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WEBS FOR TYPE Q LIE SUPERALGEBRAS

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DEPARTMENT OF MATHEMATICS

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For my grandparents

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Contents

Abstract

This dissertation consists of two parts. In the first part, we construct a monoidal supercategory $\mathfrak{q}_n\text{-}\mathbf{Web}_{\uparrow\downarrow}$, whose morphism spaces are superspaces spanned by oriented type Q webs modulo certain relations. We prove that $\mathfrak{q}_n\text{-}\mathbf{Web}_{\uparrow\downarrow}$ is monoidally equivalent to the full subsupercategory of modules over the Lie superalgebra $\mathfrak{q}_n$, tensor-generated by the symmetric powers $\mathscr{S}^k(\mathbb{C}^{n|n})$ and their duals $\mathscr{S}^k(\mathbb{C}^{n|n})^*$. This affords a diagrammatic presentation by generators and relations of the $\mathfrak{q}_n$-morphisms between these modules. In the second part, we prove that a related (not monoidal) supercategory $\mathfrak{q}\text{-}\mathbf{Web}^k_{\uparrow}$ is equivalent to the supercategory of spin permutation modules $\mathcal{M}^\lambda$ of the Sergeev superalgebra Ser_k. We develop a combinatorics of weighted supertabloids, and identify bases for the space $\mathrm{Hom}_{\mathrm{Ser}_k}(\mathcal{M}^\lambda, \mathcal{M}^\mu)$ of Ser_k-morphisms between any $\mathcal{M}^\lambda, \mathcal{M}^\mu$, in terms of both supertabloids and webs.

Chapter 1

Introduction

1.1. Overture

The subject of this dissertation is the representation theory of the Lie superalgebra $\mathfrak{q}_n$ and the Sergeev superalgebra Ser_k over the field $\mathbb{C}$ of complex numbers.[1] The two are intimately related by a result called the Schur-Weyl-Sergeev duality, so it is no surprise that they may be studied simultaneously. Roughly speaking, the prefix "super" means "$\mathbb{Z}_2$-graded", where $\mathbb{Z}_2 = \{0, 1\}$ is the group with two elements. The axioms for superalgebras are natural $\mathbb{Z}_2$-graded analogs of those for ordinary algebras, and every conceivable datum associated to the representation theory of a superalgebra (modules, homomorphisms, tensor products, . . .) has and/or respects a $\mathbb{Z}_2$-grading. Lie superalgebras first appeared in the physics of supersymmetry, and were classified into types by Kac in the 1970s [22]. The type Q Lie superalgebra, $\mathfrak{q}_n$, constitutes one such type, while Ser_k arose decades later in the work of Sergeev [37] as a byproduct of the representation theory of $\mathfrak{q}_n$.

More specifically, this dissertation is concerned with the goal of obtaining a complete diagrammatic description of the $\mathfrak{q}_n$-morphisms[2] (resp. Ser_k-morphisms)

[1]Elsewhere in the literature, the former is called the "queer" Lie superalgebra, and the latter the "Hecke-Clifford" superalgebra.

[2]We shorten "homomorphism of A-modules" to "A-morphism" for a superalgebra A.

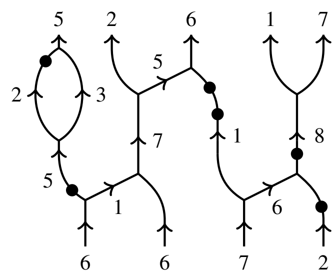
between certain $\mathfrak{q}_n$-modules (resp. Ser_k-modules). The idea is to represent each morphism by some sort of picture, with the operations of composition and – in the case of $\mathfrak{q}_n$ – tensor product translated into certain operations on the pictures. We then allow linear combinations of pictures, in order to model linear combinations of morphisms. Once a collection of modules of interest – and a basic pictorial scheme for the morphisms between them – have been decided upon, one can ask for a presentation of the morphisms by generators and relations with respect to the aforementioned operations – a complete diagrammatic description. Notably, we will make use of a single diagrammatic calculus to describe morphisms between modules over both $\mathfrak{q}_n$ and Ser_k.

There are several reasons for wanting to pursue such a diagrammatic description. Foremost among them is the fact that, with a little practice, it is significantly easier to perform and interpret calculations on morphisms when they are written in terms of diagrams instead of mathematical script. Questions like, "Are these two morphisms actually the same?", "Is this morphism invertible?", and so on are much easier to answer in the diagrammatic setting. With diagrams, previously hidden aspects of the ambient representation theory can be uncovered, and connections to the representation theories of other structures are more readily observed.

In the remainder of this chapter, we will introduce the combinatorics of oriented type Q webs, briefly sketch their origins and development, and exhibit some of the core ideas of the dissertation. Along the way, we will also describe the main results of the dissertation. We conclude by outlining the organization of the dissertation and discussing future avenues of research.

1.2. Oriented type Q webs

For the sake of exposition, we will focus in this chapter on the way oriented type Q webs describe $\mathfrak{q}_n$-morphisms rather than Ser_k-morphisms. Our $\mathfrak{q}_n$-modules of interest are tensor products of the symmetric powers $\mathscr{S}^k := \mathscr{S}^k(\mathbb{C}^{n|n})$ and their duals $\mathscr{S}^{k*} := \mathscr{S}^k(\mathbb{C}^{n|n})^*$ for $k \in \mathbb{Z}_{>0}$. (See Chapter 2 for the definitions of $\mathfrak{q}_n$ and these modules.) We call the diagrams used to describe the $\mathfrak{q}_n$-morphisms between them *oriented type Q webs*, although we will often simply refer to them as *webs*. Not allowing duals for the moment, a first example of a web is

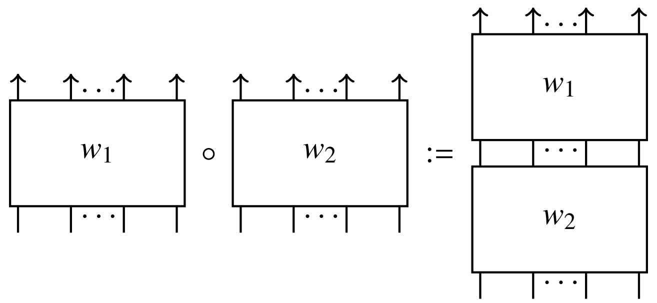


We illustrate $\mathscr{S}^k$, or equivalently the identity map $\mathscr{S}^k \to \mathscr{S}^k$, by $\uparrow$ labeled with the number k. The tensor product of modules is illustrated by horizontal concatenation, and we follow the convention of reading webs from bottom to top. Hence the above web represents a $\mathfrak{q}_n$-morphism

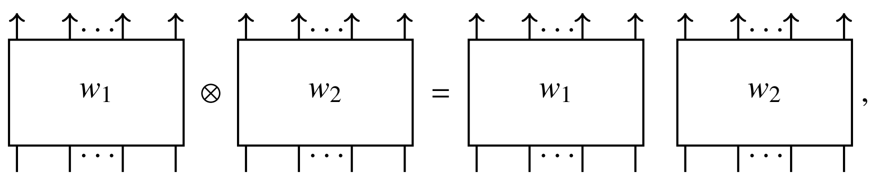
$$\mathscr{S}^6 \otimes \mathscr{S}^6 \otimes \mathscr{S}^7 \otimes \mathscr{S}^2 \quad \longrightarrow \quad \mathscr{S}^5 \otimes \mathscr{S}^2 \otimes \mathscr{S}^6 \otimes \mathscr{S}^1 \otimes \mathscr{S}^7.$$

Note that the arrowheads and numbers in a web are merely labels, so their exact locations relative to the edges they are labeling and to any dots on those edges are immaterial. We will use the terms "edge" and "strand" interchangeably, and will refer to the numerical label of a strand as its "thickness", despite the fact that all strands are drawn with the same thickness (the metaphor will nevertheless be justified shortly). Moreover, we will refer to a strand of thickness k as a "k-strand".

The bottom-to-top convention implies that composition of webs corresponds to vertical stacking, i.e. for webs w_1, w_2 we have

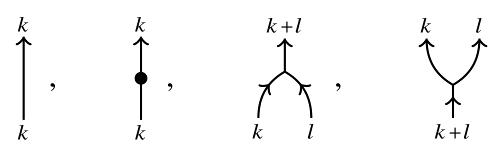


If the source of w_1 is not equal to the target of w_2, we declare $w_1 \circ w_2 = 0$. For now, one may think of the tensor product of webs as simply horizontal concatenation,



although this is not quite the truth for reasons related to the $\mathbb{Z}_2$-grading on morphisms (see Section 4.1).

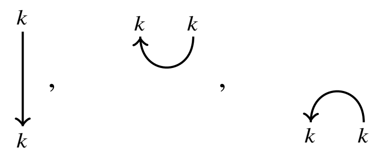
Webs which are entirely upward-oriented like the example above can be built up by taking compositions of tensor products of four basic types of webs. For this reason we call them the *upward-oriented generators*. They consist of



for $k, l \in \mathbb{Z}_{>0}$, which we call *identities*, *dots*, *merges*, and *splits*, respectively. Each is $\mathbb{Z}_2$-homogeneous but only dots are of degree 1, the rest 0. Since $\mathbb{Z}_2$-degree is additive across compositions and tensor products (see Section 2.1), every upward-oriented web is homogeneous with degree the number of dots modulo 2 (this will also be true for arbitrarily oriented webs). What $\mathfrak{q}_n$-morphism each represents will be discussed later (see Section 5.2), but their sources and targets are clear.

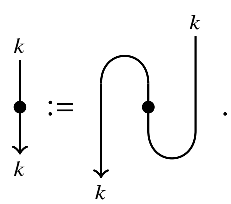
We also allow $\mathbb{C}$-linear combinations among webs having pairwise equal sources and pairwise equal targets. Moreover, compositions and tensor products are declared to be distributive with respect to addition.

As for (not necessarily upward-) oriented type Q webs, we adjoin the dual symmetric powers $\mathscr{S}^{k*}$, illustrated by $\downarrow$ labeled with the number k, and the generators

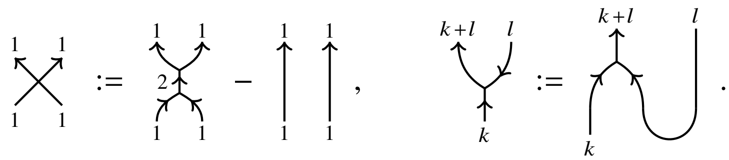


for $k \in \mathbb{Z}_{>0}$, called *identities*, *cups*, and *caps*, respectively. Each of these webs is $\mathbb{Z}_2$-homogeneous of degree 0. Note that the source of every cup and the target of every cap is the trivial $\mathfrak{q}_n$-module $\mathscr{S}^0 = \mathbb{C}$, which we draw as an empty region. This means we are implicitly viewing the $\mathfrak{q}_n$-isomorphisms $\mathbb{C} \otimes V \simeq V \simeq V \otimes \mathbb{C}$ for a $\mathfrak{q}_n$-module V as equalities.

Right-oriented cups and caps, and downward-oriented dots, merges, and splits can all be defined in terms of the existing generators, for example,



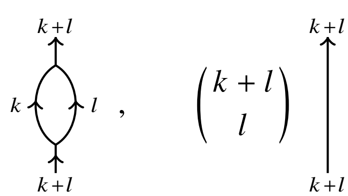
We can even define crossings and multi-oriented merges and splits, for example,



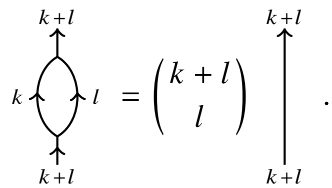
(See Section 4.5 for the definitions of crossings between strands of arbitrary orientation and thickness.) It would only make sense to define such crossings if they satisfy the Coxeter relations of the simple transpositions in the symmetric group $\mathfrak{S}_k$, which is indeed the case here. In fact, oriented Brauer diagrams (see [1]) and oriented Brauer-Clifford diagrams (see [8]) fit neatly into the framework of oriented type Q webs (see Remark 5.7).

1.3. Some history

The task of finding a complete set of relations among the morphisms being described is usually much more challenging than finding a set of generators. Let us clarify what is meant by "finding relations." It so happens, for example, that the $\mathfrak{q}_n$-morphisms represented by the webs



are equal for $k, l \in \mathbb{Z}_{>0}$, where the scalar on the right is a binomial coefficient. We should therefore declare the corresponding webs to be equal, resulting in the relation



The problem is knowing whether you have found and declared all possible relations.

This is the question Kuperberg was dealing with when he originally developed webs in the 1990s [26]. He was attempting to describe morphisms between certain modules over the quantum group $U_q(\mathfrak{sl}_n)$, an associative algebra over the field $\mathbb{C}(q)$ of rational functions in the indeterminate q. A 1932 result of Rumer, Teller, and Weyl [35], modernly interpreted, had already accomplished this for $n = 2$ in terms of what are known today as Temperley-Lieb diagrams. Kuperberg's webs subsume the Temperley-Lieb calculus, and he achieved a diagrammatic description of the relevant morphisms for $n = 3$.

As for $n > 3$, partial progress was made by Morrison [28] and Kim [23, 24], but the problem remained open until 2014 with the remarkable work of Cautis, Kamnitzer, and Morrison [9]. They noticed that when properly viewed, a powerful result in representation theory – the quantum skew Howe duality – provides all of the generators and most of the relations for free. Since then, other authors have applied this method to different types of Howe duality, obtaining similar results in other settings. Most of these have remained in type A [33, 34, 42], but of note is Sartori and Tubbenhauer's recent foray into types B, C, and D [39].

1.4. Motivation and methods

The starting point of the work in this dissertation was an attempt to apply the methods of [9] to the type Q Howe duality discovered by Cheng and Wang [10]. Let us briefly explain that approach, as it will both motivate the present work and give an indication of the methods involved. As previously, we will suppress the dual symmetric powers at first.

Given two type Q Lie superalgebras $\mathfrak{q}_m$ and $\mathfrak{q}_n$, there is a superspace

$$\mathscr{S} := \mathscr{S}(\mathbb{C}^{m|m} \circledast \mathbb{C}^{n|n})$$

on which both can act simultaneously. (Here $\circledast$ is a $\mathbb{Z}_2$-graded version of the tensor product of modules, see Section 2.1.) The actions commute with each other and are mutually centralizing [10, Corollary 3.1]; for our purposes, these statements translate respectively to the existence and surjectivity of a superalgebra homomorphism

$$\phi_m^{\uparrow}\colon U(\mathfrak{q}_m) \twoheadrightarrow \mathrm{End}_{\mathfrak{q}_n}(\mathscr{S})$$

where $U(\mathfrak{q}_m)$ is the universal enveloping superalgebra of $\mathfrak{q}_m$ and $\mathrm{End}_{\mathfrak{q}_n}(\mathscr{S})$ is the space of $\mathfrak{q}_n$-endomorphisms of $\mathscr{S}$.

It happens that $\mathscr{S}$ is a weight module of $\mathfrak{q}_m$, with weights in bijection with $\mathbb{Z}_{\geq 0}^m$. Moreoever, given a $\mathfrak{q}_m$ weight $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m) \in \mathbb{Z}_{\geq 0}^m$, the associated weight space of $\mathscr{S}$ is isomorphic *as a* $\mathfrak{q}_n$*-module* to

$$\mathscr{S}^{\lambda} := \mathscr{S}^{\lambda_1}(\mathbb{C}^{n|n}) \otimes \mathscr{S}^{\lambda_2}(\mathbb{C}^{n|n}) \otimes \cdots \otimes \mathscr{S}^{\lambda_m}(\mathbb{C}^{n|n}),$$

a tensor product of symmetric powers as seen in the previous section. The upshot is that $\phi_m^{\uparrow}$ provides correspondences

$$\{\text{elements of } U(\mathfrak{q}_m)\} \quad \longrightarrow \quad \left\{\mathfrak{q}_n\text{-morphisms } \mathscr{S}^{\lambda} \to \mathscr{S}^{\mu} \text{ for } \lambda, \mu \in \mathbb{Z}_{\geq 0}^m\right\}$$

$$\{\text{relations in } U(\mathfrak{q}_m)\} \quad \longrightarrow \quad \{\text{relations among the above } \mathfrak{q}_n\text{-morphisms}\}$$

where in the second of these, multiplication of elements in $U(\mathfrak{q}_m)$ corresponds to composition of $\mathfrak{q}_n$-morphisms. This explains our decision to focus on tensor products of symmetric powers in the first place. Note that since m determines the length of these tensor products, which we want to allow to be arbitrarily high, we must consider $\phi_m^{\uparrow}$ for all $m \in \mathbb{Z}_{>0}$.

Let $\mathfrak{q}_n$-$\mathbf{Mod}_{\mathscr{S}}$ be the monoidal category of $\mathfrak{q}_n$-modules with objects all tensor products of symmetric powers.[3] In categorical language, a diagrammatic description of $\mathfrak{q}_n$-$\mathbf{Mod}_{\mathscr{S}}$ amounts to a monoidal equivalence between $\mathfrak{q}_n$-$\mathbf{Mod}_{\mathscr{S}}$ and a monoidal category whose morphisms are defined diagrammatically. There is a natural way to translate $U(\mathfrak{q}_m)$ into webs (see Lemma 5.3), and the above correspondences tell us how to begin defining a monoidal category whose morphisms are (linear combinations of) oriented type Q webs, $\mathfrak{q}$-$\mathbf{Web}_{\uparrow}$, and a monoidal functor

$$\Psi^{\uparrow} \colon \mathfrak{q}\text{-}\mathbf{Web}_{\uparrow} \to \mathfrak{q}_n\text{-}\mathbf{Mod}_{\mathscr{S}}.$$

From the work leading up to its definition, it will be easily proved that $\Psi^{\uparrow}$ satisfies all of the properties required to be a monoidal equivalence except faithfulness. That is, it is possible there are relations in $\mathfrak{q}_n$-$\mathbf{Mod}_{\mathscr{S}}$ which do not come from $U(\mathfrak{q}_m)$ for any m, and hence have not yet been imposed in $\mathfrak{q}$-$\mathbf{Web}_{\uparrow}$.

This does turn out to be the case, but identifying the missing relations proves to be a redeemably illuminating exercise. Indeed, a series of reduction arguments

[3]Roughly speaking, a monoidal category is one which has an operation similar to the tensor product in the category of vectors spaces over a field.

imply that the missing relators are found in the kernels of the maps

$$\Xi_k \colon \operatorname{Ser}_k \twoheadrightarrow \operatorname{End}_{\mathfrak{q}_n}\left((\mathscr{S}^1)^{\otimes k}\right)$$

for sufficiently large k where $(\mathscr{S}^1)^{\otimes k} := \mathscr{S}^1 \otimes \mathscr{S}^1 \otimes \cdots \otimes \mathscr{S}^1$ (k tensorands). Crucially, Ser_k turns out to be isomorphic to the space of webs in $\mathfrak{q}$-$\mathbf{Web}_{\uparrow}$ starting and ending in k-many 1-strands.

Further arguments involving the combinatorics of shifted tableaux prove that, surprisingly, there exists a single element $e_n \in \operatorname{Ser}_k$ for $k = 1 + 2 + \cdots + (n + 1)$ with the property that setting the corresponding morphism of $\mathfrak{q}$-$\mathbf{Web}_{\uparrow}$ to zero makes $\Psi^{\uparrow}$ faithful. That is, if we define $\mathfrak{q}_n$-$\mathbf{Web}_{\uparrow}$ to be the quotient of $\mathfrak{q}$-$\mathbf{Web}_{\uparrow}$ by the additional relation $e_n = 0$ (the only relation which depends on n), then the induced functor $\Psi^{\uparrow} \colon \mathfrak{q}_n$-$\mathbf{Web}_{\uparrow} \to \mathfrak{q}_n$-$\mathbf{Mod}_{\mathscr{S}}$ is a monoidal equivalence (Theorem 5.6). This technique is very similar to one used in [42, §5].

Fortunately, extending $\Psi^{\uparrow}$ to a monoidal equivalence

$$\Psi^{\uparrow\downarrow} \colon \mathfrak{q}_n\text{-}\mathbf{Web}_{\uparrow\downarrow} \to \mathfrak{q}_n\text{-}\mathbf{Mod}_{\mathscr{S},\mathscr{S}^*}$$

where $\mathfrak{q}_n$-$\mathbf{Web}_{\uparrow\downarrow}$ includes webs of all orientations and $\mathfrak{q}_n$-$\mathbf{Mod}_{\mathscr{S},\mathscr{S}^*}$ includes dual symmetric powers, is comparatively easy. In particular, one does not need to apply special methods to find missing relations a second time. It should be noted, however, that the actual layout of this dissertation follows a slightly different route to $\Psi^{\uparrow\downarrow}$ than the one discussed above; rather, the idea behind its conception is what is on display here.

A few concluding remarks are in order. First, each piece of categorical data used in this section (e.g. category, functor) should actually be replaced by its natural super analog (e.g. supercategory, superfunctor). The latter arise from the fact that the morphism spaces being considered here are $\mathbb{Z}_2$-graded. (See Section 2.3 for definitions.) For these, we follow the recent exposition of Brundan and Ellis [3], of which the present work will be one of the first applications (another is [8]).

Second, we do *not* need the full power of the type Q Howe duality, namely the surjectivity of the maps $\phi_m^\uparrow$, for our arguments to go through. On the contrary, we need only begin with the fact that each $\phi_m^\uparrow$ exists to develop the machinery of webs, and use them and the Schur-Weyl-Sergeev duality to *prove* that the $\phi_m^\uparrow$ are surjective (Corollary 5.8). This idea was first used by Sartori and Tubbenhauer [39] to prove new Howe dualities using webs, in a cunning reversal of the usual Howe-duality-to-webs program of Cautis-Kamnitzer-Morrison. This affords a rigorous proof, using the diagrammatics of webs, of the folklore result to the effect that the relevant Schur-Weyl duality implies the Howe duality, in type Q.

1.5. Webs for spin permutation modules

Upward-oriented type Q webs can also be used to describe Ser_k-morphisms between the spin permutation modules $\mathcal{M}^\lambda$ where $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{Z}_{\geq 0}^k$ such that $\lambda_1 + \cdots + \lambda_k = k$. (See Chapter 6 for definitions of Ser_k and these modules.) Indeed, there are commuting actions of $\mathfrak{q}_n$ and Ser_k on the tensor space $(\mathbb{C}^{n|n})^{\otimes k}$, and each $\mathfrak{q}_n$ weight space is isomorphic as a Ser_k-module to some $\mathcal{M}^\lambda$. Hence we can play the same game as before, except the appropriate webs category already exists: it is the subcategory $\mathfrak{q}$-$\mathbf{Web}_\uparrow^k$ of $\mathfrak{q}$-$\mathbf{Web}_\uparrow$ whose webs have the property that the sum of the thicknesses of their strands is k. The main differences here are that $\mathfrak{q}$-$\mathbf{Web}_\uparrow^k$ is *not* monoidal and, of course, does not include downward-oriented strands; thus we will not be modeling tensor products or duals of the $\mathcal{M}^\lambda$. In particular, there are only finitely many $\mathcal{M}^\lambda$, whereas there are infinitely many tensor products of $\mathscr{S}^k$ and $\mathscr{S}^{k*}$.

Let Ser_k -$\mathbf{Mod}_{\mathcal{M}}$ be the full subcategory of Ser_k-modules with objects the spin permutation modules $\mathcal{M}^\lambda$. Unlike the situation with $\Psi^\uparrow$, the functor

$$\Gamma\colon \mathfrak{q}\text{-}\mathbf{Web}_\uparrow^k \to \mathrm{Ser}_k\text{ -}\mathbf{Mod}_{\mathcal{M}}$$

arising from the aforementioned commuting actions is already faithful and, in fact, an equivalence (Theorem 6.5). Consequently, it is much easier to produce a basis for each space $\mathrm{Hom}_{\mathrm{Ser}_k}(\mathcal{M}^\lambda, \mathcal{M}^\mu)$ of Ser_k-morphisms, by producing one for each morphism space $\mathrm{Hom}_{\mathfrak{q}\text{-}\mathbf{Web}_\uparrow^k}(\lambda, \mu)$ of $\mathfrak{q}$-$\mathbf{Web}_\uparrow^k$ and applying Γ. We do exactly this, obtaining a basis for these Ser_k-morphisms in terms of webs (Corollary 6.8). Also of note is the combinatorics of *weighted supertabloids*, which, ultimately, we develop for the purpose of proving the aforementioned basis theorem.

1.6. Outline and summary of results

The dissertation is organized as follows. First, in Chapters 2 and 3 we provide all necessary background information for the main content of the dissertation. Chapter 2 covers basic generalities needed for studying representations of superalgebras, while Chapter 3 discusses the pertinent representation theory of $\mathfrak{q}_n$ and Ser_k. Next, in Chapter 4 we develop the combinatorics of oriented type Q webs and explore their various properties.

Finally, in Chapters 5 and 6 we apply the method of [9] to two different sets of commuting actions and prove the main theorems of the dissertation. In Chapter 5 we prove the monoidal superequivalences $\mathfrak{q}_n$-$\mathbf{Web}_\uparrow \cong \mathfrak{q}_n$-$\mathbf{Mod}_{\mathscr{S}}$ and $\mathfrak{q}_n$-$\mathbf{Web}_{\uparrow\downarrow} \cong$

$\mathfrak{q}_n$-$\mathbf{Mod}_{\mathscr{S},\mathscr{S}^*}$ (Theorem 5.6), obtaining a webs description of the $\mathfrak{q}_n$-morphisms between tensor products of symmetric powers $\mathscr{S}^k$ and their duals $\mathscr{S}^{k*}$. In Chapter 6 we prove the superequivalence $\mathfrak{q}$-$\mathbf{Web}_{\uparrow}^{k} \cong \mathrm{Ser}_k$ -$\mathbf{Mod}_{\mathcal{M}}$ (Theorem 6.5), obtaining a diagrammatic description of the Ser_k-morphisms between spin permutation modules $\mathcal{M}^{\lambda}$. Also in Chapter 6, we establish bases for the morphism spaces $\mathrm{Hom}_{\mathrm{Ser}_k}(\mathcal{M}^{\lambda}, \mathcal{M}^{\mu})$ in terms of both webs and supertabloids (Corollary 6.8).

1.7. Future work

The results of this dissertation raise a number of interesting questions. Perhaps most immediate is whether there is a version of Theorem 5.6 for the quantum group $U_q(\mathfrak{q}_n)$, a deformation (or q-analog) of $U(\mathfrak{q}_n)$ which is an associative superalgebra over the field $\mathbb{C}(q)$ of rational functions in the indeterminate q. First defined by Olshanskii [32], it has since been explored, for example, in [2, 12, 13, 15, 21]. This matter is currently being investigated by Nicholas Davidson, Jonathan Kujawa, and the author, and early results look promising.

Aside from elucidating more of the representation theory of $U_q(\mathfrak{q}_n)$, there is another reason for pursuing such a result. It is known that every braided monoidal category can be viewed as a machine for producing knot invariants, in a process we avoid discussing further here. While the category of all $U_q(\mathfrak{q}_n)$-modules does not admit a braiding (see [32]), there is hope that various well-behaved subcategories do. If this is the case, then an analog of Theorem 5.6 for $U_q(\mathfrak{q}_n)$ could furnish a machine for producing type Q knot invariants, which, to the author's knowledge, have not been seen before.

One could also ask for a version of Theorem 5.6 for the type P Lie superalgebra $\mathfrak{p}_n$. The immediate problem here is that the Schur-Weyl partner of $\mathfrak{p}_n$ – called the marked Brauer algebra by Kujawa and Tharp [27, 41] and the periplectic Brauer algebra by Coulembier and Ehrig [5–7] – is not semisimple, so the techniques used in this dissertation exploiting the semisimplicity of Ser_k would not carry over. (Also, note that a quantum group $U_q(\mathfrak{p}_n)$ has yet to be discovered.)

In another direction, there is a q-analog of Ser_k called the (quantum) Hecke-Clifford superalgebra, and the question arises as to whether there are versions of Theorem 6.5 and Corollary 6.8 for it. This object also first appeared in [32], and has since been studied in, for example, [2, 13]. In particular, the corresponding analogs of the spin permutation modules $\mathcal{M}^\lambda$ were defined and studied in [13]. This matter is currently being investigated by the author, and looks promising, albeit contingent on a basic understanding of the quantum type Q webs. A fortunate consequence of such a theorem would be that morphisms between four different types of permutation modules – those of the Hecke-Clifford superalgebra, Ser_k, $\mathfrak{S}_k$, and the Iwahori-Hecke algebra $\mathscr{H}(\mathfrak{S}_k)$ – would be obtainable from a single webs calculus, by setting $q = 1$, disallowing Clifford dots, or both.

Finally, it would be interesting if, in addition to the basis of $\mathrm{Hom}_{\mathrm{Ser}_k}(\mathcal{M}^\lambda, \mathcal{M}^\mu)$ given in Corollary 6.8, there is a "semistandard" basis in the spirit of the semistandard bases of the corresponding morphism spaces for $\mathfrak{S}_k$ and $\mathscr{H}(\mathfrak{S}_k)$ (see [18, §13] and [30, §4.3], respectively). In particular, both of the latter, letting λ, μ run over the appropriate indexing set, are *cellular* bases of the associated Schur algebras in the sense of [16]. It is an open question as to whether there is an adequate notion of cellularity for superalgebras, but it seems one could be pursued in a manner similar to the way monoidal supercategories generalize the notion of monoidal categories. (See the comment about the failure of cellularity for the marked Brauer algebra in [27, §1.3].)

Chapter 2

General background

In this chapter, we provide general background information necessary for the main content of the dissertation. This includes discussions of superalgebras and their modules, (symmetric monoidal) supercategories, and locally unital superalgebras.

2.1. Basic superalgebra

First, we recall some basic facts about superalgebras and their modules. The reader is referred to [4, §2] or [11] for more information. All vector spaces will be over the field $\mathbb{C}$ of complex numbers.

We write $\mathbb{Z}_2 = \{0, 1\}$ for the group with two elements, and define a *superspace* to be a $\mathbb{Z}_2$-graded vector space $V = V_0 \oplus V_1$, where V_0 (resp. V_1) is the *even part* (resp. the *odd part*) of V. A nonzero vector $v \in V$ is ($\mathbb{Z}_2$-)*homogeneous* if $v \in V_0 \cup V_1$, and if $v \in V_i$ we denote by $\overline{v} = i \in \mathbb{Z}_2$ its ($\mathbb{Z}_2$-)*degree*, or *parity*. A *subsuperspace* of V is a superspace $W = W_0 \oplus W_1$ which is contained in V and inherits its $\mathbb{Z}_2$-grading from V, i.e. $W_0 = W \cap V_0$ and $W_1 = W \cap V_1$.

Remark 2.1. *We adopt the convention that whenever a barred vector $\overline{v}$ occurs, it is assumed that v is homogeneous. Similarly, whenever a definition is given in terms of homogeneous vectors, it should be extended to arbitrary vectors by linearity.*

The *superdimension* of a superspace V is $\operatorname{sdim} V := \dim V_0 - \dim V_1$. Up to isomorphism, finite-dimensional superspaces are of the form $\mathbb{C}^{m|n} := \mathbb{C}^m \oplus \mathbb{C}^n$, which has superdimension $m - n$.

Tensor products $V \otimes W$ and morphism spaces $\operatorname{Hom}(V, W)$ of linear maps $V \to W$ for superspaces V, W are each in turn superspaces. Indeed, the parity assignments are given by

$$\overline{v \otimes w} := \overline{v} + \overline{w}, \quad \overline{f} := i \text{ if } f(V_j) \subseteq W_{i+j},$$

respectively, for $v \in V$, $w \in W$, $f \in \operatorname{Hom}(V, W)$, and $i, j \in \mathbb{Z}_2$. Even maps (resp. odd maps) of superspaces are also called *grading-preserving* maps (resp. *grading-reversing* maps).

Tensor products of linear maps also behave differently in the super world: for $f \in \operatorname{Hom}(V, V')$ and $g \in \operatorname{Hom}(W, W')$ we have

$$(f \otimes g)(v \otimes w) := (-1)^{\overline{g} \cdot \overline{v}} f(v) \otimes g(w)$$

for $v \in V$, $w \in W$. Furthermore, for composable pairs f, h and g, k of linear maps, we have the *superinterchange law*

$$(f \otimes g) \circ (h \otimes k) := (-1)^{\overline{g} \cdot \overline{h}} (f \circ h) \otimes (g \circ k). \tag{2.1}$$

An *associative superalgebra* consists of a superspace $A = A_0 \oplus A_1$ with an associative, bilinear multiplication satisfying $A_i A_j \subseteq A_{i+j}$ for $i, j \in \mathbb{Z}_2$. (In this work, superalgebras are not necessarily unital, see §2.3.) We will use the word "superalgebra" with no adjective preceding it to mean "associative superalgebra". Every ordinary algebra A has the structure of a superalgebra concentrated in degree zero, i.e. $A_0 = A$, $A_1 = 0$.

A *subsuperalgebra* of A is a superalgebra B which is a subsuperspace of A. A *superalgebra homomorphism* (which we will often shorten to *homomorphism*) is a grading-preserving linear map $\phi\colon A \to B$ between superalgebras A, B with the property that $\phi(a_1 a_2) = \phi(a_1)\phi(a_2)$ for $a_1, a_2 \in A$.[1]

The linear endomorphisms $\mathrm{End}(V) = \mathrm{Hom}(V, V)$ of a superspace V naturally form a superalgebra under composition, since $\overline{f \circ g} = \overline{f} + \overline{g}$ for $f, g \in \mathrm{End}(V)$. Furthermore, if V is finite-dimensional, say $V \simeq \mathbb{C}^{m|n}$, then after choosing a $\mathbb{Z}_2$-homogeneous basis of V we may identify $\mathrm{End}(V)$ with the matrix superalgebra

$$M(m|n) := \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} : A \text{ is } m \times m,\ B \text{ is } m \times n,\ C \text{ is } n \times m,\ D \text{ is } n \times n \right\},$$

which has the superspace decomposition

$$M(m|n)_0 = \left\{ \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \right\}, \quad M(m|n)_1 = \left\{ \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \right\}.$$

(One can check that this is consistent with the $\mathbb{Z}_2$-grading defined on $\mathrm{End}(V)$ above.) In this case, we may emphasize this identification by denoting $\mathrm{End}(V)$ by $M(V)$. If additionally $m = n$, then we have the subsuperalgebra

$$Q(n) := \left\{ \begin{bmatrix} A & B \\ B & A \end{bmatrix} \right\} \subseteq M(n|n).$$

In this case, we may also denote $Q(n)$ by $Q(V)$. Note that

$$\dim Q(n)_i = n^2 = \tfrac{1}{2} \dim M(n|n)_i$$

[1]The reader is warned that the requirement of being grading-preserving is not always present elsewhere in the literature.

for $i \in \mathbb{Z}_2$ and consequently

$$\dim Q(n) = \frac{1}{2} \dim M(n|n).$$

A *Lie superalgebra* consists of a superspace $\mathfrak{g}$ with a bilinear multiplication $[\cdot\,,\cdot]$ satisfying

- $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$,
- $[a,b] = -(-1)^{\overline{a}\cdot\overline{b}}[b,a]$, and
- $[a,[b,c]] = [[a,b],c] + (-1)^{\overline{a}\cdot\overline{b}}[b,[a,c]]$

for $a,b,c \in \mathfrak{g}$ and $i,j \in \mathbb{Z}_2$. The even part $\mathfrak{g}_0$ is a Lie algebra in the ordinary sense. Every superalgebra A is a Lie superalgebra under the *supercommutator* $[a,b] = ab - (-1)^{\overline{a}\cdot\overline{b}}ba$ for $a,b \in A$. For a superspace V, we say two endomorphisms $X,Y \in \mathrm{End}(V)$ *supercommute* if $[X,Y] = 0$. A *homomorphism of Lie superalgebras* $\phi\colon \mathfrak{g} \to \mathfrak{h}$ between Lie superalgebras $\mathfrak{g},\mathfrak{h}$ is a linear map ϕ satisfying $[\phi(x),\phi(y)] = \phi([x,y])$ for all $x,y \in \mathfrak{g}$.

