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NK1 OF NONABELIAN GROUPS

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DENNIS RAY HARMON

Norman, Oklahoma

1984

NK_1 OF NONABELIAN GROUPS

A DISSERTATION

APPROVED FOR THE DEPARTMENT OF MATHEMATICS

By

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INTRODUCTION

Let Rng denote the category of associative rings with identity and let Z-mod denote the category of abelian groups. For a functor $F\colon \mathrm{Rng} \to Z\text{-mod}$ the functors $NF, LF\colon \mathrm{Rng} \to Z\text{-mod}$ are defined as follows: $NF(A) = \ker(\, F(A[t]) \to F(A)\,)$ induced by the augmentation $A[t] \to A$ sending t to 0, and $LF(A) = \operatorname{coker}(\, F(A[t]) \oplus F(A[t^{-1}]) \to F(A[t,t^{-1}])\,)$ induced by the obvious inclusions. For a contracted functor F such as K_0, K_1 or SK_1 there is an isomorphism [1, Cor. 7.3, p. 663]

$$F(A[t_1,\ldots,t_n,t_1^{-1},\ldots,t_n^{-1}]) \cong (\, I + 2N + L\,)^n F(A).$$

In particular, for $F = K_1$ and $A = Z\pi$ the integral group ring,

$$K_1Z[Z \times \pi] \cong K_1Z\pi \oplus NK_1Z\pi \oplus NK_1Z\pi \oplus K_0Z\pi.$$

Martin [8, Thm. 2.2] showed that $NK_1Z\pi = 0$ for π a finite abelian group of square-free order.

This thesis extends Martin's results to the case of an arbitrary finite group. The method is outlined as follows:

We first reduce to the case where π is hyper-elementary by showing that $NK_1Z\pi$ is a Frobenius module over $G_0Z\pi$. Cartesian squares then give rise to natural Mayer-Vietoris exact sequences of K-groups [9] and hence exact sequences of NK-groups. We then use the fact that $NK_iA = 0$ for a regular ring A, $i = 0,1$ [3] and $i = 2$ [10].

Symbols used throughout this thesis are: Z = the integers, Q = the rationals, and C_m = the cyclic group of order m.

At this time I would like to extend my thanks to my thesis advisor, Bruce Magurn for his invaluable guidance and support.

NK_1 OF NONABELIAN GROUPS

CHAPTER I

PRELIMINARIES

We begin by stating some definitions due to Lam [6, p. 106]. Let Sg denote the category whose objects are finite groups and whose arrows are inclusions $i: \rho \to \pi$ whenever $\rho < \pi$. A Frobenius functor is a contravariant functor $F: Sg \to$ Comm. Ring with arrow map $(-)^*$, which is also a covariant functor $Sg \to Z$-mod under an arrow map $(-)_*$, such that $i_*(i^*(x)\cdot y) = x\cdot i_*(y)$ for $\rho \to \pi$ an arrow in Sg, $x \in F(\pi)$, $y \in F(\rho)$.

Let F be a Frobenius functor. An F-module is an object map and two arrow maps $M(-)$, $(-)^*$, $(-)_*$: $Sg \to Z$-mod such that $M(-)$, $(-)^*$ is a contravariant functor, $M(-)$, $(-)_*$ is a covariant functor, $M(\pi)$ is an $F(\pi)$-module for each finite group π, and for $i: \rho \to \pi$ in Sg,

$$i^*(x\cdot y) = i^*(x)\cdot i^*(y) \quad \text{for} \quad x \in F(\pi),\ y \in M(\pi)$$
$$i_*(i^*(x)\cdot y) = x\cdot i_*(y) \quad \text{for} \quad x \in F(\pi),\ y \in M(\rho)$$
$$i_*(x\cdot i^*(y)) = i_*(x)\cdot y \quad \text{for} \quad x \in F(\rho),\ y \in M(\pi).$$

A morphism of Frobenius functors $F \to G$ is a transformation $F \to G$ natural with respect to both arrow maps. A morphism of F-modules is a transformation $f\colon M \to N$ natural with respect to both arrow maps, such that $f\colon M(\pi) \to N(\pi)$ is $F(\pi)$-linear for each finite group π.

