A Theorem of Wiener

Joshua Whitman
Department of Mathematics
Oklahoma State University

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1 Introduction

In mathematics, a Fourier series is a way to represent certain functions as the sum of simple sinusoids. More precisely, it decomposes any periodic function (a *signal*) into a (possibly infinite) sum of sines and cosines. The Fourier transform takes a signal and creates another function which gives you the amplitude and phase shift of any sinusoid present within the signal at a given frequency. Fourier analysis is hugely important, not only in mathematics but also in a vast array of scientific and engineering applications.

In this paper, we will prove a fascinating property of Fourier series that was originally discovered by Norbert Wiener. That is, if one has a non-zero function represented as a Fourier series with the property that the partial sums of the coefficients converge absolutely, then the *reciprocal* of that function may also be represented as a Fourier series with the same property. This certainly qualifies as a very pleasing but surprising result – there appears to be nothing which intuitively says that the reciprocal of a function should be so well-behaved.

In this paper, we will show that functions with the properties mentioned above are an example of a certain kind of space called a Banach algebra. Several important theorems of such spaces will be developed, which will eventually relate them to the field of complex numbers.

1.1 Representing the Fourier Series

One way we might represent the Fourier series is as a sum of sine waves with phase shifts:

$$f(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \sin\left(\frac{2\pi n x}{T} + \phi_n\right)$$

We will let the period $T = 2\pi$ in order to simplify things (so f will be 2π periodic instead of T periodic). We can use a trigonometric identity to expand the addition inside the sine function, then relabel the constants in order to write:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \sin(nt) + b_n \cos(nt))$$

It is well known as a consequence of Euler's equation that $\cos(nt) = (e^{int} + e^{-int})/2$ and $\sin(nt) = (e^{int} - e^{-int})/2i$. Thus, if we let $c_n = \frac{a_n}{2i} + \frac{b_n}{2}$ for $n \geq 1$, $c_n = \frac{-a_n}{2i} + \frac{b_n}{2}$ for $n \leq -1$, and $c_0 = \frac{a_0}{2}$, then

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}$$

This representation is clearly the most elegant of the ones we have seen.

1.2 Representing the Fourier Transform

Suppose that $f(t) = \sum\limits_{n=-\infty}^{\infty} c_n e^{int}$. Now, define $\widehat{f}\colon \mathbb{N} \to \mathbb{C}$ by

$$\widehat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-int}dt$$

If the integral of the sum equals the sum of the integral (which depends on uniform convergence), we can write

$$\widehat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} \sum_{k=-\infty}^{\infty} c_k e^{i(k-n)t}dt = \sum_{k=-\infty}^{\infty} c_k \frac{1}{2\pi}\int_0^{2\pi} e^{i(k-n)t}dt$$

If $k = n$, then $\frac{1}{2\pi}\int_0^{2\pi} e^{i(k-n)t}dt = 1$. If $k \neq n$, then since e^{it} is 2π-periodic,

$$\frac{1}{2\pi}\int_0^{2\pi} e^{i(k-n)t}dt = \frac{1}{2\pi}\int_0^{2\pi(k-n)} \frac{e^{iu}}{k-n}du = \frac{1}{2\pi}\frac{k-n}{k-n}\int_0^{2\pi} e^{iu}du$$

$$= \frac{1}{2\pi}(\int_0^{\pi} e^{iu}du + \int_0^{\pi} e^{i(u+\pi)}du) = \frac{1}{2\pi}(\int_0^{\pi} e^{iu}du + \int_0^{\pi} e^{-iu}du) = 0$$

We can conclude that $\widehat{f}(n) = c_n$.

1.3 A Theorem about Fourier Series

Theorem 1. *Suppose* $\sum_{n=-\infty}^{\infty} |c_n| < \infty$. *Then the function* $f(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}$ *converges uniformly as a series. (Since each partial sum is continuous, we then also know that* f *is continuous).*

Proof. Let $\varepsilon > 0$. Choose N so that $\sum_{|n|>N} |c_n| < \varepsilon$. If $n > N$, then for all t,

$$|f(t) - \sum_{k=-n}^{n} c_n e^{int}| = |\sum_{|k|>n} c_k e^{ikt}| \leq \sum_{|k|>n} |c_k| \leq \sum_{|k|>N} |c_k| < \varepsilon$$

□

2 The Theorem

The following theorem is the main result of this paper.

Theorem 2. *Suppose* $\sum_{n=-\infty}^{\infty} |c_n| < \infty$ *and let* $f(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}$. *Suppose* $f(t) \neq 0$ *and let* $g(t) = \frac{1}{f(t)}$. *Then there exists a sequence* (b_n) *such that* $g(t) = \sum_{n=-\infty}^{\infty} b_n e^{int}$ *and* $\sum_{n=-\infty}^{\infty} |b_n| < \infty$.

