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ONE-DIMENSIONAL RIEMANNIAN FOLIATIONS ON THE HEISENBERG GROUP

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A dissertation APPROVED FOR THE
DEPARTMENT OF MATHEMATICS

BY

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CHAPTER 0

INTRODUCTION

The purpose of this dissertation is to investigate Riemannian foliations on the Heisenberg group endowed with the natural left invariant matric. Our interest in the subject was inspired by the following two fundamental results: the classification of metric fibrations on Euclidean spaces of D.Gromoll and G.Walschap ([**11**]) and the work of D.Gromoll and K.Grove on one-dimensional Riemannian foliations on spaces of constant curvature ([**8**]).

Although some of the ideas we use in order to obtain the main results are valid for Riemannian foliations on general manifolds. our study relies heavily on properties of the Heisenberg group. In the first chapter we summarize these algebraic and geometric properties. We discuss such geometric objects as the isometry group. the Killing vector fields. the geodesics. and the Jacobi vector fields. The geodesic properties are actually essential in obtaining key information about the Riemannian foliations. In section **1.5** we derive the form of the Killing vector fields. Their importance consists in that the nonzero ones correspond to the one-dimensional Riemannian foliations. The form of the Jacobi fields is given in section **1.7** and is yet another important feature needed in our investigation of Riemannian foliations.

We would like to point out that even though most of the results in the first chapter are known, our description of the automorphism group and Killing vector fields corrects the corresponding erroneous results from [**7**].

The second chapter contains an overview of Riemannian foliations and Riemannian submersions. As the two concepts are closely related, we start this chapter with some generalities and remarks on Riemannian submersions. We also mention some results and examples found in the literature. Given our interest in homogeneous foliations, we include in section **2.3** some examples of homogeneous and nonhomogeneous foliations.

The third chapter is dedicated to the study of one-dimensional Riemannian foliation. The synopsis of well-known facts from section **3.1** is followed by the statements of two of the fundamental theorems that are the key ingredients in relating the homogeneity of one-dimensional Riemannian foliations with the properties of the mean curvature form of the leaves. We also reproduce the proof of one of these theorems in order to emphasize the construction of the Killing vector field associated to a one-dimensional Riemannian foliation with closed mean curvature form.

Our main results are listed in chapter four. An essential role in our investigations of Riemannian foliations on the Heisenberg group is played by the position of the vertical bundle with respect to the central direction. The first two propositions provide a partial answer to this problem. Based on these two results we show in Proposition 4 that the vertical bundle of a one-dimensional foliation makes a constant angle with the central direction along each leaf. In Proposition 5 we conclude that the same property is valid for basic vector fields.

Theorem 1 states that every one-dimensional Riemannian foliation on H^{2n+1} is homogeneous. In order to prove the theorem we use the results from section **2.2** and the key geometric properties mentioned above. We would like to remark that, as noted in the literature, D.Gromoll and K.Grove's result on the homogeneity of one-dimensional Riemannian foliations on spaces of constant curvature relies heavily on the constancy of the curvature. As the Heisenberg group has two-planes with both positive and negative curvature, our theorem is a first step toward extending the homogeneity property to more general manifolds.

Using a result of G.Walschap on the homogeneity of codimension one Riemannian foliations on the general Heisenberg group ([**32**], Theorem 4.4.) and Theorem 1 we conclude that any Riemannian foliation on the three-dimensional Heisenberg group is homogeneous. As a consequence of Theorem 1 we also obtain homogeneity of one-dimensional foliations on compact quotients $\Gamma\backslash H^{2n+1}$. We note that this extends a similar result of G.Walschap on three-dimensional nilmanifolds ([**31**]). We conclude the chapter by observing that there are no codimension one Riemannian foliations on $\Gamma\backslash H^{2n+1}$. This is a consequence of Theorem 4.4 in [**32**] and, when compared to the similar situation on Euclidean spaces, it shows a somewhat surprising rigidity property.

CHAPTER 1

THE HEISENBERG GROUP

1.1 Lie algebra considerations

Let $\mathcal{V}$ be a 2n-dimensional vector space and let $\{X_1, \ldots, X_n, Y_1, \ldots, Y_n\}$ be any basis of $\mathcal{V}$. Let $\mathcal{Z}$ be a one-dimensional vector space spanned by some element Z. The bracket relations

$$[X_i, Y_i] = -[Y_i, X_i] = Z, \; 1 \leq i \leq n,$$

and all other brackets zero define a Lie algebra structure on $\mathfrak{h}_{2n+1} = \mathcal{V} \oplus \mathcal{Z}$. With this structure, $\mathfrak{h}_{2n+1}$ is called the (2n+1)-dimensional Heisenberg algebra and the corresponding simply connected Lie group H^{2n+1} is called the (2n+1)-dimensional Heisenberg group.

It is well known that the Lie group exponential map $\exp : \mathfrak{h}_{2n+1} \to H^{2n+1}$ is a diffeomorphism. The group multiplication is given by the Baker-Campbel-Hausdorf formula:

$$\exp(X) \cdot \exp(Y) = \exp(X + Y + \frac{1}{2}[X, Y]).$$

So, if

$$X = \sum_{i=1}^{n} x_i X_i + \sum_{i=1}^{n} y_i Y_i + zZ, \quad \tilde{X} = \sum_{i=1}^{n} \tilde{x}_i X_i + \sum_{i=1}^{n} \tilde{y}_i Y_i + \tilde{z} Z,$$

then

$$\exp(X) \cdot \exp(Y) = \exp(\sum_{i=1}^{n}(x_i + \tilde{x}_i)X_i + \sum_{i=1}^{n}(y_i + \tilde{y}_i)Y_i + (z + \tilde{z} + \frac{1}{2}I(x, \tilde{x}))Z).$$

By definition, if $x = (x_1, \ldots, x_n, y_1, \ldots, y_n), \tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_n, \tilde{y}_1, \ldots \tilde{y}_n) \in \mathbf{R}^{2n}$,

$$I(x, \tilde{x}) = \sum_{i=1}^{n}(x_i \tilde{y}_i - \tilde{x}_i y_i) = xJ\tilde{x}^t,$$

where $J(x_1, \ldots, x_n, y_1, \ldots, y_n) = (y_1, \ldots, y_n, -x_1, \ldots, -x_n)$. Here we identify J with its corresponding matrix in the natural basis of $\mathbf{R}^{2n}$.

Consequently, H^{2n+1} can be regarded as $\mathbf{R}^{2n+1}$ with group operation given by

$$(x, z) \cdot (\tilde{x}, \tilde{z}) = (x + \tilde{x}, z + \tilde{z} + \frac{1}{2}I(x, \tilde{x})),$$

where $x, \tilde{x} \in \mathbf{R}^{2n}$ and $z, \tilde{z} \in \mathbf{R}$. Note that under this identification we have:

$$\exp(\sum_{i=1}^{n} x_i X_i + \sum_{i=1}^{n} y_i Y_i + zZ) = (x_1, \ldots, x_n, y_1, \ldots, y_n, z).$$

1.2 The structure of Aut(H^{2n+1})

In this section we give a description of $\mathrm{Aut}(H^{2n+1})$ as obtained in [**16**]. First, we introduce some notations and discuss some facts that will also be needed later.

Let $\omega = x_1 \wedge y_1 + \cdots + x_n \wedge y_n$. Define

$$O(\omega, 2n) = \{\psi \in \mathrm{Aut}(\mathbf{R}^{2n}) : \psi \cdot \omega = \lambda\omega, \lambda \in \mathbf{R}^*\}.$$

Also consider the group homomorphism $\hat{}\ : O(\omega, 2n) \to \mathbf{R}^*$ defined by requiring $\hat{\psi}$ to satisfy $\psi \cdot \omega = \hat{\psi}\omega$.

The following theorem describes the structure of the automorpshism group of H^{2n+1}. The proof is based on two fundamental facts:

1. Every $\phi \in$Aut(H^{2n+1}) descends to an automorphism $\bar{\phi}$ of $\mathbb{R}^{2n}$.
2. An automorphism $\bar{\phi}$ of $\mathbb{R}^{2n}$ lifts to an automorphism ϕ of H^{2n+1} if and only if $\bar{\phi} \in O(\omega, 2n)$.

Theorem ([16]).

$$Aut(H^{2n+1}) = Hom(\mathbb{R}^{2n}, \mathbb{R}) \rtimes O(\omega, 2n),$$

where (η, ψ) acts by

$$(\eta, \psi)(x, z) = (\psi(x), \dot{\psi} z + \eta(x)).$$

It is worthwhile mentioning that, by the theorem above, Aut(H^{2n+1}) becomes a subgroup of the group of linear transformations on $\mathbb{R}^{2n+1}$.

From now on we will identify $\eta \in$ Hom$(\mathbb{R}^{2n})$ with the dual vector $(\alpha, \beta) \in \mathbb{R}^n \times \mathbb{R}^n$ with the property that

$$\eta(x) = \sum_{i=1}^{n} (\alpha_i x_i + \beta_i y_i).$$

for any $x = (x_1, \ldots, x_n, y_1, \ldots, y_n) \in \mathbb{R}^n \times \mathbb{R}^n$.

In some sense $O(\omega, 2n)$ extends Sp$(n, \mathbb{R}) = \{A \in$ Aut$(\mathbb{R}^{2n}), \psi \cdot \omega = \psi\}$. In fact, it is not difficult to show that $O(\omega, 2n) \cong$ Sp$(n, \mathbb{R}) \times \mathbb{R}^*$. Using this observation we obtain that Aut$(H^{2n+1}) \cong \mathbb{R}^{2n} \rtimes (\mathbb{R}^* \times$ Sp$(n, \mathbb{R}))$. This description corrects the error in [**7**], where Aut(H^{2n+1}) is taken to be $\mathbb{R}^{2n+1} \times$ Sp$(n, \mathbb{R})$.

Observation.

Recall from section **1.1** that we defined the linear map $J : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ given by $J(x_1, \ldots, x_n, y_1, \ldots, y_n) = (y_1, \ldots, y_n, -x_1, \ldots, -x_n)$. Let Ω denote the matrix of J with respect to the canonical basis of $\mathbb{R}^{2n}$ and note that $\Omega \in O(\omega, 2n)$. Consequently, J lifts to an automorphism of H^{2n+1} which will also be denoted by J. It is interesting to observe that if j is the automorphism of $\mathfrak{h}_{2n+1}$ defined by $j = J_{*e}$ and if $\{X_1, \ldots, X_n, Y_1, \ldots, Y_n, Z\}$ is the original basis considered in section **1.1** then $j(X_i) = Y_i, j(Y_i) = -X_i$, and $j(Z) = Z, 1 \leq i \leq n$. Moreover, if $\mathfrak{h}_{2n+1}$ is equipped with a positive definite inner product $\langle , \rangle$ with respect to which the basis $\{X_1, \ldots, X_n, Y_1, \ldots, Y_n, Z\}$ is orthonormal then the restriction of j to $\mathcal{Z}^{\perp}$ satisfies $\langle j(X), Y\rangle = \langle [X, Y], Z\rangle$, for all $X, Y \in \mathcal{Z}^{\perp}$.

