{ij}) \supset F$, with the derivation induced by the specialization, has no new constants. Letting W_k be the Wronskian of $X(k)$ in $F\{Y_{ij}\}[X_{ij}]$ we have (Theorem 2.2.3 in Chapter 2):

Theorem 0.3.15. *Let F be as in Theorem 0.3.14. There is a specialization $Y_{ij} \mapsto f_{ij}$ with $F(X_{ij}) \supset F$ a no new constant extension if and only if there are $f_{ij} \in F$ such that all the W_k map to non-zero elements.*

In Chapter 4 of this dissertation, we offer an example of how to compute a new constant if the condition of Theorem 0.3.15 fails.

Unfortunately, Theorem 0.3.15 gives an infinite set of conditions and so far it remains an open question whether there are f_{ij} such that the images of all the W_k are simultaneously non-zero.

Finally, in our proofs of Theorems 0.3.14 and 0.3.15 the fact that the group under consideration is $\mathrm{GL}_n(C)$ does not matter. The only property of this group that is used is that it is a connected algebraic group and thus we generalize the results for such groups.

Chapter 1. Preliminary results

In this chapter we set up the notation and establish the first results that will be used in our construction of the generic extension for $\mathrm{GL}_n(C)$.

1.1 The differential field $\mathbf{F\langle Y_{ij}\rangle(X_{11},\ldots,X_{nn})}$

Let F be a differential field with algebraically closed field of constants C.

Let Y_{ij}, $i,j = 1,\ldots,n$, be differential indeterminates. For convenience, denote $Y_{ij}^{(k+1)}$ by $Y_{ij,k}$ for $k \geqslant 0$. So, in this setting, $D(Y_{ij,k}) = Y_{ij,k+1}$ for $k \geqslant 0$. As usual, $Y_{ij,0} = Y_{ij}$. For $k = 0$ we will omit the k-subindex in our notation and simply write Y_{ij}. We will consider the differential ring $R = F\{Y_{ij}\}[X_{11},\ldots,X_{nn}]$ with derivation $D(X_{ij}) = \sum_{\ell=1}^{n} Y_{i\ell}X_{\ell j}$. This derivation extends to the quotient field $Q = F\langle Y_{ij}\rangle(X_{11},\ldots,X_{nn})$ of R in the obvious way: if $p,q \in R$, $q \neq 0$, then

$$D\left(\frac{p}{q}\right) = \frac{D(p)q - pD(q)}{q^2}.$$

Henceforth we will regard the field Q as a differential field for the above derivation.

It is convenient to use single letters to abbreviate the subscripts of the coefficients of the power products in a polynomial. Thus, an expression like

$$a_\alpha X_{11}^{\alpha_{11}} \cdots X_{kl}^{\alpha_{kl}}$$

means "$a_{\alpha_{11}\ldots\alpha_{kl}} X_{11}^{\alpha_{11}} \cdots X_{kl}^{\alpha_{kl}}$". We will also write X^α to denote the power product "$X_{11}^{\alpha_{11}} \cdots X_{1n}^{\alpha_{1n}} \cdots X_{n1}^{\alpha_{n1}} \cdots X_{nn}^{\alpha_{nn}}$", as before. So a typical term of a polynomial in the X_{ij} will be denoted by "$a_\alpha X^\alpha$". We will also use this abbreviated notation for differential polynomials in the Y_{ij} or in both the X_{ij} and the Y_{ij}.

The ring $F[X_{ij}]$ is assumed to be ordered with the degree reverse lexicographical term order (degrevlex, Definition 0.2). That is, the set $\mathbb{T}^{n^2} = \{X^\beta \,|\, X = (X_{ij}), \beta = (\beta_{ij}) \in \mathbb{N}^{n^2}\}$ of the power products in the X_{ij} is ordered by $X_{11} > \cdots > X_{1n} > \cdots > X_{n1} > \cdots > X_{nn}$, and

$$X^\alpha < X^\beta \text{ if and only if } \begin{cases} \displaystyle\sum_{j=1}^{n}\sum_{i=1}^{n} \alpha_{ij} < \sum_{j=1}^{n}\sum_{i=1}^{n} \beta_{ij}, \text{ or} \\ \displaystyle\sum_{j=1}^{n}\sum_{i=1}^{n} \alpha_{ij} = \sum_{j=1}^{n}\sum_{i=1}^{n} \beta_{ij}, \text{ and the first coordina-} \\ \text{tes } \alpha_{ij} \text{ and } \beta_{ij} \text{ from the right which are different} \\ \text{satisfy } \alpha_{ij} > \beta_{ij}. \end{cases}$$

1.2 Darboux polynomials and constants in $\mathbf{F}\langle \mathbf{Y_{ij}}\rangle(\mathbf{X_{11}}, \ldots, \mathbf{X_{nn}})$

Our main goal through the rest of this chapter is to show that the field of constants $\mathcal{C}$ of $Q = F\langle Y_{ij}\rangle(X_{11}, \ldots, X_{nn})$ coincides with the field of constants C of F. In this section we will show that this can be reduced to showing that the only *Darboux polynomials* in R are, up to a coefficient in F, powers of $\det[X_{ij}]$.

Definition 1.2.1. *Let D be a derivation on the polynomial ring $A = k[Y_1, \ldots, Y_s]$. A polyomial $p \in A$ is called a* Darboux polynomial *if there is a polynomial $q \in A$ such that $D(p) = qp$. That is, p divides $D(p)$.*

Darboux polynomials correspond to generators of principal differential ideals in A. In Chapter I of J.A. Weil's Ph.D. Thesis [W95], the author discusses constants and Darboux polynomials in Differential Algebra. In particular, the following property is given (Lemma 12, Section 4.1):

Lemma 1.2.2. *If $p_1, p_2 \in A$ are Darboux polynomials for the derivation D then p_1p_2 is also a Darboux polynomial for D. Conversely, if p is a Darboux polynomial for D, then all its irreducible factors are Darboux polynomials as well.*

Proof. If $D(p_1) = q_1p_1$ and $D(p_2) = q_2p_2$, with $q_1, q_2 \in A$, then

$$D(p_1p_2) = D(p_1)p_2 + p_1D(p_2) = (q_1 + q_2)p_1p_2.$$

Conversely, suppose that $D(p) = qp$, with $q \in A$ and $p = p_1^m p_2$, where p_1 is irreducible and p_1 and p_2 are relatively prime. If we replace $D(p_1^m p_2)$ with its equivalent expression $qp_1^m p_2$, we get

$$qp_1^m p_2 = (m-1)p_1^{m-1}D(p_1)p_2 + p_1^m D(p_2).$$

Since p_1^m divides both terms on the right hand side of the above expression and p_1 and p_2 are relatively prime, p_1 has to divide $D(p_1)$. For the same reason, we have that p_2 must divide $D(p_2)$. Applying the same argument, one gets by induction that all the prime factors of p need to be Darboux polynomials as well.□

The following proposition gives a necessary condition for the existence of new constants in the extension $Q \supset F$ in terms of Darboux polynomials:

Proposition 1.2.3. *Let $p_1, p_2 \in R = F\{Y_{ij}\}[X_{11}, \ldots, X_{nn}]$ with $p_2 \neq 0$ and p_1 and p_2 relatively prime. Suppose that $D(\frac{p_1}{p_2}) = 0$, then p_1 and p_2 are Darboux polynomials. Moreover, if $q_1, q_2 \in R$ are such that $D(p_1) = q_1 p_1$ and $D(p_2) = q_2 p_2$, then $q_1 = q_2$.*

Proof. We have

$$D\left(\frac{p_1}{p_2}\right) = \frac{D(p_1)p_2 - p_1 D(p_2)}{p_2^2} = 0,$$

thus $D(p_1)p_2 - p_1 D(p_2) = 0$, that is

$$D(p_1)p_2 = p_1 D(p_2). \tag{1}$$

Since p_1 and p_2 are relatively prime, the last equation implies that p_1 divides $D(p_1)$ and p_2 divides $D(p_2)$.

Now, let $q_1, q_2 \in R$ be such that $D(p_1) = q_1 p_1$ and $D(p_2) = q_2 p_2$, respectively. Then it follows from (1) that

$$q_1 p_1 p_2 = q_2 p_1 p_2.$$

Hence, $q_1 = q_2$.□

1.3 Finding the Darboux polynomials of R

Let $p = \ell \det[X_{ij}]^a \in R$, with $\ell \in F$. We have

$$D(p) = \ell' \det[X_{ij}]^a + \ell(a \det[X_{ij}]^{a-1} D(\det[X_{ij}])).$$

But,

$$D(\det[X_{ij}]) = \mathrm{trace}[Y_{ij}] \det[X_{ij}].$$

Thus,

$$\begin{aligned} D(p) &= \ell' \det[X_{ij}]^a + \ell(a \det[X_{ij}]^{a-1} \mathrm{trace}[Y_{ij}] \det[X_{ij}] \\ &= (\frac{\ell'}{\ell} + a\,\mathrm{trace}[Y_{ij}])\ell \det[X_{ij}]^a. \end{aligned}$$

That is, $p = \ell \det[X_{ij}]^a$ with $\ell \in F$ is a Darboux polynomial in R. Our next step is to show that the only Darboux polynomials in R are exactly those of this form. Hence, no two Darboux polynomials in R are relatively prime. Then, by Proposition 1.2.3, there are no new constants in Q.

First we show that there are no non-trivial Darboux polynomials in the ring of differential polynomials $F\{Y_{ij}\}$. In order to simplify the notation we will use $h'(Y)$ instead of $D(h(Y))$, for a polynomial $h(Y) \in F\{Y_{ij}\}$.

Proposition 1.3.1. *If $h(Y) \in F\{Y_{ij}\}$ satisfies $h'(Y) = g(Y)h(Y)$, for some $g(Y) \in F\{Y_{ij}\}$ then $h(Y) \in F$.*

Proof. The subindices of $Y_{ij,k} = D^k(Y_{ij})$ are ordered triples $\{ij, k\}$ of natural numbers. Order them with the lexicographical order, namely, $\{i_1j_1, k_1\} > \{i_2j_2, k_2\}$ if and only if the first coordinates s_1 and s_2 from the left, for $s = i, j, k$ above, which are different satisfy $s_1 > s_2$.

