Betti Numbers of Edge Ideals of Cyclic Graphs

Martha Gipson
Department of Mathematics
College of Arts and Sciences
Oklahoma State University

Spring 2015

Contents

1 Introduction

The goal of this paper is to analyze the Betti numbers of the edge ideals of cyclic graphs. We will prove the values of the Betti numbers corresponding to the minimal linear first syzygies and the minimal quadratic first syzygies. We will also conjecture formulas to determine the Betti table for the edge ideal of a cyclic graph on any number of vertices.

2 Background Material

In this section we will briefly cover the mathematical definitions, theorems, and concepts that are used throughout this research paper.

2.1 Cyclic Graphs

A graph with vertex set $\mathcal{V} = \{v_1, \ldots, v_n\}$ and edge set $\mathcal{E} = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1\}$ is a *cyclic graph* C_n.

2.2 Edge Ideals

The *edge ideal* of a cyclic graph on n vertices is $I_{C_n} = (x_ix_j \mid v_iv_j \text{ is an edge in } C_n)$. To be specific, we can say that $I_{C_n} = (x_1x_2, x_2x_3, \ldots, x_{n-1}x_n, x_nx_1)$.

2.3 Free Resolutions

A *free resolution* is a way of encoding the complexity of an algebraic structure. We are particularly interested in the *minimal free resolution* of an ideal. To construct a minimal free resolution of S/I, where I is an ideal, we begin with a set of minimal generators for I. Assuming there are b_1 minimal generators, we place the module S^{b_1}, the direct sum of b_1 copies of S in step one, forming a sequence

$$S^{b_1} \xrightarrow{d_1} S \xrightarrow{\pi} S/I \longrightarrow 0,$$

where π is the usual projection map. The map d_1 has a kernel that is a submodule of R^{b_1}; suppose that kernel has a minimal generating set of d_2 elements. Then we can map R^{b_2} onto this kernel, letting d_2 be the map from R^{b_2} to R^{b_1} via the kernel of d_1. Repeating this process produces a minimal free resolution, which Hilbert's Syzygy Theorem guarantees has a finite number of steps:

$$0 \longrightarrow S^{b_r} \xrightarrow{d_r} \cdots \xrightarrow{d_3} S^{b_2} \xrightarrow{d_2} S^{b_1} \xrightarrow{d_1} S \xrightarrow{\pi} S/I \longrightarrow 0$$

We say that I has b_2 minimal first syzygies, b_3 minimal second syzygies, etc. Syzygies are just relations, S-linear combinations that sum to zero. A syzygy is called *linear* if it imposes a linear relation. For example, $b \cdot a^2 + (-a) \cdot ab = 0$, and thus there is a linear syzygy on the monomials a^2 and ab. The b_i in the minimal free resolution are called the *Betti numbers* of S/I. We have constructed the maps so that at each step, the kernel of d_i is equal to the image of d_{i+1}.

Example 2.1. Let $I = (a^2, ab, b^3)$. The minimal free resolution of S/I is:

$$0 \longrightarrow S^2 \xrightarrow{\begin{pmatrix} -b & 0 \\ a & -b^2 \\ 0 & a \end{pmatrix}} S^3 \xrightarrow{\begin{pmatrix} a^2 & ab & b^3 \end{pmatrix}} S \xrightarrow{\pi} S/I \longrightarrow 0$$

We can refine the information in Example 2.1 by computing the degrees of each of the syzygies. The notation $S(-d)$ indicates a shift in S by degree d. This shift is used so that the maps do not change the degrees of elements, and it also indicates the degrees of syzygies. The relation $b \cdot a^2 + (-a) \cdot ab = 0$ represents a syzygy of degree three since a^2b has degree three. In Example 2.1, suppressing the maps, the minimal graded free resolution looks like this:

$$0 \longrightarrow S(-3) \oplus S(-4) \longrightarrow S(-2)^2 \oplus S(-3) \longrightarrow S \xrightarrow{\pi} S/I \longrightarrow 0$$

One can read off the *graded Betti numbers* now: for example, $b_{1,2} = 2$ because there are two minimal generators of degree two, and $b_{2,4} = 1$ because there is a single minimal syzygy of degree four.

Finally, we can refine the information even further by considering the *multigraded Betti numbers*. The syzygy corresponding to the relation $b \cdot a^2 + (-a) \cdot ab = 0$ has multidegree a^2b, and $b_{2,a^2b} = 1$ for the resolution in Example 2.1.

2.4 Betti Table

In the previous section we found that the graded Betti numbers can be read off the minimal graded free resolution. The *Betti table* displays the graded Betti numbers. The graded Betti number $b_{i,i+j}$ is displayed in column i and row j where the rows and column are numbered starting from zero. For instance, in the minimal graded free resolution from Example 2.1, the corresponding Betti table is as follows:

	0	1	2
total:	1	3	2
0:	1	.	.
1:	.	2	1
2:	.	1	1

2.5 Computing the Number of Minimal First Syzygies

In [BCP], Bayer, Charalambous, and Popescu develop a useful combinatorial tool for computing multigraded Betti numbers of monomial ideals. We will use only a special case of [BCP, Theorem 2.2]. Let m be a monomial, and let I be a monomial ideal. Define $K_m(I)$ to be the simplicial complex, essentially a possibly higher-dimensional graph, formed in the following way. We consider the set of monomials:

$$\left\{t : \frac{m}{t} \in I,\ t \text{ squarefree monomial}\right\}$$

Then we let $K_m(I)$ be the sets of variables corresponding to the elements of this set of monomials.

