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DEFLECTION AND VIBRATION IN SHEAR-DEFORMABLE BEAMS AND LAMINATED COMPOSITES MADE OF BIMODULAR AND MULTIMODULAR MATERIALS

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THE UNIVERSITY OF OKLAHOMA

GRADUATE COLLEGE

DEFLECTION AND VIBRATION IN SHEAR-DEFORMABLE BEAMS AND LAMINATED COMPOSITES MADE OF BIMODULAR AND MULTIMODULAR MATERIALS

A DISSERTATION

SUBMITTED TO THE GRADUATE FACULTY

in partial fulfillment of the requirements for the

degree of

DOCTOR OF PHILOSOPHY

By

FARAMARZ GORDANINEJAD

Norman, Oklahoma

1983

DEFLECTION AND VIBRATION IN SHEAR-DEFORMABLE BEAMS
AND LAMINATED COMPOSITES MADE OF BIMODULAR
AND MULTIMODULAR MATERIALS

A DISSERTATION

APPROVED FOR THE SCHOOL OF AEROSPACE, MECHANICAL
AND NUCLEAR ENGINEERING

By

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[illegible]

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ABSTRACT

A transfer-matrix analysis is presented for determining the static and dynamic behavior of thick, orthotropic beams of "multimodular materials" (i.e., materials which have different elastic behavior in tension and compression, with nonlinear stress-strain curve approximated as piecewise linear, with four or more segments). Also, an exact solution is presented for cases in which the neutral-surface location is constant along the beam axis. Results for axial displacement, transverse deflection, bending slope, frequency, bending moment, transverse shear force, axial force, and location of neutral surface are presented for different load and boundary conditions. In addition, comparisons are made among multimodular, bimodular, and unimodular models for a beam with aramid-cord-rubber properties taken from experimental stress-strain curves.

Also, presented is a closed-form solution of the static equilibrium equation for a bimodular composite laminate in cylindrical bending. The resulting shear-stress distribution is used to obtain expressions for the Timoshenko-type shear correction coefficient and the maximum dimensionless transverse shear stress. Based on this more accurate prediction of shear correction coefficient for bimodular materials, a new theory is developed for the vibration of a bimodular sandwich beam with thick facings which agrees much better with experimental results than classical sandwich theory.

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Appreciation for the understanding and patience of his loving wife, Roya, can hardly be expressed in words.

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NOMENCLATURE

a	partial extensional stiffness
a_c, a_t	break points defined in Fig. 2.3
A, A'	extensional stiffnesses
b	partial flexural-extensional coupling stiffness
B, B', B''	flexural-extensional coupling stiffnesses
c	core thickness
$C_i (i=1,4)$	constant coefficient defined by Eqs. (2.21)
C_A^N, C_B^N C_B^M, C_D^M	stiffnesses defined in Appendix B
d_1, d_2	constant coefficients defined by Eqs. (2.21)
D, D'	flexural stiffnesses
E	longitudinal Young's modulus
E_{cc}, E_{tc}	core Young's moduli in compression and tension, respectively
f	frequency
g	transverse shear moduli ratio
g_1, g_2	approximated functions defined in Fig. A.1
G	transverse shear modulus
h	laminate or beam thickness
h_c, h_t	distances defined in Fig. G.1

h_f	facing thickness
H_c	defined by Eq. (G.22)
I	rotatory inertia coefficient
K^2	shear correction coefficient
ℓ	length of laminate or beam
$L_i(i=1,2,3,4)$	defined by Eq. (G.24)
$\Delta L_i(i=1,N_s)$	length of each element
m	mode number
M	moment resultant
$M_i(i=1,N_s)$	moment resultant at each station
N	force resultant
N_s	number of stations
P,P_1,P_2	translational inertia coefficients
$\bar{\bar{p}}$	constant defined by Eq. (5.23)
q	transverse load
q_o	amplitude of q
$\bar{\bar{q}}$	constant defined by Eq. (5.23)
Q	transverse resultant
$\bar{\bar{r}}$	constant defined by Eq. (5.23)
R	coupling inertia coefficient
S	thickness-shear stiffness
t	time
$[t_s],[t_f]$	station and field matrices
u,w	mid-plane displacements in the x and z directions
U,W	total displacements in the x and z directions
$\bar{U},\bar{W},\bar{X}$	amplitudes of u, w, ψ

x,y,z	position coordinates in the longitudinal, transverse, and downward normal directions (measured from the mid-plane)
z'	downward normal position coordinate measured from the nuetral surface
z_n	neutral-surface position measured in the z-coordinate system
z_{nx},z_{ny}	neutral-surface positions for $\varepsilon_x = 0$ and $\varepsilon_y = 0$, respectively
Z	dimensionless z_n, $Z = z_n/h$
ε	strain
$\varepsilon_f^c,\varepsilon_f^t$	defined in Fig. A.1
ε_x	normal strain in the x direction
γ	transverse shear strain
$\gamma_{xz},\tilde{\gamma}_{xz}$	engineering shear strain in xz plane: actual and uniform-shear-distribution cases
τ_{xz}	shear stress in xz plane
σ	stress
σ_x	normal stress in the x direction
ψ	bending slope
ω,Ω	frequency
Subscript	
1,2,3	layer number
(c,t)	(compression, tension)
$(\ )_{,x}$	$\partial(\ )/\partial x$

Superscript

o	midplane
1,2	section number
-	dimensionless
(c,t)	(compression, tension)
(k)	index (= c,t)
b	bimodular
p	particular solutions
L	left
R	right

SECTION I

INTRODUCTORY REMARKS

Composite materials, especially those in the form of fiber-reinforced laminates, have many applications in modern engineering structures. The major advantages include high strength/weight, high stiffness/weight, and good corrosion resistance. In addition, there is considerable design versatility due to the fact that by varying the fiber volume fraction, fiber orientation, and stacking sequence, one can tailor the material to the specific application. These characteristics have made an impressive impact on engineering, particularly in the aerospace industry.

Predicting the behavior of thick beams and laminates becomes more important as the use of such material steadily increases. It is now well known that certain materials have different stress-strain behavior when they are loaded in tension and in compression. Experimental evidence of this behavior has been found in numerous materials including cast iron, tire-cord-rubber, concrete, epoxies, rock, and soft biological tissues.

These materials not only have a different behavior under tension and compression but some of them, such as tire-cord-rubber, have a drastic nonlinear stress-strain curve which makes the analysis of these materials much more tedious. Stress-strain curves for such materials could be approximated to be bilinear with one modulus when the fibers are stretched and another when they are compressed. However, in the present work this approximation is extended to two segments in tension and two segments in compression.

The analysis of laminated bimodular material is more complicated than unimodular material (ordinary material) due to the dependency of the material stiffness on the material properties which, indeed, depend on the state of stress (i.e., tensile or compressive) in the laminate. Since stiffnesses are functions of neutral-surface position and neutral-surface location is not known à priori, an iteration procedure is needed in the general case. But, for the special cases when the axial force is zero (e.g., clamped-free beam), the position of the neutral surface is constant along the length of the beam and consequently the computations are much easier.

For these special cases, a closed-form solution can be obtained, whereas for the general case, a numerical method must be implemented. In the present work a transfer-matrix method is used. This method, which has been proven to be very efficient computationally, is applied for fairly complicated conditions of loading and boundary.

Sections II and III of the present work deal with the analysis of a thick multimodular beam with rectangular cross section and in the last two sections, emphasis is given to the transverse shear effect in bimodular laminate and sandwich beams.

SECTION II

DEFLECTION OF A THICK BEAM OF MULTIMODULAR MATERIAL†

A transfer-matrix analysis is presented for determining the static behavior of thick beams of "multimodular materials" (i.e., materials which have different elastic behavior in tension and compression, with non-linear stress-strain curves approximated as piecewise linear, with four or more segments). To validate the transfer-matrix method results, a closed-form solution is also presented for cases in which the neutral-surface location is constant along the beam axis. Numerical results for axial displacement, transverse deflection, bending slope, bending moment, transverse shear, axial force, and location of neutral surface are presented for multimodular and bimodular models of unidirectional aramid-cord-rubber. The transfer-matrix method results agree very well with the closed-form solutions.

2.1 Introduction

In 1941 Timoshenko [2.1] considered the flexural stresses in bimodular material, i.e., a bilinear material having different moduli in tension and in compression. Ambartsumyan [2.2] in 1965, introduced the terminology, "bimodulus", and extended the concept to two-dimensional analysis.

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† An abbreviated version of this section will appear in International Journal for Numerical Methods in Engineering, (Nov.,1983).

Numerous static papers appeared after this work; Marin [2.3] gave the effective modulus for stiffness of bimodular beam undergoing pure bending.

Small-deflection bending of Bernoulli-Euler beams of homogeneous, bimodular material was treated in Refs. 2.4-2.12. Large static deflections of beams of bimodular material were analyzed in Refs. 2.13-2.14. Kamiya [2.15] considered transverse-shear-deformation effects on bimodular beams for the first time. Recently, Tran and Bert [2.16] treated bending of thick beams of bimodular materials and obtained both closed-form and transfer-matrix solutions. To the best of the present author's knowledge, no previous work is available in the context of multimodular beams.

2.2 Modeling of the Stress-Strain Curve

Bert and Kumar [2.17] recently presented experimental stress-strain curves for unidirectional cord-rubber materials. In the present work, a stress-strain curve for aramid-rubber taken from [2.17] has been linearly approximated by four segments (two segments in tension and two segments in compression). For choosing the "break points" t and c (see Fig. 2.1), the area between two fitting lines and the experimental curve in each portion has been minimized to achieve the "best" fit (see Appendix A.1). To find comparable moduli for the bimodular case, one has to minimize the area between two straight lines and the experimental curve (see Appendix A.2). Finally, for the unimodular case, the "best" single straight line is used (see Appendix A.3).

2.3 Theory and Formulation

Consider a rectangular-cross-section beam of thickness h and length ℓ as shown in Fig. 2.2. The origin of the Cartesian coordinate system

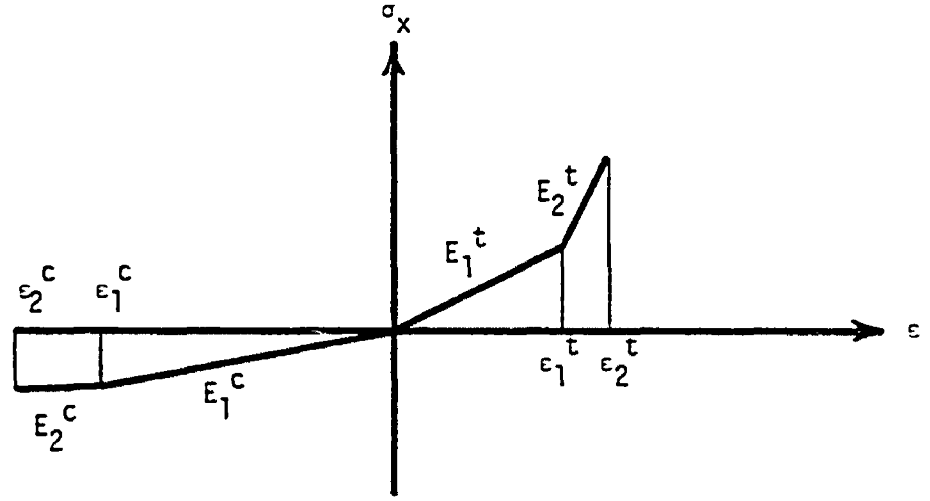


Fig. 2.1. Multimodular model.

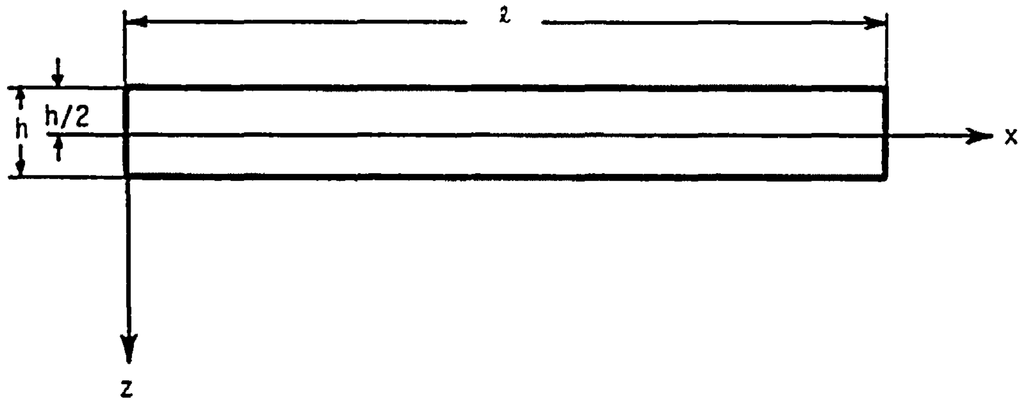


Fig. 2.2. Geometry of beam.

is located on the mid-surface of the beam with the z-axis being measured positive downward.

2.3.1 Displacement Field

The same displacement field used in classical Timoshenko beam theory is implemented here

$$U(x,z) = u(x) + z\psi(x) \quad , \quad W(x,z) = w(x) \tag{2.1}$$

where U and W are displacements in the x and z directions, respectively, u and w are corresponding displacements at the midplane, and ψ is the bending slope.

2.3.2 Stress Field

For a four-segment approximation of the normal stress-strain curve, considering the general case (i.e., when $-\frac{h}{2} < a_c$, $a_t < \frac{h}{2}$), the following stress field has been considered for the case of convex bending (see Figs. 2.1 and 2.3).

$$\sigma_x \equiv \begin{cases} E_1^c \, \varepsilon_1^c + E_2^c(\varepsilon_x - \varepsilon_1^c) & -h/2 \leq z \leq a_c \\ E_1^c \varepsilon_x & a_c \leq z \leq z_n \\ E_1^t \varepsilon_x & z_n \leq z \leq a_t \\ E_1^t \varepsilon_1^t + E_2^t(\varepsilon_x - \varepsilon_1^t) & a_t \leq z \leq h/2 \end{cases} \tag{2.2}$$

$$\tau_{xz} = G\gamma_{xz}$$

where E_1^c, E_2^c, E_1^t, E_2^t, G, ε_1^c, and ε_1^t are material constants, σ_x is the axial normal stress, ε_x is the axial normal strain, γ_{xz} is the transverse shear strain, τ_{xz} is the transverse shear stress, and z_n is the location of the neutral surface. It is noted that this material is linear elastic in shear. Comparison of Figs. 2.1 and 2.3 leads to

$$\varepsilon_x = \kappa(z - z_n) \tag{2.3}$$

$$\varepsilon_1{}^c = \kappa(a_c - z_n) \tag{2.4}$$

$$\varepsilon_1{}^t = \kappa(a_t - z_n) \tag{2.5}$$

$$\varepsilon_f{}^c = \kappa(-h/2 - z_n) \tag{2.6}$$

$$\varepsilon_f{}^t = \kappa(h/2 - z_n) \tag{2.7}$$

where $\varepsilon_f{}^c$ and $\varepsilon_f{}^t$ are the final values attained at the respective compressive and tensile outer fibers and κ is the curvature.

Using linear strain measure and the strain field of Eqs. (2.1), one obtains

$$\begin{aligned} \varepsilon_x &= U_{,x} = u_{,x} + z\psi_{,x} \\ \gamma_{xz} &= W_{,x} + U_{,z} = w_{,x} + \psi \end{aligned} \tag{2.8}$$

Comparison of Eqs. (2.3) and (2.8) gives

$$u_{,x} = -\kappa z_n \quad , \quad \psi_{,x} = \kappa \tag{2.9}$$

Note that $(\ )_{,x}$ denotes $d(\ )/dx$.

2.3.3 Constitutive Relation

For the assumed beam, the normal and transverse shear stress resultants and moment, each per unit width, are defined as

$$(N,Q) = \int_{-h/2}^{h/2} (\sigma_x, \tau_{xz})dz \quad , \quad M = \int_{-h/2}^{h/2} z\sigma_x dz \tag{2.10}$$

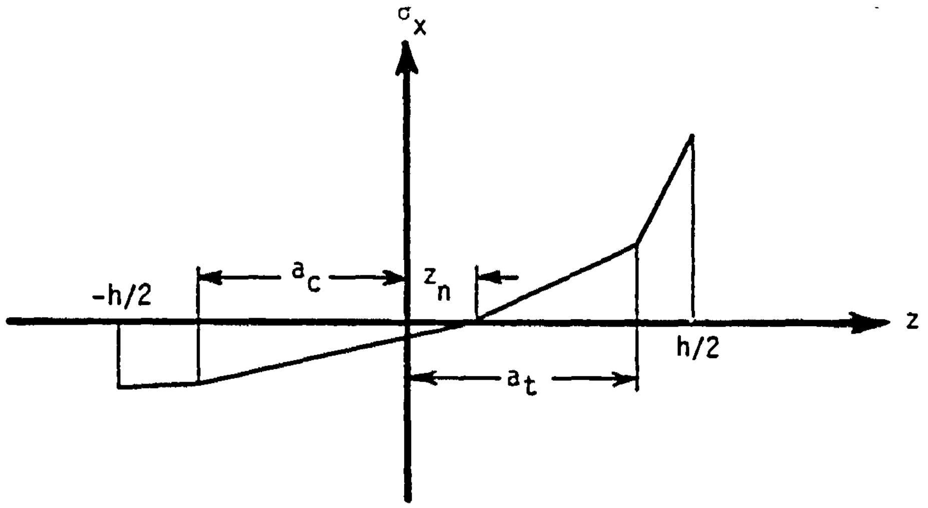


Fig. 2.3. Stress distribution for convex bending case.

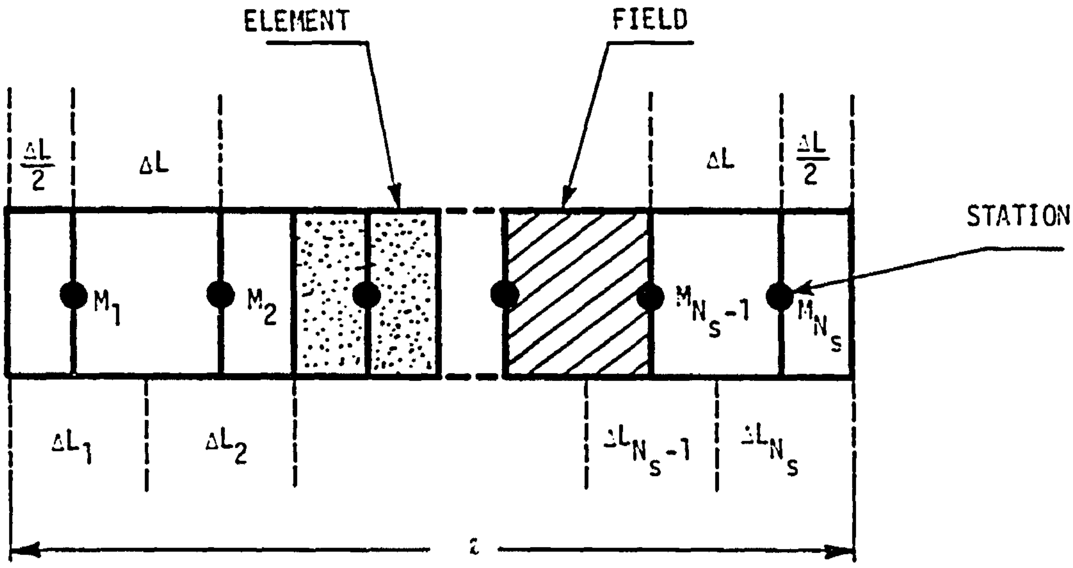


Fig. 2.4. Mesh for transfer-matrix analysis.

Using the assumed stress and displacement field, the system of Eqs. (2.10) can be written as the constitutive relation for a multimodular beam:

$$\begin{Bmatrix} N \\ M \\ Q \end{Bmatrix} = \begin{bmatrix} A + C_N^A & B + C_N^B & 0 \\ B + C_M^B & D + C_M^D & 0 \\ 0 & 0 & S \end{bmatrix} \begin{Bmatrix} u_{,x} \\ \psi_{,x} \\ w_{,x} + \psi \end{Bmatrix} \tag{2.11}$$

where A, B, D, and S denote the respective extensional, flexural-extensional coupling, flexural, and transverse shear stiffnesses defined by

$$\begin{aligned} (A,B,D) &= \int_{-h/2}^{h/2} (1,z,z^2)E_i^{(k)}dz \qquad \begin{matrix} i=1,2 \\ k=t,c \end{matrix} \\ S &= K^2 \int_{-h/2}^{h/2} G\,dz \end{aligned} \tag{2.12}$$

Here, the stiffnesses C_N^A, C_N^B, C_M^B, and C_M^D are not present in linear or bimodular materials, and are as defined in Appendix B. In Eq. (2.12), t and c denote tensile-strain and compressive-strain regions, respectively. The quantity K^2 is a shear correction coefficient which is generally taken to be 5/6 for static loading of a rectangular-section beam.

2.3.4 Equilibrium Equations

The equilibrium equations for transverse distributed loading q(x) can be written as

$$N_{,x} = 0 \quad ; \quad Q_{,x} + q(x) = 0 \quad ; \quad M_{,x} - Q = 0 \tag{2.13}$$

By substitution of Eq. (2.11) into Eqs. (2.13), one obtains the following equations of equilibrium in terms of the generalized displacements

$$
\begin{aligned}
&(A'u_{,x} + B'\psi_{,x})_{,x} = 0 \\
&[S(w_{,x} + \psi)]_{,x} = -q(x) \\
&(B''u_{,x} + D'\psi_{,x})_{,x} - S(w_{,x} + \psi) = 0
\end{aligned}
\tag{2.14}
$$

where

$$
\begin{aligned}
A' &= A + C_N^A \\
B' &= B + C_N^B \\
B'' &= B + C_M^B \\
D' &= D + C_M^D
\end{aligned}
\tag{2.15}
$$

2.4 Closed-Form Solution

A closed-form solution can be obtained only when the stiffnesses and thus neutral-surface position (z_n) do not depend on x. Therefore, neutral-surface location [2.18] must be constant

$$z_n = -u_{,x}/\psi_{,x} = \text{constant} \tag{2.16}$$

Using Eqs. (2.11) one is able to express $u_{,x}$ and $\psi_{,x}$ in terms of N, M, and the stiffnesses as follows:

$$
\begin{Bmatrix} u_{,x} \\ \psi_{,x} \end{Bmatrix} = \frac{1}{B'B'' - A'D'} \begin{bmatrix} -D' & B' \\ B'' & -A' \end{bmatrix} \begin{Bmatrix} N \\ M \end{Bmatrix}
\tag{2.17}
$$

Combining Eqs. (2.16) and (2.17), one obtains

$$z_n = (B'M - D'N)/(A'M - B''N) \tag{2.18}$$

It is obvious that z_n = const. when N = 0. Thus, for these special cases (see Appendix C)

$$z_n = B'/A' = \text{constant} \tag{2.19}$$

Now, equilibrium Eqs. (2.14) can be simplified as follows [2.19]:

$$\begin{aligned} &A'u_{,xx} + B'\psi_{,xx} = 0 \\ &S(w_{,xx} + \psi_{,x}) = -q(x) \\ &B''u_{,xx} + D'\psi_{,xx} - S(w_{,x} + \psi) = 0 \end{aligned} \tag{2.20}$$

The general solution for Eqs. (2.20) can be written as follows:

$$\begin{aligned} &u(x) = d_1 + d_2 x + \frac{3B'}{A'} C_4 x^2 + u_p(x) \\ &\psi(x) = -C_2 + \frac{6(B'B'' - A'D')}{SA'} C_4 - 2C_3 x - 3C_4 x^2 + \psi_p(x) \\ &w(x) = C_1 + C_2 x + C_3 x^2 + C_4 x^3 + w_p(x) \end{aligned} \tag{2.21}$$

where u_p, ψ_p, w_p are particular solutions (see Appendix D) and C_1, C_2, C_3, C_4, d_1, and d_2 are arbitrary constants determined by the boundary conditions of the beam. The following boundary conditions have been considered for closed-form solutions:

1. Hinged-Hinged (free to move axially at x = L)

   $u(0) = N(\ell) = 0 \quad ; \quad M(0) = M(\ell) = 0 \quad ; \quad w(0) = w(\ell) = 0$

2. Clamped-Free

   $u(0) = N(\ell) = 0 \quad ; \quad \psi(0) = M(P) = 0 \quad ; \quad w(0) = Q(P) = 0$

3. Clamped-Clamped (free to move axially at x = L)

   $u(0) = N(\ell) = 0 \quad ; \quad \psi(0) = \psi(\ell) = 0 \quad ; \quad w(0) = w(\ell) = 0$

The values of constants C_1, C_2, C_3, C_4, d_1, and d_2 are listed in Appendix D.

2.5 Transfer-Matrix Solution

As it has been shown in [2.20,2.21], in the transfer-matrix approach, the beam is divided into N_s elements, each of which is assumed to be of mass m and concentrated at the center of mass of the element. The mass center of each element is called the station. The stations are separated by fields which are taken to be massless and contain all of the stiffnesses of the beam. At the end points of the beam, there are two half fields of length $\Delta\ell/2$ (see Fig. 2.4), and between these half fields there are N_s stations separated by $(N_s - 1)$ full fields of length, $\Delta\ell$, where $\Delta\ell = \ell/N_s$ and ℓ is the length of the beam. By writing the equilibrium equations for each station and each element and connecting the elements by transfer matrices, one transfers the generalized displacements (u,w,ψ) and the forces (N,Q,M) from the left side of the beam to the right side.

Since the same procedure used in [2.16] has been used here to derive transfer matrices and state vectors $(u,w,\psi,N,Q,M)^T$, the readers are referred to this reference. Note that since the present work deals with multimodular material, some changes in the field matrix are necessary (see Appendix E). In the calculation of the stiffnesses for the cases where the axial force is not zero, the neutral-surface locations and the corresponding distances to the "break points" in the σ_x vs z curve (a_c and a_t) are not constant and not known <u>à priori</u>. Therefore, an iterative technique has been employed to compute the neutral-surface locations z_n, also a_c and a_t. One must first assume $(2N_s + 2)$ sets of values of z_n, a_c, and a_t and then compute the stiffnesses and solve the governing equations for the state vector. Finally, by using Eqs. (2.18), (C.3), and (C.4), compute new values of z_n, a_c, and a_t. Obviously, if the assumed and computed sets of z_n, a_c, and a_t are in sufficiently close agreement, the problem is solved; otherwise, assume the calculated set z_n, a_c, and a_t and repeat the procedure.

2.6 Numerical Results

In the following, numerical results are presented for a thick beam with a rectangular cross section and constructed of multimodular material (see Table 2.1): unidirectional aramid-cord-rubber, which is used in the tire industry. Various boundary conditions and loading conditions were investigated (see Tables 2.2 and 2.3). In the transfer-matrix analysis, twenty-five elements were used. Each element was of length 0.32 in. for dimensional cases and dimensionless length of 0.04 for nondimensional cases. The shear correction coefficient was taken to be 5/6.

For all cases considered, the computations are carried out for axial elongation u (or $\bar{u} = u\, E_2{}^t/q_o \ell$), transverse deflection W (or $\bar{W} = W\, E_2{}^t/q_o \ell$), bending slope ψ (or $\bar{\psi} = \psi\, E_2{}^t/q_o$) axial force N (or $\bar{N} = N/q_o \ell$), shear force Q (or $\bar{Q} = Q/q_o \ell$), bending moment M (or $\bar{M} = M/q_o \ell^2$) and neutral-surface location z_n (or $\bar{z}_n = z_n/h$), where $\bar{u}$, $\bar{W}$, $\bar{\psi}$, $\bar{N}$, $\bar{Q}$, $\bar{M}$, and $\bar{z}_n$ are nondimensional parameters.

Due to lack of comparable results in the literature, comparisons are made between the closed-form solution (CFS) and the transfer-matrix solution (TMS) developed here. Excellent agreement between CFS and TMS for the twenty-five element model has been achieved and still it can be improved by increasing the number of elements. For most of the results, the error is less than 2%. Figure 2.5 contains the plots of the dimensionless transverse deflection ($\bar{W}$) versus dimensionless position ($\bar{X} = x/\ell$) for Case 11 for multimodular, bimodular, unimodular, and average-modular

Table 2.1. Elastic Properties and Geometric Parameters for an Aramid-Cord-Rubber Beam

	Model*	Longitudinal Young's Modulus, psi x 10^{-6}				Longitudinal-Thickness Shear Modulus, psi x 10^{-3}
		Tension		Compression		Tension and Compression
Elastic Properties	M	E_2^t	E_1^t	E_1^c	E_2^c	G
		0.580	0.420	0.032	0.010	0.537
	B	E_b^t		E_b^c		
		0.470		0.018		0.537
		E		E		
	U	0.275		0.275		0.537
	A	0.244		0.244		0.537
Geometric Parameters	Beam length	8.0 in.				
	Beam depth (thickness)	0.6 in.				
	Beam width	1.0 in.				

*M ∿ Multimodular, B ∿ Bimodular, U ∿ Unimodular, A ∿ Average Modular.

Table 2.2. Summary of Cases Considered

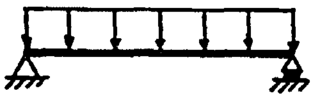
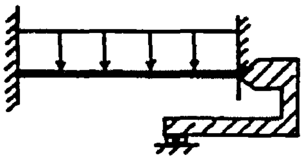
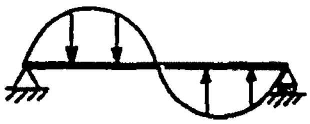
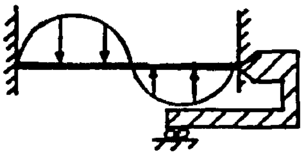
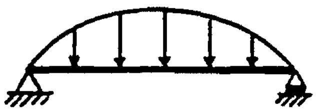
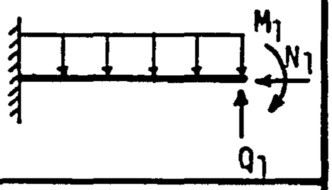
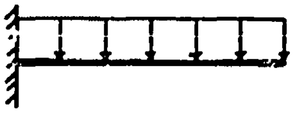
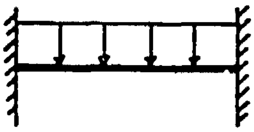
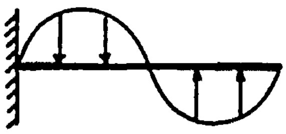
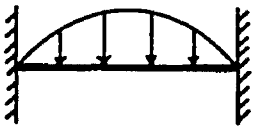
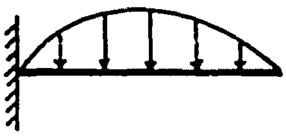
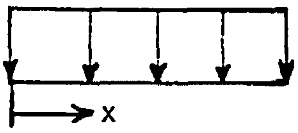
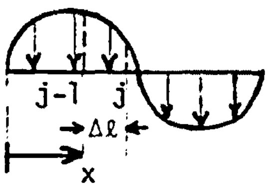
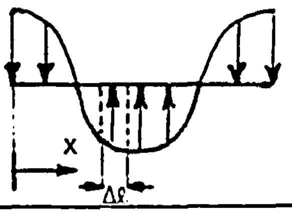
Case No.	Boundary Condition and Load Position	Case No.	Boundary Condition and Load Position
1		7	
2		8	
3		9	
4		10	
5		11	
6			

Table 2.3. Values of K_m and K_q for Various Loadings

Type of Loading	K_m	K_q
Uniform Load $q(x) = q_o$ 	$q_o(\Delta\ell)^2/2$	$q_o\Delta\ell$
Sine Load $q(x) = q_o \sin \frac{n\pi}{\ell} x$ 	$\frac{q_o\ell}{n\pi}[\ell \cos \frac{n\pi}{\ell} x_{j-1} - \frac{\ell}{n\pi}$ $(\sin \frac{n\pi}{\ell} x_j - \sin \frac{n\pi}{\ell} x_{j-1})]$	$-\frac{q_o\ell}{n\pi}(\cos \frac{n\pi}{\ell} x_j$ $- \cos \frac{n\pi}{\ell} x_{j-1})$
Cosine Load $q(x) = q_o \cos \frac{n\pi}{\ell} x$ 	$\frac{q_o\ell}{n\pi}[-\frac{\ell}{n\pi}(\cos \frac{n\pi}{\ell} x_j$ $- \cos \frac{n\pi}{\ell} x_{j-1}) - \ell \sin \frac{n\pi}{\ell} x_{j-1}]$	$\frac{q_o\ell}{n\pi}(\sin \frac{n\pi}{\ell} x_j$ $- \sin \frac{n\pi}{\ell} x_{j-1})$

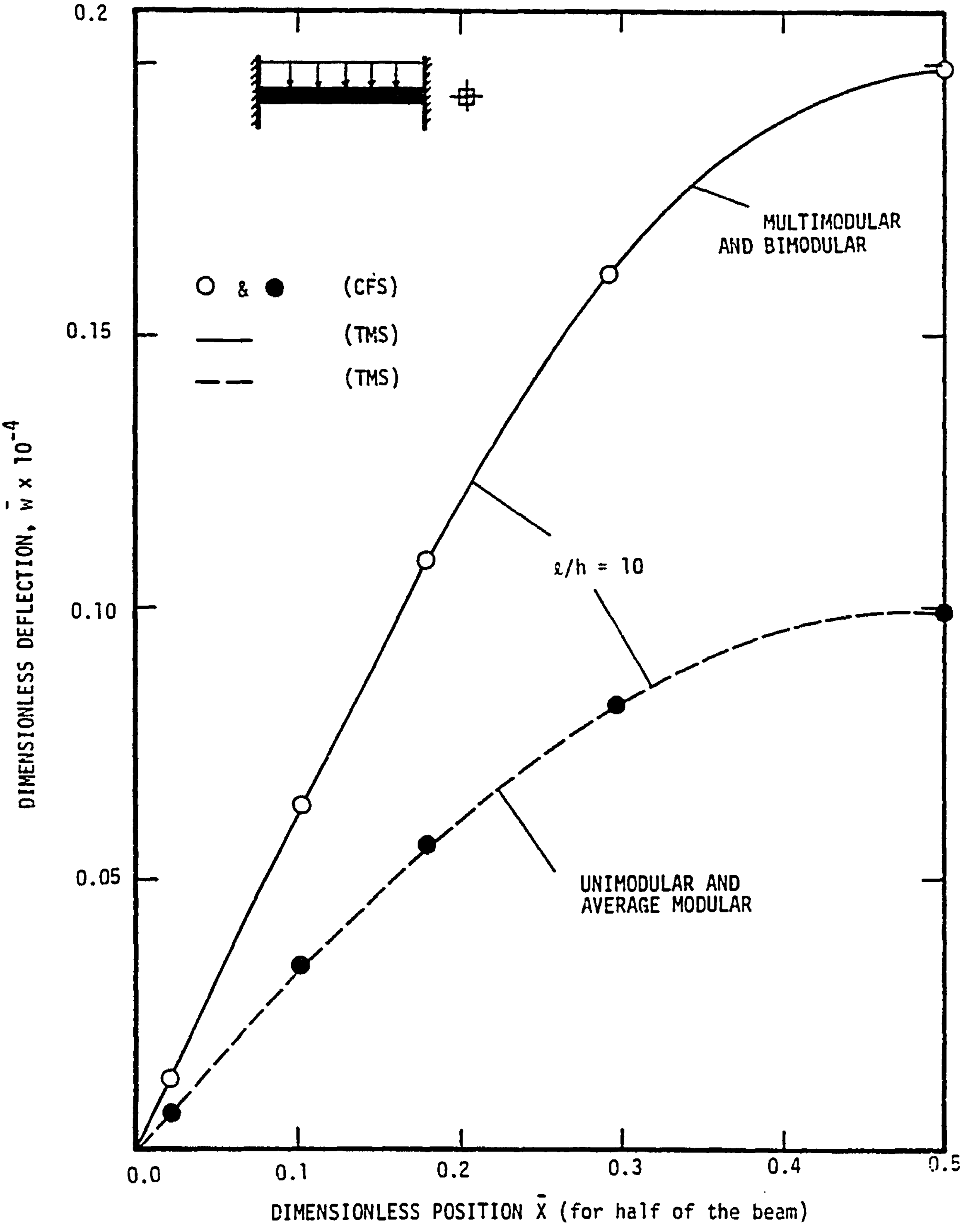


Fig. 2.5. Comparison among multimodular, bimodular, unimodular, and average modular deflection distribution for closed-form and transfer-matrix solution of clamped-clamped aramid-cord rubber beam (ℓ/h = 10).

[i.e., $E = (E_b{}^t + E_b{}^c)/2$] cases, where $\ell/h = 10$.

As one can see, there is a considerable difference between transverse deflection of multimodular and bimodular models on one hand and unimodular and average modular models on the other. Note that Case 11 is a special case because both ends are not free to move and one expects axial force to be developed due to bending-stretching coupling caused by bimodular action. However, the computed axial force is close to zero which means z_n is constant.

To validate TMS, in Figs. 2.6 through 2.9, a comparison is made between TMS and CFS. Behavior of Cases 1, 4, 10, and 11 was studied for the multimodular model considering different dimensionless parameters (ℓ/h = 5, 10, 15; $\bar{M}_1 = -1.0$; $\bar{N}_1 = -1.0$; $\bar{Q}_1 = -1.0$).

In Table 2.4, for a specific beam (see Table 2.1), dimensioned comparisons have been made between multimodular and bimodular models. Tables 2.5-2.9 again show the validity of TMS while they present the computed results for Cases 3, 5, 6, 7, and 8.

Since closed-form solutions are not available for the complicated boundary conditions considered in Cases 9, 10, and 11, only transfer-matrix results are presented for these cases. In Figs. 2.10-2.13, the behavior of a clamped-free beam (with applied moment and axial and shear forces at the free end) and a clamped-clamped beam under a uniform load is investigated. See also Table 2.10. It is of particular interest to note that in Fig. 2.10, the sign of the deflection depends upon the ℓ/h ratio, the crossover point being at $\ell/h \simeq 11$.

For cases where axial force is zero, the neutral-surface location is constant; otherwise it varies along the beam length. In Figs. 2.14-2.17,

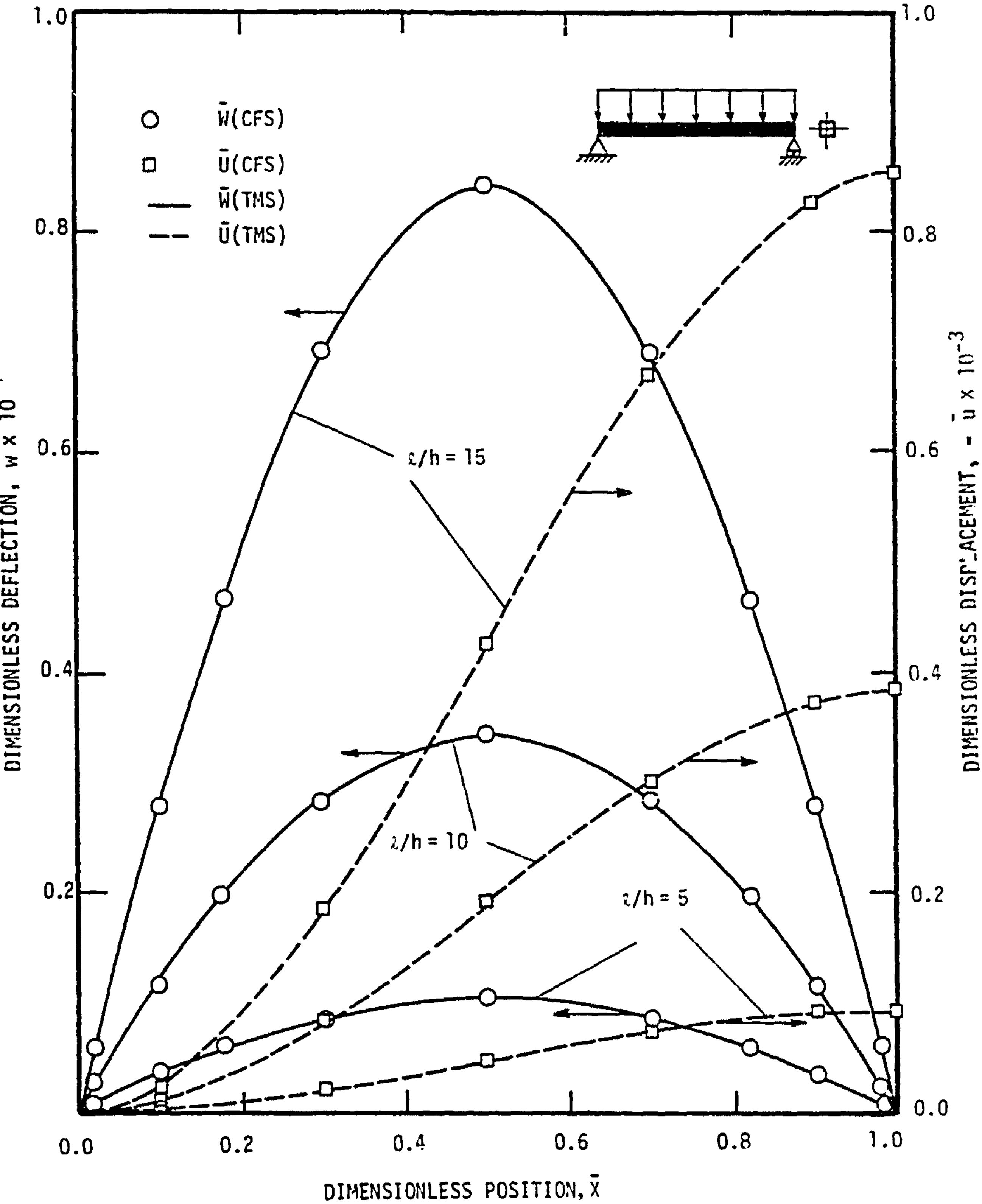


Fig. 2.6. Closed-form and transfer-matrix solutions of hinged-hinged, aramid-cord-rubber beam with rectangular cross section (ℓ/h = 5, 10, 15).