The observations above can be used to construct the map j without having an a priori orthonormal basis ([**5**]). In the presence of an inner product on $\mathfrak{h}_{2n+1}$, define $j : \mathcal{Z}^{\perp} \to \mathcal{Z}^{\perp}$ by $\langle j(X), Y\rangle = \langle [X, Y], Z\rangle$, where $Z \in \mathcal{Z}, |Z| = 1$. It is easy to see that $j^2 = -\mathrm{Id}$. Also, if $\{X_1, \ldots, X_n\}$ is a set of orthonormal vectors in $\mathcal{Z}^{\perp}$ then $\{X_1, \ldots, X_n, Y_1 = jX_1, \ldots, Y_n = jX_n, Z\}$ is an orthonormal basis of $\mathfrak{h}_{2n+1}$ such that $[X_i, Y_i] = Z$ and $[Y_i, X_i] = -Z, 1 \leq i \leq n$. In sections **1.6** and **1.7** we will use the map j constructed as above (without having an a priori basis).

1.3 Classification of left invariant metrics on H^{2n+1}

Using ideas that are similar to the ones in [**7**], in this section we give the classification of left invariant metrics on H^{2n+1}. We would like to mention that our Heisenberg group, though isomorphic to the one defined in [**7**], has the group operation defined differently.

Two Riemannian metrics g_1 and g_2 on a manifold M are said to be *equivalent* if there exists a diffeomorphism ϕ such that $g_2(X,Y) = \phi^* g_1(X,Y) := g_1(\phi_* X, \phi_* Y)$, for any vector fields X and Y on M. If M is a Lie group, a metric g on M is called *left invariant* if $L_p{}^* g = g$ for any $p \in M$, where $L_p : M \to M$ is the left translation by $p : L_p(q) = pq, q \in M$.

Note that if g is a left invariant metric on H^{2n+1} and ϕ is a diffeomorphism then $\phi^* g$ is not necessarily left invariant. In fact, as observed in [**29**], if $\phi(e) = e$, then $\phi^* g$ is left invariant if and only if ϕ is an automorphism of H^{2n+1}. Consequently, in order to classify the left invariant metrics on the Heisenberg group it is enough to find the orbit space for the action $(\phi, g) \to \phi^* g$ of $\mathrm{Aut}(H^{2n+1})$ on the set of left invariant metrics on H^{2n+1}. This action can actually be regarded as an action on the set $\mathrm{Symm}^+(2n+1, \mathbb{R})$ of positive definite, symmetric $(2n+1)\times(2n+1)$ matrices. Indeed, every left invariant metric g is uniquely determined by the matrix G with entries given by g evaluated on pairs of vectors of the basis $\mathcal{B} = \{X_1, \ldots, X_n, Y_1, \ldots, Y_n, Z\}$ of $\mathfrak{h}_{2n+1}$. In order to write the expression of this action we need to identify the group

$\mathrm{Aut}(H^{2n+1})$ with a group of matrices.

As mentioned in the previous section, $\mathrm{Aut}(H^{2n+1})$ can be regarded as a group of linear transformations on $\mathbb{R}^{2n+1}$. Hence we can define the group homomorphism $i: \mathrm{Aut}(H^{2n+1}) \to \mathrm{GL}(2n+1, \mathbb{R})$ by letting $i(\phi)$ to be the matrix of the automorphism ϕ relative to the canonical basis of $\mathbb{R}^{2n+1}$. Using the identification between H^{2n+1} and $\mathbb{R}^{2n+1}$ from section **1.1**, we have $i(\phi)(x, z)^t = (\phi(x, z))^t$, for any $(x, z) \in \mathbb{R}^{2n+1}$. One can also define a homomorphism $\tilde{i}: \mathrm{Aut}(\mathfrak{h}_{2n+1}) \to \mathrm{GL}(2n + 1, \mathbb{R})$ by assigning to an automorphism Φ of $\mathfrak{h}_{2n+1}$, the matrix of Φ relative to the basis $\mathcal{B}$.

It is well known that $*: \mathrm{Aut}(H^{2n+1}) \to \mathrm{Aut}(\mathfrak{h}_{2n+1})$ defined by $\phi \to \phi_{*e}$ is a Lie group isomorphism. In fact, based on $\exp \circ \phi_{*e} = \phi \circ \exp$ and the description of the exponential map from section **1.1** we may conclude that $i = \tilde{i} \circ *$.

By the previous remarks, $g_2 = \phi^* g_1$ if and only if $G_2 = i(\phi)^t G_1 i(\phi)$, where G_i is the matrix of g_i relative to the basis $\mathcal{B}$, $i = 1, 2$.

Consequently, the problem of classifying left invariant metrics on H^{2n+1} is equivalent to finding the orbit space for the action

$$i(\mathrm{Aut}(H^{2n+1})) \times \mathrm{Symm}^+(2n + 1, \mathbb{R}) \to \mathrm{Symm}^+(2n + 1, \mathbb{R}),$$

$$(A, G) \to A^t G A.$$

Recall that every automorphism (η, ψ) of H^{2n+1} sends $(x_1, \ldots, x_n, y_1, \ldots, y_n, z)$ to $(\psi(x_1, \ldots, x_n, y_1, \ldots, y_n), \sum_{i=1}^n (\alpha_i x_i + \beta_i y_i) + \hat{\psi} z)$. Thus,

$$i((\eta, \psi)) = \begin{pmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ \alpha & \beta & \hat{\psi} \end{pmatrix},$$

where $\bar{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ is the matrix associated to ψ relative to the canonical basis of $\mathbb{R}^{2n}$. Note that (η, ψ) belongs to $\mathrm{Aut}(H^{2n+1})$ if and only if $\bar{A}^t \Omega \bar{A} = \lambda \Omega$.

where $\lambda(=\dot{\psi}) \in \mathbb{R}^*$ and $\Omega = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ is the matrix of the 2n-form ω relative to the canonical basis of $\mathbb{R}^{2n}$.

As one can easily check, for every $G \in \mathrm{Symm}^+(2n+1, \mathbb{R})$ there exist $\alpha, \beta \in \mathbb{R}^n$ and $a \in \mathbb{R}$ such that

$$\begin{pmatrix} I_n & 0 & 0 \\ 0 & aI_n & 0 \\ \alpha & \beta & a \end{pmatrix}^t G \begin{pmatrix} I_n & 0 & 0 \\ 0 & aI_n & 0 \\ \alpha & \beta & a \end{pmatrix} = \begin{pmatrix} \tilde{G} & 0 \\ 0 & 1 \end{pmatrix},$$

with $\tilde{G} \in \mathrm{Symm}^+(2n, \mathbb{R})$. Moreover, by [**36**], any such matrix can be put in diagonal form $\mathrm{diag}[\lambda_1^2, \ldots, \lambda_n^2, \lambda_1^2, \ldots, \lambda_n^2]$. Combining the two observations above we may conclude that every $G \in \mathrm{Symm}^+(2n+1, \mathbb{R})$ can be reduced to the form $\mathrm{diag}[\lambda_1^2, \ldots, \lambda_n^2, \lambda_1^2, \ldots, \lambda_n^2, 1]$ via some matrix in $i(\mathrm{Aut}(H^{2n}))$. In fact, every orbit contains a unique such diagonal matrix.

Now we write the form of a left invariant metric g whose matrix G relative to the basis $\mathcal{B}$ is of the from $\mathrm{diag}[\lambda_1^2, \ldots, \lambda_n^2, \lambda_1^2, \ldots, \lambda_n^2, 1]$. Observe that the left invariant vectors corresponding to the elements of $\mathcal{B}$ are

$$X_i(p) = \frac{\partial}{\partial x_i}(p) - y_i \frac{\partial}{\partial z}(p), \quad Y_i(p) = \frac{\partial}{\partial y_i}(p) + x_i \frac{\partial}{\partial x_i}(p), \quad Z(p) = \frac{\partial}{\partial z}(p),$$

for any $p = (x_1, \ldots, x_n, y_1, \ldots, y_n, z) \in H^{2n+1}$. The dual 1-forms associated to the vector fields $X_1, \ldots, X_n, Y_1, \ldots, Y_n$, and Z are $dx_1, \ldots, dx_n, dy_1, \ldots, dy_n$, and $dz + \sum_{i=1}^n (x_i dy_i - y_i dx_i)$, respectively. Thus, every left invariant metric is equivalent to a metric of the form

$$g = \sum_{i=1}^n \lambda_i^2 (dx_i^2 + dy_i^2) + (dz + \sum_{i=1}^n (x_i dy_i - y_i dx_i))^2$$

1.4 The isometry group

Given a left invariant metric g on H^{2n+1}, we would like to find its isometry group $\mathrm{Iso}(H^{2n+1}, g)$. A key role in determining this group is played by the *isotropy group* $\mathrm{Iso}_e(H^{2n+1}, g)$ of g at the identity e of H^{2n+1}, i.e., the group of isometries of g fixing e. As is the case for any nilpotent Lie group equipped with a left invariant metric, $\mathrm{Iso}(H^{2n+1}, g) = H^{2n+1} \rtimes \mathrm{Iso}_e(H^{2n+1}, g)$, where H^{2n+1} is interpreted as the group of isometries given by left translations.

Based on the observations from the previous section, $\mathrm{Iso}_e(H^{2n+1}, g)$ is a subgroup of $\mathrm{Aut}(H^{2n+1})$. Recall from section **1.3** that $\mathrm{Aut}(H^{2n+1})$ acts on the set of left invariant metrics by $(\phi, g) \to \phi^* g$. This action is equivalent to the action of $i(\mathrm{Aut}(H^{2n+1}))$ on $\mathrm{Symm}^+(2n+1, \mathbb{R})$ given by $(A, G) \to A^t G A$. Thus,

$$i(\mathrm{Iso}_e(H^{2n+1}, g)) = \{A \in i(\mathrm{Aut}(H^{2n+1})), A^t G A = G\},$$

where G is the matrix of g. In the following we will consider a metric g whose matrix G is the identity. Note that for such g we have

$$\begin{aligned} i(\mathrm{Iso}_e(H^{2n+1}, g)) &= i(\mathrm{Aut}(H^{2n+1})) \cap O(2n+1, \mathbb{R}) \\ &= \{\begin{pmatrix} \bar{A} & 0 \\ 0 & \lambda \end{pmatrix}, \bar{A}^t \Omega \bar{A} = \lambda \Omega, \bar{A}\bar{A}^t = I_{2n}, \lambda = \pm 1\}. \end{aligned}$$

(The last condition above actually follows from the previous two conditions.)

