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PATTERNS OF ANALYTICAL THINKING AND KNOWLEDGE USE IN STUDENTS' EARLY UNDERSTANDING OF THE LIMIT CONCEPT

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BY

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Abstract

This study explored first-semester calculus students' early understanding of limits, relative to their function knowledge and graphing calculator use. The purpose was to identify and describe students' patterns of analytical thinking and knowledge use in determining limit situations, as a first step in developing a grounded theory of early development of intuitive limit concepts.

Over four task-based interviews, ten students progressed from examining local function behavior to analyzing increasingly difficult limit situations. Written and oral responses were analyzed relative to a four-element framework developed by the author: (a) analyzing functions locally in graphical and numerical settings; (b) conjecturing limits from representative graphs and tables; (c) understanding advantages and limitations of tables and graphs to conjecturing limits, particularly when using graphing calculators; and (d) producing multiple sources of evidence to justify a limit conjecture, and knowing whether this evidence is sufficient. Students demonstrating these four elements were deemed to have an *intuitive-analytic* understanding of the limit concept.

Students in this study had difficulty determining local function behavior, and did not always connect this to determining limit situations. They could read graphs and tables to conjecture limits, but often based such conjectures on poor tables or graphs. They learned that tables and graphs might mislead them, but rarely analyzed this, either assuming representativeness or assuming they were being misled. These students relied on formula-based expectations, graphs, or a few function values in determining limit situations. They did not know how to move from "almost certain" to certain in determining a limit situation.

These students' function knowledge and methods of analyzing local function behavior both did and did not influence their determination of limit situations. Partial analyses led them to accept non-representative behavior, which led to erroneous limit conjectures. On the other hand, even full and complete analyses did not always result in correct limit conjectures.

The graphing calculator played a significant role. Graphs and tables on the calculator were often taken as the "standard" of comparison, without analysis. Awareness of calculator limitations did not necessarily imply correct limit conjectures, due to ignoring the limitations, or erroneously assuming the effects of those limitations.

CHAPTER 1

Introduction

The concept of a limit is central to an understanding of calculus since it is the foundation on which the definitions of continuity, derivative and integral are based. At the same time, the limit concept appears in different contexts, such as with functions, sequences, and series; and involves different processes, such as secant lines approaching tangent lines, iterative root-finding procedures, and area calculations. Moreover, the mathematical definitions of limit (in each of the various contexts) differ substantially from intuitive limit ideas. The central role of the limit concept in calculus necessitates its early introduction, but its complexity raises several questions. What is a reasonable starting point for introducing the limit concept? In what ways do students' understanding of the limit concept develop? How can this development be measured? These questions provided the impetus for this research.

This study explored students' early understanding of the limit concept in first semester calculus, and identified effects of two factors influencing this development: students' knowledge and understanding of functions, and their knowledge and use of graphing calculators. Specifically, students' intuitive ideas about limits of functions at particular points were explored through task-based interviews. Over the course of four interviews, the tasks evolved from examining local behavior of functions to conjecturing the existence or non-existence of specified limits, based on tables or graphs, in increasingly difficult situations.

We begin with an analysis of some of the complexities of the limit concept. This analysis suggests several possibly crucial elements of students' early understanding of limits, providing focus for the problem addressed in this research. From there, the significance of the problem is outlined and the research questions are elaborated.

The Complexities of the Limit Concept

The limit concept is analyzed by juxtaposing the ε–δ limit definition with intuitive limit ideas. Each of these components of the limit concept contributes to its complexity. When examined side by side, however, it becomes clear that the full complexities of the limit concept are greater than the sum of the complexities of its components.

The ε–δ Limit Definition

The complexity of the $\varepsilon - \delta$ definition of a limit of a function at a particular point begins with the language of the statement itself:

> Let *f* be a function defined on some open interval that contains the number *a*, except possibly at *a* itself. Then we say that the **limit of *f(x)* as *x* approaches *a* is *L***, and we write
> $$\lim_{x \to a} f(x) = L$$
> if for every number $\varepsilon > 0$, there is a corresponding number $\delta > 0$ such that
> $|f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$ (Stewart, 1995, p. 71).

The quantifiers *for every*, *there is*, *such that*, and *whenever* cause difficulty (Cornu, 1991; Cottrill et al., 1996). The word *limit* has meanings in everyday use that interfere with the mathematical meaning. The word *approaches* (and its many mathematical "synonyms" such as *converges to*, and *tends toward*) contradicts the static nature of the definition (Monaghan, 1991).

One approach used to introduce the ε–δ limit definition is to give a specific tolerance measurement, that is, a specific ε, and ask students to find a suitable range of x-values around a that will guarantee function values within the specified tolerance measurement of L. This focuses attention on the correspondence between δ and ε in the definition. Repeating this exercise with several different values of ε focuses attention on the idea that any tolerance ε will work when the limit exists.

In practice, whether using the ε–δ limit definition or attempting an introductory ε–δ exercise, a candidate for L must already exist. In fact, the ε–δ limit definition is used primarily to *verify* whether a conjectured limit is, without a doubt, the true limit. The definition itself provides no method for *conjecturing* such an L. Certainly an educated conjecture is more efficient than a random guess, and this raises the question of how to educate students to make intelligent limit conjectures.

Intuitive Limit Ideas

One approach used to introduce the notion of limit conjectures is to begin with an informal idea of limit:

> We write
> $$\lim_{x \to a} f(x) = L$$
> and we say, "the limit of $f(x)$, as x approaches a, equals L" if we can make the values of $f(x)$ arbitrarily close to L (as close as we like) by taking x to be sufficiently close to a, but not equal to a.... Roughly speaking, this says that the values of $f(x)$ get closer and closer to the number L, as x gets closer and closer to the number a (from either side of a) but $x \neq a$ (Stewart, 1995, p. 51).

This has the advantage that it suggests a means of conjecturing a limit, by either computing appropriate function values or observing appropriate function values on a graph. The disadvantage lies in the clear loss of precision of the ε–δ limit definition and the retention of subtle language difficulties (Tall & Schwarzenberger, 1978). Moreover, the *activity* of conjecturing a limit has its own set of complexities, due to its dependence on tables and graphs, on students' function knowledge, and, in modern calculus courses, on students' use of graphing calculators.

Dependence on tables and graphs. "Correct" intuitive limit conclusions depend on the accuracy and consistency of the tables and graphs on which they are based. Both tables and graphs can fail in multiple ways to accurately represent the true nature of a function. For example, a poor sample of x-values can lead to correct, but misleading, y-values in a table. As another example, a poor sample of x-values can lead to incorrect y-values in both a table and a graph when technology is being used. In this situation, both the table and graph might be inaccurate, but still consistent with one another, which eliminates consistency of representations as a means of detecting an inaccuracy. This is when a student's knowledge and understanding of functions is most important.

Influence of function knowledge. An intuitive limit conjecture, under ideal circumstances, depends on analysis of function values and awareness of which function values to focus on. Students must analyze function values by inputting them into a process in which function values are compared to a target limit value, or compared to each other, or examined for a pattern. At the same time, this analysis is useless if students are analyzing the wrong function values. To engage in this process in both numerical and graphical settings, and, ideally, understand how the processes differ from and parallel one another, students need a firm understanding of local function behavior.

Influence of graphing calculator use. Graphing calculators allow easy access to numerous intuitive limit ideas. First, graphs and tables can be produced quickly and easily. Second, the trace feature permits a dynamic sense of the limit process, and by indicating which ordered pairs are being "landed on", provides a bridge between graphical and numerical representations. Third, the zoom feature can sometimes show how smaller viewing windows lead to more accurate limit conjectures. These features have several drawbacks, however.

There are several ways in which graphing calculators can mislead students. Certainly, calculator-produced tables and graphs can be misleading due to computational limitations, but poor input choices can also lead to trouble. For example, a poor choice of a viewing window can lead to no graph, or a misleading graph. If the interesting behavior at a particular point is "hiding" between two pixels, then the trace feature will not detect this. Or, if the limit situation involves a vertical asymptote, then repeated use of the zoom feature may not detect this, since some asymptotic behavior is only visible using a very large y-interval and a small x-interval.

Crucial Elements of Early Understanding of Limits

These analyses suggest that intuitive limit ideas are much more accessible to first semester calculus students than the ε-δ limit definition, but that intuitive limit ideas without analytical thinking can be quite misleading. Early limit ideas are appropriately characterized by several crucial elements. First, the ability to analyze functions locally in both graphical and numerical settings seems to be a prerequisite. Second, students must be able to draw correct intuitive limit conclusions from accurate graphs and tables. Third, an understanding of the advantages and disadvantages of tables and graphs to conjecturing limits, particularly when using graphing calculators is necessary. Finally, students must be able to produce multiple sources of evidence to justify a limit conjecture, and know whether their evidence leaves room for doubt. Students who possess these four elements will be deemed to have an *intuitive-analytic* understanding of the limit concept.

Evaluating whether a student actually possesses the four elements of intuitive-analytic understanding of the limit concept is a difficult endeavor. Designing problems that could detect these abilities and understandings is only part of the difficulty. Student solutions to such problems, say on homework or exams, would not necessarily reflect their thought processes, and likely would not provide a record of their uses of the graphing calculator. Observation of students while they are in the process of solving such problems and subsequent interaction with them would elicit a much richer, detailed picture of their understandings and abilities than written work alone.

Problem Statement

The problem addressed in this study was to describe how and to what extent students in a first-semester, graphing-calculator based, calculus course gain an intuitive-analytic understanding of limits. The focus was on how they gained an understanding of each of the four elements of early understanding of limits. This was accomplished through a series of task-based interviews and questionnaires, each of which focused on one or more of these four elements.

Significance of the Problem

Very little research exists on the limit concept, and most of that focuses on students' acquisition of the ε–δ definition, concluding that this acquisition proceeds very slowly (Cornu, 1991; Ervynck, 1981; Tall, 1992; Tall & Vinner, 1981; Williams, 1991). There are few descriptions of the more intuitive ideas involved in conjecturing limits based on tables and graphs. Cottril et al. (1996) suggest that such intuitive limit ideas may be quite complicated for students.

There is precedent for examining function knowledge and limit concepts in the same study, with (Lauten, Graham, Ferrini-Mundy, 1994), and without (Ferrini-Mundy & Graham, 1994) graphing calculators. These studies also used task-based interviews to develop detailed descriptions of students' understandings of function and limit, but the function tasks did not focus on the function knowledge most salient to an intuitive understanding of limits, namely local function behavior in numerical, graphical and symbolic settings. Moreover, spontaneous use of the graphing calculator in the first study was minimal.

Although there is extensive literature on functions, (see Harel & Dubinsky, 1992 for a small collection) much of it focuses on the definition of function and on how understanding of the function concept itself develops. This study focuses on function behavior and its influence on the development of students' understanding of limits.

Thus, this study will make a significant contribution to the research. In addition, the knowledge gained from this study has the potential to inform instruction. By describing patterns of analytical thinking and knowledge use employed by students in solving problems about limit situations, pedagogical strategies and mathematical problems that account for these patterns can be constructed.

Research Questions

1. Early understanding of the limit concept. How and to what extent do the four elements of intuitive-analytic understanding of the limit concept emerge and develop over the course of the study? In particular, how do students analyze local function behavior? Can they draw correct intuitive limit conclusions from accurate graphs and tables? In what ways do students develop awareness of the advantages and disadvantages of tables and graphs to conjecturing limits, particularly when using graphing calculators? Do they spontaneously produce multiple sources of evidence to justify a limit conjecture?
2. Influence of function knowledge. In what ways do students analyze local behavior of functions in graphical and numerical settings? In particular, on what aspects of tables, graphs and formulas do students focus, in analyzing local function behavior? How do they decide whether a graph and a table of the same function are consistent? How do they decide whether a table or graph reflects the true nature of the function? Do their methods of analyzing local function behavior support or hinder their success with determining a limit situation?
3. Influence of graphing calculator use. How do students spontaneously use graphing calculators in analyzing functions and making limit conjectures? To what extent are they aware of the limitations of the graphing calculator, and how do they deal with this? How convincing is this tool for them?

CHAPTER 2

Review of the Literature

The purpose of this chapter is to outline how previous research and the content of the course textbook, Stewart's *Calculus*, 3rd edition (1995), influenced the design of the interview tasks and protocols in this study. Relevant literature[1] is reported within the framework of the four elements of intuitive-analytic understanding of limits: (a) analysis of local function behavior, (b) intuitive limit conjectures, (c) limitations of numerical and graphical evidence, and (d) intuitive-analytic limit conjectures.

Analysis of Local Function Behavior

The role of analysis of local function behavior in determining limit situations has rarely been addressed directly in the literature. In the current study, there are two dimensions of particular interest: types of local function behavior and representations of local function behavior.

Types of Local Function Behavior

The interview tasks in the current study present several types of local function behavior for each of the situations in which a limit exists or does not exist. This follows

[1] ERIC and DAI databases were searched, using key words *mathematics*, *calculus*, *limits*, *functions*, and *graphing calculators*. Descriptors of found sources were used in later searches. Document references led to other literature.

Tall's (1990) suggestion that student exploration of both examples and non-examples of a concept can help a student understand general properties while avoiding narrow over-generalizations. There is promise in this suggestion, as shown by several researchers who have used this strategy successfully in settings where a teacher introduces a concept, discusses it with students, and then allows individual exploration (Blackett, 1987; Tall, 1986a; Thomas, 1988; as cited in Tall, 1990). On a practical note, the course textbook (Stewart, 1995) contains many different types of local function behavior in its introductory sections on the limit concept, suggesting that opportunities existed for students to explore the limit concept through examples and non-examples.

The types of local function behavior selected were based on examples and exercises in the course textbook (Stewart, 1995). In particular, types of local behavior implying the existence of a limit included highly oscillatory (damped) behavior and piecewise monotonic behavior associated either with a point of continuity or an isolated singularity. Types of local behavior implying the non-existence of a limit included highly oscillatory behavior (not damped), asymptotic behavior associated with a vertical asymptote, and piecewise monotonic behavior associated with a jump singularity.

The emphases placed on each of the types of local behavior stemmed from research results of others and design considerations. Types of local function behavior known to cause difficulties for students were emphasized. These include highly oscillatory behavior (Cottril et al., 1996; Ervynck, 1981; Williams, 1991), asymptotic behavior and removable singularities (Boers & Jones, 1993, as cited in Tall 1996; Williams, 1991). Some types of local function behavior have been shown to block engagement in analysis of local function behavior or limit situations, and thus, were deemphasized. These include continuous behavior (Cottril et al., 1996; Ferrini-Mundy & Graham, 1994; Williams 1991), and jump singularities (Ferrini-Mundy & Graham, 1994).

Multiple Representations of Local Function Behavior

Most interview tasks in the current study required students to make connections between two or three given representations (table, graph and/or formula) of a function's local behavior. This stems from research results suggesting that tasks which force students to make connections between representations are more effective than those which merely point out connections (Thompson 1995), especially when graphing calculators are present (Porzio, 1997). Thus, one task required matching of given graphs, tables and formulas without the use of graphing calculators, while other tasks required analysis of representativeness and consistency of representations in the presence of graphing calculators. Some follow-up questions were designed to elicit students' understanding and use of connections between representations.

Most interview tasks provided opportunities to use the graphing calculator, but none explicitly requested any particular use of the graphing calculator, and none were accompanied by graphs entered in the calculator. This is a different approach than that of Lauten, Graham, and Ferrini-Mundy (1994). They found, in a case study of one student's understanding of functions and limits, that while she used the graphing calculator when a graph was provided, "she did not independently pick up the calculator to explore an idea before answering a question" (p. 235). In the present study, opportunities to use the graphing calculator were provided in three ways. First, all but one of the interview tasks contained a function formula, allowing students to produce graphs and function values. Second, some tasks provided only one or two representations, to allow students to produce the remaining representation(s). Third, some tasks gave misleading representations, to allow students to generate better representations.

Students' preferences for particular representations and beliefs about the usefulness of different representations were explored. This was motivated by Keller and Hirsch's argument (1998) that, "If a student has a cognitive preference for a representation, it is likely that the strongest connections between representations and between concept and representation are constructed to and from the student's preferred representation" (p. 1). They found, using a pre-test/post-test study, that first semester calculus students do have preferences for specific representations, and that students using graphing calculators were more likely to express a preference for graphs than students not using technology. The current study used two questionnaires to ask students which representation they would prefer to use to (a) describe a function's behavior to a fellow student (first interview), and (b) determine a limit situation (last interview).

Summary

The interview tasks in the current study incorporated several different types of local function behavior, presenting each in graphs, tables and/or formulas. Table 1 shows the types of local function behavior and representations presented to students over the course of the four interviews. The tasks required students to analyze and connect various given and spontaneously produced representations. In addition, students' preferences for graphs, tables or formulas were explored through questionnaires.

Table 1

Local Function Behavior Present in Interview Tasks

Types of Local Function Behavior	Representations		
	Graphs	Tables	Formulas
Limit exists			
Highly oscillatory, damped			+
Monotonic[a], continuous[b]	+	+	
Monotonic, removable singularity	+	+	+
Limit does not exist			
Highly oscillatory, not damped	+	+	+
Asymptotic[c]	+	+	+
Monotonic, jump singularity	+	+	+

Note. Each + indicates that the associated type of function behavior was presented using the associated representation in at least one interview task.

[a]Piecewise monotonic. [b]Students saw continuity either in a graph or a table, not both.

[c]Asymptotic to a vertical line.

Intuitive Limit Conjectures

Most research on students' understanding of limits has focused on their difficulties making the transition from intuitive ideas to more formal ideas. Nevertheless, these studies still have much to say about students' difficulties with the intuitive ideas themselves. In particular, other researchers' observations about students' uses of tables and graphs to make intuitive limit conjectures informed several of the interview questions in this study.

Tables

Research on students' generation of and uses of numerical evidence in conjecturing limits contributed both to the design of tables presented in the interview tasks, and to the creation of questions eliciting students' understanding of the tables. In particular, special care was taken with both the content and format of the tables.

Content. There were two common characteristics of each table. First, each table contained an "ordered pair" for the point of interest, usually indicating that the y-value was undefined. This differs from the course textbook (Stewart, 1995) and from known literature. Others have shown that students tend to equate the computation of a limit with substitution of a function value (Cottril et al., 1996; Ferrini-Mundy & Graham, 1994; Williams 1991). Several first semester calculus students in Keller and Hirsh's study (1998) commented that "a table-of-values gives exact answers but may skip the value of interest" (p. 13). In the current study, it was thought that including this "ordered pair" would prevent students from believing that they simply were not given enough information to address the problem, and thus would encourage analysis of nearby function behavior. Second, the remaining ordered pairs in each table were symmetric about the point of interest with respect to x-values. Differences between adjacent x-values were usually, but not always, successive negative powers of 10. These characteristics follow those of tables given in the course textbook.

Format. The tables were oriented either horizontally or vertically (both orientations appear in Stewart (1995) the course textbook), and formatted so that x-values could be read from smallest to largest. In horizontally oriented tables, this meant that x-values could be scanned left-to-right, from smallest to largest. For vertically oriented tables, this meant that x-values could be scanned top-to-bottom, from smallest to largest. The objective was to order the x-values in the tables to match the ordering of the x-values on the x-axis of a graph.

Such orderings were not always employed in the course textbook or other literature. For example, the course textbook saved space in creating a table for an even function by putting a $\pm$ sign in front of each x-value. Williams (1991) presented a vertically oriented table in which the ordering of x-values to the right of the point of interest was different from that to the left. Both of these approaches force a "reading" of the table that is fundamentally different from the "reading" of a graph to determine one-sided limits. In the first case, the table is scanned top-to-bottom twice, and in the second case, each half of the table is scanned top-to-bottom. On a graph, there is no choice but to scan left-to-right for the left-hand limit and to scan right-to-left for the right-hand limit. Thus in the current study, the tables were formatted so that the two one-sided limits would have to be determined using different scanning directions, as must occur in a graph.

Understanding of tables. A number of questions about students' understanding of limits were created based on other studies. Cottrill et al. (1996) suggest that conjecturing a limit from numerical evidence involves a three-step process: (a) constructing a domain process with x-values approaching the point of interest, (b) constructing a range process with y-values approaching the numerical candidate for a limit value, and (c) coordinating these two processes by applying the function to the x-values in the domain process to obtain the y-values in the range process. Students in their study had considerable difficulty with this, some even unable to construct a domain process, focusing instead on a single point. That students might have difficulty even with constructing a domain process is not surprising in light of research on limits of sequences. Much of this research comments on students' difficulty with deciding whether the sequence {.9, .99, .999, .9999, ...} converges to 1 or to $.\overline{9}$, thinking that $.\overline{9}$ is just less than 1 (Ferrini-Mundy & Graham; Monaghan, 1991; Sierpinska, 1987; Tall & Schwarzenberger, 1978; Tall & Vinner, 1981).

These considerations led this researcher to ask students to describe how they "read" a table, that is, where they focused and what they looked for when using a table to determine a limit situation. In addition, since the presentation of a table did not force students to coordinate the domain and range processes through the function, students were asked to explain how they decided whether the table values were correct and represented the true nature of the function. Often, students were asked to check a few of the ordered pairs in the table, to see if they were correct.

Graphs

Research on students' generation of and uses of graphical evidence in conjecturing limits and analyzing function behavior, particularly in the presence of graphing calculators, guided the presentation of graphs in the interview tasks and formulation of questions about students' understanding of the graphs. Specifically, the format of graphs was a major consideration, as decisions had to be made regarding whether they should be formatted with respect to technology-based conventions or by-hand conventions.

Format. All graphs presented in the interview tasks were produced using *Mathematica*, but were formatted in two different ways, depending on whether students had access to graphing calculators or not. Graphs in the first interview and first task of the second interview were formatted as if they had been drawn by hand. Each of these graphs contained x- and y-axes labeled with appropriate units. Any vertical asymptotes were drawn as dashed lines. A "hole" or deleted endpoint of a piece of a graph was denoted by a small open circle. An included endpoint of a piece of a graph was denoted by a small closed circle. These were standard conventions utilized in the course textbook (Stewart, 1995). All other graphs were formatted as much as possible as if they were drawn using a graphing calculator. Each of these graphs was contained in a rectangular box with the left and bottom edges labeled with appropriate units, so students could tell what viewing window had been used. The graphs themselves were left "as is" to illustrate some of the technological limitations of calculator-based graphs. These conventions for displaying "calculator-based" graphs were also observed in the course textbook.

Understanding of graphs. Several questions about students' understanding of graphs were based on others' work. First, students were asked to describe how presented graphs did or did not reflect the true nature of the function. Tall (1992) noted that standard "by-hand" conventions for denoting holes, asymptotes, and jumps can be confusing to students, since taken at face value, such circles, disks, and dashed lines do not truly represent ordered pairs on the graph. That calculator-produced graphs do not automatically display these characteristics with the same conventions causes even more confusion (C. G. Williams, 1993).

Second, students were asked to describe how they "read" a graph, that is, where they focused and what they looked for when using a graph to determine a limit situation.

This paralleled a question asked of students using tables to determine limit situations, in an attempt to discover any parallel difficulties in using graphs to determine limit situations. There was reason to expect that students' descriptions based on graphs would refer to movement along the curve. Lauten, Graham and Ferrini-Mundy (1994) in a case study of one student's understanding of limits noted several instances in which she referred to points moving along the curve, but not quite reaching the limit point. Williams (1991) found that all ten students in his study, at some point, believed that limits involved motion along the graph.

Summary

The tables and graphs presented to students in the interview tasks of the current study were carefully constructed so that the content and format of each representation largely matched those in the course textbook (Stewart, 1995). The exceptions were to include the "ordered pair" of interest in each table, and to order table x-values from smallest to largest, to aid in comparison of tables and graphs. Some graphs were provided in a "by-hand" format and others in a "by-calculator" format. Follow-up questions were designed to elicit students' methods of interpreting numerical and graphical evidence to determine limit situations.

Limitations of Numerical and Graphical Evidence

Several limit tasks from the literature and from the course textbook (Stewart, 1995), focusing on both mathematical and technological limitations, were adapted for the current study. Typically such tasks involve presentation or generation of conflicting information, with the hope that mathematical resolution of the conflict will result in a more correct, i.e. more formal, understanding of the limit concept. Students' reactions to such tasks have been documented in other studies, and contributed significantly to the adaptations of tasks in the current study.