For the remainder of this chapter, let R be a commutative ring with identity. For π a finite group and $R\pi$ the group ring of π over R, let $\mathcal{L}^R$ denote the category whose objects are the finitely generated left $R\pi$-modules which are projective as R-modules and let $G_0^R R\pi$ denote $K_0(\mathcal{L}^R)$, its Grothendieck group. $G_0^R R[-]$ is a Frobenius functor. For M,N objects in $\mathcal{L}^R$ the multiplication in $G_0^R R\pi$ is $[M]\cdot[N] = [M \mathbin{\bar{\otimes}_R} N]$ where the bar "$-$" denotes the diagonal action of π on $M \mathbin{\bar{\otimes}_R} N$. An arrow $i\colon \rho \to \pi$ in Sg induces the homomorphisms $i_*\colon G_0^R R\rho \to G_0^R R\pi$ defined by $i_*[M] = [R\pi \otimes_{R\rho} M]$ for $[M] \in G_0^R R\rho$ and $i^*\colon G_0^R R\pi \to G_0^R R\rho$ defined by $i^*[N] = [N]$ for $[N] \in G_0^R R\pi$.

According to Lam [6, p. 114], $K_0R[-]$ and $K_1R[-]$ are $G_0^R R[-]$-modules. For $[M] \in G_0^R R\pi$, $[N] \in K_0 R\pi$ the action is $[M]\cdot[N] = [M \mathbin{\bar{\otimes}_R} N]$, and for $[N,\alpha] \in K_1 R\pi$ the action is $[M]\cdot[N,\alpha] = [M \mathbin{\bar{\otimes}_R} N,\ 1_M \otimes \alpha]$. An arrow $i\colon \rho \to \pi$ extends to a homomorphism $i\colon R\rho \to R\pi$ inducing homomorphisms on K_0 and K_1 by the usual restriction and extension of scalars.

Let $\mathcal{f}(R\pi)$ denote the category of finitely generated left $R\pi$-modules and let $G_0R\pi$ denote its Grothendieck group $K_0(\mathcal{f}(R\pi))$. For a regular ring R, $G_0R\pi \cong G_0^R R\pi$ [13, Thm. 1.2] giving us a ring structure on $G_0R\pi$ even when $R\pi$ is not commutative, i.e. when π is not abelian.

Let C_m denote the cyclic group of order m. A finite group π is hyperelementary if $C_m \triangleleft \pi$ and $[\pi : C_m] = p^n$ for some prime p, $p \nmid m$, and some integer $n \geq 0$. This is equivalent to $\pi \cong C_m \rtimes B$ with $|B| = p^n$. Let $\mathcal{H}$ be the collection of hyperelementary groups. By the work of Lam [6, p. 123],

$$M(\pi) = \sum_{\substack{\rho \in \mathcal{H} \\ \rho < \pi}} i_* M(\rho)$$

for any $G_0^R R[-]$-module $M[-]$. This technique for computing $M(\pi)$ is referred to as "hyperelementary induction" and will be used in Chapter III.

CHAPTER II

ANOTHER FROBENIUS MODULE

Let A,B be commutative rings with identity, $f\colon A \to B$ a ring homomorphism, π a finite group and $f_*\colon K_1A\pi \to K_1B\pi$ the induced homomorphism. For $[M] \in G_0^A A\pi$, $[N,\alpha] \in K_1A\pi$,

$$f_*([M]\cdot[N,\alpha]) = f_*([M \mathbin{\overline{\otimes}}_A N,\ 1_M \otimes \alpha])$$

$$= [B\pi \otimes_{A\pi} (M \mathbin{\overline{\otimes}}_A N),\ 1_{B\pi} \otimes (1_M \otimes \alpha)]$$

$$[M]\cdot f_*([N,\alpha]) = [B \mathbin{\overline{\otimes}}_A M]\cdot[B\pi \otimes_{A\pi} N,\ 1_{B\pi} \otimes \alpha]$$

$$= [(B \mathbin{\overline{\otimes}}_A M) \mathbin{\overline{\otimes}}_B (B\pi \otimes_{A\pi} N),\ (1_B \otimes 1_M) \otimes (1_{B\pi} \otimes \alpha)]$$

