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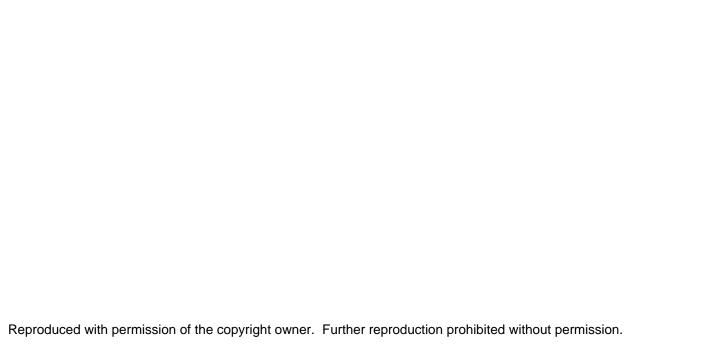
Harmon, Dennis Ray

NK1 OF NONABELIAN GROUPS

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THE UNIVERSITY OF OKLAHOMA GRADUATE COLLEGE

NK₁ OF NONABELIAN GROUPS

A DISSERTATION SUBMITTED TO THE GRADUATE FACULTY in partial fulfillment of the requirements for the degree of DOCTOR OF PHILOSOPHY

By
DENNIS RAY HARMON
Norman, Oklahoma
1984

NK₁ OF NONABELIAN GROUPS A DISSERTATION APPROVED FOR THE DEPARTMENT OF MATHEMATICS

Ву

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INTRODUCTION

Let Rng denote the category of associative rings with identity and let Z-mod denote the category of abelian groups. For a functor F: Rng + Z-mod the functors NF, LF: Rng + Z-mod are defined as follows: NF(A) = ker (F(A[t]) + F(A)) induced by the augmentation A[t] + A sending t to 0, and LF(A) = coker ($F(A[t]) + F(A[t]) + F(A[t]) + F(A[t,t^{-1}])$) induced by the obvious inclusions. For a contracted functor F such as K_0 , K_1 or SK_1 there is an isomorphism [1, Cor. 7.3, p. 663]

$$F(A[t_1,...,t_n,t_1^{-1},...,t_n^{-1}]) = (I + 2N + L)^n F(A).$$

In particular, for $F=K_1$ and $A=2\pi$ the integral group ring,

$$\mathsf{K}_1 \mathsf{Z} [\mathsf{Z} \times \pi] \ \cong \ \mathsf{K}_1 \mathsf{Z} \pi \ \oplus \ \mathsf{N} \mathsf{K}_1 \mathsf{Z} \pi \ \oplus \ \mathsf{N} \mathsf{K}_1 \mathsf{Z} \pi \ \oplus \ \mathsf{K}_0 \mathsf{Z} \pi.$$

Martin [8, Thm. 2.2] showed that $NK_1Z\pi = 0$ for π a finite abelian group of square-free order.

This thesis extends Martin's results to the case of an arbitrary finite group. The method is outlined as follows:

We first reduce to the case where π is hyperelementary by showing that $NK_1Z\pi$ is a Frobenius module over $G_0Z\pi$. Cartesian squares then give rise to natural Mayer-Vietoris exact sequences of K-groups [9] and hence exact sequences of NK-groups. We then use the fact that $NK_1A = 0$ for a regular ring A, i = 0,1 [3] and i = 2 [10].

Symbols used throughout this thesis are: $Z = \text{the integers, } Q = \text{the rationals, and } C_{m} = \text{the cyclic}$ group of order m.

At this time I would like to extend my thanks to my thesis advisor, Bruce Magurn for his invaluable guidance and support.