As one could check, the identity component corresponds to the matrix group $Sp(n, \mathbb{R}) \cap O(2n, \mathbb{R}) \cong U(n)$. So, we get the following description of the identity component $\mathrm{Iso}_0(H^{2n+1})$ of the isometry group:

$$\mathrm{Iso}_0(H^{2n+1}) \cong H^{2n+1} \rtimes U(n).$$

1.5 The Killing vector fields

Nonvanishing Killing vector fields are the main source for one-dimensional Riemannian foliations. In this section we compute the complete Killing vector fields on the Heisenberg group equipped with a left invariant metric.

By definition, a *Killing vector field* on a Riemannian manifold M is a vector field whose local flow consists of local isometries of M. The collection of complete Killing vector fields on M, i.e. Killing vector fields with a globaly defined flow, has a Lie algebra structure with bracket operation defined by the usual bracket of vector fields on M. This algebra is isomorphic to $\mathrm{iso}(M)$, the Lie algebra of the isometry group of M under the correspondence $x \to (\exp(tx) \cdot p)'(0)$, where $x \in \mathrm{iso}(M)$ and $\exp(tx) \cdot p$ is the orbit through p of the one-parameter group of isometries generated by x. We will use this isomorphism to find a basis for the Lie algebra of complete Killing vector fields on H^{2n+1}.

First we fix some notations. Let E_{ij} and F_{ij} be $n \times n$ matrices defined by:

$$E_{ij} = \begin{pmatrix} 0 & \cdots & & \cdots & & \cdots & 0 \\ \vdots & \ddots & & & & & \vdots \\ & & 0 & \cdots & 1 & & \\ \vdots & & \vdots & \ddots & \vdots & & \vdots \\ & & -1 & \cdots & 0 & & \\ \vdots & & & & & \ddots & \vdots \\ 0 & \cdots & & \cdots & & \cdots & 0 \end{pmatrix}, 1 \leq i < j \leq n.$$

and

$$F_{ij} = \begin{pmatrix} 0 & \cdots & & \cdots & & \cdots & 0 \\ \vdots & \ddots & & & & & \vdots \\ & & 0 & \cdots & 1 & & \\ \vdots & & \vdots & \ddots & \vdots & & \vdots \\ & & 1 & \cdots & 0 & & \\ \vdots & & & & & \ddots & \vdots \\ 0 & \cdots & & \cdots & & \cdots & 0 \end{pmatrix}, 1 \le i \le j \le n.$$

Also consider the $(2n+1) \times (2n+1)$ matrices

$$A_{ij} = \begin{pmatrix} E_{ij} & 0 & 0 \\ 0 & E_{ij} & 0 \\ 0 & 0 & 0 \end{pmatrix}, 1 \le i < j \le n, \; B_{ij} = \begin{pmatrix} 0 & F_{ij} & 0 \\ -F_{ij} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 1 \le i \le j \le n.$$

The collection $\{A_{ij}, 1 \le i < j \le n\} \cup \{B_{ij}, 1 \le i \le j \le n\}$ is a basis for the Lie algebra $\mathfrak{u}(n)$. The corresponding elements in the Lie algebra of $\mathrm{Iso}_e(H^{2n+1})$ will also be denoted by A_{ij} and B_{ij}, respectively. The induced one-parameter groups of isometries are:

$$A_{ij}(t) := \exp(tA_{ij}) = \begin{pmatrix} E_{ij}(t) & 0 & 0 \\ 0 & E_{ij}(t) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$B_{ij}(t) := \exp(tB_{ij}) = \begin{pmatrix} C_{ij}(t) & D_{ij}(t) & 0 \\ -D_{ij}(t) & C_{ij}(t) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

where

$$E_{ij}(t) = \begin{pmatrix} 1 & \cdots & & \cdots & & \cdots & 0 \\ \vdots & \ddots & & & & & \vdots \\ & & \cos t & \cdots & \sin t & & \\ \vdots & & \vdots & \ddots & \vdots & & \vdots \\ & & -\sin t & \cdots & \cos t & & \\ \vdots & & & & & \ddots & \vdots \\ 0 & \cdots & & \cdots & & \cdots & 1 \end{pmatrix}.$$

$$C_{ij}(t) = \begin{pmatrix} 1 & \cdots & & \cdots & & \cdots & 0 \\ \vdots & \ddots & & & & & \vdots \\ & & \cos t & \cdots & 0 & & \\ \vdots & & \vdots & \ddots & \vdots & & \vdots \\ & & 0 & \cdots & \cos t & & \\ \vdots & & & & & \ddots & \vdots \\ 0 & \cdots & & \cdots & & \cdots & 1 \end{pmatrix}.$$

and

$$D_{ij}(t) = \begin{pmatrix} 0 & \cdots & & \cdots & & \cdots & 0 \\ \vdots & \ddots & & & & & \vdots \\ & & 0 & \cdots & \sin t & & \\ \vdots & & \vdots & \ddots & \vdots & & \vdots \\ & & -\sin t & \cdots & 0 & & \\ \vdots & & & & & \ddots & \vdots \\ 0 & \cdots & & \cdots & & \cdots & 0 \end{pmatrix}.$$

The orbit of $A_{ij}(t)$ through a point $p = (x_1, \ldots, x_n, y_1, \ldots, y_n, z)$ is the curve

$$(x_1, \ldots, x_{i-1}, \cos t x_i + \sin t x_j, \ldots, x_{j-1}, -\sin t x_i + \cos t x_j, \ldots x_n,$$

$$, y_1, \ldots, y_{i-1}, \cos t y_i + \sin t y_j, \ldots, y_{j-1}, -\sin t y_i + \cos t y_j, \ldots, y_n, z),$$

$t \in \mathbb{R}$, whose tangent vector at $t = 0$ is

$$\begin{aligned} \tilde{A}_{ij}(p) &= x_j \frac{\partial}{\partial x_i} - x_i \frac{\partial}{\partial x_j} + y_j \frac{\partial}{\partial y_i} - y_i \frac{\partial}{\partial y_j} = \\ &= x_j(X_i - y_i Z) - x_i(X_j - y_j Z) + y_j(Y_i + x_i Z) - y_i(Y_j + x_j Z) = \\ &= x_j X_i - x_i X_j + y_j Y_i - y_i Y_j + 2(x_i y_j - x_j y_i) Z. \end{aligned}$$

Similarly, the orbit of $B_{ij}(t)$ through a point $p = (x_1, \ldots, x_n, y_1, \ldots, y_n, z)$ is the curve

$$(x_1, \ldots, x_{i-1}, \cos t x_i + \sin t y_j, \ldots, x_{j-1}, -\sin t y_i + \cos t x_j, \ldots x_n,$$

$$, y_1, \ldots, y_{i-1}, \cos t y_i - \sin t x_j, \ldots, y_{j-1}, \sin t x_i + \cos t y_j, \ldots, y_n, z),$$

$t \in \mathbf{R}$, whose tangent vector at $t = 0$ is

$$\bar{B}_{ij}(p) = y_j \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial y_i} - x_i \frac{\partial}{\partial y_j} =$$

$$= y_j(X_i - y_i Z) + y_i(X_j - y_j Z) - x_j(Y_i + x_i Z) - x_i(Y_j + x_j Z) =$$

$$= y_j X_i + y_i X_j - x_j Y_i - x_i Y_j - 2(x_i x_j + y_i y_j)Z.$$

Corresponding to X_i, Y_i and Z we have the following one-parameter groups of isometries:

$$t \to L_{\exp(tX_i)}, t \to L_{\exp(tY_i)}, \text{ and } t \to L_{\exp(tZ)}.$$

The associated Killing vector fields are:

$$\bar{X}_i(p) = X_i + y_i Z, \bar{Y}_i(p) = Y_i - x_i Z, \bar{Z}(p) = Z,$$

where $p = (x_1, \ldots, x_n, y_1, \ldots, y_n, z)$.

Hence $\{\bar{A}_{ij}, 1 \le i < j \le n\} \cup \{\bar{B}_{ij}, 1 \le i \le j \le n\} \cup \{\bar{X}_i, \bar{Y}_i, \bar{Z}, 1 \le i \le n\}$ is a basis for the Lie algebra of Killing vector fields on H^{2n+1}.

1.6 The geodesics

The geodesic equations are obtained in [**5**] in the more general context of two-step nilpotent Lie groups with a left invariant metric. Here we will only summarize some important properties for the geodesics of the Heisenberg group. The following proposition gives the equation of a geodesic starting at e. Since the isometry group $\mathrm{Iso}(H^{2n+1})$ is transitive, any other geodesic can be obtained by left translating the geodesic above.

Proposition. *Let γ be a unit speed geodesic with $\gamma(0) = e$ and $\dot\gamma(0) = \cos\theta Z + \sin\theta X$, where $X \perp \mathcal{Z}$, $Z \in \mathcal{Z}$, and $|X| = |Z| = 1$.*

(i) *If $\theta = \pi/2$, then $\gamma(t) = \exp(tX)$.*

(ii) *If $\theta \in (0, \pi/2)$, then $\gamma(t) = \exp(X(t) + Z(t))$, where*

$$X(t) = [\cos(t\cos\theta) - 1]\tan\theta j^{-1}X + \sin(t\cos\theta)\tan\theta X,$$

$$Z(t) = [t(1 + \frac{1}{2}\tan^2\theta)\cos\theta - \frac{1}{2}\sin(t\cos\theta)\tan^2\theta]Z.$$

(iii) *If $\theta = 0$, then $\gamma(t) = \exp(tZ)$.*

Observations.

1. If $\dot\gamma(0)$ is orthogonal to the central direction then γ minimizes the distance between any two of its points, i.e., γ is a line. Actually, these are the only lines through e.

2. In the second case above, γ minimizes up to the first conjugate point which occurs at $t = 2\pi/\cos\theta$. Moreover,

$$\dot\gamma(t) = (L_{\gamma(t)})_*[-\sin(t\cos\theta)\sin\theta j^{-1}X_e + \cos(t\cos\theta)\sin\theta X_e + Z\cos\theta].$$

As a consequence,

$$\dot\gamma(2\pi/\cos\theta) = (L_{\gamma(2\pi/\cos\theta)})_*(\dot\gamma(0)).$$

This property plays a significant role in determining the Riemannian foliations on H^{2n+1}. Another interesting property is that any two unit speed geodesics γ_1 and γ_2 with $\gamma_1(0) = \gamma_2(0) = e$ and making the same angle θ with $\mathcal{Z}^\perp$ will intersect at $\gamma_1(2\pi/\cos\theta) = \gamma_2(2\pi/\cos\theta) = \exp(2\pi(1 + \frac{1}{2}\tan^2\theta))Z$.

