UNIVERSITY OF OKLAHOMA

GRADUATE COLLEGE

EXPLORING CHILDREN'S GEOMETRIC THINKING

A Dissertation

SUBMITTED TO THE GRADUATE FACULTY

in partial fulfillment of the requirements for the

degree of

Doctor of Philosophy

By

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Norman, Oklahoma
1998

UMI Number: 9828779

EXPLORING CHILDREN'S GEOMETRIC THINKING

A Dissertation
APPROVED FOR THE
DEPARTMENT OF INSTRUCTIONAL LEADERSHIP
AND ACADEMIC CURRICULUM

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ACKNOWLEDGMENTS OF GRATITUDE

I would like to sincerely express my appreciation to everyone who has contributed to the development and completion of this study and my degree. I gratefully acknowledge:

Dr. Anne Reynolds, my committee chair, for believing in me and giving me the freedom to explore my research, her invaluable effort, patience, and commitment to revising my writing, and her guidance, thoughtful suggestions, and expertise in this field.

Dr. Jayne Fleener, for her continual advice, support, and encouragement during these few years.

Dr. Frank McQuarrie, Dr. Jay Smith, and Dr. Pamela Fry, for their friendliness, understanding, and cooperation.

Department of Instructional Leadership and Academic Curriculum, for providing me years of assistantship and experiences in teaching and supervision.

Department of Mathematics, for the opportunity to teach mathematics content courses for prospective elementary teachers.

Students in my mathematics education, methods, and content classes, for their insights in learning mathematics and effort in understanding my English accent.

Yunik, Mogen, Jorn, Emi, Taile, Elisa, and Edem, for their time, effort, and willingness to participate, as well as their commitment to make this study possible.

My mom Lim Ah Moy, dad Ng Kah Seng, sister Ng Wei Lean, grandma Tan Teng Lee, and uncle Lim Kim Eng for their unconditional love always.

My brother Ng Chin Git, for constantly inspiring me and reminding me of my goal, and for sharing this challenging journey with me with humor and laughter.

TABLE OF CONTENTS

ABSTRACT

This study examined seven 4th and 5th graders' geometric thinking, specifically their understanding in the areas of area and volume. Base Ten Blocks, Tangrams, questions on 2-D and 3-D are three categories of geometric tasks given during exploration. Data were gathered over fifteen weeks through video-taped one-on-one interview sessions as well as frequent classroom observations. Findings indicate that these students have exhibited a wide range of maturity in understanding the concepts of area and volume. Students' conceptions of area range from believing that only congruent figures have the same space, that the space occupied by figures are the same regardless of their shapes or arrangements, to the idea that the space exists in its own right whether or not it is occupied. As for volume, some students view a 3-D block as a "box" consisting of six separate sides; some recognize the interior of a block but not its connecting or shared edges; some see the block in terms of layers; one has conceptualized a 3-D cube as one coherent, integrated, and coordinated block. Moreover, the data also supports that those who are not capable of thinking multiplicatively struggle in solving problems dealing with 3-D blocks. The educational implications of this study are briefly discussed.

CHAPTER I

BACKGROUND AND THEORETICAL PERSPECTIVE

INTRODUCTION

Since our surroundings are filled with objects of various shapes, sizes, and locations, geometry seems to be an excellent source to begin learning mathematics. Our natural curiosity about geometric ideas and enthusiasm to explore the world around us starts at an early age. Geometric forms can easily be found in designs, artifacts, as well as nature. Shaw and Blake (1998) state, "As children work with positions in space and three- and two-dimensional objects, they gain implicit knowledge of forms and relationships" (p. 210). Children can "see" that the cover of a book and a cereal box have similar appearance or shape. These early experiences with rectangular objects can assist them in understanding properties of a rectangle and other related geometric concepts later (Trafton & LeBlanc, 1973). Also, as stated by the National Council of Teachers of Mathematics (1989), "Children are naturally interested in geometry and find it intriguing and motivating; their spatial capacities often exceed their numeric skills, and tapping these strengths can foster an interest in mathematics and improve number understandings and skills" (p. 48).

Geometry is often viewed as the study of terms and their definitions. Since not much geometry is included in textbooks or appears on achievement tests, to many elementary teachers, geometry often becomes unimportant and irrelevant for elementary school (Dana, 1987). However, geometry has recently gained increased recognition in the elementary school curriculum. This is evidenced by the tremendous effort of the National Council of Teachers of Mathematics to include geometry in curriculum standards for lower grades.

Clements & Battista (1992) state that geometry is important because "it offers us a way to interpret and reflect on our physical environment [and] can serve as a tool for the study of other topics in mathematics, science, [and other related fields]" (p. 457). Although many agree with them, the debate on how children learn geometry or construct geometric ideas has been controversial with increased attention being brought to this matter.

Van Hieles (1957) and Piaget, Inhelder, and Szeminska (1960) are two of the few groups who did extensive research on how students make sense of geometric concepts. Dina van Hiele-Geldof and Pierre Marie van Hiele, two Dutch educators, believe that students progress through several levels of thought in geometry, agreeing with Piaget to a certain extent. They point out that students do not learn geometry by memorizing facts, names, or rules, but abstract mathematics "from their own systematic patterns of activities." Therefore, they emphasize the role of the student in actively constructing his or her own knowledge (Clements & Battista, 1992).

Van Hieles developed a model that identifies five levels of thinking in geometry. These geometric thinking levels and their characteristics are briefly summarized below :

Level 0 Visual: Judges shapes according to their appearance.

Level 1 Analysis: Sees figures in terms of their components and discovers properties of a class of shapes empirically.

Level 2 Informal deduction: Interrelates previously discovered properties.

Level 3 Deduction: Proves theorems deductively and establishes interrelationships among networks of theorems.

Level 4 Rigor: Establishes theorems in different postulational systems.

Van Hieles' research focused on the five levels mentioned above as well as the role of appropriate instruction in assisting learners to move from one level to the next (Fuys, Geddes, & Tischler, 1988; Geddes & Fortunato, 1993). On the other hand, Piaget et al (1960) were more concerned about children's readiness to learn. Piaget's theory of intellectual development led him to believe that a child's concept of geometry begins with topological types of relationships before it becomes projective or Euclidean in nature (Piaget, Inhelder, & Szeminska, 1960). This claim has been accepted by some and widely criticized by others. Hence, without refuting Piaget's theory of intellectual development, many indicate that more research is needed (Geddes & Fortunato, 1993).

PIAGET AND STAGES OF DEVELOPMENT

Piaget, a developmental psychologist, believed that there are developmental stages in the ability of children to think mathematically. These developmental stages also apply in the area of geometry. In general, the concept of Euclidean space-shapes develops at the late preoperational period (4-7 years old), but the concept of Euclidean space-coordination of horizontal and vertical begins at a later stage, the concrete operational period (7-11 years old). While the concept of measurement-area starts at the concrete operational stage, the concepts of measurement-volume and projective geometry develop at the level of formal operations which begins around age eleven (Copeland, 1984).

Piaget often uses the phrase "stages of development." In one context, it means "periods in the life of a child"; they are sensorimotor, preoperational, concrete operational, and formal operational. In the following sections, stages are numbered I, II, III, and IV, and refer to the levels of understanding of a particular concept, the other sense in which Piaget uses the phrase. Stage I means there is no understanding for the concept; stage II refers to partial understanding; stage III and IV represent full understanding and complete understanding respectively. However, the two sets of stages mentioned above may be related. Stage I is at the preoperational level; stage II is transitional between preoperational and concrete operational level; stage III is at the concrete operational; and stage 4 is at the abstract level or formal operational stage (Copeland, 1984).

GEOMETRY

Although there are many types of geometries, Copeland (1984) points out that topology, Euclidean geometry, projective geometry, and measurement or metric geometry are those most related to children's daily experiences.

In the mathematics of topology, shapes or figures can be stretched, compressed, squeezed, flexed and bent, but not torn or glued (Doyle, 1994). Therefore, topology is sometimes called "rubber geometry" (Copeland, 1984). For instance, a triangle is topologically equivalent to a circle, since it can be deformed into a circle, without tearing or gluing. Historically, Euclidean geometry has been considered to be the child's first experiences with geometric concepts. In contrast, Piaget proposed from his research that children's first geometric concepts are more topological in nature. Hence, geometric activities for children of ages four to six are recommended that emphasizes topological relations such as proximity, separation, order, and enclosure (Piaget & Inhelder 1956, Copeland 1984).

As for Euclidean geometry, Piaget and Inhelder (1956) explained that children at stage I, in general, can only recognize or draw closed shapes, rounded shapes and shapes based on simple topological relations. At stage II (approximately age 5 to age 7), they begin to recognize the basic properties or elements of Euclidean shapes, for instance, angles of different sizes, straight and curved lines, parallels, relations between equal or unequal sides of a figure. Not until stage III are children able to construct a mental image of a geometric figure. To do that, children need to achieve "reversibility" which means they must know how to "return to a fixed point of reference" so that movements or actions can be connected and coordinated into a whole. A point of reference on the figure helps the child to check the various relationships that exist in the figure (Copeland, 1984). Piaget's research further shows that children at each stage (stage I, II, and III) are able to recognize or re-present only those shapes which they can actually reconstruct through their own actions. Therefore, the "abstraction of [a geometric] shape is achieved on the basis of coordination of the child's actions, or at least not entirely, from the object direct" (p. 43).

Projective geometry is concerned with the appearance of an object or idea from different points of view (Copeland, 1984). From a psychological standpoint, the development of projective space or systems starts when an object is to be considered in relation to how it looks from a particular point of view, but no longer viewed in isolation (Piaget & Inhelder, 1956). Children's egocentric attitude (one point of view) is one factor in why perspective appears at a relatively late stage in the psychological development of children.

The National Council of Teachers of Mathematics (1989) has included "measurement" as one of the curriculum standards for grades K-4. Measurement is important to the curriculum since it enables students to see that mathematics is useful in everyday situations (NCTM, 1989). However, children do not develop the notion of measuring as early as is commonly thought (Piaget, Inhelder, and Szeminska ,1960). These researchers state:

> ...the study of how children come to measure is particularly interesting because the operations involved in measurement are so concrete that they have their roots in perceptual activity (visual estimates of size, etc.) and at the same time so complex that they are not fully elaborated until some time between the ages of 8 and 11. Questions of measurement are closely bound up with those of conservation .... (p. vii).

Further, Copeland (1984) and Piaget et al (1960) conclude that the concept of conservation of length (the length of a measuring instrument does not change as it is moved) and the concept of subdivision are prerequisite to understanding of geometrical measurement.

As for measurement in two dimensions (area), the research of Piaget et al indicates that children develop conservation of area approximately one year after conserving the concept of length (one-dimensional measurement). The formula for determining area: Area = length x width should not be expected to develop until stage IV (eleven to twelve years old), although children at age nine are generally ready for two-dimensional measurement using alternative methods such as the method of superposition of a unit square (Copeland, 1984). For instance, given unit squares (1 cm x 1 cm) and a vertical or horizontal rectangle (3 cm x 2 cm), children are able to find out the area of the rectangles by using unit squares.

Similarly, there are various stages of understanding measurement in three dimensions -- volume. Conservation of volume is not developed at stage I and II as children at these stages believe that volume varies as the shape changes. At stage III, which begins around the age of 6 or 7 and continues until 11 or 12, children can grasp the notion of interior volume (the amount of matter contained within an object). For instance, they understand that interior volume or "room space" contained in the blocks remains the same as the exterior dimensions (length, width, and height) changes. But the conservation of volume is more advanced or elaborated at stage IV, the level of formal operations. They now understand "volume" is not just the interior "contained" by a 3-dimensional object, but that space exists in its own right regardless of whether it is occupied (Piaget et al, 1960; Copeland, 1984). Piaget explained, "...they discover for the first time that it is not merely the interior 'contained' which is invariant but the space occupied in a wider context" (p. 385).

CRITICS ON PIAGET'S THEORY

Piaget et al's work does lead to some controversial criticism. Piaget and Inhelder believe that drawing shapes or geometric figures is not an act of perception. Children are able to re-present only those shapes which they "reconstruct through their own actions" (p. 43). Hence, inaccurate drawings of geometric shapes indicate that children do not have adequate mental tools for spatial representation at that stage. However, some argue that inaccuracies in drawing may be caused by children's motor difficulties (Clements & Battista, 1992).

Regarding the absence of Euclidean notions in early stages, Peel (1959), Rosser, Horan, Mattson, & Mazzeo (1984), and Rosser, Lane, & Mazzeo (1988) argue that children at an early age are able to work with certain Euclidean ideas. Further, Clements & Battista (1992) conclude that the topological primacy theory is neither supported nor disproved totally. Contrary to Piaget's work which illustrates that children construct topological first and then projective and Euclidean ideas later, Clements & Battista (1992) think that ideas of all types may be developed over time, "becoming increasingly integrated and synthesized" (p. 426). Hence, they suggest that more research is needed in this area, the development of geometric thinking.

RECENT STUDIES ON CHILDREN'S GEOMETRIC THINKING

The previous background, Piaget's idea of developmental stages in children's geometry learning, was a basis for my study design. When I began to analyze the data collected, I realized that more literature is needed to explain these students' geometric thinking. Therefore, I continued to look for research studies in this area. The following section describes recent literature on children's geometric thinking.

Reynolds and Wheatley (1996) believe that the construction of an interable unit and the coordination of units and subunits are key elements in students' construction of area. In their study, students were asked to figure out ways for comparing areas of 3-by-5 and 12-by-30 regions. Students' construction and coordination of units is evidenced in both geometric and numerical settings (Wheatley & Reynolds, 1996). They also identified different types of unit coordination developed by children as they tried to determine the number of one-inch cubes on the outer layer of a 6" x 6" x 6" cube, a three-dimensional setting (Wheatley & Reynolds, 1993).

In Battista and Clements' (1996) recent study on students' understanding of three-dimensional rectangular arrays of cubes, they argued that students' approaches in attempting the problems were influenced by how they "see" the cubes in the rectangular blocks. If they see the blocks as uncoordinated or unstructured sets of little cubes, they usually lose track of their one by one counting. Some "see" the blocks as six separate sides or surfaces without regard to the interior. Others count cubes that appear on the outside layer and then estimate cubes in the interior. The more sophisticated strategy is to see the blocks in terms of layers, called layering strategy. Their research was conducted on a group of above average third and fifth graders. Although 60% of the fifth graders used layering strategies, only half of them used it correctly (Battista & Clements, 1998). It is common for students who use unsophisticated strategies to count more than once as they attempted to find out the number of cubes in the blocks. Ben-Chaim, Lappan, and

Houang (1985) reported that approximately 39% of fifth to eighth grade students in their study counted more than once.

These studies on unit coordination and children's understanding of 3-D arrays of cubes will help in analyzing students' geometric thinking in Chapter III and IV.

CHAPTER II

THE STUDY

PURPOSE AND SIGNIFICANCE

The purpose of this research was to explore children's problem solving process while engaged in geometric tasks. The investigation was intended to give us insight into children's construction of mathematical concepts and problem-solving process in geometry, particularly area and volume. This study would contribute to the discussion regarding children's geometry learning.

As the role of teachers changes from delivering materials to providing opportunities for students to be actively involved in the process of knowledge construction, the findings from this study will offer teachers an opportunity to understand how students make sense of their mathematical activity, especially in geometry. Since most teachers have limited opportunities to explore children's understanding in their classrooms, they are often unaware of difficulties encountered by children (Labinowicz, 1987). Steffe and D'Ambrosio (1995) elaborate a type of teaching called "constructivist teaching." In their constructivist model of teaching, teachers formulate or build models of how they "make sense of [students'] mathematical knowledge, including its construction" (p. 146). Information gathered from interaction with children in this research could give teachers a sense of how children go about solving geometric problems and further help teachers to reflect on their own teaching. It will also aid in the design of activities that will encourage students' geometric construction.

METHODS

Clinical Interview method and the teaching experiment method are helpful in finding out learners' reflective thought in mathematics. The difference between these two methods is summarized by Glasersfeld (1987) as follows:

> The teaching experiment, as I suggested before, is, however, something more than a clinical interview. Whereas the interview aims at establishing "where the child is," the experiment aims at ways and means of "getting the child on" (p. 13).

The teaching experiment generally has characteristics such as the interaction between researcher or experimenter and student, the focus on "what students do" and especially on "how they do it," and the gathering of qualitative data from teaching episodes and clinical interviews. By extending Piaget's clinical interview method to include teaching episodes, it provides the researcher with an opportunity to examine how an individual student constructs mathematical meaning (Cobb & Steffe, 1983).

According to Menchinskaya (1969a), there are two kinds of teaching experiments, macroscheme, which is more curriculum oriented, and microscheme which is psychologically oriented. Since this study was interested in how children give meaning to geometric tasks, the teaching experiment approach used was microscheme which has an element of interacting with an individual student. It allowed me, as a researcher, to take part in students' sense making process while they engaged in geometry activities.

All interview sessions were video recorded. Following each interview, I recorded my thinking immediately. In addition, all recorded interview sessions were transcribed for analysis. Furthermore, fieldnotes from observations in the participants' classrooms served as a supplement to main data collection for this study.

PARTICIPANTS

The purpose of this study was to investigate fourth and fifth graders' problem solving process while engaged in geometric tasks. I recruited seven students from two schools. Selection of students was based on their willingness to contribute their time and effort to this study. This criteria was necessary since the research involved in-depth interviews which required intensive participation and interaction. In addition, I often visited the participants in their classrooms to have a general idea of their classroom mathematical experiences. I explained my interest in learning about how children make sense of geometric problems to participants and their parents. Since this research involved minors, I involved only those students who had signed parental and participant permission slips.

SETTING

Interviews or meetings took place during or after school at the students' convenience. During each meeting, I presented each participant with one or two problems involving geometry. Participants were encouraged to make conjectures and to use whatever means they feel convenient or comfortable with to explain their thinking. That includes drawings, manipulatives, calculators, and others. My purpose in this research was to examine children's thinking in geometric tasks. Therefore, as a researcher in this study, I refrained from intentionally asking questions that would lead to right answers. However, I used questioning to challenge their ideas or assist them to probe deeper into their thinking. As suggested by Cobb and Steffe (1983), teaching is necessary in the activity of exploring how children make meaning of their mathematical experiences. They state that children's interactions with adults have an effect on their construction of mathematical concepts. Hence, they stress the importance of researchers acting as teachers. Further, I tried my best to create an atmosphere or setting that helped participants to express their "wonderful" ideas. The having of wonderful ideas, according to Duckworth (1987), is the essence of intellectual development.

GEOMETRY TASKS

According to Copeland (1984), topology, Euclidean geometry, projective geometry, and measurement are four kinds of geometries that are most related to children's daily experience. Therefore, this research selected mathematical activities or problems from the types of geometries mentioned above. Many would treat these geometries stated above as different types of geometries. But children may look at these different geometric tasks or activities as just geometry or mathematics problems. Geometric tasks given to each participant varied due to different levels or stages of understanding of the participant of geometric concepts. I looked specifically at both two- and three-dimensional geometric tasks. Furthermore, I asked questions and phrased words in various ways in order to make contact with their thinking. Piaget maintained that standardized format is not necessary since the same words may not mean the same thing for every child. Reaching the children or making contact with children's thinking is what matters (Duckworth, 1987). A list of the focus questions used with participants is given in Appendix A. Appendix B and C contain samples of material used with students. Figures of the students' work and the illustrations of their exploration are provided in Appendix D.

GUIDING QUESTIONS

When children were given geometric tasks, I was interested in learning how they responded to the problems or what their natural tendencies were (Labinowicz, 1985). Specifically, I was interested in finding out:

(A) What were children's strategies in approaching geometric tasks?

(B) To what extent did they apply their knowledge to new tasks or situations?

CHAPTER III

DESCRIPTION, INTERPRETATION, AND ANALYSIS

This chapter includes the background of each student, description and interpretation of each task, and analysis of students' geometric thinking. Geometric tasks given to participants in the following sections are grouped into three categories; Category A is for tasks involving base ten blocks, Category B is for tangrams activities, and Category C is for questions regarding the meaning of 2-D and 3-D. Figures of the students' work and the illustrations of their exploration are provided in Appendix D.

YUNIK

Yunik was a fourth grader who believed mathematics is "...numbers, you have to think about it." She told me she liked mathematics and was doing "okay" in mathematics. Addition and multiplication are the types of mathematics she liked. She could read only at second grade level, but she thought English was still easier than mathematics. The part of mathematics she thought was hard was "like 100 times something." She heard about geometry in 4th grade but did not remember what it was. Her mom often helped her with mathematics homework and was teaching her how to do multiplication and division problems during the semester I interviewed her. Yunik described her mom as, "She is good mostly in everything...." According to her, the way her mom taught her (using real life examples) helped her to "learn how to do mathematics" and like mathematics. Here is an example she gave, "...like 2 + 2, she (her mom) showed me 2, like 2 apples or 2 people, and then they met 2 other people, and then they all go to mall. How many were there first, and then how many were there now. 2 + 2 is 4...."

During the first meeting, I showed her manipulatives such as tangrams and base ten blocks and briefly explained to her what we were going to do for the rest of the semester. She showed great interest in manipulatives and kept playing with them.

The geometric tasks she explored and engaged in during interview sessions are presented below.

(A) **Base Ten Blocks**

Task (A-1): How many small cubes are there in the big cube?

I let Yunik play with cubes for a while and explained to her that the big cube was solid and made up of small cubes.

R: How many small cubes do you think are there in this big cube?

Y: (Counted small cubes on top surface) 10 hundred 80.

R: How did you get 10 hundred 80?

She pointed to the 1^{st} row as 10, 2^{nd} as 20, ..., and 10^{th} as 100 accordingly (see Figure 1). Then, turning to another surface and counting 110, 120, ..., 190, she thought for some time before she said, "200." She went on with another surface and started counting 210, 220, ..., but couldn't explain how she got '10 hundred 80' previously. She did know a flat is a 100 small cubes and made up of 10 longs. So, I placed a flat close to the big cube to see what she could figure out. She pointed to each face of the big cube and said, "100, 200, ..., 600."

R: How many of this flat will go into this big cube? You want to try?

I handed her all ten flats. She used 6 flats to build a frame of the big cube.

R: When you build this, does it mean that inside is hollow?

Y: Oh, I know that.

She seemed to realize something and piled those flats together, but then said, "I don't know." I showed her a hollow (blue) cube and a solid cube (big cube) and explained that the blue one was hollow inside. She tried to build a cube that looked like the solid cube by forming a frame before placing the rest of the flats inside the frame. Then, realizing that there were not enough flats, she said, "I can't." I piled up all 10 flats and placed them close to the big cube and asked her, "Do you think this is OK?"

Y: Yeah, it's the same. I was thinking of the top layer, bottom layer, and the sides.

I then placed the 10 flats piled up vertically and placed them side by side with the solid cube.

R: Tell me how many of these small cubes in the big cube?

Y: 600 (without hesitation).

R: How did you get 600?

She started counting each row of the top surface of the big cube, "10, 20, ..., 90" then pointed to a flat, "There is one hundred here." Looking at the big cube again, she touched all sides of the cube, "600." Then, she turned to 10 flats and counted them, "100, 200, ..., 10 hundred." She was surprised that it was over 600.

R: So, do you believe there is a thousand of these cubes there?

Y: Yeah.

R: Or 600?

She shook her head, meaning 1000 was her answer.

Interpretation: Yunik's idea of a "cube" is a box made up of six sides. This is obvious by her effort of trying to build a frame of the big cube using 6 flats at two instances, that is, preceding and following the question of whether the cube is hollow inside. She even explained that she was only thinking of the top layer, bottom layer, and the sides. Piling up 10 flats and placing them close to the big cube challenged her thinking of the cube. Although the piling up of 10 flats as well as her previous knowledge of a flat made of 100 small cubes or 10 longs helped her to see that the big cube consisted of 1000 small cubes rather than the 600 she thought earlier, it does not necessarily indicate that she has constructed the cube as a coordinated set of faces which allows for "overlapping."

Task (A-2): How many small cubes will be left if one layer of small cubes is removed from all sides of the big cube?

Yunik's initial attempt was to place a long on each side of the big cube to help keep track of how many sides she had removed. Her strategy did not seem to go anywhere, so she became frustrated. When I explained the question again, she took one flat away from a pile of 10 flats each time I mentioned one side of the big cube. With 6 flats removed, she had 4 flats left and so she gave 400 small cubes as the answer.

My explanation of the question may have misguided her, so I decided to demonstrate to her that the top and side surfaces shared one long as shown in the shaded part of Figure 2.

R: So, if I remove these top and left sides, will I be removing 200 small cubes, more than 200, or less than 200?

Without hesitation, she said, "More than 200." At this point, I knew she couldn't "see" that each edge is actually shared by two sides. I demonstrated to her again how each side could be removed and every two sides shared a long. She still believed that 600 small cubes were removed and 400 small cubes were left.

Interpretation: Yunik knew from previous task that the big cube consisted of 10 flats or 1000 small cubes. However, she believed that outer layers of the cube was made up of 6 flats and the remaining 4 flats or 400 small cubes were left inside the big cube. My attempts at showing her that each edge of the big cube is a long shared by two sides did not make any difference in her thinking. This strongly indicates her construction of the 10 x 10 x 10 cube as an uncoordinated set of faces with no "connecting" or "overlapping."

Task (A-3): How many small cubes are there in the 5 x 5 x 5 cube?

Yunik turned to each side of the cube and started counting by 5 for each column "5, 10,15, ..., 100," but did not keep track of what she was counting.

R: How many of them on the top layer?

Y: 30.

I helped her to keep track of what she was counting.

Y: 25.

R: (Holding a 5 x 5 x 1 flat) So, this is 25. Now, how many 25's are there in this cube?

She counted surfaces of the 5 x 5 x 5 cube without keeping track. Then, she placed her thumb and index finger on top and bottom, then to front and back, and then right and left of the cube. She came up with 6 sides. Using calculator, she added 25 six times to get 150.

Interpretation: Yunik had 150 small cubes for the amount of small cubes for 5 x 5 x 5 cube, but she obtained that by counting small squares that appear on six surfaces of the 5 x 5 x 5 cube. She did not think of small cubes inside the cube because she could not "see" them. Her thinking of a 5 by 5 by 5 cube was just the outer surface of the cube covered with 25 small unit cubes on each side. If I were to place five 5 x 5 x 1 flats side by side with the 5 x 5 x 5 cube, she probably would have counted it layer by layer, like she did in Task (A-1). The experience she had from Task (A-1) did not help her in this task although the two tasks had the same questions, only with different sized cubes. As with the 10 x 10 x 10 cube, she had constructed only an uncoordinated set of faces with no overlapping.

Task (A-4): A long is made up of 10 small cubes. How many longs do you need to make the big cube (10 x 10 x 10)?

I had 10 flats piled up and placed them side by side with the cube.

R: Earlier you told me that there are 10 longs in a flat, and these 10-flats cube and solid cube are the same. So, how many longs are there in this solid cube?

Yunik took the solid cube and shook it, and said, "I don't know." A second later, she said, "Wait," realizing that she could use the 10-flats cube to solve the problem and began counting the flats, "10, 20, ..., 100." To ascertain she understood the task, I had her explain the question.

Y: How many of these (longs) are in here (solid cube)?

R: Why did you count these flats when I asked how many longs there are in the solid cube?

Y: Because they are the same.

R: Are you sure of the answer?

Y: Kind of.

R: Why?

Y: Wait, I can do something. (She used the calculator to add 10 nine times)

I was curious why she added 10 nine times rather than ten times.

R: How do you know how many times you need to add?

Y: 10 times.

She had a hard time keeping track of how many times she needed to add. I assisted her and she came up with 100.