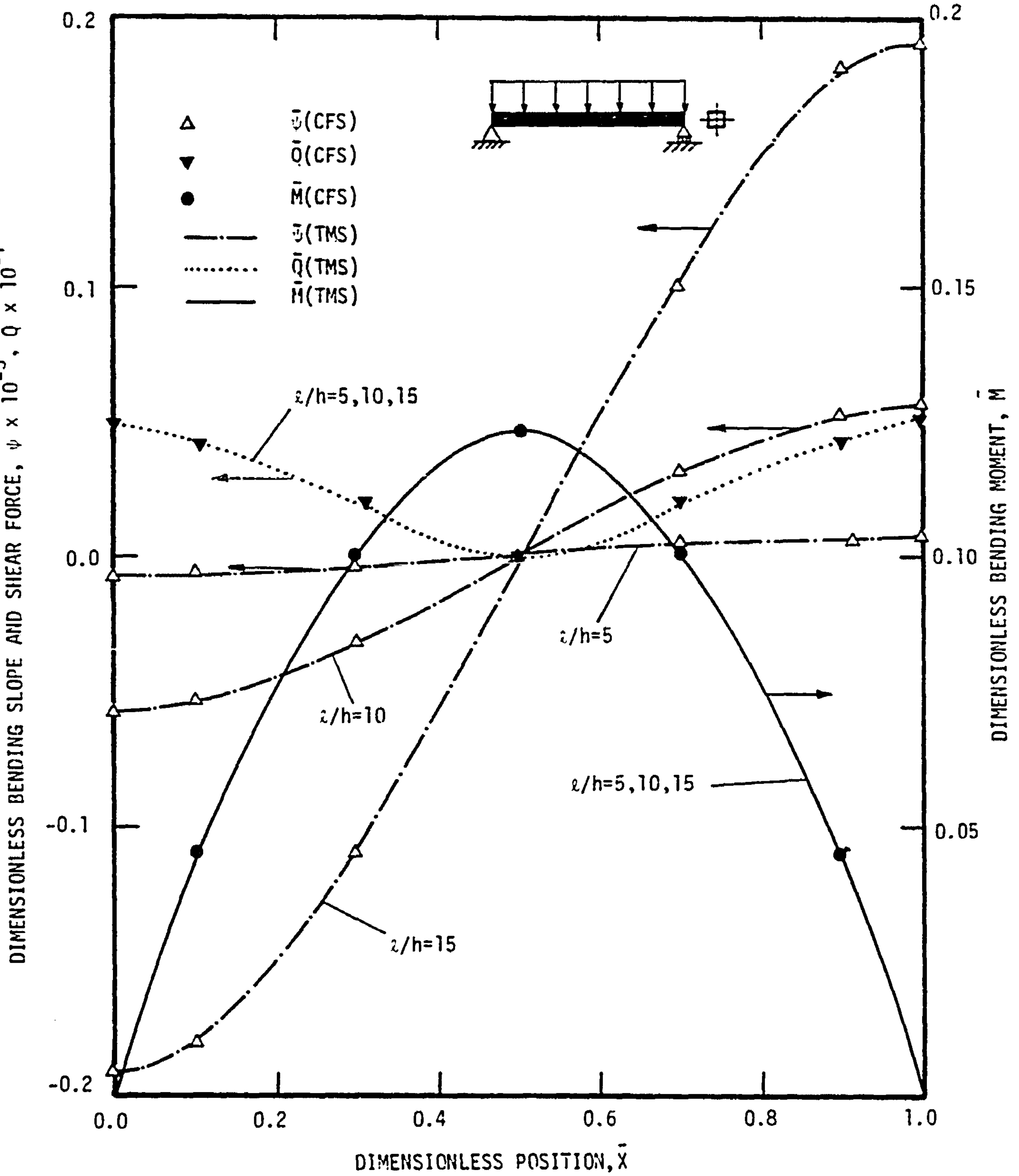


Fig. 2.7. Closed-form and transfer-matrix solutions of hinged-hinged, aramid-cord-rubber beam with rectangular cross section (ℓ/h = 5, 10, 15).

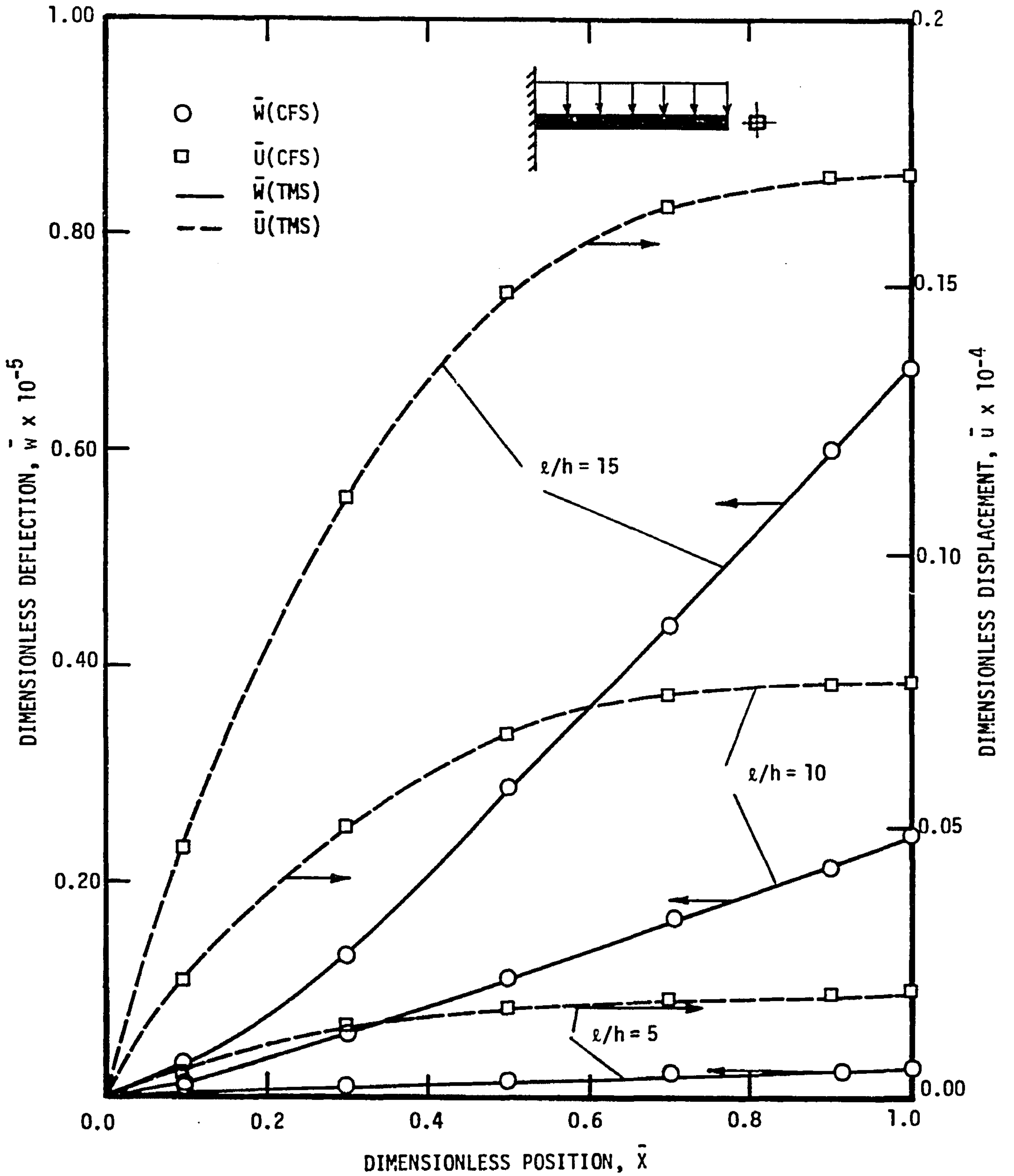


Fig. 2.8. Closed-form and transfer-matrix solutions of clamped-free, aramid-cord-rubber beam with rectangular cross section (ℓ/h = 5, 10, 15).

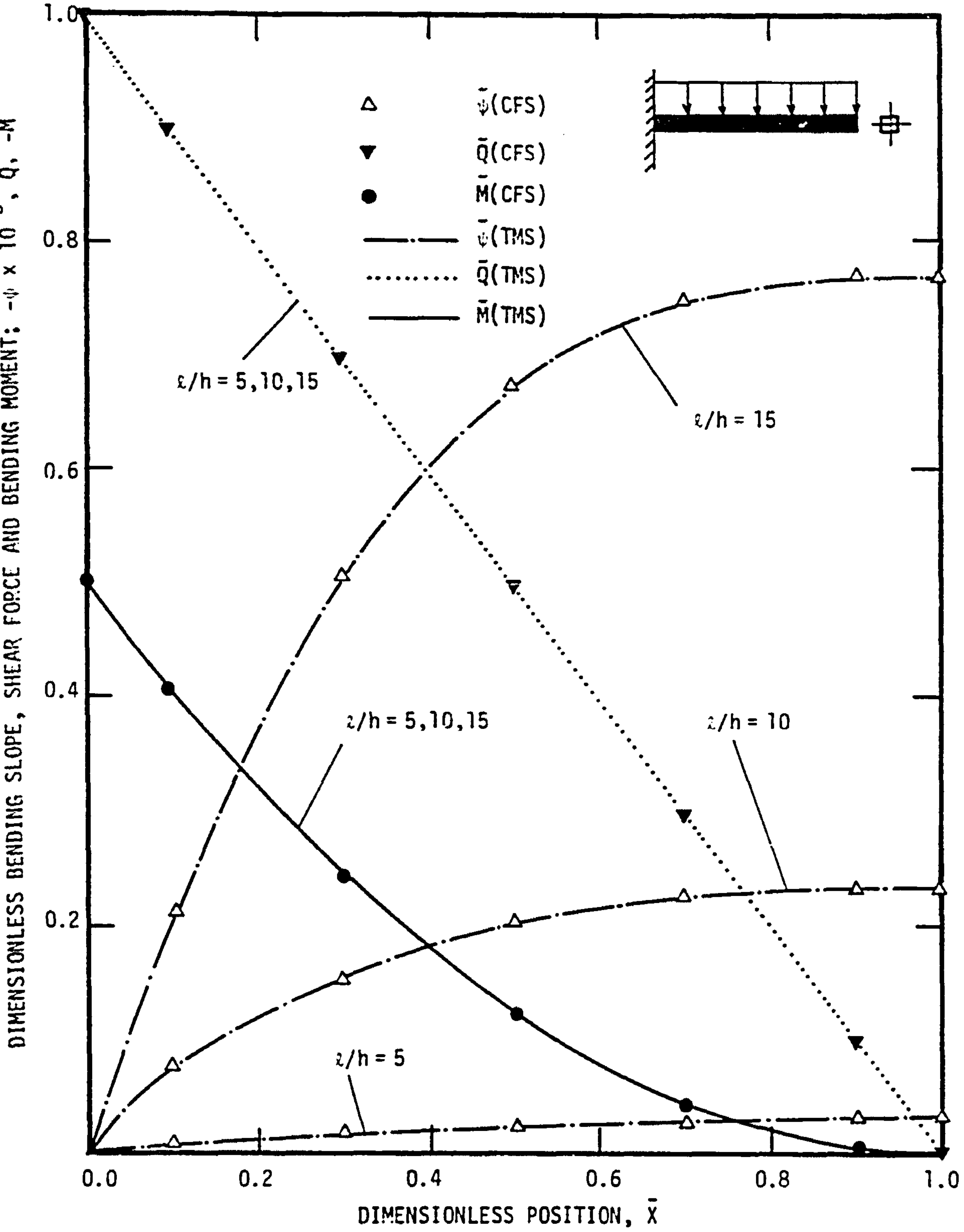


Fig. 2.9. Closed-form and transfer-matrix solutions of clamped-free, aramid-cord-rubber beam with rectangular cross section (ℓ/h = 5, 10, 15).

Table 2.4. Comparison Between Bimodular and Multimodular Cases for Both Closed-Form and Transfer-Matrix Solutions for an Aramid-Rubber Beam (Case 2)‡

x/ℓ	$u \times 10^4$, in.				$w \times 10^3$, in.				$\psi \times 10^3$, rad				M, lb-in.				Q, lb			
	B*		M*		B		M		B		M		B		M		B		M	
	CFS†	TMS†	CFS	TMS	CFS	TMS	CFS	TMS	CFS	TMS	CFS	TMS	CFS	TMS	CFS	TMS	CFS	TMS	CFS	TMS
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	-0.2277	-0.2264	-0.2286	-0.2273	0.0000	0.0000	0.0000	0.0000	0.1273	0.1273	0.1273	0.1273
0.02	-0.0036	-0.0036	-0.0036	-0.0036	0.1121	0.1117	0.1122	0.1118	-0.2259	-0.2246	-0.2268	-0.2255	0.0203	0.0203	0.0203	0.0203	0.1263	0.1263	0.1263	0.1263
0.06	-0.0323	-0.0321	-0.0318	-0.0317	0.3292	0.3272	0.3296	0.3277	-0.2117	-0.2105	-0.2126	-0.2113	0.0597	0.0597	0.0597	0.0597	0.1184	0.1184	0.1184	0.1184
0.10	-0.0878	-0.0873	-0.0865	-0.0861	0.5256	0.5222	0.5263	0.5229	-0.1842	-0.1831	-0.1850	-0.1839	0.0953	0.0953	0.0953	0.0953	0.1030	0.1030	0.1030	0.1030
0.14	-0.1666	-0.1657	-0.1643	-0.1634	0.6890	0.6844	0.6899	0.6853	-0.1451	-0.1443	-0.1457	-0.1449	0.1249	0.1249	0.1249	0.1249	0.0812	0.0812	0.0812	0.0812
0.18	-0.2639	-0.2625	-0.2601	-0.2588	0.8090	0.8036	0.8101	0.8047	-0.0969	-0.0969	-0.0963	-0.0973	0.1467	0.1467	0.1467	0.1467	0.0542	0.5422	0.0542	0.0542
0.22	-0.3734	-0.3714	-0.3681	-0.3662	0.8783	0.8724	0.8795	0.8735	-0.0427	-0.0423	-0.0428	-0.0425	0.1592	0.1592	0.1592	0.1592	0.0239	0.0239	0.0239	0.0239
0.26	-0.4884	-0.4858	-0.4815	-0.4789	0.8924	0.8863	0.8936	0.8875	0.0143	0.0143	0.0144	0.0144	0.1618	0.1618	0.1618	0.1618	-0.0080	-0.0080	-0.0080	-0.0080
0.30	-0.6015	-0.5983	-0.5930	-0.5899	0.8504	0.8446	0.8515	0.8457	0.0704	0.0701	0.0706	0.0704	0.1542	0.1542	0.1542	0.1542	-0.0393	-0.0393	-0.0393	-0.0393
0.34	-0.7057	-0.7020	-0.6957	-0.6921	0.7550	0.7498	0.7560	0.7508	0.1220	0.1215	0.1225	0.1220	0.1369	0.1369	0.1369	0.1369	-0.0682	-0.0682	-0.0682	-0.0682
0.38	-0.7945	-0.7903	-0.7832	-0.7791	0.6121	0.6079	0.6129	0.6087	0.1660	0.1652	0.1667	0.1659	0.1110	0.1110	0.1110	0.1110	-0.0928	-0.0928	-0.0928	-0.0928
0.42	-0.8622	-0.8576	-0.8500	-0.8455	0.4308	0.4278	0.4313	0.4283	0.1995	0.1986	0.2004	0.1994	0.0781	0.0781	0.0781	0.0781	-0.1116	-0.1116	-0.1116	-0.1116
0.46	-0.9046	-0.8998	-0.8918	-0.8871	0.2224	0.2208	0.2227	0.2211	0.2205	0.2195	0.2215	0.2204	0.0403	0.0403	0.0403	0.0403	-0.1233	-0.1233	-0.1233	-0.1233
0.50	-0.9190	-0.9142	-0.9060	-0.9012	0.0000	-0.0000	0.0000	-0.0000	0.2277	0.2266	0.2286	0.2275	0.0000	-0.0000	0.0000	-0.0000	-0.1273	-0.1273	-0.1273	-0.1273
0.54	-0.9046	-0.8998	-0.8918	-0.8871	-0.2224	-0.2209	-0.2227	-0.2211	0.2205	0.2195	0.2215	0.2204	-0.0403	-0.4032	-0.0403	-0.0403	-0.1233	-0.1233	-0.1233	-0.1233
0.58	-0.8622	-0.8576	-0.8500	-0.8455	-0.4308	-0.4278	-0.4313	-0.4284	0.1995	0.1986	0.2004	0.1994	-0.0781	-0.0781	-0.0781	-0.0781	-0.1115	-0.1116	-0.1116	-0.1116
0.62	-0.7945	-0.7903	-0.7832	-0.7791	-0.6121	-0.6079	-0.6129	-0.6087	0.1660	0.1652	0.1667	0.1659	-0.1110	-0.1110	-0.1110	-0.1110	-0.0928	-0.0928	-0.0928	-0.0928
0.66	-0.7057	-0.7020	-0.6957	-0.6921	-0.7545	-0.7498	-0.7560	-0.7508	0.1220	0.1215	0.1225	0.1220	-0.1369	-0.1369	-0.1369	-0.1369	-0.0682	-0.0682	-0.0682	-0.0682
0.70	-0.6015	-0.5983	-0.5930	-0.5899	-0.8504	-0.8446	-0.8515	-0.8457	0.0704	0.0701	0.0706	0.0704	-0.1542	-0.1542	-0.1542	-0.1542	-0.0393	-0.0393	-0.0393	-0.0393
0.74	-0.4884	-0.4858	-0.4815	-0.4789	-0.8924	-0.8863	-0.8936	-0.8875	0.0143	0.0143	0.0144	0.0144	-0.1618	-0.1618	-0.1619	-0.1618	-0.0080	-0.0080	-0.0080	-0.0080
0.78	-0.3734	-0.3714	-0.3681	-0.3662	-0.8783	-0.8724	-0.8795	-0.8735	-0.0427	-0.0423	-0.0428	-0.0425	-0.1592	-0.1592	-0.1592	-0.1592	0.0239	0.0239	0.0239	0.0239
0.82	-0.2639	-0.2625	-0.2601	-0.2587	-0.8090	-0.8036	-0.8101	-0.8047	-0.0969	-0.0963	-0.0973	-0.0967	-0.1467	-0.1467	-0.1467	-0.1467	0.0542	0.0542	0.0542	0.0542
0.86	-0.1666	-0.1657	-0.1643	-0.1634	-0.6890	-0.6844	-0.6899	-0.6853	-0.1451	-0.1443	-0.1457	-0.1449	-0.1249	-0.1249	-0.1249	-0.1249	0.0812	0.0812	0.0812	0.0812
0.90	-3.0878	-0.0873	-0.0865	-0.0860	-0.5256	-0.5222	-0.5263	-0.5229	-0.1842	-0.1831	-0.1850	-0.1839	-0.0953	-0.0953	-0.0953	-0.0953	0.1030	0.1030	0.1030	0.1030
0.94	-0.0323	-0.0321	-0.0318	-0.0316	-0.3292	-0.3272	-0.3296	-0.3277	-0.2117	-0.2105	-0.2126	-0.2114	-0.0597	-0.0597	-0.0597	-0.0597	0.1184	0.1184	0.1184	0.1184
0.98	-0.0036	-0.0036	-0.0036	-0.0036	-0.1121	-0.1117	-0.1122	-0.1119	-0.2259	-0.2246	-0.2268	-0.2255	-0.0203	-0.0203	-0.0203	-0.0203	0.1263	0.1263	0.1263	0.1263
1.00	-0.0000	0.0000	-0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	-0.2277	-0.2264	-0.2286	-0.2273	0.0000	0.0000	-0.0000	-0.0000	0.1273	0.1273	0.1273	0.1273

* B - bimodular and M - multimodular.

† CFS denotes closed-form solution; TMS signifies transfer-matrix solution.

‡ Axial force is zero and z_n is constant for this case [$z_n = \begin{cases} 0.1981 \text{ in. } 0 \le x < \ell/2 \\ -0.1981 \text{ in. } \ell/2 \le x \le \ell \end{cases}$ (for multimodular) and $z_n = \begin{cases} 0.2018 \text{ in. } 0 \le x < \ell/2 \\ -0.2018 \text{ in. } \ell/2 \le x \le \ell \end{cases}$ (for bimodular)]

Table 2.5. Closed-Form and Transfer-Matrix Solutions for an Aramid-Rubber Beam (Case 3)*

x/ℓ	u x 10^3, in.		w x 10^2, in.		ψ x 10^2, rad		M, lb-in.	Q, lb
	CFS**	TMS**	CFS	TMS	CFS	TMS	CFS & TMS	CFS & TMS
0.00	0.0000	0.0000	0.0000	0.0000	-0.1829	-0.1827	-0.0000	0.2546
0.02	-0.0007	-0.0007	0.0444	0.4443	-0.1825	-0.1823	0.0407	0.2541
0.06	-0.0064	-0.0064	0.1326	0.1323	-0.1797	-0.1794	0.1215	0.2501
0.10	-0.0177	-0.0177	0.2186	0.2182	-0.1704	-0.1737	0.2004	0.2422
0.14	-0.0345	-0.0344	0.3012	0.3006	-0.1655	-0.1653	0.2761	0.2304
0.18	-0.0564	-0.0563	0.3791	0.3783	-0.1544	-0.1542	0.3474	0.2150
0.22	-0.0832	-0.0831	0.4510	0.4500	-0.1409	-0.1407	0.4133	0.1962
0.26	-0.1143	-0.1142	0.5157	0.5146	-0.1252	-0.1250	0.4727	0.1743
0.30	-0.1494	-0.1492	0.5724	0.5711	-0.1075	-0.1074	0.5246	0.1497
0.34	-0.1878	-0.1876	0.6200	0.6186	-0.0881	-0.0880	0.5682	0.1227
0.38	-0.2290	-0.2287	0.6578	0.6564	-0.0673	-0.0672	0.6029	0.0937
0.42	-0.2723	-0.2719	0.6852	0.6837	-0.0455	-0.0454	0.6281	0.0633
0.46	-0.3170	-0.3166	0.7019	0.7004	-0.0229	-0.0229	0.6433	0.0319
0.50	-0.3624	-0.3619	0.7075	0.7059	-0.0000	-0.0000	0.6484	0.0000
0.54	-0.4078	-0.4073	0.7019	0.7004	0.0229	0.0229	0.6433	-0.0319
0.58	-0.4525	-0.4519	0.6852	0.6837	0.0455	0.0454	0.6281	-0.0633
0.62	-0.4958	-0.4952	0.6578	0.6564	0.0673	0.0672	0.6029	-0.0937
0.66	-0.5370	-0.5363	0.6200	0.6186	0.0881	0.0880	0.5682	-0.1227
0.70	-0.5754	-0.5747	0.5724	0.5711	0.1075	0.1074	0.5246	-0.1497
0.74	-0.6105	-0.6097	0.5157	0.5146	0.1252	0.1250	0.4727	-0.1743
0.78	-0.6416	-0.6408	0.4510	0.4500	0.1409	0.1407	0.4133	-0.1962
0.82	-0.6684	-0.6675	0.3791	0.3783	0.1544	0.1542	0.3475	-0.2150
0.86	-0.6903	-0.6894	0.3012	0.3006	0.1655	0.1653	0.2761	-0.2304
0.90	-0.7071	-0.7061	0.2186	0.2182	0.1740	0.1737	0.2004	-0.2422
0.94	-0.7184	-0.7174	0.1326	0.1323	0.1792	0.1794	0.1215	-0.2501
0.98	-0.7241	-0.7231	0.0444	0.0443	0.1825	0.1823	0.0407	-0.2541
1.00	-0.7248	-0.7238	0.0000	0.0000	0.1828	0.1827	-0.0000	-0.2546

*For Case 3, axial force N is zero and z_n is constant, equal to 0.1981 in.
**CFS ∼ closed-form solution; TMS ∼ transfer-matrix solution.

Table 2.6. Closed-Form and Transfer-Matrix Solutions for an Aramid-Rubber Beam (Case 5)*

x/ℓ	$u \times 10^3$, in.		$w \times 10$, in.		$\psi \times 10^2$, rad		M, lb-in.	Q, lb
	CFS**	TMS**	CFS	TMS	CFS	TMS	CFS & TMS	CFS & TMS
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0186	0.0000
0.02	-0.0358	-0.0358	-0.0001	-0.0001	0.0180	0.0180	1.0185	-0.0010
0.06	-0.1073	-0.1072	-0.0013	-0.0014	0.0541	0.0542	1.0171	-0.0089
0.10	-0.1786	-0.1785	-0.0038	-0.0039	0.0901	0.0901	1.0120	-0.0243
0.14	-0.2493	-0.2491	-0.0077	-0.0077	0.1258	0.1257	1.0009	-0.0462
0.18	-0.3190	-0.3188	-0.0130	-0.0130	0.1610	0.1609	0.9819	-0.0731
0.22	-0.3870	-0.3867	-0.0198	-0.0198	0.1953	0.1952	0.9537	-0.1035
0.26	-0.4527	-0.4523	-0.0280	-0.0280	0.2285	0.2283	0.9155	-0.1353
0.30	-0.5153	-0.5149	-0.0376	-0.0376	0.2601	0.2599	0.8672	-0.1667
0.34	-0.5743	-0.5738	-0.0485	-0.0486	0.2898	0.2896	0.8091	-0.1955
0.38	-0.6288	-0.6283	-0.0608	-0.0607	0.3174	0.3171	0.7425	-0.2201
0.42	-0.6784	-0.6778	-0.0740	-0.0740	0.3424	0.3421	0.6689	-0.2389
0.46	-0.7226	-0.7220	-0.0883	-0.0882	0.3647	0.3644	0.5903	-0.2506
0.50	-0.7613	-0.7606	-0.1033	-0.1032	0.3842	0.3839	0.5093	-0.2546
0.54	-0.7942	-0.7935	-0.1185	-0.1188	0.4008	0.4005	0.4282	-0.2506
0.58	-0.8215	-0.8208	-0.1349	-0.1347	0.4146	0.4143	0.3497	-0.2389
0.62	-0.8434	-0.8428	-0.1511	-0.1509	0.4257	0.4254	0.2761	-0.2201
0.66	-0.8604	-0.8599	-0.1674	-0.1671	0.4343	0.4340	0.2094	-0.1955
0.70	-0.8730	-0.8725	-0.1835	-0.1833	0.4406	0.4404	0.1514	-0.1666
0.74	-0.8819	-0.8815	-0.1995	-0.1993	0.4451	0.4449	0.1030	-0.1353
0.78	-0.8877	-0.8874	-0.2152	-0.2150	0.4480	0.4479	0.0648	-0.1035
0.82	-0.8912	-0.8909	-0.2306	-0.2304	0.4498	0.4497	0.0366	-0.0731
0.86	-0.8931	-0.8928	-0.2458	-0.2455	0.4508	0.4506	0.0177	-0.0462
0.90	-0.8939	-0.8937	-0.2606	-0.2603	0.4512	0.4510	0.0066	-0.0243
0.94	-0.8942	-0.8940	-0.2752	-0.2750	0.4513	0.4512	0.0014	-0.0089
0.98	-0.8942	-0.8940	-0.2897	-0.2895	0.4513	0.4512	0.0005	-0.0010
1.00	-0.8942	-0.8940	-0.2970	-0.2967	0.4513	0.4512	0.0000	0.0000

*For Case 5, axial force N is zero and z_n is constant, equal to 0.1981 in.

**CFS ∿ closed-form solution; TMS ∿ transfer-matrix solution.

Table 2.7. Closed-Form and Transfer-Matrix Solutions for an Aramid-Cord Rubber Beam (Case 6)*

x/ℓ	$u \times 10^2$, in.		$w \times 10$, in.		$\psi \times 10^2$, rad		M, lb-in.	Q, lb
	CFS**	TMS**	CFS	TMS	CFS	TMS	CFS & TMS	CFS & TMS
0.00	0.0000	0.0000	0.0000	0.0000	-0.0000	0.0000	-2.0372	0.5093
0.02	0.0070	0.0070	0.0033	0.0033	-0.0354	-0.0354	-1.9557	0.5088
0.06	0.0202	0.0202	0.0116	0.0116	-0.1018	-0.1018	-1.7934	0.5048
0.10	0.0322	0.0322	0.0218	0.0218	-0.1625	-0.1626	-1.6331	0.4968
0.14	0.0431	0.0431	0.0338	0.0337	-0.2176	-0.2177	-1.4759	0.4851
0.18	0.0529	0.0529	0.0472	0.0471	-0.2672	-0.2673	-1.3230	0.4696
0.22	0.0617	0.0617	0.0620	0.0619	-0.3115	-0.3115	-1.1757	0.4509
0.26	0.0695	0.0695	0.0778	0.0777	-0.3506	-0.3507	-1.0348	0.4290
0.30	0.0763	0.0763	0.0946	0.0944	-0.3849	-0.3850	-0.9014	0.4043
0.34	0.0822	0.0822	0.1121	0.1119	-0.4146	-0.4148	-0.7763	0.3773
0.38	0.0872	0.0872	0.1301	0.1299	-0.4401	-0.4402	-0.6601	0.3484
0.42	0.0914	0.0915	0.1485	0.1484	-0.4616	-0.4617	-0.5535	0.3180
0.46	0.0950	0.0950	0.1672	0.1670	-0.4794	-0.4796	-0.4567	0.2866
0.50	0.0979	0.0979	0.1860	0.1858	-0.4941	-0.4943	-0.3701	0.2546
0.54	0.1002	0.1003	0.2048	0.2047	-0.5058	-0.5061	-0.2938	0.2227
0.58	0.1020	0.1021	0.2236	0.2235	-0.5150	-0.5153	-0.2275	0.1913
0.62	0.1034	0.1035	0.2423	0.2422	-0.5220	-0.5224	-0.1712	0.1609
0.66	0.1045	0.1045	0.2609	0.2607	-0.5273	-0.5276	-0.1244	0.1320
0.70	0.1052	0.1053	0.2792	0.2791	-0.5310	-0.5313	-0.0865	0.1050
0.74	0.1057	0.1058	0.2974	0.2972	-0.5335	-0.5339	-0.0569	0.0803
0.78	0.1060	0.1061	0.3153	0.3152	-0.5351	-0.5355	-0.0348	0.0584
0.82	0.1062	0.1063	0.3330	0.3329	-0.5360	-0.5365	-0.0192	0.0396
0.86	0.1063	0.1064	0.3505	0.3505	-0.5365	-0.5370	-0.0091	0.0242
0.90	0.1063	0.1064	0.3679	0.3679	-0.5367	-0.5372	-0.0033	0.0124
0.94	0.1064	0.1064	0.3852	0.3852	-0.5368	-0.5373	-0.0007	0.0045
0.98	0.1064	0.1065	0.4024	0.4024	-0.5368	-0.5373	-0.0000	0.0005
1.00	0.1064	0.1065	0.4110	0.4110	-0.5368	-0.5373	-0.0000	0.0000

*For Case 6, axial force N is zero and $z_n = \begin{cases} -0.1981 \text{ in.} & 0 < x < 0.22\ell \\ 0.1981 \text{ in.} & 0.22\ell < x < 0.78\ell \\ -0.1981 \text{ in.} & 0.78\ell < x < \ell \end{cases}$

**CFS ∿ closed-form solutions, TMS ∿ transfer-matrix solutions.

Table 2.8. Closed-Form and Transfer-Matrix Solutions for an Aramid-Cord Rubber Beam (Case 7)*

x/ℓ	$u \times 10^4$, in.		$w \times 10^2$, in.		$\psi \times 10^3$, rad		M, lb-in.		Q, lb
	CFS**	TMS**	CFS	TMS	CFS	TMS	CFS	TMS	CFS & TMS
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	-0.5333	-0.5325	0.4000
0.02	0.1762	0.1760	0.0241	0.0241	-0.0889	-0.0888	-0.4706	-0.4698	0.3840
0.06	0.4648	0.4645	0.0733	0.0731	-0.2346	-0.2345	-0.3528	-0.3520	0.3520
0.10	0.6742	0.6740	0.1227	0.1224	-0.3403	-0.3402	-0.2453	-0.2445	0.3200
0.14	0.8118	0.8116	0.1710	0.1707	-0.4097	-0.4096	-0.1480	-0.1472	0.2880
0.18	0.8846	0.8844	0.2172	0.2168	-0.4464	-0.4464	-0.0610	-0.0602	0.2560
0.22	0.8999	0.8997	0.2603	0.2598	-0.4542	-0.4541	0.0158	0.0160	0.2240
0.26	0.8648	0.8647	0.2995	0.2989	-0.4365	-0.4364	0.0823	0.0822	0.1920
0.30	0.7866	0.7865	0.3338	0.3332	-0.3970	-0.3969	0.1387	0.1395	0.1600
0.34	0.6724	0.6723	0.3628	0.3621	-0.3394	-0.3393	0.1848	0.1856	0.1280
0.38	0.5295	0.5294	0.3859	0.3852	-0.2672	-0.2672	0.2206	0.2214	0.0960
0.42	0.3650	0.3649	0.4027	0.4020	-0.1842	-0.1842	0.2615	0.2470	0.0640
0.46	0.1861	0.1861	0.4129	0.4121	-0.9392	-0.9391	0.2616	0.2624	0.0320
0.50	-0.0000	-0.0000	0.4163	0.4156	-0.0000	-0.0000	0.2667	0.2675	-0.0000
0.54	-0.1861	-0.1861	0.4129	0.4121	0.9392	0.9390	0.2616	0.2624	-0.0320
0.58	-0.3650	-0.3649	0.4027	0.4020	0.1842	0.1842	0.2615	0.2624	-0.0640
0.62	-0.5295	-0.5294	0.3859	0.3852	0.2672	0.2672	0.2206	0.2214	-0.0960
0.66	-0.6724	-0.6723	0.3628	0.3621	0.3394	0.3393	0.1848	0.1856	-0.1280
0.70	-0.7866	-0.7865	0.3338	0.3332	0.3970	0.3969	0.1387	0.1395	-0.1600
0.74	-0.8648	-0.8647	0.2995	0.2989	0.4365	0.4364	0.0823	0.0822	-0.1920
0.78	-0.8999	-0.8997	0.2603	0.2598	0.4542	0.4541	0.0158	0.0160	-0.2240
0.82	-0.8846	-0.8844	0.2172	0.2168	0.4464	0.4464	-0.0610	-0.0602	-0.2560
0.86	-0.8118	-0.8116	0.1710	0.1707	0.4097	0.4069	-0.1480	-0.1472	-0.2880
0.90	-0.6742	-0.6740	0.1227	0.1224	0.3403	0.3402	-0.2453	-0.2445	-0.3200
0.94	-0.4648	-0.4645	0.0733	0.0731	0.2346	0.2345	-0.3528	-0.3520	-0.3520
0.98	-0.1762	-0.1760	0.0241	0.0241	0.0889	0.0888	-0.4706	-0.4698	-0.3840
1.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	-0.5333	-0.5325	-0.4000

*For Case 7, axial force N is zero and z_n constant, equal to 0.1981 in.
**CFS ∿ closed-form solutions; TMS ∿ transfer-matrix solutions.

Table 2.9. Closed-Form and Transfer-Matrix Solutions
for an Aramid-Cord Rubber Beam (Case 8)*

x/ℓ	u x 10⁴, in.		w x 10³, in.		ψ x 10³, rad		M, lb-in.		Q, lb	
	CFS**	TMS**	CFS	TMS	CFS	TMS	CFS	TMS	CFS	TMS
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	-0.0949	-0.0945	0.1510	0.1509
0.02	0.0291	0.0289	0.0911	0.0908	-0.0147	-0.0146	-0.0708	-0.0704	0.1500	0.1499
0.06	0.0622	0.0619	0.2739	0.2723	-0.0314	-0.0312	-0.0238	-0.0235	0.1421	0.1420
0.10	0.0635	0.0632	0.4455	0.4426	-0.0320	-0.0320	0.0193	0.0197	0.1267	0.1266
0.14	0.0364	0.0363	0.5926	0.5886	-0.0184	-0.0183	0.0566	0.0569	0.1049	0.1048
0.18	-0.0141	-0.0139	0.7042	0.6993	0.0071	0.0070	0.0859	0.0862	0.0779	0.0778
0.22	-0.0821	-0.0816	0.7717	0.7663	0.0414	0.0412	0.1061	0.1063	0.0476	0.0475
0.26	-0.1608	-0.1598	0.7906	0.7844	0.0812	0.0806	0.1162	0.1164	0.0157	0.0156
0.30	-0.2430	-0.2416	0.7573	0.7519	0.1227	0.1219	0.1162	0.1164	-0.0156	-0.0157
0.34	-0.3218	-0.3199	0.6754	0.6706	0.1624	0.1614	0.1065	0.1066	-0.0445	-0.0446
0.38	-0.3906	-0.3883	0.5495	0.5456	0.1971	0.1960	0.0882	0.0883	-0.0691	-0.0692
0.42	-0.4440	-0.4415	0.3876	0.3849	0.2241	0.2228	0.0629	0.0629	-0.0878	-0.0880
0.46	-0.4778	-0.4751	0.2004	0.1989	0.2412	0.2398	0.0327	0.0327	-0.0996	-0.0997
0.50	-0.4894	-0.4866	0.0000	-0.0000	0.2413	0.2456	-0.0000	-0.0000	-0.1036	-0.1037
0.54	-0.4778	-0.4751	-0.2004	-0.1989	0.2412	0.2398	-0.0327	-0.0327	-0.0996	-0.0997
0.58	-0.4440	-0.4415	-0.3876	-0.3849	0.2241	0.2228	-0.0629	-0.0629	-0.0878	-0.0880
0.62	-0.3906	-0.3883	-0.5495	-0.5456	0.1971	0.1960	-0.0882	-0.0883	-0.0691	-0.0692
0.66	-0.3218	-0.3199	-0.6754	-0.6706	0.1624	0.1614	-0.1065	-0.1066	-0.0445	-0.0446
0.70	-0.2430	-0.2416	-0.7573	-0.7519	0.1227	0.1219	-0.1162	-0.1164	-0.0156	-0.0157
0.74	-0.1608	-0.1598	-0.7906	-0.7844	0.0812	0.0806	-0.1162	-0.1164	0.0157	0.0156
0.78	-0.0821	-0.0816	-0.7717	-0.7663	0.0414	0.0412	-0.1061	-0.1063	0.0476	0.0475
0.82	-0.0141	-0.0139	-0.7042	-0.6993	0.0071	0.0070	-0.0859	-0.0862	0.0779	0.0778
0.86	0.0364	0.0363	-0.5926	-0.5886	-0.0184	-0.0183	-0.0566	-0.0569	0.1049	0.1048
0.90	0.0635	0.0632	-0.4455	-0.4426	-0.0320	-0.0320	-0.0193	-0.0197	0.1267	0.1266
0.94	0.0622	0.0619	-0.2739	-0.2723	-0.0314	-0.0312	0.0238	0.0235	0.1421	0.1420
0.98	0.0291	0.0289	-0.0911	-0.0908	-0.0147	-0.0146	-0.0708	0.0704	1.1500	0.1499
1.00	0.0000	0.0000	-0.0000	-0.0000	0.0000	0.0000	0.0949	0.0945	0.1510	0.1509

*For Case 8, axial force N is zero and $z_n = \begin{cases} -0.1981 \text{ in.} & 0 \leq x < 0.1\ell \\ 0.1981 \text{ in.} & 0.1\ell \leq x < 0.5\ell \\ -0.1981 \text{ in.} & 0.5\ell \leq x < 0.9\ell \\ 0.1981 \text{ in.} & 0.9\ell \leq x \leq \ell \end{cases}$

**CFS ∿ closed-form solutions; TMS ∿ transfer-matrix solutions.

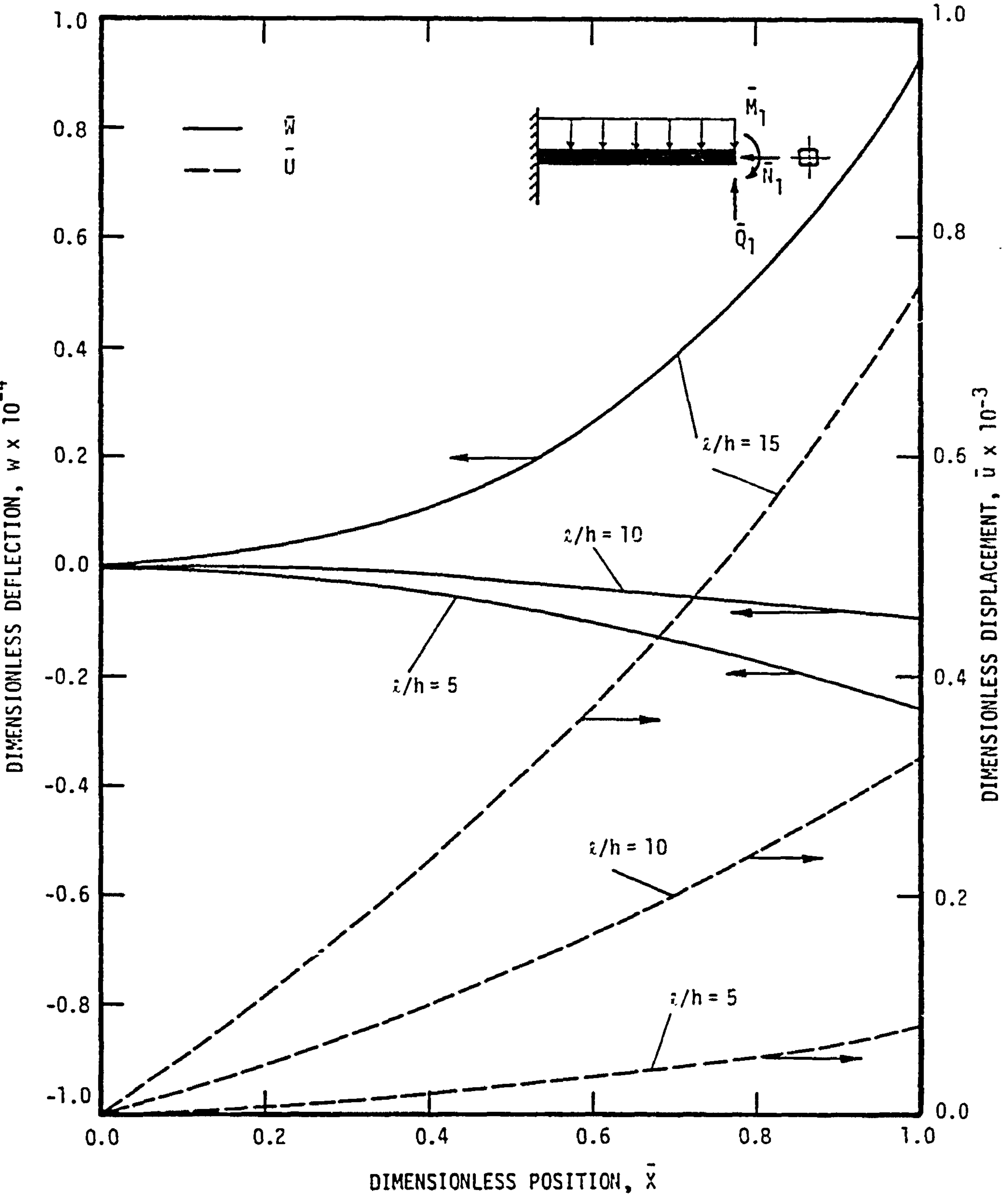


Fig. 2.10. Transfer-matrix solution of clamped-free, aramid-cord-rubber beam with rectangular cross section and applied force and moment at the free end (ℓ/h = 5, 10, 15).

