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ANITA M. RANSDELL WALKER

Norman, Oklahoma

1983

GENERALIZED ITERATIVE METHODS AND

NONLINEAR FUNCTIONAL EQUATIONS

A THESIS

APPROVED FOR THE DEPARTMENT OF MATHEMATICS

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GENERALIZED ITERATIVE METHODS AND

NONLINEAR FUNCTIONAL EQUATIONS

BY: ANITA M. RANSDELL WALKER

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Let X and Y be Banach spaces, P be a Gateaux differentiable mapping from X to Y and $c : [0,\infty) \to (0,\infty)$ be a continuous nonincreasing function for which $\int^{\infty} c(u)du = \infty$. If $P'(x)(B(0;1))$ contains $B(0; c(\|x\|))$ for each $x \in X$, then P is an open mapping of X onto Y. If the differentiability assumption on P is removed and instead P is both open and locally expansive, then $P(X) = Y$. If A is a continuous mapping from X to X satisfying for each $x \in X$, $\langle Ax - Ay, j \rangle \geq c(\max\{\|x\|, \|y\|\})\|x - y\|^2$ for some $j \in J(x - y)$, then A is a homeomorphism of X onto X. The main technique used in establishing these results is a new fixed point theorem which includes Ekeland's Theorem as a special case.

Perturbations of nonlinear operators are also investigated. If $F'(x)(B(0;1)) \supset B(0; c(\|x\|))$ and if F is perturbed by a nonlinear operator G satisfying a boundedness condition, then $F + G$ is an open mapping from X onto Y. The case where both F and G are Gateaux differentiable operators satisfying various coercive conditions again yields surjectivity results for the sum $F + G$. These proofs rely on the existence of contractor inequalities derived from the hypotheses. Finally, if G is a compact operator and $I - F$ is compact, then $F + G$ is surjective; the proof uses methods of algebraic topology.

CHAPTER I

INTRODUCTION

This thesis falls within the general framework of the study of nonlinear operators acting on Banach spaces, a study begun in the early part of this century in connection with certain boundary value problems arising in partial differential equations. More specifically, suppose P is a nonlinear mapping from a Banach space X to a Banach space Y; there are numerous approaches to studying the normal solvability of the equation $Px = y$ for $y \in Y$, many of which involve local assumptions on the operator P. This is due in large part to the fact that P often arises from a differential or an integral operator which has local smoothness or monotonicity properties. In this work we will derive mapping theorems for operators satisfying each of the above local assumptions. The theorems we present have been established as independent results in the field of normal solvability for nonlinear operators; however, the techniques can be applied in studying the normal solvability of nonlinear or quasilinear differential equations with nonlinear boundary conditions. For such applications the reader is referred to [1] and [25].

As a prelude to the results we will present a short background on the theory of normally solvable nonlinear operators. A nonlinear mapping $P:X \to Y$ is <u>Gateaux differentiable</u> if for each $x \in X$ there is an operator $P'(x)$, not necessarily linear or bounded, mapping X to Y for which

$$\lim_{t\to 0} t^{-1}\{P(x + th) - P(x)\} = P'(x)(h) \qquad (h \in X)$$

If $P'(x)$ is linear, then P is <u>normally solvable</u> if the injectivity of the set of adjoint operators $\{P'(x)^*\}$ implies the surjectivity of P.

The study of nonlinear normal solvability was pioneered by S.I. Pohozhayev [32] who showed that P is a surjection if Y is a reflexive Banach space, P(X) is weakly closed in Y and each $P'(x)^*$ is injective (see [16]). F.E. Browder (see, among others, [4], [5], [6]) improved upon and generalized these results, and eventually W.A. Kirk and J. Caristi [22] showed that $P(X) = Y$ if P(X) is closed in Y and each $P'(x)^*$ is injective.

The restriction that P have closed (weakly closed) range can be weakened by requiring that the operators $P'(x)^*$ have uniformly bounded inverses. To investigate this case further, assume momentarily that $P'(x)$ is bounded and linear, and define in the manner of Browder [3] the following quantity:

$$\alpha(P'(x)^*) = \inf \{c > 0 : \|y^*\| \leq c\|P'(x)^*y^*\|\}.$$

If no such constant exists, then set $\alpha(P'(x)^*) = \infty$. In the case that $\alpha(P'(x)^*) < \infty$, there are three major surjectivity results (see [34] for more discussion and proofs).

The first involves a uniform bound on $\alpha(P'(x)^*)$ for all $x \in X$:

THEOREM. <u>Let</u> X <u>and</u> Y <u>be Banach spaces and let</u> P <u>be a Gateaux differentiable mapping from</u> X <u>to</u> Y <u>having closed graph. Suppose there is a</u> $c > 0$ <u>such that</u> $\alpha(P'(x)^*) \leq c^{-1}$ <u>for all</u> $x \in X$. <u>Then</u> P <u>is an open mapping onto</u> Y.

If the bound on $P'(x)^*$ is allowed to vary as x ranges over bounded sets, then a comparable result is obtained if P satisfies a coercive condition. For $X = R^n$ the result is due to R.S. Palais [31]; M.A. Krasnoselskii [24] developed the result for P continuously Fréchet differentiable. A more general version due to W.J. Cramer and W.O. Ray [9] is:

THEOREM. Let X and Y be Banach spaces and let P be a Gateaux differentiable operator from X to Y having closed graph. Suppose for each $R > 0$ there is a $c(R) > 0$ such that $\alpha(P'(x)^*) \leqslant c(R)^{-1}$ whenever $\|x\| \leqslant R$, and suppose $\|Px\| \to \infty$ as $\|x\| \to \infty$. Then P is an open mapping of X onto Y.

If the coercive condition on P is replaced by one on $\alpha(P'(x)^*)$, then a similar result is:

THEOREM. Let X and Y be Banach spaces and let P be a Gateaux differentiable mapping from X to Y having closed graph. Let $c : [0,\infty) \to (0,\infty)$ be a continuous nonincreasing function for which $\int^\infty c(u)du = \infty$, and suppose for each $x \in X$ that $\alpha(P'(x)^*) \leqslant c(\|x\|)^{-1}$. Then P is an open mapping of X onto Y.

Another approach to solving $Px = y$ in a Banach space involves an iteration scheme. If $P'(x)$ has a bounded inverse, then Newton's method provides an iteration procedure for solving the equation $Px - y = 0$. However, the bounded inverse condition is often too restrictive; in fact, it may be that $P'(x)^{-1}$ does not even exist. In this case J. Moser [29] showed that Newton's method can be modified to still yield a solution of $Px = y$ if P is continuously Fréchet differentiable. The method has since been generalized in [35] to the case when P is Gateaux differentiable. In some of our work to follow we make implicit use of such an iteration scheme. Having the preceding discussion as a background for our results, we now proceed to present our conclusions. In Chapter II we prove a fixed point theorem which is a generalized version of Caristi's celebrated result. Using it we will derive two types of surjectivity theorems. The theorems involving differentiability assumptions rely on a Newton-Kantorovich scheme which was first developed by M. Altman [1] and further investigated by Cramer and Ray [9]. The theorem involving monotonicity assumptions extends results of Browder using both new and standard techniques.

In Chapter III we extend some notions of nonlinear normal solvability to perturbations of nonlinear operators. Using a result of Chapter II as a starting place, we first perturb a surjective Gateaux differentiable operator by "small" (in the sense of a norm condition) Gateaux differentiable operators and then deduce some consequences of this result. Our methods of proof employ analytical techniques such as "contractor inequalities" pioneered by Altman [1], and our own fixed point theorem of Chapter II, and the topological methods of a local degree theory.

In Chapter IV we investigate the openness and surjectivity of an accretive operator, an early version of which appears in [3]. The techniques of proof are highly analytical and are unrelated to the methods we use in the other chapters; hence we separate this result from the others.

CHAPTER II

MAPPING THEOREMS FOR GATEAUX DIFFERENTIABLE OPERATORS

1. INTRODUCTION

In 1972 I. Ekeland [14] established a variational principle that has proven to be very useful in the study of nonlinear normal solvability (see D. Downing and W.A. Kirk [13]). Four years later J. Caristi [7] independently established a fixed point theorem which Kirk [21] subsequently observed is equivalent to Ekeland's Theorem.

CARISTI'S THEOREM. *Let* (M,d) *be a complete metric space and* $g : M \to M$ *an arbitrary mapping. Suppose there is a lower semicontinuous mapping* $\phi : M \to [0,\infty)$ *for which*

$$d(x, g(x)) \leq \phi(x) - \phi(g(x)) \qquad (x \in M).$$

Then g *has a fixed point in* M.

In Section 2 we obtain a generalized version of this result and then apply it to derive surjectivity conclusions about three nonlinear operators P, namely when P is uniformly surjective, when P satisfies a pair of "contractor inequalities" and when P is locally expansive.

2. A FIXED POINT THEOREM

In this section we prove the following extension of Caristi's Theorem.

THEOREM II.2.1. *Let* (M,d) *be a complete metric space and let* ϕ *be a lower semicontinuous mapping from* M *to* $[0,\infty)$. *Let* $x_0 \in M$ *be fixed and suppose* $c : [0,\infty) \to (0,\infty)$ *is a continuous nonincreasing function for which* $\int^{\infty} c(u)du = \infty$. *If* $g : M \to M$ *is an arbitrary mapping which satisfies*

$$(2.1) \qquad c(d(x, x_0))d(x, g(x)) \leqslant \phi(x) - \phi(g(x)) \qquad (x \in M),$$

then g *has a fixed point in* M.

In Caristi's Theorem $c(u) = 1$. The basic tool in proving Theorem II.2.1 is a fundamental maximal principle of H. Brézis and F.E. Browder (Corollary 3 of [2]):

PROPOSITION B. *Let* $(M,d,\leqslant)$ *be a partially ordered metric space and let* $\phi : M \to [0,\infty)$ *be an arbitrary function. Suppose*

(2.2) *if* $x \leqslant y$ *and* $x \neq y$, *then* $\phi(y) < \phi(x)$,

(2.3) $S(x) = \{y \in M : y \geqslant x\}$ *is closed for each* $x \in M$, *and*

(2.4) *any nondecreasing sequence in* M *has compact closure*.

Then there is an $x_0 \in M$ *for which* $S(x_0) = \{x_0\}$; *that is,* x_0 *is a maximal element in* $(M,d,\leqslant)$.

It is worth observing that Theorem II.2.1 can be deduced by a direct application of Zorn's Lemma. The proposition enables us to prove our extension by investigating the behavior of sequences in the context of a partial ordering; essentially the same proof gives the desired conclusion via Zorn's Lemma if we consider instead nets. In either case, we obtain the existence of a maximal element which is the essential step in the proof. The proof we present will apply the proposition of Brézis and Browder, thereby eliminating any reference to the Axiom of Choice.

PROOF OF THEOREM II.2.1. Define $E = \{(x,s) \in M \times [0,\infty) : d(x,x_o) \leq s\}$ and define a metric ρ on E by $\rho((x,s), (y,t)) = \max\{ \| t - s \|, d(x,y)\}$.

It is easy to show that (E,ρ) is a complete metric space, for suppose $\{(x_n,s_n)\}$ is Cauchy in (E,ρ). Then for every $\varepsilon > 0$ there exists a $N > 0$ so that $m,n \geq N$ implies $\rho((x_n,s_n), (x_m,s_m)) < \varepsilon$. By definition of ρ it follows that $\| s_n - s_m \| < \varepsilon$ and $d(x_n,x_m) < \varepsilon$ for $m,n \geq N$. Thus $\{x_n\}$ and $\{s_n\}$ are Cauchy in (M,d) and $[0,\infty)$, respectively, and the completeness of both spaces implies that $s_n \to s_\infty$ and $x_n \to x_\infty$. Thus $(x_n,s_n) \to (x_\infty,s_\infty)$, giving that Cauchy sequences converge.

It remains to be verified that $d(x_\infty,x_o) \leq s_\infty$. For every n, it is true that $d(x_n,x_o) \leq s_n$. Then convergence of the sequences $\{x_n\}$ and $\{s_n\}$ implies that $d(x_\infty,x_o) \leq s_\infty$. This completes the argument that (E,ρ) is complete.

Now define a partial ordering " $\precsim$ " on E by saying that $(x,s) \precsim (y,t)$ if and only if

(2.5) $$d(x,y) \leq t - s, \text{ and}$$

(2.6) $$\int_s^t c(u)du \leq \phi(x) - \phi(y).$$

To show that " $\precsim$ " is indeed a partial ordering we must verify that it is reflexive, anti-symmetric and transitive. Reflexivity is trivial.

To see anti-symmetry, let $(x,s) \precsim (y,t)$ and $(y,t) \precsim (x,s)$. Then (2.5) implies $d(x,y) \leq t - s$ and $d(y,x) \leq s - t$. Adding gives that $2d(x,y) \leq 0$; hence $x = y$. Then (2.6) implies $s = t$, and so $(x,s) = (y,t)$.

To see that " $\precsim$ " is transitive as well, suppose that $(x,s) \precsim (y,t)$ and $(y,t) \precsim (z,r)$. Then by the triangle inequality and (2.5),

$$\begin{aligned} d(x,z) &\leqslant d(x,y) + d(y,z) \\ &\leqslant t - s + r - t \\ &= r - s. \end{aligned}$$

Applying (2.6) gives that

$$\begin{aligned} \int_s^r c(u)du &= \int_s^t c(u)du + \int_t^r c(u)du \\ &\leqslant \phi(x) - \phi(y) + \phi(y) - \phi(z) \\ &= \phi(x) - \phi(z). \end{aligned}$$

Hence $(x,s) \underset{\sim}{<} (z,r)$, and thus " $\underset{\sim}{<}$ " is a partial ordering.