3 Some Lemmas and Definitions

In this section we will list some definitions and lemmas that should be useful to us.

Definition 3.1. *$f \in C(\mathbb{T})$ if $f\colon \mathbb{R} \to \mathbb{C}$ is continuous and 2π-periodic, $f(t+2\pi) = f(t)$ for all $t \in \mathbb{R}$. $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$.*

Lemma 3. *Suppose $\sum_{n=-\infty}^{\infty} |a_n| < \infty$ and $f(t) = \sum_{n=-\infty}^{\infty} a_n e^{int}$. Then $f \in C(\mathbb{T})$ and $a_n = \widehat{f}(n)$.*

Lemma 4. *If $f \in C(\mathbb{T})$ and $\widehat{f}(n) = 0$ for all $n \in \mathbb{Z}$ then $f = 0$*

Lemma 5. *If $f \in C(\mathbb{T})$, $a_n = \widehat{f}(n)$, and $\sum |a_n| < \infty$ then $f(t) = \sum_{n=-\infty}^{\infty} a_n e^{int}$.*

Definition 3.2. *$f \in A$ if $f \in C(\mathbb{T})$ and $\sum |\widehat{f}(n)| < \infty$. Note, by the previous lemmas, $f \in A$ if and only if there exists a_n such that $\sum |a_n| < \infty$ and $f(t) = \sum a_n e^{int}$.*

Definition 3.3. *$f \in C^k(\mathbb{T})$ if $f, f', f'', ..., f^{(k)}$ are continuous and $f(t+2\pi) = f(t)$*

Lemma 6. *$f \in C^1(\mathbb{T})$ implies that $\widehat{f'}(n) = in\widehat{f}(n)$*

Corollary 6.1. *$f \in C^2(\mathbb{T})$ implies that $\widehat{f''}(n) = -n^2\widehat{f}(n)$.*

Lemma 7. *$f \in C(\mathbb{T})$ implies that $|\widehat{f}(n)| \leq \frac{1}{2\pi}\int_0^{2\pi} |f(t)|dt$.*

Corollary 7.1. *$C^2(\mathbb{T}) \subseteq A$*

Definition 3.4. *If $f \in A$ where $f = \sum a_n e^{int}$ and $\sum |a_n| < \infty$, then $\|f\|_A = \sum |a_n|$*

Lemma 8. *$\|\cdot\|_A$ is a norm on A.*

4 Banach Spaces

This section contains some propositions and definitions about Banach spaces, which we will define shortly. We know that if $\|\cdot\|$ is a norm on a vector space X then there exists an induced metric $d : X \times X \to \mathbb{R}$ defined by $d(x, y) = \|x - y\|$.

Proposition 1. *Suppose X, Y are normed vector spaces and $T : X \to Y$ is linear. The following are equivalent:*

1. *T is continuous*

2. *T is continuous at 0*

3. *There exists $c > 0$ such that $\|Tx\|_Y \leq c\|x\|_X$ for all $x \in X$*

Proposition 2. *Let X be a vector space with norm $\|\cdot\|$. Then the induced metric is complete if and only if for every sequence $x_1, ..., \in X$ with $\sum_{n=1}^{\infty} \|x_n\| < \infty$, $\sum_{n=1}^{\infty} x_n$ converges.*

We will now define what a Banach space is.

Definition 4.1. *A Banach space is a complete normed vector space.*

It turns out that in certain circumstances the quotient space of a Banach space is also a Banach space.

Definition 4.2. *Let Y be a closed subspace of a Banach space X and X/Y be the quotient space as an abelian group. Define $\|x + Y\| = \inf_{y \in Y} \|x - y\|$.*

Proposition 3. *If X is a Banach space and Y is a closed subspace, then:*

1. *$\|x + Y\|$ is well-defined*

2. *$\|\cdot\|$ is a norm on X/Y*

3. *The metric space on X/Y induced by $\|\cdot\|$ is complete*

We should also address the issue of taking a Riemann integral of a vector-valued function. Recall that a partition P of $[a, b]$ is $(x_0, ..., x_n)$ where $a = x_0 < x_1 < ... < x_n = b$, and $\|P\| = \min\{x_{j+1} - x_j : j = 0, ..., n-1\}$.

Definition 4.3. *Let f be a function from the interval $[a, b]$ to a Banach space. We say $\int_a^b f$ exists and equals I if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if P partitions $[a, b]$, $\|P\| < \delta$, and $x_{j-1} \leq t_j \leq x_j$ then $\|I - \sum_{j=1}^{n} f(t_j)(x_j - x_{j-1})\| < \varepsilon$.*

All of the properties we would expect a Riemann integral to have work out nicely for vector-valued functions.

