PATTERN AVOIDANCE CRITERIA FOR FIBER BUNDLES ON SCHUBERT VARIETIES

A thesis presented

by

Timothy R. Alland

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Approved by:

Advisor: Dr. Edward Richmond
Reader: Dr. Roger Zierau

Abstract

We show a permutation pattern avoidance criteria for when the natural projection from a flag variety to the Grassmannian, restricted to a Schubert variety, is a fiber bundle structure. We define the concept of split patterns and show that this projection is a fiber bundle if and only if the corresponding permutation avoids the split patterns $3|12$ and $23|1$. Continuing, we show that a Schubert variety has an iterated fiber bundle structure of Schubert varieties in the Grassmannian if and only if the corresponding permutation avoids the patterns 3412, 52341, and 635241. This extends the findings of Lakshmibai-Sandhya, Ryan, and Wolper, who's combined results show that a Schubert variety has such an iterated fiber bundle structure if it is smooth i.e. the corresponding permutation avoids the patterns 3412 and 4231.

Acknowledgements

I would like to first thank my advisor, Dr. Edward Richmond, for bringing me into this research and allowing me to take part in what has been an incredible introduction to mathematical research. I would like to thank him as well for his guidance throughout the preparation of this thesis and in the rest of our work together. I would like to express thanks to my second reader, Dr. Roger Zierau, for his time and for helping me find new ways to connect Schubert varieties with the material in our class. I would also like to thank my academic advisor, Dr. Lisa Mantini, for all of her support and guidance over the course of my undergraduate career in mathematics. And I would especially like to thank my parents, Mark and Mary Jane, for their endless love, support, and encouragement.

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1. Introduction

This thesis seeks to convey to an audience of readers with little to no experience in combinatorics, algebraic geometry, or indeed any level of mathematical knowledge beyond that which is expected of an undergraduate mathematics student in his or her third or fourth year of studies, the results which were produced by Dr. Edward Richmond and myself during the spring and summer of 2016. After a brief background, we will begin by defining and giving examples of various key terms which are necessary for any meaningful understanding of our research.

1.1. Background.

The study of Schubert varieties arose from the study of projective geometry in the 19th century. The study of Schubert calculus (of which Schubert varieties are a part) was also advanced in part due to Hilbert's 15th problem of his famous list of 23 problems published in 1900. More modern study of Schubert varieties builds upon the studies of mathematicians such as Ehresmann, Chevalley, Bernstein, Gelfand, Gelfand, and Demazure[BL00]. Often what is studied and considered significant are Schubert varieties' smoothness and singularities about which there is much active research today. One peculiarity about Schubert varieties is their intersection between apparently disparate fields of mathematics, including geometry, combinatorics, and representation theory. As we will do later, one can show a geometric property for a Schubert variety based on its combinatorial characteristics.

The use of pattern avoidance as a means for describing the geometry of Schubert varieties was pioneered by Lakshmibai and Sandhya who proved that a Schubert variety X_w is smooth if and only if w avoids 3412 and 4231 [LS90]. Pattern avoidance has also been used to describe many other properties of Schubert varieties such as when Schubert varieties are Gorenstein, factorial, defined by inclusions, have small resolutions, and are local complete intersections [BW03, Deo90, GR02, BMB07, WY06, UW13]. For a survey of these results, see [AB].

1.2. Flags, Grassmannians, and Fiber Bundles.

We now begin by letting $\mathbb{K}$ be an algebraically closed field and we define

$$\mathrm{F}\ell(n) := \{V_\bullet = (V_1 \subset V_2 \subset \cdots \subset V_{n-1} \subset \mathbb{K}^n) \mid \dim(V_i) = i\}$$

to be the **complete flag variety** on $\mathbb{K}^n$. In other words, let $\mathrm{F}\ell(n)$ be the collection of all complete flags in $\mathbb{K}^n$ which are themselves collections of nested subspaces of $\mathbb{K}^n$. A flag is considered **complete** if it contains an i-dimensional subspace for each $i = 1, \cdots, n-1$. A flag that is not complete is called a partial flag and is "missing" one or more i-dimensional vector spaces. A **partial flag variety** on $\mathbb{K}^n$ is defined by any subset $\mathbf{a} := \{a_1 < \cdots < a_k\} \subseteq \{1, 2, \cdots, n-1\}$.

Example 1.1. *The complete flag variety for $n = 4$ is*

$$\mathrm{F}\ell(4) := \{V_\bullet = (V_1 \subset V_2 \subset V_3 \subset \mathbb{K}^4) \mid \dim(V_i) = i\}.$$

Define $\boldsymbol{a} := \{1 < 3\} \subseteq \{1, 2, 3\}$. The partial flag variety given by

$$\mathrm{F}\ell(\{1,3\}, 4) := \{V_\bullet^{\{1,3\}} = (V_1 \subset V_3 \subset \mathbb{K}^4) \mid \dim(V_i) = i\}.$$

Now, for each $r \in \{1, \ldots, n-1\}$, the **Grassmannian** of r-dimensional subspaces of $\mathbb{K}^n$ is defined by

$$\mathrm{Gr}(r, n) := \{V \subset \mathbb{K}^n \mid \dim(V) = r\}.$$

For example,

$$\mathrm{Gr}(2,4) := \{V \subset \mathbb{K}^4 \mid \dim(V) = 2\}$$

(i.e. is the set of 2-dimensional subspaces in $\mathbb{K}^4$.) The reader may note as well that $\mathrm{Gr}(2,4) = \mathrm{F}\ell(\{2\},4)$.

Now, there is a natural projection map

$$\pi_r : \mathrm{F}\ell(n) \twoheadrightarrow \mathrm{Gr}(r,n) \tag{1}$$

given by $\pi_r(V_\bullet) = V_r$. One can see that this projection takes a collection of $n-1$ subspaces and forgets about all except the one of a specified dimension. This function is an example of a **fiber bundle** on $\mathrm{F}\ell(n)$.

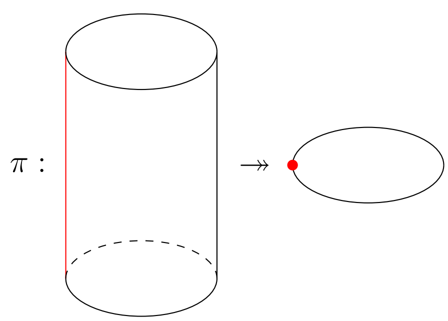
To better understand the notion of a function being a fiber bundle, we must begin with what it means to be a fiber.

Definition 1.2. *Let $\pi : A \to B$ be a function and let $b \in B$. Then the **fiber** over b is the set $\pi^{-1}(b) := \{a \in A \mid \pi(a) = b\}$.*

Definition 1.3. *Let $\pi : A \to B$ be a function. We say that π is a **fiber bundle** if for every $b_1, b_2 \in B$, the fibers over b_1 and b_2 are isomorphic and Zariski-locally trivial.*

By locally trivial, we mean to say that for any $b \in B$, there is a neighborhood, U, around b such that $\pi^{-1}(U) \simeq U \times F$ where F is the fiber over b. We say Zariski-locally trivial because we are working inside the Zariski topology. Since the type of topology considered does not come up in our proofs, we will not go over the details of the Zariski topology as they are beyond the scope of this paper.