Let $\{mn, t\}$ be the largest subindex such that $Y_{mn,t}$ occurs in h.

Let $h(Y) = \sum_\alpha a_\alpha Y_{11}^{\alpha_{11}} \cdots Y_{mn,t}^{\alpha_{mn,t}}$. Then

$$\begin{aligned} h'(Y) &= \sum_\alpha a'_\alpha Y_{11}^{\alpha_{11}} \cdots Y_{mn,t}^{\alpha_{mn,t}} + \sum_\alpha a_\alpha \alpha_{11} Y_{11}^{\alpha_{11}-1} Y_{11,1}^{\alpha_{11,1}+1} \cdots Y_{mn,t}^{\alpha_{mn,t}} \\ &\qquad + \cdots + \sum_\alpha a_\alpha \alpha_{mn,t} Y_{11}^{\alpha_{11}} \cdots Y_{mn,t}^{\alpha_{mn,t}-1} Y_{mn,t+1} \\ &= h_1(Y_{11}, \cdots, Y_{mn,t}) + \Big(\sum_\alpha a_\alpha \alpha_{mn,t} Y_{11}^{\alpha_{11}} \cdots Y_{mn,t}^{\alpha_{mn,t}-1}\Big) Y_{mn,t+1} \\ &= g(Y)h(Y). \end{aligned}$$

For $Y_{mn,t+1} = D(Y_{mn,t})$ we have $\{mn, t+1\} > \{mn, t\}$, thus it does not occur $h(Y)$ by the choice of $\{mn, t\}$. Also, it does not occur in $h_1(Y_{11}, \cdots, Y_{mn,t})$. Thus the above equation implies that $Y_{mn,t+1}$ must occur in $g(Y)$. Let $g_{t+1}(Y)$ be its coefficient in $g(Y)$ and put $h_2(Y) = \sum_\alpha a_\alpha \alpha_{mn,t} Y_{11}^{\alpha_{11}} \cdots Y_{mn,t}^{\alpha_{mn,t}-1}$, then we have

$$h(Y) g_{t+1}(Y) Y_{mn,t+1} = h_2(Y) Y_{mn,t+1}$$

or

$$h(Y)g_{t+1}(Y) = h_2(Y).$$

But the total degree of $h_2(Y)$ is strictly less than the total degree of $h(Y)$. This forces $h(Y) \in F$.□

Remarks.

1.3.2. If $X^\alpha = X_{11}^{\alpha_{11}} \cdots X_{1n}^{\alpha_{1n}} \cdots X_{n1}^{\alpha_{n1}} \cdots X_{nn}^{\alpha_{nn}}$ is an X-power product then

$$\begin{aligned} D(X^\alpha) = &\Big(\sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_{ij} Y_{ii}\Big) X^\alpha \\ &+ \sum_{i=1}^{n}\sum_{j=1}^{n}\Big(\sum_{\ell > i} \alpha_{ij} Y_{i\ell} X_{11}^{\alpha_{11}} \cdots X_{ij}^{\alpha_{ij}-1} \cdots X_{\ell j}^{\alpha_{\ell j}+1} \cdots X_{nn}^{\alpha_{nn}} \\ &\qquad + \sum_{\ell < i} \alpha_{ij} Y_{i\ell} X_{11}^{\alpha_{11}} \cdots X_{\ell j}^{\alpha_{\ell j}+1} \cdots X_{ij}^{\alpha_{ij}-1} \cdots X_{nn}^{\alpha_{nn}}\Big). \end{aligned}$$

1.3.3. Given a power product X^α, we want to know which are the X-power products in whose derivatives appear a Y-multiple of X^α. That is, if

$$X^\alpha = X_{11}^{\alpha_{11}} \cdots X_{1n}^{\alpha_{1n}} \cdots X_{n1}^{\alpha_{n1}} \cdots X_{nn}^{\alpha_{nn}}$$

then we need to find the power products X^β such that $D(X^\beta)$ contains a power product of the form

$$Y_{rt} X_{11}^{\alpha_{11}} \cdots X_{1n}^{\alpha_{1n}} \cdots X_{n1}^{\alpha_{n1}} \cdots X_{nn}^{\alpha_{nn}}.$$

From the expression obtained in Remark 1.3.2 for $D(X^\alpha)$, we see that the only such power products are the ones of the form

$$X^{\alpha_{rs,t}} = \begin{cases} X_{11}^{\alpha_{11}} \cdots X_{rs}^{\alpha_{rs}+1} \cdots X_{ts}^{\alpha_{ts}-1} \cdots X_{nn}^{\alpha_{nn}} & \text{if } r < t \\ X_{11}^{\alpha_{11}} \cdots X_{ts}^{\alpha_{ts}-1} \cdots X_{rs}^{\alpha_{rs}+1} \cdots X_{nn}^{\alpha_{nn}} & \text{if } r > t \end{cases}$$

for $r, s = 1, \ldots, n$ and $t \neq r$, and X^α itself.

1.3.4. Let $p \in R$. Since $D(X_{ij}) = \sum_{\ell=1}^{n} Y_{i\ell} X_{\ell j}$, then the total degree of p with respect to the X_{ij} does not change after differentiation. Therefore, if $D(p) = qp$ then we must have $q \in F\{Y_{ij}\}$.

Proposition 1.3.5. *Let $p = \sum_\alpha p_\alpha(Y) X^\alpha \in R$. Then for any α with $p_\alpha(Y) \neq 0$, the coefficient of X^α in $D(p)$ is*

$$p'_\alpha(Y) + p_\alpha(Y) \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{ij} Y_{ii} + \sum_{i=1}^{n} \sum_{j=1}^{n} (\alpha_{ij} + 1) \sum_{\ell \neq i} p_{\alpha_{ij,\ell}}(Y) Y_{i\ell},$$

where $\alpha_{ij,\ell}$ is the exponent vector of the power product

$$X^{\alpha_{ij,\ell}} = \begin{cases} X_{11}^{\alpha_{11}} \cdots X_{ij}^{\alpha_{ij}+1} \cdots X_{\ell j}^{\alpha_{\ell j}-1} \cdots X_{nn}^{\alpha_{nn}} & \text{if } i < \ell \\ X_{11}^{\alpha_{11}} \cdots X_{\ell j}^{\alpha_{\ell j}-1} \cdots X_{ij}^{\alpha_{ij}+1} \cdots X_{nn}^{\alpha_{nn}} & \text{if } \ell > i \end{cases}$$

as in Remark 1.3.3.

Proof. This is a direct consequence of Remarks 1.3.2 and 1.3.3.□

Proposition 1.3.6. *Let $p \in R$ and suppose that $D(p) = qp$, for some $q \in F\{Y_{ij}\}$. Then $p \in F[X_{11}, \ldots, X_{nn}]$.*

Proof. Let $p = \sum p_\alpha(Y)X^\alpha$. Then

$$\begin{aligned} D(p) &= \sum p'_\alpha(Y)X^\alpha + p_\alpha(Y)D(X^\alpha) \\ &= qp \\ &= \sum q(Y)p_\alpha(Y)X^\alpha. \end{aligned}$$

By Proposition 1.3.5, for each α such that the coefficient of the X-power product X^α in p is $p_\alpha(Y) \neq 0$ the corresponding coefficient of X^α in $D(p)$ is

$$\begin{aligned} D(p)_\alpha = p'_\alpha(Y) + p_\alpha(Y) \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_{ij}Y_{ii} \\ + \sum_{i=1}^{n}\sum_{j=1}^{n}(\alpha_{ij}+1)\sum_{\ell\neq i} p_{\alpha_{ij,\ell}}(Y)Y_{i\ell}. \end{aligned}$$

Since $D(p) = qp$, it must be $D(p)_\alpha = q(Y)p_\alpha(Y)$. That is, for each α, we have

$$\begin{aligned} q(Y)p_\alpha(Y) = p'_\alpha(Y) + p_\alpha(Y) \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_{ij}Y_{ii} \\ + \sum_{i=1}^{n}\sum_{j=1}^{n}(\alpha_{ij}+1)\sum_{\ell\neq i} p_{\alpha_{ij,\ell}}(Y)Y_{i\ell}. \end{aligned}$$

Therefore, for every α, the coefficient $p_\alpha(Y)$ of X^α in p divides

$$p'_\alpha(Y) + \sum_{i=1}^{n}\sum_{j=1}^{n}(\alpha_{ij}+1)\sum_{\ell\neq i} p_{\alpha_{ij,\ell}}(Y)Y_{i\ell}.$$

Thus, for each α, there is $u_\alpha(Y)$ such that

$$p_\alpha(Y)u_\alpha(Y) = p'_\alpha(Y) + \sum_{i=1}^{n}\sum_{j=1}^{n}(\alpha_{ij}+1)\sum_{\ell\neq i} p_{\alpha_{ij,\ell}}(Y)Y_{i\ell}.$$

As in the proof of Proposition 1.3.1, order the triples of natural numbers with the lexicographical ordering. Let $\{mn, t\}$ be the largest subindex such that $Y_{mn,t}$ occurs in p. We have $D(Y_{mn,t}) = Y_{mn,t+1}$ and $\{mn, t+1\} > \{mn, t\}$.