The *universal enveloping superalgebra* of a Lie superalgebra $\mathfrak{g}$ is the unique associative unital superalgebra $U(\mathfrak{g})$ equipped with a Lie superalgebra homomorphism $i\colon \mathfrak{g} \to U(\mathfrak{g})$ satisfying the following universal property: for every superalgebra A and Lie superalgebra homomorphism $\phi\colon \mathfrak{g} \to A$, there exists a unique superalgebra homomorphism $\psi\colon U(\mathfrak{g}) \to A$ such that $\psi \circ i = \phi$. We will use the terms "$\mathfrak{g}$-module" and "$U(\mathfrak{g})$-module", and the terms "$\mathfrak{g}$-morphism" and "$U(\mathfrak{g})$-morphism", interchangeably, as the representation theories of the two superalgebras are equivalent.

We call $M(m|n)$, when viewed as a Lie superalgebra, the *general linear Lie superalgebra* and denote it $\mathfrak{gl}_{m|n}$. Likewise we call $Q(n)$, when viewed as a Lie subsuperalgebra of $\mathfrak{gl}_{n|n}$, the *type Q Lie superalgebra* and denote it $\mathfrak{q}_n$. The reader can verify that an equivalent definition of $\mathfrak{q}_n$ is

$$\mathfrak{q}_n = \{X \in \mathfrak{gl}_{n|n} : [X, P_n] = 0\}$$

where

$$P_n := \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \in (\mathfrak{gl}_{n|n})_1 \tag{2.2}$$

and I_n denotes the $n \times n$ identity matrix. Note that $P_n^2 = -I_{2n}$.

A *supermodule* over a superalgebra A, or an A-*supermodule*, consists of a superspace V and a homomorphism $\rho\colon A \to \mathrm{End}(V)$. When ρ is understood we will simply write $a.v$ in place of $\rho(a)(v)$. The natural $M(m|n)$-supermodule is $\mathbb{C}^{m|n}$, and the natural $Q(n)$-supermodule is $\mathbb{C}^{n|n}$, where both actions are given by matrix multiplication.

An A-*supermodule homomorphism* (which we will often shorten to A-*morphism*) between A-supermodules V, W is a linear map $\phi\colon V \to W$ with the property that

$$\phi(a.v) = (-1)^{\overline{\phi}\cdot\overline{a}} a.\phi(v)$$

for $a \in A$, $v \in V$. We denote by $\mathrm{Hom}_A(V, W)$ the space of A-morphisms $V \to W$, a subsuperspace of $\mathrm{Hom}(V, W)$. Of particular interest will be the space $\mathrm{End}_A(V) = \mathrm{Hom}_A(V, V)$ of A-endomorphisms of V, a superalgebra under composition.

A *subsupermodule* of an A-supermodule V is a submodule $W \subseteq V$ in the ordinary sense which is also a subsuperspace of V. An A-supermodule V is *irreducible*, or *simple*, if it has no nontrivial subsupermodules. Irreducibles are further classified into two types:

- V is *self-associate* if it has a nontrivial submodule which is not a subsuperspace (i.e. as an ordinary module, V is reducible),

- otherwise V is *absolutely irreducible* (i.e. as an ordinary module, V remains irreducible).

By [4, Lemma 2.3], V is self-associate irreducible only if it admits an odd A-involution J. Note that this necessarily implies $\operatorname{sdim} V = 0$. For example, the natural $Q(n)$-module $\mathbb{C}^{n|n}$ is self-associate irreducible, with odd $Q(n)$-involution given by, for example, $\sqrt{-1}\, P_n$ (see (2.2)).

Given two superalgebras A, B, we make $A \otimes B$ into a superalgebra by defining the multiplication

$$(a \otimes b)(a' \otimes b') := (-1)^{\overline{a'} \cdot \overline{b}} aa' \otimes bb'$$

for $a, a' \in A$ and $b, b' \in B$. If V and W are A- and B-supermodules, respectively, then we can make the tensor product $V \otimes W$ into an $A \otimes B$-supermodule, denoted $V \boxtimes W$, by defining the action

$$(a \otimes b).(v \otimes w) := (-1)^{\overline{b} \cdot \overline{v}} a.v \otimes b.w$$

for $a \in A$, $b \in B$, $v \in V$, and $w \in W$.

If V and W are *simple* A- and B-supermodules, respectively, then there are three possibilities for the $A \otimes B$-supermodule $V \boxtimes W$ [4, Lemma 2.9]:

1. If both V and W are absolutely irreducible, then so is $V \boxtimes W$.

2. If exactly one of V or W is self-associate, then so is $V \boxtimes W$.

3. If both V and W are self-associate, then $V \boxtimes W$ is a direct sum of two isomorphic copies of an absolutely irreducible $A \otimes B$-supermodule.

We explain case (3) in greater detail, as it will occur later on. In that case, the summands of $V \boxtimes W$ are isomorphic to the (± 1)-eigenspaces of the even $A \otimes B$-involution $J_V \otimes J_W$, where J_V and J_W are odd A- and B- involutions of V and W, respectively. In this scenario we follow Kleshchev [25, §12.2] and denote by $V \circledast W$ the 1-eigenspace of $J_V \otimes J_W$. The reader is warned, however, that other authors may choose $V \circledast W$ to denote the (-1)-eigenspace. Either way, some authors denote $V \circledast W$ by $2^{-1} V \boxtimes W$ (e.g. [11]), for the reason that

$$\dim V \circledast W = \tfrac{1}{2}(\dim V \boxtimes W) = \tfrac{1}{2}(\dim V)(\dim W).$$

The kernel of a homomorphism $\phi \colon A \to B$ is a *superideal*, i.e. a two-sided ideal $I \subseteq A$ which is a subsuperspace of A. A superalgebra is *simple* if it has no nontrivial superideals. By [11, Theorem 3.1] every finite-dimensional simple superalgebra A is isomorphic to some $M(m|n)$ (in which case A is called *type* $\mathtt{M}$) or some $Q(n)$ (in which case A is called *type* $\mathtt{Q}$).

Finally, we recall the $\mathbb{Z}_2$-graded analogs of Schur's lemma and Wedderburn's theorem (see [25, §12.2]). The former states that if A is a superalgebra and V is a simple A-supermodule, then

$$\dim \operatorname{End}_A(V) = \begin{cases} 1 & \text{if } V \text{ is absolutely irreducible} \\ 2 & \text{if } V \text{ is self-associate.} \end{cases}$$

The extra dimension in the self-associate case comes from the fact that $P_n \in \operatorname{End}_A(V)$ for an appropriate choice of basis, where $n = \dim V_0 = \dim V_1$.

A superalgebra A is *semisimple* if every A-supermodule is completely reducible (isomorphic to a direct sum of simple supermodules), or, equivalently, if A is isomorphic to a direct sum of type $\mathtt{M}$ and type $\mathtt{Q}$ matrix superalgebras. In particular, if A is semisimple and $\{V^\alpha : \alpha \in \Lambda\}$ is a complete set of pairwise nonisomorphic simple A-supermodules, then we have the superalgebra decomposition

$$A \simeq \bigoplus M(V^\alpha) \oplus \bigoplus Q(V^\beta)$$

where the first sum is over all $\alpha \in \Lambda$ such that V^α is absolutely irreducible and the second is over all $\beta \in \Lambda$ such that V^β is self-associate.

2.2. Monoidal supercategories

The categories presented in this paper are (symmetric monoidal) supercategories in the sense of Brundan and Ellis [3]. We recall this terminology here. A helpful source for the ordinary versions of these objects is [14].

The category **SVec** has as objects all superspaces and as morphisms all linear maps between them. We declare that composition in **SVec** obeys the superinterchange law (2.1). As such, because of the (sometimes) negative sign on the right side of (2.1), **SVec** is *not* monoidal under the tensor product of superspaces.

The fix is to define the (not full) subcategory $\mathbf{SVec}_0$ of **SVec** with the same objects but only the even linear maps, i.e.

$$\mathrm{Hom}_{\mathbf{SVec}_0}(V, W) := \mathrm{Hom}_{\mathbf{SVec}}(V, W)_0$$

for $V, W \in \mathbf{SVec}$. No negative sign ever occurs on the right side of (2.1) in $\mathbf{SVec}_0$, and it does turn out to be monoidal. Hence we may define *supercategory* to mean $\mathbf{SVec}_0$-enriched category, i.e. a category $\mathbf{C}$ such that $\mathrm{Hom}_{\mathbf{C}}(x, y)$ is a superspace for all $x, y \in \mathbf{C}$ and the composition maps

$$\begin{aligned} \operatorname{Hom}_{\mathbf{C}}(y,z) \otimes \operatorname{Hom}_{\mathbf{C}}(x,y) &\rightarrow \operatorname{Hom}_{\mathbf{C}}(x,z) \\ f \otimes g &\mapsto f \circ g \end{aligned}$$

are even linear maps for all $x, y, z \in \mathbf{C}$. (Note that this means $\overline{f \circ g} = \overline{f} + \overline{g}$, as was already the case for $f, g \in \operatorname{End} V$ and V a superspace.)

A *subsupercategory* of a supercategory $\mathbf{C}$ is a supercategory $\mathbf{D}$ which is a subcategory of $\mathbf{C}$ with the property that $\operatorname{Hom}_{\mathbf{D}}(x,y)$ is a subsuperspace of $\operatorname{Hom}_{\mathbf{C}}(x,y)$ for all $x, y \in \mathbf{D}$. A subsupercategory $\mathbf{D}$ of $\mathbf{C}$ is *full* if it is full as an ordinary subcategory, i.e. if $\operatorname{Hom}_{\mathbf{D}}(x,y) = \operatorname{Hom}_{\mathbf{C}}(x,y)$ for all $x, y \in \mathbf{D}$.

A *superfunctor* is a functor of $\mathbf{SVec}_0$-enriched categories, i.e. a functor $F\colon \mathbf{C} \rightarrow \mathbf{D}$ between supercategories $\mathbf{C}, \mathbf{D}$ with the property that

$$\begin{aligned} F_{x,y}\colon \operatorname{Hom}_{\mathbf{C}}(x,y) &\rightarrow \operatorname{Hom}_{\mathbf{D}}(F(x), F(y)) \\ g &\mapsto F(g) \end{aligned}$$

is an even linear map for all $x, y \in \mathbf{C}$. Moreover, F is

- *full* if $F_{x,y}$ is surjective for all $x, y \in \mathbf{C}$,
- *faithful* if $F_{x,y}$ is injective for all $x, y \in \mathbf{C}$, and
- *evenly dense* if for every $d \in \mathbf{D}$ there exists $c \in \mathbf{C}$ with an even isomorphism $f \in \operatorname{Hom}_{\mathbf{D}}(F(c), d)_0$.

Altogether, F is a *superequivalence* if it is *fully faithful* (i.e. full and faithful) and evenly dense.

A *supernatural transformation* $\beta\colon F \to G$ between superfunctors $F, G\colon \mathbf{C} \to \mathbf{D}$ is a family $(\beta_x)_{x\in\mathbf{C}}$ of morphisms where $\beta_x \in \mathrm{Hom}_{\mathbf{D}}(F(x), G(x))$, satisfying the *supernaturality* condition

$$\beta_{y,i} \circ F(f) = (-1)^{i\cdot\overline{f}} G(f) \circ \beta_{x,i}$$

for $f \in \mathrm{Hom}_{\mathbf{C}}(x, y)$ and $i \in \mathbb{Z}_2$, where $\beta_x = \beta_{x,0} + \beta_{x,1}$ is the $\mathbb{Z}_2$-decomposition of β_x. A supernatural transformation is *even* if $\beta_x = \beta_{x,0}$ for all $x \in \mathbf{C}$, and a *supernatural isomorphism* if β_x is an isomorphism for all $x \in \mathbf{C}$.

The *tensor product* $\mathbf{C} \boxtimes \mathbf{D}$ of supercategories $\mathbf{C}, \mathbf{D}$ is the supercategory with objects all pairs (c, d) of objects $c \in \mathbf{C}, d \in \mathbf{D}$, and

$$\mathrm{Hom}_{\mathbf{C}\boxtimes\mathbf{D}}((c_1, d_1,), (c_2, d_2)) := \mathrm{Hom}_{\mathbf{C}}(c_1, c_2) \otimes \mathrm{Hom}_{\mathbf{D}}(d_1, d_2)$$

where composition is defined using (2.1). A *monoidal supercategory* consists of the following data:

- a supecategory $\mathbf{C}$,
- a superfunctor $(- \otimes_{\mathbf{C}} -)\colon \mathbf{C} \boxtimes \mathbf{C} \to \mathbf{C}$,
- a unit object $\mathbb{1}_{\mathbf{C}} \in \mathbf{C}$, and
- even supernatural isomorphisms $\lambda\colon (\mathbb{1}_{\mathbf{C}} \otimes_{\mathbf{C}} -) \to \mathrm{Id}_{\mathbf{C}}$, $\rho\colon (- \otimes_{\mathbf{C}} \mathbb{1}_{\mathbf{C}}) \to \mathrm{Id}_{\mathbf{C}}$ satisfying the *coherence axiom* $\rho_x \otimes_{\mathbf{C}} 1_y = 1_x \otimes_{\mathbf{C}} \lambda_y$ for all $x, y \in \mathbf{C}$.

A *monoidal subsupercategory* of a monoidal supercategory $\mathbf{C}$ is a monoidal supercategory $\mathbf{D}$ which is a subsupercategory of $\mathbf{C}$ with the same tensor product operation and unit object as $\mathbf{C}$.

In general, a monoidal supercategory must also satisfy some associativity constraints, but we omit them because all monoidal supercategories $\mathbf{C}$ in this work will have $(x \otimes_{\mathbf{C}} y) \otimes_{\mathbf{C}} z = x \otimes_{\mathbf{C}} (y \otimes_{\mathbf{C}} z)$ for all $x, y, z \in \mathbf{C}$. In addition, all of the webs supercategories in this paper will be *strict* in the sense that the components of λ and ρ are identities and we have $x \otimes_{\mathbf{C}} \mathbb{1} = x = \mathbb{1} \otimes_{\mathbf{C}} x$ for all objects x in those categories.

An *ideal* in a monoidal supercategory $\mathbf{C}$ consists of a subsuperspace $I(x, y) \subseteq \mathrm{Hom}_{\mathbf{C}}(x, y)$ for every pair of objects $x, y \in \mathbf{C}$, such that for all $x, y, z, w \in \mathbf{C}$ we have

1. $h \circ g \circ f \in I(x, w)$ whenever $f \in \mathrm{Hom}_{\mathbf{C}}(x, y)$, $g \in \mathrm{Hom}_{\mathbf{C}}(y, z)$, and $h \in \mathrm{Hom}_{\mathbf{C}}(z, w)$, and
2. $f \otimes 1_z \in I(x \otimes z, y \otimes z)$ and $1_z \otimes f \in I(z \otimes x, z \otimes y)$ whenever $f \in \mathrm{Hom}_{\mathbf{C}}(x, y)$.

The *quotient* $\mathbf{C}/I$ is the supercategory with the same objects as $\mathbf{C}$ and morphisms $\mathrm{Hom}_{\mathbf{C}/I}(x, y) := \mathrm{Hom}_{\mathbf{C}}(x, y)/I$ for $x, y \in \mathbf{C}$, which one can easily check is again a monoidal supercategory. Since an intersection of ideals is again an ideal, there is a unique minimal ideal containing a given set of morphisms X, which we call the *ideal generated by X*.

A *monoidal superfunctor* consists of a superfunctor $F\colon \mathbf{C} \to \mathbf{D}$ between monoidal supercategories $\mathbf{C}, \mathbf{D}$, even supernatural isomorphisms

$$\zeta_{x,y}\colon F(x) \otimes_{\mathbf{D}} F(y) \xrightarrow{\sim} F(x \otimes_{\mathbf{C}} y)$$

for all $x, y \in \mathbf{C}$, and an even isomorphism $\gamma \in \mathrm{Hom}_{\mathbf{D}}(\mathbb{1}_{\mathbf{D}}, F(\mathbb{1}_{\mathbf{C}}))_0$ satisfying

$$\zeta_{x \otimes y, z} \circ (\zeta_{x,y} \otimes 1_{F(z)}) = \zeta_{x, y \otimes z} \circ (1_{F(x)} \otimes \zeta_{y,z}) \tag{2.3}$$

for all $x, y, z \in \mathbf{C}$, where the subscripts $\mathbf{C}$ and $\mathbf{D}$ of the $\otimes$ were suppressed for ease of exposition (they go in the obvious places). A *monoidal superequivalence* is a monoidal superfunctor which is also a superequivalence. We note that for every monoidal superfunctor considered in this work, the maps $\zeta_{x,y}$ will be obvious, so we will omit any discussion of them.

If $F\colon \mathbf{C} \to \mathbf{D}$ is an monoidal superfunctor, then its *kernel* is the ideal Ker_F of $\mathbf{C}$ given by $\mathrm{Ker}_F(x, y) := \{f \in \mathrm{Hom}_{\mathbf{C}}(x, y) : F(f) = 0\}$ for $x, y \in \mathbf{C}$. If I is an ideal of $\mathbf{C}$ with $I(x, y)$ a subsuperspace of $\mathrm{Ker}_F(x, y)$ for all $x, y \in \mathbf{C}$ (e.g. I is generated by a set of morphisms in Ker_F), then there is an obvious induced monoidal superfunctor $F\colon \mathbf{C}/I \to \mathbf{D}$. Furthermore, there is a "first isomorphism theorem" for monoidal supercategories: $F\colon \mathbf{C}/\mathrm{Ker}_F \to \mathbf{D}$ is faithful, and $F\colon \mathbf{C}/I \to \mathbf{D}$ is full (resp. evenly dense) if and only if $F\colon \mathbf{C} \to \mathbf{D}$ is full (resp. evenly dense).

A *braiding* on a monoidal supercategory $\mathbf{C}$ is an even supernatural isomorphism $\sigma\colon (- \otimes_{\mathbf{C}} -) \to (- \otimes_{\mathbf{C}} -)^{\mathrm{opp}}$ where $(- \otimes_{\mathbf{C}} -)^{\mathrm{opp}}$ is the superfunctor given by $(x, y) \rightsquigarrow y \otimes_{\mathbf{C}} x$, satisfying

$$\sigma_{x\otimes y,z} = (\sigma_{x,z} \otimes 1_y) \circ (1_x \otimes \sigma_{y,z}), \quad \sigma_{x,y\otimes z} = (1_y \otimes \sigma_{x,z}) \circ (\sigma_{x,y} \otimes 1_z)$$

for all $x, y, z \in \mathbf{C}$. A *symmetry* is a braiding σ with the additional property of

$$\sigma_{y,x} \circ \sigma_{x,y} = 1_{x\otimes y}.$$

A *braided monoidal supercategory* (resp. *symmetric monoidal supercategory*) is a monoidal supercategory equipped with a braiding (resp. with a symmetry). Examples of symmetric monoidal supercategories include **SVec** and $\mathbf{SVec}_0$, both via the symmetry

$$\sigma_{V,W}(v \otimes w) := (-1)^{\overline{v}\cdot\overline{w}} w \otimes v$$

for superspaces V, W and $v \in V$, $w \in W$. One can check that the quotient of a braided (resp. symmetric) monoidal supercategory by an ideal is again braided (resp. symmetric).

A *braided (resp. symmetric) monoidal superfunctor* is a monoidal superfunctor $F\colon \mathbf{C} \to \mathbf{D}$ between braided (resp. symmetric) monoidal supercategories $(\mathbf{C}, \sigma^{\mathbf{C}})$, $(\mathbf{D}, \sigma^{\mathbf{D}})$ with the property that

$$F(\sigma^{\mathbf{C}}_{x,y}) \circ \zeta_{x,y} = \zeta_{y,x} \circ \sigma^{\mathbf{D}}_{F(x),F(y)}$$

for all $x, y \in \mathbf{C}$. One can check that the induced functor $F\colon \mathbf{C}/I \to \mathbf{D}$ is also braided (resp. symmetric) where I is an ideal of $\mathbf{C}$ with $I(x,y)$ a subsuperspace of $\mathrm{Ker}_F(x,y)$ for $x, y \in \mathbf{C}$.

2.3. Locally unital superalgebras

Some of the superalgebras A in this paper are *locally unital*. This means A has a system $(1_\alpha)_{\alpha\in\Lambda}$ of pairwise orthogonal idempotents along which A admits the superspace decomposition

$$A = \bigoplus_{\alpha,\beta\in\Lambda} 1_\beta A 1_\alpha.$$

A *locally unital homomorphism* is a homomorphism that takes distinguished idempotents to distinguished idempotents. Note that every locally unital homomorphism

$f\colon A \to B$ yields superspace maps

$$f_{\alpha,\beta}\colon 1_\beta A 1_\alpha \to f(1_\beta) B f(1_\alpha)$$

for all $\alpha, \beta \in \Lambda$.

Advantageously, the concepts of locally unital superalgebra and supercategory, and of locally unital homomorphism and superfunctor, are equivalent. Indeed, a locally unital superalgebra (A, Λ) corresponds to the supercategory $\mathbf{A}$ with object set Λ and morphisms

$$\mathrm{Hom}_{\mathbf{A}}(\alpha, \beta) := 1_\beta A 1_\alpha$$

for $\alpha, \beta \in \Lambda$, where composition in $\mathbf{A}$ corresponds to multiplication in A. We will refer to $\mathbf{A}$ as *the supercategory associated to A*. A locally unital homomorphism $f\colon A \to B$ corresponds to the superfunctor $F\colon \mathbf{A} \to \mathbf{B}$ with object assignment f and morphism assignments $f_{\alpha,\beta}$ for $\alpha, \beta \in \Lambda$. It's clear that

- f is injective if and only if F is faithful, and
- f is surjective if and only if F is full.

We will refer to F as *the superfunctor associated to f*.

2.4. Remark on terminology

Remark 2.2. *For convenience, we will often omit the prefix "super" from several adjectives and nouns, although the reader should always assume it to be in effect. That is, every instance in this dissertation of the word "algebra" will in fact mean, and should be taken to mean, "superalgebra", and so on.*

Chapter 3

Representation theory of $\mathfrak{q}_n$ and Ser_k

In this chapter, we provide background information specific to $\mathfrak{q}_n$ and Ser_k which will be needed for the main content of the dissertation. We refer the reader to [11, 25] for more information.

3.1. Lie superalgebra $\mathfrak{q}_n$

As stated in Section 2.1, the *type Q Lie superalgebra* is the Lie subsuperalgebra $\mathfrak{q}_n$ of $\mathfrak{gl}_{n|n}$ realized in block matrix form as

$$\mathfrak{q}_n = \left\{ \begin{bmatrix} A & B \\ B & A \end{bmatrix} : A, B \text{ are } n \times n\text{-matrices} \right\}.$$

Its $\mathbb{Z}_2$-grading is given by the superspace decomposition

$$(\mathfrak{q}_n)_0 = \left\{ \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} \right\}, \quad (\mathfrak{q}_n)_1 = \left\{ \begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix} \right\},$$

and its bracket operation is given by

$$[X, Y] = XY - (-1)^{\overline{X} \cdot \overline{Y}} YX$$

for $X, Y \in \mathfrak{q}_n$, both of which it inherits from $\mathfrak{gl}_{n|n}$. The reader can check that an equivalent definition of $\mathfrak{q}_n$ is $\mathfrak{q}_n = \{X \in \mathfrak{gl}_{n|n} : [X, P_n] = 0\}$ where P_n is as in (2.2). There is a $\mathbb{Z}_2$-homogeneous basis $\{A_{i,j}, B_{i,j} : 1 \leq i, j, \leq n\}$ of $\mathfrak{q}_n$ where $A_{i,j} \in (\mathfrak{q}_n)_0$ is the block matrix in which A has a 1 in the (i, j)-entry and zeros elsewhere and $B = 0$; the $B_{i,j} \in (\mathfrak{q}_n)_1$ are defined similarly.

To study modules over $\mathfrak{q}_n$, we will study the equivalent concept of modules over its universal enveloping algebra $U(\mathfrak{q}_n)$. According to [12, Proposition 2.1], $U(\mathfrak{q}_n)$ is the associative superalgebra generated by the even elements e_i, f_i, h_i and odd elements $e_{\bar{i}}, f_{\bar{i}}, h_{\bar{j}}$ for $1 \leq i \leq n-1$ and $1 \leq j \leq n$, subject to the relations

$$h_i h_j - h_j h_i = h_i h_{\bar{j}} - h_{\bar{j}} h_i = 0, \quad h_{\bar{i}} h_{\bar{j}} + h_{\bar{j}} h_{\bar{i}} = \delta_{i,j} 2 h_i, \tag{Q1}$$

$$h_i e_j - e_j h_i = \begin{cases} e_j & i = j \\ -e_j & i = j+1 \\ 0 & i \neq j, j+1 \end{cases}, \quad h_i f_j - f_j h_i = \begin{cases} -f_j & i = j \\ f_j & i = j+1 \\ 0 & i \neq j, j+1 \end{cases},$$

$$h_i e_{\bar{j}} - e_{\bar{j}} h_i = \begin{cases} e_{\bar{j}} & i = j \\ -e_{\bar{j}} & i = j+1 \\ 0 & i \neq j, j+1 \end{cases}, \quad h_i f_{\bar{j}} - f_{\bar{j}} h_i = \begin{cases} -f_{\bar{j}} & i = j \\ f_{\bar{j}} & i = j+1 \\ 0 & i \neq j, j+1 \end{cases}, \tag{Q2}$$

$$h_{\bar{i}}e_j - e_jh_{\bar{i}} = \begin{cases} e_{\bar{j}} & i = j \\ -e_{\bar{j}} & i = j+1 \\ 0 & i \neq j, j+1 \end{cases}, \quad h_{\bar{i}}f_j - f_jh_{\bar{i}} = \begin{cases} -f_{\bar{j}} & i = j \\ f_{\bar{j}} & i = j+1 \\ 0 & i \neq j, j+1 \end{cases},$$

$$h_{\bar{i}}e_{\bar{j}} + e_{\bar{j}}h_{\bar{i}} = \begin{cases} e_j & i = j, j+1 \\ 0 & i \neq j, j+1 \end{cases}, \quad h_{\bar{i}}f_{\bar{j}} + f_{\bar{j}}h_{\bar{i}} = \begin{cases} f_j & i = j, j+1 \\ 0 & i \neq j, j+1 \end{cases}, \tag{Q3}$$

$$\begin{aligned} e_if_j - f_je_i &= \delta_{i,j}(h_i - h_{i+1}), \\ e_{\bar{i}}f_{\bar{j}} + f_{\bar{j}}e_{\bar{i}} &= \delta_{i,j}(h_i + h_{i+1}), \\ e_{\bar{i}}f_j - f_je_{\bar{i}} &= \delta_{i,j}(h_{\bar{i}} - h_{\overline{i+1}}) = e_if_{\bar{j}} - f_{\bar{j}}e_i, \end{aligned} \tag{Q4}$$

$$\begin{gathered} e_ie_{\bar{j}} - e_{\bar{j}}e_i = e_{\bar{i}}e_{\bar{j}} + e_{\bar{j}}e_{\bar{i}} = f_if_{\bar{j}} - f_{\bar{j}}f_i = f_{\bar{i}}f_{\bar{j}} + f_{\bar{j}}f_{\bar{i}} = 0 \quad \text{if } i \neq j \pm 1, \\ e_ie_j - e_je_i = f_if_j - f_jf_i = 0 \quad \text{if } |i-j| > 1, \\ e_ie_{i+1} - e_{i+1}e_i = e_{\bar{i}}e_{\overline{i+1}} + e_{\overline{i+1}}e_{\bar{i}}, \quad e_ie_{\overline{i+1}} - e_{\overline{i+1}}e_i = e_{\bar{i}}e_{i+1} - e_{i+1}e_{\bar{i}}, \\ f_if_{i+1} - f_{i+1}f_i = f_{\bar{i}}f_{\overline{i+1}} + f_{\overline{i+1}}f_{\bar{i}}, \quad f_if_{\overline{i+1}} - f_{\overline{i+1}}f_i = f_{\bar{i}}f_{i+1} - f_{i+1}f_{\bar{i}}, \end{gathered} \tag{Q5}$$

$$\begin{gathered} e_i^{(2)}e_j - e_ie_je_i + e_je_i^{(2)} = e_{\bar{i}}e_ie_j - e_{\bar{i}}e_je_i - e_ie_je_{\bar{i}} + e_je_ie_{\bar{i}} = 0 \quad \text{if } i = j \pm 1, \\ f_i^{(2)}f_j - f_if_jf_i + f_jf_i^{(2)} = f_{\bar{i}}f_if_j - f_{\bar{i}}f_jf_i - f_if_jf_{\bar{i}} + f_jf_if_{\bar{i}} = 0 \quad \text{if } i = j \pm 1, \end{gathered} \tag{Q6}$$

where we denote by $e_i^{(j)}$, $f_i^{(j)}$ the *divided powers*

$$e_i^{(j)} := \frac{e_i^j}{j!}, \quad f_i^{(j)} := \frac{f_i^j}{j!}.$$

The canonical embedding $\mathfrak{q}_n \hookrightarrow U(\mathfrak{q}_n)$ sends

$$A_{i,i+1} \mapsto e_i, \quad A_{i+1,i} \mapsto f_i, \quad A_{j,j} \mapsto h_j,$$

$$B_{i,i+1} \mapsto e_{\bar{i}}, \quad B_{i+1,i} \mapsto f_{\bar{i}}, \quad B_{j,j} \mapsto h_{\bar{j}}.$$

For the remainder of the section, we discuss some basic tools we will need in order to study the $\mathfrak{q}_n$-modules we define in the next section.

The universal enveloping algebra $U(\mathfrak{q}_n)$ of $\mathfrak{q}_n$ is a Hopf algebra with coproduct $\Delta\colon U(\mathfrak{q}_n) \to U(\mathfrak{q}_n) \otimes U(\mathfrak{q}_n)$ and antipode $s\colon U(\mathfrak{q}_n) \to U(\mathfrak{q}_n)$ given by

$$\Delta(X) = X \otimes 1 + 1 \otimes X, \quad s(X) = -X$$

for $X \in \mathfrak{q}_n$. Thus, if V, W are $\mathfrak{q}_n$-modules, the tensor product $V \otimes W$ and the dual space $V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ are naturally $\mathfrak{q}_n$-modules via

$$X.(v \otimes w) \;:=\; \Delta(X)(v \otimes x) \;=\; X.v \otimes w + (-1)^{\overline{X}\cdot\overline{v}} v \otimes X.w,$$
$$(X.f)(v) \;:=\; (-1)^{\overline{X}\cdot\overline{f}}(f \circ s(X))(v) \;=\; -(-1)^{\overline{X}\cdot\overline{f}} f(X.v)$$

respectively for $X \in \mathfrak{q}_n$, $v \in V$, $w \in W$, and $f \in V^*$. Note $U(\mathfrak{q}_n)$ is *cocommutative*, i.e. $\mathrm{flip} \circ \Delta = \Delta$ where $\mathrm{flip}\colon U(\mathfrak{q}_n) \otimes U(\mathfrak{q}_n) \to U(\mathfrak{q}_n) \otimes U(\mathfrak{q}_n)$ is the linear map given by $\mathrm{flip}(X \otimes Y) := (-1)^{\overline{X}\cdot\overline{Y}} Y \otimes X$ for $X, Y \in U(\mathfrak{q}_n)$.

For a superspace V and $k \in \mathbb{Z}_{>0}$, we have the tensor space $V^{\otimes k} := V \otimes V \otimes \cdots \otimes V$ (k tensorands). Let $\Delta^k\colon U(\mathfrak{q}_n) \to U(\mathfrak{q}_n)^{\otimes k}$ be the map

$$\Delta^k := (\Delta \otimes 1^{\otimes(k-1)}) \circ (\Delta \otimes 1^{(k-2)}) \circ \cdots \circ \Delta$$

where 1 denotes the identity map on $U(\mathfrak{q}_n)$. (By *coassociativity* of the coproduct in any Hopf algebra, Δ^k is invariant under the choice of tensor position for each occurrence of Δ in its definition.) If V is a $\mathfrak{q}_n$-module, then $V^{\otimes k}$ is also a $\mathfrak{q}_n$-module via

$$\begin{aligned} X.(v_1 \otimes \cdots \otimes v_k) &:= \Delta^k(X)(v_1 \otimes \cdots \otimes v_k) \\ &= X.v_1 \otimes v_2 \otimes \cdots \otimes v_k \\ &\quad +(-1)^{\overline{X}\cdot\overline{v_1}} v_1 \otimes X.v_2 \otimes \cdots \otimes v_k \\ &\quad \vdots \\ &\quad +(-1)^{\overline{X}(\overline{v_1}+\cdots+\overline{v_{k-1}})} v_1 \otimes v_2 \otimes \cdots \otimes X.v_k \end{aligned} \tag{3.1}$$

for $X \in \mathfrak{q}_n$ and $v_1, \ldots, v_k \in V$. By convention, we declare $V^{\otimes 0} = \mathbb{C}$ to be the trivial $\mathfrak{q}_n$-module, in which all $X \in \mathfrak{q}_n$ act by zero.

Given a $\mathfrak{q}_n$-module V, we can form the *symmetric algebra* $\mathscr{S}(V) := T(V)/I$ where I is the two-sided ideal of the tensor algebra $T(V) := \bigoplus_{k\geq 0} V^{\otimes k}$ generated by all expressions of the form $v \otimes w - (-1)^{\overline{v}\cdot\overline{w}} w \otimes v$ for $v, w \in V$. Thus $\mathscr{S}(V) = \bigoplus_{k\geq 0} \mathscr{S}^k(V)$ inherits the $\mathbb{Z}$-grading from $T(V)$ where $\mathscr{S}^k(V) := V^{\otimes k}/(V^{\otimes k} \cap I)$. We call $\mathscr{S}^k(V)$ the k^{th} *symmetric power of* V, and note that from the definition of I we have

$$\mathscr{S}^k(V) \simeq \bigoplus_{l=0}^{k} S^l(V_0) \otimes \bigwedge\nolimits^{k-l}(V_1)$$

where $S^l(V_0)$ and $\bigwedge^{k-l}(V_1)$ are the ordinary symmetric and exterior powers of V_0 and V_1, respectively.