Example 1.4. *Let A be the surface of a cylinder of a fixed radius that extends to infinity in both directions and B be a circle of the same radius. Then the projection $\pi : A \twoheadrightarrow B$ seen below where the fiber over each point on the circle is a line on the cylinder is a fiber bundle.*



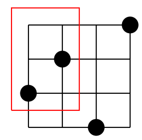
Let $V \in \mathrm{Gr}(r,n)$ be an r-dimensional subspace. In the previously introduced projection, π_r, the fiber over V is $\pi_r^{-1}(V)$, i.e. the set of elements in $\mathrm{F}\ell(n)$ that map to V. This projection is a fiber bundle because for any $V \in \mathrm{Gr}(r,n)$, the fiber over V is isomorphic to $\mathrm{F}\ell(r) \times \mathrm{F}\ell(n-r)$ and is locally trivial.

A main goal of this thesis is to show using pattern avoidance when the projection π_r, restricted to a Schubert variety of $\mathrm{F}\ell(n)$, remains a fiber bundle. Now, if we fix a basis $\{e_1, \ldots, e_n\}$ of $\mathbb{K}^n$ and let $E_i := \mathrm{span}\langle e_1, \ldots, e_i\rangle$, then a permutation $w = w(1)\cdots w(n) \in S_n$ defines the **Schubert variety**

$$X_w := \{V_\bullet \in \mathrm{F}\ell(n) \mid \dim(E_i \cap V_j) \geq r_w[i,j]\}$$

where $r_w[i,j] := \#\{k \leq j \mid w(k) \leq i\}$. This is to say that the Schubert variety X_w is equal to the set of flags in $\mathrm{F}\ell(n)$ that meet the intersection conditions mandated by the rank of the upper-left sub-matrices of the permutation w.

Example 1.5. *For example, take $w = w(1)w(2)w(3)w(4) = 3241$. Instead of expressing w as a matrix of 1's and 0's, throughout we will draw an array where the 1's are dots and placed in accordance with the ordered pairs $(w(i), i)$ and with $(1,1)$ marking the upper-left corner, similar to that of the coordinates of a matrix.*

$w = 3241 =$ 

Here, the upper-left 3×2 sub-matrix is marked by the red box. This outline shows that $r_w[3,2] = 2$ which implies $\dim(E_3 \cap V_2) \geq 2$. Since $\dim(V_2) = 2$, we have that $V_2 \subset E_3$. This turns out to be the only meaningful intersection condition for the Schubert variety X_{3241}. Hence,

$$X_{3241} = \{V_\bullet = (V_1 \subset V_2 \subset V_3 \subset \mathbb{K}^4) \mid V_2 \subset E_3\}.$$

We now give our first main theorem, a pattern avoidance criteria that determines whether the projection π_r, restricted to a Schubert variety, is a fiber bundle.

Theorem 1.6. *Let $r < n$ and $w \in S_n$. The projection π_r restricted to X_w is a Zariski-locally trivial fiber bundle if and only if w avoids the split patterns $3|12$ and $23|1$ with respect to r.*

Precise definitions of permutations and classical and split pattern avoidance will be covered in Section 2.1.

1.3. **Iterated Fiber Bundle Structures.** We will now introduce another natural projection similar to the map $\pi_r : \mathrm{F}\ell(n) \twoheadrightarrow \mathrm{Gr}(r,n)$. If $\mathbf{b} \subseteq \mathbf{a} \subseteq [n-1]$ are two subsequences, then the natural projection

$$\pi^{\mathbf{a}}_{\mathbf{b}} : \mathrm{F}\ell(\mathbf{a}, n) \twoheadrightarrow \mathrm{F}\ell(\mathbf{b}, n)$$

given by $\pi^{\mathbf{a}}_{\mathbf{b}}(V^{\mathbf{a}}_\bullet) = V^{\mathbf{b}}_\bullet$ is a fiber bundle where if $V \in V^{\mathbf{b}}_\bullet$, then $V \in V^{\mathbf{a}}_\bullet$. Note that $\pi^{\mathbf{a}}_{\mathbf{b}}$ is a more general form of the projection π_r and when $\mathbf{a} = [n-1]$ and $\mathbf{b} = \{r\}$, $\pi^{\mathbf{a}}_{\mathbf{b}} = \pi_r : \mathrm{F}\ell(n) \twoheadrightarrow \mathrm{Gr}(r,n)$.

Now, let $\sigma = \sigma(1) \cdots \sigma(n-1) \in S_n$. We may use σ to define a collection of nested subsets

$$\sigma_1 \subset \sigma_2 \subset \cdots \subset \sigma_{n-2} \subset \sigma_{n-1} = [n-1] \quad \text{where} \quad \sigma_i := \{\sigma(1), \ldots, \sigma(i)\}.$$

Each map $\pi^{\sigma_i}_{\sigma_{i-1}}$ is a fiber bundle and together forms an **iterated fiber bundle structure** on the complete flag variety given by

$$\mathrm{F}\ell(n) \overset{\pi^{[n-1]}_{\sigma_{n-2}}}{\twoheadrightarrow} \mathrm{F}\ell(\sigma_{n-2}, n) \overset{\pi^{\sigma_{n-2}}_{\sigma_{n-3}}}{\twoheadrightarrow} \cdots \overset{\pi^{\sigma_3}_{\sigma_2}}{\twoheadrightarrow} \mathrm{F}\ell(\sigma_2, n) \overset{\pi^{\sigma_2}_{\sigma_1}}{\twoheadrightarrow} \mathrm{F}\ell(\sigma_1, n) \twoheadrightarrow pt \tag{2}$$

where the fibers over each map are isomorphic to Grassmannians.

Now, just like how we were interested in when the projection π_r restricted to a Schubert variety had a fiber bundle structure, we want to know the maps $\pi^{\sigma_i}_{\sigma_{i-1}}$ restricted to a Schubert variety also give an iterated fiber bundle structure.

Definition 1.7. *Let $w \in S_n$. We say X_w has a* ***complete parabolic bundle structure*** *if there is a permutation $\sigma \in S_{n-1}$ such that the maps $\pi^{\sigma_i}_{\sigma_{i-1}}$ give an iterated fiber bundle structure on the Schubert variety*

$$X_w = X_{n-1} \overset{\pi^{[n-1]}_{\sigma_{n-2}}}{\twoheadrightarrow} X_{n-2} \overset{\pi^{\sigma_{n-2}}_{\sigma_{n-3}}}{\twoheadrightarrow} \cdots \overset{\pi^{\sigma_3}_{\sigma_2}}{\twoheadrightarrow} X_2 \overset{\pi^{\sigma_2}_{\sigma_1}}{\twoheadrightarrow} X_1 \twoheadrightarrow pt \tag{3}$$

where $X_i := \pi^{[n-1]}_{\sigma_i}(X_w) \subseteq \mathrm{F}\ell(\sigma_i, n)$. In other words, each map

$$\pi^{\sigma_i}_{\sigma_{i-1}} : X_i \twoheadrightarrow X_{i-1}$$

is a Zariski-locally trivial fiber bundle.

As before, there are some Schubert varieties that have complete parabolic bundle structures and some that do not. The smallest example for which no σ induces a complete parabolic bundle structure is X_{3421} (see Example 4.3). Ryan showed that if $\mathbb{K} = \mathbb{C}$, then any smooth Schubert variety has a complete parabolic bundle structure [Rya87]. Wolper later generalized this result for any algebraically closed field [Wol89]. Combined with the Lakshmibai-Sandhya smoothness criteria, we get the following theorem.

Theorem 1.8. ([Rya87, Wol89, LS90]) *If w avoids patterns* 3412 *and* 4231*, then X_w has a complete parabolic bundle structure.*

However, the converse of this statement does not hold true. For example, the permutation $\sigma = 213$ induces a complete parabolic bundle structure on X_{4231} (see Example 4.2. By employing Theorem 1.6, we are able to produce a stronger pattern avoidance characterization of Schubert varieties with complete parabolic bundle structure.