Now, for each α such that $Y_{mn,t}$ occurs in $p_\alpha(Y)$ we have that $Y_{mn,t+1}$ will occur in $p'_\alpha(Y)$ but not in $p_\alpha(Y)$ or in

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(\alpha_{ij}+1)\sum_{\ell\neq i} p_{\alpha_{ij,\ell}}(Y)Y_{i\ell}$$

by the choice of $\{mn, t\}$. Therefore, it must occur in $p_\alpha(Y)u_\alpha(Y)$. Let

$$p_\alpha(Y) = \sum a_\beta Y_{11}^{\beta_{11}} Y_{12}^{\beta_{12}} \cdots Y_{mn,t}^{\beta_{mn,t}}.$$

Then

$$\begin{aligned} p'_\alpha(Y) = \sum a'_\beta Y_{11}^{\beta_{11}} \cdots Y_{mn,t}^{\beta_{mn,t}} \\ &+ \sum a_\beta \beta_{11} Y_{11}^{\beta_{11}-1} Y_{11,1}^{\beta_{11,1}+1} \cdots Y_{mn,t}^{\beta_{mn,t}} + \dots \\ &+ \sum a_\beta \beta_{mn,t} Y_{11}^{\beta_{11}} \cdots Y_{mn,t}^{\beta_{mn,t}-1} Y_{mn,t+1}. \end{aligned}$$

So $Y_{mn,t+1}$ occurs in $p'_\alpha(Y)$ only in

$$\begin{aligned} \sum a_\beta \beta_{mn,t} Y_{11}^{\beta_{11}} \cdots Y_{mn,t}^{\beta_{mn,t}-1} Y_{mn,t+1} \\ &= \Big(\sum a_\beta \beta_{mn,t} Y_{11}^{\beta_{11}} \cdots Y_{mn,t}^{\beta_{mn,t}-1}\Big) Y_{mn,t+1} \\ &= v(Y) Y_{mn,t+1}. \end{aligned}$$

Since $Y_{mn,t+1}$ occurs in $p_\alpha(Y)u_\alpha(Y)$ and not in $p_\alpha(Y)$ it must occur in $u_\alpha(Y)$. Let $u_{\alpha,t+1}(Y)$ be the coefficient of $Y_{mn,t+1}$ in $u_\alpha(Y)$. Then it has to be

$$p_\alpha(Y)u_{\alpha,t+1}(Y)Y_{mn,t+1} = v(Y)Y_{mn,t+1}.$$

The above equation implies that $p_\alpha(Y)$ divides $v(Y)$. But this is impossible since the total degree of $v(Y)$ is strictly less than the total degree of $p_\alpha(Y)$. This contradiction yields the result.□

Lemma 1.3.7. *Let $p \in F[X_{11}, \ldots, X_{nn}]$ and suppose that there is $q \in F\{Y_{ij}\}$ such that $D(p) = qp$. Let X^β be any power product in p. Then the coefficient of Y_{ii} in q is $\sum_{j=1}^{n} \beta_{ij}$, for $i = 1, \ldots, n$. In particular, the sums $\sum_{j=1}^{n} \beta_{ij}$, for $i = 1, \ldots, n$, are independent of the choice of X^β.*

Proof. We have $p = \sum a_\beta X^\beta$, with $a_\beta \in F$.

Thus,

$$\begin{aligned} D(p) &= \sum a'_\beta X^\beta + a_\beta D(X^\beta) \\ &= qp \\ &= \sum q(Y)a_\beta X^\beta. \end{aligned}$$

By Proposition 1.3.5, the coefficient of X^β in $D(p)$ is

$$a'_\beta + a_\beta \sum_{i=1}^{n}\sum_{j=1}^{n} \beta_{ij}Y_{ii} + \sum_{i=1}^{n}\sum_{j=1}^{n}(\beta_{ij}+1)\sum_{\ell \neq i} a_{\beta_{ij,\ell}} Y_{i\ell}.$$

Hence, it must be

$$q(Y)a_\beta = a'_\beta + a_\beta\Big(\sum_{i=1}^{n}\sum_{j=1}^{n} \beta_{ij}Y_{ii} + \sum_{i=1}^{n}\sum_{j=1}^{n}(\beta_{ij}+1)\sum_{\ell \neq i} a_{\beta_{ij,\ell}} Y_{i\ell}\Big).$$

From this,

$$q(Y) = \frac{a'_\beta}{a_\beta} + \sum_{i=1}^{n}\sum_{j=1}^{n} \beta_{ij} Y_{ii} + \sum_{i=1}^{n}\sum_{j=1}^{n} (\beta_{ij}+1) \sum_{\ell \neq i} \frac{a_{\beta_{ij,\ell}}}{a_\beta} Y_{i\ell}.$$

The coefficient of Y_{ii} in the above expression is $\sum_{j=1}^{n} \beta_{ij}$, for $i = 1, \ldots, n$. Since this expression for q is valid for any index β, the "in particular" part follows immediately.□

Corollary 1.3.8. *Let p be as in Lemma 1.3.7. Let X^α be the leading power product of p. Let X^β be any power product with non-zero coefficient in p. Then $\sum_{j=1}^{n} \beta_{ij} = \sum_{j=1}^{n} \alpha_{ij}$, for $i = 1, \ldots, n$. Thus p is homogeneous of degree $\sum_{j=1}^{n}\sum_{i=1}^{n} \alpha_{ij}$.*

Corollary 1.3.9. *Let $p \in F[X_{11}, \ldots, X_{nn}]$ and suppose that $D(p) = qp$, for some $q \in F\{Y_{ij}\}$. Let X^α be the leading power product of p, and let $\ell \in F$ be its coefficient. Then*

$$q = \frac{\ell'}{\ell} + \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_{ij} Y_{ii}.$$

Proof. By Proposition 1.3.5 and since $D(p) = qp$, the coefficient of X^α in $D(p)$ is

$$\ell q = \ell' + \ell \Big(\sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_{ij} Y_{ii} + \sum_{i=1}^{n}\sum_{j=1}^{n} (\alpha_{ij}+1) \sum_{\ell \neq i} p_{\alpha_{ij,\ell}} Y_{i\ell} \Big). \qquad (1)$$

The $p_{\alpha_{ij},k}$ are the coefficients of the power products in p, different from X^α, whose derivatives involve it. By Remark 1.3.3, these other X-power products are those of the form

$$X^{\alpha_{rs,t}} = \begin{cases} X_{11}^{\alpha_{11}} \cdots X_{rs}^{\alpha_{rs}+1} \cdots X_{ts}^{\alpha_{ts}-1} \cdots X_{nn}^{\alpha_{nn}} & \text{if } r < t \\ X_{11}^{\alpha_{11}} \cdots X_{ts}^{\alpha_{ts}-1} \cdots X_{rs}^{\alpha_{rs}+1} \cdots X_{nn}^{\alpha_{nn}} & \text{if } r > t \end{cases},$$

all of which violate Corollary 1.3.8 for $i = r$ and $i = t$. Therefore it must be $p_{\alpha_{ij},k} = 0$, for all $i, j = 1, \ldots, n$; $k \neq i$. But now, substituting back in (1), we see that then

$$\ell q = \ell' + \ell \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{ij} Y_{ii}.$$

Hence,

$$q = \frac{\ell'}{\ell} + \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{ij} Y_{ii}. \square$$

In order to show that the Darboux polynomial p as in the previous results has the desired form we will first show that it is not reduced with respect to $\det[X_{ij}]$. For that we show that the leading power product of p is a power of the leading power product of $\det[X_{ij}]$. The first step is the following:

Lemma 1.3.10. *Let $p \in F[X_{ij}]$ be such that $D(p) = qp, q \in F\{Y_{ij}\}$. Let $lp(p) = X^\alpha$ be its leading power product. Then $\alpha_{ij} = 0$ for $j \neq n-i+1$ and $\alpha_{i,n-i+1} > 0$, $i = 1, \ldots, n$. That is, $lp(p) = X_{1n}^{\alpha_{1n}} X_{2,n-1}^{\alpha_{2,n-1}} \cdots X_{n1}^{\alpha_{n1}}$.*

Proof. To prove that $\alpha_{ij} = 0$ for $j \neq n-i+1$ we first show that $\alpha_{ij} = 0$ for $j > n-k+1$, $i \geqslant k$, $k = 2, \ldots, n$. Indeed, for $k = 2$ we have $j > n-1$, so $j = n$ and

$$D(\mathrm{lp}(p)) = \alpha_{nn} \sum_{k=1}^{n-1} Y_{nk} X_{11}^{\alpha_{11}} \cdots X_{kn}^{\alpha_{kn}+1} \cdots X_{nn}^{\alpha_{nn}-1} + \ldots$$

Since q has no Y_{ij} with $i \neq j$, each term in $D(\mathrm{lp}(p))$ containing such a Y_{ij} must be cancelled. In particular we need to cancel the terms containing

$$Y_{nj} X_{11}^{\alpha_{11}} \cdots X_{jn}^{\alpha_{jn}+1} \cdots X_{nn}^{\alpha_{nn}-1}$$

for $j = 1, \ldots, n-1$ above. For that we can only use the derivative of power products of the form

$X^{\alpha_{n\ell,j}} =$

$$X_{11}^{\alpha_{11}} \cdots X_{j\ell}^{\alpha_{j\ell}-1} \cdots X_{jn}^{\alpha_{jn}+1} \cdots X_{n1}^{\alpha_{n1}} \cdots X_{n\ell}^{\alpha_{n\ell}+1} \cdots X_{nn}^{\alpha_{nn}-1}, \qquad \ell < n,$$

all of which are strictly greater than $\mathrm{lp}(p)$. Thus they may not occur in p. As a consequence, it has to be $\alpha_{nn} = 0$. Now let $k > 2$ be such that $\alpha_{in} = 0$ for $i \geqslant k$. Then

$\mathrm{lp}(p) =$

$$X_{11}^{\alpha_{11}} \cdots X_{k-1,n}^{\alpha_{k-1,n}} \cdots X_{k,n-1}^{\alpha_{k,n-1}} X_{k+1,1}^{\alpha_{k+1,1}} \cdots X_{k+1,n-1}^{\alpha_{k+1,n-1}} \cdots X_{n,n-1}^{\alpha_{n,n-1}}$$

and

$$\begin{aligned} D(\mathrm{lp}(p)) = \alpha_{k-1,n}\Big(& \sum_{i<k-1} Y_{k-1,i} X_{11}^{\alpha_{11}} \cdots X_{in}^{\alpha_{in}+1} \cdots X_{k-1,n}^{\alpha_{k-1,n}-1} \cdots X_{n,n-1}^{\alpha_{n,n-1}} \\ & + \sum_{i>k-1} Y_{k-1,i} X_{11}^{\alpha_{11}} \cdots X_{k-1,n}^{\alpha_{k-1,n}-1} \cdots X_{in}^{\alpha_{in}+1} \cdots X_{n,n-1}^{\alpha_{n,n-1}}\Big) \\ & + \ldots \end{aligned}$$

Likewise, we need to cancel all the terms in $D(\mathrm{lp}(p))$ involving $Y_{k-1,i}$, for $i \neq k-1$. In particular, we need to cancel the terms of $D(\mathrm{lp}(p))$ involving

$$Y_{k-1,i}X_{11}^{\alpha_{11}} \cdots X_{in}^{\alpha_{in}+1} \cdots X_{k-1,n}^{\alpha_{k-1,n}-1} \cdots X_{n,n-1}^{\alpha_{n,n-1}},$$

with $i < k-1$. In order to do it, we can only use power products of the form $X^{\alpha_{k-1,\ell,i}} =$

$$X_{11}^{\alpha_{11}} \cdots X_{i\ell}^{\alpha_{i\ell}-1} \cdots X_{in}^{\alpha_{in}+1} \cdots X_{k-1,\ell}^{\alpha_{k-1,\ell}+1} \cdots X_{k-1,n}^{\alpha_{k-1,n}-1} \cdots X_{n,n-1}^{\alpha_{n,n-1}},$$

for $i < k-1$.