Example 2.2. Suppose $I = (ab, bc, cd, da)$, and let $m = abcd$. Then the set of monomials described above consists of the monomials 1, a, b, c, d, ab, ad, bc, and cd. Thus $K_m(I)$ is a graph, the 4-cycle with edges $\{a, b\}$, $\{b, c\}$, $\{c, d\}$, and $\{d, a\}$ (coincidentally the same graph for which I is the edge ideal).

We will use the following result of Bayer, Charalambous, and Popescu.

Theorem 2.3. *Let I be a monomial ideal, let K be a simplicial complex, and let $\tilde{H}_0(K)$ be the number of connected components of K minus one. The number of minimal first syzygies of S/I of degree d is equal to*

$$\sum_{\substack{\deg m = d \\ m \text{ monomial}}} \tilde{H}_0(K_m(I)).$$

3 Linear First Syzygies

In this section we will examine the minimal linear first syzygies of a cyclic graph C_n.

Theorem 3.1. *The Betti number corresponding to the minimal linear first syzygies of a cyclic graph C_n is n.*

Proof. Let i, j, k be distinct integers such that $1 \leq i, j, k \leq n$. We note that $\{x_i\} \in K_{x_i,x_j,x_k}(I_{C_n})$ if $x_j x_k \in I_{C_n}$. Throughout this proof all indices are taken mod n.

Case 1: Suppose i, j, k are three consecutive integers. Without loss of generality we may assume $j = i+1$ and $k = i+2$. We first consider

$$\frac{x_i x_j x_k}{x_i} = \frac{x_i x_{i+1} x_{i+2}}{x_i} = x_{i+1} x_{i+2}$$

and we note that $x_{i+1}x_{i+2} \in I_{C_n}$. Next we consider

$$\frac{x_i x_j x_k}{x_j} = \frac{x_i x_{i+1} x_{i+2}}{x_{i+1}} = x_i x_{i+2}$$

and we note that $x_i x_{i+2} \notin I_{C_n}$ unless $n = 3$. Lastly we consider

$$\frac{x_i x_j x_k}{x_k} = \frac{x_i x_{i+1} x_{i+2}}{x_{i+2}} = x_i x_{i+1}$$

and we note that $x_i x_{i+1} \in I_{C_n}$. From this we can see that $\{\emptyset, x_i, x_{i+2}\} = K_{x_i x_{i+1} x_{i+2}}(I_{C_n}) = K_{x_i x_j x_k}(I_{C_n})$. Thus $K_{x_i x_j x_k}(I_{C_n})$ has two connected components and contributes one to the Betti number.

Case 2: Suppose i, j, k are not three consecutive integers. Without loss of generality we may assume $i < j < k$. We note that

$$\frac{x_i x_j x_k}{x_i} = x_j x_k, \frac{x_i x_j x_k}{x_j} = x_i x_k, \text{ and } \frac{x_i x_j x_k}{x_k} = x_i x_j.$$

Subcase 1: Assume that i and j are consecutive integers. Without loss of generality, we may assume $j = i+1$. Since i, j, k are not three consecutive integers, then $k \neq i+2$ and $k \not\equiv i-1 \pmod{n}$. We see that $x_ix_j = x_ix_{i+1}$ and $x_ix_{i+1} \in I_{C_n}$ so $x_k \in K_{x_ix_{i+1}x_k}(I_{C_n})$. However x_ix_k and $x_jx_k = x_{i+1}x_k \notin I_{C_n}$, so $x_j, x_k \notin K_{x_ix_jx_k}(I_{C_n})$. Thus $K_{x_ix_jx_k}(I_{C_n})$ has only one connected component and therefore contributes zero to the Betti number.

Subcase 2: Assume that j and k are consecutive integers. That is, $k = j + 1$. Since i, j, k are not three consecutive integers, then $i \neq j - 1$ and $i \not\equiv k + 1 \pmod{n}$. We see that $x_jx_k = x_jx_{j+1} \in I_{C_n}$. Therefore $x_i \in K_{x_ix_jx_k}(I_{C_n})$. However, $x_ix_j, x_ix_k \notin I_{C_n}$, and therefore $x_j, x_k \notin K_{x_ix_jx_k}(I_{C_n})$. Thus $K_{x_ix_jx_k}(I_{C_n})$ has only one connected component and therefore contributes zero to the Betti number.

Subcase 3: Assume that i and k are consecutive integers. Since we have specified that $i < j < k$, then i and k are consecutive integers if $k \not\equiv i - 1 \pmod{n}$. That is, i and k are consecutive integers if $i = 1$ and $k = n$. Since i, j, k are not three consecutive integers, then $j \neq i + 1, k - 1$. We see that $x_ix_k \in I_{C_n}$ so $x_j \in K_{x_ix_jx_k}(I_{C_n})$. However, $x_ix_j, x_jx_k \notin I_{C_n}$ so $x_i, x_k \notin K_{x_ix_jx_k}(I_{C_n})$. Thus $K_{x_ix_jx_k}(I_{C_n})$ has only one connected component and therefore contributes zero to the Betti number.