We now verify the hypotheses of Proposition B. For (2.2) let $(x,s) \underset{\sim}{<} (y,t)$ but $(x,s) \neq (y,t)$. Then $s \neq t$, for if $s = t$, then $(x,s) \underset{\sim}{<} (y,s)$ and $(x,s) \neq (y,s)$. Hence $x \neq y$. But (2.5) implies that $d(x,y) = s - s = 0$, a contradiction. Thus we may assume $\int_s^t c(u)du > 0$. Applying (2.6) yields that $\phi(y) < \phi(x)$, and (2.2) holds.

Next observe that if $S(x,s) = \{(y,t) \in E : (y,t) \underset{\sim}{>} (x,s)\}$, then $S(x,s)$ is a closed set. If $\{(y_n,t_n)\}$ is a Cauchy sequence in $S(x,s)$, then $\{(y_n,t_n)\}$ must converge to (y_∞,t_∞) in E by completeness of this space. Then by definition of ρ, $y_n \to y_\infty$ in M and $t_n \to t_\infty$ in $[0,\infty)$. Since for every n (2.5) gives $d(y_n,x) \leqslant t_n - s$, it follows that

$$\begin{aligned} d(y_\infty,x) &= \lim_{n\to\infty} d(y_n,x) \\ &\leqslant \lim_{n\to\infty} t_n - s \\ &= t_\infty - s \end{aligned}$$

giving (2.5). In addition, for every n (2.6) implies that $\int_s^{t_n} c(u)du \leqslant \phi(x) - \phi(y_n)$. Then lower semicontinuity of ϕ implies that

$$\int_s^{t_\infty} c(u)du = \lim_{n\to\infty} \int_s^{t_n} c(u)du$$

$$\leq \underline{\lim}_{n \to \infty} \phi(x) - \phi(y_n)$$

$$\leq \overline{\lim}_{n \to \infty} \phi(x) - \phi(y_n)$$

$$= \phi(x) - \underline{\lim}_{n \to \infty} \phi(y_n)$$

$$\leq \phi(x) - \phi(y_\infty).$$

Hence (2.6) holds and $(y_\infty, t_\infty) \gtrsim (x,s)$, giving (2.3).

Finally suppose $\{(x_n, s_n)\}$ is a nondecreasing sequence in E. Since $(x_n, s_n) \leq (x_{n+1}, s_{n+1})$ for every n, (2.5) implies $\{s_n\}$ is nondecreasing in $[0,\infty)$, and by (2.6), $0 \leq \int_{s_n}^{s_{n+1}} c(u)du \leq \phi(x_n) - \phi(x_{n+1})$. So $\{\phi(x_n)\}$ is nonincreasing and bounded below by 0, and thus converges.

Since $(x_1, s_1) \leq (x_n, s_n)$ for every n, (2.6) also implies that $\int_{s_1}^{s_n} c(u)du \leq \phi(x_1) - \phi(x_n) \leq \phi(x_1) < \infty$. Hence $\lim_{n \to \infty} \int_{s_1}^{s_n} c(u)du \leq \phi(x_1)$, which implies that $s_n \to s_\infty < \infty$ since $\int_{s_1}^{\infty} c(u)du = \infty$ by hypothesis. By (2.5) $d(x_n, x_m) \leq \| s_n - s_m \|$, and so $\{x_n\}$ is Cauchy in (M,d) and $x_n \to x_\infty$. So again the definition of ρ gives that $\{(x_n, s_n)\}$ is Cauchy and hence convergent. Since nondecreasing sequences converge, (2.4) holds.

Hence we may apply Proposition B and conclude that there is a $(w,s) \in E$ for which $S(w,s) = \{(w,s)\}$.

Now consider $(g(w), s + d(w,g(w)))$. Since by the triangle inequality $d(g(w),x_o) \leq d(g(w),w) + d(w,x_o) \leq d(g(w),w) + s$, it follows that $(g(w), s + d(w,g(w))) \in E$. We will show that $(w,s) \lesssim (g(w), s + d(w,g(w)))$ and obtain our result.

Clearly (2.5) holds since $d(w,g(w)) = s + d(w,g(w)) - s$. Next using the fact that c is nonincreasing and that $d(w,x_o) \leq s$, we derive (2.6) from (2.1):

$$\int_s^{s + d(w,g(w))} c(u)du \leqslant c(s)d(w,g(w))$$

$$\leqslant c(d(w,x_0))d(w,g(w))$$

$$\leqslant \phi(w) - \phi(g(w)).$$

Thus $(w,s) \underset{\sim}{\leq} (g(w),\ s + d(w,g(w)))$, and maximality implies they are in fact equal. Hence $s = s + d(w,g(w))$, and so $d(w,g(w)) = 0$. So $w = g(w)$ and g has a fixed point. ■

3. SURJECTIVITY RESULTS

In this section we apply Theorem II.2.1 to Gateaux differentiable operators and local expansions. We do not require the Gateaux derivative $P'(x)$ of an operator P to be either continuous or linear; however, the derivative is homogeneous by definition since

$$\begin{aligned} P'(x)(\lambda h) &= \lim_{t \to 0} t^{-1}[P(x + t\lambda h) - P(x)] \\ &= \lim_{t \to 0} \lambda\lambda^{-1}t^{-1}[P(x + t\lambda h) - P(x)] \\ &= \lambda \lim_{t \to 0} (\lambda t)^{-1}[P(x + t\lambda h) - P(x)] \\ &= \lambda P'(x)(h). \end{aligned}$$

Let $B(w;r)$ denote the set $\{y : \|y - w\| \leqslant r\}$.

THEOREM II.3.1. <u>Let</u> X <u>and</u> Y <u>be</u> <u>Banach</u> <u>spaces</u> <u>and</u> P <u>be</u> <u>a</u> <u>Gateaux</u> <u>differentiable</u> <u>mapping</u> <u>from</u> X <u>to</u> Y <u>having</u> <u>closed</u> <u>graph</u>. <u>Let</u> $c : [0,\infty) \to (0,\infty)$ <u>be</u> <u>a</u> <u>continuous</u> <u>nonincreasing</u> <u>function</u> <u>satisfying</u> $\int^\infty c(u)du = \infty$, <u>and</u> <u>suppose</u> <u>for</u> <u>each</u> $x \in X$ <u>that</u>

(3.1) $P'(x)(B(0;1)) \supset B(0;c(\|x\|))$.

<u>Then</u> P <u>is</u> <u>surjective</u>.

It is worth noting that Theorem II.3.1 generalizes Theorem 4 of [38], in which it is assumed that c is the constant function. Also, Theorem 3.2 of [9] implies that P is an open mapping.

Since our assumptions in Theorem II.3.1 are infinitessimal in character on $P'(x)$, the result shows how global conclusions may be derived from infinitessimal hypotheses. In the theorem we desire to solve the equation $Px = y$; that is, $y - Px = 0$. Consider then the Newton iterate $g(x) \in x + tP'(x)^{-1}(y - Px)$ (the introduction of "t" is a standard numerical analysis technique used to accelerate the convergence of the iterates). It is noteworthy that $g(x) = x$ if and only if $y - Px = 0$. To see this note that $x = x + tP'(x)^{-1}(y - Px)$ implies $P'(x)(0) = y - Px$; homogeneity of $P'(x)$ gives the result. Thus $g(x) - x = tP'(x)^{-1}(y - Px)$, giving that $P'(x)(g(x) - x) = t(y - Px)$. Setting $h = t^{-1}(g(x) - x)$ gives $P'(x)(h) = y - Px$. Hence we are led to consider a hypothesis of the form (3.1). In addition, the sequence $\{g^n(x)\}$ is roughly speaking a sequence of Newton-Kantorovich iterates.

In Theorem II.3.1 we avoid the explicit use of the injectivity of each $P'(x)^*$. But observe that (3.1) implies $P'(x)$ is surjective, and so $Y = R(P'(x)) = N(P'(x)^*)^{\perp}$ implies $N(P'(x)^*) = \{0\}$, making $P'(x)^*$ injective.

PROOF OF THEOREM II.3.1. Define a metric ρ on the space X by $\rho(x,y) = \max\{(3/2)\|x - y\|, c(0)^{-1}\|Px - Py\|\}$. Since P has closed graph, (X,ρ) is a complete metric space (cf. proof of Theorem II.2.1).

Fix $y \in Y$ and set $\phi(x) = 3\|y - Px\|$. Note that ϕ is continuous from (X,ρ) to $[0,\infty)$ since P has closed graph.

We proceed by contradiction and assume $y \notin P(X)$. For any $x \in X$,

$c(\|x\|)\|y - Px\|^{-1}(y - Px) \in B(0;c(\|x\|))$. By (3.1) there is a $w \in B(0;1)$ for which $P'(x)(w) = c(\|x\|)\|y - Px\|^{-1}(y - Px)$. Setting $h = c(\|x\|)^{-1}\|y - Px\|w$, it follows that $P'(x)(h) = y - Px$ while $\|h\| \leq c(\|x\|)^{-1}\|y - Px\|$.

By hypothesis P is Gateaux differentiable, so we can choose $t \in (0,1]$ so small that

$$(3.2) \qquad \|P(x + th) - P(x) - tP'(x)(h)\| \leq \tfrac{1}{2}t\|y - Px\|.$$

Setting $g(x) = x + th$, it follows that $g(x) \neq x$ (since $y \neq Px$ and $t > 0$ by supposition) and (3.2) becomes

$$(3.3) \qquad \begin{aligned} \|P(g(x)) - P(x) - t(y - Px)\| &\leq \|P(g(x)) - P(x) - tP'(x)(h)\| + \\ &\quad + t\|P'(x)(h) - (y - Px)\| \\ &\leq \tfrac{1}{2} t\|y - Px\|. \end{aligned}$$

Also $\|g(x) - x\| = t\|h\| \leq tc(\|x\|)^{-1}\|y - Px\|$, or equivalently,

$$(3.4) \qquad c(\|x\|)\|g(x) - x\| \leq t\|y - Px\|.$$

Applying the triangle inequality to (3.3) gives

$$(3.5) \qquad \|P(g(x)) - P(x)\| \leq (3/2)t\|y - Px\|.$$

A second application of the triangle inequality to (3.3) gives

$$\|P(g(x)) - y\| - (1 - t)\|Px - y\| \leq \tfrac{1}{2} t\|Px - y\|, \text{ or}$$

$$(3.6) \qquad \tfrac{1}{2} t\|y - Px\| \leq \|Px - y\| - \|P(g(x)) - y\|.$$

Now, (3.5) and (3.6) together imply that

$$(3.7) \qquad \begin{aligned} \|P(g(x)) - P(x)\| &\leq 3(\|Px - y\| - \|P(g(x)) - y\|) \\ &= \phi(x) - \phi(g(x)). \end{aligned}$$

Also (3.4) and (3.6) imply

$$(3.8) \qquad \begin{aligned} (3/2)c(\|x\|)\|g(x) - x\| &\leq 3(\|Px - y\| - \|P(g(x)) - y\|) \\ &= \phi(x) - \phi(g(x)). \end{aligned}$$

Now let $x_o = 0$ be fixed in X. Consider first if $\rho(x,0) = (3/2)\|x\|$.

Then $\|x\| \leq (3/2)\|x\|$ implies that $c((3/2)\|x\|) \leq c(\|x\|)$ since c is nonincreasing. Likewise if $\rho(x,0) = c(0)^{-1}\|Px - P(0)\|$, (and thus $(3/2)\|x\| \leq c(0)^{-1}\|Px - P(0)\|$), then $\|x\| \leq c(0)^{-1}\|Px - P(0)\|$. Applying c gives $c(c(0)^{-1}\|Px - P(0)\|) \leq c(\|x\|)$. So in either case $c(\rho(x,0)) \leq c(\|x\|)$.

Now, if $\rho(x,g(x)) = (3/2)\|g(x) - x\|$, then (3.8) implies

$$\begin{aligned} c(\rho(x,0))\rho(x,g(x)) &\leq (3/2)c(\|x\|)\|g(x) - x\| \\ &\leq \phi(x) - \phi(g(x)), \end{aligned}$$

and (2.1) holds, while if $\rho(x,g(x)) = c(0)^{-1}\|Px - P(g(x))\|$, then since c is nonincreasing, (3.7) gives

$$\begin{aligned} c(\rho(x,0))\rho(x,g(x)) &\leq c(\|x\|)c(0)^{-1}\|Px - P(g(x))\| \\ &\leq \|Px - P(g(x))\| \\ &\leq \phi(x) - \phi(g(x)). \end{aligned}$$

So again (2.1) holds.

Hence Theorem II.2.1 implies that g has a fixed point, a contradiction. Hence our supposition is false and $y \in P(X)$. Therefore P is a surjection. ■

It is not difficult to see that the same proof, _mutatis mutandis_, establishes:

THEOREM II.3.2. _Let_ X _and_ Y _be Banach spaces,_ D _be a subset of_ X, P _be a mapping from_ D _to_ Y _having closed graph and_ $c : [0,\infty) \to (0,\infty)$ _be a continuous nonincreasing function for which_ $\int^{\infty} c(u)du = \infty$. _Suppose for some_ $q \in (0,1)$, $y \in Y$ _and for each_ $x \in D$ _there is an_ $\bar{x} \in D$ _and a_ $t \in (0,1]$ _for which_

(3.3') $\|P\bar{x} - Px - t(y - Px)\| \leq tq\|y - Px\|$, _and_

(3.4') $c(\|x\|)\,\|\bar{x} - x\| \leq t\,\|y - Px\|$.

Then $y \in P(D)$.