But all of them are strictly greater than $\mathrm{lp}(p)$ and therefore they cannot occur in p. Thus, it has to be $\alpha_{k-1,n} = 0$. Since this argument is valid for any $k > 2$, it follows that $\alpha_{kn} = 0$, for $k = 2, \ldots, n$. This makes the statement that $\alpha_{ij} = 0$ for $j > n-k+1$, $i \geqslant k$, true for $k = 2$.

Now assume that k is such that $\alpha_{ij} = 0$ for $j > n-k+1, i \geqslant k$. So $\mathrm{lp}(p) =$

$$X_{11}^{\alpha_{11}} \cdots X_{1n}^{\alpha_{1n}} \cdots X_{k,n-k+1}^{\alpha_{k,n-k+1}} X_{k+1,1}^{\alpha_{k+1,1}} \cdots X_{k+1,n-k+1}^{\alpha_{k+1,n-k+1}} \cdots X_{n,n-k+1}^{\alpha_{n,n-k+1}}$$

and, for all $i > k$,

$$\alpha_{i,n-k+1}Y_{ik}X_{11}^{\alpha_{11}} \cdots X_{1n}^{\alpha_{1n}} \cdots X_{k,n-k+1}^{\alpha_{k,n-k+1}+1} \cdots X_{i,n-k+1}^{\alpha_{i,n-k+1}-1} \cdots X_{n,n-k+1}^{\alpha_{n,n-k+1}}$$

occurs in $D(\mathrm{lp}(p))$. Thus we need to cancel it. For that we can only use the derivative of power products of the form $X^{\alpha_{ij,k}} =$

$$X_{11}^{\alpha_{11}} \cdots X_{kj}^{\alpha_{kj}-1} \cdots X_{k,n-k+1}^{\alpha_{k,n-k+1}+1} \cdots X_{ij}^{\alpha_{ij}+1} \cdots X_{i,n-k+1}^{\alpha_{i,n-k+1}-1} \cdots X_{n,n-k+1}^{\alpha_{n,n-k+1}}$$

with $j < n-k+1$ since $\alpha_{kj} = 0$ for all $j > n-k+1$ by hypothesis. But all such power products are strictly greater than $\mathrm{lp}(p)$ and therefore they cannot occur in p. This forces $\alpha_{i,n-k+1} = 0$ for $i > k$. We can repeat this process until $k = n$ and get that $\alpha_{ij} = 0$ for all $j > n-k+1$, $i \geqslant k$, $k = 2, \ldots, n$, that is,

$$\mathrm{lp}(p) = X_{11}^{\alpha_{11}} \cdots X_{1n}^{\alpha_{1n}} X_{21}^{\alpha_{21}} \cdots X_{2,n-1}^{\alpha_{2,n-1}} X_{31}^{\alpha_{31}} \cdots X_{n-1,2}^{\alpha_{n-1,2}} X_{n1}^{\alpha_{n1}}.$$

Now we show that $\alpha_{ij} = 0$ for $j < n-k+1$, $k = 1, \ldots, n-1$, $i \leqslant k$. The process is analogous to what we just did. First we show that $\alpha_{i1} = 0$ for $i < n$. Indeed, for each i we have for $\ell > i$ that

$$\alpha_{i1} Y_{i\ell} X_{11}^{\alpha_{11}} \cdots X_{i1}^{\alpha_{i1}-1} \cdots X_{\ell 1}^{\alpha_{\ell 1}+1} \cdots X_{n1}^{\alpha_{n1}}$$

occurs in $D(\mathrm{lp}(p))$. So, in order to cancel it, we need to use the derivative of power products of the form

$$X^{\alpha_{ij,\ell}} =$$

$$X_{11}^{\alpha_{11}} \cdots X_{i1}^{\alpha_{i1}-1} \cdots X_{ij}^{\alpha_{ij}+1} \cdots X_{\ell 1}^{\alpha_{\ell 1}+1} \cdots X_{\ell j}^{\alpha_{\ell j}-1} \cdots X_{n1}^{\alpha_{n1}}$$

with $j > 1$, all of which are strictly greater than $\mathrm{lp}(p)$ if $\ell < n$, and for $\ell = n$ we cannot simply have one of those since $\alpha_{nj} = 0$ for $j \neq 1$. Thus such power products cannot occur in p and it has to be $\alpha_{i1} = 0$ for $i < n$.

Let $k \leqslant n-1$ be such that $\alpha_{ij} = 0$ for $j < n-k+1$, $i \leqslant k$. We have

$$\mathrm{lp}(p) = X_{1,n-k+1}^{\alpha_{1,n-k+1}} \cdots X_{1n}^{\alpha_{1n}} \cdots X_{k,n-k+1}^{\alpha_{k,n-k+1}} \cdots X_{n1}^{\alpha_{n1}}$$

and for all $i < k$, $\ell > i$, we have that

$$\alpha_{i,n-k+1} Y_{i\ell} X_{1,n-k+1}^{\alpha_{1,n-k+1}} \cdots X_{i,n-k+1}^{\alpha_{i,n-k+1}-1} \cdots X_{\ell,n-k+1}^{\alpha_{\ell,n-k+1}+1} \cdots X_{n1}^{\alpha_{n1}}$$

occurs in $D(\mathrm{lp}(p))$ and in order to cancel it we only have the derivative of power products of the form
$X^{\alpha_{ij,\ell}} =$

$$X_{1,n-k+1}^{\alpha_{1,n-k+1}} \cdots X_{i,n-k+1}^{\alpha_{i,n-k+1}-1} \cdots X_{ij}^{\alpha_{ij}+1} \cdots X_{\ell,n-k+1}^{\alpha_{\ell,n-k+1}+1} \cdots X_{\ell j}^{\alpha_{\ell j}-1} \cdots X_{n1}^{\alpha_{n1}}$$

with $j > n-k+1$ since $\alpha_{ij} = 0$ for $i \leqslant k$, $j < n-k+1$.

For $\ell < k$, all this power products are strictly greater than $\mathrm{lp}(p)$ and therefore they cannot occur in p. For $\ell \geqslant k$ we cannot simply have such power products since for $\ell \geqslant k$, $\alpha_{\ell j} = 0$ if $j > n-k+1$. Thus it has to be $\alpha_{i,n-k+1} = 0$ for $i \leqslant k-1$.

We can repeat this process until $k = n-1$ and get that $\alpha_{ij} = 0$, $j < n-k+1$, $i \leqslant k$, $k = 1, \ldots, n-1$. This completes the proof for the first part of the Lemma.

To prove that $\alpha_{i,n-i+1} \neq 0$, for all $i = 1, \ldots, n$, suppose that there is i such that $\alpha_{i,n-i+1} = 0$ and let $j \neq i$ be such that $\alpha_{j,n-j+1} \neq 0$ then $D(\mathrm{lp}(p))$ will contain

$$\alpha_{j,n-j+1} Y_{ji} X_{1n}^{\alpha_{1n}} \cdots X_{j,n-j+1}^{\alpha_{j,n-j+1}-1} \cdots X_{i,n-j+1} \cdots X_{n1}^{\alpha_{n1}} + \ldots \qquad \text{if } i > j$$

or

$$\alpha_{j,n-j+1} Y_{ji} X_{1n}^{\alpha_{1n}} \cdots X_{i,n-j+1} \cdots X_{j,n-j+1}^{\alpha_{j,n-j+1}-1} \cdots X_{n1}^{\alpha_{n1}} + \ldots \qquad \text{if } i < j.$$

As noted above, since q does not contain any Y_{ij} with $i \neq j$, we need to cancel the terms in $D(p)$ involving either of the above. But that is impossible since $\alpha_{ij} = 0$ for all j and by Corollary 1.3.8 all the power products

$$X_{11}^{\beta_{11}} \cdots X_{ij}^{\beta_{ij}} \cdots X_{nn}^{\beta_{nn}}$$

in p must have $\beta_{ij} = 0$ for $j = 1, \ldots, n$. In particular, we cannot have in p the power products of the form $X^{\alpha_{j,n-j+1,i}}$ of Remark 1.3.3.□

Next we show that the α_{st} in the exponents of the X_{st} in $\mathrm{lp}(p)$ are all equal:

Lemma 1.3.11. *Let $p \in F[X_{ij}]$ be such that $D(p) = qp$, $q \in F\{Y_{ij}\}$. Let*

$$lp(p) = X_{1n}^{\alpha_{1n}} X_{2,n-1}^{\alpha_{2,n-1}} \cdots X_{n1}^{\alpha_{n1}}$$

be its leading power product. Then $\alpha_{i,n-i+1} = \alpha_{1n}$, for $i > 1$, that is, if $a = \alpha_{1n}$, then

$$lp(p) = (X_{1n} X_{2,n-1} \cdots X_{n1})^a.$$

Proof. Let ℓ be the coefficient of $\mathrm{lp}(p)$ in p. We have

$$\begin{aligned}
D(\ell X_{1n}^{\alpha_{1n}} X_{2,n-1}^{\alpha_{2,n-1}} \cdots X_{n1}^{\alpha_{n1}}) &= \Big(\sum_{i=1}^{n} \alpha_{i,n-i+1} \ell Y_{ii}\Big) X_{1n}^{\alpha_{1n}} X_{2,n-1}^{\alpha_{2,n-1}} \cdots X_{n1}^{\alpha_{n1}} \\
&\quad + \alpha_{1n} \ell \sum_{k \neq 1} Y_{1k} X_{1n}^{\alpha_{1n}-1} \cdots X_{k,n-k+1}^{\alpha_{k,n-k+1}} \cdots X_{kn} \cdots X_{n1}^{\alpha_{n1}} \\
+\ell \sum_{1<i} \alpha_{i,n-i+1} \sum_{k>i} & Y_{ik} X_{1n}^{\alpha_{1n}} \cdots X_{i,n-i+1}^{\alpha_{i,n-i+1}-1} \cdots X_{k,n-k+1}^{\alpha_{k,n-k+1}} \cdots X_{k,n-i+1} \cdots X_{n1}^{\alpha_{n1}} \\
+\ell \sum_{1<i} \alpha_{i,n-i+1} \sum_{k>i} & Y_{ik} X_{1n}^{\alpha_{1n}} \cdots X_{k,n-i+1} \cdots X_{k,n-k+1}^{\alpha_{k,n-k+1}} \cdots X_{i,n-i+1}^{\alpha_{i,n-i+1}-1} \cdots X_{n1}^{\alpha_{n1}} \\
&\qquad + \ell' X_{1n}^{\alpha_{1n}} X_{2,n-1}^{\alpha_{2,n-1}} \cdots X_{n1}^{\alpha_{n1}}.
\end{aligned}$$

In order to cancel

$$\alpha_{1n}\ell Y_{1k}X_{1n}^{\alpha_{1n}-1}\cdots X_{k,n-k+1}^{\alpha_{k,n-k+1}}\cdots X_{kn}\cdots X_{n1}^{\alpha_{n1}}, \qquad k\neq 1,$$

above, we can only use the derivative of the power product

$$X^{\alpha_{1,n-k+1,k}} = X_{1,n-k+1}\cdots X_{1n}^{\alpha_{1n}-1}\cdots X_{k,n-k+1}^{\alpha_{k,n-k+1}-1}\cdots X_{kn}\cdots X_{n1}^{\alpha_{n1}},$$

since for $j\neq n-k+1$ we have $\alpha_{kj}=0$.

Let $a_{\alpha_{1,n-k+1,k}}$ be the coefficient of $X^{\alpha_{1,n-k+1,k}}$ in p. Then

$$a_{\alpha_{1,n-k+1,k}} = -\ell\alpha_{1n} \tag{2}$$

On the other hand, in order to cancel

$$\alpha_{k,n-k+1}\ell Y_{k1}X_{1,n-k+1}\cdots X_{1n}^{\alpha_{1n}}\cdots X_{k,n-k+1}^{\alpha_{k,n-k+1}-1}\cdots X_{n1}^{\alpha_{n1}}, \qquad k\neq 1$$

above, the only power product that we can use is, again,

$$\begin{aligned} X^{\alpha_{kn,1}} &= X_{1,n-k+1}\cdots X_{1n}^{\alpha_{1n}-1}\cdots X_{k,n-k+1}^{\alpha_{k,n-k+1}-1}\cdots X_{kn}\cdots X_{n1}^{\alpha_{n1}} \\ &= X^{\alpha_{1,n-k+1,k}}, \end{aligned}$$

since $\alpha_{1j}=0$ for $j\neq n$. Thus it must be

$$a_{\alpha_{1,n-k+1,k}} = -\ell\alpha_{k,n-k+1} \tag{3}$$

as well.

From (2) and (3) it follows that, for $k\neq 1$, $\alpha_{1n}=\alpha_{k,n-k+1}$.$\square$

As a consequence of the above results we obtain the following expression for q:

Corollary 1.3.12. *Let $p \in F[X_{11}, \ldots, X_{nn}]$ and suppose that $D(p) = qp$, $q \in F\{Y_{ij}\}$. Let a be such that*

$$lp(p) = (X_{1n}X_{2,n-1} \cdots X_{n1})^a$$

and let $\ell \in F$ be the coefficient of $lp(p)$. Then

$$q = \frac{\ell'}{\ell} + a\sum_{i=1}^{n} Y_{ii}.$$

Proof. This is a consequence of Corollary 1.3.9 and Lemma 1.3.11.

It is worth noting the following:

Corollary 1.3.13. *Let p be as in Corollary 1.3.12. Then p is homogeneous of degree na.*□

Proof. This is a consequence of Corollary 1.3.8 and Lemma 1.3.11.□

Lemma 1.3.11 implies that p is not reduced with respect to $\det[X_{ij}]$. Since this is a key point in the proof of our main result we restate the Lemma in order to record this fact as the following theorem:

Theorem 1.3.14. *Let $p \in F[X_{ij}]$ be such that $D(p) = qp$, $q \in F\{Y_{ij}\}$. Then*

$$\mathrm{lp}(p) = (X_{1n}X_{2,n-1} \cdots X_{n1})^a = \mathrm{lp}(\det[X_{ij}])^a.$$

Thus p is not reduced with respect to $\det[X_{ij}]$.

Proof. This is just a restatement of Lemma 1.3.11.

Remark 1.3.15. Let $p_1, p_2 \in F[X_{ij}]$ be two polynomials such that $\mathrm{lp}(p_1) = X^\alpha = \mathrm{lp}(p_2)$. Then we can write $p_1 = fp_2 + r$ where $f \in F$ and r is reduced with respect to p_2. Indeed, since $\mathrm{lp}(p_1) = \mathrm{lp}(p_2)$, we have that $\mathrm{lp}(p_2)$ divides $\mathrm{lp}(p_1)$. So p_1 is not reduced with respect to p_2 and, using the Multivariable Division Algorithm (Chapter 0, Section 0.2), we can write $p_1 = fp_2 + r$, with $0 \neq f \in F[X_{ij}]$, r reduced with respect to p_2 and $\mathrm{lp}(p_1) = \mathrm{lp}(f)\mathrm{lp}(p_2)$. The last equation implies that $\mathrm{lp}(f) = 1$. Hence, $f \in F$.

We are ready to prove our main result on the form of Darboux polynomials in R:

Theorem 1.3.16. *Let $p \in F[X_{ij}]$ and $q \in F\{Y_{ij}\}$ be polynomials in R that satisfy the Darboux condition $D(p) = qp$. Then there is $a \in \mathbb{N}$ and $\ell \in F$ such that*

$$p = \ell \det[X_{ij}]^a$$

and

$$q = \frac{\ell'}{\ell} + a \sum_{i=1}^{n} Y_{ii}.$$

Proof. Let $q_1 = \sum_{i=1}^{n} aY_{ii}$, so that,

$$D(\det[X_{ij}]^a) = q_1 \det[X_{ij}]^a = (q - \frac{\ell'}{\ell}) \det[X_{ij}]^a.$$

By Remark 1.3.15 we can write $p = \ell \det[X_{ij}]^a + r$, with r reduced with respect to $\det[X_{ij}]^a$. Now,

$$\begin{aligned}
D(p) &= D(\ell \det[X_{ij}]^a) + D(r) \\
&= \ell' \det[X_{ij}]^a + \ell(q - \frac{\ell'}{\ell}) \det[X_{ij}]^a + D(r) \\
&= \ell' \det[X_{ij}]^a + q\ell \det[X_{ij}]^a - \ell' \det[X_{ij}]^a + D(r) \\
&= q\ell \det[X_{ij}]^a + D(r).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
D(p) &= qp \\
&= q\ell \det[X_{ij}]^a + qr.
\end{aligned}$$

Therefore, it has to be $D(r) = qr$. But r is reduced with respect to $\det[X_{ij}]^a$. It follows, by Theorem 1.3.14, that $r = 0$. The statement about the form of q is just the content of Corollary 1.3.12.□

Next we show that the result in Theorem 1.3.16 does not depend on the $\mathrm{GL}_n(C)$-equivariant derivation chosen on R.

Theorem 1.3.17. *Let F be a differential field with algebraically closed field of constants C. Let $D_{E(ij)} \in Lie(GL_n(C))$ be the a derivation given by multiplication by the matrix $E(ij)$, with 1 in position (i,j) and zeros elsewhere. Let $\mathcal{D}_{st}$, $s, t = 1, \ldots, n$, be any other basis of $Lie(GL_n(C))$. Define a derivation in the ring $R = F\{Y_{ij}\}[X_{11}, \ldots, X_{nn}]$ by $\mathcal{D} = \sum Y_{st}\mathcal{D}_{st}$. Let p and q be polynomials in R that satisfy the Darboux condition $\mathcal{D}(p) = qp$. Then* *there is a positive integer a and a scalar $\ell \in F$, such that $p = \ell \det[X_{ij}]^a$ and $q = \frac{\ell'}{\ell} + a\sum_{i=1}^{n} Y_{ii}$.*

Proof. Since the $D_{E(ij)}$ are a basis of $\mathrm{Lie}(\mathrm{GL}_n(C))$ we have

$$\mathcal{D}_{st} = \sum c_{st,ij} D_{E(ij)},$$

with $c_{st,ij} \in C$. Thus,

$$\begin{aligned}
\mathcal{D} &= \sum_{s,t} Y_{st}\mathcal{D}_{st} \\
&= \sum_{s,t} Y_{st} \sum_{i,j} c_{st,ij} D_{E(ij)} \\
&= \sum_{i,j}\sum_{s,t} c_{st,ij} Y_{st} D_{E(ij)} \\
&= \sum_{i,j} Z_{ij} D_{E(ij)},
\end{aligned}$$

where $Z_{ij} = \sum_{s,t} c_{st,ij} Y_{st}$. Now, $[c_{st,ij}]$ is a matrix of change of basis so it is invertible. Also the $c_{st,ij}$ are contants for D, thus the map $Z_{ij,k} \to Y_{ij,k}$ is a differential bijection. In other words, the differential rings

$$R = F\{Y_{ij}\}[X_{11}, \ldots, X_{nn}], D$$

and

$$R' = F\{Z_{ij}\}[X_{11}, \ldots, X_{nn}], \mathcal{D}$$

are isomorphic and therefore we can apply Theorem 1.3.16 to R'. $\square$

Chapter 2. A Generic Picard-Vessiot Extension for $\mathbf{GL_n(C)}$ and some specialization properties

In this chapter we construct a Picard-Vessiot extension with differential Galois group $\mathrm{GL}_n(C)$ for the differential field $F\langle Y_{ij}\rangle$. The inverse differential Galois problem for $\mathrm{GL}_n(C)$ consists in determining, given a differential base field F with algebraically closed field of constants C, what differential extensions of F are Picard-Vessiot extensions with differential Galois group $\mathrm{GL}_n(C)$. We prove that if such a Picard-Vessiot extension of F exists then it has to be a homomorphic image of the one for $F\langle Y_{ij}\rangle$. We also provide existence conditions for such a homomorphism.