NK, OF NONABELIAN GROUPS

CHAPTER I

PRELIMINARIES

We begin by stating some definitions due to Lam [6, p. 106]. Let Sg denote the category whose objects are finite groups and whose arrows are inclusions i: $\rho \to \pi$ whenever $\rho < \pi$. A Frobenius functor is a contravariant functor F: Sg + Comm. Ring with arrow map (-)*, which is also a covariant functor Sg + Z-mod under an arrow map (-)*, such that $i_*(i^*(x)\cdot y) = x\cdot i_*(y)$ for $\rho \to \pi$ an arrow in Sg, $x \in F(\pi)$, $y \in F(\rho)$.

Let F be a Frobenius functor. An F-module is an object map and two arrow maps M(-), $(-)^*$, $(-)_*$: Sg + Z-mod such that M(-), $(-)^*$ is a contravariant functor, M(-), $(-)_*$ is a covariant functor, $M(\pi)$ is an $F(\pi)$ -module for each finite group π , and for i: $\rho + \pi$ in Sg,

$$i^*(x \cdot y) = i^*(x) \cdot i^*(y)$$
 for $x \in F(\pi)$, $y \in M(\pi)$
 $i_*(i^*(x) \cdot y) = x \cdot i_*(y)$ for $x \in F(\pi)$, $y \in M(\rho)$
 $i_*(x \cdot i^*(y)) = i_*(x) \cdot y$ for $x \in F(\rho)$, $y \in M(\pi)$.

A morphism of Frobenius functors F + G is a transformation F + G natural with respect to both arrow maps. A morphism of F-modules is a transformation $f \colon M \to N$ natural with respect to both arrow maps, such that $f \colon M(\pi) \to N(\pi)$ is $F(\pi)$ -linear for each finite group π .

For the remainder of this chapter, let R be a commutative ring with identity. For π a finite group and $R\pi$ the group ring of π over R, let \mathbf{Z}^R denote the category whose objects are the finitely generated left $R\pi$ -modules which are projective as R-modules and let $G_0^R R\pi$ denote $K_0(\mathbf{Z}^R)$, its Grothendieck group. $G_0^R R[-]$ is a Frobenius functor. For M,N objects in \mathbf{Z}^R the multiplication in $G_0^R R\pi$ is $[M] \cdot [N] = [M \ \overline{\Theta} \ N]$ where the bar $[M] \cdot [N] = [M \ \overline{\Theta} \ N]$ where the bar $[M] \cdot [R] \cdot [R$

According to Lam [6, p. 114], $K_0R[-]$ and $K_1R[-]$ are $G_0^RR[-]$ -modules. For [M] ϵ $G_0^RR\pi$, [N] ϵ $K_0R\pi$ the action is [M]·[N] = [M $\overline{\Theta}$ N], and for [N, α] ϵ $K_1R\pi$ the action is [M]·[N, α] = [M $\overline{\Theta}$ N, 1_M Θ α]. An arrow i: $\rho + \pi$ extends to a homomorphism i: $R\rho \to R\pi$ inducing homomor-

phisms on K_0 and K_1 by the usual restriction and extension of scalars.

Let $f(R\pi)$ denote the category of finitely generated left $R\pi$ -modules and let $G_0R\pi$ denote its Grothendieck group $K_0(f(R\pi))$. For a regular ring R, $G_0R\pi \cong G_0^RR\pi$ [13, Thm. 1.2] giving us a ring structure on $G_0R\pi$ even when $R\pi$ is not commutative, i.e. when π is not abelian.

Let C_m denote the cyclic group of order m. A finite group π is hyperelementary if $C_m \triangleleft \pi$ and $[\pi:C_m]=p^n$ for some prime $p, p \nmid m$, and some integer $n \geq 0$. This is equivalent to $\pi \in C_m \rtimes B$ with $|B|=p^n$. Let $\mathcal H$ be the collection of hyperelementary groups. By the work of Lam [6, p. 123],

$$M(\pi) = \sum_{\substack{\rho \in \mathcal{H} \\ \rho < \pi}} i_{\mathcal{H}} M(\rho)$$

for any $G_0^R[-]$ -module M[-]. This technique for computing M(π) is referred to as "hyperelementary induction" and will be used in Chapter III.