Interpretation: Initially, Yunik did not know how to find the number of longs in the solid big cube. Placing 10 flats side by side with the big cube helped her to see that both are equal in size. Although she could not break down the big cube into longs physically, she used the pile of 10 flats to solve the problem. From her prior experience, she knew that there are 10 longs in a flat, therefore, she counted each flat by 10, 20, ..., 100, meaning there are 100 longs total in 10 flats.

Task (A-5): Imagine the big cube is a hotel (Hotel A), with each small cube being a room, how many rooms are there? Imagine 2 stacks of 5 flats attached side by side is also a hotel (Hotel B), which hotel has more space (Figure 3)?

Yunik had a hard time figuring out the number of rooms in the hotel (big cube). I handed her a flat and asked her questions such as, "How many longs in a flat? ... How many flats in the cube?" I believed she recalled the task earlier and used the flat to move along the big cube and said, "10 hundred, ugh, 100 ten." No matter how I arranged the Hotel B, which was made up of 10 flats in 2 stacks, vertically or horizontally, she still believed that both hotels had the same number of rooms, 1000.

Interpretation: From her previous tasks, Yunik saw the big cube as a box made up of 6 sides with or without anything inside. That perception made it difficult for her to figure out how many unit cubes or rooms were in the hotel (big cube). Another explanation could be her ability to relate unit cubes and the big cube to rooms and hotel respectively. However, the questions "How many longs in a flat" and "How many flats in the big cube" reminded her of the prior experience. That experience helped her to use a flat as a tool to figure out the number of units. Also, she had seen 10 flats placed side by side with the big cube a few times in earlier tasks. That experience led her to believe that both have the same size. Therefore, regardless of how the 10-flats were arranged, she concluded later that both hotels had the same number of rooms.

Task (A-6): How many layers do you need to remove so that the big cube will become 4 x 4 x 4 cube?

A 10 x 10 x 10 cube and a 4 x 4 x 4 cube were provided to her. Within a few seconds, Yunik placed the 4.x 4 x 4 cube at the center of the top surface of 10 x 10 x 10 cube (see Figure 4). At first she thought there would be 300 cubes (3 flats) all around 4 sides of the 4 x 4 x 4 cube. Then, she looked at me with surprise, "Wait, if you take off 3 here and 3 here, these 2 will still be here (A and B of Figure 5)." She was confused when she realized she could not take off 3 flats from each side evenly.

R: If you take off 300 here (right) and there (left), do you still have 300 here (A) and here(B)?

Y: I don't know.

R: Remember my question?

Y: How can that big one (10 x 10 x 10) turn into small one (4 x 4 x 4)? It's hard. You can bend it.

She then took longs to make a 4 x 4 x 10 rectangular shape and drew a line all around the middle of the shape. She explained to me that the 4 x 4 x 10 bent at the pencil mark would look like 4 x 4 x 4 cube provided.

Interpretation: Yunik thought of removing layers as removing "complete" flats all around. This thinking confused her since she could only see herself removing 3 vertical flats on right and left sides. She did not consider region A and B shown in Figure 5 as part of "complete" horizontal flats but just the "leftover." However, she might not have comprehended the question given. This is indicated by the way she rephrased the question as "How can that big one turn into small one (4 x 4 x 4)?" Her interpretation of the question was further indicated by her demonstration of 4 x 4 x 4 cube with a rectangular shape of 4 x 4 x 10. As with the earlier questions, she failed to coordinate overlapping cubes.

(B) **Tangrams**

Task (B-1): Figures 6, 7, 8, 9, and 10 are square, rectangle, triangle, parallelogram, and trapezoid respectively. Do all these shapes have the same space?

Yunik did not seem to respond to this problem, so I rephrased the question.

R: If I were to use these tangrams - like triangles, to fill up all these shapes. Do you believe that same number of triangles will fill up each of these 5 shapes (see Figure 6-10)?

Y: Probably.

At this point, I decided to modify the task and have her explore with smaller pieces. I gave her three tangram pieces - middle triangle, parallelogram, and square. I asked her, "Do you think these three have the same space?"

Y: I don't get what you mean.

I tried to use terms or words that I thought she could understand and find ways to reach her thinking. Later, I realized that she probably did not have experience in associating space with shapes. I gave her two small triangles to explore. She made a square. I had her place the two triangles on top of the blue square.

R: So, do these 2 triangles have the same space as this blue square?

Y: No, but they can fit in there.

I realized she did not comprehend the word 'space,' but she did use the term 'fit in.' Therefore, I rephrased my question.

R: So these 2 small triangles fit in this blue square. Now, do you think they can fit in this red triangle (middle triangle)?

She took quite awhile to make a mid-sized triangle with 2 small triangles (see Figure 11). She did not flip over the 2-pieced triangle to fit in the mid-sized triangle, instead she separated the small triangles and attached them again with its base aligned with the edge of the table (Figure 12). She used that table edge as a reference point. Then, she demonstrated how triangles fit in by placing one on top of the other.

R: It can fit in these two (square, triangle), so, do you think they have the same space?

She agreed and without hesitation, she also showed me how two small triangles could fit in the parallelogram. The previous two experiences might have helped her to believe that "it is possible to fit those in." Therefore she tried without hesitancy. She later believed that all three have the same space since they can all be fitted in with the same two pieces.

To see whether she could apply what she had learned from previous experience to a new task, I gave her a set of tangrams and 5 shapes as shown in Figures 6-10. Her task was to determine whether these 5 shapes had the same space. First, I had her make a 5-pieced triangle (see Figure 13). By adding 2 big triangles, she was able to make a square (Figure 6) and a rectangle (Figure 7) with little assistance. I then asked her, "Do these shapes (Figure 6 and 7) have the same space?"

Y: Yes.

R: Why do you think so?

Y: This is a square (Figure 6) right? And this is a rectangle (Figure 7). It's just like a square being pulled out.

She used her two hands to demonstrate to me how a square can be 'pulled out' to make a rectangle. Using the same seven tangram pieces was not an indication to her that both figures had the same space. Later, I had her try to fit in the same seven pieces to a right-angled triangle (Figure 8). Instead of asking, "Do these shapes have the same space?" like before, I rephrased it to, "These seven pieces can fit in this square, rectangle, and triangle (Figures 6, 7, and 8). Now, what can you tell me from that?"

Y: They have the same space.

Interpretation: Using the same seven tangram pieces to make a square (Figure 6) and a rectangle (Figure 7) was not an indication that both figures have the same space. Instead, Yunik believed they have the same space because square can be "pulled out" to form the rectangle. Also, the term "fit in" rather than "same space" makes more sense to her. However, after exploring different tangram shapes and moving around pieces to fit in one another, she seemed to equate "fit in" with "same space." This was clearly shown by her believing that Figures 6, 7, and 8 (square, rectangle, and triangle) have the same space because they can be fitted in by all 7-pieced tangrams set.

Analysis

Battista and Clements (1996) suggest from their findings that students' initial conception of a 3-D cube arrays is an uncoordinated set of views. It is appropriate to categorize Yunik's construction at this stage. Her idea of a "cube" as a "box" made up of six sides is consistently demonstrated in all tasks of category A. This is evidenced by her using 6 flats to build a frame of the 10 x 10 x 10 cube. Her counting of "squares" that appear on 6 surfaces of the 5 x 5 x 5 cube to give 150 unit cubes in Task (A-3) is another strong indication. In Task (A-6), she struggled to find out why she could not take off 3 full flats from each side. This suggests the absence of coordination of "overlapping" or "sharing" of units in her construction of the 3-D cubes.

During the course of investigating tasks in category A, she had demonstrated her additive thinking rather than multiplicative thinking. One good example is she used the calculator to add 10 ten times to get 100 in Task (A-4). The absence of multiplicative thinking could be explained by her inability to construct composite units and her lack of coordination.

Task (B-1) is designed to determine one's notion of area. In this activity, Yunik did not think that Figure 6 (a square) and Figure 7 (a rectangle) had the same space although both can be made with the same 7 tangrams pieces. This implies that she does not conserve area at this point. Area is not just the interior occupied by shapes but the space exists in its own right whether or not it is occupied by pieces of shapes. The term "fit it" made more sense to her at this stage. Her idea of area is pieces of shapes fitting in rather than the space itself.

MOGEN

Mogen was a fourth grader who liked to read mystery and funny books. For her, mathematics is "...you add numbers, word problems...just numbers." She said she was "pretty good" in mathematics and liked it because she liked to work with numbers. Her mother encouraged her to bring home mathematics homework so that she could help her. At home, she sometimes played games on multiplication and division on the computer. Although she believed that mathematics could be found in a lot of places like grocery stores, books, schools, she did not think that there was any mathematics in TV programs. Geometry was a term she heard from her older brother but did not have any idea what it was.

During interview sessions, she had the opportunity to explore and engage in a variety of geometric tasks. Some of those are presented below.

(A) Base Ten Blocks

Task (A-1): How many small cubes are there in the big cube?

M: Probably 1000, because if there is 10 of this (long) in here (all 6 surfaces of the big cube), then probably a whole lot.

R: How did you get 1000?

M: Just a guess.

Mogen later went on to count 6 surfaces of the big cube, "100, 200, ..., 600."

R: When you say 600, does it mean inside or outside?

M: Outside.

R: This is a solid cube, so there are a lot of cubes inside too. If you also want to include inside, altogether, how many will it be?

M: If it got something inside, may be, it will be 1000.

I then encouraged her to try other strategies like using flats to build the big cube. She played with flats and later had 10 flats placed side by side with the big cube. She then counted those 10 layers of flats, "100, 200, ..., 1000."

Interpretation: From her prior experience in third grade, Mogen came to know that a long has 10 unit cubes and a flat has 100 unit cubes. Her counting of 6 surfaces of the big cube gave her 600 unit cubes, the number of unit cubes she could see covering the cube. After explaining to her that there were also small cubes inside the big cube, she estimated that there would be 1000, a number she associated with a large quantity. By exploring with flats, she was able to figure out the number of unit cubes by counting layers of flats. Placing 10 flats side by side with the big cube may have challenged her initial picture of a cube, but, it is possible that it has not challenged her thinking of 600 cubes on the surface. Her solution of 1000 is more "procedural" and less based in understanding.

Task (A-2): How many small cubes will be left if you remove one layer of small cubes from all surfaces?

M: 1000 takes away 100 is 900.

R: I meant remove 1 layer of every side.

M: Take away 100 from every side, 1000 - 600 = 400.

I explained to her that 2 sides, for instance top and side layers, shared or joined by a long (Figure 2). Mogen counted 100 for top surface, 90 for front side, and kept reducing 10 for each side. She reduced 10 each time because 1 long has 10 small cubes.

Interpretation: Mogen interpreted the meaning of removing one layer of small cubes from all surfaces as reducing a flat or 100 unit cubes from all six sides of the cube. Therefore, she had 400 small cubes as the answer for the first attempt. This strategy may have been influenced by her thinking of the big cube as 6 flats covering all outer layers of the cube. After demonstrating to her that each edge of the cube was a long shared by two sides, she kept her previous strategy but reduced 10 each time in moving from one side to the other. This task may not make sense to her since she had difficulty seeing all outer layers connecting to one another vertically or horizontally. My explanation has no meaning for her because she has not constructed any idea of "overlap." At this stage, she still thinks that the frame or outer layers of the big cube is formed by six separate flats.

Task (A-3): How many equal number of layers from each side should you remove in order to have a cube of size 4 x 4 x 4?

Mogen placed the 4 x 4 x 4 cube at the center of the top surface of the big cube (see Figure 4).

M: Need to take off 6 layers, 3 from top, 3 from bottom. But 3 times around it.

Interpretation: Placing the 4 x 4 x 4 cube at the center of the top surface enabled her to see there were three "rounds" of unit cubes before reaching the four edges of the top surface of the big cube (top view of Figure 4). Her saying of "... three times around it" may possibly be interpreted as removing all longs (in vertical fashion) around the 4 x 4 x 4 cube. That removal will leave a rectangular prism of size of 10 x 4 x 4 rather than of a 4 x 4 x 4 cube. Although, further probing questions such as "what do you mean by three times around it?" or "In what directions will the big cube shrink to a smaller size?" may push her to think further, it is apparent from her previous responses that coordination of the overlapping units is not part of her scheme.

Task (A-4): How many small cubes are there in the 4 x 4 x 4 cube?

M: About 50.

Mogen looked at one surface of the cube and counted column by column, "4, 8, 12, 16." (see Figure 14) Since there were 6 sides, she added 16 six times and came up with 87. I gave her a flat of 4 x 4 x 1 (see Appendix B) and asked her how many of those were in the 4 x 4 x 4 cube.

M: Four.

R: How many small cubes are there in this flat?

She added 16 four times with calculator and came up with 64.

Interpretation: Counting the number of unit cubes by columns on one surface and adding 6 times for six surfaces gave her a total number of unit cubes in the 4 x 4 x 4 cube. Employing this strategy shows that her idea of a cube is a "frame" with six surfaces covered with unit cubes. Providing her a flat of (4 x 4 x 1) gave her an opportunity to see the 4 x 4 x 4 cube from a different perspective. She was able to see that four 4 x 4 x 1 could in fact make a 4 x 4 x 4 cube. Therefore, she added 16 unit cubes 4 times to come up with 64, the amount of unit cubes in the 4 x 4 x 4 cube. This task is similar to Task (A-1) where she was asked to find the number of unit cubes in the big cube (10 x 10 x 10). She employed exactly the same approach for these two tasks. In both instances, she counted the number of unit cubes that appear on the outer layer. But after providing her with flats, she changed her strategy for solving the problem.

Task (A-5): How many 4 x 4 x 4 cubes are there in 10 x 10 x 10 cube?

M: 24. ... 4 on each side, and there are 6 sides, 24.

R: If you put it like this, does it share right here and here?

Mogen seemed to realize my point. She was thinking hard, turning the big cube and counting the small cubes in the big cube, but did not know how to go on. At that point, I guided her by putting a 4 x 4 x 4 cube at an upper corner of the big cube. She took quite awhile to figure out there were 4 on top and 4 at the bottom.

Interpretation: It was not surprising that she multiplied 4 by 6 sides to get 24, since her construction of a cube is a "frame" of six uncoordinated surfaces.

Task (A-6): How many longs (10 x 1 x 1) are there in the big cube?

M: 160... each side has 10, 6 sides, so 60, and inside you have even more.

R: Outside you said it has 60, how about inside?

M: 100 inside.

To engage Mogen in further thinking, I asked her how many longs in a flat.

M: 100...you have 10 in this (flat), ...like if you count this (a stack of 10 flats placed side by side with a big cube) 10, 20, ..., 100. So, there is 100 of this (long). ... I just count by 10's.

R: So, is it 160 or 100?

M: 100, because there are 10 of these (flats) in here (big cube), and there is 10 of these (longs) in this (flat). So, that has to be 100.

R: Why 160 before?

M: I counted it wrong. I counted outside 60, and inside 100 (which she estimated). I should have counted inside and outside together...because inside it's just with it.

Interpretation: Mogen knew from previous experience in third grade that there were 10 longs in a flat and 10 unit cubes in a long. With a stack of 10 flats placed side by side with a big cube, she believed there were 100 longs altogether. She obtained 100 first by addition, that is, counting by tens, then by multiplication. At her initial attempt, she considered a cube as a combination of two disconnected entities. This is obvious by her estimation of 60 longs covering outside of the cube and 100 longs making up the inside of the cube. The question "How many longs in a flat?" did make her think otherwise, as she said, " I should have counted inside and outside together, because inside it's just with it."

Task (A-7): If part B is removed, how many small cubes will be there (see Figure 15)?

R: Do you know how many small cubes in this big cube?

M: 1000.

R: If I were to cut part B away, how many small cubes will be left?

Mogen didn't know what to respond. I encouraged her to estimate and she came up with 500 unit cubes left.

R: Do you think it's more than half, less than half, or exactly 500, if I were to cut everything across.

M: Exactly… because there is same amount of cubes over here (part A) and here (part B). So, if you cut 500 away, then there is still 500 left.

R: You see, I don't cut it straight, I cut it in a crooked way, like a staircase. Do you think this is still half?

M: There will still be half.

Interpretation: Mogen looked at part A and part B as two equal parts. Therefore, she confidently explained that if 500 small cubes on part B were cut away, there would be 500 unit cubes left for part A. Although I have at one point tried to encourage her to think further and implied that there might be more or less than 500 by using words such as "I don't cut it straight, I cut it in a crooked way, like a staircase" she still did not change her mind. Having her count the unit cubes on the surface layer of part A and B could possibly help her to think otherwise.

(B) Tangrams

Task (B-1): Do Figures 6-10 have the same area?

R: What is area?

M: … like if you have … (she drew a quadrilateral and shaded inside) it will be like the area inside the shape.

R: Can I find area in this room?

M: …like all around the room, the floor, the wall, area inside the board (hanging on wall).

R: Any other area examples in real life?

M: …like you're building a house or something, you will need a ground to build on…like dad is park ranger, like at the park, that kind of area there, when mom is writing on paper.

R: Any other word to substitute for area? Like grocery store, I could say food store. How would you explain area to younger kids?

M: Probably draw a shape, and show them the inside of the shape.

R: Do you think all these (see Figures 6-10) have the same area?

M: Not the same.

R: Is there any way you can measure and find out whether they have the same area?

Mogen then used a ruler to measure the perimeters of Figure 6 (square) and Figure 7 (rectangle). That did not work for her. So, I suggested whether she could use a small triangle to measure Figure 6-10.

M: May be you can count how many you can fit in.

R: Is it the same like measure the surrounding?

M: Not the same, those are just like the perimeter.

I then challenged her to add two large triangles to Figure 13 (5-pieced triangle) to form Figures 6-10. Although she had some difficulty for Figure 8, she was able to accomplish the task.

R: Do they (Figure 6-10) have the same area?

M: (nodding) ... because the same amount and the shapes fits each of them.

Interpretation: Before I proceeded with the task, I tried to find out how much Mogen knew about area. She defined area as "inside of the shape." Examples she gave for area from the dialogue seemed to demonstrate that she had a good grasp of the notion of area. However, determining the area of the shapes shown in Figures 6-10 was a challenge to her. Although she believed that counting the number of small triangles to fit in Figure 6-10 rather than measuring their perimeters would help her to determine whether those shapes have the same space, she did use the ruler to measure the perimeters of Figure 6 and 7 during her first attempt. I believe she did so because the ruler was available to her at that moment and she was experimenting with the tool provided to solve a problem she had not seen before. After exploring the 7-pieced tangrams set to fit in Figures 6, 7, 8, 9, and 10, she concluded that they have the same space. The phrase "... because the same amount and the shapes fits each of them" means the tangrams set still has the same seven shapes and still contains the same amount of space regardless how they were arranged to fit in those figures. If the same set could fit in all Figures 6-10, then those figures also have the same space. Her reasoning indicates that she conserved space in terms of area.

(C) 3-Dimensional

Task (C-1): What is 3-D?

M: 3-dimensional. It's not flat, kind of like cube...

R: What are examples of 3-D?

Mogen showed me the big cube we used often during interview sessions, unit cubes, and a cup with a lid. The cube she drew was shown in Figure 16.

R: How about pencil?

M: ...(thinking) ... yeah, I guess it could be. It's kind of like cube, ... round.

According to her, the cross that she wore, shirt, paper, and books were not 3-D because they were "flat".

R: Anything that is 3-D here?

M: The cabinet.

R: But it is flat in front.

M: If you look at it, there is a side on it.

Interpretation: Mogen understood "3-D" as something that looks like a cube. Therefore, the big cube and the unit cube are in the form of a cube. Her 3-D category also includes objects not in a "cube" shape but having "sides" added to a flat front surface. This is clearly indicated by how she described the cabinet, "... if you look at it, there is a side on it." Her drawing of a cube also indicated at this point. She first drew the front, a shape of a "square", then she extended her so called sides by drawing lines attached to the front "square" as shown in Figure 16. However, she had doubts about a pencil which looks more like "round" than "cube" to her. Although books have "sides," she still categorized them as "flat" like she did for the cross, shirt, and paper. Therefore, in order to be included in 3-D category, she believed the objects needed to look like a "cube" or were thick enough for her to call "sides."

Analysis

Mogen's strategies demonstrated in all activities of category A strongly suggest her conception of 10 by 10 by 10 and 4 by 4 by 4 cubes as uncoordinated sets of faces (Battista & Clements, 1996). For instance, she had 100 unit cubes in each 6 surfaces for the total number of unit cubes in 10 x 10 x 10 cube. Also, she reduced 10 unit cubes each time moving from side to side in Task (A-2) which indicated that she had not constructed any idea of "overlap." Counting the number of unit cubes in one surface and multiplying it by 6 surfaces in Task (A-4) is another example.

Furthermore, there is no indication of her multiplicative thinking during the course of exploration for all tasks. A multiplicative thinker is one who is able to think about units of one and units of more than one simultaneously (Clark & Kamii, 1996). This may explain why she had difficulty in viewing 3-D cubes as coordinating sets of faces since she was unable to construct composite units which is necessary in multiplicative thinking.

Her construction of a 3-D cube array is a "frame" of six uncoordinated surfaces covered with unit cubes. Often times, she was able to change her strategy and solved the problems following my probing questions and her exploring with flats provided. The change of strategy which leads to right solutions may not be significant at this point since coordination of the overlapping units is not part of her scheme. But it did challenge her to view the 3-D cubes from another perspective.

JORN

Jorn was a fourth grader who liked to draw, read, and write. His favorite subjects were art, science, and social studies. His love for drawing comes from his parents who are artists. He said he did fine in mathematics. According to him, mathematics was about "adding numbers, estimating, getting the answer." He thought it was good to get the right answer, but if it did not happen, he said, "...it's not the end of the world." His parents helped him with mathematics and they sometimes spent time doing problems together. He heard of geometry from friends and books, but did not know what it was.

During the interview sessions, I observed that he usually did not spend too much time on reasoning. He said he liked to use "hunches" when solving problems and never asked himself any questions such as "why." Nevertheless, he would push himself to think deeper and further if I challenged him with a question.

The following are part of the geometric tasks he explored during the interview sessions.

(A) Base Ten Blocks

Task (A-1): How many small cubes are there in the big cube?

J: 100

R: How do you know it's 100?

J: We used that in third grade ... we (he and his classmates) call it a hundred cube.

Jorn told me that his teacher had them work with base ten manipulatives for about a week in third grade. They used that for adding purposes. He then counted the top surface of the big cube row by row, "10, 20, ..., 100."

I then showed him a flat. He said, "This is 10." To engage him in thinking further, I asked him, "How many small cubes in this flat?" He said 10 immediately, but then realized that the flat was too big for just 10 cubes. He then showed me a long and said, "This is 10, and that (pointing to a flat) is 50."

R: Would you like to check it?

He placed 5 longs on top of the flat and counted the other 5 columns of the flat (not covered with longs) as well, and said, "This must be 100." But he seemed unsure and started counting 10 small cubes of the first row and then rows across the flat (see Figure 17).

J: I am confused. (Pointed to the big cube) I forgot what it is worth.

I encouraged him to find ways to figure it out instead of trying to remember what they called it in third grade.

J: I don't think we learned this before.

R: You said there were 10 in this (long), so, how many of this (long) in this one (flat)?

He figured out 10 without any difficulty and agreed that a flat has 100 small cubes. I then challenged him to use the flat to find the number of small cubes in the big cube. He placed the flat vertically on the top surface and moved it across column by column. "A thousand," he said.

R: Are you sure?

He repeated what he did a second ago and said, "1000."

Interpretation: Because Jorn had some experience with adding using base ten manipulatives while he was in third grade, he tried to recall how many small cubes there were in a long, a flat, and a big cube, which he thought were 10, 50, and 100 respectively. But placing 5 longs on top of a flat and seeing that some of the flat's surface was not covered, he realized that 50, the quantity of small cubes in a flat he thought earlier, was too small. So, he changed his amount to 100. He then checked it multiplicatively by using the number of unit cubes of the first row and number of rows across the flat as shown in Figure 17. His effort at figuring out the number of unit cubes in the big cube was again interrupted by his effort to remember what he learnt a year ago. His words, "I forgot what it is worth" and "I don't think we learned this before" were an obvious indication of his trying to remember. Questions such as "How many longs in one flat?" and "How many small cubes in one flat?" guided and led him to use flats to get a thousand small cubes in the big cube. Without those guiding questions, he was unable to think simultaneously from 10 unit cubes in one long, 10 longs in one flat, 100 unit cubes in a flat, and finally to 10 flats in the big cube. All of these overwhelmed him and he said, "I am confused." This might be another explanation of why he turned to his recollection of what he memorized previously.

Task (A-2): How many longs are there in the big cube?

J: There are 10 (longs) in this (flat), and 10 of this (flat) in here (big cube) ... I multiply 10 by 10 and got 100.

R: Why did you multiply?

J: Just a faster way to get the answer.

Interpretation: From the first task, Jorn had opportunities to explore the number of longs in a flat by placing a few longs on the surface of a flat, and the number of flats in the big cube by moving a flat vertically row by row on the top surface of the big cube. That experience helped him tremendously in this task. Although he thought using multiplication was "a faster way to get the answer", his approach of multiplying 10 longs in a flat with 10 flats in a big cube rather than adding 10 longs ten times may indicate his beginning stage of multiplicative thinking.

Task (A-3): How many small cubes will be left if one layer is removed from all surfaces?

Jorn counted the number of faces of the cube, and said, "4."

R: 4 what?

J: 400.

R: 400 what?

J: This little (showed me a small cube) or 4 of this (he meant flat).

R: Okay, what if you remove 2 layers?

J: 2 left.

Interpretation: At his first attempt, Jorn counted 6 sides of the big cube and thought that removing 1 layer for each side, he would have 4 layers or 400 small cubes remaining. His idea of the outer surface of the big cube was layers of flats covering 6 sides with no joined or shared edges. I was interested in seeing how he would respond if 2 layers were removed since 2 flats for 6 sides would need 12 flats. Unexpectedly he counted only four sides (top, bottom, right, and left), although he did use the same approach as earlier. I reminded him that a cube has six sides, but he insisted that only two layers would be left. To ascertain that he did not misinterpret the question, I demonstrated to him that every two sides, vertically or horizontally, was joined with a long. After my explanation, he then tried to find out the number of unit cubes covering the outer layer of the big cube. To get a total number of small cubes for each surface, he multiplied the number of small cubes that made up the length and the width. This may indicate that he had an understanding of the concept of area. He had 10 x 10 for front side, 9 x 10 for top, 9 x 9 for both right and left, 8 x 9 for the back, and 8 x 8 for the bottom surface. He did not have 10 x 10 throughout because he realized that all edges were joined by a long. Although he had difficulty in keeping track of surfaces he had counted, with some assistance, he was able to list the total number of small cubes for each side, as shown in Figure 18. With a calculator, he subtracted the total of each side from 1000 and obtained 512 left as the answer. The approach he employed in finding the totals for each side and the number of unit cubes remaining implies that he was able to see the outer layer of the big cube from a different perspective, that is, every edge is a long shared by 2 sides.

Task (A-4): How many equal number of layers from each side should you remove in order to have a cube of size 4 x 4 x 4 remaining?

Jorn placed the 4 x 4 x 4 cube in the middle of a surface of the big cube. He then counted the number of longs surrounding or not covered by 4 x 4 x 4 cube (Figure 4) and said "3 times." I asked him where would the 4 x 4 x 4 cube be placed to have layers removed 3 times. He replied, "inside" confidently.

Interpretation: Jorn saw three "rounds" of unit cubes surrounding the 4 x 4 x 4 cube after placing it at the center of the top surface of the big cube. His saying "three times" may mean removing all longs vertically around 4 x 4 x 4 cube three times which will have a rectangular prism (10 x 4 x 4) remains. It could also mean that the removal involves right, left, front, and back of the big cube only. Therefore, I asked him about the location of the 4 x 4 x 4 cube after the removal. His response of "inside" indicates that his saying of "three times" meant taking off three layers all around the big cube and leaving a 4 x 4 x 4 cube.