<u>Lemma 2.1</u>: $B\pi \otimes_{A\pi} (M \mathbin{\overline{\otimes}}_A N) \cong (B \mathbin{\overline{\otimes}}_A M) \mathbin{\overline{\otimes}}_B (B\pi \otimes_{A\pi} N)$ as $B\pi$-modules.

Proof: Define $\phi\colon B\pi \otimes_{A\pi} (M \mathbin{\overline{\otimes}}_A N) \to (B \mathbin{\overline{\otimes}}_A M) \mathbin{\overline{\otimes}}_B (B\pi \otimes_{A\pi} N)$ by $(bg\otimes(m\otimes n)) = (b\otimes gm)\otimes(1\otimes gn)$ for $bg \in B\pi$, $m \in M$, $n \in N$. Then ϕ is π-linear:

for $h \in \pi$,

$$\phi(h\cdot(bg\otimes(m\otimes n))) = \phi(bhg\otimes(m\otimes n))$$
$$= (b\otimes hgm)\otimes(1\otimes hgn)$$
$$= h\cdot((b\otimes gm)\otimes(1\otimes gn))$$

by the diagonal π-actions and the trivial action of π on B. Thus ϕ is B-linear.

Define $\psi\colon (B \overline{\otimes}_A M) \overline{\otimes}_B (B\pi \otimes_{A\pi} N) \to B\pi \otimes_{A\pi} (M \overline{\otimes}_A N)$

by $\psi((b\otimes m)\otimes(rg\otimes n)) = rbg\otimes(g^{-1}m\otimes n)$ for $b \in B$, $rg \in B\pi$, $m \in M$, and $n \in N$. Then ψ is π-linear:
for $h \in \pi$,

$$\psi(h\cdot((b\otimes m)\otimes(rg\otimes n))) = \psi((b\otimes hm)\otimes(rhg\otimes n))$$
$$= rbhg\otimes(r(hg)^{-1}hm\otimes n)$$
$$= rbhg\otimes(rg^{-1}m\otimes n)$$
$$= h\cdot(rbg\otimes(g^{-1}m\otimes n))$$

So ψ is $B\pi$-linear as well. Checking composites,

$$\phi\psi((b\otimes m)\otimes(rg\otimes n)) = \phi(rbg\otimes(g^{-1}m\otimes n))$$
$$= (rb\otimes g(g^{-1}m))\otimes(1\otimes gn)$$
$$= (rb\otimes m)\otimes(1\otimes gn)$$
$$= (b\otimes m)\otimes(rg\otimes n)$$

$$\begin{aligned}\psi\phi(bg\otimes(m\otimes n)) &= \psi((b\otimes gm)\otimes(1\otimes gn)) \\ &= b\otimes(gm\otimes gn) \\ &= bg\otimes(m\otimes n).\end{aligned}$$

Thus. ϕ and ψ are inverses and the isomorphism has been proved.

By the commutative square

$$\begin{array}{ccc} B\pi \underset{A\pi}{\otimes} (M \underset{A}{\overline{\otimes}} N) & \xrightarrow{1\otimes(1\otimes\alpha)} & B\pi \underset{A\pi}{\otimes} (M \underset{A}{\overline{\otimes}} N) \\ \phi \downarrow \uparrow \psi & & \phi \downarrow \uparrow \psi \\ (B \underset{A}{\overline{\otimes}} M) \underset{B}{\overline{\otimes}} (B\pi \underset{A\pi}{\otimes} N) & \xrightarrow[(1\otimes1)\otimes(1\otimes\alpha)]{} & (B \underset{A}{\overline{\otimes}} M) \underset{B}{\overline{\otimes}} (B\pi \underset{A\pi}{\otimes} N) \end{array}$$