Note that $\mathscr{S}^1(V) = V$ and, again by convention, $\mathscr{S}^0(V) = \mathbb{C}$. In particular, we have $\operatorname{sdim} \mathscr{S}^k(V) = 0$ if $\operatorname{sdim} V = 0$, e.g. if $V = \mathbb{C}^{n|n}$.

We can also define the *exterior algebra* $\mathcal{E}(V)$ by changing the generators of I to all expressions $v \otimes w + (-1)^{\overline{v}\cdot\overline{w}} w \otimes v$. Continuing as before, we would obtain the *exterior powers* $\mathcal{E}^k(V)$, which in particular have the property that

$$\mathcal{E}^k(V) \simeq \bigoplus_{l=0}^{k} S^l(V_1) \otimes \bigwedge^{k-l}(V_0).$$

However, for the $\mathfrak{q}_n$-modules V we will be interested in, $\mathcal{E}^k(V)$ is isomorphic to $\mathscr{S}^k(V)$ for all $k \in \mathbb{Z}_{>0}$ (see Remark 3.1), so we will not consider them further.

The standard Cartan subalgebra of $\mathfrak{q}_n$, denoted $\mathfrak{h}$, consists of block matrices with A and B diagonal. By *weight module* over $\mathfrak{q}_n$ we mean a $\mathfrak{q}_n$-module V which is $\mathfrak{h}_0$-semisimple, i.e. V admits the superspace decomposition

$$V = \bigoplus_{\lambda \in \mathfrak{h}_0^*} V_\lambda, \quad V_\lambda := \{v \in V : H.v = \lambda(H)v \text{ for all } H \in \mathfrak{h}_0\}.$$

Whether or not V is a weight module, we call V_λ the λ *weight space of* V for $\lambda \in \mathfrak{h}_0^*$.

For $\lambda \in \mathfrak{h}_0^*$, a weight module V over $\mathfrak{q}_n$ is *of highest weight* λ if there exists $v \in V_\lambda$ such that $U(\mathfrak{q}_n).v = V$ and v is annihilated by all $A_{i,j}$ and $B_{i,j}$ with $i < j$. Note that by identifying $\mathfrak{h}_0^*$ with $\mathbb{C}^n$ in the obvious way, we can rewrite each V_λ as

$$V_\lambda = \{v \in V : A_{i,i}.v = \lambda_i v,\ 1 \leq i \leq n\}.$$

3.2. Symmetric powers $\mathscr{S}^k(V_n)$

In this section we describe our $\mathfrak{q}_n$-modules of interest, the symmetric powers $\mathscr{S}^k(V_n)$ of the natural $\mathfrak{q}_n$-module V_n and their duals $\mathscr{S}^{k*}(V_n)$.

First we define the indexing sets

$$I(n|0) := \{1, \ldots, n\}, \quad I(0|n) := \{\overline{1}, \ldots, \overline{n}\}, \quad I(n|n) := I(n|0) \cup I(0|n).$$

There is an involution on $I(n|n)$ interchanging $I(n|0)$ and $I(0|n)$ given by barring every element, where we declare that bars cancel each other in pairs, e.g. $\overline{\overline{5}} = 5$. For $i \in I(n|n)$, let $\underline{i} \in I(n|0)$ denote the unbarred version of i, e.g. $\underline{5} = \underline{\overline{5}} = 5$; additionally, define $\delta(i) \in \mathbb{Z}_2$ to be 0 if $i \in I(n|0)$ and 1 if $i \in I(0|n)$.

Let V_n denote the natural $\mathfrak{q}_n$-module and V_n^* the dual of V_n, both of which are isomorphic as superspaces to $\mathbb{C}^{n|n}$. We denote the standard basis of V_n by $v_1, \dots, v_n, v_{\overline{1}}, \dots, v_{\overline{n}}$ where v_i is even if $i \in I(n|0)$ and odd if $i \in I(0|n)$, and the dual standard basis of V_n^* by $g_1, \dots, g_n, g_{\overline{1}}, \dots, g_{\overline{n}}$, i.e. $g_i(v_j) = \delta_{i,j}$ for $i, j \in I(n|n)$. In particular, $\overline{v_i} = \overline{g_i}$. The actions of $U(\mathfrak{q}_n)$ on V_n and V_n^* are given by

$$e_i.v_k = \delta_{i,\underline{k}-1} v_{k-1}, \quad f_i.v_k = \delta_{i,\underline{k}} v_{k+1}, \quad h_j.v_k = \delta_{j,\underline{k}} v_k,$$

$$e_{\overline{i}}.v_k = \delta_{i,\underline{k}-1} v_{\overline{k-1}}, \quad f_{\overline{i}}.v_k = \delta_{i,\underline{k}} v_{\overline{k+1}}, \quad h_{\overline{j}}.v_k = \delta_{j,\underline{k}} v_{\overline{k}},$$

$$e_i.g_k = -\delta_{i,\underline{k}} g_{k+1}, \quad f_i.g_k = -\delta_{i,\underline{k}-1} g_{k-1}, \quad h_j.g_k = -\delta_{j,\underline{k}} g_k,$$

$$e_{\overline{i}}.g_k = \delta_{i,\underline{k}} (-1)^{\delta(k)+1} g_{\overline{k+1}}, \quad f_{\overline{i}}.g_k = \delta_{i,\underline{k}-1} (-1)^{\delta(k)+1} g_{\overline{k-1}},$$

$$h_{\overline{j}}.g_k = \delta_{j,\underline{k}} (-1)^{\delta(k)+1} g_{\overline{k}}$$

for $1 \le i \le n-1$, $1 \le j \le n$, and $k \in I(n|n)$.

Let $\Lambda(n|n)$ be the set of $2n$-tuples $\lambda = (\lambda_1, \dots, \lambda_n, \lambda_{\overline{1}}, \dots, \lambda_{\overline{n}})$ with $\lambda_1, \dots, \lambda_n \in \mathbb{Z}_{\ge 0}$ and $\lambda_{\overline{1}}, \dots, \lambda_{\overline{n}} \in \mathbb{Z}_2$. For $k \in \mathbb{Z}_{\ge 0}$ let

$$\Lambda(n|n, k) := \{\lambda \in \Lambda(n|n) : \lambda_1 + \dots + \lambda_n + \lambda_{\overline{1}} + \dots + \lambda_{\overline{n}} = k\}.$$

Then the symmetric power $\mathscr{S}^k(V_n)$ admits the monomial basis

$$B_k := \{v_\lambda : \lambda \in \Lambda(n|n,k)\}, \quad v_\lambda := v_1^{\lambda_1} \cdots v_n^{\lambda_n} v_{\overline{1}}^{\lambda_{\overline{1}}} \cdots v_{\overline{n}}^{\lambda_{\overline{n}}}$$

where we omitted the tensor product symbol $\otimes$ for brevity (and will often do so). Moreover, the dual symmetric power $\mathscr{S}^{k*}(V_n) := (\mathscr{S}^k(V_n))^*$ admits the dual monomial basis $B_k^* := \{g^\lambda : \lambda \in \Lambda(n|n,k)\}$ given by $g^\lambda(v_\mu) = \delta_{\lambda,\mu}$ for $\lambda, \mu \in \Lambda(n|n,k)$. Both bases are $\mathbb{Z}_2$-homogeneous with

$$\overline{v_\lambda} = \overline{g^\lambda} \equiv \lambda_{\overline{1}} + \cdots + \lambda_{\overline{n}} \mod 2$$

for $\lambda \in \Lambda(n|n,k)$, and we have $\operatorname{sdim} \mathscr{S}^k(V_n) = \operatorname{sdim} \mathscr{S}^{k*}(V_n) = 0$ as discussed in the previous section.

Remark 3.1. *For $k \in \mathbb{Z}_{\geq 0}$ the linear map $P_n^{\otimes k} \colon V_n^{\otimes k} \to V_n^{\otimes k}$ induces a $\mathfrak{q}_n$-isomorphism $\mathscr{S}^k(V_n) \xrightarrow{\sim} \mathcal{E}^k(V_n)$, where P_n is as in* (2.2). *We sketch the proof here, leaving details to the reader. First, using the fact that $[P_n, X] = 0$ for all $X \in \mathfrak{q}_n$, one can check that $[P_n^{\otimes k}, \Delta^k(X)] = 0$ for all $X \in \mathfrak{q}_n$, implying that $P_n^{\otimes k}$ is a $\mathfrak{q}_n$-morphism. Next, a direct calculation shows that the composition of $P_n^{\otimes k}$ with the projection $V_n^{\otimes k} \twoheadrightarrow \mathcal{E}^k(V_n)$ factors through $\mathscr{S}^k(V_n)$. Finally, since $P_n^2 = -1_{V_n}$, $P_n^{\otimes k}$ is invertible with inverse $\pm P_n^{\otimes k}$, depending on whether k is even or odd.*

The actions of $\mathfrak{q}_n$ on $\mathscr{S}^k(V_n)$ and $\mathscr{S}^{k*}(V_n)$ can be deduced from its action on V_n using the coproduct Δ and antipode s of $U(\mathfrak{q}_n)$. Nevertheless, we provide explicit formulas for these actions for the reader's convenience. In order to do so, we require more shorthand notation.

For $\lambda \in \Lambda(n|n,k)$, $1 \le i \le n-1$, and $1 \le j \le n$, define

$$\lambda \pm \alpha_i := (\lambda_1, \dots, \lambda_i \pm 1, \lambda_{i+1} \mp 1, \dots, \lambda_{\overline{n}}),$$

$$\lambda \pm \alpha_{\overline{i}} := (\lambda_1, \dots, \lambda_{\overline{i}} \pm 1, \lambda_{\overline{i+1}} \mp 1, \dots, \lambda_{\overline{n}}),$$

$$\lambda \pm \epsilon_{\overline{i}} := (\lambda_1, \dots, \lambda_i \pm 1, \dots, \lambda_{\overline{i}} \mp 1, \dots, \lambda_{\overline{n}}),$$

where in each case only two entries of λ have been altered. Note that these need not lie in $\Lambda(n|n,k)$, e.g. $\lambda + \alpha_i \notin \Lambda(n|n,k)$ if $\lambda_{i+1} = 0$. If $\lambda \pm \alpha_i \in \Lambda(n|n,k)$ then $v_{\lambda \pm \alpha_i}$ and $g_{\lambda \pm \alpha_i}$ are in the monomial bases of $\mathscr{S}^k(V_n)$ and $\mathscr{S}^{k*}(V_n)$; if not we set $v_{\lambda \pm \alpha_i} = g_{\lambda \pm \alpha_i} = 0$, and similarly for $\lambda \pm \alpha_{\overline{i}}$ and $\lambda \pm \epsilon_{\overline{i}}$. Moreover, we allow iterations of these "additions" in the natural way, except when passing to the monomial bases we only set $v_\mu = 0$ if the entire iterated sum μ is not in $\Lambda(n|n,k)$. For example, while we set $v_{\lambda+\alpha_i} = 0$ if $\lambda_{i+1} = 0$, we set $v_{\lambda+\alpha_i+\epsilon_{\overline{i+1}}} = 0$ only if $\lambda_{\overline{i+1}} = 0$.

The actions of $U(\mathfrak{q}_n)$ on $\mathscr{S}^k(V_n)$ and $\mathscr{S}^{k*}(V_n)$ are given by

$$e_i.v_\lambda = \lambda_{i+1} v_{\lambda+\alpha_i} + v_{\lambda+\alpha_{\overline{i}}}, \quad f_i.v_\lambda = \lambda_i v_{\lambda-\alpha_i} + v_{\lambda-\alpha_{\overline{i}}}, \quad h_j.v_\lambda = (\lambda_j + \lambda_{\overline{j}}) v_\lambda,$$

$$e_{\overline{i}}.v_\lambda = (-1)^{\lambda_{\overline{1}}+\cdots+\lambda_{\overline{i}}} (\lambda_{i+1} v_{\lambda+\alpha_i-\epsilon_{\overline{i}}} + v_{\lambda+\alpha_{\overline{i}}+\epsilon_{\overline{i}}}),$$

$$f_{\overline{i}}.v_\lambda = (-1)^{\lambda_{\overline{1}}+\cdots+\lambda_{\overline{i-1}}} (\lambda_i v_{\lambda-\alpha_i-\epsilon_{\overline{i+1}}} + v_{\lambda-\alpha_{\overline{i}}+\epsilon_{\overline{i+1}}}),$$

$$h_{\overline{j}}.v_\lambda = (-1)^{\lambda_{\overline{1}}+\cdots+\lambda_{\overline{j-1}}} (\lambda_j v_{\lambda-\epsilon_{\overline{j}}} + v_{\lambda+\epsilon_{\overline{j}}}),$$

$$e_i.g_\lambda = -((\lambda_{i+1}+1) g_{\lambda-\alpha_i} + g_{\lambda-\alpha_{\overline{i}}}), \quad f_i.g_\lambda = -((\lambda_i+1) g_{\lambda+\alpha_i} + g_{\lambda+\alpha_{\overline{i}}}),$$

$$h_j.g_\lambda = -(\lambda_j + \lambda_{\overline{j}}) g_\lambda,$$

$$e_{\overline{i}}.g_\lambda = -(-1)^{\lambda_{\overline{i+1}}+\cdots+\lambda_{\overline{n}}} ((\lambda_{i+1}+1) g_{\lambda-\alpha_i+\epsilon_{\overline{i}}} + g_{\lambda-\alpha_{\overline{i}}-\epsilon_{\overline{i}}}),$$

$$f_{\bar{i}}.g_\lambda = -(-1)^{\lambda_{\bar{i}}+\cdots+\lambda_{\bar{n}}}((\lambda_i + 1)v_{\lambda+\alpha_i+\epsilon_{\overline{i+1}}} + v_{\lambda+\alpha_{\bar{i}}-\epsilon_{\overline{i+1}}}),$$

$$h_{\bar{j}}.g_\lambda = -(-1)^{\lambda_{\bar{j}}+\cdots+\lambda_{\bar{n}}}((\lambda_j + 1)v_{\lambda+\epsilon_{\bar{j}}} + v_{\lambda-\epsilon_{\bar{j}}})$$

for $\lambda \in \Lambda(n|n,k)$, $1 \le i \le n-1$, and $1 \le j \le n$.

3.3. Categories $\mathfrak{q}_n$-$\mathbf{Mod}_{\mathscr{S}}$ and $\mathfrak{q}_n$-$\mathbf{Mod}_{\mathscr{S},\mathscr{S}^*}$

We now define the categories of $\mathfrak{q}_n$-modules whose morphisms we will describe with webs in Chapter 5.

Denote by $\mathfrak{q}_n$-**Mod** the category with objects all $\mathfrak{q}_n$-modules and morphisms all $\mathfrak{q}_n$-morphisms between them. It is a monoidal supercategory under the tensor product of $\mathfrak{q}_n$-modules. Since $U(\mathfrak{q}_n)$ is cocommutative (see Section 3.1), the linear maps $\sigma_{V,W}(v \otimes w) := (-1)^{\bar{v}\cdot\bar{w}} w \otimes w$ for $\mathfrak{q}_n$-modules V, W and $v \in V$, $w \in W$ are $\mathfrak{q}_n$-morphisms. Hence they constitute a symmetry on $\mathfrak{q}_n$-**Mod**, making it a symmetric monoidal supercategory.

Let $\mathbb{Z}^*_{\ge 0}$ be the set of symbols $\{0*, 1*, 2*, \dots\}$.

Definition 3.2. *We define* $\mathfrak{q}_n$-$\mathbf{Mod}_{\mathscr{S}}$ *to be the full subsupercategory of* $\mathfrak{q}_n$-**Mod** *with objects all tensor products of the form*

$$\mathscr{S}^{a_1}(V_n) \otimes \cdots \otimes \mathscr{S}^{a_l}(V_n)$$

for $a_1, \dots, a_l \in \mathbb{Z}_{\ge 0}$ *and* $l \in \mathbb{Z}_{>0}$. *We define* $\mathfrak{q}_n$-$\mathbf{Mod}_{\mathscr{S},\mathscr{S}^*}$ *to be the full subsupercategory of* $\mathfrak{q}_n$-**Mod** *with objects all tensor products of the form*

$$\mathscr{S}^{b_1}(V_n) \otimes \cdots \otimes \mathscr{S}^{b_l}(V_n)$$

for $b_1, \dots, b_l \in \mathbb{Z}_{\geq 0} \cup \mathbb{Z}^*_{\geq 0}$ *and* $l \in \mathbb{Z}_{>0}$. *Hence both* $\mathfrak{q}_n$-$\mathbf{Mod}_{\mathscr{S}}$ *and* $\mathfrak{q}_n$-$\mathbf{Mod}_{\mathscr{S},\mathscr{S}^*}$ *are symmetric monoidal supercategories with symmetry* σ.

Note that $\mathfrak{q}_n$-$\mathbf{Mod}_{\mathscr{S}}$ is a full subsupercategory of $\mathfrak{q}_n$-$\mathbf{Mod}_{\mathscr{S},\mathscr{S}^*}$; said differently, the natural inclusion functor $\mathfrak{q}_n$-$\mathbf{Mod}_{\mathscr{S}} \hookrightarrow \mathfrak{q}_n$-$\mathbf{Mod}_{\mathscr{S},\mathscr{S}^*}$ is full. By Remark 3.1, we lose no information by not including the exterior powers $\mathcal{E}^k(V_n)$ and their duals in these categories. (This is in contrast to the situations of [9, 34, 42], in which symmetric and exterior powers had to be dealt with separately.)

3.4. Superalgebra Ser_k and duality

In this section, we discuss some of the representation theory of Ser_k, as well as its relationship to $\mathfrak{q}_n$ in the form of the Schur-Weyl-Sergeev duality.

The *Sergeev algebra* Ser_k is the associative, unital superalgebra generated by the even elements $s_1, \dots, s_{k-1}$ and odd elements $c_1, \dots, c_k$ subject to the relations

$$\begin{gathered} c_i^2 = 1, \quad c_i c_j = -c_j c_i, \\ s_i^2 = 1, \quad s_i s_j = s_j s_i \quad \text{if } i \neq j \pm 1, \quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \\ c_i s_j = s_j c_i \quad \text{if } i \neq j, j+1, \quad s_i c_i = c_{i+1} s_i, \quad s_i c_{i+1} = c_i s_i \end{gathered} \tag{3.2}$$

for admissible i, j. Two important subalgebras of Ser_k are the *Clifford algebra* C_k generated by $c_1, \dots, c_k$, and the group algebra $\mathbb{C}\mathfrak{S}_k$ of the symmetric group $\mathfrak{S}_k$ on k letters, generated by $s_1, \dots, s_{k-1}$. (We think of s_i as the simple transposition of $\mathfrak{S}_k$ interchanging i and $i+1$.) We have canonical embeddings $\mathrm{Ser}_k \hookrightarrow \mathrm{Ser}_l$ for $k < l$ obtained by mapping $c_i \mapsto c_i$ and $s_j \mapsto s_j$ for $1 \leq i \leq k$ and $1 \leq j \leq k-1$.

From the relations, it is easy to prove that Ser_k admits the homogeneous basis

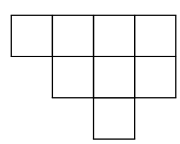
$$\{c_1^{a_1} \cdots c_k^{a_k} \sigma \mid a_1, \ldots, a_k \in \{0,1\}, \sigma \in \mathfrak{S}_k\}$$

(see [25, §13]), which we will refer to as the *standard basis*. Also crucial to the present work is the fact that Ser_k is semisimple [4, Lemma 3.6].

In order to state further results about Ser_k and $\mathfrak{q}_n$, we introduce some of the combinatorics attached to them. A *strict partition* is a nonincreasing sequence of nonnegative integers $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$ such that $\lambda_i = \lambda_{i+1}$ implies $\lambda_i = 0$. If in addition $\sum_i \lambda_i = k$, we say λ is a *strict partition of* k. We let $\mathcal{SP}(k)$ (resp. $\mathcal{SP}$) be the set of all strict partitions of k (resp. all strict partitions).

For $\lambda \in \mathcal{SP}(k)$ the *size* of λ is $|\lambda| = k$, and we say λ has *length* $l(\lambda)$ if the number of nonzero parts in λ is $l(\lambda)$. For example, $\lambda = (4,3,1,0,\ldots) \in \mathcal{SP}(8)$ with $l(\lambda) = 3$ and $|\lambda| = 8$. We will usually omit the trailing zeros of a strict partition, e.g. $\lambda = (4,3,1)$.

To every $\lambda \in \mathcal{SP}$ we associate the *shifted frame* $[\lambda]$, the array of squares with λ_i squares in row i for $1 \leq i \leq l(\lambda)$, such that row i has been shifted to the right $i-1$ units from being left-justified, e.g.

$[(4,3,1)] =$  .

For strict partitions λ, μ, we write $\lambda \subseteq \mu$ if $\lambda_i \leq \mu_i$ for all i, or equivalently if $[\lambda]$ is contained $[\mu]$.

For $\lambda \in \mathcal{SP}$, let $\delta(\lambda) \in \{0,1\}$ be 1 if $l(\lambda)$ is odd and 0 if $l(\lambda)$ is even. It is known [11, Theorem 2.18] that for every $\lambda \in \mathcal{SP}(k)$ with $l(\lambda) \leq n$ (hence we may identify $\lambda \in \mathbb{Z}^n$ as an element of $\mathfrak{h}_0^*$), there is a unique finite-dimensional, irreducible $\mathfrak{q}_n$-module of highest weight λ, which we denote $L_n(\lambda)$. Furthermore, $L_n(\lambda)$ is self-associate if and only if $\delta(\lambda) = 1$. For example, by [10, Proposition 3.1] $L_n(\lambda)$ where $\lambda = (k,0,0,\dots) = (k)$ is isomorphic to $\mathscr{S}^k(V_n)$ for $k \in \mathbb{Z}_{>0}$.

There is action of Ser_k on the tensor space $V_n^{\otimes k}$ via

$$c_i.(w_1 \otimes \cdots \otimes w_k) := (-1)^{\overline{w_1}+\cdots+\overline{w_{i-1}}} \sqrt{-1}\, w_1 \otimes \cdots \otimes P_n(w_i) \otimes \cdots w_k,$$

$$s_j.(w_1 \otimes \cdots \otimes w_k) := (-1)^{\overline{w_j}\cdot\overline{w_{j+1}}} w_1 \otimes \cdots \otimes w_{j+1} \otimes w_j \otimes \cdots \otimes w_k$$

for $w_1,\dots,w_k \in V_n$, $1 \le i \le k$, and $1 \le j \le k-1$, where P_n is as in (2.2). This action commutes with that of $\mathfrak{q}_n$ given in (3.1), so we have, for example, a homomorphism

$$\Xi_k \colon \mathrm{Ser}_k \to \mathrm{End}_{\mathfrak{q}_n}(V^{\otimes k}).$$

What's more, we have the following *Schur-Weyl-Sergeev duality* established by Sergeev [37].

Theorem 3.3. *As a $U(\mathfrak{q}_n) \otimes \mathrm{Ser}_k$-module, $V_n^{\otimes k}$ admits the decomposition*

$$V_n^{\otimes k} \simeq \bigoplus_{\substack{\lambda \in \mathcal{SP}(k) \\ l(\lambda) \le n}} L_n(\lambda) \circledast D^\lambda$$

where each D^λ is an irreducible Ser_k-module, self-associate if and only if $\delta(\lambda) = 1$. Moreover, $\{D^\lambda : \lambda \in \mathcal{SP}(k)\}$ is a complete, irredundant set of isomorphism classes of irreducible Ser_k-modules.

Corollary 3.4. *The homomorphism $\Xi_k \colon \mathrm{Ser}_k \to \mathrm{End}_{\mathfrak{q}_n}(V_n^{\otimes k})$ is surjective for all $k,n \in \mathbb{Z}_{>0}$; it is injective if and only if $k < 1 + 2 + \cdots + (n+1)$.*

Proof. Since Ser_k is semisimple, Theorem 3.3 together with Wedderburn's theorem implies that

$$\mathrm{Ser}_k \simeq \bigoplus_{\substack{\lambda\in\mathcal{SP}(k)\\ \delta(\lambda)=0}} M(D^\lambda) \oplus \bigoplus_{\substack{\lambda\in\mathcal{SP}(k)\\ \delta(\lambda)=1}} Q(D^\lambda). \tag{3.3}$$

Also by Theorem 3.3, we have

$$\mathrm{End}_{\mathfrak{q}_n}(V^{\otimes k}) \simeq \bigoplus_{\substack{\lambda\in\mathcal{SP}(k)\\ l(\lambda)\le n}} \mathrm{End}_{\mathfrak{q}_n}(L_n(\lambda)\circledast D^\lambda) \simeq \bigoplus_{\substack{\lambda\in\mathcal{SP}(k)\\ \delta(\lambda)=0\\ l(\lambda)\le n}} M(D^\lambda) \oplus \bigoplus_{\substack{\lambda\in\mathcal{SP}(k)\\ \delta(\lambda)=1\\ l(\lambda)\le n}} Q(D^\lambda)$$

where the last isomorphism uses Schur's lemma. Since Ξ_k can be thought of as quotienting Ser_k by the matrix superalgebras over D^λ with $l(\lambda) > n$, it is surjective. For the second claim, we simply note that the unique $\lambda \in \mathcal{SP}$ with $l(\lambda) > n$ and $|\lambda|$ minimal is $\lambda = (n+1, n, \ldots, 2, 1)$, which has $|\lambda| = 1 + 2 + \cdots + (n+1)$. ■

3.5. Quasi-idempotents of Ser_k

Lastly, we recall from [38] certain quasi-idempotents $e_\lambda \in \mathrm{Ser}_k$ parameterized by $\mathcal{SP}(k)$. They are the analogs in type Q of the Young symmetrizers which project $\mathbb{C}\mathfrak{S}_k$ onto a copy of the associated (irreducible) Specht module.

For $1 \le i < j \le k-1$, define the elements $s_{i,j}, \tau_{i,j}, \pi_1, \pi_j \in \mathrm{Ser}_k$ by letting $s_{i,j} \in \mathbb{C}\mathfrak{S}_k$ be the transposition interchanging i and j and setting

$$\tau_{i,j} := \frac{1}{\sqrt{-2}}(c_i - c_j)s_{i,j}, \quad \pi_1 := 0, \quad \pi_j := \tau_{1,j} + \tau_{2,j} + \cdots + \tau_{j-1,j}.$$

The π_j are odd analogs in Ser_k of the *Jucys-Murphy elements* $x_1 = 0$, $x_j = s_{1,j} + s_{2,j} + \cdots + s_{j-1,j}$ in $\mathbb{C}\mathfrak{S}_k$. Note that $s_i = s_{i,i+1}$ for $1 \le i \le k-1$.

For $\lambda \in \mathcal{SP}(k)$, let T_λ be the tableau of shape λ obtained by filling the boxes of $[\lambda]$ with $1, 2, \ldots, k$ from left to right in each row, starting from the top and working down, e.g.

$$T_{(4,3,1)} = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline \multicolumn{1}{c|}{} & 5 & 6 & 7 \\ \cline{2-4} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 8 & \multicolumn{1}{|c}{} \\ \cline{3-3} \end{array}\,.$$

Define $a_\lambda \in \mathrm{Ser}_k$ by

$$a_\lambda := \prod_{i=1}^{k} \left(\frac{\mathrm{col}(i)(\mathrm{col}(i)+1)}{2} - \pi_i^2 \right)$$

where $\mathrm{col}(i)$ is the number of the column occupied by i in T_λ, e.g.

$$a_{(4,3,1)} = 1 \cdot (3-\pi_2^2)(6-\pi_3^2)(10-\pi_4^2)(3-\pi_5^2)(6-\pi_6^2)(10-\pi_7^2)(6-\pi_8^2).$$

Let also $b_\lambda \in \mathrm{Ser}_k$ be

$$b_\lambda := \sum_{\sigma \in R_\lambda} \sigma$$

where $R_\lambda \subset \mathfrak{S}_k$ consists of all permutations fixing the rows of T_λ. Finally, define $e_\lambda \in \mathrm{Ser}_k$ to be

$$e_\lambda := a_\lambda b_\lambda.$$

Unfortunately, a simpler form of e_λ is unknown, and not for lack of trying (see [19, 20, 31, 38]).

By [38, Corollary 3.3.4] each e_λ is nonzero and quasi-idempotent, the latter of which means $e_\lambda^2 = c_\lambda e_\lambda$ for some nonzero $c_\lambda \in \mathbb{C}$. Further, the left ideal of Ser_k generated by e_λ is isomorphic as a Ser_k-module to an isotypic sum of copies of D^λ. The exact multiplicities of these sums are known [31], but not of consequence here. What is material is that, in the decomposition (3.3), e_λ belongs to $M(D^\lambda)$ if $\delta(\lambda) = 0$ and to $Q(D^\lambda)$ if $\delta(\lambda) = 1$.

Corollary 3.5. *The kernel of* $\Xi_k\colon \mathrm{Ser}_k \to \mathrm{End}_{\mathfrak{q}_n}(V^{\otimes k})$ *is the two-sided ideal of*

Ser_k *generated by the elements* e_λ *with* $l(\lambda) > n$.

Proof. From the proof of Corollary 3.4, we know the kernel of Ξ_k is exactly

$$\bigoplus_{\substack{\lambda\in\mathcal{SP}(k)\\ \delta(\lambda)=0\\ l(\lambda)>n}} M(D^\lambda) \oplus \bigoplus_{\substack{\lambda\in\mathcal{SP}(k)\\ \delta(\lambda)=1\\ l(\lambda)>n}} Q(D^\lambda).$$

Since every summand here is a *simple* superalgebra, a set consisting of one nonzero element from each summand constitutes a generating set. The claim is proved. ■

There is one quasi-idempotent which will be especially useful later on. For $n \in \mathbb{Z}_{>0}$ define the strict partition $\lambda(n) := (n+1, n, \ldots, 2, 1)$, and let $e_n := e_{\lambda(n)}$ be the corresponding element of Ser_k. Its utility will derive from the fact that for every $\mu \in \mathcal{SP}$ with $l(\mu) > n$, we have $\mu \supseteq \lambda(n)$. Hence by Corollary 3.5, $\ker \Xi_k$ is generated by the elements e_μ with $\mu \supseteq \lambda(n)$.

Chapter 4

Oriented type Q webs

In this chapter, we lay out the diagrammatics of oriented type Q webs in full detail, by defining various webs supercategories and exploring their intrinsic properties. As in Chapter 1, we first focus solely on upward-oriented webs, and then extend them to arbitrarily oriented webs.

4.1. Definition of $\mathfrak{q}$-$\mathbf{Web}_{\uparrow}$

We define a strict monoidal supercategory $\mathfrak{q}$-$\mathbf{Web}_{\uparrow}$ as follows, under the assumption that the reader is familiar with the contents of Section 1.2.

The category $\mathfrak{q}$-$\mathbf{Web}_{\uparrow}$ has as objects the set $\langle\uparrow\rangle$ of all finite-length sequences with entries in $\{0\uparrow, 1\uparrow, 2\uparrow, 3\uparrow, \dots\}$,[1] including the empty sequence $\varnothing$. Tensor product of objects is given by horizontal concatenation with $\varnothing$ acting as strict tensor unit, i.e. if $\lambda = (\lambda_1\uparrow, \dots, \lambda_k\uparrow)$ and $\mu = (\mu_1\uparrow, \dots, \mu_l\uparrow)$ are nonempty sequences in $\langle\uparrow\rangle$ then

$$\lambda \otimes \mu := (\lambda_1\uparrow, \dots, \lambda_k\uparrow, \mu_1\uparrow, \dots, \mu_l\uparrow),$$

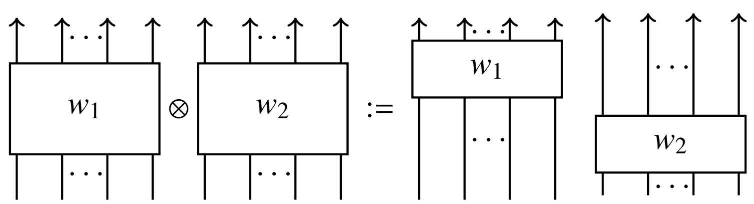
[1]The element $0\uparrow$ is included solely for technical reasons concerning the definition of a certain functor in Section 5.2; see Remark 4.1 and Lemma 5.3.

and if ν is any sequence in $\langle \uparrow \rangle$ (empty or nonempty) then

$$\varnothing \otimes \nu := \nu, \quad \nu \otimes \varnothing := \nu.$$

Remark 4.1. *It will often be convenient to omit some of the edge labels within a web. We will do so only if no ambiguity is possible. Furthermore, we follow the conventions of erasing edges labeled with a zero, and declaring webs containing an edge labeled by a negative integer to be zero. (The latter occurs in some of the formulas below as a matter of convenience only.)*

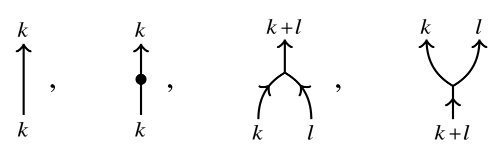
The morphism spaces of $\mathfrak{q}$-**Web**$_\uparrow$ – given below by generators and relations – are superspaces spanned by upward-oriented type Q webs modulo certain relations. Composition of morphisms in $\mathfrak{q}$-**Web**$_\uparrow$ is by vertical concatenation of webs, extended by linearity, where we declare the composition of two webs with incompatible source and target to be zero. Tensor product of morphisms is given by linearly extending the following rule for webs w_1, w_2:



(4.1)

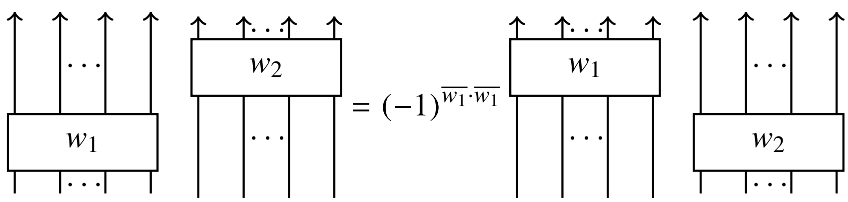
(The difference in heights on the right is to respect the superinterchange law (2.1).)