Task (A-5): How many 4 x 4 x 4 cubes can you find in 10 x 10 x 10 cube?

R: If I can cut this big cube (10 x 10 x 10) into this size (4 x 4 x 4), how many can I get?

Jorn placed a 4 x 4 x 4 cube on the front surface of the big cube, moved and measured it very carefully. He first said 2 ½ , then 6, then 5. He even said that the remaining strip would make one 4 x 4 x 4 cube.

R: How about ignore the remaining, just get the exact size of 4 x 4 x 4.

J: That will be 4.

R: How about this side? (I pointed to another side of the big cube.)

He realized he left out some, so he started to measure the top surface of the big cube with the 4 x 4 x 4 cube and found out that there were eight.

R: Why 8?

J: You got 4 from the top, 4 from the bottom, but there is not enough in the middle to make a whole (shaded portion of Figure 19).

R: You have 4 on top, 4 at bottom, how about the side?

J: No, you don't have enough to go there.

Interpretation: Looking at the front surface of the big cube with an area of 10 units by 10 units, Jorn had the 4 x 4 x 4 cube placed four times on the 8 units by 10 units surface area. He came up with five cubes because he estimated that the remaining strips would make an extra 4 x 4 x 4 cube. During his initial attempt, he was trying hard to find out how many one surface area (bottom part of the 4 x 4 x 4 cube) he could get from a larger surface (front side of 10 x10 x 10 cube). He did not realize that he was dealing with two 3-dimensional figures. My question reminded him of that. He then changed his approach by placing the 4 x 4 x 4 cube on the four corners of the top surface. His response of "4 from the top, 4 from the bottom" means he could get four exact size of 4 x 4 x 4 cubes from the upper as well as the lower corners. He was again paying particular attention to the remaining strips (shaded portion of Figure 19). However, his perception led him to believe that these remaining strips did not appear to be as large as the one he saw earlier as he said, "... there is not enough in the middle to make a whole."

Task (A-6): How many small cubes are there in the 4 x 4 x 4 cube?

J: 18. Each has 4, and there are 4 rows. So, 4 by 4 is 18.

R: Would you like to check?

Jorn checked it with calculator and had 16 as an answer.

R: OK, let's see whether you can use 16 (small cubes) to make this 4 x 4 x 4 cube.

He used 16 small cubes provided to form a 4 by 4 square. He realized that 16 was just enough for one layer. Then, he examined the 4 x 4 x 4 cube carefully and came up with 64 by counting the number of longs (4 x 1 x 1) in the cube.

Interpretation: Jorn multiplied the length and the width, 4 x 4 = 16, to find the number of small cubes in the 4 x 4 x 4 cube. He employed the same strategy to get the number of units in the 10 x 10 x 10 in earlier tasks. Knowing that there were 16 vertical longs in the 4 x 4 x 4 cube, and each long represented 4 unit cubes, he multiplied 16 by 4 to get the total number of unit cubes. His approach to this task indicates he is beginning to make distinctions between area and volume.

Task (A-7): How many longs (10 x 1 x 1) are there in the big cube (10 x 10 x 10)?

Jorn counted 10 down and 10 across the front surface of the big cube and had 100.

R: 100 of what?

J: Of these little blocks (which he meant 10 x 1 x 1 longs)

Interpretation: Jorn's response of "these little blocks" by which he meant longs, convinced me that he was counting the number of longs in the big cube rather than the number of unit cubes in a flat, although both have 100 in them. He counted the number of "squares" that appear on the front side of the big cube. Each "square" is a long of size 10 x 1 x1. He figured out the number of longs by multiplying the length, 10 longs counting down, and the width, 10 longs counting across. This strategy, which he frequently used in the previous tasks, clearly shows his multiplicative thinking in 2-D settings as well as his capability of understanding the notion of interior area.

Task (A-8): Imagine the big cube is a hotel, with each small cube being a room. How many rooms are there?

J: 100

R: How did you get that?

J: 10 of this (small cubes) up, 10 across. Oh, wait, 1000, I think.

Jorn counted downwards layer by layer, "100, 200, …, 1000."

Interpretation: This task is similar to Task (A-1) where he was asked to determine the number of unit cubes rather than rooms. Changing the task to finding the number of rooms did not prevent him from applying the same problem solving strategy as before. As in Task (A-1), he still multiplied 10 by 10 to get 100 for his first attempt. Then, realizing 100 was just for 1 layer, he added 10 times to get 1000. Having him explain how he obtained 100 helped him to probe deeper and being able to realize that 100 was too small indicates that he was using his number sense.

Task (A-9): Suppose there is another hotel with 20 small cubes tall, 10 small cubes wide, and 5 small cubes in length. How will the hotel look and how many rooms are there?

Before I gave Jorn this task, I provided him with 2 big cubes (10 x 10 x 10) and explained to him that a big cube is 10 small cubes tall, 10 wide, and 10 long (he called it back or fat).

J: It's 100 blocks.

R: How did you get 100 blocks?

J: 5 back, 20 tall, I just multiply 5 by 20.

R: Why did you multiply, not add?

J: I guess it's just a hunch.

He further explained that multiply would result in a bigger number. I stacked up two big cubes and explained the question again.

R: Now, tell me, how does this new hotel look?

J: Bigger and thinner…wait, taller and thinner, and just as wide…what it means by thinner is, it doesn't have much back.

R: OK, how many rooms are there?

He explained that side 1 to side 6 (Figure 20) is 50 each, so the total was 300.

R: But I heard you said before that side 5 was 100.

J: But you took off 5 back, so it's only 50.

To engage him in thinking further, I gave him a flat which I think he was very comfortable with. I then placed that flat side by side with face 5 of the big cube.

R: You said this flat is 100 and that (face 5) is 50.

My intention in asking the above question was to let him compare and see the difference. He looked a little confused but came up with 3000 by multiplying 500 with 6 sides. He did not feel comfortable with the answer because he thought 3000 was too large and he said, "This is hard."

I then encouraged him to build the new hotel with flats. He placed flats side by side with big cubes and figured out it was "exactly 1000." He explained that each flat was 100, with 5 layers thick, it would be 500. Since there were 2 big cubes with 5 layers thick, the total would be 1000.

R: Why did you think that was 3000 or 300 before?

J: I don't know. I guess I just use hunches. I just sort of know it, I never ask myself any questions. I am not a very good explainer.

Interpretation: The information of a new hotel with 20 unit cubes tall, 10 unit cubes wide, and 5 unit cubes long overwhelmed him. However, 20 tall and 5 long seemed to get his attention. Although I stacked up 2 big cubes which have 20 unit cubes in height, 10 in width, and 10 in length, he was still able to see the new hotel as "taller and thinner, and just as wide" compared to a big cube. Because he kept thinking of the new hotel as half thinner, he estimated 50 unit cubes, which is half of 100, for each side. His statement "you took off 5 back, so it's only 50" led me to believe that he was thinking of the number of unit cubes in one layer (one surface) rather than the whole big cube. Building the "new hotel" with flats helped him to actually see its dimensionality. Taking a flat, which consists of 100 small cubes, as a start, he was able to organize his thoughts and figure out that there would be exactly 1000 unit cubes or "rooms."

(B) Tangrams

Task (B-1): Do the middle triangle, the square, and the parallelogram (see Appendix C) have the same space?

R: Which of these shapes have the same space?

Jorn believed that the two small triangles, the two large triangles have the same space, but the middle triangle, the square, and the parallelogram did not. I then encouraged him to try two small triangles with the square. A few seconds later, he was surprised that 1 small triangle could fit in half of the middle triangle. So, he placed the other small triangle on the other half to cover the middle triangle. He was able to fit in two small triangles on the square and parallelogram easily.

R: So, what can you tell me with all these pieces?

J: These 2 (small triangles) fit in all of them.

R: Do they have the same space?

He believed that they have the same space even though they did not look alike. I stopped at that point since we ran out of time. One week later, I gave him the same task again.

R: What is area?

J: Space ... like... (he drew a triangle) all of these in the triangle. This is area in between the triangle.

R: Any space in real life?

J: Sometimes. If you're a farmer, you have a fence, you can have the area that you want. We use area everyday, like regular fence in the playground.

R: Now, back to tangrams. Which pieces have exactly the same space?

He showed me two small triangles with square, and two big triangles.

R: Can these 3 pieces (the middle triangle, the square, and the parallelogram) have the same space?

He moved the shapes around and put one on top of the other and said it was impossible.

R: If I want to find out whether these 3 have the same space, how am I going to find out?

He suggested that I use a ruler to measure from point A to B (see Figure 21). I challenged him whether he could measure those by using two small triangles.

J: ... measure how many times it goes across it.

He used the base of one small triangle and measured point A to B.

R: Some people they try to fit in these 2 small triangles on those three pieces. Would you like to try this method?

He tried and discovered that all 3 pieces could be fitted in completely with the same 2 small triangles and said, "They all have the same size." He believed that his previous method could also work, although the latter was more accurate.

Interpretation: Since the middle triangle, the square, and the parallelogram do not have the same shape, Jorn did not think that they have the same space, although, he did agree otherwise after successfully fitting in two pieces of triangles in all of them. One week later, he again thought that the middle triangle, the square, and the parallelogram had different space since they did not look alike. This could be explained by the fact that it was not his idea to use the two small triangles to determine the space of those three shapes. Although he believed that the two small triangles was a more accurate way to determine their space, his method of measuring using a ruler or the base of a small triangle could also work. This indicates he does not clearly differentiate between "area" and "length."

Task (B-2): Do these shapes (Figure 6-10) have the same space?

J: May be, let's see.

Jorn used a big triangle to measure Figure 8 and the others. He was trying to find out how many times the big triangle could fit in the 7-pieced tangrams triangle. I then encouraged him to also try other strategies. First, I had him make a 5-pieced triangle (Figure 13). Then, his task was to add two big triangles to the 5-pieced triangle to form shapes in Figure (6-10). He was able to fit in seven tangram pieces to all figures without any difficulties.

R: So, what do you think about all these shapes (Figure 6-10)?

J: They all have the same size.

R: Now, do you think these 5 shapes have the same space?

He agreed that they did, although he looked surprised.

Interpretation: Jorn's effort in determining how many times the big triangle could fit in Figure (6-10) indicates that he recognized this task was similar to the previous one, and he was thinking of applying the same approach used in previous task. Seven tangrams pieces fitting in all Figure (6-10) led him to conclude that "they all have the same size." Accommodating a new idea into existing schema takes time. In the process of accepting this new thinking, he was still struggling with not believing his old idea, that is, different shapes have different space or sizes. This explains why he looked surprised.

(C) 2-D and 3-D

Task (C-1): What are 3-D and 2-D?

J: 3-D is this (showed me the big cube). Sort of comes out to you. (He then drew a rectangle) That's a 2-D.

R: What is the difference?

J: 2-D is a flat surface. 3-D is not a flat surface (holding the 10 x 10 x 10 cube), it's got sides, shape. ... I taught myself ... from reading books.

R: What other things here are 3-D?

J: Pencil is a 3-D. I'm a 3-D. Camera is 3-D. Button is 3-D.

R: So, what is 2-D here?

J: Writing on the wall, the numbers on the clock, screen on the TV. Well, the screen sort of come out a little bit.

R: So, is it 3D or 2D?

J: 3D.

R: How about paper?

J: Paper is 2-D.

R: Why?

J: Anything you write on it doesn't come out.

R: How about the paper itself?

J: The paper is pretty much 3-D. (Holding the paper) if you hold it straight, it's 2-D. (Folding paper slightly into half) then, it's 3-D.

R: How about your parents' drawing?

J: They are 2-D. Well, unless you make a wave of it, then it's 3-D.

R: Can you draw 3-D and 2-D?

Jorn's drawing in Figure 22 shows how he differentiated 3-D from 2-D.

J: It (b) is actually a 2-D, but it's a drawing of 3D square. But the 3-D square is actually a 2-D ... it's on a flat surface, doesn't come out. That (a) is a 2-D square.

R: How about hair? (I gave him a strand of my hair.)

J: (Thinking for a few seconds) I guess it's 2-D... I think it's 3-D, ... because it curves (he showed me how he could curve his hair.)

R: What if I make it straight?

J: It's 2-D.

Interpretation: Jorn's idea of 3-D was objects that have sides, shapes, and "sort of comes out." Hair, paper, and his parents' drawing did not match his description of 3-D because they looked more "flat" than "shape" to him, unless they could be made like "wave". Believing that the dimensions of the objects change with bending or curving implies that he had not yet completely grasped the idea of volume.

Analysis

Jorn tended to think in 2-D rather than 3-D although he explained that 3-D were objects that have "sides and shapes." This is evidenced on numerous occasions where he just counted the number of unit cubes in one layer to determine the total number of unit cubes of 3-D cubes. However, with further probing questions, he was able to figure out and proceed with the tasks.

He demonstrated his multiplication skills in many instances such as applying the formula of length x width to determine the quantity of small cubes in a layer of various dimensions. His ability to think of "squares" as unit cubes and 10-unit cubes or longs indicates multiplicative thinking in 2-D setting. However, his lack of coordination of units, that is, not seeing an edge shared by two sides, suggests that he is not a multiplicative thinker in 3-D setting.

EMI

Emi was a very well-mannered and polite fourth grader who liked animals, arts and crafts, played basketball, and joined her church choir. She loved animals so much that she watched Discovery Channel often and wanted to be a veterinarian one day. Like Taile, she was quite familiar with manipulatives-base ten blocks and tangrams. She had played with base ten blocks in second, third, and fourth grades for addition and multiplication, and tangrams for geometry. According to her, mathematics "...are addition, subtraction, multiplication, division, and geometry. ... is challenging but is fun too." She said mathematics could be found in the park, classroom, or even doing the cross-stitches. She believed she was "pretty good" at mathematics but thought that division and long multiplication were hard. "Shapes ..like sphere, pyramid" were the words she described for geometry which she had learned in third grade. She said her dad was "really good at math" and she always asked him for help with math homework.

The following are part of the geometric tasks she explored during the interview sessions.

(A) **Base Ten Blocks**

Task (A-1): How many small cubes are there in the big cube?

E: 1000 (very quickly and confidently)...because this (flat) has a 100 in it, and 10 of these (flats) make one of this (big cube).

Interpretation: As with Taile, Emi had many experiences with base ten blocks in early grades. Her familiarity with these manipulatives helped her in solving this problem.

Knowing that there were 100 small cubes in a flat, she figured out 10 flats or 10 hundreds gave 1000.

Task (A-2): How many longs are there in the big cube?

E: (Thinking hard) 100? 100 of these (longs)...there are 10 of these (longs) in this (flats), and there are 10 of these (small cubes) in this (long)...you have 100 x 10 is 1000. 1000 divided by 2 is 500...oh, gosh.

Those numbers confused her and she lost track of what she was doing. So, I decided to explain the task again.

E: 10 of this (long) in here (flat) and 10 of this (flat) in here (big cube), so 10 x 10 is 100.

Interpretation: Emi solved the problem mentally and said there were 100 longs in the big cube. In the process of trying to explain her strategy, she spelled out all possible details but could not keep track of the ones she had used. Again, she used a flat as a reference point. She first found out the number of longs in a flat, then the number of flats in the big cube, the same strategy she applied in the previous task.

Task (A-3): How many small cubes will be left if you remove one layer of small cubes from all surfaces?

E: (Demonstrating with flats) take off 2 sides (left and right), take off top and bottom layers, and these 2 (front and back). So you took 600, you will have 400 left."

I explained to Emi that each edge of the big cube was a long shared by 2 sides.

She drew a 10 x 10 grid so that she could "divide it up into 100 and then take them off...." She erased the four edges of the grid (see Figure 23). Instead of counting the number of squares left on the grid, she turned to the top side of the big cube and obtained 32 for the perimeter (the four edges of the top surface). Then, she counted unshaded part of a flat and obtained 64 (see Figure 24).

E: ..then you have 8 layers left, because you take all these layers off, 64 x 8 = 512

R: Should you count inside or perimeter?

E: Inside, because you want to know how many left.

Interpretation: As with some previous participants, Emi first thought the outer layer of the big cube was made up of 6 full flats. Unlike the other participants who often turned around the big cube to make sense of the task given, she drew a 10 x 10 grid mirroring a surface of the big cube. The grid gave her more flexibility in removing unwanted parts which she did by erasing the four edges of the grid (Figure 23). She recognized that the outer layer all around the big cube had to be removed. That was evidenced by her explanation which said, "...have 8 layers left, ...64 x 8 = 512."

Task (A-4): How many equal number of layers from each side should you remove in order to have a cube of size 4 x 4 x 4 remaining?

E: (Using calculator) 1000 - 16 = 984, 984 divided by 10 = 98.4 ...98.4 doesn't go well.

I gave Emi a flat (4 x 4 x 1) and asked, "If you remove one layer (4 x 4 x 1), what will happen?" She placed the 4 x 4 x 4 cube on top of the flat and counted "1,2,3" from 4 sides (see Figure 4).

E: 3 times. ... you have to take 3 layers off.

R: How about the bottom layer?

E: (Thinking) 3 times all around... because if you placed it (4 x 4 x 4 cube) on top, bottom, sides, it's all the way around.

Interpretation: Emi chose 1000 take away 16 because she thought there were 1000 and 16 small cubes in the big cube and 4 x 4 x 4 cube respectively. However, a flat of 4 x 4 x 1 made her realize that 16 is only a portion of 4 x 4 x 4 cube. Her explanation of "three times all around" may mean removing three layers from all sides and leaving a 4 x 4 x 4 cube as the question indicated. However, it may also be interpreted as removing three times all around each surface for all six surfaces without thinking of leaving behind a cube of size 4 x 4 x 4.

Task (A-5): How many 4 x 4 x 4 cubes are there in the big cube?

E: (Using calculator) 1000 divided by 16 = 6.25

R: Could you explain to me what the question asked?

E: You want to know how many times this (4 x 4 x 4) can go into this (big cube)? So, you want to know how many 16 can go into 1000.

R: What if I meant how many of this cube (4 x 4 x 4) can go into this big cube?

Emi marked four 4 x 4's on the top side of the big cube and drew a line of length 4 small cubes counting down from upper edge (Figure 19).

E: 4 times on the top, 8 times (altogether).

R: Any extra or remainders of small cubes?

E: May be a couple.

She measured again and discovered that there were four flats in the middle (see Figure 19).

Interpretation: Emi associated 4 x 4 x 4 with 16 which she got from 4 x 4. This is evidenced by her subtracting 16 from 1000 in previous task and dividing 16 into 1000 in this task. This could also indicate that she was not imaging a 4 x 4 x 4 cube. However, her latter strategy of finding the number of 4 x 4 x 4 cubes in the big cube shows that she could 'see' 4 x 4 x 4 cube positioned in each corner of the big cube. Marking four squares of 4 by 4 on the top surface and the height of four small cubes gives her four cubes on the top. She knew the bottom half would also give another four which she described, "4 times on the top, 8 times [altogether]."

Task (A-6): Imagine the big cube is a hotel with each small cube being a room (10 tall, 10 wide, 10 long). Build another hotel that is 20 tall, 10 wide, and 5 long. Also, build another one with 20 long, 6 wide, and 10 tall.

Emi placed the blue cube (10 x 10 x 10) on top of the big cube and counted, "1,2,3,4,5," to show me that it was 20 tall, 10 wide, and 5 long. I then had her draw the two hotels (Figure 25).

R: How many rooms are there in the new hotel (20 x 10 x 5)?

E: It's going to be 1000 too. See, put this (shaded part of Figure 26) on top.

R: Do they have the same rooms?

E: Yes (with confidence).

R: Can you build another hotel that has 1000 rooms?

She arranged 10 flats in various ways.

R: How many ways can you build a 1000-room hotel?

E: A lot of ways.

She then showed me several arrangements with the same 10 flats.

R: How about build one that has 20 long, 6 wide, and 10 tall.

She had the 6 flats join with 4 flats as shown in Figure 27 and said, "There are still 2 flats here (marked with arrow)."

R: How many rooms will be there?

Using calculator, she came up with 12 x 100 = 1200. But she was surprised with this big number, so, she used the big cube as a guide.

E: This is (big cube) 1000. This one (10 flats) and 2 imaginary flats (arrow shown in Figure 27) has 1000 + 2 flats, it will be 1200.

Interpretation: Placing a blue cube (10 x 10 x 10) on top of the big cube to get the height of 20 units tall and counting 5 units across for the width demonstrated that Emi knew how a rectangular prism of 20 x 10 x 5 looked. Her drawing of the 20 x 10 x 5 shape was another indication. She believed both 10 x 10 x10 cube and 20 x 10 x 5 rectangular prism had the same number of rooms. Using 10 flats provided, she was able to manipulate them in different forms and concluded that they still contained 1000 rooms regardless of how they were arranged. In the case of the 20 x 6 x 10 hotel, her prior knowledge of a hundred unit cubes in a flat and 10 flats or 1000 unit cubes in the big cube enabled her to build the 20 x 6 x 10 new hotel and come up with 1200 unit cubes or rooms. In her effort of trying to have 2 stacks of 6 flats joining together to build the 20 x 6 x 10 hotel, she recognized she was short of two flats, since only 10 flats were provided. Therefore, she thought there should be 2 additional flats which she called "2 imaginary flats" joining the stack of 4 flats as shown in Figure 27. A flat, 100 unit cubes, was used repeatedly as a "reference point" in this task as well as in the previous tasks.

Task (A-7): How many small cubes are there in 5 x 5 x 5 cube?

E: ...25, then 6 sides...150?

R: You have 25 for 1st layer, how about other layers?

E: ...25, 50, 75, 100, 125, oh, it's 125.

R: 125 or 150?

Emi believed it was 125. I gave her a 4 x 4 x 4 cube to compare with.

R: You said this (4 x 4 x 4) is 64, this (5 x 5 x 5) is 125?

E: That doesn't make sense, because you just take a layer off.

She placed a 4 x 4 x 4 cube on top of a 5 x 5 x 5 cube and was surprised that they differed greatly in the number of small cubes but not in size. She even found out that 4 x 4 x 4 cube had 80 small cubes (125 - 25 - 20 = 80) instead of 64 she calculated by counting the 4 layers of 16 each. She was confused.

Interpretation: At her initial attempt, Emi looked at the 5 x 5 x 5 cube as 6 surfaces of 25 unit cubes. This strategy was different than the one she tried in Task (A-1) where she used a flat as a "reference" to find the number of unit cubes in the 10 x 10 x 10 cube. She recognized that removing a layer all around a 5 x 5 x 5 cube would give a 4 x 4 x 4 cube but she was curious as to why the number of unit cubes in the 4 x 4 x 4 was only half of those in the 5 x 5 x 5 cube as she described, "That doesn't make sense, because you just take a layer off." In her effort of finding out the big difference between a 4 x 4 x 4 and a 5 x 5 x 5 cubes, she subtracted 25 (5 x 5), left layer, and 20 (4 x 5), four-fifth of front layer from 125. That subtraction left a rectangular prism of 4 x 4 x 5, which has 80 unit cubes rather than a cube of 4 x 4 x 4, which has 64 unit cubes.

Task (A-8): If the block is cut across along the line, how many small cubes will be left for the lower portion (see Figure 15)?

E: 500, it will be totally in half.

R: But I don't cut it straight, it's crooked.

E: If it is this way (dividing the cube in half vertically), it would still be 500. So, if it goes this way (Figure 15) it's still be 500.

R: But I see this (portion A of Figure 15) has a little bit more, don't you think?

E: I still think it is 500. I do believe it still be 500.

Interpretation: Emi was very confident that the big cube would have half of its unit cubes left whether it was cut vertically or diagonally in stair-case form as shown in Figure 15. However, this does not indicate her inability to solve this task with further interaction. Questions such as "Would you like to count them?" would probably be able to engage her further.

(B) Tangrams

Task (B-1): Do these shapes in Figure 6-10 have the same space?

R: What is area?

E: It's like number of square units in a shape.

R: Do these shapes (Figure 6-10) have the same area?

E: No, some of them are bigger than another, and they are all different shapes.

I had Emi make a 5-pieced triangle (Figure 13) and add 2 large triangles to it. She was able to use all 7 pieces to fit in all those shapes without much difficulty.

E: They do have the same area... these (the set of tangrams) can fit in all of these shapes (Figures 6-10). These (tangrams) don't change shape at all, they just change the position.

Interpretation: The term "square units" in her definition of area might have misguided her in believing that, shapes that look bigger would contain more square units, and so they have a larger area. This indicates that she has no idea what a "square unit" is. It is simply a term she "knows" from school. After fitting 7-piece tangrams to various shapes (Figures 6-10), she realized that the position or arrangement of those tangram pieces could make shapes appear differently. However, she believed they still had the same area since they contained the same tangrams pieces.

(C) 2-D and 3-D

Task (C-1): What do you know about 3-D and 2-D?

E: (Holding a 4 x 4 x 4 cube) 3-D…is like 3-dimensional, you can see all sides of it, the sides and front. It's not flat like a piece of paper…is a shape, not flat, it's got dimension to it."

R: Would you like to give some examples?

Emi showed me a small cube, but couldn't decide whether the long and the flat were 3-D.

E: …I guess it could be, because it has sides.

R: Pencil?

E: Yeah, I guess so.

R: Paper?

E: No (very confident) it's flat. 3-D has shape, it's not flat, it's got shape to it, like body.

R: If paper is not 3-D, is it 2-D?

E: I'm not really sure. I haven't learned a whole lot about that (2-D).

R: TV and TV screen?

E: TV screen? I guess that could be 2D. It looks like it has shape to it but it doesn't.

R: TV set itself?

E: Yeah, it could be 3-D because it has shape to it.

Interpretation: Emi's idea of 3-D was objects having obvious thickness which she described as "sides." Therefore, she did not think that paper was 3-D since it looked "flat" to her. Although a flat which she used often in previous tasks, has a thickness of 1 unit cube, it still appeared "flat" to her.

Analysis

Due to her familiarity with base ten blocks, Emi could determine the number of unit cubes in a flat and a big cube with ease. Sometimes she had ideas on how to approach the problems but found herself lost in the process of explaining her strategy. This may indicate her lack of experience in explaining or keeping track of her thoughts. This is evidenced by how she explained her strategy in Task (A-2).

Battista and Clements (1996) point out that many students "conceive of their personally constructed procedure as having two distinct steps" (p. 291). They first determine the quantity of cubes in a layer, then the number of layers. Emi applied the above procedure to most of the tasks in category A. In finding the number of longs in the big cube, she did not see that the "square" could represent one unit cube as well as a 10-units long. Rather, she approached it step by step. First, she determined the number of longs in a flat, then the number of flats in the big cube. This may indicate her beginning stage of multiplicative thinking in 2-D setting.

As mentioned earlier, she had some experience in dealing with base ten blocks in mathematics classrooms. This may explain why she could see the big cube in terms of layers more comfortably. However, this is not the case for 4 by 4 by 4 and 5 by 5 by 5 blocks. She did not image them as cubes but rather 6 faces of 4 by 4 and 5 by 5 layers. Therefore, I would think that she was in a transition stage, where restructuring is local rather than global (Battista & Clements, 1996).

Her investigation in Task (B-1) and Task (A-6) shows that she understood the space in different shapes and rectangular blocks may remain unchanged regardless of their arrangements. However, her interpretation of 3-D in Task (C-1) suggests that she was not yet at a stage where she could see a 3-D object in terms of one coherent unit of length, width, and height.

TAILE

Taile was a fourth grader who liked to collect things related to science and baseball. He played baseball and liked collecting rocks because he thought they were pretty. In school, he liked mathematics, social studies, and P.E. According to him, mathematics is "a subject in school, an important part of life." He said, "You have to use mathematics so many times a day, buy candy bar, buy other stuff, adding tax, ...we use mathematics in every subject, like social studies." He liked mathematics because he loved to figure things out. He further added, "...I learn stuff in math so that I can then show it off." He believed that he was good in mathematics because he used to be in mathematics enrichment class before. His mom taught him division and often helped him with his mathematics homework. He said he did not know much about geometry and defined it as "shape stuff."