5 Banach Algebras

Definition 5.1. *A Banach algebra is a complex Banach space A with a multiplication operator defined making A into a ring, such that $\lambda(xy) = (\lambda x)y = x(\lambda y)$ for $\lambda \in \mathbb{C}$ and $x, y \in A$. Multiplication also has the following property:*

$$\|xy\| \leq \|x\|\|y\|$$

If our space has an identity e, we will assume $\|e\| = 1$. We will use the abbreviation CBAID for a commutative complex Banach algebra with identity. Assume from now on that we are working in a CBAID called A.

Lemma 9. *Multiplication is continuous. That is, if $x_n \to x$ and $y_n \to y$ then $x_n y_n \to xy$.*

Proof. Given $\varepsilon > 0$, let $\varepsilon_0 = \min(\|x\| + \|y\|, \frac{\varepsilon}{2(\|x\|+\|y\|)})$ and choose N such that $m > N$ implies $\|x_m - x\| < \varepsilon_0, \|y_m - y\| < \varepsilon_0$. Now,

$$\|x_m y_m - xy\| = \|x_m y_m - xy_m - xy + xy_m\|$$

$$\leq \|x_m - x\|\|y_m\| + \|x\|\|y_m - y\| \leq \varepsilon_0(\|y_m - y\| + \|y\|) + \varepsilon_0\|x\| \leq \varepsilon_0^2 + \varepsilon_0(\|x\| + \|y\|)$$

$$\leq \frac{\varepsilon}{2(\|x\| + \|y\|)}(\|x\| + \|y\|) + \frac{\varepsilon}{2(\|x\| + \|y\|)}(\|x\| + \|y\|) = \varepsilon$$

. ∎

Proposition 4. *If $\|x\| < 1$ then $\sum_{n=0}^{\infty} x^n$ converges to $(e - x)^{-1}$. (Define $x^0 = e$).*

Proof.

$$(e - x) \lim_{N\to\infty} \sum_{n=0}^{N} x^n = \lim_{N\to\infty} \sum_{n=0}^{N} (x^n - x^{n+1}) = \lim_{N\to\infty} (x^0 - x^{N+1}) = e$$

because $\|\cdot\|$ is continuous so $\|\lim_{N\to\infty} x^N\| = \lim_{N\to\infty} \|x^N\| = \lim_{N\to\infty} \|x\|^N = 0$ and so $\lim_{N\to\infty} x^N = 0$. ∎

Corollary 9.1. *If x is invertible and $\|y\| < \|x^{-1}\|^{-1}$ then $x - y$ is invertible and $(x - y)^{-1} = x^{-1} \sum (x^{-1}y)^n$.*

Proof.
$$\|x^{-1}y\| \le \|x^{-1}\|\|y\| < \|x^{-1}\|\|x^{-1}\|^{-1} = 1$$
Therefore $(e - x^{-1}y)^{-1} = \sum(x^{-1}y)^n$. But $(e - x^{-1}y)^{-1}x^{-1}(x - y) = (e - x^{-1}y)^{-1}(e - x^{-1}y) = e$, so $(x - y)^{-1} = (e - x^{-1}y)^{-1}x^{-1} = x^{-1}\sum(x^{-1}y)^n$. □

In a Banach algebra without an identity, we include "closed under scalar multiplication" to the definition of "ideal". Note that in a CBAID this is redundant: $\lambda \in C, x \in I$ (I an ideal) implies $\lambda x = (\lambda e)x \in I$.

Lemma 10. *If I is a closed ideal of A, then A/I is a CBAID.*

Proof. From Proposition 3 we know that A/I is a Banach space. We also know that it forms a commutative ring with scalar multiplication working correctly. There exists an identity if A has an identity.

Finally, we will show that $\|(x+I)(y+I)\| \le \|x+I\|\|y+I\|$ for $x, y \in A$. Given $\varepsilon > 0$ there exists $z_1, z_2 \in I$ such that $\|x - z_1\| < \|x + I\| + \varepsilon$ and $\|y - z_2\| < \|y + I\| + \varepsilon$. Now, $\|(x+I)(y+I)\| = \|xy+I\| = \inf_{z\in I}\|xy+z\| \le \|xy + (z_1y + xz_2 + z_1z_2)\| = \|(x + z_1)(y + z_2)\| \le (\|x + I\| + \varepsilon)(\|y + I\| + \varepsilon)$. Taking the limit as $\varepsilon \to 0$, we have $\|(x + I)(y + I)\| \le \|x + I\|\|y + I\|$. □

Theorem 11. *If I is a maximal ideal of A, then I is closed.*

Proof. We must have $\bar{I} = I$ or $\bar{I} = A$, because $\bar{I}$ an ideal and $I \subseteq \bar{I}$. Now, suppose $s \in I$ such that $\|e-s\| < 1$. Then by Proposition 4, $(e-(e-s)) = s$ is invertible, so then $I \supseteq (s) = A$. This is impossible, therefore $B(e, 1) \cap I = \emptyset$. Hence $e \notin \bar{I}$, so $\bar{I} = I$. □

Definition 5.2. *Define the spectrum of x to be*

$$\sigma(x) = \{\lambda \in \mathbb{C} : x - \lambda e \textit{ is not invertible}\}$$

Note that if A were the set of square matrices, then $\sigma(X)$ would be the set of eigenvalues of X.