Mathematical Limitations

Two of the interview tasks in the current study included tables or graphs that were intentionally misleading, due to mathematical limitations. A mathematical limitation arises when idiosyncrasies of the function are improperly accounted for in producing a table of values or a graph. This occurs when attention is restricted to x-values insufficiently close to the point of interest, or when a poor sample of x-values yields a non-representative table or graph. Such tasks have appeared in the literature (Williams, 1991) and also are contained in the course textbook (Stewart, 1995).

Insufficiently close x-values. One interview task, adapted from Williams (1991), provided a rational function with a vertical asymptote at $x = 0$ which was not apparent in the given table and graph, due to x-values chosen too far from 0. The objective was to predict the existence or non-existence of a limit as x approached 0. In this case, the limitation imposed by x-values insufficiently close to 0 manifested itself in the calculator-based graph as well as in the table, although in the graph, this was also tied to pixel limitations. Williams's version presented only the table and the formula, each initially independent of the other. After his students had reached contradictory limit conjectures, he informed them that the formula matched the table. Some rejected this, and others automatically changed their (correct) formula-based conjectures to match their table-based conjectures. The adaptation for the current study presented students with the formula, table and graph of the function, in the interest of determining whether the presence of all three representations would assist students in recognizing the misleading nature of the table and graph.

Poor sample of x-values. One interview task, adapted from an example in the course textbook (Stewart, 1995), provided students with the function $\sin(\pi/x)$, and a table and graph to use in predicting the limit as x approached 0. The table was misleading in the sense that all negative x-values had corresponding y-values of 1, and all positive x-values had corresponding y-values of -1. Stewart's version presented function values that all turned out to be 0, generated using "typical" x-values approaching 0, such as .1, .01, .001, etc. The adaptation for the current study used a different table, in the interest of seeing whether students would reject it, and produce their own function values using the "typical" x-values.

Technological Limitations

Three interview tasks included poor tables or graphs, due to limitations of the graphing calculator. Both subtraction inaccuracies (Tall, 1992) and pixel limitations (Tall, 1990) have been cited as primary causes of poor tables and graphs that need to be pointed out to students. In addition, the course textbook (Stewart, 1995) contains both examples and exercises intended to elucidate these points.

Subtraction inaccuracies. One interview task, adapted from an exercise in the course textbook (Stewart, 1995), raised the issue of subtraction inaccuracies. Subtraction errors occur when two numbers very close to one another are subtracted. If two numbers with identical decimal expansions past the carrying capacity of the calculator are subtracted, the calculator will erroneously compute the difference as 0. Subtraction inaccuracies can manifest themselves in both calculator-based tables and in calculator-based graphs. This led to two adaptations of Stewart's exercise for the current study, one numerical and the other graphical.

Pixel limitations. Two interview tasks (described in the section on mathematical limitations, adapted from Stewart (1995) and Williams (1991)) raised the issue of pixel limitations. Pixel limitations occur when interesting function behavior is "hidden" between two pixels on a calculator-based graph. In one case, a graph with highly oscillatory behavior near the y-axis was only somewhat well-represented, and in the other case, a vertical asymptote at $x = 0$ did not appear on the graph.

It is possible for pixel limitations to manifest themselves in calculator-based tables, if the tables are generated from x- and y-coordinates displayed while using the trace feature. Such tables were not presented in any tasks in the current study, but students' propensity to use the trace feature to generate numerical information was tracked. This is an attempt to address Lauten, Graham and Ferrini-Mundy's questions (1994) about whether a tendency to trace along a curve creates difficulty in interpreting y-values as vertical distances on graphs, or whether it contributes to an image of ordered pairs as points moving along a curve.

Reactions to Mathematical Conflict

The current study's tasks on limitations of tables and graphs intentionally presented mathematically conflicting information. These mathematical conflicts were explicitly pointed out to students, and practical (non-mathematical) resolution of those conflicts was discouraged. Each task in this study was introduced and motivated with a short paragraph suggesting that resolution of unexpected or conflicting information might be necessary in determining a limit situation based on a table or graph. When a task presented conflicting information, this was explicitly stated. This was followed by questions asking for explanation of the conflicting information either on the task itself or in subsequent interaction. In addition, students were asked if the given table and/or graph could be augmented, modified, or "fixed" in such a way that the conflict was resolved.

This approach to mathematical conflict was based on difficulties other researchers have had with provoking mental conflict and mathematical resolution. Williams (1991) investigated second semester calculus students' understanding of limits of functions by attempting to provoke conflict, and found that provoking conflict was extremely difficult to accomplish due to students' beliefs that mathematics is a collection of arbitrary and disconnected facts, formulas, rules and procedures, with no real cohesive harmony. Moreover, on the few occasions when conflict was evoked, students responded in unexpected ways, for example, by deciding that the conflicts represented anomalous circumstances, and so they need not concern themselves with resolving the conflict. Sierpinska (1987) used this same approach with similar results, suggesting the modification employed in the current study.

Summary

The interview tasks in the current study were designed to point out several different causes of limitations of both numerical and graphical evidence in analyzing limit situations. This involved presenting conflicting representations and requiring students to analyze and resolve the conflicts. Table 2 shows the instances of poor representations presented to students over the course of the four interviews, and the causes of each.

Table 2

Numerical and Graphical Limitations Present in Interview Tasks

Causes	Representations: Graphs	Tables
Mathematical limitations		
Insufficiently close x-values	+	+
Poor sample of x-values		+
Technological limitations		
Subtraction inaccuracies	+	+
Pixel limitations	+	

Note. Each + indicates that the associated limitation was presented using the associated representation in at least one interview task.

Intuitive-Analytic Limit Conjectures

To evaluate students' acquisition of the fourth element of intuitive-analytic understanding of limits, the final interview task in the current study was designed to detect whether students would spontaneously produce multiple sources of evidence in determining a limit situation. In fact, students had the opportunity to determine the limit situation for certain, using the Squeeze Theorem, allowing questions eliciting students' beliefs and understanding about the mathematical certainty of their evidence.

Multiple Sources of Evidence

Students in the current study were given only the formula of a function with highly oscillatory, damped behavior around the origin, and asked to determine the limit as x approached 0. Some researchers have found that students often rely on only one source of evidence in making a limit conjecture, with a strong preference for ad hoc analysis of the function formula (Williams, 1991), or analysis of the graph (Ervynck, 1981; Lauten, Graham & Ferrini-Mundy, 1994, Williams, 1991). Thus, the final interview task in the current study was designed so that if students produced only one source of evidence, they were asked to produce more evidence, either supporting their original conjecture or suggesting a modified conjecture.

Mathematical Certainty

Students were asked in several different ways how certain they were about their answer. First, students were asked to respond to two multiple-choice questions:

(1) How certain are *you* that your conclusion is correct?
___ absolutely certain
___ fairly certain, but there is room for doubt
___ not at all certain
(2) How certain do you expect *everyone else* should be that your conclusion is correct?
___ absolutely certain
___ fairly certain, but there is room for doubt
___ not at all certain

These multiple-choice questions were inspired by Mason, Burton and Stacey's approach (1982, as cited in Tall, 1992) to eliciting increasing levels of conviction from students about their conjectures. The choices were adapted from Fishbein, Tirosh and Melamed's questions asking students for their level of confidence about conjectures involving the notion of infinity. Second, students were asked whether they felt they had produced an estimated guess or had determined the limit for certain. Finally, students who expressed doubts were asked to explain the source(s) of their doubts.

It was anticipated that students might not articulate their sense of certainty or doubts in terms of the mathematical soundness of their methods. This expectation stemmed from other researchers' observations about students' perceptions of mathematical truth. Williams (1991) found among students in his study that,

> It was an article of faith that no general description of limit worked for all cases: "I don't think there is a definition that is going to fulfill every function there is." Mathematical truth, then, was truth for particular cases (p. 232).

He also found that some students held incomplete, but very robust conceptions of limit which could be reconciled with many counterexamples "by considering them either as exceptions or as cases to which limits would not apply" (p. 232). Lauten, Graham and Ferrini-Mundy (1994) and Ferrini-Mundy and Graham (1994) had similar findings.

Summary

The interview tasks in the current study culminated in a problem designed to detect students' spontaneous methods for determining a limit situation, and to elicit their beliefs about the mathematical certainty of their methods. Sources of each student's apparent certainty or doubt were sought as comparisons to those revealed in the existing literature.

CHAPTER 3

Method

This research employed a qualitative design involving four interviews with each of 10 first-semester calculus students. During each interview, students worked on mathematical tasks involving ideas related to limits. The primary data consist of students' written work and oral comments elicited during the interviews. These data were analyzed for patterns of analytical thinking and knowledge use employed by each student, within each interview and across all four interviews.

Participants

Sampling

Participants were students in a single section (128 students) of a first-semester calculus class (described below) at a large comprehensive state university. Students were selected on the basis of a background questionnaire (see Appendix A) given during the first week of the fall 1998 semester. The background questionnaire asked students to indicate previous mathematics courses, experience with graphing calculators, experience with the topic of mathematical limits, and whether they would be "willing to participate in a study about how first-semester calculus students develop an understanding of mathematical limits". The criteria for selection, in addition to a willingness to participate, were

(a) no previous calculus course, or

(b) little to no experience with the topic of mathematical limits (determined by a response of: "none", "briefly introduced", or "I have learned the techniques but no theory" to the questionnaire item asking about prior experience with the topic of mathematical limits.)

Among the 46 students indicating a possible willingness to participate, 24 students satisfied one or both criteria above, and were given more information about the study (see Appendix B). Of these 24 students, 14 agreed to participate in the study, and 10 were included in the final sample[1]. Among these 10 students, 5 were enrolled in the researcher's assigned discussion sections, and 5 were students in another teaching assistant's discussion sections[2].

Protection of Human Subjects

The use of human subjects in this research was approved (see Appendix C) by the University of Oklahoma's Institutional Review Board.

To protect their confidentiality, subjects are not identifiable from raw or refined data. Subjects' names and student identification numbers on the background questionnaires were removed. Each student was assigned a numerical code, which was the only identifying code on his or her background survey, written interview work, and audio cassette labels. In addition, each student was assigned a pseudonym to be used in

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[1] Four students were dropped. Two could not schedule their first interviews within the third week. One student was not a native speaker of English, making it difficult to distinguish his language problems from his mathematical difficulties. The fourth student exhibited extraordinary difficulty with even very basic questions, suggesting he was inappropriately placed in calculus, and thus not a member of the population of interest.

this dissertation and any subsequent publications.

Subjects in this study were volunteers. To assure them that refusal to participate would involve no penalty to their course grade, students were informed of this three times: on the background survey, information handout, and consent form (see Appendix D). To support the assurance that students need not fear grade penalties, they were informed on the information handout that the course instructor would not know the names of the study participants.

Subjects were informed both on the information handout and the consent form that time devoted to the interviews might cost them study time. They were assured that if they felt the time commitment was having a negative impact on their course work, then they were free to discontinue their participation, with no penalty to their grade.

Course Description

This course followed a lecture-discussion format. Fifty-minute lectures were presented to the entire class on Mondays, Wednesdays and Fridays by a full professor, and 50-minute discussions were held for groups of 20-25 students by the researcher and another graduate teaching assistant. All six discussion sections met between the Wednesday and Friday lectures, primarily to discuss homework, which was collected weekly on Fridays. Specific emphases varied from one discussion section to another each week, depending on students' questions.

Course topics roughly followed those in the textbook, Stewart's *Calculus*, 3rd edition (1995). A review of functions was presented over the first five lectures. Then the tangent line and velocity problems were introduced, which motivated the limit concept.

[2] Each teaching assistant was responsible for grading homework of his/her own students. Overall, these homework assignments counted 5% towards students' final course grades.

The limit concept was first treated informally, followed by limit laws. The ε–δ definition of a limit was *not* covered. Then concepts of continuity and differentiability were presented, followed by applications of derivatives.

Graphing calculators were required of all students in the course, and were used during lectures and discussions, primarily to compute function values and draw graphs of functions, often in tandem. Graphs and numerical data were nearly always analyzed in terms of expected behavior, function properties and the limitations of technology. The textbook supported the use of graphing calculators by presenting examples and exercises that illuminated their advantages and disadvantages. Graphing calculator-based homework problems were assigned on a regular basis to reinforce these ideas.

The professor was approached before the study for permission to conduct the study with his students. This particular professor was known to use graphing calculators in previous semesters, and in particular, to focus attention on their weaknesses. He was informed that the study would investigate students' intuitive notions of the limit concept through interviews and would require no effort on his part. Initially, he expressed reservations about how helpful his instruction would be to students involved in such a study, since he had no intention of covering the ε–δ definition of limit, and didn't really perceive his instruction in this course to involve "teaching limits". After assurances that only intuitive limit ideas were being investigated, and that the omission of the ε–δ definition of limit would not negatively impact the study, he agreed.

He was not requested to design his instruction or tailor homework assignments to parallel tasks in the study. He did see the tasks in the study, but not until the third week of lectures, when the first interviews were already in progress, and most of the material on intuitive limit ideas had already been presented in the course.

Interview Tasks

Four sets of tasks were constructed, evolving from examining local behavior of functions to conjecturing the existence or non-existence of specified limits, based on tables or graphs, in increasingly difficult situations. The four interviews roughly followed the four elements of intuitive-analytic understanding of the limit concept, with each interview focusing on one or more of these elements. Graphing calculators were allowed on the last three sets of tasks.

Several common threads linked the tasks to one another. First, consistency and representativeness of graphs, tables and formulas were emphasized throughout. Second, numerous examples of local function behavior were provided, but for most limit situations, the functions were not defined at the point in question. Third, each task was introduced with a motivating idea.

Interview 1 – Multiple Representations of Local Function Behavior

This interview focused on the first element of intuitive-analytic understanding of limits, namely, the ability to analyze functions locally in both graphical and numerical settings. The objectives were to examine students' methods of recognizing the same local function behavior in graphical, numerical, and symbolic representations; and determine students' preferences for graphs, tables, or function formulas.

The task (see Appendix E) involved five three-way matching problems. Students were given formulas for five functions, all undefined at $x = 3$, but with different behavior near $x = 3$:

$$f(x) = \frac{1}{x-3}$$

$$g(x) = \frac{1}{(x-3)^2}$$

$$h(x) = \frac{x^2 - 5x + 6}{x - 3}$$

$$p(x) = \begin{cases} x-2 & \text{if} \quad x < 3 \\ x-3 & \text{if} \quad x > 3 \end{cases}$$

$$t(x) = \sin\left(\frac{1}{x-3}\right)$$

Students (a) chose which graph (among 8 choices) matched each function, (b) chose which table (among 8 choices) matched each function, and (c) explained in 3-5 written sentences how they chose the graph and table for a particular function.

There were five versions of this task: in part (c), each student explained his or her reasoning for only one of the five functions. All tables contained the same x-values, starting with $x = 2$, ending with $x = 4$, and intermediate x-values approaching 3. Each table matched one of the graphs, so it was possible to match a table and a graph, and for neither to match the formula.

The first of two questionnaires was given at the end of this interview. Four multiple-choice items comprised this questionnaire (see Appendix F), each requesting a brief explanation. These items asked for preferences for graphs, tables, or formulas; and for beliefs about the usefulness of these representations when describing a function to a fellow student.

Interview 2 – Intuitive Limit Conjectures and their Qualifying Factors

This interview focused on both the second and third elements of intuitive-analytic understanding of limits. First, the ability to draw correct intuitive limit conclusions from accurate graphs and tables was addressed. Second, the possibility was raised that graphs and tables can be inaccurate. The objectives were to explore how students conjectured a limit's existence or non-existence based on a representative graph or table; examine how students decided whether a graph or table represented the true nature of a function's local behavior; and determine if students recognized causes of poor graphs or tables.

Two different tasks were given, each designed in graphical and numerical versions (see Appendices G and H). Students completed either the graphical versions, or the numerical versions of both tasks. The first task[3], presented a representative graph (table of ordered pairs) of a rational function, without the function formula. Students (a) conjectured the existence or non-existence of three limits, and (b) for each potential limit, either conjectured a limit value or explained why the limit did not exist.

The second task[4] presented three graphs (tables of ordered pairs) of the function

$$g(x) = \frac{\tan(x) - x}{x^3},$$

along with the function formula. Only one graph (table) represented the true nature of the function near the origin. Students (a) chose the most representative graph (table), (b) determined what the chosen graph (table) implied about the limit as x approached 0, and (c) explained how the other graphs (tables) misled them about $\lim_{x \to 0} g(x)$.

Interview 3 – Multiple Representations of Limit Situations

This interview continued the focus on the third element of intuitive-analytic understanding of the limit concept, and began to focus on the fourth element. Here, the possibility was raised that graphs and tables can be inconsistent with one another, or

[3] This problem was not explicitly adapted from any source, but is quite similar to several problems in the textbook, (Stewart, 1995).
[4] This problem was adapted from an exercise in the textbook (Stewart, 1995). See Discussion 4 in Appendix O for Stewart's version.

consistent with one another but still inaccurate. In addition, the issue of multiple sources of evidence to justify a limit conjecture was introduced. The objectives were to examine how students decided, in light of conflicting or consistent representations, when a graph or table represented the true nature of a function's local behavior; and see if students generated new and better information about particular limit situations in the face of poor and conflicting information.

Two tasks comprised this interview (see Appendix D). In the first task[5], students were given the formula

$$t(x) = \sin\left(\frac{\pi}{x}\right),$$

a misleading table of ordered pairs, and a fairly representative graph of a function. The table and graph clearly contradicted one another. Students (a) decided if the table was representative, (b) decided if the graph was representative, and (c) decided what was true about $\lim_{x \to 0} t(x)$.

In the second task[6], students were given the formula

$$h(x) = x + 1 + \frac{1}{10^{20} x},$$

a misleading table of ordered pairs, and a misleading graph of a function. The table and graph did not contradict one another, except at the point at which the limit was to be determined. Students (a) decided if the table and graph contradicted one another, (b) decided if the table was representative, (c) decided if the graph was representative, and (d) decided what was true about $\lim_{x \to 0} h(x)$.

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[5] This problem was adapted from an example in the textbook, (Stewart, 1995).

[6] This problem was adapted from a task given by Williams (1991).

Interview 4 – Intuitive-Analytic Limit Conjectures

This interview focused on the fourth element of an intuitive-analytic understanding of limits, namely the ability to produce multiple sources of evidence to justify a limit conjecture, and know whether this evidence leaves room for doubt. The objectives were to describe students' chosen strategies in analyzing a limit situation; and evaluate students' convictions about their conjecture, and about the value of intuitive limit ideas in analyzing limit situations.

The task involved one multi-part problem[7] (see Appendix J). Given just the function formula

$$f(x) = x^2 \cos\left(\frac{1}{x}\right),$$

students decided what was true about $\lim_{x \to 0} f(x)$, producing their own evidence. In addition, students indicated whether they were convinced their conjecture was right, and whether others should be convinced their conjecture was right. In the event that students failed to produce multiple sources of evidence to justify their conjecture, they were asked to produce additional evidence to either support their original conjecture or to indicate that their original conjecture should be modified.

The second of two questionnaires was given at the end of this interview (see Appendix K). This was a follow-up to the first questionnaire, with four parallel items. These items asked students for their preferences for graphs, tables, or function formulas; and for their beliefs about the usefulness of these representations when making educated guesses about limit situations.

[7] This problem was not explicitly adapted from any source, but is quite similar to several problems in the textbook (Stewart, 1995).

Data Collection

Random assignment. The five versions of the task in interview 1 and the two versions of the tasks in interview 2 were counterbalanced, and 14 file folders were produced, each containing one of the ten possible collections of interview tasks. These 14 collections of interview tasks were then randomly assigned to subjects prior to the first interview.

Interview protocols. For each phase of interviews, a pre-established protocol was used (see Appendix L) to ensure timely completion of the interviews and uniformity of oral instructions and baseline oral questions. The baseline oral questions for the second, third and fourth sets of interviews were produced after preliminary analyses of the previous set of interviews. This allowed for follow-up questions and probes to issues that arose in earlier interviews. In addition to following the interview protocols, the researcher audiotaped each interview, took notes during each interview, and used a stopwatch to note time periods devoted to task completion, and to interactions.

Class notes. To provide context to students' responses in the interview tasks, the researcher took extensive notes during all lectures, writing what the instructor wrote on the blackboard, describing graphing calculator tasks, writing the instructor's questions and especially writing comments indicative of his viewpoints on graphing calculator-based conclusions.

Data Preparation

Demographics. The background data were summarized by gender and by high school graduating class. In addition, demographics on the final sample of students and the entire class were compiled for comparison purposes.

Transciptions. Interview tapes were transcribed as completely and as accurately as possible. In particular, all verbal interactions were transcribed, brief pauses (usually shorter than 5 seconds) were indicated with ellipsis marks (...), and long pauses (usually at least 5 seconds) were timed with a stopwatch so the length of a pause could be noted. Students' recorded comments were compared with their written work and the researcher's interview notes, leading to parenthetical remarks within the transcripts describing students' actions during the interaction. In some cases, there were brief periods of audible but indeterminable dialogue. These periods were indicated with blanks (____).

Content summaries. Class notes were summarized to establish the course content to which students were exposed prior to each interview. To validate their accuracy, these summaries were compared to notes taken by the other teaching assistant. In addition, the content was cross-referenced with students' actual interview dates, to detect possible differences due to different content exposure.

Data Analysis

Data Matrices

Three data matrices were designed to help organize the task-response data for each student. These three matrices (described in detail below) focused on (a) intuitive-analytic understanding of limits, (b) influence of function knowledge, and (c) influence of graphing calculator use within each of the four interviews. Each cell of each data matrix contained relevant evidence about a particular student's responses, in a particular interview, pertaining to a specific element of that matrix's theme.

The evidence in the data matrices consisted of students' analytical thinking and knowledge use exhibited in their written and oral responses, or indicated in the researcher's interview notes. *Analytical thinking* is taken to mean students' thought-processes, whether correct and appropriate or not, employed in making decisions during the course of solving a problem. *Knowledge use* is taken to mean a student's collection of "knowledge", correct and appropriate or not, on which his or her thought processes depend.

Matrix on intuitive-analytic understanding of limits. This matrix contained a column for each interview, and a row for each of the four elements of intuitive-analytic understanding of limits. The first interview did not address the second, third, or fourth elements of intuitive-analytic understanding of limits. The second interview did not address the fourth element of intuitive-analytic understanding of limits. Thus there were four empty cells in this matrix.

Matrix on influence of function knowledge. This matrix contained a column for each of interviews 2, 3, and 4, and four rows addressing the sub-questions of research question 2: (a) analysis of local behavior, (b) consistency among representations (c) accuracy of representations, and (d) evaluation of the influence of function knowledge. A separate cell with data from interview 1 provided context for the data on function knowledge in subsequent interviews.

Matrix on influence of graphing calculator use. This matrix contained five columns, one for each task on which a student might have used a graphing calculator: task 2 of interview 2, tasks 1 and 2 of interview 3, questions 1-3 and questions 4-6 of interview 4. The four rows addressed the sub-questions of research question 3. Each cell of the first row indicated whether and when the student used graphing calculator, by marks in a checklist: none, before writing, during writing, and/or after writing (during interaction). Each cell of the second row described how the student used the graphing calculator and whether those uses were spontaneous or prompted, by marks in a checklist: draw graph, set window, zoom in/out, trace, other, compute values, unknown use. The third row was devoted to a written description of the order of events for that task. Finally, the fourth row was devoted to evaluation of the influence of graphing calculator use.

Detection, Validation and Use of Patterns

By student. For each student, data were examined within the three task-based matrices to generate preliminary categories of analytical thinking and knowledge use. These preliminary categories were modified and refined by re-analyzing the written work, transcripts, and interview notes to identify supporting or contradictory data.

Each student's responses to the two questionnaires were transferred to a fourth data matrix for comparison of responses to parallel questions. Relevant transcript data, whether supportive or contradictory, were included in the matrix to allow a comparison of questionnaire responses to spontaneous comments during the interviews.

Across students. The three sets of task-based data matrices were analyzed in a three-step procedure. First, the data were analyzed within interviews. Patterns of analytical thinking and knowledge use within each interview provided evidence addressing the research objectives underlying that interview. Second, the matrices were analyzed within matrix sub-themes. Patterns of analytical thinking and knowledge use within each matrix sub-theme provided evidence addressing the sub-questions of the overall research questions. Finally, the evidence accumulated in the first two steps was analyzed to detect emerging relationships between the answers to the sub-questions, providing a plausible picture of these students' intuitive-analytic understanding of limits, and the influence of their function knowledge and graphing calculator use on this understanding.