Theorem 1.9. *The permutation w avoids patterns* 3412*,* 52341 *and* 635241 *if and only if the Schubert variety X_w has a complete parabolic bundle structure.*

The key notion used to prove both Theorems 1.6 and 1.9 is that of a Billey-Postnikov (BP) decomposition whose definition is given in Proposition 2.10.

2. Preliminaries

2.1. **Permutations and Patterns.** Permutations play a key role in defining Schubert varieties. Indeed, each permutation indexes a unique Schubert variety. We will look at permutations both for their role in indexing Schubert varieties as well as their ability to tell us when the projection π_r restricted to a Schubert variety has a fiber bundle structure and when a Schubert variety has a complete parabolic bundle structure (our results).

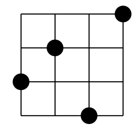
For two integers, $m < n$, define the interval $[m, n] := \{m, m+1, \cdots, n\}$. For $m = 1$, we write $[n] := \{1, 2, \cdots, n\}$. We will regard elements $w \in S_n$ as permutations $w : [n] \to [n]$. The symmetric group S_n has simple generators $s_1, \cdots, s_{n-1}$ with Coxeter relations

$$s_i^2 = 1, \quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \quad \text{and} \quad s_i s_j = s_j s_i \quad \text{if } |i - j| > 1$$

where s_i is the simple transposition of the elements $(i, i+1)$.

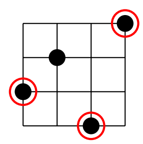
Common ways we will be writing permutations is in one-line notation where for $w \in S_n$, we write $w = w(1)w(2)\cdots w(n)$. It is also common to express permutations in terms of arrays, briefly described before (see Example 1.5) and depicted below.

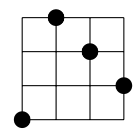
Example 2.1. *Take $w \in S_4$. Below is an example showing equivalent ways of expressing a permutation in one-line notation and in an array.*

$w = 3241 =$ 

Definition 2.2. *Let $k < n$, $w \in S_n$, and $p \in S_k$ with $w = w(1)w(2)\cdots w(n)$, and $p = p(1)p(2)\cdots p(k)$. Then w is said to **contain the pattern** p if there exists a subsequence, $\boldsymbol{a} = (a_1 < \cdots < a_k)$ in w such that $w(a_1)w(a_2)\cdots w(a_k)$ has the same relative order as p. If there is no such subsequence, then w **avoids** p.*

Example 2.3.

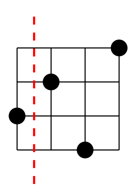




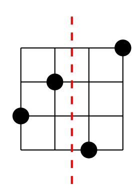
$w = 3241$ *contains* 231 $\qquad$ $w = 4123$ *avoids* 231

Definition 2.4. *Let $r, k < n$, $w \in S_n$, and $p \in S_k$ with $w = w(1)w(2)\cdots w(n)$. Then w is said to **contain the split pattern** $p = p_1|p_2 = p(1)\cdots p(j)|p(j+1)\cdots p(k)$ with respect to r if there exists a subsequence, $\boldsymbol{a} = (a_1 < \cdots < a_k)$ in w such that $w(a_1)w(a_2)\cdots w(a_k)$ has the same relative order as p and $a_j \leq r < a_j + 1$. If there is no such subsequence, then w **avoids** p with respect to r.*

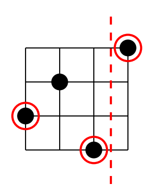
Example 2.5. *Let $p = 23|1$, a split pattern.* 3241 *contains the split pattern with respect to $r = 3$, but avoids it with respect to $r = 1, 2$.*



$r = 1$



$r = 2$



$r = 3$

Lemma 2.6. *A permutation contains a pattern in the classical sense if and only if it contains a split version of the same pattern with respect to at least one $r < n$.*

This follows clearly from the definitions and is left as an exercise to the reader.

We will now go over some properties and notation of S_n as a Coxeter group. Let $S = \{s_1, \cdots, s_{n-1}\}$, the set of simple generators. For any $w \in S_n$, a **word** for w is an expression of simple generators $s_{i_1} \cdots s_{i_k} = w$. Further, $s_{i_1} \cdots s_{i_k}$ is a **reduced word** if the word cannot be expressed in a fewer number of simple generators. It is known that if $w = s_{i_1} \cdots s_{i_k}$ and $w = s_{j_1} \cdots s_{j_k}$ are two reduced words for some $w \in S_n$, then if $s \in \{s_{i_1}, \cdots, s_{i_k}\}$ then $s \in \{s_{j_1}, \cdots, s_{j_k}\}$. Therefore, for any $w \in S_n$, we may define

$$S(w) := \{s \in S \mid s \text{ is in the reduced word of } w\}$$

to be the **support** of w. If the expression $w = s_{i_1} \cdots s_{i_k}$ is reduced then we say $\ell(w) := k$ is the **length** of w. Now, for any $w \in S_n$, we define

$$D_L(w) := \{s \in S \mid \ell(sw) < \ell(w)\}$$
$$D_R(w) := \{s \in S \mid \ell(ws) < \ell(w)\}$$

to be the sets of left and right descents, respectively. In other words, the sets of left and right descents are the sets of simple generators that, when composing them on the left and right, respectively, decrease the length of the reduced word. Descents earned their name because whenever $w(i) > w(i+1)$, $s_i \in D_R(w)$.

Example 2.7. *Let $w = 4213$. There are two instances where $w(i) > w(i+1)$ and this is for $i = 1, 2$. Hence, $D_R(w) = \{s_1, s_3\}$. If we check the reduced word for w, we find $w = s_3s_2s_1s_2$. Consider*

$$\begin{aligned} w \cdot s_1 &= (s_3s_2s_1s_2)s_1 \\ &= (s_3s_1s_2s_1)s_1 && \textit{(since } s_is_{i+1}s_i = s_{i+1}s_is_{i+1}\textit{)} \\ &= s_3s_1s_2 && \textit{(since } s_i^2 = 1.\textit{)} \\ \textit{Also, } w \cdot s_2 &= (s_3s_2s_1s_2)s_2 \\ &= s_3s_2s_1 && \textit{(since } s_i^2 = 1.\textit{)} \end{aligned}$$

Since s_3 cannot be moved completely to the right, it is not a right descent. This confirms that $D_R(w) = \{s_1, s_2\}$.

This leads us to introduce what is called a **parabolic decomposition**. Let $J \subseteq S$ and $W = S_n$. Define $W_J \subseteq W$ to be the subgroup generated by J and W^J to be the set of minimal length representatives of the cosets given by W/W_J. As such, for any $v \in W^J$ and $u \in W_J$, $D_R(v) \cap J = \emptyset$ and $S(u) \subseteq J$. Then, for any $w \in W$, and $J \subset W$, there is a unique parabolic decomposition $w = vu$ where $v \in W^J$ and $u \in W_J$. For our projection $\pi_r : X_w \twoheadrightarrow \mathrm{Gr}(r, n)$, we will define $J := S \setminus \{s_r\}$.