Lemma 12. *$\sigma(x)$ is a compact subset of $\mathbb{C}$.*

Proof. Say $\lambda \in \mathbb{C} \setminus \sigma(x)$. This means $x - \lambda e$ is invertible. Corollary 11.1 says that if $|\delta| = \|\delta e\| < \|(x - \lambda e)^{-1}\|^{-1} = r$ then $x - \lambda e - \delta e = x - (\lambda + \delta)e$ is invertible. Thus $B(\lambda, r) \subseteq \mathbb{C} \setminus \sigma(x)$. This means that $\sigma(x)$ is closed.

Suppose $|\lambda| > \|x\|$. λe is invertible and $\|x\| < \|(\lambda e)^{-1}\|^{-1}$, so $\lambda e - x$ is invertible. Therefore $\lambda \in \mathbb{C} \setminus \sigma(x)$. So $\sigma(x)$ is bounded. □

6 The Hahn-Banach Theorem

6.1 The Real Version

Theorem 13. *Let $(X, \|\cdot\|)$ be a real normed vector space, S be a subspace of X, and $\Lambda : S \to \mathbb{R}$ be a linear function with a certain $c > 0$ such that $|\Lambda x| \le c\|x\|$ for all $x \in S$. Then there exists a linear function $\bar{\Lambda} : X \to \mathbb{R}$ where $\bar{\Lambda}\Big|_S = \Lambda$ and $|\bar{\Lambda}x| \le c\|x\|$ for all $x \in X$.*

Proof. Let $\mathcal{O} = \{(Y, \lambda) : Y$ a subspace of $X, S \subseteq Y, \lambda : Y \to \mathbb{R}$ is linear with $\lambda\Big|_S = \Lambda$ and $|\lambda x| \le c\|x\|$ $(\forall x \in Y)\}$. We want there to exist $(X, \cdot) \in \mathcal{O}$.

Define $(Y_1, \lambda_1) \le (Y_2, \lambda_2)$ if $Y_1 \subseteq Y_2$, and $\lambda_2\Big|_{Y_1} = \lambda_1$. (We say that (Y_2, λ_2) is an extension of (Y_1, λ_1)). We will use Zorn's Lemma to show that $\mathcal{O}$ has a maximal element.

Suppose there is a chain $\mathcal{Y} \subseteq \mathcal{O}$, that is, a totally-ordered subset of a partially-ordered set. Let $Y = \bigcup_{Y_\alpha \in \mathcal{Y}} Y_\alpha$. If $x \in Y$, there exists $Y_\alpha \in \mathcal{Y}$ with $x \in Y_\alpha$; define $\lambda : Y \to \mathbb{R}$ by $\lambda(x) = \lambda_\alpha(x)$. Since $S \subseteq Y_\alpha \subseteq Y$, we know that $\lambda\Big|_S = \lambda_\alpha\Big|_S = \Lambda$ and we know $|\lambda x| = |\lambda_\alpha x| \le c\|x\|$. Given $x, y \in Y$, we know $x \in Y_\alpha$, $y \in Y_{\alpha'}$ for some $Y_\alpha, Y_{\alpha'} \in \mathcal{Y}$. WLOG let $Y_{\alpha'} \subseteq Y_\alpha$ using total-ordering. Then $x, y \in Y_\alpha$ and any linear comb $ax + by \in Y_\alpha$. Then $\lambda(ax + by) = \lambda_\alpha(ax + by) = a\lambda_\alpha(x) + b\lambda_\alpha(y) = a\lambda(x) + b\lambda(y)$ and so λ is linear. Therefore $(Y, \lambda) \in \mathcal{O}$. By definition $\lambda\Big|_{Y_\alpha} = \lambda_\alpha$, so (Y, λ) is an upper bound for $\mathcal{O}$.