CHAPTER 4

Data Summaries

Data are summarized in three sections, beginning with data from the background questionnaire. Interview data are summarized by interview, and illustrated using individual students' responses. Finally, interview questionnaire data are summarized by item.

Students' Backgrounds

Data from the background questionnaire (Appendix A) are presented in Appendix M. Each table reports demographics by gender and high school graduating class, and includes totals for the entire course and for the final sample of students in the study.

Relative to the entire course, several biases appeared in the final sample, some unintentional and others due to the selection criteria. First, the final sample was unintentionally biased in favor of male participants, against engineering majors, and in favor of little graphing calculator experience, as is shown in Tables M1, M2, and M4, respectively. Second, the final sample was intentionally biased towards students with no prior calculus courses, and with little experience with limits, as is shown in Tables M3 and M6, respectively.

On the remaining questionnaire items, the distributions of responses were roughly comparable for the entire course and the final sample. In particular, both groups were primarily freshmen with few indicating an intention to major or minor in math (see Table M2). Most students expected the graphing calculator to be very helpful to them in learning calculus, liked computers and calculators a lot, felt they had average to high aptitude with computers and calculators, and felt they had average to high aptitude with math (see Tables M4, M5 and M6).

Interviews

Interviews were conducted between the third and eighth weeks of the fall 1998 semester (see Appendix N for a schedule). The interview data are presented sequentially. To provide context for each interview, the course content up to that point in the semester is briefly summarized (see Appendix O for full details on lecture and homework content). Overall results for each interview are then presented, along with selected student responses to illustrate or qualify general conclusions.

Interview 1

Context. The lectures prior to interview 1 involved functions, intuitive limit ideas, limit laws, slopes of tangent lines, and instantaneous velocity. The graphing calculator was introduced in the first week with a worksheet (see Appendix P), and from that point on, was used to compute function values and draw graphs, nearly always in tandem. Graphing calculator-based graphs were nearly always analyzed with respect to computed function values, expected function behaviors, or related graphs, often for the purpose of pointing out the limitations of the graphing calculator.

General results. Students' approaches to this triple-matching task were characterized by two elements. First, their formula-based expectations were minimal, especially with asymptotic and oscillatory behaviors. Either they did not immediately recognize a function's local behavior, or one feature dominated their analysis. Second, ordered pairs ruled their choices, at least initially. Jason and Brad's approaches to this task illustrate these conclusions.

Students' difficulties with asymptotic and oscillatory behaviors can be seen in their graph-table choices for the functions

$$f(x)=\frac{1}{x-3} \text{ and } t(x)=\sin\left(\frac{1}{x-3}\right).$$

Table 3 shows the distribution of students' triple-matches on this task. Each table cell represents one of the 64 possible graph-table choices. Graphs (rows) and tables (columns) are listed so that the eight main diagonal cells represent the eight correct graph-table matches. In addition, the first five main diagonal cells represent the graph-table pairs corresponding to the functions f, g, h, p, and t, respectively. A letter within a cell indicates that a student matched that function's formula with that cell's graph-table pair. For example, the upper left cell contains four f's, indicating that four students matched the formula for function f to graph C and table H. The table also shows initial choices in parentheses. In every case, an initial choice was changed to a correct choice.

Notice that only four of the ten students selected the correct graph-table pair for the function f, and two of those four initially chose incorrect graphs. Four students selected graph A: a piecewise linear graph with a jump at $x = 3$. Three students (two, initially) selected graph G, a highly oscillatory graph. One student could not decide on a graph. The conclusion that students did not expect asymptotic behavior from this function is reinforced by their apparent success with the function g, which also has asymptotic behavior.

Table 3

Distribution of Triple Matches in Interview 1 Task

	Tables								
Graphs	H	F	A	B	D	G	E	C	Blank
C	ffff	(g)					g		
H	(f*)	ggggg gg							
D			hhhhh hhhh p	(h)(h)					
E			(h)	ppppp pppp					
G	(f)(f)f	t			ttttt				
B		g							
F		g			t		t		
A	f				h			fff	
Blank	f								tt

Note. A letter within a cell indicates that a student matched that function's formula with that cell's graph-table pair. Each main diagonal cell represents a matching table and graph, the first five representing correct matches for the functions *f*, *g*, *h*, *p* and *t*, respectively. Letters in parentheses represent initial (or second*) choices. Letters without parentheses represent final choices.

Notice that only five students selected the correct graph-table pair for the function t. One student selected the correct graph, and two others selected the other oscillatory graph. The remaining two students could not decide between the two oscillatory graphs. Some students expected oscillatory behavior since the function formula involved the sine function, but simply guessed between the two oscillatory graphs.

Students' dependence on computing ordered pairs was evident from both their problem-solving approaches and their explanations of their choices shown in Table 4 and Table 5. Nine students actually computed ordered pairs with pencil and paper with at least one function while completing this task. Five students' explanations referred to "plugging in points" or "substituting values".

Table 4

Written Explanations for Correct Triple Matches in Interview 1 Task

Students	Explanations
	$g(x)=\dfrac{1}{(x-3)^2}$
Brad graph H table F	To find the graph, I found the graph that skyrocketed as it approached '3'. Since I know that when you take decimals and square them and then divide very small #'s by 1 you get huge outputs.
Laura graph C, H table F	The table I figured out by plugging in points and using the process of elimination. For a while, I though [sic] the graph was C because I figured that the higher the x-value (after 3) the closer the graph would be to zero. Then (as I was writing this) I realized that the same would be true with negative values with a high absolute value. Because there is a a [sic] square in the denominator and a positive number on top the graph can never cross the x-axis and become negative so it is H. The numbers really close to 3 result in high values.

(table continues)

Students	Explanations
	$h(x)=\dfrac{x^2-5x+6}{x-3}$
Brandon graph D table A	I factored the polynomial into $\dfrac{(x-3)(x-2)}{x-3}$ and reduced it to $(x-2)$ to help me find out what the line looked like. But I kept in mind that the function was undefined at $x=3$.
Matt graph D table A	My first reaction was plugin [sic] numbers from tables. Then it was factored out on top and $(x-3)$ was cancelled on top and bottom leaving $(x-2)$. Then I used the table to find the graph.
Paul graph D table B, A	I factored the function into $\dfrac{(x-3)(x-2)}{x-3}$. I then proceeded to cancel out the 2 (x-3)'s, leaving me with the equation $h(x)=(x-2)$. Then, I plugged in points to find the table and graph, while seeing if the answers I got made logical sense with what I thought the graph would look like.
	$p(x)=\begin{cases} x-2 & if \quad x<3 \\ x-3 & if \quad x>3 \end{cases}$
Josh graph E table B	First, I saw that there was a hole at 3. I substituted values for the first equation and looked it [sic] its graph. Then I did the same for the second equation.
	$t(x)=\sin\left(\dfrac{1}{x-3}\right)$
Jason graph G table D	Using the first equation, I noticed that y rose quicker the closer it got to 3 which would make the period on the last equation shorter as it approached 3.
Mike graph G table D	I used the info that $0.84147=\sin 1$ so I guessed that $-0.84147=\sin\text{-}1$. Using this I picked table D and by using the stats on table D I chose graph G.

Table 5

Written Explanations for Incorrect Triple Matches in Interview 1 Task

Students	Explanations
	$f(x) = \dfrac{1}{x-3}$
Mark graph A table C	First, I just put the variables into the functions and tried to find the set of numbers that held true throughout the entire set. After I picked out those, I used them to see if those points matched the lines on the graph. The t(x) function has me stumped because I just can't remember what sin-1 is.
	$g(x) = \dfrac{1}{(x-3)^2}$
Alan graph C table E	I tried to simplify the numerator first. Next I substituted x values. Finely [sic] I made an educated guess as to the graph.

Jason's approach. Ultimately, Jason made correct matches for each function, but he had trouble with the graph of $f(x) = 1/(x-3)$. He began by choosing graph G, with the highly oscillatory behavior, for this function. In fact, Jason began the task by computing ordered pairs: he computed the ordered pairs (0,-1/3), (1,-1/2), and (2,-1), all of which appear to be points on graph G. Jason was one of three students who did this. All three were convinced (at least initially) by their ordered pairs that this was the correct graph of *f*.

In explaining his choices for the trigonometric function, Jason appeared to have made a reasonably good connection between the functions $f(x) = 1/(x-3)$ and $t(x) = \sin(1/(x-3))$, writing,

> Using the first equation, I noticed that y rose quicker the closer it got to 3 which would make the period on the last equation shorter as it aproached [sic] 3.

But there were two oscillatory graphs, and his explanation does not indicate why he chose one over the other. When asked about this, he revealed that he wasn't really sure about his answer, saying,

> I knew it was one of these two [pointing to graphs F and G] and I kind of guessed it was that one [graph G] because that one looked more like a sine wave than that one... but I think it may have been that one too. I don't know. It was one of the two.

Upon asking him what he meant by "this one looks more like a sine wave", he replied

> A sine wave kind of goes like this: [drawing a graph of sin(x) on [$-\pi$, π] without axes]... and that one [pointing to graph F], it started out, it kind of was a W or something.

Essentially, Jason's focus was on the oscillatory behavior of the function, and he appeared to apply his knowledge of sin(x) by way of a global comparison with graphs F and G.

Brad's approach. Brad made three incorrect choices. Like Jason, Brad selected graph G for the function $f(x) = 1/(x-3)$, based on ordered pairs he had computed. However, Brad did not recognize this error, and it subsequently caused erroneous choices for the function $t(x) = \sin(1/(x-3))$.

Upon asking Brad if he was able to use his knowledge of some functions to help him figure out some of the others, he responded,

> I used process of elimination to figure out which one the sine one was. Because I knew it had to be one of these, either F or G. Since I already put G for the first one, I just chose this other one, and then I figured out the corresponding table.

He clearly knew the trigonometric function had to match either graph F or graph G, and yet saw no contradiction in having matched graph G to the function $f(x) = 1/(x-3)$. In fact, at no point during the interaction did this occur to him.

Brad's written explanation of his choices for the function $g(x) = 1/(x-3)^2$ makes it clear that his understanding of this function's local behavior was based on computing ordered pairs:

> To find the graph I found the graph that skyrocketed as it approached '3'. Since I know that when you take decimals and square them and then divide very small #'s by 1 you get huge outputs.

In fact, he did compute ordered pairs for the function g as well as for the functions f and p, the piecewise linear function. At the same time, it is clear that, although he knew the graph of g "skyrocketed" as it approached 3, this knowledge did not transfer to the graph of f. Brad's responses to this task suggest that he has little or no sense of what type of local function behavior to expect based on a function formula.

Interview 2

Context. At this point in the semester, all of the major limit ideas had been presented, including limit laws, the Squeeze Theorem, and computation of derivatives using limits. The graphing calculator had been used in class to point out limitations of both graphs and tables in predicting limits. Students had just completed the third homework assignment the previous Friday, which covered intuitive limit ideas both graphically and numerically. In particular, students saw a problem similar to task 2 in lecture, and were assigned the textbook problem on which task 2 was based in homework. See lecture 7 and discussion 4 summaries in Appendix O.

General results. First, in conjecturing limits from tables and graphs, students initially tended to restrict their attention to particular locations in the table or graph. For example, with tables, several looked at ordered pairs only to one side of the point in question, or, even more restrictive, looked at just one ordered pair next to the point in question. For graphs, some first looked for a point or hole at the pertinent x-value. Second, many students turned to the graphing calculator to draw a graph to compare to the given graphs or tables in the second task. These students took the calculator's graph as the standard. Third, students' understanding of the causes of poor graphs or tables was very vague. They had a general notion that round-off errors would cause problems, and believed that the calculator could not "handle" small numbers. The effects of these limitations on numerical computations seemed quite believable (although mysterious) to most students, but they seemed unable to grasp the effects that computational limitations would have on the calculator's ability to draw accurate and representative graphs.

Students' written responses to the three limit situations in this interview's first task are presented Table 6, Table 7, and Table 8, respectively. Table 6 indicates that all nine students "correctly" determined the existence of a limit at the removable singularity, but Mark believed this was simply due to the function being undefined at $x = 1$. Eight students "correctly" determined that the limit was 6, but Laura determined this by an "averaging" process based on the function values in the table closest to $x = 1$. Table 7 indicates that eight students "correctly" determined the non-existence of a limit at the vertical asymptote, but most explained this by saying the left hand limit was different from the right hand limit, with Josh and Paul not recognizing the vertical asymptote. Mark reiterated his notion that the non-existence of a function value at the point of interest implies the limit must exist. Table 8 shows some variety in responses, with six students believing the limit existed at the point of continuity. Mark and Brad decided the limit did not exist because the function was defined at $x = 4$. Matt decided the limit did not exist, based on a hasty perusal of the table: by focusing only on the negative and (understood) positive signs on the function values, he concluded this was the same as the previous limit situation. Notice that Laura refers to her "averaging" process used in the first limit situation, and that Paul seems to have only looked at the limit from the left.

Table 6

Written Responses to Removable Singularity Limit Situation in Interview 2 Task 1

Students	E	DNE	Explanation or suggested limit value for $\lim_{x \to 1} f(x)$
			Numerical version
Jason	✓		6
Josh	✓		6 because the left and right hand limits approach 6
Laura	✓		6 both .999 and 1.000 are equal distance from 1 and their corresponding y values are equidistance [sic] from 6
Mark	✓		I think that this exists since at the x-value 1 it is undefined. The limit is undefined as it approaches from the left and the right.
Matt	✓		The y values approach 6 from both sides
Paul	✓		6 As x gets closer to 1, the y values get progressively closer to 6. At 1, x is actually undefined.
			Graphical version
Brad	✓		6
Brandon	✓		6
Mike	✓	(✓)	the whole [sic] on the graph at $x = 1$ tells me that $f(x)$ is undefined at $x = 1$ (at least I'm not shown a value on the graph in this range) but $\lim_{x \to 1} = 6$

Note. E = exists. DNE = does not exist. (✓) indicates an initial choice, later changed.

Table 7

Written Responses to Vertical Asymptote Limit Situation in Interview 2 Task 1

Students	E	DNE	Explanation or suggested limit value for $\lim_{x\to 2} f(x)$
			Numerical version
Jason		✓	Because $\lim_{x\to 2^-} f(x) \neq \lim_{x\to 2^+} f(x)$
Josh		✓	because the left and right hand limits are different $\lim_{x\to 2^-} f(x)$ closer to +8000 $\lim_{x\to 2^+} f(x)$ closer to -8000
Laura		✓	it looks like there is an asymtopte [sic] at that point and $\lim_{x\to 2^-} f(x) = \infty$ $\lim_{x\to 2^+} f(x) = -\infty$
Mark	✓		I also believe that this exists since it is undefined at the x-value of 2. The only difference is that the graph does not continue up, but decreases
Matt		✓	The y values approach different numbes [sic] depending on what side it comes from
Paul	(✓)	✓	~~as x gets closer to 2 it seems like y gets closer to 8000.~~ [student's strike-through] The left and right limits are not equal?
			Graphical version
Brad	(✓)	✓	both one sided limits are different
Brandon		✓	The $\lim_{x\to 2^+}$ is differs [sic] from $\lim_{x\to 2^-}$
Mike		✓	the y-value can get as large as desired by taking a value of x that is close enough to 2 (from the left) and the exact opposite from the right so $\lim_{x\to 2}$ is undefined

Note. E = exists. DNE = does not exist. (✓) indicates an initial choice, later changed.

Table 8

Written Responses to Point of Continuity Limit Situation in Interview 2 Task 1

Students	E	DNE	Explanation or suggested limit value for $\lim_{x \to 4} f(x)$
			Numerical version
Jason	✓		0 [Initially wrote 4, but during interaction indicated he had been thinking 0]
Josh	✓		because the left and right hand limits are the same both approach 0
Laura	✓		0 for the same reason at [sic] the first question
Mark		✓	I don't believe that this limit exists because as it approaches 4 and when it reaches 4 it has the value zero. This is why I do not believe this exists.
Matt		✓	same as above [Presumably, he is referring to his explanation for the limit as x approaches 2: The y values approach different numbes [sic] depending on what side it comes from].
Paul	✓		0 as x gets closer to 4, the y value becomes a smaller and smaller negative #. Eventually it will reach 0
			Graphical version
Brad		✓	at $x = 4$ there is a y-value; it doesn't skip
Brandon	✓		0
Mike	✓		as x approaches 4 from both sides $f(x)$ approaches 0 from both sides $\lim_{x \to 4} f(x) = 0$

Note. E = exists. DNE = does not exist.

Students' written responses to this interview's second task are presented in Table 9. Although five students selected the "correct" graph or table, namely A, each of these students relied on a calculator-based graph to determine his or her choice. Three students could not explain how the other tables/graphs would be misleading.

Table 9

Written Responses to Interview 2 Task 2

Students	Table/graph A	B	C	$\lim_{x\to 0} g(x)$	Explanation of tables/graphs
				Numerical version	
Jason			✓	$\lim_{x\to 0} g(x) = 0$	It appears that it's approaching $.\overline{33}$ in A and .32 in B.
Josh	✓			the limit as $x\to 0$ of $g(x)$ = 1/3	they suggest the calculator is powerful enough to calculate these small numbers accurately. but it really is not
Laura	✓			1/3	~~for table C for ta The x values are so close Table C: the values are so~~ [student's strike-through] I don't know
Mark		✓		I think it shows that this limit exists.	The other table causes me to believe that this limit does not exist and that as x approaches zero that it goes to infinity.
Matt	✓			It has a limit of 1/3	They round off too much
Paul	✓			That limit = $.\overline{3}$	Table B: Ø [It doesn't] show a smooth, steady upward curve like my graph seems to indicate. It jumps at some points. Table C: Shows the graph is practically horizontal, which isn't the case according to my graph.
				Graphical version	
Brad		✓		it = 0	no clue
Brandon	✓			.4	B – no limit C – 0
Mike			✓	because close to $x = 0$ the curve $f(x)$ is at 0	[blank]

Paul's approach (numerical version). Paul's written response to the limit as x approached 1 (there was a hole at $x = 1$) seemed quite reasonable:

> As x gets closer to 1, the y-values get progressively closer to 6. At 1, x is actually undefined.

However, he had difficulty with the limit as x approached 2 (there was an asymptote at $x = 2$), initially believing it was 8000. Now it appeared he was only paying attention to one side of the table, and focusing on the ordered pair next to $x = 2$. This one-sided bias was also apparent in his written response to the limit as x approached 4 (there was a root at $x = 4$):

> As x gets closer to 4, the y-value becomes a smaller and smaller negative #. Eventually it will reach 0.

After writing the above answer, Paul crossed out his explanation for the limit as x approached 2, and wrote:

> The left and right limits are not equal?

I questioned why he originally thought the limit existed and what made him change:

> Paul: I was looking at it [the limit as x approached 2] too one-sidely, and well, I, I just recently learned in calculus that the, that the right- and left-hand limits need to be, need to be the same in order for that limit to exist.
> I: Okay.
> Paul: And so, in, in all of my recent math experience, I hadn't been informed of that, and so
> I: Okay.
> Paul: I was just looking too narrowly, and just looking as it got close to 2 from one side, because that's all I've ever done before
> I: Okay.
> He: until last week.

His focus on the ordered pair next to $x = 2$ in the table arose later in response to a question about the limit as x approached 2 from the left.

> Paul: As x approaches 2 from the left, it looks like it gets real large, like about 8000.

I: Okay, would you say the limit is equal to 8000?
Paul: Um, I'd uh, I suppose so. Yes. Yeah.
I: Okay, now why do you say that?
Paul: Just... maybe for the same reason that I said that, it seems like, like the closer it gets to 2 it starts to level off at about 8000, because that's ____ that's getting closer and closer to that, but it doesn't exceed.

Essentially Paul treated the table as if it were complete, not thinking to extend the behavior exhibited in the table to ordered pairs not displayed. When this possibility was raised, then he re-evaluated his initial guess.

I: What would you expect, so, for example, if I were to stick an extra x-value and y-coordinate in there
Paul: Mhm.
I: and the x-value is 1.9999, four nines,
Paul: Okay.
I: What would you expect a y-value to be,
Paul: Um
I: based on this graph, uh, based on the table?
Paul: Based on the table... I would expect, well, looks like I made a mistake, now that you say that, with respect to, it looks like the decimal place moves. Um... I would expect it to be [pause of 13 seconds] I don't know, get larger? Because it almost, it almost seems now that I look at this a second time,
I: Mhm.
Paul: that it's just getting incredibly huge, even as these, these little numbers
I: Mhm.
Paul: get smaller.
I: Mhm.
Paul: So I would expect it to jump up. It would be an _____, so it looks like I did have it wrong.
I: So, so then, tell me again, so what do you think now about the limit as x goes to 2 from the left?
Paul: Um, it's probably infinite.

In the second task in this interview, Paul began by drawing a graph of the function

$$g(x) = \frac{\tan(x) - x}{x^3}$$

on his graphing calculator in the window [-5, 5] x [-5, 5]. Initially he was confused by the graph produced by the calculator, so he checked to make sure that the function was entered correctly. It was suggested that he think about what x-values were in the tables, and he tentatively decided to zoom in at the origin. After zooming in a second time, he still felt that the graph did not seem to make sense, because the function was supposed to be undefined at $x = 0$. Then a discussion ensued, largely dominated by the researcher, about how the graphing calculator draws graphs and that it will draw a few points and connect them with straight lines, often drawing through holes in the graph. After all of this (7.75 minutes was devoted to this entire interaction) it came out that he expected a vertical asymptote at $x = 0$, based on his observation of the x^3 in the denominator of the function formula.

> Paul: Yeah, I was looking, I was looking for, like, a, um, something like this, [draws a pair of axes and small piece of a vertical asymptote to the left of the y-axis] something that that approached, but didn't quite get, I think that's it
> I: [Interrupting] Oh I see, so you're thinking maybe there was a vertical asymptote at 0.
> Paul: That's what I was thinking, yeah.
> I: Okay.
> Paul: And I was hoping that, yeah.

Now, earlier in the dialog, Paul indicated his intended strategy very clearly:

> I'm going, what I was planning on doing was, um, using, graphing this out and so decide, and if that proves it, that, that some of the tables can be misleading. Then, um, look at each of the tables, at some values and try and get a sense of what the graph looked like it was going to be.

So he strongly expected asymptotic behavior at the origin, drew the graph (perhaps to ascertain the nature of the asymptotic behavior) intending for the calculator-based graph to be his standard of comparison, and was derailed by the graph that actually appeared. After finally accepting that there might be a hole at 0, he was able continue with the task.

Ultimately, Paul chose table A (the correct table) because it matched the graph on his graphing calculator, saying,

> I chose A because um, because it's uh, it looked, when I looked at the graph, it seemed like it was a, it was a steady parabola shape, upwards parabola shape, and um, this had the only values that seemed to fit with that.

He responded to question 3, "How do the other tables mislead you about the $\lim_{x \to 0} g(x)$?" by describing graphs that would match those tables, and comparing those imaginary graphs to his calculator-based graph. When pressed, he responded that table C would imply the limit as x approached 0 was 0, and ventured that table B might imply the limit was .3, but he really wasn't sure.

When asked why the calculator would give incorrect output values, he responded,

> Possibly because the calculator just isn't capable of handling values that small, or maybe if we, if we zoom in so close in attention, like, extremely close, or too close, maybe it just starts to do, like, funky things, that, like, computationally, but when you look at it as a whole, it doesn't.

About a minute later, he seemed to have a new insight:

> "I bet it's 'cause, it's 'cause the rounding off. The calculator is rounding off to significant figures, like on, like on how the tangent affects the x, because these values are a lot smaller, ____."

Paul never seemed to have more than a vague sense that round off errors and lack of ability to handle small numbers would cause computational problems. He still believed the graph would be right. The message that tables can be misleading was not lost on Paul, as will be clear in his responses to interview 3.

Interview 3

Context. At this point in the semester, students had been working on computations of tangent line slopes, velocities, instantaneous rates of change, and derivatives using limits. They had taken the first exam, which included four limit problems, one requiring the computation of values and one requiring the Squeeze

Theorem. These four limit problems were gone over in lecture after the exam. The Squeeze Theorem was revisited in order to show

$$\lim_{x \to 0} \frac{\sin x}{x} = 1,$$

and this result was used to compute limits of other functions involving trigonometric functions.