3. If $\dot{\gamma}(0) = Z$ then γ minimizes up to the first conjugate point which occurs at 2π. From now on, every geodesic tangent to the central direction will be called central.

4. Every geodesic makes a constant angle with the central direction.

For the proof of the conjugacy properties as well as other interesting observations regarding cut points, one is referred to [**31**].

1.7 The Jacobi vector fields

In this section we describe the Jacobi vector fields along geodesics γ in H^{2n+1}. We use slightly more general versions of the formulas in [**14**].

Let γ be a geodesic in H^{2n+1} with $\gamma(0) = e$ and let J be a Jacobi vector field along γ. Depending on the angle made by γ with the central direction we have the following possibilities:

(i) If $\dot{\gamma}(0) = X$, $|X| = 1$, $X \in Z^{\perp}$ then any Jacobi field along γ has the form:

$$(1) \qquad J(t) = f(t)Z \circ \gamma(t) + g(t)Y \circ \gamma(t) + \sum_{i=2}^{n} (f_i(t)X_i \circ \gamma(t) + g_i(t)Y_i \circ \gamma(t)),$$

where X, Y, X_i and Y_i are left invariant vector fields whose values at e (also denoted by X, Y, X_i, and Y_i, respectively) are defined such that $Y = j(X), Y_i = j(X_i)$, $i = 2, \ldots, n$, and $\{X, Y, X_2, \ldots, X_n, Y_2, \ldots, Y_n, Z\}$ is an orthonormal basis of $\mathfrak{h}_{2n+1}$.

The coefficients f, g, f_i, and g_i are given by the following formulas:

$$f(t) = f(0) + tf'(0) + g'(0)t^2 + \frac{2}{3}(2g(0) - f'(0))t^3$$

$$g(t) = g(0) + tg'(0) + (2g(0) - f'(0))t^2.$$

$$f_i(t) = f_i(0) + tf_i'(0), \quad g_i(t) = g_i(0) + tg_i'(0), \quad i = 2, \dots, n.$$

(ii) If $\dot\gamma(0) = \cos\theta Z + \sin\theta X_1 \in \mathcal{Z} \oplus \mathcal{Z}^{\perp}, |X_1| = |Z| = 1, \theta \in (0, \pi/2)$, construct an orthonormal basis $\{X_1, \dots, X_n, Y_1 = jX_1, \dots, Y_n = jX_n\}$ of $\mathcal{Z}^{\perp}$. Any Jacobi field along γ has the form:

(2)
$$J = \sum_{k=1}^{n} (a_k E_k + \bar a_k \bar E_k).$$

where

(3)
$$E_1(t) = \sin\theta[e^{tj(Z\cos\theta)} X_1 \circ \gamma(t) - \tan\theta Z \circ \gamma(t)],$$
$$\bar E_1(t) = \sin\theta[e^{tj(Z\cos\theta)} j(Z\cos\theta) X_1 \circ \gamma(t)],$$

while for $k \geq 2$

(4)
$$E_k(t) = e^{tj(Z\cos\theta)} X_k \circ \gamma(t),$$
$$\bar E_k(t) = e^{tj(Z\cos\theta)} j(Z\cos\theta) X_k \circ \gamma(t).$$

Notation:

$$e^{tj(Z\cos\theta)} := \sum_{i=0}^{\infty} \frac{t^i}{i!} j^i(Z\cos\theta) = \cos(t\cos\theta)I + \sin(t\cos\theta)j.$$

For $k = 1$ the coefficients are given by

(5)
$$a_1(t) = a_1(0) + t(a_1'(0) + c) + \bar a_1'(0)(1 - \cos(\cos\theta t)) - c\frac{\sin(\cos\theta t)}{\cos\theta},$$
$$\bar a_1(t) = \bar a_1(0) + \bar a_1'(0)\frac{\sin(\cos\theta t)}{\cos\theta} + c\frac{1 - \cos(\cos\theta t)}{\cos^2\theta}.$$

where $c = \bar{a}_1(0)\sin^2\theta - a_1'(0)(1 + \tan^2\theta)$.

For $k \geq 2$

$$
(6) \qquad
\begin{aligned}
a_k(t) &= a_k(0) + \bar{a}_k'(0)(1 - \cos(\cos\theta t)) + a_k'(0)\frac{\sin(\cos\theta t)}{\cos\theta}, \\
\bar{a}_k(t) &= \bar{a}_k(0) + a_k'(0)\frac{\sin(\cos\theta t)}{\cos\theta} - a_k'(0)\frac{1 - \cos(\cos\theta t)}{\cos^2\theta}.
\end{aligned}
$$

Note that

$$
(7) \qquad
\begin{aligned}
E_k(2\pi/\cos\theta) &= L_{\gamma(2\pi/\cos\theta)_*}E_k(0), \\
\bar{E}_k(2\pi/\cos\theta) &= L_{\gamma(2\pi/\cos\theta)_*}\bar{E}_k(0),
\end{aligned}
$$

for $1 \leq k \leq n$.

(iii) If $\dot{\gamma}(0) = Z$, then

$$
J(t) = h(t)Z \circ \gamma(t) + \sum_{i=1}^{n}(f_i(t)X_i \circ \gamma(t) + g_i(t)Y_i \circ \gamma(t)),
$$

where $X_i, Y_i, i = 1, \ldots, n$ are constructed as in (i) and

$h(t) = at + b$, $f_i(t) = a_i\cos t + b_i\sin t + c_i$, $g_i(t) = a_i\sin t - b_i\cos t + d_i$, $i = 1, \ldots, n$.

CHAPTER 2

RIEMANNIAN FOLIATIONS

2.1 Generalities

A *Riemannian foliation* (*metric foliation*) on a Riemannian manifold M is a foliation on M with locally equidistant leaves. The most basic example is provided by the foliation on the metric product $M_1 \times M_2$ with leaves given by the submanifolds $\{p\} \times M_2, p \in M_1$. The leaves can be regarded as preimages of points $p \in M_1$ via the projection map $\pi : M_1 \times M_2 \to M_1$.

In fact, the leaves of a Riemannian foliation are locally preimages via a Riemannian submersion of which the projection map above is an example. Consequently, the local study of Riemannian foliations is closely related to that of Riemannian submersions. Considered as generalizations of isometries $\pi : M \to B$ for the case $\dim(M) > \dim(B)$, Riemannian submersions are dual in some sense to isometric immersions. Even though the latter have been studied for a long time, a thorough investigation of the former has been started only fairly recently.

Riemannian foliations and Riemannian submersions occur frequently in geometry. The most natural examples are the ones induced by the projections $\pi : G \to G/H$ of a Riemannian homogeneous space.

In one of the first papers to give an in depth perspective on Riemannian foliations,

B. O'Neill ([**19**]) shows that their complexity, at least at the local level, can be characterized by two tensors. While isometric immersions can be fully described by the second fundamental form, Riemannian submersions are determined by the integrability ($A-$) tensor and second fundamental ($S-$) form of the fibers. They measure how far the induced foliation is from a metric product (*split*) foliation. More exactly, if both A and S vanish, the foliation splits, at least locally.

Interesting examples of Riemannian foliations can be obtained by requiring one of the two tensors above to vanish. If $A \equiv 0$, the normal bundle becomes integrable. Warped products $B \times_f F$ fall in this category. If $S \equiv 0$, the leaves of the foliation are totally geodesic. Among the most famous examples of such submersions are the Hopf fibrations of spheres. In fact, Riemannian submersions with totally geodesic fibers of spheres and complex projective spaces have been classified ([**24**]). Partial results in this direction have also been obtained on compact simple Lie groups ([**23**]).

It is worthwhile mentioning that under suitable curvature restrictions, the vanishing of one of the structural tensors implies the vanishing of the other. Indeed, a totally geodesic metric foliation on a nonpositively curved space and a *flat*, i.e. $A \equiv 0$, metric foliation on a nonnegatively curved space are split foliations ([**31**]).

The fundamental equations for Riemannian submersions are derived in [**19**]. They are the analogues of the Gauss-Codazzi equations for immersions. As an important consequence, one obtains that Riemannian submersions are curvature nondecreasing. This property has been used extensively to construct metrics of nonnegative curvature on manifolds such as $\mathbb{C}P^2 \# -\mathbb{C}P^2$ ([**3**]) , some exotic spheres

([**10**], [**27**]), and the tangent bundle of the n-sphere ([**4**]).

Yet another geometrically appealing property obtained by O'Neill is the fact that a geodesic starting perpendicular to a fiber remains perpendicular to any fiber it intersects ([**20**]). While this had been observed earlier in the more general context of metric foliations ([**26**]), [**20**] provides an extensive study of the geodesic behavior of the top manifold as compared to that of the base. In particular, conjugacy and index comparison theorems are derived.

Among the many applications of Riemannian submersions we would like to mention the Soul Theorem ([**21**]) which shows that every open nonnegatively curved manifold M can be regarded as the top space of a Riemannian submersion with base space the soul of M.

We end this section with an observation concerning the topology of the leaves of a Riemannian foliations. In [**26**], B.Reinhardt shows that the leaves of a Riemannian foliation on a compact manifold have the same universal covering. In fact, the same is true on arbitrary manifolds for complete foliations. Note that if the foliation is given by a global submersion, the holonomy displacement map (which will be defined in section **2.2**) gives a homeomorphism between any two leaves.

Stronger topological results were obtained for one-dimensional foliations and they will be discussed in section **3.1**.

2.2 Introduction

For a detailed treatment of metric foliations and Riemannian submersions the reader is referred to [**19**], [**20**], and [**30**].

Let M, B be differentiable manifolds with and let $\pi : M \to B$ be a submersion, i.e., π is a surjective differentiable map of maximal rank. For any $b \in B$, $\pi^{-1}(b)$ is a submanifold of M of dimension $\dim(M) - \dim(B)$. Consequently, in the presence of a Riemannian metric on M, for each $m \in M$ one has a decomposition of the tangent space M_m into a vertical subspace $\mathcal{V}_m$ tangent to $\pi^{-1}(\pi(m))$ and a horizontal space $\mathcal{H}_m = \mathcal{V}_m^{\perp}$.