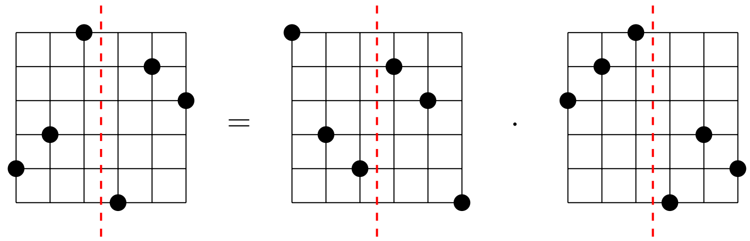
Lemma 2.8. *Let $w \in S_n$ and write*

$$w = w_1|w_2 = w(1) \cdots w(r)|w(r+1) \cdots w(n).$$

Let $w = vu$ be the parabolic decomposition with respect to $J = S \setminus \{s_r\}$. Then

(1) $v = v_1|v_2$ where v_1 and v_2 respectively consist of the entries of w_1 and w_2 arranged in increasing order.
(2) $u = u_1|u_2$ where u_1 and u_2 are respectively the unique permutations on $[1, r]$ and $[r+1, n]$ with relative orders of w_1 and w_2.

Example 2.9. *Let $w = 541|623$. If $w = vu$ is the parabolic decomposition with respect to $J = S \setminus \{s_3\}$, then $v = 145|236$ and $u = 321|645$.*



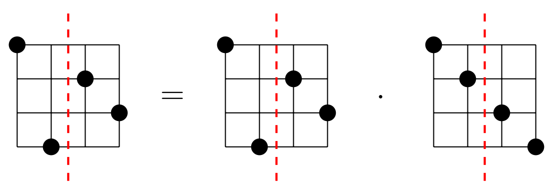
541|623 145|236 321|645

We can see in the arrays above that the array of u looks like that of w where all the nodes on the left and right side, respectively, have been pushed as high and low as they can. In many ways, the following proposition reflects a key starting point for our research and introduces a combinatorial condition for when π_r is a fiber bundle.

Proposition 2.10. ([RS, Theorem 3.3, Proposition 4.2]) *Let $w \in S_n$, $r < n$, and $w = vu$ be the parabolic decomposition with respect to $J = S(w) \setminus \{s_r\}$. Then the following are equivalent.*

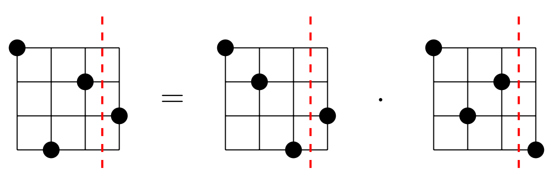
*(1) $w = vu$ is a **BP decomposition** with respect to J.*
(2) $S(v) \cap J \subseteq D_L(u)$.
(3) The projection $\pi_r : X_w \twoheadrightarrow X_v^J$ is a Zariski-locally trivial fiber bundle.

Example 2.11. *Let $w = 1423 = s_3 s_2 \in S_4$ and pick $r = 2$. Then the parabolic decomposition with respect to $J = S(w) \setminus \{s_2\}$ is*



1423 1423 1234

*Here, $S(v) \backslash \{s_2\} = \{s_3\}$ and $D_L(u) = \emptyset$. Clearly $\{s_3\} \not\subseteq \emptyset$, so this is **not** a BP decomposition. However, take the parabolic decomposition with respect to $J = S(w) \backslash \{s_3\}$. Then,*



1423 1243 1324

*Here, $S(v) \backslash \{s_3\} = \emptyset$ and $D_L(u) = \{s_2\}$. Clearly $\emptyset \subset \{s_2\}$, so this **is** a BP decomposition.*

We also say that w has a **complete BP decomposition** if we can write

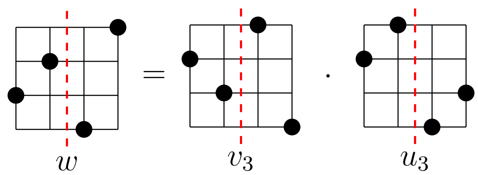
$$w = v_k \cdots v_1$$

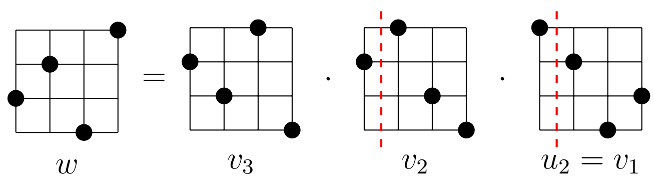
where for every $i \in [k-1]$, $|S(v_i \cdots v_1)| = i$ and $v_i(v_{i-1} \cdots v_1)$ is a BP decomposition with respect to $J = S \setminus \{s_{r_i}\}$ where s_{r_i} is the unique simple transposition in $S(v_i) \setminus S(v_{i-1} \cdots v_1)$.

We now give the correlation between complete BP decompositions and complete parabolic bundle structures on Schubert varieties.

Proposition 2.12. ([RS, Lemma 4.3, Corollary 3.7]) *Let $w \in S_n$. w has a complete BP decomposition if and only if X_w has a complete parabolic bundle structure.*

Example 2.13. *Consider $w = 3241 \in S_4$. Then a complete BP decomposition given by is $w = v_3v_2v_1 = (s_1s_2)(s_1)(s_3)$ and $\sigma = 213$ induces a complete parabolic bundle structure on X_w. Depicted below are the arrays of the complete BP decomposition as it is being solved.*





See Example 4.1 for more details.

3. Main Proofs

It is in this section that we will prove Theorems 1.6 and 1.9 as well as introduce and prove a key proposition needed to prove Theorem 1.9. Before we do so, however, we introduce and prove the following lemmas which play a key role in the proof of Theorem 1.6.

Lemma 3.1. *Let $W = S_n$, $r < n$, $J = S \setminus \{s_r\}$, and $v = v_1|v_2 = v(1)\cdots v(r)|v(r+1)\cdots v(n) \in W^J$. Then*

$$S(v) = \{s_l \cdots s_{m-1}\},$$

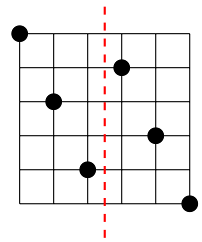
where $l = v^{-1}(r+1)$ is the smallest entry on the right and $m = v^{-1}(r)$ is the greatest entry on the left of the division after $v(r)$.

Proof. Since m is the greatest element on the left, any entry whose value is greater than m is on the right and since each side is in increasing order, they must therefore also be in the same positions as in the identity. Since no value greater than m must be moved in order to achieve v, all simples, if any, of value greater than or equal to m are not present in $S(v)$.

Similarly, since l is the smallest element on the right, all elements whose value is less than l must then be on the left in increasing order and therefore in the same positions as in the identity. Since no value less than l must be moved in order to achieve v, all simples, if any, of value less than l are not present in $S(v)$.

We now must show that the simple reflections $s_l, \cdots, s_{m-1} \in S(v)$. Since $s_l, \cdots, s_r$ are needed simply to move l from the l-th position, as in the identity, to the $(r+1)$-th, as in v, and $s_r, \cdots, s_{m-1}$ are needed simply to move m from the m-th position, as in the identity, to the r-th, as in v, we have that together, all $s_l, \cdots, s_{m-1} \in S(v)$. Thus $S(v) = \{s_l, \cdots, s_{m-1}\}$, as required. $\square$

Example 3.2. *Let* $v = 135|246 \in S_6$*. In the array below, we can see that 5 is the greatest entry on the left and that 2 is the least entry on the right. Since both 1 and 6 remain in positions 1 and 6, respectively,* $s_1, s_5 \notin S(v)$*. Now, moving 5 from the 5th position to the 3rd would require simples* s_4 *and* s_3*, so* $s_3, s_4 \in S(v)$*. Likewise, moving 2 from the 2nd position to the 4th would require simples* s_2 *and* s_3*, so* $s_2, s_3 \in S(v)$*. Thus, we find* $S(v) = \{s_2, s_3, s_4\}$*, as in Lemma 3.1. Indeed,* $v = s_2 s_4 s_3$*.*