General Results. For many students, the calculator-based graph was still the standard by which they decided whether a given graph or table represented the true nature of a function's behavior. At the same time, there is evidence that values given in tables were rejected outright, but values produced with a graphing calculator were accepted as true. Students' did not always generate new information about particular limit situations, and when they did, it was not always better information. Ultimately, many based their decisions about limit situations on poor information.

Students' written responses to this interview's first task are presented in Table 10. Notice that each of the three students incorrectly determining the representativeness of the table and graph initially believed the table was not representative. Even among those who correctly identified the representativeness of the table and graph, several used incorrect reasoning to determine that the limit did not exist.

Students' written responses to this interview's second task are presented in Table 11. The four students deciding that the graph and table did contradict one another cited the discrepancy at the origin as their reason. Notice that nearly every student conjectured that the limit existed and equaled 1. No one except Mike seemed to have any expectation that this function should have a vertical asymptote at $x = 0$.

Table 10

Written Responses to Interview 3 Task 1

	Table		Graph		
Students	Y	N	Y	N	Response to $\lim_{x \to 0} t(x)$
					"Incorrect" evaluations of representativeness
Alan	✓	(✓)		✓	It should not be as random as it aproaches [sic] 0.
Laura[a]	✓	(✓)			does not exist $\lim_{x \to 0^-} = 1 \quad \lim_{x \to 0^+} = -1$
Mike	✓	(✓)		✓	it's undefined
					"Correct" evaluations of representativeness
Brad		✓	✓		it goes to zero
Brandon	(✓)	✓	✓		It doesn't exist
Jason		✓	✓		It is undefined because $\lim_{x \to 0^-} t(x) \neq \lim_{x \to 0^+} t(x)$
Josh		✓	✓		it does not exist
Mark		✓	✓		I don't think that this limit exists because it is hard to tell from the graph and the table doesn't give me any reason to believe it exists.
Matt		✓	✓		When it approaches 0 the graph shows the varience [sic] while the table shows a straight line that breaks at 0 and continues on the negative side.
Paul		✓	✓		I think it approaches 0. When I put in really small decimal values of x into the equation I get zero, and the graph seems to ossilate [sic] around that point.

Note. (✓) indicates an initial choice, later changed.

[a]Laura's written response to the representativeness of the graph is unclear.

Table 11

Written Responses to Interview 3 Task 2

Students	Table Y	Table N	Graph Y	Graph N	Response to $\lim_{x\to 0} h(x)$
					Yes: table and graph contradict one another
Brad	✓		✓		it = 1
Josh	✓	(✓[a])		✓	it approaches 1
Mark	✓			✓	I think the limit exists according to the table, but I am not for sure because of what the graph depicts.
Paul[b]	✓			✓	I suppose it is 1, at least looking at the apparent limits from the right and left sides.
					No: table and graph do not contradict one another
Alan[c]	✓		✓		It exists and is approximately $y = x$
Brandon	(✓)	✓	✓		That it is 1
Jason	✓		✓		$\lim_{x\to 0} h(x) = 1$
Laura	✓		✓		$\lim_{x\to 0} = 1$
Matt	✓		✓		the limit exist. it approaches 1 but is not defined at $x = 0$.
Mike		✓		✓	it is undefined

Note. (✓) indicates an initial choice, later changed.

[a]Josh initially chose Yes, then No, before deciding Yes for the table. [b]Paul initially thought the graph and table did not contradict one another. [c]Alan initially thought the graph and table did contradict one another.

Paul's approach. Paul began by drawing a graph of the function. He responded to whether the table reflected the true nature of the function by writing, "I doubt it…[his ellipsis marks]. I've learned to rarely trust these values." As for the graph, he wrote, "I trust the graph a little more however… [his ellipsis marks]. I believe this to be more accurate." His distrust of values in the given table did not, however, extend to values he produced on the calculator.

Distrusting the given table, he fell into the trap of entering his own values of x, to decide what was true about $\lim_{x \to 0} t(x)$, writing,

> I think it approaches 0. When I put really small decimal values of x into the equation, I get zero, and the graph seems to oscillate around that point.

Note that there was no analysis at all of whether the values in the given table or his own values actually represented the true nature of the function. Moreover, he took the oscillation around 0 as "supporting" evidence that his conjecture is correct.

Josh's approach. Josh also began by drawing a graph on the graphing calculator, but he rejected the table based on his belief that calculator could not calculate the y-values correctly, saying,

> I looked at the table and I saw that this side was just saying 1, 1, 1, 1 and um, the other side's saying negative 1, so I thought, because uh, you know, it can't be 0, so that's why I ____. All the calculator's doing is taking really, really, really small numbers close to 0 and probably all the significant digits are reporting the number that it's taking the sine of as 0, so the sine of 0 is going to be 1. So that's why the graph was reporting ____ because, like, it doesn't have the power to calculate the real digits.

When asked why he thought that the graph *did* reflect the true nature of the function, he said,

> Because I knew that it was, like, a sine curve and I just kind of looked at it and thought, since it is a sine curve, it's going to have up and down, maximum and minimum spots, kind of thing. It couldn't be a straight line.

His reasoning behind his conjecture that the limit did not exist was even more interesting.

> I looked at the graph, and I couldn't tell by the graph, because it's all together, and then I looked at the table, and it looked like it was, you know, undefined at 0 and it was approaching negative 1 from the right and approaching 1 from the left, so I just determined it didn't exist based on that.

So, despite his earlier conclusion that the table did not reflect the true nature of the function, he used the table to predict the limit situation. Eventually, he was able to describe how the graph was oscillating more and more, why it was doing that, and that it didn't seem like you could pick a number that it was going to. This insight that infinitely many oscillations implied the limit did not exist would reappear in interview 4.

Josh exhibited quite different difficulties in the second task of this interview. After initially misreading the x-values in the table, he decided (correctly) that the graph and table did match one another except at $x = 0$. He decided (erroneously) that the table did match the true nature of the function by comparing it to the graph.

> Josh: I just matched it with the graph... and, like, realized that it matched the graph and everything, except for at 0. I knew it couldn't be defined at 0, so,
> I: Okay.
> Josh: That's pretty much ____.
> I: Okay. Um, okay, so are, uh... Are all of these, these x and y-values, these are all correct?
> He: ... Mmm. Seems so.
> I: Okay. Let's check a few of them and see.

He checked two values on his calculator. Then ensued a discussion of the effect of the 10^{20} term in the formula. When asked if it was possible to make the fractional expression of the formula very large, he correctly responded, "You'd have to multiply by something really, really little," but he had great difficulty carrying this out. He initially changed his mind and decided the fraction could not be made larger than 1, re-evaluated and decided that 10^{-20} would create 1, and then repeatedly confused the relative sizes of the numbers 10^{20} and 10^{-20}. Ultimately, he did correctly compute the y-value of approximately 2 corresponding to 10^{-20} and recognized that 10^{-21} would result in an even larger y-value, hence the table was not reflecting the true nature of the function.

After deciding that function must have an asymptote at $x = 0$, Josh was asked if it were possible to draw the graph on the graphing calculator so that the vertical asymptote would appear. Initially, he suggested zooming in. After some prompting which included a reminder about the y-value of 2 at $x = 10^{-20}$, he amended this, saying that a larger y-range and smaller x-range were needed. Having decided this, he then had trouble entering the function formula, entering range values, and choosing a suitable viewing window. Initially he graphed the function on the x-range $[-10^{-7}, 10^{-7}]$, which gave what appeared to be the constant function 1. Then he tried the x-range $[-10^{-23}, 10^{-23}]$, which gave no graph at all. He thought this might be due to the power of the calculator. After more discussion on the possibility of using the x-range $[-10^{-20}, 10^{-20}]$, he turned to enter this in the calculator, and asked, "This will be, like, a bigger x-value, right? Than 10 to the minus 30th, or something like that?" After more discussion about why 10^{-30} was the wrong direction to go, he finally was able to draw a correct graph.

At all times in this second task, Josh exhibited a very weak sense of numbers, and this tremendously hindered his ability to deal with this function. In addition, he had considerable trouble with the mechanics of entering information into the graphing calculator. In fact at one point, he asked how to enter a number like 10^{-23}. According to his background questionnaire, Josh had no experience with graphing calculators prior to this course.

Summary. Both of these students made correct intuitive limit conjectures. Paul correctly based the limit conjecture of 0 on his calculated values of 0 in task 1. Josh correctly determined the limit situation from the given table of values in task 1; and correctly conjectured that the limit was 1 in task 2, based on the given table and graph. That their conjectures were based on bad sources of information was not really clear to either student in task 1, and quite difficult for Josh to recognize in task 2. Essentially, these students relied on easy sources of data, and their "supporting" evidence was minimal.

Interview 4

Context. At this point in the semester, students were finishing the last homework assignment containing limit problems, which focused on limits involving trigonometric functions. The Squeeze Theorem had been used in lectures 8, 9, 15, and 16 and a limit problem similar to the one in this task was given on the first hour exam (between lectures 14 and 15). The lectures at this point were devoted to computations of derivatives, using the derivative rules.

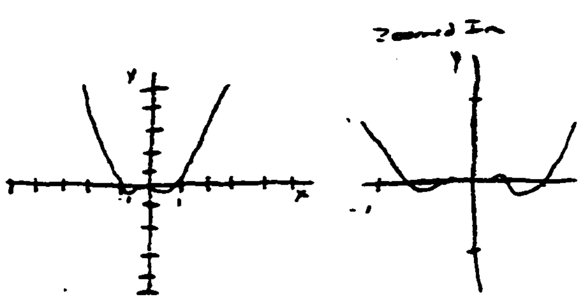
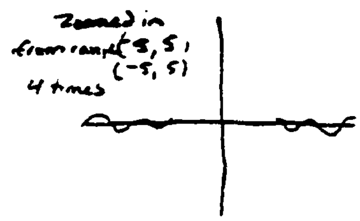
General Results. Students employed several strategies in analyzing this limit situation, including deciding what to expect based on the formula, drawing a graph and computing function values. For the most part, each student's initial impression dominated his or her solution. No one used the Squeeze Theorem, although several students wrote similar ideas. They were aware that their techniques gave them educated guesses rather than certain answers, and many believed that it was possible to determine the limit situation for certain, but they did not know how one would go about doing that.

Most students' doubts stemmed from lack of confidence in their own abilities rather than from understanding of the lack of mathematical soundness of the techniques they were using.

Students' written responses to the limit situation in this task are presented in Table 12, Table 13 and Table 14. Notice that most students describe the behavior of the function, or focus on one aspect of the function's behavior in their initial explanations. When additional evidence is requested however, several students refer to tables and/or graphs.

Table 12

Incorrect Written Responses Requiring Additional Evidence in Interview 4 Task

Explanation about $\lim_{x \to 0} f(x)$	Additional evidence
Josh (wrong→wrong)	
(+ small #)(-1 →+1) The limit does not exist because as you get closer to 0, the quantity 1/x will get larger causing the cosine function to just keep on repeating so then you just multiply the small number squared by the quantity cos(1/x).  	as the function's x value approaches 0, the maximum values and minimum values seem to be different because the value of (x^2) will constantly change although the cosine function will repeat 

(table continues)

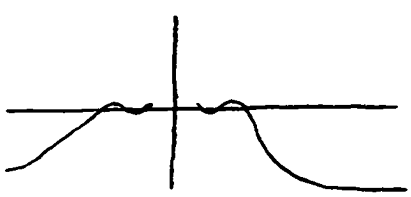
Explanation about $\lim_{x \to 0} f(x)$	Additional evidence
Laura (right→wrong)	
~~infinity~~ $\lim_{x \to 0^-} f(x) = -\infty \quad \lim_{x \to 0^+} f(x) = \infty \quad 0$ ~~-10000000~~ $\cancel{\lim_{x \to 0} f(x) = \text{undefined}}$ $\lim_{x \to 0} f(x) = 0$ $\cos\left(\frac{1}{x}\right)$ as $x \to 0$ is between -1 and 1 but x^2 will be smaller and smaller as $x \to 0$ and so the value of f(x) will grow increasingly smaller	if x = .005 f(x) = 1.25×10^{-5} if x = .001 f(x) = 5.6×10^{-7} if x = .0005 9×10^{-8} if x = .0001 1×10^{-8} if x = .00005 2×10^{-9} if x = .00001 -1×10^{-10} if x = .000005 5×10^{-6} if x = .000001 -1×10^{-10} if x = .0000005 5×10^{-7} if x = .0000001 -9×10^{-15} if x = .00000005 5×10^{-8} there is no limit $x \to 0^+ \Rightarrow \lim_{x \to 0} f(x)$ is undefined
Mark (wrong→right)	
I don't think that this exists. I feel this way because by looking at the graph of the function as $x \to 0$ from the left and as x approaches zero from the right, I think that there is a different values for each one. This is why I do not think that the limit exists. After zooming in on the graph it appears that the is [sic] not continuous and that the limit still does not exist. 	Something that indicates that I need to change my conclusion is that the calculator is unable to portray the true graph since the values as $x \to 0$ are so small. This leads me to believe that the limit does not exist. Knowing that this is possible it causes me to believe that the limit does exist. If I also put values in that become closer to zero I find that I get y values getting closer and closer to zero. This also helps me believe that there is a limit.

Table 13

Correct Written Responses Requiring Additional Evidence in Interview 4 Task

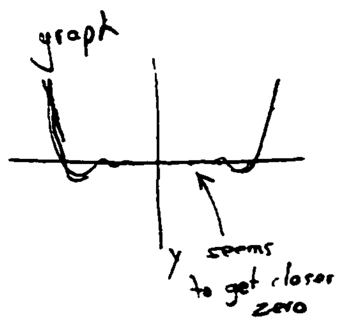
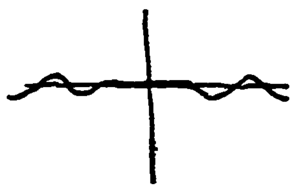
Explanation about $\lim_{x \to 0} f(x)$	Additional evidence
Jason (right→right)	
I think $\lim_{x \to 0} f(x) = 0$ because as x approaches 0, x^2 approaches 0 and 0 times any thing is 0.	$-1 \le \cos\left(\frac{1}{x}\right) \le 1$ and the graphing calculator appears to approach 0 too. If you put numbers in for x that approach 0, $f(x)$ approaches 0.
Mike (right→right)	
$f(x) = x^2 \cos\left(\frac{1}{x}\right)$ x drives this part to ∞ if $x \to 0$ cos keeps $\cos\left(\frac{1}{x}\right)$ between 1 and -1 x^2 drives function to zero as $x \to 0$ so $\lim_{x \to 0}$ ~~should be close to zero~~ $= 0$	x y .5 -.10403 .1 -.00839 .001 5.62×10^{-7} .0000001 -9.072×10^{-15} 
Paul (right→right)	
I think that the $\lim_{x \to 0} x^2 \cos\left[\frac{1}{x}\right] = 0$, because the x^2 term in front of makes whatever value you get out of $\cos\left[\frac{1}{x}\right]$ very small.	I graphed out this function and found that y is undefined where $x = 0$. That makes sense. But on either side of 0, at a very small value of x, I see that y equals a very small number as well.

Table 14

Correct Written Responses in Interview 4 Task

Explanation about $\lim_{x \to 0} f(x)$
Brad
$\lim_{x \to 0} = 0$ 1) I manually pluged [sic] in small numbers 2) graphed it and zoomed in a lot
Brandon
$\lim_{x \to 0} x^2 \cos\left(\frac{1}{x}\right) = 0$ I conclude as x gets smaller and smaller at points very close to 0 the y value get very close to 0 and it is true for the other side because $\cos x = \cos(-x)$ and $(-x)^2$ is the same as x^2 
Matt
I believe the $\lim_{x \to 0} f(x)$ is 0. When x becomes smaller the x^2 also becomes a very small number and by multiplying it with $\cos \frac{1}{x}$ then a small number is returned.

Students' responses to questions 2 and 3, and if applicable, questions 5 and 6 are presented in Table 15. Each cell in the table represents one possible combination of responses to questions 2 and 3 (rows) and to questions 5 and 6 (columns). For example, Jason initially responded that he was absolutely certain (A) of his own conclusion and that everyone else should be fairly certain (F) of his conclusion. After producing more evidence, Jason at first responded the same way (A-F) but changed his mind and responded A-A, believing now that everyone else should be absolutely certain of his conclusion. Notice that "fairly certain" was the "favorite" response.

Table 15

Student Certainty of Conclusion Correctness in Interview 4 Task

	Later certainty (self-others)				
Early certainty (self-others)	A-A	A-F	F-A	F-F	N.A.
A-A				Laura: r → w	
A-F	Jason: r → r Mike: r → r	(Jason)	(Josh)		
F-A					Brad: r
F-F	(Mike)			Mark: w → r Josh: w → w	Brandon: r Matt: r
F-N				Paul: r → r	

Note. A = absolutely certain, F = fairly certain, N = not at all certain, N.A. = not applicable. Names in parentheses indicate initial choices, later changed.

Josh's approach. Josh's approach was dominated by his newfound knowledge that infinitely many oscillations meant the limit could not exist. He began by drawing a graph on the graphing calculator, and zooming in. He wrote quite a bit, and drew the two graphs (see Table 12) he produced on his calculator.

> (+ small #)(-1 → +1)
> The limit does not exist because as you get closer to 0, the quantity $1/x$ will get larger causing the cosine function to just keep on repeating so then you just multiply the small number squared by the quantity $\cos(1/x)$.

It seemed that Josh both understood the local behavior of this function near the origin, and did not understand how this related to the limit situation. To understand this better, Josh was asked about the graph.

> I: Okay. So, what does the graph on the calculator lead you to believe?
> Josh: It makes you think that, just as you get closer, well, I graphed it on the calculator, and it looks like it'd be approaching somewhere just below the x-axis,
> I: Okay.
> Josh: like, ___ , or some small number. Then I zoomed in, and it looks like it's just going to repeat, keep on bouncing up and down.
> I: Okay. How many times do you think it's going to bounce up and down?
> Josh: Probably an infinite number of times.
> I: Okay.
> Josh: Cosine is just going to keep on repeating.
> I: Okay. If it bounces up and down infinitely many times, is it, is it bouncing up and down to the same values at the top and bottom?
> Josh: ... No, because you're multiplying your, the cosine of it, even though your cosine is repeating, you're still multiplying by a smaller, a different x-value as you're
> I: Mhm.
> Josh: x is getting... constantly changing.
> I: Okay. So, what is it that makes you think that the limit does not exist? Is it that bouncing up and down that makes you think it doesn't exist?
> Josh: Pretty much, because that's, that's the thing. It just says, I got close, but even if I took smaller and smaller values, it's never going to get to a certain number
> I: Okay.
> Josh: because the smaller numbers you take, it's going to just
> I: Okay.
> He: It's different on each side.

After asking Josh to continue with the problem, he graphed the function in the window [-5, 5] x [-5, 5] and zoomed in four times, concluding that his original conjecture was correct. He described the apparent lack of a curve in a neighborhood of 0 as being possibly due to the calculator's limited ability to handle small numbers. At the same time, he acknowledged that the graph was "acting like it's so close to the *x*-axis you can't see it," and that it was still going up and down towards 0 but very close to the *x*-axis, "because your x-squared is making it get smaller and smaller." He correctly indicated that, based on the graph alone, the limit would be 0, but followed this with,

> I still think it does not exist because your, the cosine formula is just going to keep on repeating, so it's not really going to a set value… and you're just changing the ___ by a small x-value.

How does he reconcile these two things? This was revealed at the end of the interview.

> I: Do you understand the limit situation now more, or less, or about the same after you tried to generate additional evidence?
> Josh: Mmm. I understand it more, because I just went back and looked at that again and made sure that, you know, as I got closer, and then I looked at the calculator and that helped me because I figured that thing just can't compute the real small, because when I zoomed in four times,
> I: Mhm.
> Josh: from a range of –5 ____.

From Josh's perspective, the graph helped him by convincing him that the graphing calculator could not give him a good graph, and therefore, he could safely ignore the graphical evidence and rely on his initial impression.

Paul's approach. Paul started out with an initial guess that the limit was 0, based on his analysis of the function formula, writing

> I think that the $\lim_{x \to 0} x^2 \cos\left[\frac{1}{x}\right] = 0$, because the x^2 term in front of makes whatever value you get out of $\cos\left[\frac{1}{x}\right]$ very small.

It's not clear from this that he understands the boundedness of the $\cos(1/x)$ is a consideration. In any event, after making his initial conjecture, he computed three values before being convinced.

In answering questions 2 and 3, Paul indicated that he was fairly certain of his conclusion, but thought everyone else should be not at all certain. When I asked him where the doubts come in, he responded, "I really don't know what I'm doing", "I know that just plugging in values doesn't always work", and "I know that there is different ways to look at things and I just focused on the one that I see, which one's obvious."

In the second part of the task, he immediately turned to the graphing calculator to draw a graph, zoomed in four times and traced, writing,

> I graphed out this function and found that y is undefined where $x = 0$. That makes sense. But on either side of 0, at a very small value of x, I see that y equals a very small number as well.

I asked him if the graph was a surprise to him or was he expecting what he saw?

> Paul: It was... I thought I knew what the limit was, as x approaches 0, but the rest of the graph seems kind of, I didn't exactly know what to expect as far as what the whole graph looked like, but I don't know where it seemed to ____.
> I: Okay. So how would you describe the way that the graph behaves overall?
> Paul: overall?
> I: Yeah.
> (Pause of 10 seconds)
> Paul: I don't know. It just seems to oscillate, like a smaller and smaller
> I: Uh huh.
> Paul: period,
> I: Yeah.
> Paul: until it, until it gets to the, to the 0 point where it, and then it, it doesn't exist there.
> I: Okay, so what is it about the function formula that, that leads you to believe that's the correct way for the graph to behave?
> Paul: that's the correct way for the graph to behave... Um... um, this number right here, of x, um, affects the, the range of the function and as it gets smaller, then the range probably should as well,
> I: Okay.
> Paul: as well as the fact that it's undefined at 0 which this graph ____.

Notice, that Paul did not comment on the $\cos(1/x)$ piece of the function formula, or on how the oscillations came to have "smaller and smaller periods". This element of the function seems to be either irrelevant or very mysterious to him.

Summary. Both Paul and Josh started out with an expectation that dominated everything else. Paul focused the damping effects of the x^2 term, and Josh focused on the oscillatory effects of the $\cos(1/x)$ term. Each of them made correct intuitive limit conjectures, but in Paul's case, he initially used only three function values, and Josh simply rejected his graphical evidence. Both initially produced only one other source of evidence.

Questionnaire Data

All 10 students completed the first questionnaire[1] (see Appendix F) at the end of interview 1 (week 3), and 9 students completed the second questionnaire (see Appendix K) at the end of interview 4 (week 8). There were differences between the first and second questionnaires on both preferences (items 1 and 2) and beliefs (items 3 and 4). First, an overwhelming initial preference for graphs changed to a preference for function formulas. Second, early beliefs that graphs, tables and formulas suffice to describe a function were replaced by beliefs that graphs and tables could be misleading. Details for each pair of parallel items on the two questionnaires are presented below.

Item 1

Which one of the following would you prefer to use (Q1) when describing how a function behaves to a fellow student, and (Q2) when analyzing a limit situation? (a) a graph, (b) a table of ordered pairs, or (c) the function formula. Please explain in a sentence or two why you chose the one you did.

Table 16 shows the preference changes on this item. On the first questionnaire, nine students preferred graphs. On the second questionnaire, five weeks later, six students' preferences had switched to formulas.

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[1]Throughout this section, Q1 = questionnaire 1 and Q2 = questionnaire 2.

Table 16

Preference Changes on Item 1

Q1 Preference	Q2 Preference			
	Graph	Table	Formula	No Response
Graph	2	1	5	1
Table	0	0	1	
Formula	0	0	0	

Students' explanations of their choices in item 1 are presented in Table 17. In questionnaire 1, students generally referred to the visual nature of graphs, although few articulated why or how the visual nature of graphs helped them in understanding functions. By the second questionnaire, the practicality of being able to produce tables and graphs given the function formula took precedence.