2.1 Generic Picard-Vessiot extension for $\mathbf{GL_n(C)}$ and the inverse problem

Let F be a differential field with algebraically closed field of constants C. Consider the previously defined differential ring $R = F\{Y_{ij}\}[X_{11}, \ldots, X_{nn}]$ with derivation $D(X_{ij}) = \sum_{\ell=1}^{n} Y_{i\ell}X_{\ell j}$ where the Y_{ij} are differential indeterminates over F. Let $Q = F\langle Y_{ij}\rangle(X_{11}, \ldots, X_{nn})$ be the differential field resulting from extending the derivation on R to its quotient field. We have:

Theorem 2.1.1. *$Q \supset F\langle Y_{ij}\rangle$ is a Picard-Vessiot extension with differential Galois group $GL_n(C)$. Moreover, Q is a generic Picard-Vessiot extension in* *the following sense:*

Let $E \supseteq F$ be any Picard-Vessiot extension of F with group $GL_n(C)$, then E is isomorphic to $F(X_{ij})$, the function field of $GL_n(C)$ extended to F, as a $GL_n(C)$-module and as an F-module and there are $f_{ij} \in F$ such that the derivation on E is $D_E = \sum_{\ell=1}^{n} f_{i\ell} X_{\ell j}$ and the Picard-Vessiot extension $F(X_{ij}) \supseteq F$ is obtained from the Picard-Vessiot extension $C\langle f_{ij}\rangle(X_{ij}) \supseteq C\langle f_{ij}\rangle$ by extension of scalars to F over C.

Proof. To show that $Q \supset F\langle Y_{ij}\rangle$ is Picard-Vessiot we will use the characterization of Proposition 0.3.2. We have:

(1) $Q = F\langle Y_{ij}\rangle\langle V\rangle$, where $V \subset Q$ is the finite-dimensional vector space over C spanned by the X_{ij};

(2) There is a group G=$\mathrm{GL}_n(C)$ of differential automorphisms of Q with $G(V) \subseteq V$ and $Q^G = F\langle Y_{ij}\rangle$. This follows from the fact that Q is the function field of $\mathrm{GL}_n(C)$ extended to $F\langle Y_{ij}\rangle$.

(3) $Q \supseteq F\langle Y_{ij}\rangle$ has no new constants.

To see (3) notice that an element of Q can be written as $\frac{p_1}{p_2}$, with $p_1, p_2 \in R$ relatively prime. By Proposition 1.2.3, if such an element is a constant, that is $D(\frac{p_1}{p_2}) = 0$, then both p_1 and p_2 are Darboux polynomials for the same polynomial q ($D(p_1) = qp_1$ and $D(p_2) = qp_2$). By Theorem 1.3.16, the only Darboux polynomials in R are those of the form $p = \ell \det[X_{ij}]^a$ with $\ell \in F$,

$a \in \mathbb{N}$, which are not relatively prime. Therefore, if $c \in Q$ is such that $D(c) = 0$ then $c \in F\langle Y_{ij}\rangle$. In fact, by Propositions 1.2.3 and 1.3.1, $c \in F$.

By Kolchin's Structure Theorem for Picard-Vessiot extensions (Theorem 5.12 in [M]), we have that if $E \supseteq F$ is any Picard-Vessiot extension of F with group $\mathrm{GL}_n(C)$, then E is isomorphic to $F(X_{ij})$, the function field of $\mathrm{GL}_n(C)$ extended to F. On the other hand, the partial derivatives with respect to the X_{ij} are a basis of the Lie algebra of $\mathrm{GL}_n(C)$ extended to F. Thus, the derivation D_E on $F(X_{ij})$ can always be given by $D_E(X_{ij}) = \sum_{\ell=1}^{n} f_{i\ell}X_{\ell j}$ for some $f_{ij} \in F$. Since $E \supset F$ is a Picard-Vessiot extension for $\mathrm{GL}_n(C)$, then so is $C\langle f_{ij}\rangle(X_{ij}) \supset C\langle f_{ij}\rangle$, the derivation on $C\langle f_{ij}\rangle(X_{ij})$ being the corresponding restriction of D_E. From this Picard-Vessiot extension we retrieve $F(X_{ij}) \supset F$ by extension of scalars to F over C.$\square$

Thus, solving the inverse differential Galois problem for $\mathrm{GL}_n(C)$ can be reduced to finding $f_{ij} \in F$ such that the constants of $C\langle f_{ij}\rangle(X_{ij})$ are the constants of $C\langle f_{ij}\rangle$. Then we can construct a Picard-Vessiot extension of F with group $\mathrm{GL}_n(C)$ by extension of scalars on $C\langle f_{ij}\rangle(X_{ij}) \supset C\langle f_{ij}\rangle$.

2.2 Specializing to a Picard-Vessiot extension of F

Let F and C be as above. In this section we give necessary and sufficient conditions for the differential homomorphism $Y_{ij} \mapsto f_{ij}$, with

$$C\langle f_{ij}\rangle(X_{ij}) \supset C\langle f_{ij}\rangle$$

a Picard-Vessiot extension, to exist. We restrict ourselves to the case that F has finite transcendence degree over C.

We consider first the case $F = C(t)$, where t is transcendental over C. The derivation on F is given by $D(t) = 1$, $D(c) = 0$, for $c \in C$.

On $F\langle Y_{ij}\rangle(X_{ij})$ consider the derivation given by $D(X_{ij}) = \sum_{\ell=1}^{n} Y_{i\ell}X_{\ell j}$. We are looking for a specialization (homomorphism) from $C\{Y_{ij}\}$ to $C(t)$ given by $Y_{ij} \mapsto f_{ij}$ so that $C\langle f_{ij}\rangle(X_{ij}) \supset C\langle f_{ij}\rangle$, with the derivation induced by this homomorphism, has no new constants. We have:

Proposition 2.2.1. *Let X_{ij} be algebraically independent over $C(t)$. Define a derivation on $C(t)(X_{ij})$ by $D(X_{ij}) = \sum f_{i\ell}X_{\ell j}$, with $f_{i\ell} \in C(t)$, and $D(t) = 1$. Let $\mathcal{C}$ be the field of constants of $C(t)(X_{ij})$. Then, $\mathcal{C} = C$ if and only if the X_{ij} are algebraically independent over $\mathcal{C}$.*

Proof. First note that t is algebraically independent over $\mathcal{C}$ since for every m we have that the Wronskian determinant

$$W(1,\ldots,t^m) = \begin{vmatrix} 1 & t & \ldots & t^m \\ 0 & 1 & \ldots & mt^{m-1} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & m! \end{vmatrix} = 0!1!\cdots m! \neq 0$$

Now we use a transcendence degree count argument. Suppose that the transcendence degree of the extension $C \subset \mathcal{C}$ is at least one. For the tower of fields $C \subset \mathcal{C} \subset \mathcal{C}(X_{ij}) \subset C(t)(X_{ij})$ we have that the transcendence degree of the extension $\mathcal{C} \subset \mathcal{C}(X_{ij})$ is n^2 and the transcendence degree of $C \subset C(t)(X_{ij})$ is $n^2 + 1$. So, in particular, the transcendence degree of $C \subset \mathcal{C}$ can be at most one. If that is the case then $C(t)(X_{ij})$ is algebraic over $\mathcal{C}(X_{ij})$. This means that there are f_i, $g \in \mathcal{C}(X_{ij})$ with $g \not\equiv 0$ such that

$$t^m + \frac{f_{m-1}(X_{ij})}{g(X_{ij})}t^{m-1} + \cdots + \frac{f_0(X_{ij})}{g(X_{ij})} = 0.$$

Let $\alpha \in \mathcal{C}^{n^2}$ be such that $g(\alpha) \neq 0$ then

$$t^m + \frac{f_{m-1}(\alpha)}{g(\alpha)}t^{m-1} + \cdots + \frac{f_0(\alpha)}{g(\alpha)} = 0$$

which contradicts the fact that t is algebraically independent over $\mathcal{C}$. This shows that $\mathcal{C} = C$.

For the converse we only need to point out that by construction the X_{ij} are algebraically independent over C.□

We can generalize this proposition as follows:

As before, let C be an algebraically closed field.

Theorem 2.2.2. *Let $F = C(t_1, \ldots, t_m)[z_1, \ldots, z_k]$ where the t_i are algebraically independent over C and the z_i are algebraic over $C(t_1, \ldots, t_m)$. Assume that the derivation on F has field of constants C and that it extends* *to* $F(X_{ij})$ *so that* $D(f \otimes X_{ij}) = D(f) \otimes X_{ij} + f \otimes \sum_{\ell=1}^{n} f_{i\ell} X_{\ell j}$ *on* $F \otimes C[X_{ij}]$. *Let* $\mathcal{C}$ *be the field of constants of* $F(X_{ij})$. *Then* $\mathcal{C} = C$ *if and only if the* X_{ij} *are algebraically independent over* $\mathcal{C}$.