The assumptions (3.3') and (3.4') replace (3.3) and (3.4) in Theorem II.3.1 and are motivated by differentiability assumptions on the operator P. Assumptions of type (3.3') and (3.4') are central in the study of normal solvability (see, for example, [4], [5], [6], [13], [16], [21], [22], [32]). They are also central to M. Altman's theory of contractor directions (see, for example, [1], [9]).

The next theorem deals with a locally expansive operator and extends a result of Browder ([3], Theorem 4.10; see also [39]).

THEOREM II.3.3. Let X and Y be Banach spaces, P be an open mapping from X to Y having closed graph and $c : [0,\infty) \to (0,\infty)$ be a continuous nonincreasing function for which $\int^{\infty} c(u)du = \infty$. Suppose for each $x \in X$ there is an $\varepsilon > 0$ such that $\|x - \bar{x}\| \leq \varepsilon$ implies

(3.9) $c(\|x\|)\,\|x - \bar{x}\| \leq \|Px - P\bar{x}\|$.

Then $P(X) = Y$.

In [3] it is assumed that P is a local homeomorphism and (3.9) is strengthened to : if $u, v \in B(x;\varepsilon)$, then $c(\max\{\|u\|, \|v\|\})\,\|u - v\| \leq \|Pu - Pv\|$. We note that besides obtaining a more general result, our approach to Theorem II.3.3 is somewhat more elementary and direct than in [3].

PROOF OF THEOREM II.3.3. Define a metric ρ on X by $\rho(x,y) = \max\{\|x - y\|, c(0)^{-1}\|Px - Py\|\}$, and observe that (X,ρ) is complete. Fix $y \in Y$ and set $\phi(x) = \|y - Px\|$. As in Theorem II.3.1, ϕ is continuous. We proceed by supposing $y \notin P(X)$ and obtain a contradiction.

Fix $x \in X$ and choose $\varepsilon > 0$ so small that (3.9) holds. Since P is an open mapping,

$$P(B(x;\varepsilon)) \cap \{tPx + (1 - t)y : 0 \leqslant t < 1\} \neq \emptyset,$$

and hence there is a $g(x) \in B(x;\varepsilon)$ such that $P(g(x)) \in \{tPx + (1 - t) : 0 \leqslant t < 1\}$. Since

$$\begin{aligned} \| P(g(x)) - Px \| &= \| Px - y \| - \| P(g(x)) - y \| \\ &= \phi(x) - \phi(g(x)) \end{aligned}$$

and since $\| x - g(x) \| \leqslant \varepsilon$, it follows by an argument analogous to that in Theorem II.3.1 that either

$$\begin{aligned} c(\rho(x,0))\rho(x,g(x)) &\leqslant c(\| x \|)\| x - g(x) \| \\ &\leqslant \| Px - P(g(x)) \| \\ &= \phi(x) - \phi(g(x)) \end{aligned}$$

or

$$\begin{aligned} c(\rho(x,0))\rho(x,g(x)) &\leqslant c(\| x \|)c(0)^{-1} \| Px - P(g(x)) \| \\ &\leqslant \| Px - P(g(x)) \| \\ &= \phi(x) - \phi(g(x)). \end{aligned}$$

So in either case (2.1) holds. Thus for some x, $g(x) = x$, contradicting that $P(g(x)) \in \{tPx + (1 - t)y : 0 \leqslant t < 1\}$. ■

As in [3] we observe that Theorem II.3.3 has a consequence for operators of the accretive type. This will comprise Chapter IV.

CHAPTER III

PERTURBATIONS OF NONLINEAR OPERATORS

1. INTRODUCTION

The purpose of the present chapter is to extend the notion of normal solvability to perturbations of nonlinear operators. In the current literature, compact perturbations of nonexpansive, identity, monotone and accretive mappings have received much attention (see, for example, [3], [11], [18], [19], [20]), and the results deal primarily with the existence of fixed points or the existence of zeros of such operators. The methods of proof vary from the application of standard fixed point theorems to the development and application of degree theories.

Section 2 is devoted to two different perturbations of the nonlinear operator P presented in Theorem II.3.1. The proofs rely on the fixed point result derived in Theorem II.2.1. In section 3 we perturb a version of a surjectivity result of Cramer and Ray [9], Theorem 3.4 , while in section 4 we employ degree theory to obtain a result for the compact perturbation of a fairly general nonlinear operator F.

2. BOUNDED PERTURBATIONS

In this section we begin with a surjective Gateaux differentiable operator F which we perturb by a "small" Gateaux differentiable operator

G. Again, as in Chapter II, section 3, we avoid the explicit use of injective adjoint operators and concentrate instead on the uniform surjectivity of F. Under the assumptions we make on $G'(x)$ we demonstrate, via a transfinite Newton's method, the surjectivity of the operator $F + G$. Hence Newton's method is still sufficiently regular that small perturbations of a differentiable operator do not hinder the convergence of the iterates.

As before, the primary tool in deriving our results is the extension of Caristi's Theorem formulated in Theorem II.2.1. Before presenting our main theorems in this section we first verify a pair of "contractor inequalities" (cf. [1]) that are instrumental in the proofs to be presented. We begin with the case that $G'(x)$ satisfies a boundedness condition. The inequalities are given by:

LEMMA III.2.1. Let X and Y be Banach spaces and let F and G be Gateaux differentiable mappings from X to Y. Let $c : [0,\infty) \to (0,\infty)$ be a continuous nonincreasing function for which $\int^{\infty} c(u)du = \infty$. Suppose for each $x \in X$ that

(2.1) $G'(x)$ is a bounded and linear operator from X to Y, and

(2.2) $F'(x)(B(0;1)) \supset B(0;c(\|x\|))$.

Suppose, in addition, for some $\mu \in (0,1)$ and each $x \in X$ that

(2.3) $c(\|x\|)^{-1}\|G'(x)\| \leq \mu$.

Let $P : X \to Y$ be defined by $P \equiv F + G$. Then there is a $q \in (\mu,1)$ such that for each $y \in Y$ there is a $t \in (0,1]$ and an $\bar{x} \in X$ such that

(2.4) $\|P\bar{x} - Px - t(y - Px)\| \leq qt\|y - Px\|$, and

(2.5) $\|\bar{x} - x\| \leq c(\|x\|)^{-1} t\|y - Px\|$.

We remark here that similar inequalities can be deduced from injectivity hypotheses on $P'(x)^*$ (see the survey [34] for a discussion). It is noteworthy that the uniform surjectivity hypothesis (2.2) is somewhat weaker than the corresponding assumptions on the adjoint $P'(x)^*$.

PROOF OF LEMMA III.2.1. Fix $y \in Y$ and $q \in (\mu,1)$. If $y = Px$ for some x, then choose $\bar{x} = x$ and the conclusions follow for any t.

So without loss of generality we may assume $y \notin P(X)$. For each $x \in X$, $c(\|x\|)\|y - Px\|^{-1}(y - Px) \in B(0;c(\|x\|))$. By (2.2) there exists a $w \in B(0;1)$ so that $F'(x)(w) = c(\|x\|)\|y - Px\|^{-1}(y - Px)$. Set $h = c(\|x\|)^{-1}\|y - Px\|\cdot w$. Then the homogeneity of $F'(x)$ implies that $F'(x)(h) = y - Px$ with $\|h\| \leq c(\|x\|)^{-1}\|y - Px\|$.

By hypothesis both F and G are Gateaux differentiable, so we may choose $t \in (0,1]$ so small that

(2.6) $\quad \|F(x + th) - F(x) - tF'(x)(h)\| \leq \frac{1}{2}(q - \mu)t\|y - Px\|$, and

(2.7) $\quad \|G(x + th) - G(x) - tG'(x)(h)\| \leq \frac{1}{2}(q - \mu)t\|y - Px\|$.

Setting $\bar{x} = x + th$ and combining (2.6) and (2.7) yields, via the triangle inequality and hypotheses (2.1) and (2.3), that

$$\begin{aligned}
\|P\bar{x} - Px - t(y - Px)\| &\leq \|F\bar{x} - Fx - tF'(x)(h)\| + \\
&\quad \|G\bar{x} - Gx - tG'(x)(h)\| + t\|F'(x)(h) - (y - Px)\| + t\|G'(x)(h)\| \\
&\leq \tfrac{1}{2}(q - \mu)t\|y - Px\| + \tfrac{1}{2}(q - \mu)t\|y - Px\| \\
&\qquad + 0 + t\|G'(x)\|\cdot\|h\| \\
&\leq (q - \mu)t\|y - Px\| + tc(\|x\|)\mu c(\|x\|)^{-1} * \\
&\qquad * \|y - Px\| \\
&= qt\|y - Px\|,
\end{aligned}$$

giving (2.4).

To derive (2.5) observe that

$$\|\bar{x} - x\| = t\|h\|$$
$$\leq c(\|x\|)^{-1}t\|y - Px\|. \blacksquare$$

We now state three of our main results and then prove them.

THEOREM III.2.2. Let X and Y be Banach spaces and let F and G be Gateaux differentiable mappings from X to Y. Let $c : [0,\infty) \to (0,\infty)$ be a continuous nonincreasing function for which $\int^{\infty} c(u)du = \infty$. Suppose for each $x \in X$ that hypotheses (2.1), (2.2) and (2.3) of Lemma III.2.1 hold. If the mapping $P = F + G$ has closed graph, then P is an open mapping from X onto Y.

As a simple consequence it is enough to assume that F and G each has closed graph.

THEOREM III.2.3. Let F and G be given as in Theorem III.2.2. Define the Gateaux differentiable operator $P : X \to Y$ by $P = F + G$. If F and G have closed graph, then so does P; in particular, P is an open mapping from X onto Y.

The proof that P is open in Theorem III.2.2 and Theorem III.2.3 follows readily from a direct application of Theorem 2.1 of [9], while both surjectivity and openness use Lemma III.2.1. As in Theorem II.3.1 we desire to solve the equation $Px = y$, where $P = F + G$. So we consider the iterates $\nu(x) \in x + t(F'(x))^{-1}(y - Px)$ and observe that $\nu(x) = x$ if and only if $y - Px = 0$.

When specialized to the case that the function c(u) in Theorem

III.2.2 is a constant, the above result extends a result of I. Rosenholtz and W.O. Ray [38].

THEOREM III.2.4. _Let_ X _and_ Y _be Banach spaces and let_ F _and_ G _be Gateaux differentiable mappings from_ X _to_ Y. _Suppose for each_ $x \in X$ _that_

(2.8) $G'(x)$ _is a bounded and linear operator from_ X _to_ Y, _and_

(2.9) $F'(x)(B(0;1)) \supset B(0;\delta)$ _for some_ $\delta > 0$.

Suppose, in addition, for some $\mu \in (0,1)$ _that_

(2.10) $\delta^{-1} \| G'(x) \| \leq \mu$.

If the operator $P = F + G$ _has closed graph, then_ P _is an open surjection. In particular, if_ F _and_ G _each have closed graph, then_ P _is an open surjection._

We will omit the proof of Theorem III.2.4 since it is completely analogous to those of Theorems III.2.2 and III.2.3.

PROOF OF THEOREM III.2.2. We begin by demonstrating the surjectivity of $P = F + G$.

Define a metric ρ on X by $\rho(x,y) = \max\{(1 + q)\| x - y \|, c(0)^{-1} * * \| Px - Py \|\}$. Since P has closed graph, (X,ρ) is a complete metric space (cf. Theorem II.2.1).

Fix $y \in Y$ and $q \in (\mu,1)$. Set $\phi(x) = (1 + q)(1 - q)^{-1} \| y - Px \|$. Then ϕ is continuous from (X,ρ) to $[0,\infty)$ since P has closed graph.

We proceed by contradiction, so suppose $y \notin P(X)$. Then by Lemma III.2.1, defining $g : X \to X$ by $g(x) = \bar{x}$, it follows that

(2.11) $$\| P(g(x)) - Px - t(y - Px) \| \leq qt \| y - Px \|, \text{ and}$$

(2.12) $$c(\| x \|)\| g(x) - x \| \leq t \| y - Px \| .$$

Note that since $y \neq Px$, it follows that $g(x) \neq x$ for each $x \in X$. Applying the triangle inequality to (2.11) gives

$$(2.13) \qquad \| P(g(x)) - Px \| \leqslant (1 + q)t \| y - Px \|.$$

A second application of the triangle inequality to (2.11) gives

$$\|P(g(x)) - y \| - (1 - t) \|Px - y \| \leqslant qt \|y - Px\| \text{, or}$$

$$(2.14) \qquad t \|y - Px \| \leqslant (1 - q)^{-1}(\|Px - y \| - \|P(g(x)) - y \|).$$

Now, (2.13) and (2.14) together imply

$$(2.15) \qquad \| P(g(x)) - Px \| \leqslant (1 + q)(1 - q)^{-1}(\|Px - y \| - \|P(g(x)) - y \|)$$
$$= \phi(x) - \phi(g(x)).$$

Also (2.12) and (2.14) imply

$$(2.16) \quad (1 + q)c(\|x \|)\|g(x) - x \| \leqslant (1 + q)(1 - q)^{-1}(\|Px - y \| - \|P(g(x)) - y \|)$$
$$= \phi(x) - \phi(g(x)).$$

Now let $x_o = 0$ be fixed in X. Consider first if $\rho(x,0) = (1 + q)* \|x \|$. Then $\|x \| \leqslant (1 + q)\|x \|$ implies that $c((1 + q)\|x \|) \leqslant c(\|x \|)$ since c is nonincreasing. Likewise, if $\rho(x,0) = c(0)^{-1} \|Px - P(0) \|$, (and thus $(1 + q)\|x \| \leqslant c(0)^{-1} \|Px - P(0)\|$) then $\|x \| \leqslant c(0)^{-1} \|Px - P(0) \|$; so $c(c(0)^{-1} \|Px - P(0) \|) \leqslant c(\|x \|)$. Hence in either case $c(\rho(x,0)) \leqslant c(\|x \|)$.