CHAPTER II

ANOTHER FROBENIUS MODULE

Let A,B be commutative rings with identity, $f\colon A\to B \text{ a ring homomorphism, } \pi \text{ a finite group and}$ $f_{\star}\colon K_1A\pi \to K_1B\pi \text{ the induced homomorphism. For}$ $[M] \in G_0^AA\pi, \quad [N,\alpha] \in K_1A\pi,$

$$\begin{split} \mathbf{f}_{\star}([\mathbf{M}] \cdot [\mathbf{N}, \alpha]) &= \mathbf{f}_{\star}([\mathbf{M} \ \overline{\Theta} \ \mathbf{N}, \ \mathbf{1}_{\mathbf{M}} \ \Theta \ \alpha]) \\ &= [\mathbf{B}\pi \ \Theta \ (\mathbf{M} \ \overline{\Theta} \ \mathbf{N}), \ \mathbf{1}_{\mathbf{B}\pi} \ \Theta \ (\mathbf{1}_{\mathbf{M}} \ \Theta \ \alpha)] \\ [\mathbf{M}] \cdot \mathbf{f}_{\star}([\mathbf{N}, \alpha]) &= [\mathbf{B} \ \overline{\Theta} \ \mathbf{M}] \cdot [\mathbf{B}\pi \ \Theta \ \mathbf{N}, \ \mathbf{1}_{\mathbf{B}\pi} \ \Theta \ \alpha] \\ &= [(\mathbf{B} \ \overline{\Theta} \ \mathbf{M}) \ \overline{\Theta} \ (\mathbf{B}\pi \ \Theta \ \mathbf{N}), \ (\mathbf{1}_{\mathbf{B}} \ \Theta \ \mathbf{1}_{\mathbf{M}}) \ \Theta \ (\mathbf{1}_{\mathbf{B}\pi} \ \Theta \ \alpha)] \end{split}$$

Lemma 2.1: $B\pi \otimes (M \otimes N) \approx (B \otimes M) \otimes (B\pi \otimes N)$ as $B\pi$ - $A\pi A B A\pi$ modules.

Proof: Define $\phi: B\pi \otimes (M \overline{\otimes} N) \rightarrow (B \overline{\otimes} M) \overline{\otimes} (B\pi \otimes N)$ $A\pi A A B A\pi$ by $(bg\Theta(m\Theta n)) = (b\Theta gm)\Theta(l\Theta gn)$ for $bg \in B\pi$, $m \in M$, $n \in N$. Then ϕ is π -linear: for h ε π,

$$\phi(h \cdot (bg@(m@n))) = \phi(bhg@(m@n))$$

$$= (b@hgm)@(1@hgn)$$

$$= h \cdot ((b@gm)@(1@gn))$$

by the diagonal $\pi\text{-actions}$ and the trivial action of π on B. Thus φ is B -linear.

Define ψ : $(B \ \overline{\otimes} \ M) \ \overline{\otimes} \ (B\pi \ \Theta \ N) \rightarrow B\pi \ \Theta \ (M \ \overline{\Theta} \ N)$ $A \quad B \quad A\pi \quad A\pi \quad A$ by $\psi((b\Theta m)\Theta(rg\Theta n)) = rbg\Theta(g^{-1}m\Theta n)$ for $b \in B$, $rg \in B\pi$, $m \in M$, and $n \in N$. Then ψ is π -linear:
for $h \in \pi$,

$$\psi(h \cdot ((b \otimes m) \otimes (rg \otimes n))) = \psi((b \otimes hm) \otimes (rhg \otimes n))$$