The morphisms of $\mathfrak{q}$-**Web**$_\uparrow$ are generated with respect to composition, tensor product, and linear combination by the webs



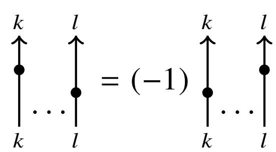
(4.2)

for $k, l \in \mathbb{Z}_{>0}$, which we refer to collectively as the *upward-oriented generators* and respectively as *identities*, *dots*, *merges*, and *splits*. We declare each generator to be $\mathbb{Z}_2$-homogeneous, but only dots to have odd parity, the rest even. Since parity is additive across compositions and tensor products (see Sections 2.1 and 2.2), the parity $\overline{w}$ of an individual web w is the number of dots modulo 2. Hence, to respect the superinterchange law (2.1), we declare that

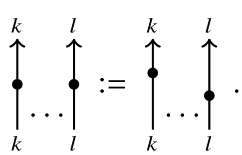


(4.3)

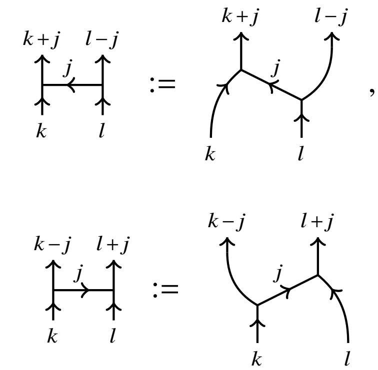
for webs w_1, w_2, which we refer to as the *superinterchange*. In particular, within an individual web each dot must lie at a unique height, and we have



for two dots lying at adjacent heights on different strands. For ease of illustration we will sometimes draw multiple dots at the same height within an individual web, resolving the resulting ambiguity by declaring that

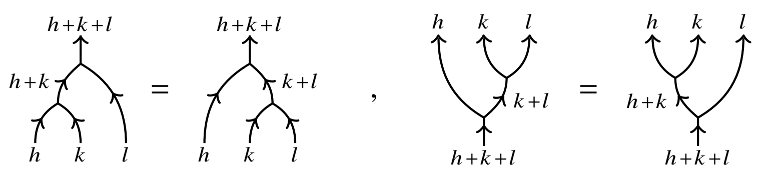


The morphisms of $\mathfrak{q}$-$\mathbf{Web}_{\uparrow}$ are subject to a number of relations in addition to (4.3). We make one piece of shorthand before stating the relations. A *ladder* is a web of the form

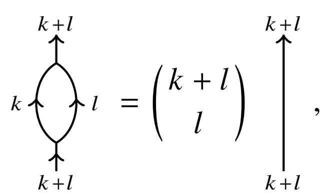


for $j, k, l \in \mathbb{Z}_{>0}$, where dots are also allowed. We call the horizontal edge of a ladder the *rung*.

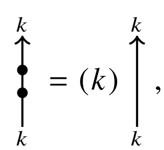
In addition to (4.3), the generating webs of $\mathfrak{q}$-**Web**$_\uparrow$ are subject to the relations



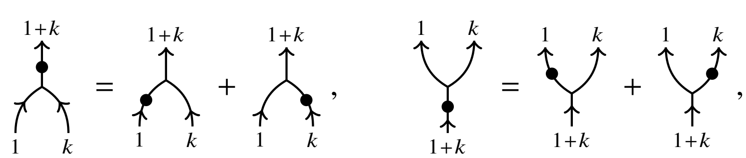
(4.4)



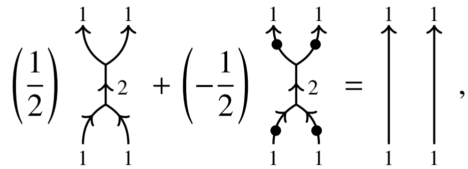
(4.5)



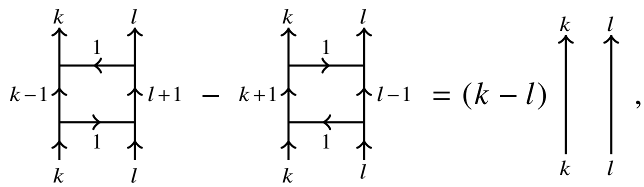
(4.6)



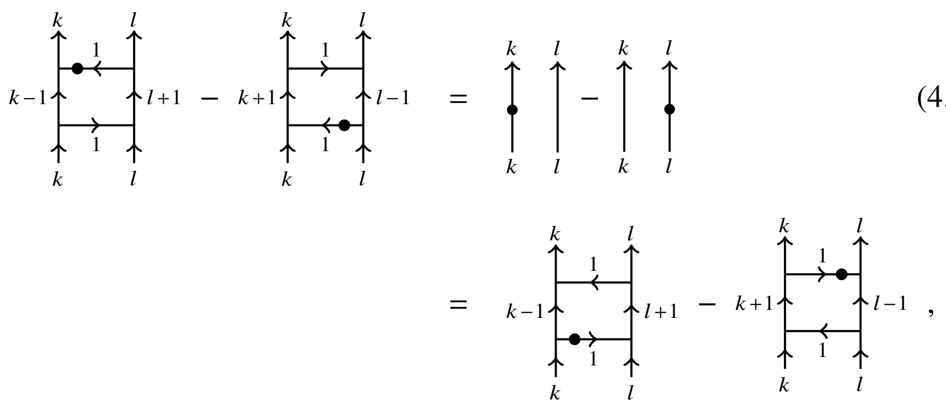
(4.7)



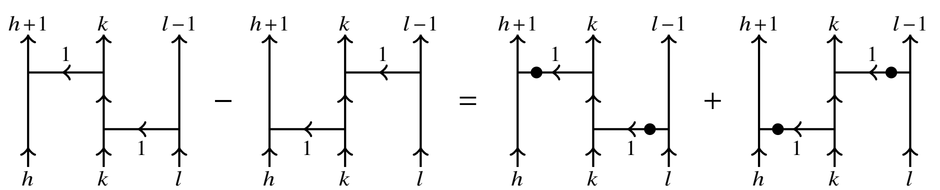
(4.8)



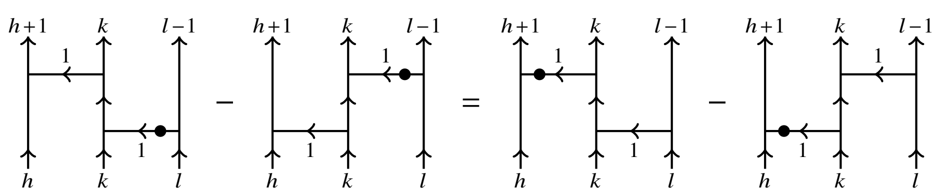
(4.9)



(4.10)



(4.11)



(4.12)

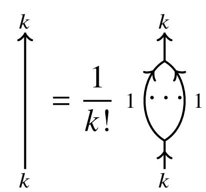
for $h, k, l \in \mathbb{Z}_{>0}$, along with the relations obtained by reflecting the webs in (4.7) across a vertical axis, and by reversing all rung orientations of the ladders in (4.11) and (4.12). Note that reversing rung orientations changes the target, but not the source, of a ladder. We refer to (4.3)-(4.12) collectively as the *upward-oriented* *relations*, and call (4.4) *associativity*, (4.5) *digon removal* (or, reading right to left, *strand explosion*), (4.8) the *dumbbell relation*, and (4.9) *square switch*. Note that setting $k = 1$ or $l = 1$ in (4.9) and (4.10) gives additional dumbbell relations by erasing edges labeled zero.

From these definitions, it's clear that $\mathfrak{q}$-$\mathbf{Web}_{\uparrow}$ is a strict monoidal supercategory.

4.2. First steps in $\mathfrak{q}$-$\mathbf{Web}_{\uparrow}$

In this section, we prove some first results about $\mathfrak{q}$-$\mathbf{Web}_{\uparrow}$. For starters, we note that both summands on the left side of the dumbbell relation are idempotent, a fact which is easily proved by direct computation using (4.3), (4.5), and (4.6).

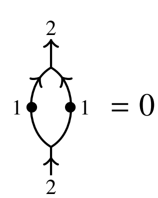
Another immediate and important consequence of the relations is



(4.13)

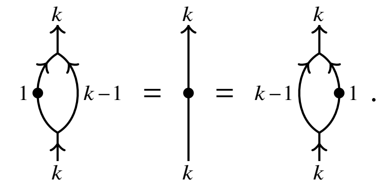
for $k \in \mathbb{Z}_{>0}$, where the dots indicate that the k-strand has been completely "exploded" into k separate 1-strands by repeatedly applying (4.5). By associativity, there is no ambiguity in the web on the right. In other words, there is only one way to split a k-strand into k-many 1-strands, and ditto for merging k-many 1-strands into a single k-strand.

Lemma 4.2. *We have*



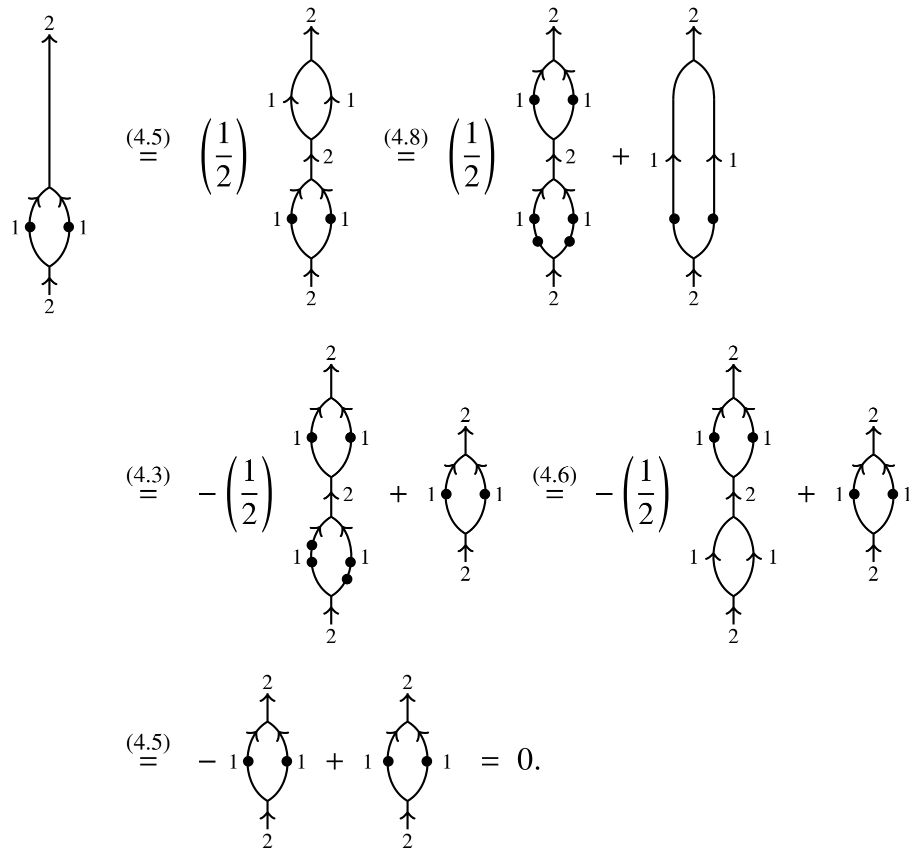
(4.14)

and, for $k \in \mathbb{Z}_{>0}$*,*

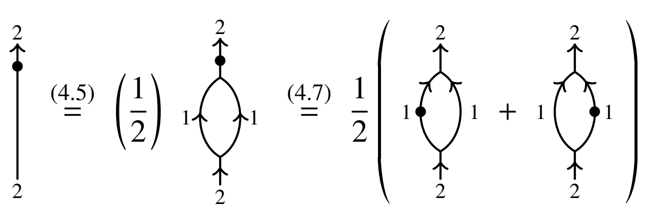


(4.15)

Proof. For (4.14), we compute that

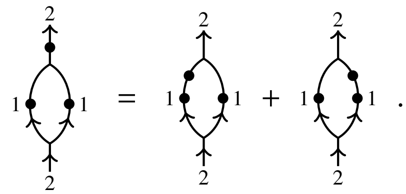


For (4.15), we first prove the case of $k = 2$. We start by computing that

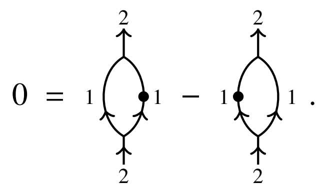


(4.16)

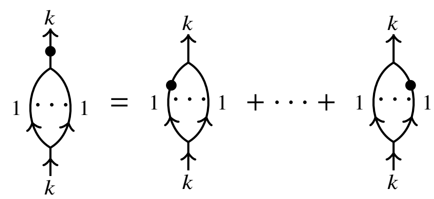
Next, we compose the left side of (4.7) on bottom with ↑↑ followed by a split to get



Using (4.14) on the left, and superinterchange and dot collision on the right, this becomes

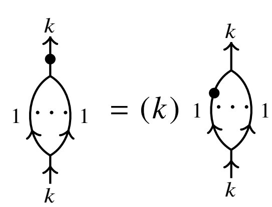


Combining the above with (4.16) and symmetry, we have (4.15) in case $k = 2$. For general k, we use (4.7) repeatedly to get



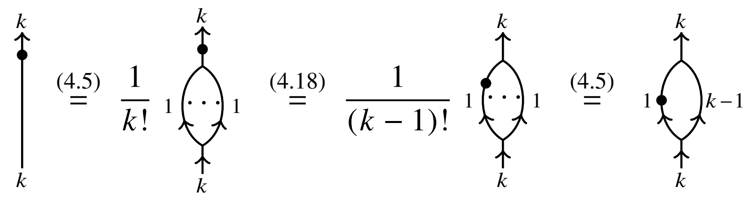
(4.17)

where the sum is over the k different webs with a dot on a unique 1-strand. By associativity and the $k = 2$ case, the summands are pairwise equal and we have, for example,



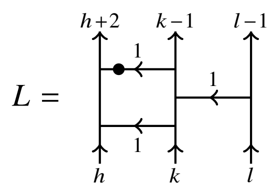
(4.18)

where on the right, only the leftmost 1-strand has a dot. We finish the proof by computing that

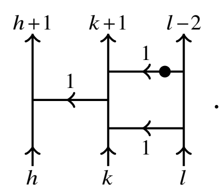


and noting that the other side of (4.15) follows by symmetry. ■

We introduce further shorthand for the following lemma. A *2-ladder* is a ladder with rungs between three adjacent vertical edges, for example

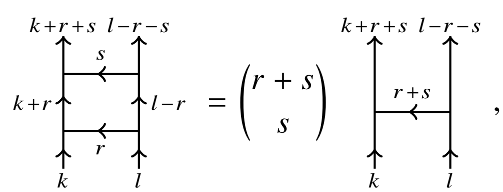


(see also (4.11) and (4.12)). We define the *rung swap* of a 2-ladder L to be the 2-ladder L' with the same source as L, which is obtained by moving each rung of L from one pair of vertical edges to the other. For example, the rung swap of the 2-ladder above is

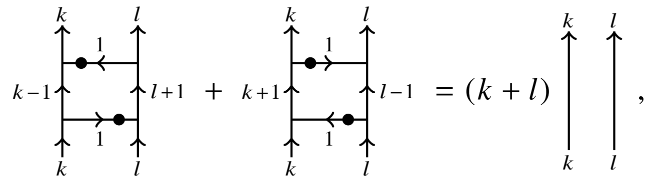


Note that, as in the above example, L and L' may have different targets.

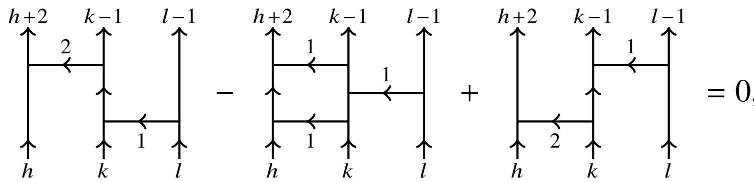
Lemma 4.3. *For $h, k, l, r, s \in \mathbb{Z}_{>0}$ we have*



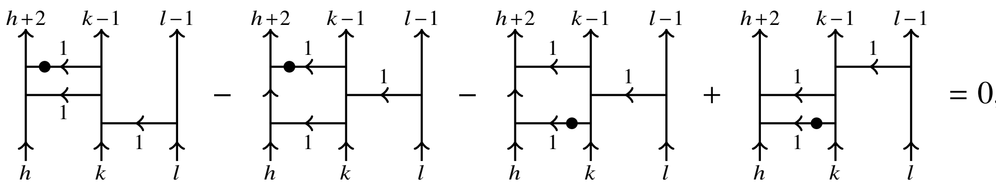
(4.19)



(4.20)



(4.21)



(4.22)

along with the equations obtained by

- *reversing all rung orientations of the ladders in* (4.19)*,*
- *reversing all rung orientations of the ladders in* (4.21) *and* (4.22)*,*
- *performing rung swaps on the ladders in* (4.21) *and* (4.22)*, and*
- *performing rung swaps and then reversing rung orientations of the ladders in* (4.21) *and* (4.22)*.*

Hence (4.19) *represents two separate equations, and each of* (4.21) *and* (4.22) *represents four separate equations.*

Proof. We leave these as exercises for the reader, but offer the following guidelines.

Part (4.19) follows from associativity and a digon removal.

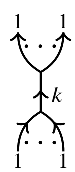
Part (4.20) is similar to [42, Lemma 2.10(b)]. Its proof involves square switches on the edges labeled $l + 1$ and $k + 1$ in the webs on the left, followed by two digon removals and the dumbbell relation.

Parts (4.21) and (4.22) are similar to [42, Lemma 2.10(c)]. The former involves the dumbbell relation on the parallel ladder rungs in the middle web, followed by associativity, (4.14), a square switch, and two digon removals. The proof of the latter is similar, but it also requires (4.12). ■

4.3. Clasp idempotents

For utility as well as for independent interest, we introduce some idempotent morphisms in $\mathfrak{q}$-$\mathbf{Web}_\uparrow$. For $k \in \mathbb{Z}_{>0}$ let $\uparrow^k \in \langle\uparrow\rangle$ denote the sequence $(1\uparrow, \ldots, 1\uparrow) \in \langle\uparrow\rangle$ of length k.

Definition 4.4. *For $k \in \mathbb{Z}_{>0}$ we define the k^{th}* clasp *to be*

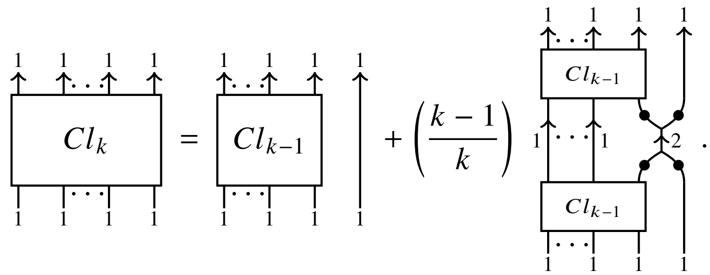
$$Cl_k = \frac{1}{k!}$$


$$\in \mathrm{End}_{\mathfrak{q}\text{-}\mathbf{Web}_\uparrow}(\uparrow^k)$$

where the dots indicate k separate 1*-strands. By associativity and digon removal, Cl_k is idempotent.*

The following lemma shows that clasps admit a recursion similar to that of the *Jones-Wenzl projectors* in the Temperley-Lieb algebra (see [43]).

Lemma 4.5. *For $k \in \mathbb{Z}_{>1}$ we have*

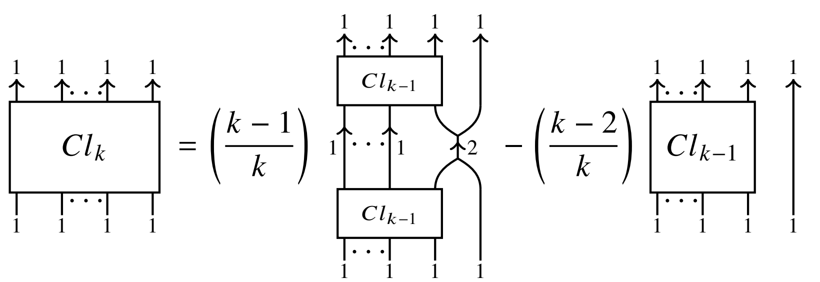


Proof. The proof is similar to that of [34, Lemma 2.12], so we leave it as an exercise to the reader. See also [42, Lemma 2.12]. ■

Note that in case $k = 2$, Lemma 4.5 is equivalent to the dumbbell relation.

It will be convenient later on to have a version of Lemma 4.5 which has no dots. This is easily obtained by applying the dumbbell relation to the rightmost web, which yields:

Corollary 4.6. *For $k \in \mathbb{Z}_{>1}$ we have*



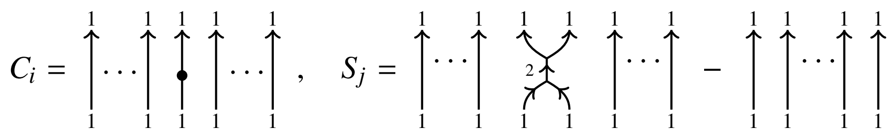
4.4. Sergeev and permutation diagrams

In this section, we prove the existence of a surjective homomorphism

$$\xi_k : \operatorname{Ser}_k \twoheadrightarrow \operatorname{End}_{\mathfrak{q}\text{-}\mathbf{Web}_{\uparrow}}(\uparrow^k)$$

for $k \in \mathbb{Z}_{>0}$, which will later be shown to be an isomorphism (see Corollary 5.5). Even without injectivity, the usefulness of ξ_k cannot be overstated, as will be seen. We start with a definition.

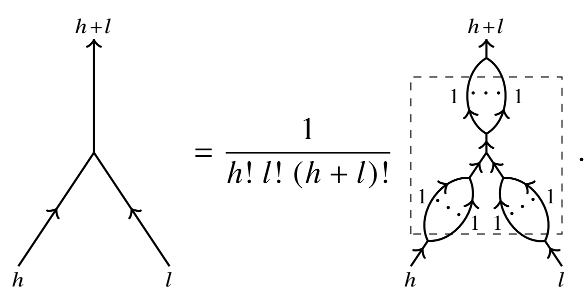
Definition 4.7. *For $k \in \mathbb{Z}_{>0}$ we define the morphisms*



in $\mathrm{End}_{\text{q-}\mathbf{Web}_\uparrow}(\uparrow^k)$ *for* $1 \le i \le k$ *and* $1 \le j \le k-1$*, where the dot is on the* i^{th} *strand of* C_i *and the dumbbell merges the* j^{th} *and* $(j+1)^{st}$ *strands of* S_j*.*

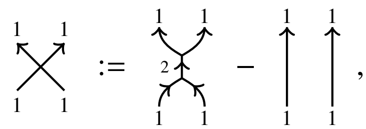
Lemma 4.8. *The algebra* $\mathrm{End}_{\text{q-}\mathbf{Web}_\uparrow}(\uparrow^k)$ *is generated by* $C_1, \ldots, C_k, S_1, \ldots, S_{k-1}$*.*

Proof. It suffices to prove that every individual web $w \in \mathrm{End}_{\text{q-}\mathbf{Web}_\uparrow}(\uparrow^k)$ can be written as a linear combination of webs containing only dumbbells of thickness 2 and dotted 1-strands. Equation (4.15) ensures that every dot in w can be moved onto a 1-strand. Next, for every merge in w, we completely explode its three edges into 1-strands using (4.13) as follows:

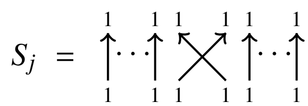


By associativity, the web enclosed by the dashed rectangle above is $(h+l)!\,Cl_{h+l}$, which, after finitely many iterations of the recursion in Lemma 4.5 (or Corollary 4.6), can be written in the desired form. Doing the same for the edges in every split of w finishes the proof. ■

We define an *upward crossing* (of 1-strands) to be



which in particular implies that

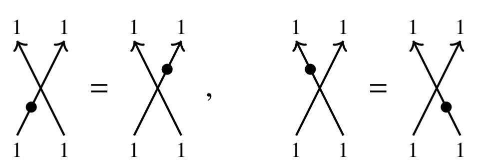


for $k \in \mathbb{Z}_{>0}$ and $1 \leq j \leq k-1$, where j^{th} and $(j+1)^{\text{st}}$ strands are crossed. We are justified in doing so by the following result.

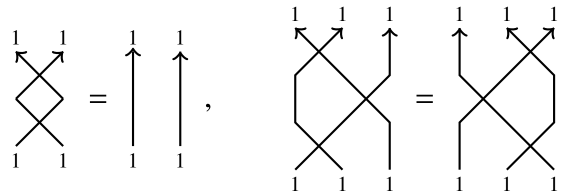
Lemma 4.9. *For $k \in \mathbb{Z}_{>0}$ we have a surjective homomorphism*

$$\begin{aligned} \xi_k : \mathrm{Ser}_k &\twoheadrightarrow \mathrm{End}_{\mathfrak{q}\text{-}\mathbf{Web}_\uparrow}(\uparrow^k) \\ c_i &\mapsto C_i \\ s_j &\mapsto S_j \end{aligned}$$

for $1 \leq i \leq k$ and $1 \leq j \leq k-1$. In particular we have



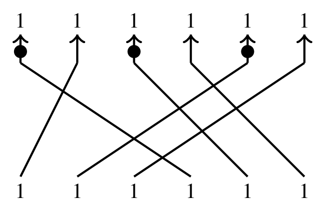
(4.23)

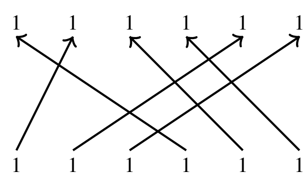


(4.24)

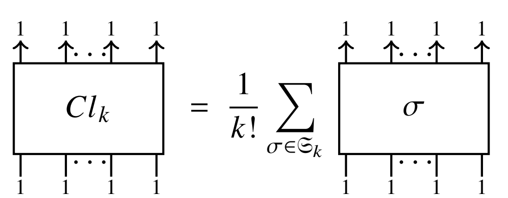
Proof. By Lemma 4.8, ξ_k is surjective. That it is a well-defined homomorphism, i.e. that relations (3.2) hold in $\mathrm{End}_{\mathrm{q}\text{-}\mathbf{Web}_\uparrow}(\uparrow^k)$, can be checked by direct calculations that we leave to the reader. ■

We will abuse notation and denote an element $\xi_k(w) \in \mathrm{End}_{\mathrm{q}\text{-}\mathbf{Web}_\uparrow}(\uparrow^k)$ simply by w for $w \in \mathrm{Ser}_k$. Moreover, we will refer to the images under ξ_k of elements of the standard basis of Ser_k as *Sergeev diagrams*, and in particular to images of elements of $\mathfrak{S}_k \subset \mathrm{Ser}_k$ as *permutation diagrams*. Examples of a Sergeev diagram and a permutation diagram, respectively, in $\mathrm{End}_{\mathrm{q}\text{-}\mathbf{Web}_\uparrow}(\uparrow^6)$ are

$\xi_6(c_1c_3c_5s_1s_4s_3s_5s_4s_2s_3) =$  ,

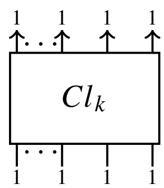
$\xi_6(s_1s_4s_3s_5s_4s_2s_3) =$  .

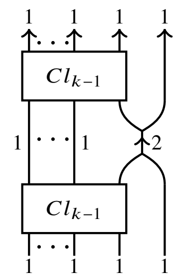
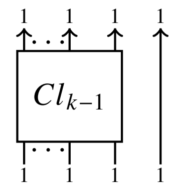
Lemma 4.10. *For $k \in \mathbb{Z}_{>0}$ we have*

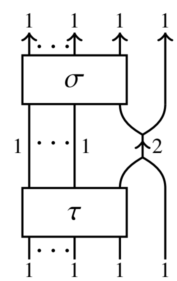


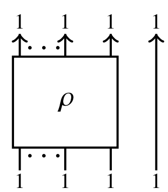
(4.25)

Proof. We prove this by induction on k. The base case of $k = 1$ is immediate because Cl_1 is a single identity strand of thickness 1; also, the case of $k = 2$ amounts to the definition of an upward crossing. Assuming the lemma is true for $k - 1$, we use Corollary 4.6 to compute that

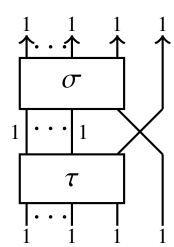
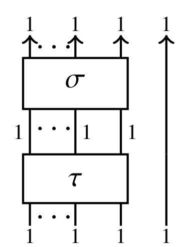


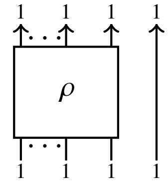
$= \left(\frac{k-1}{k}\right)$  $- \left(\frac{k-2}{k}\right)$ 

$= \frac{k-1}{k((k-1)!)^2} \sum_{\sigma,\tau \in \mathfrak{S}_{k-1}}$ 

$- \frac{k-2}{k(k-1)!} \sum_{\rho \in \mathfrak{S}_{k-1}}$  .

Applying the definition of an upward crossing on the dumbbell, this becomes

$\frac{k-1}{k((k-1)!)^2} \sum_{\sigma,\tau \in \mathfrak{S}_{k-1}} \Bigg($  $+$  $\Bigg)$

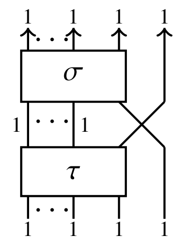
$- \frac{k-2}{k(k-1)!} \sum_{\rho \in \mathfrak{S}_{k-1}}$  .

Now the web inside the parentheses on the right is $\xi_{k-1}(\sigma\tau)$. Since the map $\sigma \mapsto \sigma\tau$ is a set bijection $\mathfrak{S}_{k-1} \to \mathfrak{S}_{k-1}$ for each $\tau \in \mathfrak{S}_{k-1}$, we have

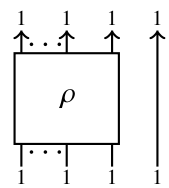
$$\sum_{\sigma,\tau\in\mathfrak{S}_{k-1}} \sigma\tau \;=\; (k-1)! \sum_{\rho\in\mathfrak{S}_{k-1}} \rho$$

and the previous becomes

$$\frac{k-1}{k((k-1)!)^2} \sum_{\sigma,\tau\in\mathfrak{S}_{k-1}}$$



$$+\left(\frac{(k-1)(k-1)!}{k((k-1)!)^2} - \frac{k-2}{k(k-1)!}\right) \sum_{\rho\in\mathfrak{S}_{k-1}}$$

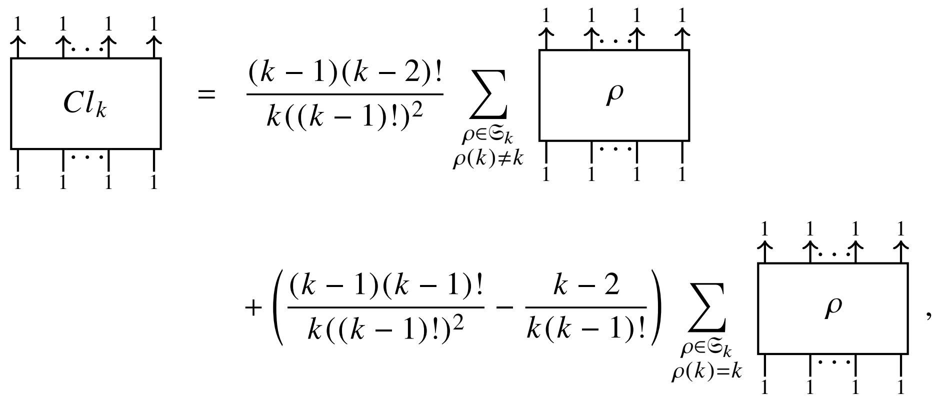


.

We can view the sum on the right as being over all $\rho \in \mathfrak{S}_k$ such that $\rho(k) = k$. Meanwhile, the summands on the left are $\xi_k(\sigma s_{k-1}\tau)$. Viewing each $\sigma s_{k-1}\tau$ as an element $\rho \in \mathfrak{S}_k$ such that $\rho(k) \neq k$, we claim

$$\sum_{\sigma,\tau\in\mathfrak{S}_{k-1}} \sigma s_{k-1}\tau \;=\; (k-2)! \sum_{\substack{\rho\in\mathfrak{S}_k \\ \rho(k)\neq k}} \rho.$$

Indeed, it's easy to see that every $\rho \in \mathfrak{S}_k$ with $\rho(k) \neq k$ can be expressed as $\rho = \sigma s_{k-1}\tau$ for some $\sigma, \tau \in \mathfrak{S}_{k-1}$, and that the other expressions of ρ in this form are $\rho = \sigma\chi s_{k-1}\chi^{-1}\tau$ for any $\chi \in \mathfrak{S}_k$ such that $\chi(k-1) = k-1$ and $\chi(k) = k$. There are $(k-2)!$ such elements χ, so the claim is proved. Altogether we have shown that



where the coefficient of both sums is $1/k!$. This obtains. ■

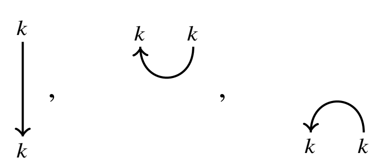
4.5. Definition of $\mathfrak{q}$-$\mathbf{Web}_{\uparrow\downarrow}$

In this section, we define a strict monoidal supercategory $\mathfrak{q}$-$\mathbf{Web}_{\uparrow\downarrow}$ which contains $\mathfrak{q}$-$\mathbf{Web}_{\uparrow}$ as a monoidal subsupercategory. We will omit parts of the definition which were already stated in the definition of $\mathfrak{q}$-$\mathbf{Web}_{\uparrow}$, or are immediately deducible from it.

The category $\mathfrak{q}$-$\mathbf{Web}_{\uparrow\downarrow}$ has as objects the set $\langle\uparrow,\downarrow\rangle$ of all finite-length sequences over the set

$$\{0\uparrow, 1\uparrow, 2\uparrow, 3\uparrow, \dots\} \cup \{1\downarrow, 2\downarrow, 3\downarrow, \dots\},$$

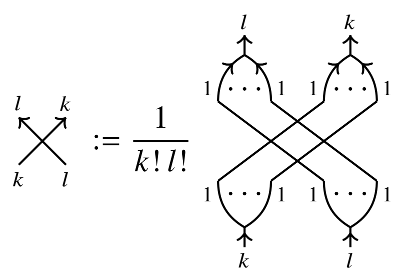
including the empty sequence $\varnothing$. The morphisms of $\mathfrak{q}$-$\mathbf{Web}_{\uparrow\downarrow}$ are generated with respect to composition, tensor product, addition, and scalar multiplication by the upward-oriented generators (4.2) along with



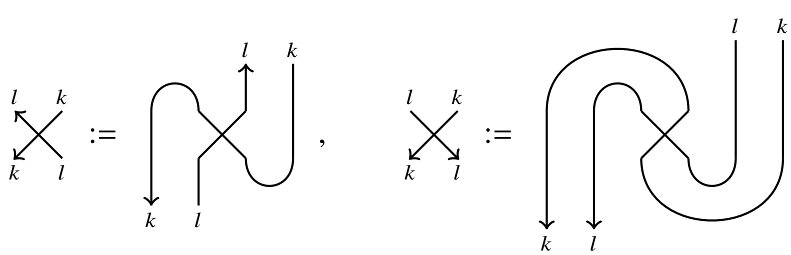
for $k \in \mathbb{Z}_{>0}$, which we call *identities*, *cups*, and *caps*, respectively. Each of these three types of generators is homogeneous of even parity. Note that the source of every cup, and the target of every cap, is the empty sequence.