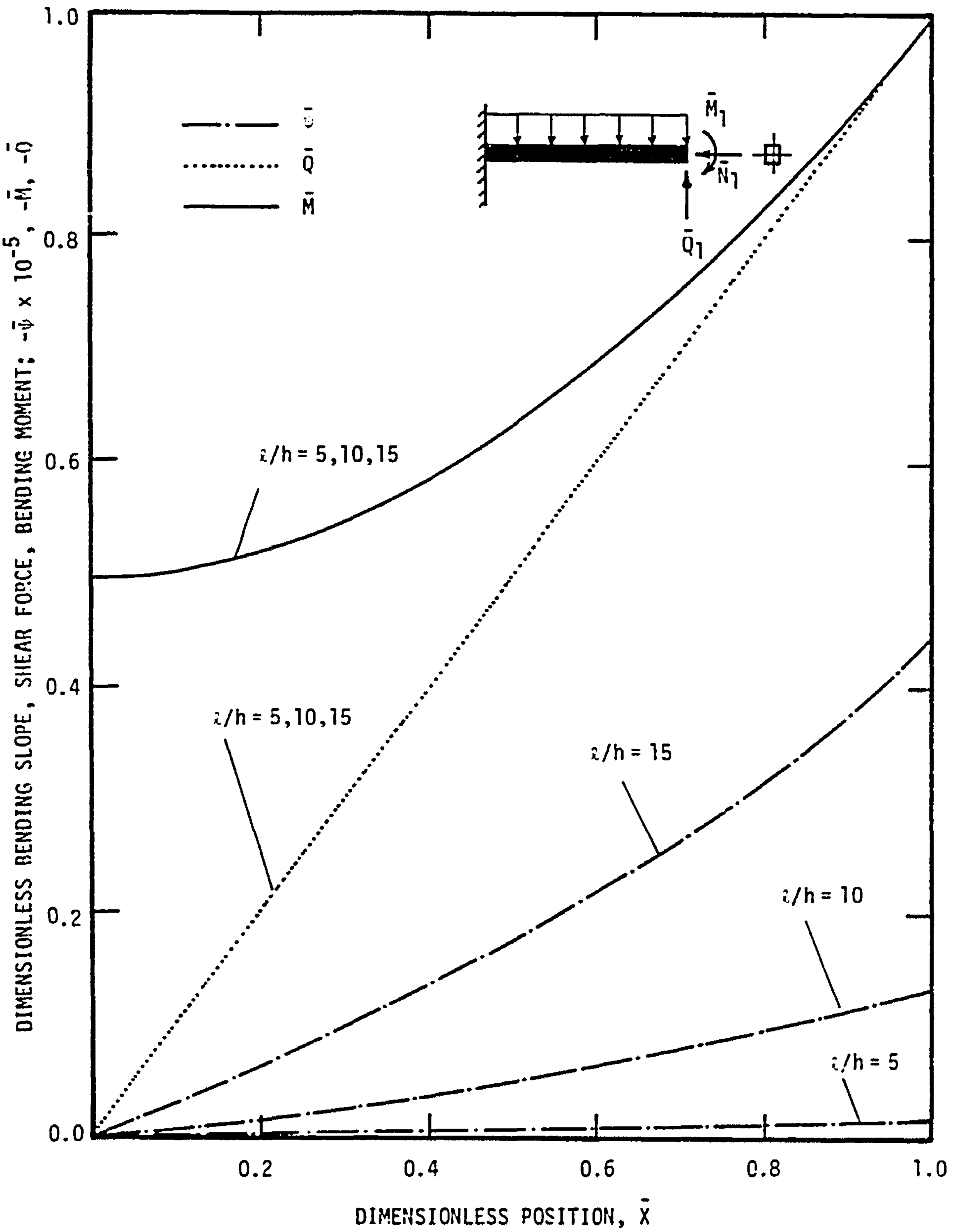


Fig. 2.11. Transfer-matrix solution of clamped-free, aramid-cord-rubber beam with rectangular cross section and applied force and moment at the free end (ℓ/h = 5, 10, 15).

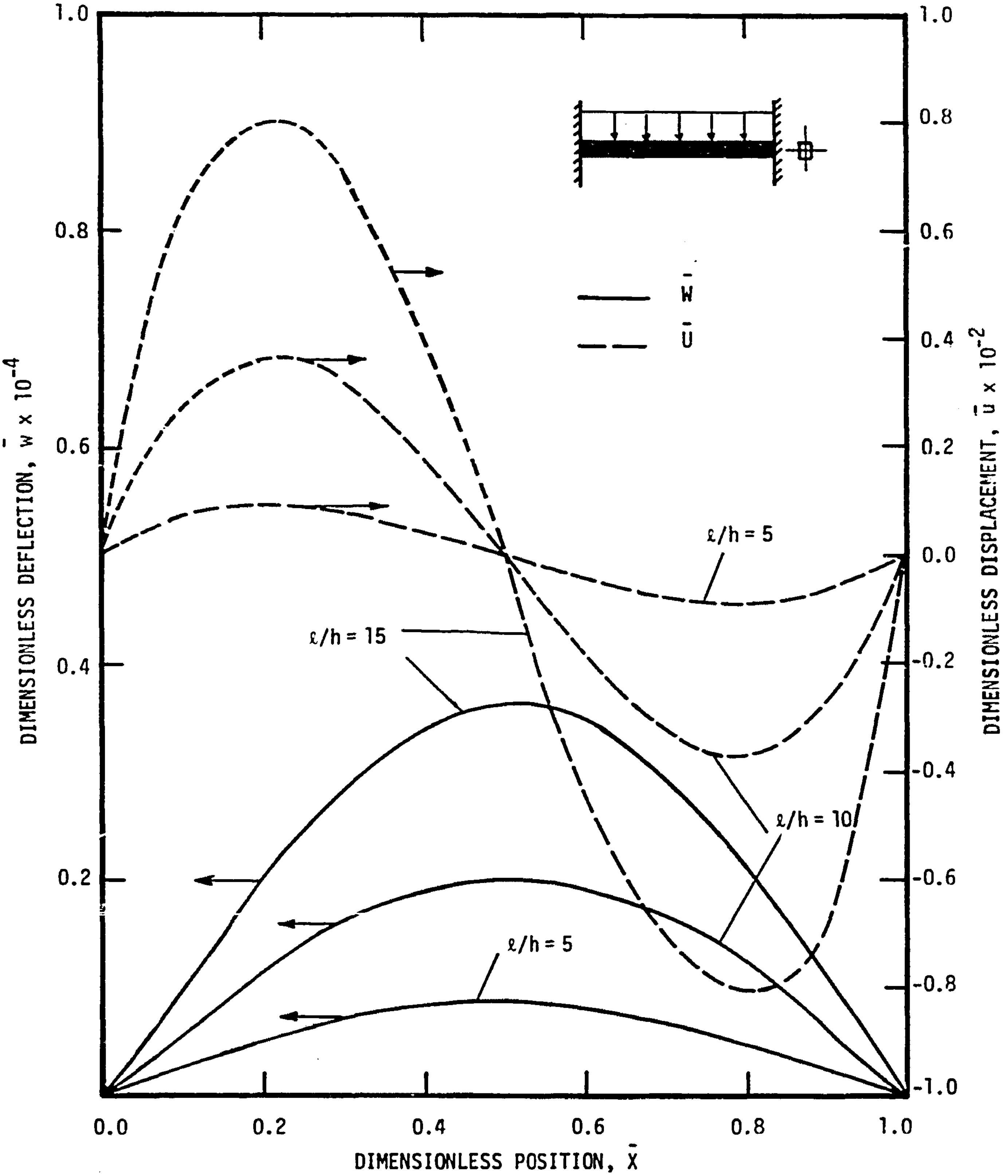


Fig. 2.12. Transfer-matrix solution of clamped-clamped, aramid-cord-rubber beam with rectangular cross section (ℓ/h = 5, 10, 15).

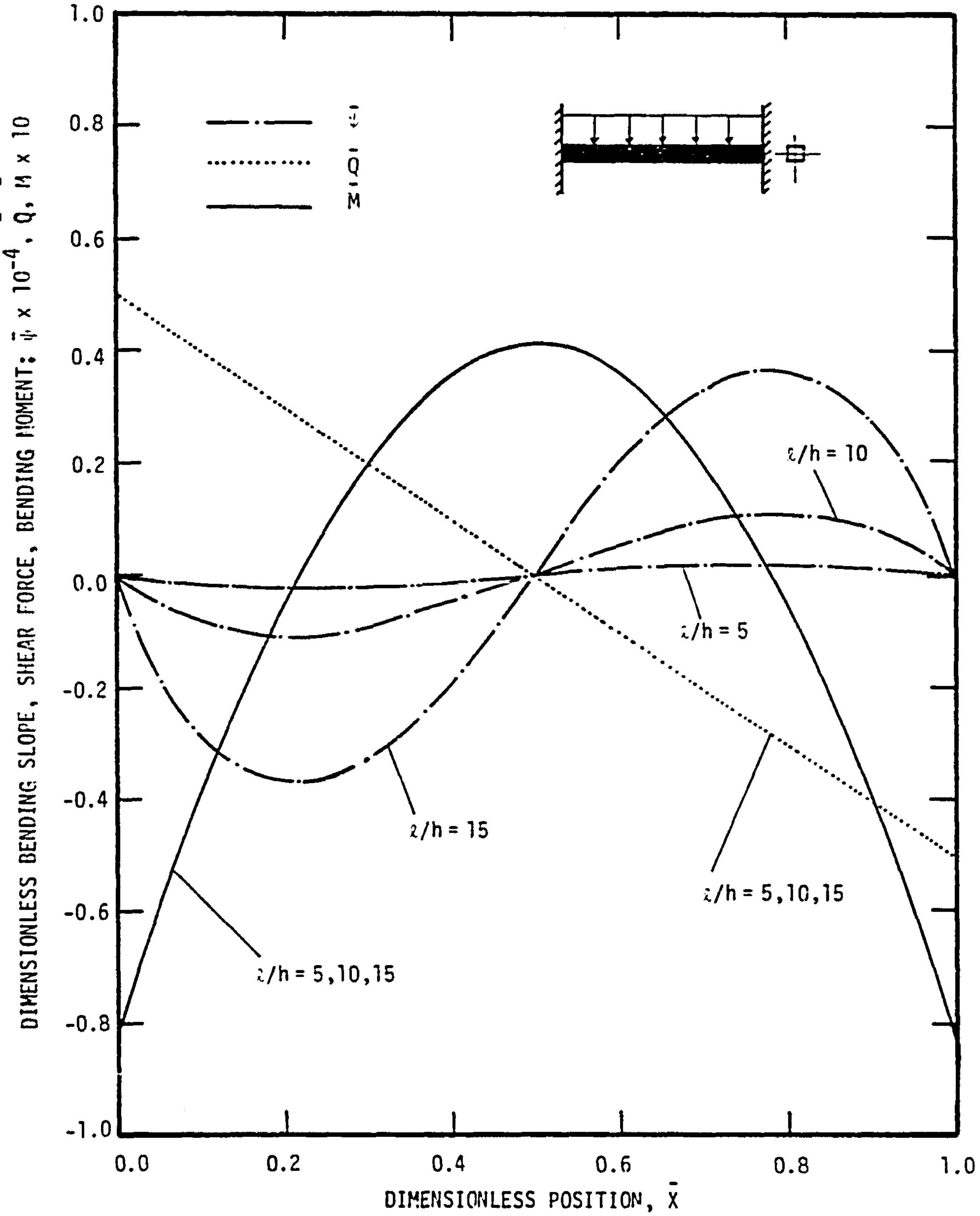


Fig. 2.13. Transfer-matrix solution of clamped-clamped, aramid-cord-rubber beam with rectangular cross section (ℓ/h = 5, 10, 15).

Table 2.10. Transfer-Matrix Solutions for an Aramid-Cord Rubber Beam (Case 11)*

x/ℓ	$u \times 10^4$, in.	$w \times 10^2$, in.	$\psi \times 10^3$, rad	M, lb-in.	Q, lb
0.00	0.0000	0.0000	0.0000	-0.4123	0.2546
0.02	0.1376	0.0157	-0.0695	-0.3716	0.2541
0.06	0.3702	0.0499	-0.1868	-0.2908	0.2501
0.10	0.5467	0.0867	-0.2759	-0.2119	0.2422
0.14	0.6689	0.1247	-0.3376	-0.1362	0.2304
0.18	0.7395	0.1626	-0.3732	-0.0648	0.2150
0.22	0.7619	0.1992	-0.3845	0.0011	0.1962
0.26	0.7403	0.2335	-0.3737	0.0642	0.1743
0.30	0.6797	0.2642	-0.3430	0.1123	0.1497
0.34	0.5854	0.2907	-0.2955	0.1560	0.1227
0.38	0.4637	0.3121	-0.2341	0.1906	0.0937
0.42	0.3210	0.3278	-0.1620	0.2158	0.0633
0.46	0.1641	0.3374	-0.0828	0.2311	0.0319
0.50	-0.0000	0.3406	-0.0000	0.2362	-0.0000
0.54	-0.1641	0.3374	0.0828	0.2311	-0.0319
0.58	-0.3210	0.3278	0.1620	0.2158	-0.0633
0.62	-0.4637	0.3121	0.2341	0.1906	-0.0937
0.66	-0.5854	0.2907	0.2955	0.1560	-0.1227
0.70	-0.6797	0.2642	0.3430	0.1123	-0.1497
0.74	-0.7403	0.2335	0.3737	0.0642	-0.1743
0.78	-0.7619	0.1992	0.3845	0.0011	-0.1962
0.82	-0.7395	0.1626	0.3732	-0.0648	-0.2150
0.86	-0.6689	0.1247	0.3376	-0.1362	-0.2304
0.90	-0.5467	0.0867	0.2759	-0.2119	-0.2422
0.94	-0.3702	0.0499	0.1868	-0.2908	-0.2501
0.98	-0.1376	0.0157	0.0695	-0.3716	-0.2541
1.00	0.0000	0.0000	0.0000	-0.4123	-0.2546

*For Case 11, axial force due to bending-stretching coupling (N) is -0.3070×10^{-4} lb (compressive), also z_n = 0.1918 in.

MULTIMODULAR BEAM*   BIMODULAR BEAM*

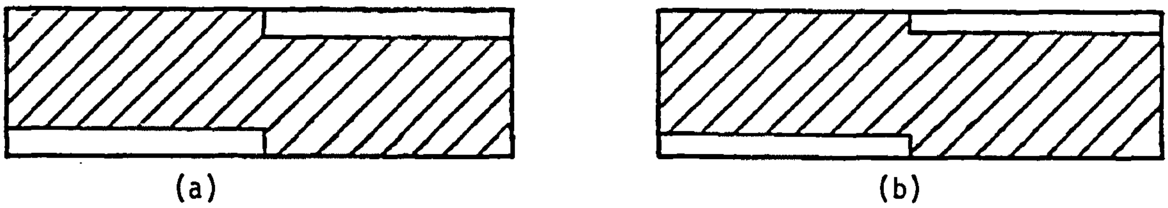


Fig. 2.14. Neutral-surface location (case 2).

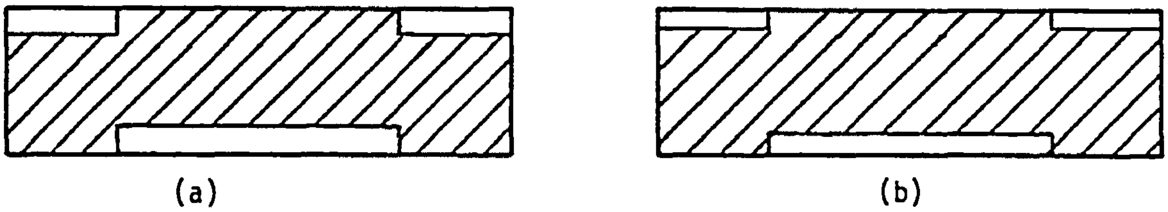


Fig. 2.15. Neutral-surface location (cases 7, 10, 11).

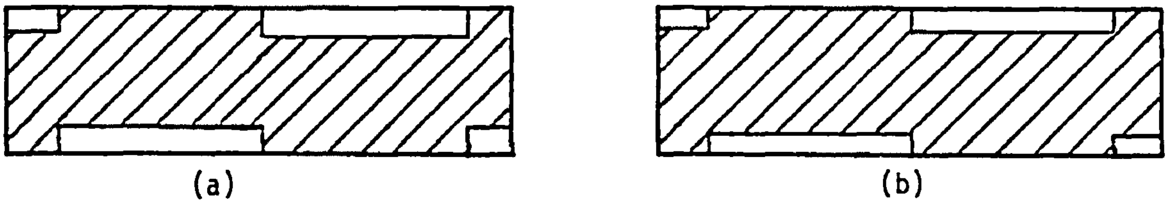


Fig. 2.16. Neutral-surface location (case 8).

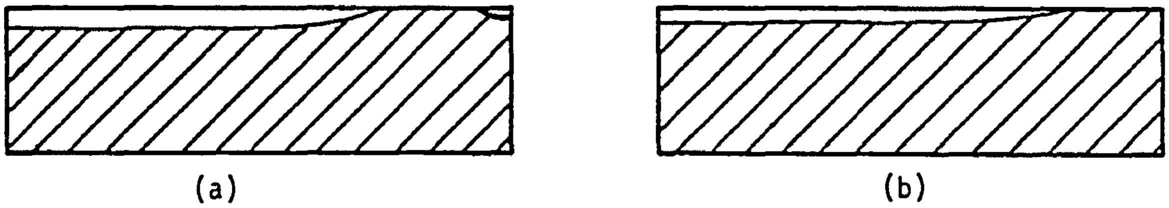
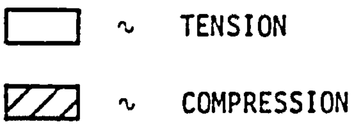


Fig. 2.17. Neutral-surface location (case 9**).



*Computations are applicable to a specific beam (see Table 1).
**For Case 9: $M_1 = -0.1$ lb-in., $N_1 = -1.0$ lb, and $Q_1 = -0.1$ lb.

the shapes of neutral-surface curves for multimodular and bimodular models have been shown for several cases.

2.7 Conclusions

Analyses of the bending deflection of multimodular thick beams with rectangular cross section based on shear-deformable-beam theory are presented. In this study, both dimensionless and dimensioned results of transfer-matrix, as well as closed-form, solutions for a rectangular multimodular beam of aramid-cord rubber are presented. The transfer-matrix and the closed-form solutions are found to agree very well.

Results of analysis of bimodular and multimodular models show that there is not a drastic difference between the two models. Although the multimodular model is a better one for approximating the stress-strain curve, the bimodular approximation is less complicated. Closed-form solutions are available only for a number of loading/boundary conditions (in which the axial force is identically zero), but the transfer-matrix method can be applied to more complicated geometry, loading, and boundary conditions. In this work, results for several boundary and loading conditions are investigated. The transfer-matrix method is found to be very effective in terms of computational time and also gives results which agree quite well with the closed-form solutions.

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SECTION III

FORCED VIBRATION OF TIMOSHENKO BEAMS MADE OF MULTIMODULAR MATERIALS†

In this section, a transfer-matrix analysis for determining the sinusoidal vibration response of a thick, rectangular-cross-section beam made of multimodular materials is presented. To validate the transfer-matrix results, a closed-form solution is also presented for the special case in which the neutral-surface location is uniform along the length of the beam. Effects of translatory and rotatory inertia coefficients on axial force are investigated for a clamped-clamped beam. Moreover, natural frequencies associated with the first three modes of a clamped-free beam are presented. Transfer-matrix results agree very well with closed-form results for the corresponding material model (one, two, or four segments).

3.1 Introduction

Many materials have different elastic behavior in tension and compression. A few examples of such materials are concrete, rock, tire-cord-

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†This section is an abbreviated version of a paper accepted by Journal of Vibration, Acoustics, Stress, and Reliability in Design (to appear, 1984), and presented orally at the 9th Biennial Vibrations Conference, Sept. 11-14, 1983, Dearborn, Michigan.

rubber, and soft biological tissues. As early as 1864, St. Venant [3.1] recognized this behavior by analyzing the pure bending behavior of a beam having different stress-strain curves in tension and compression. Timoshenko [3.2] originated the concept of bimodulus (or bimodular) materials in 1941 by considering the flexural stresses in such a material undergoing pure bending. Ambartsumyan [3.3] in 1965 renewed interest in the analysis of bimodular materials, i.e., materials having different moduli in tension and compression. Since then, there have been numerous investigations on the static behavior of bimodular beams; these were surveyed by Tran and Bert [3.4]. Recently, Bert and Gordaninejad [3.5] studied bending of thick beams of "multimodular" materials. Only a few studies have been made on vibration of bimodular beams. Bert and Tran [3.6] worked on transient response of thick beams of bimodular materials.

In this section, the transfer-matrix method [3.7], which computationally is very efficient, is applied. Also, the beam is modeled as a Timoshenko beam, i.e., both transverse shear deformation and rotatory inertia are considered.

3.2 Closed-Form Solution

The general equation of motion, if z_n is constant along the beam, is

$$
\begin{aligned}
&A'u_{,xx} + B'\psi_{,xx} = PU_{,tt} + R\psi_{,tt} \\
&S(w_{,xx} + \psi_{,x}) = Pw_{,tt} - q(x,t) \qquad (3.1) \\
&(B''u_{,xx} + D'\psi_{,xx}) - S(w_{,x} + \psi) = Ru_{,tt} + I\psi_{,tt}
\end{aligned}
$$

where

$$(P,R,I) = \int_{-h/2}^{h/2} \rho(1,z,z^2)dz$$

and ρ is the density of material.

For guided-guided boundary condition, i.e.

$$u(0,t) = u(\ell,t) = 0 \quad ; \quad \psi(0,t) = \psi(\ell,t) = 0 \qquad Q(0,t) = Q(\ell,t) = 0 \tag{3.2}$$

if

$$q(x,t) = q_o \cos \alpha x \cos \Omega t \tag{3.3}$$

then the following sets of functions satisfy the equations of motion†

$$u(x,t) = \bar{U} \sin \alpha x \cos \Omega t \quad ; \quad \psi(x,t) = \bar{X} \sin \alpha x \cos \Omega t \qquad w(x,t) = \bar{W} \cos \alpha x \cos \Omega t \tag{3.4}$$

where

$$\Omega = 2\pi f \quad , \quad \alpha = m\pi/\ell \qquad (m=1,2,3,\ldots) \tag{3.5}$$

f ≡ circular frequency of the excitation, and

$$\bar{U} = \frac{q_o(S\alpha)(B'\alpha^2-R\Omega^2)}{(S\alpha^2-P\Omega^2)[B'\alpha^2-R\Omega^2)(B''\alpha^2-R\Omega^2)-(D'\alpha^2+S-I\Omega^2)(A'\alpha^2-P\Omega^2)]+(S\alpha)^2(A'\alpha^2-P\Omega^2)}$$

$$\bar{X} = \frac{A'\alpha^2 - P\Omega^2}{B'\alpha^2 - R\Omega^2}\bar{U} \quad ; \quad \bar{W} = \frac{1}{S\alpha^2 - P\Omega^2}\left[q_o - \frac{(S\alpha)(A'\alpha^2 - P\Omega^2)}{B'\alpha^2 - R\Omega^2}\right]\bar{U} \tag{3.6}$$

Since from Eqs. (2.16)

$$z_n = - u_{,x}/\psi_{,x} \tag{3.7}$$

then

$$z_n = \frac{B'\alpha^2 - R\Omega^2}{A'\alpha^2 - P\Omega^2} = \text{constant} \tag{3.8}$$

3.3 Transfer-Matrix Solution

The transfer-matrix model used in the present study is the same as

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† See page 107

that employed in [3.

which here includes

for the assumed beam

$$[S]_{N_s+1} = [T$$

where $[T_f]_i$ is the f

number of stations,

ends of the beam, $\Delta\ell$

$[S]_{N_s+1}$, $[S]_0$ are st

of the beam.

In the calculat

force is not zero, t

distances to the "br

constant and not kno

been employed to com

a_t. One must first

then compute the sti

state vector. Final

the new values of z_n

sets of z_n, a_c, and

solved; otherwise, as

procedure.

The numerical re

with a rectangular cr

be aramid cord-rubber which is used in automobile tires (see Table 3.1). Four different boundary conditions are investigated (see Table 3.2) and comparisons are made between multimodular, bimodular, and unimodular models for each set of boundary conditions. In this study, a mesh of twenty-five elements is used with each element being of length of 0.32 in. The shear correction coefficient is taken to be 5/6.

In order to validate the transfer-matrix solution (TMS), in Fig. 3.1, a comparison is made between the closed-form solution (CFS) and the TMS for a guided-guided beam with cosine load distribution (case 1). Also, a comparison is made among unimodular, bimodular, and multimodular (static and dynamic) cases (see also Table 3.3) for f = 100 Hz. As one can see, there is excellent agreement between the TMS and CFS results. However, this agreement can be improved even further by increasing the number of elements.

For the other cases (2-4), the CFS is not available; therefore, in Figs. 3.2, 3.4, and 3.6 comparisons between different models (one, two, and four segment approximations) are made. As one might notice in all four cases, there is considerable difference between transverse deflection of multimodular and bimodular beams on one hand and that of the unimodular model on the other hand. In contrast, there is no substantial difference between multimodular and bimodular results.

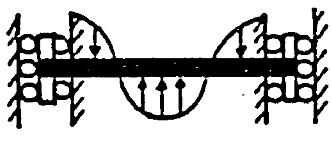
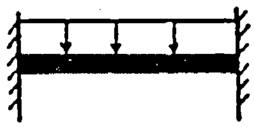
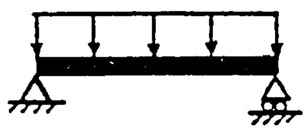
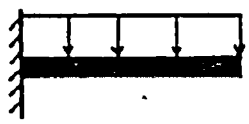
Another interesting observation in Fig. 3.2 is that for f = 100 Hz, the unimodular beam is in the range of its first mode, whereas the bimodular and multimodular beams are between their first and second modes. The explanation for this is that, for this case (2), most of the layers of the beam are under compression and since E_1^c, E_2^c, and E_b^c are much smaller

Table 3.1. Elastic Properties and Geometric Parameters for an Aramid-Cord-Rubber Beam

		Longitudinal Young's Modulus, MPa (psi x 10^{-6})				Londitudinal-Thickness Shear Modulus, MPa (psi x 10^{-3})
	Model*	Tension		Compression		Tension and Compression
Elastic Properties	M	E_2^t	E_1^t	E_1^c	E_2^c	G
		4000 (0.580)	2896 (0.420)	221 (0.032)	71 (0.01)	3.70 (0.537)
	B	E_b^t		E_b^c		
		3240 (0.470)		124 (0.018)		3.70 (0.537)
	U	E		E		
		1896 (0.275)		1896 (0.275)		3.70 (0.537)
Geometric Parameter	Beam length	20.32 cm (8.0 in.)				
	Beam thickness	1.52 cm (0.6 in.)				
	Beam width	2.54 cm (1.0 in.)				

*M ∿ multimodular, B ∿ bimodular, U ∿ unimodular.

Table 3.2. Summary of Cases Considered

CASE NO.	BOUNDARY CONDITION AND LOAD POSITION	CASE NO.	BOUNDARY CONDITION AND LOAD POSITION
1		3	
2		4	

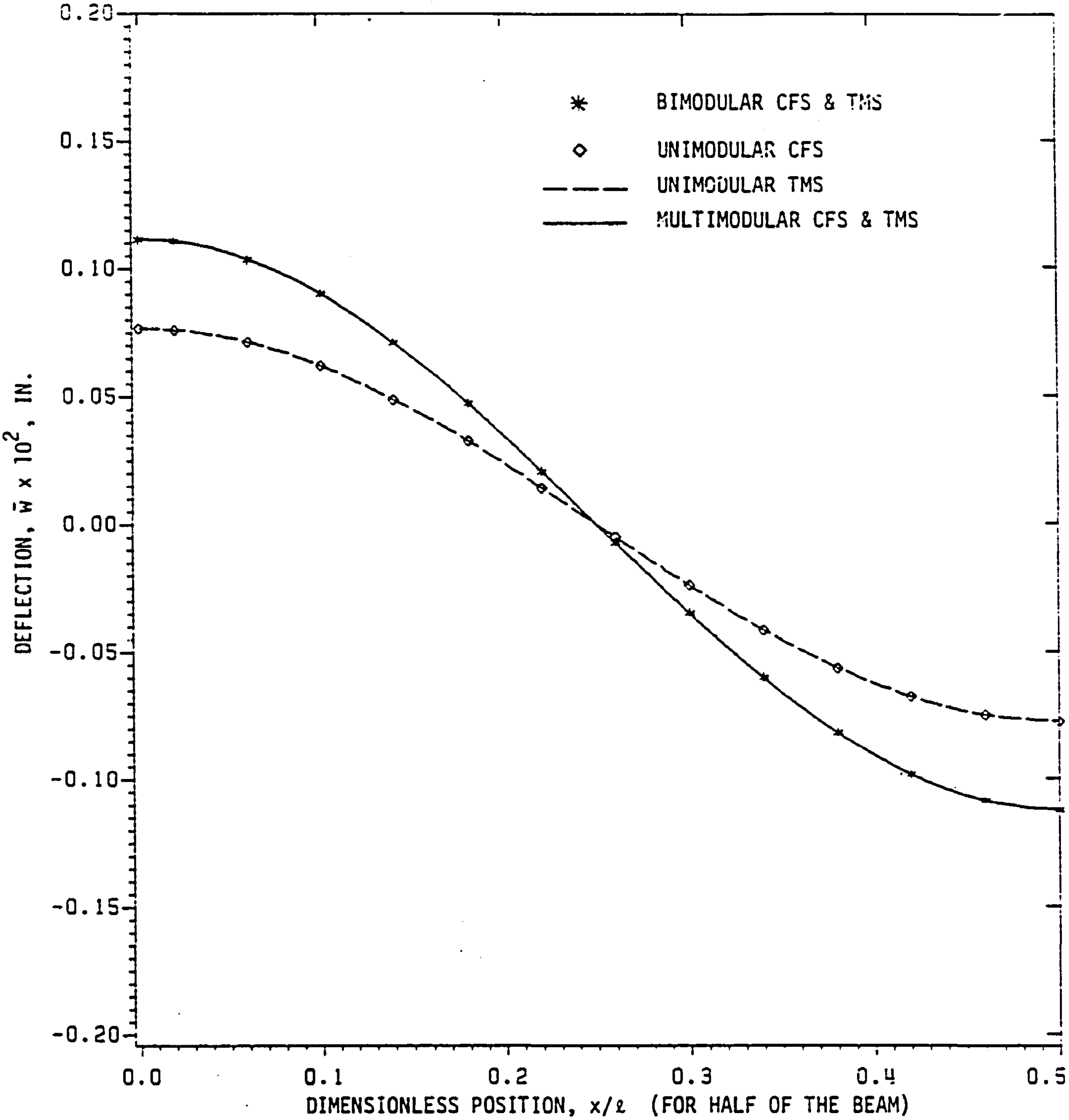


Fig. 3.1. Comparison among multimodular, bimodular, and unimodular deflection distribution for closed-form and transfer-matrix solutions of guided-guided aramid-cord rubber beam (f = 100 Hz)

Table 3.3. Comparison Between CFS* and TMS for an Aramid-Cord-Rubber Beam (Case 1)†, f = 100 Hz

x/ℓ	$\bar{N} \times 10^2$, lb		$\bar{Q} \times 10$, lb		$\bar{M}$, lb-in.	
	CFS	TMS	CFS	TMS	CFS	TMS
0.00	-0.169	-0.164	0.000	0.000	0.203	0.203
0.02	-0.168	-0.162	-0.198	-0.200	0.201	0.202
0.06	-0.157	-0.152	-0.585	-0.587	0.189	0.189
0.10	-0.137	-0.132	-0.935	-0.937	0.164	0.165
0.14	-0.108	-0.104	-1.226	-1.228	0.129	0.130
0.18	-0.072	-0.069	-1.141	-1.143	0.086	0.087
0.22	-0.032	-0.031	-1.565	-1.566	0.038	0.038
0.26	0.011	0.010	-1.590	-1.591	-0.013	-0.013
0.30	0.052	0.050	-1.514	-1.516	-0.063	-0.063
0.34	0.091	0.087	-1.344	-1.346	-0.109	-0.109
0.38	0.123	0.119	-1.089	-1.091	-0.148	-0.148
0.42	0.148	0.143	-0.766	-0.768	-0.178	-0.178
0.46	0.164	0.158	-0.394	-0.396	-0.196	-0.197
0.50	0.169	0.163	0.000	0.000	-0.203	-0.204
0.54	0.164	0.158	0.394	0.396	-0.196	-0.197
0.58	0.148	0.143	0.766	0.768	-0.178	-0.178
0.62	0.123	0.119	1.089	1.091	-0.148	-0.148
0.66	0.091	0.087	1.344	1.346	-0.109	-0.109
0.70	0.052	0.050	1.514	1.516	-0.063	-0.063
0.74	0.011	0.010	1.590	1.591	-0.013	-0.013
0.78	-0.032	-0.031	1.565	1.566	0.038	0.038
0.82	-0.072	-0.069	1.141	1.143	0.086	0.087
0.86	-0.108	-0.104	1.226	1.228	0.129	0.130
0.90	-0.137	-0.132	0.935	0.937	0.164	0.165
0.94	-0.157	-0.152	0.585	0.587	0.189	0.189
0.98	-0.168	-0.162	0.198	0.200	0.201	0.202
1.00	-0.169	-0.164	0.000	0.000	0.203	0.203

*CFS ∿ closed-form solution; TMS ∿ transfer-matrix solution

†For case 1, $Z = z_n/h$ is piecewise constant, equal to $\begin{cases} +\,0.3305 & 0 \le x/\ell < 0.22 \\ -\,0.3305 & 0.22 \le x/\ell < 0.78 \\ +\,0.3305 & 0.78 \le x/\ell \le 1.00 \end{cases}$

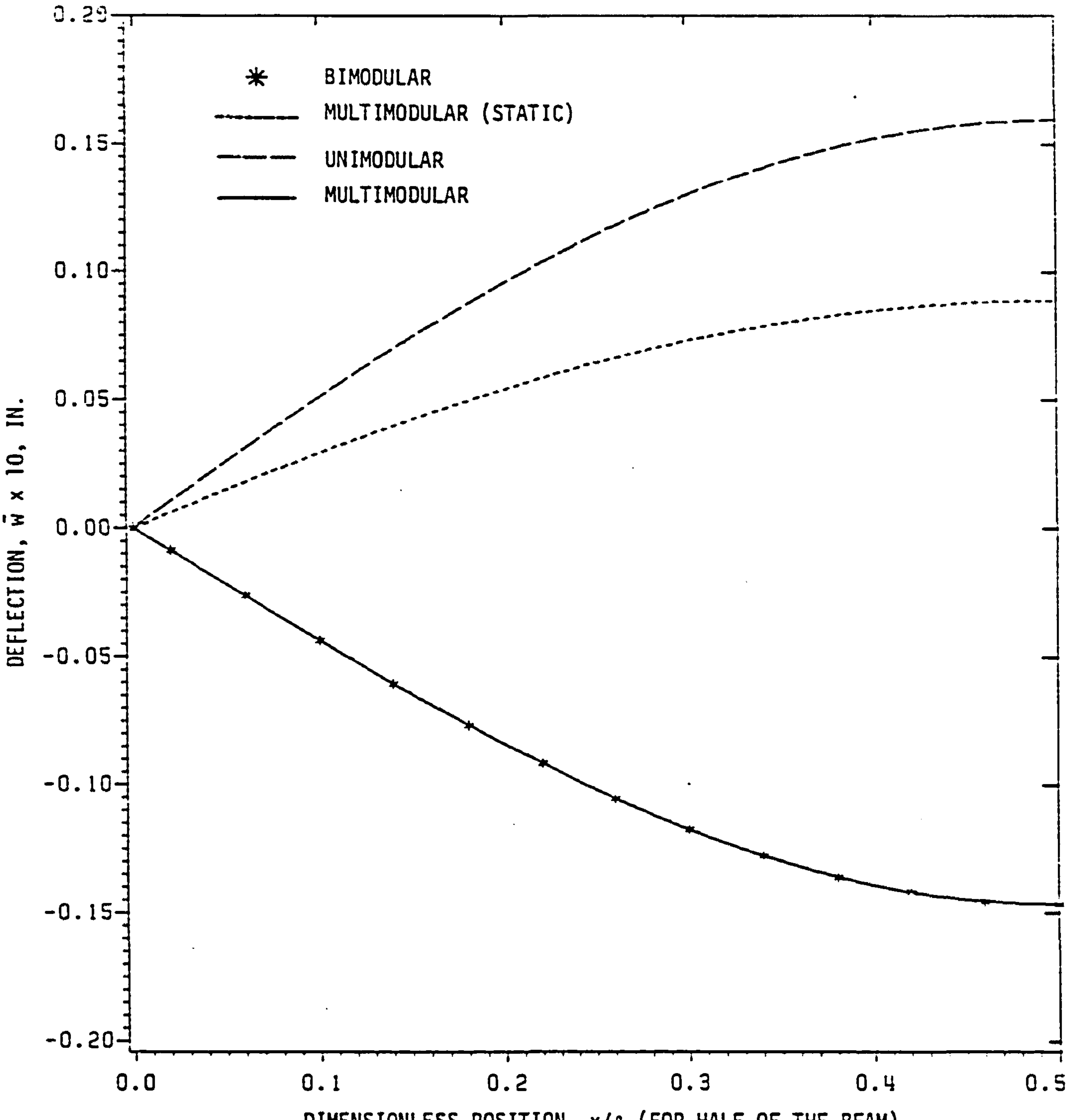


Fig. 3.2. Comparison among multimodular, bimodular, and unimodular deflection distribution for transfer-matrix solution of hinged-hinged, aramid-cord rubber beam (f = 100 Hz)

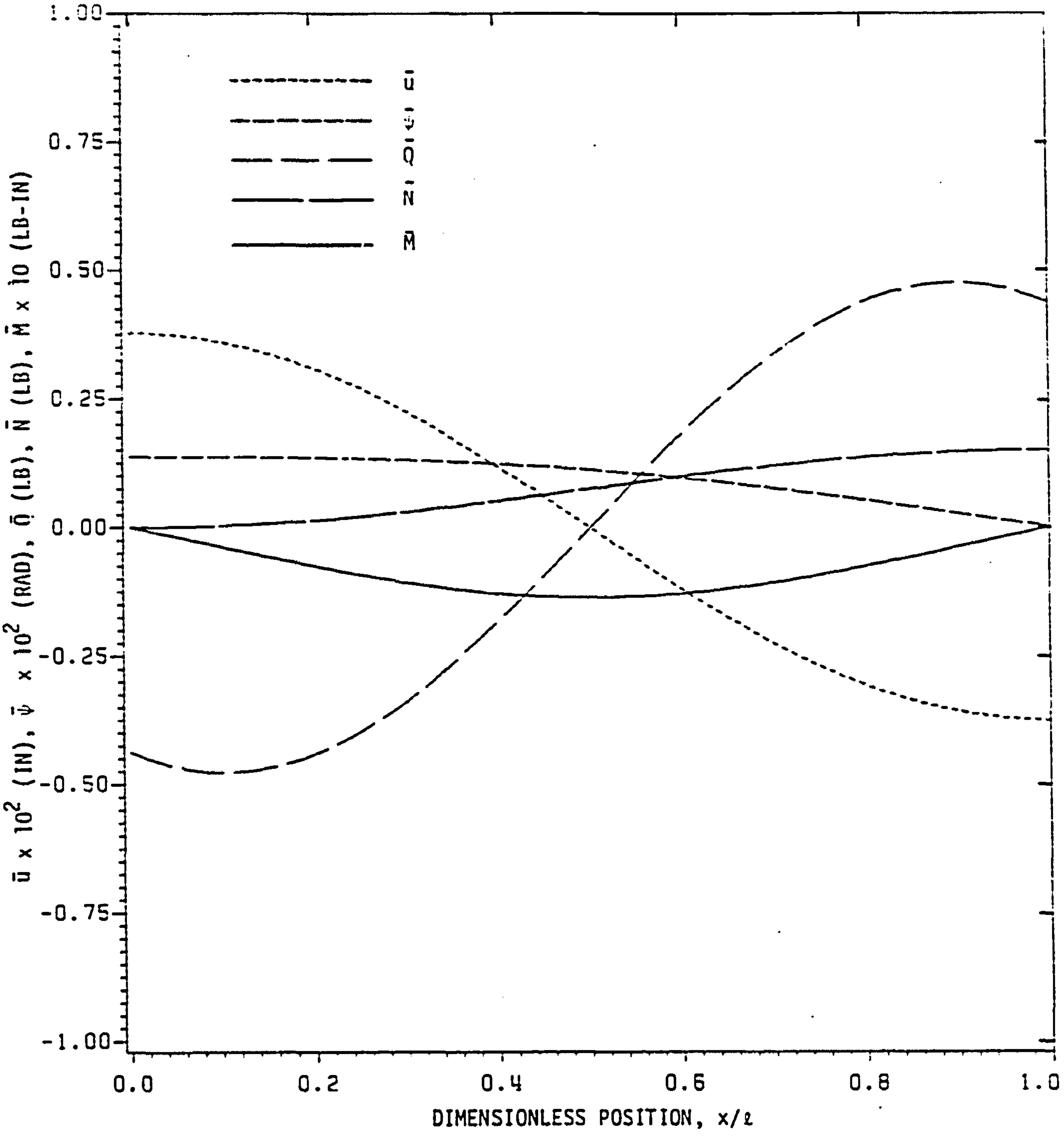


Fig. 3.3. Transfer-matrix solution of hinged-hinged, aramid-cord rubber beam (f = 100 Hz)

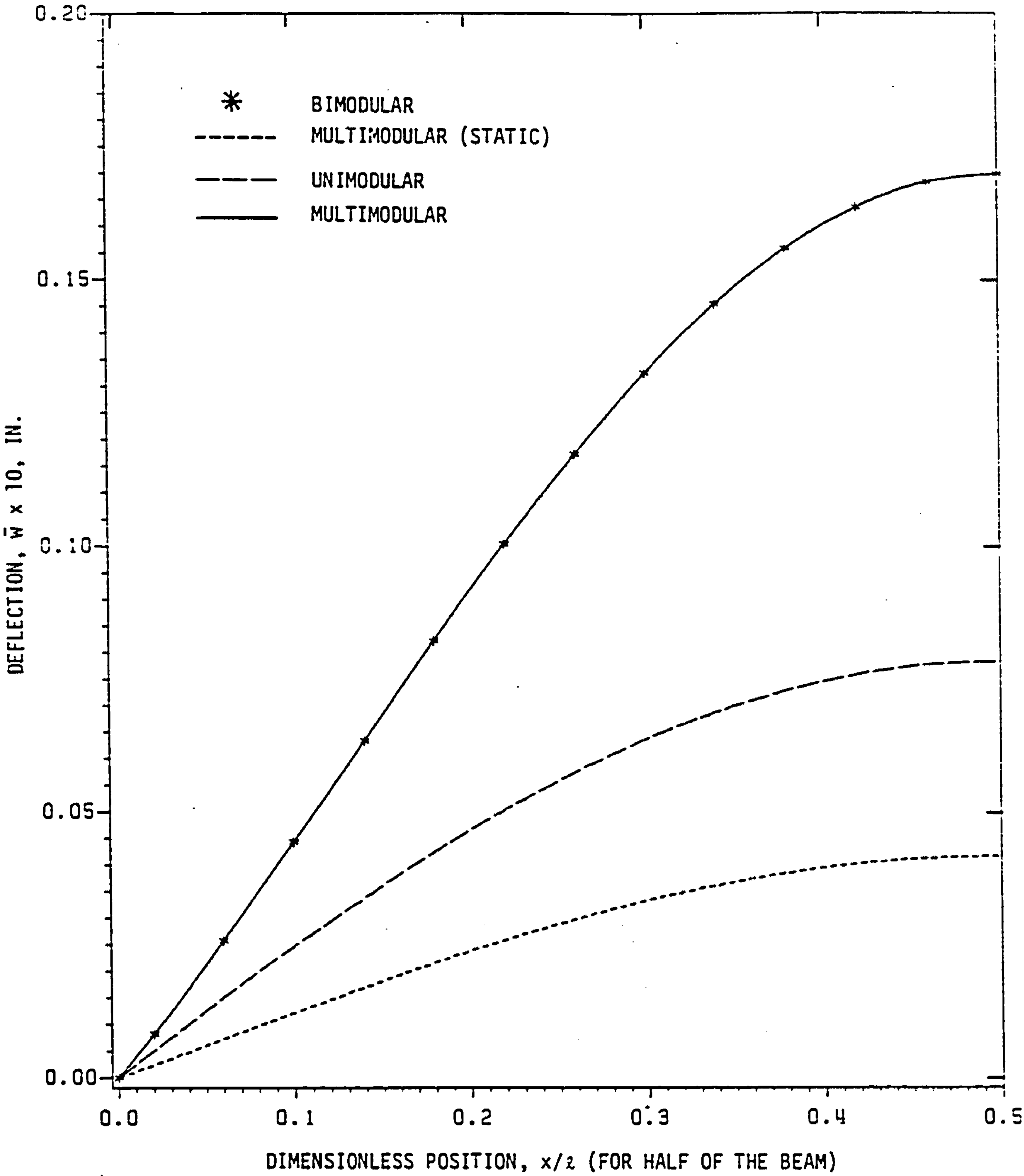


Fig. 3.4. Comparison among multimodular, bimodular, and unimodular deflection distribution for transfer-matrix solution of clamped-clamped aramid-cord rubber beam (f = 100 Hz)