I was told that he had ADD (Attention Deficit Disorder). That explains why he was distracted very easily. I sometimes found it extremely difficult to get him engaged in activities. I constantly had to challenge him by giving him new tasks because he became bored in a short period of time.

The following are part of the geometric tasks he explored during the interview sessions.

(A) **Base Ten Blocks**

Task (A-1): How many longs (10 x 1 x 1) are there in the big cube (10 x 10 x 10)?

Taile started telling me all he knew about base ten blocks before I gave him any task.

T: This a 1000 (big cube). This is 100 (flat). This is 10 (long). I learned this many times in school. 2nd grade, 3rd grade.

R: How did you get 1000?

T: This (flat) has 10 of this (long), that is 100, and you have 10 of this (flat) in here (big cube). Each of this will be 100. 100, 200, ..., 1000.

R: So, how many of this (long) in this big cube?

T: (Taking a long look at the big cube, he took a flat) 10 in this. 10 ten's is a hundred. 10, 20, 30, ..., 100. 100 tens.

He called longs as tens because there were 10 small cubes in a long.

Interpretation: Due to his experience with base ten blocks in previous years, Taile was quick to call the big cube "a thousand," the flat "a hundred," and the long "a ten," by which he meant there were 1000, 100, and 10 small cubes in a big cube, a flat, and a long respectively. Knowing that there were 10 longs in a flat, not only helped him to find the number of small cubes in a big cube, but also the quantity of longs in a big cube. In both cases, he added 10 times, that is, 10 one hundreds to give 1000 small cube and 10 ten-longs to give 100 longs in a big cube.

Task (A-2): How many small cubes will be left if you remove one layer of small cubes from all surfaces?

Taile was so excited and jumped in to explain before I had a chance to finish the question.

T: Yeah, because if you took this off from here, here ... (every side of the big cube), it will be another box. Then if you took this off again, again, and again, it will be nothing. It will just be gone.

R: What if you just remove 1 layer of every side?

T: You see, these (shaded part of Figure 2) are connected. This layer is connected to this layer too.

He went on to explain every layer was connected to another layer by a long vertically and horizontally.

R: So, how many small cubes left?

T: See, this is gone, this is gone ... (shaded part of Figure 28), you only have 8 in each row. (He then wrote down 8 x 8 = 64.)

R: But you see 64 is only for this side (referred to one surface of the big cube).

T: (Moving the cube around) Oh, I did not think of that. A lot. You see, I am just talking about this one side here (non-shaded part of Figure 28), but there are so many sides on here (other faces of Figure 28) Can I show you something?

He took 10 flats and stacked them up, placed them side by side with the big cube and said, "these are the same." He seemed to forget the question, so I repeated my question again. He pointed to the top layer of the big cube and explained that there were 64 small cubes for one layer, and then counted down 8 times, "64, 64, ..., 64." Then he multiplied 64 by 8 to get 512 (shaded portion of the front surface of Figure 28).

R: Why did you multiply by 8 not 10?

T: Because, remember, you took off these layers (top and bottom layers). These layers are gone, you only have 8 layers left.

Interpretation: Obviously Taile liked this problem which explains why he interrupted before I finished describing the task. Unlike the previous 3 participants, he was able to see the big cube as "connecting layers." He also noticed that the big cube would decrease in size and disappear if one layer were removed from all surfaces continuously. Phrases such as "it will be another box" and "it will just be gone" explain his observation mentioned above. To find the number of small cubes left behind, he first found out how many remained for 1 layer (64) and multiplied by 8, the number of layers left from the big cube, which gives 64 x 8 = 512. His choosing of 64 x 8 (Figure 28) explains that he excluded the outer layer of the big cube. It also shows that he was able to see the big cube as 10 separate layers as well as connecting parts simultaneously.

Task (A-3): How many equal number of layers from each side should you remove in order to have a cube of size 4 x 4 x 4?

Taile demonstrated to me by placing 4 x 4 x 4 cube in the middle of the front side of the big cube (Figure 4), and said, "three layers from each side" without any hesitation.

Interpretation: His explanation of the big cube becoming another box if one layer was removed in Task (A-1) led me to believe that he could "see" the cube reduced in size to a 4 x 4 x 4 cube in this task. Therefore, his saying "three layers from each side" may mean three outer layers of small cubes all around the big cube. However, it could also mean three layers from top, bottom, right, and left sides which leaves a rectangular prism of 4 x 4 x 10.

Task (A-4): How many 4 x 4 x 4 cubes are there in the big cube?

Taile placed a 4 x 4 x 4 cube on the front side of the big cube as shown in Figure 4 and said, "You have some remainders."

R: If you ignore the remainders, how many can you find?

T: There are 4 (showing the front side of the big cube).

R: How about the top?

T: Same thing, 4, 4, ... (touching all sides of the cube).

I then suggested to him one way where a 4 x 4 x 4 cube could be found in the big cube by placing the 4 x 4 x 4 cube in an upper corner of the big cube. He thought for awhile and found two 4 x 4 x 4 cubes on the upper corner and said, "See, remainders, they are everywhere." He continued his exploration and found out there were eight altogether, four from the top, and four from the bottom (see Figure 19).

Interpretation: At his initial attempt, Taile believed there were twenty-four 4 x 4 x 4 cubes, four from each surface, in the big cube. It is possible that he might have overlooked the 4 x 4 x 4 cube as two dimensional figure of 4 by 4. Although he was able to figure out the number of 4 x 4 x 4 cubes in the big cube, he was still very concerned about small cubes left as remainders which were not part of any eight 4 x 4 x 4 cubes.

Task (A-5): Build a "hotel" that is 20 small cubes tall, 10 small cubes long, and 5 small cubes wide. Suppose the big cube (10 x 10 x 10) is also a "hotel", do both hotels have the same space?

First, Taile tried by placing five longs on top of another five longs. He seemed to pay more attention to the "five small cubes wide" part of the question asked and so he made a joke, "Can I cut these (flats) into halves?" He then took 10 flats and piled them up. He also tried to place 10 longs vertically on top of the big cube. Finally, he took 5 flats and placed them on another set of 5 flats (Figure 26) and said confidently, "This is 20 tall, 10 long, and 5 wide."

R: Now, do they (the big cube and the "hotel" he had just built) have the same space?

T: This (big cube) is wider, that is taller. They are even. This is 1000 and that is 1000.

R: Same space?

T: Yeah. Look, if I took that (5 flats on top) and put them down here, and push it together, it will look like this (pointing to big cube).

Interpretation: As Taile struggled to figure out how "five small cubes wide" looked, the idea of cutting flats into halves crossed his mind and it seemed like there was no other solution. However, the availability of 10 flats enabled him to explore further. Taking 5 flats from a pile of 10 flats and placing them on top of the remaining 5 flats helped him to realize that both stacks have 1000 small cubes in it. Although one looked taller and another looked wider, he believed both had the same space regardless of how they were arranged.

Task (A-6): Do the big cube and the blue cube (an hollow 10 x 10 x 10 cube) have the same space?

I handed Taile a big cube and a blue (hollow inside) cube. Before I even had a chance to ask him any questions, he started talking continuously about the cubes.

T: This (blue) is hollow, and it's much lighter than this (solid). This one (hollow) has 600 (pointing to all sides of the hollow cube). You know why? Because it has no center. See, (tapping the hollow cube with a pencil and listening to the sound produced) hollow. See, (tapping the solid cube with a pencil) solid. Since, this has all of it, 1000. That is 600.

R: 600 what?

T: 600 little cubes. (Pointing to the solid cube) This has 1000. This has center, center cubes. This (hollow blue cube) is not, this one is hollow. ... Wait a minute, this is a trick. 300, understand that these are not full cubes. It's half of cube.

I stopped interviewing because he suddenly felt bored and stopped responding.

Interpretation: Due to the experience he frequently had with the solid big cube, he was very sure of the number of small cubes in the big cube. However, a hollow big cube posed a big challenge to him. He obtained 600 small cubes because he thought there were 6 sides and each side had 100, assuming each surface is 1 small cube thick. The idea of sharing edges did not cross his mind this time. Later, he reduced the number to half, believing that the thickness of each surface was equivalent to half of a small cube. This is evidenced by what he described, "... these are not full cubes. It's half a cube." He referred to the space in these big cubes as the number of small cubes contained in each figure. This clearly indicated that he had not yet grasped the idea of space in three-dimensional setting or he has a different interpretation of "space."

(C) 2-D and 3-D

Task (C-1): What is 3-D and 2-D?

T: 3-D is like this, I can draw (he drew a rectangular block).

Taile further added that he taught himself to draw 3-D and he had no idea what 2-D was.

T: (Holding a 4 x 4 x 4 cube) this is a geometrical cube. See, all these (keys on calculator) sticking out, they are 3-D. If it is flat (holding and showing side and front of paper) like this, it is not 3-D.

R: How about TV screen?

T: 2-D I guess, but it just depends. There are some 3-D on TV, like video game, 3-D stuff on supernintendo.

R: How about TV set itself?

T: The TV itself not what on the screen? That is 3-dimensional.

Interpretation: He used the term "sticking out" to differentiate between 2-D and 3-D. Therefore, he categorized keys on calculator and cubes as 3-D but excluded paper which looked flat to him. His term of "sticking out" was not just limited to real life objects but also extended to visual effects of video games seen in TV.

Analysis

Taile and Emi were classmates. Both had experienced base ten blocks numerous times before. Besides his familiarity with the manipulatives, Taile was able to view the big cube as one coherent block as well as connecting layers which made up the block. This is evidenced by his construction of the block decreased in size and disappeared if one layer all around were removed continuously in Task (A-2), and the big cube reduced in size to form a 4 by 4 by 4 cube in Task (A-4).

Task (A-2) has been presented to many elementary children and elementary education majors in college. So far, Taile is one of the very few who could "see" by himself that every layer shared a long vertically and horizontally. Choosing 64, the number of unit cubes of 1 layer ignoring shared edges, multiplied by eight layers left strongly demonstrated his construction of 8 by 8 by 8 cube after removing 1 layer.

When he was asked to compare the space in an hollow (10 x 10 x 10) cube and a solid (10 x 10 x 10) cube, he believed an hollow cube has only 600 (assuming each surface is 1 small cube in thickness) or 300 (assuming its surface is half a cube thick) unit cubes. By referring the space as the number of small cubes contained in the big cube indicates his understanding of interior volume, interior "contained" by some three-dimensional object (Copeland, 1984). He would need more experience or exploration to construct "volume" of 3-D object, that is, the space exists in its own right whether occupied or not occupied by unit cubes.

ELISA

Elisa was an active fifth grader who played violin and attended art class after school. According to her, mathematics is "figuring things out, putting things together, like a puzzle pieces." She liked mathematics, especially multiplication, because she did well in multiplication in fourth grade, but she did not like decimal problems because she always got them wrong. Her dad has high expectations for her and wanted her to keep up with her grade in mathematics. He checked her mathematics homework very often. She had heard of geometry before but did not know what it was.

She always came to the library on time for the interview sessions. I also found her very committed to complete the tasks given even if she was tired. Often times, she was eager to solve the problems herself and liked to try different strategies. One of the explanations could be she wanted to make sure that the problems were done correctly. Her versatility in trying different approaches and her commitment to engage in geometric tasks during interview sessions were two unique characteristics that set her apart from other participants. When I asked her whether base ten blocks problems were difficult for her since she had never experienced it before, she responded, "... it depends on how hard you try."

The following are geometric tasks she explored during the interview sessions.

(A) Base Ten Blocks

Task (A-1): How many small cubes are there in the big cube?

Elisa counted down 10 and across 10 and said, "Wouldn't it be 100 cubes?"

R: How many of these small cubes are there in the big cube?

L: Oh, wait. This is 1000 cube.

R: How did you get 1000?

L: Because this is 10 for each row, 10 by 10 is 100, then 100 in each one (referring to each layer), then times 10, 1000.

She was very confident that the big cube has a 1000 small cubes.

Interpretation: Measuring 10 down and across, Elisa obtained 100 by multiplying 10 by 10 mentally. My question of asking her how many unit cubes there were in the big cube made her rethink about her answer. Looking at the cube, she realized that 100 was only a part of the big cube. She then multiplied 100 by 10 layers to get 1000 unit cubes. Her love and confidence in multiplication was clearly shown in this task. She multiplied 10 by 10 and 100 by 10 mentally within a flash. As she told me before, she had already "mastered" her multiplication skills in fourth grade.

Task (A-2): How many small cubes will be left if you remove one layer of small cubes from all surfaces?

L: One layer, do you mean a whole layer?

I explained to Elisa that after removing a layer from all surfaces, the cube became smaller.

L: One layer here, 1 layer there, ... there should be minus 600 (counting 6 sides of the cube) which would be 400 left.

R: So, you think if take off a layer of every side, it would be 400 left. How about 2 layers from every side?

L: Oh, there will be about...(turning to each side of the big cube) 2, 4, 6, 8, 10, probably 12, ... could be none.

She was surprised she obtained 12 layers since a big cube has only 10 flats in total.

L: (Looking at the cube) let me see, so, I take off 1 layer, another layer, kind of confusing ... if you take off 1 layer, you will be taking out the other.

R: What do you mean?

L: See, when I take off this, I am taking off some of this piece, some of that piece. I am not exactly for sure how you will be able to take a layer off at all without being able to take off some of these, unless you take it off the same side...

She seemed to be very confused by the question asked, hence I decided to take another approach.

R: Let's say you take off this layer (top), what will happen?

L: Then, there is going to be a 90 out of this side (right).

R: Okay, 90 out of that, and then what?

She suddenly realized something, grabbing a pencil, started counting and writing. She had 200 for left and right layers, 160 for front and back. Then she counted 8 x 8 = 64 for top layer and explained that the bottom layer would also be 64. The total was 488.

R: Is 488 outside or inside?

L: You took it off 488. For 1000, minus 488, (computing on paper) 512.

R: Now, which one do you feel more confident with, 400 or 512?

L: 512, because I actually took the time to think it out.

Interpretation: Elisa could be confused by the term "layer" used in this question or she had not previously thought about the idea of cubes being "shared" in this way. She thought of "a layer" as a whole layer which is a flat of 100 unit cubes in it. To be able to remove 2 layers (from her understanding of layers) from each side, the big cube needs to have 12 layers. But she knew there were only 10 layers in the big cube from previous tasks. That "12 layers" helped her to rethink the meaning of "layer" in this task. Although she could see that each layer was connected to another as she said, "...if you take off 1 layer, you will be taking out the other." She did not know how to proceed with the problem. In an effort to guide her from her confusion with the term "layer", I asked her what would happen if the top layer was removed. That question helped her to focus on just one small part at a time. From that point on, she saw the light and was able to apply her knowledge of multiplication, addition, and subtraction to figure out that 488 and 512 unit cubes were removed and remained respectively. This strategy allowed her to take care of the confusion she had earlier and be able to look at "layer" from different perspective, that is, taking off a whole layer (100 unit cubes) from one side means there will be less than 100 unit cubes per layer on other sides.

Task (A-3): How many equal number of layers from each side should you remove in order to have a cube of size 2 x 2 x 2 remaining?

L: 1000 minus 8 (2 x 2 x 2), I need to take off 992 to have 8 left.

I assumed that Elisa misunderstood the question, therefore I explained it again. That did not help her much. Since I had a 4 x 4 x 4 cube with me, I modified the question by changing 2 x 2 x 2 cube to 4 x 4 x 4 cube. Placing 4 x 4 x 4 cube in the middle of a surface of the big cube (Figure 4), I asked her how many equal number of layers needed to be removed to have a cube of 4 x 4 x 4 left.

L: (Placing 3 flats on top and bottom of the 4 x 4 x 4 cube, she filled up the right and left sides of the cubes with longs.) You won't be able to remove the exact layer.

Interpretation: As in Task (A-2), Elisa defined "layer" as a whole flat with 100 unit cubes. With that in mind, the phrase "equal number of layers from each side" from the question could be interpreted as equal number of whole flats from each side. Therefore, she could only see three layers on top and bottom as she said, "you won't be able to remove the exact layer [on the right and left sides]."

Task (A-4): How many longs (10 x 1 x 1) in the big cube?

Elisa said, "100" in a second. She demonstrated three approaches to solving the problem. First, she placed a long vertically on the top surface of the big cube, 10 down and 10 across. Then, she counted the number of 'squares' on that surface, each square represents 1 long. Finally, she said there were 10 longs in a flat, and 10 flats in the big cube, and that gave 100 in total.

Interpretation: In her first and second strategy, Elisa counted the number of "squares" appearing on one surface of unit cube, to represent longs. In her final approach, she multiplied 10 longs to 10 flats. Her approaches in this task reflected her versatility and multiplicative thinking.

Task (A-5): Suppose the big cube is a hotel, with each small cube being a room, build another hotel that is 20 small cubes tall, 10 long, and 5 wide. Do both hotels have the same number of rooms or space? Build your hotel that has 1000 rooms. Build a 2 x 3 x 5 block.

Elisa tried to stack up flats on top of the others to get 20 small cubes tall but could not find enough flats. She placed two flats vertically on top of three flats. They did not seem to look like the one she had in mind, so, she snapped them together and said, "It's like this but double." I had her explain further about her statement. She first placed five flats vertically on top of the big cube, but then she found another five flats. So, she stacked them up on the previous five flats (Figure 26).

R: Do you think both hotels have the same space or number of rooms?

L: Yes (taking 10 flats and put it side by side with the big cube).

R: Can you build another hotel that has 1000 rooms?

She took 10 flats and arranged them enthusiastically in various ways. She figured out how it came up to 1000 small cubes by referring to the dimension of the big cube.

R: How many ways can you build a 1000 rooms hotel?

L: Many many ways, as long as I used these (10 flats), no matter how I arrange them, it's going to be 1000.

R: How about building a 2 x 3 x 5 block with small cubes?

She stacked two 5-flats up (Figure 26) and used that as a guide to build 2 x 3 x 5 block. She piled up 15 small cubes in a 3 by 5 fashion and said, "It's like this but twice." After asking her to explain further, she demonstrated it by placing three longs side by side and said, "It's like this, cut into half and bend it over."

Interpretation: The big cube, 10 x 10 x 10, was used often as a reference to build blocks with other dimensions. Elisa also recognized that 10 flats would always give 1000 rooms regardless of how they were arranged. This could reflect her notion of conservation of interior volume which explains that the "room space" contained in the blocks does not change as the exterior dimensions varies.

Task (A-6): How many small cubes are there in this rectangular block with 15 small cubes tall, 5 wide, and 13 long.

L: (Using calculator) 5 x 15 = 75, 75 x 13 = 975 Each plate right here is 75, if 13 of these plates, you have to multiply it.

Interpretation: Elisa recognized that each layer, which she called plate, has 75 unit cubes. To get 13 layers of 75 each, she multiplied them, although she might have used the 10 x 10 x 10 big cube as a reference to solve this problem, that is, 10 flats of 100 unit cubes in each flat. Her explanation in 5 x 15 = 75, then 75 x 13 = 975 reflected her familiarity with multiplication.

Task (A-7): If the big cube is cut across from point A to C, how many small cubes will be left for the lower portion (ADC)?

L: I think it will be half, they kind of mirror itself.

Elisa was not sure. She started counting squares from A to B, B to C, and multiplied them, 8 x 9 = 72. Then, she multiplied again with 10 because each square represents a long which has 10 small cubes (72 x 10 = 720). She was not pleased with 720, she multiplied the number of squares from AD with DC which was 10 x10 = 100. With each square represents a long, she had 100 x 10 = 1000 and said, "That cannot be right, it cannot be 1000 itself." A few seconds later, she switched to another approach by counting the number of squares of the lower portion (ADC) by ten's, "10, 20,30, ..., 550."

R: Does that make sense to you? The whole thing is 1000, half of it is 500. I just draw a line, here it is, 50 more.

L: Because there is more, you are not dividing exactly right in the middle ... here you got extra 20 (referring to s and t in Figure 29).

R: Where did you get extra 50?

L: (Counting number of squares for top and lower portions of the cube) 1, 2, ..., 45. 1, 2, 3, ..., 55. I don't know why.

I stopped at that point because she seemed to look tired.

Interpretation: In many instances, Elisa used a "square" to represent a long to solve the problem. Being able to look at a "square" and interpret it as one unit cube, one long, or 10 unit cubes simultaneously indicates that she understood the meaning of ten as a unit as well as 10 small units. This conservation of ten could enhance her multiplicative thinking. Knowing that a whole big cube contains 1000 unit cubes, cutting across approximately half of the big cube could not possibly give 1000 as she said, "That cannot be right, it cannot be 1000 itself." That reflects her number sense, her knowing of the magnitude of 1000. According to her, one portion has more than 500 (55 longs) and the other has less than 500 (45 longs) because the big cube was not divided exactly right in the middle diagonally. However, she could not figure out where the extra 50 unit cubes came from.

(B) Tangrams

Task (B-1): Which of these pieces (tangram pieces) have the same area?

Elisa picked out two small triangles and two large triangles.

R: How about these 3 (the middle triangle, the square, and the parallelogram)?

L: (Placing the parallelogram on top of the middle triangle as shown in Figure 30) Cut that off (shaded-part) and put it right here (A), then they have the same area.

She then tried the square and the middle triangle. She believed cutting the shaded-part off and placing it on (B) as shown in Figure 30 would give the same area. Then, she tried the parallelogram on the square.

L: Looks like all of these have the same area.

Then, she decided to try another approach. She used the square and the parallelogram and drew them on Figure 8 (seven-pieced tangrams triangle). After drawing, she said, " I don't think they have the same area…I can't tell." Then, she started counting and found out both have 9 pieces, "They are equal."

R: Equal in what?

L: Equal in area, and I bet there are some other easier way I can find out without doing these.

She believed that the middle triangle, square, and parallelogram have the same space. I gave her two small triangles as another approach to determine whether they have the same area. She was able to fit two small triangles into those three shapes easily and concluded that they had the same area.

Interpretation: Like most participants, Elisa believed that only shapes congruent to each other would give the same area. Therefore, at her initial attempt, she chose the two congruent small triangles and the two congruent large triangles. As in other previous tasks, she always wanted to make sure that she did the problem correctly. This could be one of the reasons why she liked to try different approaches to solve a problem. In this task, both the first approach, placing one shape on top of the other to conjecture their areas, and the second approach, figuring out how many each shape could fit into Figure 8, did not allow her to measure with accuracy but rather with approximation. Therefore, she said, "... I bet there are some other easier way I can find out without doing these." She felt more confident with the suggested approach because she could actually measure the middle triangle, the square, and the parallelogram with the two small triangles.

(C) 2-D and 3-D

Task (C-1): What do you know about 3-D and 2-D?

L: (Showing me a 10 x 10 x 10 cube) 3-D ...it has shapes, not flat, like things sticking out...you can reach in and touch it...like star war movies, you can see things coming out. Disney has some movies, like pops out at you.

R: How about TV?

L: TV is 2-D because it's flat, it doesn't like come out at you.

R: You meant TV screen, okay, how about TV set?

L: All things in real life is 3-D, but TV screen is 2-D, TV set is 3-D.

R: Paper?

L: 3-D because ...(couldn't explain)...it depends, like paper in TV is 2-D.

Elisa said she learned about 3-D from TV, books, and found out more by asking friends and her parents.

Interpretation: Although she believed that "all things in real life" are 3D, she did not include a TV screen because it looked flat to her. Another distinction between 2-D and 3-D was objects with "pops out" or "coming out" feature. Hence, according to her, Disney movies that have "pop out" features were considered as 3-D, but paper in TV would be 2-D.

Analysis

In finding the number of unit cubes in the big cube (Task A-1) and rectangular block (Task A-6), Elisa used layering strategies which means she multiplied the number of layers with the quantity of unit cubes in each layer. Battista and Clements (1996) believe that "Those who complete a global restructuring of the array" (p. 290) implement layering strategies, that means Elisa no longer sees the 3-D or block as an "uncoordinated set of faces."

Her using of layering strategies could be linked to her understanding of the concept of volume. At this stage, she was still not able to measure volume by seeing its relationships in terms of length x width x height (Copeland, 1984). However, her ability to build blocks of different dimensions to give 1000 rooms and believe that the quantity of rooms remains unchanged regardless of their arrangement, indicated her conservation of interior volume.

Consistently, she defined "layer" as a full flat which consists of 100 unit cubes. Her interpretation of "layer" confused her on various occasions. Nevertheless, she often was able to proceed with the problem solving after making sense of the questions I asked. For instance, she knew that it was impossible to have 12 layers in 1000-unit-cubes block since each layer has 100 as explained in Task (A-2).

She exhibited her ability to think multiplicatively throughout tasks in category A. For example, each "square" on the surface of the big cube could represent one unit cube or a long as needed in different situations. This multiplicative thinking might have enabled her to see the big cube or other rectangular block in terms of layers easily which Battista and Clements (1996) classified as layering strategies as mentioned earlier.

EDEM

Edem was a fifth grader who believed that geometry has something to do with mathematics and shapes. He was in advanced mathematics class with Elisa. He said he liked mathematics, "It's something I am good at, numbers are very easy for me to work with." However, when asked for what kind of mathematics he liked, he responded, "I like plain problem, like 8 x 10 straight out. Word problems are kind of confusing because you need to translate it into numbers in your head. That takes too long." He also learned 2-D and 3-D from watching TV, a mathematics program called "Square One." He enjoyed that program because it has humor in it and it let viewers figure out the mathematics problems before showing the answer.

While interacting with him during interview sessions, I observed a few interesting characteristics about him. He paid attention to every detail that I said and made use of the information as much as he could. After carefully listening to or reading each task given, he immersed himself in deep thinking. Then, he would speak slowly and explain patiently to me, again, never leaving out any detail that he could remember. He always tried to add, subtract, multiply, and divide mentally. If necessary, he would use pencil and paper to do computation but refused to use a calculator, as he said, "That is cheating." During his investigation with various tasks provided, he sometimes felt pressure when not being able to give the right answers. There were two possible explanations. One could be his high confidence in mathematics. Another could be his popularity in school; teachers believed that he was smart and far above average and his friends thought that he knew everything.

Below are geometric tasks he explored during the interview sessions.

(A) Base Ten Blocks

Task (A-1): How many small cubes are there in the big cube?

D: Since this (a flat) is 100, and this (the big cube) is made up of ... 10 sets of 100's, probably a thousand.

R: Probably? So, you are not sure?

D: I might have counted it wrong.

Edem then used his index finger to count flats that made up the thickness of the big cube. He obtained 10 flats and said, "It is a thousand."

R: A thousand of what?

D: (Showing me 1000) a thousand of small cubes.

Interpretation: He knew that a flat has 100 unit cubes, therefore, he counted number of flats that made up the thickness. He came up with 10 x 100 = 1000 like he said earlier, 10 sets of 100's.

Task (A-2): How many small cubes will be left if you remove one layer of small cubes from all surfaces?

D: 400... each layer is 100 small cubes, there are 6 sides, 1 layer off of each side, then 600 small cubes were taken off.

I explained to Edem that the top and side layers shared edges. Holding the cube in front of him, he stared at it and thought hard. He came up with 512.

D: What I figured is the sharing first. I figured out how many cubes each layer didn't share with, so, 64 x 6, I got 384.

R: What is 384?

D: 384 is how many that each layer doesn't share, and that's all of them add together.

He then explained that he added 4 horizontal edges (4 x 10) and the remaining 8 edges (8 x 8).

R: So, which one, 400 or 512?

D: This one (512), because I worked it out more thoroughly.