The morphisms of $\mathfrak{q}$-$\mathbf{Web}_{\uparrow\downarrow}$ are subjects to (4.3)-(4.12) and several other relations. The first is simply the extension of the superinterchange to include the fact that cups and caps can exchange heights with merges, splits, and dots (with whom they share no strands) with an individual web, owing to their even parity. We will continue to refer to the entirety of the superinterchange as merely (4.3), since it is the only interesting case.

We need some more definitions before stating the other relations of $\mathfrak{q}$-$\mathbf{Web}_{\uparrow\downarrow}$. We define *upward crossings* by



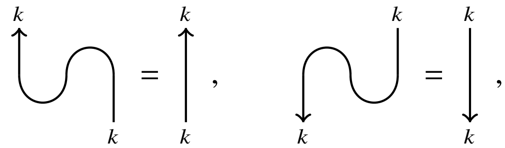
for $k, l \in \mathbb{Z}_{>0}$, where each crossing of 1-strands on the right side is as defined in the previous section. (Hence, the $k = l = 1$ case here agrees with that definition.) Using these, we define *left* and *downward crossings* by



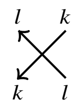
(4.26)

for $k, l \in \mathbb{Z}_{>0}$, respectively.

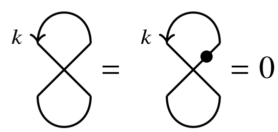
In addition to (4.3)-(4.12), the morphisms of $\mathfrak{q}$-$\mathbf{Web}_{\uparrow\downarrow}$ are subject to the relations



(4.27)

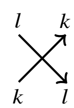


is invertible, (4.28)

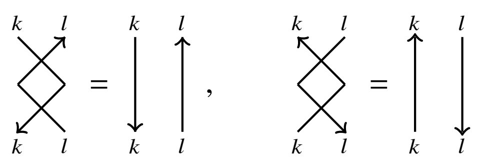


(4.29)

for $h, k, l \in \mathbb{Z}_{>0}$. Relation (4.28) is really the statement that there is an additional generator

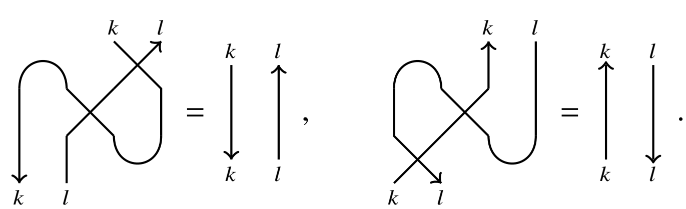


for $k, l \in \mathbb{Z}_{>0}$, which we call a *right crossing*, and further relations



(4.30)

We call (4.27) the *isotopy relations*. Note that by applying the definition (4.26) of a left crossing, the two equations of (4.30) state that

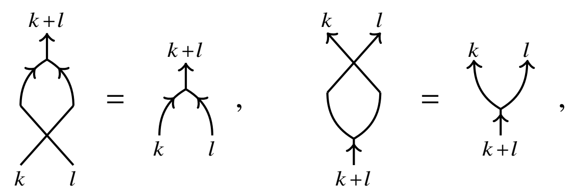


It is clear from these definitions that $\mathfrak{q}$-$\mathbf{Web}_{\uparrow\downarrow}$ is a strict monoidal supercategory.

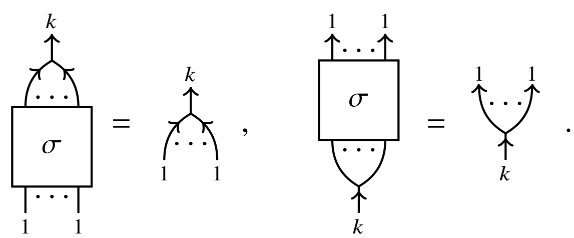
4.6. Fullness of $\mathfrak{q}$-**Web**$_\uparrow$ and symmetry

In this section, we prove $\mathfrak{q}$-**Web**$_\uparrow$ is full in $\mathfrak{q}$-**Web**$_{\uparrow\downarrow}$, and that both are symmetric.

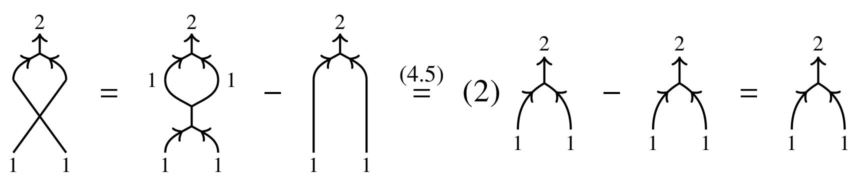
Lemma 4.11. *For $k, l \in \mathbb{Z}_{>0}$ we have*



and for a permutation diagram $\sigma \in \mathrm{End}_{\mathfrak{q}\text{-}\mathbf{Web}_\uparrow}(\uparrow^k)$,



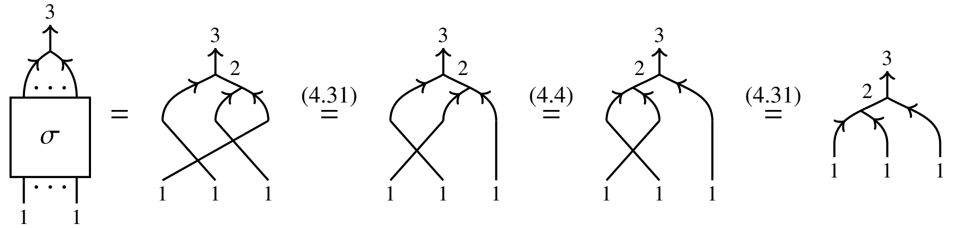
Proof. First we prove the third equality. We compute:



(4.31)

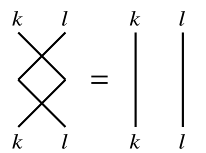
Since every permutation diagram σ is a composition of s_i, this combined with associativity proves the third equation of the lemma. For example, if $\sigma = s_2 s_1 \in$

$\mathrm{End}_{\text{q-}\mathbf{Web}_\uparrow}(\uparrow^3)$ then

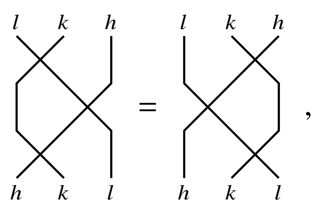


The fourth equality is proved similarly, while the first and second follow from the third and fourth plus associativity, once one writes them using the definition of an upward crossing. ∎

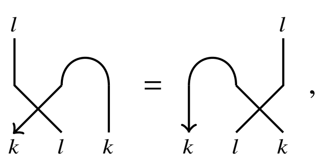
Lemma 4.12. *For $h, k, l \in \mathbb{Z}_{>0}$ we have*



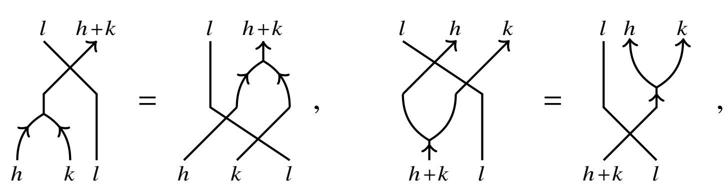
(4.32)



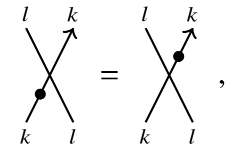
(4.33)



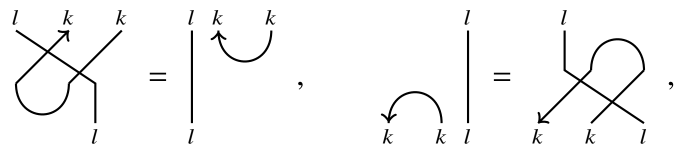
(4.34)



(4.35)



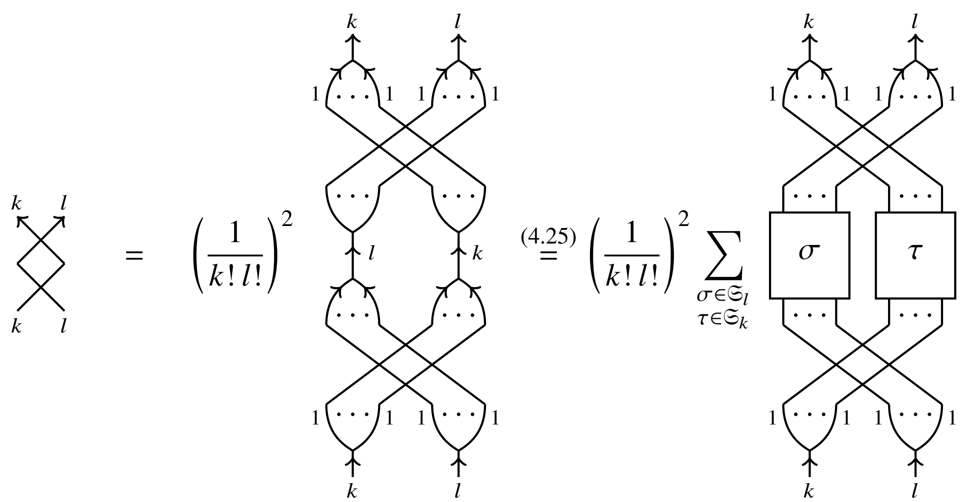
(4.36)

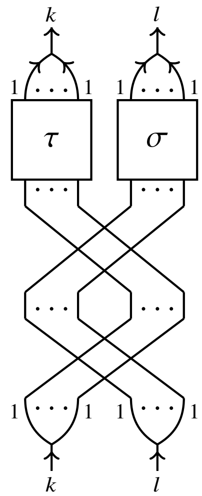
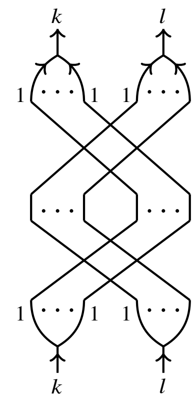


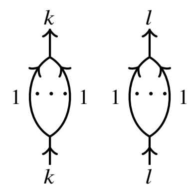
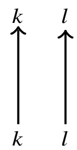
(4.37)

along with the relations obtained by reflecting the webs in (4.34) *and* (4.37) *about a horizontal axis, and by reflecting the webs in* (4.35) *and* (4.36) *about a vertical axis. In each case, any consistent orientation of the unoriented strands is permissible.*

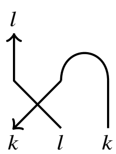
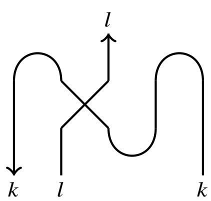
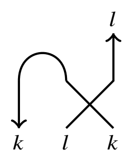
Proof. We prove (4.32) in case both strands are upward-oriented; the other cases can be deduced from it and the isotopy relations or from (4.30). For upward-oriented strands, the case of $k = l = 1$ is a consequence of Lemma 4.9. Using this, the computation for arbitrary $k, l \in \mathbb{Z}_{>0}$ is as follows:



$$\overset{(4.24)}{=} \left(\frac{1}{k!\,l!}\right)^2 \sum_{\substack{\sigma\in\mathfrak{S}_l\\ \tau\in\mathfrak{S}_k}}$$  $$\overset{(4.11)}{=} \left(\frac{1}{k!\,l!}\right)^2 \sum_{\substack{\sigma\in\mathfrak{S}_l\\ \tau\in\mathfrak{S}_k}}$$ 

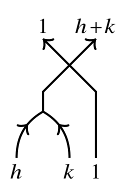
$$\overset{(4.24)}{=} \left(\frac{1}{k!\,l!}\right)^2 \sum_{\substack{\sigma\in\mathfrak{S}_l\\ \tau\in\mathfrak{S}_k}}$$  $$\overset{(4.13)}{=}$$  .

For (4.34) we observe that

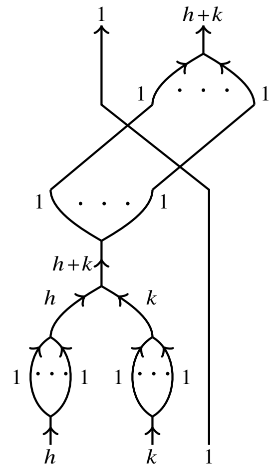
 $$\overset{(4.26)}{=}$$  $$\overset{(4.27)}{=}$$  ,

and the other cases are proved similarly.

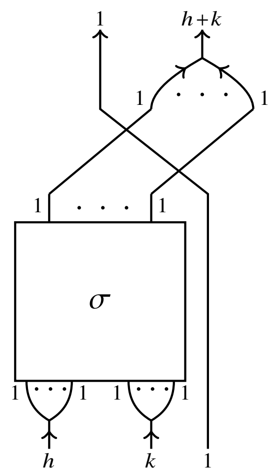
For (4.35) we first prove the equality on the left in case the l-strand is upward-oriented with $l = 1$:



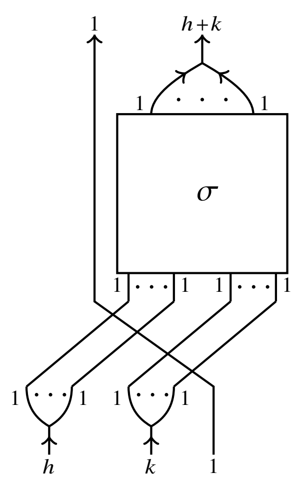
$$\overset{(4.13)}{=} \frac{1}{h!k!(h+k)!}$$

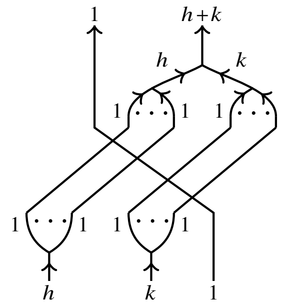
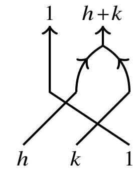


$$\overset{(4.25)}{=} \frac{1}{h!k!(h+k)!} \sum_{\sigma \in \mathfrak{S}_{h+k}}$$

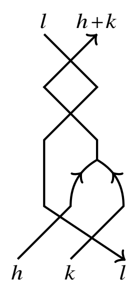
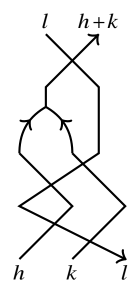


$$\overset{(3.2)}{=} \frac{1}{h!k!(h+k)!} \sum_{\sigma \in \mathfrak{S}_{h+k}}$$



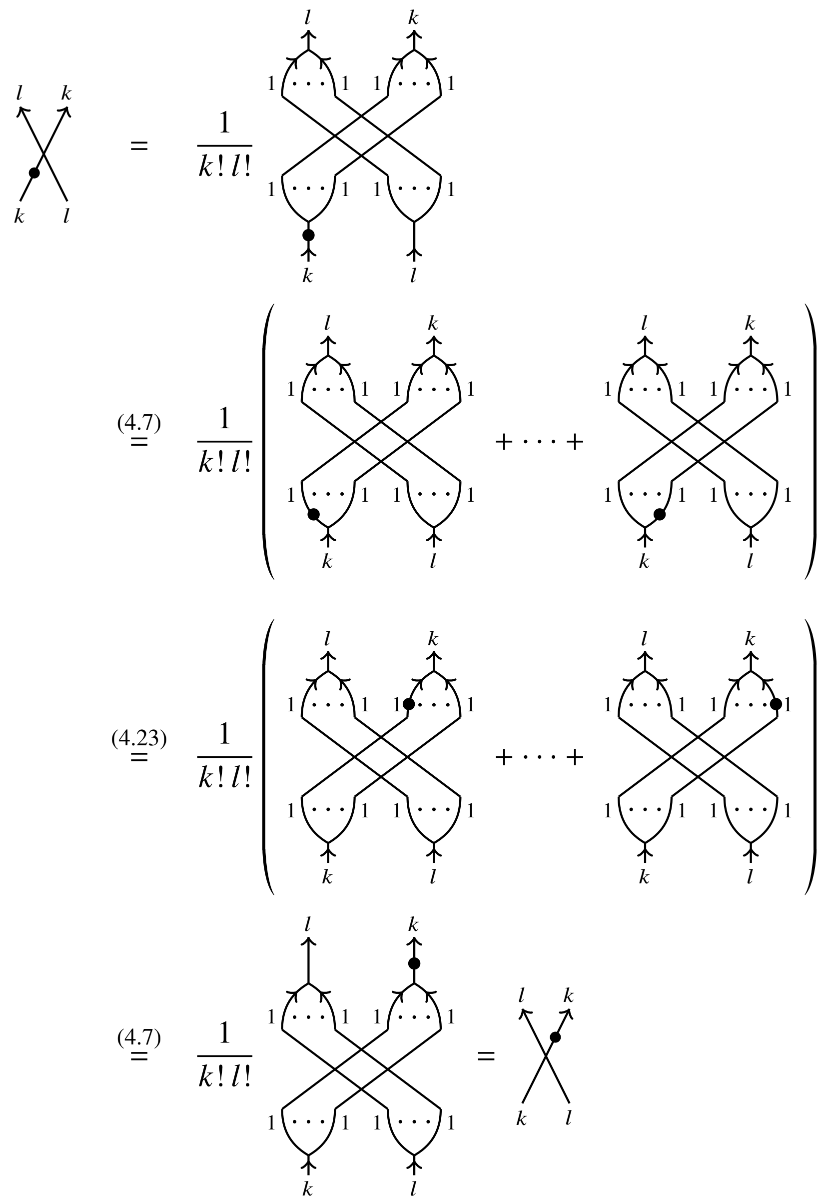
$\overset{(4.11)}{=} \quad \dfrac{1}{h!k!(h+k)!} \displaystyle\sum_{\sigma \in \mathfrak{S}_{h+k}}$  $=$  .

Similar proofs take care of all cases of (4.35), excepting the two equations pictured when the l-strand is downward-oriented (because they involve right crossings). We can prove the equality on the left in that case by composing on bottom with  and on top with  , both of which are invertible by (4.32), to get the equivalent relation

 $=$  . (4.38)

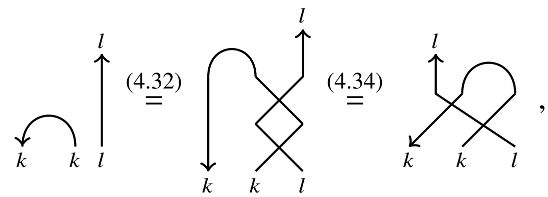
Applying (4.32) to both sides here obtains a previously established case of (4.35), so it holds. The proof of the equation on the right when the l-strand is downward-oriented is similar.

For (4.36), we first prove the identity pictured when the l-strand is upward-oriented. The case of $k = l = 1$ is a consequence of Lemma 4.9. Using this, the computation for arbitrary $k, l \in \mathbb{Z}_{>0}$ is as follows:



where the two sums are over the k different webs with a dot on a unique 1-strand. The other cases of (4.36) are proved similarly, or are otherwise easily dispatched by direct calculation, except for the equation pictured in (4.36) when the l-strand is downward-oriented. In that case, a trick similar to the one used in (4.38) suffices.

For (4.37) we compute that



and the other cases are proved similarly.

Finally for (4.33), we leave the proof to the reader but offer the following hints. The case of all upward-oriented strands is proved similarly to (4.32) in the case of all upward-oriented strands. The other seven cases are proved by direct computation using the first case and other existing relations. The proof is complete. ■

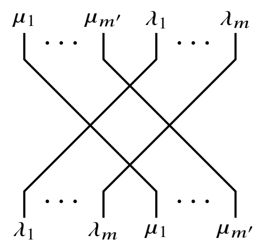
Proposition 4.13. *The supercategory* $\mathfrak{q}$-$\mathbf{Web}_{\uparrow}$ *is a full subsupercategory of* $\mathfrak{q}$-$\mathbf{Web}_{\uparrow\downarrow}$. *In other words, we have*

$$\mathrm{Hom}_{\mathfrak{q}\text{-}\mathbf{Web}_{\uparrow}}(\lambda, \mu) = \mathrm{Hom}_{\mathfrak{q}\text{-}\mathbf{Web}_{\uparrow\downarrow}}(\lambda, \mu)$$

for all $\lambda, \mu \in \langle \uparrow \rangle$.

Proof. It suffices to show that every individual web $w \in \mathrm{Hom}_{\mathfrak{q}\text{-}\mathbf{Web}_{\uparrow\downarrow}}(\lambda, \mu)$ can be expressed in terms of the upward-oriented generators. Suppose w contains a cap c. Then since $\lambda \in \langle \uparrow \rangle$, the head of c must eventually connect to the tail of a cup c', after crossing finitely many other strands. Using (4.34) and (4.37), the strand connecting the head of c to the tail of c' may be contracted until an application of (4.27) or (4.29) resolves it, c, and c' into an upward identity or zero, respectively. Finitely many iterations of this process resolves any cups and caps of w, as well as any right crossings, since their lower right ends must eventually connect to the tail of a cup. This completes the proof. ■

We are now ready to prove that $\mathfrak{q}$-$\mathbf{Web}_{\uparrow\downarrow}$, and hence $\mathfrak{q}$-$\mathbf{Web}_{\uparrow}$, are symmetric. For every pair of sequences $\lambda, \mu \in \langle\uparrow,\downarrow\rangle$ we define the morphism

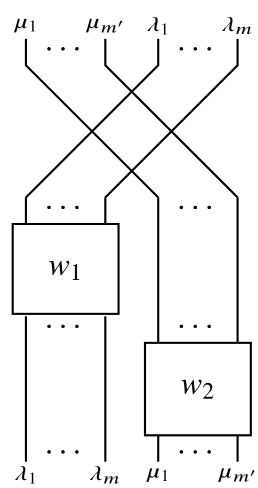
$\Sigma_{\lambda,\mu} :=$ 

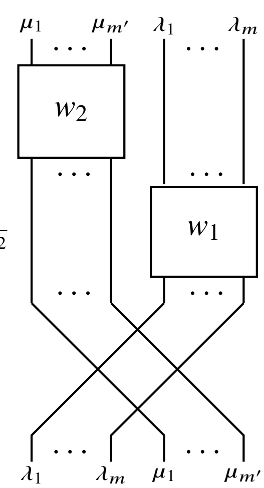
$\in \mathrm{Hom}_{\mathfrak{q}\text{-}\mathbf{Web}_{\uparrow\downarrow}}(\lambda \otimes \mu, \mu \otimes \lambda)$

where the strands are oriented according to λ and μ. By Proposition 4.13, $\Sigma_{\lambda,\mu} \in \mathrm{Hom}_{\mathfrak{q}\text{-}\mathbf{Web}_{\uparrow}}(\lambda \otimes \mu, \mu \otimes \lambda)$ if $\lambda, \mu \in \langle\uparrow\rangle$.

Proposition 4.14. *The morphisms $(\Sigma_{\lambda,\mu})_{\lambda,\mu\in\langle\uparrow,\downarrow\rangle}$ constitute a symmetry on $\mathfrak{q}$-$\mathbf{Web}_{\uparrow\downarrow}$. Thus by Proposition 4.13, the morphisms $(\Sigma_{\lambda,\mu})_{\lambda,\mu\in\langle\uparrow\rangle}$ constitute a symmetry on $\mathfrak{q}$-$\mathbf{Web}_{\uparrow}$, and both are symmetric monoidal supercategories.*

Proof. For $\lambda, \mu, \nu \in \langle\uparrow,\downarrow\rangle$, the axioms $\Sigma_{\lambda\otimes\mu,\nu} = (\Sigma_{\lambda,\nu} \otimes 1_\mu) \circ (1_\lambda \otimes \Sigma_{\mu,\nu})$ and $\Sigma_{\lambda,\mu\otimes\nu} = (1_\mu \otimes \Sigma_{\lambda,\nu}) \circ (\Sigma_{\lambda,\mu} \otimes 1_\nu)$ are true by the definition of Σ. From (4.33) we know $\Sigma_{\mu,\lambda} \circ \Sigma_{\lambda,\mu} = 1_{\lambda\otimes\mu}$, and each $\Sigma_{\lambda,\mu}$ is even by definition. Hence it remains to show that Σ is natural, i.e. that



$= (-1)^{\overline{w_1}\cdot\overline{w_2}}$


for individual webs $w_1 \in \mathrm{End}_{\mathrm{q}\text{-}\mathbf{Web}_{\uparrow\downarrow}}(\lambda)$ and $w_2 \in \mathrm{End}_{\mathrm{q}\text{-}\mathbf{Web}_{\uparrow\downarrow}}(\mu)$, where $\overline{w_1}, \overline{w_2}$ denote the parities of w_1, w_2, respectively (i.e. the number of dots modulo 2). This is true by (4.35)-(4.37) and the superinterchange. ■

Chapter 5

Webs for symmetric powers

This chapter constitutes the first of two applications of type Q webs, in this case to the $\mathfrak{q}_n$-morphisms between tensor products of the symmetric powers $\mathscr{S}^k(V_n)$ and their duals $\mathscr{S}^{k*}(V_n) := (\mathscr{S}^k(V_n))^*$. Recall from Definition 3.2 the categories $\mathfrak{q}_n$-$\mathbf{Mod}_{\mathscr{S}}$ and $\mathfrak{q}_n$-$\mathbf{Mod}_{\mathscr{S},\mathscr{S}^*}$, both of which are symmetric monoidal supercategories. The main result of this chapter is Theorem 5.6, in which we prove the existence of superequivalences

$$\Psi^{\uparrow}\colon \mathfrak{q}\text{-}\mathbf{Web}_{\uparrow} \to \mathfrak{q}_n\text{-}\mathbf{Mod}_{\mathscr{S}}, \quad \Psi^{\uparrow\downarrow}\colon \mathfrak{q}\text{-}\mathbf{Web}_{\uparrow\downarrow} \to \mathfrak{q}_n\text{-}\mathbf{Mod}_{\mathscr{S},\mathscr{S}^*},$$

thereby obtaining webs descriptions of the aforementioned $\mathfrak{q}_n$-morphisms.

5.1. Commuting actions and functors Φ_m

As discussed in Chapter 1, the process that ends with type Q webs describing $\mathfrak{q}_n$-morphisms begins with commuting actions of $\mathfrak{q}_m$ and $\mathfrak{q}_n$, where we regard n as fixed and m as variable. In this section we describe these actions and their consequences, paving the way for a connection between webs and morphisms. We refer the reader to Chapters 2 and 3 for all relevant background information.

We start by considering the superalgebra $U(\mathfrak{q}_m)\otimes U(\mathfrak{q}_n)$ and its module $V_m \boxtimes V_n$.

The latter admits the homogeneous even $U(\mathfrak{q}_m) \otimes U(\mathfrak{q}_n)$-involution P given by

$$P := \sqrt{-1}\, P_m \otimes P_n = \sqrt{-1} \begin{bmatrix} 0 & I_m \\ -I_m & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix},$$

where the matrices P_m and P_n are defined with respect to the standard bases of V_m and V_n, respectively. Indeed, we have

$$\overline{P} = \overline{P_m \otimes P_n} = \overline{P_m} + \overline{P_n} = 1 + 1 = 0 \in \mathbb{Z}_2,$$

and P commutes with the action of $U(\mathfrak{q}_m) \otimes U(\mathfrak{q}_n)$ because P_m and P_n commute with all elements of $\mathfrak{q}_m$ and $\mathfrak{q}_n$, respectively. Thus P is an even $U(\mathfrak{q}_m) \otimes U(\mathfrak{q}_n)$-endomorphism of $V_m \boxtimes V_n$, and it is an involution because

$$\begin{aligned} P^2 &= (\sqrt{-1}\, P_m \otimes P_n)^2 \\ &= \sqrt{-1}^{\,2} (P_m \otimes P_n) \circ (P_m \otimes P_n) \\ &= -\sqrt{-1}^{\,2}\, P_m^2 \otimes P_n^2 \\ &= -\sqrt{-1}^{\,2}\, 1_{V_m} \otimes 1_{V_n} \\ &= 1_{V_m \boxtimes V_n}, \end{aligned}$$

where the negative in the third line comes from the superinterchange law. We may therefore denote by $V_m \circledast V_n$ the 1-eigenspace of P, which has half the dimension of $V_m \boxtimes V_n$ and admits the homogeneous basis

$$\{x_{i,j}, y_{i,j} : i \in I(m|0), j \in I(n|0)\}$$

where

$$x_{i,j} := v_i \otimes v_j + \sqrt{-1}\, v_{\bar{i}} \otimes v_{\bar{j}}, \quad y_{i,j} := v_i \otimes v_{\bar{j}} - \sqrt{-1}\, v_{\bar{i}} \otimes v_j.$$

One can check by direct calculations that we have a $U(\mathfrak{q}_m)$-isomorphism

$$\begin{array}{rcl} V_m \circledast V_n & \xrightarrow{\sim} & \bigoplus_{j=1}^{n} V_m \\ x_{i,j} & \mapsto & v_i \ \text{(in the } j^{\text{th}} \text{ summand)} \\ y_{i,j} & \mapsto & -\sqrt{-1}\, v_{\bar{i}} \ \text{(in the } j^{\text{th}} \text{ summand)} \end{array} \tag{5.1}$$

and a $U(\mathfrak{q}_n)$-isomorphism

$$\begin{array}{rcl} V_m \circledast V_n & \xrightarrow{\sim} & \bigoplus_{i=1}^{m} V_n \\ x_{i,j} & \mapsto & v_j \ \text{(in the } i^{\text{th}} \text{ summand)} \\ y_{i,j} & \mapsto & v_{\bar{j}} \ \text{(in the } i^{\text{th}} \text{ summand)} \end{array} \tag{5.2}$$

viewing $U(\mathfrak{q}_m)$ as the subsuperalgebra $U(\mathfrak{q}_m) \otimes 1$ of $U(\mathfrak{q}_m) \otimes U(\mathfrak{q}_n)$ and similarly for $U(\mathfrak{q}_n)$. It is clear from these isomorphisms that the actions of $U(\mathfrak{q}_m)$ and $U(\mathfrak{q}_n)$ on $V_m \circledast V_n$ commute with each other.

We now form the symmetric algebra

$$\mathscr{S} := \mathscr{S}(V_m \circledast V_n)$$

by quotienting the tensor algebra $T(V_m \circledast V_n) = \bigoplus_{k \in \mathbb{Z}_{\geq 0}} (V_m \circledast V_n)^{\otimes k}$ by the two-sided ideal generated by all expressions of the form $w \otimes z - (-1)^{\overline{w} \cdot \overline{z}} z \otimes w$ for $w, z \in V_m \circledast V_n$.

The algebra $\mathscr{S}$ is a $U(\mathfrak{q}_m) \otimes U(\mathfrak{q}_n)$-module via the coproduct

$$1 \otimes \text{swap} \otimes 1 \circ (\Delta_m \otimes \Delta_n) \colon U(\mathfrak{q}_m) \otimes U(\mathfrak{q}_n) \to (U(\mathfrak{q}_m) \otimes U(\mathfrak{q}_n))^{\otimes 2}$$

where Δ_m and Δ_n are the coproducts of $U(\mathfrak{q}_m)$ and $U(\mathfrak{q}_n)$, respectively, and $1 \otimes \text{swap} \otimes 1$ denotes the signed transposition of the middle two tensor factors. From isomorphisms (5.1) and (5.2), the actions of $U(\mathfrak{q}_m)$ and $U(\mathfrak{q}_n)$ on $\mathscr{S}$ commute and we have, for example, a homomorphism

$$\phi_m \colon U(\mathfrak{q}_m) \to \text{End}_{\mathfrak{q}_n}(\mathscr{S}). \tag{5.3}$$

Lemma 5.1. *For $\lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{Z}^m_{\geq 0}$, the $\mathfrak{q}_m$ weight space of $\mathscr{S}$ associated to λ is isomorphic as a $\mathfrak{q}_n$-module to*

$$\mathscr{S}^\lambda := \mathscr{S}^{\lambda_1}(V_n) \otimes \cdots \otimes \mathscr{S}^{\lambda_m}(V_n).$$

Proof. Denote by $\mathscr{S}_\lambda$ the $\mathfrak{q}_m$ weight space of $\mathscr{S}$ corresponding to $\lambda \in \mathbb{Z}^m_{\geq 0}$ so that we are trying to prove $\mathscr{S}_\lambda \simeq \mathscr{S}^\lambda$ as $\mathfrak{q}_n$-modules. By definition, $\mathscr{S}_\lambda$ consists of all $v \in \mathscr{S}$ such that $A_{i,i}.v = \lambda_i v$ for $1 \leq i \leq m$. This combined with the definition of the $\mathfrak{q}_m$-action on $\mathscr{S}$ implies that a pure tensor in $\mathscr{S}_\lambda$ is a product of $x_{i,j}$ and $y_{i,j}$ in which the number of tensorands with first subscript i (and any second subscript j) is λ_i. Using the relation $w \otimes z = (-1)^{\overline{w} \cdot \overline{z}} w \otimes z$ in $\mathscr{S}$, we can reorder the tensorands of each pure tensor by their first subscripts. The isomorphism $\mathscr{S}_\lambda \to \mathscr{S}^\lambda$ is then given by sending $x_{i,j} \mapsto v_j$ and $y_{i,j} \mapsto v_{\overline{j}}$ and then extending these assignments across tensor products. ■

From the proof of Lemma 5.1, it's clear that $\mathscr{S} = \bigoplus_{\lambda \in \mathbb{Z}^m_{\geq 0}} \mathscr{S}_\lambda$. Hence $\mathscr{S}$ is a

$\mathfrak{q}_m$ weight module and we have

$$\begin{aligned}
\operatorname{End}_{\mathfrak{q}_n}(\mathscr{S}) &= \operatorname{End}_{\mathfrak{q}_n}\left(\bigoplus_{\lambda\in\mathbb{Z}^m_{\geq 0}} \mathscr{S}_\lambda\right) \\
&= \bigoplus_{\lambda,\mu\in\mathbb{Z}^m_{\geq 0}} \operatorname{Hom}_{\mathfrak{q}_n}(\mathscr{S}_\lambda, \mathscr{S}_\mu) \\
&= \bigoplus_{\lambda,\mu\in\mathbb{Z}^m_{\geq 0}} \pi_\mu \circ \operatorname{End}_{\mathfrak{q}_n}(\mathscr{S}) \circ \pi_\lambda
\end{aligned}$$

where $\pi_\nu\colon \mathscr{S} \to \mathscr{S}_\nu$ denotes the projection of $\mathscr{S}$ onto its ν weight space for $\nu \in \mathbb{Z}^m_{\geq 0}$. Since the π_ν are pairwise orthogonal idempotents, the above implies that $\operatorname{End}_{\mathfrak{q}_n}(\mathscr{S})$ is locally unital with distinguished idempotents $(\pi_\nu)_{\nu\in\mathbb{Z}^m_{\geq 0}}$. By Lemma 5.1, the supercategory associated to $\operatorname{End}_{\mathfrak{q}_n}(\mathscr{S})$ in the sense of Section 2.3 is isomorphic to the full subcategory of $\mathfrak{q}_n$-$\mathbf{Mod}_{\mathscr{S}}$ with objects the $\mathscr{S}^\lambda$ for $\lambda \in \mathbb{Z}^m_{\geq 0}$.