and Lemma 2.1, $f_*([M]\cdot[N,\alpha]) = [M]\cdot f_*([N,\alpha])$ in $K_1B\pi$. For $i\colon \rho \to \pi$ in Sg, the $B\pi$-linear isomorphisms

$$B\pi \underset{B\rho}{\otimes} (B\rho \underset{A\rho}{\otimes} N) \cong B\pi \underset{A\rho}{\otimes} N \cong B\pi \underset{A\pi}{\otimes} (A\pi \underset{A\rho}{\otimes} N)$$

yield $i_*f_* = f_*i_*$. Also, $i^*f_* = f_*i^*$ so f is natural with respect to both arrow maps. Thus we have proved, Theorem 2.2: For a homomorphism $f: A \to B$ of commutative rings, the induced homomorphism $f_*: K_1A[-] \to K_1B[-]$ is a morphism of $G_0^A A[-]$-modules.

Corollary 2.3: $NK_1Z[-]$ is a $G_0Z[-]$-module.

Proof: The category of $G_0^A A[-]$-modules is an abelian category [6, p. 108], so $\ker f_*$ is a $G_0^A A[-]$-module. The ring homomorphism $Z \to A$ induces a morphism $G_0Z[-] \cong G_0^Z Z[-] \to G_0^A A[-]$, making $\ker f_*$ a $G_0Z[-]$-module. Now apply Theorem 2.2 with $A = Z[t]$, $B = Z$, and f = augmentation.

Corollary 2.4: For any finite group π,

$$NK_1Z\pi = \sum_{\substack{\rho \in \mathcal{H} \\ \rho < \pi}} i_* NK_1 Z\rho$$

where $\mathcal{H}$ is the class of hyperelementary groups.

CHAPTER III

A VANISHING RESULT

We are now in a position to prove the main result of this thesis.

Theorem 3.1: If π is a finite group of square-free order then $NK_1Z\pi = 0$.

Proof: By Corollary 2.4 we can assume that π is hyper-elementary. Let $\pi = C_m \rtimes B$ with $|B| = p$, p a prime and $p \nmid m$. Let $C_m = \langle a \rangle$, $C_p = B = \langle b \rangle$. Then π has the presentation

$$\pi = \langle\, a,b:\ a^m,\ b^p,\ bab^{-1}a^{-\alpha}\ \rangle$$

where $\alpha^p \equiv 1 \pmod{m}$. If $\alpha \equiv 1 \pmod{m}$ then π is abelian and thus $NK_1Z\pi = 0$ [8, Thm. 2.2].

For a divisor d of m, let ζ_d denote a complex primitive d^{th} root of unity. The twisted group algebra $Q(\zeta_d)\circ B$ has the additive structure of a free right $Q(\zeta_d)$-module based on B and multiplicative structure determined by that in $Q(\zeta_d)$, by that in B, and the rule $ba = a^{\alpha}b$.

For any collection M of positive divisors of m, let $\mathcal{O}(M)$ denote the image of $Z\pi$ under the composite

$$Z\pi \hookrightarrow Q\pi \xrightarrow{\cong} \bigoplus_{d|m} Q(\zeta_d)\circ B \longrightarrow \bigoplus_{d\varepsilon M} Q(\zeta_d)\circ B$$

For each $d \varepsilon M$ there is a Cartesian square

$$\begin{array}{ccc} \mathcal{O}(M) & \xrightarrow{\phi} & \mathcal{O}(M-\{d\}) \\ \psi\downarrow & & \downarrow \\ \mathcal{O}(d) & \longrightarrow & \mathcal{O}(d)/J \end{array}$$