Interpretation: As with other previous participants, Edem thought removing 1 layer of all surfaces meant taking away six full flats, one from each side. Explanation on shared edges made him rethink his initial approach. He had an interesting way of figuring out the outer layer. As shown in Figure 31, he counted 6 sides of 8 by 8 which did not include all 12 edges, 4 vertical and 8 horizontal. That gave 8 x 8 x 6 = 384 unit cubes on outer layer, which he explained, "384 is how many that each layer doesn't share ...." Then, he added 4 longs (10 x 1 x 1 horizontal longs of Figure 31) which gave 4 x 10 = 40. The remaining edges, 4 vertical and 4 horizontal 8-unit-cubes; that again gave 64 unit cubes. Therefore, the outer layer consists of 384, 40, and 64 unit cubes which makes a total of 488. Taking away 488 from 1000 means there would be 512 unit cubes left behind.

Task (A-3): How many equal number of layers from each side should you remove in order to have a cube of size 2 x 2 x 2 remaining?

D: (Looking at front and right sides of the cube and counting toward the middle.) Four. If you take away 4 rows from each side, it leaves an area of 4 small cubes right there in the middle. So, 4 by 4 by 4 is 4 small cubes on each side.

R: So, is it 4 by 4 by 4 or 2 by 2 by 2?

D: 2 by 2 by 2 would be 4 on each side, because 10 by 10 by 10 is one hundred on each side.

R: 2 by 2 by 2 is actually how many left? I mean how many small cubes.

D: 10 by 10 by 10 (thinking and looking at the cube) eight.

R: Can you draw 2 by 2 by 2 for me?

D: (Drawing and talking softly) I didn't draw it very well. I can only get to 3 sides (Figure 32).

Interpretation: Unlike in Task (A-3), Edem counted 4 rows rather than 4 layers (4 flats of 100 unit cubes each) from each side. He used 10 x 10 x 10 big cube as a reference to find out the dimensions and number of unit cubes of the remaining cube left behind, that is a 2 x 2 x 2 cube with eight unit cubes. His drawing in Figure 32 showed that he had an idea on how a 2 x 2 x 2 cube looks.

Task (A-4): How many longs are there in this big cube?

D: (Looking at the big cube, counting mentally) 100. This (the big cube) is 1000 smaller cubes here, because it takes 100 ten's to make 1000. So that would be 100 of this (long) to make this (big cube) 1000.

Interpretation: Unlike most of the participants who used 10 longs in a flat as a starting point to figure out the quantity of longs in 10 flats, which is equivalent to 1 big cube, Edem multiplied 100 by 10 mentally to get 1000. This shows his familiarity with multiplication facts. However, he demonstrated later that it took 100 ten's to make 1000 and 100 longs to make the big cube. He was able to associate the "long" as a whole unit long as well as "ten", 10 unit cubes. This viewing could indicate his multiplicative thinking.

Task (A-5): Build a "hotel" using these longs or/and flats. The "hotel" is 20 small cubes tall, 10 small cubes long, and 5 small cubes wide.

D: (Taking 2 sets of 5 flats, putting one set on top of the other as shown in Figure 26) This is the easiest way I can think of. This is 5 wide, 10 long, and 20 tall. I could build it with these (longs), but it would be a lot harder to hold together.

I had Edem draw the "hotel" he built. While drawing, he said, "Since it is 10 long, 20 tall, so it is half as long as it is tall" as shown in Figure 33.

Interpretation: He had a great sense of how a "hotel" of 20 x 10 x 5 size looks, therefore, he rearranged the 10 flats into 2 sets of 5 flats with ease. His thinking of how the building should be "half as long as it is tall", although not accurately shown in his drawing, indicated his number sense.

Task (A-6): Imagine the big cube is also a hotel, with each small cube being a room. Do both hotels have the same space?

D: Each of this (flat) is 100. 10 of them, 10 x 100 is 1000, and this (big cube) has 1000 small cubes. But an easier way would be to snap them together and put them side by side with the big cube, and compare them.

Interpretation: Edem recognized from the previous task that the new "hotel" with dimensions of 20 x 10 x 5 was made up of 10 flats. With that in mind, he knew both of the "hotels" were the same.

Task (A-7): Build "hotels" that has 1000 rooms.

Edem took 9 flats and 10 longs and arranged them in various ways. Here are 2 of his designed "hotels" (Figure 34) and a list of 1000-roomed hotels with various dimensions (Figure 35).

Interpretation: He applied his multiplication skills to aim for 1000 with 3 different numbers. He came up with those dimensions, as shown in Figure 35, mentally. His drawings and selection of dimensions strongly demonstrated his ability to conserve 1000-roomed hotel.

Task (A-8): How many 5 x 5 x 5-sized cubes are there in the big cube?

D: (Using mental calculation) it would fit into this (the big cube) 8 times. ... the cube is 5 by 5 by 5, 5 x 25 is 125. I divided 1000 by 125. Actually, I tried to multiply 125 by different numbers, ... I started with 10 first, although I know that number wouldn't work, then I tried 5, 6, 7, then 8.

R: Can you fit in exactly the same size here?

D: You mean, take this (5 x 5 x 5) cube and fill them up here (the big cube)?

Edem was still puzzled over my question. So, I decided to explain by demonstration. I placed a 5 x 5 x 5 cube on an upper corner of the big cube and asked him, "How many of these (5 x 5 x 5) cubes can I put inside this big cube?"

D: Oh, put inside? I think it would take 8 of these.

R: How can you see 8, you can't even put inside?

D: It's just the way the mathematics set up. I am confident ... the way ... and the answer I got when I multiplied them.

I demonstrated to him one more time. This time, I placed it on upper 2 corners.

D: This half up, you have 4, this half down, will be 4. So, still equals 8.

R: How about 4 x 4 x 4 cube?

D: (Without using pencil or calculator) 8 x 4 = 32, 32 x 20 = 64, 640, 32 x 30 = 960, 32 x 31 = 992 ... doesn't go in evenly, I don't think ... 32 x 32 goes all the way to ... 992 + 32 will be 1024 ... wait 4 x 4 is 16. I need to redo it again, 16 x 4 = 64.

R: My question is to put the whole cube, exactly the same size, how many 4 x 4 x 4 cubes will go inside there. Like the one you did before with 5 x 5 x 5 cube, you need 8 of those to build the big cube. So, how many 4 x 4 x 4 cubes I need to build the big cube?

D: (Using multiplication as shown below) 64 x 20 ... 960, still doesn't fit in evenly. It fits in 15 times.

R: Let's compare your answer with this (using a 4 x 4 x 4 cube to measure (10 x 10 x 10), see whether you can fit that in 15 times.

D: (Placing the 4 x 4 x 4 cube on top corners of the big cube, as shown in Figure 19) With the actual shape, it doesn't seem to fit in because there is this one ... empty strip, all the way round, not used (shaded part of Figure 19). If you split these empty strip, you can get more. But with the actual shape, you can only fit in 8 times.

Interpretation: In both instances, Edem did not look at the 5 x 5 x 5 and 4 x 4 x 4 cubes as "cubes" or their actual shapes, but rather the number of unit cubes contained in each cube. Therefore, he divided 1000 by 125 and 64, the quantity of unit cubes in 5 x 5 x 5 and 4 x 4 x 4 cubes respectively. These two activities suggest that he may be manipulating numbers by multiplication and has not necessarily built meaningful connection between his number manipulation and the cube. This is not surprising since mathematics he has in the classroom is algorithm driven.

Task (A-9): If this rectangular block is 5 small cubes wide, 15 small cubes tall, 13 small cubes long, how many small cubes are there?

D: (He multiplied 5 by 13, then by 15, still refusing to use calculator) 975.

R: Why do you multiply these 3 numbers?

D: Because they are the dimension of the ... they are the amount of squares in each side. They are 5 across here 15 times, 5, 5, ..., 15 times. So, 5 x 15 to find out that number, and then they are however many of these times 13. So, whatever this number was times 13.

Interpretation: Edem applied the same strategy used in Task (A-1). According to him, there were 15 rows of 5 unit cubes, that is 15 x 5 = 75 unit cubes for one layer. Then, he multiplied 75 by 13 because there were 13 layers of 75.

Task (A-10): How many of these 1 by 1 tiles are needed to fill in the shaded part (see Figure 36)?

D: There is 14 here, there are 10 in the middle, 2 left on each side, so 40 here, and 40 here (right and left vertical shade). Then, 20 here and 16 here, so, there are 4 left, so 2 here, 2 here. 2 x 10 = 20 and 20 here (top and bottom horizontal shade). So, 120 tiles.

R: OK, let me see whether you agree with another method. What if I used 20 x 14 and then take away 10 x 16. Will that work?

D: Yeah, because this is how much space you can't put it in (10 x 16), and this is how much space you would be able to put it in (20 x 14), so you subtract this (10 x 16) from that (20 x 14).

Interpretation: I was amazed by Edem's ability to solve this problem mentally. He divided the shaded parts into 4 portions, two 2 by 20 on right and left, two 2 by 10 on top and bottom. He also recognized that 20 by 14 rectangular shape was the amount of space to fill in 1 by 1 tiles while 10 by 16 rectangular shape was the amount of space not to have any tiles. By that, he believed that 20 x 14 take away 10 x 16 would be the shaded portion filled with 1 by 1 tiles. His approach in this task reflected his understanding of area, the space exists in its own right whether or not it is occupied.

Task (A-11): If I were to cut across the cube through the "staircase" line, how many small cubes will be left for the lower block/portion? Exactly half, more than half, or less than half of the 1000 small cubes (Figure 15)?

D: It would be more than half because of this cube here (A) and this cube here (B). So, you have a whole row here (C) and a whole row here (D) that this (upper block) doesn't have (Figure 37). .

R: So, how many small cubes will be left then?

D: 10, 9, 8, ... see, from 1 to 10 and each one will be 10 deep. So, there is 100 here, 90 here, 80 here, ... (adding up 100, 90, 80, 70, 60, 50, 40, 30, 20, and 10 mentally) 540, wait a minute, 550.

Interpretation: Unlike other participants, Edem noticed immediately that the lower block has 2 additional longs, what he called as "whole row" C and D. He knew that each "square" appeared on the block representing a long which he described as "each one will be 10 deep." Amazingly, he got a total of 550 unit cubes for lower block by adding 100, 90, 80, 70, 60, 50, 40, 30, 30, and 10 mentally. As usual, he refused to use a calculator.

(B) Tangrams

Task (B-1): Which tangrams pieces have the same area?

R: What is area?

D: Area is the size, it is how much space it takes up ... it's like when they measure the area of the house, they usually measure in the square feet. These (taking 2 large triangle pieces) are about the same area. (Then, taking 2 small triangles to form a square) And if you put these together, you have the same area as this one (the square).

R: Do you think these 3 (the middle triangle, square, and parallelogram) have the same area?

D: (Using 2 small triangles to fit in the three shapes) These two (the middle triangle and the square) have the same area because these shapes (two small triangles) fit in with each other. (Trying with the parallelogram) Yeah, they all have the same area, because they fit in with each other.

R: Before you tried with these 2 small triangles, do you think they have the same area?

D: I thought may be ... because if you (holding the parallelogram) kind of change the shape a little, I thought may be they have the same area.

Interpretation: Using the two small triangles to fit into the middle triangle, the square, and the parallelogram, Edem convinced himself that three of them have the same area. His strategy in Task (A-10) and this task indicated his understanding of area.

(C) 2-Dimensional and 3-Dimensional

Task (C-1): What are 3-D and 2-D?

D: 2-D means 2 dimensional, it has width and height. 3-D means 3 dimensional, it has width, height, and depth.

R: What kind of things are 2-D or 3-D here?

D: (Holding the big cube) This is 3-D. Here is width, height, and depth. Each of this (small cube) is 3-D because it is an object type (it has width, height, and depth). (Looking at a flat) Even though the depth is not nearly as much as width and height, there is still a depth. Paper, ... well, there is not much depth, but a little bit of depth, probably still 3-D too.

R: So, all of the things here are 3-D?

D: Yeah, but say if I draw a square. The square that I draw is not 3-D but 2-D. But now if I do this (adding depth to his square) it's still 2-D, but it appears to be 3-D.

R: Would you like to explain a little bit about what you meant by "it is still 2-D although it appears to be 3-D" ?

D: Because if you look at it, the lines are bent, at a different angle, it gives the illusion of it moving back, just like with this perspective thing here (drawing a rail road track), a rail road track going off the distance, it gets smaller as they go on ...

R: Why do you think this is 2-D although it appears like 3-D?

D: Because it's not coming up, it doesn't have a real depth. (Holding a piece of paper) if you look on the other side of the paper, it's not pushing out there (showing the back of the paper) and it's not popping up here (showing the front of the paper). It's still 2-D.

R: How about 1-D?

D: 1-D I think is one dimension. It's what they call geometric line, only have one dimension, either have only long or only wide.

R: Can you see it here?

D: You can't actually see it, because it has no length or width.

R: Can you draw it?

D: I can draw something to represent it but whatever I draw will have at least some width. I can draw to represent geometric line but it will have some width.

R: How about a dot?

D: (Drawing a dot) it has a little bit from here to here, a little bit from here to here. So, it's still a 2-dimension.

Interpretation: Edem's understanding of 3-D, 2-D, and 1-D far exceeds his classmates, probably is compatible with high school students. He believed that all things that were "object type" as he described them were 3-D even though they might have a very "little" depth such as paper. However, drawings that appear like 3-D were in fact 2-D because he said, "... it doesn't have a real depth." He used the drawing of a rail road track as an example to further illustrate his point, that is, how a bending line gave an illusion of an object "moving back." He meant that the bent line would appear to be the side of an object. His explanation and drawing of the rail-road track certainly demonstrated his understanding of perspective relationship. As mentioned in earlier chapter, perspective is a part of projective geometry (Copeland, 1984). He also realized that 1-D was imaginary as he explained "...whatever I draw will have at least some width."

Analysis

In approaching geometric tasks provided, Edem was able to apply his multiplicative thinking, his viewing of the block as connecting layers and cubes, and his notions of area and volume to solve the problems.

To think multiplicatively means being able to think about units of one as well as composite units (Steffe, 1992). Edem had, in many occasions demonstrated that he was a consistent multiplicative thinker. For instance, a "square" appearing on the surface of the block could represent one unit cube or one long (10 unit cubes) in appropriate situations. His ability to think multiplicatively enabled him to construct units more than one and coordinate units in layers, rows, and columns. This ability certainly helped him to solve geometric tasks in category A.

His enumeration strategies in dealing with 3-D problems of base ten blocks indicated his ability to conceptualize the block as connecting unit-cubes, layers, as well as one coherent, integrated, and coordinated block. His strategies suggested that his restructuring of the block was global rather than local (Battista & Clements, 1996). Therefore, it is not surprising that he could see each edge, which is a long, shared by two surfaces and each vertice was a unit cube shared by three surfaces. He took those joint edges and cubes into consideration when solving Task (A-2). His understanding of area was strongly demonstrated in his strategy applied in Task (A-11). He was able to obtain area of different regions by multiplying length and width, a developmental level which Piaget (1960) classified as stage 4. He understood that area was not just how many 1 by 1 tiles needed to fill in that shaded region but also the space occupied by those tiles as he described earlier, "... this is how much space you would be able to put it [tile] in ..."

As for conservation of volume, his selection of different dimensions for 1000-roomed hotels indicated his grasp of the concept of interior volume, a type of conservation. That means he had the idea that the "room space" contained in the blocks was constant as the exterior dimensions (length, width, and height) were changed (Copeland, 1984). However, "volume" is not just limited to the interior occupied by exterior dimensions, but it is the space exists in its own right (Piaget, 1960). Although, the tasks given to him earlier were not designed to determine his conservation of volume as described above, a type of conservation, which Piaget believed would occur only at stage 4, that is, when children are at formal operations stage. I believe that he would be able to conserve the notion of volume in a deeper sense based on his mathematical experiences and ideas he brought with him or constructed throughout interview sessions, especially how he made sense of 1-D, 2-D, and 3-D.

CHAPTER IV

CONCLUSION

The research in this study provides an insight on seven 4th and 5th grade students' geometric thinking, specifically their understanding in area and volume. These students, although just one year apart, have exhibited a wide range of maturity in understanding the concepts of area and volume from exploring tasks designed in three categories: Base Ten Blocks, Tangrams, questions on 2-D and 3-D. Maturity, in this research, is defined as level of understanding of a certain concept. Below are discussions regarding participants' different levels of understanding in area and volume as well as multiplicative structure in 2-D and 3-D settings.

AREA

For Yunik, "area" is a term that has little meaning to her. She believed only congruent figures, figures that have exactly the same size and shapes, had the same space. In Task (B-1), she struggled to believe that different figures could have the same space although they consisted of the same seven tangram pieces.

Mogen had a better understanding of "area" than Yunik. She defined "area" as "inside of the shape." Some of the real life examples given were space occupied by the floor, wall, park, and the board. Although she did attempt to measure the "space" using a ruler, which indicates a linear way of looking at area, she believed that some small pieces of shapes could be used as a measuring tool to determine the area of larger figures with various shapes. Her remarks on Task (B-1), "... because the same amount and the shapes fits each of them" indicates her conservation of interior area, that is, the space occupied by a shape or a figure is independent of its arrangement as demonstrated in Task (B-1).

As with Mogen, Jorn, Emi, Taile, and Elisa also recognized that the arrangement or position of the tangram pieces could yield different appearance but still hold the same area. Besides having constructed the concept of interior area, the latter four participants extended their notions of area to tasks in category A. For instance, they multiplied the numbers of unit squares in a row and a column to determine the total unit squares that appear on one surface of Diene's blocks. With these applications, Jorn, Emi, Taile, and Elisa have demonstrated a better grasp of the concept of interior area.

Edem's ability to solve Task (A-10) sets him apart from other participants. His approach in that task strongly reflects his construction of the concept of area, that space exists in its own right whether or not it is occupied. As explained earlier in Edem's analysis section, he understood that area was not just the number of 1 by 1 tiles needed to fill in some shaded regions, but the space occupied by those tiles.

VOLUME

Yunik's view of a "cube" is a "box" consisting of six separate sides. Her construction of the "cube" as six uncoordinated layers is demonstrated consistently in all tasks using Diene's blocks. Using 6 flats of 10 x 10 x 1 each to build a frame of the 10 x 10 x 10 cube is one clear example of her construction. As for Mogen, she had difficulty in seeing all outer layers of a 10 x 10 x 10 cube connecting to one another vertically or horizontally. Her construction of a 3-D cube array is also a "frame" of six uncoordinated surfaces of unit cubes. Their construction at this stage are paralleled to Battista and Clements' findings (1998) which suggest that students' initial conception of a 3-D cube arrays is an uncoordinated set of views.

Battista and Clements (1996) also found that many students "conceive of their personally constructed procedure as having two distinct steps" (p. 291). They see 3-D cubes as more than just an uncoordinated set of faces . First, they determine the number of small cubes (1 x 1 x 1) in a layer, then the number of layers of the 3-D block. Emi and Jorn applied this strategy to most of the tasks in category A. In addition, both believed that 3-D objects have "sides and shapes," but their interpretations in Task (C-1) still did not show clear distinctions between 2-D and 3-D. Jorn believed that a strand of hair and a piece of paper were 2-D if they were held "straight" but 3-D if they were in wave-like form. Believing that the dimensions of the objects change with curving or bending suggest that he had not yet completely grasped the idea of interior volume. Their strategies used in tasks of category A also indicate that they tended to think in 2-D rather than 3-D. As described by Battista and Clements (1996), these students are in a transition stage, "whose restructuring is local rather than global" (p. 258).

Both Elisa and Taile frequently used a layering strategy in approaching tasks associated with 3-D cubes. According to Battista and Clement (1996, p. 290), "Those who complete a global restructuring of the array" implement layering strategies. Their abilities in building a thousand-roomed hotel of 20 x 10 x 5 in Task (A-5) reflect their notion of conservation of interior volume, which explains that the "room space" contained in the blocks does not change as the exterior dimensions varies (Copeland, 1984). Although both were able to view the big cube (10 x 10 x 10) as one coherent block as well as connecting layers which made up the block, Taile's strategy in Task (A-2) and his ability to 'see' by himself that every layer shared a long (10 x 1 x 1) vertically and horizontally indicate that his understanding of volume is more "mature" than Elisa's.

Edem's enumeration strategies in approaching 3-D tasks in category A indicate his ability to conceptualize a 3-D cube as one coherent, integrated, and coordinated block. In Task (A-2), he recognized overlapping or connecting edges, and that each unit cube positioned at the eight corners was shared by three surfaces. His drawing of 3-D cubes show that he had a great sense of how "hotels" or blocks look with different dimensions. His more sophisticated reasoning and strategy used in Task (A-11) as well as his constructions of 1-D, 2-D, and 3-D ideas clearly indicate his maturity and set him apart from other participants. For instance, he argued that his drawing of a square with added depth was 2-D although it "appears to be 3-D [because] it doesn't have a real depth." His explanation of a geometric line as 1-D, "... I can draw to represent geometric line but it will have some width" is another indication. Based on his mathematical experiences and interactions throughout the interview sessions, I believe Edem has constructed the concept of volume, the space that exists in its own right rather than just the interior occupied by exterior dimensions (Piaget, 1960).

MULTIPLICATIVE STRUCTURE IN AREA (2-D) AND VOLUME (3-D)

Although these students' geometric activities were designed and analyzed for evidence of concepts of area and volume, the data suggests that students who were unable to think multiplicatively or were at an early stage of multiplicative thinking had difficulty in viewing 3-D cubes as coherent, coordinating sets of unit cubes, and thus struggled in solving tasks in category A which involved base ten blocks.

The construction of the idea of area and volume parallels the construction of multiplicative structure in 2-D and 3-D settings respectively. This parallel is elaborated below.

The "X" or any "square" in Figure 38 represents an element of a row and the column associated with that row. Hence, in order to determine the area or the total number of "squares" of the surface by counting the number of squares across and down (or up) from any corner (marked "O"), a person needs to be able to look at "O" as a member of the row and column associated with it simultaneously. I call this "multiplicative structure in a 2-D setting." Similarly, X in Figure 39 is an element of the row as well as the column and the "depth" associated with it. For instance, to determine the total unit cubes of the 3-D block in Figure 39 by counting across, downwards, and backwards from the same position (Y in Figure 39), a person needs to be able to think of that unit cube "Y" as a member of the length, width, and height simultaneously. This is a complex and sophisticated multiplicative thinking. I call this "multiplicative structure in 3-D setting."

During the course of exploring tasks in category A, Yunik and Mogen had demonstrated their additive thinking rather than multiplicative thinking. Reynolds and Wheatley (1996) in their discussion of the emergence of multiplicative reasoning argue that: "The construction of an element as simultaneously a member of a row and a column in an array is important if students are to think beyond a repeated addition model" (p. 328). Both Yunik and Mogen had not developed that construction and thus they determined the surface area by adding repeatedly, that is counting row by row or column by column. They saw a "square" as a member of a row or a column, but not both simultaneously, which is the thinking of the coordination of composite units, a necessary component in multiplicative reasoning (Steffe, 1994).

Although Emi was able to apply a layering strategy in tasks in category A, she could not see that the "squares" that appear on the surface of the 3-D block represent unit cubes as well as 10-unit longs (10 x 1 x 1). Her strategy of approaching the task step by step, that is, finding the number of longs in a flat (10 x 10 x 1) and then multiplying it by the number of flats in the big cube, indicates that she was in a transition stage to multiplicative thinking in 2-D setting.

Jorn had developed multiplicative structure in 2-D setting. This was evidenced by his ability to determine area by seeing its relationships in terms of length and width, as well as to think of "squares" as both unit cubes and 10-unit cubes or longs (10 x 1 x 1) simultaneously. However, his viewing of the big cube (10 x 10 x 10) in terms of layers rather than one coherent unit of length, width, and height indicates the absence of the construction of multiplicative structure in 3-D setting. This further explains why he struggled in seeing "overlapping" or "sharing" edges of 3-D cubes when solving tasks in category A.

As for Elisa and Taile, they were able to determine area in terms of length and width with ease. This indicates that they have developed multiplicative structure in a 2-D setting. Although they implemented layering strategy in solving 3-D tasks in category A, they recognized that each layer or long was connected to each other vertically or horizontally as Elisa said, "... if you take off one layer, you will be taking out the other." They were in a transition stage to multiplicative structure in 3-D setting. Taile's construction of the 10 x 10 x10 block as decreasing in size and disappearing as one layer all around was removed continuously is another indication of his seeing the 3-D block as one coherent connected set of cubes.

Edem is a solid multiplicative thinker. The data suggests that he has developed multiplicative structure in both 2-D and 3-D settings. His explanation in Task (C-1) shows clear distinctions for 1-D, 2-D, and 3-D. He was able to clarify 1-D, 2-D, and 3-D by seeing their relationships in terms of length (1-D), length and width (2-D), and length and width as well as height (3-D) respectively. Further, his enumeration strategies in solving 3-D tasks in category A indicates that his idea of a 3-D cube is a coherent, integrated, and coordinated block.

REFLECTING ON THIS STUDY

Piaget's idea of developmental stages and Copeland's interpretation of Piaget's work in geometry, specifically in area and volume, made me aware of students' level of understanding when solving geometric tasks. With that awareness, I therefore put forth extra effort to reach their thinking at different levels by using probing questions. Words used varied with each student. As Duckworth (1982) points out, "It is important to vary the words used until they make contact with the [student's] thinking" (p. 27). Moreover, the guiding questions, children's approaches in solving geometric tasks and the extent they apply their knowledge to new tasks, have consistently reminded me to pay attention to students' strategies. The strategies used, from the least to the most sophisticated, reflect their level of understanding in each category as explained in the previous sections.

When students were presented with various tasks, I challenged them with probing questions which often have encouraged them to view the problems from different perspectives. They were also given opportunities to compare how different strategies could be used to solve the problems. Mogen's progress in Task (A-6) is one example. When she was asked to determine the number of longs (10 x 1 x 1) in the big cube, she had 160 longs in total, that is, 60 for outside and 100 inside the big cube. After presenting her with 10 flats and asking her a few probing questions, she came up with 100 longs in the big cube and said, " ... I should have counted inside and outside together ... because inside is just with it." These exchanging of ideas and interaction could promote the construction of high-level thinking (Piaget, 1967).

This research does not suggest that students need to learn the concept of multiplication before area and volume or vice-versa. Data indicates that students showed various levels of understanding for the concepts of area, volume, and multiplicative reasoning structure. I believe as students progress from one stage to another, their ideas of area, volume, and multiplicative thinking do mature and become more integrated. The parallelism in the construction of area, volume, and multiplicative reasoning structure as described earlier is one good indication.

This study has important implications for designing classroom instructions. Information gathered from the interaction with students in this research give us, mathematics teachers and educators, a sense of how children go about solving geometric tasks. Knowledge of their levels of understanding as well as strategies used, from the simple to the sophisticated, can help us to design activities that will encourage students' geometric construction. For instance, Diene's blocks are useful tools for exploring concepts of area, volume, and multiplicative structure in 2-D and 3-D settings. Activities on unit coordination can help students in developing the notion of area (Reynolds & Wheatley, 1996). Reynolds and Wheatley (1997) further suggest that, "... posing problems for which the construction of arrays is likely can facilitate the development of multiplicative reasoning."

It is a common practice for teachers to "teach" concepts of area and volume with a brief introduction and then go straight to using formula. Length x Width (formula for determining area) and Length x Width x Height (formula for determining volume) are just symbols used to represent the concepts of area and volume. They are meaningless to students unless they have constructed the idea of area and volume. This may explain why so many elementary education college students memorize formula without understanding. They struggle with the notion of volume and fail to see the connection with Length x Width x Height (the multiplicative structure in a 3-D setting).

The pedagogical implication of this finding is that, before introducing standard algorithms or prescribed rules or formulas for determining area and volume, it is vital to let students explore and investigate the meaning of area, volume, and multiplicative structure through meaningful or worthwhile tasks, " ... tasks that are likely to promote the development of students' understandings of concepts ..." (NCTM, 1991, p. 25). By not allowing students the opportunity for exploration, which is an integral element of intellectual development, we are simply robbing their innate ability to create and construct meaning for themselves. What else can be more important than "teaching" and guiding students to think on their own?