In light of the last paragraph, we'd like to replace $U(\mathfrak{q}_m)$ with a locally unital superalgebra in such a way that the ϕ_m become locally unital homomorphisms. Such an algebra exists, and is called the *idempotented version* $\dot{U}(\mathfrak{q}_m)$. Let $\epsilon_i \in \mathbb{Z}^m$ for $1 \leq i \leq m$ be the m-tuple $(0,\dots,0,1,0,\dots,0)$ with 1 in the i^{th} spot and zeros elsewhere, and let $\alpha_j = \epsilon_j - \epsilon_{j+1} = (0,\dots,0,1,-1,0,\dots,0)$ be the j^{th} *simple root* for $1 \leq j \leq m-1$. We adjoin to $U(\mathfrak{q}_m)$ the homogeneous even *weight idempotents* 1_λ for $\lambda \in \mathbb{Z}^m$, and let I be the two-sided ideal generated by all relations of the form

$$\begin{gathered}
1_\lambda 1_\mu = \delta_{\lambda,\mu} 1_\lambda, \quad h_j 1_\lambda = \lambda_j 1_\lambda, \quad e_i 1_\lambda = 1_{\lambda+\alpha_i} e_i, \quad f_i 1_\lambda = 1_{\lambda-\alpha_i} f_i, \\
h_{\bar{j}} 1_\lambda = 1_\lambda h_{\bar{j}}, \quad e_{\bar{i}} 1_\lambda = 1_{\lambda+\alpha_i} e_{\bar{i}}, \quad f_{\bar{i}} 1_\lambda = 1_{\lambda-\alpha_i} f_{\bar{i}}
\end{gathered} \tag{5.4}$$

for $1 \leq i \leq m-1$ and $1 \leq j \leq m$. Then $\dot{U}(\mathfrak{q}_m)$ is defined by

$$\dot{U}(\mathfrak{q}_m) := \left(\bigoplus_{\lambda,\mu \in \mathbb{Z}^m} 1_\mu U(\mathfrak{q}_m) 1_\lambda \right) \Big/ I.$$

From this we immediately have

$$\dot{U}(\mathfrak{q}_m) = \bigoplus_{\lambda,\mu \in \mathbb{Z}^m} 1_\mu \dot{U}(\mathfrak{q}_m) 1_\lambda$$

as superspaces, so this combined with the first equation of (5.4) implies $\dot{U}(\mathfrak{q}_m)$ is locally unital with distinguished idempotents $(1_\lambda)_{\lambda \in \mathbb{Z}^m}$. Note that as a consequence of the second equation in the first line of (5.4), we need not include the h_j in a list of generators of $\dot{U}(\mathfrak{q}_m)$.

Knowing as we do the distinguished idempotents $(\pi_\nu)_{\nu \in \mathbb{Z}^m_{\geq 0}}$ of $\mathrm{End}_{\mathfrak{q}_n}(\mathscr{S})$, we will be more interested in the quotient $\dot{U}(\mathfrak{q}_m)_{\geq 0}$ of $\dot{U}(\mathfrak{q}_m)$ obtaining by setting $1_\lambda = 0$ for all $\lambda \in \mathbb{Z}^m$ containing a negative entry; it is locally unital with distinguished idempotents $(1_\lambda)_{\lambda \in \mathbb{Z}^m_{\geq 0}}$.

Since $\mathscr{S}$ is a weight module over $\mathfrak{q}_m$, ϕ_m can be adapted to a locally unital homomorphism

$$\phi_m \colon \dot{U}(\mathfrak{q}_m)_{\geq 0} \to \mathrm{End}_{\mathfrak{q}_n}(\mathscr{S}),$$

by sending $1_\lambda \mapsto \pi_\lambda$ for $\lambda \in \mathbb{Z}^m_{\geq 0}$. Indeed, the relations of $\dot{U}(\mathfrak{q}_m)$ are designed precisely for this purpose. Denoting by $\dot{\mathbf{U}}(\mathfrak{q}_m)_{\geq 0}$ the supercategory associated to $\dot{U}(\mathfrak{q}_m)_{\geq 0}$ in the sense of Section 2.3, we have established the following.

Proposition 5.2. *For $m \in \mathbb{Z}_{>0}$ there exists a superfunctor*

$$\Phi_m \colon \dot{\mathbf{U}}(\mathfrak{q}_m)_{\geq 0} \quad \to \quad \mathfrak{q}_n\text{-}\mathbf{Mod}_{\mathscr{S}}$$

$$1_\lambda \rightsquigarrow \mathscr{S}^\lambda$$
$$1_\mu x 1_\lambda \rightsquigarrow \phi_m(x) \colon \mathscr{S}^\lambda \to \mathscr{S}^\mu$$

for $\lambda \in \mathbb{Z}^m_{\geq 0}$, $x \in \dot{U}(\mathfrak{q}_m)$.

Note that by postcomposing with the inclusion $\mathfrak{q}_n\text{-}\mathbf{Mod}_{\mathscr{S}} \to \mathfrak{q}_n\text{-}\mathbf{Mod}_{\mathscr{S},\mathscr{S}^*}$, we can view Φ_m as going $\dot{\mathbf{U}}(\mathfrak{q}_m)_{\geq 0} \to \mathfrak{q}_n\text{-}\mathbf{Mod}_{\mathscr{S},\mathscr{S}^*}$.

5.2. Functors Π_m and $\Psi^{\uparrow\downarrow}$

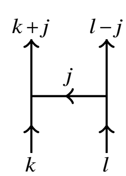
We now factor Φ_m through $\mathfrak{q}\text{-}\mathbf{Web}_\uparrow$, by defining superfunctors Π_m and monoidal superfunctors $\Psi^\uparrow$ and $\Psi^{\uparrow\downarrow}$ so that the diagrams

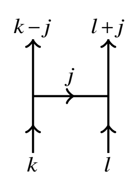
$$\begin{array}{ccc} \dot{\mathbf{U}}(\mathfrak{q}_m)_{\geq 0} & \xrightarrow{\Phi_m} & \mathfrak{q}_n\text{-}\mathbf{Mod}_{\mathscr{S}} \\ \Pi_m \downarrow & \nearrow_{\Psi^\uparrow} & \\ \mathfrak{q}\text{-}\mathbf{Web}_\uparrow & & \end{array} \quad , \quad \begin{array}{ccc} \dot{\mathbf{U}}(\mathfrak{q}_m)_{\geq 0} & \xrightarrow{\Phi_m} & \mathfrak{q}_n\text{-}\mathbf{Mod}_{\mathscr{S},\mathscr{S}^*} \\ \Pi_m \downarrow & \nearrow_{\Psi^{\uparrow\downarrow}} & \\ \mathfrak{q}\text{-}\mathbf{Web}_{\uparrow\downarrow} & & \end{array} \tag{5.5}$$

commute for $m \in \mathbb{Z}_{>0}$. The Π_m on the right is obtained from the Π_m on the left by postcomposing with the inclusion $\mathfrak{q}\text{-}\mathbf{Web}_\uparrow \to \mathfrak{q}\text{-}\mathbf{Web}_{\uparrow\downarrow}$. Also in this section, we prove that the homomorphisms $\xi_k \colon \mathrm{Ser}_k \to \mathrm{End}_{\mathfrak{q}\text{-}\mathbf{Web}_\uparrow}(\uparrow^k)$ are isomorphisms.

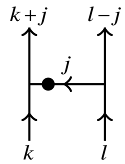
For $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{Z}^m_{\geq 0}$, let $\lambda\uparrow$ denote the sequence $(\lambda_1\uparrow, \ldots, \lambda_m\uparrow) \in \langle\uparrow\rangle$.

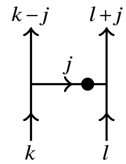
Lemma 5.3. *For* $m \in \mathbb{Z}_{>0}$ *there exists a superfunctor* $\Pi_m \colon \dot{\mathbf{U}}(\mathfrak{q}_m)_{\geq 0} \to \mathfrak{q}\text{-}\mathbf{Web}_\uparrow$ *given on objects by* $1_\lambda \rightsquigarrow \lambda\uparrow$ *for* $\lambda \in \mathbb{Z}^m_{\geq 0}$, *and on generating morphisms by*

$\Pi_m(e_i^{(j)} 1_\lambda) =$ 

, $\Pi_m(f_i^{(j)} 1_\lambda) =$ 

,

$\Pi_m(e_{\bar{i}}^{(j)}1_\lambda) =$ 

, $\Pi_m(f_{\bar{i}}^{(j)}1_\lambda) =$ 

, $\Pi_m(h_{\bar{i}}1_\lambda) =$ 

where $k := \lambda_i$, $l := \lambda_{i+1}$, and each of the five webs has the appropriate identity strands to its left and right. Moreover each Π_m is full.

Proof. To show Π_m is well-defined, we must show that the relations (Q1)-(Q6) of $\dot{\mathbf{U}}(\mathfrak{q}_m)$ hold in $\mathfrak{q}$-$\mathbf{Web}_\uparrow$. The first two equations in (Q3) are easily obtained from (4.7) by composing with various dotted and undotted merges and splits to obtain the appropriate ladders. The rest are proved by the superinterchange law, by relations (4.3), (4.6), and (4.9)-(4.12), and by Lemma 4.3.

As for fullness of Π_m, it suffices to show that every web $w \in \mathrm{Hom}_{\mathfrak{q}\text{-}\mathbf{Web}_\uparrow}(\lambda, \mu)$, where $\lambda, \mu \in \langle\uparrow\rangle$ are sequences of length at most m, can be expressed as a composition of images under Π_m of the generating morphisms of $\dot{\mathbf{U}}(\mathfrak{q}_m)_{\geq 0}$. Since merges and splits can be realized as ladders with certain edges labeled zero, this is clear. ■

As with Φ_m, we may view Π_m as going $\dot{\mathbf{U}}(\mathfrak{q}_m)_{\geq 0} \to \mathfrak{q}$-$\mathbf{Web}_{\uparrow\downarrow}$ by postcomposing with the inclusion $\mathfrak{q}$-$\mathbf{Web}_\uparrow \to \mathfrak{q}$-$\mathbf{Web}_{\uparrow\downarrow}$.

In order to define $\Psi^{\uparrow\downarrow}$, we identify some particular morphisms of $\mathfrak{q}_n$-$\mathbf{Mod}_{\mathscr{S},\mathscr{S}^*}$. Recall from Section 3.2 the monomial basis $B_k = \{v_\lambda : \lambda \in \Lambda(n|n,k)\}$ of $\mathscr{S}^k(V_n)$ and its dual basis $B_k^* = \{g^\lambda : \lambda \in \Lambda(n|n,k)\}$ of $\mathscr{S}^{k*}(V_n)$. For $k \in \mathbb{Z}_{>0}$ we define the k^{th} *evaluation* and *coevaluation maps* by

$$\begin{aligned} \mathrm{ev}_k : \mathscr{S}^{k*} \otimes \mathscr{S}^k &\to \mathbb{C} \\ g^\lambda \otimes v_\mu &\mapsto g^\lambda(v_\mu) = \delta_{\lambda,\mu} \end{aligned}$$

for $\lambda, \mu \in \Lambda(n|n,k)$ and

$$\begin{aligned} \mathrm{coev}_k \colon \mathbb{C} &\to \mathscr{S}^k \otimes \mathscr{S}^{k*} \\ 1 &\mapsto \sum_{\lambda \in \Lambda(n|n,k)} v_\lambda \otimes g^\lambda, \end{aligned}$$

respectively.

Recall from Section 3.3 the symmetry σ on $\mathfrak{q}_n$-$\mathbf{Mod}_{\mathscr{S},\mathscr{S}^*}$, which is given by $\sigma_{V,W}(v \otimes w) := (-1)^{\overline{v}\cdot\overline{w}} w \otimes w$ for $V, W \in \mathfrak{q}_n$-$\mathbf{Mod}_{\mathscr{S},\mathscr{S}^*}$ and $v \in V$, $w \in W$.

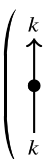
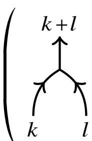
Proposition 5.4. *There exists a monoidal superfunctor*

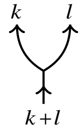
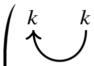
$$\Psi^{\uparrow\downarrow} \colon \mathfrak{q}\text{-}\mathbf{Web}_{\uparrow\downarrow} \to \mathfrak{q}_n\text{-}\mathbf{Mod}_{\mathscr{S},\mathscr{S}^*},$$

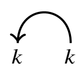
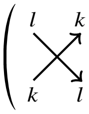
which restricts to a monoidal superfunctor

$$\Psi^{\uparrow} \colon \mathfrak{q}\text{-}\mathbf{Web}_{\uparrow} \to \mathfrak{q}_n\text{-}\mathbf{Mod}_{\mathscr{S}},$$

given on objects by extending the assignments $(k\uparrow) \rightsquigarrow \mathscr{S}^k(V_n)$, $(k\downarrow) \rightsquigarrow \mathscr{S}^{k*}(V_n)$ across tensor products for $k \in \mathbb{Z}_{>0}$, and on morphisms by extending the assignments

$\Psi^{\uparrow\downarrow}\Big($  $\Big) = \Phi_1(h_{\bar{1}} 1_{(k)}),$ $\Psi^{\uparrow\downarrow}\Big($  $\Big) = \Phi_2(e_1^{(l)} 1_{(k,l)}),$

$\Psi^{\uparrow\downarrow}\Big($  $\Big) = \Phi_2(f_1^{(l)} 1_{(k+l,0)}),$ $\Psi^{\uparrow\downarrow}\Big($  $\Big) = \mathrm{coev}_k,$

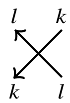
$\Psi^{\uparrow\downarrow}\left(\right.$  $\left.\right) = \mathrm{ev}_k,\qquad \Psi^{\uparrow\downarrow}\left(\right.$  $\left.\right) = \sigma_{\mathscr{S}^k,\mathscr{S}^{l*}}$

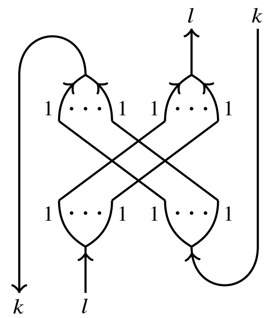
across tensor products for $k, l \in \mathbb{Z}_{>0}$. *Moreover, diagrams* (5.5) *commute, both functors are symmetric, and both functors are evenly dense.*

Proof. First we note that (5.5) will commute automatically if $\Psi^{\uparrow\downarrow}$ is well-defined because of the natural embeddings $\mathfrak{q}_{m'} \hookrightarrow \mathfrak{q}_m$ for $m' \leq m$. Next we argue $\Psi^{\uparrow\downarrow}$ is well-defined, i.e. that the images of the relations of $\mathfrak{q}$-$\mathbf{Web}_{\uparrow\downarrow}$ hold in $\mathfrak{q}_n$-$\mathbf{Mod}_{\mathscr{S},\mathscr{S}^*}$. By Lemma 5.3, the upward-oriented relations (4.3)-(4.12) hold in $\mathfrak{q}_n$-$\mathbf{Mod}_{\mathscr{S},\mathscr{S}^*}$ because they are images under Π_m of relations in $\dot{\mathbf{U}}(\mathfrak{q}_m)_{\geq 0}$ (for m sufficiently large), which hold in $\mathfrak{q}_n$-$\mathbf{Mod}_{\mathscr{S},\mathscr{S}^*}$ by Proposition 5.2. The remaining relations, (4.27)-(4.30), amount to straightforward calculations; we perform said calculation in the case of the left side of (4.27), leaving the others to the reader. We do so by checking that $(1_k \otimes \mathrm{ev}_k) \circ (\mathrm{coev}_k \otimes 1_k)(v_\lambda) = v_\lambda$ for a basis vector $v_\lambda \in \mathscr{S}^k(V_n)$ where $\lambda \in \Lambda(n|n,k)$, $k \in \mathbb{Z}_{>0}$, and 1_k denotes the identity map on $\mathscr{S}^k(V_n)$:

$$
\begin{aligned}
(1_k \otimes \mathrm{ev}_k) \circ (\mathrm{coev}_k \otimes 1_k)(v_\lambda) &= (1_k \otimes \mathrm{ev}_k)\left(\sum_{\mu\in\Lambda(n|n,k)} v_\mu \otimes g^\mu \otimes v_\lambda\right)\\
&= \sum_{\mu\in\Lambda(n|n,k)} g^\mu(v_\lambda) v_\mu\\
&= \sum_{\mu\in\Lambda(n|n,k)} \delta_{\mu,\lambda} v_\mu\\
&= v_\lambda.
\end{aligned}
$$

To show $\Psi^{\uparrow\downarrow}$ is symmetric, we claim $\Psi^{\uparrow\downarrow}(\Sigma_{\lambda,\mu}) = \sigma_{\mathscr{S}^\lambda,\mathscr{S}^\mu}$ for $\lambda, \mu \in \langle\uparrow,\downarrow\rangle$. Comparing the definitions of Σ and σ, it suffices to prove that $\Psi^{\uparrow\downarrow}$ maps a single crossing of strands of arbitrary thickness and orientation to the appropriate σ. One of the four cases is dispatched by the definition of $\Psi^{\uparrow\downarrow}$; we prove the most illuminating of the remaining three, namely that of

$X_{k,l} :=$ 

$= \left(\dfrac{1}{k!\,l!}\right)$ 

for $k, l \in \mathbb{Z}_{>0}$, leaving the others to the reader.

First we need a bit of notation. For $\lambda \in \Lambda(n|n,k)$ and $\rho \in \mathfrak{S}_k$, let $v_{\rho(\lambda)} \in V_n^{\otimes k}$ be the pure tensor obtained by first mapping v_λ along the inclusion $\mathscr{S}^k(V_n) \hookrightarrow V_n^{\otimes k}$ which simply forgets the symmetric structure, and second acting on the resulting pure tensor by ρ in the sense of Section 3.4. In particular we have $\overline{v_{\rho(\lambda)}} = \overline{v_\lambda}$. We check that $\Psi^{\uparrow\downarrow}(X_{k,l}) = \sigma_{\mathscr{S}^{k*},\mathscr{S}^l}$ by mapping a basis vector $g^\lambda \otimes v_\mu$ through it in stages for $\lambda \in \Lambda(n|n,k)$ and $\mu \in \Lambda(n|n,l)$, simplifying as we go. Reading from bottom to top, and ignoring the $1/k!l!$ for the moment, we have

$$
\begin{aligned}
g^\lambda \otimes v_\mu \quad &\overset{\text{cup}}{\longmapsto} \quad \sum_{\omega \in \Lambda(n|n,k)} g^\lambda \otimes v_\mu \otimes v_\omega \otimes g^\omega \\
&\overset{\text{splits}}{\longmapsto} \quad \sum_{\omega \in \Lambda(n|n,k)} g^\lambda \otimes \left(\sum_{\tau \in \mathfrak{S}_l} v_{\tau(\mu)} \right) \otimes \left(\sum_{\rho \in \mathfrak{S}_k} v_{\rho(\omega)} \right) \otimes g^\omega \\
&= \sum_{\omega \in \Lambda(n|n,k)} g^\lambda \otimes \left(\sum_{\substack{\tau \in \mathfrak{S}_l \\ \rho \in \mathfrak{S}_k}} v_{\tau(\mu)} \otimes v_{\rho(\omega)} \right) \otimes g^\omega
\end{aligned}
$$

$$\xrightarrow{\text{crossings}} \sum_{\omega\in\Lambda(n|n,k)} g^{\lambda}\otimes\left(\sum_{\substack{\tau\in\mathfrak{S}_l\\ \rho\in\mathfrak{S}_k}}(-1)^{\overline{v_{\tau(\mu)}}\cdot\overline{v_{\rho(\omega)}}}v_{\rho(\omega)}\otimes v_{\tau(\mu)}\right)\otimes g^{\omega}$$

$$= \sum_{\omega\in\Lambda(n|n,k)}(-1)^{\overline{v_\mu}\cdot\overline{v_\omega}}g^{\lambda}\otimes\left(\sum_{\rho\in\mathfrak{S}_k}v_{\rho(\omega)}\right)\otimes\left(\sum_{\tau\in\mathfrak{S}_l}v_{\tau(\mu)}\right)\otimes g^{\omega}$$

$$\xrightarrow{\text{merges}} \sum_{\omega\in\Lambda(n|n,k)}(-1)^{\overline{v_\mu}\cdot\overline{v_\omega}}g^{\lambda}\otimes\left(\sum_{\rho\in\mathfrak{S}_k}v_{\omega}\right)\otimes\left(\sum_{\tau\in\mathfrak{S}_l}v_{\mu}\right)\otimes g^{\omega}$$

$$= k!\,l!\sum_{\omega\in\Lambda(n|n,k)}(-1)^{\overline{v_\mu}\cdot\overline{v_\omega}}g^{\lambda}\otimes v_\omega\otimes v_\mu\otimes g^{\omega}$$

$$\xrightarrow{\text{cap}} k!\,l!\sum_{\omega\in\Lambda(n|n,k)}(-1)^{\overline{v_\mu}\cdot\overline{v_\omega}}g^{\lambda}(v_\omega)v_\mu\otimes g^{\omega}$$

$$= k!\,l!\,(-1)^{\overline{v_\mu}\cdot\overline{v_\lambda}}v_\mu\otimes g^{\omega}$$

$$= k!\,l!\,\sigma_{\mathscr{S}^{k*},\mathscr{S}^{l}}(g^{\lambda}\otimes v_\mu).$$

Scaling this by $1/k!l!$ proves the claim.

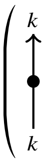
To show $\Psi^{\uparrow\downarrow}$ is evenly dense, we note that for every $\lambda\in\langle\uparrow,\downarrow\rangle$ we have the canonical even $\mathfrak{q}_n$-isomorphism $\mathscr{S}^{\lambda}\xrightarrow{\sim}\mathscr{S}^{\lambda^+}=\Psi^{\uparrow\downarrow}(\lambda^+)$ where λ^+ is the result of deleting every entry of $0\uparrow$ in λ. This proves the proposition for $\Psi^{\uparrow\downarrow}$; similar arguments prove it for $\Psi^{\uparrow}$, except we must note that $\Psi^{\uparrow}$ is symmetric because $\Psi^{\uparrow\downarrow}$ is and Proposition 4.13 holds. ∎

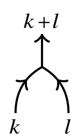
Using the definitions of Φ_m and $\Psi^{\uparrow}$ above, we can give explicit definitions of the images under $\Psi^{\uparrow}$ of merges, splits, and dots. To do so, we introduce some new notation. For $k,l\in\mathbb{Z}_{>0}$, $\lambda\in\Lambda(n|n,k)$, and $\mu\in\Lambda(n|n,l)$, define the pure tensor

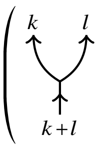
$v_{\lambda+\mu} \in \mathscr{S}^{k+l}(V_n)$ to be the concatenation of v_λ and v_μ, i.e.

$$v_{\lambda+\mu} := v_1^{\lambda_1} v_{\bar{1}}^{\lambda_{\bar{1}}} \cdots v_n^{\lambda_n} v_{\bar{n}}^{\lambda_{\bar{n}}} v_1^{\mu_1} v_{\bar{1}}^{\mu_{\bar{1}}} \cdots v_n^{\mu_n} v_{\bar{n}}^{\mu_{\bar{n}}}.$$

The tensorands of $v_{\lambda+\mu}$ can be reordered using the relation $u \otimes w = (-1)^{\bar{u}\cdot\bar{w}} w \otimes u$ of $\mathscr{S}^{k+l}(V_n)$ for $u, w \in V_n$, so that, up to a negative sign, $v_{\lambda+\mu}$ is equal to some monomial basis vector. Then using the isomorphisms (5.1), (5.2), one can verify that

$\Psi^{\uparrow}\Big($  $\Big)(v_\lambda) = \sum_{i=1}^{n} (-1)^{\lambda_{\bar{1}}+\cdots+\lambda_{\overline{i-1}}} (\sqrt{-1}\,\lambda_i v_{\lambda-\epsilon_{\bar{i}}} - \sqrt{-1}\, v_{\lambda+\epsilon_{\bar{i}}}),$

$\Psi^{\uparrow}\Big($  $\Big)(v_\lambda \otimes v_\mu) = v_{\lambda+\mu},$

$\Psi^{\uparrow}\Big($  $\Big)(v_\omega) = \sum_{\substack{\lambda\in\Lambda(n|n,k)\\ \mu\in\Lambda(n|n,l)\\ v_{\lambda+\mu}=v_\omega}} v_\lambda \otimes v_\mu$

for $\omega \in \Lambda(n|n, k+l)$ (see Section 3.2 for the definition of $\lambda \pm \epsilon_{\bar{i}}$).

Recall from Sections 3.4 and 4.4 the homomorphisms $\Xi_k\colon \mathrm{Ser}_k \to \mathrm{End}_{\mathfrak{q}_n}(V_n^{\otimes k})$ and $\xi_k\colon \mathrm{Ser}_k \to \mathrm{End}_{\mathfrak{q}\text{-}\mathbf{Web}_\uparrow}(\uparrow^k)$ for $k \in \mathbb{Z}_{>0}$, respectively. By Corollary 3.4 and Lemma 4.9, both are surjective. Let $\Psi_k^{\uparrow}\colon \mathrm{End}_{\mathfrak{q}\text{-}\mathbf{Web}_\uparrow}(\uparrow^k) \to \mathrm{End}_{\mathfrak{q}_n}(V_n^{\otimes k})$ denote the homomorphism induced by the functor $\Psi^{\uparrow}$.

Corollary 5.5. *For $k \in \mathbb{Z}_{>0}$, the homomorphism $\xi_k\colon \mathrm{Ser}_k \to \mathrm{End}_{\mathfrak{q}\text{-}\mathbf{Web}_\uparrow}(\uparrow^k)$ is an isomorphism, and the linear map $\Psi_k^{\uparrow}\colon \mathrm{End}_{\mathfrak{q}\text{-}\mathbf{Web}_\uparrow}(\uparrow^k) \to \mathrm{End}_{\mathfrak{q}_n}(V_n^{\otimes k})$ is surjective.*

Proof. We claim we have the commutative diagram

$$\begin{array}{ccc} \mathrm{Ser}_k & \xrightarrow{-\Xi_k} & \mathrm{End}_{\mathfrak{q}_n}(V_n^{\otimes k}) \\ {\scriptstyle \hat{\xi}_k}\downarrow & \nearrow {\scriptstyle \Psi_k^{\uparrow}} & \\ \mathrm{End}_{\mathfrak{q}\text{-}\mathbf{Web}_{\uparrow}}(\uparrow^k) & & \end{array}$$

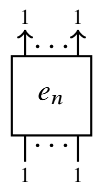
where $\hat{\xi}_k$ is given by $c_i \mapsto C_i$, $s_j \mapsto -S_j$ for $1 \leq i \leq k$, $1 \leq j \leq k-1$. It is clear from the relations of Ser_k that $\hat{\xi}_k$ is a well-defined homomorphism. Commutativity of the diagram follows from an examination of the definitions, Proposition 5.4, and the above discussion. This implies $\Psi_k^{\uparrow}$ is surjective, being the last map in a composition which is surjective.

To show ξ_k is an isomorphism it remains to show it's injective, which we do by proving the equivalent statement that $\hat{\xi}_k$ is injective. To do so, we note that the above diagram commutes for all $n \in \mathbb{Z}_{>0}$, and that Ξ_k is an isomorphism for n sufficiently large by Corollary 3.4. Thus commutativity of the diagram for such an n implies $\hat{\xi}_k$ is injective, being the first map in a composition which is injective. Hence ξ_k is also injective and therefore an isomorphism. ■

5.3. Main theorem

In this section, we prove the first main theorem of the dissertation, which obtains webs presentations of $\mathfrak{q}_n$-$\mathbf{Mod}_{\mathscr{S},\mathscr{S}^*}$ and $\mathfrak{q}_n$-$\mathbf{Mod}_{\mathscr{S}}$ by generators and relations.

Recall from Section 3.5 the quasi-idempotent $e_n \in \mathrm{Ser}_k$ where $k = |\lambda(n)| = 1 + 2 + \cdots + (n+1)$. We define the supercategory $\mathfrak{q}_n$-$\mathbf{Web}_{\uparrow\downarrow}$ (resp. $\mathfrak{q}_n$-$\mathbf{Web}_{\uparrow}$) to be the quotient of $\mathfrak{q}$-$\mathbf{Web}_{\uparrow\downarrow}$ (resp. $\mathfrak{q}$-$\mathbf{Web}_{\uparrow}$) by the extra relation



$= 0$,

viewing e_n as its image under the homomorphism $\xi_k \colon \operatorname{Ser}_k \to \operatorname{End}_{\mathfrak{q}\text{-}\mathbf{Web}_\uparrow}(\uparrow^k)$. Thus $\mathfrak{q}_n$-$\mathbf{Web}_{\uparrow\downarrow}$ and $\mathfrak{q}_n$-$\mathbf{Web}_{\uparrow}$ are symmetric monoidal supercategories, and we have the induced functors $\Psi^{\uparrow\downarrow} \colon \mathfrak{q}_n\text{-}\mathbf{Web}_{\uparrow\downarrow} \to \mathfrak{q}_n\text{-}\mathbf{Mod}_{\mathscr{S},\mathscr{S}^*}$ and $\Psi^{\uparrow} \colon \mathfrak{q}_n\text{-}\mathbf{Web}_{\uparrow} \to \mathfrak{q}_n\text{-}\mathbf{Mod}_{\mathscr{S}}$. From the proof of Corollary 5.5, we have the commutative diagram

$$\begin{array}{ccc} \operatorname{Ser}_k & \xrightarrow{-\Xi_k} & \operatorname{End}_{\mathfrak{q}_n}(V_n^{\otimes k}) \\ {\scriptstyle \hat{\xi}_k}\big\downarrow & \nearrow{\scriptstyle \Psi_k^{\uparrow}} & \\ \operatorname{End}_{\mathfrak{q}_n\text{-}\mathbf{Web}_\uparrow}(\uparrow^k) & & \end{array} \tag{5.6}$$

for all $k \in \mathbb{Z}_{>0}$, where the original $\hat{\xi}_k$ has been postcomposed with the quotient map $\operatorname{End}_{\mathfrak{q}\text{-}\mathbf{Web}_\uparrow}(\uparrow^k) \twoheadrightarrow \operatorname{End}_{\mathfrak{q}_n\text{-}\mathbf{Web}_\uparrow}(\uparrow^k)$.

Theorem 5.6. *The superfunctors*

$$\Psi^{\uparrow\downarrow} \colon \mathfrak{q}_n\text{-}\mathbf{Web}_{\uparrow\downarrow} \to \mathfrak{q}_n\text{-}\mathbf{Mod}_{\mathscr{S},\mathscr{S}^*}, \quad \Psi^{\uparrow} \colon \mathfrak{q}_n\text{-}\mathbf{Web}_{\uparrow} \to \mathfrak{q}_n\text{-}\mathbf{Mod}_{\mathscr{S}}$$

are superequivalences of symmetric monoidal supercategories.

Proof. We focus on $\Psi^{\uparrow\downarrow}$, as its proof will subsume the proof for $\Psi^{\uparrow}$. By Proposition 5.4, $\Psi^{\uparrow\downarrow}$ is symmetric monoidal and evenly dense, so it remains to show that it's fully faithful, i.e. that the linear maps

$$\Psi^{\uparrow\downarrow}_{\lambda,\mu} \colon \operatorname{Hom}_{\mathfrak{q}_n\text{-}\mathbf{Web}_{\uparrow\downarrow}}(\lambda,\mu) \to \operatorname{Hom}_{\mathfrak{q}_n}(\mathscr{S}^{\lambda},\mathscr{S}^{\mu})$$

are isomorphisms for all $\lambda, \mu \in \langle\uparrow,\downarrow\rangle$. We do this by first proving that the above maps are isomorphisms if and only if the maps

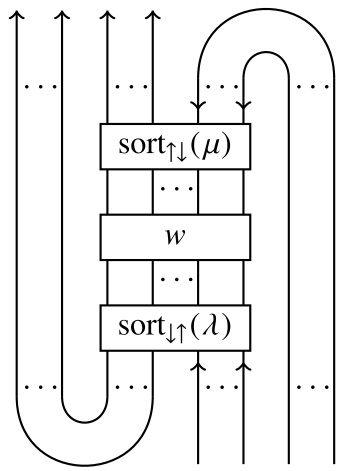
$$\Psi^{\uparrow}_{k} \colon \operatorname{End}_{\mathfrak{q}_n\text{-}\mathbf{Web}_{\uparrow}}(\uparrow^{k}) \to \operatorname{End}_{\mathfrak{q}_n}(V_n^{\otimes k})$$

are isomorphisms for all $k \in \mathbb{Z}_{>0}$.

To do so, we start by forming the webs $\mathrm{sort}_{\downarrow\uparrow}(\lambda)$ and $\mathrm{sort}_{\uparrow\downarrow}(\mu)$ defined by the following properties:

1. Both $\mathrm{sort}_{\downarrow\uparrow}(\lambda)$ and $\mathrm{sort}_{\uparrow\downarrow}(\mu)$ consist only of crossings,
2. the target of $\mathrm{sort}_{\downarrow\uparrow}(\lambda)$ is λ and the source of $\mathrm{sort}_{\uparrow\downarrow}(\mu)$ is μ,
3. the source of $\mathrm{sort}_{\downarrow\uparrow}(\lambda)$ has all down strands to the left of all up strands,
4. the target of $\mathrm{sort}_{\uparrow\downarrow}(\mu)$ has all up strands to the left of all down strands, and
5. $\mathrm{sort}_{\downarrow\uparrow}(\lambda)$ and $\mathrm{sort}_{\uparrow\downarrow}(\mu)$ have the minimal number of crossings necessary for (3) and (4).

Property (5) simply ensures well-definedness. We can then transform an individual web $w \in \operatorname{Hom}_{\mathfrak{q}_n\text{-}\mathbf{Web}_{\uparrow\downarrow}}(\lambda,\mu)$ into the web w' whose edges along the bottom and top are all upward-oriented via

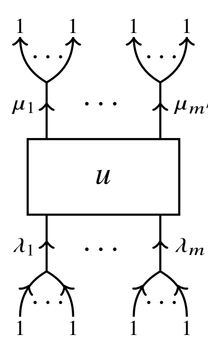
$w' :=$  .

By (4.27) and (4.32), the assignment $w \mapsto w'$ has a natural inverse and is therefore one-to-one. This combined with Proposition 4.13 implies we have reduced the problem to showing that

$$\Psi^{\uparrow}_{\lambda,\mu} \colon \operatorname{Hom}_{\mathfrak{q}_n\text{-}\mathbf{Web}_{\uparrow}}(\lambda,\mu) \to \operatorname{Hom}_{\mathfrak{q}_n}(\mathscr{S}^{\lambda},\mathscr{S}^{\mu})$$

is an isomorphism for all $\lambda, \mu \in \langle\uparrow\rangle$.