Zorn's lemma implies the existence of a maximal element $(Y_0, \lambda) \in \mathcal{O}$. We claim that $Y_0 = X$. Suppose $x_0 \in X \backslash Y_0$. Let $Y' = \operatorname{span}(Y_0, x_0)$. Every $y \in Y'$ can be represented uniquely as $y = y_0 + ax_0$ $(y_0 \in Y_0, a \in \mathbb{R})$. Check: if $y_0 + ax_0 = y_0' + a'x_0$ then $(y_0 - y_0') + (a - a')x_0 = 0$, so $a = a'$ (otherwise $x_0 = \frac{1}{a'-a}(y_0 - y_0') \in Y_0$), so $y_0 = y_0'$. Now, define $\lambda' : Y' \to \mathbb{R}$ by

$$\lambda'(y_0 + ax_0) = \lambda(y_0) + a\lambda'(x_0)$$

which is linear. We need to show that there exists a suitable value for $\lambda'(x_0)$. Suppose $y_1, y_2 \in Y_0$. Then $\lambda'(y_2) - \lambda'(y_1) = \lambda(y_2 - y_1) \le c\|y_2 - y_1\| \le c\|y_2 + x_0\| + c\|y_1 + x_0\|$, so $-c\|y_1 + x_0\| - \lambda'(y_1) \le c\|y_2 + x_0\| - \lambda'(y_2)$. We can choose $\lambda'(x_0)$ such that

$$\sup_{y_1 \in Y_0} (-c\|y_1 + x_0\| - \lambda'(y_1)) \le \lambda'(x_0) \le \inf_{y_2 \in Y_0} (c\|y_2 + x_0\| - \lambda'(y_2))$$

. Now, for $a \geq 0$,

$$-c\|y_0/a + x_0\| - \lambda'(y_0/a) \leq \lambda'(x_0) \leq c\|y_0/a + x_0\| - \lambda'(y_0/a)$$

$$-c\|y_0 + ax_0\| - \lambda'(y_0) \leq a\lambda'(x_0) \leq c\|y_0 + ax_0\| - \lambda'(y_0)$$

$$|\lambda'(y_0 + ax_0)| = |\lambda'(y_0) + a\lambda'(x_0)| \leq c\|y_0 + ax_0\|$$

A similar proof can be done for $y_0 - ax_0$. This means that $(Y', \lambda') \in \mathcal{O}$. But then (Y_0, λ) would not be maximal, so by contradiction we know that $Y_0 = X$. □

6.2 The Complex Version

Theorem 14. *Let $(X, \|\cdot\|)$ be a complex normed vector space, S be a subspace of X, and $\Lambda : S \to \mathbb{C}$ be a linear function with a certain $c \in \mathbb{R}, c > 0$ such that $|\Lambda x| \leq c\|x\|$ for all $x \in S$. Then there exists a linear function $\bar{\Lambda} : X \to \mathbb{C}$ where $\bar{\Lambda}\Big|_S = \Lambda$ and $|\bar{\Lambda}x| \leq c\|x\|$ for all $x \in X$.*

Proof. Define $\lambda : S \to \mathbb{R}$ by $\lambda x = \Re(\Lambda x)$. λ is $\mathbb{R}$-linear and $|\lambda x| \leq c\|x\|$. So the real Hahn-Banach Theorem implies that there exists $\mathbb{R}$-linear function $\bar{\lambda} : X \to \mathbb{R}$ where $\bar{\lambda}\Big|_S = \lambda$ and $|\bar{\lambda}x| \leq c\|x\|$ for all $x \in X$.

Let $\bar{\Lambda}x = \bar{\lambda}x - i\bar{\lambda}(ix)$. From the $\mathbb{R}$-linearity of $\bar{\lambda}$, we know that $\bar{\Lambda}(x+y) = \bar{\Lambda}x + \bar{\Lambda}y$ for $x, y \in X$ and that $\bar{\Lambda}(cx) = a\bar{\Lambda}(x)$ for $a \in \mathbb{R}$. Now, $\bar{\Lambda}(ix) = \bar{\lambda}(ix) - i\bar{\lambda}(i^2x) = \bar{\lambda}(ix) - i\bar{\lambda}(-x) = \bar{\lambda}(ix) + i\bar{\lambda}(x) = i(\bar{\lambda}(x) - i\bar{\lambda}(ix)) = i\bar{\Lambda}(x)$. Therefore $\bar{\Lambda}(ax) = a\bar{\Lambda}(x)$ for $a \in \mathbb{C}$.

Let $s \in S$. Then $\bar{\Lambda}(s) = \bar{\lambda}(s) - i\bar{\lambda}(is) = \lambda(s) - i\lambda(is) = \Re(\Lambda s) - i\Re(\Lambda(is)) = \Re(\Lambda s) - i\Re(i\Lambda s) = \Re(\Lambda s) - i(-\Im(\Lambda s)) = \Lambda s$. So $\bar{\Lambda}\Big|_S = \Lambda$.