where ϕ,ψ are projections and $J = \psi(\ker\phi)$ is the ideal of $\mathcal{O}(d)$ generated by $\prod_{e\varepsilon M-\{d\}} \Phi_e(\zeta_d)$, and $\Phi_e(x)$ is the minimal polynomial of ζ_e over Q [7, pp. 403-4]. Then $\Phi_e(\zeta_d)$ is a unit of $Z[\zeta_d]$ if neither d/e nor e/d is a power of a prime, and is associate in $Z[\zeta_d]$ to a prime q if $e/d = q^r$ for some $r > 0$ [7, Lemma 9.3]. Choosing d minimal in M and using the fact that m is square-free we obtain

$$\prod_{e\varepsilon M-\{d\}} \Phi_e(\zeta_d) \sim \prod_{\substack{p_i \text{ prime} \\ p_i d\varepsilon M}} \Phi_{p_i d}(\zeta_d) \sim \prod_{\substack{p_i \text{ prime} \\ p_i d\varepsilon M}} p_i = \gamma$$

where " $\sim$ " means associate to. So for a minimal divisor $d \in M$, J is the ideal generated by γ.

If d is such that $B \to \mathrm{Aut}(Q(\zeta_d))$ has a non-trivial kernel then $\mathcal{O}(d)$ is in fact the group ring $Z[\zeta_d]B$ and $NK_1 Z[\zeta_d]B = 0$ [8, Thm. 2.2]. This occurs exactly when $d \mid (m, \alpha-1)$.

Otherwise, B acts faithfully on $Q(\zeta_d)$. Let $F = Q(\zeta_d)^B$, the subfield of $Q(\zeta_d)$ left fixed by the action of B. The cyclic algebra $Q(\zeta_d)\circ B =$ $(Q(\zeta_d)/F, b, 1)$ [11, p. 259] is a crossed-product algebra with trivial factor set and hence isomorphic to $M_p(F)$ [11, Cor. 29.8]. Then $\dim_F Q(\zeta_d) = p$ and by choosing the integral basis $\{1, \zeta_d, \ldots, \zeta_d^{p-1}\}$ for $Q(\zeta_d)/F$, the isomorphism $Q(\zeta_d)\circ B \to M_p(F)$ above restricts to $Z[\zeta_d]\circ B = \mathcal{O}(d) \to M_p(R)$ where R is the ring of integers in $Q(\zeta_d)^B$.

For a nonzero prime $P \triangleleft R$, $P\cdot Z[\zeta_d] = P_1^e \cdots P_g^e$ where $P_1, \ldots, P_g$ are distinct maximal ideals in the Dedekind domain $Z[\zeta_d]$ and e is the ramification index. Since $e \mid p$, $e = 1$ or $e = p$. If $e = 1$ then P is unramified in $Z[\zeta_d]$. Otherwise $e = p$; but the only primes $P \triangleleft R$ that ramify are those for which $q \mid d$ where $Z \cap P = qZ$, and $p \nmid d$ since $p \nmid m$. Hence $e \neq 0$ in Z/qZ and P is tamely ramified in $Z[\zeta_d]$. So $Z[\zeta_d]\circ B$ is hereditary [11, Thm. 40.15] and hence regular. Thus $NK_i Z[\zeta_d]\circ B = 0$ for $i = 0,1,2$.

$M_p(R)$ is a maximal order containing $\mathcal{O}(d)$ and $d\cdot M_p(R) \subset \mathcal{O}(d)$, so $\mathcal{O}(d)/(\gamma) \cong M_p(R/(\gamma))$ [7, Prop. 10.2]. For each pair of primes p_i, p_j in the factorization of γ, $p_iZ + p_jZ = Z$ and so $p_iR + p_jR = R$. By the Chinese Remainder Theorem $R/(\gamma) \cong \bigoplus_i R/(p_i)$ and thus $M_p(R/(\gamma)) \cong \bigoplus_i M_p(R/(p_i))$. Since $p_i \nmid d$, (p_i) is unramified in $Z[\zeta_d]$, hence in R as well. So $(p_i) = \prod_j P_{ij}$ where $\{P_{ij}\}$ are distinct maximal ideals in R. Thus $\bigoplus_i M_p(R/(p_i)) \cong \bigoplus_{i,j} M_p(R/P_{ij})$, a direct sum of matrix rings over fields, hence regular. So $NK_i(\mathcal{O}(d)/(\gamma)) = 0$ for $i = 0,1,2$.