Subcase 4: No two of i, j, k are consecutive integers. That is, $j \neq i+1, k-1$ and $k \not\equiv i - 1 \pmod{n}$. Then $x_ix_j, x_jx_k, x_ix_k \notin I_{C_n}$ and so $K_{x_ix_jx_k}(I_{C_n}) = \{\emptyset\}$. Thus $K_{x_ix_jx_k}(I_{C_n})$ has only one connected component and therefore contributes zero to the Betti number.

From these cases we see that $K_{x_ix_jx_k}(I_{C_n})$ contributes to the Betti number only if and only if i, j, k are three consecutive integers. As noted in Case 1, if i, j, k are three consecutive integers, then $K_{x_ix_jx_k}(I_{C_n})$ has two connected components and therefore contributes one to the Betti number. For C_n we have simplicial complexes $K_{x_1x_2x_3}, ..., K_{x_nx_1x_2}$. There are n simplicial complexes, each of which contributes one to the Betti number. We conclude that the Betti number is n. $\square$

4 Quadratic First Syzygies

In this section we will examine the minimal quadratic first syzygies of a cyclic graph C_n.

Theorem 4.1. *The Betti number corresponding to the minimal quadratic first syzygies of a cyclic graph C_n is $\frac{n(n-5)}{2}$ for $n \geq 6$.*

Proof. Let i, j, k, ℓ be distinct integers such that $1 \leq i, j, k, \ell \leq n$. Define $m = x_i x_j x_k x_\ell$. We will denote the simplicial complex with maximal faces $F_1, \ldots, F_r$ by $K_m(I_{C_n}) = \langle F_1, \ldots, F_r \rangle$. We recall that $x_i x_j \in K_m(I_{C_n})$ if and only if $\frac{m}{x_i x_j} \in I_{C_n}$. Throughout this proof all indices are taken mod n.

Case 1: Assume that none of i, j, k, ℓ are consecutive integers. There is no combination of two of x_i, x_j, x_k, x_ℓ by which we can divide m and result in an element of I_{C_n}. Thus $K_m(I_{C_n}) = \{\emptyset\}$. Since $K_m(I_{C_n})$ has only one connected component it contributes zero to the Betti number.

Case 2: Assume that exactly two of i, j, k, ℓ are consecutive. Without loss of generality we may assume that i and j are consecutive integers and that $j = i + 1$. We note that $k, \ell \neq i + 2, i - 1$ and moreover $\ell \neq k - 1, k + 1$. We first consider the pair $x_k x_\ell$. We see that $\frac{m}{x_k x_\ell} = x_i x_{i+1}$ and $x_i x_{i+1} \in I_{C_n}$, so $x_k x_\ell \in K_m(I_{C_n})$. No other combination M of two of x_i, x_{i+1}, x_j, x_k satisfies $\frac{m}{M} \in I_{C_n}$, so there are no other elements in $K_m(I_{C_n})$. Since $K_m(I_{C_n}) = \langle x_k x_\ell \rangle$ has only one connected component it therefore contributes zero to the Betti number.

Case 3: Assume that there are exactly two pairs of two consecutive integers among i, j, k, ℓ. Without loss of generality we may assume that $k = i + 1$ and $\ell = j + 1$. We further restrict j so that $j \neq i + 2, i - 2$. Consider first the pair $x_j x_{j+1}$. We see that $\frac{m}{x_j x_{j+1}} = x_i x_{i+1}$ and $x_i x_{i+1} \in I_{C_n}$, so $x_j x_{j+1} \in K_m(I_{C_n})$. Now consider the pair $x_i x_{i+1}$. We see that $\frac{m}{x_i x_{i+1}} = x_j x_{j+1}$ and $x_j x_{j+1} \in I_{C_n}$, so $x_i x_{i+1} \in K_m(I_{C_n})$. No other combination M of two of x_i, x_{i+1}, x_j, x_{j+1} satisfies $\frac{m}{M} \in I_{C_n}$, so there are no other elements in $K_m(I_{C_n})$. Since $K_m(I_{C_n}) = \langle x_i x_{i+1}, x_j x_{j+1}\rangle$ has two connected components it therefore contributes one to the Betti number.

Case 4: Assume that there are exactly three consecutive integers among i, j, k, ℓ. Without loss of generality we may assume $k = i+1$ and $\ell = i+2$. We restrict j so that $j \neq i+3, i-1$. Consider the pair $x_i x_j$. We see that $\frac{m}{x_i x_j} = x_{i+1}x_{i+2}$ and $x_{i+1}x_{i+2} \in I_{C_n}$, so $x_i x_j \in K_m(I_{C_n})$. Next consider the pair $x_{i+2}x_j$. We see that $\frac{m}{x_{i+2}x_j} = x_i x_{i+1}$ and since $x_i x_{i+1} \in I_{C_n}$ then $x_{i+2}x_j \in K_m(I_{C_n})$. There is no other combination M of two of x_i, x_{i+1}, x_{i+2}, x_j such that $\frac{m}{M} \in I_{C_n}$, so there are no other elements of $K_m(I_{C_n})$. Since $K_m(I_{C_n}) = \langle x_i x_j, x_{i+2}x_j\rangle$ has only one connected component it therefore contributes zero to the Betti number.