If M and B are Riemannian manifolds then a differentiable map $\pi : M \to B$ is called a *Riemannian submersion* if π is a submersion and π_* preserves the length of horizontal vectors, that is $|\pi_* x| = |x|$, for all $m \in M$ and $x \in \mathcal{H}_m$.

One can easily check every Riemannian submersion $\pi : M \to B$ determines a metric foliation whose leaves are given by the preimages of points in B. The converse is also true locally. Thus, the following definitions and remarks, formulated in the language of Riemannian submersions, can be extended for metric foliations.

As noted in [**19**], the crucial role in the understanding of a Riemannian submersion is played by the integrability tensor A and the second fundamental form S given by:

$$A : \mathcal{H} \times \mathcal{H} \to \mathcal{V}, \qquad A_X Y = (\nabla_X Y)^v.$$

$$S : \mathcal{H} \times \mathcal{V} \to \mathcal{V}, \qquad S_X V = -(\nabla_V X)^v.$$

The mean curvature form κ is the horizontal one-form defined by $\kappa(E) = \mathrm{tr}(S_{E^h})$. If the leaves are one dimensional, we have $\kappa(X) = \langle S_X V, V\rangle$, where $X \in \mathcal{H}$ and $V \in \mathcal{V}$ with $|V| = 1$.

A horizontal vector field X on M is called *basic* if $\pi_* X = \tilde{X} \circ \pi$, where $\tilde{X}$ is a vector field on B. If X is a horizontal vector field along $\pi^{-1}(b), b \in B$, X will still be called *basic* if $\pi_{*m} X_m = \pi_{*m'} X_{m'}$, for all $m, m' \in \pi^{-1}(b)$. Finally, a horizontal one-form on M is called basic if its dual vector field is basic.

Let $\tilde{\gamma}$ be a geodesic in B with $\dot{\tilde{\gamma}}(0) = \tilde{x}$ and let X be the unique basic vector field along $\pi^{-1}(b)$ with $\pi_* X = \tilde{x}$. For each $m \in \pi^{-1}(b)$ consider the geodesic γ_m of M starting at m in direction X_m. This way we can define a diffeomorphism $h^t_{\tilde{\gamma}} : \pi^{-1}(\tilde{\gamma}(0)) \to \pi^{-1}(\tilde{\gamma}(t))$, called the *holonomy displacement map*. Note that every curve c in $\pi^{-1}(\tilde{\gamma}(0))$ gives rise to a geodesic variation $H_{\tilde{\gamma}}$ of $\gamma := \gamma_{c(0)}$ given by $H_{\tilde{\gamma}}(t, s) := h^t_{\tilde{\gamma}}(c(s))$. The corresponding Jacobi vector field J along γ is vertical and $J(t) = (H_{\tilde{\gamma}})_*(\frac{\partial}{\partial s})|_{(t,0)}$. Moreover,

$$J' = J'^v + J'^h = -S_{\dot{\gamma}} J - A^*_{\dot{\gamma}} J,$$

where $A^*_{\dot{\gamma}}$ is the adjoint of $A_{\dot{\gamma}}$.

Note that if the leaves have dimension one and if $|J(0)| = |\dot{c}(0)| = 1$ then

$$\langle J(0), J'(0)\rangle = -\langle J(0), S_{\dot{\gamma}(0)} J(0)\rangle = -\kappa(\dot{\gamma}(0)). \tag{8}$$

The formula above is a special case a more general result for k-dimensional Riemannian foliations due to Rummler ([**28**]). Let $\{V_1, \cdots, V_k\}$ be an orthonormal basis for the vertical bundle at $p = \gamma(0)$ and let $V_i(t) = h^t_{\gamma *} V_i$. Also, consider the function $\phi(t) := \det \langle V_i(t), E_j(t)\rangle, 1 \leq i, j \leq k$, where $\{E_1(t), \cdots, E_k(t)\}$ is an orthonormal basis for the vertical bundle at $\gamma(t)$. We have

$$\phi'(t) = -\kappa(\dot{\gamma}(t))\phi(t).$$

One can say that κ measures the growth in the volume form of the leaves under holonomy dispacement.

2.3 Examples of Riemannian foliations

2.3.1 Metric product foliations

Let B and F be two Riemannian manifolds. The projection $\pi : B \times F \to B$ from $B \times F$ equipped with the product metric onto B is a trivial Riemannian submersion which induces a foliation on $B \times F$. A typical leaf of this foliation is $\{b\} \times F, b \in B$. Note that the leaves are totally geodesic and the normal bundle is integrable. These are the simplest examples of Riemannian foliations.

2.3.2 Warped products

The previous example can be generalized by altering the product metric g on $B \times F$ along the leaves. More exactly, consider a positive function $h : F \to \mathbb{R}$ and define a metric $\tilde{g}$ on $B \times F$ by

$$\tilde{g}_{(b,f)}(X_1 + Y_1, X_2 + Y_2) = g_{(b,f)}(X_1, X_2) + h(f)g_{(b,f)}(Y_1, Y_2),$$

where $(0, Y_1), (0, Y_2)$ are tangent and $(X_1, 0), (X_2, 0)$ are orthogonal to the leaf through (b, f). Observe that π is still a Riemannian submersion since the metric has been modified only along the leaves. While in general the leaves will not be totally geodesic, the normal bundle is always integrable.

2.3.3 Homogeneous foliations

Let M be a Riemannian manifold and H a subgroup of the isometry group of M. If the orbits of H have the same dimension, then they are the leaves of a foliation. By definition, any foliation obtained by the procedure above is called *homogeneous*.

It is easy to see that any homogeneous foliation is Riemannian. Indeed, let $\mathcal{O}_1$ and $\mathcal{O}_2$ be the leaves through m_1 and m_2, respectively. Also consider a geodesic γ realizing the (local) distance between m_1 and $\mathcal{O}_2$. For any $h \in H, h \cdot m_1 \in \mathcal{O}_1$, and $h \cdot \gamma$ is a geodesic of the same length as γ realizing the (local) distance between $h \cdot m_1$ and $\mathcal{O}_2$ and the foliation is Riemannian.

The following example is a special case of a homogeneous foliation. Let G be a Lie group endowed with a left invariant metric and let H be a subgroup of G. The left action $(h, g) \to hg$ of H on G is an isometric action since left translations are isometries. Consequently, the cosets $\{Hg | g \in G\}$ are the leaves of a homogeneous foliation. If H is closed (so that $H\backslash G$ has a manifold structure), by a standard procedure the left invariant metric on G descends to a metric on $H\backslash G$ and the submersion $\pi : G \to H\backslash G$ becomes Riemannian.

A notable example of the situation above occurs on

$$G = S^3 = \{(z_1, z_2) \in \mathbb{C} \times \mathbb{C}, |z_1|^2 + |z_2|^2 = 1\},$$

regarded as the group of unit quaternions. Consider the subgroup $H = \{(e^{i\lambda t}, e^{i\mu t}), t \in \mathbb{R}\}$, where λ and μ are real numbers. The left action of H on S^3 equipped with the left invariant metric induces a homogeneous foliation with orbits given by $\{(e^{i\lambda t} z_1, e^{i\mu t} z_2), t \in \mathbb{R}\}$.

If λ/μ is rational, H is closed and all leaves are diffeomeorphic to S^1. If λ/μ is irrational, the foliation contains exactly two closed leaves: $z_1 = 0$ and $z_2 = 0$. Any other leaf is dense in a torus with equation $z_1 = k$ for some $0 < k < 1$. It is well-known that these are the only one-dimensional Riemannian foliations on S^3 [**9**].

It is also interesting to consider the right action $(h, g) \to gh^{-1}$ of H on G. In general, the cosets $\{gH, g \in G\}$ are not the leaves of a Riemannian foliation. As pointed out in [**32**], under suitable conditions, the foliation does become Riemannian. This happens, for example, if H is normal or if the metric on G is Ad_H-invariant. Under the weaker condition that Ad_H is an isomorphism when restricted to the Lie algebra h, the metric on G projects to a metric on G/H. Since the metric on G is not in general right invariant, this construction is a possible source for nonhomogeneous Riemannian foliation. In fact, this idea was used in [**32**] to construct a nonhomogeneous Riemannian flow on $SL(2, \mathbb{R})$. We will present this example together with some other examples of nonhomogeneous flows in section **2.3**.

We would also like to make some comments on Riemannian foliations on quotients of Lie groups by lattices. As before, consider a left invariant metric on a Lie group G and the homogeneous foliation induced by the left action of a closed subgroup H. Also, consider a discrete subgroup Γ acting on the right on G. Observe that Γ preserves the foliation on G. Moreover, if Γ is a subgroup of the isometry group of G, the induced foliation on G/Γ is Riemannian. In fact, the induced foliation is homogeneous since every isometry preserving the foliation on G will generate an isometry on G/Γ.

2.3.4 Nonhomogeneous metric flows

a) Let $\mathfrak{g}$ denote the metric Lie algebra with orthonormal basis $\{V, X, Y\}$ and bracket relations given by:

$$[V, X] = Y - V, [V, Y] = -X + V, [X, Y] = -X - Y + V.$$

It is easy to see that $\mathfrak{g}$ is the Lie algebra $SL(2, \mathbb{R})$. The metric flow associated to the left invariant vector field V is metric but not homogeneous. As mentioned in [**32**], in order to show the metric condition it is enough to check that $< [V, E], E >= 0$ for any $E \in V^{\perp}$. The condition is entirely algebraic and is satisfied in this case.

The following example is due to Y. Carrière ([**2**]) and it provides a nonhomogeneous one-dimensional Riemannian foliation on a compact manifold. We would like to mention that nonhomogeneity is considered in a weaker sense here. It means that there are no Riemannian metrics on the manifold for which the leaves of the foliation are orbits of an isometric group action.

b) Let A be a matrix in $SL(2, \mathbb{Z})$ with trace$(A) > 2$. The automorphism A has two real eigenvalues λ_1 and λ_2, and $0 < \lambda_1 < 1 < \lambda_2$, $\lambda_1\lambda_2 = 1$. Let V_1 and V_2 be the eigenvectors corresponding to λ_1 and λ_2, respectively. Let F_1 and F_2 be the projections to $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ of the flows induced by V_1 and V_2 on $\mathbb{R}^2$. The leaves of both F_1 and F_2 are dense and A induces a diffeomorphism of T^2 preserving these leaves. Note that if $\mathbb{R}^2$ and $\mathbb{R}$ are given the usual Euclidean metric, the flows F_1 and F_2 are Riemannian; in fact they are (locally) homogeneous. Now let F_1' and F_2' be the flows on $T^2 \times \mathbb{R}$ corresponding to F_1 and F_2. Consider the action of $\mathbb{Z}$ on $T^2 \times \mathbb{R}$ given by $(n, (x, t)) \to (A^n x, t + n)$. Since F_1' and F_2' are preserved by this action, they descend to two flows F_1 and F_2 on the quotient manifold $(T^2 \times \mathbb{R})/\mathbb{Z}$. These flows are Riemannian but not homogeneous. The proof of this fact can be found in [**2**].