Now if $\rho(x,g(x)) = (1 + q) \|x - g(x) \|$, then (2.16) implies

$$c(\rho(x,0))\rho(x,g(x)) \leqslant c(\|x \|)(1 + q) \|x - g(x) \|$$
$$\leqslant \phi(x) - \phi(g(x)),$$

while if $\rho(x,g(x)) = c(0)^{-1} \|Px - P(g(x)) \|$, then (2.15) implies

$$c(\rho(x,0))\rho(x,g(x)) \leqslant c(\|x \|)c(0)^{-1} \|Px - P(g(x)) \|$$
$$\leqslant \|Px - P(g(x)) \|$$
$$\leqslant \phi(x) - \phi(g(x)).$$

Thus Theorem II.2.1 applies to give a fixed point of g, a contradiction. Hence $y \in P(X)$ and P is a surjection.

Now to show P is open as well, fix $w \in X$ and let $\delta > 0$. It suffices to show that $B(Pw;\varepsilon) \subset P(B(w;\delta))$ for a sufficiently small choice of ε.

So let $y \in B(Pw;\varepsilon)$; then $\|y - Pw\| \leq \varepsilon$. Define a mapping $\bar{P}$ from $B(w;\delta)$ into Y by $\bar{P}(x) = y - Px$. We will show that $0 \in \bar{P}(B(w;\delta))$, thereby completing the proof of Theorem III.2.2. We accomplish this by applying Theorem 2.1 of [9].

By hypothesis, for $q \in (\mu,1)$ fixed, there is a $t \in (0,1]$ and an $\bar{x} \in X$ for which (2.4) and (2.5) hold. Hence (2.4) implies

$$\begin{aligned}\|\bar{P}\bar{x} - (1-t)\bar{P}x\| &= \|y - P\bar{x} - (1-t)(y - Px)\| \\ &= \|Px - P\bar{x} + t(y - Px)\| \\ &\leq qt\|y - Px\| \\ &= qt\|\bar{P}x\|,\end{aligned}$$

while (2.5) implies

$$\begin{aligned}\|\bar{x} - x\| &\leq c(\|x\|)^{-1}t\|y - Px\| \\ &= c(\|x\|)^{-1}t\|\bar{P}x\|.\end{aligned} \tag{2.17}$$

Now observe that $x \in B(w;\delta)$ implies $\|x - w\| \leq \delta$. So $\|x\| - \|w\| \leq \|x - w\|$ yields that $\|x\| \leq \delta + \|w\|$. Applying the function c gives $c(\delta + \|w\|) \leq c(\|x\|)$. So (2.17) becomes

$$\|\bar{x} - x\| \leq c(\delta + \|w\|)^{-1}t\|\bar{P}x\|. \tag{2.18}$$

In order to apply Theorem 2.1 of [9], we must verify that

$$(1-q)^{-1}\int_0^a s^{-1}B(s)ds \leq \alpha \tag{2.19}$$

for appropriate choices of α, $B : [0,\infty) \to [0,\infty)$ and a. If we choose $\alpha = \delta$, $M = c(\delta + \|w\|)^{-1}$, $B(s) = s$ and $a = c(\delta + \|w\|)^{-1}\|\bar{P}w\|e^{1-q}$, and if $0 < \varepsilon \leq \delta(1-q)c(\delta + \|w\|)e^{q-1}$, then (2.19) follows:

$$
\begin{aligned}
(1-q)^{-1}\int_0^a s^{-1}B(s)ds &= (1-q)^{-1}\int_0^a ds \\
&= (1-q)^{-1}a \\
&= (1-q)^{-1}c(\delta+\|w\|)^{-1}\|\bar{P}w\|e^{1-q} \\
&\leqslant (1-q)^{-1}c(\delta+\|w\|)^{-1}\varepsilon e^{1-q} \\
&\leqslant (1-q)^{-1}c(\delta+\|w\|)^{-1}\delta(1-q)c(\delta+\|w\|)* \\
&\qquad *e^{q-1}e^{1-q} \\
&= \delta \\
&= \alpha.
\end{aligned}
$$

Hence Theorem 2.1 applies to give the existence of an $x_o \in B(w;\delta)$ for which $\bar{P}x_o = 0$. Thus $y = Px_o$ and $y \in P(B(w;\delta))$, as required. ■

Theorem III.2.3 is an easy consequence of the Mean Value Theorem.

PROOF OF THEOREM III.2.3. Let $\{x_n\}$ be a sequence in X for which $x_n \to x_\infty$ and $Px_n \to y$; that is, $y = \lim_{n\to\infty} Fx_n + Gx_n$.

Since $\{x_n\}$ is Cauchy, for every $\varepsilon > 0$ there exists an $N > 0$ so that $m,n \geqslant N$ implies $\|x_n - x_m\| < \varepsilon\mu^{-1}c(0)^{-1}$. Applying the Mean Value Theorem of McLeod [27] (see also [28]) to G yields that

$$G(x_n) - G(x_m) \in \overline{co}\,\{G'(tx_n + (1-t)x_m)(x_n - x_m) : 0 < t < 1\},$$

from which it follows via (2.3) that

$$
\begin{aligned}
\|Gx_n - Gx_m\| &\leqslant \|x_n - x_m\|\sup\{\|G'(tx_n + (1-t)x_m)\| : 0 < t < 1\} \\
&\leqslant \|x_n - x_m\|\sup\{\mu c(\|tx_n + (1-t)x_m\|) : 0 < t < 1\} \\
&\leqslant \|x_n - x_m\|\mu c(0) \\
&< \varepsilon,
\end{aligned}
$$

if $m,n \geqslant N$.

Hence $\{Gx_n\}$ is Cauchy in Y, so the completeness of Y implies $\{Gx_n\}$ converges. By assumption G has closed graph, and so $\lim_{n\to\infty} Gx_n = Gx_\infty$.

Since $Fx_n + Gx_n \to y$, it follows that $Fx_n \to y - Gx_\infty$. But F has closed graph, so $Fx_\infty = y - Gx_\infty$. Therefore $y = Fx_\infty + Gx_\infty = Px_\infty$, giving P closed graph, as desired.

Now by Theorem III.2.2, P is an open surjection. ∎

Our second perturbation of Theorem II.3.1 involves a Lipschitz operator G. An operator $G : X \to Y$ is <u>Lipschitzian</u> if there is a finite $K \geq 0$ such that $\|Gx - Gy\| \leq K\|x - y\|$ for every $x,y \in X$. Let $\|G\|$ denote the least such constant K; then $\|\cdot\|$ is a seminorm on the space of Lipschitz operators from X to Y. As in Lemma III.2.1 we obtain a pair of "contractor inequalities" for G Lipschitzian.

LEMMA III.2.5. *Let F be a Gateaux differentiable mapping from a Banach space X to a Banach space Y, and let $G : X \to Y$ be a Lipschitz operator. Let $c : [0,\infty) \to (0,\infty)$ be a continuous nonincreasing function for which $\int^\infty c(u)du = \infty$. Suppose for each $x \in X$ that*

(2.20) $F'(x)(B(0;1)) \supset B(0;c(\|x\|))$.

Suppose, in addition, for some $\mu \in (0,1)$ and each $x \in X$ that

(2.21) $c(\|x\|)^{-1}\|G\| \leq \mu$.

Define $P : X \to Y$ by $P = F + G$. Then there is a $q \in (\mu,1)$ so that for each $y \in Y$ there is a $t \in (0,1]$ and an $\bar{x} \in X$ for which

(2.22) $\|P\bar{x} - Px - t(y - Px)\| \leq qt\|y - Px\|$, *and*

(2.23) $\|\bar{x} - x\| \leq c(\|x\|)^{-1}t\|y - Px\|$.

PROOF. Fix $y \in Y$ and $q \in (\mu,1)$. If $y \in P(X)$, then there is an $x \in X$ for which $y = Px$. Choosing $\bar{x} = x$ yields the result for all t.

So without loss of generality, assume $y \notin P(X)$. For any $x \in X$,

$c(\|x\|)\|y - Px\|^{-1}(y - Px) \in B(0;c(\|x\|))$. By hypothesis (2.20) there exists a $w \in B(0;1)$ for which $F'(x)(w) = c(\|x\|)\|y - Px\|^{-1} *$ $*(y - Px)$. Setting $h = c(\|x\|)^{-1}\|y - Px\|\cdot w$ gives that $F'(x)(h) =$ $y - Px$ and $\|h\| \leq c(\|x\|)^{-1}\|y - Px\|$.

The operator F is Gateaux differentiable, so we may choose $t \in (0,1]$ so small that

$$(2.24) \quad \|F(x + th) - F(x) - tF'(x)(h)\| \leq (q - \mu)t\|y - Px\|.$$

Setting $\bar{x} = x + th$ and applying the triangle inequality gives

$$(2.25) \quad \begin{aligned} \|P\bar{x} - Px - t(y - Px)\| &\leq \|F\bar{x} - Fx - tF'(x)(h)\| + \|G\bar{x} - Gx\| + \\ &\qquad t\|F'(x)(h) - (y - Px)\| \\ &\leq (q - \mu)t\|y - Px\| + \|G\|\cdot\|\bar{x} - x\| + 0. \end{aligned}$$

Now observe that $\|\bar{x} - x\| = t\|h\| \leq c(\|x\|)^{-1}t\|y - Px\|$, giving (2.23). Hence (2.25) becomes, via (2.21),

$$\begin{aligned} \|P\bar{x} - Px - t(y - Px)\| &\leq (q - \mu)t\|y - Px\| + c(\|x\|)\mu c(\|x\|)^{-1} * \\ &\qquad * t\|y - Px\| \\ &= qt\|y - Px\|, \end{aligned}$$

giving (2.22). ■

Our main results for this class of operators follow.

THEOREM III.2.6. _Let_ F _and_ G _be as in Lemma_ III.2.5. _If the mapping_ $P = F + G$ _has closed graph, then_ P _is an open mapping from_ X _onto_ Y.

Since the proof of Theorem III.2.6 is identical to that of Theorem III.2.2, we omit the details. Once again we observe that Theorem III.2.6 is true if each of F and G has closed graph.

THEOREM III.2.7. _Let_ F _and_ G _be as in Lemma_ III.2.5. _If_ F _and_ G _have closed graph, then_ $P = F + G$ _has closed graph; in particular,_ P _is an open mapping from_ X _onto_ Y.

PROOF. Let $\{x_n\}$ be a sequence in X for which $x_n \to x_\infty$ and $Px_n \to y$; then $\lim_{n \to \infty} Fx_n + Gx_n = y$.

Since $\{x_n\}$ is Cauchy, for every $\varepsilon > 0$ there is an $N > 0$ so that $m,n \geq N$ implies $\|x_n - x_m\| < \varepsilon\mu^{-1}c(0)^{-1}$. Then since G is Lipschitzian, it follows by (2.21) that

$$\begin{aligned} \|Gx_n - Gx_m\| &\leq \|G\|\cdot\|x_n - x_m\| \\ &\leq \mu c(0)\,\|x_n - x_m\| \\ &< \varepsilon . \end{aligned}$$

So $\{Gx_n\}$ is Cauchy in Y and thus convergent. By hypothesis G has closed graph, and so $Gx_n \to Gx_\infty$.

Now since $Fx_n + Gx_n \to y$, it follows that $Fx_n \to y - Gx_\infty$. But F has closed graph, so $y - Gx_\infty = Fx_\infty$. Therefore, $y = Fx_\infty + Gx_\infty = Px_\infty$, giving P closed graph.

By Theorem III.2.6 P is an open surjection. ∎

We conclude this section by remarking that most of the conclusions presented above are fairly direct consequences of earlier results which have used inequalities such as those of Lemma III.2.1 and Lemma III.2.5 as their main assumptions. Thus, for example, surjectivity is a special case of Theorem II.3.2 while openness was inferred directly from Theorem 2.1 of [9]. The main goal of this section has been to expose a further class of operators to which these more general results apply. The "contractor inequalities" have come to play a central role in the theory of nonlinear normal solvability, and the theorems of this section show that the inequalities and the mapping theorems they imply are stable under "small" perturbations.

3. COERCIVE PERTURBATIONS

We now return to the case where both F and G are Gateaux differentiable operators. We consider a version of Theorem 3.4 of [9] in which it is proven that an operator F is surjective if each $F'(x)$ is surjective and if F satisfies a coercive condition. By slightly altering the hypotheses of this result and then perturbing F by a "small" operator G also satisfying a coercive condition, we are able to retain the surjectivity conclusion, this time of $F + G$.

To prove our main results we rely on the following theorem.

THEOREM III.3.1. *Let* X *and* Y *be Banach spaces*, $D \subset X$, $P : X \to Y$ *an operator having closed graph and* $g : D \to (0,\infty)$ *a mapping which sends bounded sets to bounded sets. Suppose there is a* $q \in (0,1)$ *such that corresponding to each* $x \in D$, *there is an* $\bar{x} \in D$ *and a* $t \in (0,1]$ *for which*

(3.1) $\|P\bar{x} - (1 - t)Px\| \leqslant qt\|Px\|$, *and*

(3.2) $\|\bar{x} - x\| \leqslant tg(x)\|Px\|$.

Suppose also that

(3.3) $\lim\limits_{\|x\|\to\infty} \|Px\| = \infty$.

Then $0 \in P(D)$.