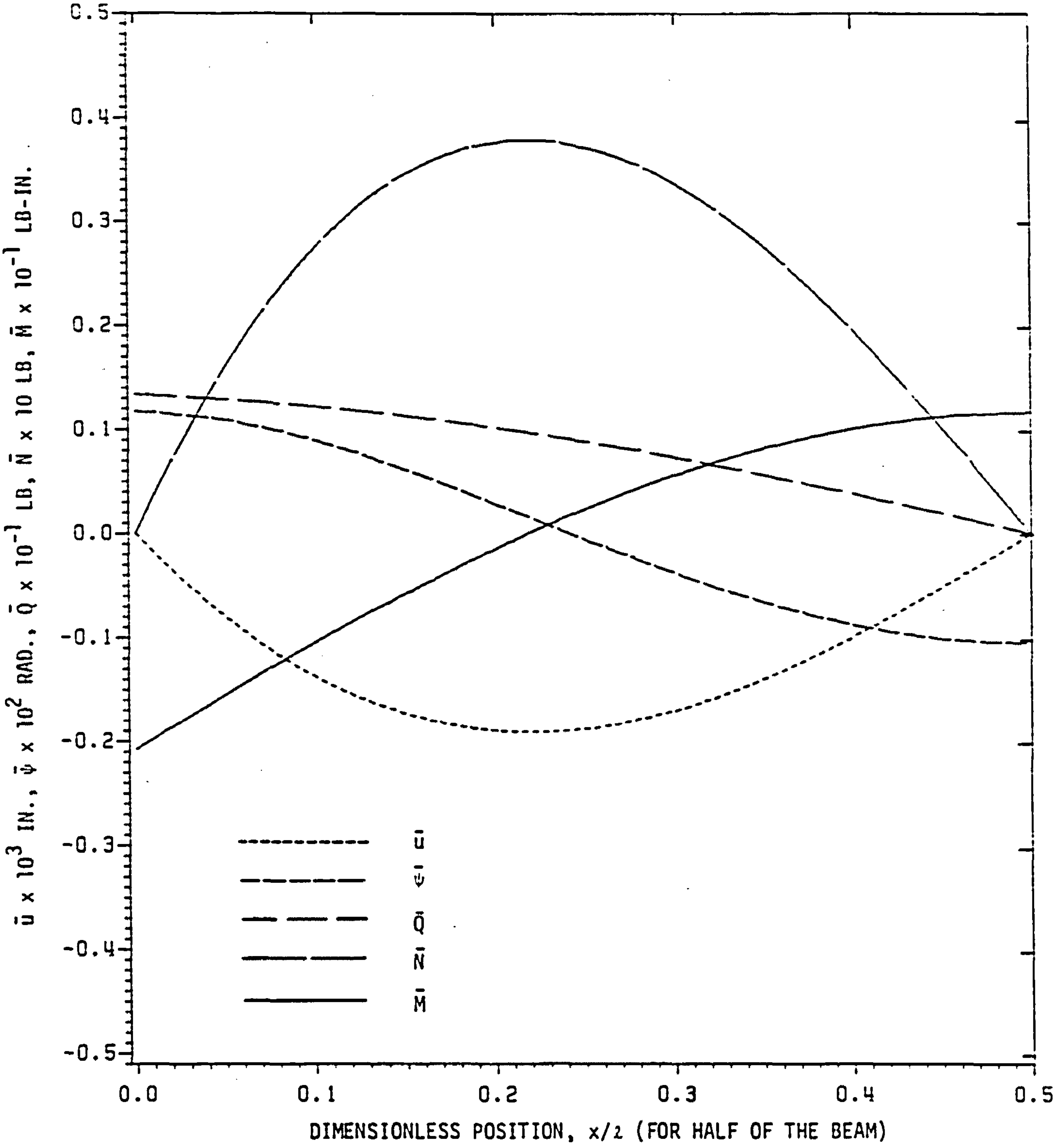


Fig. 3.5. Transfer-matrix solution of clamped-clamped, aramid-cord rubber beam (f = 100 Hz)

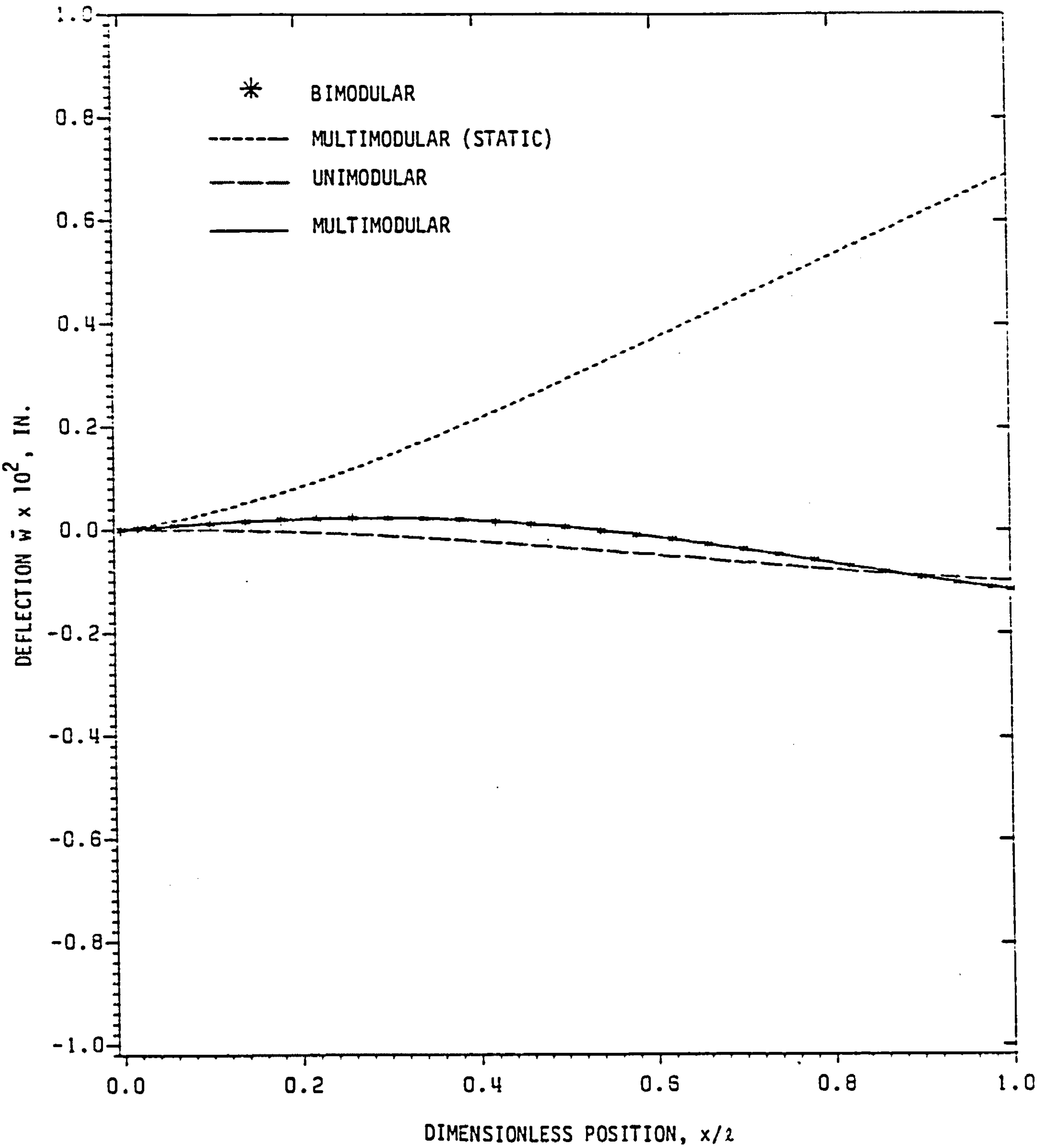


Fig. 3.6. Comparison among multimodular, bimodular, and unimodular deflection distribution for transfer-matrix solution of clamped-free aramid-cord rubber beam (f = 100 Hz)

than E (See Table 3.1), then the unimodular beam is stiffer than the other two. Therefore, the fundamental frequency of the unimodular beam is higher than those of the bimodular and multimodular beams.

Also, the distributions of $\bar{u}$ (axial displacement), $\bar{\psi}$ (bending slope), $\bar{N}$ (axial force), $\bar{Q}$ (transverse shear force), and $\bar{M}$ (bending moment) are shown graphically in Figs. 3.3, 3.5, and 3.7 for f taken to be 100 Hz. Note that for cases 1 and 3, this frequency is less than the fundamental frequency, whereas for cases 2 and 4, the frequency of 100 Hz is in the range of the first and second modes. The first three mode shapes of a clamped-free beam of multimodular material is investigated in Fig. 3.8. For this case, the natural frequencies associated with the first three modes are f_1 = 30.9 Hz, f_2 = 131.1 Hz, and f_3 = 278.4 Hz.

Finally, by rewriting the equations of motion in a new form ($N_{,x} = P_1 u_{,tt}$, $M_{,x} - Q = I\psi_{,tt}$, $Q_{,x} = P_2 u_{,tt} - q(x,t)$), the effect of translatory and rotatory inertia coefficients on axial force for a thick multimodular clamped-clamped beam (f = 100 Hz) is studied (see Table 3.4). The results show the significant effect of I and slight effect of P_1 as one looks at it through full theory (vibration) as compared to the static case ($I = P_1 = P_2 = 0$).

3.5 Conclusions

An analysis of forced vibration of a thick beam with a rectangular cross section and made of "multimodular" material is presented. In this study, numerical results obtained by both the closed-form and transfer-matrix methods are given for a beam made of aramid-cord rubber.

Comparisons are made on one hand between closed-form and transfer-

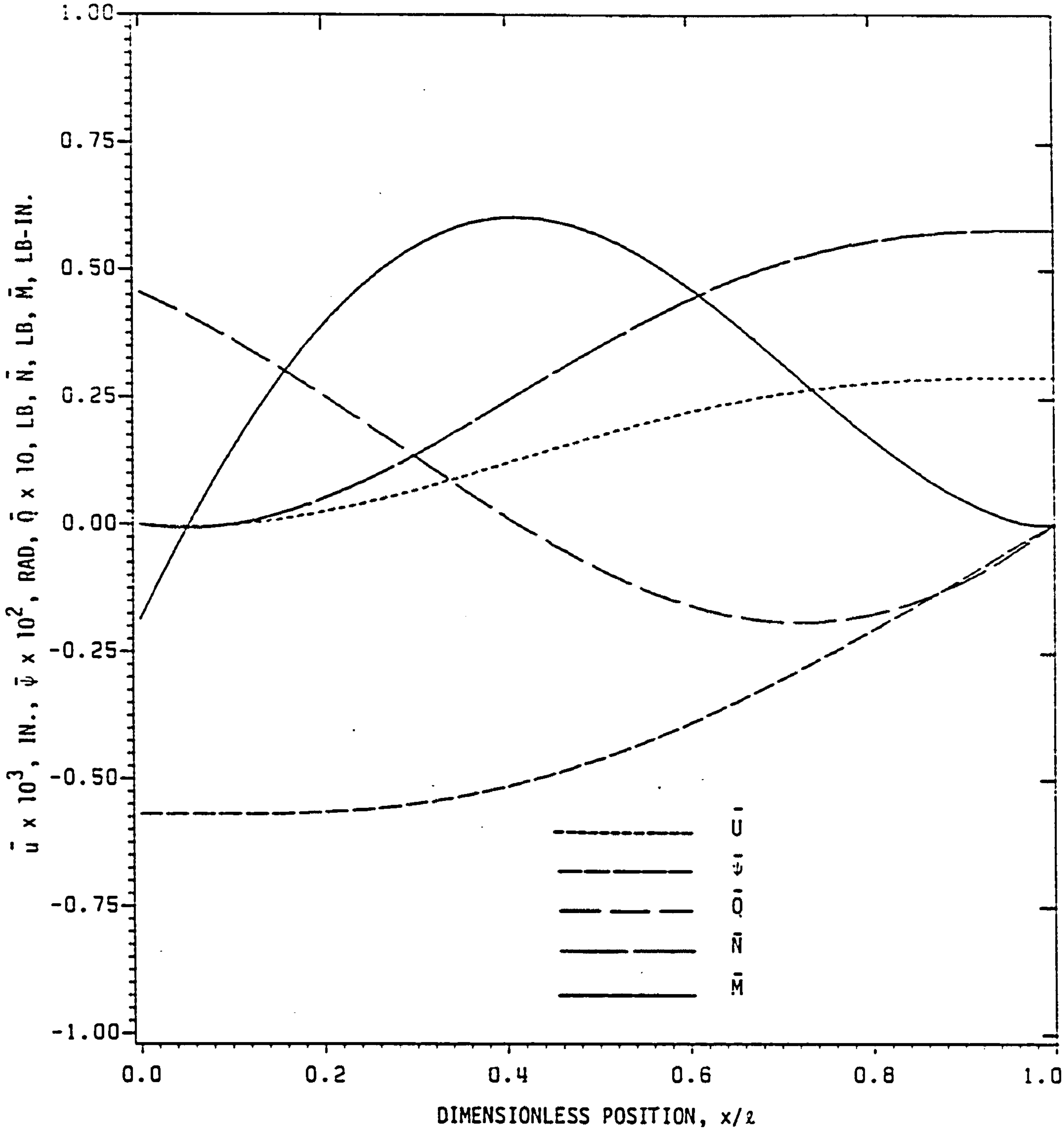


Fig. 3.7. Transfer-matrix solution of clamped-free aramid-cord rubber beam (f = 100 Hz)

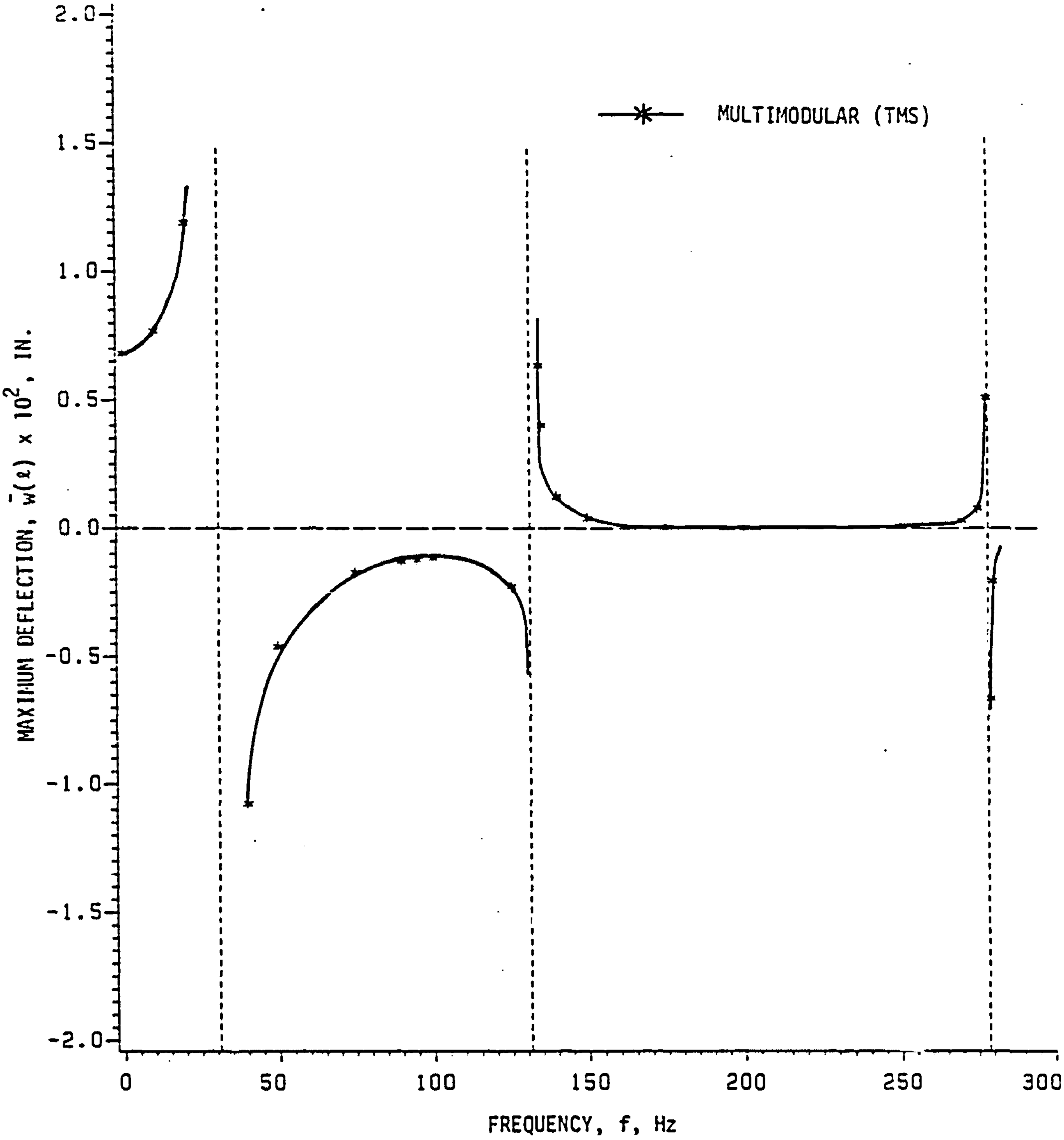


Fig. 3.8. Investigation of the first three mode shapes of thick multimodular, clamped-free aramid-cord rubber beam with rectangular cross section.

Table 3.4. Effect of Translatory and Rotatory Inertia Coefficients on Axial Force for a Thick Multimodular Cantilever Beam (f = 100 Hz)

x/L	Axial force, lb x 10^3*				
	$I, P_1, P_2 \neq 0$ Full Theory	$I=0 \quad P_1 \& P_2 \neq 0$	$I \& P_1 = 0 \quad P_2 \neq 0$	$P_1 = 0 \quad I \& P_2 \neq 0$	$I, P_1, P_2 = 0$ Static
0.00	11.74	11.76	-0.1399	-0.1621	-0.0699
0.02	11.49	11.51	-0.1399	-0.1621	-0.0699
0.06	10.56	10.58	-0.1399	-0.1621	-0.0699
0.10	8.90	8.92	-0.1399	-0.1621	-0.0699
0.14	6.69	6.73	-0.1399	-0.1621	-0.0699
0.18	4.15	4.19	-0.1399	-0.1621	-0.0699
0.22	1.44	1.48	-0.1399	-0.1621	-0.0699
0.26	-1.26	-1.21	-0.1399	-0.1621	-0.0699
0.30	-3.82	-3.76	-0.1399	-0.1621	-0.0699
0.34	-6.09	-6.03	-0.1399	-0.1621	-0.0699
0.38	-7.97	-7.91	-0.1399	-0.1621	-0.0699
0.42	-9.38	-9.31	-0.1399	-0.1621	-0.0699
0.46	-10.25	-10.18	-0.1399	-0.1621	-0.0699
0.50	-10.54	-10.47	-0.1399	-0.1621	-0.0699

*1 lb = 4.448 newtons

matrix results and on the other hand, among unimodular, bimodular, and multimodular models. These results show a considerable difference between the unimodular and bimodular models and a slight difference between the bimodular and multimodular models. Therefore, although a four-segment model is a better approximation, the two-segment approximation gives nearly the same results. This proves that the bimodular model precision is a good approximation.

The values of the first three mode shapes for the clamped-free case are presented. Finally, the effects of axial translatory and rotatory inertia coefficients on axial force for a clamped-clamped beam are discussed.

The transfer-matrix method is found to be very effective in terms of computational time and also in terms of the accuracy of results, which agree very well with the closed-form solution.

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SECTION IV

TRANSVERSE SHEAR EFFECTS IN BIMODULAR COMPOSITE LAMINATES†

A closed-form solution for the Timoshenko-type shea
efficient (K^2) governing the deflection of bimodular com
in cylindrical bending is presented. The bending-stress
a laminate constructed of bimodular materials (which ha
elastic moduli in tension and compression) is used in th
equilibrium equation to obtain the transverse shear-stre
This shear-stress distribution is used to obtain express
correction coefficient (based on equivalent shear strain
maximum dimensionless transverse shear stress $(\bar{\tau}_{xz})_{max}$.
effects of the elastic-constant ratios on the neutral-su
shear correction coefficient for laminates consisting of
or bimodular materials are studied.

4.1 Introduction

Materials which have different moduli in tension an
are called bimodular materials. Rock, concrete, cord-ru

†This section formed a paper published in Journal of Com
July 1983, and presented orally at the 20th Annual Meet
Engineering Science, August 22-24, 1983, University of
Delaware.

and certain biological tissues are examples of such materials. Even aramid-fiber, polymer-matrix composites exhibit some bimodularity. The analysis of laminated bimodular material is more complicated than unimodular material (ordinary material) due to the dependency of the material stiffness on the material properties, which indeed depend on the state of stress (i.e., tensile or compressive) in the laminate.

Although transverse shear deformations have been considered in the analyses of bimodular laminates [4.1-4.4] in recent years, there has been no effort to include the effect of bimodularity of the material on the Timoshenko-type shear correction coefficient, K^2. In other words, it has been tacitly assumed that the value of K^2 is the same as that of an ordinary-material laminate. The use of such a shear correction factor in predicting the deflection of shear deformable, ordinary-material laminates in cylindrical bending is well established [4.5,4.6]. It is noted that Whitney [4.5] extended Chow's symmetric-laminate work [4.7] to arbitrary laminates. Also, it can be shown that the resulting expressions in [4.5,4.6] are algebraically equivalent.

In this paper, a straight-forward approach analogous to that used in elementary shear theory for single-layer ordinary materials is employed. It may be considered to be a generalization of the work of Bert [4.6] from ordinary-material laminates to bimodular-material laminates.

4.2 Theory and Formulation

Consider a rectangular-cross-section laminated beam of thickness h. The origin of the Cartesian coordinate system is located on the midsurface of the beam with the x axis being along the length of the beam and the z axis being measured positive downward. The same displacement field as used in shear deformable beam theory is implemented here:

$$
\begin{aligned}
(x,z) &= u(x) + z\psi(x) \\
(x,z) &= w(x)
\end{aligned}
\tag{4.1}
$$

The axial normal strain and transverse shear strain are given by

$$\varepsilon_x = u_{,x} + z\psi_{,x} \quad ; \quad \gamma_{xz} = \psi + w_{,x} \tag{4.2}$$

where a comma denotes the derivative with respect to the quantity following it. The longitudinal bending stress at any distance z from the mid-plane of the laminate is

$$\sigma_x = E^{(k)}\varepsilon_x \qquad k=c,t \tag{4.3}$$

Also

$$\tau_{xz} = G^{(k)}\gamma_{xz} \tag{4.4}$$

The longitudinal stress resultant and stress couple are defined as

$$(N,M) = \int_{-h/2}^{h/2} (1,z)\sigma_x \, dz \tag{4.5}$$

With the aid of Eqs. (4.2), (4.3), and (4.5), one can derive the following laminate constitutive relation

$$\begin{Bmatrix} N \\ M \end{Bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{Bmatrix} u_{,x} \\ \psi_{,x} \end{Bmatrix} \tag{4.6}$$

where

$$(A,B,D) = \int_{-h/2}^{h/2} E^{(k)}(1,z,z^2)\, dz \qquad k=c,t$$

In the absence of body force, the two-dimensional static equilibrium equation for forces in the direction along the length of the beam is

$$\sigma_{x,x} + \tau_{xz,z} = 0 \tag{4.7}$$

Integrating Eq. (4.7) with respect to z and using Eqs. (4.3) and (4.4), one has

$$\gamma_{xz} = -[G^{(k)}(AD - B^2)]^{-1} [(Da - Bb)N_{,x} + (Ab - Ba)M_{,x}] \tag{4.8}$$

where a and b are "partial stiffnesses" for stretching and bending-stretching coupling defined by the following expressions for the convex bending case (top layers in compression and bottom layers in tension)

$$(a,b) = \begin{cases} \displaystyle\int_{-h/2}^{z} (1,z)E^c \, dz & z_n > z > -h/2 \\[2ex] \displaystyle\int_{-h/2}^{z_n} (1,z)E^c \, dz + \int_{z_n}^{z} (1,z)E^t \, dz & h/2 > z > z_n \end{cases} \tag{4.9}$$

Multiply Eq. (4.7) by unity and z, respectively, and then integrating through the thickness, one obtains the following equilibrium equations, which coincide with those of elementary beam theory,

$$M_{,x} = Q \quad , \quad N_{,x} = 0 \tag{4.10}$$

where

$$Q = \bar{\gamma}_{xz} \int_{-h/2}^{h/2} G^{(k)} \, dz \tag{4.11}$$

Therefore, Eq. (4.8) simplifies as follows:

$$\gamma_{xz} = -Q(Ab - Ba)[G^{(k)}(AD - B^2]^{-1} \tag{4.12}$$

Extending the definition of the shear correction coefficient as used in [4.6], which is based on the shear strain energy, to the case of bimodular material, one has

$$K^2 = \frac{(AD - B^2)^2}{\left[\int_{-h/2}^{h/2} G^{(k)}\, dz\right]\left[\int_{-h/2}^{h/2} (1/G^{(k)})(Ab - Ba)^2 dz\right]} \tag{4.13}$$

If G is constant throughout the thickness of the beam, Eq. (4.13) can be simplified to

$$K^2 = \frac{(AD - B^2)^2}{h \int_{-h/2}^{h/2} (Ab - Ba)^2\, dz} \tag{4.14}$$

By changing the coordinate from z measured from the midplane to z' measured from the neutral-surface position, one can simplify Eq. (4.14) as follows:

$$K^2 = \frac{D^2}{h \int_{-h_c}^{h_t} b^2\, dz'} \tag{4.15}$$

To evaluate the stiffnesses (A,B, and D), one needs to know about the location of the neutral-surface position. A closed-form solution can be obtained only when the neutral-surface position (z_n) and thus, the stiffnesses do not depend on x. The same criterion to define z_n as used in [4.4] is applied here

$$\varepsilon_x = 0$$

Using Eqs. (4.2) and (4.5) and assuming* that

$$N = 0 \tag{4.17}$$

one gets

$$z_n = B/A = \text{constant} \tag{4.18}$$

Equation (4.18) can be solved explicitly for z_n.

It is convenient to introduce the dimensionless transverse shear stress defined as follows:

$$\bar{\tau}_{xz} = \frac{\tau_{xz}}{Q/h} \tag{4.19}$$

Combining Eqs. (4.4) and (4.12) and measuring z' from the neutral-surface location, one obtains

$$\bar{\tau}_{xz} = bh/D \tag{4.20}$$

Detailed derivations for K^2 and $(\bar{\tau}_{xz})_{max}$ are given in Appendix G.

4.3 Numerical Results

Three cases of laminates constructed of bimodular materials are considered here: single-layer, two-layer, and three-layer. Numerical results obtained from closed-form solutions for the shear correction coefficient and the maximum dimensionless shear stress for different bimodular-material parameters are presented. Also, the effects of elastic-constant ratios on neutral-surface position and K^2 for both unimodular and bimodular laminates (up to three layers) are studied. The laminates are assumed to be made of equal-thickness layers and the effect of the sign of the longitudinal normal

*Note that this assumption is made only for the bimodular case (not unimodular).

Table 4.1 Effect of Bimodular Ratio on Neutral-Surface Location for Single-Layer Bimodular Material

E^c/E^t	h_t/h
0	0
1/2	0.414
1	0.500
2	0.586
5	0.691
10	0.760
100	0.909
1000	0.969
∞	1

Table 4.2 Effect of Bimodular Ratios on Neutral-Surface Location for Two-Layer Bimodular Laminates ($h_1/h_2 = 1$)

$E_2{}^c/E_2{}^t$	h_t/h		
	$E_1{}^c/E_2{}^t = 1/2$	$E_1{}^c/E_2{}^t = 1$	$E_1{}^c/E_2{}^t = 2$
0	0.411	0	0.5811
1/4	0.413	0.333	0.5816
1/2	0.414	0.414	0.5822
3/4	0.415	0.464	0.5827
1	0.417	0.500	0.5833
2	0.421	0.586	0.5858
3	0.424	0.634	0.5886
4	0.427	0.666	0.5917
5	0.430	0.691	0.5955
10	0.439	0.760	0.6667

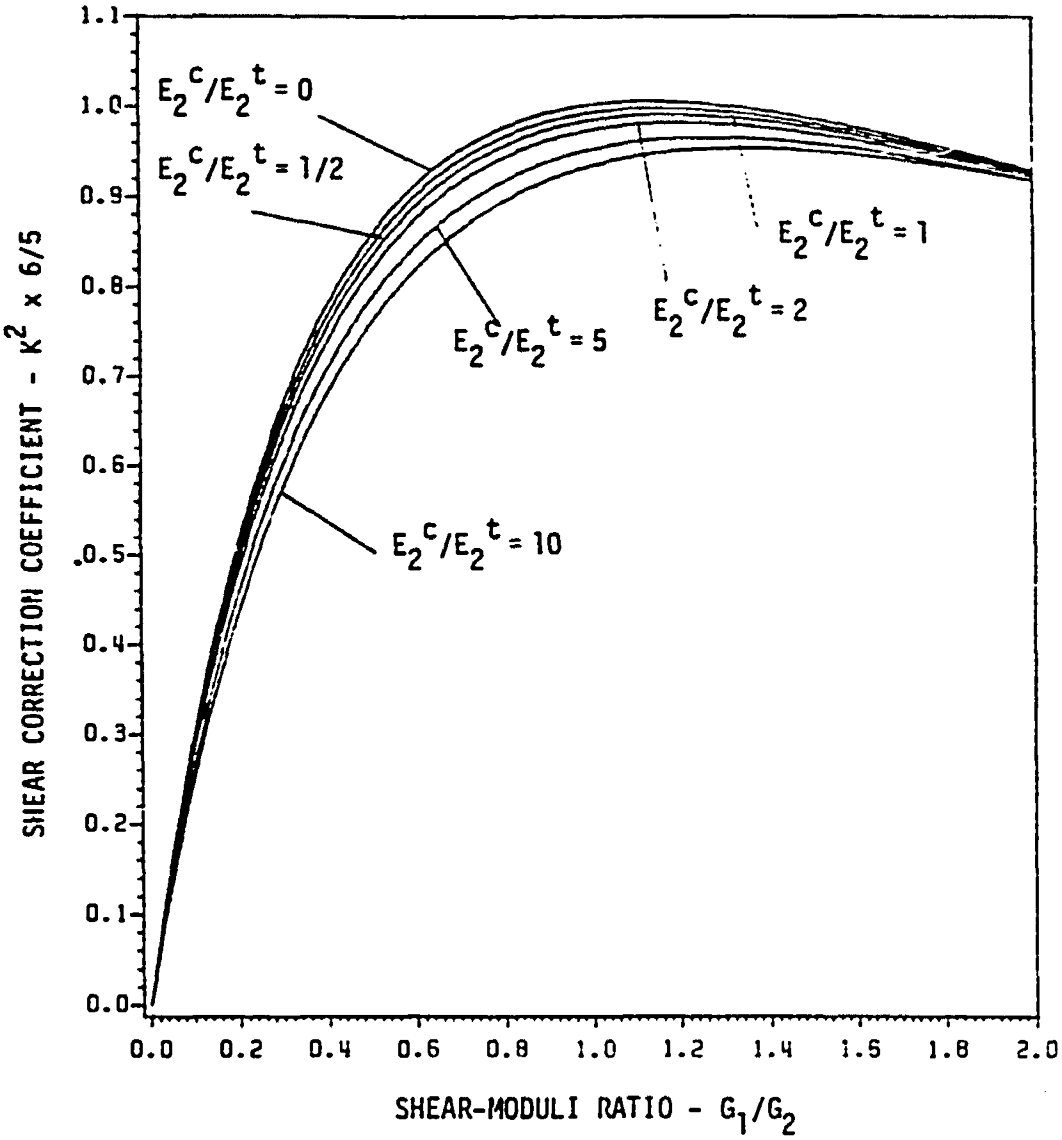

Fig. 4.1. Effect of shear modulus ratio on shear correction coefficient for various bimodulus ratios ($E_2{}^c/E_2{}^t$) for two-layer bimodular laminates ($h_1/h_2 = 1$, $E_1{}^c/E_2{}^t = 1/2$).

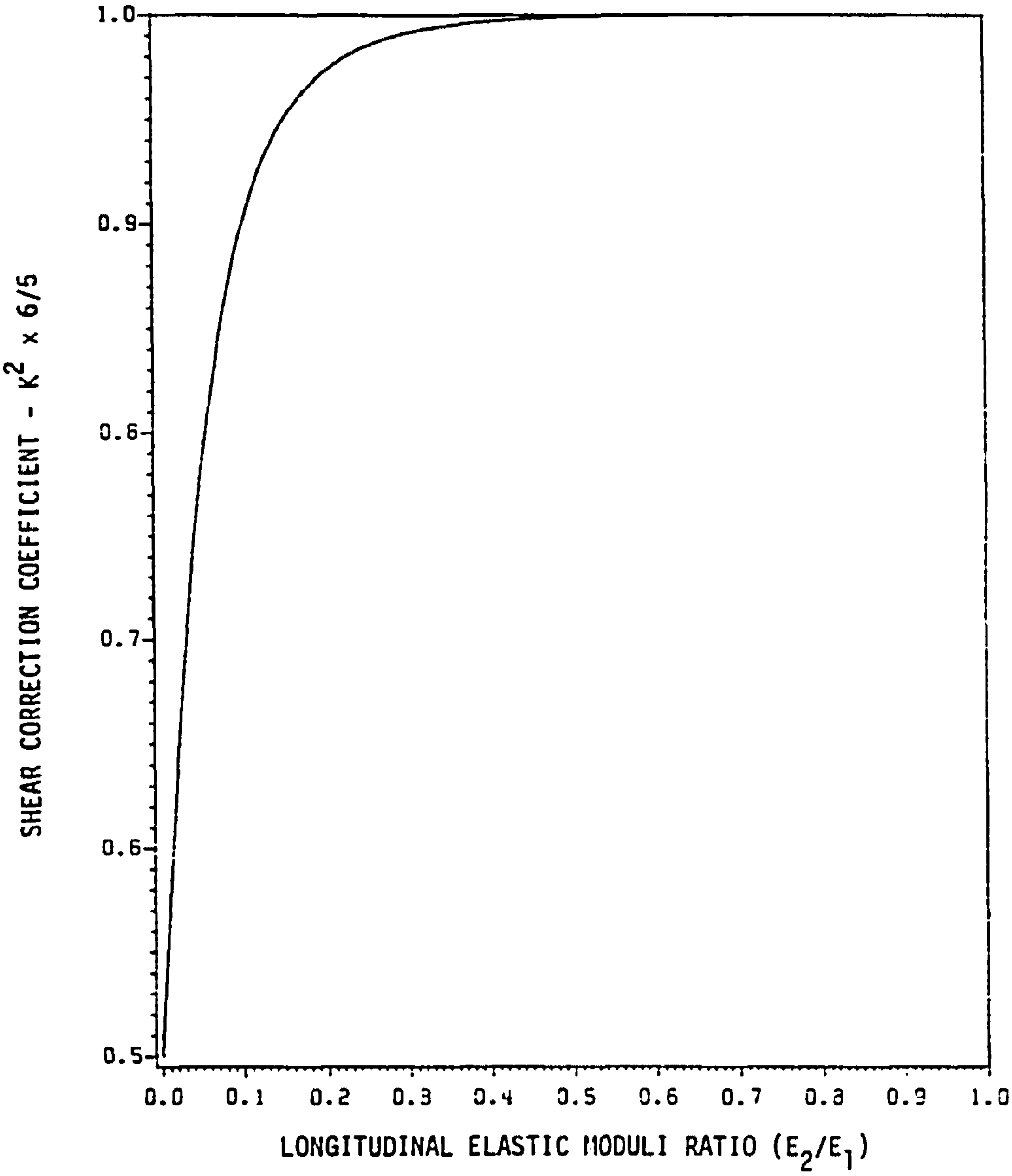

Fig. 4.2. Effect of longitudinal Young's modulus ratio on shear correction coefficient for two-layer unimodular materials ($h_1/h_2 = 1$, $G_1/G_2 = 1$).

Table 4.3 Effect of Elastic-Moduli Ratio on Dimensionless Maximum Shear Stress for a Two-Layer Unimodular-Material Laminate ($h_1/h_2 = 1$)

E_1/E_2	$(2/3)(\bar{\tau}_{xz})_{max}$
0	0
1/4	0.548
1/2	0.727
3/4	0.870
1	1.000
2	1.456
5	2.791
10	6.241
100	7.087
1000	7.897
∞	8.000

Table 4.4 Effect of Bimodular Ratios on Neutral-Surface Location for Three-Layer Bimodular Laminates ($h_1/h_2 = h_3/h_2 = 1$, $E_1{}^c/E_2{}^c = E_3{}^c/E_2{}^c = 1$)

$E_3{}^t/E_2{}^c$	h_t/h		
	$E_2{}^t/E_2{}^c = 1/2$	$E_2{}^t/E_2{}^c = 1$	$E_2{}^t/E_2{}^c = 2$
0	0.724	0.833	0.609
1/4	0.643	0.722	0.573
1/2	0.586	0.633	0.541
3/4	0.542	0.561	0.513
1	0.508	0.500	0.488
2	0.418	0.333	0.414
5	0.310	0.119	0.309
8	0.267	0.033	0.266
10	0.250	0	0.249

strain on shear moduli is ignored (that is, $G^c/G^t = 1$).

In the case of single-layer bimodular material (Appendix G), it is interesting to note that both the transverse shear correction coefficient and the maximum transverse shear stress are unaffected by the bimodular ratio E^c/E^t. However, this does not imply that the distribution of shear stress is the same for unimodular and bimodular materials (see Fig. G.1). The effect of bimodular ratio on the location of the neutral surface is studied in Table 4.1. For values of $E^c < E^t$ (e.g., aramid-cord/rubber), h_t falls within the lower half of the thickness; whereas for materials such as rock ($E^c > E^t$), h_t is in the upper half of the beam (for the convex bending case).

In bending of a two-layer bimodular laminate since one of the two layers is always in compression (or tension), only three of the four elastic moduli belonging to these layers are pertinent [see Fig. G.2]. This means that the location of the neutral surface depends on only two elastic-moduli ratios. The effects of these two ratios on h_t are studied for a wide range of $E_2{}^c/E_2{}^t$ in Table 4.2. The shear correction coefficient varies drastically with the ratio of the shear moduli of the two layers (G_1/G_2), whereas it changes only a little for a wide range of $E_2{}^c/E_2{}^t$ for $E_1{}^c/E_2{}^t = 1$, as shown in Fig. 4.1.

The shear correction coefficient for two-layer unimodular laminates increases rapidly from 50% to 98% of the classical value (5/6) as the elastic-modulus ratio E_1/E_2 varies from 0 to 0.25 as shown in Fig. 4.2. For ratios greater than 0.5, a value of 5/6 would be a good approximation for K^2. The dimensionless maximum shear stress increases from 0 to 12 as E_1/E_2 changes from o to infinity as tabulated in Table 4.3.

Three-layer bimodular laminates which have facings (top and bottom layers) made of the same material are considered here. For the case in which $E_1{}^c/E_2{}^c = 1$, the effect of bimodular ratio $E_3{}^t/E_2{}^c$ on h_t/h for different ratios of $E_2{}^t/E_2{}^c$ (1/2, 1, and 2) are investigated in Table 4.4. As $E_3{}^t/E_2{}^t$ increases from 0 to 10, h_t/h decreases for all three values of $E_2{}^t/E_2{}^c$. Also, a plot of K^2 vs $E_1{}^c/E_2{}^c$ is given in Fig. 4.3 which shows the shear correction factor as a function of the bimodularity of the material.

Finally, the effect of elastic moduli ratio E_2/E_1 on K^2 and $(\bar{\tau}_{xz})_{max}$ are studied (see Fig. 4.4 and Table 4.5) for a three-layer unimodular material laminates with identical facings. For different values of G_1/G_2, K^2 increases rapidly for E_1/E_2 less than 0.05 but after this point, K^2 changes very slowly. The maximum dimensionless shear stress varies from 0 to 40.5 for $0 < E_1/E_2 < \infty$.

4.4 Conclusions

Closed-form solutions for the Timoshenko-type shear correction coefficient and the maximum dimensionless transverse shear stress are presented for bimodular laminates undergoing cylindrical bending. These solutions depend on bimodular ratio(s) for different laminations. However, it is interesting to note that both the transverse shear correction coefficient and the maximum dimensionless transverse shear stress $(\bar{\tau}_{xz})_{max}$ for a single-layer material are unaffected by bimodular ratio, E^c/E^t.

For two- and three-layer laminates, both K^2 and $(\bar{\tau}_{xz})_{max}$ depend upon the bimodular ratios and transverse shear moduli ratios. According to the results presented, in some cases K^2 is less than the classical value of 5/6.