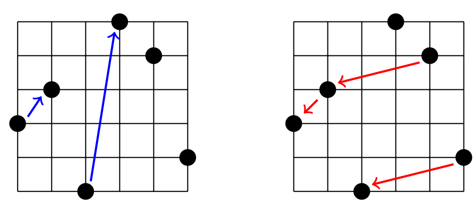
Lemma 3.3. *Let* $u = u(1) \cdots u(n) \in S_n$*. Then*

$$D_L(u) = \{s_k \mid u^{-1}(k+1) < u^{-1}(k)\}.$$

In other words, s_k *is a left descent of* u *if and only if when moving from the node in the* k*-th row to the node in the* $(k+1)$*-th row of the permutation matrix, we move to the left.*

Proof. This follows directly from the fact that $D_L(u) = D_R(u^{-1})$.Computationally, we can use the fact that if A is the permutation matrix for u, then $A^{-1} = A^t$. Therefore, instead of rewriting A^{-1}, we may switch to treating the columns as rows and vice-versa. □

Example 3.4. *Take the permutation used before,* $u = 436125$ *which corresponds to the permutation array below. In the first array, we mark the right descents and in the second array, we marks the left descents.*



Hence, $D_R(u) = \{s_1, s_3\}$ *and* $D_L(u) = \{s_2, s_3, s_5\}$*.*

Before we begin the proof of Theorems 1.6 and 1.9, we will go over key terminology in relation to our permutation arrays. Let A be the permutation array of $w = w(1) \cdots w(n)$. We say a region, R, of A is **decreasing** if for all nodes $(w(i), i), (w(j), j)$ in R, $w(i) > w(j)$ whenever $i < j$. While visually counter-intuitive, if a region of an array is decreasing, then its nodes are moving up as it progresses from left to right and therefore may be marked with a northeast arrow. We also say that R is **empty** if there are no nodes of the form $(w(i), i)$ in its interior and mark this with a shaded background (see Figures 2 and 3). Finally, we say a pair of nodes, $(w(i), i)$, $(w(j), j)$, is **increasing** if $i < j$ and $w(i) < w(j)$.

Proof of Theorem 1.6. Fix $r < n$ and let $w \in S_n$. Let $w = vu$ be the parabolic decomposition with respect to $J = S \setminus \{s_r\}$. By Proposition 2.10, it suffices to prove that w avoids the split patterns 3|12 and 23|1 with respect to position r if and only if $S(v) \cap J = S(v) \setminus \{s_r\} \subseteq D_L(u)$. Let

$$m := \max\{w(k) \mid k \leq r\} \qquad \text{and} \qquad l := \min\{w(k) \mid k > r\}.$$

The nodes $(m, w^{-1}(m))$ and $(l, w^{-1}(l))$ partition the permutation array of w into regions labeled $A - H$ as in Figure 1. By definition of m and l, the regions D and E must be empty. Moreover, Lemma 2.8 part (1) and Lemma 3.1 imply that

$$S(v) = \{s_k \mid l \leq k < m\}. \tag{4}$$

Similarly, the permutation array of u partitions into regions $A' - H'$ as in Figure 1. By Lemma 2.8 part (2), the nodes in each region labeled $A - H$ maintain the same relative order of those in $A' - H'$ respectively. In particular, $(r, w^{-1}(m))$ and $(r+1, w^{-1}(l))$ are nodes in the permutation array of u. Furthermore, since regions D and E are empty, the sizes of regions A and H are the same as the size of regions A' and H'.

Now suppose w avoids the patterns 3|12 and 23|1 with respect to position r. Then regions B, G must be empty and regions C, F must be decreasing in the permutation array of w. Thus regions B', G' are empty and regions C', F' are decreasing in the permutation array of u (See Figure 2). Now Lemma 3.3 and Equation (4) imply that $D_L(u)$ contains $S(v) \setminus \{s_r\}$ and hence $w = vu$ is a BP decomposition.

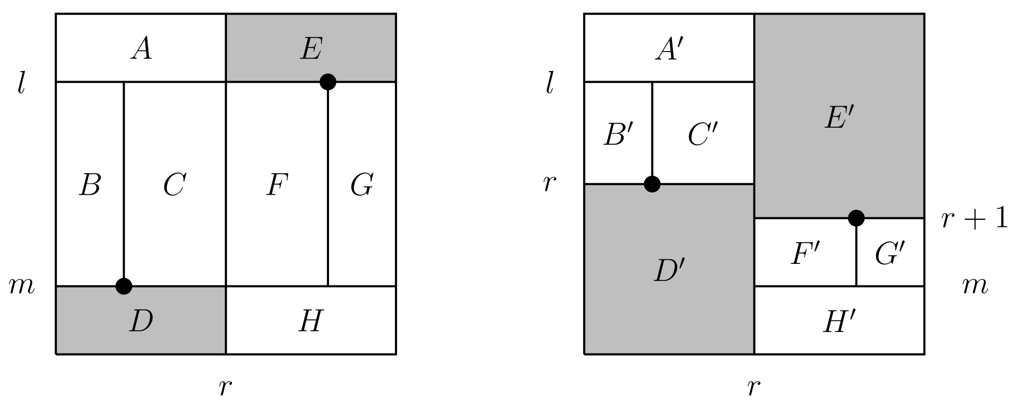


FIGURE 1. Permutation arrays of w and u partitioned by nodes at $(m, w^{-1}(m))$ and $(l, w^{-1}(l))$.

Conversely, suppose $S(v) \setminus \{s_r\} \subseteq D_L(u)$. In particular, Lemma 3.3 and Equation (4) say that $u^{-1}(k+1) < u^{-1}(k)$ for all $k \in [l, r-1] \sqcup [r+1, m-1]$. This implies that regions B', G' are empty and regions C', F' are decreasing in the permutation array of u. Hence regions B, G are empty and regions C, F are decreasing in the permutation array of w. Thus w avoids both split patterns 3|12 and 23|1 with respect to position r. This completes the proof.

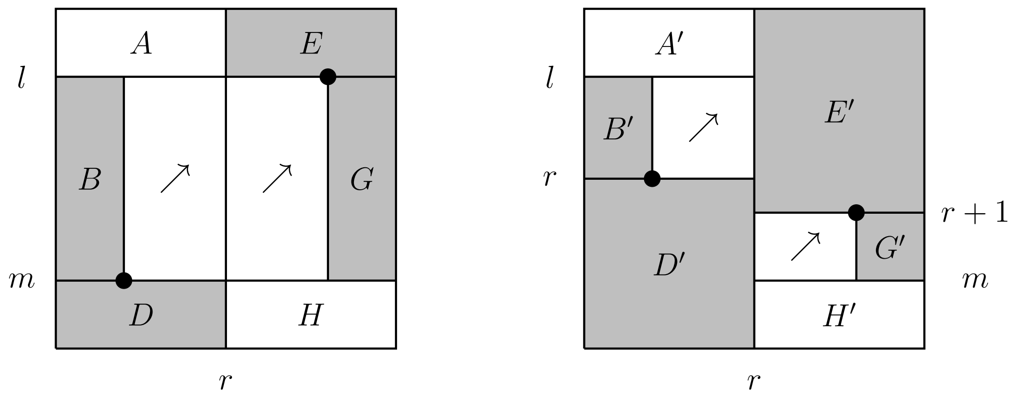


Figure 2. Permutation arrays of w and u with w avoiding $3|12$ and $23|1$ with respect to position r or equivalently, $S(v) \setminus \{s_r\} \subseteq D_L(u)$.

$\square$

Before we prove Theorem 1.9, we need the following proposition.