$$= rbhg \otimes (r(hg)^{-1}hm \otimes n)$$

$$= rbhg \otimes (rg^{-1}m \otimes n)$$

$$= h \cdot (rbg \otimes (g^{-1}m \otimes n))$$

So ψ is $B\pi\text{-linear}$ as well. Checking composites,

$$\phi\psi((b@m)@(rg@n)) = \phi(rbg@(g^{-1}m@n))$$

$$= (rb@g(g^{-1}m))@(1@gn)$$

$$= (rb@m)@(1@gn)$$

$$= (b@m)@(rg@n)$$

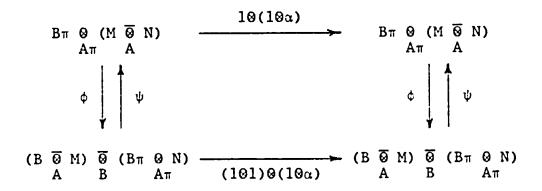
$$\psi\phi(bg\Theta(m\Thetan) = \psi((b\Thetagm)\Theta(1\Thetagn))$$

$$= b\Theta(gm\Thetagn)$$

$$= bg\Theta(m\Thetan).$$

Thus. $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ are inverses and the isomorphism has been proved.

By the commutative square



and Lemma 2.1, $f_*([M] \cdot [N,\alpha]) = [M] \cdot f_*([N,\alpha])$ in $K_1B\pi$. For i: $\rho \to \pi$ in Sg, the B π -linear isomorphisms

$$B\pi$$
 Θ $(B\rho$ Θ $N)$ \cong $B\pi$ Θ N \cong $B\pi$ Θ $(A\pi$ Θ $N)$ $B\rho$ $A\rho$ $A\rho$ $A\pi$ $A\rho$

yield $i_*f_* = f_*i_*$. Also, $i^*f_* = f_*i^*$ so f is natural with respect to both arrow maps. Thus we have proved,

Theorem 2.2: For a homomorphism $f: A \to B$ of commutative rings, the induced homomorphism $f_*: K_1A[-] \to K_1B[-]$ is a morphism of $G_0^A[-]$ -modules.

Corollary 2.3: $NK_1Z[-]$ is a $G_0Z[-]$ -module.

Proof: The category of $G_0^A[-]$ -modules is an abelian category [6, p. 108], so $\ker f_*$ is a $G_0^A[-]$ -module. The ring homomorphism $Z \to A$ induces a morphism $G_0Z[-] = G_0^Z[-] + G_0^A[-]$, making $\ker f_*$ a $G_0Z[-]$ -module. Now apply Theorem 2.2 with A = Z[t], B = Z, and f = augmentation.

Corollary 2.4: For any finite group π ,

$$NK_{1}Z\pi = \sum_{\substack{\rho \in \mathcal{H} \\ \rho < \pi}} i_{*}NK_{1}Z_{\rho}$$

where ${\cal H}$ is the class of hyperelementary groups.

CHAPTER III

A VANISHING RESULT

We are now in a position to prove the main result of this thesis.

Theorem 3.1: If π is a finite group of square-free order then $NK_1Z\pi = 0$.

Proof: By Corollary 2.4 we can assume that π is hyperelementary. Let $\pi = C_m \rtimes B$ with |B| = p, p a prime and $p \nmid m$. Let $C_m = \langle a \rangle$, $C_p = B = \langle b \rangle$. Then π has the presentation

$$\pi = \langle a,b : a^{m}, b^{p}, bab^{-1}a^{-\alpha} \rangle$$

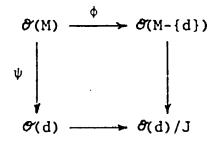
where $\alpha^p \equiv 1 \pmod m$. If $\alpha \equiv 1 \pmod m$ then π is abelian and thus $NK_1Z\pi = 0$ [8, Thm. 2.2].