It is reasonable to expect prospective teachers to have grasped the notion of area and volume before they begin their teaching career. However, my experience of teaching mathematics methods and content courses designed for teachers indicates that a large number of the prospective teachers still have vague understanding in area and volume. This further implies the need for teacher education programs to design instructional activities that could provide plenty of opportunities for prospective teachers to build or extend on their current mathematical meanings in area and volume.

The present study, although intensively and extensively conducted, involved only seven students. In the future, I plan to continue to research on children's geometric thinking but on a larger scale. Furthermore, I strongly believe that teachers' understanding of mathematical concepts has a great impact on how they approach teaching. Therefore, future studies should also focus on the prospective teachers' geometric thinking.

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APPENDICES

Appendix A: Sample of Students' Activities

Appendix B: Base Ten Blocks / Diene's Blocks

Appendix C: A Set of Tangrams

Appendix D: Students' Work and Illustrations of Students' Exploration

APPENDIX A

Sample of Students' Activities

1. This is a 10 x 10 x 10 big cube made up of small cubes.

a. How many small cubes are there in this big cube?

b. How many small cubes will be left if you remove one layer of small cubes from all surfaces?

c. How many equal number of layers from each side should you remove in order to have a cube of size 2 x 2 x 2 remaining ?

d. Here is a solid shape. It is made up of 10 small cubes. How many of these do you need to make this big cube (10 x 10 x 10)?

e. Suppose you want to build a "hotel" using these shapes (show students the 10 x 1 solid) This "hotel" is 20 small cubes tall, 10 small cubes long, and 5 small cubes wide.

f. Imagine the big cube is also a hotel, with each small cube being a room. Do both hotels have the same space?

g. If this rectangular block is 5 small cubes wide, 15 small cubes tall, and 13 small cubes long, how many small cubes are there?

h. How many of these 1 by 1 tiles are needed to fill in the shaded part (Figure 36)?

i. If I were to cut across the cube through the "staircase" line, how many small cubes will be left for the lower block/portion? Exactly half, more than half, or less than half of the 1000 small cubes (Figure 15)?

2. This is a set of tangrams.

a. Make a triangle or square with these 5 smaller pieces.

b. Add these 2 big triangles to this 5 pieced-triangle (or 5 pieced-square) to form other shapes such as parallelogram, trapezoid, triangle, rectangle, and square.

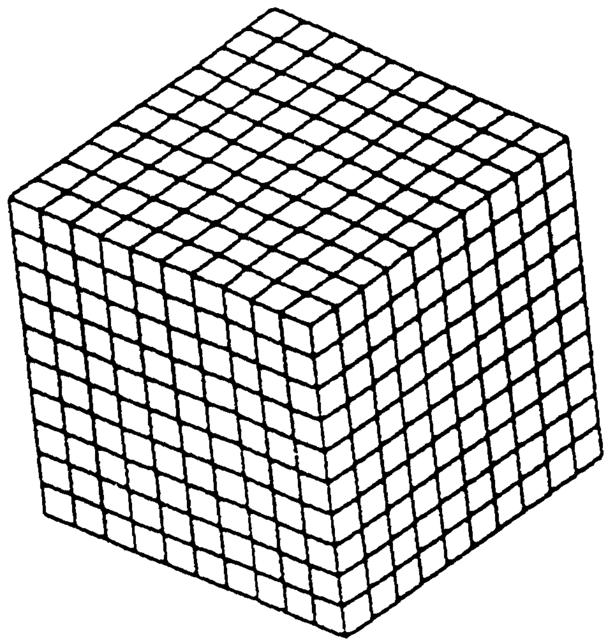
c. Which tangrams pieces have the same area?

d. Do these shapes in Figure 6-10 have the same space?

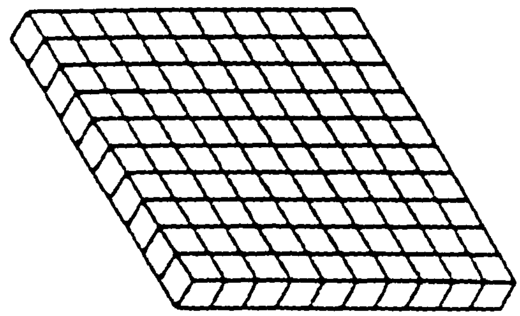
3. What are 2-D and 3-D?

APPENDIX B

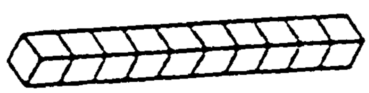
Base Ten Blocks / Diene's Blocks



big cube (10 x 10 x 10)

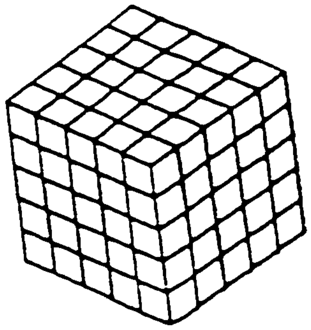


flat (10 x 10 x 1)

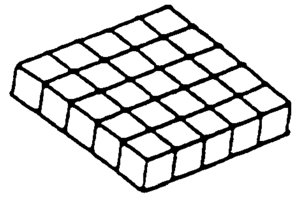


long (10 x 1 x 1)

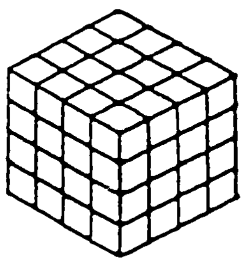
unit cube or small cube
(1 x 1 x 1)



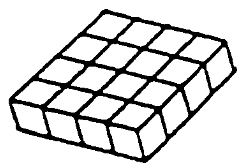
5 x 5 x 5 cube



5 x 5 x 1 flat



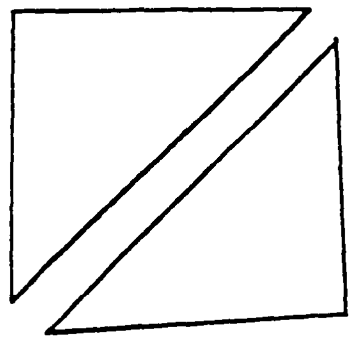
4 x 4 x 4 cube



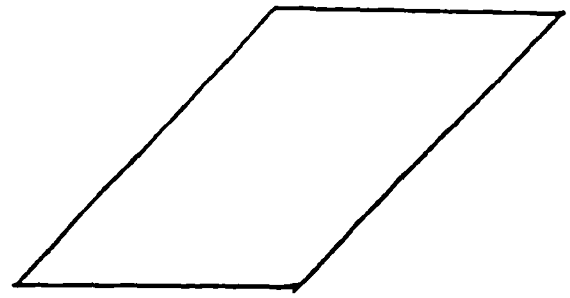
4 x 4 x 1 flat

APPENDIX C

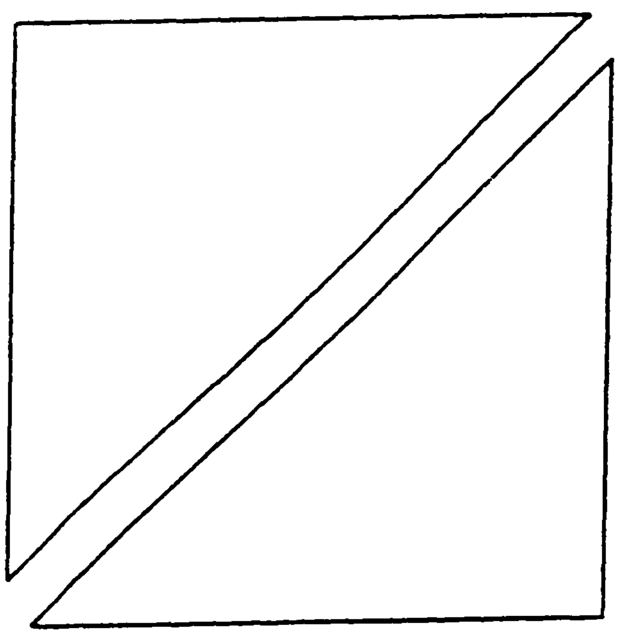
A SET OF TANGRAMS



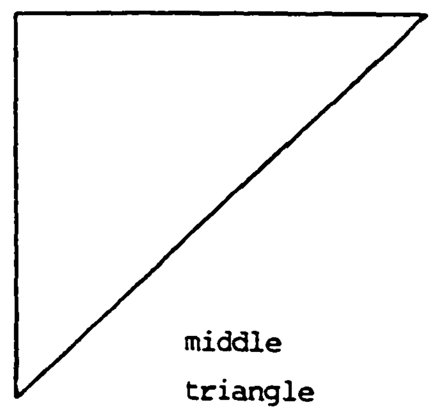
2 small triangles



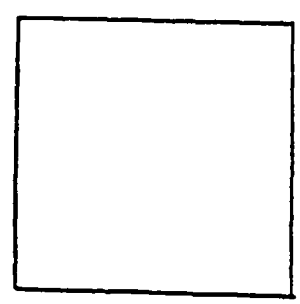
parallelogram



2 large triangles



middle
triangle



square

APPENDIX D

STUDENTS' WORK

AND

ILLUSTRATIONS OF STUDENTS' EXPLORATION

Figure 1 : Yunik's strategy of counting row by row of the outer layer

of the big cube

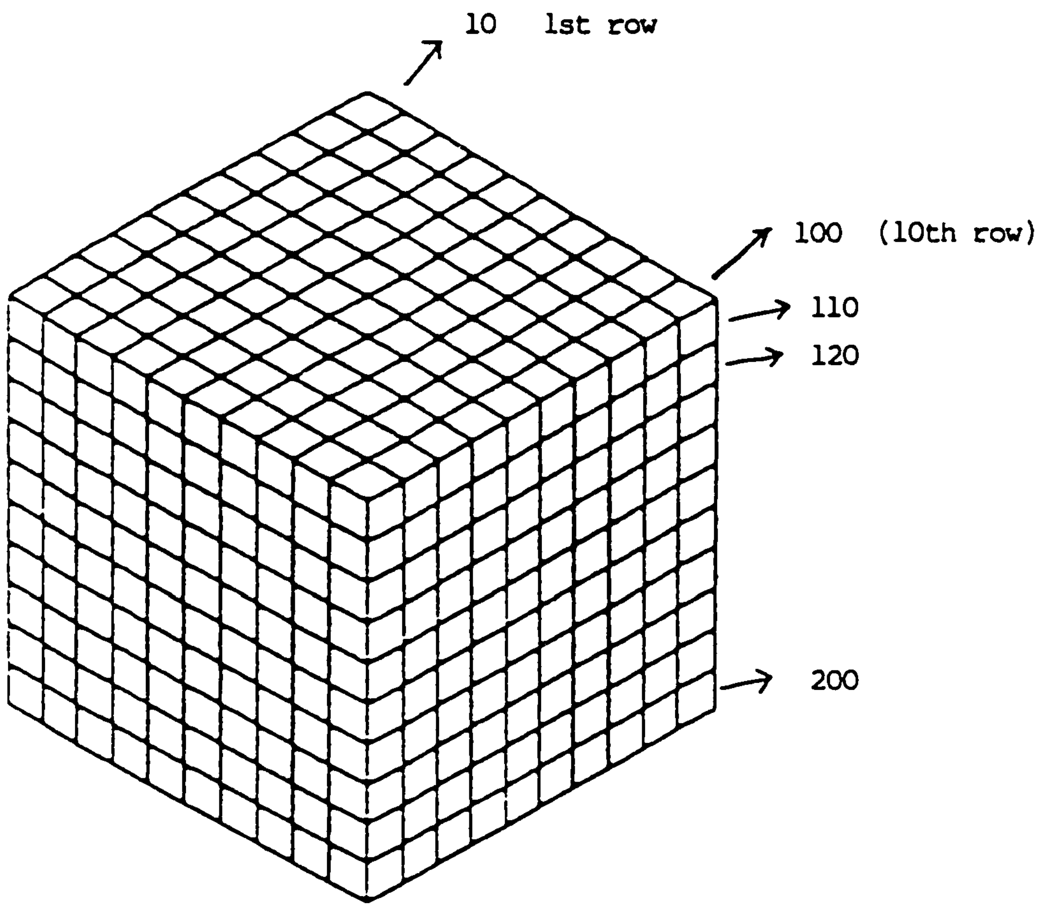


Figure 2 : Top and side surfaces share a long

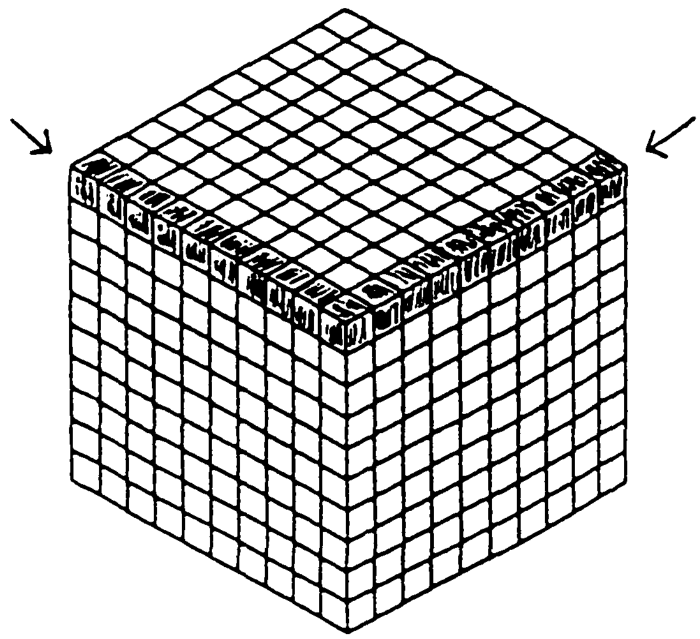


Figure 3 : Comparing the number of rooms in Hotel A (big cube) and Hotel B (2 stacks of 5 flats)

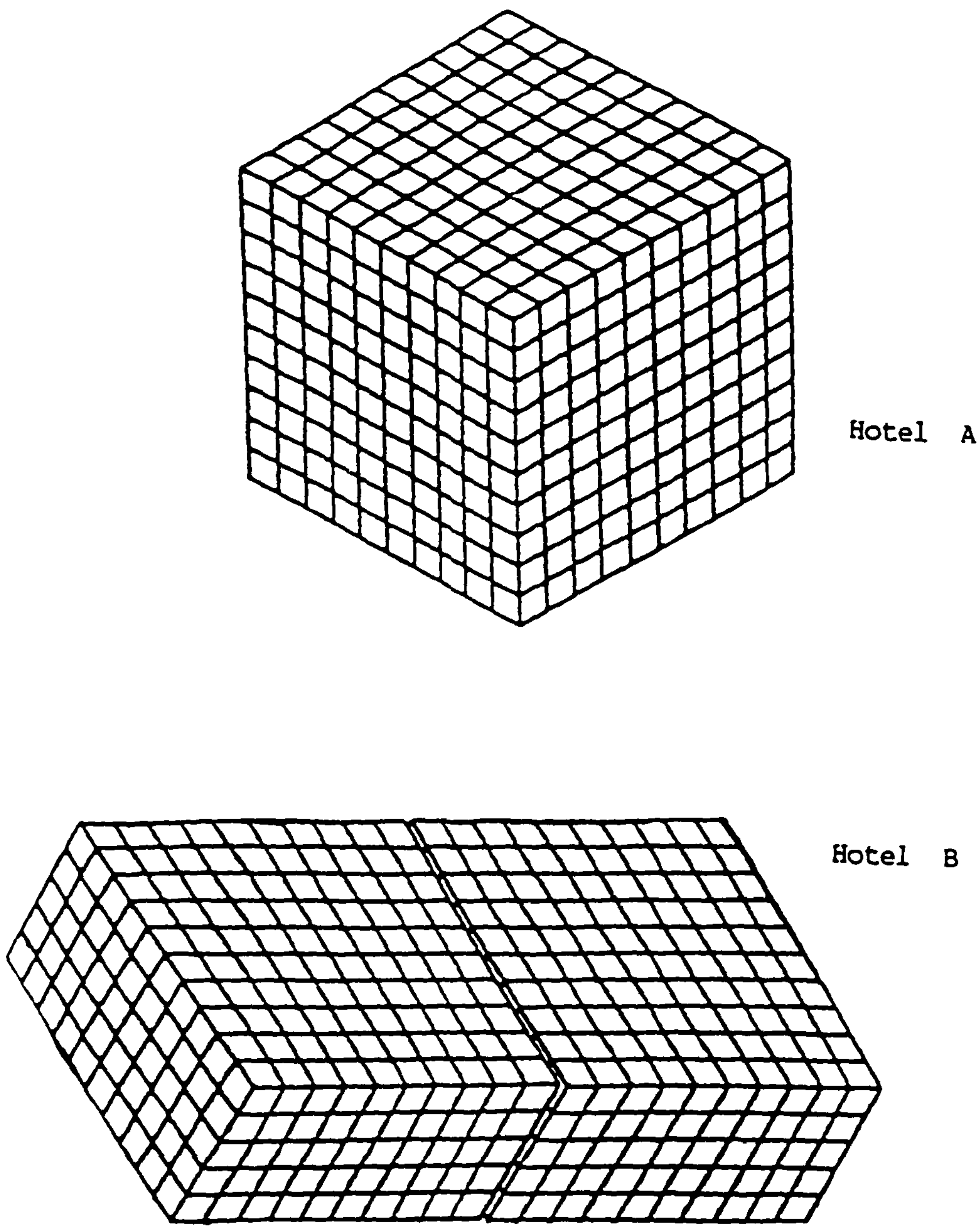


Figure 4 : Placing 4 x 4 x 4 cube at the center of the top surface of the 10 x 10 x 10 cube

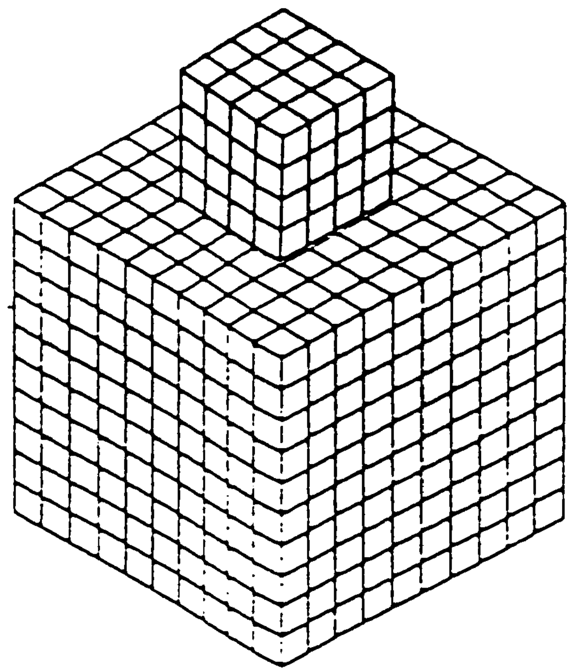


Figure 5 : Portions A and B remain after removing 3 flats from left and right sides of the big cube