Proposition 3.5. *If $w \in S_n$ avoids 3412, 52341 and 635241, then there exists $r < n$ such that w avoids $3|12$ and $23|1$ with respect to position r. Further, if $S(w) \neq \emptyset$, then we can choose r such that $s_r \in S(w)$.*

Proof. We prove the first part of the proposition by contradiction. Therefore, let $w \in S_n$ such that w avoids 3412, 52341 and 635241 and assume for contradiction that w contains either the pattern $3|12$ or $23|1$ with respect to every position $r < n$. In particular, w must contain $3|12$ with respect to position 1. Let $w(1)w(i)w(j)$ have the same relative order as 312. In so doing, we partition w into regions $A - K$ as seen in Figure 3. Furthermore, we may choose $w(i), w(j)$ so as to force regions E, F, and J to be empty. Since w avoids 3412, region D must also be empty and regions C and I must be decreasing. We will show by a series of cases which vary based on if region I is empty or nonempty and if w contains $3|12$ or $23|1$ with respect to position i that w must contain the pattern 3412, 52341, or 635241.

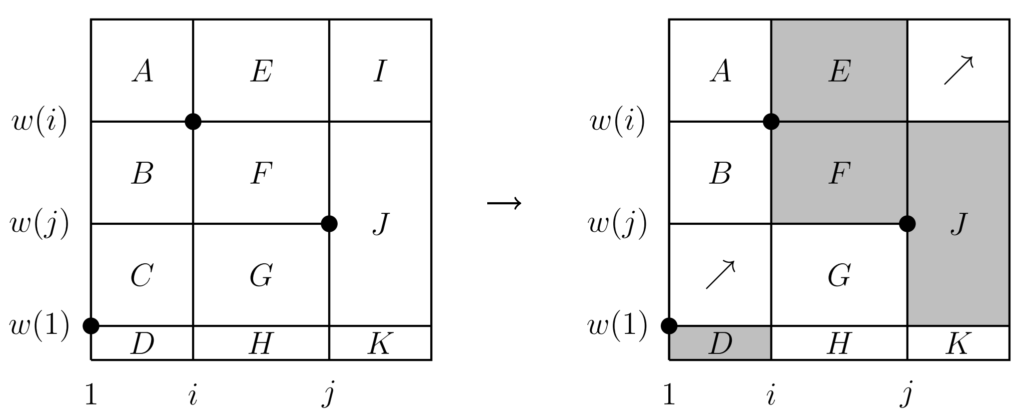


Figure 3. Permutation array of w containing $3|12$ with respect to position $r = 1$.

Case 1: First, assume region I is nonempty and that w contains $3|12$ with respect to position i. Then, region G must have two increasing nodes. However, this implies that w contains 52341 as seen in Figure 4, a contradiction.

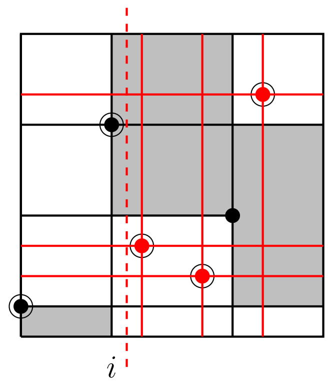


FIGURE 4. Permutation array of w containing $3|12$ with respect to $r = i$ and region I is nonempty.

Case 2: Now, assume that region I is nonempty, but that w contains the pattern $23|1$ with respect to position i. There are three ways for this to take place, each shown in Figure 5. Either there is a node in region A whose value is greater than that of at least one node in region I, or there are two increasing nodes in region B or $B \cup C$. In the first two scenarios, w contains 52341 and in the latter, w contains 635241, a contradiction.

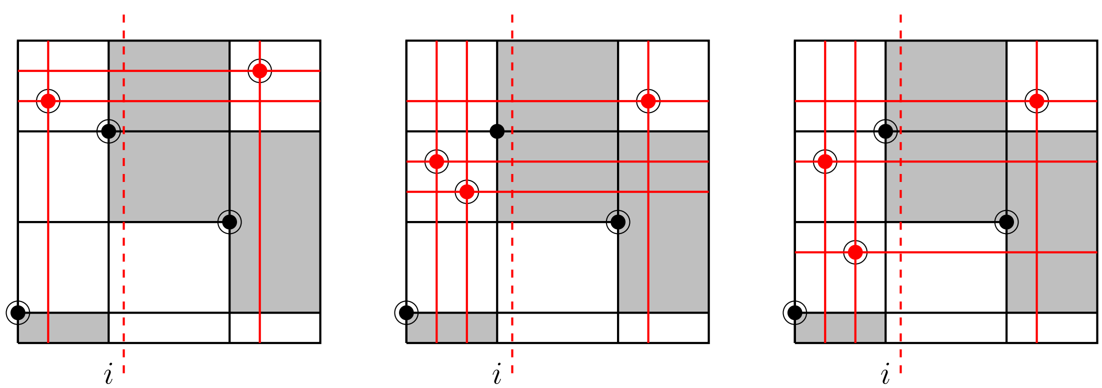


FIGURE 5. Permutation array of w containing $23|1$ with respect to $r = i$ and region I is nonempty.

Case 3: Now, assume that region I is empty. It then is not possible for w to contain $23|1$ with respect to position i, so we assume that w contains $3|12$ with respect to position i. By so doing, we assume that there are two increasing nodes, noted $w(i'), w(j')$ in region $F \cup G$ as shown in Figure 6. As before, we may now partition region $F \cup G$ into regions $A' - H'$ and choose $w(i'), w(j')$ such that regions E', F', and H' are empty. Since w avoids 3412 and 52341, regions A' and D' must also be empty and regions C' and I' must be decreasing.

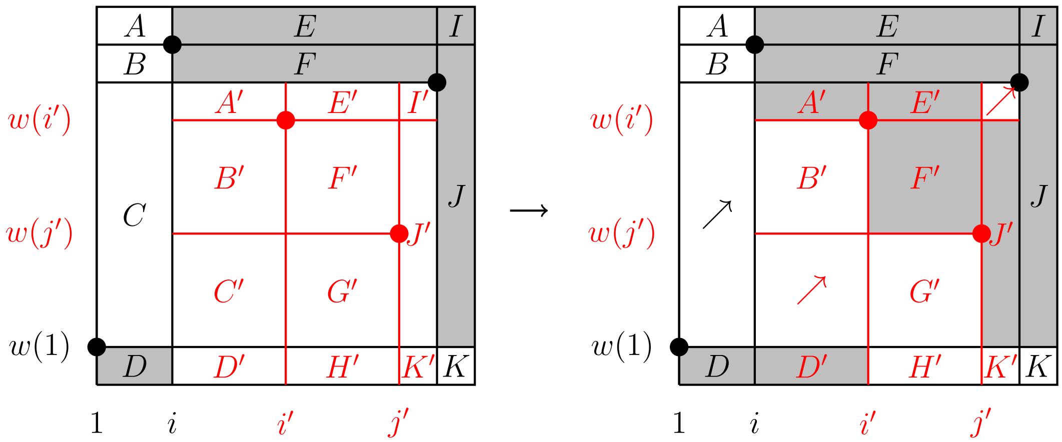


Figure 6. Permutation array of w containing $3|12$ with respect to position $r = i$ and region I is empty.

Subcase 3a: We now assume that w contains $3|12$ with respect to position i'. Since regions D, D' are empty, and I' decreasing, we must have two increasing nodes in region G'. However, this implies that w contains 52341, a contradiction (see Figure 7.