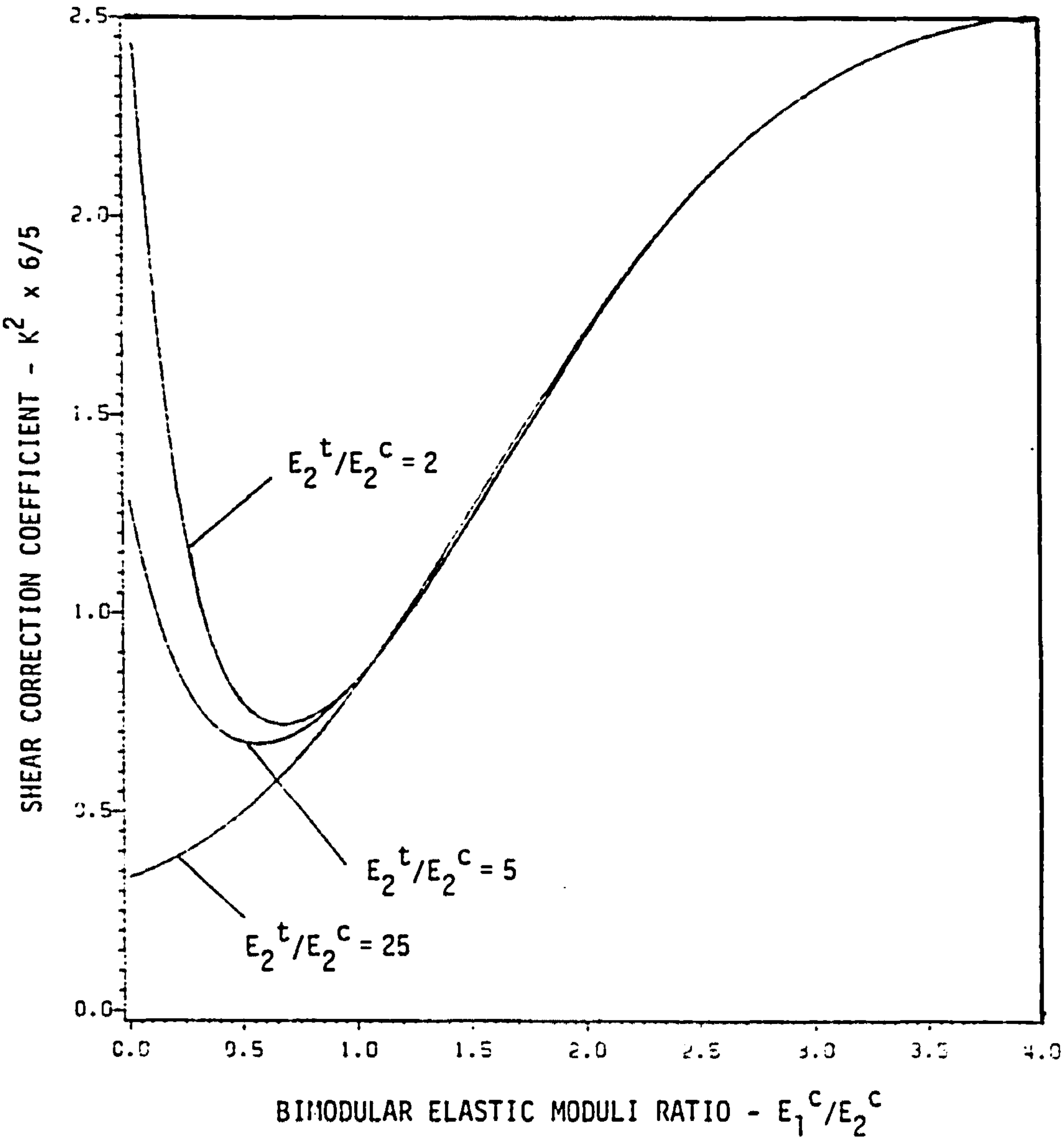

Fig. 4.3. Effect of bimodular elastic moduli ratio $E_1{}^c/E_2{}^c$ on shear correction coefficient for a three-layer bimodular laminated ($h_1/h_2=h_3/h_2=1$, $G_1/G_2=G_3/G_2=1$, $E_3{}^t/E_2{}^c=2$).

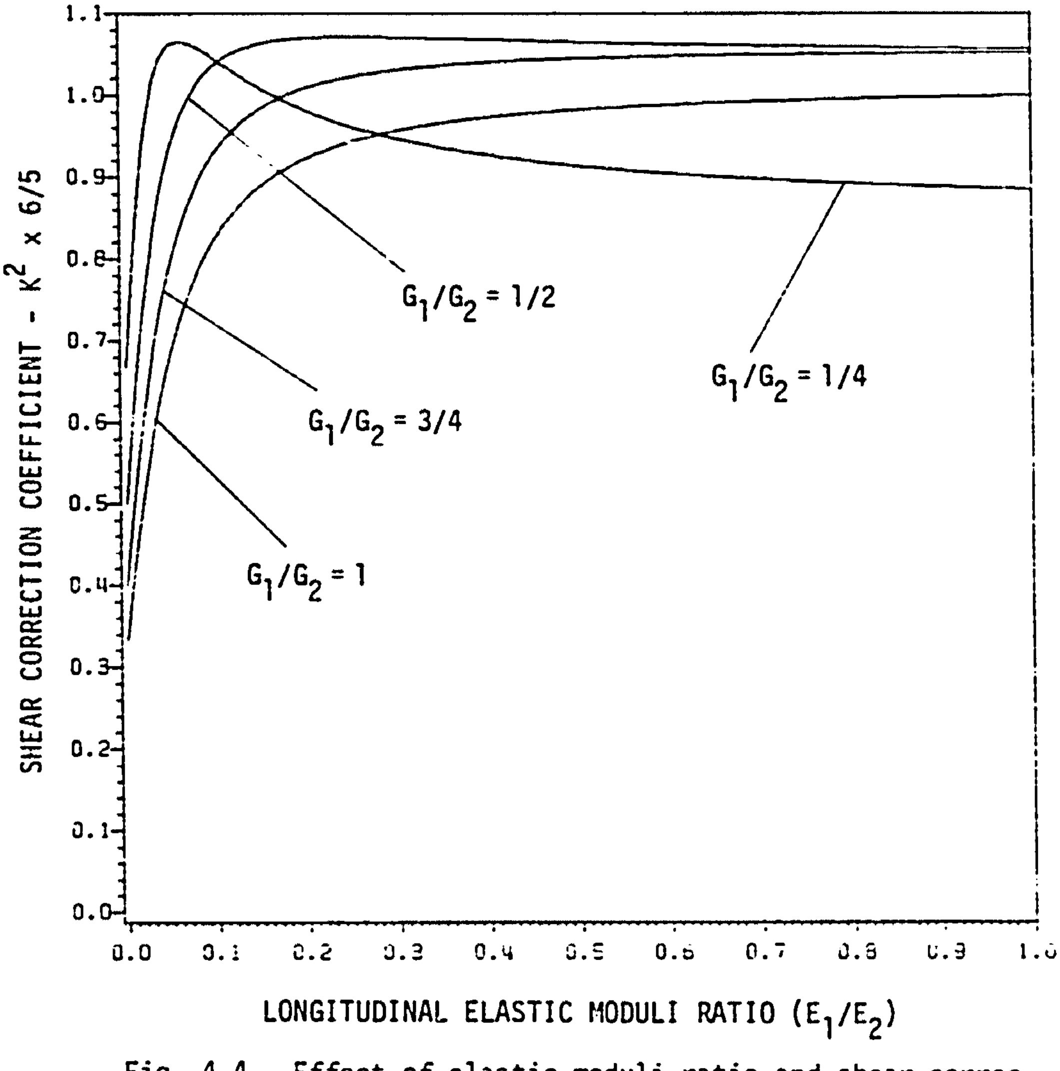

Fig. 4.4. Effect of elastic moduli ratio and shear correction coefficient for various shear-moduli ratios for three-layers unimodular-material laminates (h_1/h_2=h_3/h_2=1, E_1/E_3=1, G_1/G_3=1).

Table 4.5 Effect of Elastic-Moduli Ratio on Dimensionless Maximum Shear Stress for Three-Layer Laminate of Unimodular Materials ($h_1/h_2 = h_3/h_2 = 1$, $E_1/E_3 = 1$)

E_1/E_2	$(2/3)(\bar{\tau}_{xz})_{max}$
0	0
1/4	0.257
1/2	0.509
3/4	0.757
1	1.000
2	1.928
5	4.355
10	7.500
100	21.428
1000	26.316
∞	27.000

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SECTION V

A NEW THEORY FOR VIBRATION OF BIMODULAR SANDWICH BEAMS WITH THICK FACINGS†

This study deals with analytical investigations of three-layer beams with cores of polyurethane foam and facings of unidirectional cord-rubber. Both of these materials are bimodular (i.e., having different behavior in compression as compared to tension). The new theory presented is a shear-flexible laminate version of the well-known Timoshenko beam theory, which, due to the bending-stretching coupling present in the bimodular case, results in a coupled sixth-order system of differential equations. In this theory, a separate derivation is presented for the shear correction factor. Due to the discontinuities in the normal-stress distribution and the bimodularity, the shear correction factor is much different than the classical homogeneous-material value of 5/6. Results are presented for the frequencies of the first three modes of vibration for a pin-ended beam without axial restraint.

5.1 Introduction

Studies on vibrations of beams were carried out as early as the 1850's. Bresse [5.1] was the first to include both the rotatory inertia

†This section is part of a paper in Journal of Sound and Vibration (to appear, October 1983).

and shear flexibility effects (in 1859). His theory is, however, referred to as the Timoshenko-beam theory. Rayleigh [5.2] included the rotatory inertia effect while later, the effect of shear stiffness was added by Timoshenko [5.3]. The first correct boundary conditions for the Timoshenko beam were derived by Kruszewski [5.4] in 1949. Another equally correct form of boundary conditions was derived in 1951 by Dengler and Goland [5.5].

The effective stiffness of a bimodular beam in pure bending was analyzed by Marin [5.6]. Khachatryan [5.7] worked with the vibration of Bernoulli-Euler bimodular beams. Tran and Bert [5.8,5.9] studied the bending of thick beams of bimodular material, while more recently Bert and Rebello [5.10] investigated thick, laminated bimodular material beams.

This analysis includes bimodular constituent materials, transverse shear deformations, and coupling and rotatory inertia effects. The theory developed here is a sixth-order theory which is an extension of the fourth-order Timoshenko beam theory to the case of a three-layer laminated beam of bimodular materials in free vibration.

5.2 The New Theory

The coordinate system used for the three-layer beam is shown in Fig. 5.1; x and z are the Cartesian position corrdinates along the length of the beam and normal to it, measured from the midplane of the beam.

By Hooke's law for any point on the beam, the axial normal stress and transverse shear stress are respectively given by:

$$\begin{aligned} \sigma_x &= E\,\varepsilon_x \\ \tau_{xz} &= G\gamma \end{aligned} \tag{5.1}$$

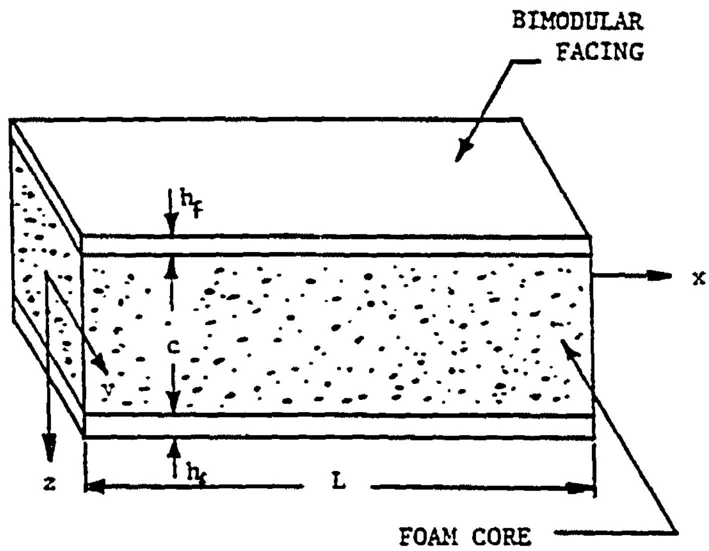

Fig. 5.1. Three-layer sandwich beam.

Let the midplane displacements be u and w in the x and z directions, respectively, and the slope function be ψ. Then, the displacements are given by:

$$\begin{aligned} U(x,t) &= u(x,t) + z\psi(x,t) \\ W(x,t) &= w(x,t) \end{aligned} \tag{5.2}$$

In the usual notations of the normal and transverse-shear stress resultants and stress couple, each per unit width, for a beam of total thickness $c + 2h_f$, one has

$$[N,Q] = \int_{-(h_f+c/2)}^{h_f+c/2} [\sigma_x, \tau_{xz}]\, dz \tag{5.3}$$

$$M = \int_{-(h_f+c/2)}^{h_f+c/2} \sigma_x\, z\, dz \tag{5.4}$$

For a bimodular sandwich beam, one substitutes the expressions (5.1) into Eqs. (5.3) and (5.4) and perform the integrations to obtain:

$$\begin{Bmatrix} N \\ M \end{Bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \kappa \end{Bmatrix} \tag{5.5}$$

$$Q = S\gamma \tag{5.6}$$

where

$$\begin{aligned} \text{midplane strain} \quad \epsilon_x &= u_{,x} \\ \text{curvature} \quad \kappa &= \psi_{,x} \\ \gamma &= \psi + w_{,x} \end{aligned} \tag{5.7}$$

and $(\ )_{,x}$ denotes the partial derivative with respect to x. The extensional, flexural-extensional coupling, and flexural stiffnesses (per unit width) for a laminated beam are respectively defined by:

$$[A,B,D] = \int_{-(h_f+c/2)}^{h_f+c/2} [1,z,z^2]\, E\, dz \tag{5.8}$$

The transverse shear stiffness (also per unit width) is

$$S = K^2 \int_{-(h_f+c/2)}^{h_f+c/2} G\, dz \tag{5.9}$$

In the case of a bimodular sandwich beam, the integrations in Eqs. (5.8) and (5.9) must be carried out in a piecewise fashion from one layer to the next, taking into account tension or compression within appropriate portions of a facing.

The equations of motion with coupling and rotatory inertias taken into account can be written as:

$$\begin{aligned} N_{,x} &= Pu_{,tt} + R\psi_{,tt} \\ Q_{,x} &= Pw_{,tt} \\ M_{,x} - Q &= I\psi_{,tt} + Ru_{,tt} \end{aligned} \tag{5.10}$$

Here, P, R, and I are the translational, coupling, and rotatory inertia coefficients per unit midplane area and are defined as:

$$[P,R,I] = \int_{-(h_f+c/2)}^{h_f+c/2} [1,z,z^2]\rho\, dz \tag{5.11}$$

Now, substituting Eqs. (5.7) into Eqs. (5.5) and (5.6), we obtain N, M, Q in terms of the displacements and slope:

$$N = Au_{,x} + B\psi_{,x}$$
$$M = Bu_{,x} + D\psi_{,x} \qquad (5.12)$$
$$Q = S(\psi + w_{,x})$$

Using Eq. (5.12) in Eqs. (5.10), we get the equations of motion as:

$$Au_{,xx} + B\psi_{,xx} = Pu_{,tt} + R\psi_{,tt} \qquad (5.13)$$

$$S(\psi_{,x} + w_{,xx}) = Pw_{,tt} \qquad (5.14)$$

$$Bu_{,xx} + D\psi_{,xx} - S(\psi + w_{,x}) = I\psi_{,tt} + Ru_{,tt} \qquad (5.15)$$

The boundary conditions for a freely supported beam are as follows:

At $\quad x = 0 \quad , \quad x = \ell$

$$N = 0 \quad , \quad w = 0 \quad , \quad M = 0$$

The governing equations and boundary conditions are exactly satisfied in closed form by the following set of functions:

$$u = \bar{U} \cos \alpha x \cos \omega t \qquad (5.16)$$

$$w = \bar{W} \sin \alpha x \cos \omega t \qquad (5.17)$$

$$\psi = \bar{X} \cos \alpha x \cos \omega t \qquad (5.18)$$

where $\bar{U}$, $\bar{W}$, $\bar{X}$ are independent of x and t and for the m^{th} mode $\alpha = m\pi/L$.

By using the first of Eqs. (5.2) along with the first of Eqs. (5.7), we find that the neutral-surface position associated with $\varepsilon_x = 0$ has a constant value given by:

$$z_n{}^* = - u_{,x}/\psi_{,x}$$

*For normal uniaxial loading with a 0° or a 90° fiber orientation, $z_{nx} = z_{ny} = z_n$ (see [5.11]).

and by Eqs. (5.16) and (5.18),

$$z_n = -\,\bar{U}/\bar{X} = B/A \tag{5.19}$$

Thus, A, B, and D are constant and Eqs. (5.16)-(5.18) can be substituted into Eqs. (5.13)-(5.15) to obtain:

$$\begin{bmatrix} A\alpha^2 - P\omega^2 & 0 & B\alpha^2 - R\omega^2 \\ 0 & S\alpha^2 - P\omega^2 & S\alpha \\ B\alpha^2 - R\omega^2 & S\alpha & S + D\alpha^2 - I\omega^2 \end{bmatrix} \begin{Bmatrix} \bar{U} \\ \bar{W} \\ \bar{X} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{5.20}$$

which implies by Eq. (5.19) that:

$$z_n = \frac{B\alpha^2 - R\omega^2}{A\alpha^2 - P\omega^2} \tag{5.21}$$

Equation (5.20) can be solved by setting the determinant of the coefficients to zero. Hence, one obtains an equation in terms of the frequency as follows:

$$\omega^6 + \bar{\bar{p}}\omega^4 + \bar{\bar{q}}\omega^2 + \bar{\bar{r}} = 0 \tag{5.22}$$

where

$$\begin{aligned} \bar{\bar{p}} &= \frac{\alpha^2\{R(SR + 2BP) - P(PD + IA + IS)\} - P^2S}{P(IP - R^2)} \\ \bar{\bar{q}} &= \frac{\alpha^2[\alpha^2\{IAS - 2BRS + P(AD + SD - B^2)\} + PAS]}{P(IP - R^2)} \\ \bar{\bar{r}} &= \frac{S\alpha^6(B^2 - AD)}{P(IP - R^2)} \end{aligned} \tag{5.23}$$

An iterative technique is used to find the resonant frequencies of the beam. An initial value of z_n (say $z_n = 0$) is assumed and value of the lowest positive root, ω^2 is determined. With this value of ω^2, a new value of z_n is calculated and checked with the previous value of z_n. If the agreement between these two are not good within a permissible error range, this new value of z_n would be the next guess. The cycle is then repeated until z_n converges. At that stage, the lowest positive root of the polynomial is the natural frequency for the given mode. After a number of iterations, if no convergence is observed, the percentage error is increased and the iterations are repeated until z_n converges.

5.3 Discussion of Results

In the following, theoretical results are presented for a thick ($\ell/(c+2h_f) = 5.3$) three-layer beam with a rectangular cross section. The core is constructed of polyurethane foam and the facings are made of aramid-cord-rubber (see [5.11]). The shear correction coefficients were taken to be: (1) 0.6913 for the beam with 0° facings, (2) 1.0487 for the beam with 90° facings (see Appendix H).

Comparisons are made between the closed-form solution and experimental data [5.11] for a freely-supported three-layer beam.

0°-Facing Beam

For the 0°-facing beam, the present theory predicts natural frequencies for the three modes measured that is in excellent agreement with the experimental values (see Fig. 5.2). The theory predicts that the neutral surface lies within the core and moves away from the middle surface as the frequency increases (see Table 5.1).

90°-Facing Beam

In the case of the 90°-facing beam, Fig. 5.3 indicates that the theory predicts much lower natural frequencies than the experimental values, with the neutral axis falling within the core. Here, again the

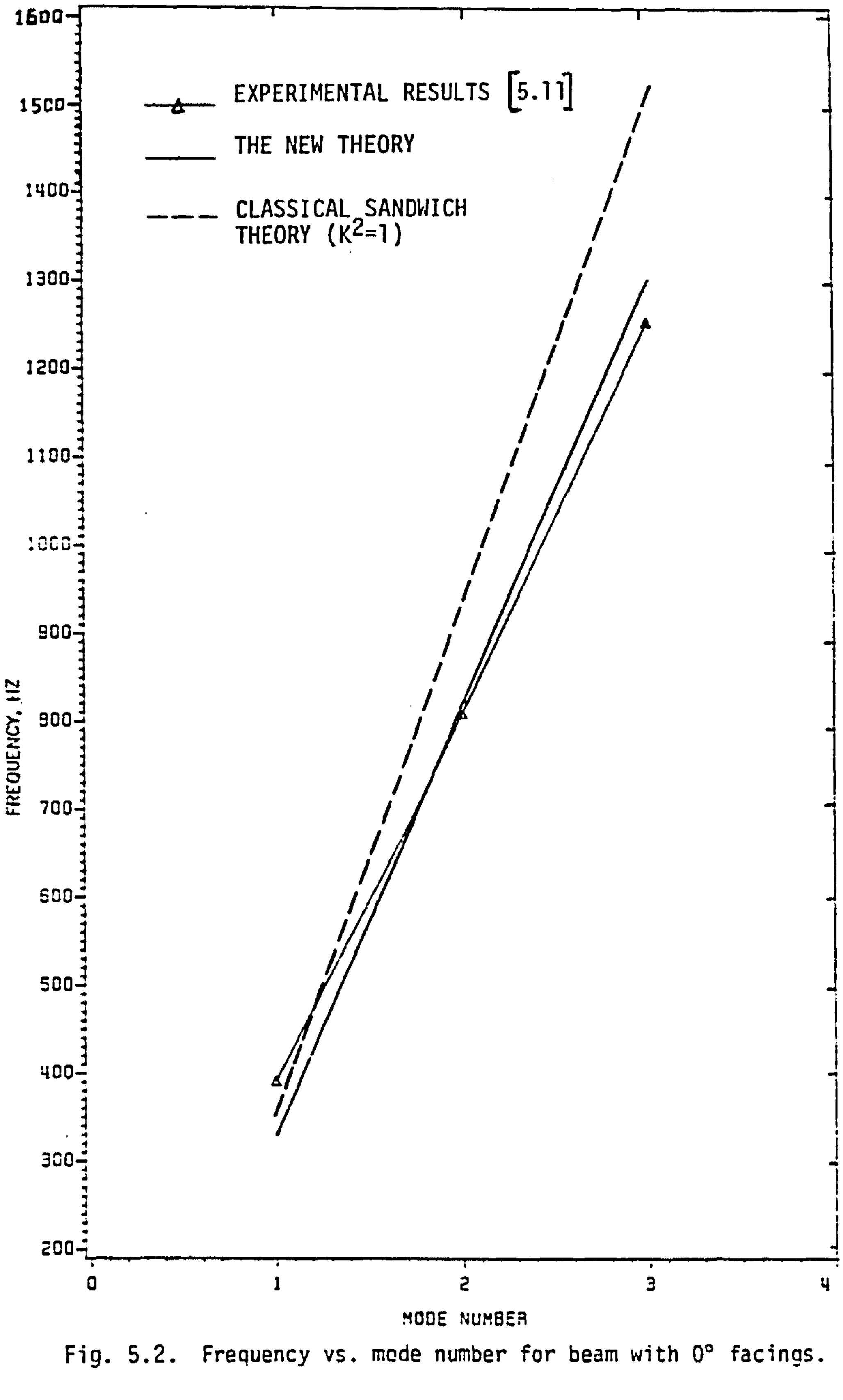

Fig. 5.2. Frequency vs. mode number for beam with 0° facings.

Table 5.1. Effect of Frequency on Neutral-Surface Location

0° Facing		90° Facing	
Freq., Hz	z_n/H	Freq., Hz	z_n/H
329.6	0.1990	108.8	0.0100
820.7	0.2002	386.8	0.0104
1301.	0.2006	756.2	0.0108

Table 5.2. Effect of Shear Correction Coefficient on Resonant Frequencies (0° Facings)

K^2	Resonant Frequency, Hz		
	1st mode	2nd mode	3rd mode
0.1	163.1	341.0	516.0
0.2	218.5	474.9	724.7
0.3	254.6	572.8	881.3
0.4	280.9	651.5	1010
0.5	301.2	717.6	1122
0.6	317.4	774.7	1220
0.7	330.6	824.8	1308
0.8	341.7	869.4	1388
0.9	351.1	909.3	1462
1.0	356.8	940.3	1524

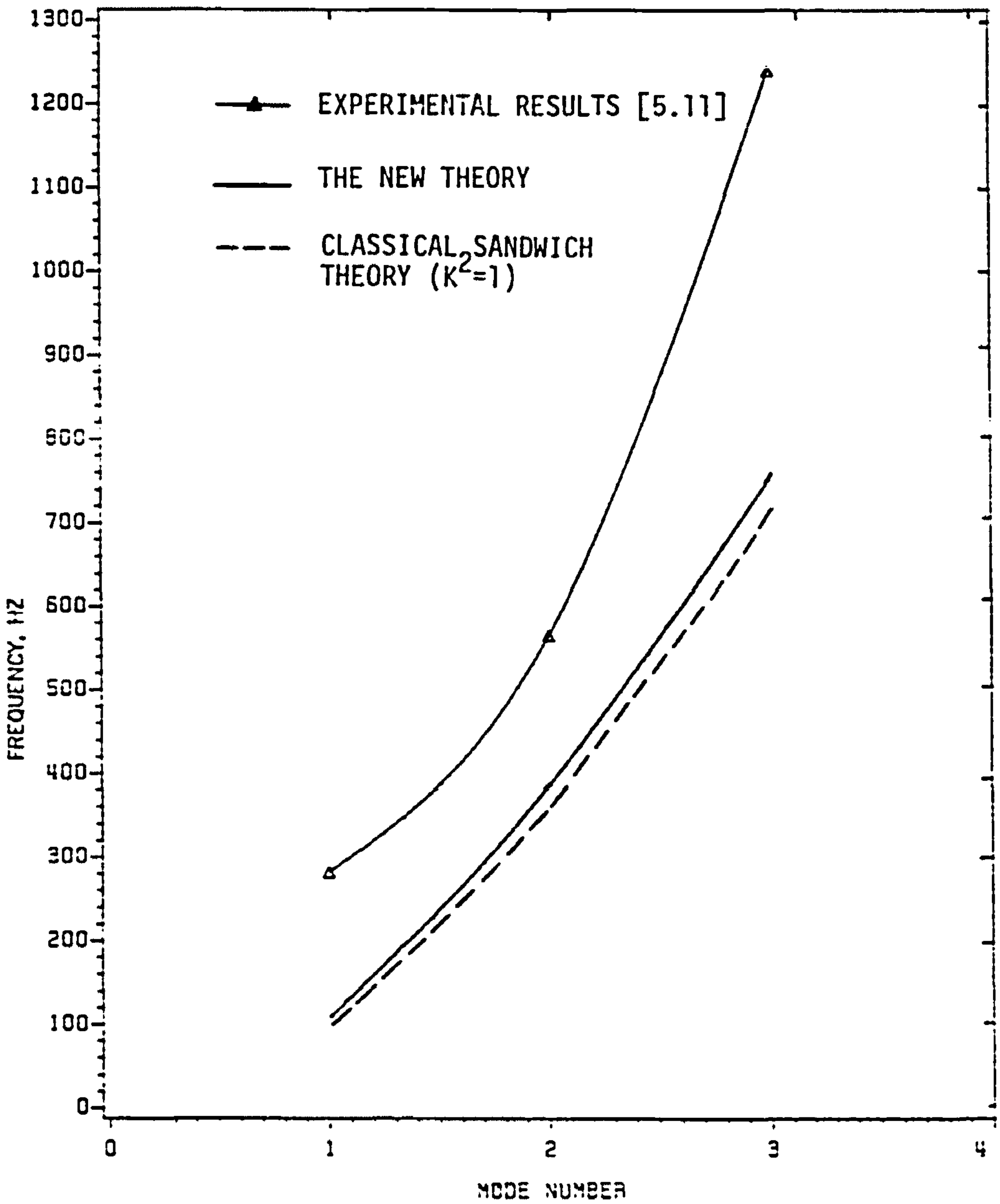

Fig. 5.3. Frequency vs. mode number for beam with 90° facings.

neutral surface moves away from the middle surface of the beam as the frequency increases (Table 5.1).

The dynamic behavior of the rubber matrix was investigated for this orientation of the facing fibers. Rough values for the dynamic higher resonant frequencies (at any given mode) for the 0°-facing sandwich beam than for the 90°-facing beam.

Finally, the effect of shear correction coefficient on the natural frequencies was studied. For the case of 90° facings, no drastic change in frequency was found whereas for the 0°-facing case, a considerable difference was predicted by the theory as one can see in Table 5.2.

5.4 Conclusions

The vibration of a thick sandwich beam made of bimodular material with rectangular cross section was studied. A new sixth-order theory based on shear-deformable-beam theory was offered for a freely supported beam. In addition, a static shear correction coefficient was applied. Experimental data for the first three modes agree with the closed-form solution in the case of 0° facings. For 90° facings, the computed resonant frequencies are considerably lower than those measured experimentally. The reason for the lower predictions could be attributed to the dynamic change in Young's modulus for rubber (facings). The effect of static shear correction coefficient was investigated for the 0° facings.

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SECTION VI

CONCLUSIONS

A transfer-matrix formulation of equations governing a thick, rectangular-cross-section, multimodular beam is presented. To check the transfer-matrix results, a closed-form solution is developed for the special cases in which the neutral-surface location is constant along the beam axis. It is noted that the transfer-matrix formulation presented here does not have loading and edge condition limitations. The transfer-matrix solutions are found to be in close agreement with the closed-form solution for a twenty-five-element mesh.

Also, closed-form solutions for the Timoshenko-type shear correction coefficient and the maximum dimensionless transverse shear stress are presented for bimodular laminates undergoing cylindrical bending. These solutions depend on bimodular ratio(s) for different laminations. Based on this more accurate prediction of shear correction coefficient for bimodular materials, a new theory is developed for the vibration of a bimodular sandwich beam with thick facings. Experimental results have shown much better agreement with this new theory than classical sandwich theory.

APPENDIX A

LINEAR APPROXIMATIONS TO THE STRESS-STRAIN CURVE

1. Multimodular Case

Consider the nonlinear stress-strain curve shown in Fig. A.1. For any arbitrary point (ε^t,σ^t) in the tension region $(\varepsilon \geq 0)$, there are two straight lines such that

$$g(\varepsilon) \equiv \begin{cases} (\sigma^t/\varepsilon^t)\varepsilon \\ [(\sigma^t - \sigma_f{}^t)/(\varepsilon^t - \varepsilon_f{}^t)](\varepsilon - \varepsilon_f{}^t) + \sigma_f{}^t \end{cases} \tag{A.1}$$

The equation of a stress-strain curve as expressed in [2.17] is

$$\sigma(\varepsilon) = K\varepsilon^n \quad ; \quad \varepsilon \geq 0 \tag{A.2}$$

where K and n are constants depending on the material. To find the proper "break point" (ε^t,σ^t), the area between the approximated curve $g(\varepsilon)$ and the actual experimental curve $\sigma(\varepsilon)$ has to be minimized. The mentioned area can be expressed

$$A = \int_0^{\varepsilon^t} [g_1(\varepsilon) - \sigma(\varepsilon)]d\varepsilon + \int_{\varepsilon^t}^{\varepsilon_f{}^t} [g_2(\varepsilon) - \sigma(\varepsilon)]d\varepsilon \tag{A.3}$$

Substitution of Eqs. (A.1) and (A.2) into Eq. (A.3) and taking the integrations gives

$$A = \frac{1}{2}\sigma^t\varepsilon^t - \frac{K}{n+1}(\varepsilon^t)^{n+1} + \frac{1}{2}(\sigma_f{}^y + \sigma^t)(\varepsilon_f{}^t - \varepsilon^t) - \frac{K}{n+1}[(\varepsilon_f{}^t)^{n+1} - (\varepsilon^t)^{n+1}] \tag{A.4}$$

By searching in the region of $\Omega \equiv (0,\varepsilon_t{}^f) \times (0,\sigma_t{}^f)$, one is able to find a point (ε^t,σ^t) such that A is minimized locally. Note that a few other methods (e.g., least-squares method) have been tried but it turned out

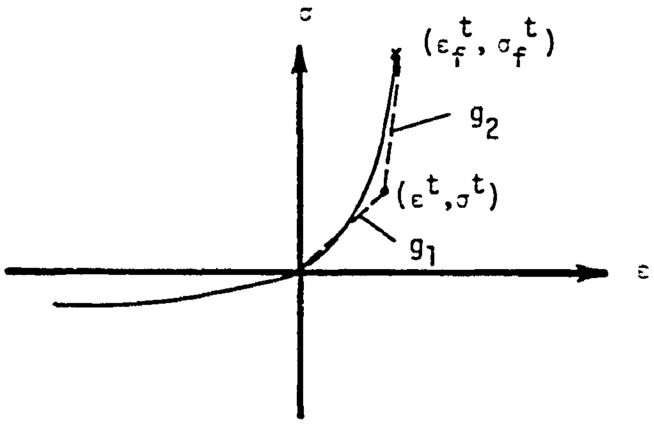

Fig. A.1 Multimodular model.

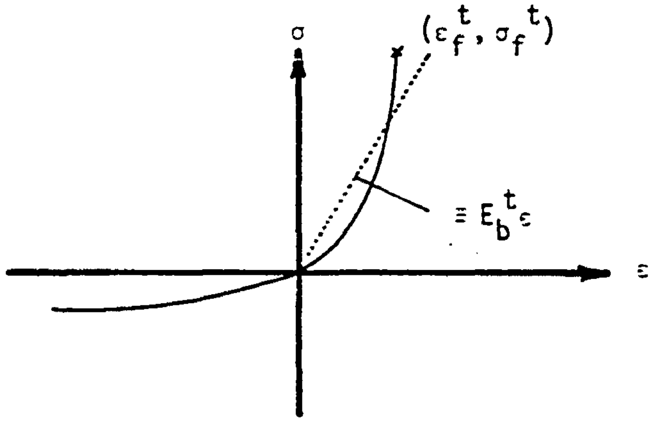

Fig. A.2 Bimodular model.

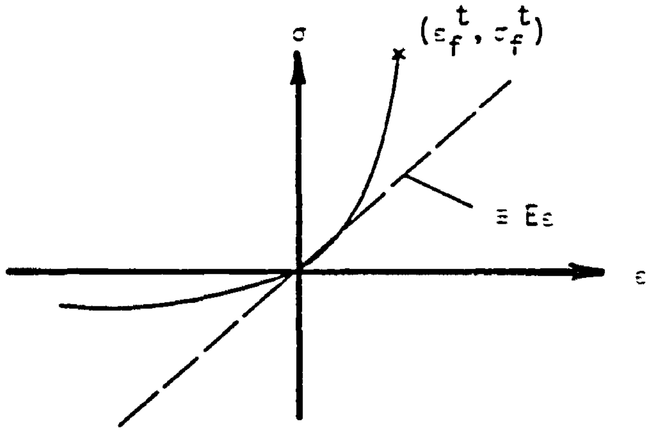

Fig. A.3 Unimodular model.

that the absolute minimum point was outside of the region Ω.

2. Bimodular Case

For this case, the least-squares method has been used. As shown in Fig. A.2, there is a line such that

$$I = \int_0^{\varepsilon_f^t} [E_b^t \varepsilon - K\varepsilon^n]^2 \, d\varepsilon \tag{A.5}$$

can be minimized in Ω. Here, E_b^t is the slope of that line. By taking the derivative of Eq. (A.5) and equating it to zero, one has

$$\frac{dI}{dE_b^t} = 2 \int_0^{\varepsilon_f^t} [E_b^t \varepsilon - K\varepsilon^n]\varepsilon \, d\varepsilon = 0 \tag{A.6}$$

By solving Eq. (A.6) for E_b^t, one obtains

$$E_b^t = \frac{3K}{n+2} (\varepsilon_f^t)^{n-1} \tag{A.7}$$

For example, for aramid-rubber in the tension region (see [2.17]), the following parameters are found:

$n_t = 1.22$

$K_t = 1.1 \times 10^6$ psi

$\varepsilon_f^t = 0.029$

$E_b^t = \frac{(3)(1.1 \times 10^6)}{1.22 + 2} (0.029)^{1.22-1} = 0.47 \times 10^6$ psi

An analogous calculation can be applied for the compression side of the bend, i.e., E_b^c can be found, provided that K_c, n_c, and ε_f^c are known.

3. Unimodular Case

Using the same method as in Case 2 and assuming only one line, which passes through the origin, to approximate both tension and compression regions (Fig. A.3), one has

$$I = \int_{\varepsilon_f^c}^{0} [K_c\varepsilon^{n_c} - E\varepsilon]^2 d\varepsilon + \int_{0}^{\varepsilon_f^t} [K_t\varepsilon^{n_t} - E\varepsilon]^2 d\varepsilon \tag{A.8}$$

and, then

$$\frac{dI}{dE} = 2\varepsilon[\int_{\varepsilon_f^c}^{0} (K_c\varepsilon^{n_c} - E\varepsilon)\varepsilon\, d\varepsilon + \int_{0}^{\varepsilon_f^t} (K_t\varepsilon^{n_t} - E\varepsilon)\varepsilon\, d\varepsilon] = 0 \tag{A.9}$$

Solving Eq. (A.9) for E, one obtains

$$E = 2[\frac{K_c}{n_c+1}(\varepsilon_f^c)^{n_c+1} - \frac{K_t}{n_t+1}(\varepsilon_f^t)^{n_c+1}]/[(\varepsilon_f^c)^2 - (\varepsilon_f^t)^2] \tag{A.10}$$

NOTE: In the present computations, the values of ε_f^c and ε_f^t considered are as follows:

$$\varepsilon_f^c = -0.046 \;;\quad \varepsilon_f^t = 0.029$$

The constants K_c, K_t, n_c, and n_t are listed in [2.17].

APPENDIX B

THE BEAM STIFFNESSES FOR RECTANGULAR-SECTION BEAMS OF MULTIMODULUAR MATERIALS

For the assumed four-segment model, there are two different bending cases in general, convex downward and concave downward bending. In convex downward bending, the top layer of a beam is in compression and the bottom layer in tension. Conversely, in concave downward bending, the top layer of the beam is in tension and the bottom layer is in compression.

Depending on the location of z_n, a_c, and a_t in σ_x vs z, eight different cases might occur. For example, for convex downward bending, consider the case when z_n, a_c, and a_t are in the range of -h/2 and h/2 (see Fig. 2.3). Substitution of Eq. (2.2) into Eq. (2.10) and using Eqs. (2.3), (2.4), and (2.5) leads to

$$N = \int_{-h/2}^{a_c} [\kappa E_1{}^c(a_c - z_n) + \kappa E_2{}^c(z - a_c)]dz + \int_{a_c}^{z_n} \kappa E_1{}^c(z - z_n)dz + \int_{z_n}^{a_t} \kappa E_1{}^t(z - z_n)dz$$

$$+ \int_{a_t}^{h/2} [\kappa E_1{}^t(a_t - z_n) + \kappa E_2{}^t(z - a_t)]dz \qquad (B.1)$$

and

$$M = \int_{-h/2}^{a_c} [\kappa E_1{}^c(a_c - z_n) + \kappa E_2{}^c(z - a_c)]z\,dz + \int_{a_c}^{z_n} \kappa E_1{}^c(z - z_n)z\,dz$$

$$+ \int_{z_n}^{a_t} \kappa E_1{}^t(z - z_n)z\,dz + \int_{a_t}^{h/2} [\kappa E_1{}^t(a_t - z_n) + \kappa E_2{}^t(z - a_t)]z\,dz \qquad (B.2)$$

Equations (B.1) and (B.2) can be written in the following form

$$N = (-\kappa z_n)\ \{[\int_{-h/2}^{a_c} E_2{}^c dz + \int_{a_c}^{z_n} E_1{}^c dz + \int_{z_n}^{a_t} E_1{}^t dz + \int_{a_t}^{h/2} E_2{}^t dz]$$

$$+ [-\int_{-h/2}^{a_c} E_2{}^c dz + \int_{-h/2}^{a_c} E_1{}^c dz + \int_{a_t}^{h/2} E_1{}^t dz - \int_{a_t}^{h/2} E_2{}^t dz]\}$$

$$+ (\kappa)\ \{[\int_{-h/2}^{a_c} E_2{}^c z\, dz + \int_{a_c}^{z_n} E_1{}^c z\, dz + \int_{z_n}^{a_t} E_1{}^t z\, dz + \int_{a_t}^{h/2} E_2{}^t z\, dz]$$

$$+ [\int_{-h/2}^{a_c} E_2{}^c a_c dz - \int_{-h/2}^{a_c} E_1{}^c a_c dz + \int_{a_t}^{h/2} E_1{}^t a_t dz - \int_{a_t}^{h/2} E_2{}^t a_t dz]\} \tag{B.3}$$

$$M = (-\kappa z_n)\ \{[\int_{-h/2}^{a_c} E_2{}^c z\, dz + \int_{a_c}^{z_n} E_1{}^c z\, dz + \int_{z_n}^{a_t} E_1{}^t z\, dz + \int_{a_t}^{h/2} E_2{}^t z\, dz]$$

$$+ [-\int_{-h/2}^{a_c} E_2{}^c z\, dz + \int_{-h/2}^{a_c} E_1{}^c z\, dz + \int_{a_t}^{h/2} E_1{}^t z\, dz - \int_{a_t}^{h/2} E_2{}^t z\, dz]\}$$

$$+ (\kappa)\ \{[\int_{-h/2}^{a_c} E_2{}^c z^2 dz + \int_{a_c}^{z_n} E_1{}^c z^2 dz + \int_{z_n}^{a_t} E_1{}^t z^2 dz + \int_{a_t}^{h/2} E_2{}^t z^2 dz]$$

$$+ [\int_{-h/2}^{a_c} E_2{}^c a_c dz - \int_{-h/2}^{a_c} E_1{}^c a_c dz + \int_{a_t}^{h/2} E_1{}^t a_t dz - \int_{a_t}^{h/2} E_2{}^t a_t dz]\} \tag{B.4}$$

Combining Eqs. (2.9) and (2.11), one gets

$$N = (-\kappa z_n)A' + \kappa B' \tag{B.5}$$

$$M = (-\kappa z_n)B'' + \kappa D' \tag{B.6}$$

Comparison of Eqs. (B.3) and (B.5) with Eqs. (B.4) and (B.6) and considering Eqs. (2.12) and (2.15), one finds that

$$\left.\begin{aligned}
C_A^N &= \int_{-h/2}^{a_c} (E_1^c - E_2^c)dz + \int_{a_t}^{h/2} (E_1^t - E_2^t)dz \\
C_B^N &= \int_{-h/2}^{a_c} (E_2^c - E_1^c)a_c dz + \int_{a_t}^{h/2} (E_1^t - E_2^t)a_t dz \\
C_B^M &= \int_{-h/2}^{a_c} (E_1^c - E_2^c)z\, dz + \int_{a_c}^{h/2} (E_1^t - E_2^t)z\, dz \\
C_D^M &= \int_{-h/2}^{a_c} (E_2^c - E_1^c)a_c z\, dz + \int_{a_c}^{h/2} (E_1^t - E_2^t)a_t z\, dz
\end{aligned}\right\} \tag{B.7}$$

As mentioned before, eight cases may occur depending on the location of z_n, a_c, and a_t. These cases have been analyzed as the same as the general case as follows (for convex downward bending)

<u>Case 1:</u>

$$\sigma_x \equiv E_1^t \varepsilon_1^t + E_2^t(\varepsilon_x - \varepsilon_1^t) \qquad -h/2 \le z \le h/2 \tag{B.8}$$

$$\left.\begin{aligned}
A' &= hE_1^t \\
B' &= ha_t(E_1^t - E_2^t) \\
B'' &= 0 \\
D' &= h^3E_2^t/24
\end{aligned}\right\} \tag{B.9}$$

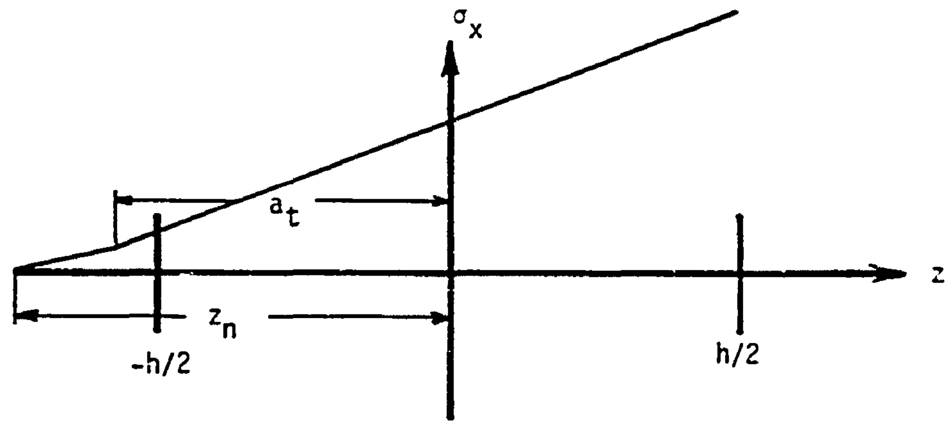

Fig. B.1 Stress distribution for Case 1.