CHAPTER 3

ONE-DIMENSIONAL RIEMANNIAN FOLIATIONS

3.1 Generalities

The existence of a one-dimensional Riemannian foliation on a manifold imposes strong restrictions on the geometry of the manifold. Consequently, not all manifolds admit such foliations. For example, there are no Riemannian foliations on compact manifolds with negative Ricci curvature ([**25**]). On the other hand, as noted in [**8**], hyperbolic spaces admit an abundance of such foliations (most of which are nonhomogeneous) with little rigidity.

One has a very good description of Riemannian foliations on flat manifolds. For example, the fibrations of Euclidean spaces are all homogeneous ([**11**],[**12**]). Moreover, on compact flat manifolds every Riemannian foliation splits. While open positively curved manifolds admit no Riemannian fibrations ([**33**]), one-dimensional metric fibrations on open nonnegatively curved symmetric spaces are either homogeneous or the base of the fibration splits (locally) isometrically ([**33**]). On manifolds M^n with constant positive sectional curvature every metric fibration of M is congruent to a generalized Hopf fibration ([**9**],[**35**]). If the fibration is one-dimensional, then it is homogeneous.

Some authors define homogeneous foliations as foliations for which there exists a metric with respect to which the leaves of the foliation are orbits of an isometric action. With this definition, the existence of one-dimensional Riemannian foliations imposes topological restrictions on the manifold. Indeed, the homogeneity condition for foliations of dimension one is equivalent to the nonvanishing of the basic cohomology space of maximal degree. The same condition is also equivalent to the existence of a metric for which the leaves are geodesics.

While a classification of one-dimensional Riemannian foliations on manifolds of a given dimension may be very difficult, one does have such a classification on three-dimensional manifolds. Y. Carrière showed that such foliations are either homogeneous or they are conjugated to the foliation described in **2.3.4** b) ([**1**]).

3.2 Fundamental properties and theorems

As noted in section **2.2**, the mean curvature form κ plays a fundamental role in understanding the structure of the foliation. In fact, the homogeneity of one-dimensional Riemannian foliations is characterized entirely in terms of the properties of κ. More precisely, we have the following:

Theorem 1. [**8**] *A one-dimensional metric foliation $\mathcal{F}$ is (locally) homogeneous if and only if κ is closed.*

Proof. Assume $\mathcal{F}$ is (locally) homogeneous. Let T be the (local) Killing vector field induced by the isometric action and $L = |T|$. Since $V \neq 0$, there exists a (local) unit vector field $V = \frac{1}{L}T$ tangent to the leaves. If X is a (local) basic vector field, using the fact that T is Killing we have

$$\langle \nabla_X LV, V \rangle + \langle \nabla_V LV, X \rangle = 0, \text{ which implies}$$

$$X(L) + \langle \nabla_X T, T \rangle + T(L) + L\langle \nabla_V V, X \rangle = 0.$$

As any Killing vector field has constant length along its integral curves, we have $X(L) + L\langle \nabla_V V, X \rangle = 0$. For $\Phi = -\ln(L)$, we get $\kappa(X) = X(\Phi) = d\Phi(X)$. Consequently, κ is closed.

For the converse, assume that κ is closed and note that $\kappa = d\Phi$, for some locally defined function Φ. Moreover, since κ vanishes on the vertical bundle, Φ is (locally) constant along leaves. If $L = \exp(-\Phi)$ and if V is a (local) unit vector field tangent to the foliation then $T = LV$ is a Killing vector field. Indeed, it is easy to see that $\langle \nabla_V LV, V \rangle = V(L) = LT(\Phi) = 0$, for any horizontal X. The only condition left to check is $\langle \nabla_V LV, X \rangle + \langle \nabla_X LV, V \rangle = 0$. But the left side equals $L\kappa(X) + X(L)$ which cancels by the definition of L and the fact that κ is closed. $\square$

Observe that on a simply connected manifold κ is closed if and only if the foliation is globally homogeneous. Φ is now globally defined and the arguments are the same.

Theorem 2. [**34**] *Let $\mathcal{F}$ be a one-dimensional metric foliation on a manifold with sectional curvature bounded either from below or from above. If κ is basic then it is also closed.*

Thus, in order to show that a one-dimensional metric foliation on a space with bounded sectional curvature is homogeneous, it is enough to show that the mean curvature form κ is basic.

CHAPTER 4

THE MAIN RESULTS

In preparation for our first proposition we first prove the following:

Lemma. *Let $\mathcal{W}$ be an m-dimensional inner product space, $\mathcal{V}$ a k-dimensional subspace of $\mathcal{W}$, $1 \le k < m-1$, and $Z \in \mathcal{W}$ such that Z is neither orthogonal to $\mathcal{V}$ nor contained in $\mathcal{V}$. Let θ_0 be the angle made by Z with $\mathcal{V}^\perp$ and let $0 < \theta_0 < \theta < \pi/2$. There exists a basis $\{h_1, h_2, \ldots, h_{m-k}\}$ of $\mathcal{V}^\perp$ such that h_i makes an angle θ with Z, for any $1 \le i \le m-k$.*

Proof. Let Z^h be the orthogonal projection of Z onto $\mathcal{V}^\perp$ and note that the conclusion of the lemma is equivalent to the existence of a basis as above for which each element of the basis makes an angle α with Z^h, where $\alpha = \cos^{-1}(\cos\theta/\cos\theta_0)$. In order to achieve this consider a basis $\{\bar{h}_1, \bar{h}_2, \cdots, \bar{h}_{m-k-1}\}$ of $(\mathcal{V} \dotplus \operatorname{span}(Z^h))^\perp$ and let $\bar{h}_{m-k} = -\bar{h}_1$. It is easy to check that the set $\{h_1, h_2, \cdots, h_{m-k}\}$ with $h_i = \cos\alpha Z^h + \sin\alpha \bar{h}_i$, $1 \le i \le m-k$ satisfies the requirements above. $\square$

Proposition 1. *Let $\mathcal{F}$ be a k-dimensional Riemannian foliation on H^{2n+1} with vertical bundle $\mathcal{V}$. If γ is a horizontal geodesic making an angle $\theta \in (0, \frac{\pi}{2})$ with Z then*

$$\mathcal{V}_{\gamma(2\pi/\cos\theta)} = L_{\gamma(2\pi/\cos\theta)_*}\mathcal{V}_{\gamma(0)}$$

Proof. Let θ_0 be the angle made by $\mathcal{Z}$ with the horizontal space at $\gamma(0)$ and assume that $1 \leq k \leq 2n-1$. If $\theta > \theta_0$, by the lemma above one can find a set of $2n+1-k$ linearly independent horizontal vectors $\{h_1, h_2, \dots, h_{2n+1-k}\}$ at $\gamma(0)$ making the same angle θ with $\mathcal{Z}$. Consider the geodesics γ_i starting at $\gamma(0)$ in direction h_i and observe that $\gamma_i(2\pi/\cos\theta) = \gamma_j(2\pi/\cos\theta)$, for any $i, j = 1, 2, \dots, 2n+1-k$. Moreover, $\dot{\gamma}_i(2\pi/\cos\theta) = L_{\gamma_i(2\pi/\cos\theta)_*}(h_i)$ for any $1 \leq i \leq 2n+1-k$. This implies that the set $\{\dot{\gamma}_i(2\pi/\cos\theta), i = 1, 2, \dots, 2n+1-k\}$ consists of $2n+1-k$ linearly independent vectors. Also note that since geodesics which are horizontal at one point stay horizontal for all time, the set above is actually a basis for the horizontal space at $\gamma_i(2\pi/\cos\theta)$. Thus, the horizontal space at $\gamma_i(2\pi/\cos\theta)$ is the left translation of the horizontal space at $\gamma(0)$. Consequently, the vertical spaces are in the same relation.

If $\theta = \theta_0$, consider the sequence $\theta_n \to \theta_0, n \geq 1$ with $\theta_n > \theta_0$ and the geodesics γ_n making angles θ_n with $\mathcal{Z}$, respectively. Since $\mathcal{V}_{\gamma_n(2\pi/\cos\theta_n)} \to \mathcal{V}_{\gamma(2\pi/\cos\theta_0)}$ and $\mathcal{V}_{\gamma_n(2\pi/\cos\theta_n)} = L_{\gamma_n(2\pi/\cos\theta_n)_*}\mathcal{V}_{\gamma_n(0)}$, using a limit type of argument we may conclude that $\mathcal{V}_{\gamma(2\pi/\cos\theta_0)} = L_{\gamma(2\pi/\cos\theta_0)_*}\mathcal{V}_{\gamma(0)}$.

If $k = 2n$ then $\dot{\gamma}(0)$ and $\dot{\gamma}(2\pi/\cos\theta)$ generate the horizontal spaces at $\gamma(0)$ and $\gamma(2\pi/\cos\theta)$. The conclusion of the theorem holds based on the left invariance property mentioned above. □

Using Proposition 1 we show that if $1 \leq k \leq 2n-1$ then the vertical bundle of a k-dimensional Riemannian foliation is left invariant along geodesics in central direction.

Proposition 2. *Let $\mathcal{F}$ be a k-dimensional Riemannian foliation on H^{2n+1} with vertical bundle $\mathcal{V}$ and $1 \le k \le 2n-1$. Then*

$$\mathcal{V}_{p\cdot\exp tZ} = L_{\exp tZ_*}\mathcal{V}_p.$$

for any $p \in H^{2n+1}$ and $t \in \mathbb{R}$.

Proof. Without loss of generality we may assume that $p = e$. Also assume that $\mathcal{Z}$ is neither horizontal nor vertical at e and let θ_0 be the angle made by $\mathcal{Z}$ with $\mathcal{H}_e$. As we mentioned in Observation 2 from section **1.6**, if γ is a geodesic making an angle θ with $\mathcal{Z}$ and if $\gamma(0) = e$ then $\gamma(2\pi/\cos\theta) = \exp(2\pi(1 + \frac{1}{2}\tan^2\theta)Z)$. Thus, for any $t \in \mathbb{R}$ with $|t| > 2\pi(1 + \frac{1}{2}\tan^2\theta_0)$ there exists $\theta > \theta_0$ such that $\exp(tZ) = \gamma(2\pi/\cos\theta)$, where γ is some unit speed horizontal geodesic with $\gamma(0) = e$. If $|t| > 2\pi(1 + \frac{1}{2}\tan^2\theta_0)$ then the result follows as an immediate consequence of Proposition 1. In order to prove the proposition for $|t| \le 2\pi(1 + \frac{1}{2}\tan^2\theta_0)$ it is enough to repeat the same argument with e replaced by $\exp(t_0Z)$ for some t_0 with $|t_0| > 4\pi(1 + \frac{1}{2}\tan^2\theta_0)$.