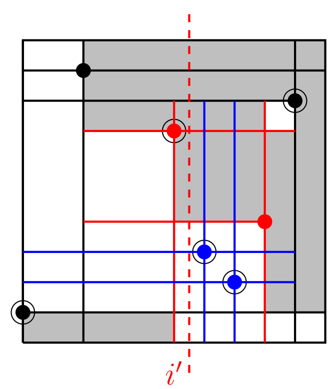


Figure 7. Permutation array of w containing $3|12$ with respect to position $r = i'$.

Subcase 3b: Having shown w cannot contain $3|12$ with respect to i', we now assume that w contains $23|1$ with respect to i'. There are four possible ways for this to happen. w must have two increasing nodes in either regions B', $B' \cup C'$, $C \cup B'$, or $C \cup C'$. As shown in Figure 8, if the two increasing nodes are in region B' or $B' \cup C'$ then w contains 52341 or 635241, respectively, a contradiction.

Finally, if w contains increasing nodes in regions $C \cup B'$ or $C \cup C'$, then we have the following three possibilities as shown in Figure 9.

Collectively, the three possibilities imply that w contains 3412, 52341, or 635241, a contradiction. This completes the first part of the proof.

For the second part of the proof, we show that we can choose an $r < n$ such that w avoids $3|12$ and $23|1$ with respect to position r and $s_r \in S(w)$. We proceed by

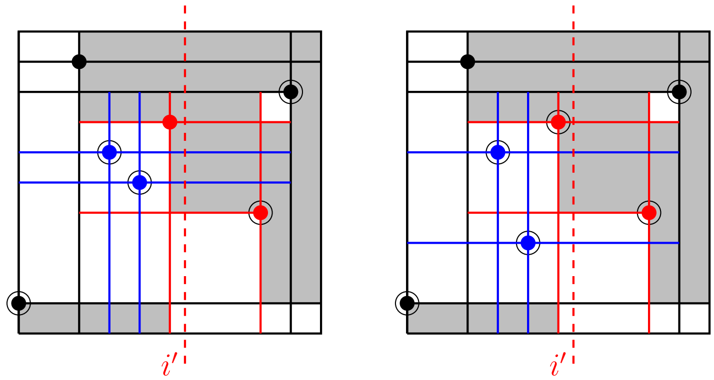


FIGURE 8. Permutation array of w containing $23|1$ with respect to position $r = i'$ using regions B' and $B' \cup C'$.

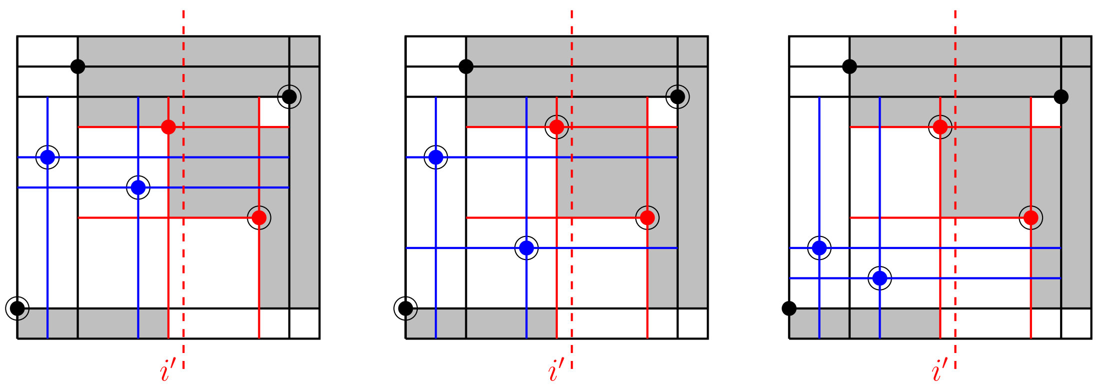


FIGURE 9. Permutation array of w containing $23|1$ with respect to position $r = i'$ using regions $C \cup B'$ and $C \cup C'$.

induction on the size of the permutation, n. If $n = 2$, then the Proposition is true for $w = s_1$ and $r = 1$ and is vacuously true for w equal to the identity. Now, if $w \in S_n$ avoids 3412, 52341, and 635241, then there exists an $r < n$ such that the parabolic decomposition $w = vu$ with respect to $J = S \setminus \{s_r\}$ is a BP decomposition. If $s_r \in S(w)$, then we are done. If not, then $s_r \notin S(w)$ which implies $w = u$. Write

$$w = w_1|w_2 = w(1)\cdots w(r)|w(r+1)\cdots w(n).$$

Since w avoids 3412, 52341, and 635241, w_1 and w_2 must also avoid them as permutations in S_r and S_{n-r}. Since either r or $n-r$ is greater than 1, we may assume without loss of generality that $r > 1$ and $S(w_1) \neq \emptyset$. By induction, there exists an $r' < r$ such that w_1 avoids $3|12$ and $23|1$ with respect to r' and $s_{r'} \in S(w_1)$. However, since $w = u$, w must also avoid $3|12$ and $23|1$ with respect to r'. Lastly, since $S(w_1) \subseteq S(w)$, $s_{r'} \in S(w)$. This completes the proof. $\square$

Proof of Theorem 1.9. By Proposition 2.12, it suffices to show that $w \in S_n$ avoids the patterns 3412, 52341 and 635241 if and only if w has a complete BP decomposition. First assume the $w \in S_n$ avoids the patterns 3412, 52341 and 635241. We will show w has a complete BP decomposition by induction on $\ell(w)$. First note that if $w = e$, then the theorem is vacuously true. If $w \neq e$, then by Theorem 1.6 and Proposition

3.5, there exists $r < n$ such that $s_r \in S(w)$ and the parabolic decomposition $w = vu$ with respect to $J = S \setminus \{s_r\}$ is a BP decomposition. Lemma 2.8 implies that u also avoids the patterns 3412, 52341 and 635241. Since $s_r \in S(w) \setminus S(u)$, we have that $\ell(u) < \ell(w)$ and by induction, we are done.

Conversely, assume w has a complete BP decomposition. In particular, there exists $r < n$ such that the parabolic decomposition $w = vu$ with respect to $J = S \setminus \{s_r\}$ is a BP decomposition. Note that if $s_r \notin S(w)$, then $w = u$. Without loss of generality, we can assume $s_r \in S(w)$ and hence $\ell(u) < \ell(w)$. By induction on $\ell(w)$, it suffices to show that if u avoids 3412, 52341 and 635241, then w avoids those same patterns. Write $w = w_1|w_2$ and $u = u_1|u_2$ with respect to position r as in Lemma 2.8. Since u avoids 3412, 52341 and 635241, both w_1 and w_2 must also avoid these patterns. Hence, if w contains one of 3412, 52341 or 635241, then it must contain the pattern using entries in both w_1 and w_2. But then w must contain either $3|12$ or $23|1$ with respect to position r in which case Theorem 1.6 implies that w cannot have a BP decomposition with respect to J which is a contradiction. This completes the proof. ☐

4. Examples

We will now give some examples to help illustrate the implications of Theorems 1.6 and 1.9. For each, we write out the corresponding fiber bundle structures (or lack thereof) and describe in greater detail, the complete parabolic bundle structures for the examples that have them. In each of these examples, we fix a flag $E_\bullet := E_1 \subset E_2 \subset E_3 \subset \mathbb{K}^4$ where $\dim(E_i) = i$ off of which the Schubert varieties in $\mathrm{F}\ell(4)$ are defined.