For a divisor d of m, let ζ_d denote a complex primitive dth root of unity. The twisted group algebra $Q(\zeta_d)$ B has the additive structure of a free right $Q(\zeta_d)$ -module based on B and multiplicative structure determined

by that in $Q(\zeta_d)$, by that in B, and the rule ba = $a^{\alpha}b$. For any collection M of positive divisors of m, let $\Theta(M)$ denote the image of $Z\pi$ under the composite

$$Z\pi \longrightarrow Q\pi \xrightarrow{\Xi} \bigoplus_{\mathbf{d}\mid \mathbf{m}} Q(\zeta_{\mathbf{d}}) \circ B \longrightarrow \bigoplus_{\mathbf{d}\in \mathbf{M}} Q(\zeta_{\mathbf{d}}) \circ B$$

For each $d \in M$ there is a Cartesian square



where ϕ, ψ are projections and $J = \psi(\ker \phi)$ is the ideal of $\mathcal{O}(d)$ generated by $\prod_{e \in M-\{d\}} \Phi_e(\zeta_d)$, and $\Phi_e(x)$ is the minimal polynomial of ζ_e over Q [7, pp. 403-4]. Then $\Phi_e(\zeta_d)$ is a unit of $Z[\zeta_d]$ if neither d/e nor e/d is a power of a prime, and is associate in $Z[\zeta_d]$ to a prime q if e/d = q^r for some r > 0 [7, Lemma 9.3]. Choosing d minimal in M and using the fact that m is square-free we obtain

where " \sim " means associate to. So for a minimal divisor d ϵ M, J is the ideal generated by γ .

If d is such that $B \to Aut(Q(\zeta_d))$ has a non-trivial kernel then $\mathcal{O}(d)$ is in fact the group ring $Z[\zeta_d]B$ and $NK_1Z[\zeta_d]B = 0$ [8, Thm. 2.2]. This occurs exactly when $d \mid (m,\alpha-1)$.

Otherwise, B acts faithfully on $Q(\zeta_d)$. Let $F = Q(\zeta_d)^B$, the subfield of $Q(\zeta_d)$ left fixed by the action of B. The cyclic algebra $Q(\zeta_d) \circ B = (Q(\zeta_d)/F, b, 1)$ [11, p. 259] is a crossed-product algebra with trivial factor set and hence isomorphic to $M_p(F)$ [11, Cor. 29.8]. Then $\dim_F Q(\zeta_d) = p$ and by choosing the integral basis $\{1, \zeta_d, \ldots, \zeta_d^{p-1}\}$ for $Q(\zeta_d)/F$, the isomorphism $Q(\zeta_d) \circ B \to M_p(F)$ above restricts to $Z[\zeta_d] \circ B = \mathcal{O}(d) \to M_p(R)$ where R is the ring of integers in $Q(\zeta_d)^B$.

For a nonzero prime $P \triangleleft R$, $P \cdot Z[\zeta_d] = P_1^e \cdots P_g^e$ where P_1, \ldots, P_g are distinct maximal ideals in the Dedekind domain $Z[\zeta_d]$ and e is the ramification index. Since $e \mid p$, e = 1 or e = p. If e = 1 then P is unramified in $Z[\zeta_d]$. Otherwise e = p; but the only primes $P \triangleleft R$ that ramify are those for which $q \mid d$ where $Z \cap P = qZ$, and $p \nmid d$ since $p \nmid m$. Hence $e \neq 0$ in Z/qZ and P is tamely ramified in $Z[\zeta_d]$. So $Z[\zeta_d] \circ B$ is hereditary [11, Thm. 40.15] and hence regular. Thus $NK_i Z[\zeta_d] \circ B = 0$ for i = 0,1,2.