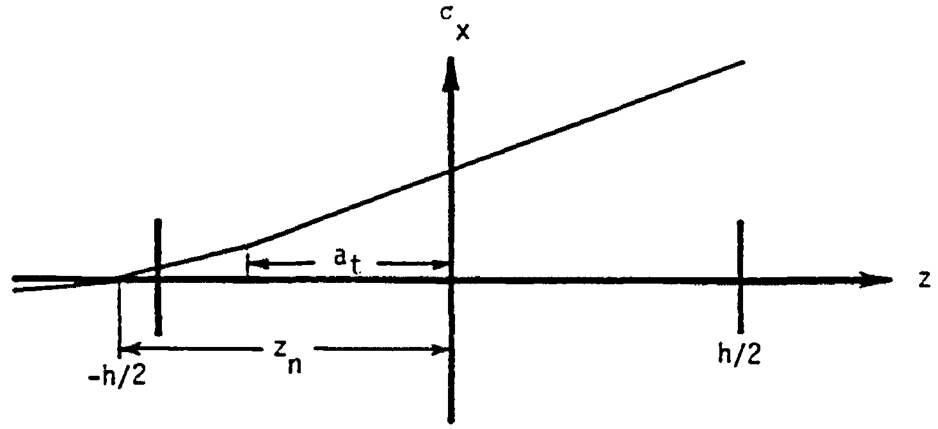

Fig. B.2 Stress distribution for Case 2.

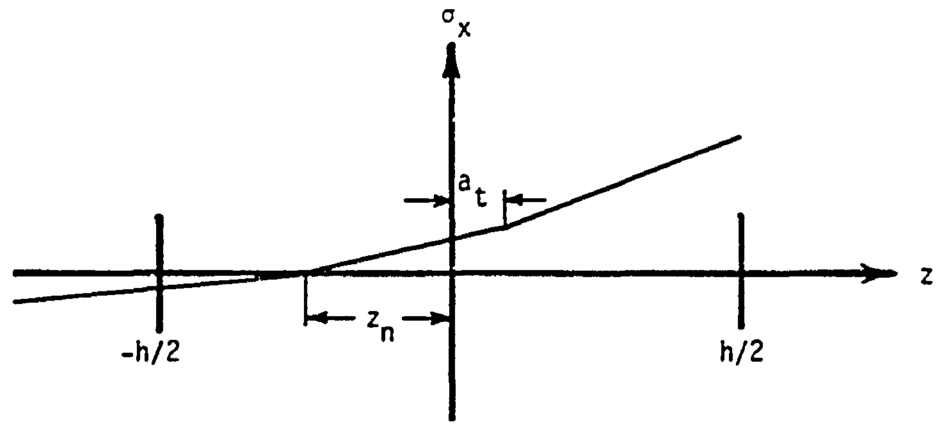

Fig. B.3 Stress distribution for Case 3.

Case 2:

$$\sigma_x \equiv \begin{cases} E_1{}^t\varepsilon_1{}^t + E_2{}^t(\varepsilon_x - \varepsilon_1{}^t) & a_t \leq z \leq h/2 \\ E_1{}^t\varepsilon_x & -h/2 \leq z \leq a_t \end{cases} \tag{B.10}$$

$$\begin{aligned} A' &= hE_1{}^t \\ B' &= -(E_1{}^t - E_2{}^t)(h/2 - a_t)^2/2 \\ B'' &= 0 \\ D' &= [h^3(E_1{}^t + E_2{}^t)/8 + a_t{}^3(E_2{}^t - E_1{}^t)/2]/3 + h^2a_t(E_1{}^t - E_2{}^t)/8 \end{aligned} \tag{B.11}$$

Case 3:

$$\sigma \equiv \begin{cases} E_1{}^t\varepsilon_1{}^t + E_2{}^t(\varepsilon_x - \varepsilon_1{}^t) & a_t \leq z \leq h/2 \\ E_1{}^t\varepsilon_x & z_n \leq z \leq a_t \\ E_1{}^c\varepsilon_x & -h/2 \leq z \leq z_n \end{cases} \tag{B.12}$$

$$\begin{aligned} A' &= h(E_1{}^c + E_1{}^t)/2 + z_n(E_1{}^c - E_1{}^t) \\ B' &= [h^2(E_2{}^t - E_1{}^c)/2 + a_t(h - a_t)(E_1{}^t - E_2{}^t) + z_n{}^2(E_1{}^c - E_1{}^t)]/2 \\ B'' &= (E_1{}^t - E_1{}^c)(h^2/4 - z_n{}^2)/2 \\ D' &= [h^3(E_1{}^c + E_2{}^t)/8 + a_t{}^3(E_2{}^t - E_1{}^t)/2 + z_n{}^3(E_1{}^c - E_1{}^t)]/3 \\ &\quad + h^2a_t(E_1{}^t - E_2{}^t)/8 \end{aligned} \tag{B.13}$$

Case 4:

This case is the general case (see Fig. 2.3) which has been discussed in detail earlier in this Appendix.

Case 5:

$$\sigma_x \equiv \begin{cases} E_1{}^t\varepsilon_x & z_n \leq z \leq h/2 \\ E_1{}^c\varepsilon_x & a_c \leq z \leq z_n \\ E_1{}^c\varepsilon_1{}^c + E_2{}^c(\varepsilon_x - \varepsilon_1{}^c) & -h/2 \leq z \leq a_c \end{cases} \tag{B.14}$$

$$
\begin{aligned}
A' &= h(E_1{}^t + E_1{}^c) + z_n(E_1{}^c - E_1{}^t) \\
B' &= [h^2(E_1{}^t - E_2{}^c)/4 + z_n{}^2(E_1{}^c - E_1{}^t) + a_c(a_c + h)(E_1{}^c - E_2{}^c)]/2 \\
B'' &= (E_1{}^t - E_1{}^c)(h^2/4 - z_n{}^2)/2 \\
D' &= [h^3(E_1{}^t + E_2{}^c) + z_n{}^3(E_1{}^c - E_1{}^t) + a_c{}^3(E_1{}^c - E_2{}^c)/2]/3 \\
&\quad - h^3 a_c(E_1{}^c - E_2{}^c)/8
\end{aligned}
\qquad (B.15)
$$

<u>Case 6:</u>

$$
\sigma_x \equiv \begin{cases} E_1{}^c \varepsilon_x & a_c \leq z \leq h/2 \\ E_1{}^c \varepsilon_1{}^c + E_2{}^c(\varepsilon_x - \varepsilon_1{}^c) & -h/2 \leq z \leq a_c \end{cases} \qquad (B.16)
$$

$$
\begin{aligned}
A' &= hE_1{}^c \\
B' &= (E_1{}^c - E_2{}^c)(h/2 + a_c)^2/2 \\
B'' &= 0 \\
D' &= [h^3(E_1{}^c + E_2{}^c)/8 + a_c{}^3(E_1{}^c - E_2{}^c)]/3 - h^2 a_c(E_1{}^c - E_2{}^c)/8
\end{aligned}
\qquad (B.17)
$$

<u>Case 7:</u>

$$
\sigma_x \equiv E_1{}^c\ {}_1{}^c + E_2{}^c(\varepsilon_x - \varepsilon_1{}^c) \qquad -h/2 \leq z \leq h/2 \qquad (B.18)
$$

$$
\begin{aligned}
A' &= hE_1{}^c \\
B' &= ha_c(E_1{}^c - E_2{}^c) \\
B'' &= 0 \\
D' &= h^3 E_2{}^c/24
\end{aligned}
\qquad (B.19)
$$

<u>Case 8:</u>

$$
\sigma_x \equiv \begin{cases} E_1{}^t \varepsilon_x & z_n \leq z \leq h/2 \\ E_1{}^c \varepsilon_x & -h/2 \leq z \leq z_n \end{cases} \qquad (B.20)
$$

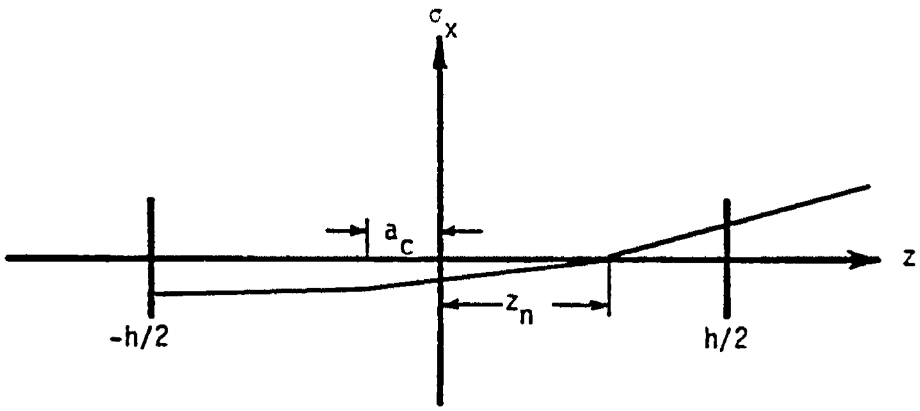

Fig. B.4 Stress distribution for Case 5.

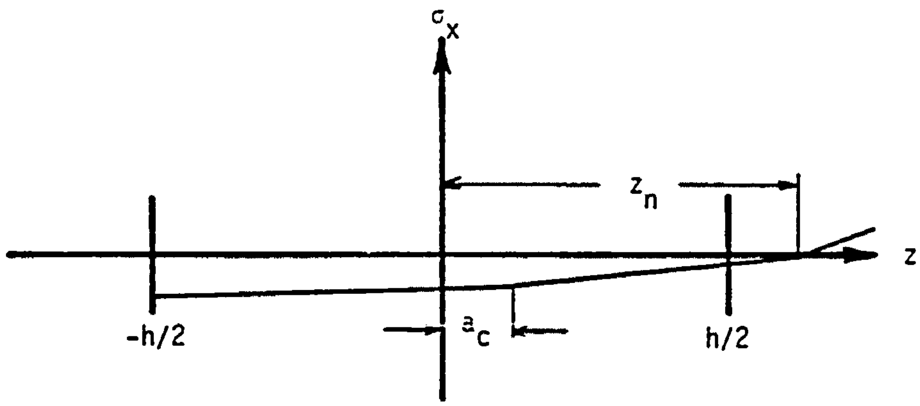

Fig. B.5 Stress distribution for Case 6.

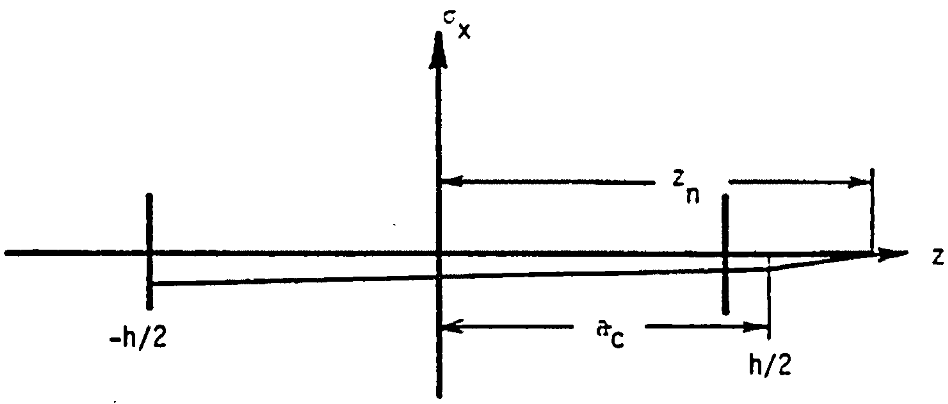

Fig. B.6 Stress distribution for Case 7.

$$
\begin{aligned}
A' &= h(E_1{}^t + E_1{}^c) + z_n(E_1{}^c - E_1{}^t) \\
B' &= (E_1{}^t - E_1{}^c)(h^2/4 - z_n{}^2)/2 \\
B' &= (E_1{}^t - E_1{}^c)(h^2/4 - z_n{}^2)/2 \\
D' &= [h^3(E_1{}^t + E_1{}^c)/8 + z_n{}^3(E_1{}^c - E_1{}^t)]/3
\end{aligned}
\qquad (B.21)
$$

For concave downward bending, one is able to derive similar equations for stiffnesses by converting as follows:

$$
\begin{array}{lll}
E_c{}^2 \longrightarrow E_2{}^t & E_1{}^t \longrightarrow E_1{}^c & a_c \longrightarrow a_t \\
E_1{}^c \longrightarrow E_1{}^t & E_2{}^t \longrightarrow E_2{}^c & a_t \longrightarrow a_c
\end{array}
$$

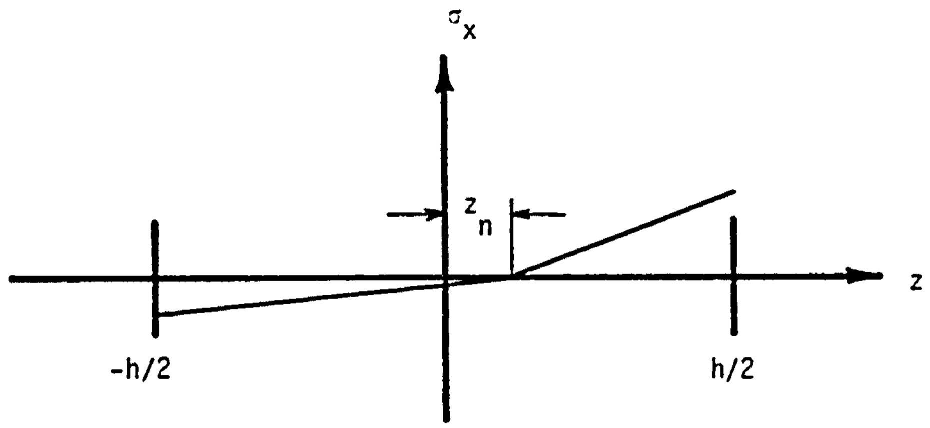

Fig. B.7 Stress distribution for Case 8.

APPENDIX C

COMPUTATION OF z_n, a_c, AND a_t

For multimodular beams, the following equation is not sufficient to determine the neutral-surface location z_n

$$z_n = \frac{B'M - D'N}{A'M - B''N} \tag{C.1}$$

even for cases where N = 0

$$z_n = B'/A' \tag{C.2}$$

Two more equations are needed for computing z_n because the stiffnesses are not only dependent on z_n but they are functions of a_c and a_t as well. Dividing Eq. (2.4) by Eq. (2.6) and Eq. (2.5) by Eq. (2.7) and solving for a_c and a_t, one can get (for the convex downward case)

$$a_c = (\varepsilon_1{}^c/\varepsilon_f{}^c)(h/2 + z_n) - z_n \tag{C.3}$$

$$a_t = (\varepsilon_1{}^t/\varepsilon_f{}^t)(h/2 - z_n) + z_n \tag{C.4}$$

For the concave downward case

$$a_c = (\varepsilon_1{}^c/\varepsilon_f{}^c)(h/2 - z_n) + z_n \tag{C.5}$$

$$a_t = (\varepsilon_1{}^t/\varepsilon_f{}^t)(h/2 + z_n) - z_n \tag{C.6}$$

The system of nonlinear Eqs. (C.1), (C.3), and (C.4) for the convex downward case, or Eqs. (C.1), (C.5), and (C.6) for the concave downward case, can be solved by using iteration of the Gauss-Seidel type [2.22].

APPENDIX D

ARBITRARY CONSTANTS AND PARTICULAR SOLUTIONS

The values of constants C_1, C_2, C_3, C_4, d_1, and d_2 for the various boundary conditions considered are listed below.

1. Hinged-Hinged (free to move axially at $x = L$)

$$
\begin{aligned}
C_1 &= -u_p(0) \\
C_2 &= -(C_3\ell + C_4\ell^2) - [w_p(\ell) - w_p(0)]/\ell \\
C_3 &= [\psi_{p,x}(\ell)/2] - 3C_4\ell \\
C_4 &= \frac{A'}{6\ell(A'D' - B'B'')} \{B'[u_{p,x}(\ell) - u_{p,x}(0)] + D'[\psi_{p,x}(\ell) - \psi_{p,x}(0)]\}
\end{aligned}
\qquad (D.1)
$$

$$
\begin{aligned}
d_1 &= -u_p(0) \\
d_2 &= -[u_{p,x}(\ell) + (6B'/A')C_4\ell]
\end{aligned}
$$

2. Clamped-Free

$$
\begin{aligned}
C_1 &= -w_p(0) \\
C_2 &= -w_{p,x}(\ell) - \psi_p(\ell) + \psi_p(0) \\
C_3 &= \psi_{p,x}(\ell)/2 + \frac{A'S\ell}{2(B'B'' - A'D')} [w_{p,x}(\ell) + \psi_p(\ell)] \\
C_4 &= \frac{A'S}{6(B'B'' - A'D')} [C_2 - \psi_p(0)]
\end{aligned}
\qquad (D.2)
$$

$$
\begin{aligned}
d_1 &= -u_p(0) \\
d_2 &= -u_{p,x}(\ell) + \frac{B'S\ell}{B'B'' - A'D'} [w_{p,x}(\ell) + \psi_p(\ell)]
\end{aligned}
$$

3. Clamped-Clamped (free to move axially at $x = L$)

$$C_1 = - w_p(0)$$

$$C_2 = \frac{6(B'B'' - A'D')}{SA'} C_4 + \psi_p(0) \tag{D.3}$$

$$C_3 = [u_p(0) - u_p(\ell) - (3B'/A')C_4\ell^2]/2\ell$$

$$C_4 = \frac{SA'}{SA'\ell^2 - 12(B'B'' - A'D')} \{\psi_p(0) + \psi_p(\ell) - 2[w_p(0) - w_p(\ell)]/\ell\}$$

$$d_1 = - u_p(0)$$

$$d_2 = [u_p(0) - u_p(\ell) - (3B'/A')C_4\ell^2]/\ell$$

The particular solutions for uniform and sinusoidal normal load are as listed below.

For uniform normal load $q(x) = q_o$:

$$u_p(x) = \frac{B'q_o}{6(A'D' - B'B'')} x^3$$

$$\psi_p(x) = - \frac{q_o}{S} x - \frac{A'q_o}{6(A'D' - B'B'')} x^3 \tag{D.4}$$

$$w_p(x) = \frac{A'q_o}{24(A'D' - B'B'')} x^4$$

For normal load $q(x) = q_o \sin \alpha x$, where $\alpha \equiv n\pi/\ell$:

$$u_p(x) = \frac{B'q_o}{\alpha^3(A'D' - B'B'')} \cos \alpha x$$

$$\psi_p(x) = - \frac{A'q_o}{\alpha^3(A'D' - B'B'')} \cos \alpha x \tag{D.5}$$

$$w_p(x) = \frac{q_o}{\alpha^2} [\frac{1}{S} + \frac{A'}{\alpha^2(A'D' - B'B'')} \sin \alpha x]$$

APPENDIX E

TRANSFER MATRICES FOR DEFLECTION OF A MULTIMODULAR BEAM

The equilibrium equations for each station can be written in matrix notation as follows:

$$\begin{Bmatrix} U \\ W \\ \psi \\ N \\ Q \\ M \\ 1 \end{Bmatrix}_i^R = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & q_s \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}_i \begin{Bmatrix} U \\ W \\ \psi \\ N \\ Q \\ M \\ 1 \end{Bmatrix}_i^L \qquad \text{(E.1)}$$

where q_s is the concentrated load at each station. In more compact form

$$[S]_i^R = [T_S]_i [S]_i^L \qquad \text{(E.2)}$$

The matrix $[T_S]_i$ is known as the station matrix. In matrix notation the equilibrium equation for each field under a distributed load $q(x)$ is

$$\begin{Bmatrix} U \\ W \\ \psi \\ N \\ Q \\ M \\ 1 \end{Bmatrix}_{i+1} = \begin{bmatrix} 1 & 0 & 0 & \frac{B'\Delta\ell}{\gamma} & \frac{B'(\Delta\ell)^2}{2\gamma} & \frac{-D'\Delta\ell}{\gamma} & \frac{-B'\Delta\ell}{2\gamma} K_m \\ 0 & 1 & -\Delta\ell & \frac{A'(\Delta\ell)^2}{2\gamma} & [\frac{\Delta\ell}{S} + \frac{A'(\Delta\ell)^3}{4\gamma}] & \frac{-B'(\Delta\ell)^2}{2\gamma} & \frac{-A'(\Delta\ell)^2}{4\gamma} K_m - \frac{\Delta\ell}{2S} K_q \\ 0 & 0 & 1 & \frac{-A'\Delta\ell}{\gamma} & \frac{-A'(\Delta\ell)^2}{2\gamma} & \frac{B'\Delta\ell}{\gamma} & \frac{A'\Delta\ell}{2\gamma} K_m \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -K_q \\ 0 & 0 & 0 & 0 & \Delta\ell & 1 & -K_m \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}_i \begin{Bmatrix} U \\ W \\ \psi \\ N \\ Q \\ M \\ 1 \end{Bmatrix}_i \tag{E.3}$$

where

$$\gamma \equiv B'B'' - A'D'$$

$$K_q \equiv \int_0^{\Delta\ell} q(\xi)d\xi \tag{E.4}$$

$$K_m \equiv \int_0^{\Delta\ell} \xi q(\xi)d\xi$$

Values of K_m and K_q for various loadings are listed in Table 3. Equation (E.4) also can be written as

$$[S]^L_{i+1} = [T_j]_i[S]^R_i \tag{E.5}$$

The matrix $[T_j]_i$ is called the field matrix.

APPENDIX F

TRANSFER MATRIX FORMULATION FOR VIBRATION OF A MULTIMODULAR BEAM

Under harmonic exitation, the steady-state-response displacements u, w and ψ are assumed to be hamonic in time. Therefore, Eqs. (3.1) can be written as

$$\bar{N}_{,x} = -\Omega^2 P\bar{u} - \Omega^2 R\bar{\psi} \quad ; \quad \bar{Q}_{,x} = -\Omega^2 P\bar{w} - q(x,t)$$
$$\bar{M}_{,x} - \bar{Q} = -\Omega^2 R\bar{u} - \Omega^2 I\bar{\psi} \tag{F.1}$$

where all of the barred quantities are amplitudes, i.e., $N(x,t) = \bar{N}(x)$ $\sin \Omega t$, etc. The continuity at each station implies

$$\bar{u}_i^R = \bar{u}_i^L \quad , \quad \bar{w}_i^R = \bar{w}_i^L \quad , \quad \bar{\psi}_i^R = \bar{\psi}_i^L \tag{F.2}$$

(R and L denote right and left, respectively)

Also, Eq. (F.1) in finite-differential form for each station i is

$$\bar{N}_i^R = \bar{N}_i^L - \Omega^2 P\bar{u}_i^L \quad ; \quad \bar{Q}_i^R = \bar{Q}_i^L - \Omega^2 P\bar{w}_i^L - \bar{Q}_i$$
$$\bar{M}_i^R = \bar{M}_i^L - \Omega^2 R\bar{u}_i^L - \Omega^2 I\bar{\psi}_i^L \tag{F.3}$$

where Q_i is the concentrated load amplitude at station i.

Equations (F.2) and (F.3) are written in matrix notation as

$$\begin{Bmatrix} \bar{u} \\ \bar{w} \\ \bar{\psi} \\ \bar{N} \\ \bar{Q} \\ \bar{M} \\ 1 \end{Bmatrix}_i^R = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -\Omega^2 P & 0 & -\Omega^2 R & 1 & 0 & 0 & 0 \\ 0 & -\Omega^2 P & 0 & 0 & 1 & 0 & Q_i \\ -\Omega^2 R & 0 & -\Omega^2 I & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}_i \begin{Bmatrix} \bar{u} \\ \bar{w} \\ \bar{\psi} \\ \bar{N} \\ \bar{Q} \\ \bar{M} \\ 1 \end{Bmatrix}_i^L \tag{F.4}$$

or

$$[S]_i^R = [T_s]_i \; [S]_i^L$$

where $[T_s]_i$ represents station matrix at station i. It should be noted that, in actuality, due to the nonlinearity of the material, the genralized displacements are not harmonic in time.

APPENDIX G

DETAILED DERIVATIONS

In the following, detailed derivations for unimodular and bimodular laminates are given. Three types of lamination are considered: Single-layer, two-layer, and three-layer. In the latter two cases, all layers are assumed to be of the same thickness. Also, in the bimodular cases, the effect of bimodularity on elastic-shear moduli is ignored (since for most of materials $G^t/G^c = 1$). Further, in the bimodular analyses, axial force is assumed to be zero.

1. Single-Layer (Bimodular Material)

Normal and shear stress distributions for single-layer, bimodular material are shown in Fig. G.1. Since the axial force is taken to be zero,

$$N = \int_{-h_c}^{h_t} \sigma_x \, dz' = \kappa \left[\int_{-h_c}^{0} E^c z' \, dz' + \int_{0}^{h_t} E^t z' \, dz' \right] = 0 \qquad \text{(G.1)}$$

Using the fact that

$$h = h_t + h_c \qquad \text{(G.2)}$$

one can integrate Eq. (G.1) to obtain

$$h_c/h = \frac{1}{1 + \sqrt{\beta}} \quad , \quad h_t/h = \frac{\sqrt{\beta}}{1 + \sqrt{\beta}} \qquad \text{(G.3)}$$

where β is the bimodular elastic-moduli ratio defined as

$$\beta = E^c/E^t$$

From Eqs. (4.6) and (4.9), one has

$$D = (E^c h_c^{\,3} + E^t h_t^{\,3})/3 \tag{G.4}$$

$$b = \frac{1}{2}\begin{cases} E^c(z')^2 - E^c h_c^{\,2} & -h_c < z' < 0 \\ E^t(z')^2 - E^c h_c^{\,2} & 0 < z' < h_t \end{cases} \tag{G.5}$$

substitution of Eqs. (G.4) and (G.5) into Eq. (4.15) and using Eq. (G.3) leads to

$$K^2 = 5/6 \tag{G.6}$$

Also, using Eqs. (4.19), (4.20), (G.3), (G.4), and (G.5), one has

$$(\bar{\tau}_{xz})_{max} = 3/2 \tag{G.7}$$

Although the maximum shear stress remains unaffected by bimodularity ratio, the shear-stress distribution does depend on β. Combining Eqs. (4.19), (G.3), (G.4), and (G.5) gives the following shear-stress distribution

$$\bar{\tau}_{xz} = \frac{3}{2}\begin{cases} \sqrt{\beta} - \sqrt{\beta}\,(\sqrt{\beta}+1)^2(z/h)^2 & -\dfrac{1}{\sqrt{\beta}} < z'/h < 0 \\ \sqrt{\beta} - \dfrac{1}{\sqrt{\beta}}\,(\sqrt{\beta}+1)^2(z/h)^2 & 0 < z'/h < \dfrac{\sqrt{\beta}}{\sqrt{\beta}+1} \end{cases} \tag{G.8}$$

2. Two-Layer Laminates

(a) Bimodular Material

Each individual layer has different elastic moduli in tension and compression. However, since one layer is only in either a tension or a compression state, only three of the four elastic moduli for two-layer analysis come into the picture (see Fig. G.2). A similar procedure to that used for the single-layer case is used here. To determine the bending and the partial bending-stretching coupling stiffnesses, one is able to obtain the neutral-surface position from the following quadratic equation

$$N = \kappa\left[\int_{-h_c}^{-h_c+h/2} E_1{}^c z'dz' + \int_{-h_c+h/2}^{0} E_2{}^c z'dz' + \int_{0}^{h_t} E_2{}^t z'dz'\right] = 0 \qquad \text{(G.9)}$$

Knowing the neutral-surface location, one can easily determine the following stiffnesses

$$D = \{E_1{}^c[(h/2 - h_c)^3 + h_c{}^3] + E_2{}^c(h/2 - h_c)^3 + E_2{}^t h_t{}^3\}/3 \qquad \text{(G.10)}$$

and

$$b = \frac{1}{2}\begin{cases} E_1{}^c(z')^2 - E_1{}^c\, h_c{}^2 & -h_c < z' < -h_c + h/2 \\ E_2{}^c(z')^2 - E_2{}^t\, h_t{}^2 & -h_c + h/2 < z' < 0 \\ E_2{}^t(z')^2 - E_2{}^t\, h_t{}^2 & 0 < z' < h_t \end{cases} \qquad \text{(G.11)}$$

Substituting Eqs. (G.10) and (G.11) into Eq. (4.15), one has

$$K^2 = \frac{5}{6} \cdot \frac{16\, g_1[(\alpha+\beta)(h/2 - h_c)^3 + \alpha h_c^3 + h_t^3]^2}{h(g_1+1)[3(g_1\alpha^2 - \beta^2)(h/2-h_c)^5 - 10(g_1\alpha^2 h_c^2 - h_t^2)(h/2-h_c)^3 + 15(g_1\alpha^2 h_c^4 - h_t^4)(h/2-h_c) + 8(g_1\alpha^2 h_c^5 + h_t^5)]} \qquad \text{(G.12)}$$

where

$$\alpha = E_1{}^c/E_2{}^t \quad , \quad \beta = E_2{}^c/E_2{}^t \quad \text{and} \quad g_1 = G_1/G_2$$

Since h_c/h and h_t/h depend upon α and β, K^2 is a function of g_1, α, and β for the case of equal-thickness layers. The maximum dimensionless shear stress also is a function of the bimodular ratios (α,β)

$$(\bar{\tau}_{xz})_{max} = \frac{3}{2}\cdot\frac{h}{\alpha[(h/2-h_c)^3+h_c^3]+\beta(h/2-h_c)^3+h_t^3}\begin{cases}\alpha[h_c^2-(h/2-h_c)^2] & -h/c<z'<-h_c+h/2 \\ h_t^2 & 0<z'<h_t\end{cases}$$

(G.13)

(b) Unimodular Material

A similar lamination as used in part (a) is considered here. The normal and shear stress distributions are shown in Fig. G.3. Since the flexural-extensional coupling stiffness (b) does not vanish, one must use Eq. (4.13) in order to compute K^2. Let the transverse shear moduli be the same for both layers ($G_2/G_1 = 1$) and $\alpha = E_2/E_1$. Then, the stiffnesses can be expressed as

$$\begin{aligned} A &= E_1 h(\alpha+1)/2 \\ B &= E_1 h^2(\alpha-1)/8 \\ D &= E_1 h^3(\alpha+1)/24 \end{aligned} \tag{G.14}$$

and

$$(a,b) = E_1 \begin{cases} [(z+h/2)\ ,\ (z^2-h^2/4)/2] & -h/2 < z < 0 \\ [(\alpha z+h/2)\ ,\ (\alpha z-h^2/4)/2] & 0 < z < h/2 \end{cases} \tag{G.15}$$

Substitution of Eqs. (G.14) and (G.15) into Eq. (4.13) gives

$$K^2 = \frac{5}{6}\cdot\frac{(\alpha^2+14\alpha+1)^2}{2[\alpha^4+12\alpha^3+102\alpha^2+12\alpha+1]} \tag{G.16}$$

It is interesting to note that K^2 as given by Eq. (G.16) gives the same result for $\alpha = c$ and $\alpha = 1/c$, where c is an arbitrary constant. Combining Eqs. (4.4), (4.12), (4.19), (G.14), and (G.15), one gets

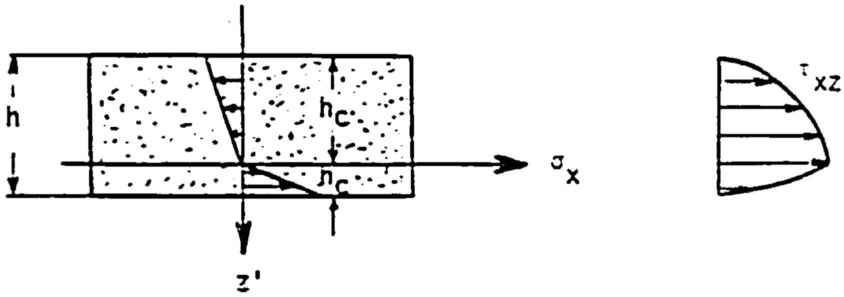

Figure G.1 Normal and shear stress distribution in a single-layer bimodular material.

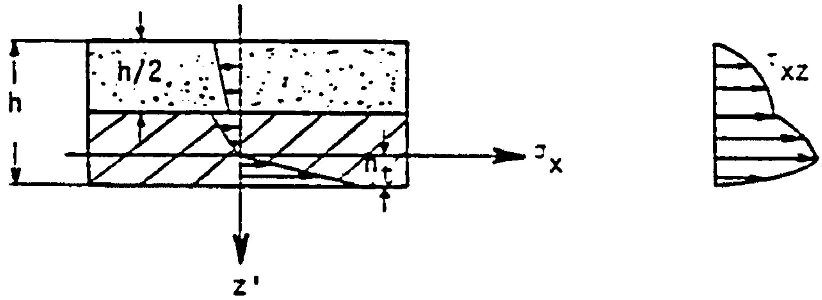

Figure G.2 Normal and shear stress distribution in a two-layer bimodular material laminate.

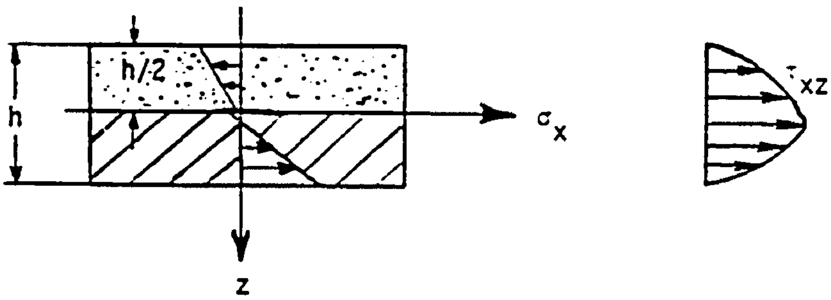

Figure G.3 Normal and shear stress distribution in a two-layer unimodular material laminate.

$$(\bar{\tau}_{xz})_{max} = \frac{3}{2} \cdot \frac{8\alpha(\alpha+1)}{\alpha^2+14\alpha+1} \tag{G.17}$$

which shows that $(\bar{\tau}_{xz})_{max}$ depends on elastic-moduli ratio (E_2/E_1), only.

3. Three-Layer Laminate

(a) Bimodular Material

Consider a three-layer laminate with the top and bottom layers (facings) made of the same material (i.e., $E_1{}^c = E_3{}^c$, $E_1{}^t = E_3{}^t$) as shown in Fig. G.4 in the convex downward bending position. One assumes that the neutral-surface position is within the middle layer (which happens for most practical cases). The axial force vanishes if

$$\int_{-h_c}^{-h_c+h/3} E_1{}^c z'dz' + \int_{-h_c+h/3}^{0} E_2{}^c z'dz' + \int_{0}^{h_t-h/3} E_2{}^t z'dz' + \int_{h_t-h/3}^{h_t} E_1{}^t z'dz' = 0 \tag{G.18}$$

Equation (G.18) can be used to determine the neutral-surface position. Having the values of h_t and h_c enables one to calculate K^2 with the same procedure as used for the two-layer case

$$D = \frac{E_2{}^c}{3} \{\alpha[h_c^3 + (h/3-h_c)^3] - (h/3-h_c)^3 + \beta(h_t-h/3)^3 + \gamma[h_t^3 - (h_t-h/3)^3]\} \tag{G.19}$$

$$\int_{-h_c}^{h_t} G^{(k)}dz = G_2h[2g_1+1]/3 \tag{G.20}$$

$$\int_{-h_c}^{h_t} (G^{-1})^{(k)} b^2 dz = \frac{(E_2{}^c)^2}{4G_2} \{\{ g_1^{-1}\alpha^2\{\tfrac{1}{5}[(h/3-h_c)^5+h_c{}^5]-\tfrac{2}{3}h_c^2[(h/3-h_c)^3+h_c^3]+hh_c^4/3\}$$

$$-[\tfrac{1}{5}(h/3-h_c)^5+\tfrac{2}{3}H_c^2(h/3-h_c)^3+(H_c^2)^2(h/3-h_c)]$$

$$+[\tfrac{\beta^2}{5}(h_t-h/3)^5+\tfrac{2\beta}{3}H_c^2(h_t-h/3)^3+(H_c^2)^2(h_t-h/3)]$$

$$+g_1^{-1}\gamma^2\{\tfrac{1}{5}[h_t^5-(h_t-h/3)^5]-\tfrac{2}{3}h_t^2[h_t^3-(h_t-h/3)^3]+hh_t^4/3\}\}\tag{G.21}$$

where

$$H_c{}^2 = (h/3-h_c)^2(\alpha-1) - h_c{}^2 \tag{G.22}$$

$$\alpha = E_1{}^c/E_2{}^c \quad , \quad \beta = E_2{}^t/E_2{}^c \quad , \quad \gamma = E_3{}^t/E_2{}^c, \quad \text{and} \quad g_1 = G_1/G_2$$

By substituting Eqs. (G.19), (G.20), and (G.21) into Eq. (4.15), one can determine K^2. Also, using Eqs. (G.19) and the following expression for b_{max} (the maximum value of partial coupling stiffness b) in the respective regions, one has the maximum dimensionless shear stress by Eq. (4.19).

$$b_{max} = \frac{1}{2}\begin{cases} L_1E_1{}^c & -h_c < z' < -h_c+h/3 \\ L_1E_1{}^c - L_2E_2{}^c & -h_c+h/3 < z' < 0 \\ L_1E_1{}^c - L_2E_2{}^c + L_3E_2{}^t & 0 < z' < h_t-h/3 \\ L_4E_1{}^t & h_t-h/3 < z' < h_t \end{cases} \tag{G.23}$$

where

$$L_1 = (h-h_c)^2 - h_c{}^2 \qquad L_3 = (h-h_t)^2$$
$$L_2 = L_1 + h_c{}^2 \qquad L_4 = L_3 - h_t{}^2 \tag{G.24}$$

(b) Unimodular Material

Shear, bending, and partial stretching-bending stiffnesses for the three-layer symmetric laminate with equal-thickness layers as shown in Fig. G.5 are ($g_1 = G_1/G_2$, $\alpha = E_1/E_2$)

$$S = \frac{hG_2}{3}(1 + 2g_1) \quad , \tag{G.25}$$

$$D = \frac{E_2h^3}{324}(26\alpha + 1) \quad , \tag{G.26}$$

and

$$\int_{-h/2}^{h/2} (G^{-1})^{(k)} b^2 dz' = \frac{E_2^{\,2}h^5}{29160}[102\alpha^2 + g_1(120\alpha^2 + 20\alpha + 1)] \tag{G.27}$$

Substituting these stiffnesses into Eq. (4.13) leads to

$$K^2 = \frac{5}{6} \cdot \frac{[26\alpha + 1]^2 g_1}{(1+2g_1)[102\alpha^2 + g_1(120\alpha^2 + 20\alpha + 1)]} \tag{G.28}$$

Absolute dimensionless maximum shear stress occurs at the midplane. Using the same procedure as used before, one obtains

$$(\bar{\tau}_{xz})_{max} = (\bar{\tau}_{xz})_{z=0} = \frac{3}{2} \cdot \frac{3(8\alpha + 1)}{26\alpha + 1} \tag{G.29}$$

and the shear stress at the interfaces is

$$(\bar{\tau}_{xz})_{z = \pm h/6} = \frac{3}{2} \cdot \frac{24\alpha}{26\alpha + 1} \tag{G.30}$$

It is interesting to note that

$$\frac{(\bar{\tau}_{xz})_{max}}{(\bar{\tau}_{xz})_{z = \pm h/6}} = 1 + (1/8\alpha) \tag{G.31}$$

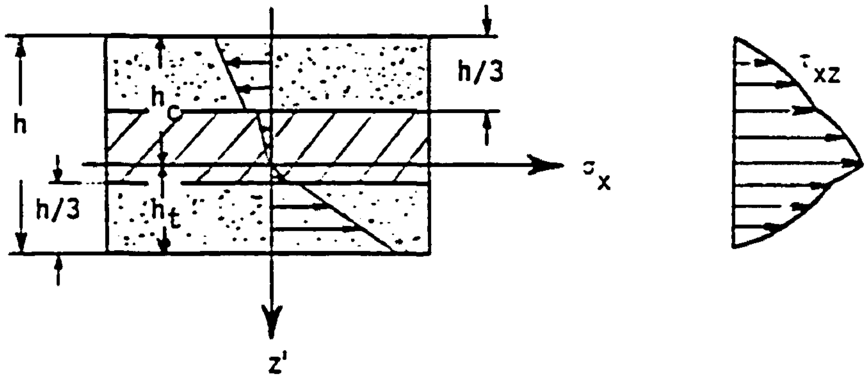

Figure G.4 Normal and shear stress distribution in a three-layer bimodular material laminate.

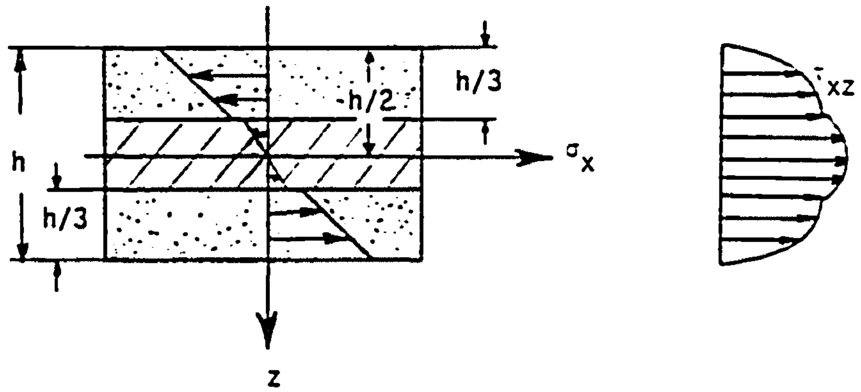

Figure G.5 Normal and shear stress distribution in a three-layer unimodular material laminate.

APPENDIX H

DETERMINATION OF THE SHEAR CORRECTION COEFFICIENT

In the following, a static shear correction coefficient is determined for the assumed beam. Considering the convex downward bending (bottom layers of the beam in tension and top layers are in compression) in which the neutral-surface position is within the core. The coordinate system and key dimensions are as shown in Figs. H.1 and H.2.

The stress distribution is

$$\sigma_x = \begin{cases} (E_c)(\kappa z') & -h_c < z' < -h_c + h \\ (E_{cc})(\kappa z') & -h_c + h < z' < 0 \\ (E_{tc})(\kappa z') & 0 < z' < h_t - h \\ (E_t)(\kappa z') & h_t - h < z' < h_t \end{cases} \tag{H.1}$$

Let us neglect the axial force, which would be small due to the absence of axial restraint. Then,

$$\int_{-h_c}^{h_t} \sigma_x \, dz' = 0 \tag{H.2}$$

Substitution of Eq. (H.1) into Eq. (H.2) gives

$$(E_c - E_{cc})(h - h_c)^2 + (E_{tc} - E_t)(h_t - h) + E_t h_t^2 - E_{tc} h_c^2 = 0 \tag{H.3}$$

Also,

$$h_c + h_t = 2h + c$$

Equations (H.3) and (H.4) form a quadratic equation to evaluate h_c or h_t. Using the definition of static shear correction coefficient from

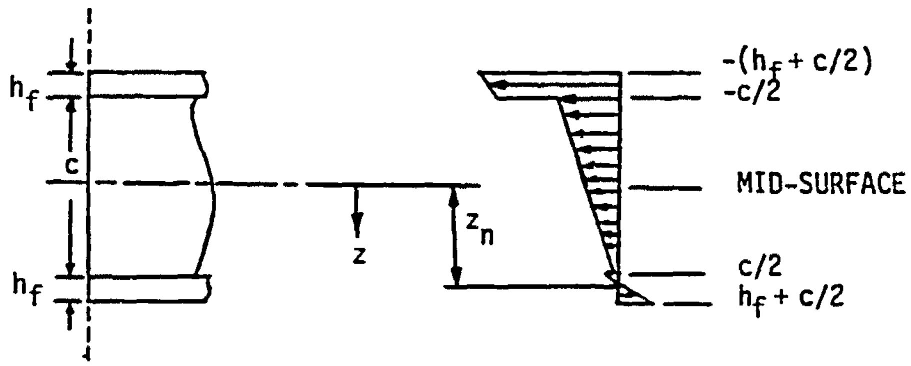

Fig. H.1. Geometry and stress distribution for sandwich beam when the neutral surface is within a facing.