Table 17

Explanations of Graph → Formula Preferences in Item 1

Students	Questionnaire 1	Questionnaire 2
	Graph → Formula	
Brad	You can personify the graph and tell what it did – most people are visual learners.	If I have a formula then I can derive a table and graph and then have all those.
Jason	You can visually see "what is happening".	I can get a graph and table if I have the function and you know the true nature if you have it.

(*table continues*)

Students	Questionnaire 1	Questionnaire 2
	Graph → Formula	
Josh	When you see a graph, it gives you a visible reference to show how the function increases, decreases, or is undefined at certain points	A graph can be misleading by not showing you how a function acts in a small enough interval. A table of ordered pairs may also mislead you by not showing a small enough interval. The function can allow you to compute any number and see some properties of that function
Laura	It is easier to see what the function is doing and to see a pattern even though it can be very difficult to draw a graph by just looking at a function.	You can plug it into the graphing calculator and zoom in a lot.
Mike	Because it is a visual way of showing the behavior whereas a table or formula are not always easily decipherable.	Because you can get the graph and ordered pairs from the formula, but you may not really understand these two things without the formula.
	Graph → Graph	
Brandon	Graphs have every point on them so you get a better understanding of what the function does.	In a graph, you have infinite amounts of plotted points. It helps you get a better view of what the function is doing at all places.
Paul	Because visual representations are easier to understand, in my opinion. Plus they are more precise.	It gives the most information I can manipulate (if I have the trace function). It also gives a visual representation of what's happening, which is helpful to me.

(*table continues*)

Students	Questionnaire 1	Questionnaire 2
	Graph → Table	
Mark	I think most people would learn better if they can actually visualize something.	Because graphs can be very misleading and if I only see a function formula I cannot picture what is happening.
	Graph → No Response	
Alan	Because I am a visual learner and graphs help me more than data.	[Not applicable]
	Table → Formula	
Matt	A table describes a function along a line letting you know exact points	With the function formula, a student can make a graph and table, or plug and chug.

Item 2

If you could choose two things to help you (Q1) tell a fellow student how a function behaves, and (Q2) analyze a limit situation, which pair would you prefer? (a) a graph and a table of ordered pairs, (b) a graph and the function formula, or (c) a table of ordered pairs and the function formula. Please explain in a sentence or two why you chose the pair you did.

Table 18 shows the preference changes on this item. The graph-formula pair was the overwhelming favorite: eight students selected this pair at least once, and six selected it twice. Every student selected a representation pair that included his or her preferred solo representation, as indicated in item 1.

Table 18

Preference Changes in Item 2

	Q2 Preference			
Q1 Preference	Graph-Table	Graph-Formula	Table-Formula	No Response
Graph-Table	1	0	0	1
Graph-Formula	0	6	1	
Table-Formula	0	1	0	

Students' explanations of their choices in item 2 are presented in Table 19. To see how these choices relate to their choices in item 1, students' preference changes on item 1 are indicated below their names. Notice that two students changed dramatically. Jason originally preferred the graph most of all, and by the fourth interview, this was his least preferred representation. Similarly, Matt's most preferred representation at the beginning, tables, was his least preferred by the end of the study.

Table 19

Explanations of Graph-Formula → Graph-Formula Preferences in Item 2

Students	Questionnaire 1	Questionnaire 2
	Graph-Formula → Graph-Formula	
Brandon graph → graph	I chose them because a table of ordered pairs is hard to get a mental picture of. With the graph and the function you can relate the two together.	With the function formula, you can tell what values might be plugged in that would cause the graph to be misleading.
Paul graph → graph	I like the graph for the previously stated reason, and if my fellow student understands the graph, the student can best learn about the function by seeing how the graph is accrued from the formula.	I can infer a table of ordered pairs from those two anyway, and I prefer the graph most, second only to the formula.
Brad graph → formula	The ordered pair is just a representation of the function. The function layout is more important.	I would rather have the primary source than a table already completed.
Josh graph → formula	This shows you how the function relates to the graph. The table is not needed because you can substitute and solve for values.	I would choose the graph b/c it will show you many ordered pairs of a function at once (not all) and with the function you could compute nearly all plus see properties of the function.

(table continues)

Students	Questionnaire 1	Questionnaire 2
	Graph-Formula → Graph-Formula	
Laura graph→formula	The graph is a good beginning but the function formula would probably be more acceptable on a test.	[No explanation given.]
Mike graph→formula	A graph is just a different type of table so with this combination, I get all three.	Because ordered pairs can be misleading because of their "missing links".
	Graph-Table → Graph-Table	
Mark graph→table	This way they can see what it looks like and also how the numbers are being manipulated	This way I could see if the graph matched the table and if they are telling me the same information.
	Graph-Table→ No Response	
Alan graph→NR	I think many people are visual learners	[Not applicable]
	Graph-Formula → Table-Formula	
Jason graph→formula	Without the formula, it is difficult to get accurate values for the variables.	The table shows some pairs a graph might not.
	Table-Formula → Graph-Formula	
Matt table→formula	The table shows the answers to the formula, letting you know where the function is at every step of the way.	A graph can help you visualize the function and the formula can let you make a table.

Item 3

Would using all three (a graph, a table of ordered pairs, and the function formula) help more than using just two of the three in (Q1) telling a fellow student how a function behaves, and (Q2) analyzing a limit situation? (a) Always, (b) Sometimes, or (c) Never. Why do you think this?

Table 20 shows the belief changes in this item. At the beginning, seven students believed that all three representations would *always* help more than just two. By the end of the study, only four students expressed this belief.

Table 20

Belief Changes in Item 3

	Q2 Beliefs*			
Q1 Beliefs	Always	Sometimes	Never	No Response
Always	3	3	0	1
Sometimes	1	1	0	
Never	0	1	0	

Students' explanations for their beliefs in item 3 are presented in Table 21. Notice that in questionnaire 2, the possibility of misleading tables or graphs is cited to justify both "sometimes" responses and "always" responses.

Table 21

Explanations for Beliefs in Item 3

Students	Questionnaire 1	Questionnaire 2
	Always → Sometimes	
Josh	You would have a quick reference for the information to relate it to each other.	As long as the information corresponds and each thing doesn't tend to lead you to a different answer.
Matt	When all three tools are at hand, the student can grasp an understanding of the function from 3 different angles.	Some things can be misleading. You cannot trust all info given to you, but with the formula, you can make your own graphs and tables.
Paul	It works out the steps, it teaches each little part of how to do the problem. This allows students to know what they are doing, and best understand what the math means.	It can help clarify situations... to show <u>exactly</u> what's going on. [Student's ellipsis marks and underlining.]
	Sometimes → Sometimes	
Jason	The more information you have the easier it is to explain.	The more information you have the better, but the information sometimes is misleading.
	Never → Sometimes	
Mike	I think that if an accurate graph is present than [sic] a table is not necessary.	I would say never, but since I haven't encountered every limit situation, I'll give the ordered pairs the benefit of the doubt.

(*table continues*)

Students	Questionnaire 1	Questionnaire 2
	Always → Always	
Brad	You can memorize how a graph looks and then remember the kind of function that lead [sic] to this graph so as to remember the graph and corresponding function for the test.	You can cooberate [sic] your answer to be certain of correctness.
Laura	Because a student will probably encounter all 3 and should be familiar w/ all of them.	It gives you the most information to work with.
Mark	I think most of the time it all helps because all students are different and one might better understand one process better than the other.	Because a graph and a table of ordered pairs can both be misleading. If you have all three you can use each of them to obtain a conclusion.
	Sometimes → Always	
Brandon	The graph I believe is always going to be useful but the function formula sometimes can get confusing. It should still always be included though.	It just gives you more facts to support your theory and help evaluate your conclusion.
	Always → No Response	
Alan	It gives more proof for a problem's solution.	[Not applicable]

Item 4

(Q1) Is it possible that these three (a graph, a table of ordered pairs, and a function formula) wouldn't provide enough good information for you to tell a fellow student how a function behaves? (a) Yes, or (b) No? Why do you think this? (Q2) What are some of the drawbacks of relying on these three (a graph, a table of ordered pairs, and the function formula) when analyzing a limit situation?

Table 22 shows students responses to item 4 in both questionnaires. Initially, students were evenly divided about the sufficiency of three representations to describe a function's behavior. Six students responded "no" to item 4 in questionnaire 1, but Mark's explanation suggests that he really believes the answer is "yes". By the end of the study, all nine students completing the second questionnaire acknowledged the misleading nature of graphs and tables. Three students even elaborated upon how graphs, tables and even formulas can be misleading.

Table 22

Explanations for Item 4

Students	Questionnaire 1	Questionnaire 2
	No responses in questionnaire 1	
Brad	I can't think of any other way to explain the concept	Sometimes they "lie".
Brandon	If I can look at the graph know all the points and see how the formula works I think I would be able to give a pretty good detailed report on the function.	In a table of ordered pairs, there might be gaps in between the plotted points where the function is moving or becoming discontinuous. The function formula can become a drawback if you assume something that is fake. A graph can be misleading because it can not tell you you have to find out if there are changes at infinitely small points.
Jason	You can see what is happening at any point as long as you have the function.	The graphs and tables can be misleading under certain situations.
Josh	This should be adequate information to solve and explain the functions.	Sometimes the graph and table of ordered pairs may be misleading.
Mark	I think if the information is thoroughly explained that it would be sufficient, but it is still going to be a little confusing to anyone.	Each of them can give you information that can cause you to misinterpret what is actually trying to be represented.
Matt	The 3 of these together can define a function down to the teeth.	A graph or table can be misleading if they are analyzing too close or far away from the limit. I don't see much drawback to the formula.

(*table continues*)

Students	Questionnaire 1	Questionnaire 2
	Yes responses in questionnaire 1	
Alan	If the data a [sic] graphs become extremely detailed and confusing.	[Not applicable.]
Laura	Some functions don't seem to do what you would expect.	Sometimes they are deceptive and don't represent every odd result the function might have.
Mike	Because I haven't been exposed to many functions I'm sure.	The graphs are many times misleading. Ordered pairs have "missing links". Function formulas can be cumbersome and sometimes cannot prove a limit. None of these can be used in every limit situation to find a limit.
Paul	Especially for complicated functions, there needs to be some explaining as to <u>why</u> something happens, not just that it does. [Student's underlining.]	They can be misleading in their information, one must pick the best way to find the limit.

CHAPTER 5

Results

Results are presented in four subsections. First, emerging patterns of analytical thinking and knowledge use are summarized. These patterns lead to conclusions about each of the three main research questions.

Emerging Patterns

Several patterns of analytical thinking and knowledge use emerged from these students' interview data. At the same time, certain mathematically correct and relevant strategies and knowledge were noticeably absent from students' written and oral responses. The categories of analytical thinking and knowledge use, both present and absent in these students' responses are described below and summarized in Table 23.

Analytical Thinking Categories

Specific instances of analytical thinking present in students' solutions fell into two categories: (a) partial analytic strategies, and (b) inappropriate dismissal of evidence. At the same time, students' consistently failed to analyze the representativeness of calculator-based evidence.

Partial analytic strategies. Many of these students employed partial analytic strategies. That is, they made conclusions based on only part of a formula, table or graph.

For example, Paul, in interview 2 decided that the x^3 term in the denominator of the function formula implied there must be an asymptote. He also initially thought, in interview 2, that observing values in one half of the table would tell him the limit. Josh, in interview 3 decided that because the graph oscillated it must match the function with the sine in it.

Inappropriate dismissal of evidence. Several students inappropriately dismissed evidence. This occurred when they dismissed evidence (good or bad) without analysis or with erroneous reasoning. For example, Paul, in the third interview immediately rejected the given table of values in the first task, without any analysis. Josh rejected that same table based on his assumption that the calculator could not calculate the values correctly. Josh also initially rejected the given graph of $\sin(\pi/x)$ in interview 3 because it was difficult to interpret. He rejected his calculator-based graph in interview 4, based on his belief that the calculator could not compute the ordered pairs near the origin correctly.

Analysis of graphing calculator evidence. Most students failed to analyze the representativeness of the graphs and function values they produced on the calculator. This does not mean that no analysis occurred. Quite a few of them zoomed in and traced along a graph, or selected a small number of domain values to substitute into the function to generate data about a limit situation. However, this was almost never preceded by an analysis of whether the exhibited behavior was reasonable and represented the true nature of the function.

Knowledge Use Categories

Students seemed to draw upon three categories of knowledge in approaching these tasks: (a) naïve theorems, (b) false assumptions, and (c) out-of-context knowledge.

Noticeably absent from students' knowledge use were: (a) numerical knowledge, or (b) mathematical theorems.

Naïve theorems. Many students applied naïve theorems. These were students' own if-then statements that they came to believe were true. Sometimes, these naïve theorems were outright false, and other times they were nearly correct. For example, one student Brad, declared that if a function is defined at the point of approach, then the limit does not exist. He explained the truthfulness of this statement by saying a limit is approached but not reached. As another example, Josh believed that if a function oscillated infinitely many times on approaching the relevant point, then the limit does not exist. He explained the truthfulness of this statement by describing how the function just kept going back and forth and did not go to any one number.

False assumptions. Many conclusions were based on false assumptions. For example, some students assumed the evidence they were analyzing was complete and correct. Paul exhibited this assumption in determining a one-sided limit from a table when he failed to extend the values of the table to ordered pairs not shown. As another example, many students assumed the calculator would produce good graphical and numerical information, turning to this tool after rejecting presented tables and/or graphs. False assumptions differ from naïve theorems in their lack of a clear if-then structure. They often served as (apparently unconscious) additional hypotheses in students' approaches.

Out-of-context knowledge. Many students drew upon correct or nearly correct knowledge in related but not necessarily relevant contexts. Often this was the final step, either aimed at "confirming" a conjecture, or generating an answer when all other knowledge seemed to be insufficient. For example, Jason, in interview 1 tried to equate the shape of the $\sin(x)$ graph to that of $\sin(1/(x-3))$. Paul, in interview 3, attributed significance to the oscillation of $\sin(\pi/x)$ *around* 0 in his conjecture that the limit as x approached 0 was 0. Josh, in reference to the non-existence of the limit of $x^2\cos(1/x)$ as x approached 0 in interview 4, commented at the end, "It's different on each side."

Numerical knowledge. Quite a few students appeared to make no use of basic numerical knowledge. Some examples that could have been applied to the tasks in the current study are: common values of $\sin(x)$ when x is given in radians, radian-degree conversions, approximate sizes of numbers given in radians, relative sizes of numbers with different numbers of digits after the decimal point, absolute sizes of a numbers like 10^{-n}, and relative sizes of numbers like 10^{-n} and 10^{-m}. Students rarely brought these facts to bear in their own solutions, and had extreme difficulty coping with these ideas during interaction. Only after extensive interaction did this knowledge come to the surface, suggesting that students did possess the knowledge, but were largely unable to access it.

Mathematical theorems. Students rarely made correct use of mathematical theorems. Some attempted to use a naïve version of the theorem that a limit exists if and only if the left- and right-hand limits exist and are equal to show when a limit did not exist. Often, they forgot the hypothesis that the one-sided limits must exist. No one referred to the Squeeze Theorem in interview 4.

Table 23

Patterns of Knowledge Use and Analytical Thinking

Knowledge Use	Analytical Thinking
Present	
• Naive theorems • False assumptions • Out-of-context knowledge	• Partial analytic strategies • Inappropriate dismissal of evidence
Absent	
• Numerical knowledge • Mathematical theorems	• Analysis of graphing calculator evidence

Early Understanding of the Limit Concept

Research questions. How and to what extent do the four elements of intuitive-analytic understanding of the limit concept emerge and develop over the course of the study? In particular, how do students analyze local function behavior? Can they draw correct intuitive limit conclusions from accurate graphs and tables? In what ways do students develop awareness of the advantages and disadvantages of tables and graphs to conjecturing limits, particularly when using graphing calculators? Do they spontaneously produce multiple sources of evidence to justify a limit conjecture?

Analysis of local function behavior. These students had little success at determining local function behavior. Initially, they resorted to ordered pairs. When they had a calculator available, they sometimes relied on the calculator to show them the "right" graph or the "right" function values. When they had erroneous formula-based expectations, it was very difficult for them to consider alternative possibilities.

Intuitive limit conjectures. Students were able to read graphs and tables to make correct intuitive limit conjectures by the end of the study. Often however, these conjectures were based on non-representative tables or graphs.

Limitations of numerical and graphical evidence. The problem involving the three different tables (or graphs) of the function

$$g(x) = \frac{\tan(x) - x}{x^3}$$

was a turning point for many students. Several of them "learned" that all tables have "bad" data, and in the next interview automatically rejected the table. By the end of the study, all of the students understood that tables and graphs might mislead them, but they rarely analyzed this, especially when the graph or table was produced on their calculator. Typically, they either assumed representativeness or assumed they were being misled. There seemed to be no middle ground for these students.

Intuitive-analytic limit conjectures. Decisions about limit situations seemed to stem from focusing on a particular part of the formula and applying a naïve theorem. For example, the x^2 in the formula $x^2\cos(1/x)$ combined with the naïve theorem that small numbers multiplied by anything give small numbers leads to the conclusion that the limit of $x^2\cos(1/x)$ as x approaches 0 is 0. On the other hand, the $\cos(1/x)$ in the formula $x^2\cos(1/x)$ combined with the naïve theorem that infinitely many oscillations implies a limit does exist leads to the conclusion that the limit of $x^2\cos(1/x)$ as x approaches 0 does not exist.

These students found many ways to "support" whatever their initial idea was about a limit situation, if they had one, including graphs primarily and function values occasionally. Mathematical theorems rarely came into play.

When asked whether they had made an educated guess or determined the limit situation for certain in interview 4, most students indicated that it was an educated guess, although they were "almost certain". They readily acknowledged that it was probably possible to determine the limit situation for certain, but indicated that they did not know how to do that.

Influence of Function Knowledge

Research questions. In what ways do students analyze local behavior of functions in graphical and numerical settings? In particular, on what aspects of tables, graphs and formulas do students focus, in analyzing local function behavior? How do they decide whether a graph and a table of the same function are consistent? How do they decide whether a table or graph reflects the true nature of the function? Do their methods of analyzing local function behavior support or hinder their success with determining a limit situation?

Focus of analysis. Students tended to focus on only part of each table, graph or formula they encountered, either ignoring the rest, or deciding the rest was not relevant. They seemed to focus on whatever was the easiest part for them to comprehend. For example, in analyzing the function $\sin(1/(x-3))$ in the first interview, several students focused on the "sin" part of the formula and looked for a graph that oscillated since they knew that $\sin(x)$ is an oscillatory function.

Consistency decisions. Students' decisions about consistency of tables and graphs depended on whether both graph and table were given, or whether one of the representations was generated on the calculator. In the first case, the decision usually involved checking that a few ordered pairs from the table lay on the graph. Some students were more careful with this than others, and noticed that the "ordered pair" (0,undefined) given in the table of the function

$$h(x) = x + 1 + \frac{1}{10^{20}x}$$

did not match the apparent ordered pair (0,1) on the graph of *h*. In the second case, with a given table and a calculator-based graph, some students traced along the curve, comparing the ordered pairs on the calculator screen with the ordered pairs in the given table. One student, Paul, in interview 2, indicated that he imagined what the graphs matching the given tables should look like, and compared those imaginary graphs to the calculator-based graph.

Representativeness decisions. Students "detected" representativeness in many different ways. Some students believed that consistency of representations implied representativeness. For example, in interview 3, numerous students contended that both the graph and table given for

$$h(x) = x + 1 + \frac{1}{10^{20}x}$$

represented the true nature of the function because they did not contradict one another. Other students believed the only the table represented the true nature of this function h, because the table *mostly* matched the graph and had the "ordered pair" (0,infinity), which fit the formula. In other tasks, if a student's calculator-based graph did not contradict the given graph or table, then the given graph or table was considered representative.

Influence on determination of limits. These students' function knowledge and methods of analyzing local function behavior both did and did not influence their determination of limit situations. Partial analyses led them to accept non-representative behavior, which led to erroneous limit conjectures. On the other hand, even full and complete analyses did not always result in correct limit conjectures.

Influence of Graphing Calculator Use

Research questions. How do students spontaneously use graphing calculators in analyzing functions and making limit conjectures? To what extent are they aware of the limitations of the graphing calculator, and how do they deal with this? How convincing is this tool for them?

Spontaneous uses. Overwhelmingly, students' approached the interview tasks by drawing graphs on their calculators, often in a standard viewing window, such as [-10,10] x [-10,10] or [-5,5] x [-5,5]. Zooming in repeatedly was their standard method for detecting the local behavior of a function. This was even their first suggestion on finding an appropriate viewing window in which to see the vertical asymptote of the function

$$h(x) = x + 1 + \frac{1}{10^{20}x}$$

in interview 3. Occasionally, students used the trace feature or computed function values in the home screen to generate numerical evidence.

Awareness of limitations. Most students accepted that calculators have limited computational abilities, but could not articulate how those limitations would arise, other than to say they were due to round-off errors. As a consequence of this and the fact that the calculator's deficiencies were always illuminated by a limit as x approached 0, some students came to believe that the computational limitations would always arise in computations involving x-values near 0. Only one student seemed to connect the calculator's computational limitations to its graphical limitations. The rest could not fathom how the calculator might produce an incorrect ordered pair for a function's graph, despite acknowledging the calculator's limited computational abilities.

Awareness of pixel limitations was mixed. Only some students could explain how a calculator-based graph might not show a hole in the graph. Those who could not explain this also could not explain the basic process by which a calculator draws a graph: namely it plots points and connects them with "straight" lines. For these students, the mechanics of the calculator's production of graphs seemed to be a complete mystery.

Students dealt with the calculator's limitations in extreme ways. Either they ignored the possibility, and assumed the calculator-based information was correct, or they assumed the calculator was incorrectly computing numbers, and ignored the calculator-based information. Thus, awareness of calculator limitations did not necessarily imply correct limit conjectures.

How convincing. Most students took the graphs and tables produced by the calculator as the "standard" of comparison, without analysis. For many students, the calculator-based information was more convincing than the information presented in the interview tasks. Students seemed to feel some sense of ownership of the information they produced on the graphing calculator. Hence, they were more willing to believe the calculator-based information than the information presented in the tasks.

CHAPTER 6

Discussion

The data generated from these interview tasks provides rich and detailed descriptions of how and to what extent these students gained an intuitive-analytic understanding of limits. However, much remains to be said about how these data in the current study compare to what was expected, what others have found in similar situations, and what might have happened under different circumstances. In particular, several aspects of the design seemed to influence the quantity and quality of data generated. The data actually generated, in turn, influenced the validity of the conclusions reached. Thus the discussion centers on design and validity issues.

Design Issues

Three elements of the design appeared to influence students' approaches to the tasks: (a) the lack of explicit requests for written explanations, (b) the choices of examples and non-examples, and (c) the lack of precise directions. Each aspect affected the data gathered, and suggested changes for future research and teaching efforts.

Written Explanations

The lack of explicit requests for written explanations in some tasks limited some students' engagement in the thinking process. It was thought at the outset, that explanations could be elicited in the interactive phase of the interviews, but there was evidence that this allowed students to avoid thinking through their answers. This was evident in cases when students wrote answers without explanations, followed by long pauses in response to requests for explanations, and a willingness to switch their answers.

The inclusion of explicit requests for written explanations in some tasks boosted some students' engagement in the thinking process. Some students made several tries at a written explanation, either erasing or crossing out earlier versions, and occasionally changing their answer in the process. The positive effect of writing on the thinking process was most clear in Laura's explanation in interview 1of her choices of a graph and table for the function

$$g(x) = \frac{1}{(x-3)^2}.$$

> The table I figured out by plugging in points and using the process of elimination. For a while, I thought the graph was C because I figured that the higher the x-value (after 3) the closer the graph would be to zero. Then (as I was writing this) I realized that the same would be true with negative values with high absolute value. Because there is a square in the denominator and a positive number on top the graph can never cross the x-axis and become negative so it is H. The numbers really close to 3 result in high values.

Wahlberg (1999), who studied second semester calculus students' understanding of limits through writing assignments, found a similar positive effect of writing. This suggests greater use of written explanations in future research, as it will generate more data for validation purposes and it may contribute to a more uniformly high engagement in interview tasks.