The Mayer-Vietoris exact sequence resulting from the Cartesian square is

$$NK_2\mathcal{O}(d)/(\gamma) \to NK_1\mathcal{O}(M) \to NK_1\mathcal{O}(M-\{d\}) \oplus NK_1\mathcal{O}(d) \to NK_1\mathcal{O}(d)/(\gamma)$$

and thus $NK_1\mathcal{O}(M) \cong NK_1\mathcal{O}(M-\{d\})$.

By iterating this procedure, starting with M the set of all divisors of m and peeling off the minimal divisor $d \in M$, we obtain $NK_1Z\pi \cong NK_1\mathcal{O}(M) = 0$, proving the theorem.

Higher N's are defined recursively by $N^{j+1}K_1 = N(N^jK_1)$ for $j = 1,2,...$ Using Theorem 2.2 with

$A = Z[s,t]$, $B = Z[t]$ and $f: s \to 0$ we can use hyper-elementary induction to compute $N^2K_1Z\pi$. For π a hyperelementary group of square-free order, we tensor the Cartesian squares in the proof of Theorem 3.1 with $Z[t]$, producing new Cartesian squares and preserving regularity [2, Thm. 9.5]. Thus $NK_1Z\pi[t] = 0$. Continuing inductively we obtain

Corollary 3.2: $N^jK_1Z\pi = 0$ for π a finite group of square-free order, $j = 1,2,...$

Theorem 2.2 remains valid when K_1 is replaced by K_0 and so $NK_0Z[-]$ is also a $G_0Z[-]$-module. The Mayer-Vietoris sequences in the proof of Theorem 3.1 hold for NK_0 and $NK_0A = 0$ for A regular. Thus we have

Corollary 3.3: $N^jK_0Z\pi = 0$ for π a finite group of square-free order, $j = 1,2,...$

CHAPTER IV

A NONVANISHING RESULT

Theorem 4.1: Let R be the ring of integers in a number field F, π a finite group and $\alpha: \pi \to \rho$ a surjective homomorphism with ρ abelian. Then the induced homomorphism $\alpha_*: NK_1R\pi \to NK_1R\rho$ is surjective.

Proof: Let $e = (1/n)\cdot \sum_{x\in K} x$ where $K = \ker \alpha$ and $n = |K|$. Then e is a central idempotent in $F\pi$, and from the idempotents e and $1-e$ we obtain a Cartesian square

$$\begin{array}{ccc} R\pi & \xrightarrow{\beta} & \Lambda \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\delta} \\ R\rho & \xrightarrow[\gamma]{} & (R/nR)\rho \end{array}$$

where $\Lambda = R\pi/(\sum_{x\in K} x)$. We get a Mayer-Vietoris sequence

$$NK_1R\pi \xrightarrow{(\alpha_*,\beta_*)} NK_1R\rho \oplus NK_1\Lambda \xrightarrow{\gamma_*-\delta_*} NK_1(R/nR)\rho \ .$$

By inspection of the sequence it suffices to show $\gamma_*(NK_1R\rho) \subset \delta_*(NK_1\Lambda)$.

For any commutative ring A, the decomposition $K_1A \cong SK_1A \oplus U(A)$ gives $NK_1A \cong NSK_1A \oplus NU(A)$ where U is the functor $A \to$ units of A. If $f: A \to B$ is a homomorphism of commutative rings then the induced homomorphism $f_*: NK_1A \to NK_1B$ takes NSK_1A to NSK_1B and $NU(A)$ to $NU(B)$. Furthermore, $NU(A) \cong 1 + t\cdot Nil(A)[t]$ where $Nil(A)$ is the nilradical of A.