If $\mathcal{Z}$ is either horizontal or vertical at e then the proposition follows as a consequence of the previous case. $\square$

It is interesting to remark that in the case of a one-dimensional foliation, Proposition 2 implies that if $\mathcal{Z}$ is vertical at p then the leaf through p is the geodesic through p in the central direction. We will use this observation in the proof of Theorem 1.

Proposition 3. *Let $\mathcal{F}$ be a one-dimensional Riemannian foliation on H^{2n+1}. If J is a holonomy Jacobi vector field along a horizontal geodesic γ making an angle $\theta \in (0, \pi/2)$, with $\mathcal{Z}$ then*

$$J(2\pi/\cos\theta) = L_{\gamma(2\pi/\cos\theta)_*} J(0).$$

Proof. As before, we may assume that $\gamma(0) = e$. Let $\dot\gamma(0) = \cos\theta Z + \sin\theta X_1$ with X_1 unit, orthogonal to $\mathcal{Z}$ and assume that θ is not equal to θ_0, the angle made by the horizontal space at $\gamma(0)$ with $\mathcal{Z}$. By Proposition 1, we have that $J(0) - L_{\gamma(-2\pi/\cos\theta)_*} J(2\pi/\cos\theta)$ is vertical at $\gamma(0) = e$. Using the observation on the form of the Jacobi vector fields mentioned in section **1.7** (ii), the holonomy Jacobi vector field J along γ can be written as

$$J = \sum_{k=1}^{n} (a_k E_k + a_k E_k),$$

with E_k, E_k, a_k, and a_k given by (3)–(6). Also, by (5) and (6), a_1 and, for $k \geq 2$, a_k and a_k are periodic with period $2\pi/\cos\theta$. Using (7), the observation above translates to $(a_1(2\pi/\cos\theta) - a_1(0))E_1(0)$ being vertical at e. But $E_1(0)$ is not vertical at e since this would imply $\theta = \theta_0$. Consequently, $a_1(0) = a_1(2\pi/\cos\theta)$ and a_1 is $2\pi/\cos\theta-$periodic, thus proving the proposition in this case. Note that we also obtain that $|J(t)|$ is periodic with the same period. Moreover, $\kappa(\dot\gamma(t)) = -1/2(|J(t)|^2)'$ is periodic as well and $\kappa(\dot\gamma(0)) = \kappa(\dot\gamma(2\pi/\cos\theta))$.

If $\theta = \theta_0$, consider a sequence $\{h_n\}_{n\geq 1}$ of unit horizontal vectors with $h_n \to \dot\gamma(0)$. We may also assume that $\theta_n > \theta_0$, where θ_n is the angle made by h_n with $\mathcal{Z}$. By a limit argument similar to the one used in Proposition 1 we obtain

$$\kappa(\dot\gamma(0)) = \lim_{n\to\infty} \kappa(\dot\gamma_n(0)) = \lim_{n\to\infty} \kappa(\dot\gamma_n(2\pi/\cos\theta_n)) = \kappa(\dot\gamma(2\pi/\cos\theta)),$$

where γ_n are geodesics with $\gamma_n(0) = e$ and $\dot\gamma_n(0) = h_n$.

Now consider the holonomy Jacobi field J along γ with $|J(0)| = 1$. Since $\theta = \theta_0$, we have $\dot\gamma(0) = \frac{1}{|Z^h|}Z^h$ and, following the notation from section **1.7**, $J(0) = E_1(0)$. Consequently, $a_1(0) = 1$ while the rest of the coefficients appearing in the formula for J cancel at $t = 0$ (and, by periodicity, at any integer multiple of $2\pi/\cos\theta$).

Using relation (8) from section **2.2** and the observations above we obtain

$$\langle J(0), J'(0)\rangle = -\kappa(\dot\gamma(0))|J(0)|^2 = a_1'(0) \text{ and}$$

$$\langle J(2\pi/\cos\theta), J'(2\pi/\cos\theta)\rangle = -\kappa(\dot\gamma(2\pi/\cos\theta))|J(2\pi/\cos\theta)|^2 =$$

$$= a_1(2\pi/\cos\theta)a_1'(2\pi/\cos\theta).$$

As noted above, $\kappa(\dot\gamma(0)) = \kappa(\dot\gamma(2\pi/\cos\theta))$ and the previous relations imply $a_1'(0)a_1^2(2\pi/\cos\theta) = a_1(2\pi/\cos\theta)a_1'(2\pi/\cos\theta)$. But $|a_1(2\pi/\cos\theta)| = |J(2\pi/\cos\theta)|$ cannot cancel and we get $a_1'(0)a_1(2\pi/\cos\theta) = a_1'(2\pi/\cos\theta)$. Using the periodicity of a_1' and relation (5) from section **1.7** we obtain

$$a_1'(0)(1 - a_1(2\pi/\cos\theta)) = \frac{2\pi\tan^2\theta}{\cos\theta}(a_1'(0))^2 = 0.$$

Thus, $a_1'(0) = 0$ and, using (5) again, a_1 is periodic. This proves the proposition. □

It is important to note that the proof of the proposition above actually shows that $\kappa(Z^h) = 0$, which will be essential in the proof of the next proposition.

Proposition 4. *If $\mathcal{F}$ is a one-dimensional Riemannian foliation on H^{2n+1} then $\mathcal{Z}$ makes a constant angle with $\mathcal{V}$ along L, for any leaf L.*

Proof. Let $Z = Z^v + Z^h \in \mathcal{V} \oplus \mathcal{H}$ and let's assume that $\mathcal{V}$ makes an angle different from 0 or $\pi/2$ with $\mathcal{Z}$ at some point on L. Let $V = \frac{1}{|Z^v|} Z^v$. We have the following:

$$V\langle V, Z\rangle = \langle \nabla_V V, Z\rangle + \langle V, \nabla_V Z\rangle =$$

$$= \langle \nabla_V V, Z^h\rangle + \langle \nabla_V V, Z^v\rangle = \kappa(Z^h) = 0,$$

where the second equality follows from the fact that Z is a Killing vector field and the third one follows from $\langle \nabla_V V, Z^v\rangle = |Z^v|\langle \nabla_V V, V\rangle = 0$. The last equality follows form Proposition 3. Hence $\langle V, Z\rangle$ is constant along L. The other two cases follow as a consequence of the case above. $\square$

Proposition 5. *Let $\mathcal{F}$ be a one-dimensional Riemannian foliation on H^{2n+1} and let X be a basic vector field along a leaf L. Then, along L, X makes a constant angle with the central direction $\mathcal{Z}$.*

Proof. We may assume that $\mathcal{Z}$ is not tangent to L since, by Proposition 4, the conclusion is true if $\mathcal{Z}$ is tangent to L. Note that it is enough to prove the proposition for basic vector fields that are orthogonal to $\mathcal{Z}$ at some point $p \in L$. Indeed, let $X_1, \ldots X_{2n-1}, H$ be (local) basic fields along L such that $\{X_1(p), \cdots, X_{2n-1}(p), H(p)\}$ is an orthonormal basis of $\mathcal{H}_p$, where $X_i(p) \perp Z(p)$, $H(p) = \frac{1}{|Z^h(p)|} Z^h(p)$, and $Z \in \mathcal{Z}, |Z| = 1$. As we will show below, X_i remains orthogonal to Z along L for any $1 \leq i \leq 2n-1$ and it follows that $H(q) = \frac{1}{|Z^h(q)|} Z^h(q)$ for any $q \in L$. Using the fact that $\mathcal{Z}$ makes a constant angle with $\mathcal{V}$ (and, thus, with $\mathcal{H}$) we may conclude that $|Z^h|$ is constant along L. Hence, since H is basic, we may conclude that Z^h is basic as well. The result follows easily since any unit basic vector field along L is can be written as $X = aZ^h + \sum_{i=1}^{2n-1} a_i X_i$, where a and a_i are constants. Consequently, $|\cos(\measuredangle(X, \mathcal{Z})| = |a||Z^h|^2$ is constant.

Now let X be a basic vector field along L such that $X_p \perp \mathcal{Z}$ for some $p \in L$ and let us show that X is orthogonal to $\mathcal{Z}$ along L. If $\tilde{p}$ is another point of L, consider the horizontal geodesics γ and $\tilde{\gamma}$ starting at p and $\tilde{p}$ in direction X_p and $X_{\tilde{p}}$, respectively. By contradiction, assume that $X_{\tilde{p}}$ is not orthogonal to $\mathcal{Z}$ and let J and $\tilde{J}$ be holonomy Jacobi fields along γ and $\tilde{\gamma}$. Using the form of the Jacobi fields along geodesics orthogonal to $\mathcal{Z}$ given in **1.7**(i), it is easy to see that $\cos(\measuredangle(J(t), \mathcal{Z}))$ is the quotient of a polynomial by the square root of another polynomial. Consequently, as $t \to \infty$, $\cos(\measuredangle(J(t), \mathcal{Z}))$ converges to some $\alpha \in [0, 1]$. But, by Proposition 3, $t \to \cos(\measuredangle(\tilde{J}(t), \mathcal{Z}))$ is a periodic function. This is a contradiction since, by Proposition 4, $J(t)$ and $\tilde{J}(t)$ make the the same angle with $\mathcal{Z}$. $\square$

Remark. Let γ be a horizontal geodesic for which $\dot{\gamma}(0) \perp \mathcal{Z}$ and let J be a Jacobi vector field along γ. As noted above, the coefficient of Z in the expression for J is, in general, a degree three polynomial. We claim that if J is a holonomy Jacobi field then this polynomial has degree at most two. To see this, choose a sequence of horizontal geodesics γ_k with $\dot{\gamma}_k(0) \to \dot{\gamma}(0)$ and $\measuredangle(\dot{\gamma}_k(0), \mathcal{Z}) < \pi/2$. Let J_k be the holonomy Jacobi field along γ_k with $J_k(0) = J(0)$. Combining the form of J_k along γ_k (see section **1.7**(ii)) with the additional restrictions imposed by Proposition 3, it is easy to check that

$$f(t) = \langle J(t), Z\rangle = \lim_{k\to\infty} \langle J_k(t), Z\rangle$$

is a polynomial in t of degree at most two.