Let $x \in X, x \neq 0$ and $a = \text{conj}(\bar{\Lambda}x)/|\bar{\Lambda}x|$. Then $|\bar{\Lambda}x| = a\bar{\Lambda}x = \bar{\Lambda}(ax) = \bar{\lambda}(ax) - i\bar{\lambda}(iax)$. Since $\bar{\lambda}$ is real, $-i\bar{\lambda}(iax)$ is $\Im(|\bar{\Lambda}x|) = 0$. So $(|\bar{\Lambda}x| = \bar{\lambda}(ax) \leq c\|ax\| = c|a|\|x\| = c\|x\|$. □

Corollary 14.1. *Let X be a complex normed vector space and $x_0 \in X$. There exists linear $\Lambda : X \to \mathbb{C}$ with $|\Lambda x| \leq \|x\|$ for all $x \in X$ and $\Lambda x_0 = \|x_0\|$.*

Proof. Let $S = \text{span}(x_0)$ and define $\Lambda : S \to \mathbb{C}$ by $\Lambda(ax_0) = a\|x_0\|$. Then $|\Lambda x| = |\Lambda(ax_0)| = |a\|x_0\|| = \|ax_0\| = \|x\|$. Using the complex HBT with $c = 1$, there exists linear $\bar{\Lambda} : X \to \mathbb{C}$ with $\bar{\Lambda}\Big|_S = \Lambda$ and $|\bar{\Lambda}x| \leq c\|x\| = \|x\|$ for $x \in X$. □

Corollary 14.2. *Let X be a complex normed vector space and $x_0 \in X$. If $\Lambda x_0 = 0$ for every continous linear function $\Lambda : X \to \mathbb{C}$, then $x_0 = 0$.*

7 Some Complex Analysis

Let $\Omega \subseteq \mathbb{C}$ be open and $f : \Omega \to \mathbb{C}$.

Definition 7.1. *We say f is analytic if f is differentiable on Ω. (That is, for all $z \in \Omega$, $f'(z) = \lim\limits_{h \to 0} \frac{f(z+h)-f(z)}{h}$ exists).*

Definition 7.2. *We say f is holomorphic in Ω if it has a power series representation. (That is, for all $z_0 \in \Omega$, there exists a series (c_n) and $r > 0$ such that $f(z) = \sum\limits_{n=0}^{\infty} c_n (z - z_0)^n$ for $|z - z0| < r$.*

One of the main results of complex analysis is that these two concepts are equivalent for single-variable complex functions.

Theorem 15. *If $f \in H(\mathbb{C})$ (f is "entire") and $\lim\limits_{z \to \infty} f(z) = 0$ then $f = 0$.*

Proof. This theorem can be proven from a consequence of the Maximum Modulus Theorem or Liouville's Theorem. □

Let's extend this concept to Banach spaces.

Definition 7.3. *Say X is a Banach space and that $\Omega \in \mathbb{C}$ is open. We say $f \in H(\Omega, X)$ if $f : \Omega \to X$ and for all $z \in \Omega$, $\lim\limits_{h \to 0} \frac{f(z+h)-f(z)}{h}$ exists.*

Corollary 15.1. *Suppose X is a Banach space, $f \in H(\mathbb{C}, X)$, and $\lim\limits_{z \to \infty} f(z) = 0$. Then $f = 0$.*

Proof. Let $\lambda : X \to \mathbb{C}$ be a bounded linear function. We claim that $\lambda \circ f \in H(\mathbb{C})$. Since $f \in H(\mathbb{C}, X)$, $f'(z)$ exists for all z. Now,

$$\left\| \lambda(f'(z)) - \frac{\lambda(f(x+h)) - \lambda(f(z))}{h} \right\|_X = \left\| \lambda \left(f'(z) - \frac{f(x+h) - f(z)}{h} \right) \right\|_X$$
$$\leq \|\lambda\| \left\| f'(z)) - \frac{f(x+h) - f(z)}{h} \right\|_X$$

which $\to 0$ as $h \to 0$. So the "derivative" of $\lambda \circ f$ exists for all z. We know that for $\varepsilon > 0$ there exists A such that $|z| > A$ implies $\|f(z)\|_X < \varepsilon / \|\lambda\|$. So $|\lambda(f(z))| \leq \|\lambda\| \|f(z)\|_X < \varepsilon$. By Theorem 15, $\lambda \circ f = 0$. By Corollary 14.2, $f(z) = 0$. □

8 The Spectrum of x is Non-Empty

Let A be a CBAID throughout. Let $G = \{x \in A : x \text{ is invertible}\}$. Note that G is open ($x \in G$ and $\|y\| < \|x^{-1}\|^{-1}$ implies $x - y \in G$).