Case 5: Assume that i, j, k, ℓ are four consecutive integers. Without loss of generality we may assume that $j = i+1$, $k = i+2$, and $\ell = i+3$. First consider the pair $x_i x_{i+1}$. We see that $\frac{m}{x_i x_{i+1}} = x_{i+2}x_{i+3}$, and since $x_{i+1}x_{i+2} \in I_{C_n}$, then $x_i x_{i+1} \in K_m(I_{C_n})$. Next consider the pair $x_i x_{i+3}$. We see that $\frac{m}{x_i x_{i+3}} = x_{i+1}x_{1+2}$, and since $x_{i+1}x_{1+2} \in I_{C_n}$, then $x_i x_{i+3} \in K_m(I_{C_n})$. Lastly consider the pair $x_{i+2}x_{i+3}$. We see that $\frac{m}{x_{i+2}x_{i+3}} = x_i x_{i+1}$, and since $x_i x_{i+1} \in I_{C_n}$, then $x_{i+2}x_{i+3} \in K_m(I_{C_n})$. No other combinations M of x_i, x_{i+1}, x_{1+2}, x_{i+3} satisfy $\frac{m}{M} \in I_{C_n}$, so there are no other elements of $K_m(I_{C_n})$. Since $K_m(I_{C_n}) = \langle x_i x_{i+1}, x_i x_{1+3}, x_{i+2}x_{1+3}\rangle$ has only one connected component it therefore contributes zero to the Betti number.

From these five cases we see that $K_m(I_{C_n})$ contributes to the Betti number if and only if i, j, k, ℓ are exactly two pairs of two consecutive integers. In that case, $K_m(I_{C_n})$ has two connected components and contributes one to the Betti number. We must determine how many different arrangements x_i, x_{i+1}, x_j, x_{j+1} there are for C_n. Recall that the graph C_n has n vertices $\{v_1, v_2, ..., v_n\}$ and n edges $\{v_1v_2, v_2v_3, ..., v_{n-1}v_n, v_nv_1\}$. For each edge v_iv_{i+1} we exclude the two vertices v_i and v_{i+1} as well as adjacent vertices v_{i-1} and v_{i+2}. In total we exclude four vertices, so we have remaining $n-4$ vertices and $n-5$ edges. Thus there are $n(n-5)$ arrangements of two pair of adjacent vertices that do not have a vertex in common. However since we have counted each pair twice we must divide this number by 2 resulting in $\frac{n(n-5)}{2}$ distinct pairs of adjacent vertices that do not have a vertex in common. Therefore there are $\frac{n(n-5)}{2}$ distinct arrangements of $m = x_i x_{i+1} x_j x_{j+1}$, and thus $\frac{n(n-5)}{2}$ distinct $K_m(I_{C_n})$, each of which contribute one to the Betti number. We conclude that the Betti number is $\frac{n(n-5)}{2}$. □

5 Conjectures

In this section we will develop methods to determine the structure of the Betti table for the edge ideal of a cyclic graph C_n. We begin by writing n as either $3q-1$, $3q$, or $3q+1$ for some $q \in \mathbb{N}$.

Conjecture 5.1. *The Betti table for S/I_{C_n} will have $q+1$ rows labeled 0, 1, 2, ... , q.*

In section 3 we proved that the entry in column 2 of row 1 is n, and in section 4 we proved that the entry in column 2 of row 2 is $\frac{n(n-5)}{2}$. Although we have no formal proofs, we now wish to make conjectures regarding the entries in all but the last row of the Betti table. Our conjectures are based on computational evidence from Macaulay 2 [GS].

Conjecture 5.2. *Let p be an integer with $p < q$. Then row p will have $p+1$ entries. The first entry of row p will be in column p, and the last entry will be in column $2p$. Moreover, for an integer $i = \{0, 1, 2, ..., p\}$ the entry in the $p+i$ column of row p is given by $\binom{p}{i}M(p)$ where*

$$M(p) = \frac{n(n-(2p+1)) \cdot \ . \ . \ . \ \cdot (n-(3p-1))}{p!}$$

Now that we have a conjecture for the entries in any row that is not the last row, we turn our attention to the last row of the Betti table, row q.

Conjecture 5.3. *The structure of row q depends on whether n is written as $3q-1$, $3q$, or $3q+1$.*

Case 1: If $n = 3q-1$ then the entry in the $2q-1$ column of row q is 1 and all other entries are empty.

Case 2: If $n = 3q$ then for an integer $i = 0, 1, ..., q-1$ the entry in the $q+i$ column is $3\binom{q}{i}$ and the entry in the $2q$ column is 2.

Case 3: If $n = 3q+1$ then for an integer $i = 0, 1, ..., q$ the entry in the $q+i$ column is $n\binom{q}{i}$ and the entry in the $2q+1$ column is 1.

Conjecture 5.4. *The regularity reg$(S/I_{C_n}) = q$, and the regularity increases at $n = 3q+2$.*

Conjecture 5.5. *The projective dimension of I_{C_n} depends on whether n is written as $3q-1$, $3q$ or $3q+1$.*

Case: If $n = 3q-1$ then the projective dimension is $2q-1$.