PROOF. We will proceed by contradiction and suppose $0 \notin P(D)$; so by replacing q by any $\bar{q} \in (q,1)$ we may assume that (3.1) is a strict inequality. Next, define a metric ρ on D by $\rho(x,y) = \max\{\|x - y\|, \|Px - Py\|\}$; note that (D,ρ) is a complete metric space and $P : (D,\rho) \to Y$ is continuous.

Now fix $x \in D$ and define $\Lambda(x) = \{y \in D :$ there is a $t \in (0,1)$ for which $\|Py - (1 - t)Px\| < t\bar{q}\|Px\|\}$; since $\bar{x} \in \Lambda(x)$, it follows that $\Lambda(x) \neq \emptyset$ for each $x \in D$. Thus we can define a sequence $\{s(x;n)\}_{n=0}^{\infty} \subset D$ recursively by $s(x;0) = x$ and $s(x;n+1) \in \Lambda(s(x;n))$. For any such sequence we obtain the following estimate from (3.1):

$$\|P(s(x;n+1)) - P(s(x;n))\| \leq (1 + \bar{q})(1 - \bar{q})^{-1}(\|P(s(x;n))\| - \|P(s(x;n+1))\|)$$

from which it follows via the triangle inequality that

$$\begin{aligned}\|P(s(x;n)) - Px\| &\leq (1 + \bar{q})(1 - \bar{q})^{-1}(\|Px\| - \|P(s(x;n))\|) \\ &\leq (1 + \bar{q})(1 - \bar{q})^{-1}\|Px\|.\end{aligned}$$

Thus the image of any such recursive sequence is contained in the ball centered at Px with radius $(1 + \bar{q})(1 - \bar{q})^{-1}\|Px\|$. Hence if $\sum(x)$ denotes the collection of elements of D obtained by taking the union of all such possible sequences, then $P(\sum(x))$ is bounded, and so (3.3) implies that $\sum(x)$ is bounded as well. Consequently, if we define

$$\bar{g}(x) = \sup\{g(y) : y \in \textstyle\sum(x)\},$$

then it follows that $\bar{g}(x) < \infty$ for all $x \in D$. Notice also that, since $\bar{x} \in \Lambda(x)$,

$$\begin{aligned}\bar{g}(\bar{x}) &= \sup\{g(y) : y \in \textstyle\sum(\bar{x})\} \\ &\leq \sup\{g(y) : y \in \textstyle\sum(x)\} \\ &= \bar{g}(x),\end{aligned}$$

and also that

$$g(x) \leq \sup\{g(y) : y \in \textstyle\sum(x)\} = \bar{g}(x).$$

We now show that $\bar{g}$ is lower semicontinuous on (D,ρ). We will use the following characterization: $\bar{g}$ is lower semicontinuous if $\underline{\lim}_{y \to x} \bar{g}(y) \geqslant \bar{g}(x)$.

We will need the following claim as well:

(3.4) for each $x \in D$ and $w \in \Lambda(x)$ there is a $\delta > 0$ such that $w \in \Lambda(y)$ whenever $\rho(x,y) < \delta$.

If (3.4) is not true, then there is a sequence $\{y_n\}$ in D such that $\rho(x,y_n) \to 0$ as $n \to \infty$, but $w \notin \Lambda(y_n)$. So, in particular,

$$\| Pw - (1 - t)Py_n \| \geqslant t\bar{q} \| Py_n \|$$

for each n, where t is chosen such that $\| Pw - (1 - t)Px \| < t\bar{q} \| Px \|$. Since $\rho(x,y_n) \to 0$, it follows that $Py_n \to Px$. Hence passing to the limit implies that $\| Pw - (1 - t)Px \| \geqslant t\bar{q} \| Px \|$; thus $w \notin \Lambda(x)$, a contradiction. This establishes (3.4).

Now fix $x \in D$ and $t > 0$. Choose $w \in \sum(x)$ such that $\bar{g}(x) - t < g(w)$. Since $w \in \sum(x)$, $w = s(x;n)$ for some sequence $\{s(x;m)\}_{m=0}^{\infty}$. Now, corresponding to $s(x;1)$ there is a $\delta > 0$ such that $\rho(x,y) < \delta$ implies $s(x;1) \in \Lambda(y)$ by (3.4). So if $\rho(x,y) < \delta$, then $w \in \sum(y)$, and so $g(w) \leqslant \bar{g}(y)$. Hence transitivity implies $\bar{g}(x) - t \leqslant \bar{g}(y)$ whenever $\rho(x,y) < \delta$. This shows, since t was arbitrary, that $\bar{g}$ is lower semicontinuous.

Now define a partial ordering on D by saying $x \lesssim y$ if and only if

(3.5) $\quad \| Px - Py \| \leqslant (1 + \bar{q})(1 - \bar{q})^{-1}(\| Px \| - \| Py \|)$,

(3.6) $\quad \| x - y \| \leqslant \bar{g}(x)(1 - \bar{q})^{-1}(\| Px \| - \| Py \|)$, and

(3.7) $\quad \bar{g}(y) \leqslant \bar{g}(x)$.

Set $\phi(x) = \| Px \|$ and observe that (2.2) of Proposition B of

Chapter II, Section 2, holds via (3.6).

To see (2.3) of the proposition, let $S(x) = \{y \in D : y \underset{\sim}{>} x\}$. Then $\{x_n\}$ Cauchy in $S(x)$ implies that $x_n \to x_\infty$ in (D, ρ). Since P is continuous, $Px_n \to Px_\infty$ as well. We must show $x_\infty \in S(x)$.

For every n, $x_n \underset{\sim}{>} x$; also $x_n - x \to x_\infty - x$ and $Px_n - Px \to Px_\infty - Px$. Thus (3.5) implies

$$\begin{aligned}\|Px_\infty - Px\| &= \lim_{n\to\infty} \|Px_n - Px\| \\ &\leqslant \lim_{n\to\infty} (1+\bar{q})(1-\bar{q})^{-1}(\|Px\| - \|Px_n\|) \\ &= (1+\bar{q})(1-\bar{q})^{-1}(\|Px\| - \|Px_\infty\|),\end{aligned}$$

giving (3.5). Also, (3.6) implies

$$\begin{aligned}\|x_\infty - x\| &= \lim_{n\to\infty} \|x_n - x\| \\ &\leqslant \lim_{n\to\infty} \bar{g}(x)(1-\bar{q})^{-1}(\|Px\| - \|Px_n\|) \\ &= \bar{g}(x)(1-\bar{q})^{-1}(\|Px\| - \|Px_\infty\|),\end{aligned}$$

establishing (3.6). Finally, the lower semicontinuity of $\bar{g}$ implies via (3.7) that

$$\bar{g}(x_\infty) \leqslant \varliminf_{x_n \to x_\infty} \bar{g}(x_n) \leqslant \varliminf_{x_n \to x_\infty} \bar{g}(x) = \bar{g}(x)$$

Thus $x_\infty \in S(x)$ as desired; so $S(x)$ is closed.

To verify (2.4) of the proposition, let $\{x_n\}$ be any nondecreasing sequence in $(D, \rho, \underset{\sim}{<})$. For $n \leqslant m$, (3.5) implies

$$\begin{aligned}\|Px_m - Px_n\| &\leqslant \sum_{j=n}^{j=m-1} \|Px_j - Px_{j+1}\| \\ &\leqslant \sum_{j=n}^{j=m-1} (1+\bar{q})(1-\bar{q})^{-1}(\|Px_j\| - \|Px_{j+1}\|)\end{aligned}$$

$$= (1 + \bar{q})(1 - \bar{q})^{-1} (\|Px_n\| - \|Px_m\|).$$

Since (3.5) implies $\{ \|Px_n\|\}$ is nonincreasing and since the sequence is bounded below by 0, $\{ \|Px_n\|\}$ converges in $(D,\rho,\lesssim)$; thus $\{ \|Px_n\|\}$ is Cauchy. Combining (3.6) and (3.7) yields for $n \leqslant m$ that

$$\|x_m - x_n\| \leqslant \sum_{j=n}^{m-1} \|x_j - x_{j+1}\|$$

$$\leqslant \sum_{j=n}^{m-1} \bar{g}(x_j)(1 - \bar{q})^{-1}(\|Px_j\| - \|Px_{j+1}\|)$$

$$\leqslant \bar{g}(x_1)(1 - \bar{q})^{-1}(\|Px_n\| - \|Px_m\|).$$

Hence $\{x_n\}$ is Cauchy in $(D, \|\cdot\|)$. Thus $\{x_n\}$ is Cauchy in (D,ρ) also. By completeness of (D,ρ), $\{x_n\}$ converges.

Therefore Proposition B applies to give an $x_o \in D$ for which $S(x_o) = \{x_o\}$. Since by hypothesis there exists an $\bar{x}_o \in D$ for which

$$\|P\bar{x}_o - (1 - t)Px_o\| \leqslant qt \|Px_o\| < \bar{q}t \|Px_o\|$$

and

$$\|\bar{x}_o - x_o\| \leqslant tg(x_o) \|Px_o\| < t\bar{g}(x_o) \|Px_o\|,$$

we obtain

$$\|P\bar{x}_o - Px_o\| \leqslant (1 + \bar{q})t \|Px_o\|$$

and

$$(1 - \bar{q})t \|Px_o\| \leqslant \|Px_o\| - \|P\bar{x}_o\|.$$

Combining inequalities yields

$$\|P\bar{x}_o - Px_o\| \leqslant (1 + \bar{q})(1 - \bar{q})^{-1}(\|Px_o\| - \|P\bar{x}_o\|)$$

and

$$\|\bar{x}_o - x_o\| \leqslant \bar{g}(x_o)(1 - \bar{q})^{-1}(\|Px_o\| - \|P\bar{x}_o\|).$$

Since we have already established that $\bar{g}(\bar{x}_o) \leq \bar{g}(x_o)$, it follows that $x_o \lesssim \bar{x}_o$, and by maximality of x_o it must be that $x_o = \bar{x}_o$. Since $q \in (0,1)$, (3.1) implies that $Px_o = 0$.■

We will make use of the more general case of the above theorem where $D = X$.

THEOREM III.3.2. <u>Let</u> F <u>and</u> G <u>be Gateaux differentiable operators from a Banach space</u> X <u>to a Banach space</u> Y, <u>and suppose both</u> F <u>and</u> G <u>have closed graph</u>. <u>Let</u> $\delta : [0,\infty) \to (0,\infty)$ <u>be an arbitrary mapping sending bounded sets to bounded sets, and define</u> $\hat{\delta} : [0,\infty) \to (0,\infty)$ <u>by</u> $\hat{\delta}(R) = \sup\{\delta(r) : 0 \leq r \leq R\}$. <u>Suppose each of the following</u>:

(3.8) <u>for every</u> $x \in X$, $F'(x)(B(0;1)) \supset B(0;\hat{\delta}(\|x\|)^{-1})$,

(3.9) <u>for every</u> $x \in X$, $G'(x)$ <u>is a bounded and linear operator from</u> X <u>to</u> Y,

(3.10) $\|G'(x)\| \leq \mu\hat{\delta}(\|x\|)^{-1}$ <u>for some</u> $\mu \in (0,1)$ <u>fixed</u>,

(3.11) $\lim\limits_{\|x\| \to \infty} \|Fx\| = \infty$, <u>and</u>

(3.12) $\overline{\lim\limits_{\|x\| \to \infty}} \|Gx\| = M$, <u>for some</u> M <u>finite</u>.

<u>Then if</u> $P = F + G$, $0 \in P(X)$ <u>and thus</u> P <u>is a surjection</u>.

The proof is rather straightforward in that we must verify all the hypotheses of the Theorem III.3.1.

PROOF. First observe that $\hat{\delta}$ is a nondecreasing mapping, and set $g(x) = \hat{\delta}(\|x\|)$. Then $g : X \to (0,\infty)$ sends bounded sets to bounded sets and g is "nondecreasing" in the sense that if $\|x\| \leq \|y\|$, then $g(x) = \hat{\delta}(\|x\|) \leq \hat{\delta}(\|y\|) = g(y)$.

Now, to show that $P = F + G$ has closed graph, let $\{x_n\}$ be a sequence in X for which $x_n \to x_\infty$ and $Px_n \to y$; then $Fx_n + Gx_n \to y$. Since $\{x_n\}$ is Cauchy, for every $\varepsilon > 0$ there is an $N > 0$ so that $m,n \geq N$ implies $\|x_n - x_m\| < \varepsilon\mu^{-1}\hat{\delta}(0)$. Applying the Mean Value Theorem of McLeod to G gives

$$Gx_n - Gx_m \in \overline{co}\ \{G'(tx_n + (1-t)x_m)(x_n - x_m) : 0 < t < 1\},$$

from which it follows that

$$(3.13)\quad \|Gx_n - Gx_m\| \leq \|x_n - x_m\| \sup \{\|G'(tx_n + (1-t)x_m)\| \ : 0 < t < 1\}.$$

Simplifying (3.13) via (3.9) and (3.10) gives that

$$\begin{aligned}\|Gx_n - Gx_m\| &\leq \|x_n - x_m\| \sup \{\mu\hat{\delta}(\|tx_n + (1-t)x_m\|)^{-1} : 0 < t < 1\} \\ &\leq \|x_n - x_m\| \mu\hat{\delta}(0)^{-1} \\ &< \varepsilon.\end{aligned}$$

Therefore $\{Gx_n\}$ is a Cauchy sequence in Y, and since G has closed graph, we conclude that $Gx_n \to Gx_\infty$.

Hence $Fx_n = Px_n - Gx_n$ converges to $y - Gx_\infty$, and since F has closed graph, $Fx_\infty = y - Gx_\infty$. Therefore, $y = Fx_\infty + Gx_\infty = Px_\infty$, giving P closed graph.