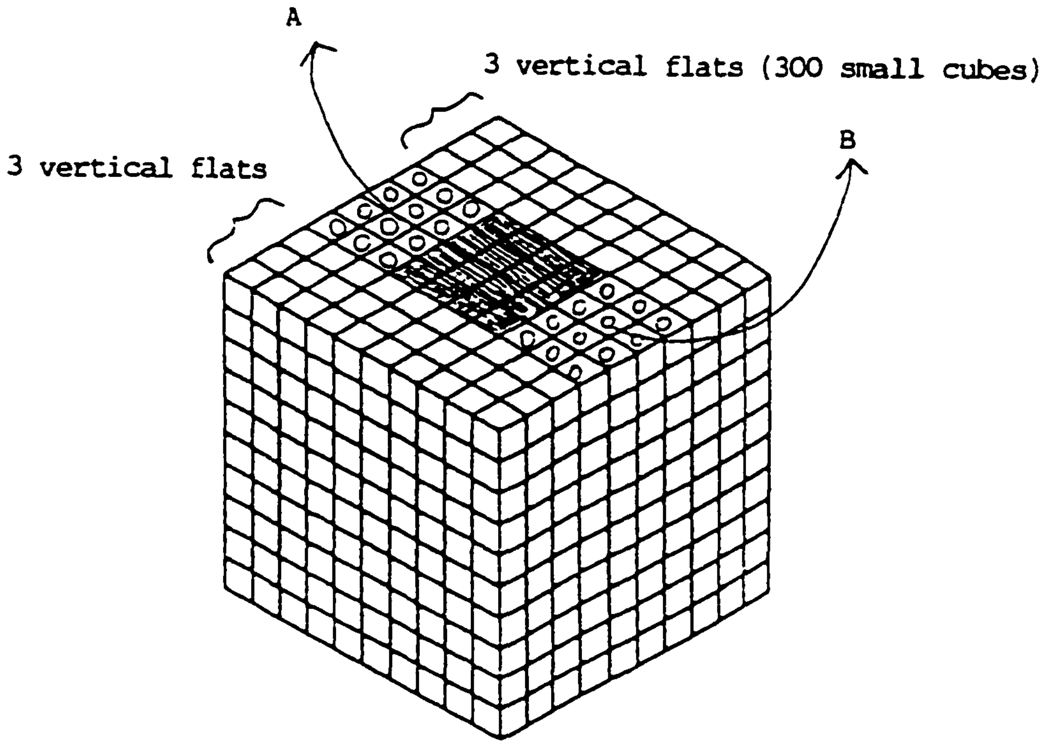


Figure 6 : Seven-pieced tangrams square

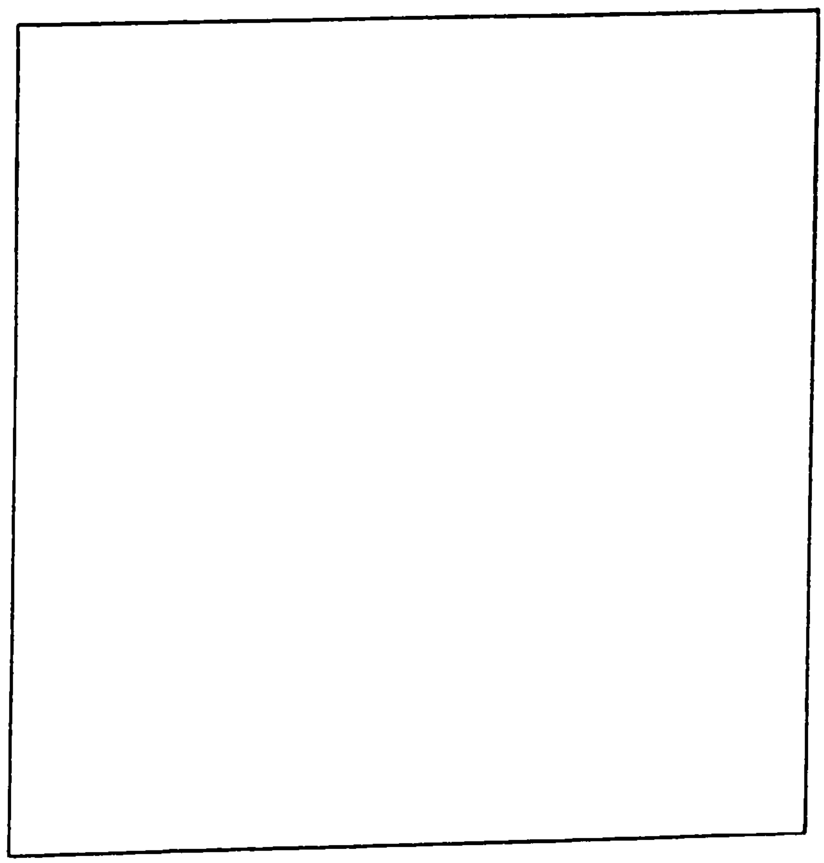


Figure 7 : Seven-pieced tangrams rectangle

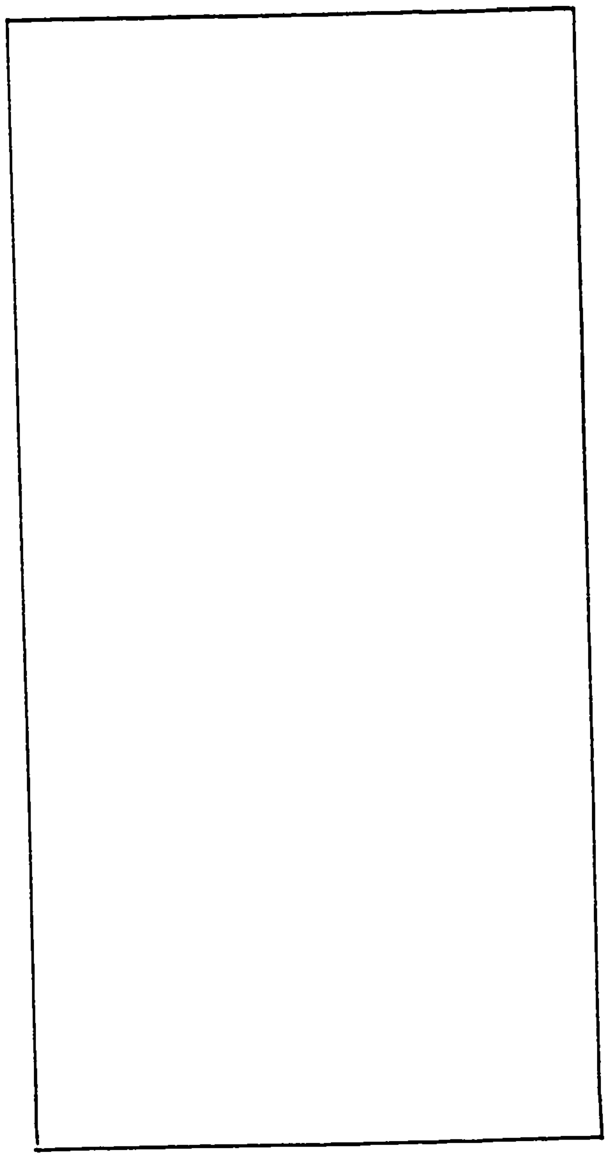


Figure 8 : Seven-pieced tangrams triangle

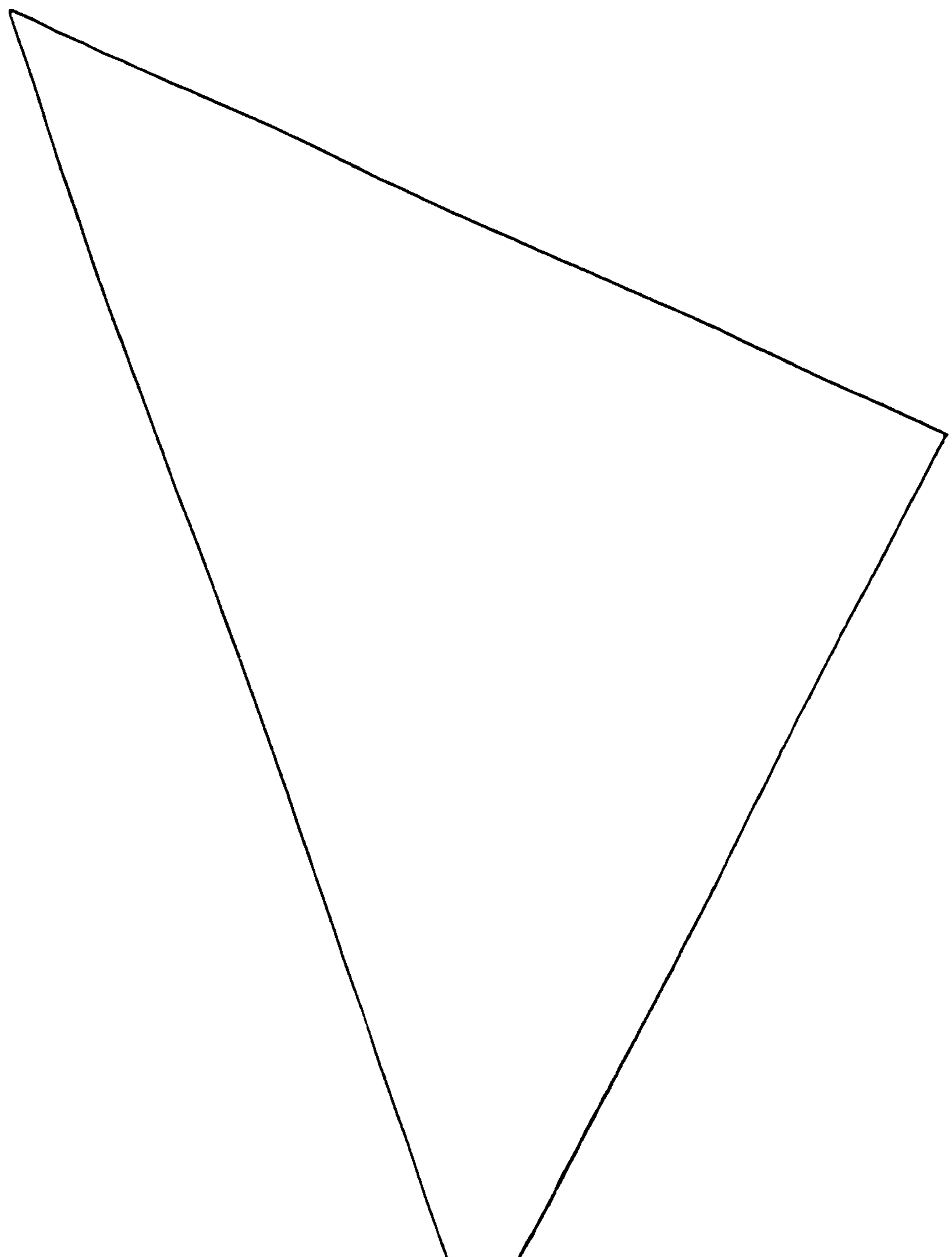


Figure 9 : Seven-pieced tangrams parallelogram

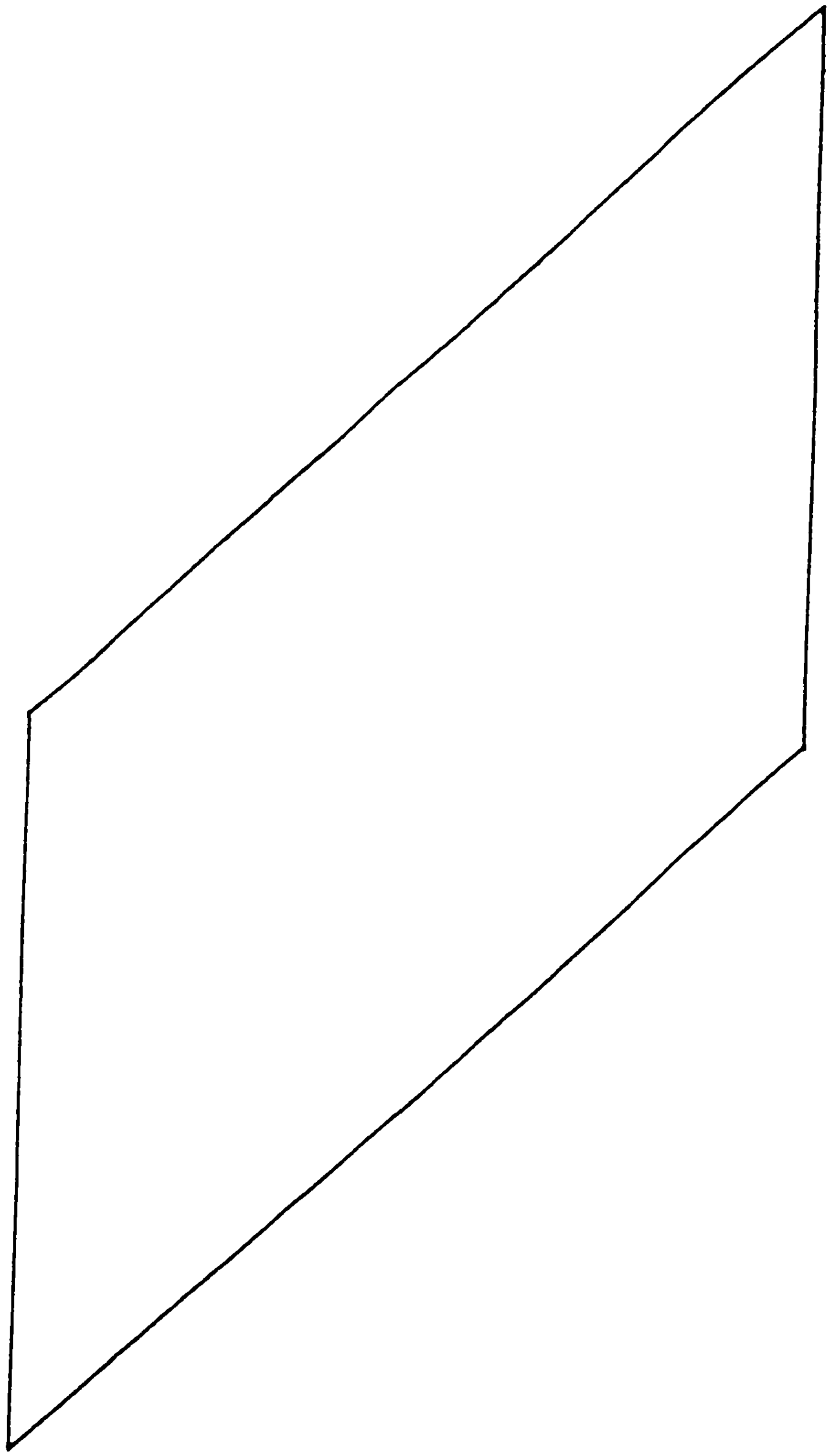


Figure 10 : Seven-pieced tangrams trapezoid

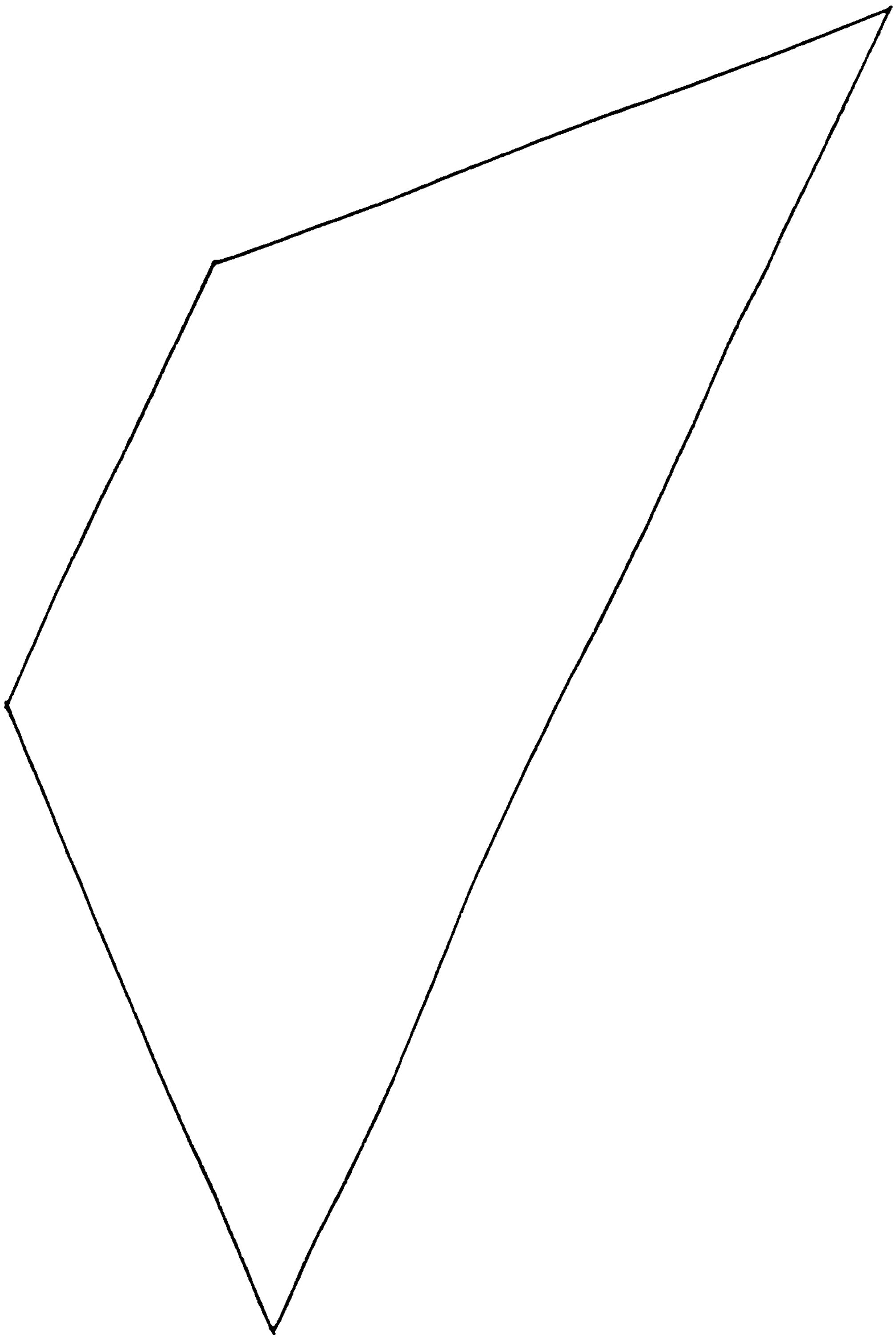


Figure 11 : Making the middle triangle with 2 small triangles

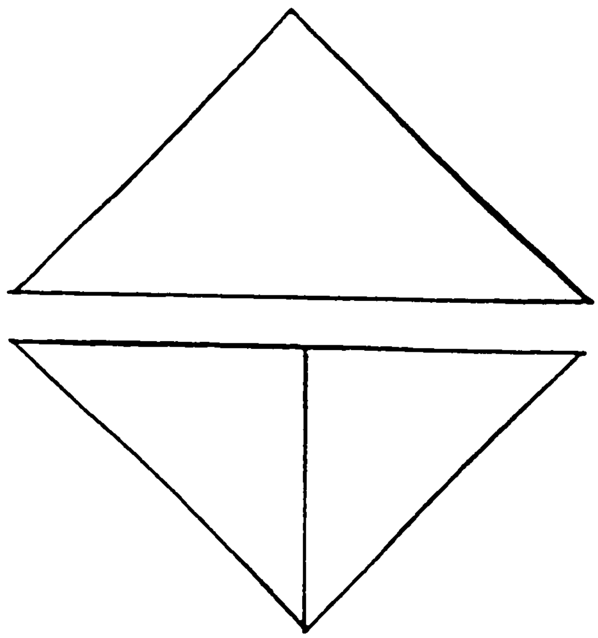


Figure 12 : Making the middle triangle with table edge as a point of reference

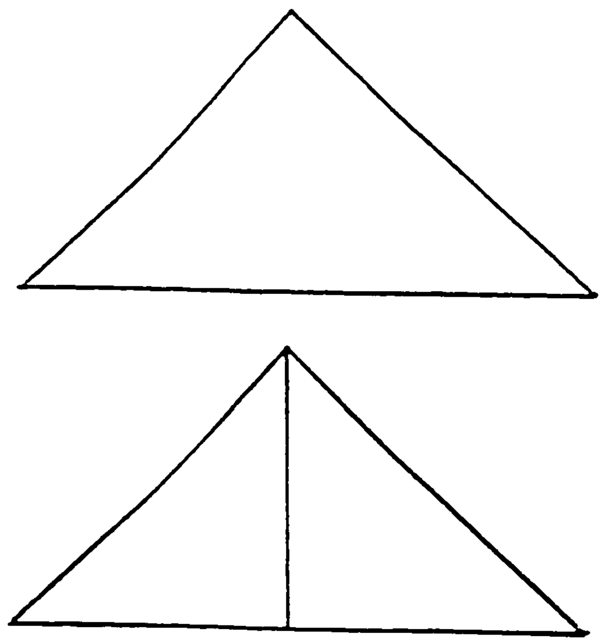


Figure 13 : Five-pieced triangle

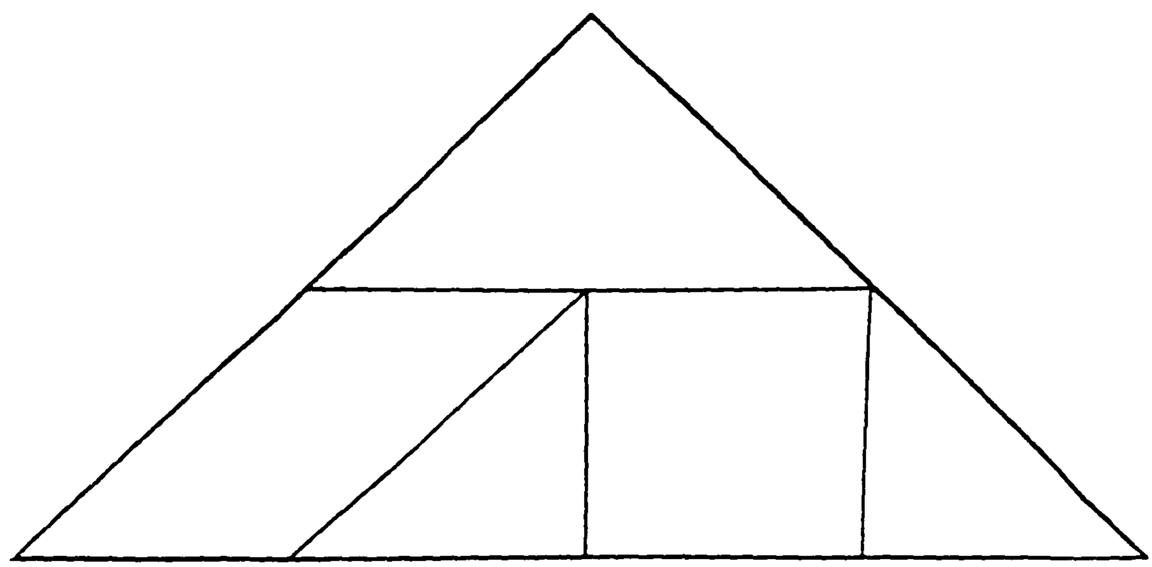


Figure 14 : Counting one surface of the 4 x 4 x 4 cube column by column

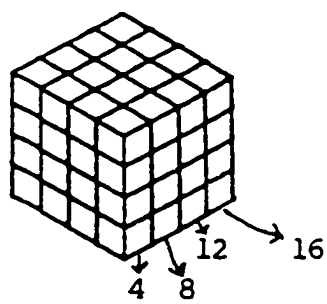


Figure 15 : Determining the number of small cubes left in portion A after removing portion B

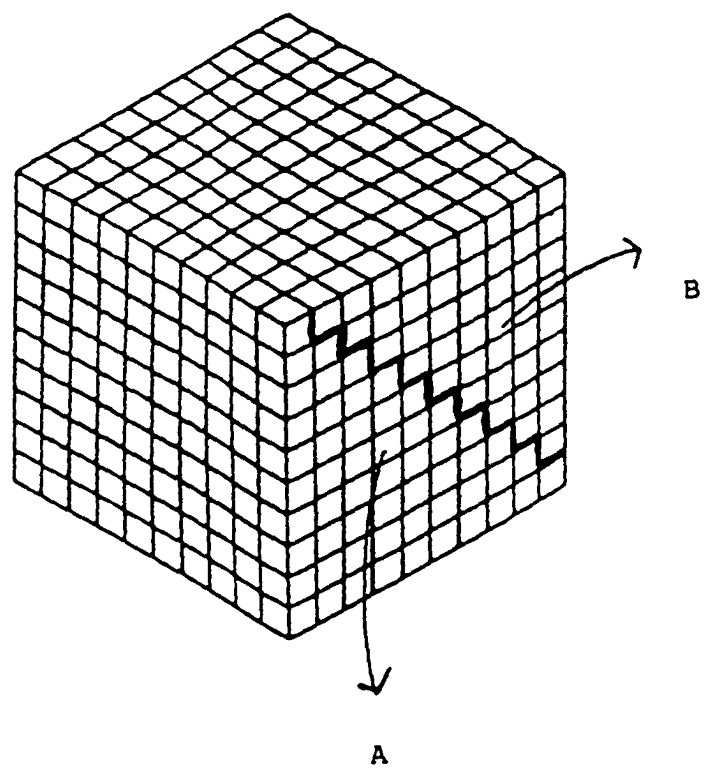


Figure 16 : Mogen's drawing of a cube as an example of 3-D

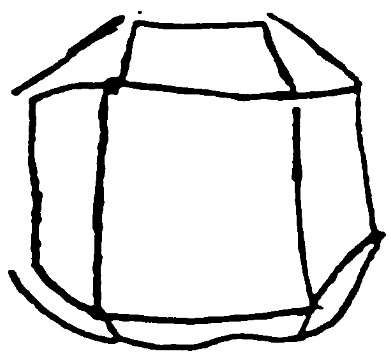


Figure 17 : Determining the number of small cubes in a flat by counting 10 small cubes of the first row, and then 10 rows down

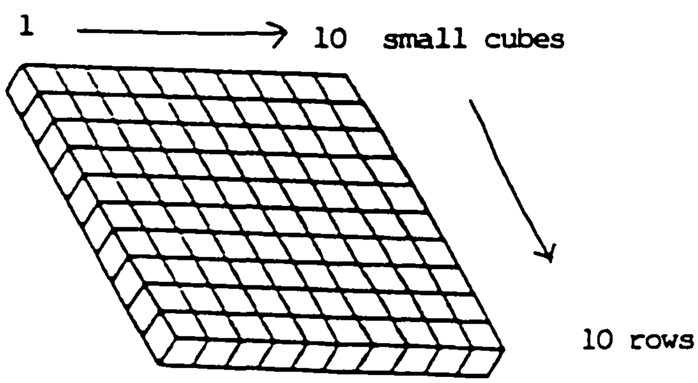


Figure 18 : Jon's strategy of determining the number of small cubes left after removing one layer all around the big cube

100
90
81
81
72 $8 \times 9 = 72$
64 $8 \times 8 = 64$

512 left

Figure 19 : The shaded strips are parts not occupied by eight 4 x 4 x 4 cubes positioned at top and bottom corners

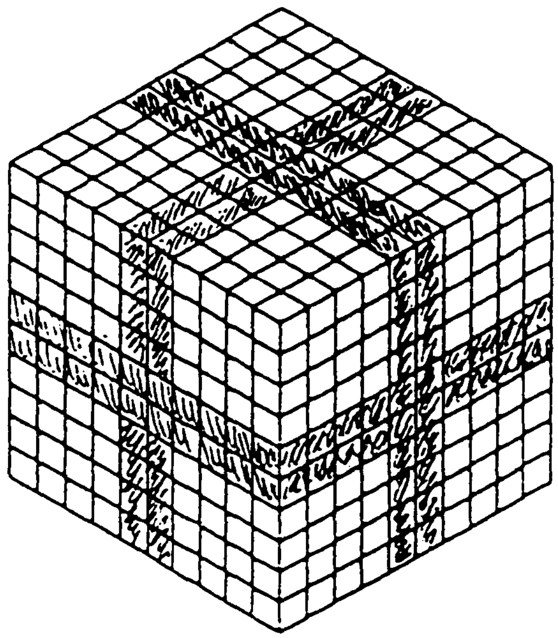


Figure 20 : Determining the number of rooms in a hotel (20 x 10 x 5) with a 20 x 10 x 10 block given

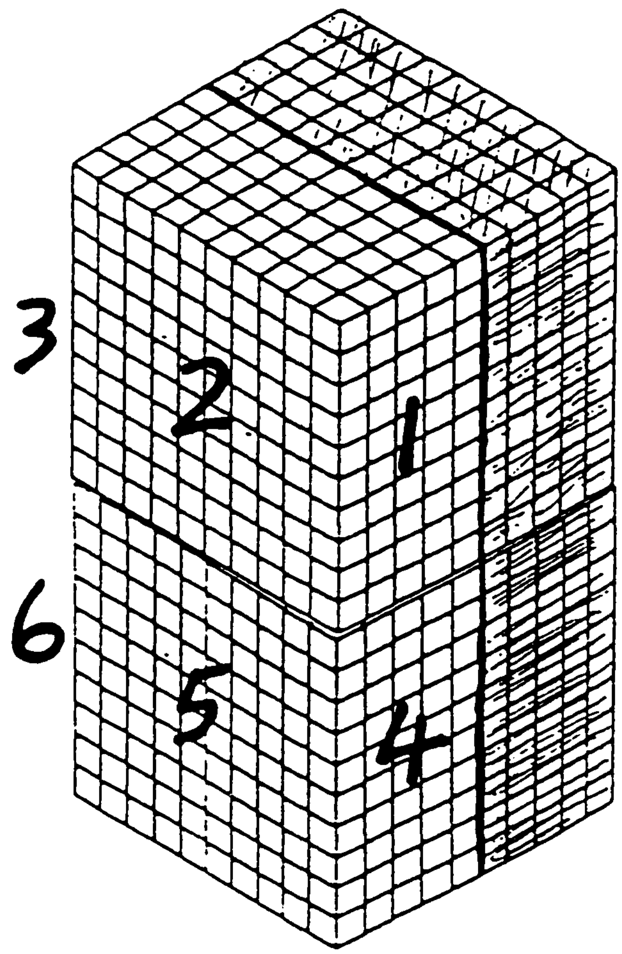
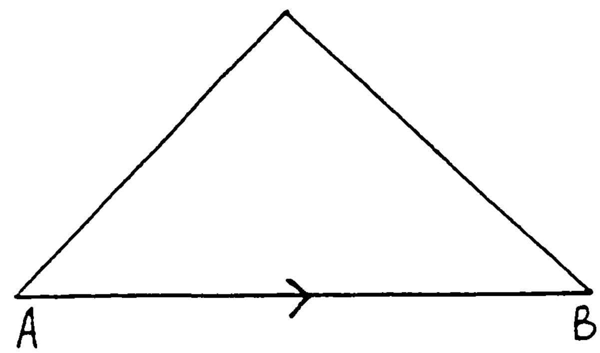


Figure 21 : Determining the area of the parallelogram and the middle triangle by measuring the distance from point A to B with a ruler



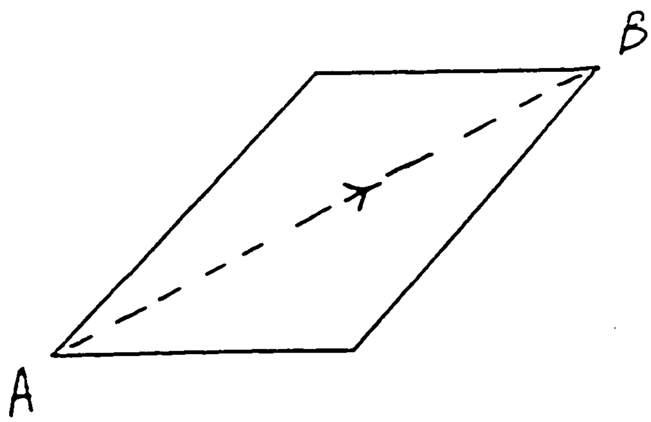
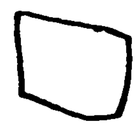
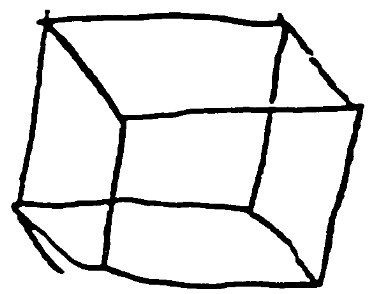


Figure 22 : Jon's drawing of a 2-D (a) and a 3-D (b) square



(a)



(b)

Figure 23 : Emi's 10 x 10 grid with outer edges erased (dotted lines)

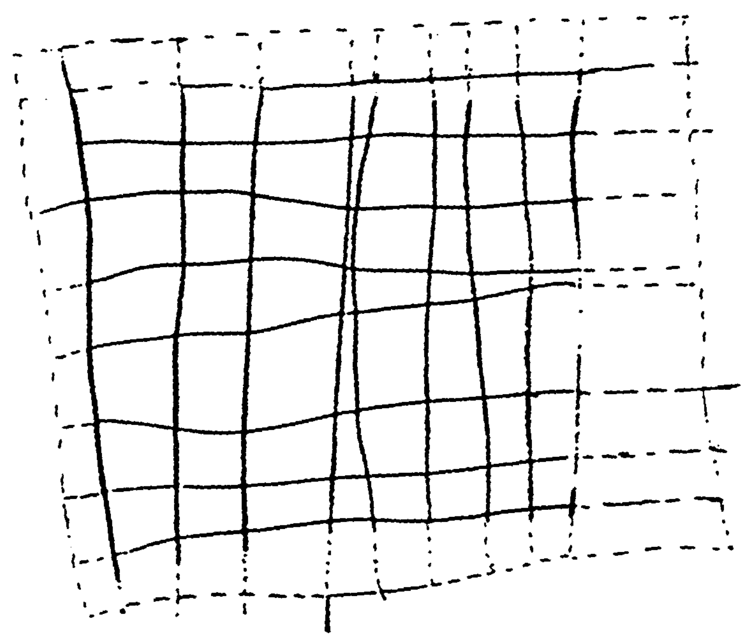


Figure 24 : Emi's strategy in determining the number of small cubes left in a layer. Shaded parts represent outer layer removed.