 $M_p(R)$ is a maximal order containing $\mathfrak{G}(d)$ and $d \cdot M_p(R) \subset \mathcal{O}(d)$, so $\mathcal{O}(d)/(\gamma) \cong M_p(R/(\gamma))$ [7, Prop. 10.2]. For each pair of primes p_i, p_j in the factorization of γ , $p_i Z + p_j Z = Z$ and so $p_i R + p_j R = R$. By the Chinese Remainder Theorem $R/(\gamma) \approx \bigoplus_{i} R/(p_i)$ and thus $M_p(R/(\gamma)) \approx \bigoplus_i M_p(R/(p_i))$. Since $p_i \nmid d$, (p_i) is unramified in $Z[\zeta_d]$, hence in R as well. So $(p_i) = \prod_{i} P_{ij}$ where $\{P_{ij}\}$ are distinct maximal ideals in R. Thus $\bigoplus_{i} M_{p}(R/(p_{i})) \cong \bigoplus_{i,j} M_{p}(R/P_{ij})$, a direct sum of matrix rings

over fields, hence regular. So $NK_{i}(\sigma(d)/(\gamma)) = 0$ for i = 0,1,2.

The Mayer-Vietoris exact sequence resulting from the Cartesian square is

 $NK_2 \sigma(d)/(\gamma) + NK_1 \sigma(M) + NK_1 \sigma(M-\{d\}) \oplus NK_1 \sigma(d) + NK_1 \sigma(d)/(\gamma)$

and thus $NK_1 \mathcal{O}(M) = NK_1 \mathcal{O}(M-\{d\})$.

By iterating this procedure, starting with M. the set of all divisors of m and peeling off the minimal divisor d ϵ M, we obtain $NK_1Z\pi = NK_1\mathcal{O}(M) = 0$, proving che theorem.

Higher N's are defined recursively by $N^{j+1}K_1 = N(N^jK_1)$ for j = 1,2,... Using Theorem 2.2 with A = Z[s,t], B = Z[t] and f: s + 0 we can use hyper-elementary induction to compute $N^2K_1Z\pi$. For π a hyperelementary group of square-free order, we tensor the Cartesian squares in the proof of Theorem 3.1 with Z[t], producing new Cartesian squares and preserving regularity [2, Thm. 9.5]. Thus $NK_1Z\pi[t] = 0$. Continuing inductively we obtain

Corollary 3.2: $N^{j}K_{1}Z\pi = 0$ for π a finite group of square-free order, j = 1, 2, ...

Theorem 2.2 remains valid when K_1 is replaced by K_0 and so $NK_0Z[-]$ is also a $G_0Z[-]$ -module. The Mayer-Vietoris sequences in the proof of Theorem 3.1 hold for NK_0 and $NK_0A = 0$ for A regular. Thus we have

Corollary 3.3: $N^{j}K_{0}Z\pi = 0$ for π a finite group of square-free order, j = 1, 2, ...

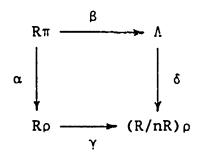
CHAPTER IV

A NONVANISHING RESULT

Theorem 4.1: Let R be the ring of integers in a number field F, π a finite group and $\alpha: \pi \to \rho$ a surjective homomorphism with ρ abelian. Then the induced homomorphism $\alpha_*: NK_1R\pi \to NK_1R\rho$ is surjective.

Proof: Let $e = (1/n) \cdot \sum_{x \in K} x$ where $K = \ker \alpha$ and n = |K|. Then e is a central idempotent in $F\pi$, and

from the idempotents e and 1-e we obtain a Cartesian square



where $\Lambda = R\pi/(\sum x)$. We get a Mayer-Vietoris sequence $x \in K$

$$NK_1R\pi \xrightarrow{(\alpha_*, \beta_*)} NK_1R\rho \oplus NK_1\Lambda \xrightarrow{\gamma_*-\delta_*} NK_1(R/nR)\rho$$
.