Examples and Non-Examples

The examples and non-examples seemed to influence students' thought processes substantially in subsequent interviews, and not always to their benefit. In particular, students "learned" several unintended lessons. For example, Paul "learned" during interview 2 that tables themselves are not to be trusted. He did not "learn" to distrust the *source* of the poor tables in interview 2, namely the graphing calculator. Josh also "learned" during interview 2 that tables can be misleading. He also seemed to "learn" that constant, apparently exact function values in a table could only come from computational mistakes made by the graphing calculator, because he immediately, without analysis, assumed that the table in task 1 of interview 3 had incorrect y-values.

Davis and Vinner (1986), in a study of high school calculus students' understanding of sequences, found that students often "misinterpreted their own experience", inappropriately attributing generality to details salient in a particular example. Zaslavsky (1989, as cited in Leinhardt, Zaslavsky & Stein, 1990) found that both correct and incorrect notions presented in examples were remembered, and that weak students reasoned from specifics of remembered examples rather than formal definitions. This suggests that the use of examples and non-examples is quite powerful, but can contribute to misconceptions. It is conceivable that if earlier examples and non-examples had been revisited and re-examined as the "lessons learned" exhibited themselves, then students might have developed appropriate interpretations of earlier experiences.

Precise Directions

Lack of precision in many of the task directions was deliberate, to prevent students from detecting the interviewer's "desired" answer, but this resulted in many different interpretations of the directions. For example, Josh, in describing how the two other tables for the function

$$g(x) = \frac{\tan(x) - x}{x^3}$$

could mislead someone about the limit as x approached 0, described how the presence of the function values in the tables could lead one to believe that the calculator was capable of calculating the function values correctly. This was an unexpected but very interesting response. Would the presence of the function values *on the graphing calculator screen* lead him to believe that the calculator is capable of calculating the function values correctly? Does he think that the calculator should be returning something different, like the word "error", if it can not compute the function values correctly? In this case and others, different interpretations led to students not engaging in the intended tasks, yet the unexpected interpretations also gave valuable insights. Given that these were task-based interviews, it was possible to follow up during interaction and pose more focused questions. However, if written work were to comprise the data in a future study, then serious thought would have to be given to the trade-off between precise questions and those more open to interpretation.

Validity Issues

Two characteristics of the data influenced the validity of conclusions about students' patterns of analytical thinking and knowledge use: (a) oral responses, and (b) graphing calculator use observations. The first aids in the validity of the conclusions reached about students' thought processes, and the second raises questions about the validity of conclusions reached about the influence of students' graphing calculator use.

Oral Responses.

Oral responses contributed positively to the validity of students' patterns of analytical thinking and knowledge use. Students' comments during interaction were often unexpected, revealing quite different thinking strategies than would have been assumed from their written work alone. For example, some students wrote "correct" answers, but their oral explanations revealed entirely erroneous lines of reasoning. Other students wrote "correct" answers and nearly correct explanations, but when pressed further, indicated uncertainty or switched their answers. Other researchers (Tall, 1990, 1992; Tall & Vinner, 1981) have noted similar phenomena.

Graphing Calculator Use Observations

Observations of students' graphing calculator use were limited. Whether a student turned to the graphing calculator first was always noted. As much as possible, the interviewer noted the actions students took with the graphing calculator. While students worked on their own though, it was often impossible to observe any more than that the student drew a graph. Occasionally, the interviewer interrupted to ask to see the graph and what viewing window was chosen, especially if the student seemed to be at a standstill. Sometimes during interaction, students were asked what they did with the calculator, and in this way, their calculator use could be reconstructed. However, neither of these techniques was systematically employed. In annotating and summarizing students' graphing calculator use, instances when students' graphing calculator actions were unknown were noted as such. Thus, the graphing calculator use reported in this study should be treated with caution.

Directions for Future Research

There are several directions in which future research on limits could go. First, it would be useful to know if the suggested changes in the interview tasks result in data that tells a different story than unfolded here. Second, there is a question of whether the results that appear to be true of these students generalize to the larger population of first time calculus students. Finally, given that other researchers have concluded that understanding of limits develops slowly, it would be interesting to know the stability of the patterns of analytical thinking and knowledge use found in the present study.

Conclusion

This study provided detailed descriptions of ten students' early understanding of the limit concept, suggesting that these students made small gains in developing an intuitive-analytic understanding of limits. Most developed the ability to draw correct intuitive conclusions from graphs and tables, but had difficulty determining whether a table or graph reflected the true nature of the function's local behavior. They tended to accept graphing calculator output at face value, despite an increased awareness of the possibility of non-representative graphs and tables. Some could produce multiple sources of evidence to justify a limit conjecture, but many of their doubts stemmed from second-guessing themselves rather than an evaluation of the mathematical soundness of their arguments. These results partially stemmed from an inability to accept limitations of the graphing calculator, and partially from weaknesses in their function knowledge.

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Appendix A

Background Questionnaire - Math 1823 Section 200 - Fall 1998

Please complete the following short questionnaire. This will help your instructor have some background information on the students in the class. This information is not used for your grade.

1. Name ______________________________

2. ID Number ___________________________

3. Male ____ Female ____

4. How are you classified?

    ____ Freshman ____ Sophomore ____Junior ____Senior

    ____ Other (What? _____________)

5. Are you considering a major or minor in math? Yes ____ No ____

    If you have decided on a major, what is it? ____________________

6. When did you graduate from high school? 19____

7. How many students were in your high school graduating class? ______ students

8. What mathematics classes did you have **in high school** and when?

____ Algebra I in 19____

____ Geometry in 19____

____ Algebra II in 19____

____ Trigonometry in 19____

____ Advanced Math in 19____

____ Precalculus in 19____

____ Calculus in 19____

____ Other (________________) in 19____

____ Other (________________) in 19____

9. Which of the following mathematics classes (if any) did you have **at the University of Oklahoma** before, when, and what was your course grade?

____ Intro. to Elementary Functions (Math 1503) in Fall/Spring/Summer of 19____

Course Grade: A B C D F W

____ Elementary Functions (Math 1523) in Fall/Spring/Summer of 19____

Course Grade: A B C D F W

____ Calculus I (Math 1823) in Fall/Spring/Summer of 19____

10. What mathematics classes (if any) did you have **at other colleges** before, when and where?

____ Beginning Algebra at __________________ in Fall/Spring/Summer of 19____

____ Intermediate Algebra at __________________ in Fall/Spring/Summer of 19____

____ College Algebra at __________________ in Fall/Spring/Summer of 19____

____ Trigonometry at __________________ in Fall/Spring/Summer of 19____

____ Precalculus at __________________ in Fall/Spring/Summer of 19____

____ Intro. to Elementary Functions at ___________ in Fall/Spring/Summer of 19____

____ Elementary Functions at __________________ in Fall/Spring/Summer of 19____

____ Calculus I at __________________ in Fall/Spring/Summer of 19____

____ Other (________________) at _____________ in Fall/Spring/Summer of 19____

____ Other (________________) at _____________ in Fall/Spring/Summer of 19____

11. Which graphing calculator (if any) do you already know how to use? __________

12. Which of the following best describes your experience with graphing calculators?

____ None

____ I have seen one or two demonstrations

____ I have seen more than two demonstrations

____ I have used graphing calculators once or twice

____ I have used graphing calculators frequently in (how many?) ____ of my classes

____ Other (Please explain)

13. How helpful do you expect the graphing calculator to be to your understanding and learning of calculus in this class?

____ Very unhelpful

____ Somewhat unhelpful

____ Undecided

____ Somewhat helpful

____ Very helpful

14. What is your general feeling about calculators and computers?

____ I like them a lot

____ They are okay

____ I can take them or leave them

____ I do not like them

____ I really hate them

15. How would you rate your aptitude with computers and calculators?

____ low

____ somewhat below average

____ average

____ somewhat above average

____ high

16. How would you rate your math aptitude?

____ low

____ somewhat below average

____ average

____ somewhat above average

____ high

17. Which of the following best describes your experience with the topic of **mathematical limits**? (Check any that apply.)

____ None

____ Briefly introduced

____ I have learned the techniques but no theory

____ I have learned both the techniques and the theory

____ I have been using mathematical limits for one year

____ I have been using mathematical limits for more than one year

____ Other (Please explain)

18. Please write any additional comments that can help determine your background in mathematics.

19. Would you be willing to participate in a study about how first-semester calculus students develop an understanding of mathematical limits? It would involve 4 short interviews with one of the teaching assistants, which should take a total of 2-3 hours of your time outside of class. (Saying yes now does not commit you to participating,

only that you will be contacted with further information. Saying no will involve no penalty to your grade, and means that you will not be asked again.)

____ Yes, I might be willing to participate in the study, after finding out more details.

____ No, I am not willing to participate in the study.

Appendix B

Information Handout

August 26, 1998

Hi,

Thank you for your willingness to consider participating in my Ph.D. dissertation research study. This letter will provide you with more information about the topic of the study, why you were selected, and what your participation would involve. If you still have questions about the study after reading this letter, you may contact me by phone at 321-1929 or by e-mail at bergthold@ou.edu.

Let me begin by saying that this is a study about how first semester calculus students develop an understanding of limits. In particular, I want to determine what makes it difficult for students to understand limits, and how I might help them overcome those difficulties, so that instruction of this topic can be improved.

Your background questionnaire indicated that you either have not had a calculus course before this semester, or that you have had little or no exposure to the topic of mathematical limits, (or both). This makes you an ideal candidate for my study because I am particularly interested in the successes and difficulties students have when they are just *beginning* to learn about the limit concept. By participating, you may acquire a better understanding of limits than you would have gained otherwise, since you will be spending extra time studying this topic. In addition, your participation may potentially result in better instruction on the topic of limits for future students.

Your participation would involve four interviews with me, each lasting from 20 to 30 minutes, in my office (Physical Sciences 929). . Each interview will be recorded on an audiocassette tape. During each interview, you will be asked to (a) fill out a short questionnaire, (b) attempt some math problems involving or related to limits, and (c) discuss any difficulties you are having with these problems. Sometimes you will solve the problems with the help of a graphing calculator and other times without the help of a graphing calculator.

The interviews will take place mostly on Mondays and Tuesdays over the course of 7 weeks, beginning next week (the week of August 31st to September 4th) and ending the week of October 12th to the 16th. The interview schedule is included on the back. I would prefer to do all of the interviews on Mondays and Tuesdays, but other days of the week can be accommodated if necessary. You will need to bring your graphing calculator to the last three interviews.

These interviews are completely independent of your work in the Calculus I course in which you are enrolled. The professor does not know who the research participants are, and will not see any of the work you do in the interviews. All of the records (written work and audiocassettes) will be kept confidential. To ensure this, all of your records will be labeled with a random numerical code, rather than with your name or other identifying code.

Having said all of this, let me now say that you are under *no* obligation to participate, and there will be no penalty to your grade in your Calculus I course if you choose not to participate. Moreover, if you *do* choose to participate, you may discontinue your participation at any time, with no questions asked and no penalty to your grade.

I would be very grateful to have your participation in this research study. Thank you for your consideration.

Sincerely, Trisha Bergthold

Please check the appropriate box below and return this letter to me or to Gabriel Matney at the end of lecture on Wednesday or during your discussion section on Wednesday or Thursday.

____ Yes, I really do want to participate in the study, and I've indicated on the back my top three preferences for when to do the first interview next week.

____ No, I think I would rather not participate in the study.

Monday	Tuesday	Wednesday	Thursday	Friday
24 August	25	26	27	28
31 *Interview 1*	1 September *Interview 1*	2	3	4
7 Labor Day Holiday	8	9	10	11
14 *Interview 2*	15 *Interview 2*	16	17	18
21 Exam 1 in Calculus I	22	23	24	25
28 *Interview 3*	29 *Interview 3*	30	1 October	2
5	6	7	8	9
12 *Interview 4*	13 *Interview 4*	14	15	16
19 Exam 2 in Calculus I	20	21	22	23

Interview Times for week of August 31st to September 4th. Please indicate your top three choices for an interview time on Monday or Tuesday by putting numbers in the appropriate boxes. If necessary, choose a time on Wednesday or Thursday. You will be contacted about your interview time on Thursday or Friday.

Monday 8-31	Tuesday 9-1	Wednesday 9-2	Thursday 9-3	Friday 9-4
10:30-11	10:30-11	10:30-11	10:30-11	XXXXXXXX XXXXXXXX
XXXXXXXX XXXXXXXX	11:30-12	11:30-12	11:30-12	XXXXXXXX XXXXXXXX
12:30-1	12:30-1	12:30-1	12:30-1	XXXXXXXX XXXXXXXX
1:30-2	1:30-2	XXXXXXXX XXXXXXXX	1:30-2	XXXXXXXX XXXXXXXX
2:30-3	2:30-3	XXXXXXXX XXXXXXXX	2:30-3	XXXXXXXX XXXXXXXX
3:30-4	3:30-4	3:30-4	3:30-4	XXXXXXXX XXXXXXXX
4:30-5	XXXXXXXX XXXXXXXX	XXXXXXXX XXXXXXXX	XXXXXXXX XXXXXXXX	XXXXXXXX XXXXXXXX
5:30-6	5:30-6	5:30-6	5:30-6	XXXXXXXX XXXXXXXX

Appendix C

Human Subjects Permission

The University of Oklahoma

OFFICE OF RESEARCH ADMINISTRATION

August 20, 1998

Ms. Trisha A. Bergthold
711 1/2 West Brooks Street
Norman OK 73069-4602

Dear Ms. Bergthold:

Your research proposal, "The Development of the Limit Concept in First Semester Calculus Students," has been reviewed by Dr. E. Laurette Taylor, Chair of the Institutional Review Board, and found to be exempt from the requirements for full board review and approval under the regulations of the University of Oklahoma-Norman Campus Policies and Procedures for the Protection of Human Subjects in Research Activities.

Should you wish to deviate from the described protocol, you must notify me and obtain prior approval from the Board for the changes. If the research is to extend beyond 12 months, you must contact this office, in writing, noting any changes or revisions in the protocol and/or informed consent form, and request an extension of this ruling.

If you have any questions, please contact me.

Sincerely yours,

Karen

Karen M. Petry
Administrative Officer
Institutional Review Board

KMP:pw
FY99-29

cc: Dr. E. Laurette Taylor, Chair, IRB
Dr. Curtis McKnight, Mathematics

Appendix D

Informed Consent Form for Participation in a Research Project

Being Conducted Under the Auspices of the University of Oklahoma – Norman Campus

Study Title: The Development of the Limit Concept in First Semester Calculus Students

Sponsor: Dr. Curtis McKnight, Mathematics Department

Principal Investigator: Trisha Bergthold, Mathematics Department

Description of the Study

The purpose of this research is to learn how students in first semester calculus develop an understanding of the limit concept. In particular, we want to determine what makes it difficult for students to understand limits, and how we might help them overcome those difficulties, so that we can improve our instruction of this topic. Your participation in this research project will involve 4 interviews outside of class time with the principal investigator. Each interview will last approximately 20-30 minutes, and will be recorded on an audiocassette tape. During each interview, you will be asked to (a) fill out a short questionnaire, (b) attempt some math problems involving or related to limits, and (c) discuss any difficulties you are having with these problems. Sometimes you will solve the problems with the help of a graphing calculator and other times without the help of a graphing calculator.

Potential Risks and Benefits of Participation

The time it takes you to participate in the 4 interviews may cost you study time for this or other classes. If, at any point, you feel that the time required to participate in this study is having a negative affect on your course work, you are free to discontinue your participation.

Your participation may potentially result in you acquiring a better understanding of limits than you would have gained otherwise, since you will be spending extra time studying this topic. In addition, your participation may potentially result in better instruction on the topic of limits for future students.

Subject's Assurances

Your participation is voluntary. Refusal to participate will involve no penalty to your course grade and you may discontinue participation at any time without penalty to your course grade.

All of the records (written work and audiocassettes) will be kept confidential. To ensure this, all of your records will be labeled with a random numerical code, rather than with your name or other identifying code.

If you have questions either about the research itself or about your rights as a research subject, you may contact Trisha Bergthold by phone at (405) 321-1929 or (405) 325-6711 or by e-mail at bergthold@ou.edu.

Signature ______________________________________ Date: ______________

Appendix E

Task for Interview 1, Version 1 of 5

In calculus, we will need to analyze and describe some rather complicated functions. Sometimes it helps to examine graphs and tables of ordered pairs of functions when we do this.

All of the functions below are undefined at x = 3, but each function "behaves" very differently at x-values *near* x = 3. Match each function formula to its corresponding graph and table.

Function formula	Corresponding Graph	Corresponding Table
$f(x)=\dfrac{1}{x-3}$		
$g(x)=\dfrac{1}{(x-3)^2}$		
$h(x)=\dfrac{x^2-5x+6}{x-3}$		
$p(x)=\begin{cases} x-2 & \text{if} \quad x<3 \\ x-3 & \text{if} \quad x>3 \end{cases}$		
$t(x)=\sin\left(\dfrac{1}{x-3}\right)$		

Please write, in 3-5 sentences, how you figured out the graph and table for the function

$$f(x)=\frac{1}{x-3},$$

or, if you didn't figure them out, then write what you tied.

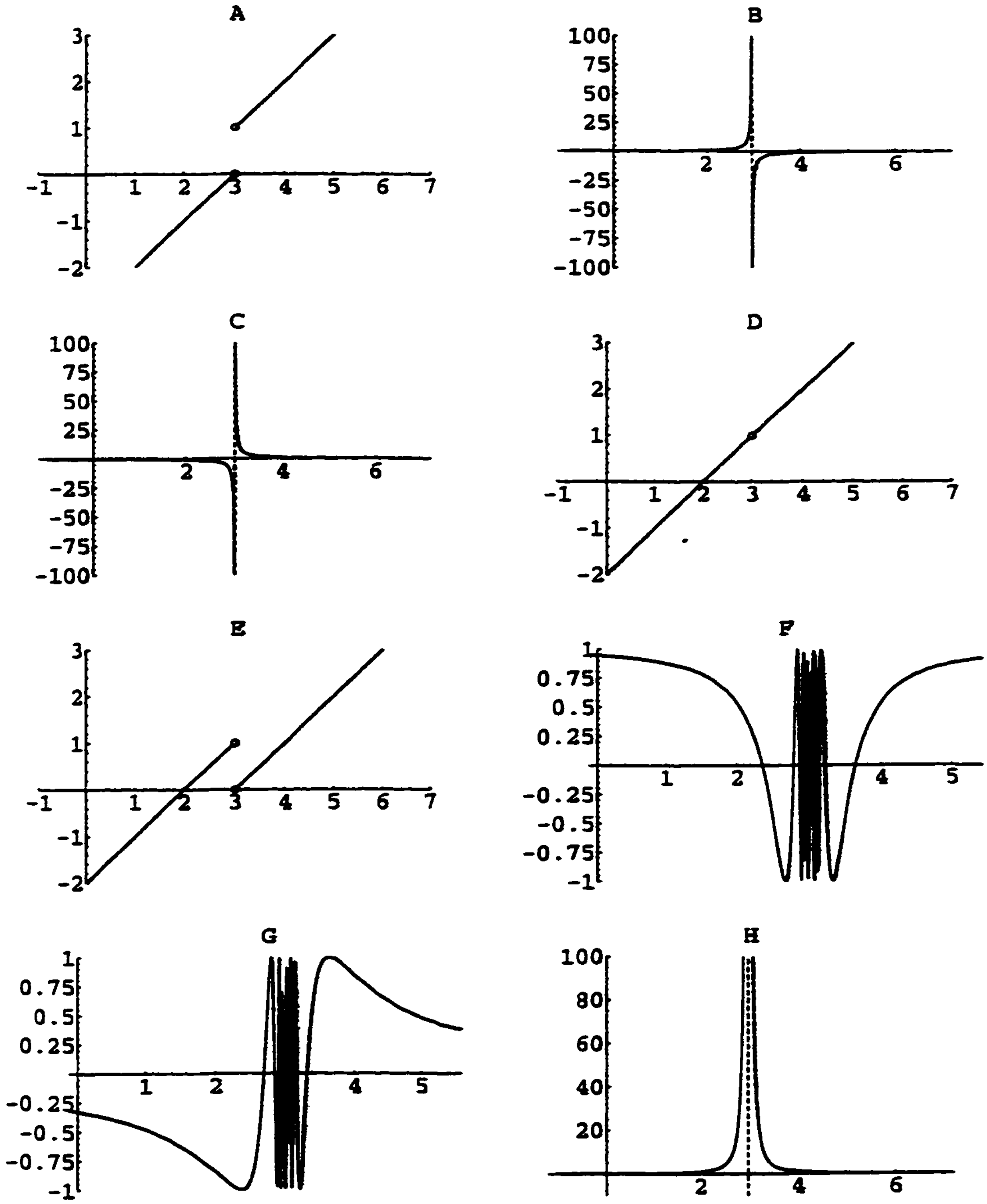

A
3
2
1
-1
-2
-1
1
2
3
4
5
6
7
B
100
75
50
25
-25
-50
-75
-100
2
4
6
C
100
75
50
25
-25
-50
-75
-100
2
4
6
D
3
2
1
-1
-2
-1
1
2
3
4
5
6
7
E
3
2
1
-1
-2
-1
1
2
3
4
5
6
7
F
1
0.75
0.5
0.25
-0.25
-0.5
-0.75
-1
1
2
4
5
G
1
0.75
0.5
0.25
-0.25
-0.5
-0.75
-1
1
2
4
5
H
100
80
60
40
20
2
4
6

Table A

x	2	2.5	2.9	2.99	2.999	3	3.001	3.01	3.1	3.5	4
y	0	0.5	0.9	0.99	0.999	undefined	1.001	1.01	1.1	1.5	2

Table B

x	2	2.5	2.9	2.99	2.999	3	3.001	3.01	3.1	3.5	4
y	0	0.5	0.9	0.99	0.999	undefined	0.001	0.01	0.1	0.5	1

Table C

x	2	2.5	2.9	2.99	2.999	3	3.001	3.01	3.1	3.5	4
y	-1	-0.5	-0.1	-0.01	-0.001	undefined	1.001	1.01	1.1	1.5	2

Table D (y-values in this table are approximated to 6 decimal places.)

x	2	2.5	2.9	2.99	2.999	3	3.001	3.01	3.1	3.5	4
y	-0.841471	-0.909297	0.544021	0.506366	-0.826880	undefined	0.826880	-0.506366	-0.544021	0.909297	0.841471

Table E (y-values in this table are approximated to 6 decimal places.)

x	2	2.5	2.9	2.99	2.999	3	3.001	3.01	3.1	3.5	4
y	0.540302	-0.416147	-0.839072	0.862319	0.562379	undefined	0.562379	0.862319	-0.839072	-0.416147	0.540302

Table F

x	2	2.5	2.9	2.99	2.999	3	3.001	3.01	3.1	3.5	4
y	1	4	100	10,000	1,000,000	undefined	1,000,000	10,000	100	4	1

Table G

x	2	2.5	2.9	2.99	2.999	3	3.001	3.01	3.1	3.5	4
y	1	2	10	100	1000	undefined	-1000	-100	-10	-2	-1

Table H

x	2	2.5	2.9	2.99	2.999	3	3.001	3.01	3.1	3.5	4
y	-1	-2	-10	-100	-1000	undefined	1000	100	10	2	1

Appendix F

Questionnaire 1

Graphs, tables of ordered pairs, and function formulas each tell us different things about functions. By answering the following questions, you will indicate how helpful each of these sources of information is *to you* in understanding functions.

1. Which one of the following would you prefer to use when describing how a function behaves to a fellow student?
   ____ a graph
   ____ a table of ordered pairs
   ____ the function formula

Please explain in a sentence or two why you chose the one you did.

2. If you could choose two things to help you tell a fellow student how a function behaves, which *pair* would you prefer?
   ____ a graph and a table of ordered pairs
   ____ a graph and the function formula
   ____ a table of ordered pairs and the function formula

Please explain in a sentence or two why you chose the pair you did.

3. Would using all three (a graph, a table of ordered pairs, and the function formula) help more than using just two of the three in telling a fellow student how a function behaves?
   ____ Always
   ____ Sometimes
   ____ Never

Why do you think this?