We now verify the contractor inequalities (3.1) and (3.2) To this end let $q \in (\mu,1)$ be fixed. If $0 \in P(X)$, then choose $\bar{x} = x$, where $0 = Px$, and the inequalities hold for all t.

So without loss of generality, suppose $0 \notin P(X)$. Fix $x \in X$ and observe that $0 \neq Px$. Since $\hat{\delta}(\|x\|)^{-1}\|Px\|^{-1}(-Px) \in B(0;\hat{\delta}(\|x\|)^{-1})$, (3.8) implies there is a $w \in B(0;1)$ for which $F'(x)(w) = \hat{\delta}(\|x\|)^{-1} * \|Px\|^{-1}(-Px)$. Setting $h = \hat{\delta}(\|x\|)\|Px\| \cdot w$ and applying $F'(x)$ gives $F'(x)(h) = -Px$ and $\|h\| \leq \hat{\delta}(\|x\|)\|Px\|$.

By hypothesis both F and G are Gateaux differentiable, so we may choose $t \in (0,1]$ so small that

(3.14) $\| F(x + th) - F(x) - tF'(x)(h) \| \leq \frac{1}{2}(q - \mu)t \| Px \|$, and

(3.15) $\| G(x + th) - G(x) - tG'(x)(h) \| \leq \frac{1}{2}(q - \mu)t \| Px \|$.

Setting $\bar{x} = x + th$ and combining (3.14) and (3.15) via (3.9) and (3.10) gives that

$$\begin{aligned}
\| P\bar{x} - (1 - t)Px \| &\leq \| F\bar{x} - Fx - tF'(x)(h) \| + \| G\bar{x} - Gx - tG'(x)(h) \| + \\
&\qquad t \| F'(x)(h) - (-Px) \| + t \| G'(x)(h) \| \\
&\leq \tfrac{1}{2}(q - \mu)t \| Px \| + \tfrac{1}{2}(q - \mu)t \| Px \| + 0 + \\
&\qquad t \| G'(x) \| \cdot \| h \| \\
&\leq (q - \mu)t \| Px \| + t\mu\hat{\delta}(\| x \|)^{-1}\hat{\delta}(\| x \|) \| Px \| \\
&= qt \| Px \| ,
\end{aligned}$$

giving (3.1).

Inequality (3.2) is immediate since

$$\begin{aligned}
\| \bar{x} - x \| &= t \| h \| \\
&\leq t\hat{\delta}(\| x \|) \| Px \| \\
&= tg(x) \| Px \| .
\end{aligned}$$

To complete the proof, let $\| x \| \to \infty$. Then (3.11) and (3.12) imply that $\| Fx \| \to \infty$ and $\| Gx \| \to M$. Since $\| Px \| \geq \| Fx \| - \| Gx \|$ and since $\| Fx \| - \| Gx \| \to \infty$ as well, it follows that $\| Px \| \to \infty$. So $\lim_{\|x\| \to \infty} \| Px \| = \infty$, establishing (3.3).

Hence $0 \in P(X)$ and P is a surjection. ∎

If we replace (3.11) and (3.12) with two other coercive conditions, the result of Theorem III.3.2 is still valid. Since the proof of the following theorem follows that of Theorem III.3.2 verbatim up to the verification of hypothesis (3.3), we will omit the repetition.

We begin with a definition. An operator G is quasibounded if G maps bounded sets to bounded sets and if $|||G||| = \lim_{\|x\| \to \infty} \|x\|^{-1} \|Gx\| < \infty$. The quasinorm of G is denoted by $|||\cdot|||$. We now state our second result.

THEOREM III.3.3. *Let F and G be given as in Theorem III.3.2 and suppose hypotheses (3.8), (3.9) and (3.10) hold. In addition suppose*

(3.11') $\lim_{\|x\| \to \infty} \|x\|^{-1} \|Fx\| = \infty$, *and*

(3.12') G *is quasibounded.*

Then $0 \in P(X)$ and thus P is a surjection.

PROOF. We will only verify hypothesis (3.3) of Theorem III.3.1. To this end let $\|x\| \to \infty$, and let $\varepsilon > 0$ be given. Since G is quasibounded there exists a $R_\varepsilon > 0$ for which $\|Gx\| \leq (|||G||| + \varepsilon)\|x\|$ for all $x \in X$ with $\|x\| \geq R_\varepsilon$. Since we are interested in $\|x\|$ large, assume without loss of generality that $R_\varepsilon \geq 1$.

For $\|x\| \geq R_\varepsilon$, observe that

$$\|Px\| \geq \|Fx\| - \|Gx\| \geq \|Fx\| - (|||G||| + \varepsilon)\|x\|.$$

Dividing through by $\|x\|$ gives

$$\|x\|^{-1}\|Px\| \geq \|x\|^{-1}\|Fx\| - (|||G||| + \varepsilon).$$

Since the right-hand side diverges as $\|x\| \to \infty$, so must $\|x\|^{-1}\|Px\|$.

Since we can assume $\|x\| \geq 1$, it follows that $\|x\|^{-1} \leq 1$, and so $\|Px\| \geq \|x\|^{-1}\|Px\|$. Thus as $\|x\| \to \infty$, $\|Px\| \to \infty$ by comparison. Therefore (3.3) holds.

Thus Theorem III.3.1 applies to give that $0 \in P(X)$, making P a surjection. ■

4. COMPACT PERTURBATIONS

In this section we prove a compact perturbation result using the Leray-Schauder local topological degree for mappings of the form $I - T$ where I is the identity transformation from a Banach space X to itself and $T : X \to X$ is a compact transformation.

Let $\bar{D}$ denote the closure of a bounded open set in X, let $f : \bar{D} \to X$ be continuous and let $p \in D$ be such that $p \notin f(\bar{D} - D)$; that is, p is not in the image of the boundary ∂D of D under f. We will denote the Leray-Schauder degree by $d[f,\bar{D},p]$. The properties of the degree which we shall require are summarized below (cf. [11]):

PROPOSITION L-S 1. *The Leray-Schauder local degree satisfies the following properties of any local degree:*

(4.1) *the local degree of the identity mapping relative to* $\bar{D}$ *at any point* $x \in D$ *is* $+1$;

(4.2) *the local degree of the constant mapping with range value* y *relative to* $\bar{D}$ *at any point* $x \neq y$ *is* 0; *and*

(4.3) *if* f *and* g *defined on* $\bar{D}$ *are homotopic via a homotopy* H *with domain* $\bar{D} \times [0,1]$ *and if* $y \notin H(x,t)$ *for all* $(x,t) \in \partial D \times [0,1]$, *then the local degrees of* f *and* g *relative to* $\bar{D}$ *at* y *are equal.*

We now state our result.

THEOREM III.4.1. *Let* F *and* G *be continuous mappings from a Banach space* X *to itself and let* $I : X \to X$ *denote the identity operator. Suppose each of the following:*

(4.4) G <u>is a compact operator,</u>

(4.5) $|||\, G \,||| = \lim\limits_{\|x\| \to \infty} \|x\|^{-1} \|Gx\| < 1$, <u>and</u>

(4.6) $I - F$ <u>is a compact operator</u>.

<u>Suppose, in addition, there is a</u> $\bar{R} > 0$ <u>so that</u>

(4.7) $\|x\| \leqslant \bar{R}$ <u>implies</u> $\|(I - F + G)(x)\| < \bar{R}$.

<u>Then</u> $0 \in (I - G)(X)$ <u>and</u> $0 \in (F - G)(X)$; <u>hence</u> $I - G$ <u>and</u> $F - G$ <u>are surjections</u>.

In proving the result for $I - G$ we will need the following definition: let X be a Banach space, $D \subset X$ a closed set with nonempty interior, $P : D \to X$ and $w \in \text{int}(D)$. Then P satisfies the <u>Leray-Schauder condition</u> on the boundary ∂D of D if

$$Px - w = \lambda(x - w) \quad \text{for} \quad x \in \partial D \quad \text{implies} \quad \lambda \leqslant 1.$$

This condition is known to imply the existence of fixed points for a wide range of mappings, among them being compact operators.

We will also need several definitions and properties before we can proceed with the proof of Theorem III.4.1. We credit the following definitions and properties to J. Cronin [11].

Let H be a mapping from $[0,1]$ into the set of compact transformations of a subset of a normed linear space X into X; that is, corresponding to each $t \in [0,1]$ there is a compact transformation $H(t)$ of a subset E of X into X. The mapping H is a <u>homotopy of compact transformations on E</u> if: given $\varepsilon > 0$ and an arbitrary set $M \subset E$, there is a $\delta > 0$ such that $|t_1 - t_2| < \delta$ implies $\|H(t_1)(x) - H(t_2)(x)\| < \varepsilon$ for all $x \in M$.

We now further characterize the Leray-Schauder degree.

PROPOSITION L-S 2. *Let* D *be a bounded open set in* X, *and let* ∂D *denote the boundary of* D. *If* T *is a compact operator from* X *to* X *and if* $0 \notin (I - T)(\partial D)$, *then the Leray-Schauder degree* $d[I - T,\bar{D},0]$ *satisfies two properties.*

(4.8) *If* $d[I - T,\bar{D},0] \neq 0$, *then there is an* $x \in D$ *for which* $(I - T)(x) = 0$. (Note this property implies the existence of a fixed point for T.)

(4.9) *Invariance under homotopy* : *If* $H(t)$ *is a homotopy of compact transformations on* $\bar{D}$ *and if for all* $x \in \partial D$ *and for all* $t \in [0,1]$, $(I - H(t))(x) \neq 0$, *then for all* $t \in [0,1]$ *the degrees* $d[I - H(t),\bar{D},0]$ *exist and have the same value.*

For a discussion of local degree theory and for proofs of the above propositions, see [11]. We now prove Theorem III.4.1.

PROOF OF THEOREM III.4.1. We begin with $I - G$. First observe that $0 \in (I - G)(X)$ if and only if G has a fixed point. We will show that G satisfies the Leray-Schauder condition on ∂D where $D = \{x : \|x\| \leqslant R\}$ for an appropriate choice of $R > 0$.

By hypothesis $|||\, G \,||| < 1$, so for $\varepsilon = 1 - |||\, G \,||| > 0$ there exists an $R_\varepsilon > 0$ for which

(4.10) $\quad \|Gx\| \leqslant (|||\, G \,||| + \varepsilon) \|x\| \quad$ for all $\|x\| \geqslant R_\varepsilon$.

So choose $R \geqslant R_\varepsilon$ and let D be as defined above. Suppose for $x \in \partial D$ that $Gx - 0 = \lambda(x - 0)$. Since $\|x\| = R$, clearly (4.10) holds. So substituting $Gx = \lambda x$ into (4.10) gives

(4.11) $$|\lambda| \cdot \|x\| \leqslant (|||\, G \,||| + \varepsilon) \|x\|$$

Since $\|x\| > 0$, divide (4.11) through by $\|x\|$ to get

$$|\lambda| \leqslant |||\, G \,||| + \varepsilon = 1.$$

So clearly $\lambda \leqslant 1$ and G satisfies the Leray-Schauder condition.

Hence there exists a fixed point of G, giving the result.

Now let $D = B(0;\bar{R})$ be the open ball centered at 0 of radius $\bar{R}$. Define $H : [0,1] \to \{$compact transformations from $\bar{D}$ into $X\}$ by $H(t)(x) = t(I - F + G)(x)$. Then $H(t)$ defines a homotopy between the zero operator and $I - F + G$. We will verify that $H(t)$ satisfies the conditions of (4.9); this will suffice to give the desired conclusions. To see why, observe that $I - H(0) = I - 0 = I$, where 0 is the zero operator. Then by (4.1) it follows that $d[I - H(0),\bar{D},0] = +1$. So an application of (4.9) gives that

$$\begin{aligned} 0 \neq 1 &= d[I - H(0),\bar{D},0] \\ &= d[I - H(1),\bar{D},0] \\ &= d[I - (I - F + G),\bar{D},0] \\ &= d[F - G,\bar{D},0]. \end{aligned}$$

Hence (4.8) implies the existence of an $x \in D$ for which $(I - H(1))(x) = 0$. But this translates to $(F - G)(x) = 0$. Thus $0 \in (F - G)(X)$, making $F - G$ surjective as desired.

We must therefore verify each of the following:

(4.12) $H(t) : \bar{D} \to X$ is compact for every $t \in [0,1]$;

(4.13) H is a homotopy of compact transformations on $\bar{D}$; and

(4.14) $0 \notin (I - H(t))(\partial D)$ for each $t \in [0,1]$.

First observe that hypotheses (4.4) and (4.6) imply that $I - F + G$ is a compact operator. Certainly $I - F + G$ is continuous since its constituent parts are continuous. Now let $\{x_n\}$ be a bounded sequence in $\bar{D}$. Then for each $t \in [0,1]$, $\{H(t)(x_n)\} = \{t(x_n - Fx_n + Gx_n)\}$.

Since G is compact by (4.4), there exists a subsequence $\{x_n'\}$ for which $\{Gx_n'\}$ converges to some $y \in X$. Now by (4.6) $I - F$ is compact, and since $\{x_n'\}$ is necessarily a bounded sequence itself, there is a subsequence $\{(x_n')'\} = \{x_n''\}$ for which $(I - F)(x_n'') \to z \in X$. (It is worth notice that $Gx_n'' \to y$.) Hence $x_n'' - Fx_n'' + Gx_n'' \to z + y \in X$. Since $t \in [0,1]$, it follows that $t(I - F + G)(x_n'') \to t(z + y) \in X$ as well. Thus $\{H(t)(x_n)\}$ is precompact, establishing (4.12).