By inspection of the sequence it suffices to show $\gamma_{\star}(NK_{1}R\rho)\ \ C\ \delta_{\star}(NK_{1}\Lambda)\,.$

For any commutative ring A, the decomposition $K_1A \cong SK_1A \oplus U(A)$ gives $NK_1A \cong NSK_1A \oplus NU(A)$ where U is the functor $A \rightarrow units$ of A. If $f: A \rightarrow B$ is a homomorphism of commutative rings then the induced homomorphism $f_*: NK_1A \rightarrow NK_1B$ takes NSK_1A to NSK_1B and NU(A) to NU(B). Furthermore, $NU(A) \cong 1 + t \cdot Nil(A)[t]$ where Nil(A) is the nilradical of A.

Since R ρ is contained in the semisimple group algebra (Q 0 R) ρ , R ρ is reduced and so NU(R ρ) = 0. (R/nR) ρ is a commutative Artinian ring and hence NSK₁(R/nR) ρ = 0 [1, Prop. 10.1, p. 685]. Therefore, $\gamma_*: NK_1R\rho \to NK_1(R/nR)\rho$ is the zero homomorphism and the assertion follows.

Corollary 4.2: Let π be a finite group and π^{ab} its abelianization. Then $NK_1Z_{\pi} \rightarrow NK_1Z_{\pi}^{ab}$ is surjective.

Martin [8, Thm. 3.12] showed that $NK_1ZC_pr \neq 0$ for p an odd prime, $r \geq 2$ or p = 2, $r \geq 3$. So if π is a finite group with quotient C_pr as above, then $NK_1Z\pi \neq 0$.

 $\text{If } NK_1Z[C_2\times C_2] \neq 0 \text{ then for n even,} \\ NK_1ZD_n \neq 0 \text{ since } D_n^{ab} = C_2\times C_2. \text{ Here } D_n \text{ denotes the}$

dihedral group of order 2n. Whether $NK_1Z[C_2 \times C_2]$ vanishes or not is unknown at present. Knowledge of NK_2ZC_n would help in answering questions of this type.

We now observe that $NK_1Z[-]$ is in fact a Green module over $G_0Z[-]$:

For A a commutative ring and π a finite group, $K_1A[t]\pi \quad \text{has the Mackey subgroup property:} \quad \text{for any groups}$ $\rho,\rho'<\pi \quad \text{the composite}$

$$K_1A[t]\rho' \xrightarrow{i_*} K_1A[t]\pi \xrightarrow{i^*} K_1A[t]\rho$$

is equal to the sum, over all double cosets $\ \rho g \rho ' \ \boldsymbol{c} \ \pi$ of the composite

$$K_1A[t]\rho' \xrightarrow{i^*} K_1A[t](g^{-1}\rho g n \rho') \xrightarrow{conj} K_1A[t](\rho n g \rho' g^{-1}) \xrightarrow{i_*} K_1A[t]\rho$$

This follows by the extension of the Mackey Subgroup Theorem to the case of an arbitrary commutative ring [4, p. 237]. Since $NK_1A\pi$ is a summand of $K_1A[t]\pi$ it inherits the above property, and with the $G_0^AA\pi$ -module structure becomes a module over the Green functor $G_0^AA[-]$. By [5, Prop. 1.1' and 1.2],

where \mathcal{H} is the collection of hyperelementary subgroups and the limits are taken with respect to inclusion and conjugation among subgroups in \mathcal{H} .

 $\label{eq:local_state} \mbox{ It is worth noting that the above observations hold for NK_0 as well.}$

CHAPTER V

NK, OF FREE PRODUCTS

Let Λ be an R-ring, i.e. R is a subring of Λ and there is a ring homomorphism $\epsilon\colon\Lambda\to R$ such that $\epsilon(r)=r$ for $r\in R$. The homomorphism $K_1R\to K_1\Lambda$ is injective, and denote the cokernel by $K_1(\Lambda;R)$.

Note that $\Lambda[t]$ is an R[t]-ring. The augmentation $t \to 0$ induces homomorphisms $K_1 \Lambda[t] \to K_1 \Lambda$ and $K_1 R[t] \to K_1 R$, thus inducing a homomorphism $K_1(\Lambda[t];R[t]) \to K_1(\Lambda;R)$. Denote the kernel by $NK_1(\Lambda;R)$.