Since $R\rho$ is contained in the semisimple group algebra $(Q \otimes R)\rho$, $R\rho$ is reduced and so $NU(R\rho) = 0$. $(R/nR)\rho$ is a commutative Artinian ring and hence $NSK_1(R/nR)\rho = 0$ [1, Prop. 10.1, p. 685]. Therefore, $\gamma_*: NK_1R\rho \to NK_1(R/nR)\rho$ is the zero homomorphism and the assertion follows.

<u>Corollary 4.2</u>: Let π be a finite group and π^{ab} its abelianization. Then $NK_1Z\pi \to NK_1Z\pi^{ab}$ is surjective.

Martin [8, Thm. 3.12] showed that $NK_1ZC_{p^r} \neq 0$ for p an odd prime, $r \geq 2$ or $p = 2$, $r \geq 3$. So if π is a finite group with quotient C_{p^r} as above, then $NK_1Z\pi \neq 0$.

If $NK_1Z[C_2 \times C_2] \neq 0$ then for n even, $NK_1ZD_n \neq 0$ since $D_n^{ab} = C_2 \times C_2$. Here D_n denotes the dihedral group of order 2n. Whether $NK_1Z[C_2 \times C_2]$ vanishes or not is unknown at present. Knowledge of NK_2ZC_n would help in answering questions of this type.

We now observe that $NK_1Z[-]$ is in fact a Green module over $G_0Z[-]$:

For A a commutative ring and π a finite group, $K_1A[t]\pi$ has the Mackey subgroup property: for any groups $\rho, \rho' < \pi$ the composite

$$K_1A[t]\rho' \xrightarrow{i_*} K_1A[t]\pi \xrightarrow{i^*} K_1A[t]\rho$$

is equal to the sum, over all double cosets $\rho g \rho' \subset \pi$ of the composite

$$K_1A[t]\rho' \xrightarrow{i^*} K_1A[t](g^{-1}\rho g \cap \rho') \xrightarrow{\text{conj.}} K_1A[t](\rho \cap g\rho' g^{-1}) \xrightarrow{i_*} K_1A[t]\rho$$

This follows by the extension of the Mackey Subgroup Theorem to the case of an arbitrary commutative ring [4, p. 237]. Since $NK_1A\pi$ is a summand of $K_1A[t]\pi$ it inherits the above property, and with the $G_0^A A\pi$-module structure becomes a module over the Green functor $G_0^A A[-]$. By [5, Prop. 1.1' and 1.2],

$$NK_1Z\pi \cong \varinjlim_{\substack{\rho<\pi \\ \rho \in \mathcal{H}}} NK_1Z\rho \cong \varprojlim_{\substack{\rho<\pi \\ \rho \in \mathcal{H}}} NK_1Z\rho$$

where $\mathcal{H}$ is the collection of hyperelementary subgroups and the limits are taken with respect to inclusion and conjugation among subgroups in $\mathcal{H}$.

It is worth noting that the above observations hold for NK_0 as well.

CHAPTER V

NK_1 OF FREE PRODUCTS

Let Λ be an R-ring, i.e. R is a subring of Λ and there is a ring homomorphism $\varepsilon\colon \Lambda \to R$ such that $\varepsilon(r) = r$ for $r \in R$. The homomorphism $K_1R \to K_1\Lambda$ is injective, and denote the cokernel by $K_1(\Lambda;R)$.

Note that $\Lambda[t]$ is an $R[t]$-ring. The augmentation $t \to 0$ induces homomorphisms $K_1\Lambda[t] \to K_1\Lambda$ and $K_1R[t] \to K_1R$, thus inducing a homomorphism $K_1(\Lambda[t];R[t]) \to K_1(\Lambda;R)$. Denote the kernel by $NK_1(\Lambda;R)$.

We consider the case $R = Z$ and $\Lambda = Z\pi$ for any group π. Consider the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & NK_1Z & \to & K_1Z[t] & \to & K_1Z & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & NK_1Z\pi & \to & K_1Z[t]\pi & \to & K_1Z\pi & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & NK_1(Z\pi;Z) & \to & K_1(Z[t]\pi;Z[t]) & \to & K_1(Z\pi;Z) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

All rows and columns are exact, and $NK_1 Z = 0$ since Z is regular. Thus $NK_1(Z\pi;Z) \cong NK_1 Z\pi$.