Proof. Since the t_i are algebraically independent over C for every s the Wronskian of the set of power products of total degree s or less in the t_i is different from zero (this Wronskian is determined up to a sign). Therefore, they are also algebraically independent over $\mathcal{C}$. We have the tower of fields

$$C \subset \mathcal{C} \subset \mathcal{C}(X_{ij}) \subset F(X_{ij})$$

where the transcendence degree of $\mathcal{C} \subset \mathcal{C}(X_{ij})$ is n^2 and the transcendence degree of $C \subset F(X_{ij})$ is $n^2 + m$. Suppose that the transcendence degree r of $C \subset \mathcal{C}$ is at least one. Then the transcendence degree ℓ of $\mathcal{C}(X_{ij}) \subset F(X_{ij})$ has to be $\ell < m$. But this implies that there is an algebraic relation among the t_i over $\mathcal{C}(X_{ij})$. Let $g(X_{ij}), f_i(X_{ij}) \in \mathcal{C}(X_{ij})$, $g(X_{ij}) \not\equiv 0$, be such that

$$t^{\delta_s} + \frac{f_{s-1}(X_{ij})}{g(X_{ij})} t^{\delta_{s-1}} + \cdots + \frac{f_0(X_{ij})}{g(X_{ij})} = 0.$$

If $\alpha \in \mathcal{C}^{n^2}$ is such that $g(\alpha) \neq 0$, then

$$t^{\delta_s} + \frac{f_{s-1}(\alpha)}{g(\alpha)} t^{\delta_{s-1}} + \cdots + \frac{f_0(\alpha)}{g(\alpha)} = 0$$

is an algebraic relation among the t_i over $\mathcal{C}$. This contradiction shows that $\mathcal{C} = C$.

For the converse, once again, we point out that by construction the X_{ij} are algebraically independent over C.$\square$

Now to *check* whether the X_{ij} are algebraically independent over C, we let $X(k)$ denote the set of power products in the X_{ij} of total degree k or less. Then we have that the X_{ij} are algebraically independent over C if and only if for each k, the set $X(k)$ is linearly independent over C. Fix any ordering in $X(k)$ and let W_k denote the Wronskian of $X(k)$ relative to that ordering (of course a Wronskian computed using any other ordering of the elements of $X(k)$ will only differ from this one by a sign). So the above condition means that in turn W_k is non-zero for all k. Now go back to $C\{Y_{ij}\}[X_{ij}]$ and let W_k be the Wronskian of $X(k)$ here. Then, the condition of Theorem 2.2.2 for finding a specialization $Y_{ij} \mapsto f_{ij}$ so that $C\langle f_{ij}\rangle(X_{ij}) \supset C\langle f_{ij}\rangle$ has no new constants can be expressed as follows:

Theorem 2.2.3. *There is a specialization of the Y_{ij} with no new constants if and only if there are f_{ij} such that all W_k map to non-zero elements.*

Chapter 3. The theorems for an arbitrary connected algebraic group G

3.1 Specializing to a Picard-Vessiot extension

The proofs of Theorems 2.2.2 and 2.2.3 in Section 2.2, Chapter 2, do not make any particular use of the fact that $G = \mathrm{GL}_n$. If we replace the X_{ij} by X_i and the Y_{ij} by Y_i we can formulate the following more general results for an arbitrary connected algebraic group G whose function field is $C(G) = C(X_1, \ldots, X_n)$. Let $\{D_1, \ldots, D_n\}$ be a basis for Lie(G), $D = \sum Y_i D_i$. Let W_k be the Wronskian of the monomials of degree k or less in the X_i for a given ordering.We have:

Theorem 3.1.1. *Let C be as above, let $F = C(t_1, \ldots, t_m)[z_1, \ldots, z_k]$ where the t_i are algebraically independent over C and the z_i are algebraic over $C(t_1, \ldots, t_m)$. Let C be the field of constants of $F(X_1, \ldots, X_n)$. Then C $= C$ if and only if the X_i are algebraically independent over C.*

Theorem 3.1.2. *There is a specialization of the Y_i with no new constants if and only if there are f_i such that all W_k map to non-zero elements.*

For the proofs of Theorems 3.1.1 and 3.1.2 we only need to replace the X_{ij} with X_i , the Y_{ij} with Y_i and n^2 with n in the proofs of Theorems 2.2.2 and 2.2.3.□

Note. Observe that the proofs of Theorems 3.1.1 and 3.1.2 do not use the fact that $C(X_1, \ldots, X_n)$ is the function field of G. However, this hypothesis is used in the following theorem to show that $F(X_1, \ldots, X_n) \supset F$ is a Picard-Vessiot extension with group G.

Under the hypothesis (and notation) of Theorems 3.1.1 and 3.1.2 we have:

Theorem 3.1.3. *Let $F = C(t_1, \ldots, t_m)[z_1, \ldots, z_k]$ where the t_i are algebraically independent over C and the z_i are algebraic over $C(t_1, \ldots, t_m)$. Then $F(X_1, \ldots, X_n) \supset F$ is a Picard-Vessiot extension with Galois group G if and only if the X_i are algebraically independent over the field of constants C of $F(X_1, \ldots, X_n)$.*

Proof. By Theorem 3.1.1, $F(X_1, \ldots, X_n) \supset F$ is a no new constant extension. On the other hand, we obtain $F(X_1, \ldots, X_n)$ by the extension of scalars:

$$\begin{aligned} F(X_1, \ldots, X_n) &= q.f.(F \otimes_C C(X_1, \ldots, X_n)) \\ &= q.f.(F \otimes_C C[G]) \end{aligned}$$

and G acts on $F \otimes_C C[G]$ fixing F. So, $G \subseteq G(F(X_1, \ldots, X_n)/F)$. Counting dimensions we get that $G = G(F(X_1, \ldots, X_n)/F)$ since $C(X_1, \ldots, X_n) = C(G)$. Finally, $F(X_1, \ldots, X_n) = F\langle V\rangle$, where V is the finite-dimensional vector space over C spanned by the X_i. This proves the converse implication. For the direct one we have that if $F(X_1, \ldots, X_n) \supset F$ is a Picard-Vesiot extension then the field of constants of $F(X_1, \ldots, X_n)$ coincides with the one of F. So we can apply Theorem 3.1.1 and get the result. $\square$

Applying Theorems 3.1.2 and 3.1.3 we also obtain:

Theorem 3.1.4. *There is a specialization of the Y_i such that $F(X_1, \ldots, X_n) \supset F$ is a Picard-Vessiot extension if and only if there are f_i such that all W_k map to non-zero elements.*

Chapter 4. Computing new constants

Let C be an algebraically closed field with trivial derivation. Let $F = C(t_1,\ldots,t_m)[z_1,\ldots,z_k]$ where the t_i are algebraically independent over C and the z_i are algebraic over $C(t_1,\ldots,t_m)$. Assume that the derivation on F has field of constants C and that it extends to $F(X_{ij})$ so that $D(f\otimes X_{ij}) = D(f)\otimes X_{ij} + f\otimes \sum_{\ell=1}^{n} f_{i\ell}X_{\ell j}$ on $F\otimes C[X_{ij}]$, for certain $f_{ij}\in F$. By Theorem 2.2.2 in Section 2.2, if there is an algebraic relation among the X_{ij} over the field of constants $\mathcal{C}$ of $F(X_{ij})$ then $\mathcal{C}$ properly contains C. In this section we will produce a new constant from such an algebraic relation. We will restrict ourselves to the case $n = 2$ and use a particular linear dependence relation.

Extend the derivation on F to $F(X_{11},X_{12},X_{21},X_{22})$ by letting $D(X_{ij}) = \sum_{\ell=1}^{2} f_{i\ell}X_{\ell j}$, where the f_{ij} are such that the Wronskian $W_1 = W(X_{11},X_{12},X_{21},X_{22}) = 0$, that is, the X_{ij} are linearly dependent over $\mathcal{C}$. Furthermore, assume that the linear relation among the X_{ij} is such that there are $\beta_{12},\beta_{21},\beta_{22}\in\mathcal{C}$ with

$$X_{11} = \beta_{12}X_{12} + \beta_{21}X_{21} + \beta_{22}X_{22} \tag{1}$$

and that X_{12},X_{21} and X_{22} are linearly independent. In order to simplify the computations we will also assume that $\det[f_{ij}] = 0$.

We want to find a,b,c such that $p = aX_{12} + bX_{21} + cX_{22}$ is a Darboux polynomial in $F[X_{ij}]$, that is $D(aX_{12}+bX_{21}+cX_{22}) = q(aX_{12}+bX_{21}+cX_{22})$ for certain $q \in F$.