We consider the case R=Z and $\Lambda=Z\pi$ for any group π . Consider the diagram

All rows and columns are exact, and $NK_1Z = 0$ since Z is regular. Thus $NK_1(Z\pi;Z) = NK_1Z\pi$.

Let ρ,η be arbitrary groups and let $\rho*\eta$ denote their free product. The integral group ring $Z[\rho*\eta]$ is isomorphic to the free product of rings $Z\rho*Z\eta$. For a complete discussion of free products Z see [12, p. 355] or perhaps [1, p. 198].

$$(2[\rho * \eta])[t] \approx (2\rho * Z\eta)[t]$$

$$\approx (2\rho * Z\eta) \otimes Z[t]$$

$$\approx (2\rho \otimes Z[t]) * (2\eta \otimes Z[t]) [1, p. 202]$$

$$\approx (2\rho \otimes Z[t]) * Z\eta[t]$$

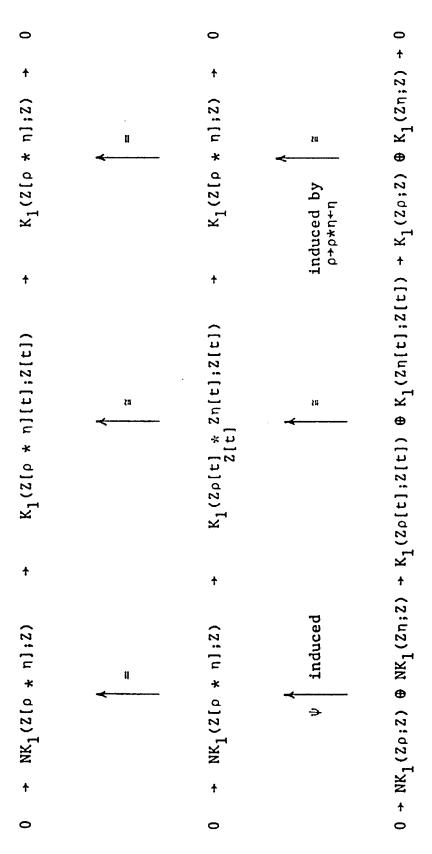
$$\approx Z\rho[t] * Z\eta[t]$$

We state a result of Stallings [13, Cor. 6.2.1]:

Lemma 5.1: If R is a regular commutative ring, and if A and B are kernels of retractions of the R-rings A and Γ onto R, and if A \otimes B is a flat R-module, then $K_1(\Lambda * \Gamma;R) \cong K_1(\Lambda;R) \oplus K_1(\Gamma;R)$.

We shall apply this result to the cases $R = Z_1Z_1$, $\Lambda = Z_0, Z_0[t]$ and $\Gamma = Z_1, Z_1[t]$.

Consider the diagram,



All rows are exact by definition. The two bottom-right vertical maps are isomorphisms by Lemma 5.1, and commute with the horizontal maps by the naturality of the isomorphisms of Bass and Stallings. Thus ψ is an isomorphism and we have proved,

Theorem 5.2: Let ρ, η be arbitrary groups. Then $NK_1Z[\rho * \eta] \cong NK_1Z\rho \oplus NK_1Z\pi$.

Corollary 5.3: $NK_1Z[F] = 0$ for any free group F of finite rank.

Proof: $NK_1Z[F] \cong NK_1Z[Z * \cdots * Z]$ $\cong NK_1Z[Z] \oplus \cdots \oplus NK_1Z[Z]$ $\cong NK_1Z[t,t^{-1}] \oplus \cdots \oplus NK_1Z[t,t^{-1}]$ = 0

by the regularity of $Z[t,t^{-1}]$ [2, Cor. 9.7].

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