To see (4.13) let $\varepsilon > 0$ be given and let M be an arbitrary bounded subset of $\bar{D}$. Then there exists an R ($\leq \bar{R}$) such that $\|x\| \leq R$ for every $x \in M$. So by (4.7) $\|x - Fx + Gx\| < \bar{R}$ for every $x \in M$. Choose $\delta = \varepsilon(\bar{R})^{-1}$. Then $|t_1 - t_2| < \delta$ implies

$$\begin{aligned}\|H(t_1)(x) - H(t_2)(x)\| &= \|t_1(x - Fx + Gx) - t_2(x - Fx + Gx)\| \\ &= |t_1 - t_2| \cdot \|x - Fx + Gx\| \\ &< \varepsilon(\bar{R})^{-1}(\bar{R}) \\ &= \varepsilon.\end{aligned}$$

So by definition, (4.13) is valid.

Finally, let $x \in \partial D$; then $\|x\| = \bar{R}$. Since for $t \in [0,1]$, $(I - H(t))(x) = x - H(t)(x) = x - t(x - Fx + Gx)$, taking the norm of both ends and applying the triangle inequality gives

$$\begin{aligned}(4.15) \qquad \|(I - H(t))(x)\| &= \|x - t(x - Fx + Gx)\| \\ &\geq \|x\| - t\|x - Fx + Gx\| \\ &= \bar{R} - t\|x - Fx + Gx\|.\end{aligned}$$

By (4.7), $\|x - Fx + Gx\| < \bar{R}$, so (4.15) becomes

$$\|(I - H(t))(x)\| > \bar{R} - t\bar{R} = (1 - t)\bar{R}.$$

As t ranges from 0 to 1, $(1 - t)\bar{R}$ ranges from $\bar{R}$ to 0. So for every $t \in [0,1]$, $\|(I - H(t))(x)\| > 0$. Therefore, $(I - H(t))(x) \neq 0$ for all $x \in \partial D$. This establishes (4.14) and concludes the proof of Theorem III.4.1. ■

We end this section by remarking that the use of analytical methods of proof such as the application of modified Newton iterates used in the earlier sections failed to give any conclusions about our compact perturbation problem. Thus we resorted to topological techniques to derive our result.

CHAPTER IV

APPLICATIONS TO ACCRETIVE OPERATORS

In this chapter we prove a version of Theorem II.3.3 for operators of the accretive type. We begin with several definitions and facts about accretive operators that will prove useful in our result.

Let X be a Banach space and let $J : X \to 2^{X^*}$, where 2^{X^*} is the power set of the dual space of X, be defined by

$$J(x) = \{j \in X^* : \langle x,j \rangle = \|x\|^2 = \|j\|^2\}.$$

Here $\langle \cdot , \cdot \rangle$ denotes the "duality pairing"; that is, for $x \in X$ and $j \in X^*$, $\langle x,j \rangle = j(x)$. An operator $A : X \to X$ is <u>accretive</u> if for each $x,y \in X$, there is a $j \in J(x - y)$ for which $\langle Ax - Ay,j \rangle \geq 0$. Since the operators P for which $P - \omega I$ are accretive have important applications to a wide range of nonlinear problems arising both in ordinary and partial differential equations, there has been considerable interest in mapping theorems for operators of this type (see, for example, [3], [10], [26], [27], [37]). Our theorem for accretive operators belongs to this latter class of results.

THEOREM IV.1.1. *Let X be a Banach space, P be a continuous mapping from X to itself and $c : [0,\infty) \to (0,\infty)$ be a continuous nonincreasing function for which $\int^\infty c(u)du = \infty$. Suppose for each $x,y \in X$ that there is a $j \in J(x - y)$ for which*

(1.1) $$< Px - Py, j > \geq c(\max\{\|x\|, \|y\|\}) \|x - y\|^2.$$

Then P *is a homeomorphism of* X *onto* X.

An earlier version Theorem IV.1.1 appears in [3], Theorem 4.11, where it is assumed that P is locally Lipschitzian and where (1.1) holds locally. Theorem IV.1.1 is an immediate consequence of Theorem II.3.3 and the fact that P is an open mapping, the proof of which will follow.

THEOREM IV.1.2. *Let* X *be a Banach space,* P *be a continuous mapping from* X *to itself and* $c : [0,\infty) \to (0,\infty)$ *be a continuous nonincreasing function for which* $\int^{\infty} c(u)du = \infty$. *Suppose for each* $x,y \in X$ *there is a* $j \in J(x - y)$ *for which*

(1.2) $$< Px - Py, j > \geq c(\max\{\|x\|, \|y\|\}) \|x - y\|^2.$$

Then P *is an open mapping of* X *into* X.

Before proving this latter result, we prove Theorem IV.1.1 assuming Theorem IV.1.2.

PROOF OF THEOREM IV.1.1. To see that P is onto, fix $\varepsilon > 0$ and set $\bar{c}(u) = c(u + \varepsilon)$. Then by (1.1), if $\|x - y\| \leq \varepsilon$, then

(1.3) $$\|Px - Py\| \cdot \|x - y\| \geq < Px - Py, j >$$
$$\geq c(\max\{\|x\|, \|y\|\}) \|x - y\|^2.$$

Now observe that $\|x - y\| \leq \varepsilon$ implies $\|y\| \leq \|x\| + \varepsilon$. So if $\max\{\|x\|, \|y\|\} = \|x\|$, then $\|y\| \leq \|x\| \leq \|x\| + \varepsilon$, while if $\max\{\|x\|, \|y\|\} = \|y\|$, then $\|x\| \leq \|y\| \leq \|x\| + \varepsilon$. So in either case $c(\max\{\|x\|, \|y\|\}) \geq c(\|x\| + \varepsilon)$, since c is nonincreasing. Hence (1.3) becomes

$$\|Px - Py\| \cdot \|x - y\| \geq c(\|x\| + \varepsilon)\|x - y\|^2,$$

and so

$$\|Px - Py\| \geq \bar{c}(\|x\|)\|x - y\|.$$

Thus (3.9) of Theorem II.3.3 follows from (1.1) and P is a surjection.

Relation (1.1) clearly implies that P is injective since j is a linear functional and since c is nonzero for all u in the domain of c.

Finally, let U be open in X, the range space of $P^{-1} : X \to X$. Then since $(P^{-1})^{-1} = P$, it follows that $(P^{-1})^{-1}(U) = P(U)$ is open. Thus P^{-1} is continuous, completing the proof. ■

We now proceed to verify that P in Theorem IV.1.1 is an open mapping. The techniques involved are unrelated to those in the previous chapters; thus we present the material separately. The methods rely heavily on ideas of K. Deimling [12] and R.H. Martin [27].

Before beginning the actual proof we collect some more facts that will be of later use. If $A : X \to X$ is continuous and accretive, then the initial value problem

$$\dot{u} = -Au \qquad u(0) = x$$

has a unique solution $u(t,x)$ on $[0,\infty)$ for all $x \in X$ (see, for example, Theorem 6.1, p.247 of [27]).

If X is a Banach space we can define for $x,y \in X$

$$m_+[x,y] = \lim_{t \downarrow 0} t^{-1}\{\|x + ty\| - \|x\|\}$$

and

$$m_-[x,y] = \lim_{t \uparrow 0} t^{-1}\{\|x + ty\| - \|x\|\}$$

It is routine to verify that for each $x \in X$, $j \in J(x)$ and each $y \in X$

$$\|x\| \cdot m_- [x,y] \leq \langle y,j \rangle \leq \|x\| \cdot m_+ [x,y]$$

and

$$m_- [x,-y] = -m_+ [x,y].$$

We next present three lemmas which we will use in the proof of Theorem IV.1.2. For $x \in X$, take $u(t,x)$ to be the solution to the initial value problem above. We will examine the asymptotic behavior of the solution. The first two lemmas are extensions of well-known facts about accretive operators, and the proofs are based upon that of Theorem 6.1, pp. 247-248 of [27].

LEMMA IV.1.3. $\left\| \frac{d}{dt} u(t,x) \right\| \leq \|Ax\| \cdot \exp\{ -\int_0^t c(\|u(s,x)\|)ds\}$.

PROOF. Fix $h \geq 0$, and for $t \geq h$ define $p(t) = \|u(t,x) - u(t-h,x)\|$. Then for $t \geq h$, the upper negative derivate $p'_-(t)$ of $p(t)$ is given by

$$\begin{aligned} p'_-(t) &= \overline{\lim_{s \downarrow 0}}\; s^{-1}[p(t) - p(t-s)] \\ &= \overline{\lim_{s \downarrow 0}}\; -s^{-1}[p(t-s) - p(t)]. \end{aligned}$$

Since $p(t-s) - p(t) = \|u(t-s,x) - u(t-h-s,x)\| - \|u(t,x) - u(t-s,x)\|$, adding and subtracting $u(t,x)$, $u(t-h,x)$ and $s(Au(t,x) - Au(t-h,x))$ inside the first pair of norms and then applying the triangle inequality yields

$$\begin{aligned} p'_-(t) \leq\; & m_- [u(t,x) - u(t-h,x),\; -(Au(t,x) - Au(t-h,x))] \\ & + \overline{\lim_{s \downarrow 0}}\; \|s^{-1}[u(t-s,x) - u(t,x)] - Au(t,x)\| \\ & + \overline{\lim_{s \downarrow 0}}\; \|s^{-1}[u(t-h-s,x) - u(t-h,x)] - Au(t-h,x)\|. \end{aligned}$$

Since the last two terms vanish by the initial value problem ($\dot{u} = -Au$),

$$p'_-(t) \leq -m_+[u(t,x) - u(t-h,x),\ Au(t,x) - Au(t-h,x)]$$
$$= -\| u(t,x) - u(t-h,x) \|^{-1} \langle Au(t,x) - Au(t-h,x),\ j \rangle$$
$$\leq -\| u(t,x) - u(t-h,x) \|^{-1} c(\max\{ \|u(t,x)\|, \|u(t-h,x)\| \}) *$$
$$* \| u(t,x) - u(t-h,x) \|^2$$
$$= -c(\max\{\|u(t,x)\|, \|u(t-h,x)\|\}) \cdot p(t).$$

Multiplying this differential inequality through by

$$\exp\{\int_h^t c(\max\{\|u(q,x)\|, \|u(q-h,x)\|\})dq\}$$

gives

$$f'_-(t) = [p(t)\exp\{\int_h^t c(\max\{\|u(q,x)\|, \|u(q-h,x)\|\})dq\}]'_- \leq 0$$

for all $t \geq h$.

If we can show that $f(t) \leq f(0)$ for all $t \geq 0$, then the proof will be complete. To this end let $\varepsilon > 0$ be arbitrary and define $g(t) = f(t) - \varepsilon t$. Then $g'_-(t) = f'_-(t) - \varepsilon \leq -\varepsilon$ for all t. Thus

$$\overline{\lim_{s \downarrow 0}}\ -s^{-1}[g(t-s) - g(t)] \leq -\varepsilon.$$

Now fix $T > 0$ and set $S = \{t \in [0,T] : t \leq s \leq T$ implies $g(s) \geq g(T)\}$. Then it is straightforward to verify that

S is closed and

if $t \in S$ and $t > 0$, then there is a $\delta > 0$ for which $t - \delta \in S$.

Finally, let $\sigma = \inf S$. Since S is closed, $\sigma \in S$. Then $\sigma = 0$, for if $\sigma > 0$, there is a $\delta > 0$ for which $\sigma - \delta \in S$, a contradiction.

Therefore $g(t) \leq g(0)$ for every $t \geq 0$. Then $f(t) - \varepsilon t \leq f(0)$ as well, and since ε is arbitrary, $f(t) \leq f(0)$ as desired. Thus

$$p(t)\ \exp\{ \int_h^t c(\max\{ \|u(q,x)\|, \|u(q-h,x)\|\})dq\} \leq$$
$$p(0)\ \exp\{ \int_h^0 c(\max\{ \|u(q,x)\|, \|u(q-h,x)\|\})dq\}$$

Dividing by h and letting h decrease to zero gives the result. ∎

Since the proof of the following lemma is completely analogous to that of Lemma IV.1.3, we only sketch it.

LEMMA IV.1.4. $\|u(t,x) - u(t,y)\| \leq \|x - y\| \exp\{-\int_0^t c(\max\{\|u(s,x)\|, *$

$* \|u(s,y)\|\})ds$.

PROOF. Fix $h \geq 0$ and for $t \geq h$ define $p(t) = \|u(t,x) - u(t,y)\|$. Proceeding as in the previous lemma, it follows that

$$p'_-(t) = \overline{\lim_{s \downarrow 0}} -s^{-1} [p(t - s) - p(t)]$$

$$\leq m_- [u(t,x) - u(t,y), -(Au(t,x) - Au(t,y))]$$

$$\leq -c(\max\{\|u(t,x)\|, \|u(t,y)\|\})\cdot p(t).$$

Solving this differential inequality gives the result. ∎

LEMMA IV.1.5. <u>Let</u> $g : [0,\infty) \to [0,\infty)$ <u>be a continuous nonincreasing function for which</u> $\int^\infty g(s)ds = \infty$, <u>and let</u> $\delta > 0$. <u>Then</u>

(1.4) ϕ <u>solves the initial value problem</u>

$$(*) \quad \dot{\phi} = \delta - \int_0^\phi g(s)ds \qquad \phi(0) = 0$$

<u>if and only if</u> ϕ <u>solves the initial value problem</u>

$$(**) \quad \phi'(t) = \delta \exp\{-\int_0^t g(\phi(s))ds\} \qquad \phi(0) = 0;$$

(1.5) <u>if</u> $M(\delta) > 0$ <u>satisfies</u> $\delta = \int_0^{M(\delta)} g(s)ds$, <u>then all solutions to</u> (*) <u>satisfy</u> $\phi(t) \leq M(\delta)$; <u>and</u>

(1.6) <u>the problem</u> (*) <u>has a unique solution on</u> $[0,\infty)$.