Example 4.1. The Schubert variety indexed by $w = 3241$ is

$$X_{3241} = \{V_\bullet = (V_1 \subset V_2 \subset V_3 \subset \mathbb{K}^4) \mid V_2 \subset E_3\}.$$

It is easy to check that w avoids the split patterns $3|12$ and $23|1$ with respect to positions $r = 1, 2$, but contains $23|1$ with respect to $r = 3$. By Theorem 1.6 we have that the projection maps π_1 and π_2 are fiber bundle maps on X_w while π_3 is not. In particular, we have fibers

$$\begin{aligned}
\pi_1^{-1}(V_1) &= \{(V_2 \subset V_3) \mid V_1 \subset V_2 \subset E_3\} \simeq X_{1342}\\
\pi_2^{-1}(V_2) &= \{(V_1 \subset V_3) \mid V_1 \subset V_2 \subset V_3\} \simeq \mathrm{F}\ell(2) \times \mathrm{F}\ell(2)\\
\pi_3^{-1}(V_3) &= \{(V_1 \subset V_2) \mid V_2 \subset E_3 \cap V_3\} \simeq \begin{cases} \mathrm{F}\ell(2) & \text{if} \quad \dim(V_3 \cap E_3) = 2\\ \mathrm{F}\ell(3) & \text{if} \quad V_3 = E_3. \end{cases}
\end{aligned}$$

Since $w = 3241$ avoids 3421, 52341, and 635241, Theorem 1.9 implies that X_w has a complete parabolic bundle structure. Indeed, the permutations $231, 213$ and 123 all induce complete parabolic bundle structures on X_w while permutations 312, 321 and 132 do not. For example, if $\sigma = 231$ then the corresponding complete BP decomposition of w in accordance to Proposition 2.12 is $w = (s_1s_2)(s_3)(s_1)$ and the complete parabolic bundle structure on X_w is

$$X_w \overset{\pi^{\{1,2,3\}}_{\{2,3\}}}{\twoheadrightarrow} X_2 \overset{\pi^{\{2,3\}}_{\{2\}}}{\twoheadrightarrow} X_1 \twoheadrightarrow pt$$

where $X_2 = \{V_\bullet^{\{2,3\}} \mid V_2 \subset E_3\}$ and $X_1 = \{V_\bullet^{\{2\}} \mid V_2 \subset E_3\}$. The fibers of each of the maps $\pi^{\{1,2,3\}}_{\{2,3\}}$ and $\pi^{\{2,3\}}_{\{2\}}$ are both isomorphic to $\mathrm{F}\ell(2)$ while X_1, which is the entire fiber since it is projected onto a single point, is isomorphic to $\mathrm{Gr}(2,3)$.

Example 4.2. The Schubert variety indexed by $w = 4231$ is

$$X_{4231} = \{V_\bullet = (V_1 \subset V_2 \subset V_3 \subset \mathbb{K}^4) \mid \dim(V_2 \cap E_2) \geq 1\}.$$

In this case w contains $3|12$ with respect to $r = 1$, contains $23|1$ with respect $r = 3$ and avoids both split patterns with respect to $r = 2$. Hence π_1, π_3 are not fiber bundle maps while π_2 is a fiber bundle map on X_{4231}. In particular,

$$\pi_1^{-1}(V_1) = \{(V_2 \subset V_3) \mid \dim(V_2 \cap E_2) \geq 1\} \simeq \begin{cases} X_{1342} & \text{if} \quad \dim(V_1 \cap E_2) = 0 \\ \mathrm{F}\ell(3) & \text{if} \quad V_1 \subset E_2. \end{cases}$$

$$\pi_2^{-1}(V_2) = \{(V_1 \subset V_3) \mid V_1 \subset V_2 \subset V_3\} \simeq \mathrm{F}\ell(2) \times \mathrm{F}\ell(2)$$

$$\pi_3^{-1}(V_3) = \{(V_1 \subset V_2) \mid \dim(V_2 \cap E_2) \geq 1\} \simeq \begin{cases} X_{3124} & \text{if} \quad \dim(V_3 \cap E_2) = 1 \\ \mathrm{F}\ell(3) & \text{if} \quad E_2 \subset V_3. \end{cases}$$

Again, Theorem 1.9 implies that X_w has a complete parabolic bundle structure. For example, if $\sigma = 213$, then the complete parabolic bundle structure on X_w is

$$X_w \overset{\pi^{\{1,2,3\}}_{\{1,2\}}}{\twoheadrightarrow} X_2 \overset{\pi^{\{1,2\}}_{\{2\}}}{\twoheadrightarrow} X_1 \twoheadrightarrow pt$$

where $X_2 = \{V_\bullet^{\{1,2\}} \mid \dim(V_2 \cap E_2) \geq 1\}$ and $X_1 = \{V_\bullet^{\{2\}} \mid \dim(V_2 \cap E_2) \geq 1\}$. The fibers of each of the maps $\pi^{\{1,2,3\}}_{\{1,2\}}$ and $\pi^{\{1,2\}}_{\{2\}}$ are both isomorphic to $\mathrm{F}\ell(2)$ while X_1 is isomorphic to $X_{2413}^{\{s_1,s_3\}} \subset \mathrm{Gr}(2,4)$. The corresponding complete BP decomposition of w is $w = (s_1 s_3 s_2)(s_1)(s_3)$.

Example 4.3. The Schubert variety indexed by $w = 3412$ is

$$X_{3412} = \{V_\bullet = (V_1 \subset V_2 \subset V_3 \subset \mathbb{K}^4) \mid V_1 \subset E_3,\ E_1 \subset V_3\}.$$

In this case w contains either $3|12$ or $23|1$ with respect to every r. Hence all projections π_r are not fiber bundle maps on X_{3412}. In particular,

$$\pi_1^{-1}(V_1) = \{(V_2 \subset V_3) \mid E_1 \subset V_3\} \simeq \begin{cases} X_{1342} & \text{if} \quad \dim(V_1 \cap E_1) = 0 \\ \mathrm{F}\ell(3) & \text{if} \quad V_1 = E_1. \end{cases}$$

$$\pi_2^{-1}(V_2) \simeq \begin{cases} X_{1234} & \text{if} \quad \dim(V_2 \cap E_3) = 1,\ \dim(V_2 \cap E_1) = 0 \\ X_{1243} & \text{if} \quad E_1 \subset V_2,\ \dim(V_2 \cap E_3) = 1 \\ X_{2134} & \text{if} \quad V_2 \subset E_3,\ \dim(V_2 \cap E_1) = 0 \\ \mathrm{F}\ell(2) \times \mathrm{F}\ell(2) & \text{if} \quad E_1 \subset V_2 \subset E_3 \end{cases}$$

$$\pi_3^{-1}(V_3) = \{(V_1 \subset V_2) \mid V_1 \subset E_3\} \simeq \begin{cases} X_{3124} & \text{if} \quad \dim(V_3 \cap E_3) = 2 \\ \mathrm{F}\ell(3) & \text{if} \quad V_3 = E_3. \end{cases}$$

Theorem 1.9 implies that X_{3412} has no complete parabolic bundle structure. It is easy to check that $\pi^{\{1,2,3\}}_{\{2,3\}}$ and $\pi^{\{1,2,3\}}_{\{1,2\}}$ are not fiber bundle maps, however $\pi^{\{1,2,3\}}_{\{1,3\}}$ is a fiber bundle map with fiber isomorphic to $\mathrm{F}\ell(2)$. In an attempt to continue the iterated fiber bundle, we see that neither of the projection maps $\pi^{\{1,3\}}_{\{1\}}, \pi^{\{1,3\}}_{\{3\}}$ induce fiber bundle structures on $\pi^{\{1,2,3\}}_{\{1,3\}}(X_{3412})$. Hence, we confirm that X_{3412} has no complete parabolic bundle structure.

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