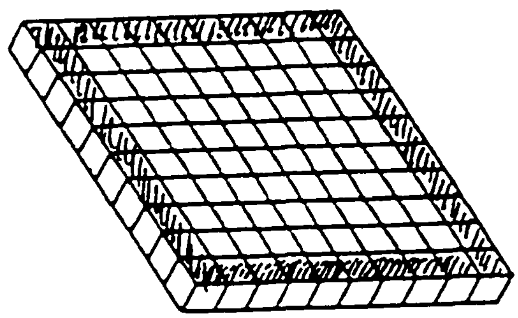


Figure 25 : Emi's drawings of the 10 x 10 x 10 and 20 x 10 x 5 hotels

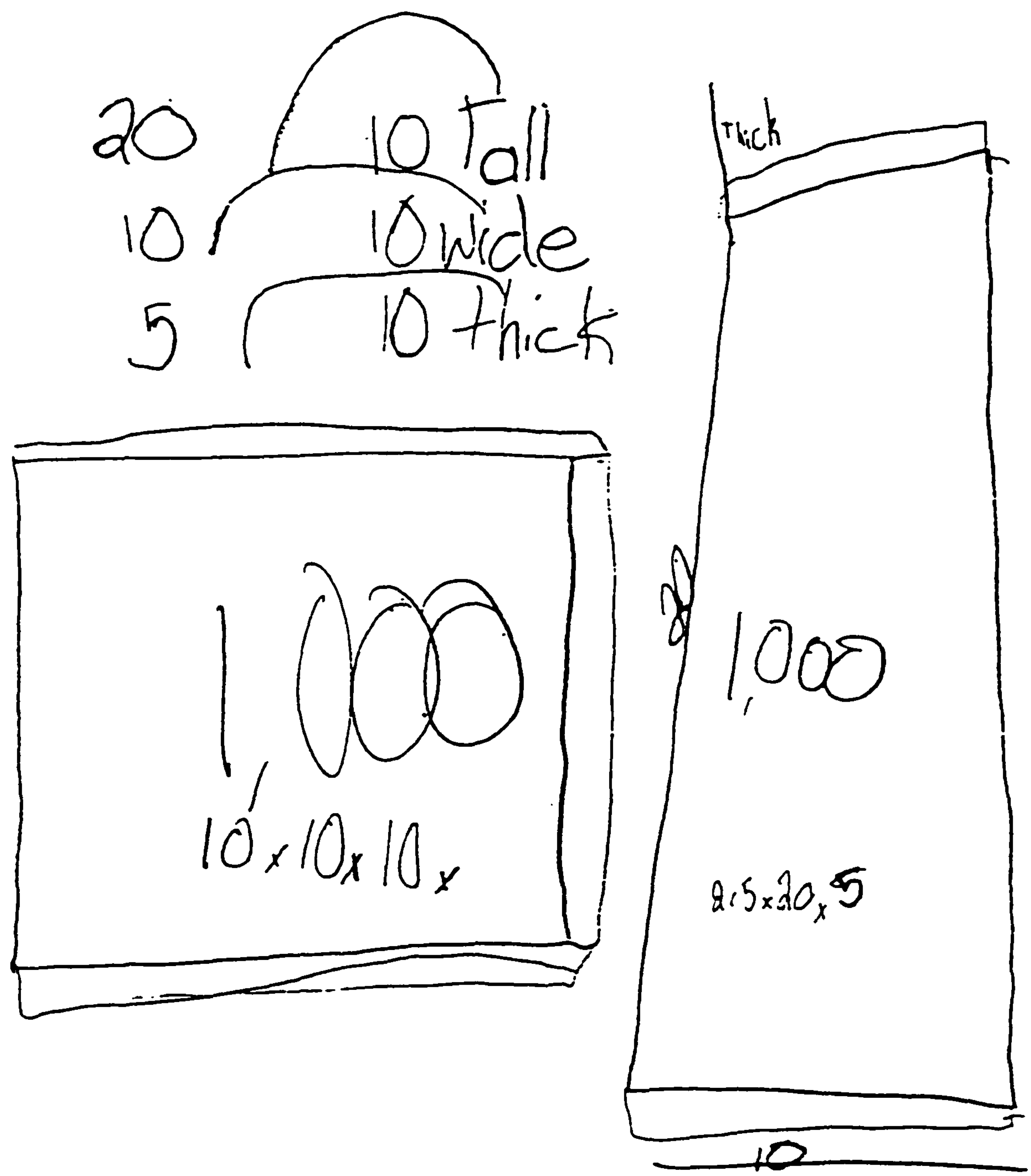


Figure 26 : Placing the shaded portion on top of the other gives a 20 x 10 x 5 block / hotel

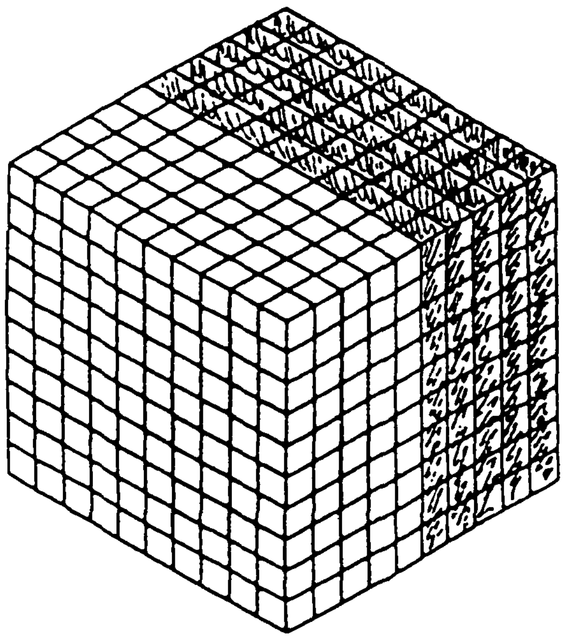


Figure 27 : Emi's illustration of a 20 x 6 x 10 hotel

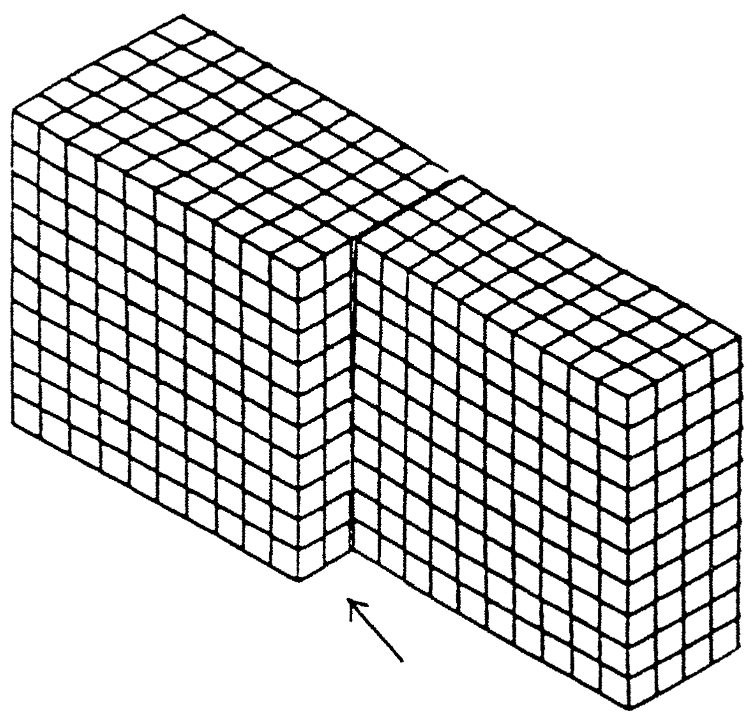


Figure 28 : Taile's strategy of determining the number of small cubes after removing 1 layer all around the big cube (8 layers of 64)

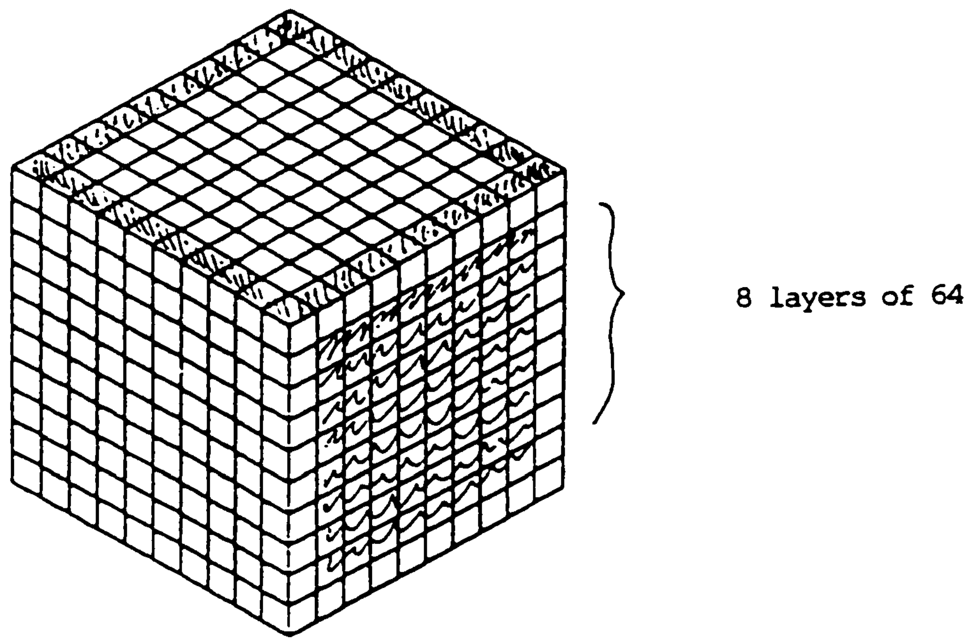


Figure 29 : Determining the number of small cubes left for portion ADC with ABCD represents the front view of the big cube.

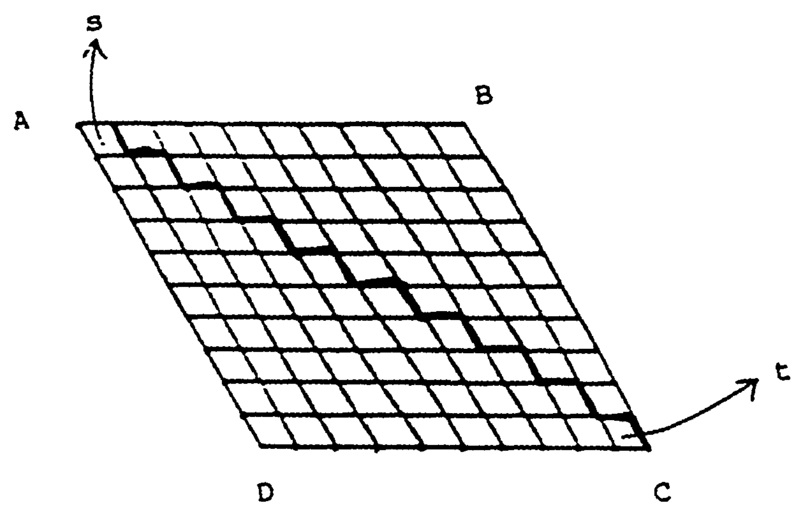


Figure 30 : Comparing the areas of the middle triangle, square, and parallelogram

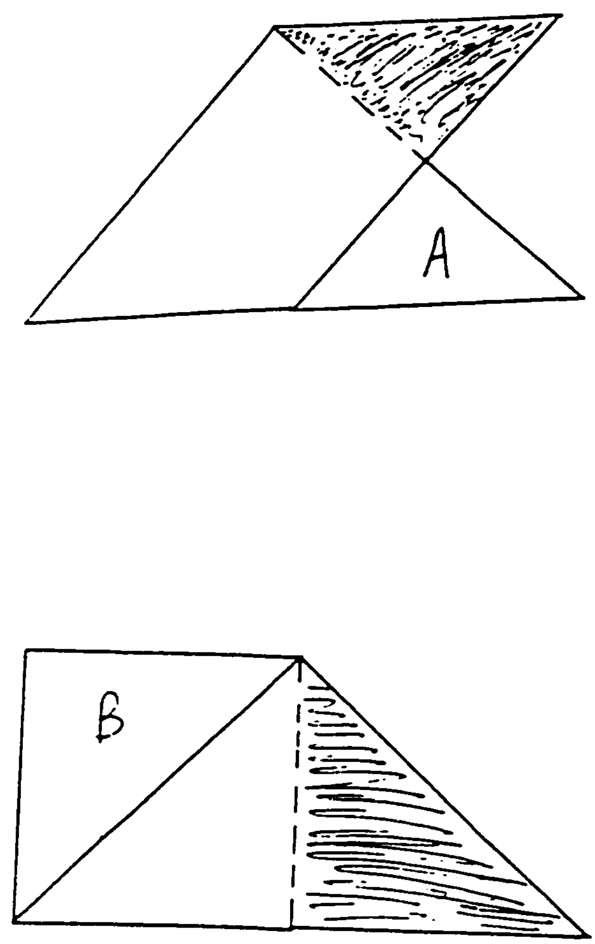
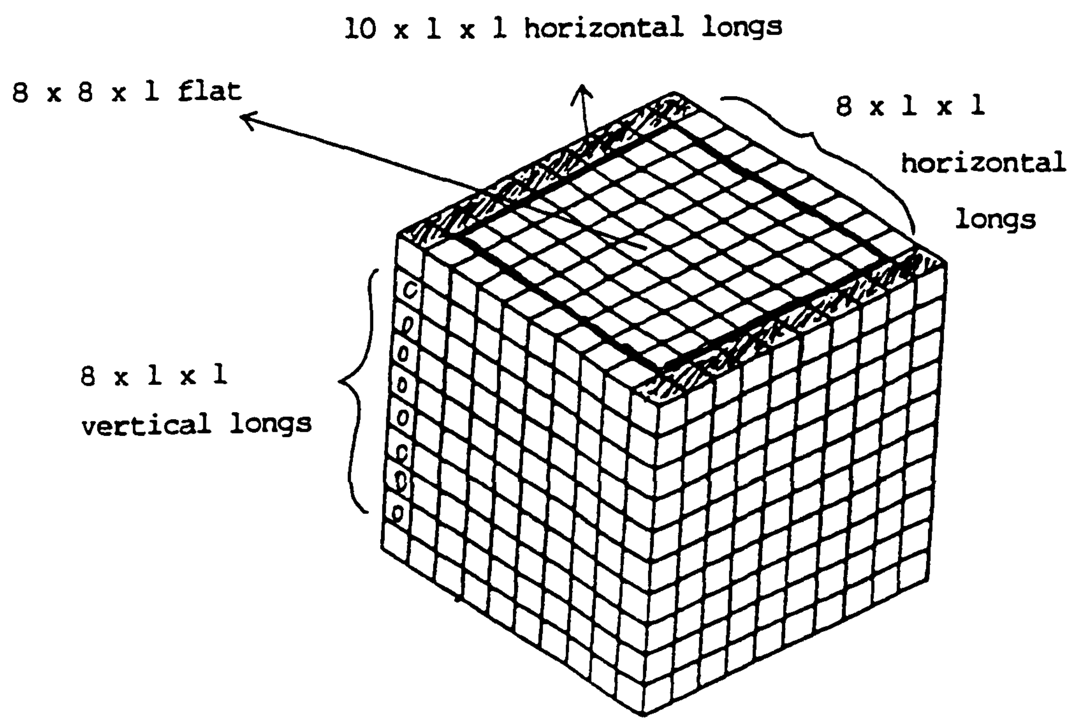


Figure 31 : Edem's strategy of determining the number of small cubes on the outer layer of the big cube



6 faces of (8 x 8 x 1) flat	=	384
4 horizontal longs (10 x 1 x 1)	=	40
4 vertical longs (8 x 1 x 1)	=	32
4 horizontal longs (8 x 1 x 1)	=	32
		488

Figure 32 : Edem's drawing of 2 x 2 x 2 cube

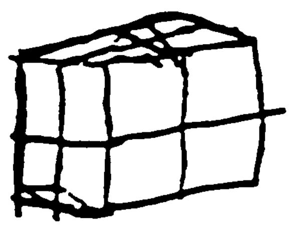


Figure 33 : Edem's drawing of a 20 x 10 x 5 "hotel"

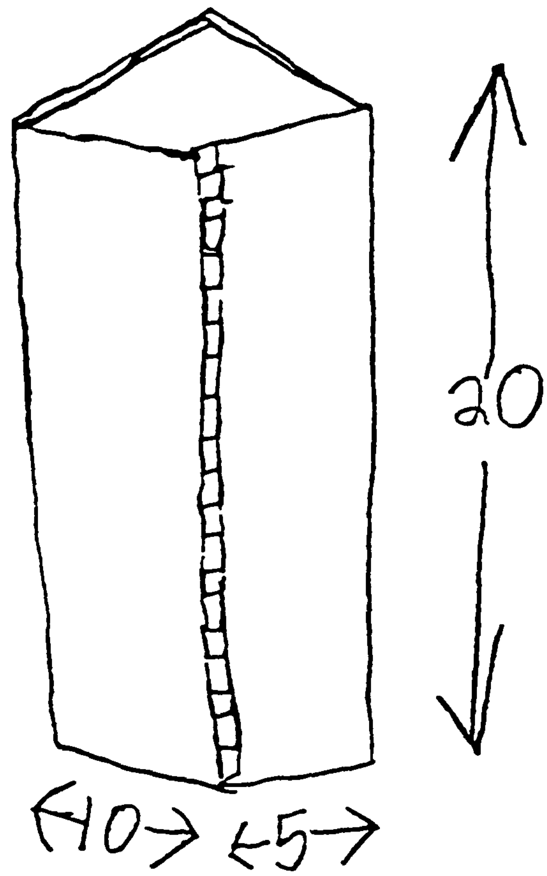


Figure 34 : Edem's drawings of a 40 x 25 x 1 and a 5 x 50 x 4 blocks

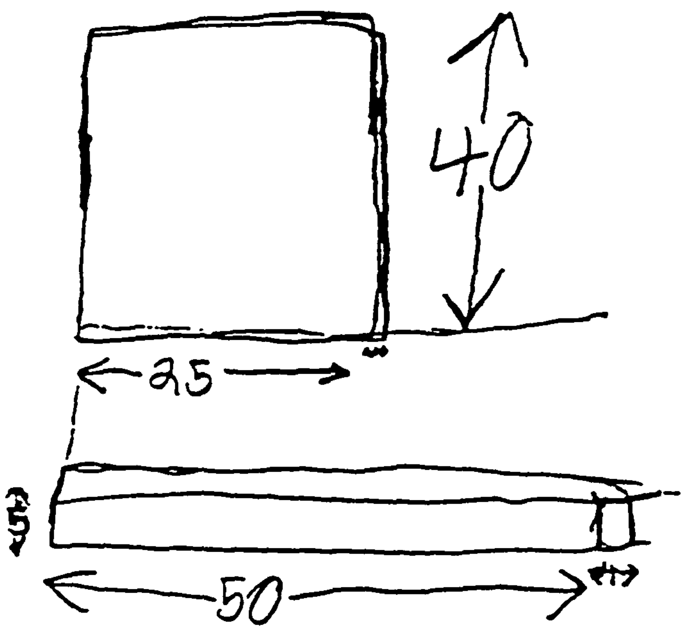


Figure 35 : Edem's 1000-roomed hotels with various dimensions

10	10	10
10	5	20
50	4	5
25	1	40
500	1	2
1	200	5
25	10	4
2	10	50
5	100	2
1	1	1000

Figure 36 : Determining the number of 1 x 1 tiles on the shaded portion

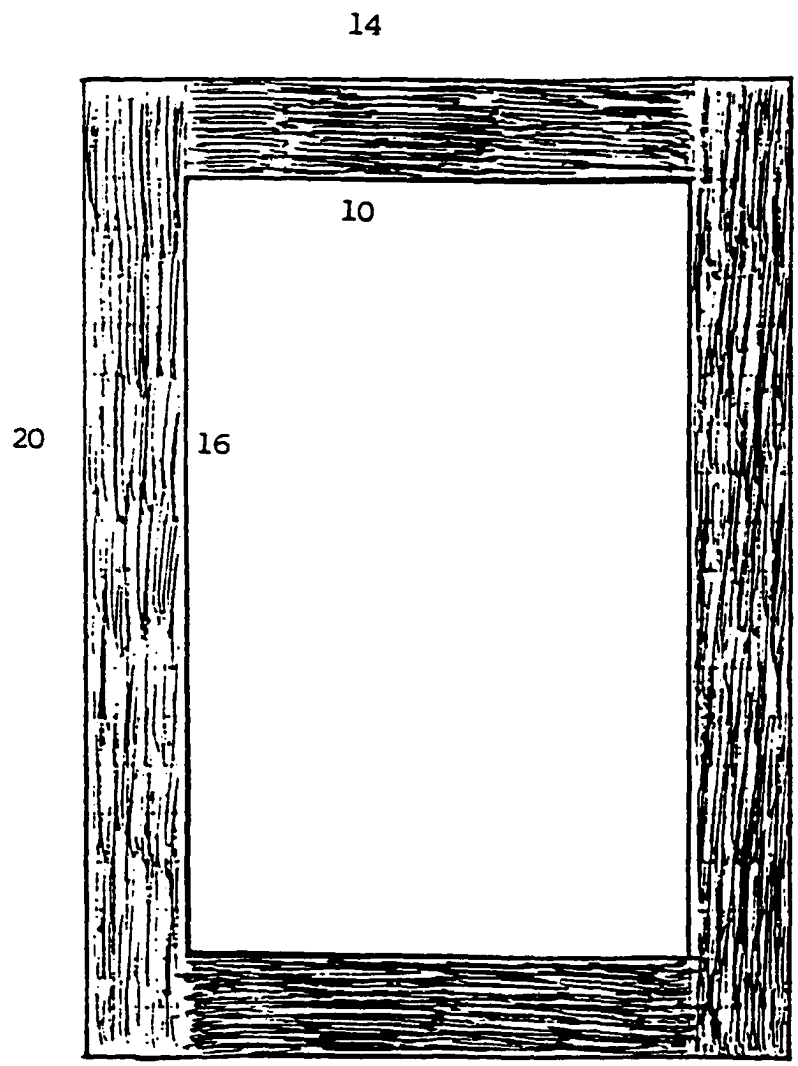


Figure 37 : Edem's strategy of determining the number of small cubes at the lower portion

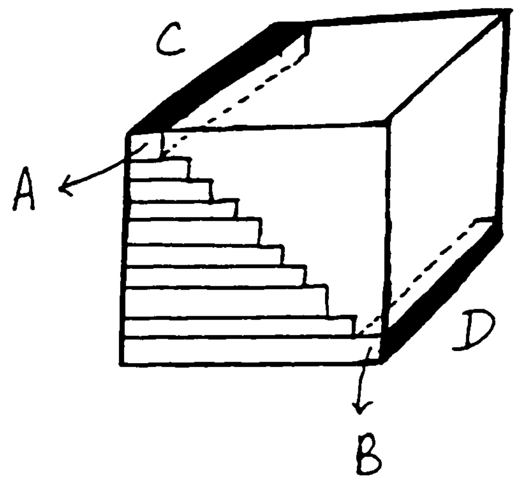


Figure 38 : Multiplicative structure in a 2-D setting

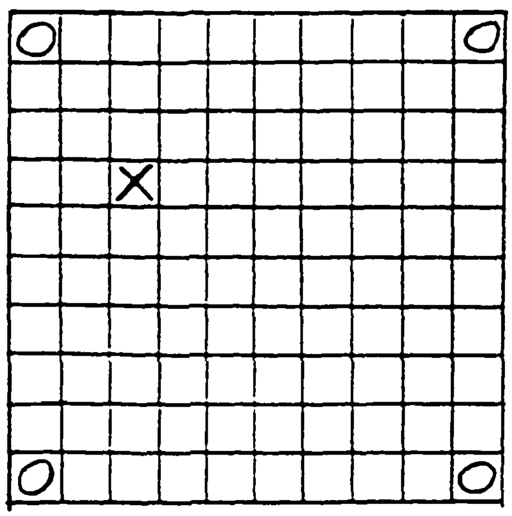


Figure 39 : Multiplicative structure in a 3-D setting

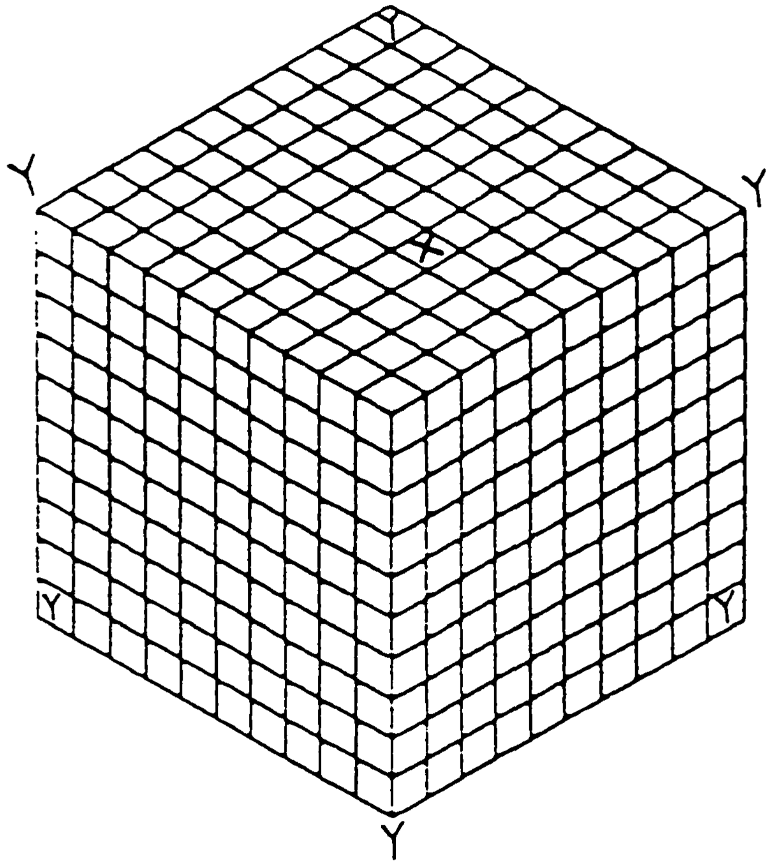
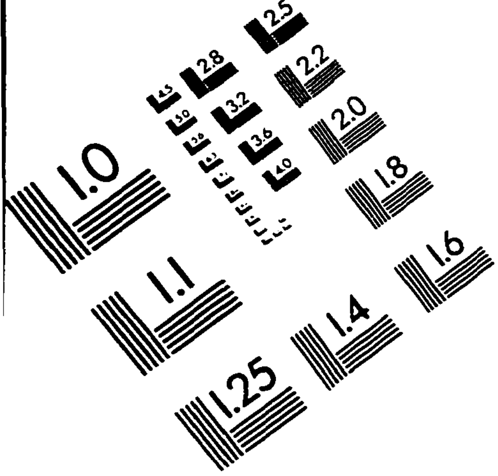
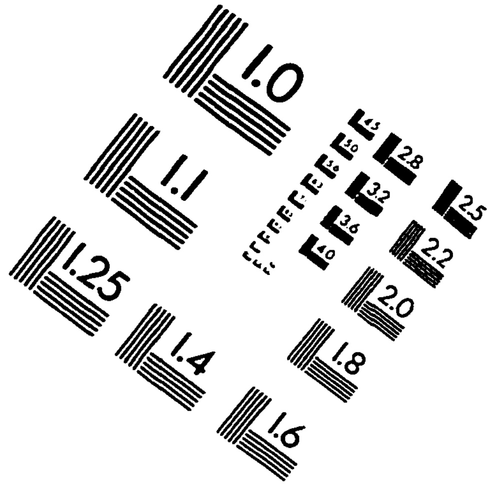
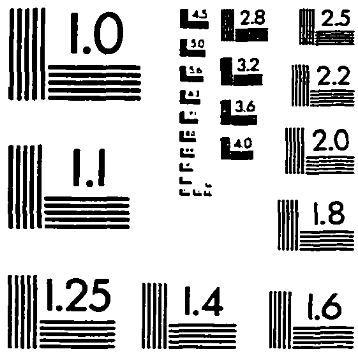


IMAGE EVALUATION TEST TARGET (QA-3)

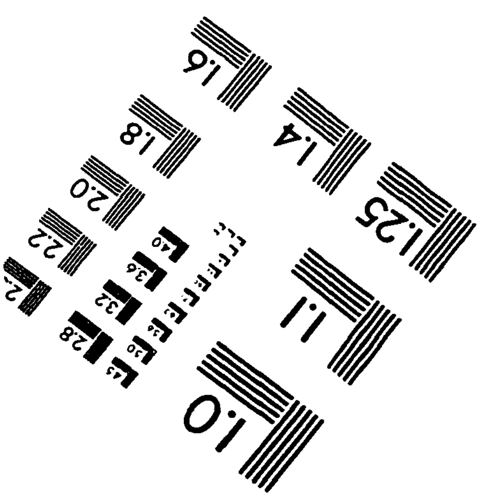


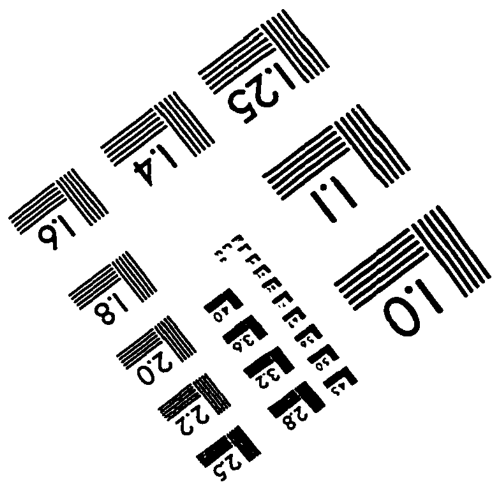




150mm

6″





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