PROOF. First observe that the Picard Existence Theorem implies that (*) has a unique solution ϕ since the function $F(y) = \delta - \int_0^y g(s)ds$

is Lipschitz continuous. To see this use the fact that g is continuous and nonincreasing to get for $x \leqslant y$ that

$$\begin{aligned} \| F(x) - F(y) \| &= \| \int_x^y g(s)ds \| \\ &\leqslant g(x) \| x - y \| \\ &\leqslant g(0) \| x - y \| . \end{aligned}$$

Now let $[0,\sigma)$ be the maximal interval of existence for ϕ. In view of (*) $\phi'(t)$ is differentiable, and so $\phi''(t) = -g(\phi(t))\phi'(t)$; that is, $\phi''(t) + g(\phi(t))\phi'(t) = 0$. Multiplying this equation through by $\exp\{ \int_0^t g(\phi(s))ds \}$ and integrating the resulting equation

$$\frac{d}{dt} [\exp\{ \int_0^t g(\phi(s))ds\}\phi'(t)] = 0$$

from 0 to t gives that ϕ solves (**).

For the reverse implication of (1.4) let $\tilde{\phi}$ solve (**). (Note there is at least one such solution since the solution of (*) solves (**) as well.) By (**), $\tilde{\phi}'(t)$ is differentiable and

$$\begin{aligned} \tilde{\phi}''(t) &= -\delta g(\tilde{\phi}(t))\exp\{ -\int_0^t g(\tilde{\phi}(s))ds\} \\ &= -g(\tilde{\phi}(t))\tilde{\phi}'(t). \end{aligned}$$

Integrating both sides from 0 to t gives that

$$\tilde{\phi}'(t) - \delta = \int_0^t -g(\tilde{\phi}(s))\tilde{\phi}'(s)ds,$$

and performing the change of variable $u = \tilde{\phi}(s)$ yields that $\tilde{\phi}$ solves (*).

Now, to show (1.5), let $M(\delta) > 0$ satisfy $\delta = \int_0^{M(\delta)} g(s)ds$ and let ϕ be a solution to (*). Then (**) implies $\phi'(t) \geqslant 0$ and (1.4) implies $\phi'(t) = \delta - \int_0^{\phi(t)} g(s)ds$. Thus, since g is nonincreasing,

$$0 \leqslant \phi'(t) = \delta - \int_0^{\phi(t)} g(s)ds = \int_0^{M(\delta)} g(s)ds - \int_0^{\phi(t)} g(s)ds = \int_{\phi(t)}^{M(\delta)} g(s)ds.$$

Therefore $\phi(t) \leqslant M(\delta)$.

Finally, (1.5) shows that ϕ is bounded on $[0,\sigma)$, and so ϕ extends to $[0,\infty)$, giving (1.6). ■

We now prove Theorem IV.1.2.

PROOF OF THEOREM IV.1.2. For $w \in X$ it suffices to show that if $\delta, \varepsilon > 0$ are sufficiently small, then $B(Pw;\varepsilon\delta) \subset P(B(w;\varepsilon))$. So for $\delta > 0$ take $\varepsilon(\delta)$ to be the number in $(0,\delta)$ such that if $\|w - x\| \leqslant \varepsilon(\delta)$, then $\|Pw - Px\| \leqslant (1 - \varepsilon)\delta$. Let $M(\delta)$ be the unique solution to the equation $\int_0^M c(s + \varepsilon(\delta) + \|w\|)ds = \delta$. Now choose δ so small that

(1.7) $$\delta - c(M(\delta) + \varepsilon(\delta) + \|w\|) \leqslant 0.$$

We will show for such a δ and $\varepsilon = \varepsilon(\delta)$, that $B(Pw;\varepsilon\delta) \subset P(B(w;\varepsilon))$.

So fix $z \in B(Pw;\varepsilon\delta)$ and set $Ax = Px - z$. Note that (1.3) holds with A replacing P since

$$\langle Ax - Ay, j \rangle = \langle Px - z - Py + z, j \rangle = \langle Px - Py, j \rangle.$$

Also if $\|x - w\| \leqslant \varepsilon$, then $\|Ax\| \leqslant \|Px - Pw\| + \|Pw - z\| \leqslant (1 - \varepsilon)\delta + \varepsilon\delta = \delta$.

Fix $x \in B(w;\varepsilon)$ and consider the system

$$\dot{u} = -Au \qquad u(0) = x;$$

let $u(t,x)$ be its solution. Set $g(t) = c(t + \varepsilon + \|w\|)$, a continuous nonincreasing function, and let ϕ be the solution to

$$(**) \quad \phi'(t) = \delta\exp\{ -\int_0^t g(\phi(s))ds\} \qquad \phi(0) = 0.$$

Set

$$\psi(t) = \delta\int_0^t \exp\{ -\int_0^s g(\|u(p,x) - x\|)dp\}ds$$

so that

$$\psi'(t) = \delta\exp\{ -\int_0^t g(\|u(p,x) - x\|)dp\}.$$

By Lemma IV.1.3 and the initial condition $u(0,x) = x$,

$$\|u(t,x) - x\| = \|\int_0^t \frac{d}{ds}(u(s,x))ds\|$$

$$\leq \int_0^t \|\frac{d}{ds} u(s,x)\| \, ds$$

$$\leq \int_0^t \|Ax\| \exp\{-\int_0^s c(\|u(p,x)\|)dp\}ds$$

$$\leq \delta\int_0^t \exp\{-\int_0^s c(\|u(p,x)\|)dp\}ds.$$

(1.8)

By the triangle inequality, $\|u(p,x)\| \leq \|u(p,x) - x\| + \|x - w\| + \|w\|$ $\leq \|u(p,x) - x\| + \varepsilon + \|w\|$ and since c is nonincreasing, it follows that

$$-c(\|u(p,x)\|) \leq -c(\|u(p,x) - x\| + \varepsilon + \|w\|)$$
$$= -g(\|u(p,x) - x\|).$$

So in particular,

$$\|u(t,x) - x\| \leq \delta\int_0^t \exp\{-\int_0^s g(\|u(p,x) - x\|)dp\}ds$$
$$= \psi(t).$$

Since g is nonincreasing, applying g to this inequality gives

$$-g(\|u(t,x) - x\|) \leq -g(\psi(t)),$$

and thus

$$\psi'(t) \leq \delta\exp\{-\int_0^t g(\psi(p))dp\}.$$

Hence $\psi(t) \leq \phi(t)$. Thus, by transitivity,

(1.9) $$\|u(t,x) - x\| \leq \phi(t)$$

when $\|x - w\| \leq \varepsilon$.

We obtain slightly more information from the above argument if $x = w$. First redefine $\psi(t) = \varepsilon\delta\int_0^t \exp\{-\int_0^s g(\|u(p,w) - w\|)dp\}ds$ so that

$$\psi'(t) = \varepsilon\delta\exp\{-\int_0^t g(\|u(p,w) - w\|)dp\}.$$

Then by Lemma IV.1.3 and the fact that $\|Aw\| \leq \varepsilon\delta$, it follows that

$$\| u(t,w) - w\| = \| \int_0^t \frac{d}{ds} u(s,w)ds \|$$

$$\leq \varepsilon\delta\int_0^t \exp\{ -\int_0^s c(\| u(p,w)\|)dp \}ds.$$

By the triangle inequality,

$$\|u(p,w)\| \leq \|u(p,w) - w\| + \| w \| \leq \| u(p,w) - w\| + \varepsilon + \| w \|$$

and so

$$-c(\|u(p,w)\|) \leq -c(\|u(p,w) - w \| + \varepsilon + \|w \|)$$

$$= -g(\|u(p,w) - w \|).$$

Thus, $\|u(t,w) - w\| \leq \varepsilon\delta \int_0^t \exp\{ -\int_0^s g(\|u(p,w) - w \|)dp\}ds = \psi(t)$.

Applying g yields $-g(\|u(t,w) - w\|) \leq -g(\psi(t))$, and thus

$$\psi'(t) \leq \varepsilon\delta \exp\{ -\int_0^t g(\psi(p))dp\}.$$

Now let $\xi(t)$ be the solution to

(1.10) $$\xi'(t) = \varepsilon\delta\exp\{ -\int_0^t g(\xi(s))ds\}.$$

Then $\psi(t) \leq \xi(t)$, and thus $\|u(t,w) - w \| \leq \xi(t)$ as well.

Since $0 < \varepsilon \leq 1$ is small, $\varepsilon\xi(t) \leq \xi(t)$, so $\xi(t) \leq \varepsilon^{-1}\xi(t)$. Applying g gives that $-g(\xi(t)) \leq -g(\varepsilon^{-1}\xi(t))$. Thus (1.10) implies

$$\varepsilon^{-1}\xi'(t) = \delta\exp\{ -\int_0^t g(\xi(s))ds\}$$

$$\leq \delta\exp\{ -\int_0^t g(\varepsilon^{-1}\xi(s))ds\},$$

and so comparing the solutions ξ and ϕ gives that $\varepsilon^{-1}\xi(t) \leq \phi(t)$; that is, $\xi(t) \leq \varepsilon\phi(t)$. Hence

(1.11) $$\| u(t,w) - w \| \leq \psi(t) \leq \xi(t) \leq \varepsilon\phi(t),$$

and we have a better estimate than when $x \neq w$.

Hence, for $x,y \in B(w;\varepsilon)$, (1.8) and (1.9) together imply that

$$\|u(t,x)\| \leqslant \phi(t) + \varepsilon + \|w\| .$$

Note that the right-hand side is independent of the choice of x. Now apply c to get

$$\begin{aligned} -c(\|u(t,x)\|) &\leqslant -c(\phi(t) + \varepsilon + \|w\|) \\ &= -g(\phi(t)). \end{aligned}$$

Applying this observation and Lemma IV.1.4 gives

$$\begin{aligned} (1.12)\ \|u(t,x) - u(t,y)\| &\leqslant \|x - y\| \exp\{ -\int_0^t c(\max\{ \|u(s,x)\|, \|u(s,y)\|\})ds\} \\ &\leqslant \|x - y\| \exp\{ -\int_0^t g(\phi(s))ds\} \end{aligned}$$

for $x,y \in B(w;\varepsilon)$.

Now since the exponential factor is less than or equal to 1, and since $\|x - y\| \leqslant \varepsilon < 1$, it follows that $\{u(t,\cdot)\}$ is a semigroup of contraction mappings. Finally, this semigroup leaves $B(w;\varepsilon)$ invariant: for $x \in B(w;\varepsilon)$, it follows from (1.11) and (1.12) that

$$\begin{aligned} (1.13)\qquad \|u(t,x) - w\| &\leqslant \|u(t,x) - u(t,w)\| + \|u(t,w) - w\| \\ &\leqslant \varepsilon\exp\{ -\int_0^t g(\phi(s))ds\} + \varepsilon\phi(t) \\ &\leqslant \varepsilon\delta^{-1}\phi'(t) + \varepsilon\phi(t) \\ &= \varepsilon\delta^{-1}(\phi'(t) + \delta\phi(t)). \end{aligned}$$

If $f(t) = \phi'(t) + \delta\phi(t)$, then $f(0) = \delta$ by Lemma IV.1.5, and

$$\begin{aligned} f'(t) &= \phi''(t) + \delta\phi'(t) \\ &= -g(\phi(t))\phi'(t) + \delta\phi'(t) \\ &= \phi'(t)\,[\delta - c(\phi(t) + \varepsilon + \|w\|)] \\ &\leqslant \phi'(t)\,[\delta - c(M(\delta) + \varepsilon + \|w\|)]. \end{aligned}$$

By (1.7) the last quantity is nonpositive, and so $f'(t) \leqslant 0$ for $t \geqslant 0$; thus $f(t) \leqslant \delta$ for $t \geqslant 0$. Hence (1.13) becomes

$$\|u(t,x) - w\| \leqslant \varepsilon \qquad \text{if } \|x - w\| \leqslant \varepsilon .$$

Hence the Contraction Mapping Theorem of Banach-Cacciopoli implies the existence of a unique fixed point $x_t \in B(w;\varepsilon)$ for each $u(t,\cdot)$ with $t \geq 0$. Indeed, $x_s = x_t$ for every s and t. To see this, let $u(t,x_t) = x_t$ and let $s \geq 0$. Then the semigroup property implies

$$u(s,x_t) = u(s,u(t,x_t)) = u(s+t,x_t) = u(t,u(s,x_t)).$$

Thus $u(s,x_t)$ is a fixed point of $u(t,\cdot)$, and by uniqueness it follows that $u(s,x_t) = x_t$. But, x_s is the unique fixed point of $u(s,\cdot)$, and so $x_s = x_t$ by uniqueness. So call this unique fixed point $\bar{x}$.

Since $\dot{u} = -Au$, it follows that $\frac{d}{dt} u(t,\bar{x}) = -Au(t,\bar{x}) = -A\bar{x}$. Then since $u(t,\bar{x})$ is constant, $-A\bar{x} = 0$. Thus $P\bar{x} - z = 0$ by definition of A. Since $z \in B(Pw;\varepsilon\delta)$ was chosen arbitrarily, we have showed that $B(Pw;\varepsilon\delta) \subset P(B(w;\varepsilon))$, establishing that P is an open mapping. ■

Consequently, the proof of Theorem IV.1.1 is complete.

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