4. Is it possible that these three (a graph, a table of ordered pairs, and a function formula) *wouldn't* provide enough good information for you to tell a fellow student how a function behaves?
   ____ Yes
   ____ No

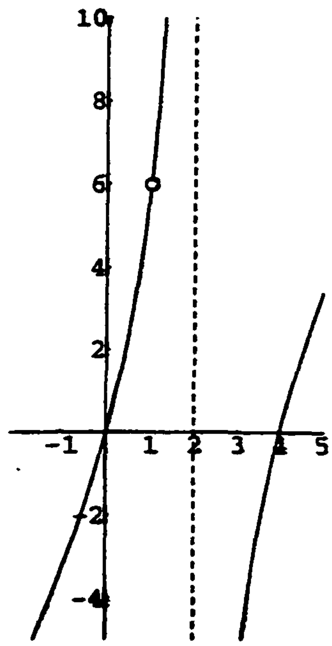
Why do you think this?

Appendix G

Tasks for Interview 2, Graphical Version*

Sometimes, a graph of a function can suggest whether certain limits exist or not, and, if so, what those limits might equal. This is only possible, though, if the graph actually reveals the true nature of the function.

The graph below is one that *does* show the true nature of a function *f*.



1. Use the graph to make educated guesses whether the limits below exist or not.
2. If a limit exists, then suggest what its value might be. If a limit does not exist, then explain in a few sentences how the graph shows this.

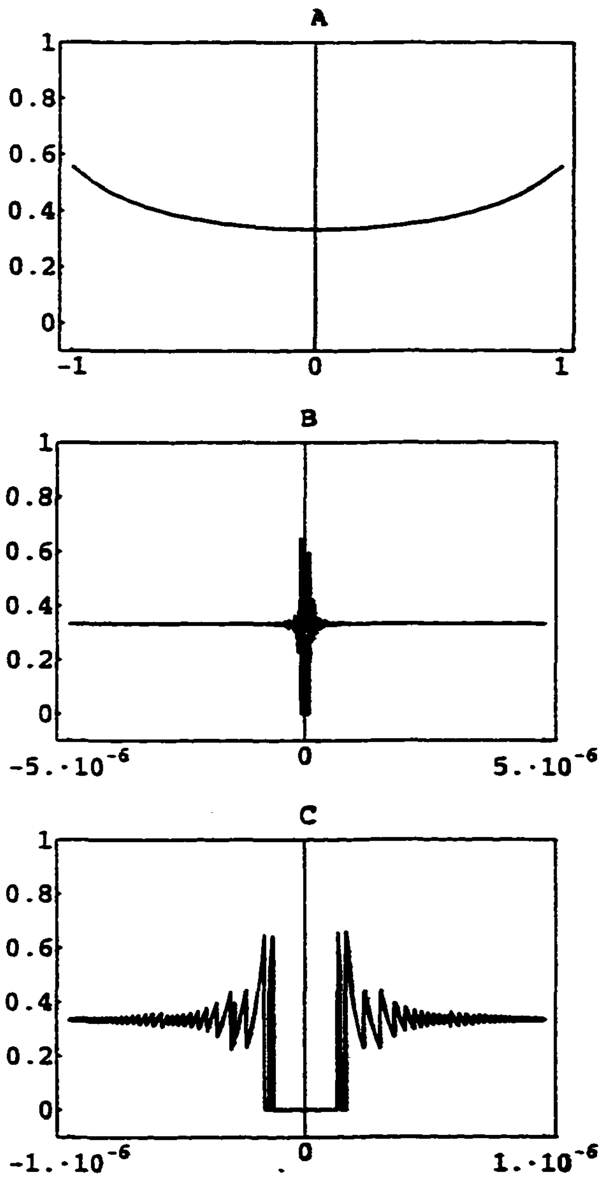
Possible Limit	Suggestion	Explanation or Suggested Limit Value
$\lim_{x \to 1} f(x)$	This graph of *f* suggests that this limit ____ exists ____ does not exist	
$\lim_{x \to 2} f(x)$	This graph of *f* suggests that this limit ____ exists ____ does not exist	
$\lim_{x \to 4} f(x)$	This graph of *f* suggests that this limit ____ exists ____ does not exist	

It *can* happen that a graph of a function does *not* reflect the function's true behavior, especially when a calculator is used to draw the graph. Then the graph can actually be misleading, suggesting conclusions that turn out to be false.

Below are three *different* graphs of the *same* function

$$g(x) = \frac{\tan(x) - x}{x^3}$$

for x-values near 0. Each graph was drawn using a computer algebra system. Graphing g in the same viewing rectangles on your graphing calculator would produce similar graphs. Only *one* graph reveals the true nature of this function for x-values near 0.



1. Which graph best reflects the true nature of the function g for x-values near 0?

2. What does your chosen graph lead you to conclude about $\lim_{x \to 0} g(x)$?

3. How do the other graphs mislead you about $\lim_{x \to 0} g(x)$?

*Task 2, on this page, was adapted from Stewart (1995).

Appendix H

Tasks for Interview 2, Numerical Version*

Sometimes, a table of ordered pairs of a function can suggest whether certain limits exist or not, and, if so, what those limits might equal. This is only possible, though, if the table of ordered pairs actually reveals the true nature of the function.

The table below (given in three pieces) is one that *does* show the true nature of a function *f*, for x-values near 1, x-values near 2, and x-values near 4. (The y-values in italic are approximated to 6 decimal places. All other y-values are exact.)

x	y
0	0
0.5	*2.333333*
0.9	*5.072727*
0.99	*5.900792*
0.999	*5.990008*
1	undefined
1.001	*6.010008*
1.01	*6.100808*
1.1	*7.088889*
1.5	15

x	y
1.5	15
1.9	79.8
1.99	799.98
1.999	7999.998
2	undefined
2.001	-7999.998
2.01	-799.98
2.1	-79.8
2.5	-15
3	-6

x	y
3.5	*-2.333333*
3.9	*-0.410526*
3.99	*-0.040101*
3.999	*-0.004001*
4	0
4.001	*0.003999*
4.01	*0.039900*
4.1	*0.390476*
4.5	1.8
5	*3.333333*

1. Use the table to make educated guesses whether the limits below exist or not.
2. If a limit exists, then suggest what its value might be. If a limit does not exist, then explain in a few sentences how the table shows this.

Possible Limit	Suggestion	Explanation or Suggested Limit Value
$\lim_{x \to 1} f(x)$	This table of ordered pairs of *f* suggests that this limit ____ exists ____ does not exist	
$\lim_{x \to 2} f(x)$	This table of ordered pairs of *f* suggests that this limit ____ exists ____ does not exist	
$\lim_{x \to 4} f(x)$	This table of ordered pairs of *f* suggests that this limit ____ exists ____ does not exist	

It *can* happen that a table of ordered pairs of a function does *not* reflect the function's true behavior, especially when a calculator is used to compute y-values for ordered pairs of the function. Then the table can actually be misleading, suggesting conclusions that turn out to be false.

Below are three *different* tables of the *same* function

$$g(x) = \frac{\tan(x) - x}{x^3}$$

for x-values near 0. The y-values were computed using a calculator. The y-values in italic are calculator output, rounded to 6 decimal places. All other y-values are identical to the calculator's output. Only *one* of the tables reveals the true nature of this function for x-values near 0.

Table A

x	y
-0.5	*0.370420*
-0.1	*0.334672*
-0.05	*0.333667*
-0.01	*0.333347*
-0.005	*0.333337*
-0.001	*0.333334*
0	undefined
0.001	*0.333334*
0.005	*0.333337*
0.01	*0.333347*
0.05	*0.333667*
0.1	*0.334672*
0.5	*0.370420*

Table B

x	y
-0.0005	*.333333*
-0.0001	.33333
-0.00005	.333336
-0.00001	.333
-0.000005	.3336
-0.000001	.3
-0.0000005	.32
0	undefined
0.0000005	.32
0.000001	.3
0.000005	.3336
0.00001	.333
0.00005	.333336
0.0001	.33333
0.0005	*.333333*

Table C

x	y
-0.0000001	0
-0.00000005	0
-0.00000001	0
-0.000000005	0
-0.000000001	0
-0.0000000005	0
0	undefined
0.0000000005	0
0.000000001	0
0.000000005	0
0.00000001	0
0.00000005	0
0.0000001	0

4. Which table best reflects the true nature of the function g for x-values near 0?

5. What does your chosen table lead you to conclude about $\lim_{x\to 0} g(x)$?

6. How do the other tables mislead you about $\lim_{x\to 0} g(x)$?

*Task 2, on this page, was adapted from Stewart (1995).

Appendix I

Tasks for Interview 3*

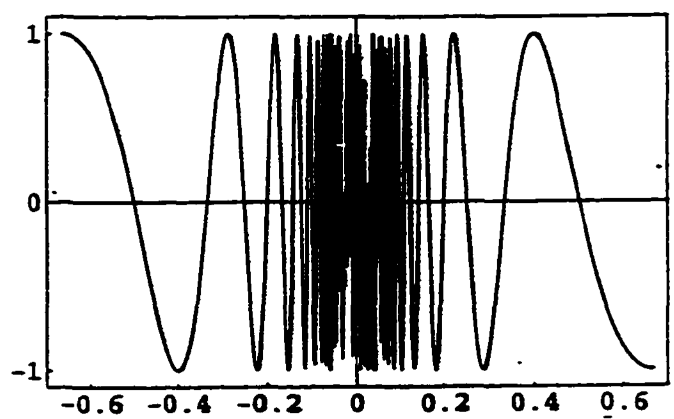
A limit conjecture is most likely to be true if it is based on a table or a graph that reflects the true nature of the function. So, analyzing a limit situation involves analyzing whether the table or graph is misleading.

A graph and a table of ordered pairs of the function

$$t(x) = \sin\left(\frac{\pi}{x}\right)$$

for x-values near 0 are given below. However, the table and graph contradict one another.

X	y
-2/3	1
-2/7	1
-2/11	1
-2/15	1
-2/19	1
-2/23	1
0	undefined
2/23	-1
2/19	-1
2/15	-1
2/11	-1
2/7	-1
2/3	-1



1. Does the table reflect the true nature of the function t for x-values near 0?

2. Does the graph reflect the true nature of the function t for x-values near 0?

3. What do you think is true about $\lim_{x \to 0} t(x)$?

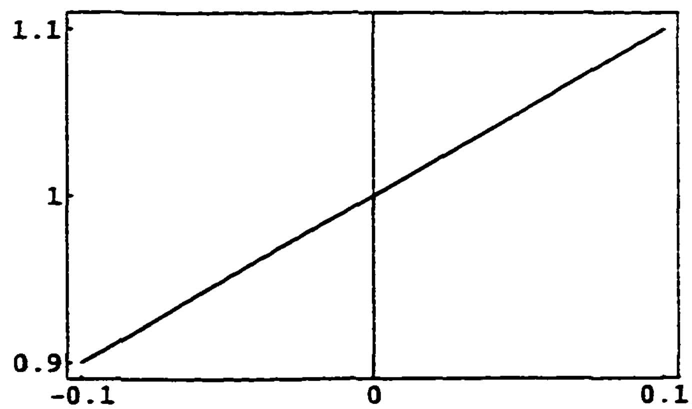
*Task 1, on this page, was adapted from Stewart (1995).

A graph and a table of ordered pairs of the function

$$h(x) = x + 1 + \frac{1}{10^{20}x}$$

for x-values near 0 are given below.

x	y
-0.1	0.9
-0.01	0.99
-0.001	0.999
-0.0001	0.9999
-0.00001	0.99999
-0.000001	0.999999
0	undefined
0.000001	1.000001
0.00001	1.00001
0.0001	1.0001
0.001	1.001
0.01	1.01
0.1	1.1



1. Do the graph and table contradict one another?

2. Does the table reflect the true nature of the function h for x-values near 0?

3. Does the graph reflect the true nature of the function h for x-values near 0?

4. What do you think is true about $\lim_{x \to 0} h(x)$?

*Task 2, on this page, was adapted from Williams (1991).

Appendix J

Tasks for Interview 4*

Sometimes a limit cannot be determined by manipulating symbols from the function formula. We must then use other methods to analyze the situation, make an educated guess and, if possible, determine conclusively whether the limit exists.

Consider the function

$$f(x) = x^2 \cos\left(\frac{1}{x}\right).$$

1. What do you think is true about $\lim_{x \to 0} f(x)$? Explain how you arrived at your conclusion.

2. How certain are *you* that your conclusion is correct?
   ____ absolutely certain
   ____ fairly certain, but there is room for doubt
   ____ not all certain

3. How certain do you expect *everyone else* should be that your conclusion is correct?
   ____ absolutely certain
   ____ fairly certain, but there is room for doubt
   ____ not all certain

*Similar to exercises in Stewart (1995).

4. **Produce some additional evidence that either indicates your original conclusion is correct, or indicates that you need to modify your conclusion.**

5. Now, how certain are *you* that your (possibly modified) conclusion is correct?
   ____ absolutely certain
   ____ fairly certain, but there is room for doubt
   ____ not all certain

6. Now, how certain do you expect *everyone else* should be that your (possibly modified) conclusion is correct?
   ____ absolutely certain
   ____ fairly certain, but there is room for doubt
   ____ not all certain

Appendix K

Questionnaire 2

Graphs, tables of ordered pairs, and function formulas each tell us different things about limits of functions. By answering the following questions, you will indicate how helpful each of these sources of information is *to you* in understanding limits of functions.

1. Which one of the following would you prefer to use when analyzing a limit situation?
    ____ a graph
    ____ a table of ordered pairs
    ____ the function formula

Please explain in a sentence or two why you chose the one you did.

2. If you could choose two things to help you analyze a limit situation, which *pair* would you prefer?
    ____ a graph and a table of ordered pairs
    ____ a graph and the function formula
    ____ a table of ordered pairs and the function formula

Please explain in sentence or two why you chose the pair you did.

3. Would using all three (a graph, a table of ordered pairs, and the function formula) help more than using just two of the three in analyzing a limit situation?
    ____ Always
    ____ Sometimes
    ____ Never

Why do you think this?

4. What are some drawbacks of relying on these three (a graph, a table of ordered pairs, and the function formula) when analyzing a limit situation?

Appendix L

Interview Protocols

Interview 1 Protocol

1. Thank you for being willing to participate in my study. Before we begin, you need to read this informed consent form and sign it. The green copy is for you to keep.
2. I am going to turn on the tape recorder now, but please try to ignore it.
3. During each of these interviews, you will work on some mathematics problems. They will likely be unfamiliar to you and somewhat difficult, but don't worry about this. I am more interested in your thought processes than your solutions, so I would like you to think out loud as much as possible. At first, you will work on your own (thinking out loud) until you feel you've done everything you can to solve the problem. During that time, I will watch and take notes. Then I'll ask a few questions about the problem and your thought processes. Finally, there is a brief questionnaire to complete at the end of the interview. Is all of this clear?
4. Here is the problem. There are three one-sided pages. You may write on all of them, but you should put your answers on the first page. Here is scratch paper in case you need it. Go ahead and begin, and, once again, please think out loud.
5. Follow-up questions
    (a) Do you see how each function is undefined at $x = 3$?
    (b) Do you see how each function behaves differently at x-values near $x = 3$?
    (c) Did you use your knowledge of some functions to help you on the other functions?
    (d) What does it mean for the table to correspond to the formula?

(e) What does it mean for the graph to correspond to the formula?

(f) What does it mean for the graph to correspond to the table?

(g) Other:

(h) Other:

(i) Other:

(j) Other:

6. Please fill out this short questionnaire.

7. Thank you very much. That is all I need from you today. May I sign you up now to do the next interview?

Participant ____ Interview ____ Date ________________ Start Time _______

Duration _______

Interview 2 Protocol

1. I am going to turn on the tape recorder now, but please try to ignore it, as before.
2. During this interview, there are two problems, each with three parts. You may use your graphing calculator, but you may find that this is unnecessary. I'll let you work on the first problem until you feel you've done everything you can do, and then I'll ask you a few questions about it. Then we'll repeat this with the second problem. Is this clear?
3. Here is the first problem. Make sure you read from the beginning.
4. Time ________________
5. Follow-up questions
    (a) Is it difficult to tell from this table/graph what the limit situations are?
    (b) How do you actually read the table/graph? Where are your eyes focusing, and what are you looking for?
    (c) What can you say about the limit of f(x) as x approaches 2 from the left?
6. Okay, let's set this problem aside for now. Here is the second problem. Again, make sure you read from the beginning.
7. Time ________________
8. Follow-up questions
    (a) How did you choose your table?
    (b) Why do you suppose the other tables turned out to be so misleading?

(c) Aren't closer x-values supposed to give you better information about the corresponding y-values?

9. Time ______________

10. That's all for today. The third interviews are in two weeks, and I would like to do the fourth interviews the week immediately after that. Is this okay, and may I schedule you for both of those now?

Participant ____ Interview ____ Date ________________ Start Time _______

Duration _______

Interview 3 Protocol

1. I am going to turn on the tape recorder now, but please try to ignore it, as before.
2. During this interview, there are two problems, each with several parts. You may use your graphing calculator. I'll let you work on the first problem until you feel you've done everything you can do, and then I'll ask you a few questions about it. Then we'll repeat this with the second problem. Is this clear?
3. Here is the first problem. Make sure you read from the beginning.
4. Time _______________
5. Follow-up questions
   (a) How did you decide whether the table reflects the true nature of the function?
   (b) Do you think that the y-values in the table are correct? (Try computing a few y-values to see.)
   (c) Can you point to places on the graph that correspond to some of the ordered pairs in the table? (For example, where is the ordered pair (-2/3,1) located on the graph?)
   (d) If you *only* had *this* table of ordered pairs for this function, what would you expect the graph to look like?
   (e) How did you decide whether the graph reflects the true nature of the function?
   (f) If you *only* had *this* graph for this function, what might some ordered pairs in the table look like? Can you "fix" the table here so that it better matches the graph?

(g) Up here (point to statement), I said that the table and graph contradict one another. Do you believe that's true?

(h) How did you figure out the limit situation in problem 3?

(i) What can you say about the limit of t(x) as x approaches 0 from the left? From the right?

6. Okay, let's set this problem aside for now. Here is the second problem. Again, make sure you read from the beginning.

7. Time _______________

8. Follow-up questions

   (a) How did you decide whether the table and graph contradict one another?

   (b) Do the table and graph include the same range of x-values and y-values?

   (c) What does the graph tell you about h(0)? What does the table tell you about h(0)? Which of these situations is right? How do you know?

   (d) Does the graph match the table for the rest of the ordered pairs?

   (e) How did you decide whether the table reflects the true nature of the function?

   (f) Could you "fix" the table here so that it better reflects the true nature of the function?

   (g) How did you decide whether the graph reflects the true nature of the function?

   (h) Could you draw a graph that would better reflect the true nature of the function?

   (i) Now, after fixing the table and graph, what do you think is true about the limit of h(x) as x approaches 0?

9. Time _______________

Participant ____ Interview ____Date ________________ Start Time _______

Duration _______

Interview 4 Protocol

1. I am going to turn on the tape recorder now, but please try to ignore it, as before.
2. During this interview, there is one problem with several parts. You may use your graphing calculator. I'll let you work on the first parts of the problem until you feel you've done everything you can do. Then, I'll ask you a few questions about your work, and give you the remaining parts of the problem. When we are done discussing your work on those parts, there will be a short questionnaire. Is this clear?
3. Here is the problem. Make sure you read from the beginning.
4. Time ______________
5. Follow-up questions
    (a) At the beginning, I wrote that sometimes a limit cannot be determined by manipulating symbols from the function formula. Can this limit be determined by manipulating symbols?
    (b) What other methods can be used to analyze a limit situation?
    (c) The first thing you tried was a graph/table. Are you certain that the graph/table reflects the true nature of the function?
    (d) Do you think that the best you can do in this situation is to make an educated guess about the limit situation?
    (e) Is it possible for some educated guesses to be more reasonable than others?
    (f) Do you think it's possible to determine for sure what the limit situation must be?

6. Okay, let's set this aside for now. Here are the remaining parts of the problem. These refer to the same problem.

7. Time _______________

8. Follow-up questions

   (a) Your evidence in this part agrees/disagrees with your evidence in the first part. Can you explain this?

   (b) Do you understand the limit situation more or less now that you have tried to generate additional evidence?

   (c) Can you think of a way to determine for sure what the limit situation must be?

9. Thank you for your efforts. I have a short questionnaire for you to fill out and then you will be done.

10. Time ________________

Appendix M

Background Questionnaire Data

Table M1

Gender and High School (HS) Class Demographics for Entire Course and Volunteers (V)

Group	Entire course	V	V & met criteria	Agreed to participate	Final Sample
Male					
1998 HS graduate	77	31	15	9	8
Pre-1998 HS graduate	13	3	3	2	1
(93-97 graduating classes)					
Females					
1998 HS graduate	24	9	4	2	1
Pre-1998 HS graduate	7	3	2	1	0
(90, 96-97 graduating classes)					

Note. Enrollment: 128 students. Completed questionnaires: 121.

Table M2

College Classification and College Major Demographics (Items 4 & 5)

Group	Males		Females		Total	
	1998 HS Class	Pre-1998 HS Class	1998 HS Class	Pre-1998 HS Class	Entire Course	Final Sample
College Classification						
Freshman	77	2	24	2	105	9
Sophomore		7		2	9	0
Junior		3		2	5	1
Senior		1		0	1	0
Other		0		1	1	0
College Major						
No Response or Undecided	15	3	5	3	26	1
Engineering[a]	44	2	6	3	55	4
Science/Tech.[b]	21	9	13	0	43	7
Humanities[c]	0	0	2	1	3	0
Maj/min in math?						
Yes	8	1	4	0	13	2
No	64	11	18	6	99	8
No Response	5	1	1	0	7	0
Not sure	0	0	1	1	2	0

Note. Every major listed was counted once. Six students indicated double majors. [a]Majors listed were engineering, aerospace, chemical, civil, computer, electrical, environmental, industrial, mechanical, and petroleum. [b]Majors listed were architecture, astronomy, astrophysics, computer science, construction science, chemistry, geology, geophysics, management information sciences, mathematics education, meteorology, pre-pharmacy, and pre-medicine. [c]Majors listed were sociology, English, and public relations.

Table M3

Math Background Demographics (Items 8-10)

Prior Math Courses	Males 1998 HS Class	Males Pre-1998 HS Class	Females 1998 HS Class	Females Pre-98 HS Class	Total Entire Course	Total Final Sample
High school calculus	57	2	18	1	78	4
Math 1503 at OU		7		3	10	0
Math 1523 at OU		9		5	14	0
Math 1823 at OU		2		0	2	0
College courses elsewhere						
College Algebra		3		1	4	1
Trigonometry		3		1	4	1
Pre-calculus		1		0	1	0

Note. Each pre-calculus course listed was counted once. Some students took more than one pre-calculus course. OU = University of Oklahoma.

Table M4

Graphing Calculator Use, Experience and Expectations Demographics (Items 11- 13)

	Males		Females		Total	
Graphing Calculator Category	1998 HS Class	Pre-1998 HS Class	1998 HS Class	Pre-1998 HS Class	Entire Course	Final Sample
Models in use						
No Response	12	5	2	3	22	4
TI	73	7	24	3	107	6
Hewlett Packard	3	1	2	0	6	0
Casio	3	0	3	1	7	0
More than one	10	0	5	0	15	0
Experience						
None	3	1	0	0	4	1
Seen 1-2 demos	1	1	1	2	5	1
Seen >2 demos	1	0	1	2	4	0
Used 1-2 times	11	7	0	1	19	5
Used often (*M*)[a]	61(2.7)	4(2.7)	22(2.9)	2 (3)	89(2.8)	3(1.3)
Expect helpful?						
Very unhelpful	10	2	3	1	16	1
S. unhelpful	8	1	2	0	11	0
Undecided	7	0	1	3	11	1
S. helpful	24	5	4	1	34	1
Very helpful	28	4	14	2	48	7

Note. The graphing calculator experience among the 24 students who met the criteria was: none – 1, seen 1-2 demos – 3, seen >2 demos – 1, used 1-2 times – 7, used frequently – 12 (2.9). One male, pre-1998 HS class, did not respond to item 13. TI = Texas Instruments. S. = Somewhat.

[a]*M* = Average, within cell, of prior classes with graphing calculator use.

Table M5

Attitudes about Calculators and Computers (Items 14 & 15)

Attitude Category	Males 1998 HS Class	Males Pre-1998 HS Class	Females 1998 HS Class	Females Pre-1998 HS Class	Total Entire Course	Total Final Sample
Feelings						
I like them a lot	51	9	13	5	78	8
They are okay	21	2	9	2	34	2
I can take them or leave them	5	1	1	0	7	0
I do not like them	0	0	1	0	1	0
I really hate them	0	0	0	0	0	0
Aptitude						
Low	1	2	1	0	4	0
S. below average	4	0	1	0	5	0
Average	23	3	11	4	41	5
S. above average	25	3	10	3	41	3
High	24	4	2	0	30	2

Note. One male, pre-1998 HS class, did not respond to these items. S. = Somewhat.