We have,

$$
\begin{aligned}
&D(aX_{12} + bX_{21} + cX_{22}) \\
&\quad = a(f_{11}X_{12} + f_{12}X_{22}) + b(f_{21}X_{11} + f_{22}X_{21}) + c(f_{21}X_{12} + f_{22}X_{22}) \\
&\quad = bf_{21}X_{11} + (af_{11} + cf_{21})X_{12} + bf_{22}X_{21} + (af_{12} + cf_{22})X_{22} \\
&\quad = bf_{21}(\beta_{12}X_{12} + \beta_{21}X_{21} + \beta_{22}X_{22}) + (af_{11} + cf_{21})X_{12} + bf_{22}X_{21} \\
&\qquad + (af_{12} + cf_{22})X_{22} \\
&\quad = (af_{11} + bf_{21}\beta_{12} + cf_{21})X_{12} + b(f_{22} + f_{21}\beta_{12})X_{21} \\
&\qquad + (af_{12} + bf_{21}\beta_{22} + cf_{22})X_{22} \\
&\quad = qaX_{12} + qbX_{21} + qcX_{22}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&[a(f_{11} - q) + bf_{21}\beta_{12} + cf_{21}]X_{12} + b(f_{22} + f_{21}\beta_{12} - q)X_{21} \\
&\qquad\qquad + (af_{12} + bf_{21}\beta_{22} + c(f_{22} - q)X_{22} = 0. \qquad (2)
\end{aligned}
$$

Since we are assuming that X_{12}, X_{21} and X_{22} are linearly independent their coefficients in (2) must be equal to zero. So we have the following homogeneous linear system in a, b, c:

$$
\begin{array}{llll}
(f_{11} - q)a & +f_{21}\beta_{12}b & +f_{21}c & = 0 \\
 & (f_{22} + f_{21}\beta_{12} - q)b & & = 0 \\
f_{12}a & +f_{21}\beta_{22}b & +(f_{22} - q)c & = 0
\end{array}
$$

In order for the above system to have non-trivial solutions we need that

$$\det \begin{bmatrix} f_{11}-q & f_{21}\beta_{12} & f_{21} \\ 0 & f_{22}+f_{21}\beta_{12}-q & 0 \\ f_{12} & f_{21}\beta_{22} & f_{22}-q \end{bmatrix} = 0.$$

But,

$$\det \begin{bmatrix} f_{11}-q & f_{21}\beta_{12} & f_{21} \\ 0 & f_{22}+f_{21}\beta_{12}-q & 0 \\ f_{12} & f_{21}\beta_{22} & f_{22}-q \end{bmatrix}$$

$$= (f_{22}+f_{21}\beta_{12}-q)\det \begin{bmatrix} f_{11}-q & f_{21} \\ f_{12} & f_{22}-q \end{bmatrix}$$

$$= (f_{22}+f_{21}\beta_{12}-q)(\det[f_{ij}] - \mathrm{trace}[f_{ij}]q + q^2)$$

$$= 0.$$

This gives,

$$f_{22}+f_{21}\beta_{12}-q = 0 \tag{3}$$

or

$$\det[f_{ij}] - \mathrm{trace}[f_{ij}]q + q^2 = 0. \tag{4}$$

From (3) and (4)we get

$$q = f_{22}+f_{21}\beta_{12} \tag{5.1}$$

or

$$q = \frac{\mathrm{trace}[f_{ij}] \pm \sqrt{\mathrm{trace}[f_{ij}]^2 - 4\det[f_{ij}]}}{2} \tag{5.2}$$

Since we are assuming that $\det[f_{ij}] = 0$, (5.2) becomes:

$$q = \begin{cases} \mathrm{trace}[f_{ij}] \\ 0 \end{cases} \tag{5.3}$$

Choose $q = \text{trace}[f_{ij}]$ and assume that $q \neq 0, f_{22} + f_{21}\beta_{12}$. Then the second equation in the system implies that $b = 0$. So we have the equivalent system:

$$\begin{array}{rrl} -f_{22}a & +f_{21}c & = 0 \\ f_{12}a & -f_{11}c & = 0 \end{array}.$$

If $f_{22} \neq 0$ then the above system has the general solution

$$a = \frac{f_{21}}{f_{22}}c, \text{ where } c \in \mathcal{C}.$$

In particular, if we choose $c = 1$ then $p = \frac{f_{21}}{f_{22}}X_{12} + X_{22}$ satisfies

$$D\left(\frac{f_{21}}{f_{22}}X_{12} + X_{22}\right) = \text{trace}[f_{ij}]\left(\frac{f_{21}}{f_{22}}X_{12} + X_{22}\right).$$

On the other hand we also have that

$$D(\det[X_{ij}]) = \text{trace}[f_{ij}]\det[X_{ij}].$$

Let

$$\theta = \frac{\frac{f_{21}}{f_{22}}X_{12} + X_{22}}{\det[X_{ij}]}.$$

We have,

$$\begin{aligned} D(\theta) &= D\left(\frac{\frac{f_{21}}{f_{22}}X_{12} + X_{22}}{\det[X_{ij}]}\right) \\ &= \frac{D\left(\frac{f_{21}}{f_{22}}X_{12} + X_{22}\right)\det[X_{ij}] - \left(\frac{f_{21}}{f_{22}}X_{12} + X_{22}\right)D(\det[X_{ij}])}{\det[X_{ij}]^2} \\ &= \frac{\text{trace}[f_{ij}]\left(\frac{f_{21}}{f_{22}}X_{12} + X_{22}\right)\det[X_{ij}] - \left(\frac{f_{21}}{f_{22}}X_{12} + X_{22}\right)\text{trace}[f_{ij}]\det[X_{ij}]}{\det[X_{ij}]^2} \\ &= 0. \end{aligned}$$

That is, θ is a *new* constant in $F(X_{ij})$.

Now we show that under the restrictions that we imposed on the f_{ij} it is possible to find a non-zero f_{22}.

Since we have a linear dependence relation among the X_{ij}, the Wronskian W_1 must be equal to zero. This Wronskian can be expressed, up to a sign, as the following product of determinants:

$$W_1 = \begin{vmatrix} 1 & 0 & 0 & 1 \\ f_{11} & f_{12} & f_{21} & f_{22} \\ A & B & E & F \\ C & D & G & H \end{vmatrix} \begin{vmatrix} X_{11} & X_{12} & 0 & 0 \\ X_{21} & X_{22} & 0 & 0 \\ 0 & 0 & X_{11} & X_{12} \\ 0 & 0 & X_{21} & X_{22} \end{vmatrix} = M(f_{ij}) \det[X_{ij}]^2,$$

where

$$A = f'_{11} + f^2_{11} + f_{12}f_{21},$$

$$B = f'_{12} + f_{11}f_{12} + f_{12}f_{22}$$

$$C = f_{11}A + f_{21}B + A'$$

$$= 3f_{11}f'_{11} + 2f_{11}f_{12}f_{21} + 2f'_{12}f_{21} + f''_{11} + f_{12}f'_{21} + f^3_{11},$$

$$D = f_{12}A + f_{22}B + B'$$

$$= 2f'_{11}f_{12} + f^2_{11}f_{12} + f^2_{12}f_{21} + f_{21}f^2_{22} + 2f'_{12}f_{22} + f_{11}f'_{12}$$

$$+ f_{12}f'_{22} + f''_{12} + f_{11}f_{12}f_{22},$$

$$E = f'_{21} + f_{21}f_{11} + f_{22}f_{21},$$

$$F = f'_{22} + f_{12}f_{21} + f^2_{22},$$

$$G = f_{11}E + f_{21}F + E'$$

$$= 2f'_{21}f_{11} + f_{21}f_{11}^2 + f_{22}f_{21}f_{11} + 2f'_{22}f_{21} + f_{12}f_{21}^2$$

$$+ f_{22}^2 f_{21} + f''_{21} + f_{21}f'_{11} + f_{22}f'_{21},$$

$$H = f_{22}F + f_{12}E + F'$$

$$= f_{21}f_{11}f_{12} + 2f_{22}f_{21}f_{12} + +3f_{22}f'_{22} + 2f_{12}f'_{21} + f'_{12}f_{21}$$

$$+ f_{22}^3 + f''_{22}.$$

and

$$M(f_{ij}) = \begin{vmatrix} 1 & 0 & 0 & 1 \\ f_{11} & f_{12} & f_{21} & f_{22} \\ A & B & E & F \\ C & D & G & H \end{vmatrix}.$$

We have after simplifying using the hypothesis that $\det[f_{ij}] = 0$,

$$M(f_{ij}) = (f_{22} - f_{11})(f'_{12}f''_{21} - f'_{21}f''_{12}) + (f'_{22} - f'_{11})(f''_{12}f_{21} - f_{12}f''_{21})$$

$$- f'_{12}f'_{21}(f_{11} - f_{22})^2 - f_{12}f_{21}(f'_{11} - f'_{22})$$

$$+ f_{12}f'_{21}(f_{11}f'_{11} + f_{22}f'_{22} - f'_{11}f_{22} - f_{11}f'_{22} + f''_{22} - f''_{11} + f_{12}f'_{21} - f'_{12}f_{21})$$

$$+ f'_{12}f_{21}(f_{11}f'_{11} + f_{22}f'_{22} - f'_{11}f_{22} - f_{11}f'_{22} + f''_{11} - f''_{22} + f'_{12}f_{21} - f_{12}f'_{21}).$$

Of course, computing $M(f_{ij})$ involved a great deal of computations. First we computed the determinant directly and then we checked the result using Dogson's method ([D], [RR]).

The Wronskian $W_1 = 0$ if and only if $M(f_{ij}) = 0$. Now, observe that if $f_{12} = 0$ then $f'_{12} = 0$ which implies that $B = 0$ and $D = 0$ as well. Therefore

$M(f_{ij}) = 0$. So, if we let $M(Y_{ij})$ be the differential polynomial in the Y_{ij} whose specialization to the f_{ij} is $M(f_{ij})$ then $M(Y_{ij})$ is in the differential ideal

$$\begin{aligned}\mathcal{I} &= \{\det[Y_{ij}], Y_{12}\} \\ &= \{Y_{11}Y_{22} - Y_{12}Y_{21}, Y_{12}\} \\ &= \{Y_{11}Y_{22}, Y_{12}\}\end{aligned}$$

of $C\{Y_{11}, Y_{12}, Y_{21}, Y_{22}\}$. It is easy to see that Y_{22} is not in $\mathcal{I}$. Indeed, suppose that

$$Y_{22} = pY_{11}Y_{22} + qY_{12} + r, \tag{6}$$

where $p, q \in C\{Y_{11}, Y_{12}, Y_{21}, Y_{22}\}$, $r = \sum_{i,j}(r_i(Y_{11}Y_{22})^{(i)} + q_jY_{12}^{(j)})$ and $p_i, q_j \in C\{Y_{11}, Y_{12}, Y_{21}, Y_{22}\}$.

Now, consider the map

$$\begin{aligned}\psi : C\{Y_{11}, Y_{21}, Y_{22}\} &\to C[Y_{11}, Y_{21}, Y_{22}] \\ Y_{22} &\mapsto Y_{22} \\ Y_{ij} &\mapsto 0 \qquad \text{for } i, j \neq 2.\end{aligned}$$

Let $\overline{p} = \psi(p)$, $\overline{q} = \psi(q), \overline{r} = \psi(r)$. We have that $\overline{r} = 0$ and (6) becomes

$$Y_{22} = 0. \tag{7}$$

which is impossible.

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