Next, given an individual web $u \in \operatorname{Hom}_{\mathfrak{q}_n\text{-}\mathbf{Web}_{\uparrow}}(\lambda,\mu)$, we can form the web $u'' \in \operatorname{Hom}_{\mathfrak{q}_n\text{-}\mathbf{Web}_{\uparrow}}(1^{|\lambda|}, 1^{|\mu|})$ defined by

$u'' :=$ 

where $|\lambda| := \lambda_1 + \cdots + \lambda_m \in \mathbb{Z}_{\geq 0}$ and similarly for $|\mu|$. By (4.5), the assignment $u \mapsto u''$ has a natural inverse and is therefore one-to-one. Further, if $|\lambda| = k \neq l = |\mu|$ then $\mathrm{Hom}_{\mathfrak{q}_n\text{-}\mathbf{Web}_\uparrow}(\lambda, \mu) = 0$ and by Theorem 3.3 we have $\mathrm{Hom}_{\mathfrak{q}_n}(V_n^{\otimes k}, V_n^{\otimes l}) = 0$. Thus we may assume $|\lambda| = |\mu| = k$, and have reduced the problem to showing that

$$\Psi_k^\uparrow \colon \mathrm{End}_{\mathfrak{q}_n\text{-}\mathbf{Web}_\uparrow}(\uparrow^k) \to \mathrm{End}_{\mathfrak{q}_n}(V_n^{\otimes k})$$

is an isomorphism for all $k \in \mathbb{Z}_{>0}$, as claimed. By Corollary 5.5 each is surjective.

Now fix $k \in \mathbb{Z}_{>0}$; we'll show $\ker \Psi_k^\uparrow = 0$. If $k < |\lambda(n)| = 1 + 2 + \cdots + (n+1)$ then Ξ_k is injective by Corollary 3.4, which in turn implies $\Psi_k^\uparrow$ is injective by commutativity of (5.6). Hence, assume $k \geq |\lambda(n)|$. By commutativity of (5.6) and surjectivity of $\xi_k \colon \mathrm{Ser}_k \to \mathrm{End}_{\mathfrak{q}_n\text{-}\mathbf{Web}_\uparrow}(\uparrow^k)$, it suffices to show $\ker \Xi_k \subseteq \ker \xi_k$. By Corollary 3.4, $\ker \Xi_k$ is generated by the quasi-idempotents e_ν for $\nu \in \mathcal{SP}(k)$ with $l(\nu) > n$, so it suffices to show $\xi_k(e_\nu) = 0$ for all such ν. By Corollary 5.13, which we prove in the next section, the simple highest weight $\mathfrak{q}_m$-module $L_m(\nu)$ is isomorphic to a direct summand of $L_m(\lambda(n)) \otimes V_m^{\otimes l}$ where $l := |\nu| - |\lambda(n)|$, for any $m \geq l(\nu)$. Choosing one such direct summand, this implies

$$\Xi_k(e_\nu) = \pi_\nu \circ (\Xi_{|\lambda(n)|}(e_n) \otimes 1_m^{\otimes l}) \in \mathrm{End}_{\mathfrak{q}_m}(V_m^{\otimes k})$$

where π_ν is the projection of $V_m^{\otimes k}$ onto the direct summand and 1_m is the identity map of V_m. By Corollary 3.4, Ξ_k is an isomorphism for m sufficiently large, so that

$$e_\nu = \Xi_k^{-1}(\pi_\nu) \circ e_n \in \mathrm{Ser}_k$$

by applying Ξ_k^{-1} to both sides of the previous equation. Here we're viewing $e_n \in \mathrm{Ser}_k$ by taking its image under the canonical embedding $\mathrm{Ser}_{|\lambda(n)|} \hookrightarrow \mathrm{Ser}_k$. Applying ξ_k to both sides here yields

$$\xi_k(e_\nu) = \xi_k(\Xi_k^{-1}(\pi_\nu)) \circ \xi_k(e_n) = \xi_k(\Xi_k^{-1}(\pi_\nu)) \circ 0 = 0.$$

Thus $\ker \Xi_k \subseteq \ker \xi_k$ and $\Psi_k^{\uparrow}$ is injective for all $k \in \mathbb{Z}_{>0}$. The proof is complete. ∎

We conclude this section with some corollaries of Theorem 5.6 which are of independent interest.

Remark 5.7. *We note that there exists a symmetric monoidal superequivalence*

$$\Theta\colon \mathcal{OBC} \to \mathfrak{q}\text{-}\mathbf{Web}^1_{\uparrow\downarrow}$$

where $\mathcal{OBC}$ is the oriented Brauer-Clifford supercategory *of [8], and $\mathfrak{q}$-$\mathbf{Web}^1_{\uparrow\downarrow}$ is the full subsupercategory of $\mathfrak{q}$-$\mathbf{Web}_{\uparrow\downarrow}$ with objects all sequences in $\langle\uparrow,\downarrow\rangle$ whose strands all have thickness 1. Indeed, Θ is the obvious assignments on objects and morphisms, and the only assertion not immediately evident is that Θ is fully faithful. By an argument similar to one in the proof of Theorem 5.6, Θ is fully faithful if and only if the linear maps $\mathrm{End}_{\mathcal{OBC}}(\uparrow^k) \to \mathrm{End}_{\mathfrak{q}\text{-}\mathbf{Web}^1_{\uparrow\downarrow}}(\uparrow^k)$ induced by Θ are isomorphisms for $k \in \mathbb{Z}_{>0}$. By [8, Corollary 3.5] and Proposition 5.5, both source and target of each is isomorphic to* Ser_k*, and under these identifications each is the identity map.*

Recall from (5.3) the locally unital homomorphisms $\phi_m\colon U(\mathfrak{q}_m) \to \mathrm{End}_{\mathfrak{q}_n}(\mathscr{S})$ where $\mathscr{S}$ is the symmetric algebra $\mathscr{S} := \mathscr{S}(V_m \circledast V_n)$. We can now recover the following result of Cheng-Wang [10, Corollary 3.1], which may be thought of as a *type Q Howe duality*. (The original Howe duality concerns commuting actions of $\mathfrak{gl}_m$ and $\mathfrak{gl}_n$ on $S(\mathbb{C}^m \otimes \mathbb{C}^n)$; see [17] for more information.)

Corollary 5.8. *For $m \in \mathbb{Z}_{>0}$ the locally unital homomorphism*

$$\phi_m : \dot{U}(\mathfrak{q}_m)_{\geq 0} \to \mathrm{End}_{\mathfrak{q}_n}(\mathscr{S})$$

is surjective.

Proof. Recall the functor $\Phi_m : \dot{\mathbf{U}}(\mathfrak{q}_m)_{\geq 0} \to \mathfrak{q}_n\text{-}\mathbf{Mod}_{\mathscr{S}}$ from Proposition 5.2, which is the superfunctor associated to ϕ_m in the sense of Section 2.3. By Lemma 5.3 and Proposition 5.4, we have $\Phi_m = \Psi^{\uparrow} \circ \Pi_m$ with both $\Psi^{\uparrow}$ and Π_m full, so Φ_m is also full. This is equivalent to ϕ_m being surjective. ■

5.4. Appendix on shifted tableaux

In this section we prove Corollary 5.13, which was used in the proof of Theorem 5.6. The subject of the corollary is the composition multiplicities of certain $\mathfrak{q}_n$-modules, but the corollary itself is a consequence of the shifted Littlewood-Richardson (LR) rule [40, Theorem 8.3]. The latter is a statement about the Schur P-functions P_λ for $\lambda \in \mathcal{SP}$. These are symmetric functions in n variables over $\mathbb{Z}$ which arise in the study of projective representations of symmetric groups, and which turn out to (almost) be the characters of the highest weight irreducible $\mathfrak{q}_n$-modules (see [29]). The shifted LR rule is stated in terms of the combinatorics of shifted tableaux, so we begin by describing said combinatorics, referring the reader to [40] for details.

Let $\mathbb{A}$ denote the ordered alphabet $\mathbb{A} := \{1' < 1 < 2' < 2 < \cdots\}$. We say the letters $1', 2', 3', \ldots$ are *marked*, and use the notation $|a|$ to denote the unmarked version of any $a \in \mathbb{A}$.

Definition 5.9. *For $\lambda \in \mathcal{SP}$, a* shifted tableau of shape λ *is a filling of the boxes of the shifted frame* $[\lambda]$ *with elements of* $\mathbb{A}$ *in such a way that*

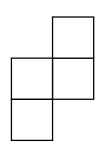
- *the entries in each row are nondecreasing,*
- *the entries in each column are nondecreasing,*
- *each row has at most one a' for $a = 1, 2, 3, \ldots$, and*
- *each column has at most one a for $a = 1, 2, 3, \ldots$.*

An example of a shifted tableau of shape $(4, 3, 1)$ is

$$\begin{array}{|c|c|c|c|} \hline 1 & 2' & 3 & 3 \\ \hline & 2' & 4' & 4 \\ \hline & & 5' & \\ \hline \end{array}\ .$$

Given a shifted tableau of shape λ and $i \in \mathbb{Z}_{>0}$, let ν_i be the number of entries a in R such that $|a| = i$. The *content* of T is then defined to be $\nu := (\nu_1, \nu_2, \ldots)$, although trailing zeros may be suppressed. For example, the content of the shifted tableau above is $(1, 2, 2, 2, 1)$.

If $\lambda \subseteq \mu$ are strict partitions, then the *skew shifted frame* $[\mu/\lambda]$ is the array of boxes obtained by removing $[\lambda]$ from $[\mu]$. For example, if $\mu = (4, 3, 1)$ and $\lambda = (3, 1)$, then we have

$[\mu/\lambda] =$  .

A *shifted tableau of shape μ/λ* is a filling of the boxes of $[\mu/\lambda]$ with elements of $\mathbb{A}$ in such a way that the four conditions of Definition 5.9 are satisfied.

To state the shifted LR rule, we still need a few more definitions. The *word* $w = w(T) = w_1 w_2 \cdots$ associated to a (possibly skew) shifted tableau T is the sequence of elements of $\mathbb{A}$ obtained by reading the rows of T from left to right, starting with the bottom row and working up. For example, the word of the shifted tableau above is $5'2'4'412'33$. Let $|w| := |w_1||w_2|\cdots$ denote the unmarked version of a word w.

Given a finite-length word $w = w_1 \cdots w_n$ in the alphabet $\mathbb{A}$, we define a series of statistics $m_i(j)$ for $i \in \mathbb{A}$ as follows:

- $m_i(j)$ = multiplicity of i among $w_{n-j+1}, \ldots, w_n$, for $0 \leq j \leq n$, and
- $m_i(n+j) = m_i(n)$ + multiplicity of i' among $w_1, \ldots, w_j$, for $0 < j \leq n$.

In particular, $m_i(0)$ is defined to be zero for all $i \in \mathbb{A}$. Here is a way to conceive of these multiplicities. Read the word w twice: first from right to left, and then from left to right. In the first reading, m_i monitors the accumulation of i, and in the second, the accumulation of i'. Note, however, that the count is not reset between the first and second reading.

Definition 5.10. *We say a word $w = w_1 \cdots w_n$ has the* lattice property *if whenever $m_i(j) = m_{i-1}(j)$ we have*

(1) $w_{n-j} \neq i, i'$ if $0 \leq j < n$ and

(2) $w_{j-n+1} \neq i-1, i'$ if $n \leq j < 2n$.

Note that either w_{n-j} or w_{j-n+1} is the letter of w to be read after the j^{th} step.

Without going into unnecessary detail, we simply reiterate that for every $\lambda \in \mathcal{SP}$ with $l(\lambda) < n$ there is a *Schur P-function* $P_\lambda = P_\lambda(x_1, \ldots, x_n)$, a symmetric function in the variables $x_1, \ldots, x_n$ variables over $\mathbb{Z}$. Let $\{f^{\mu}_{\lambda,\nu} : \lambda, \nu, \mu \in \mathcal{SP}\} \subseteq \mathbb{Z}$ denote the structure constants of the Schur P-functions under multiplication in the ring of symmetric functions, i.e.

$$P_\lambda P_\nu = \sum_\mu f^\mu_{\lambda,\nu} P_\mu.$$

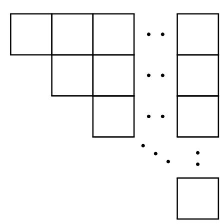
Theorem 5.11. [Shifted LR rule] *For $\lambda, \mu, \nu \in \mathcal{SP}$ the coefficient $f^\mu_{\lambda,\nu}$ is the number of shifted tableaux T of shape μ/λ and content ν such that*

(a) the word $w = w(T)$ satisfies the lattice property, and

(b) the leftmost i of $|w|$ is unmarked in w for $1 \le i \le l(\nu)$.

We call a shifted tableau T satisfying (a) and (b) a shifted LR tableau*, so that $f^\mu_{\lambda,\nu}$ is the number of shifted LR tableaux of shape μ/λ and content ν.*

Recall from Section 3.5 the strict partition $\lambda(n) := (n+1, n, \ldots, 2, 1)$, which has shifted frame

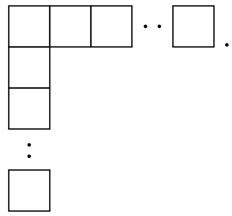
$[\lambda(n)] =$  .

Note that every $\mu \in \mathcal{SP}$ with $l(\mu) > n$ has $\mu \supseteq \lambda(n)$.

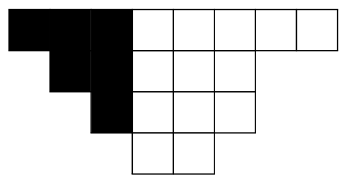
Corollary 5.12. *For every $\mu \in \mathcal{SP}$ with $l(\mu) > n$, there exists $\nu \in \mathcal{SP}$ such that $l(\nu) \le l(\mu)$ and $f^\mu_{\lambda(n),\nu} > 0$.*

Proof. We prove this by constructing, for every $\mu \in \mathcal{SP}$, a shifted LR tableau $T_{\mu,n}$ of shape $\mu/\lambda(n)$ whose content ν is a strict partition with $l(\nu) \le l(\mu)$.

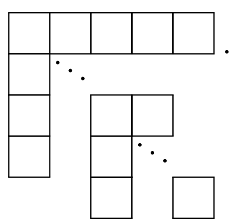
First, we define a *hook* to be a left-justified array of boxes in which only the first row may have more than one box:



Given the shape of $[\lambda(n)]$, the skew shape $[\mu/\lambda(n)]$ can be thought of as consisting of a series of hooks wedged inside each other. For example, if $\mu = (8, 5, 4, 2)$ and $n = 2$ then

$[\mu/\lambda(2)] =$ 

where $\lambda(2)$ has been blacked out for convenience. Here is a picture showing the hooks of $[\mu/\lambda(2)]$ and how they've been wedged together:



We number the hooks of $[\mu/\lambda(n)]$ $1^{\text{st}}, 2^{\text{nd}}, 3^{\text{rd}}, \ldots$ from the upper left to the lower right. Arranged in this order, it is clear that the number of boxes in each hook must be strictly decreasing.

We define $T_{\mu,n}$ to be the shifted tableau of shape $[\mu/\lambda(n)]$ whose i^{th} hook has the form

$$\begin{array}{lllll} \boxed{i'} & \boxed{i} & \boxed{i} & \cdots & \boxed{i} \\ \boxed{i'} & & & & \\ \boxed{i'} & & & & \\ \vdots & & & & \\ \boxed{i'} & & & & \\ \boxed{i} & & & & \end{array}.$$

The unmarked i at the very bottom takes priority over all of the i', so that if the i^{th} hook has only one row then every entry will be i. (This is to ensure that property (b) of the shifted LR rule is satisfied.) Returning to the example of $\mu = (8,5,4,2)$ and $n = 2$, we have

$$T_{\mu,2} = \begin{array}{lllll} \boxed{1'} & \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} \\ \boxed{1'} & \boxed{2'} & \boxed{2} & & \\ \boxed{1'} & \boxed{2'} & \boxed{3} & & \\ \boxed{1} & \boxed{2} & & & \end{array}$$

whose corresponding word is $w(T_{\mu,2}) = 121'2'31'2'21'1111$.

Clearly $T_{\mu,n}$ satisfies the conditions of Definition 5.9 and is a shifted tableau. Since ν_i is just the number of boxes in the i^{th} hook, the content $\nu = (\nu_1, \nu_2, \dots)$ of $T_{\mu,n}$ is a strict partition with $l(\nu) \leq l(\mu)$. And, as was previously observed, property (b) of the shifted LR rule is satisfied.

It remains to show that $T_{\mu,n}$ has the lattice property. That condition (1) of Definition 5.10 is satisfied is easy to see: reading $w(T_{\mu,n})$ from right to left, the first $i-1$ always appears before the first i, and at no point have as many i been passed as $i-1$ (except at the very start). In particular, $m_i(n) \leq m_{i-1}(n) - 1$. This inequality, combined with the facts that between every pair of i' in $w(T)$ is an $(i-1)'$, and that every i' is followed by an $(i+1)'$ or i when reading left to right, ensure that condition (2) is also met. This completes the proof. ■

Recall from Section 3.4 the simple highest weight $\mathfrak{q}_n$-modules $L_n(\lambda)$ for $\lambda \in \mathcal{SP}$

with $l(\lambda) \leq n$, and that $\delta(\lambda) \in \mathbb{Z}_2$ is defined to 0 if $l(\lambda)$ is even and 1 if $l(\lambda)$ is odd.

Corollary 5.13. *Suppose $\nu \in \mathcal{SP}$ with $m := l(\nu) > n$. Then the $\mathfrak{q}_m$-module $L_m(\nu)$ is isomorphic to a direct summand of $L_m(\lambda(n)) \otimes V_m^{\otimes l}$ where $l := |\nu| - |\lambda(n)|$.*

Proof. First we note that since $m = l(\nu) > n$, we have $l(\lambda(n)) = n + 1 \leq m$ and $L_m(\lambda(n))$ is a well-defined $\mathfrak{q}_m$-module.

We'd like to compute $\mathrm{ch}(L_m(\lambda(n)) \otimes V_m^{\otimes l})$. The *character* of a weight module $M = \bigoplus_{\lambda \in \mathfrak{h}_0^*} M_\lambda$ over $\mathfrak{q}_n$ is the formal power series

$$\mathrm{ch}\, M := \sum_{\lambda=(\lambda_1,\dots,\lambda_n)\in\mathbb{Z}^n} \dim M_\lambda \cdot x_1^{\lambda_1} \cdots x_n^{\lambda_n}$$

in the variables $x_1, \dots, x_n$ over $\mathbb{Z}$ (although clearly the coefficients all lie in $\mathbb{Z}_{\geq 0}$). We omit a rigorous discussion of the character theory for $\mathfrak{q}_n$, but refer the interested reader to [11]. We do note, however, that for weight modules M, N over $\mathfrak{q}_n$, both $M \otimes N$ and $M \oplus N$ are in turn weight modules and we have

$$\mathrm{ch}(M \otimes N) = \mathrm{ch}\, M \cdot \mathrm{ch}\, N, \quad \mathrm{ch}(M \oplus N) = \mathrm{ch}\, M + \mathrm{ch}\, N.$$

Furthermore, there exists a $\mathfrak{q}_n$-isomorphism $M \simeq N$ if and only if $\mathrm{ch}\, M = \mathrm{ch}\, N$.

Now by [11, Theorem 3.48] we have

$$\mathrm{ch}\, L_n(\lambda) := 2^{-\frac{l(\lambda)-\delta(\lambda)}{2}} Q_\lambda$$

for $\lambda \in \mathcal{SP}$ with $l(\lambda) < n$, where $Q_\lambda = Q_\lambda(x_1, \dots, x_n)$ is the *Schur Q-function*, another symmetric function in n variables over $\mathbb{Z}$. On the other hand, It is well known (e.g. [40, §6]) that $Q_\lambda = 2^{l(\lambda)} P_\lambda$ for $\lambda \in \mathcal{SP}$, so we have

$$\operatorname{ch} L_n(\lambda) = 2^{\frac{l(\lambda)+\delta(\lambda)}{2}} P_\lambda.$$

Returning to the statement of the corollary, recall from Theorem 3.3 the $U(\mathfrak{q}_m) \otimes \mathrm{Ser}_l$-isomorphism

$$V_m^{\otimes l} \simeq \bigoplus_{\substack{\mu \in \mathcal{SP}(l) \\ l(\mu) \leq m}} L_m(\mu) \circledast D^\mu$$

where D^μ is an irreducible Ser_l-module. Viewing both sides just as $U(\mathfrak{q}_m)$-modules, we have an isomorphism

$$V_m^{\otimes l} \simeq \bigoplus_{\substack{\mu \in \mathcal{SP}(l) \\ l(\mu) \leq m}} 2^{-\delta(\mu)} (\dim D^\mu) L_m(\mu).$$

Under the assumptions in the statement of the corollary, we can now calculate that

$$\begin{aligned}
&\operatorname{ch}(L_m(\lambda(n)) \otimes V_m^{\otimes l}) \\
= \; & \operatorname{ch} L_m(\lambda(n)) \cdot \operatorname{ch}\left(\bigoplus_{\substack{\mu \in \mathcal{SP}(l) \\ l(\mu) \leq m}} 2^{-\delta(\mu)} (\dim D^\mu) L_m(\mu) \right) \\
= \; & \sum_{\substack{\mu \in \mathcal{SP}(l) \\ l(\mu) \leq m}} 2^{-\delta(\mu)} (\dim D^\mu) \operatorname{ch} L_m(\lambda(n)) \cdot \operatorname{ch} L_m(\mu) \\
= \; & \sum_{\substack{\mu \in \mathcal{SP}(l) \\ l(\mu) \leq m}} 2^{-\delta(\mu)} (\dim D^\mu) 2^{\frac{l(\lambda(n))+\delta(\lambda(n))}{2}} 2^{\frac{l(\mu)+\delta(\mu)}{2}} P_{\lambda(n)} P_\mu
\end{aligned}$$

$$= \quad 2^{\frac{l(\lambda(n))+\delta(\lambda(n))}{2}} \sum_{\substack{\nu\in\mathcal{SP}\\ \mu\in\mathcal{SP}(l)\\ l(\mu)\leq m}} 2^{\frac{l(\mu)-\delta(\mu)}{2}} (\dim D^{\mu}) f^{\nu}_{\lambda(n),\mu} P_{\nu}.$$

Consequently, for $\nu \in \mathcal{SP}$ with $m := l(\nu) > n$ the coefficient of $\operatorname{ch} L_m(\nu) = 2^{\frac{l(\nu)+\delta(\nu)}{2}} P_{\nu}$ in the above is

$$\sum_{\substack{\mu\in\mathcal{SP}(l)\\ l(\mu)\leq m}} 2^{\frac{l(\lambda(n))+\delta(\lambda(n))+l(\mu)+\delta(\mu)-l(\nu)-\delta(\nu)}{2}} (\dim D^{\mu}) f^{\nu}_{\lambda(n),\mu}.$$

Now $L_m(\nu)$ is isomorphic to a direct summand of $L_m(\lambda(n)) \otimes V_m^{\otimes l}$ if and only if this coefficient is positive, which occurs if and only if $f^{\nu}_{\lambda(n),\mu} > 0$ for at least one $\mu \in \mathcal{SP}$ with $l(\mu) \leq m$. This indeed happens by Corollary 5.12 (although the roles of μ and ν have interchanged), so the claim is proved. ■

Chapter 6

Webs for spin permutation modules

In this chapter, we use our existing theory of type Q webs to investigate the spin permutation modules $\mathcal{M}^\lambda$ of Ser_k. To the author's knowledge, these first appeared in [38], and have since appeared in [13, 44] ([13] actually concerns a quantum analog of $\mathcal{M}^\lambda$). But in each case they were used to study other objects, and have yet to be explored in their own right.

In particular, we prove a superequivalence $\mathfrak{q}\text{-}\mathbf{Web}_\uparrow^k \cong \mathrm{Ser}_k\text{-}\mathbf{Mod}_{\mathcal{M}}$ where

- $\mathfrak{q}\text{-}\mathbf{Web}_\uparrow^k$ is the full (not monoidal) subsupercategory of $\mathfrak{q}\text{-}\mathbf{Web}_\uparrow$ whose webs have total thickness k, and
- $\mathrm{Ser}_k\text{-}\mathbf{Mod}_{\mathcal{M}}$ is the full subsupercategory of Ser_k-modules with objects the spin permutation modules $\mathcal{M}^\lambda$.

This obtains a diagrammatic description of $\mathrm{Ser}_k\text{-}\mathbf{Mod}_{\mathcal{M}}$, as was done previously for $\mathfrak{q}_n\text{-}\mathbf{Mod}_{\mathscr{S}}$ and $\mathfrak{q}_n\text{-}\mathbf{Mod}_{\mathscr{S},\mathscr{S}^*}$. Along the way, we develop the combinatorics of λ-supertabloids to describe the elements of $\mathcal{M}^\lambda$, and of λ-supertabloids of weight μ to describe Ser_k-morphisms $\mathcal{M}^\lambda \to \mathcal{M}^\mu$. We then identify bases for the morphism spaces $\mathrm{Hom}_{\mathfrak{q}\text{-}\mathbf{Web}_\uparrow^k}(\lambda,\mu)$, which in turn produces bases for $\mathrm{Hom}_{\mathrm{Ser}_k}(\mathcal{M}^\lambda,\mathcal{M}^\mu)$ via the superequivalence, both in terms of webs and of weighted supertabloids.

6.1. Projective representations of symmetric groups

The subject of this chapter may actually be thought of as the projective, or spin, representation theory of the symmetric groups. We give a brief explanation of this connection here, which will serve as further motivation for studying representations of Ser_k.

A *projective representation* of a finite group G is a choice of vector space V and group homomorphism $\rho\colon G \to PGL(V)$. Here $GL(V)$ is the group of linear automorphisms of V and $PGL(V)$ is the projective general linear group $PGL(V) = GL(V)/\mathbb{C}^\times$. In pioneering work [36], Schur established that studying projective representations of $\mathfrak{S}_k$ is equivalent to studying ordinary representations of its *twisted group algebra*, the associative algebra $\mathcal{T}_k$ with generators $t_1, \dots, t_{k-1}$ and relations

$$t_i^2 = 1, \quad t_i t_j = -t_j t_i \quad \text{if } i \neq j \pm 1, \quad t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}. \tag{6.1}$$

The reader is referred to Stembridge's excellent treatment [40, §1] for details in this direction.

Unfortunately, the method of parabolic induction – used to great effect in the representation theory of $\mathfrak{S}_k$ – is not easy to define for $\mathcal{T}_k$ (see [40, §4]). However, a solution to this problem was found by Brundan and Kleshchev [4]. They viewed $\mathcal{T}_k$ as an associative *super*algebra, declaring the generators $t_1, \dots, t_{k-1}$ to be odd. They then established a "super Morita equivalence" between $\mathcal{T}_k$ and Ser_k [4, Corollary 3.5], and subsequently focused on representations of Ser_k instead. The latter is ripe for parabolic induction (see [25, §13]). Hence, altogether, studying projective representations of $\mathfrak{S}_k$ is equivalent to studying ordinary representations of Ser_k.

We follow the convention of replacing the word "projective" in this context with "spin", partly to avoid confusion with projective modules in the homological sense. The spin permutation modules $\mathcal{M}^\lambda$ of Ser_k and their λ-supertabloids introduced in this chapter may be thought of as spin analogs of the permutation modules M^λ of $\mathfrak{S}_k$ and their λ-tabloids, hence the naming conventions. The reader is referred to [18, §4] for more on the latter.

6.2. Spin permutation modules of Ser_k

For $m \in \mathbb{Z}_{>0}$ let

$$\Lambda(m,k) := \{\lambda \in \mathbb{Z}^m_{\geq 0} : |\lambda| = k\}, \quad \Lambda(k) := \bigcup_{m=1}^{k} \Lambda(m,k)$$

be the sets of *compositions of k of length m* and of *compositions of k*, respectively, where in the latter we treat compositions as equal up trailing zeros, e.g. $(3,2,0) = (3,2,0,0)$. Since the actions of $\mathfrak{q}_m$ and Ser_k on $V_m^{\otimes k}$ commute (see Section 3.4), the $\mathfrak{q}_m$ weight space $(V_m^{\otimes k})_\lambda$ is a Ser_k-module for $\lambda \in \Lambda(m,k)$. From now on we denote this Ser_k-module

$$\mathcal{M}^\lambda := (V_m^{\otimes k})_\lambda$$

and call it the *spin permutation module of shape λ*.

By an argument similar to that used in Lemma 5.1, $\mathcal{M}^\lambda$ has a monomial basis consisting of all pure tensors $v_t := v_{t_1} \otimes \cdots \otimes v_{t_k}$, $t := (t_1, \ldots, t_k) \in I(n|n)^k$, with the property that the number of t_i with $\underline{t_i} = j$ is λ_j for $1 \leq i \leq k$ and $j \in I(n|0)$.

Remark 6.1. *Let λ^+ denote the result of deleting all entries of λ which are zero, e.g. if $\lambda = (4,0,0,3,0,1)$ then $\lambda^+ = (4,3,1)$. From the action of* Ser_k *on $\mathcal{M}^\lambda$, it's* *clear that we have an even* Ser_k*-isomorphism* $\mathcal{M}^\lambda \simeq \mathcal{M}^\mu$ *if and only if* λ^+ *and* μ^+ *are equal up to a rearrangement of their entries (in particular, only if* $|\lambda^+| = |\mu^+|$*). Hence, when speaking of spin permutation modules* $\mathcal{M}^\lambda, \mathcal{M}^\mu, \ldots$*, no information is lost by assuming that* $\lambda, \mu, \ldots$ *have only nonzero entries.*

There is also a combinatorial interpretation of $\mathcal{M}^\lambda$, which we now construct. Recall from Section 5.4 the ordered set $\mathbb{A} := \{1' < 1 < 2' < 2 < \cdots\}$, where we say the letters $1', 2', 3', \ldots$ are *marked*, and use the notation $|a|$ to denote the unmarked version of any $a \in \mathbb{A}$. There is an involution on $\mathbb{A}$ given by marking every element, where we declare that $a'' = a$ for an unmarked $a \in \mathbb{A}$. For $k \in \mathbb{Z}_{>0}$ let $\mathbb{A}_k := \{1, 1', 2, 2', \ldots, k, k'\} \subset \mathbb{A}$.

Definition 6.2. *For* $\lambda \in \Lambda(k)$*, a* supertabloid of shape λ*, or a* λ-supertabloid*, is an arrangement* T *of* k*-many elements of* $\mathbb{A}_k$ *into* $l(\lambda)$ *left-justified rows, such that*

1. *for* $1 \le i \le l(\lambda)$*, the* i^{th} *row of* T *has* λ_i *entries, and*
2. *for* $1 \le j \le k$*, exactly one element of* $\{j, j'\} \subset \mathbb{A}_k$ *appears in* T*.*

We define the parity $\overline{a} \in \mathbb{Z}_2$ *of an entry* a *in* T *to be* 0 *if* a *is unmarked and* 1 *if* a *is marked. The* parity $\overline{T} \in \mathbb{Z}_2$ *of* T *is defined to be sum of the parities of its entries modulo 2. By the* i^{th} entry of T *we mean the entry* a *of* T *with* $|a| = i$ *for* $1 \le i \le k$*.*

Below are some examples of $(2,1,3)$-supertabloids.

$$\begin{array}{l} \hline 3' \; 6' \\ \hline 2 \\ \hline 1 \; 4' \; 5 \\ \hline \end{array} \qquad \begin{array}{l} \hline 6' \; 3' \\ \hline 2 \\ \hline 5 \; 1 \; 4' \\ \hline \end{array} \qquad \begin{array}{l} \hline 4 \; 6 \\ \hline 1' \\ \hline 3 \; 2 \; 5 \\ \hline \end{array} \qquad \begin{array}{l} \hline 3 \; 5' \\ \hline 4 \\ \hline 6 \; 2' \; 1' \\ \hline \end{array}$$

We declare two λ-supertabloids T, T' to be *equivalent* if they are identical up to permutations of the positions of the entries which stabilize the rows. Thus the two leftmost $(2,1,3)$-supertabloids above are equivalent, otherwise they are pairwise inequivalent. For convenience, we will usually display supertabloids such that the entries of each row are arranged in the increasing order of $\mathbb{A}$, as in the leftmost $(2,1,3)$-supertabloid above.

There is a parity-preserving bijection between elements v_t of the monomial basis of $\mathcal{M}^\lambda$ and equivalence classes of λ-supertabloids. Indeed, we construct the λ-supertabloid T associated to v_t by placing an i (resp. an i') in row $\underline{t_i}$ if $t_i \in I(n|0)$ (resp. if $t_i \in I(0|n)$ for $1 \le i \le k$, and then rearranging the entries of each row into the increasing order of $\mathbb{A}$. An example of this correspondence for $k = 6$ is

$$v_3 v_2 v_{\overline{1}} v_{\overline{3}} v_3 v_{\overline{1}} \quad \longleftrightarrow \quad \begin{array}{l} \hline 3' \ 6' \\ \hline 2 \\ \hline 1 \ 4' \ 5 \\ \hline \end{array} \ .$$

That these assignments are bijective is clear, and we have established the following.