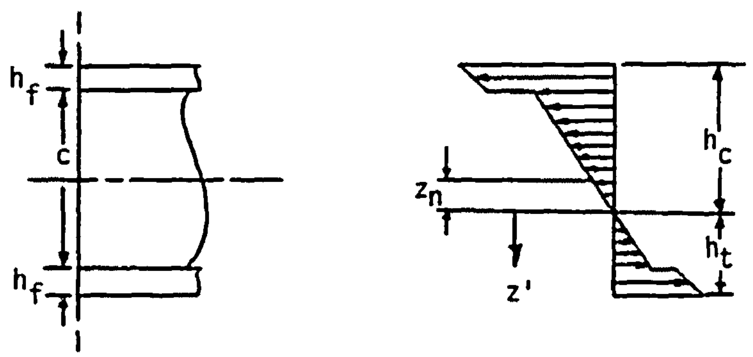

Fig. H.2. Stress distribution for sandwich beam when the neutral surface is within the core.

[5.12], for this particular problem one has

$$K^2 = \frac{D^2 \int_{-h_c}^{h_t} G\, dz'}{h^2 \int_{-h_c}^{h_t} Gb^2\, dz'} \tag{H.5}$$

where

$$b = \int_{-h_c}^{z'} E_z\, dz' = \begin{cases} \int_{-h_c}^{z'} E_c z' dz' & -h_c < z' < -h_c + h \\ \int_{-h_c}^{-h_c + h} E_c z' dz' + \int_{-h_c + h}^{z'} E_{cc} z' dz' & -h_c + h < z' < 0 \\ \int_{-h_c}^{-h_c + h} E_c z' dz' + \int_{-h_c + h}^{0} E_{cc} z' dz' + \int_{0}^{z'} E_{tc} z' dz' & 0 < z' < h_t - h \\ \int_{-h_c}^{-h_c + h} E_c z' dz' + \int_{-h_c + h}^{0} E_{cc} z' dz' + \int_{0}^{h_t - h} E_{tc} z' dz' + \int_{h_t - h}^{z'} E_t z' dz' & h_t - h < z' < h_t \end{cases} \tag{H.6}$$

Note that -b/H is the dimensionless transverse shear stress, the distribution of which is shown in Fig. H.3 for the case of 0° facings.

For the material used in [5.11], the resulting values of shear correction factor are:

1. For 0° facings, $K^2 = 0.6913$
2. For 90° facings, $K^2 = 1.0487$

(H.7)

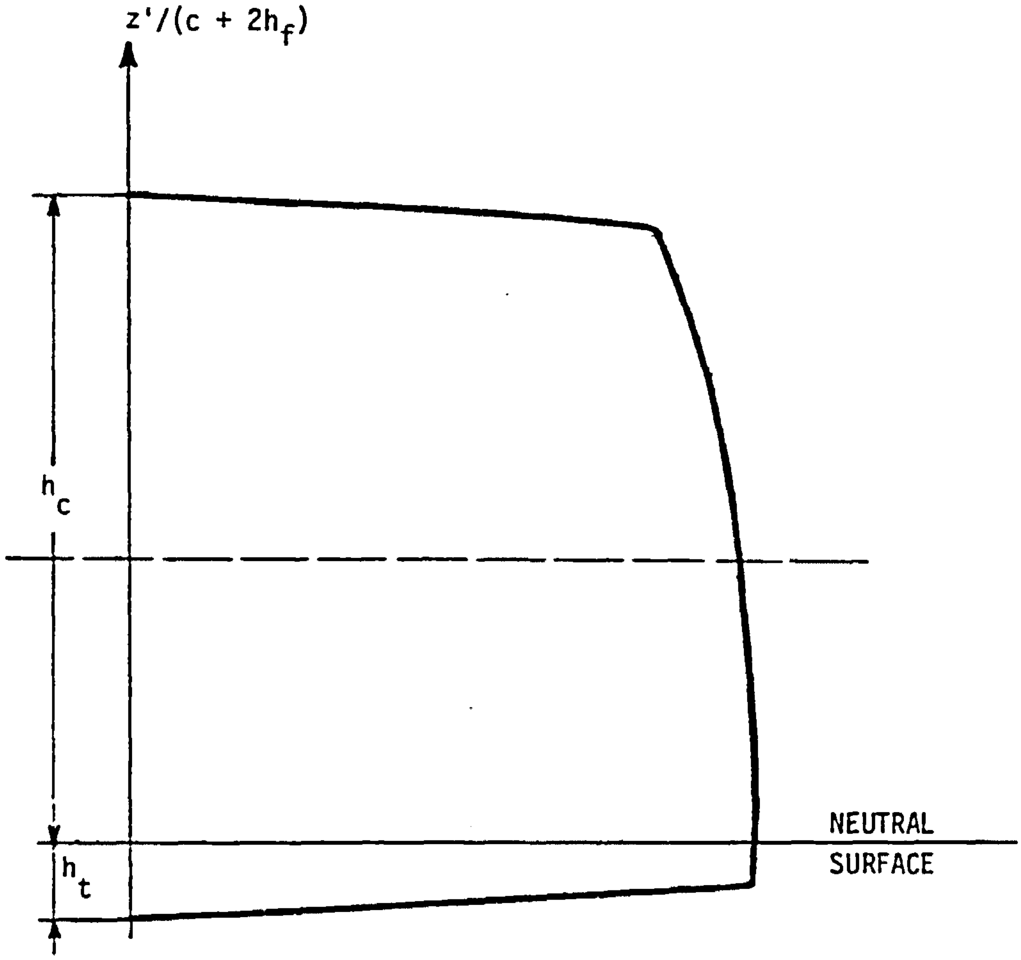

Fig. H.3. Distribution of transverse shear stress for the beam having 0° facings.