Lemma 16. *The map $x \mapsto x^{-1}$ is continuous on G.*

Proof. If $\|y\| < \|x^{-1}\|^{-1}$ then $(x-y)^{-1} = x^{-1}(e-x^{-1}y)^{-1} = x^{-1}\sum(x^{-1}y)^n = x^{-1} + x^{-2}y + x^{-3}y^2 + \ldots$ so $(x-y)^{-1} - x^{-1} = x^{-2}y(e + x^{-1}y + (x^{-1}y)^2 + \ldots)$. Therefore

$$\|(x-y)^{-1} - x^{-1}\| \leq \|x^{-2}\|\|y\|(\|e\| + \|x^{-1}\|\|y\| + (\|x^{-1}\|\|y\|)^2 + \ldots) = \frac{\|x^{-2}\|\|y\|}{1 - \|x^{-1}\|\|y\|}$$

which $\to 0$ as $\|y\| \to 0$. □

Lemma 17. *If $x, x-y, y \in G$ then $(x^{-1} - (x-y)^{-1})y^{-1} = -x^{-1}(x-y)^{-1}$ (analagous to $\frac{1}{x} - \frac{1}{x-y} = \frac{-y}{x(x-y)}$).*

Proof.

$$(x^{-1} - (x-y)^{-1})x(x-y) = (e - x(x-y)^{-1})(x-y) = (x-y) - x = -y$$

□

Corollary 17.1. *If $x \in G$, then (analogous to $\left(\frac{1}{t}\right)' = \frac{-1}{t^2}$),*

$$\lim_{y \to 0, y \in G} (x^{-1} - (x-y)^{-1})y^{-1} = -x^{-2}$$

Lemma 18. *If $\|x\| < 1$ then $\|(e-x)^{-1}\| \leq \frac{1}{1-\|x\|}$.*

Proof.

$$\|(e-x)^{-1}\| = \left\|\sum_{n=0}^{\infty} x^n\right\| \leq \sum_{n=0}^{\infty} \|x\|^n = \frac{1}{1-\|x\|}$$

□

Theorem 19. *If A is a CBAID and $x \in A$ then $\sigma(x) \neq \emptyset$.*

Proof. Suppose $\sigma(x) = \emptyset$. Define $F : \mathbb{C} \to A$ by $F(\lambda) = (x - \lambda e)^{-1}$. Then F is a differentiable and in fact $F'(\lambda) = (x - \lambda e)^{-2}$. Noting that if $h \in \mathbb{C}, h \neq 0$ then $he \in G$, we can show this:

$$\lim_{h \to 0} \frac{F(\lambda + h) - F(\lambda)}{h} = \lim_{h \to 0}((x - (\lambda + h)e)^{-1} - (x - \lambda e)^{-1})(he)^{-1}$$

$$= \lim_{h \to 0}(((x-\lambda e)-he)^{-1}-(x-\lambda e)^{-1})(he)^{-1} = \lim_{y \to 0, y \in G}(((x-\lambda e)-y)^{-1}-(x-\lambda e)^{-1})y^{-1}$$

$$= (x - \lambda e)^{-2}$$

Now, if $\lambda \in \mathbb{C}, |\lambda| > \|x\|$, then $\left\|\frac{x}{\lambda}\right\| < 1$, so by Lemma 18

$$\|F(\lambda)\| = \|(x - \lambda e)^{-1}\| = |\lambda^{-1}| \left\| \left(\frac{x}{\lambda} - e\right)^{-1} \right\| \leq |\lambda^{-1}| \frac{1}{1 - \left\|\frac{x}{\lambda}\right\|}$$

So $\lim_{\lambda \to \infty} F(\lambda) = 0$. Since F is differentiable, Corollary 15.1 implies that $F(\lambda) = 0$ for all λ. But $(x - \lambda e)^{-1} = 0$ is false, therefore $\sigma(x) \neq \emptyset$. □

As a consequence we have the following:

Corollary 19.1. *If A is a CBAID and every non-zero element of A is invertible then $A = \{\lambda e : \lambda \in \mathbb{C}\}$, so $A \cong \mathbb{C}$*

Proof. Let $x \in A$. Since $\sigma(x) \neq \emptyset$, there exists $\lambda \in \mathbb{C}$ so $x - \lambda e$ is not invertible. Hence $x - \lambda e = 0$. □

9 Complex Homomorphisms

Definition 9.1. *A complex homomorphism of A is a linear ring homomorphism $\varphi : A \to \mathbb{C}$ such that $\varphi(e) = 1$. The set of all complex homomorphisms of A is known as the "maximal ideal space" of A and is denoted $\widehat{A}$.*

The following can be called the "Fundamental Theorem" of CBAID.

Theorem 20. *Suppose A is a CBAID and $x \in A$. Then x is not invertible if and only if there exists $\varphi \in \widehat{A}$ such that $\varphi(x) = 0$.*

Proof. [$\Longleftarrow$] If $\varphi(x) = 0$ then for all y, $\varphi(xy) = \varphi(x)\varphi(y) = 0 \neq 1 = \varphi(e)$, so $xy \neq e$.