Proposition 6.3. *For $\lambda \in \Lambda(m,k)$, $\mathcal{M}^\lambda$ is isomorphic as a superspace to the span of the equivalence classes of λ-supertabloids T.*

We can use the above isomorphism to express the action of Ser_k on $\mathcal{M}^\lambda$ in terms of supertabloids. Let $\lambda \in \Lambda(m,k)$ and $T \in \mathcal{M}^\lambda$ be a λ-supertabloid. For $1 \le i \le k$ define $\delta(i) \in \mathbb{Z}_2$ to be 0 if the i^{th} entry of T is unmarked and 1 if it's marked. Let $T_{i \to i'}$ denote the λ-supertabloid obtained from T by marking the i^{th} entry of T, and for $1 \le j \le k-1$ let $T_{j \leftrightarrow j+1}$ denote the λ-supertabloid obtained from T by interchanging the positions of the j^{th} and $(j+1)^{\text{st}}$ entries. Then one can check that the action of Ser_k on λ-supertabloids is given by

$$c_i.T = (-1)^{\delta(1)+\cdots+\delta(i)+1} \sqrt{-1}\, T_{i \to i'}, \quad s_j.T = (-1)^{\delta(j)\delta(j+1)} T_{j \leftrightarrow j+1}.$$

For example, if $k = 6$ and $m \geq 3$ then

$$c_4.\left(\begin{array}{l}\hline 3'\ 6' \\ \hline 2 \\ \hline 1\ 4'\ 5 \\ \hline\end{array}\right) = -\sqrt{-1}\ \begin{array}{l}\hline 3'\ 6' \\ \hline 2 \\ \hline 1\ 4\ 5 \\ \hline\end{array}\,, \qquad s_5.\left(\begin{array}{l}\hline 3'\ 6' \\ \hline 2 \\ \hline 1\ 4'\ 5 \\ \hline\end{array}\right) = \begin{array}{l}\hline 3'\ 5 \\ \hline 2 \\ \hline 1\ 4'\ 6' \\ \hline\end{array}\,.$$

We conclude this section by highlighting two key examples of spin permutation modules, namely $\mathcal{M}^{1^k}$ and $\mathcal{M}^{(k)}$. The former is isomorphic to Ser_k itself. Indeed, we have a Ser_k-isomorphism $\mathrm{Ser}_k \xrightarrow{\sim} \mathcal{M}^{1^k}$ by sending a standard basis element $w = c_1^{a_1} \cdots c_k^{a_k}\sigma$ to $w.T_0$ where T_0 is the 1^k-supertabloid with entries $1, 2, \ldots, k$ from top to bottom. For example, if $k = 3$ then

$$c_2(1\ 3) \quad \mapsto \quad c_2(1\ 3).\left(\begin{array}{c}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}\right) = -\sqrt{-1}\ \begin{array}{c}\hline 3 \\ \hline 2' \\ \hline 1 \\ \hline\end{array}\,.$$

Meanwhile $\mathcal{M}^{(k)}$, known in the literature as the *basic spin module*, is isomorphic to the Clifford algebra C_k when the latter is thought of as a Ser_k-module via

$$c_i.c_{i_1}c_{i_2}\cdots = c_i c_{i_1} c_{i_2} \cdots, \qquad \sigma.c_{i_1}c_{i_2}\cdots = c_{\sigma(i_1)}c_{\sigma(i_2)}\cdots$$

for $1 \leq i, i_1, i_2, \cdots \leq k$ and $\sigma \in \mathfrak{S}_k$. Indeed, a Ser_k-isomorphism $C_k \xrightarrow{\sim} \mathcal{M}^{(k)}$ is given by $w \mapsto w.T_1$ where T_1 is the (k)-supertabloid with entries $1, 2, \ldots, k$ from left to right. For example, if $k = 3$ then

$$c_1c_3 \quad \mapsto \quad c_1c_3.\left(\begin{array}{c}\hline 1\ 2\ 3 \\ \hline\end{array}\right) = \left(-\sqrt{-1}\right)^2\ \begin{array}{c}\hline 1'\ 2\ 3' \\ \hline\end{array} = -\ \begin{array}{c}\hline 1'\ 2\ 3' \\ \hline\end{array}\,.$$

It's straightforward to see that both maps are isomorphisms of superspaces which preserve the actions of Ser_k.[1]

[1]The interested reader may also verify that $\mathcal{M}^{(k)}$ is irreducible, and is none other than $D^{(k)}$, i.e. the submodule of Ser_k which is the left ideal generated by the quasi-idempotent $e_{(k)}$.

6.3. Functors Ω_m and Γ

From now on we regard k as fixed and m as variable. In this section, we define a (not monoidal) subsupercategory $\mathfrak{q}$-$\mathbf{Web}_\uparrow^k$ of $\mathfrak{q}$-$\mathbf{Web}_\uparrow$ and a superequivalence $\Gamma\colon \mathfrak{q}\text{-}\mathbf{Web}_\uparrow^k \to \mathrm{Ser}_k\text{-}\mathbf{Mod}_{\mathcal{M}}$ where $\mathrm{Ser}_k\text{-}\mathbf{Mod}_{\mathcal{M}}$ is the category of spin permutation modules $\mathcal{M}^\lambda$ of Ser_k. In order to do so, we must first consider the type Q Schur superalgebra.

Following [12], we define the *type Q Schur superalgebra* to be

$$\mathcal{Q}(m,k) := \mathrm{End}_{\mathrm{Ser}_k}(V_m^{\otimes k}) = \mathrm{Hom}_{\mathrm{Ser}_k}(V^{\otimes k}, V^{\otimes k}).$$

Since $V_m^{\otimes k}$ is a weight module over $\mathfrak{q}_n$ with weights in bijection with $\Lambda(m,k)$, we can rewrite the above as

$$\begin{aligned}\mathcal{Q}(m,k) &= \bigoplus_{\lambda,\mu\in\Lambda(m,k)} \mathrm{Hom}_{\mathrm{Ser}_k}((V_m^{\otimes k})_\lambda, (V_m^{\otimes k})_\mu) \\ &= \bigoplus_{\lambda,\mu\in\Lambda(m,k)} \mathrm{Hom}_{\mathrm{Ser}_k}(\mathcal{M}^\lambda, \mathcal{M}^\mu).\end{aligned}$$

By the Schur-Weyl-Sergeev duality (see Theorem 3.3), $U(\mathfrak{q}_m)$ surjects onto $\mathcal{Q}(m,k)$, so the latter can be viewed as a quotient of the former. Further, $\mathcal{Q}(m,k)$ admits an idempotented presentation $\dot{\mathcal{Q}}(m,k)$ which embeds as a locally unital subalgebra of $\dot{U}(\mathfrak{q}_m)_{\geq 0}$ [12, Theorem 4.10]. Indeed, we have

$$\dot{U}(\mathfrak{q}_m)_{\geq 0} = \bigoplus_{k\geq 0} \dot{\mathcal{Q}}(m,k), \quad \dot{\mathcal{Q}}(m,k) = \left(\bigoplus_{\lambda,\mu\in\Lambda(m,k)} 1_\mu \dot{U}(\mathfrak{q}_m)_{\geq 0} 1_\lambda\right),$$

and in particular, this combined with the definition of $\mathcal{Q}(m,k)$ implies

$$1_\mu \dot{U}(\mathfrak{q}_m)_{\geq 0} 1_\lambda \simeq \mathrm{Hom}_{\mathrm{Ser}_k}(\mathcal{M}^\lambda, \mathcal{M}^\mu).$$

To summarize, we have canonical superspace isomorphisms

$$\omega_{\lambda,\mu} \colon 1_\mu \dot{U}(\mathfrak{q}_m)_{\geq 0} 1_\lambda \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Ser}_k}(\mathcal{M}^\lambda, \mathcal{M}^\mu)$$

for $\lambda, \mu \in \Lambda(m,k)$ by letting an element $x \in 1_\mu \dot{U}(\mathfrak{q}_m) 1_\lambda$ act on $V_m^{\otimes k}$.

Let Ser_k -$\mathbf{Mod}_{\mathcal{M}}$ be the full (not monoidal) subsupercategory of Ser_k-modules with objects the $\mathcal{M}^\lambda$ for $\lambda \in \Lambda(k)$. Let also $\dot{\mathbf{U}}(\mathfrak{q}_m)^k_{\geq 0}$ be the supercategory associated to the locally unital subsuperalgebra

$$\dot{U}(\mathfrak{q}_m)^k_{\geq 0} := \bigoplus_{\lambda,\mu \in \Lambda(m,k)} 1_\mu \dot{U}(\mathfrak{q}_m)_{\geq 0} 1_\lambda = \dot{\mathcal{Q}}(m,k)$$

of $\dot{U}(\mathfrak{q}_m)_{\geq 0}$ in the sense of Section 2.3. In other words, $\dot{\mathbf{U}}(\mathfrak{q}_m)^k_{\geq 0}$ is the full subsupercategory of $\dot{\mathbf{U}}(\mathfrak{q}_m)_{\geq 0}$ with objects $\lambda \in \Lambda(m,k)$. In light of the previous paragraph, we have established the following.

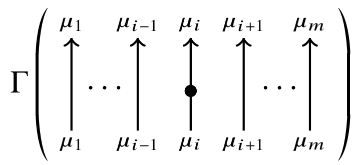
Proposition 6.4. *For $m \in \mathbb{Z}_{>0}$ there exists a fully faithful superfunctor*

$$\begin{aligned} \Omega_m \colon \dot{\mathbf{U}}(\mathfrak{q}_m)^k_{\geq 0} &\to \mathrm{Ser}_k\text{ -}\mathbf{Mod}_{\mathcal{M}} \\ \lambda &\rightsquigarrow \mathcal{M}^\lambda \\ 1_\mu x 1_\lambda &\mapsto \omega_{\lambda,\mu}(x) \colon \mathcal{M}^\lambda \to \mathcal{M}^\mu \end{aligned}$$

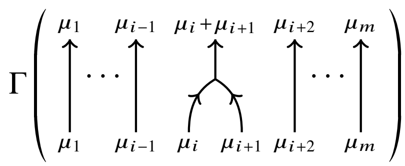
for $\lambda, \mu \in \Lambda(m,k)$, $x \in \dot{U}(\mathfrak{q}_m)^k_{\geq 0}$.

Let $\mathfrak{q}$-$\mathbf{Web}^k_\uparrow$ be the full (not monoidal) subsupercategory of $\mathfrak{q}$-$\mathbf{Web}_\uparrow$ with objects the set of all $\lambda = (\lambda_1{\uparrow}, \dots, \lambda_l{\uparrow}) \in \langle \uparrow \rangle$ such that $\lambda_1 + \cdots + \lambda_l = k$ and $l \leq k$.

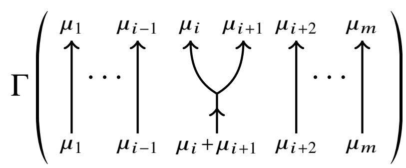
Theorem 6.5. *There exists a superequivalence* $\Gamma \colon \mathfrak{q}\text{-}\mathbf{Web}_{\uparrow}^{k} \to \mathrm{Ser}_k\text{-}\mathbf{Mod}_{\mathcal{M}}$ *given on objects by* $\lambda \rightsquigarrow \mathcal{M}^{\lambda}$ *for* $\lambda \in \mathfrak{q}\text{-}\mathbf{Web}_{\uparrow}^{k}$, *and on morphisms by*



$$= \Omega_m(h_{\bar{i}} 1_{\mu}),$$



$$= \Omega_m(e_i^{(\mu_{i+1})} 1_{\mu}),$$



$$= \Omega_m(1_{\mu} f_i^{(\mu_{i+1})}).$$

Proof. First, we argue that Γ is well-defined. Since, up to the obvious isomorphisms in $\mathrm{Ser}_k\text{-}\mathbf{Mod}_{\mathcal{M}}$ (see Remark 6.1), we have the commutative diagram

$$\begin{array}{ccc} \dot{\mathbf{U}}(\mathfrak{q}_m)_{\geq 0}^{k} & \xrightarrow{\Omega_m} & \mathrm{Ser}_k\text{-}\mathbf{Mod}_{\mathcal{M}} \\ {\scriptstyle \Pi_m}\downarrow & \nearrow_{\Gamma} & \\ \mathfrak{q}\text{-}\mathbf{Web}_{\uparrow}^{k} & & \end{array}$$

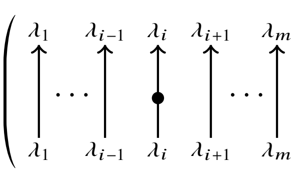
for $m \leq k$, this is true by an argument similar to the one for $\Psi^{\uparrow}$ in Proposition 5.4. Since each Ω_m is full by Proposition 6.4, commutativity of the diagram implies Γ is full as the last functor in a composition which is full. Also by Remark 6.1, Γ is evenly dense, so it remains to show Γ is faithful. By an argument similar to that in the proof of Theorem 5.6, Γ is faithful if and only if the homomorphism

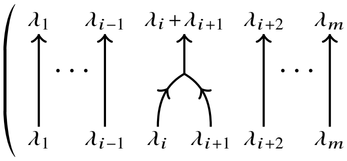
$$\Gamma_k \colon \operatorname{End}_{\mathfrak{q}\text{-}\mathbf{Web}^k_\uparrow}(\uparrow^k) \twoheadrightarrow \operatorname{End}_{\mathrm{Ser}_k}(\mathcal{M}^{1^k})$$

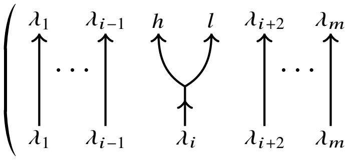
induced by Γ is injective (it is surjective because Γ is full). By Corollary 5.5 and the fullness of $\mathfrak{q}$-$\mathbf{Web}^k_\uparrow$ in $\mathfrak{q}$-$\mathbf{Web}_\uparrow$, the source of Γ_k is isomorphic to Ser_k; meanwhile the target is isomorphic to $\operatorname{End}_{\mathrm{Ser}_k}(\mathrm{Ser}_k)$ via the isomorphism $\mathcal{M}^{1^k} \simeq \mathrm{Ser}_k$ as Ser_k-modules (see the discussion at the end of the previous section). From basic representation theory we know the dimensions of A and $\operatorname{End}_A(A)$ are equal for a finite-dimensional algebra A, so this combined with the surjectivity of Γ_k implies Γ_k is injective. This completes the proof. ■

Using the definitions of Ω_m and Γ above, we can give explicit descriptions of the images under Γ of merges, splits, and dots. To do so, we introduce some new notation.

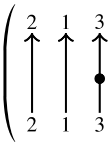
For $\lambda \in \Lambda(m,k)$ and $T \in \mathcal{M}^\lambda$ a λ-supertabloid, let $\mathrm{merge}_i(T)$ be the supertabloid of shape $(\lambda_1, \ldots, \lambda_{i-1}, \lambda_i + \lambda_{i+1}, \lambda_{i+2}, \ldots, \lambda_m)$ obtained by merging rows i and $i+1$ of T for $1 \le i \le m-1$. Let $\mathrm{split}_i^{h,l}(T)$ be the sum of supertabloids of shape $(\lambda_1, \ldots, \lambda_{i-1}, h, l, \lambda_{i+1}, \ldots, \lambda_m)$ obtained by splitting row i of T into (on top) a row of length h and (on bottom) a row of length l, for $h, l \in \mathbb{Z}_{>0}$ with $h + l = \lambda_i$. Recall also the notation $\delta(j) \in \mathbb{Z}_2$ and the supertabloid $T_{j\to j'}$ for $1 \le j \le k$ from the previous section. One can check that

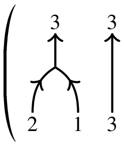
$\Gamma\Bigg($  $\Bigg)(T) = \displaystyle\sum_{\substack{1\le j\le k \\ j \text{ or } j' \text{ in row } i}} (-1)^{\delta(1)+\cdots+\delta(j-1)} T_{j\to j'},$

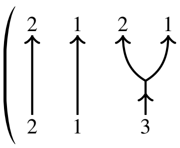
$$\Gamma\Bigg($$  $$\Bigg)(T) = \mathrm{merge}_i(T),$$

$$\Gamma\Bigg($$  $$\Bigg)(T) = \mathrm{split}_i^{h,l}(T).$$

For example, if $k = 6$ and $m = 3$ then

$$\Gamma\Bigg($$  $$\Bigg)\left(\begin{array}{l} 3'\ 6' \\ 2 \\ 1\ 4'\ 5 \end{array}\right) = \begin{array}{l} 3'\ 6' \\ 2 \\ 1'\ 4'\ 5 \end{array} - \begin{array}{l} 3'\ 6' \\ 2 \\ 1\ 4\ 5 \end{array} + \begin{array}{l} 3'\ 6' \\ 2 \\ 1\ 4'\ 5' \end{array},$$

$$\Gamma\Bigg($$  $$\Bigg)\left(\begin{array}{l} 3'\ 6' \\ 2 \\ 1\ 4'\ 5 \end{array}\right) = \begin{array}{l} 2\ 3'\ 6' \\ 1\ 4'\ 5 \end{array},$$

$$\Gamma\Bigg($$  $$\Bigg)\left(\begin{array}{l} 3'\ 6' \\ 2 \\ 1\ 4'\ 5 \end{array}\right) = \begin{array}{l} 3'\ 6' \\ 2 \\ 1\ 4' \\ 5 \end{array} + \begin{array}{l} 3'\ 6' \\ 2 \\ 1\ 5 \\ 4' \end{array} + \begin{array}{l} 3'\ 6' \\ 2 \\ 4'\ 5 \\ 1 \end{array}.$$

6.4. A basis for $\mathrm{Hom}_{\mathrm{Ser}_k}(\mathcal{M}^\lambda, \mathcal{M}^\mu)$

In this section, we develop the combinatorics of λ-*supertabloids of weight* μ for $\lambda, \mu \in \Lambda(k)$, and prove that they index a basis of $\mathrm{Hom}_{\mathrm{Ser}_k}(\mathcal{M}^\lambda, \mathcal{M}^\mu)$.

Definition 6.6. *For $\lambda, \mu \in \Lambda(k)$, a λ-*supertabloid of weight μ *is similar to a λ-supertabloid T except that, for $1 \le i \le k$,*

1. *the number of entries a in T with $|a| = i$ is λ_i, and*

2. *for every row of T, no more than one marked i can occur in that row.*

In particular, what were previously called λ-supertabloids are also λ-supertabloids of weight 1^k.

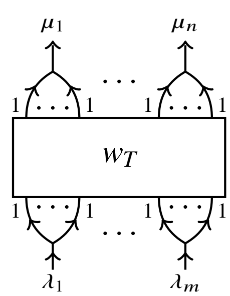
Below are some $(2,1,3)$-supertabloids of weight $(4,2)$, $(1,1,1,2,1)$, (6), and (1,3,2), respectively.

$$\begin{array}{lll} \hline 1' & 2' & \\ \hline 1 & & \\ \hline 1 & 1 & 2' \\ \hline \end{array} \qquad \begin{array}{lll} \hline 4' & 5' & \\ \hline 4 & & \\ \hline 1 & 2' & 3 \\ \hline \end{array} \qquad \begin{array}{lll} \hline 1 & 1 & \\ \hline 1' & & \\ \hline 1 & 1 & 1 \\ \hline \end{array} \qquad \begin{array}{lll} \hline 2 & 3' & \\ \hline 3 & & \\ \hline 1' & 2' & 2 \\ \hline \end{array}$$

We declare two λ-supertabloids T, T' of weight μ to be *equivalent* if they are identical up to permutations of the positions of the entries which stabilize the rows. We denote by $\mathcal{M}^{\lambda,\mu}$ the superspace spanned by all equivalence classes of λ-supertabloids of weight μ, where the parity of each is again the sum of the parities of its entries modulo 2. In particular, $\mathcal{M}^{\lambda,1^k}$ is the spin permutation module $\mathcal{M}^\lambda$.

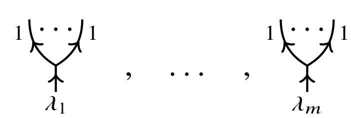
Recall that for $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{Z}^m_{\geq 0}$, $\lambda\uparrow$ denotes the sequence $(\lambda_1\uparrow, \ldots, \lambda_m\uparrow) \in \langle\uparrow\rangle$. For $m, n \in \mathbb{Z}_{>0}$, $\lambda \in \Lambda(m,k)$, $\mu \in \Lambda(n,k)$, and a supertabloid $t \in \mathcal{M}^{\lambda,\mu}$, define

$\Delta_{\lambda,\mu}(t) :=$

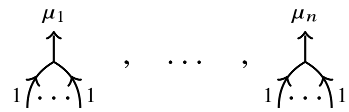


$\in \mathrm{Hom}_{\mathfrak{q}\text{-}\mathbf{Web}^k_\uparrow}(\lambda\uparrow, \mu\uparrow)$

where w_T is the Sergeev diagram defined as follows. Let us refer to the webs



as the λ-*clamps* and the webs

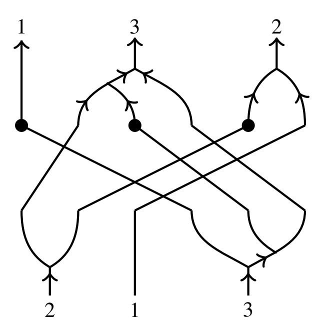


as the μ-*clamps*. Reading from bottom to top, every strand of w_T starts in one of the λ-clamps and ends in one of the μ-clamps. We think of the strands beginning in the i^{th} λ-clamp as representing the entries in the i^{th} row of T, for $1 \le i \le m$. For every entry a in the i^{th} row of T with $|a| = l$, a strand of w_T ends in the l^{th} μ-clamp for $1 \le l \le n$. By Lemma 4.11, this determines a unique $(\mathfrak{S}_\lambda, \mathfrak{S}_\mu)$-double coset of $\mathfrak{S}_k$, where $\mathfrak{S}_\mu := \mathfrak{S}_{\mu_1} \times \cdots \times \mathfrak{S}_{\mu_n}$ is the *Young subgroup* of $\mathfrak{S}_k$ associated to μ and similarly for $\mathfrak{S}_\lambda$. We do this for every strand of w_T in the unique way such that

- there are no crossings of two strands starting in the same λ-clamp, and
- there are no crossings of two strands ending in the same μ-clamp.

In other words, the permutation diagram part of w_T is the $(\mathfrak{S}_\lambda, \mathfrak{S}_\mu)$-double coset representative of minimal length, which is well known to be unique (see [30, §4.1]). Finally, we place a dot at the top of every strand of w_T representing an odd entry of T, placing it on the leftmost strand which starts in the same λ-clamp and ends in the same μ-clamp as it. For example,

$$\Delta_{(2,1,3),(1,3,2)}\left(\begin{array}{lll} \overline{2\ \ 3'} & & \\ \overline{3} & & \\ \overline{1'\ \ 2'\ \ 2} & & \end{array}\right) =$$



.

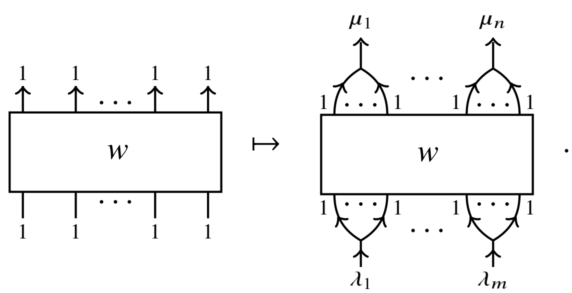
From the definitions it's clear that if $T, T' \in \mathcal{M}^{\lambda,\mu}$ are equivalent then $\Delta_{\lambda,\mu}(T) =$

$\Delta_{\lambda,\mu}(T')$. Hence we get a linear map $\Delta_{\lambda,\mu}\colon \mathcal{M}^{\lambda,\mu} \to \mathrm{Hom}_{\mathfrak{q}\text{-}\mathbf{Web}^k_\uparrow}(\lambda\uparrow, \mu\uparrow)$, which is by definition a $\mathbb{Z}_2$-homogeneous map of even parity.

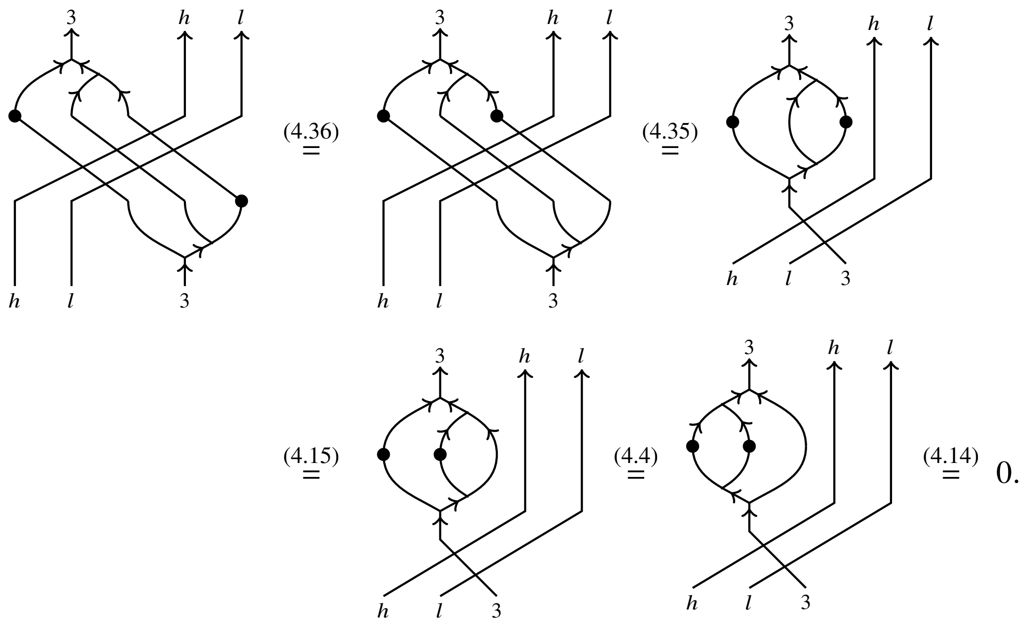
Theorem 6.7. *For $\lambda, \mu \in \Lambda(k)$, the map $\Delta_{\lambda,\mu}\colon \mathcal{M}^{\lambda,\mu} \to \mathrm{Hom}_{\mathfrak{q}\text{-}\mathbf{Web}^k_\uparrow}(\lambda\uparrow, \mu\uparrow)$ is an isomorphism of superspaces.*

Proof. First we prove that $\Delta_{\lambda,\mu}$ is surjective. Consider the map

$$\mathrm{End}_{\mathfrak{q}\text{-}\mathbf{Web}^k_\uparrow}(\uparrow^k) \quad \to \quad \mathrm{Hom}_{\mathfrak{q}\text{-}\mathbf{Web}^k_\uparrow}(\lambda\uparrow, \mu\uparrow)$$

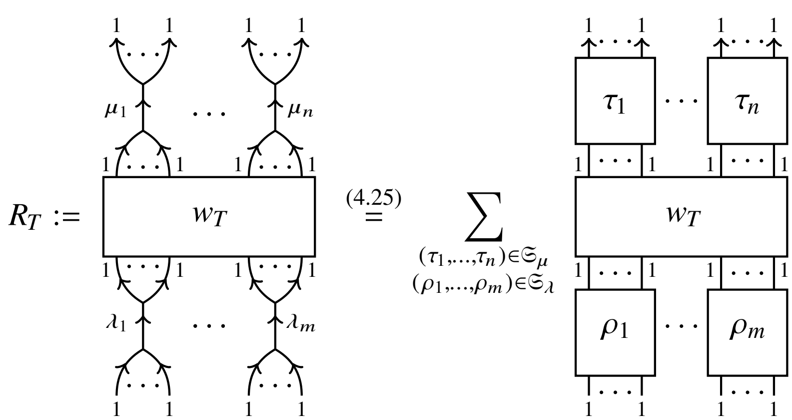


By (4.5), every morphism in the target has an obvious preimage under this map, so it is surjective. We claim that every web in $\mathrm{Hom}_{\mathfrak{q}\text{-}\mathbf{Web}^k_\uparrow}(\lambda\uparrow, \mu\uparrow)$ of the form on the right above is equal to $\Delta_{\lambda,\mu}(T)$ for some $T \in \mathcal{M}^{\lambda,\mu}$ up to a scalar (possibly zero), which would imply that $\Delta_{\lambda,\mu}$ is surjective. Indeed, crossings of strands ending in the same μ-clamp can be untied by Lemma 4.11, as can crossings of strands starting in the same λ-clamp, and the heights of dots can be permuted up to a sign by the superinterchange. To satisfy property (2) of Definition 6.6, we must consider the possibility that there are two strands which begin in the same λ-clamp, end in the same μ-clamp, and both have a dot. In that case the entire web is actually zero, as in the following example for arbitrary $h, l \in \mathbb{Z}_{>0}$:



This proves the claim and $\Delta_{\lambda,\mu}$ is surjective.

For injectivity, we define



and completely split every strand on top and bottom of $\Delta_{\lambda,\mu}(T)$ to get a map

$$\begin{aligned} \mathcal{M}^{\lambda,\mu} &\rightarrow \mathrm{End}_{\mathrm{q}\text{-}\mathbf{Web}^k_\uparrow}(\uparrow^k) \\ T &\mapsto R_T. \end{aligned}$$

We make the following claims.

1. When expressing R_T in the standard basis of Ser_k, the coefficient of w_T is a positive integer.

2. For inequivalent $T', T \in \mathcal{M}^{\lambda,\mu}$, the coefficient of $w_{T'}$ in the expression of R_T in the standard basis is zero.

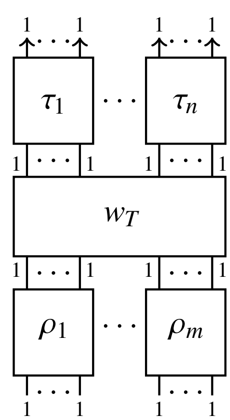
These together would imply that the map given by $T \mapsto R_T$ has trivial kernel and is injective, implying that $\Delta_{\lambda,\mu}$ is injective as the first map in an injective composition. Indeed, a linear combination $\sum_{T\in X} b_T\, T \neq 0$ of pairwise inequivalent $T \in \mathcal{M}^{\lambda,\mu}$ with coefficients $b_T \in \mathbb{C}$ would then map to

$$\begin{aligned}\sum_{T\in X} b_T\, R_T \quad &= \quad \sum_{T\in X} (l_T\, b_T) w_T \\ &\quad + \text{a linear combination of } w \in \mathrm{Ser}_k \text{ with } w \neq w_{T'} \text{ for all } T' \in \mathcal{M}^{\lambda,\mu}\end{aligned}$$

for some positive integers l_T, which is nonzero.

For shorter notation, let $\tau := (\tau_1, \ldots, \tau_n) \in \mathfrak{S}_\mu$, $\rho := (\rho_1, \ldots, \rho_m) \in \mathfrak{S}_\lambda$, and

$\tau w_T\, \rho :=$



$= R_t.$

A pair (τ, ρ) with $\tau w_T\, \rho = w_T$ can only permute strands of w_T starting in the same λ-clamp and ending in the same μ-clamp (although the clamps aren't present), and further must consist of pairs (τ_i, ρ_j) which cancel each other out. Since the total number of dots on each grouping of these strands is 0 or 1, no negative signs are incurred when sliding them to the top of $\tau w_T\, \rho$ in order to write it in the standard basis of Ser_k. This combined with the fact that $1_{\mathfrak{S}_\mu} w_T 1_{\mathfrak{S}_\lambda} = w_T$ proves claim (1).

Now suppose $T', T \in \mathcal{M}^{\lambda,\mu}$ are inequivalent. If w_T and $w_{T'}$ don't have equivalent underlying permutation diagrams (i.e. ignoring dots they are still inequivalent), then based on the previous paragraph there is no pair (τ, ρ) with $\tau w_{T'} \rho = w_T$. Likewise if w_T and $w_{T'}$ differ only in their dots, then there is no pair (τ, ρ) with $\tau w_{T'} \rho = w_T$ because (τ, ρ) can only cross/uncross groupings of strands where at most one dot is present and which (if present) lies on the leftmost strand. This proves claim (2) and the theorem. ■

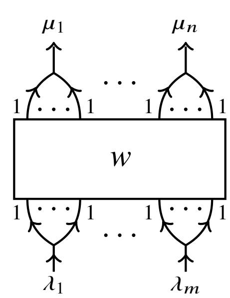
Corollary 6.8. *For $\lambda, \mu \in \Lambda(k)$, the set*

$$\{\Delta_{\lambda,\mu}(T) : T \in \mathcal{M}^{\lambda,\mu}\}$$

is a $\mathbb{Z}_2$-homogeneous basis of the morphism space $\mathrm{Hom}_{\mathfrak{q}\text{-}\mathbf{Web}^k_\uparrow}(\lambda\uparrow, \mu\uparrow)$. *Hence by Theorem 6.5 the set*

$$\{\Gamma(\Delta_{\lambda,\mu}(T)) : T \in \mathcal{M}^{\lambda,\mu}\}$$

is a $\mathbb{Z}_2$-homogeneous basis of $\mathrm{Hom}_{\mathrm{Ser}_k}(\mathcal{M}^\lambda, \mathcal{M}^\mu)$. *Explicitly, the former consists of all webs of the form*



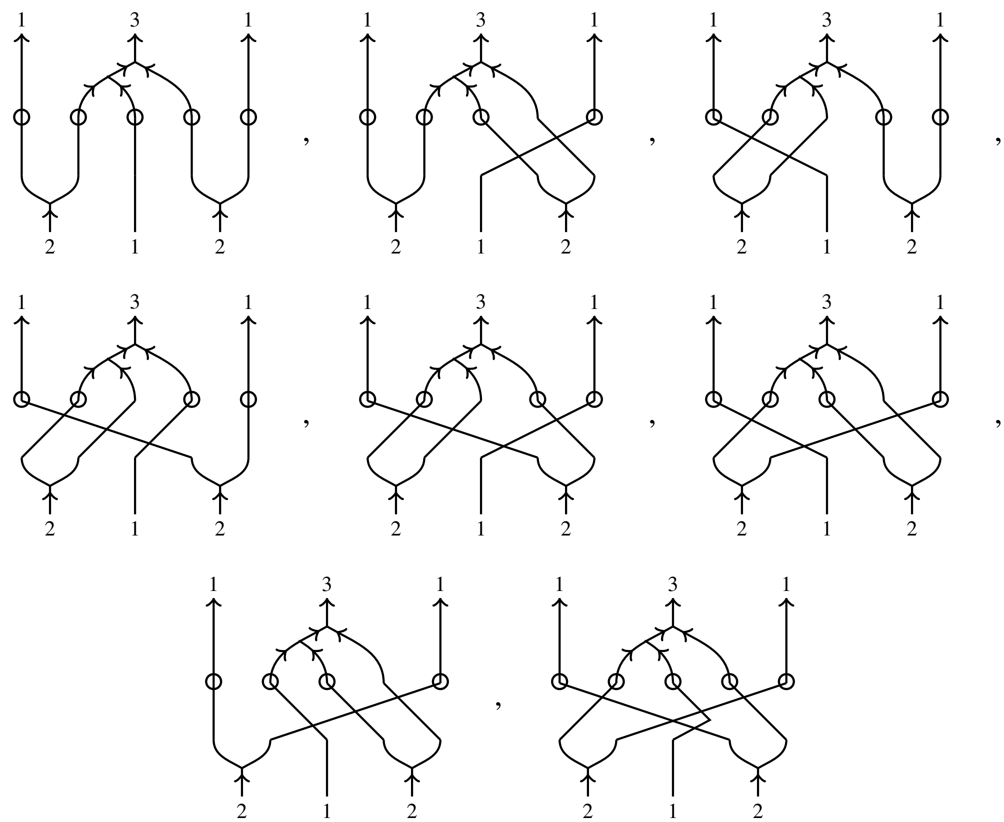
where w is a Sergeev diagram such that

1. *no two strands cross which start in the same λ-clamp,*

2. *no two strands cross which end in the same μ-clamp, and*

3. *for every (λ-clamp, μ-clamp) pair, there is at most one strand connecting them which has a dot, which is the leftmost strand.*

For example, a basis of $\mathrm{Hom}_{\text{q-}\mathbf{Web}^{6}_{\uparrow}}((2\uparrow, 1\uparrow, 2\uparrow), (1\uparrow, 3\uparrow, 1\uparrow))$ is given by the webs



where the symbol ○ denotes a location where a dot is permissible. Thus we have

$$\begin{aligned} \dim \mathrm{Hom}_{\mathrm{Ser}_6}(\mathcal{M}^{(2,1,2)}, \mathcal{M}^{(1,3,1)}) &= \dim \mathrm{Hom}_{\text{q-}\mathbf{Web}^{6}_{\uparrow}}((2\uparrow, 1\uparrow, 2\uparrow), (1\uparrow, 3\uparrow, 1\uparrow)) \\ &= 2\left(2^5\right) + 6\left(2^4\right) = 160. \end{aligned}$$

It is a fulfilling exercise to determine what Ser_6 morphisms these webs represent.

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