APPENDIX I

COMPUTER PROGRAMS

```
      COMMON /ANON/ P
      COMMON /XMOMNT/ XM1,XM2
      COMMON /XMODUL/ XE1C, XE1T
      COMMON /BMODUL/ BE1C, BE1T
      COMMON /DISTAN/ X1(27), Y(27),XZN(27)
      COMMON / ABD / A1 ,B1 ,D1 ,CAN,CBN ,CBM ,CDM,XL,XM
      COMMON /ALM/ AMX(27), QX(27),ANX(27)
      COMMON /NOE/ NE2, NE1, NE
      COMMON / DIMEN / BD,BL , S, Q0,AP
      COMMON / AMODUL / E1C, E1T,E2C , E2T,G
      COMMON /BC / NBC, NBI, SCL, DIM ,SWH , EPS
      COMMON /TNK / A, B, ZN, ZNB,XA,XB
      COMMON / STAIN / XFC, EC1, ET1, XFT
      COMMON /STIF/ A11, B11, B12, D11
      COMMON  / FRASTN / C, T
      COMMON  / COFI/ C1, C2, C3 ,C4 , DX1, DX2
      COMMON /SOLU/ UP(27), VP(27), WP(27), UPX(27), VPX(27), WPX(27)
      COMMON /FSOLU/ U(27), V(27), W(27), UX(27), VX(27), WX(27)
      COMMON /SANDWH/  ECP, ETP, ECC, ETC, GP, GC
      DATA  ECP, ETP, ECC, ETC, GP, GC   /32.E+03, 41.E+04, 758.,814.,
     1  665., 634. /
      DATA  SFT, SBD, SBL  /.0625 ,1.25, 6.656 /
      DATA  E2C, E1C, E1T, E2T, G  /10.3E+03 , 31.9E+03 , .42E+06   ,
     1  .59E+06, .5366E+03  /
      DATA  XFC, EC1, ET1, XFT /-.046 , -.011, .014 , .029 /
      DATA  XE1C, XE1T  /.18E+05 , .47E+06 /
      DATA  Q0 , XM1, XM2, NE / .1, .0, .0, 25 /
      DATA XA,XB, ZN,ZNB /0.04 , 0.270, 0.19, 0.22/
      DATA  BD, BL /0.6, 8.0/
      DATA  P     /.2/
      DATA  AF /0.0/
      DATA  BE1T, BE1C /5.193E+05, 1.740E+03/
      DATA  NBC,DIM, SCL , SWH  /1 ,2., 1., 0.   /
      EPS = .0000001
      AK = 5./6.
      AK = 0.7817
      C = EC1/XFC
      T = ET1/XFT
      HPD = BD/2.
      S =    AK*BD*G
      NE1 = NE + 1
      NE2 = NE + 2
      IF ( DIM .EQ. 2 ) THEN DO
      BD = P
      BL = 1.
      HPD = BD/2.
      Q0 = 1.
      XE1C = XE1C/XE1T
      XE1T = 1.
      SM= AK*P*G  /E2T
      E1T = E1T/E2T
      E1C = E1C/E2T
      E2C = E2C/E2T
      E2T = 1.
      SB= AK*P*G  /BE1T
      BE1C = BE1C / BE1T
      BE1T = 1.
      ENDIF
      IF (DIM  .EQ. 2. ) GO TO 598
      PRINT 400,BL, BD, AP,NE
      GO TO 599
  598 CONTINUE
      PRINT 411,    P, AP, NE
  599 CONTINUE
```

```
      PRINT 430, XE1C, XE1T, G
  400 FORMAT (1H1, ///, 20X, 'L =', F10.5, /, 20X, 'H =', F10.5, /, 20X,
     1'N = ', F10.5, /, 20X, 'NE='    , I5  )
  411 FORMAT (     ///, 20X, 'P =', F10.5, /, 20X,
     1'N = ', F10.5, /, 20X, 'NE='    , I5  )
  430 FORMAT (//, 20X, 'BIMODULUS PROPERTIES:', //, 20X, 'EC =', E12.4,
     1/,20X,'ET =',E12.4, /, 21X, 'G =', E12.4)
      PRINT 120 , Q0, XM1, XM2
 120  FORMAT (//,20X, 'APPLIED FORCE:'  , //, 20X, 'Q0 =', F10.5,/,
     1  20X, 'M1 =', F10.5, /, 20X, 'M2 =', F10.5 , ///)
      DO  100  NB = 1,3
      DO  110  LD = 1,2
      IF (SWH  .EQ. 1. )  THEN DO
      ZNB = .6
      BD = SBD
      BL = SFL
      XA = BD/2 - SFT
      XB = XA
      S = AK*( GF*(BD - 2.*XA) + GC*2.*XA)
      IF ( ZNB .LT.  XB          )  GO TO 90
      E2C = ECF
      E1C = ECC
      E1T = ECF
      E2T = ETF
      GO TO 91
  90  CONTINUE
      E1T = ETC
      E1C = ECC
      E2C = ECF
      E2T = ETF
  91  CONTINUE
      ENDIF
      A = XA
      B = XB
      ZN = ZNB
      NBI = 1
      IF ( DIM .EQ. 2.)  S = SM
      CALL  ZNCOM (A, B,ZN , E2C, E1C, E1T, E2T)
      CALL  EXPSN  (A11, B11, B12, D11)
      PRINT 410,E2C, E1C, E1T, E2T, G, EC1, ET1, XFC, XFT
  410 FORMAT (//, 20X, 'MULTIMODULUS PROPERTIES:', //, 20X, 'E2C=',
     1E12.4, /, 20X, 'E1C=', E12.4, /, 20X, 'E1T=', E12.4, /, 20X,
     2'E2T=' , E12.4, /,22X,  'G=',E12.4,//,20X,'EC1 =',E12.4,/,20X,
     3'ET1 =', E12.4, /, 20X, 'EC2 =', E12.4, /, 20X, 'ET2 =', E12.4)
      CALL  COEFFI (A11 ,B11 ,B12 ,D11 )
      CALL  FEXSOL (A11 ,B11 ,B12 ,D11 )
      CALL  WRITES
      IF (SWH  .EQ.  1.) GO TO 109
      NBI = 2
      IF ( DIM .EQ. 2.)  S = SB
      CALL  ZNCOM(HBD,HBD,ZN , XE1C, XE1C, XE1T, XE1T )
      CALL  EXPSN ( A11, B11, B12, D11)
      CALL  COEFFI( A11, B11, B12, D11 )
      CALL  FEXSOL( A11, B11, B12, D11 )
      CALL  WRITES
  109 CONTINUE
      A = XA
      B = XB
      ZN = ZNB
      SCL = SCL + 1.
 110  CONTINUE
      A = XA
      B = XB
      ZN = ZNB
      NBC = NBC + 1
      SCL = 1.
```

```
100   CONTINUE
      STOP
      END
      SUBROUTINE  ZNCCM  (A, B, ZN ,E2C, E1C, E1T, E2T )
      COMMON /DISTAN/ X1(27), Y(27),XZN(27)
      COMMON / ABD / A1 ,B1 ,D1 ,CAN,CBN ,CBM ,CDM,XL,XM
      COMMON /NOE/ NE2, NE1, NE
      COMMON / DIMEN / BD,BL , S, QO,AP
      COMMON /BC / NBC, NBI, SCL, DIM ,SWH , EPS
      COMMON /XMODUL/ XE1C, XE1T
      COMMON  / FRASIN / C, T
      COMMON / STAIN / XFC, EC1, ET1, XFT
      COMMON /STIF/ A11, B11, B12, D11
281   CONTINUE
      CALL  STIFF ( A, B, E2C, E1C, E1T, E2T, ZN, BD)
      IF ( (NBC  .EQ.  2)  .AND.  (SCL  .EQ.  1.)  .AND. (SWH  .EQ.  0.)
     1  )  THEN DO
      CALL  STIFF ( B, A, E2T, E1T, E1C, E2C, ZN, BD)
      ENDIF
      IF (SWH  .NE. 0. )  GO TO 285
      A11 = A1 + CAN
      B11 = B1 + CBN
      B12 = B1 + CBM
      D11 = D1 + CDM
      GO TO 286
285   CONTINUE
      A11 = A1
      B11 = B1
      B12 = B1
      D11 = D1
286   CONTINUE
      ZN1 = B11/A11
      IF ( (NBI  .EQ.  2)  .OR.  (SWH  .NE.  0.) )  GO TO 289
      AA = C* ( BD/2. + ZN ) - ZN
      A = AA
      BB = T* (BD/2. - ZN ) + ZN
      B = BB
      IF  ((NBC  .EQ.  2)  .AND.  (SCL  .EQ.1.) )  THEN DO
      AA = C* ( BD/2. - ZN ) + ZN
      A = AA
      BB = T* (BD/2. + ZN ) - ZN
      B = BB
      ENDIF
289   CONTINUE
      IF  ( ABS(ZN - ZN1)  .LE.  EPS      )  GO TO 282
      ZN = ZN1
      GO TO 281
282   CONTINUE
      RETURN
      END
      SUBROUTINE  STIFF  ( A , B , E2C , E1C, E1T, E2T, ZN , BD)
      COMMON / ABD / A1 ,B1 ,D1 ,CAN,CBN ,CBM ,CDM,XL,XM
      COMMON /BC / NBC, NBI, SCL, DIM ,SWH , EPS
      COMMON /SANDWH/  ECF, ETF, ECC, ETC, GF, GC
      COMMON /STIF/ A11, B11, B12, D11
      IF    (SWH .EQ. 0) GO TO 802
      IF    (ZN  .GT.  B)  GO TO 804
      IF  ( (ZN  .LE.  B)  .AND.  (ZN  .GT.  -A) )  GO TO 802
      IF    (ZN  .LE. -A)  GO TO 800
804   CONTINUE
      IF ( (NBC  .EQ.  2)  .AND.  (SCL  .EQ.  1.)  .AND. (SWH  .EQ.  1.)
     1  )  GO TO 805
806   CONTINUE
      E2C = ECF
      E1C = ECC
      E1T = ECC
```

```
      E2T = ETP
      A1 = E2C* ( BD/2 + ZN - 2*A ) + 2*A*E1C + (BD/2 - ZN )*E2T
      B1 = .5* (E2T - E2C ) * ( ZN*ZN - BD*BD/4 )
      D1 = E2C*(BD*BD*BD/8 + ZN*ZN*ZN -2*A*A*A ) /3 + 2/3*A*A*A*E1C
     1 +E2T*( BD*BD*BD/8 - ZN*ZN*ZN )/3
      GO TO 803
  800 CONTINUE
      IF ( (NBC  .EQ.  2)  .AND.  (SCL  .EQ.  1.)  .AND. (SWH  .EQ.  1.)
     1  )  GO TO 806
 805  CONTINUE
      E2C = ETP
      E1C = ECP
      E1T = ECC
      E2T = ECP
      A1 = E2C* ( BD/2 - ZN - 2*A ) + 2*A*E1C + (BD/2 + ZN )*E2T
      B1 = .5* (E2T - E2C ) * ( ZN*ZN - BD*BD/4 )
      D1 = E2C*(BD*BD*BD/8 - ZN*ZN*ZN -2*A*A*A ) /3 + 2/3*A*A*A*E1C
     1 +E2T*( BD*BD*BD/8 + ZN*ZN*ZN )/3
      GO TO 803
  802 CONTINUE
      IF (SWH  .EQ.  1.)  THEN DO
      E2C = ECP
      E1C = ECC
      E1T = ETC
      E2T = ETP
      ENDIF
      IF ( (NBC  .EQ.  2)  .AND.  (SCL  .EQ.  1.)  .AND. (SWH  .EQ.  1.)
     1  )  THEN DO
      E2C = ETP
      E1C = ETC
      E1T = ECC
      E2T = ECP
      ENDIF
      A1 =  E2C* (BD/2. - A) + E1C* (ZN + A) + E1T* (B - ZN) + E2T* (BD
     1  /2. - B)
      B1 =  0.5* ( E2C*(A*A - BD*BD/4.) + E1C*(ZN*ZN - A*A) + E1T*(
     1  B*B - ZN*ZN) + E2T*(BD*BD/4. - B*B) )
      D1 =  1./3. * (E2C*(BD*BD*BD/8. - A*A*A) + E1C*(ZN*ZN*ZN + A*A*A)
     1 +E1T*(B*B*B - ZN*ZN*ZN) + E2T*(BD*BD*BD/8. - B*B*B) )
      CAN =  (E1C - E2C) * (BD/2. - A) + (E1T - E2T) *(BD/2. - B)
      CBN = -(E1C - E2C) * (BD/2. - A)*A+(E1T - E2T) *(BD/2. - B)*B
      CBM =  0.5* ( (E1C - E2C)* (A*A - BD*BD/4.) + (E1T - E2T)*(BD*BD
     1  /4. - B*B) )
      CDM =  0.5* (-(E1C - E2C)* (A*A - BD*BD/4.)*A+(E1T - E2T)*(BD*BD
     1  /4. - B*B)*B )
 803  CONTINUE
      RETURN
      END
      SUBROUTINE  EXPSN (A11, B11, B12, D11 )
      COMMON /BMODUL/ BE1C, BE1T
      COMMON /NOE/ NE2, NE1, NE
      COMMON / DIMEN / BD,BL , S, Q0,AP
      COMMON /BC / NBC, NBI, SCL, DIM ,SWH , EPS
      COMMON / ABD / A1 ,B1 ,D1 ,CAN,CBN ,CBM ,CDM,XL,XM
      COMMON /XMODUL/ XE1C, XE1T
      COMMON / AMODUL / E1C, E1T,E2C , E2T,G
      COMMON /UNK / A, B, ZN, ZNB,XA,XB
      COMMON /SOLU/ UP(27), VP(27), WP(27), UPX(27), VPX(27), WPX(27)
      DEN =  A11*D11 - B11*B12
      HBD = BD/2.
      X = 0.
      IF(SCL .EQ. 2.)GO TO 310
      DO 300  I = 1,NE2
      UP( I )  =  (B11  *Q0) /(6.*DEN) *X*X*X
      VP( I )  =  -(Q0/S)* X - (A11 *Q0) /(6*DEN)  *X*X*X
      WP( I )  =  (A11*Q0) /(24.*DEN) *X*X*X*X
```

```
      UPX(I )   =   (B11  *Q0) /(2.*DEN) *X*X
      VPX(I )   =   -(Q0/S)     - (A11 *Q0) /(2*DEN) *X*X
      WPX(I )   =   (A11*Q0) /(6.0*DEN) *X*X*X
      X = X + BL/NE
      IF ( I .EQ. 1)  X = BL/(2.*NE)
      IF(I .EQ. NE1)  X = BL
300   CONTINUE
      GO TO 320
310   CONTINUE
      X = 0.
      AL = 2.*3.141592654/BL
      DO 330  I = 1,NE2
      IF  ( ( (NBC  .EQ.  1)  .OR.  (NBC .EQ.  3) )  .AND.  (SWH .EQ.
     1  0.) .AND.  (X  .GT. BL/2))   THEN DO
      A = XA
      B = XB
      ZN = ZNB
      IF  (NBI  .EQ.  2)  GO TO 340
      IF (NBC  .EQ.  1)  XZ = 1.
      IF (NBC  .EQ.  3)  XZ = 3.
      NBC = 2
      SCL = 1.
      CALL  ZNCCM (A, B,ZN , E2C, E1C, E1T, E2T)
      SCL = 2.
      IF (XZ  .EQ.  1.)  NBC = 1
      IF (XZ  .EQ.  3.)  NBC = 3
      GO TO 350
 340  CONTINUE
      CALL  ZNCCM(HBD,HBD,ZN , XE1T, XE1T, XE1C, XE1C )
 350  CONTINUE
      ENDIF
      IF  ( ( (NBC  .EQ.  1)  .OR.  (NBC .EQ.  3) )  .AND.  (SWH .EQ.
     1  1.) .AND.  (X  .GT. BL/2))   THEN DO
      IF (NBC  .EQ.  1)  XZ = 1.
      IF (NBC  .EQ.  3)  XZ = 3.
      NBC = 2
      SCL = 1.
      CALL  ZNCCM (A, B,ZN , E2C, E1C, E1T, E2T)
      SCL = 2.
      IF (XZ  .EQ.  1.)  NBC = 1
      IF (XZ  .EQ.  3.)  NBC = 3
      ENDIF
      XM = 1.
      DEN =   A11*D11 - B11*B12
      UP( I )   =   (B11  *Q0) *COS(AL*X) / (AL*AL*AL*DEN)
      VP( I )   =-  (A11  *Q0) *COS(AL*X) / (AL*AL*AL*DEN)
      WP( I )   =   Q0*SIN(AL*X) / (AL*AL) *(1./S + A11/(AL*AL*DEN)   )
      UPX(I )   = -(B11  *Q0) *SIN(AL*X) / (AL*AL*AL*DEN)*AL
      VPX(I )   =   (A11  *Q0) *SIN(AL*X) / (AL*AL*AL*DEN)*AL
      WPX(I )   =   Q0*COS(AL*X) / (AL*AL) *(1./S + A11/(AL*AL*DEN)   )*AL
      X = X + BL/NE
      IF ( I .EQ. 1)  X = BL/(2.*NE)
      IF (I .EQ. NE1)  X = BL
330   CONTINUE
320   CONTINUE
      RETURN
      END
      SUBROUTINE  COEFFI (A11, B11, B12, D11)
      COMMON /XMOMNT/ XM1,XM2
      COMMON /NOE/ NE2, NE1, NE
      COMMON / DIMEN / BD,BL , S, Q0,AP
      COMMON /BC / NBC, NBI, SCL, DIM ,SWH , TPS
      COMMON  / COFI/ C1, C2, C3 ,C4 , DX1, DX2
      COMMON /SOLU/ UP(27), VP(27), WP(27), UPX(27), VPX(27), WPX(27)
      DEN =   A11*D11 - B11*B12
      IF (NBC .EQ. 2) GO TO 700
```

```
      IF ( NBC  .EQ. 3 ) GO TO 702
      C1 = -WP(1)
      C4 =  S*A11/( S*A11*BL*BL + 12*DEN) *( VP(1) + VP(NE2) - 2./BL
     1  *( WP(1) - WP(NE2)))
      C2 = -6.*DEN/(S*A11)*C4 + VP(1)
      C3 = 1./(2.*BL) * ( VP(NE2) - VP(1) - 3.*C4*BL*BL )
      DX1 = -UP(1)
      DX2 = 2.*B11*C3/A11 - ( UPX (NE2) + VPX (NE2)*B11/A11 )
      GO TO 701
  700 CONTINUE
      C1 =  -WP(1)
      DX1 = -UP(1)
      C2 = -WPX(NE2) - VP(NE2) + VP(1)
      C4 =-S*A11/(6.*DEN) * (C2 - VP(1) )
      DX2 = -1./DEN *( DEN * UPX(NE2) + S*B11*BL* ( WPX(NE2) + VP(NE2)))
      C3 =+1./(2.*DEN)*(DEN* VPX(NE2) - S*A11*BL* ( WPX(NE2) + VP(NE2)))
      GO TO 701
  702 CONTINUE
      C1 = -WP(1)
      DX1 = -UP(1)
      C4 = A11* ( B11* (UPX(NE2) - UPX(1) ) + D11* ( VPX(NE2) - VPX(1) )
     1  + XM1 - XM2 ) / (6.*BL*DEN)
      C3 = .5*VPX(NE2) - 3.*C4*BL- .5*XM2*A11/DEN
      C2 = -C3*BL - C4*BL*BL-(WP(NE2) - WP(1)) / BL
      DX2 = -6.*B11*C4*BL/A11 - UPX(NE2)  - XM2*B11/DEN
  701 CONTINUE
      RETURN
      END
      SUBROUTINE  PEXSOL (A11, B11, B12, D11)
      COMMON / DIMEN / BD,BL , S, Q0,AP
      COMMON /ALM/ AMX(27), QX(27),ANX(27)
      COMMON  / COFI/ C1, C2, C3 ,C4 , DX1, DX2
      COMMON /XSTIF/ XA11, XB11, XB12, XD11
      COMMON /NOE/ NE2, NE1, NE
      COMMON /SOLU/ UP(27), VP(27), WP(27), UPX(27), VPX(27), WPX(27)
      COMMON /ESOLU/ U(27), V(27), W(27), UX(27), VX(27), WX(27)
      DEN =  A11*D11 - B11*B12
      X = .0
      DO 100 I = 1,NE2
      U ( I ) = DX1 + DX2*X +3.*B11/A11*C4*X*X + UP(I)
      V ( I ) = -C2 - 6.*DEN/(S*A11)*C4 - 2.*C3*X - 3.*C4*X*X  + VP(I)
      W ( I) =  C1 + C2*X+ C3*X*X + C4*X*X*X + WP(I)
      UX (I) = DX2 + 6.*B11/A11       *C4*X + UPX(I)
      VX (I) = -2*C3 -6.*C4*X +VPX(I)
      WX (I) = C2 + 2.*C3*X + 3.*C4*X*X + WPX(I)
      AMX (I) = B12*UX(I) + D11*VX(I)
      QX (I) = S*( WX(I) + V(I) )
      ANX(I) = A11*UX(I) + B11*VX(I)
      X = X + BL/NE
      IF ( I .EQ. 1)  X = BL/(2.*NE)
      IF (I .EQ. NE1)  X = BL
  100 CONTINUE
      RETURN
      END
      SUBROUTINE  WRITES
      COMMON / AMODUL / E1C, E1T,E2C , E2T,G
      COMMON /DISTAN/ X1(27), Y(27),XZN(27)
      COMMON /XMODUL/ XE1C, XE1T
      COMMON /ANON/ P
      COMMON /ALM/ AMX(27), QX(27),ANX(27)
      COMMON /NOE/ NE2, NE1, NE
      COMMON / DIMEN / BD,BL , S, Q0,AP
      COMMON /BC / NBC, NBI, SCL, DIM ,SWH , EPS
      COMMON /PSOLU/ U(27), V(27), W(27), UX(27), VX(27), WX(27)
      COMMON /UMK / A, B, ZN, ZNB,XA,XB
      COMMON / STAIN / XPC, EC1, ET1, XPT
```

```
      IF (SWH  .EQ. 1. ) GO TO 590
      IF ( NBI .EQ. 2 ) GO TO 590
      PRINT 420 ,A, B
 420  FORMAT(//,20X,'A = ', F10.5, / , 20X , 'B = ' , F10.5 )
      IF (NBC .EQ. 2) GO TO 500
      IF (NBC  .EQ.  3)  GO TO 533
      IF (SCL .EQ. 2 )GO TO 511
      PRINT 510
 510  FORMAT ( 1H1, // , 50X , 'CLAMPED-CLAMPED.....UNIFORM LOAD' , / ,
     1  50X , '--------------------------------  ' , / , 60X ,
     2  ' MULTIMODULUS ' , ///)
      GO TO 515
 511  CONTINUE
      PRINT 512
 512  FORMAT ( 1H1, // , 50X , 'CLAMPED-CLAMPED........SINE LOAD' , / ,
     1  50X , '--------------------------------  ' , / , 60X ,
     2  ' MULTIMODULUS ' , ///)
      GO TO 515
 500  CONTINUE
      IF(SCL .EQ. 2 )GO TO 521
      PRINT 520
 520  FORMAT ( 1H1, // , 50X , 'CLAMPED-FREE........UNIFORM LOAD' , / ,
     1  50X , '--------------------------------  ' , / , 60X ,
     2  ' MULTIMODULUS ' , ///)
      GO TO 515
 521  CONTINUE
      PRINT 522
 522  FORMAT ( 1H1, // , 50X , 'CLAMPED-FREE........   SINE LOAD' , / ,
     1  50X , '--------------------------------  ' , / , 60X ,
     2  ' MULTIMODULUS ' , ///)
      GO TO 515
 533  CONTINUE
      IF (SCL .EQ. 2 )GO TO 536
      PRINT 537
 537  FORMAT ( 1H1, // , 50X , 'HINGED-HINGED.......UNIFORM LOAD' , / ,
     1  50X , '--------------------------------  ' , / , 60X ,
     2  ' MULTIMODULUS ' , ///)
      GO TO 515
 536  CONTINUE
      PRINT 538
 538  FORMAT ( 1H1, // , 50X , 'HINGED-HINGED.......   SINE LOAD' , / ,
     1  50X , '--------------------------------  ' , / , 60X ,
     2  ' MULTIMODULUS ' , ///)
 515  CONTINUE
      CALL  WRITE1
      IF ( NBI .EQ. 1) GO TO 540
 590  CONTINUE
      IF ( NBC .EQ. 2) GO TO 550
      IF ( NBC .EQ. 3) GO TO 530
      IF (SCL .EQ. 2 )GO TO 592
      PRINT 560
 560  FORMAT ( 1H1, // , 50X , 'CLAMPED-CLAMPED.....UNIFORM LOAD' , / ,
     1  50X , '--------------------------------  ' , / , 60X ,
     2       'BIMODULUS ' , ///)
      GO TO 591
 592  CONTINUE
      PRINT 593
 593  FORMAT ( 1H1, // , 50X , 'CLAMPED-CLAMPED........SINE LOAD' , / ,
     1  50X , '--------------------------------  ' , / , 60X ,
     2       'BIMODULUS ' , ///)
      GO TO 591
 550  CONTINUE
      IF (SCL .EQ. 2 )GO TO 571
      PRINT 570
 570  FORMAT ( 1H1, // , 50X , 'CLAMPED-FREE........UNIFORM LOAD' , / ,
     1  50X , '--------------------------------  ' , / , 60X ,
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```
     2       'BIMODULUS ' , ///)
      GO TO 591
 571  CONTINUE
      PRINT 572
 572  FORMAT ( 1H1, // , 50X , 'CLAMPED-FREE...........SINE LOAD' , / ,
     1 50X , '-------------------------------- ' , / , 60X ,
     2       'BIMODULUS ' , ///)
      GO TO 591
 530  CONTINUE
      IF (SCL .EQ. 2 )GO TO 531
      PRINT 532
 532  FORMAT ( 1H1, // , 50X , 'HINGED-HINGED.......UNIFORM LOAD' , / ,
     1  50X , '--------------------------------  ' , / , 60X ,
     2       'BIMODULUS ' , ///)
      GO TO 591
 531  CONTINUE
      PRINT  534
 534  FORMAT ( 1H1, // , 50X , 'HINGED-HINGED..........SINE LOAD' , / ,
     1  50X , '--------------------------------  ' , / , 60X ,
     2       'BIMODULUS ' , ///)
 591  CONTINUE
      CALL WRITE1
 540  CONTINUE
      RETURN
      END
      SUBROUTINE  WRITE1
      COMMON /DISTAN/ X1(27), Y(27),XZN(27)
      COMMON /UNK / A, B, ZN, ZNB,XA,XB
      COMMON /ALM/ AMX(27), QX(27),ANX(27)
      COMMON / DIMEN / BD,BL , S, Q0,AF
      COMMON /BC / NBC, NBI, SCL, DIM ,SWH , EPS
      COMMON /NOE/ NE2, NE1, NE
      COMMON /FSOLU/ U(27), V(27), W(27), UX(27), VX(27), WX(27)
      IF(DIM .EQ. 2) GO TO 630
      PRINT 600
 600  FORMAT( 7X  , ' X/L ' , 11X , ' U ' , 14X , ' W ' , 14X , ' V ' ,
     1  15X , 'M ' , 12X , ' Q ',12X, ' N '  , 12X, ' ZN '          )
      GO TO 640
 630  CONTINUE
      PRINT 650
 650  FORMAT (  7X , ' X/L ' , 10X , 'U/L' , 10X , 'W/L' , 12X , ' V ' ,
     1  9X , ' M/Q0*L*L ' , 5X , ' Q/Q0*L  ' , 9X , 'ZN/L ' )
 640  CONTINUE
      Y ( 1 ) = 0.
      DO  610  I = 1,NE2
      X1 ( I ) = Y(I) / BL
      XZN(I) = ZN
      IF  (  ( (NBC  .EQ.1  )  .OR.  (NBC  .EQ.  3) )  .AND.  ((NBI  .EQ
     1 .  1)  .OR.  (NBI  .EQ.  2))  ) THEN DO
      IF ( ( X1(I)  .LE.  .50)  .AND.  (SCL  .EQ. 2.)  .AND.  (SWH .EQ.
     1  0.) )  XZN(I) = -ZN
      IF ( ( X1(I)  .LE.  .50)  .AND.  (SCL  .EQ. 2.)  .AND.  (SWH .EQ.
     1  1.) )  XZN(I) = -ZN
      ENDIF
      PRINT 620, X1(I), U(I), W(I), V(I), AMX(I), QX(I),  ANX(I),XZN(I)
 620  FORMAT  ( / ,  3X , F10.5 , 5X , E12.4 ,5X , E12.4 ,5X , E12.4 ,
     1  5X , F10.5 , 5X ,F10.5 ,5X , F10.5 , 5X, F10.5 )
      IF ( I .EQ. NE2 ) GO TO 610
      Y (I + 1 ) = Y (I ) + BL/NE
      IF( I .EQ. 1 )  Y(2) = BL/(2.*NE)
      IF ( I .EQ. NE1)  Y(NE2) = BL
 610  CONTINUE
      RETURN
      END
```

```
C     MAIN PROGRAM
C     LODTYP= 1:UNIFORM LOAD----LODTYP=2:SINUSOIDAL LOAD
C     LODTYP= 3...COSINE LOAD
C     NBC = 1 : CLAMPED-FREE
C     NBC = 1 : CLAMPED-CLAMPED
C     IF NBC=4 ..HINGED-HINGED WITH APPIED END MOMENT
      REAL A,B,D,S,XX1,XX2,ZAVG,SAI
      COMMON/TIME/XMN1(25),XMN2(25),XKONQ(25),XUNMA(25),XWNMA(25),XPNMA(
     125),X1,X5
      COMMON/UDAV/UDISN(25),UACN(25),UVELN(25)
      COMMON/WDAV/WDISN(25),WACN(25),WVELN(25)
      COMMON/PDAV/PDISN(25),PACN(25),PVELN(25)
      COMMON/SMAT/SM(7,7)
      COMMON /PPPTY/E2C,E1C,E1T,E2T,G,XPC,EC1,ET1,XPT
      COMMON/PMAT/PM(7,7)
      COMMON/AVE/SV(364)
      COMMON /ZNVEC/ZA(52),ZC(364),XCA(52),XCC(364),XTA(52),XTC(364)
      COMMON /STIPST/ASTO(52),BSTO1(52),BSTO2(52),DSTO(52),SSTO(52)
      COMMON/COP/WM(7,7),TM(364,7)
      COMMON/XSIGN/SIGN1(52),SIGN2(52)
      DIMENSION BDAPY(4),BMINV2(4),SLOE(27)
      DIMENSIONBMN(7,7),WSV1(7),UX(52),PSAIX(52)
      DIMENSION PM(7,7),WSV(7,7),ENPM(7,7),DELZ(52)
      DIMENSION BMINV(4,4),WAREA(28),BM1(4,4),RCMAT(7,7)
      DIMENSION XDELT(50)
      DATA BT,BL,ALOAD,QNO/0.6,8.00,0.0,0.1/
      XNSTAT = 25.
      NTSTEP = 1
      READ(5,200)  (XDELT(I),I=1,NTSTEP)
      XMN=0.
      YMN=0.
      XON=0.
      XM1=-.1
      NTTAL=0.
      NTTAL=1
      NSCAN=1
      NSCAN=0
      MNOTRI= 70
      MNOTRI=0
      MNOTRI= 7000
      NPERT=0
      NPERT=1
      ALFA= 2.*(3.1415927/BL)
      DO 2 I=1,25
      UDISN(I)= 0.
      UACN(I)= 0.
      UVELN(I)= 0.
      WDISN(I)= 0.
      WACN(I)= 0.
      WVELN(I)= 0.
      PDISN(I)= 0.
      PACN(I)= 0.
      PVELN(I)= 0.
    2 CONTINUE
      READ(5,212)  E2C,E1C,E1T,E2T,G
      READ(5,212)  XPC,EC1,ET1,XPT
      READ(5,220) NSTAT,NBC,LOADP
      NXLOE= NSTAT+1
      KNSTAT = NSTAT+1
      FIELD= BL/XNSTAT
      HFIELD= FIELD/2.
      DO 4 I=1,KNSTAT
      SLOE(I)= FIELD
      IF(I.EQ.1.OR.I.EQ.KNSTAT) SLOE(I)= HFIELD
```

```
    4 CONTINUE
      LODTYP=3
      LODTYP=2
      LODTYP=1
      RO=0.0
      RO=0.037*0.32
      RMAS=0.
      RMAS=RO*BT/(12.0*32.20)
      XIMAS=(RO*(BT**3)/12.)/(12.*32.20)
      NOTM=(NSTAT*2)+1
      NOSV=NOTM+1
      READ(5,200)  (ZA(I),I=1,NOSV)
      READ(5,200)   (XCA(I),I=1,NOSV)
      READ(5,200)   (XTA(I),I=1,NOSV)
      AK=5./6.
      NZA=1
      NOTRI=1
      II=1
      N=1
      NN=NSTAT+1
      IZ=1
      NSYM=(NSTAT/2)+1
      ICNCT=-1
      DO 31 I=1,NOSV
      SIGN2(I)=0.
      SIGN1(I)=1.
   31 CONTINUE
C     COMPUTE TRANSFER MATRICES
      L1=7
      M1=7
      N1=7
      IF=7
      IW=7
      IP=7
      XTIME=0.
      DO 188 KD=1,NTSTEP
      INCR=7
      DELT=XDELT(KD)
      DELT=1.
   10 CONTINUE
      XX1=0.
      XX2=0.
      NST= 1
   12 CONTINUE
      XLOE=SLOF(N)
      CALL STIFF(NZA,AK,BT,A11,B11,B12,D11,S,ZAVG,XCAVG,XTAVG,SAI)
      IF(N.GT.1) GO TO 30
      CALL FFMAT(XLOE,A11,B11,B12,D11,S,QNO,XX1,XX2,ALFA,LODTYP)
      CALL COPY(II,PM)
      GO TO 40
   30 CALL FEMAT(XLOE,A11,B11,B12,D11,S,QNO,XX1,XX2,ALFA,LODTYP)
      CALL VMULFF(FM,WM,L1,M1,N1,IF,IW,PM,IP,IER)
      CALL COPY(II,PM)
      IF(NN.GT.NSTAT.AND.N.EQ.NN) GOTO 20
   40 XLOAD=0.
      IF(N.EQ.LOADP) XLOAD=XLOAD+ALOAD
      CALL STMAT(XLOAD,PMAS,RMAS,XIMAS,NST,DELT)
      CALL VMULFF(SM,WM,L1,M1,N1,IP,IW,PM,IP,IER)
      CALL COPY(II,PM)
      IF (NN.EQ.NSTAT.AND.N.EQ.NN) GO TO 20
      N=N+1
      GO TO 12
   20 CONTINUE
C     IMPOSE BOUNDARY CONDITION
      IF(NBC.EQ.1) GO TO 50
      IF(NBC.EQ.2) GO TO 60
```

```
      IF(NBC.EQ.3) GO TO 916
      IF(NBC.EQ.4) GO TO 900
      IF(NBC.EQ.5) GO TO 900
      IF(NBC.EQ.6) GO TO 906
 50   CONTINUE
      DO 70 I1=1,4
      I2=I1+3
      DO 70 J1=1,4
      J2=J1+3
 70   BM1(I1,J1) =PM(I2,J2)
      DO 90 I4=1,3
      DO 90 I5=1,4
      I6=I5+3
 90   BMN(I4,I5)=PM(I4,I6)
      IF(XMN.NE.0.0.OR.XQN.NE.0.0.OR.XNN.NE.0.0) GO TO 54
      XLOE=SLOE(1)
      XLOE=XLOE-0.001
      CALL FEMAT(XLOE,A11,B11,B12,D11,S,QNO,XX1,XX2,ALFA,LODTYP)
      MT1=(NOSV-2)*7
      DO 52 IE=1,7
      MT2=MT1+1
      DO 52 JE=1,7
 52   WM(IE,JE)= TM(MT2,JE)
      CALL VMULFF(FM,WM,L1,M1,N1,IF,IW,BNPM,IP,IER)
      GO TO 760
 54   CONTINUE
      BDARY(1)=XMN
      BDARY(2)=XQN
      BDARY(3)=XNN
      BDARY(4)=1.
      GO TO 760
 60   CONTINUE
      DO 62 M=1,3
      DO 62 N5=1,4
      N6=N5+3
 62   BM1(M,N5)=PM(M,N6)
      DO 63 NNX=1,4
      N7= NNX+3
 63   BM1(4,NNX)= PM(7,N7)
      DO 66 N8=1,3
      NP8=N8+3
      DO 66 N9=1,4
      NP9=N9+3
 66   BMN(N8,N9)= PM(NP8,NP9)
      GO TO 760
906   CONTINUE
      DO 907 I=1,4
      I907=I+1
      IF(I.GE.3) I907=I+3
      DO 907 J=1,4
      J907=J+3
907   BM1(I,J) =PM(I907,J907)
      DO 908 I=1,3
      I908=I
      IF(I.GT.1) I908=I+2
      DO 908 J=1,4
      J908=J+3
908   BMN(I,J)=PM(I908,J908)
      GO TO 760
916   CONTINUE
      DO 917 I=1,4
      I917=I*2
      IF(I.EQ.4) I917=I+3
      DO 917 J=1,4
      J917=J+2
      IF(J.GT.1) J917=J+3
```

```
917   BM1(I,J)=PM(I917,J917)
      DO 918 I=1,3
      I918=2*(I-1)+1
      DO 918 J=1,4
      J918=J+2
      IF(I.GT.1) J918=J+3
918   BMN(I,J)=PM(I918,J918)
      GO TO 760
 800  CONTINUE
      DO 810 I=1,4
      I810 =I
      IF(I.EQ.3) I810= I+1
      IF(I.EQ.4) I810= I+3
      DO 810 J=1,4
      IF(J.EQ.1) J810= J+2
      IF(J.GE.2) J810= J+3
 810  BM1(I,J)=PM(I810,J810)
      DO 820 I=1,3
      IF(I.EQ.1) I820= I+2
      IF(I.GE.2) I820= I+3
      DO 820 J=1,5
      J820 = J+2
 820  BMN(I,J)= PM(I820,J820)
      BDARY(1)= -XM1*PM(1,4)
      BDARY(2)= -XM1*PM(2,4)
      BDARY(3)= XMN-XM1*PM(4,4)
      BDARY(4)= 1-XM1*PM(7,4)
      GO TO 760
 900  CONTINUE
      DO 910 I=1,4
      I910= I
      IF(I.EQ.2) I910= I+1
      IF(I.EQ.3) I910= I+2
      IF(I.EQ.4) I910= I+3
      DO 910 J=1,4
      IF(J.EQ.1) J910= J+1
      IF(J.EQ.2) J910= J+2
      IF(J.GE.3) J910= J+3
 910  BM1(I,J)= PM(I910,J910)
      DO 920 I=1,3
      IF(I.EQ.1) I920 = I+1
      IF(I.EQ.2) I920 = I+2
      IF(I.EQ.3) I920 = I+3
      DO 920 J=1,4
      IF(J.EQ.1) J920= J+1
      IF(J.EQ.2) J920= J+2
      IF(J.GE.3) J920= J+3
 920  BMN(I,J)= PM(I920,J920)
 760  CONTINUE
      NB=4
      IB=4
      IDGT=3
      CALL LINV2F(BM1,NB,IB,BMINV,IDGT,WAREA,IER)
      IF(NBC.NE.4.AND.NBC.NE.1) GO TO 840
      IF(XMN.EQ.0.0.AND.XQN.EQ.0.0.AND.XNN.EQ.0.0) GO TO 840
      IDGT=1
      CALL VMULFF(BMINV,BDARY,NB,IB,IDGT,NB,NB,BMINV2,NB,IER)
      IF(NBC.EQ.1) GO TO 56
      WSV(2,1)= XM1
      DO 830 I=1,4
      I830=I
      IF(I.GE.2) I830=I+1
 830  WSV(I830,1)= BMINV2(I)
      L2=3
      M2=5
      N2=1
```

```
      GO TO 841
   56 CONTINUE
      DO 58 I=1,4
   58 WSV(I,1)=BMINV2(I)
      L2=3
      M2=4
      N2=1
      GO TO 841
  840 CONTINUE
C     LOOP 80 IS USED FOR NBC=1,2,3,5,6 WHICH ARE HOMOGENEOUS BC
      DO 80 I3=1,4
   80 WSV(I3,1)=BMINV(I3,4)
      L2=3
      M2=4
      N2=1
  841 CONTINUE
      CALL VMULFF(BMN,WSV,L2,M2,N2,IP,IW,PM,IP,IER)
      NOESV=NOSV*7
      NDUM1=NOESV-7
      NDUM2=NDUM1+3
      NDUM3=NDUM2+2
      NDUM4=NDUM1+4
      NDUM5= NDUM1+2
      NDUM6= NDUM1+6
      NDUM0=NDUM1+1
      MX=1
      NX=1
C     LOOP 100 TAKE CARE OF STATE VECTORS AT BOUNDARIES
      DO 100 I8=1,NOESV
      IF(NBC.EQ.3) GO TO 98
      IF(NBC.EQ.4) GO TO 850
      IF(NBC.EQ.5) GO TO 930
      IF(NBC.EQ.6) GO TO 123
      IF(I8.LE.7.AND.I8.GE.4) GO TO 109
      IF(NBC.EQ.1) GO TO 92
      IF(NBC.EQ.2) GO TO 94
   92 CONTINUE
      IF(I8.GT.NDUM1.AND.I8.LE.NDUM2) GO TO 110
      IF(I8.GT.NDUM2) GO TO 111
      GO TO 96
   94 CONTINUE
      IF(I8.GT.NDUM2) GO TO 110
   96 CONTINUE
      SV(I8)=0.
      GO TO 100
  109 SV(I8)=WSV(MX,1)
      MX=MX+1
      GO TO 100
  110 SV(I8)= PM(NX,1)
      NX=NX+1
      GO TO 100
  111 CONTINUE
      SV(NDUM2+1)=XMN
      SV(NDUM2+2) = XQN
      SV(NDUM2+3) = XNN
      SV(NDUM2+4) = 1.
      GO TO 100
   98 CONTINUE
      SV(I8)=0.
      IF(I8.EQ.3) SV(I8)= WSV(1,1)
      IF(I8.GE.5.AND.I8.LE.7) SV(I8)= WSV(I8-3,1)
      IF(I8.EQ.NDUM0)   SV(I8)=PM(1,1)
      IF(I8.EQ.NDUM2)   SV(I8)=PM(2,1)
      IF(I8.EQ.NDUM3)   SV(I8)=PM(3,1)
      IF(I8.GT.NDUM6)   SV(I8)=1.
      GO TO 100
```

```
 123  CONTINUE
      SV(I8)=0.
      IF(I8.GE.4.AND.I8.LE.7)   SV(I8)=WSV(I8-3,1)
      IF(I8.EQ.NDUM0)   SV(I8)=PM(1,1)
      IF(I8.EQ.NDUM4)   SV(I8)=PM(2,1)
      IF(I8.EQ.NDUM3)   SV(I8)=PM(3,1)
      IF(I8.GT.NDUM6)   SV(I8)=1.
      GO TO 100
 850  CONTINUE
      SV(I8)= 0.
      IF(I8.GE.3.AND.I8.LE.7) SV(I8)=WSV(I8-2,1)
      IF(I8.EQ.NDUM2) SV(I8)= PM(1,1)
      IF(I8.GE.NDUM3) SV(I8)= PM(I8-NDUM2,1)
      IF(I8.EQ.NDUM4) SV(I8)= XMN
      GO TO 100
 930  CONTINUE
      SV(I8) = 0.
      IF(I8.EQ.2) SV(I8) = WSV(1,1)
      IF(I8.EQ.4) SV(I8)= WSV(2,1)
      IF(I8.EQ.6) SV(I8)= WSV(3,1)
      IF(I8.EQ.7) SV(I8)= WSV(4,1)
      IF(I8.EQ.NDUM5) SV(I8)= PM(1,1)
      IF(I8.EQ.NDUM4) SV(I8)= PM(2,1)
      IF(I8.EQ.NDUM6) SV(I8)= PM(3,1)
      IF(I8.GT.NDUM6) SV(I8)=1.
 100  CONTINUE
C     COMPUTE FORCE AND DISPLACEMENT
      DO 120 I9=1,7
 120  WSV1(I9)=SV(I9)
      JJ=0
      DO 131 LP=1,NOSV
C     RESET A,B,D,TO COMPUTE UX,PSAIX,HENCE ZN
      A11=ASTO (LP)
      B11=BSTO1(LP)
      B12=BSTO2(LP)
      D11=DSTO (LP)
      LOESV=(LP-1)*7
      IF(LP.EQ.1) GO TO 150
      IF(LP.EQ.NOSV.AND.NBC.GE.2) GO TO 150
      IF(LP.EQ.NOSV.AND.NBC.EQ.1) GO TO 153
      DO 140 LR=1,7
      JJ=JJ+1
      DO 140 LC=1,7
 140  RCMAT(LR,LC)=TM(JJ,LC)
      N3=1
      CALL VMULFF(RCMAT,WSV1,L1,M1,N3,IF,IW,PM,IP,IER)
      DO 141 K=1,7
      LOESV=LOESV+1
 141  SV(LOESV)=PM(K,1)
      GO TO 151
C     COMPUTE NETRAL SURFACE LOCATION
 150  MLSV=LOESV+4
      NLSV=LOESV+6
      GO TO 152
 151  MLSV=LOESV-3
      NLSV=LOESV-1
 152  UX(LP)=-(B11*SV(MLSV)-D11*SV(NLSV))
      PSAIX(LP)=(A11*SV(MLSV)-B12*SV(NLSV))
      GO TO 130
 153  CONTINUE
      IF(XMN.NE.0.0.OR.XQN.NE.0.0.OR.XNN.NE.0.0) GO TO 150
      CALL VMULFF(RNPM,WSV1,L1,M1,N3,IF,IW,PM,IP,IER)
      UX(LP)=+(B11*PM(4,1) -D11*PM(6,1) )
      PSAIX(LP)=-(A11*PM(1,4)-B12*PM(6,1))
 130  CONTINUE
      IF(NBC.EQ.3) GO TO 137
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      ZC(LP)=-UX(LP)/PSAIX(LP)
      GO TO 138
 137  ZC(LP)= B11*ALFA*ALFA/(A11*ALFA*ALFA - PMAS*OMEGA*OMEGA)
 138  CONTINUE
      HT=HT/2.
      IF(SIGN1(LP).LT.0.) GO TO 132
      XTC(LP)= ET1/XPT*(HT-ZC(LP))+ZC(LP)
      XCC(LP)=-EC1/XPC*(HT+ZC(LP))+ZC(LP)
      GO TO 131
 132  CONTINUE
      XTC(LP)=-ET1/XPT*(HT+ZC(LP))+ZC(LP)
      XCC(LP)= EC1/XPC*(HT-ZC(LP))+ZC(LP)
 131  CONTINUE
C     CONVERGENCE CRITERIA
      ERR=0.
      DO 160 I10=1,NOSV
  160 ERR=ERR+((ZC(I10)-ZA(I10))/ZC(I10))**2
      RMS=SQRT(ERR)
      DO 191 I=1,NOSV
  191 DELZ(I)=ZC(I)-ZA(I)
      NZA=1
      II=1
      N=1
      CALL SIGN(NOSV,STEST)
      IF(STEST.GT.0.001.OR.STEST.LT.-0.001) GO TO 193
      IF(RMS.LE.0.005) GO TO 180
  193 CONTINUE
      IF(NOTRI.GT.MNOTRI) GO TO 188
      IF(NSCAN.EQ.1.AND.NTTAL.EQ.0) GO TO 170
      IF(NTTAL.EQ.1) GO TO 172
      IF(MNOTRI.EQ.1) GO TO 170
      PRINT 280,RMS,NOTRI
      PRINT 211,(ZA(IX),DELZ(IX),IX=1,NOSV)
      CALL ZNSYM(ZA,ZC,IZ,NOSV,IZNCT)
      CALL ZNSYM(XCA,XCC,IZ,NOSV,IZNCT)
      CALL ZNSYM(XTA,XTC,IZ,NOSV,IZNCT)
      IZ=IZ+1
      IF(IZ.GT.NSYM) GO TO 170
      GO TO 192
  172 CONTINUE
      PRINT 280,RMS,NOTRI
      DO 174 I11=1,NOSV
      XCA(I11)=XCC(I11)
      XTA(I11)=XTC(I11)
  174 ZA(I11)=ZC(I11)
  192 NOTRI=NOTRI+1
      GO TO 10
  170 CONTINUE
      IZ=1
      IZNCT=-1
      PRINT 280,RMS,NOTRI,XTIME,SV(177),QNO,ZA(25),XCA(25),XTA(25)
      PRINT 211,(ZA(IX),DELZ(IX),XCA(IX),XTA(IX),IX=1,NOSV)
      IF(NSCAN.EQ.3) GO TO 171
      READ(5,200) (ZA(I),ZA(NOSV-I+1),I=6,7)
      READ(5,200)  (XCA(I),XCA(NOSV-I+1),I=6,7)
      READ(5,200)  (XTA(I),XTA(NOSV-I+1),I=6,7)
  211 FORMAT(/,1X,E12.4,12X,E12.4,12X,E12.4,12X,E12.4)
  171 NOTRI=NOTRI+1
      GO TO 10
  180 CONTINUE
      XTIME=XTIME+DELT
      PRINT 280,RMS,NOTRI,XTIME,SV(177),QNO,ZA(25),XCA(25),XTA(25)
      NOTRI=NOTRI+1
  188 CONTINUE
      NOTRI=NOTRI-1
      PRINT 241
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      PRINT 242,BL,BT
      PRINT 291,NBC,RO,XTIME
  291 FORMAT(/,5X,'NBC=',I5,5X,'RO=',F8.5,5X,'TIME=',F10.7,/)
      PRINT 293,XM1,XMN,XQN,XNN
  293 FORMAT(/,1X,'M1=',F5.2,2X,'MN=',F5.2,2X,'Q=',F5.2,2X,'N=',F5.2,/)
      PRINT 292,LODTYP,ALFA
  292 FORMAT(/,5X,'LODTYP=',I5,10X,'ALFA=',F10.5,/)
      IF(NPC.NE.4) GO TO 191
  191 PRINT 247,QNO
      PRINT 249,NSTAT
      PRINT 250
      PRINT 300,(SLOE(I),I=1,NXLOE)
      PRINT 240
      PRINT 248
      IF(NPERT.EQ.1) GO TO 400
      PRINT 300, (SV(KX),KX=1,NOESV)
      GO TO 440
  400 NEIV=7
      NCAOT=((NOSV-2)/2)+2
      CALL AVERAG(NCAOT,NEIV,SV)
      NPRIN=NCAOT*NEIV
      PRINT 300,(SV(I),I=1,NPRIN)
  440 CONTINUE
      IF(NPERT.EQ.1) GO TO 420
      PRINT 260
      PRINT 211,(ZA(IX),DELZ(IX),XCA(IX),XTA(IX),IX=1,NOSV)
      GO TO 460
  420 NEIV=1
      PRINT 211,(ZA(IX),DELZ(IX),XCA(IX),XTA(IX),IX=1,NOSV)
      PRINT 260
      CALL AVERAG(NCAOT,NEIV,ZC)
      NPRIN=NCAOT*NEIV
      PRINT 210, (SV(I),I=1,NPRIN)
  460 CONTINUE
      PRINT 280,RMS,NOTRI
  241 FORMAT('1',1X,'MATERIAL IS ARAMID RUBBER')
  242 FORMAT(/,1X,'BEAM LENGTH=',F6.3,5X,'BEAM THICKNESS=',1X,F5.3)
  247 FORMAT(/,1X,'LOAD TYPE.- DISTRIBUTED LOAD P(X)=',F10.3)
  248 FORMAT(/,5X,'U',13X,'W',13X,'PSAI',12X,'M',13X,'Q',13X,'N')
  249 FORMAT(/,1X,'NUMBER OF ELEMENTS=',1X,I5)
  240 FORMAT(/,1X,'STATE VECTORS U,W,PSAI,M,Q,N')
  260 FORMAT(/,5X,'N-S LOCATIONS')
  220 FORMAT(14I5)
  210 FORMAT(/,1X,E11.4)
  200 FORMAT(F12.4,6F10.4)
  300 FORMAT(/,7(1X,E11.4,3X),/)
  250 FORMAT(/,1X,'DISTANCES BETWEEN STATIONS')
  280 FORMAT(////,5X,'RMS=',3X,F10.5,5X,'N.O.T=',3X,I5,F10.5,3X,F10.6,
     13X,4(E11.4,3X),/)
  212 FORMAT(6E15.10)
      STOP
      END
      SUBROUTINE STIFF(NZA,AK,BT,A11,B11,B12,D11,S,ZAVG,XCAVG,XTAVG,SAI)
      COMMON /PRPTY/E2C,E1C,E1T,E2T,G,XPC,EC1,ET1,XPT
      COMMON /INVEC/ZA(52),ZC(364),XCA(52),XCC(364),XTA(52),XTC(364)
      COMMON /STIFST/ASTO(52),BSTO1(52),BSTO2(52),DSTO(52),SSTO(52)
      COMMON/XSIGN/SIGN1(52),SIGN2(52)
      ZTTAL=0.0
      XCTAL=0.0
      XTTAL=0.0
      SAI=0.0
      DO 10 I=1,2
      SAI=SAI+SIGN1(NZA)
      Z= ZA(NZA)/BT
      XC=XCA(NZA)/BT
      XT=XTA(NZA)/BT
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      ZTTAL=ZTTAL+Z
      XCTAL=XCTAL+XC
      XTTAL=XTTAL+XC
      SSTO(NZA)=G*BT*AK
      S=SSTO(NZA)
      IF(SIGN1(NZA).LT.0.0)GO TO 20
      CALL  STIFFC(Z,XC,XT,E2C,E1C,E1T,E2T,    ASTO(NZA),BSTO1(NZA),
     1BSTO2(NZA),DSTO(NZA),BT)
      GO TO 10
20    CONTINUE
      CALL  STIFFC(Z,XT,XC,E2T,E1T,E1C,E2C,    ASTO(NZA),BSTO1(NZA),
     1BSTO2(NZA),DSTO(NZA),BT)
10    NZA=NZA+1
      ZAVG= ZTTAL/2.
      XCAVG=XCTAL/2.
      XTAVG=XTTAL/2.
      IF(SAI.GE.0.9) GO TO 30
      IF(SAI.GE.-1.) GO TO 40
30    CONTINUE
      CALL  STIFFC(Z,XC,XT,E2C,E1C,E1T,E2T,    A11,B11,B12,D11,BT)
      GO TO 50
40    CONTINUE
      CALL  STIFFC(Z,XT,XC,E2T,E1T,E1C,E2C,    A11,B11,B12,D11,BT)
50    CONTINUE
      RETURN
      END
      SUBROUTINE  STIFFC(Z,XC,XT,E2C,E1C,E1T,E2T,A11,B11,B12,D11,BT)
      IF(XC .LT. -.5 .AND. Z .LT. -.5 .AND. XT .LE. -.5) GO TO 30
      IF(XC .LT. -.5 .AND. Z .LE. -.5 .AND. XT .GT. -.5) GO TO 40
      IF(XC .LE. -.5 .AND. Z .GT. -.5 .AND. XT .LT. +.5) GO TO 50
      IF(XC .GT. -.5 .AND. Z .GT. -.5 .AND. XT .LT. +.5) GO TO 60
      IF(XC .GT. -.5 .AND. Z .LT. +.5 .AND. XT .GE. +.5) GO TO 70
      IF(XC .LT. +.5 .AND. Z .GE. +.5 .AND. XT .GT. +.5) GO TO 80
      IF(XC .GE. +.5 .AND. Z .GT. +.5 .AND. XT .GT. +.5) GO TO 90
      IF(XC .LE. -.5 .AND. Z .GT. -.5 .AND. XT .GE. +.5) GO TO 100
30    CONTINUE
      A11        =BT*E1T
      B11        =BT**2*XT*(E1T-E2T)
      B12        =0.
      D11        =BT**3*E2T/24.
      GO TO 10
40    CONTINUE
      A11        =BT*E1T
      B11        =BT**2/2.*(1./2.-XT)**2*(E2T-E1T)
      B12        =0.
      D11        =BT**3/3.*((E1T+E2T)/8.+XT**3/2.*(E2T-E1T))+BT**3/8.*XT
     1 *(E1T-E2T)
      GO TO 10
50    CONTINUE
      A11        =BT*((E1C+E1T)/2.+Z*(E1C-E1T))
      B11        =BT**2/2.*((E2T-E1C)/4.+XT*(1.-XT)*(E1T-E2T)    +Z*Z*(E1C
     1 -E1T))
      B12        =BT**2/2.*(1./4.-Z*Z)*(E1T-E1C)
      D11        =BT**3/3.*((E1C+E2T)/8.+XT**3/2.*(E2T-E1T)+Z**3*(E1C-E1T
     1 ))+BT**3/8.*XT*(E1T-E2T)
      GO TO 10
60    CONTINUE
      A11        =BT*((E1C+E1T)/2.+Z*(E1C-E1T))
      B11        =BT**2/2.*((E2T-E2C)/4.+XT*(1.-XT)*(E1T-E2T)    +Z*Z*(E1C
     1 -E1T)+XC*(1.+XC)*(E1C-E2C))
      B12        =BT**2/2.*(1./4.-Z*Z)*(E1T-E1C)
      D11        =BT**3/3.*((E2C+E2T)/8.+XT**3/2.*(E2T-E1T)+Z**3*(E1C-E1T
     1 )+XC**3*(E1C-E2C)/2)+BT**3/8.*(XT*(E1T-E2T)-XC*(E1C-E2C))
      GO TO 10
70    CONTINUE
      A11        =BT*((E1C+E1T)/2.+Z*(E1C-E1T))
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      B11        =BT**2/2.*((E1T-E2C)/4.+XC*(1.+XC)*(E1C -E2C)+Z*Z*(E1C
     1 -E1T))
      B12        =BT**2/2.*(1./4.-Z*Z)*(E1T-E1C)
      D11        =BT**3/3.*((E1T+E2C)/8.+XC**3/2.*(E1C-E2C)+Z**3*(E1C-E1T
     1 ))-BT**3/8.*XC*(E1C-E2C)
      GO TO 10
80    CONTINUE
      A11        =BT*E1C
      B11        =BT**2/2.*(1./2.+XC)**2*(E1C-E2C)
      B12        =0.
      D11        =BT**3/3.*((E1C+E2C)/8.+XC**3/2.*(E1C-E2C))-BT**3/8.*XC
     1 *(E1C-E2C)
      GO TO 10
90    CONTINUE
      A11        =BT*E1C
      B11        =BT**2*XC*(E1C-E2C)
      B12        =0.
      D11        =BT**3*E2C/24.
      GO TO 10
100   CONTINUE
      A11        =BT*((E1T+E1C)/2.+Z*(E1C-E1T))
      B12        =BT**2*(1./4.-Z*Z)*(E1T-E1C)/2.
      B11        =BT**2*(1./4.-Z*Z)*(E1T-E1C)/2.
      D11        =BT**3/3.*((E1T+E1C)/8.+Z**3*(E1C-E1T))
10    CONTINUE
      RETURN
      END
      SUBROUTINE FFMAT(XLOE,A,B,B2,D,S,QNO,XX1,XX2,ALFA,LODTYP)
      COMMON/FMAT/FM(7,7)
      DO 10 I=1,7
      DO 10 J=1,7
      IF(I.EQ.J) FM(I,J)=1.
 10   IF(I.NE.J) FM(I,J)=0.0
      GAMA=(B*B2)-A*D
      FM(1,4)=(B*XLOE)/GAMA
      FM(1,5)=B*(XLOE**2)/(2.*GAMA)
      FM(1,6)= -(XLOE*D)/GAMA
      FM(2,3)= -XLOE
      FM(2,4)=A*(XLOE**2)/(2.*GAMA)
      FM(2,5)= A*(XLOE**3)/(4.*GAMA)+XLOE/S
      FM(2,6)=-B2*(XLOE**2)/(2.*GAMA)
      FM(3,4)= -(A*XLOE)/GAMA
      FM(3,5)= -A*(XLOE**2)/(2.*GAMA)
      FM(3,6)=(B2*XLOE)/GAMA
      FM(4,5)= XLOE
      IF(LODTYP.EQ.2.OR.LODTYP.EQ.3) GO TO 20
      FM(1,7)= -B*QNO*(XLOE**3)/(4.*GAMA)
      FM(2,7)= -QNO*(A*(XLOE**4)/(8.*GAMA)+(XLOE**2)/(2.*S))
      FM(3,7)= A*(XLOE**3)*QNO/(4.*GAMA)
      FM(4,7)= -QNO*(XLOE**2)/2.
      FM(5,7)= -QNO*XLOE
      GO TO 50
 20   CONTINUE
      XX2=XX2+XLOE
      XX1= XX2-XLOE
      CX1=COS(ALFA*XX1)
      CX2=COS(ALFA*XX2)
      SX1=SIN(ALFA*XX1)
      SX2= SIN(ALFA*XX2)
      CONST=QNO/ALFA
      IF(LODTYP.EQ.2) GO TO 22
      IF(LODTYP.EQ.3) GO TO 24
 22   CONTINUE
      SITM = CONST*(XLOE*CX1-(1./ALFA)*(SX2-SX1))
      SITQ= CONST*(CX2-CX1)
      GO TO 26
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24  CONTINUE
    SIGQ=-CONST*(SX2-SX1)
    SIGM = CONST*(-XLOE*SX1-(1./ALPA)*(CX2-CX1))
26  CONTINUE
    PM(1,7) = -(B*XLOE*SIGM )/(2.*GAMA)
    PM(2,7) = -A*SIGM *(XLOE**2)/(4.*GAMA)+(SIGQ*XLOE)/(2.*5)
    PM(3,7) = (A*SIGM*XLOE)/(2*GAMA)
    PM(4,7) =-SIGM
    PM(5,7) = SIGQ
50  CONTINUE
    RETURN
    END
    SUBROUTINE COPY(LN,MTC)
    REAL MTC(7,7)
    COMMON/COP/WM(7,7),TM(364,7)
    DO 10 I=1,7
    DO 20 J=1,7
20  TM(LN,J)=MTC(I,J)
10  LN=LN+1
    DO 30 K=1,7
    DO 30 L=1,7
30  WM(K,L)=MTC(K,L)
    RETURN
    END
    SUBROUTINE STMAT(XLOAD,PMAS,RMAS,XIMAS,NST,DELT)
    COMMON/SMAT/SM(7,7)
    COMMON/TIME/XMN1(25),XMN2(25),XKONQ(25),XUNMA(25),XWNMA(25),XPNMA(
   125),X1,X5
    COMMON/UDAV/UDISN(25),UACN(25),UVELN(25)
    COMMON/WDAV/WDISN(25),WACN(25),WVELN(25)
    COMMON/PDAV/PDISN(25),PACN(25),PVELN(25)
    OMEGA= (2.*3.1415927)*278.5
    DO 10 I=1,7
    DO 10 J=1,7
    IF(I.EQ.J) SM(I,J)=1.
10  IF(I.NE.J) SM(I,J)=0.
    SM(5,7)=XLOAD
    NEMAR=1
    NEMAR=0
    BETANM= 0.5
    GAMANM=0.5
    BETANM=0.6
    UDIS= UDISN(NST)
    UAC = UACN(NST)
    WDIS= WDISN(NST)
    UVEL= UVELN(NST)
    WAC=WACN(NST)
    WVEL= WVELN(NST)
    PDIS=PDISN(NST)
    PAC= PACN(NST)
    PVEL = PVELN(NST)
    IF(NEMAR.EQ.1) GO TO 20
    SM(4,1) = -PMAS*(OMEGA**2)
    SM(4,3) = -XIMAS*(OMEGA**2)
    SM(5,2) = -PMAS*(OMEGA**2)
    SM(6,1) =-PMAS*(OMEGA**2)
    SM(6,3) =-RMAS*(OMEGA**2)
    GO TO 30
20  CONTINUE
    X1=BETANM*(DELT**2)
    X2= BETANM*DELT
    X3= (1./2.*BETANM) -1.
    X4= (1.-GAMANM)*DELT
    X5= GAMANM*DELT
    XMN1(NST)= (UDIS/X1)+(UVEL/X2)+(UAC*X3)
    XMN2(NST)=(PDIS/X1)+(PVEL/X2)+(PAC*X3)
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      XKONQ(NST)= (WDIS/X1)+(WVEL/X2)+(WAC*X3)
      XUNMA(NST)= UVEL+X4*UAC
      XWNMA(NST)= WVEL+X4*WAC
      XPNMA(NST)= PVEL+ X4*PAC
      SM(4,1)= RMAS/X1
      SM(4,3)= XIMAS/X1
      SM(4,7)= -RMAS*XMN1(NST)-XIMAS*XMN2(NST)
      SM(5,2)= PMAS/X1
      SM(5,7)= -XKONQ(NST)*PMAS
      SM(6,1)= PMAS/X1
      SM(6,3)= RMAS/X1
      SM(6,7)= -PMAS*XMN1(NST)-RMAS*XMN2(NST)
      NST=NST+1
   30 CONTINUE
      RETURN
      END
      SUBROUTINE AVERAG(NCAOT,NEIV,MTAV)
      REAL MTAV(364)
      COMMON/AVE/SV(364)
C     USE SV TO STORE BOTH SV AND ZA AFTER AVERAGING
      NSV=1
      NMTAV=1
      DO 10 NPOV=1,NCAOT
      IF(NPOV.EQ.1.OR.NPOV.EQ.NCAOT) GO TO 20
      DO 40 K=1,NEIV
      SV(NSV)= (MTAV(NMTAV)+MTAV(NMTAV+NEIV))/2.
      NMTAV=NMTAV+1
      NSV=NSV+1
   40 CONTINUE
      NMTAV=NMTAV+NEIV
      GO TO 10
   20 DO 30 J=1,NEIV
      SV(NSV)=MTAV(NMTAV)
      NSV=NSV+1
      NMTAV=NMTAV+1
   30 CONTINUE
   10 CONTINUE
      RETURN
      END
      SUBROUTINE ZNSYM(ZA,ZC,IZ,NOSV,IZNCT)
      DIMENSION ZA(20),ZC(20)
      NNZ=NOSV-IZ+1
      IF(IZ.GT.1) GO TO 10
      ZA(IZ)= ZC(IZ)
      ZA(NNZ)=ZC(NNZ)
      GO TO 20
   10 CONTINUE
      N=IZ
      IF(IZ.GT.2) N=N+IZNCT
      IF(IZ.GT.2) NNZ=NNZ-IZNCT
      ZA(N)=ZC(N)
      ZA(N+1)=ZC(N+1)
      ZA(NNZ)=ZC(NNZ )
      ZA(NNZ-1)=ZC(NNZ-1)
   20 CONTINUE
      IZNCT=IZNCT+1
      RETURN
      END
      SUBROUTINE SIGN(NOSV,STEST)
      COMMON/XSIGN/SIGN1(52),SIGN2(52)
      COMMON/AVE/SV(364)
      M=2
      DO 10 I=1,NOSV
      IF(SV(M).LT.0.0) SIGN1(I)=-1.
      IF(SV(M).GE.0.0) SIGN1(I)=1.
      M=M+7
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      SIGN1(I)=1.
   10 CONTINUE
      STEST=0.
      DO 20 I=1,NOSV
      STEST=STEST+(SIGN1(I)-SIGN2(I))
   20 CONTINUE
      STEST=0.
      DO 30 I=1,NOSV
      SIGN2(I)= SIGN1(I)
   30 CONTINUE
      RETURN
      END
      SUBROUTINE NEMARK(NSTAT,INCR)
      COMMON/AVE/SV(364)
      COMMON/TIME/XMN1(25),XMN2(25),XKONQ(25),XUNMA(25),XWNMA(25),XPNMA(
     125),X1,X5
      COMMON/UDAV/UDISN(25),UACN(25),UVELN(25)
      COMMON/WDAV/WDISN(25),WACN(25),WVELN(25)
      COMMON/PDAV/PDISN(25),PACN(25),PVELN(25)
      DO 10 I=1,NSTAT
      UDISN(I)= SV(INCR+1)
      WDISN(I)=SV(INCR+2)
      PDISN(I)= SV(INCR+3)
      UACN(I)= (UDISN(I)/X1)-XMN1(I)
      WACN(I)=(WDISN(I)/X1)-XKONQ(I)
      PACN(I)= (PDISN(I)/X1)-XMN2(I)
      UVELN(I)= XUNMA(I)+X5*UACN(I)
      WVELN(I)= XWNMA(I)  + X5*WACN(I)
      PVELN(I)  = XPNMA(I)  + X5*PACN(I)
      INCR=INCR+14
   10 CONTINUE
      RETURN
      END
```

```
C   PROGRAM TO FIND THE FIRST THREE NATURAL FREQUENCIES BY THE
C        SIXTH - ORDER THEORY
C   ZN IS THE POSITION OF THE NEUTRAL AXIS
C
        REAL I
C
C   E IS THE PERCENTAGE ERROR PERMITTED IN EVALUATING ZN
C   L IS THE NUMBER OF ITERATIONS FOR ZN
C   MM IS THE NUMBER OF ITERATIONS FOR THE ERROR, E
C   GR IS THE ACCELERATION DUE TO GRAVITY, IN/SEC SEC
C   ADD IS THE INCREMENT IN E
C
        READ(5,066)E, L, GR,MM,ADD
 066    FORMAT(F10.4,I5,F10.1,I5,F10.4)
C
C   TL IS THE LENGTH OF THE BEAM
C   ROF=DENSITY OF FACING,LB/(IN IN IN)
C   ROC=DENSITY OF CORE,LB/(IN IN IN)
C
        READ(5,160) H,C,TL,ROF,ROC
 160    FORMAT(3F10.5,2F10.7)
C
C   T= 1/2 THE THICKNESS OF BEAM
C
        T=H+C/2.
C
C   ALPHA = M*PI/TL  AND FOR M=1
C
        ALPHA=3.1416/TL
C
C   I = MASS M.I. OF BEAM, LB SEC SEC/IN
C
        I=(1./GR)*(ROC*C*C*C/12.+2.*(ROF/3.)*(T*T*T-(C/2.)*(C/2.)*
     &  (C/2.)))
C
C   S = THICKNESS SHEAR STIFFNESS, LB/IN
C   R=INPLANE COEFF, LB SEC/(IN IN)
C   FOR THIS CASE OF IDENTICAL THICKNESS FACINGS, R=0.0
C
        R=0.0
C
C   DENOTE NORMAL COEFF,P BY PI, LB SEC SEC/(IN IN IN)
C
        PI=(1./GR)*(ROC*C+2.*ROF*H)
        WRITE (6,156)
 156    FORMAT(//10X,'1-DIRECTION IMPLIES FIBERS LONGITUDINAL'/
     &  10X,'2-DIRECTION IMPLIES FIBERS  TRANSVERSE'/)
        WRITE (6,605)
 605    FORMAT(10X,'JOB=0 IMPLIES ZN IS WITHIN CORE'/10X,
     &  'JOB=1 IMPLIES ZN IS WITHIN FACING'//)
        WRITE(6,138) H,C,TL,ROF,ROC
 138    FORMAT(10X,'H=',F8.5,2X,'C=',F6.3,2X,'TL=',F6.3,2X,'ROF=',
     &  F10.7,2X,'ROC=',F10.7)
        WRITE (6,161) I,R,PI
 161    FORMAT (3X,'ROTATORY INERTIA COEFF, I=',E13.6/3X,
     &  'COUPLING INERTIA, R=',E9.2/3X,'TRANSL INERTIA, P=',E13.6/)
        WRITE (6,012)
 012    FORMAT(/3X,'*********************************************************
     &*************************************'/)
C
C   E=YOUNGS MODULI, T CORRESP TO TENSILE, C CORRESP COMPRESSIVE
C   1 CORRESP FIBER DIRN, 2 CORRESP TRANSVERSE DIRN, C CORRESP CORE
C   G=SHEAR MODULI, C CORRESP CORE
C
```

```
      READ(5,501) ET1,EC1,ET2,EC2,ETC,ECC
 501  FORMAT(6F10.1)
      READ(5,502) GF,GC
 502  FORMAT(2F10.2)
      DO 565 JA=1,MM
      E=E+ADD
      WRITE(6,81) E
 81   FORMAT(//12X,'E = ',F7.4//)
C
C  N=1 IS FOR FIBERS IN LONGITUDINAL
C  N=2 IS FOR FIBERS IN TRANSVERSE
C
     XK=0.0
     DO 333 KK=1,10
     XK=XK+0.1
      DO 150 N=1,2
      IF(N.EQ.2) GO TO 606
      ET=ET1
      EC=EC1
      GO TO 607
 606  ET=EC2
      EC=ET2
 607  EA=EC+ET
      EB=EC-ET
      ED=ET-EC+2.*ECC
      EF=ETC+ECC
      EG=ETC-ECC
C
C  CHECK IF ZN FALLS WITHIN CORE, IE. JOB=0
C
      JOB=0
      DA=-EG/3.
      DB=H*EA+(C/2.)*EF
      DC=H*(H+C)*EB-(C*C/4.)*EG
      ZA=-(2.*DB)/(6.*DA)
      ZB=ZA*ZA-DC/(3.*DA)
      IF(ZB.LT.0.0) GO TO 602
      ZB=SQRT(ZB)
      ZN=ZA-ZB
      IF(ABS(ZN).LT.(C/2.)) GO TO 603
      ZN=ZA+ZB
      IF(ABS(ZN).GT.(C/2.)) GO TO 602
      GO TO 603
C
C  ZN WITHIN FACING IF.JOB=1
C
 602  JOB=1
      ZC=-(2.*H*EA+C*ED)/(2.*EB)
      ZD=(ZC*ZC-T*T)
      ZD=SQRT(ZD)
      ZN=ABS(ZC-ZD)
      IF(ZN.LT.T.AND.ZN.GT.(C/2.)) GO TO 603
      ZN=ABS(ZC+ZD)
 603  WRITE(6,155) N,ZN,JOB
 155  FORMAT(//15X, 'FIBERS IN ',I2,'-DIRECTION',2X,
     & 'WITH INITIAL ZN=',F7.4/15X,'ZN FOLLOWS JOB=',I2/)
      WRITE(6,509) ET,EC,ETC,ECC
 509  FORMAT(/3X,'YOUNGS MODULI, ET=',F8.1,3X,'EC=',F8.1,3X,
     & 'ETC=',F8.1,3X,'ECC=',F8.1)
      WRITE(6,511) GF,GC
 511  FORMAT(3X,'SHEAR MODULI, GF=',F6.1,3X,'GC=',F8.1/)
      WRITE (6,912)
      WRITE(6,95)
 95   FORMAT (/15X,'H',4X,'K',6X,'A',10X,'B',10X,'D',8X,'OMEGA HZ'
     & ,6X,'ZN'//)
      WRITE (6,912)
```

```
      DO 100 J=1,3
      AL= J*ALPHA
      DO 10 K=1,L
      IF(ABS(ZN).GT.(C/2.)) GO TO 701
 579  JOB=0
      DA=-EG/3.
      DB=H*EA+(C/2.)*EF
      DC=H*(H+C)*EB-(C*C/4.)*EG
      DD=(1./3.)*(H*(H*H+1.5*H*C+.75*C*C)*EA+EF*C*C*C/8.)
      AA=3.*DA
      AB=DB
      BA=-1.5*DA
      BB=-DB
      BC=-DC/2.
      GO TO 578
 701  JOB=1
C
C  IF ZN FALLS WITHIN A FACING, IT MUST BE POSITIVE (SEE APPENDIX II)
C
      ZN=ABS(ZN)
      AA=EB
      AB=H*EA+(C/2.)*ED
      BA=.5*EB
      BB=0.0
      BC=(-.5)*EB*T*T
      DA=(1./3.)*EB
      DB=0.0
      DC=0.0
      DD=(1./3.)*(H*(H*H+1.5*H*C+.75*C*C)*EA+(C*C*C/8.)*ED)
 578  A=AA*ZN+AB
      B=BA*ZN*ZN+BB*ZN+BC
      D=DA*ZN*ZN*ZN+DB*ZN*ZN+DC*ZN+DD
      S=0.69134*(GC*C+2.*H*GF)
      IF (N.EQ.2) S=S*1.0487/0.69134
      P1=FI*(I*FI-B*B)
      AL2=AL*AL
      AL6=AL2*AL2*AL2
      P=(AL2*(F*(G*P+2.*B*FI)-FI*(FI*D+I*A+I*S))-FI*FI*S)/P1
      Q=AL2*(AL2*(S*(I*A-2.*B*B)+FI*(A*D+S*D-B*B))+FI*A*S)/P1
      R1=S*AL6*(B*B-D*A)/P1
C
C  TO FIND ROOTS OF   W**6+P*W**4+Q*W**2+R1 = 0
C  OR ROOTS OF   X**3 = G*X+HI
C  WHERE X = W**2+P/3 AND G AND HI ARE DEFINED BELOW
C
      G =P*P/3.-Q
      HI=P*Q/3.-2.*P*P*P/27.-R1
      U=G/3.
      V=HI/2.
C
C  LET CI=V**2-U**3
C
      CI=V*V-U*U*U
      IF (CI) 15,20,25
C
C  LET Y = (OMEGA)**2 HENCE Y = X-P/3
C
 15   TI=ARCOS(V/(U*SQRT(U)))
      Y1=2.*SQRT(U)*COS(TI/3.)-P/3.
      Y2=2.*SQRT(U)*COS(TI/3.+ 3.1416*2./3.)-P/3.
      Y3=2.*SQRT(U)*COS(TI/3. +3.1416*4./3.)-P/3.
      IF (Y1.LT.0.0.AND.Y2.LT.0.0) GO TO 70
      IF (Y1.GE.0.0.AND.Y2.GE.0.0) GO TO 75
      IF (Y1.GE.0.0) Y=Y1
      IF (Y2.GE.0.0) Y=Y2
      GO TO 80
```

```
75      IF (Y1.LE.Y2) GO TO 85
        Y=Y2
        GO TO 90
85      Y=Y1
90      IF (Y3.GE.0.0.AND.Y3.LE.Y) Y=Y3
        GO TO 30
70      IF (Y3.LT.0.0) GO TO 35
        Y=Y3
        GO TO 30
20      Y1=2.*(V**.333)-P/3.
        Y2=-(V**.333)-P/3.
        IF (Y1.LT.0.0.AND.Y2.LT.0.0) GO TO 35
        IF (Y1.GE.0.0.AND.Y2.GE.0.0) GO TO 60
        IF (Y1.GE.0.0) Y=Y1
        IF (Y2.GE.0.0) Y=Y2
        GO TO 30
60      IF (Y1.LE.Y2) GO TO 65
        Y=Y2
        GO TO 30
65      Y=Y1
        GO TO 30
25      Y=(V+SQRT(CI))**.333+(V-SQRT(CI))**.333-P/3.
        IF (Y.LT.0.0) GO TO 35
C
C    TO FIND  TRUE ZN
C    DENOTE NEW VALUE OF ZN BY ZNEW
C
30      ZNEW=(B*AL2-P*Y)/(A*AL2-FI*Y)
        IF (ABS((ZN-ZNEW)/ZN)-E) 45,45,50
45      IF(ZNEW.GT.T.OR.ZNEW.LT.(-C/2.)) GO TO 721
        GO TO 46
C
C    STATEMENT AFTER 721 IS PERFORMED INORDER TO CHANGE THE INITIAL
C    GUESS IF ZN DOES NOT FALL WITHIN ITS LIMITS
C    IF ZN FALLS WITHIN A FACING, IT MUST BE POSITIVE (SEE APPENDIX II)
C
721     IF(K.EQ.L) GO TO 35
        ZNEW=-ZNEW/4.
50      ZN=ZNEW
        IF(K.EQ.L) GO TO 35
10      CONTINUE
46      OMEGA=SQRT(Y)/(2.*3.1416)
        WRITE (6,55)  J, K, A, B, D, OMEGA, ZNEW, JOB
55      FORMAT(14X, I2,2X, I4,2X, F8.0,2X, F8.0,3X,
     &  F8.0,3X, F8.1,4X, F7.4,2X,'** JOB=',I2/)
        GO TO 100
35      WRITE (6,40)  J, K, Y, ZNEW
40      FORMAT(//10X,'END ITERATION FOR M=',I2,' WITH'/20X,'K=',I4
     &  /20X,'(OMEGA)**2=',F14.0/20X,'ZN=',F10.4/)
        IF(ZNEW.GT.T.OR.ZNEW.LT.(-C/2.)) ZN=-ZNEW/4.
100     CONTINUE
        WRITE (6,012)
150     CONTINUE
333    CONTINUE
565     CONTINUE
      STOP
        END
```