[$\Longrightarrow$] Suppose x is not invertible. Let I be the ideal generated by x, $I = \{xy : y \in A\}$. If x is not invertible then $e \notin I$ so I is a proper ideal. So $I \subseteq J$, where J is a maximal (proper) ideal. Now we showed in Theorem 11 that J is closed, hence by Lemma 10 A/J is a CBAID. We also know that since J is maximal that A/J is a field: every non-zero element of A/J is invertible. Hence by the previous corollary $A/J \cong \mathbb{C}$. Therefore there exists $\varphi \in \widehat{A}$ with $J = \varphi^{-1}(0)$. This means $\varphi(x) = 0$ since $x \in J$. □

Here is another important fact about complex homomorphisms:

Theorem 21. *If A is a CBAID and $\varphi \in \widehat{A}$ then $|\varphi(x)| \leq \|x\|$ for all $x \in A$.*

Proof. Suppose $\|x\| < 1$. Then $e - x$ is invertible, so $\varphi(e - x) \neq 0$, hence $\varphi(x) \neq 1$. So, $\|x\| < 1 \implies \varphi(x) \neq 1$. This shows that $|\varphi(x)| \leq \|x\|$ for all x, because if $|\varphi(x)| > \|x\|$ then we would have $y = x/\varphi(x)$ with $\|y\| < 1$ and $\varphi(y) = 1$. □

10 Final Proofs

Let

$$A = \left\{ f : f = \sum_{n=-\infty}^{\infty} c_n e^{int}, \sum_{n=-\infty}^{\infty} |c_n| < \infty, c_n \in \mathbb{C} \right\}$$

And define $\|f\|_A = \sum |c_n|$. A is a CBAID. (We know the norm is complete from Theorem 1 and Proposition 2).

If $f \in A$, define $S_n = \sum_{k=-n}^{n} c_k e^{ikt}$. We know $\|f - S_n\|_A \to 0$ as $n \to \infty$ because $\sum_{|k|>n} |c_k| \to 0$. Let $\varphi \in \widehat{A}$. From Theorem 21 we know φ is continuous, so $\varphi(S_n) \to \varphi(f)$ as $n \to \infty$.

Define $e_n = e^{int} \in A$. Note $e_n = (e_1)^n$. Now, $\varphi(e_n) = (\varphi(e_1))^n$. Let $\alpha = \varphi(e_1)$. Then $|\alpha| \leq \|e_1\| = 1$ and $|1/\alpha| = |\varphi(e_{-1})| \leq \|e_{-1}\| = 1$. Therefore $|a| = 1$. Say $\alpha = e^{it}$, some $t \in \mathbb{T}$.

Lemma 22. *If $f \in A$ and $\varphi \in \widehat{A}$, there exists $t_0 \in \mathbb{T}$ such that $\varphi(f) = f(t_0)$*

Proof. There exists $t_0 \in \mathbb{T}$ such that $e^{it_0} = \varphi(e_1)$. Therefore $\varphi(e_n) = e^{int_0}$ for all $n \in \mathbb{Z}$. Suppose $f = \sum_{k=-\infty}^{\infty} c_k e^{ikt} = \sum_{k=-\infty}^{\infty} c_k e_k$. Then $S_n = \sum_{k=-n}^{n} c_k e_k$ and $\varphi(S_n) = \sum_{k=-n}^{n} c_k \varphi(e_k) = \sum_{k=-n}^{n} c_k e^{ikt_0} = S_n(t_0)$. Finally, $\varphi(f) = \lim_{n\to\infty} \varphi(S_n) = f(t_0)$. □

Now, the moment of truth:

Theorem 23. *Say $f \in A$ and $f(t) \neq 0$ for all $t \in \mathbb{T}$. Let $g = \frac{1}{f}$. Then $g \in A$.*

Proof. Say $\varphi \in \widehat{A}$. There exists $t \in \mathbb{T}$ such that $\varphi(f) = f(t) \neq 0$. So $\varphi(f) \neq 0$ for all $\varphi \in \widehat{A}$. Therefore by Theorem 20, f is invertible: there exists $g \in A$ such that $fg = 1$. □

11 Conclusion

We have shown that which was to be demonstrated. This result may have at first seemed surprising, but by now it should seem much more reasonable. We showed that the set of functions which can be represented by Fourier series with the property that the partial sums of their coefficients converge absolutely form a special kind of space called a commutative Banach algebra with identity. We also showed that such spaces are deeply connected to the field of complex numbers. By studying the invertibility of elements of a CBAID and relating them the to the invertibility of complex numbers, we were able to show that every non-zero function in the space has an inverse.