Case 2: If $n = 3q$ then the projective dimension is $2q$.

Case 3: If $n = 3q+1$ then the projective dimension is $2q+1$.

To illustrate these examples we will examine S/I_{C_n}.

Example 5.6. To determine the Betti table for $S/I_{C_{10}}$ we begin by writing $10 = 3(3)+1$. Based on Conjecture 5.1, the Betti table for S/I_{C_n} will have 4 rows, which we will label as row 0, row 1, row 2, and row 3. Row 0 will have a single entry: a 1 in column 0. Row 1 will have two entries: a 10 in column 1 and, based on Theorem 3.1, a 10 in column 2. Conjecture 5.2 tells us that row 2 will have three entries, with the first entry in column 2 and the last entry in column 4. Based on Theorem 4.1, the entry in column 2 will be $\dfrac{10(10-5)}{2} = 25$. We note that from Conjecture 5.2, $M(2) = 25$ as well. Then by Conjecture 5.2 the entry in column 3 is $\binom{2}{1}M(2) = \binom{2}{1}25 = 50$, and the entry in column 4 is $\binom{2}{2}M(2) = \binom{2}{2}25 = 25$. Lastly we will determine the entries of row 3. By Conjecture 5.4, the entries in columns 3, 4, 5, and 6 are, respectively, $10\binom{3}{0} = 10$, $10\binom{3}{1} = 30$, $10\binom{3}{2} = 30$, and $10\binom{3}{3} = 10$. Furthermore, the entry in column 7 is 1. Thus the Betti table for $S/I_{C_{10}}$ is as follows:

	0	1	2	3	4	5	6	7
total:	1	10	35	60	55	30	10	1
0:	1	.	.	.	.	.	.	.
1:	.	1	10	.	.	.	.	.
2:	.	.	25	50	25	.	.	.
3:	.	.	.	10	30	30	10	1

The Betti tables provided in the appendix will confirm that this is indeed the Betti table for S/I_{C_n}.

6 Appendix

Included in this section are the Betti tables for S/I_{C_3} through $S/I_{C_{30}}$.

Betti table for S/I_{C_3} :

	0	1	2
total:	1	3	2
0:	1	.	.
1:	.	3	2

Betti table for S/I_{C_4} :

	0	1	2	3
total:	1	4	4	1
0:	1	.	.	.
1:	.	4	4	1

Betti table for S/I_{C_5} :

	0	1	2	3
total:	1	5	5	1
0:	1	.	.	.
1:	.	5	5	.
2:	.	.	.	1

Betti table for S/I_{C_6} :

	0	1	2	3	4
total:	1	6	9	6	2
0:	1	.	.	.	.
1:	.	6	6	.	.
2:	.	.	3	6	2

Betti table for S/I_{C_7} :

	0	1	2	3	4	5
total:	1	7	14	14	7	1
0:	1	.	.	.	.	.
1:	.	7	7	.	.	.
2:	.	.	7	14	7	1

Betti table for S/I_{C_8} :

	0	1	2	3	4	5
total:	1	8	20	24	12	1
0:	1	.	.	.	.	.
1:	.	8	8	.	.	.
2:	.	.	12	24	12	.
3:	.	.	.	.	.	1

Betti table for S/I_{C_9} :

	0	1	2	3	4	5	6
total:	1	9	27	39	27	9	2
0:	1	.	.	.	.	.	.
1:	.	9	9	.	.	.	.
2:	.	.	18	36	18	.	.
3:	.	.	.	3	9	9	2

Betti table for $S/I_{C_{10}}$:

	0	1	2	3	4	5	6	7
total:	1	10	35	60	55	30	10	1
0:	1	.	.	.	.	.	.	.
1:	.	10	10	.	.	.	.	.
2:	.	.	25	50	25	.	.	.
3:	.	.	.	10	30	30	10	1

Betti table for $S/I_{C_{11}}$:

	0	1	2	3	4	5	6	7
total:	1	11	44	88	99	66	22	1
0:	1	.	.	.	.	.	.	.
1:	.	11	11	.	.	.	.	.
2:	.	.	33	66	33	.	.	.
3:	.	.	.	22	66	66	22	.
4:	.	.	.	.	.	.	.	1

Betti table for $S/I_{C_{12}}$:

	0	1	2	3	4	5	6	7	8
total:	1	12	54	124	165	132	58	12	2
0:	1	.	.	.	.	.	.	.	.
1:	.	12	12	.	.	.	.	.	.
2:	.	.	42	84	42	.	.	.	.
3:	.	.	.	40	120	120	40	.	.
4:	.	.	.	.	3	12	18	12	2

Betti table for $S/I_{C_{13}}$:

	0	1	2	3	4	5	6	7	8	9
total:	1	13	65	169	260	247	143	52	13	1
0:	1	.	.	.	.	.	.	.	.	.
1:	.	13	13	.	.	.	.	.	.	.
2:	.	.	52	104	52	.	.	.	.	.
3:	.	.	.	65	195	195	65	.	.	.
4:	.	.	.	.	13	52	78	52	13	1

Betti table for $S/I_{C_{14}}$:

	0	1	2	3	4	5	6	7	8	9
total:	1	14	77	224	392	434	308	140	35	1
0:	1	.	.	.	.	.	.	.	.	.
1:	.	14	14	.	.	.	.	.	.	.
2:	.	.	63	126	63	.	.	.	.	.
3:	.	.	.	98	294	294	98	.	.	.
4:	.	.	.	.	35	140	210	140	35	.
5:	.	.	.	.	.	.	.	.	.	1

Betti table for $S/I_{C_{15}}$:

	0	1	2	3	4	5	6	7	8	9	10
total:	1	15	90	290	570	723	605	330	105	15	2
0:	1	.	.	.	.	.	.	.	.	.	.
1:	.	15	15	.	.	.	.	.	.	.	.
2:	.	.	75	150	75	.	.	.	.	.	.
3:	.	.	.	140	420	420	140	.	.	.	.
4:	.	.	.	.	75	300	450	300	75	.	.
5:	.	.	.	.	.	3	15	30	30	15	2

Betti table for $S/I_{C_{16}}$:

	0	1	2	3	4	5	6	7	8	9	10	11
total:	1	16	104	368	804	1152	1112	720	300	80	16	1
0:	1	.	.	.	.	.	.	.	.	.	.	.
1:	.	16	16	.	.	.	.	.	.	.	.	.
2:	.	.	88	176	88	.	.	.	.	.	.	.
3:	.	.	.	192	576	576	192	.	.	.	.	.
4:	.	.	.	.	140	560	840	560	140	.	.	.
5:	.	.	.	.	.	16	80	160	160	80	16	1

Betti table for $S/I_{C_{17}}$:

	0	1	2	3	4	5	6	7	8	9	10	11
total:	1	17	119	459	1105	1768	1938	1462	748	255	51	1
0:	1	.	.	.	.	.	.	.	.	.	.	.
1:	.	17	17	.	.	.	.	.	.	.	.	.
2:	.	.	102	204	102	.	.	.	.	.	.	.
3:	.	.	.	255	765	765	255	.	.	.	.	.
4:	.	.	.	.	238	952	1428	952	238	.	.	.
5:	.	.	.	.	.	51	255	510	510	255	51	.
6:	.	.	.	.	.	.	.	.	.	.	.	1

Betti table for $S/I_{C_{18}}$:

	0	1	2	3	4	5	6	7	8	9	10	11	12
total:	1	18	135	564	1485	2628	3231	2790	1683	690	171	18	2
0:	1	.	.	.	.	.	.	.	.	.	.	.	.
1:	.	18	18	.	.	.	.	.	.	.	.	.	.
2:	.	.	117	234	117	.	.	.	.	.	.	.	.
3:	.	.	.	330	990	990	330	.	.	.	.	.	.
4:	.	.	.	.	378	1512	2268	1512	378	.	.	.	.
5:	.	.	.	.	.	126	630	1260	1260	630	126	.	.
6:	.	.	.	.	.	.	3	18	45	60	45	18	2

Betti table for $S/I_{C_{19}}$:

	0	1	2	3	4	5	6	7	8	9	10	11	12	13
total:	1	19	152	684	1957	3800	5187	5054	3515	1710	551	114	19	1
0:	1	.	.	.	.	.	.	.	.	.	.	.	.	.
1:	.	19	19	.	.	.	.	.	.	.	.	.	.	.
2:	.	.	133	266	133	.	.	.	.	.	.	.	.	.
3:	.	.	.	418	1254	1254	418	.	.	.	.	.	.	.
4:	.	.	.	.	570	2280	3420	2280	570	.	.	.	.	.
5:	.	.	.	.	.	266	1330	2660	2660	1330	266	.	.	.
6:	.	.	.	.	.	.	19	114	285	380	285	114	19	1

Betti table for $S/I_{C_{20}}$:

	0	1	2	3	4	5	6	7	8	9	10	11	12	13
total:	1	20	170	820	2535	5364	8060	8760	6915	3920	1554	420	70	1
0:	1	.	.	.	.	.	.	.	.	.	.	.	.	.
1:	.	20	20	.	.	.	.	.	.	.	.	.	.	.
2:	.	.	150	300	150	.	.	.	.	.	.	.	.	.
3:	.	.	.	520	1560	1560	520	.	.	.	.	.	.	.
4:	.	.	.	.	825	3300	4950	3300	825	.	.	.	.	.
5:	.	.	.	.	.	504	2520	5040	5040	2520	504	.	.	.
6:	.	.	.	.	.	.	70	420	1050	1400	1050	420	70	.
7:	.	.	.	.	.	.	.	.	.	.	.	.	.	1

Betti table for $S/I_{C_{21}}$:

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
total:	1	21	189	973	3234	7413	12173	14619	12936	8393	3927	1281	259	21	2
0:	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1:	.	21	21	.	.	.	.	.	.	.	.	.	.	.	.
2:	.	.	168	336	168	.	.	.	.	.	.	.	.	.	.
3:	.	.	.	637	1911	1911	637	.	.	.	.	.	.	.	.
4:	.	.	.	.	1155	4620	6930	4620	1155	.	.	.	.	.	.
5:	.	.	.	.	.	882	4410	8820	8820	4410	882	.	.	.	.
6:	.	.	.	.	.	.	196	1176	2940	3920	2940	1176	196	.	.
7:	.	.	.	.	.	.	.	3	21	63	105	105	63	21	2