Let ρ,η be arbitrary groups and let $\rho * \eta$ denote their free product. The integral group ring $Z[\rho * \eta]$ is isomorphic to the free product of rings $Z\rho \underset{Z}{*} Z\eta$. For a complete discussion of free products see [12, p. 355] or perhaps [1, p. 198].

$$\begin{aligned}(Z[\rho * \eta])[t] &\cong (Z\rho \underset{Z}{*} Z\eta)[t] \\ &\cong (Z\rho \underset{Z}{*} Z\eta) \underset{Z}{\otimes} Z[t] \\ &\cong (Z\rho \underset{Z}{\otimes} Z[t]) \underset{Z[t]}{*} (Z\eta \underset{Z}{\otimes} Z[t]) \quad [1, \text{p. } 202] \\ &\cong Z\rho[t] \underset{Z[t]}{*} Z\eta[t]\end{aligned}$$

We state a result of Stallings [13, Cor. 6.2.1]:

<u>Lemma 5.1</u>: If R is a regular commutative ring, and if A and B are kernels of retractions of the R-rings Λ and Γ onto R, and if $A \underset{R}{\otimes} B$ is a flat R-module, then $K_1(\Lambda * \Gamma;R) \cong K_1(\Lambda;R) \oplus K_1(\Gamma;R)$.

We shall apply this result to the cases $R = Z, Z[t]$, $\Lambda = Z\rho, Z\rho[t]$ and $\Gamma = Z\eta, Z\eta[t]$.

Consider the diagram,

$$
\begin{array}{ccccccccc}
0 & \to & NK_1(Z[\rho * \eta];Z) & \to & K_1(Z[\rho * \eta][t];Z[t]) & \to & K_1(Z[\rho * \eta];Z) & \to & 0 \\
 & & \uparrow{\scriptstyle =} & & \uparrow{\scriptstyle \cong} & & \uparrow{\scriptstyle =} & & \\
0 & \to & NK_1(Z[\rho * \eta];Z) & \to & K_1(Z\rho[t] \underset{Z[t]}{*} Z\eta[t];Z[t]) & \to & K_1(Z[\rho * \eta];Z) & \to & 0 \\
 & & {\scriptstyle \psi}\uparrow{\scriptstyle \text{induced}} & & \uparrow{\scriptstyle \cong} & & {\scriptstyle \text{induced by } \rho \to \rho * \eta \leftarrow \eta}\uparrow{\scriptstyle \cong} & & \\
0 & \to & NK_1(Z\rho;Z) \oplus NK_1(Z\eta;Z) & \to & K_1(Z\rho[t];Z[t]) \oplus K_1(Z\eta[t];Z[t]) & \to & K_1(Z\rho;Z) \oplus K_1(Z\eta;Z) & \to & 0
\end{array}
$$

All rows are exact by definition. The two bottom-right vertical maps are isomorphisms by Lemma 5.1, and commute with the horizontal maps by the naturality of the isomorphisms of Bass and Stallings. Thus ψ is an isomorphism and we have proved,

<u>Theorem 5.2</u>: Let ρ,η be arbitrary groups. Then $NK_1Z[\rho * \eta] \cong NK_1Z\rho \oplus NK_1Z\pi$.

<u>Corollary 5.3</u>: $NK_1Z[F] = 0$ for any free group F of finite rank.

Proof:
$$\begin{aligned} NK_1Z[F] &\cong NK_1Z[Z * \cdots * Z] \\ &\cong NK_1Z[Z] \oplus \cdots \oplus NK_1Z[Z] \\ &\cong NK_1Z[t,t^{-1}] \oplus \cdots \oplus NK_1Z[t,t^{-1}] \\ &= 0 \end{aligned}$$
by the regularity of $Z[t,t^{-1}]$ [2, Cor. 9.7].

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