Theorem 1. *Let $\mathcal{F}$ be a one-dimensional foliation on H^{2n+1} equipped with a left invariant metric. $\mathcal{F}$ is Riemannian if and only if it is homogeneous.*

Proof. Let p and $\tilde{p}$ be two points on the same leaf. Also, let X and $\tilde{X}$ be the values of a basic vector field at p and $\tilde{p}$, respectively. We want to show that $\kappa(X) = \kappa(\tilde{X})$. Observe that X (and $\tilde{X}$) may be chosen to be orthogonal to $\mathcal{Z}$ since, for $Z \in \mathcal{Z}$ with $|Z| = 1$, we already have that Z^h is basic and $\kappa(Z^h)$ is constant along individual leaves.

Let γ and $\tilde{\gamma}$ be horizontal geodesics with $\dot{\gamma}(0) = X$ and $\dot{\tilde{\gamma}}(0) = \tilde{X}$. Recall that if J is the holonomy Jacobi field along γ with $|J(0)| = 1$ then, by (8) in section **2.2**, we have $\kappa(X) = -\langle J'(0), J(0)\rangle$. Using Proposition 4 and the form of the Jacobi fields given in 1.7(i), we obtain

$$\frac{|f(t)|}{|J(t)|} = |\cos(\measuredangle(J(t), Z))| = |\cos(\measuredangle(\tilde{J}(t), Z))| = \frac{|\tilde{f}(t)|}{|\tilde{J}(t)|},$$

where $\tilde{J}$ is the holonomy Jacobi field along $\tilde{\gamma}$ with $|\tilde{J}(0)| = 1$. The relation above implies

$$f^2(t)|\tilde{J}(t)|^2 = \tilde{f}^2(t)|J(t)|^2. \tag{9}$$

Let's denote $|J(t)|^2$ by $h(t)$ and $|\tilde{J}(t)|^2$ by $\tilde{h}(t)$. Note that h and $\tilde{h}$ cannot have any real roots since a holonomy Jacobi field cannot have any zeroes. Also note that f

and $\tilde{f}$ are polynomials of degree at most two, while h and $\tilde{h}$ have degree at most four.

We will show that $h \equiv \tilde{h}$, provided that f is not identically zero. Note that $h \equiv \tilde{h}$ is enough to conclude that κ is basic since $\kappa(X_p) = -h'(0)$ and $\kappa(\tilde{X}) = -\tilde{h}'(0)$.

If $f \equiv 0$, in order to show that κ is basic we will adapt the technique used in Theorem 1.1 from [**11**]. First, assume f is not identically zero and consider the following cases:

(1) f and $\tilde{f}$ are not relatively prime

By (9) and the observations above, we must have $f = a\tilde{f}$, for some real number a. Indeed, since f and $\tilde{f}$ are not relatively prime they must have the same degree since otherwise either J or $\tilde{J}$ will have a zero. For the same reason, the common factor of f and $\tilde{f}$ must be of degree two. Thus $f = a\tilde{f}$, which implies $\tilde{h} = h/a^2$. Since $h(0) = \tilde{h}(0) = 1$, we have $h \equiv \tilde{h}$.

(2) f and $\tilde{f}$ are relatively prime

In this case we get that f^2 divides h. Consequently, f^2 divides $h - f^2$. Note that if $h(0) = f^2(0)$ then the $\mathcal{Z}$ is vertical at p and the leaf through p is the geodesic in the central direction along which κ is basic. If $h(0) \neq f^2(0)$, then the degree of $h - f^2$ is less than or equal to two. By degree count, f must have degree at most one. But f cannot have degree one since f^2 divides h and h has no real roots. Thus, f is constant. A similar argument shows that $\tilde{f}$ must be constant. If $f \neq 0$, as before, we obtain $h \equiv \tilde{h}$.

Let's discuss the case when f is identically zero. By **1.7**(i) and (9), $g, \tilde{f}$, and $\tilde{g}$ are also identically zero. This implies that the vertical bundles at p and $\tilde{p}$ are orthogonal to span$\{X, jX, Z\}$ and span$\{\tilde{X}, j\tilde{X}, Z\}$, respectively.

Let $\lambda = \kappa(X)$ and consider the vector field $J_0(t) = (1 - \lambda t)E \circ \gamma(t)$ along γ, where E is the left invariant vector field defined by the condition that E_p is vertical of unit length. Based on the observations above, J_0 is a Jacobi vector field. Moreover, $J_0'(0) = -\lambda J_0(0) = -S_{\dot{\gamma}(0)}J_0(0)$. Thus, if π is the Riemannian submersion locally defining $\mathcal{F}$, J_0 projects to a Jacobi field $\pi_* J_0$ along $\pi \circ \gamma$ (see [**20**]). Assuming $\lambda \neq 0, \pi \circ \gamma$ has a conjugate point at $1/\lambda$. By Lemma 1 in [**20**], there exists a unique Jacobi field $\tilde{J}_0$ along $\tilde{\gamma}$ such that $\pi_* J_0 = \pi_* \tilde{J}_0$, $\tilde{J}_0(1/\lambda) = 0$, and $\tilde{J}_0'^v + S_{\dot{\tilde{\gamma}}}\tilde{J}_0^v + A_{\dot{\tilde{\gamma}}}\tilde{J}_0^h = 0$. But $\tilde{J}_0 = (1 - \lambda t)\tilde{E} \circ \tilde{\gamma}$, where $\tilde{E}$ is the left invariant vector field for which $\tilde{E}_{\tilde{p}}$ is vertical and of length one. To see this, recall that $\tilde{J}_0(0)$ is orthogonal to $\tilde{X}, j\tilde{X}$, and Z. Thus, the coefficients of Z and $\tilde{Y} = j\tilde{X}$ in the expression of $\tilde{J}_0$ from **1.7**(i) will cancel at $t = 0$. Since they also cancel at $t = 1/\lambda$, they must be identically zero. Based on **1.7**(i) again and the fact that $\tilde{J}_0$ cancels at $t = 1/\lambda$, we may conclude that $\tilde{J}_0$ has the form above. Now, if we let $t = 0$ in $\tilde{J}_0'^v + S_{\dot{\tilde{\gamma}}}\tilde{J}_0^v + A_{\dot{\tilde{\gamma}}}\tilde{J}_0^h = 0$, we get $\kappa(\tilde{X}) = - < S_{\tilde{X}}\tilde{J}_0(0), \tilde{J}_0(0) >= \lambda |\tilde{J}_0(0)|^2$. Consequently, $\kappa(\tilde{X}) = \lambda$. In view of the above, the same must be true if $\lambda = 0$. Consequently κ is basic and we are done. □

In [**32**], G.Walschap shows that every codimension one Riemannian foliation on H^{2n+1} is actually left invariant and is generated by an ideal of the Lie algebra $\mathfrak{h}_{2n+1}$. Thus, the foliation is given by cosets of the form $\{gK, g \in H^{2n+1}\}$, where

K is a normal subgroup of H^{2n+1}. As noted in section **2.3**, all such foliations are homogeneous.

Using the result mentioned above in conjunction with Theorem 1, we obtain the following:

Theorem 2. *All Riemannian foliations on H^3 are homogeneous.*

The following theorem generalizes a result of G.Walschap ([**31**]) regarding one-dimensional Riemannian foliations on three-dimensional nilmanifolds.

Theorem 3. *Let Γ be a lattice in H^{2n+1}. There exists a unique one-dimensional Riemannian foliation $\mathcal{F}$ on $\Gamma\backslash H^{2n+1}$. $\mathcal{F}$ is homogeneous and its lift to H^{2n+1} has vertical bundle $\mathcal{Z}$.*

Proof. Let $\tilde{\mathcal{F}}$ be the lifted foliation. Note that $\tilde{\mathcal{F}}$ is a one-dimensional Riemannian foliation on H^{2n+1}. By Theorem 1, any such foliation is homogeneous. But then $\mathcal{F}$ is also homogeneous since the isometric action defining $\tilde{\mathcal{F}}$ descends to an isometric action on $\Gamma\backslash H^{2n+1}$.

According to [**5**], the identity component of $\mathrm{Iso}(\Gamma\backslash H^{2n+1})$ is $C/C\cap\Gamma$, where C is the center of H^{2n+1}. If $\pi_1 : C \to C/C\cap\Gamma$ and $\pi_2 : H^{2n+1} \to \Gamma\backslash H^{2n+1}$ denote the projection maps then the action of $C/C\cap\Gamma$ on $\Gamma\backslash H^{2n+1}$ is given by $(\pi_1(c), \pi_2(h)) \to \pi_2(ch)$. Consequently, the leaves of the lifted foliation are orbits of C acting on the left on H^{2n+1} and the conclusion follows. Note that we obtain a principal circle bundle over T^{2n}. $\square$

Theorem 2 remains valid for Riemannian foliations on $\Gamma\backslash H^3$. This is due to the fact that there are no two-dimensional Riemannian foliations on $\Gamma\backslash H^3$. Indeed, if $\mathcal{F}$ is such a foliation, by Theorem 2, its lift must be homogeneous. But then $\mathcal{F}$ must also be homogeneous and this is impossible because the dimension of the isometry group of $\Gamma\backslash H^{2n+1}$ is one. The same type of argument used in conjunction with the homogeneity of Riemannian foliations of codimension one on H^{2n+1} implies the following:

Proposition 6. *Let Γ be a lattice in H^{2n+1}. There are no Riemannian foliations of codimension one on $\Gamma\backslash H^{2n+1}$.*

CHAPTER 5

CONCLUDING REMARKS

The main goal of our dissertation is to show that one-dimensional Riemannian foliations on the Heisenberg group are homogeneous. In section **1.5** we describe the corresponding Killing vector fields associated to these foliations. Using our result and the homogeneity of codimension one foliations ([**32**]), we conclude that every Riemannian foliation on the three-dimensional Heisenberg group is homogeneous. As noted in [**31**], the same is true on $\Gamma\backslash H^3$, where Γ is a lattice in H^3. In chapter **4** we give another proof for this theorem and we improve it by showing that there are no codimension one foliations on $\Gamma\backslash H^{2n+1}$. We also show that the only one-dimensional Riemannian foliation on the space above occurs as the projection of the foliation on H^{2n+1} with leaves tangent to the center.

Some interesting geometric properties for Riemannian foliations of any dimension on the Heisenberg group are also derived. It would be interesting to investigate whether all these foliations are homogeneous. If valid, the previous statement would imply that there is only one Riemannian foliation on $\Gamma\backslash H^{2n+1}$. This is the one-dimensional foliation described above.

We expanded our inquiries to one-dimensional Riemannian foliations on semisimple Lie groups and showed that they are also homogeneous provided the leaves are closed. This result is not included in the dissertation and will soon be submitted for publication.

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