Betti table for $S/I_{C_{22}}$:

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
total:	1	22	209	1144	4070	10054	17930	23606	23177	16962	9152	3542	924	154	22	1
0:	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1:	.	22	22	.	.	.	.	.	.	.	.	.	.	.	.	.
2:	.	.	187	374	187	.	.	.	.	.	.	.	.	.	.	.
3:	.	.	.	770	2310	2310	770	.	.	.	.	.	.	.	.	.
4:	.	.	.	.	1573	6292	9438	6292	1573	.	.	.	.	.	.	.
5:	.	.	.	.	.	1452	7260	14520	14520	7260	1452	.	.	.	.	.
6:	.	.	.	.	.	.	462	2772	6930	9240	6930	2772	462	.	.	.
7:	.	.	.	.	.	.	.	22	154	462	770	770	462	154	22	1

Betti table for $S/I_{C_{23}}$:

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
total:	1	23	230	1334	5060	13409	25829	37030	39997	32637	19987	9016	2898	644	92	1
0:	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1:	.	23	23	.	.	.	.	.	.	.	.	.	.	.	.	.
2:	.	.	207	414	207	.	.	.	.	.	.	.	.	.	.	.
3:	.	.	.	920	2760	2760	920	.	.	.	.	.	.	.	.	.
4:	.	.	.	.	2093	8372	12558	8372	2093	.	.	.	.	.	.	.
5:	.	.	.	.	.	2277	11385	22770	22770	11385	2277	.	.	.	.	.
6:	.	.	.	.	.	.	966	5796	14490	19320	14490	5796	966	.	.	.
7:	.	.	.	.	.	.	.	92	644	1932	3220	3220	1932	644	92	.
8:	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1

Betti table for $S/I_{C_{24}}$:

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
total:	1	24	252	1544	6222	17616	36476	56616	66789	60192	41316	21336	8106	2184	372	24	2
0:	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1:	.	24	24	.	.	.	.	.	.	.	.	.	.	.	.	.	.
2:	.	.	228	456	228	.	.	.	.	.	.	.	.	.	.	.	.
3:	.	.	.	1088	3264	3264	1088	.	.	.	.	.	.	.	.	.	.
4:	.	.	.	.	2730	10920	16380	10920	2730	.	.	.	.	.	.	.	.
5:	.	.	.	.	.	3432	17160	34320	34320	17160	3432	.	.	.	.	.	.
6:	.	.	.	.	.	.	1848	11088	27720	36960	27720	11088	1848	.	.	.	.
7:	.	.	.	.	.	.	.	288	2016	6048	10080	10080	6048	2016	288	.	.
8:	.	.	.	.	.	.	.	.	3	24	84	168	210	168	84	24	2

Betti table for $S/I_{C_{25}}$:

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
total:	1	25	275	1775	7575	22830	50600	84600	108325	106975	81455	47450	20800	6650	1450	200	25	1
0:	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1:	.	25	25	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
2:	.	.	250	500	250	.	.	.	.	.	.	.	.	.	.	.	.	.
3:	.	.	.	1275	3825	3825	1275	.	.	.	.	.	.	.	.	.	.	.
4:	.	.	.	.	3500	14000	21000	14000	3500	.	.	.	.	.	.	.	.	.
5:	.	.	.	.	.	5005	25025	50050	50050	25025	5005	.	.	.	.	.	.	.
6:	.	.	.	.	.	.	3300	19800	49500	66000	49500	19800	3300	.	.	.	.	.
7:	.	.	.	.	.	.	.	750	5250	15750	26250	26250	15750	5250	750	.	.	.
8:	.	.	.	.	.	.	.	.	25	200	700	1400	1750	1400	700	200	25	1

Betti table for $S/I_{C_{26}}$:

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
total:	1	26	299	2028	9139	29224	69069	123838	171184	184002	154089	100074	49803	18564	4992	936	117	1
0:	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1:	.	26	26	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
2:	.	.	273	546	273	.	.	.	.	.	.	.	.	.	.	.	.	.
3:	.	.	.	1482	4446	4446	1482	.	.	.	.	.	.	.	.	.	.	.
4:	.	.	.	.	4420	17680	26520	17680	4420	.	.	.	.	.	.	.	.	.
5:	.	.	.	.	.	7098	35490	70980	70980	35490	7098	.	.	.	.	.	.	.
6:	.	.	.	.	.	.	5577	33462	83655	111540	83655	33462	5577	.	.	.	.	.
7:	.	.	.	.	.	.	.	1716	12012	36036	60060	60060	36036	12012	1716	.	.	.
8:	.	.	.	.	.	.	.	.	117	936	3276	6552	8190	6552	3276	936	117	.
9:	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1

Betti table for $S/I_{C_{27}}$:

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
total:	1	27	324	2304	10935	36990	92907	177930	264276	307407	281070	201582	112455	48006	15282	3492	513	27	2
0:	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1:	.	27	27	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
2:	.	.	297	594	297	.	.	.	.	.	.	.	.	.	.	.	.	.	.
3:	.	.	.	1710	5130	5130	1710	.	.	.	.	.	.	.	.	.	.	.	.
4:	.	.	.	.	5508	22032	33048	22032	5508	.	.	.	.	.	.	.	.	.	.
5:	.	.	.	.	.	9828	49140	98280	98280	49140	9828	.	.	.	.	.	.	.	.
6:	.	.	.	.	.	.	9009	54054	135135	180180	135135	54054	9009	.	.	.	.	.	.
7:	.	.	.	.	.	.	.	3564	24948	74844	124740	124740	74844	24948	3564	.	.	.	.
8:	.	.	.	.	.	.	.	.	405	3240	11340	22680	28350	22680	11340	3240	405	.	.
9:	.	.	.	.	.	.	.	.	.	3	27	108	252	378	378	252	108	27	2

Betti table for $S/I_{C_{28}}$:

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
total:	1	28	350	2604	12985	46340	123312	251360	399476	500332	496370	390012	241360	116256	42732	11592	2163	252	28	1
0:	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1:	.	28	28	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
2:	.	.	322	644	322	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
3:	.	.	.	1960	5880	5880	1960	.	.	.	.	.	.	.	.	.	.	.	.	.
4:	.	.	.	.	6783	27132	40698	27132	6783	.	.	.	.	.	.	.	.	.	.	.
5:	.	.	.	.	.	13328	66640	133280	133280	66640	13328	.	.	.	.	.	.	.	.	.
6:	.	.	.	.	.	.	14014	84084	210210	280280	210210	84084	14014	.	.	.	.	.	.	.
7:	.	.	.	.	.	.	.	6864	48048	144144	240240	240240	144144	48048	6864	.	.	.	.	.
8:	.	.	.	.	.	.	.	.	1155	9240	32340	64680	80850	64680	32340	9240	1155	.	.	.
9:	.	.	.	.	.	.	.	.	.	28	252	1008	2352	3528	3528	2352	1008	252	28	1

Betti table for $S/I_{C_{29}}$:

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
total:	1	29	377	2929	15312	57507	161675	349653	592383	795354	851556	728103	495523	266133	111099	35148	8091	1305	145	1
0:	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1:	.	29	29	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
2:	.	.	348	696	348	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
3:	.	.	.	2233	6699	6699	2233	.	.	.	.	.	.	.	.	.	.	.	.	.
4:	.	.	.	.	8265	33060	49590	33060	8265	.	.	.	.	.	.	.	.	.	.	.
5:	.	.	.	.	.	17748	88740	177480	177480	88740	17748	.	.	.	.	.	.	.	.	.
6:	.	.	.	.	.	.	21112	126672	316680	422240	316680	126672	21112	.	.	.	.	.	.	.
7:	.	.	.	.	.	.	.	12441	87087	261261	435435	435435	261261	87087	12441	.	.	.	.	.
8:	.	.	.	.	.	.	.	.	2871	22968	80388	160776	200970	160776	80388	22968	2871	.	.	.
9:	.	.	.	.	.	.	.	.	.	145	1305	5220	12180	18270	18270	12180	5220	1305	145	.
10:	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1

Betti table for $S/I_{C_{30}}$:

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
total:	1	30	405	3280	17940	70746	209600	479550	863220	1237560	1423239	1316580	978175	580170	271560	98436	26865	5310	685	30	2
0:	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1:	.	30	30	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
2:	.	.	375	750	375	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
3:	.	.	.	2530	7590	7590	2530	.	.	.	.	.	.	.	.	.	.	.	.	.	.
4:	.	.	.	.	9975	39900	59850	39900	9975	.	.	.	.	.	.	.	.	.	.	.	.
5:	.	.	.	.	.	23256	116280	232560	232560	116280	23256	.	.	.	.	.	.	.	.	.	.
6:	.	.	.	.	.	.	30940	185640	464100	618800	464100	185640	30940	.	.	.	.	.	.	.	.
7:	.	.	.	.	.	.	.	21450	150150	450450	750750	750750	450450	150150	21450	.	.	.	.	.	.
8:	.	.	.	.	.	.	.	.	6435	51480	180180	360360	450450	360360	180180	51480	6435	.	.	.	.
9:	.	.	.	.	.	.	.	.	.	550	4950	19800	46200	69300	69300	46200	19800	4950	550	.	.
10:	.	.	.	.	.	.	.	.	.	.	3	30	135	360	630	756	630	360	135	30	2

7 Acknowledgements

I would to thank Dr. Christopher Francisco for serving as my thesis director. Creating this Honors thesis has been a thouroughly rewarding and enjoyable experience, due greatly to Dr. Francisco's invaluable support. Without his guidance and encouragment this research would not have been possible.

I would like to thank Dr. Jay Schweig for serving as my second reader. I'm deeply grateful that he lent his expertise to my research.

Lastly, I would like to thank all the faculty from the Department of Mathematics from whom I've had the privilege of taking classes. Through them I have built my understanding of mathematics and strengthened my love of this discipline.

References

[BCP] D. Bayer, H. Charalambous, and S. Popescu, Extremal Betti numbers and applications to monomial ideals. *J. Algebra* **221** (1999), no. 2, 497–512.

[GS] D. R. Grayson and M. E. Stillman, Macaulay 2, a software system for research in algebraic geometry. `http://www.math.uiuc.edu/Macaulay2/`.