Table M6

Perceived Math Aptitude and Experience with Mathematical Limits (Items 16 & 17)

Experience Level	Males		Females		Total	
	1998 HS Class	Pre-1998 HS Class	1998 HS Class	Pre-98 HS Class	Entire Course	Final Sample
Math Aptitude						
Low	0	0	0	0	0	0
S. below average	1	1	1	1	4	1
Average	9	3	5	2	19	2
S. above average	49	6	14	4	73	6
High	18	3	5	0	26	1
Limits Experience						
None	8	5	3	2	18	4
Brief introduction	12	4	5	1	22	1
Techniques, but no theory	9	0	4	2	15	4
Techniques and the theory	15	2	3	1	21	1[a]
Used limits 1 year	23	2	7	0	32	1
Used >1 year	10	0	3	1	14	0
No Response	1	1	1	0	3	0

Note: Each level checked was counted once. Some students checked two levels.

[a]This student checked both this level and "used limits 1 year".

Appendix N

Interview and Course Content Schedule

Monday	Tuesday	Wednesday	Thursday	Friday
17 August Lecture 1	18	19 Lecture 2 Discussion 1	20 Discussion 1	21 Lecture 3
24 Lecture 4	25	26 Lecture 5 Discussion 2	27 Discussion 2	28 Lecture 6 Homework 1 due
31 Lecture 7 Week for Interview 1 8 students MT, 2 students WTh	1 September	2 Lecture 8 Discussion 3	3 Discussion 3	4 Lecture 9 Homework 2 due
7 Holiday	8	9 Lecture 10 Discussion 4	10 Discussion 4	11 Lecture 11 Homework 3 due
14 Lecture 12 Week for Interview 2 7 students MT, 2 students WTh	15	16 Lecture 13 Discussion 5	17 Discussion 5	17 Lecture 14 Homework 4 due
21 Exam 1	22	23 Lecture 15 Discussion 6	24 Discussion 6	25 Lecture 16 Homework 5 due
28 Lecture 17 Week for Interview 3 10 students MT	29	30 Lecture 18 Discussion 7	1 October Discussion 7	2 Lecture 19 Homework 6 due
5 Lecture 20 Week for Interview 4 7 students MT, 2 students WTh	6	7 Lecture 21 Homework 7 due Discussion 8	8 Discussion 8	9 Holiday

Figure N1

Note. One student did not do interviews 2 and 4.

Appendix O

Course Content Summary

Lecture 1. A review of functions was begun. A function was presented as a rule that assigns numerical values to each number in a certain set. As an example, a table of world population estimates during certain years was presented, followed by a graph of the ordered pairs in the table. The instructor pointed out that the graph gave more information than the table, and said, "I want you to always have a graph in mind when you see a function." The second example was

$$g(x) = x^2 - 2 \text{ for } 0 \le x \le 4,$$

given with the acknowledgement that many functions would be given by formulas, but this was not necessary. To illustrate this, a third example was given: for each natural number *n*,

$$h(n) = \text{the digit in the } n\text{th decimal place of } \sqrt{2}.$$

Lecture 2. Three more examples of functions were presented, each a piecewise function with a different context. The first example involved no context, and was presented as an illustration that some functions require more than one formula:

$$j(x) = \begin{cases} x+2 & if \quad x \le 1 \\ x^2 & if \quad x > 1 \end{cases}.$$

The second example was the cost of renting a car as a function of miles driven, assuming that a car rental company charges $50 for the first 100 miles and 10 cents per mile thereafter. Given this context, students, interacting with the instructor, constructed the function:

$$k(x)=\begin{cases} 50 & \text{if} \quad x\le 100 \\ 50+\dfrac{1}{10}(x-100) & \text{if} \quad x>100 \end{cases}.$$

The third example was the absolute value function: a function whose formula can be re-expressed as two piecewise-defined formulas. Domain and range were presented and illustrated using the six examples of functions presented earlier.

Discussion 1. A worksheet (see Appendix P) was given introducing students to basic graphing calculator operations and limitations. In particular, students learned how to enter numerical expressions for computation and function formulas for drawing of graphs. Zooming in on a point and tracing along a curve were touched upon, and issues related to viewing window dimensions were raised.

Lecture 3. The notion of instantaneous speed was discussed in an attempt to lead students to think of limit ideas. Ideas were elicited from the instructor as to how one would calculate such a quantity. Student suggestions led to calculating (with the calculator) average velocities over smaller and smaller intervals. This raised the issue of division by zero in relation to calculating instantaneous speed, which the instructor promised would be revisited in weeks to come. Then he returned to the ideas of domain and range in the context of creating new functions by addition, subtraction, multiplication, division and composition of other functions, presenting several examples.

Lecture 4. Families of functions were introduced, and examined by simultaneously graphing several curves of each family, using graphing calculators, and discussing properties of each family or functions within a family. Polynomials were presented first, and discussed with respect to evenness and oddness. Students were told to be familiar with graphs similar to x, x^2, x^3, x^4, and x^5, both individually and as a family and how they relate to one another, as that was the whole point to graphing them simultaneously. Trigonometric functions were presented next, and discussed with respect to periodicity. The vehicle for this discussion was a simultaneous graph of sin(x) and cos(x) in the viewing window [-6.4,6.4]x[-4,4] on the graphing calculator. The discussion of this pair of graphs began with the instructor's comment "The calculator should not know *anything* you don't know." He then proceeded to ask several questions. What is sin($n\pi$) for all n? What is cos($n\pi$) for all n? Where do they cross? At what height? What is the domain of each function? The range? What is the period of each function?

Lecture 5. Families of functions were touched upon again, this time focusing on exponential and logarithmic functions. The ideas of shifting and scaling were presented, with examples based on trigonometric functions. The instructor introduced the tangent and velocity problems, commenting, "Finding slopes of tangents and understanding what is meant by velocity are the two problems which motivate almost everything we'll do in calculus I. In this section, we'll see how both require us to study limits."

Discussion 2. Students were finishing the first homework assignment, due in lecture 6. That assignment contained 9 problems, primarily focusing on domain and range of functions. This assignment was due the Friday before most of the first interviews were conducted. It should be noted that among the functions in this assignment were:

$$g(x) = \frac{2}{3x-5} \text{ and } f(x) = \frac{x^2+5x+6}{x+2}$$

both of which were very similar to functions in the matching task in interview 1. In addition, the assignment included a piecewise defined function, an example of which was also included in the matching task in interview 1.

Lecture 6. This lecture began with an example. The problem of finding the slope of the tangent to x^3 at $(1,1)$ was approached using approximations with secant lines. First, the slopes of a few secant lines were computed by hand. Then, the secant slope function was drawn on the graphing calculator to, "see if we can see any limiting behavior", as the instructor put it. After drawing the secant slope function on the graphing calculator, the "zoom in" and "trace" features were used to gather what the instructor called "experimental" evidence, leading the class to conjecture that 3 was the slope of the tangent line. The instructor commented, "Not only does the experimental evidence indicate the slope is 3, but in fact there is *no doubt* that the slope is 3." To show that this was so, the instructor wrote the following passage on the board.

> Considerations such as these lead us to realize that the required slope is given by the limit

$$\lim_{x\to 1}\frac{x^3-1}{x-1}=\lim_{x\to 1}\frac{(x-1)(x^2+x+1)}{(x-1)}=\lim_{x\to 1}(x^2+x+1)=3$$

Next, came an example involving velocity. Given that the distance traveled after time t was $s=t^2$, the velocity at time $t=7$ was calculated directly using an algebraic technique similar to the previous example. These two examples, both handled with algebraic techniques, were then used to set up the next example, which could not be handled in this way. The instructor suggested that one way to handle a problem like

$$\lim_{x\to 0}(1+x)^{1/x}$$

was to use the calculator to generate a table of values or a graph, although this would give *experimental* evidence only. After finding function values for $x = .1, .01, .001$, and $.00000001$, it was pointed out that this limit was equal to the number e, something that numerical and graphical evidence might hint at, but could not reveal definitively.

Lecture 7. This lecture was primarily devoted to examples of limits for which the graphing calculator would fail in some way. The first example to be considered was

$$\lim_{x\to 0}\left(\sin(x+1)+\cos\frac{2\pi}{x}-1\right).$$

Several test values, such as $x = .000001, .0000001$, etc., were entered in the calculator, resulting in values very close to .84147, at which point the instructor asked, "Is there any reason why we should not believe that?" Upon graphing the function in the viewing window [-2,2]x[-2,2], and zooming in, it was revealed that the function appeared to be "bouncing around a lot", contradicting the numerical information. What was causing the trouble? The trouble stemmed from poor choices of x-values: substituting x-values of the form $\pm 10^{-n}$ into $\cos\frac{2\pi}{x}$ always results in the number 1, causing the calculator to produce values close to sin(1) for the function. It was suggested that less deceiving numbers like $x = .000001\pi$ would be more helpful. The example was summed up in the following way by the instructor: " Putting in values like x = .00001 seems to indicate the limit is about .84. Drawing the graph would make us a little suspicious. In fact, the limit does not exist. You should write yourself a few more notes to convince yourself of this. I looked at the graph to see if the numbers are telling the whole story. Every time I use the calculator, I'm suspicious, and I don't want just a little evidence, but a lot of evidence."

The next example considered was

$$\lim_{x \to 0} \frac{\sqrt{x^2+1}-1}{x^2}.$$

This limit was ultimately calculated algebraically, but first numerical evidence was generated. For x-values .01, .001, .0001 and .00001, the function values were very near .5, but for x-values 10^{-13} and 10^{-16}, the calculator returned 0. A graph was then drawn, and the trace feature suggested that .5 was the limit. In fact, students were told, the limit was equal to .5 and this was shown algebraically. What was the problem with the calculator? The instructor answered, "It can't deal with small numbers because it must round off. The calculator is almost doomed to failure because we are always interested in small values divided by small values. If you are relying on the calculator, you need *lots* of evidence. A graph is helpful but only if you believe what you see. You have seen the dangers of experimentation but sometimes this is all you can do."

Next, one-sided limits, including infinite limits, were explained by means of a graphical example. The instructor commented, "When you take a limit, you want to know what is the behavior of this function nearby.... One message you should be getting is that graphs are very valuable." Finally, the limit laws were introduced, with the comments that they had already been used in many examples, and that one had to ensure that all of the individual limits made sense.

Lecture 8. This lecture was part of the pre-interview 1 experience for 2 of the 10 students. Limit laws were continued and two theorems were presented. First, the theorem that a limit exists if and only if the left and right hand limits exist and are equal was presented along with the comments, "This shows that the value of a limit can be found by calculating the corresponding one-sided limits. It also gives a way to prove that certain limits do not exist, in particular, if the corresponding one-sided limits are not equal." Second, the Squeeze Theorem was presented as a means of calculating a limit for a "nasty" function g, if one could find simple functions f and h meeting the hypotheses of the theorem. Exercises from the textbook were presented as examples to illustrate the limit laws. Extensive time was devoted to understanding

$$\lim_{x\to 0} x\sin\left(\frac{1}{x}\right).$$

First, an ad hoc analysis of the function led to the conclusion that $\sin(1/x)$ was bouncing back and forth very rapidly, but x is trying to get to 0. Then the function was graphed on the graphing calculator in the window [-3,3]x[-2,2], and everyone zoomed in at the origin three times, upon which the professor suggested that perhaps the squeeze theorem would be of help. After applying the Squeeze Theorem to show that the limit was 0, the professor commented, "Keep this example in mind because it is kind of generic because sine or cosine of *anything* is between –1 and 1." Further analysis of this function involved finding out why it appeared to level out as x-values became larger, and finding x-values at which the function equaled x or $-x$.

Discussion 3. Students were finishing the second homework assignment due in lecture 9. This assignment contained 9 problems focusing on function transformations and predicting tangent line slopes and instantaneous velocities from numerical data generated about secant line slopes and average velocities.

Lecture 9. This lecture began with a recap of results obtained the previous lecture about the function $x\sin(1/x)$, including a reminder that the limit as x approached 0 had been determined to equal 0 by the Squeeze Theorem. Continuity was introduced and the Intermediate Value Theorem was presented followed by an application involving finding a root of $\cos x = x$, correct to one decimal place. It was pointed out to students that this process of finding a root was the basis of what their calculators were doing in solving an equation. The professor commented, "I want you to understand what your calculator is doing." (The Monday following lecture 9 was a holiday.)

Lecture 10. The ideas of tangent line slope, velocity, and instantaneous rate of change were reviewed, using several exercises from the textbook as examples. First, the slope of the tangent line to

$$y = 1 - 2x - 3x^2$$

at the point (-2, -7) was computed using limits. This led to an equation of the tangent line. Both the function and the tangent line were graphed on the graphing calculator in the window [-10,10]x[-10,10], and the zoom-in feature was used around the point (-2, -7) to see that the two functions are indistinguishable after zooming in close enough.

In the second example, the slope of the tangent line to

$$y = x^3 - 4x + 1$$

at $x = a$ was computed, leading to straightforward computations producing equations of two different tangent lines. After determining what the graph of the function ought to look like for large positive and large negative x-values, it was graphed on the graphing calculator along with the two tangent lines in the window [-3,3]x[-3,3]. This graph was used to illustrate that a tangent line can cross a curve at a point of tangency; that is, tangents have nothing to do with staying on one side or the other of the curve, so one must always use the ideas of limits.

Finally, a velocity problem based only on a graph of the position function was presented. This involved determining from the graph what the initial velocity was, whether the car was going faster at one point or another, whether it was slowing down or speeding up at certain points, and what was happening between certain points. Students were informed that there was a lot of information in this curve and that it would be a good exercise to plot the velocity curve based on the graph of the position curve.

Discussion 4. Students were finishing the third homework assignment, due in lecture 11. The assignment contained 16 problems, focusing on limits, continuity and the intermediate value theorem. Specifically, students computed limits (two-sided and one-sided) of one function given graphically (no formula), which contained a cusp, a jump, a vertical asymptote and a hole. This was very much like the function given in task 1 of interview 2, both in the graphical and numerical versions. A piecewise function, given as a multi-piece formula, required a graph be drawn and several limit situations be determined. A rational function was given, with instructions to find the vertical asymptotes and draw the graph. Most important, the problem on which task 2 of interview 2 was based was assigned. Here is the full statement of the problem.

(a) Evaluate $h(x) = \dfrac{(\tan x - x)}{x^3}$ for x = 1, 0.5, 0.1, 0.05, 0.01, 0.005.

(b) Guess the value of $\lim\limits_{x \to 0} \dfrac{(\tan x - x)}{x^3}$.

(c) Evaluate h(x) for successively smaller values of x until you finally reach 0 values for h(x). Are you still confident that your guess in part (b) is correct?

Next, students were asked to estimate the value of

$$\lim_{x \to 0} \frac{6^x - 2^x}{x}$$

by graphing the function $y = (6^x - 2^x)/x$, and state the answer to two decimal places. These problems were followed by five limit exercises utilizing the limit laws. In one case, the limit did not exist because of a vertical asymptote, and in another case, the limit did not exist because the left and right limits were not equal.

Next, two piecewise functions were given, requesting graphs and explanations for why they were discontinuous at a given point. One piecewise function was given with a constant c appearing in both pieces of the formula, and directions to find the constant c making the function continuous on the real line.

Finally, two applications of the intermediate value theorem were given, the second requiring students to use their graphing calculators to find an interval of length 0.01 containing a root of the function in question.

Lecture 11. Derivatives were introduced in terms of limits with the admonition, "You should never forget that you are computing a limit when you find a derivative." Examples of computing derivatives using limits were given, based on

$$f(x) = ax + b \text{ and } f(x) = \frac{2}{\sqrt{3-x}}.$$

The geometric interpretation of a derivative was presented by showing a graph of a function (no formula) and drawing a plausible graph of the derivative function. The theorem that differentiability implies continuity was proved, and a counterexample to the converse, namely the absolute value function, was presented.

Lectures 12. Derivative computations dominated this lecture. First, the derivative at $x = a$ was computed, using limits, for the function

$$f(x) = x - \frac{2}{x}.$$

A graph of $y = x - 2/x$ was drawn on the graphing calculator, and it was pointed out that when x is very big, the function looks like $y = x$. Then, the graph was analyzed to determine what the derivative *should* look like, and the derivative function was then graphed on the graphing calculator to confirm the predicted behavior. Differentiation formulas were introduced next, followed by several computational examples.

Lecture 13. This lecture was part of the pre-interview 2 experience for 2 of the 9 students who did this interview. The hour was strictly devoted to presenting examples of derivative computations using the derivative rules.

Discussion 5. Students were finishing the fourth homework assignment, due in lecture 14, and also preparing for the first exam. The assignment contained seven problems, focusing on computations of tangent line slopes, velocities and instantaneous rates of change, using limits.

Lecture 14. This hour was devoted to answering questions in preparation for the exam to be given the following Monday. Topics raised were domain and range, continuity, and the Intermediate Value Theorem.

Lecture 15. This was the lecture following the first hour exam, so a considerable amount of time was devoted to going over the exam problems. The exam included one problem requiring students to calculate the following four limits, with instructions to "use a calculator only if you know of no other technique and then give the answer to three decimal places".

$$\lim_{x\to 1}\frac{x^5-1}{x-1},\ \lim_{x\to 0}(1-x)^{1/x},\ \lim_{x\to 0}\frac{\sqrt{x^2+4}-2}{x^2}\text{ and }\lim_{x\to 0}x^3\cos\frac{1}{x}.$$

Calculator use was necessary for the second limit, and the fourth limit required the Squeeze Theorem. Notice that the function in the fourth limit is similar to the function used in the task for interview 4.

After going over the exam, derivatives of trigonometric functions were introduced by showing geometrically that

$$\cos\theta < \frac{\sin 6}{6} < 1,$$

and hinting that the Squeeze Theorem could be used to find the derivative of sin(x) at the origin.

Discussion 6. Students were finishing the fifth homework assignment, due in lecture 16. The assignment contained 17 problems, focusing on computing derivatives by the limit definition and by differentiation rules, recognizing or drawing graphs of derivative functions based on graphs of functions, and applications of derivatives.

Lectures 16, 17 and 18. The 16$^{\text{th}}$ lecture began by applying the Squeeze Theorem to the inequality established in the last lecture, to arrive at the conclusion

$$\lim_{\theta \to 0} \frac{\sin 6}{6} = 1.$$

This led to limit computations of the derivatives of sin(x) and cos(x), computations of derivatives of other trigonometric functions using these results and the differentiation rules, and computations of other limits dependent on the one above. Lecture 17 continued with computations of derivatives based on the rules for differentiating trigonometric functions. Lecture 18 introduced the chain rule and presented several computational examples.

Discussion 7. Students were finishing the sixth homework assignment, due in lecture 19, and also beginning to think about the seventh homework assignment due the following week in lecture 21 (a Wednesday). Assignment 6 contained 10 problems, focusing on limits involving trigonometric functions, and derivatives of functions containing trigonometric functions. Assignment 7 contained 8 problems, all involving computations of derivatives, using the chain rule.

Lectures 19, 20 and 21. Lecture 19 presented more chain rule derivative examples, and introduced implicit differentiation. Lecture 20 provided examples of implicit differentiation, including orthogonal trajectories. Lecture 21 was part of the pre-interview 4 experience for 2 of the 9 students who did that interview. That lecture presented higher order derivatives, including computations and graphical interpretations.

Appendix P

Graphing Calculator Worksheet

TI-85 Introduction

Primary Reading: TI-85 Guidebook chapters 1,4,12 and 14

Stewart: Review and Preview Chapter, section 3, pp. 26-31

Essential Terminology: viewing rectangle, xrange, yrange, xmin, xmax, xscale, ymin, ymax, yscale, zooming in, zooming out, aspect ratio.

1. Enter the expression $\sqrt{4}+3^3+2\sin\pi$ in the home screen. Press ENTER. Is the result what you expected? Explain in a sentence or two.

2. Use the LIST command to find the square roots of the numbers 1, 2, 4, 9, and 16 all at once. Is the result what you expected? Explain in a sentence or two.

3. Graph $h(x) = x$ with the default viewing rectangle of [-10, 10] x [-10, 10]. Now try the following viewing rectangles. Describe what you see in each case, and explain why the graph appears so differently in each viewing rectangle.

   a. [-10, 10] x [-1, 1]

   b. [-1, 1] x [-10, 10]

   c. [0, 1] x [-1, 0]

4. Graph $f(x) = \sin(x)$ with the viewing rectangle [-2π, 2π] x [-1, 1].

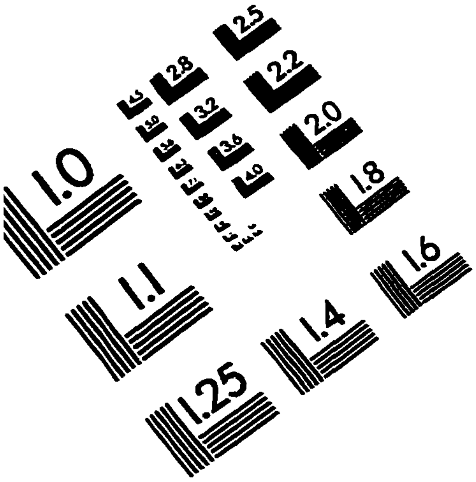
   a. Use TRACE to approximate a maximum *value* of *f* and a positive *root* of *f*. What are the *exact* answers?

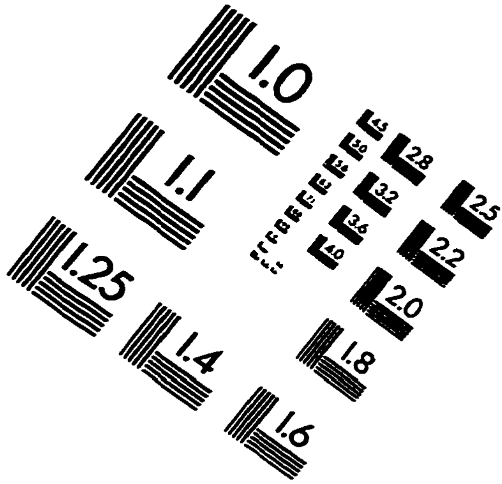
   b. Use ZOOM to zoom in on the graph near the point (π,0). What do you see? Is this what you expected to see?

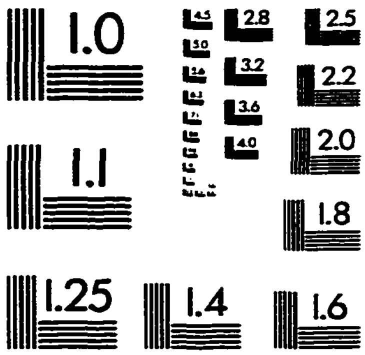
5. Graph $g(x) = x^4 - 18x^3 + 107x^2 - 210x$ in the viewing rectangle [-10, 10] x [-10, 10]. Note, $g(0) = 0$, but the graph doesn't show this. Now graph $g(x)$ in the viewing rectangle [-1, 8] x [-130, 10]. Is this a better viewing rectangle? Why or why not?

6. By graphing, estimate the domain and range of $f(x) = \sqrt{110 + 100x - 10x^2}$. Find the exact answer analytically.

IMAGE EVALUATION TEST TARGET (QA-3)







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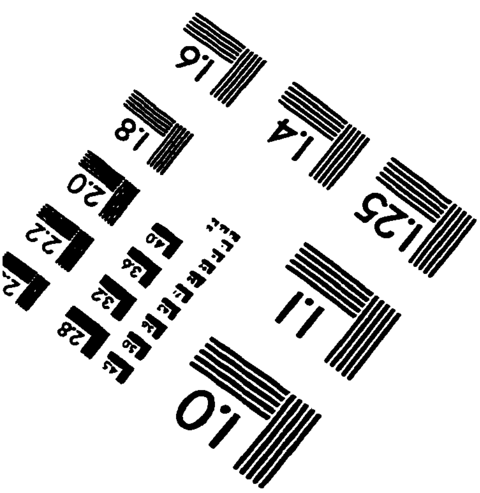

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